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Dissertation

Critically frustrated signed graphs

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Abstract

Many theories on ordinary graphs have been extended to signed graphs in the last years. In most of them, balanced parts behave similarly to the unsigned case, while the unbalanced parts are the main source of the differences with respect to the unsigned case. Therefore, understanding the source of unbalance of a signed graph may be a key step to get a better understanding.

In the first part of this work, this is done by studying criticality with respect to the frustration index. The critically k-frustrated signed graphs are characterized and their structural properties are studied. We define and characterize a special family of critically k-frustrated signed graphs where each pair of negative circuits share edges. Moreover, we prove that the number of non-decomposable and irreducible critically k-frustrated signed graphs, for $k \leq 3$, is finite.

In the second part, we generalize old approaches of vertex-coloring of signed graphs, where the strong influence of the unbalance parts on the coloring can be seen, and we give a result relating frustration index and our definition of chromatic number.

Zusammenfassung

Viele Theorien über gewöhnliche Graphen wurden in den letzten Jahren auf signierte Graphen erweitert. In den meisten dieser Theorien verhalten sich balancierte Teile ähnlich wie im nicht signierten Fall, während die unbalancierten Teile die Hauptursache für die Unterschiede im Vergleich zum nicht signierten Fall sind. Daher kann das Verständnis der Quelle der Unbalanciertheit eines signierten Graphen ein wichtiger Schritt sein, um ein besseres Verständnis zu erlangen.

Im ersten Teil dieser Arbeit wird dies durch die Untersuchung der Kritikalität in Bezug auf den Frustrationsindex erreicht. Die kritisch k-frustrierten signierten Graphen werden charakterisiert und ihre strukturellen Eigenschaften

werden untersucht. Wir definieren und charakterisieren eine spezielle Familie von kritisch k-frustrierten signierten Graphen, bei denen jedes Paar negativer Knoten gemeinsame Kanten hat. Außerdem beweisen wir, dass die Anzahl der nicht zerlegbaren und irreduziblen kritisch k-frustrierten signierten Graphen für $k \leq 3$ endlich ist.

Im zweiten Teil verallgemeinern wir alte Ansätze der Knotenfärbung signierter Graphen, wobei der starke Einfluss des unbalancierten Teils auf die Färbung deutlich wird. Weiterhin geben wir ein Ergebnis an, das den Frustrationsindex und unsere Definition der chromatischen Zahl in Beziehung setzt.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

Graphs are the most natural way to describe networks, that is sets of relations and interactions among given objects or living beings. Basically, actors are represented by vertices and the presence of a relation between two actors is represented by an edge between the two corresponding vertices. The analysis of these graphs is a key point for the understanding of mechanisms acting across networks, for their optimization, and for a wide range of applications in different fields (see [1, 8, 15]). Of course relationships can be of various kinds. In order to cope with this problem, properties like weights, directions etc. can be added to the edges.

Of particular relevance is also the quality of a relation: Being an enemy of someone can neither be considered the same as being a friend nor as having no relationship.

In [17] Heider developed a psychological theory of social balance starting with the observation of small networks. These networks consisted of three actors, say A, B, and C, and the possible friendship/enmity relations among them. Heider noticed that if A and B are enemies, but C is friends with both A and B, some changes in the network may be expected, since it is difficult to be friends with two persons who do not like each other. Hence, it is possible

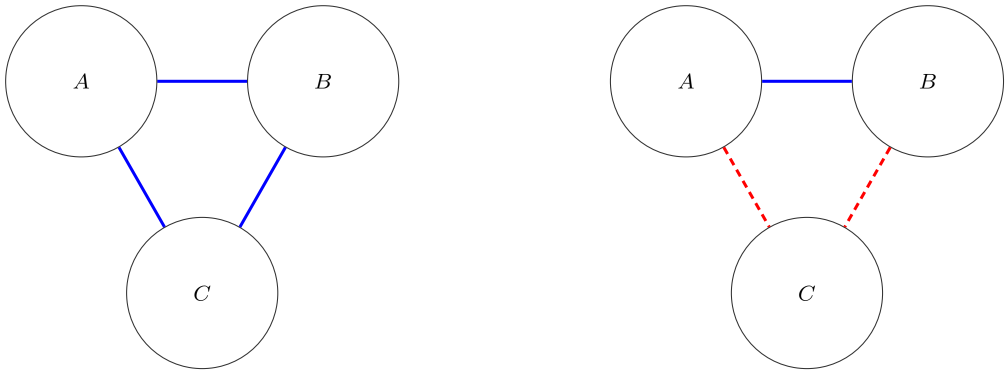


Figure 1.1: The two possible balanced triangles, where the dotted edges are the negative edges

that either C will become enemy of one of his friends, or that A and B will become friends. Similarly, if A, B, and C are enemies, it may be convenient that two of them become friends since they share the same enemy. These two situations are said to be unbalanced because of their instability. In contrast, if A and B are friends and C is the enemy of both, there is no reason for them to change the network -disregarding external factors. This is also the case if all three are friends. These two configurations are said to be balanced.

Heider's theory [17] states that networks tend towards balance, that is to networks where the rules "the enemy of my enemy is my friend" and "the friend of my friend is my friend" hold. This concept can easily be defined with the usage of graphs and can be extended to general networks. Therefore, motivated by the development of a formal basis for the psychological theory of social balance of Heider ([17]), from 1953, Cartwright and Harary developed the concept of signed graphs [9, 16].

A *signed graph* is a graph whose edges are either positive or negative. In particular, the positive and negative edges can be used to represent friendship and enmity, respectively. In this way, the networks from [17] can be easily represented (see Figure 1.1, 1.2). Note that, in this, work, we always represent positive edges with solid lines, and negative edges with dashed lines.

Harary observed that balanced circuits correspond to circuits having an

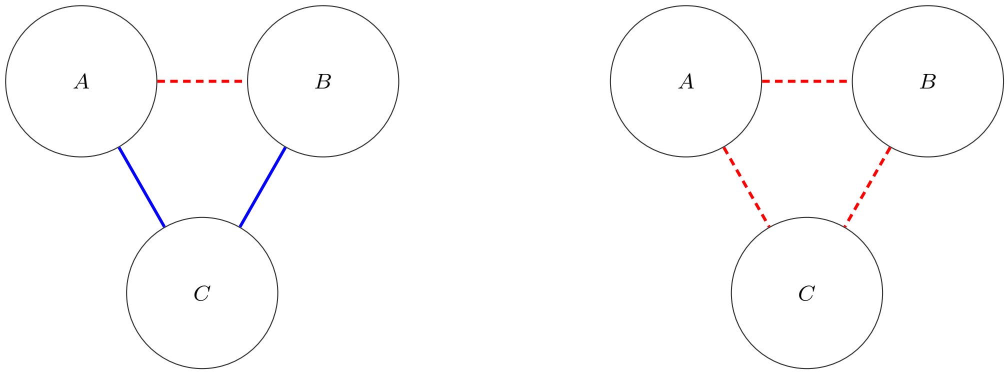


Figure 1.2: The two possible unbalanced triangles, where the dotted edges are the negative edges

even number of negative edges, as we can also see in 1.1. He defined a circuit to be *positive* if it has an even number of negative edges, and *negative* otherwise. In general, a signed graph is said to be *balanced* if all of its circuits are positive.

Balance is not a common state for a network, so it is interesting to understand how far a signed graph is from being balanced, or how "unbalanced" the signed graph is. One of the most common parameters to answer this question is the *frustration index*. Computing the frustration index of a signed graph is a NP-complete problem [1]. Furthermore, this parameter is neither strongly related to the number of negative edges of the graph nor to its size. Hence, to deal with this problem we focus on the source of frustration in a signed graph, that is we try to define which structures have the strongest impact on the frustration index of a signed graph and which have no relevance. For that, we define criticality with respect to the frustration index: A signed graph is *critically frustrated* if, for each edge e of the signed graph, the removal of e decreases the frustration index.

In the first part of this thesis, we focus on the study of critically frustrated signed graphs. In the next sections of this chapter we give basic definitions and results. In Chapter 2 we characterize such signed graphs and give some examples. Here, we also define two fundamental operations, i.e. decomposition and subdivision, on critically frustrated signed graphs. We show that to study critically frustrated signed graphs it is enough to focus on signed graphs which are not the result of a subdivision. Therefore, for each positive integer k we define the family $\mathcal{L}(k)$ to be the family of the critically k-frustrated signed graphs having no subdivision, and $\mathcal{L} = \bigcup_{k\geq 1} \mathcal{L}(k)$. Of particular interest is also the family of critically k-frustrated signed graphs in $\mathcal{L}(k)$ which are non-decomposable. We denote such a family with $\mathcal{L}^*(k)$, with $\mathcal{L}^* = \bigcup_{k\geq 1} \mathcal{L}^*(k)$. We then describe $\mathcal{L}(k)$ and $\mathcal{L}^*(k)$ for $k \in \{1, 2\}$. The study of k-critical signed graphs becomes more complicated for $k \geq 3$. We give a first hint to extend such studies by analyzing the family of non-decomposable critically 3-frustrated signed planar graphs, which turns out to be significantly larger than the previously shown families.

In Chapter 3 we give general results for signed graphs in $\mathcal{L}$ and in $\mathcal{L}^*$. In particular, we first prove that, despite $\mathcal{L}(k)$ being small for $k \in \{1, 2\}$, for each $k \geq 3$ it contains an infinite amount of elements. Nevertheless, the infinitely large families which we provide are not contained in $\mathcal{L}^*(k)$. We later provide a construction to build non-decomposable critical signed graphs from other critical signed graphs with smaller frustration index. We also give some structural results for critical signed graphs, with particular focus on signed graphs having no $-K_5$-minor.

In Chapter 4 we use some of the previously given results to prove that $\mathcal{L}^*(3)$ has a finite number of elements.

Lastly, in Chapter 5 we focus on a special subfamily of $\mathcal{L}^*$, i.e. critically frustrated signed graphs where each critically frustrated subgraph is non-decomposable. We denote this family with $\mathcal{S}^*$. We characterize $\mathcal{S}^*$, and later we build critically k-frustrated signed graphs belonging to this family for each value of k. We also entirely describe the elements in $\mathcal{S}^*$ having frustration index 3.

This part is fundamental to have a better understanding of signed graphs as balanced signed graphs behave similar to unsigned graphs, and therefore we focus on the structures which create the main differences between the unsigned and the signed cases. This also becomes helpful in the second part of this work where we study coloring of signed graphs. Coloring is a wide field of research on unsigned graphs. Hence, extending it to signed graphs is a natural step. Surprisingly, this step is not as smooth as one may expect: While in the unsigned case the set of colors chosen has no influence on the coloring, in the signed case it plays a crucial role. In Chapter 6 we suggest a new solution to this problem by defining new possible color sets. We use this approach to extend previous results of unsigned cases and we prove some new properties provided by this choice of the color sets. In Chapter 7 we conclude this work by suggesting some open questions and conjectures.

Some of the results in this work have already been published, in particular:

- The results in Chapter 2, except for Section 2.4 and the results in Chapter 5, except for Section 5.3, are published in

  [6] C. Cappello and E. Steffen. Frustration-critical signed graphs. *Discrete Applied Mathematics*, 322:183-193, 2022.

- The results in Section 2.4, Chapter 3 except for Section 3.3, and Section 5.3 are published in

  [5] C.Cappello, R. Naserasr, E. Steffen, and Z. Wang. Critically 3-frustrated signed graphs. *arXiv e-print*, arXiv:2304.10243, 2023.

- The results in Chapter 6, except for the Subsection 6.1.3 are published in

  [7] C. Cappello and E. Steffen. Symmetric Set Coloring of Signed Graphs. *Annals of Combinatorics*, pages 1-17, 2022.

1.2 Signed graphs and balance

Given a graph G, we denote with $V(G)$ the set of vertices of G, and with $E(G)$ the set of edges of G. We also allow graphs to have multiedges and loops. There are two formal definitions for signed graphs, depending on how negative edges are defined.

Definition 1.2.1. *A signed graph* (G, Σ) *is a graph* G *together with a set of negative edges* Σ*, called the signature of* G*.*

In this case, $e \in E(G)$ is *positive* if $e \notin \Sigma$.

Similarly, a signed graph can be defined as follows.

Definition 1.2.2. *A signed graph* (G, σ) *is a graph* G *together with a map* $\sigma : E(G) \to \{+1, -1\}$*, called the signature of* G*.*

In the latter definition, an edge $e \in E(G)$ is *positive* if $\sigma(e) = +1$, and *negative* otherwise. Clearly, the two definitions are equivalent, since $\Sigma = \sigma^{-1}(-1)$. In both cases, G is called the *underlying graph.*

Even if the two definitions of signature are equivalent, depending on the topic one or the other definition may be more convenient. In this work, we consider a signed graph (G, Σ) as a graph together with a set of edges, with exception for Section 1.4 and Chapter 6, where we work on coloring and the second definition makes notation easier to read.

Given a graph G and $U \subseteq V(G)$, we denote with $G[U]$ the subgraph of G induced by U. If H_1 and H_2 are subgraphs of G, we may denote with $H_1 \cup H_2$ (resp. $H_1 \cap H_2$) the graph having vertex set $V(H_1) \cup V(H_2)$ (resp. $V(H_1) \cap V(H_2)$) and edge set $E(H_1) \cup E(H_2)$ (resp. $E(H_1) \cap E(H_2)$).

A *thread* in G is a path whose internal vertices are all of degree 2 in G. If the length of the path is k, then we refer to it as a k-thread. A *theta-graph* is a graph consisting of two vertices which are connected by three internally vertex-disjoint paths. Given a graph G and a subgraph H of G, we denote with $G - H$ the subgraph induced by the vertices in G but not in H. A *circuit* is a connected 2-regular graph. If H is a subgraph of (G, Σ), then we may denote with (H, Σ) the signed graph $(H, \Sigma \cap E(H))$. The *degree* of a vertex $v \in V(G)$ is the number of edges incident with v, where loops are counted twice. It is denoted as $d_G(v)$.

An *edge-cut* of a graph G is a set of edges $\partial_G(U) = \{uv \in E(G) : u \in U, v \notin U\}$. The cardinality of $\partial_G(U)$ is denoted by $d_G(U)$. If $U = \{u\}$ and G has no loops, it holds that $d_G(\{u\})$ is equal to the degree of u. In this cases we may write $\partial_G(u)$ and $d_G(u)$ for $\partial_G(\{u\})$ and $d_G(\{u\})$, respectively. Furthermore, if (G, Σ) is a signed graph, we denote with $d^-_{(G,\Sigma)}(U) = |\partial_G(U) \cap \Sigma|$ and with $d^+_{(G,\Sigma)}(U) = d_G(U) - d^-_{(G,\Sigma)}(U)$. An edge-cut of a signed graph (G, Σ) is *equilibrated* under Σ if $d^-_{(G,\Sigma)}(U) = d^+_{(G,\Sigma)}(U)$. If there is no ambiguity we may omit the expression "under Σ" or the indices G and (G, Σ). The signed graph obtained from a graph H by replacing every edge with two edges, one negative and one positive, is denoted by $\pm H$ and it is called the *signed extension* of H.

Given a signed graph (G, Σ), we define a set B of edges to be *negative* if $|B \cap \Sigma|$ is odd, and positive otherwise. A signed graph (G, Σ) is said to be *balanced* if all of its circuits are positive. Otherwise it is called *unbalanced*.

Two signatures Σ and Γ on G are *equivalent* if there is an edge-cut $\partial(U)$ in G such that $\Gamma = \Sigma \Delta \partial(U)$, where Δ denotes the symmetric difference of two sets [16]. Hence, two signatures on G are equivalent if they have the same set of negative circuits [34]. If Σ and Γ are equivalent signatures on G, then we also say that Γ is a signature of (G, Σ) and it is obtained by switching at $\partial_G(U)$. As a consequence, a balanced signed graph (G, Σ) is switching equivalent to an all-positive graph $(G, \emptyset)$, that is the graph with no negative edges. If a signed graph (G, Σ) is switching equivalent to the all-negative graph $(G, E(G))$, then (G, Σ) is called *antibalanced*. Moreover, the signed graph $(G, E(G))$ is also denoted as $-G$.

Note that, if we consider a signed graph as a pair (G, σ), with $\sigma : E(G) \to \{+1, -1\}$, switching at a vertex v means to negate the sign of all edges incident with v, that is to define a new signature $\sigma' : E(G) \to \{+1, -1\}$ such that $\sigma'(e) = -\sigma(e)$ if $e = xv$, for a certain $x \in V(G)$, and $\sigma'(e) = \sigma(e)$ otherwise. In this case, if (G, σ) and (G, σ') are switching equivalent, we say that (G, σ) is given by switching (G, σ') at a set of vertices U. Furthermore, given a graph G we denote with $(G, +)$ (resp., with $(G, \text{-})$) the signature $+ : E(G) \to \{+1\}$ (resp., $\text{-} : E(G) \to \{-1\}$).

1.3 Frustration index and criticality

Balance is a specific state for a signed graph. Hence, it comes natural to look for a parameter which describes the distance of a signed graph from being balanced. One of the most popular parameters is the *frustration index.*

Definition 1.3.1. *The frustration index of a signed graph* (G, Σ)*, is defined as*

$$\ell(G, \Sigma) = \min\{|\Pi| : (G, \Pi) \text{ is switching equivalent to } (G, \Sigma)\}.$$

If $\ell(G, \Sigma) = k$, then (G, Σ) is said to be *k-frustrated.*

Another closely related approach to quantifying the lack of balance in a signed graph is based on the notion of *negative-circuit cover*: That is a set of edges that contains at least one edge of each negative circuit of (G, Σ). Given a signed graph, the fact that the order of a minimum negative-circuit cover and the frustration index are the same is a consequence of the following lemma for which we provide a simple proof.

Lemma 1.3.2. *There is a one-to-one correspondence between the set of minimal negative-circuit covers of a signed graph* (G, Σ) *and equivalent minimal signatures of* (G, Σ)*.*

Proof. First, note that any signature, in particular a minimal signature, is a negative-circuit cover. Let E' be a minimal negative-circuit cover. We claim that E' is a signature switching equivalent to Σ. To this end, observe that $(G - E', \Sigma)$ is balanced and thus, it can be changed to $(G - E', \emptyset)$ by switching at an edge-cut $\partial(X)$. Then, when switching (G, Σ) at $\partial(X)$, the signature of the resulting signed graph has to be E'. Otherwise, the new signature is a proper subset E'' of E'. Since E'' is also a negative-circuit cover, it contradicts the minimality of E'. This also implies that a negative-circuit cover provided by a minimal signature is minimal, as for otherwise, an included minimal negative-circuit cover would be a smaller signature. □

Therefore, in the following, we can think of the frustration index $\ell(G, \Sigma)$ as the minimum number of edges needed to remove in order to make the signed graph balanced.

Note that, by definition, switching has no impact on the frustration index, since it always changes the sign of an even number of edges in a circuit. It is easy to observe that a signed graph is balanced if and only if there exists a bipartition of its vertices into two sets A and B, such that $\Sigma = \partial(A)$ [16]. Hence, if we have an all-negative graph, the problem of computing the frustration index is equivalent to the Max-Cut Problem.

In this work, we investigate the true source of frustration in a since graph, since neither the number of negative edges nor the size of the graph are reliable indicators of the frustration index. As an example we can observe, on one side, that a large bipartite all-negative graph is always balanced, independently from the size of the graph. On the other side, a signed graph consisting of one vertex and k negative loops has frustration index k. To deal with this problem, we define *critically k-frustrated* signed graphs.

Definition 1.3.3. *A signed graph (G, Σ) is critically k-frustrated if $\ell(G, \Sigma) = k$ and, for each edge $e \in E(G)$, it holds that $\ell(G - e, \Sigma) = k - 1$.*

If it is clear from the context, we may use k-critical to refer to a critically k-frustrated signed graph. Given a signed graph (G, Σ), for each positive integer t, if $|\Sigma| = t$, we call Σ a *t-signature*. Clearly, it holds that $\ell(G, \Sigma) \leq |\Sigma|$. If it holds that $\ell(G, \Sigma) = |\Sigma|$, then we say that Σ is a *minimum signature*. The following Lemma is a fundamental tool.

Lemma 1.3.4. *Let (G, Σ) be a k-frustrated signed graph. If Γ is a set of k edges such that $(G - \Gamma, \Sigma)$ is balanced, then Γ is a signature of (G, Σ).*

Proof. Let $(G', \Sigma') = (G - \Gamma, \Sigma)$. Since (G', Σ') is balanced there is an edge-cut $\partial_{G'}(U)$ in G' such that $\partial_{G'}(U) = \Sigma'$. Since $|\Gamma| = k$, it follows that $(E(G) \setminus \partial_G(U)) \cap \Sigma = (E(G) \setminus \partial_G(U)) \cap \Gamma$. Thus, $e \in (\Sigma \setminus \partial_G(U)) \cup (\partial_G(U) \setminus \Sigma)$ if and only if $e \in \Gamma$ and therefore, $\partial_G(U) \Delta \Sigma = \Gamma$. $\square$

A trivial upper bound for the frustration index of a loopless signed graph

(G, Σ) is $\frac{|E(G)|}{2}$. As shown by the signed graph $\pm H$, this bound is sharp. Hence, given a k-signature of a k-frustrated signed graph, the following statement holds.

Lemma 1.3.5. *If Σ is a minimum signature of a k-frustrated signed graph (G, Σ), then $d^-_{(G,\Sigma)}(U) \leq d^+_{(G,\Sigma)}(U)$ for every $U \subseteq V(G)$. Furthermore, if G is n-edge-connected, then $G - \Sigma$ is $\lceil \frac{n}{2} \rceil$-edge-connected.*

Proof. If $d^+_{(G,\Sigma)}(U) < d^-_{(G,\Sigma)}(U)$, then $|\Sigma \Delta \partial_G(U)| < |\Sigma|$. Hence, at most half of the edges of an edge-cut are in Σ and the statements follow. $\square$

Edge-cuts are a fundamental part in the study of critical graphs. In Chapter 2 we provide a characterization of critical graphs by using equilibrated edge-cuts. This characterization is a key tool in most of the proofs related to the frustration index.

1.4 Coloring of signed graphs

It is not always easy to extend parameters from unsigned graphs to the signed case, since negative edges often have a strong impact on the problem. One of the most meaningful examples is represented by coloring. Recall that, in this section, in order to make the notation easier for the reader, we consider the signature to be a map $\sigma : E(G) \to \{+1, -1\}$.

A *coloring* c of a graph G is a map $c : V(G) \to S$, where S is a set of colors. A coloring is *proper* if for every edge $u = vw$ it holds that $c(u)$ is different from $c(v)$. If $|S| = k$, with $k \geq 1$, then c is called a *k-coloring*. Given a graph G, the *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G has a proper k-coloring. Note that loops cannot be colored. Therefore, in this section and in Chapter 6 we only consider loopless multigraphs. Furthermore, as we only consider proper colorings, we may omit the adjective "proper".

Coloring problems are a wide area of research in graph theory [13]. Hence, extending it to signed graphs is a natural step. Given a signed graph (G, σ), a *proper coloring* of (G, σ) is a map $c : V(G) \to S$ such that, for each edge

$e = uv$, $c(v)$ is different from $\sigma(e)c(u)$. If (G, σ) admits a proper coloring with elements from S, then (G, σ) is *S-colorable.* While the coloring-condition for positive edges remains unchanged with respect to the unsigned case, the condition on a negative edge $e = vw$ requires that $c(v) \neq -c(w)$. It implies that $-s \in S$ for each $s \in S$. Two different cases may occur: Either $s \in S$ is a *non-self-inverse* element, i.e. $s \neq -s$, or s is a *self-inverse* element, i.e. $s = -s$. Hence, while in the unsigned case the color set S has no influence on the colorings since all elements play the same role, in the signed case the choice of the color set influences the number of colors required. On one hand, non-self-inverse elements can be assigned to adjacent vertices if they are connected by negative edges, so one color can be assigned to all of the vertices in large antibalanced subgraphs. On the other hand, if $s \in S$, it is then required that $-s \in S$, that is non-self-inverse elements have to be taken in pairs.

The aim of the second part of this work is to define a coloring for signed graphs which does not depend on the number of allowed self-inverse elements. For that purpose, we first observe that a color set requires to be *symmetric.*

Definition 1.4.1. *A set S together with a sign "$-$" is a* symmetric set *if it satisfies the following conditions:*

1. *$s \in S$ if and only if $-s \in S$.*
2. *If $s = s'$, then $-s = -s'$.*
3. *$-(-s) = s$.*

Symmetric sets may contain both non-self-inverse and self-inverse elements. We denote as S_{2k}^t a symmetric set with t self-inverse elements, say $0_1, ..., 0_t$, and $2k$ non-self-inverse elements, say $\pm 1, ..., \pm k$. Clearly, $|S_{2k}^t| = t + 2k$.

A further natural requirement on signed graph coloring is that switching equivalent signed graphs should have the same coloring properties. We show that our symmetric sets fulfill this requirement.

Let (G, σ') be obtained from (G, σ) by switching at a vertex u. If (G, σ) admits a proper coloring c with elements from S_{2k}^t, then c' with $c'(u) = -c(u)$

and $c'(v) = c(v)$ for $v \neq u$ is a proper coloring of (G, σ') with elements from S^t_{2k}.

Proposition 1.4.2. *Let (G, σ) and (G, σ') be equivalent signed graphs. Then (G, σ) admits a proper S^t_{2k}-coloring if and only if (G, σ') admits a proper S^t_{2k}-coloring.*

Schweser and Stiebitz [28] used the term symmetric set for subsets $Z \subseteq \mathbb{Z}$ with the property that $Z = -Z$, where $-Z = \{-z : z \in Z\}$. In the case of finite sets this gives symmetric sets with t self-inverse elements for $t \in \{0, 1\}$. Examples for symmetric sets with more than one self-inverse element are subsets Z' of $\mathbb{Z}^k_{2n}$, with $Z' = -Z'$. Here the vectors whose entries are either 0 or n are self-inverse.

Self-inverse elements in a certain way nullifies the effect of the sign, so the following proposition trivially holds.

Proposition 1.4.3. *Every signed graph (G, σ) has a proper $S^{\chi(G)}_0$-coloring.*

A major issue of coloring with symmetric sets is that its cardinality and the number of its self-inverse elements have the same parity. This has some surprising consequences as it occur that the set of colors has more elements than the vertex set of the graph. This issue has been addressed in several ways (see e.g. [31]).

Zaslavsky [33, 35] first considered two different sets, $M_{2k} = \{\pm 1, \ldots, \pm k\}$ and $M_{2k+1} = \{0, \pm 1, \ldots, \pm k\}$. He worked on the chromatic polynomial by distinguishing two cases: The 0-free coloring with elements from M_{2k}, and the coloring with elements from M_{2k+1}, i.e. a coloring where 0 is allowed.

Based on this coloring, Máčajová, Raspaud, and Škoviera [23] introduced the signed chromatic number $\chi_{\pm}(G, \sigma)$ to be the smallest integer n for which (G, σ) admits a proper M_n-coloring.

Kang and Steffen [20] introduced the cyclic coloring of signed graphs using cyclic groups $\mathbb{Z}_n$ as the set of colors. The cyclic chromatic number, denoted by $\chi_{mod}(G, \sigma)$, is the smallest integer n such that (G, σ) admits a proper coloring with elements of $\mathbb{Z}_n$.

Interestingly, these approaches do not only provide different chromatic numbers for the same signed graphs but they even have different general bounds. For example, on one hand an antibalanced triangle is colorable with M_2 by assigning 1 to all its vertices but it is not $\mathbb{Z}_2$-colorable. On the other hand, the signed extension of the complete graph on 4 vertices, $\pm K_4$, has a $\mathbb{Z}_6$-coloring but no M_6-coloring. Note that here, in both types of coloring, the set of colors contains more elements than the vertex set of the graph. The reason for this is given by the different number of self-inverse elements allowed: In the coloring defined by Zaslavsky we can have either 0 or 1 self-inverse element, while the cyclic coloring uses 1 or 2 self-inverse elements.

Definition 1.4.4. *Let (G, σ) be a signed graph and $t \in \{0, \dots \chi(G)\}$ be fixed. The* symset t-chromatic number *(or t-chromatic number for short) of (G, σ) is the minimum $\lambda_t = t + 2k$ for which (G, σ) admits an S^t_{2k}-coloring, and it is denoted by $\chi^t_{sym}(G, \sigma)$.*

By Proposition 1.4.2, if (G, σ) and (G, σ') are equivalent, then $\chi^t_{sym}(G, \sigma) = \chi^t_{sym}(G, \sigma')$. If a graph has t-chromatic number $\chi^t_{sym}(G, \sigma) = \lambda_t$, we say that (G, σ) is *λ_t-chromatic.*

Clearly, Zaslavsky's coloring is equivalent to coloring with elements from S^0_{2k} or S^1_{2k} and cyclic coloring is equivalent to coloring with elements from S^1_{2k} and S^2_{2k}. Consequently, the chromatic numbers studied in [20, 23] can be defined as the minimum between two specific t-chromatic numbers of signed graphs.

Proposition 1.4.5. *Given a signed graph (G, σ), it holds*
$\chi_{\pm}(G, \sigma) = \min\{\chi^0_{sym}(G, \sigma), \chi^1_{sym}(G, \sigma)\}$ *and*
$\chi_{mod}(G, \sigma) = \min\{\chi^1_{sym}(G, \sigma), \chi^2_{sym}(G, \sigma)\}$.

Thus, for a signed graph (G, σ), $\chi_{\pm}(G, \sigma)$ and $\chi_{mod}(G, \sigma)$ depend on the number of allowed self-inverse colors. In general, fixing the number of self-inverse elements (instead of choosing from different cases) causes some issues due to parity. For instance, the all-positive complete graph on $2n$ vertices has a proper $S_{2j}^{2(n-j)}$-coloring for each $j \in \{0, \dots, n\}$, and in particular $\chi_{sym}^{2j}(K_{2n}, +) = 2n$. However, for each $j \in \{0, \dots, n\}$ it also holds $\chi_{sym}^{2j+1}(K_{2n}, +) = 2n + 1$.

Since $S_0^{\chi(G)} \subset S_{2k}^t$ for all $t \geq \chi(G)$, it follows by Proposition 1.4.3 that every signed graph (G, σ) has a proper S_{2k}^t-colorings for all $t \geq \chi(G)$. For this reason, in general we assume $t \leq \chi(G)$.

If an antibalanced subgraph of (G, σ) which is induced by a non-self-inverse color is bipartite, then the non-self-inverse color can be replaced by two self-inverse colors. In that case, (G, σ) has an S_{2k}^t- and an $S_{2(k-1)}^{t+2}$-coloring.

Let $N = \min\{\chi_{sym}^t(G, \sigma) \colon 0 \leq t \leq \chi(G)\}$. The above examples show that N is not necessarily associated with a unique symmetric set S for which (G, σ) admits a minimum proper S-coloring. To overcome this problem we define the *symset chromatic number* of a signed graph (G, σ) in the following way.

Definition 1.4.6. *Let (G, σ) be a signed graph, then the symset-chromatic number of (G, σ) is defined as*

$$\chi_{sym}(G, \sigma) = \max_t \min\{\chi_{sym}^t(G, \sigma) \colon 0 \leq t \leq \chi(G)\}.$$

Furthermore, we say that an S_{2k}^t-coloring is minimum if $\chi_{sym}(G, \sigma) = t+2k$. By Proposition 1.4.2 it follows that equivalent signed graphs have the same symset chromatic number.

Proposition 1.4.7. *For every signed graph (G, σ) it holds $\chi_{sym}(G, \sigma) \leq \chi(G)$. Furthermore, if (G, σ) and (G, σ') are equivalent, then $\chi_{sym}(G, \sigma) = \chi_{sym}(G, \sigma')$. In particular, if (G, σ) is equivalent to $(G, +)$, then $\chi_{sym}(G, \sigma) = \chi(G)$.*

Although the symset-chromatic number is the main aim of our research, the study of the symset-t-chromatic number is a fundamental step to obtain a better understanding of this coloring. Hence, we first prove some results regarding the symset-t-chromatic number. These allow us to study the behavior of the symset-chromatic number.

Chapter 2

Critically frustrated signed graphs

In this chapter we provide some basic results to work with critical signed graphs. We give some examples and entirely describe the family of 1- and 2-critical signed graphs.

The chapter is organized as follows: In Section 2.1 we characterize critically frustrated signed graphs and give some examples of such signed graphs. In Section 2.2 we define decomposition and subdivision for critical signed graphs. These tools are fundamental to deal with this kind of criticality: Subdividing a k-critical signed graph provides a new k-critical signed graph. Hence, each k-critical signed graph generates infinitely many k-critical signed graphs which are, from our point of view, all the same. In other words, this family shows in some way the same behavior as the original signed graph, that is the signed graph without subdivisions. Thus, we can focus on signed graphs having no subdivision. Similarly, if we subdivide some signed graphs and we combine them, depending on the subdivision we may obtain infinitely many signed graphs which are, again, essentially the same from our point of view. In Section 2.3 we describe the families of 1- and 2-critical signed graphs. Starting from $k \geq 3$ the description of the families of k-critical signed graphs becomes more complex. To see that, in Section 2.4 we focus our work on the 3-critical planar signed graphs and we prove that, excluding decomposable signed graphs and signed graphs resulting from subdivision, there exist ten signed graphs in this family - which is significantly more than in the families with smaller frustration index.

2.1 Characterization and families of critical signed graphs

The results of this section have been published in [6].

Let (G, Σ) be a signed graph and k a positive integer. (G, Σ) is said to be *critically k-frustrated* (or k-critical) if $\ell(G, \Sigma) = k$ and, for each edge $e \in E(G)$, it holds $\ell(G - e, \Sigma) = k - 1$. The graph with one vertex and k loops is denoted by kC_1. Clearly, $-kC_1$ is k-critical, and every subgraph with at least one edge is j-critical, for $1 \leq j \leq k$. We begin with an easy but important result.

Proposition 2.1.1. *Let $k \geq 1$. A k-frustrated signed graph contains an m-critical subgraph for every $m \in \{1, \dots, k\}$.*

Proof. By Lemma 1.3.4 we can assume that Σ is a k-signature. Let $E \subseteq \Sigma$ and $|E| = k - m$. Then $\ell(G - E, \Sigma) \leq m$. Furthermore, $\ell(G - E, \Sigma) < m$ implies $\ell(G, \Sigma) < k$, a contradiction. Hence, $(G - E, \Sigma)$ is m-frustrated. In order to obtain an m-critical subgraph of (G, Σ) remove step-wise those edges whose removal does not decrease the frustration index. $\square$

Next we give a characterization of critical signed graphs, which is a fundamental tool in the entire work.

Theorem 2.1.2. *Let $k \geq 1$ be an integer and (G, Σ) be a k-frustrated signed graph. The following statements are equivalent.*

1. *(G, Σ) is k-critical.*

2. *Every edge is contained in a k-signature of (G, Σ).*

3. *If* Γ *is a* k*-signature of* (G,Σ)*, then every positive edge is contained in an equilibrated edge-cut of* (G,Γ)*.*

Proof. ($1 \to 2$) Let $e \in E(G)$, $G' = G - e$ and $\Sigma' = \Sigma \setminus \{e\}$ (it might be that $e \notin \Sigma$). Then $\ell(G',\Sigma') = k-1$. Let Γ be a set of $k-1$ edges of G' such that $(G'-\Gamma,\Sigma)$ is balanced. By Lemma 1.3.4, Γ is a signature of (G',Σ') and therefore, $\Gamma \cup \{e\}$ is a signature of (G,Σ).

($2 \to 3$) Let Γ be a k-signature of (G,Σ) and $e \notin \Gamma$. There is a k-signature Γ' of (G,Σ) which contains e. Thus, there is an edge-cut $\partial_G(U)$ such that $\partial_G(U)\Delta\Gamma = \Gamma'$. By Lemma 1.3.5, $d^-_{(G,\Gamma)}(U) \leq d^+_{(G,\Gamma)}(U)$. But if $d^-_{(G,\Gamma)}(U) < d^+_{(G,\Gamma)}(U)$, then Γ' is a t-signature with $t > k$, a contradiction. Hence, $\partial_G(U)$ is equilibrated under Γ.

($3 \to 1$) (G,Σ) is k-frustrated. Thus, it has a k-signature Γ, by Lemma 1.3.4. There is a k-signature Γ_e with $e \in \Gamma_e$ for every $e \in E(G)$, since every positive edge is contained in an equilibrated edge-cut. Thus, $\ell(G-e,\Sigma) < k$ for every $e \in E(G)$. □

In the following, Theorem 2.1.2 will be used as a basic tool (and many times) in most of the proofs. Hence, in the later chapters we may do not always refer to it when applying the theorem.

Let $\lambda(G)$ denote the edge-connectivity of a graph G. The following corollary is an immediate consequence of Theorem 2.1.2.

Corollary 2.1.3. *Let* $k \geq 1$ *and* $G \neq C_1$*. If* (G,Σ) *is a* k*-critical signed graph, then* $2 \leq \lambda(G) \leq 2k$*.*

Next, we show that for each positive integer k, there exist critically k-frustrated signed graphs. In the following we provide some examples of families of critical signed graphs.

Proposition 2.1.4. *Let* G *be a plane triangulation. If* G *has* $n \geq 3$ *vertices, then* $-G$ *is* $(n-2)$*-critical. Furthermore,* $-G$ *has three pairwise disjoint* $(n-2)$*-signatures.*

Proof. $-G$ has $2n-4$ triangles and $3n-6$ edges. Since every edge is in at most two triangles, $\ell(-G) \geq n-2$. Let G^* be the plane dual of G. G^* is cubic and bridgeless. Hence, it is 3-edge colorable by the 4-Color Theorem. Each color class C_i^* contains $n-2$ edges, and $G^* - C_i^*$ is Eulerian. Let C_i be the set of edges of G which corresponds to C_i^* in G^*. Thus, $G - C_i$ is bipartite. By Lemma 1.3.4, C_i is a signature and thus, $\ell(-G) = n-2$. The Kempe chains in G^*, which are induced by two color classes C_{i+1}^* and C_{i+2}^* (indices in $\mathbb{Z}_3$) correspond to equilibrated edge-cuts in $-G$, with signature C_i. Thus, $-G$ has three pairwise disjoint $(n-2)$-signatures. Therefore, $-G$ is $(n-2)$-critical. $\square$

An *odd wheel* W_{2k+1} ($k \geq 1$) is the graph which consists of an odd circuit C_{2k+1} and a vertex v which is connected to every vertex of C_{2k+1}.

Proposition 2.1.5. *The antibalanced odd wheel $-W_{2k+1}$ is $(k+1)$-critical.*

Proof. Let $v_0, \ldots, v_{2k}$ be the vertices of C_{2k+1} in this order. For $i \in \mathbb{Z}_{2k+1}$, if k is even, then let $V_i = \{v_i, v_{i+2}, \ldots, v_{i+k}, v_{i+k+1}, v_{i+k+3}, \ldots, v_{i-2}\}$, and if k is odd, then let $V_i = \{v_{i+1}, v_{i+3}, \ldots, v_{i+k}, v_{i+k+1}, v_{i+k+3}, \ldots, v_{i-1}\}$. For each edge $e \in E(W_{2k+1})$ there exists i such that $e \in \partial(V_i)$. Furthermore $|\partial(V_i)| = 3k+1$ and therefore, $E(G)\backslash\partial(V_i)$ is a $(k+1)$-signature of W_{2k+1}. Every edge of W_{2k+1} is in precisely two elements of $\{T^1, \ldots, T^{2k+1}, C_{2k+1}\}$, where $T^1, \ldots, T^{2k+1}$ are the $2k+1$ triangles of W_{2k+1}. It follows that $\ell(-W_{2k+1}) = k+1$. Thus, W_{2k+1} is $(k+1)$-critical. $\square$

The *projective cube* H_k of dimension $k \geq 1$ can be constructed as follows: Each vertex v is labeled with a $(0,1)$-string $s(v)$ of length k and $s(v) \neq s(w)$ if $v \neq w$. Two vertices are adjacent if the Hamming distance of their labels is 1 or k.

The *signed projective cube* of dimension k is the signed graph (H_k, Σ) with negative edges connecting vertices whose labels have Hamming distance k.

Signed projective cubes play an exceptional role in the study of signed graph homomorphism, see [24]. We show that they are critical.

Proposition 2.1.6. *Let $k \geq 1$ be an integer. The signed projective cube (H_k, Σ) of dimension k is 2^{k-1}-critical. Furthermore, it has $k+1$ pairwise disjoint 2^{k-1}-signatures.*

Proof. The set Σ is a perfect matching of H_k and $\ell(H_k, \Sigma) \leq |\Sigma| = 2^{k-1}$. The j^{th} digit in $s(v)$ is denoted by $s_j(v)$. For $i \in \{1, \ldots, k\}$ let $U_i = \{v \colon s_i(v) = 0\}$. Then $d_{H_k}(U_i) = 2^k$ and $\Sigma \subseteq \partial_{H_k}(U_i)$. Hence, $\partial_{H_k}(U_i) \Delta \Sigma = B_i$ is a 2^{k-1}-signature and $B_1, \ldots, B_k, \Sigma$ are $k+1$ pairwise disjoint 2^{k-1}-signatures of (H_k, Σ). Thus, $\ell(H_k, \Sigma) = 2^{k-1}$ and (H_k, Σ) is 2^{k-1}-critical, by Theorem 2.1.2. □

It is not hard to see that the antibalanced complete graph of order n is $\lfloor \frac{(n-1)^2}{4} \rfloor$-critical. Since K_n has $\frac{n(n-1)}{2}$ edges, it follows that it cannot have three pairwise disjoint signatures if $n \geq 5$.

2.2 Decomposition and subdivision of critical signed graphs

The results of this section have been published in [6].

By combining critical signed graphs or making some operations on them, we can find an infinite number of critical signed graphs which may be actually given by the same "basic" critical graph. In this section we study these operations so that we can later focus on the principal structures causing criticality.

Let $n \geq 2$. A k-critical signed graph (G, Σ) is $(k_1, \ldots, k_n)$*-decomposable* if it contains pairwise edge-disjoint k_i-critical subgraphs (H_i, Σ_i) and $k = k_1 + \cdots + k_n$. If the numbers $k, k_1, \ldots, k_n$ are irrelevant we just say that (G, Σ) is decomposable.

Proposition 2.2.1. *Let (G, Σ) be a k-critical signed graph.*

1. *If (G, Σ) contains a negative loop or two vertices which are connected by a positive and a negative edge, then (G, Σ) is $(1, k-1)$-decomposable.*

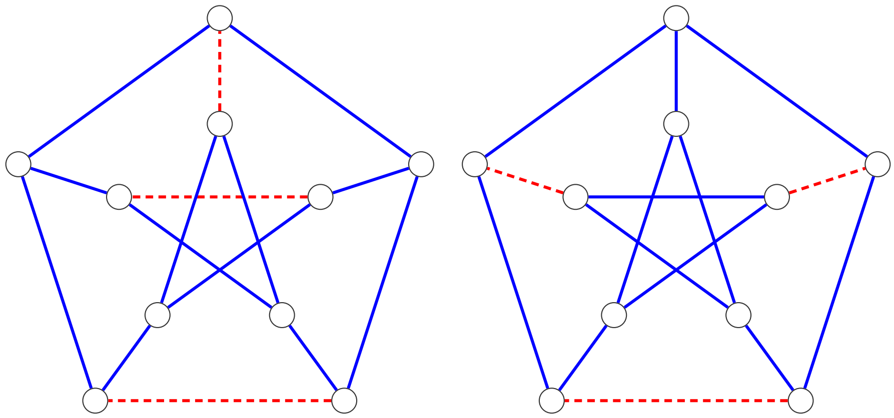


Figure 2.1: The dotted lines represent Σ_1 on the left and Σ_2 on the right

2. *If G is connected and (G, Σ) is (k_1, k_2)-decomposable into (H_1, Σ_1) and (H_2, Σ_2), then $V(H_1) \cap V(H_2) \neq \emptyset$. Furthermore, if $v \in V(H_1) \cap V(H_2)$, then $d_G(v) \geq 4$.*

3. *If (G, Σ) is $(k_1, \ldots, k_n)$-decomposable into $(H_1, \Sigma_1), \ldots, (H_n, \Sigma_n)$, then $\bigcup_{i=1}^{n} E(H_i) = E(G)$.*

Proof. 1. Let e^+ and e^- be the two edges which are incident with the same vertices or let e^l be a negative loop. By Lemma 1.3.4 there is a k-signature Γ. Clearly, $e^l \in \Gamma$ and precisely one of e^+, e^- is in Γ. Since every equilibrated edge-cut in (G, Γ) gives an equilibrated edge-cut in $(G - \{e^+, e^-, e^l\}, \Gamma)$, the statement follows by Theorem 2.1.2.

2. If $V(G_1)$ and $V(G_2)$ are disjoint, then there is an edge $e \in E(G) \setminus (E(H_1) \cup E(H_2))$. Hence, $\ell(G - e, \Sigma) = \ell(G, \Sigma)$, a contradiction. By Corollary 2.1.3, every vertex of H_i has degree at least 2 in H_i, so the statement follows. Statement 3. is proved similar to statement 2. $\square$

By Proposition 2.2.1, k-critical signed cubic graphs are non-decomposable. For example, see the 3-critical signed Petersen graphs (P, Σ_1) and (P, Σ_2) in Figure 2.1.

Let (G, Σ) be a signed graph and $t \geq 1$ be an integer. A *t-multiedge* between two vertices v, w is a set of t edges between v and w and it is denoted by E_{vw}. A t-multiedge has a sign if all edges of E_{vw} have the same sign. That is, it is positive/negative if all edges of E_{vw} are positive/negative.

Let (G, Σ) be a signed graph and E_{xy} be a t-multiedge having a sign. Let (G', Σ') be obtained from $(G - E_{xy}, \Sigma)$ by adding a vertex v and a positive t-multiedge E^{+}_{vx} and a t-multiedge E^{Σ}_{vy} which has the same sign as E_{xy}. We say that (G', Σ') is obtained from (G, Σ) by *subdividing* a multiedge. Furthermore, (H', Γ') is a *subdivision* of (H, Γ) if $(H', \Gamma') = (H, \Gamma)$ or (H', Γ') is obtained from (H, Γ) by a sequence of multiedge subdivisions. If (H', Γ') is a subdivision of (H, Γ) and $(H', \Gamma') \neq (H, \Gamma)$, then (H', Γ') is called a *proper subdivision* of (H, Γ). Note that the order of the subdivisions has an influence on the resulting graph (see Figure 2.2).

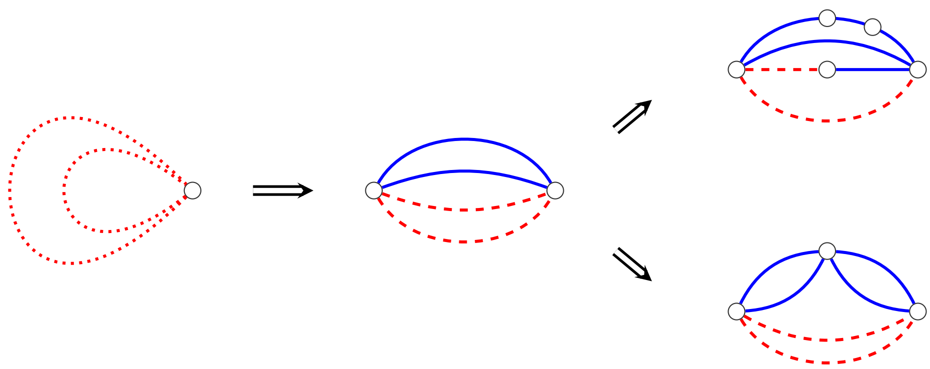


Figure 2.2: An example of how the resulting signed graph depends on the order of the subdivisions

Theorem 2.2.2. *Let $k \geq 1$ and let (H, Γ), (G, Σ) be two signed graphs such that (H, Γ) is a subdivision of (G, Σ). Then the following statements hold:*

1. *$\ell(G, \Sigma) = \ell(H, \Gamma)$.*

2. *(G, Σ) is k-critical if and only if (H, Γ) is k-critical.*

3. *(G, Σ) is decomposable if and only if (H, Γ) is decomposable.*

Proof. Let (H,Γ) be obtained from (G,Σ) by subdividing a t-multiedge E_{xy} ($t \geq 1$), and let v be the only vertex of $V(H) \setminus V(G)$ which subdivides E_{xy}. Since $G - E_{xy} = H - v$, we consider an edge e of $G - E_{xy}$ also as an edge of $H - v$. Furthermore, let $\ell(G,\Sigma) = k$ and Σ be a k-signature and Γ be the signature on H which is obtained from Σ by the subdivision of multiedge E_{xy}.

1. By definition, $\ell(H,\Gamma) \leq k$. Suppose to the contrary that $\ell(H,\Gamma) < k$. Then there is an edge-cut $\partial_H(U)$ with $d^+_{(H,\Gamma)}(U) \leq d^-_{(H,\Gamma)}(U)$. If $x, y \in U$ or $\in V(G) \setminus U$, then $\partial_H(U) = \partial_G(U)$, a contradiction. Hence, $x \in U$ and $y \notin U$. Both cases whether E_{xy} is negative or not, are covered when $v \in U$. But then $\partial_G(U \setminus \{v\})$ is an edge-cut in (G,Σ) with more negative than positive edges, a contradiction. Hence, $\ell(H,\Gamma) = \ell(G,\Sigma)$.

2. ($\rightarrow$) Let (G,Σ) be k-critical. By Theorem 2.1.2 there is a k-signature Σ_1 such that $E_{xy} \subseteq \Sigma_1$. By construction, $\Gamma_1 = (\Sigma_1 \setminus E_{xy}) \cup E^{\Sigma_1}_{vy}$ is a k-signature of (H,Γ) and $E^+_{vx} \subset \Gamma_1 \Delta \partial_H(v)$, which is also a k-signature of (H,Γ).

Every edge $e \notin E^+_{vx} \cup E^{\Sigma_1}_{vy}$, can be considered as an edge of $G - E_{xy}$. If e is positive, then there is an edge-cut $\partial_G(U)$ equilibrated under Σ containing e. If $x \in U$ and $y \notin U$, then $\partial_H(U')$ with $U' = U \cup \{v\}$ is the corresponding equilibrated edge-cut in (H,Γ). If both x, y are in U or not in U, then there is nothing to prove. Hence, every positive edge of (H,Γ) is contained in an equilibrated edge-cut and the statement follows by Theorem 2.1.2.

The other direction of this statement is proved similarly.

3. Note, by the definition, a decomposable signed graph is k-critical. Therefore, by 2., we assume that (G,Σ) and (H,Γ) are k-critical. Let E_{xy}, E_{xv} and E_{yv} be t-multiedges in the respective signed graphs.

($\leftarrow$) Let (H,Γ) be (k_1,k_2)-decomposable into (H_1,Γ_1) and (H_2,Γ_2), $k_1 + k_2 = k$. For $i \in \{1,2\}$ let $E^i_{xv} = E_{xv} \cap E(H_i)$ and $E^i_{yv} = E_{yv} \cap E(H_i)$. All four of these multiedges have a sign in their respective graphs and $|E^i_{xv}| = |E^i_{yv}| = t_i$ with $t_1 + t_2 = t$ and $t_i \geq 0$.

Let E^i_{xy} be a set of t_i edges of $E_{x,y}$, so that $E^1_{xy} \cap E^2_{xy} = \emptyset$.

For $i \in \{1,2\}$ let (G_i,Σ_i) be the subgraph of (G,Σ) with $E(G_i) = E(H_i -$

$v) \cup E^i_{xy}$ and $\Sigma_i = \Sigma \cap E(G_i)$. Then every edge of G is contained in precisely one of G_1, G_2. Furthermore, (H_i, Γ_i) is a subdivision of (G_i, Σ_i). Thus, by 2., (G_i, Σ_i) is k_i-critical and (G, Σ) is (k_1, k_2)-decomposable into (G_1, Σ_1) and (G_2, Σ_2). The opposite direction is proved similarly starting with a decomposition of (G, Σ). □

We ask a non-empty graph to have a non-empty vertex set. The signed graph with two vertices which are connected by t positive and t negative edges is a subdivision of $-tC_1$. All other connected critical frustrated signed graphs have at least three vertices. As one might expect, it is easy to decide whether a critical signed graph is a subdivision of another one.

Theorem 2.2.3. *Let (G, Σ) be a non-decomposable k-critical signed graph with at least three vertices. Then (G, Σ) is a proper subdivision of a signed graph (H, Γ) if and only if G has a vertex with precisely two neighbors.*

Proof. ($\leftarrow$) By Lemma 1.3.4 we can assume that Σ is a k-signature. Let v be a vertex with precisely two neighbors x and y. Let E_{vx} and E_{vy} be the set of edges between v and x and between v and y, respectively. Since (G, Σ) is non-decomposable, it follows by Proposition 2.2.1 that E_{vx} and E_{vy} have a sign. Since (G, Σ) is critical, we additionally can assume that E_{vy} is negative.

If $|E_{vx}| < |E_{vy}|$, then $|\partial(v)\Delta\Sigma| < k$, a contradiction. If $|E_{vy}| < |E_{vx}|$, then there is a k-signature Σ_1 with $E_{vx} \subseteq \Sigma_1$. But then $|\partial(v)\Delta\Sigma_1| < k$, a contradiction. Thus, $|E_{vx}| = |E_{vy}| = t$.

Let (H, Γ) be the signed graph with $V(H) = V(G) \setminus \{v\}$ and $E(H) = E(G - v) \cup E_{xy}$, where E_{xy} is a set of t edges between x and y, and $\Gamma = (\Sigma \cap E(G - v)) \cup E_{xy}$. Now it is easy to see that (G, Σ) is a proper subdivision of (H, Γ) (which indeed is also k-critical by Theorem 2.2.2). The other direction of the statement is trivial. □

Note that, as we said at the beginning of this chapter, given a k-critical graph, one edge can be subdivided infinite times, and it would produce an infinitely large family of k-critical graphs. In the following, we focus on signed graphs which are not a proper subdivision of other graphs. In particular, we call such signed graphs *irreducible*.

2.3 Characterization of 1- and 2-critical signed graphs

The results of this section have been published in [6].

For positive integers $k \geq 3$, the number of irreducible and non-decomposable k-critical graphs is huge, and it may be even difficult to establish whether it is finite. However, for $k = 1, 2$ we can provide a list.

For $k \geq 1$, we denote with $\mathcal{L}(k)$ the family of irreducible critically k-frustrated signed graphs, and with $\mathcal{L} = \bigcup_{k\geq 1} \mathcal{L}(k)$. Similarly, we denote with $\mathcal{L}^*(k)$ the set of signed graphs in $\mathcal{L}(k)$ which are also non-decomposable, and $\mathcal{L}^* = \bigcup_{k\geq 1} \mathcal{L}^*(k)$.

Therefore, a signed graph is k-critical if and only if it is a subdivision of an element of $\mathcal{L}(k)$.

We denote with $-C_1 \mathbin{\dot\cup} -C_1$ the disjoint union of two negative circuits of length 1.

Theorem 2.3.1. $\mathcal{L}(1) = \mathcal{L}^*(1) = \{-C_1\}$

Proof. Clearly, $-C_1$ is 1-critical.

Let (G, Σ) be 1-critical. Hence, (G, Σ) contains a negative circuit C. If there is an edge e which is not an edge of C, then $\ell(G - e, \Sigma) = 1$, since C is a subgraph of $G - e$, a contradiction. Since (G, Σ) is irreducible it follows that $(G, \Sigma) = -C_1$. □

Theorem 2.3.2. $\mathcal{L}(2) = \{-C_1 \mathbin{\dot\cup} -C_1, -2C_1, -K_4\}$*, and* $\mathcal{L}^*(2) = \{-K_4\}$*.*

Proof. ($\leftarrow$) Clearly, the elements of $\mathcal{L}(2)$ are 2-critical.

($\rightarrow$) We can assume that Σ is a 2-signature, $\Sigma = \{e_1, e_2\}$ and $e_i = x_i y_i$. If (G, Σ) has less than four positive edges, the statement is trivial. So we assume that G has at least six edges.

Suppose that (G, Σ) contains a multiedge E_{xy}. If it does not have a sign, then it contains a positive and a negative edge. Hence (G, Σ) is decomposable by Proposition 2.2.1 and it contains a subdivision of $-C_1 \dot\cup -C_1$ or $-2C_1$. If it has a sign, then it contains precisely two edges. The graph $G - E_{xy}$ is 2-edge-connected, since for otherwise (G,Σ) would have an edge-cut with two negative edges and at most one positive edge, a contradiction to Lemma 1.3.5. Hence there are two edge-disjoint paths between x and y in $G - E_{xy}$. Thus, (G,Σ) is a subdivision of $-2C_1$.

We assume that G is simple in the following. If G contains a divalent vertex, then, by Theorem 2.2.3, it is a subdivision of a 2-critical signed graph. So we can assume that $d_G(v) \geq 3$ for every vertex v.

If G has a 2-edge-cut E_2, then $G - E_2$ has (precisely) two components H_1, H_2. There is a 2-signature Σ_1, which contains precisely one edge of E_2. The other edge of Σ_1 is in $E(H_1)$ or $E(H_2)$, say $E(H_2)$. Since H_1 contains a vertex, G is bridgeless and $d_G(v) \geq 3$ for every vertex v there is a balanced circuit C_b in (G,Σ) with $E(C_b) \subset E(H_1)$. Since the second edge of Σ_1 is in H_2, it follows that there is a negative circuit C_u in H_2 which is vertex-disjoint from C_b. Every 2-signature Σ_2 which contains an edge of C_b contains at least two edges of C_b. Hence, $\Sigma_2 \cap E(C_u) = \emptyset$, a contradiction. Therefore, for every 2-signature and in particular for Σ, $G - \Sigma$ is 2-edge connected and every equilibrated edge-cut of (G,Σ) contains precisely four edges.

Since $G-\Sigma$ is 2-edge-connected, there are two edge-disjoint paths $P_1(x_i,y_i), P_2(x_i,y_i)$ $(i \in \{1,2\})$ between x_i and y_i. Every equilibrated edge-cut contains e_1 and e_2 and, therefore, each of the two positive edges is contained in one of $P_1(x_1,y_1), P_2(x_1,y_1)$ and in one of $P_1(x_2,y_2), P_2(x_2,y_2)$. Hence, $E(P_1(x_1,y_1)) \cup E(P_2(x_1,y_1)) = E(P_1(x_2,y_2)) \cup E(P_2(x_2,y_2))$. If $x_2, y_2 \in V(P_1(x_1,y_1))$, then (G,Σ) is $-2C_1$, i.e. e_2 is incident to one of x_1, y_1. Thus $x_2 \in V(P_1(x_1,y_1))$ if and only if $y_2 \in V(P_2(x_1,y_1))$. Hence, (G,Σ) contains a subdivision of $-K_4$. Thus, $(G,\Sigma) = -K_4$, since (G,Σ) is irreducible. $\square$

Theorem 2.3.2 also follows from the following result.

Theorem 2.3.3. [14] *Let (G,Σ) be a k-frustrated signed graph. If (G,Σ) contains no $-K_4$-subdivision, then (G,Σ) contains k edge-disjoint negative cir-*

cuits.

Anyway, we gave the proof of Theorem 2.3.2 since this is easier to prove than Theorem 2.3.3 and it still implies many results of this work. Furthermore, we believe that Theorem 2.3.2 may imply Theorem 2.3.3.

2.4 The family of critically 3-frustrated planar signed graphs

The results of this section have been published in [5].

As we have seen, the families of 1- and 2-critical signed graphs are easy to describe. For critically k-frustrated signed graphs, where $k \geq 3$, this is not the case.

In this section, we approach this problem by considering the family of irreducible and non-decomposable 3-critical planar signed graphs, which we denote with $\mathcal{P}^*(3)$. We prove that this family consists of ten elements (up to isomorphism), depicted in Figure 2.6.

In the following, given a plane graph G, if F is a face of G, then we denote with C_F the facial circuit consisting of the boundary of F. With a bit of abuse of notation, we may say that an edge belongs to a face to indicate that such an edge belongs to the facial circuit of such a face. Furthermore, given a positive integer $n \geq 2$, we define a n-vertex of G to be a vertex of degree n.

We begin with a result which is a consequence of the characterization of $\mathcal{L}(2)$.

Lemma 2.4.1. *Let C_1, C_2, and C_3 be three negative circuits of a signed graph (G, Σ). If $E(C_1) \cap E(C_2) \cap E(C_3) = \emptyset$, then the signed subgraph induced by C_1, C_2, and C_3 contains either a $-K_4$-subdivision or two edge-disjoint negative circuits.*

Proof. Since $E(C_1) \cap E(C_2) \cap E(C_3) = \emptyset$, the frustration index of the signed subgraph induced by $C_1 \cup C_2 \cup C_3$ is at least 2. Hence, it contains a critically 2-frustrated subgraph. The statement then follows from Theorem 2.3.2. $\square$

Noting that each edge of a plane graph is in two faces, we have the following fact, which implies that an element of $\mathcal{P}^*(3)$ has at most six negative faces.

Lemma 2.4.2. *Every k-critical signed plane graph has at most $2k$ negative facial circuits. Moreover, if there are $2k$ negative facial circuits, then they are the only facial circuits.*

Next we show that each signed plane graph in $\mathcal{P}^*(3)$ has exactly six negative faces. In fact, we prove this for a larger class of critically 3-frustrated signed graphs which are not necessarily irreducible.

Theorem 2.4.3. *Let (G, Σ) be a non-decomposable 3-critical signed plane graph. Then (G, Σ) consists of six negative facial circuits.*

Proof. Since (G, Σ) is not decomposable, by Theorem 2.3.3 (G, Σ) contains a $-K_4$-subdivision (H, Σ) as a subgraph. Let e_1 be an edge of $G - E(H)$, noting that it is not an empty set because $\ell(G, \Sigma) = 3$. Without loss of generality, we assume that Σ is a minimum signature where e_1 is assigned to be negative. We observe that the other two negative edges of Σ are on the $-K_4$-subdivision (H, Σ). We also consider G together with a planar embedding.

To prove the theorem it suffices to show that each facial circuit of (G, Σ) contains at most one negative edge. That is because, this together with the fact that $\ell(G, \Sigma) = 3$ would imply the existence of six negative faces. The claim then follows from Lemma 2.4.2.

As there are only two negative edges in (H, Σ), say e_2 and e_3, no face of (H, Σ) contains two negative edges. Thus in (G, Σ) no face contains three negative edges.

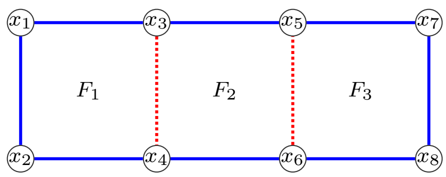


Figure 2.3: F_1, F_2 and F_3

It remains to show that no face of (G,Σ) contains exactly two negative edges. Suppose to the contrary that F_2 is such a face. As the negative edges cannot be e_2 and e_3, and by the symmetry between these two labels, we may assume that e_1 and e_2 are the negative edges of F_2. Let F_1 and F_3 be the other faces such that $e_1 \in E(C_{F_1})$ and $e_2 \in E(C_{F_3})$. Observe that e_3 neither belongs to C_{F_1} nor to C_{F_3}, as for otherwise (G,Σ) has only two negative faces, contradicting the fact that it contains a $-K_4$-subdivision. See Figure 2.3 for illustration, where a solid x_ix_j connection presents a positive path some of which could be of length 0, dashed connections each shows a negative path, thus each of length at least 1. We first claim that C_{F_1} and C_{F_3} have no common edge. Otherwise, a common edge e' together with e_1 and e_2 forms an edge-cut, and by switching at this edge-cut we have a signature with only 2 negative edges.

Let C_{F_4} and C_{F_5} be the two negative facial circuits of (G,Σ) such that $e_3 \in E(C_{F_4} \cap C_{F_5})$. Observe that each of C_{F_4} and C_{F_5} must share at least one edge with either C_{F_1} or C_{F_3}. Otherwise we would have a set of three edge-disjoint negative circuits, contradicting with the assumption that (G,Σ) is non-decomposable. We now consider the following two cases:

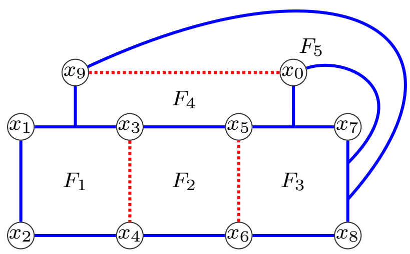


Figure 2.4: Case (α)

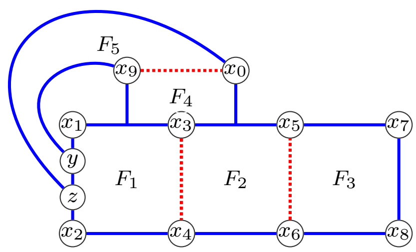


Figure 2.5: Case (β)

Case (1): C_{F_4} shares a common edge with (at least) one of C_{F_1} and C_{F_3}, and C_{F_5} shares a common edge with the other.

By symmetry, we assume that C_{F_4} has a common edge with C_{F_1}, and hence C_{F_5} has a common edge with C_{F_3}. See Figure 2.4. Then there is an edge-cut crossing the faces F_4, F_1, F_2, F_3, F_5, and F_4 in this order containing two positive edges and three negative edges, a contradiction with $\ell(G, \Sigma) = 3$.

Case (2): Each of C_{F_4} and C_{F_5} shares a common edge with the same C_{F_i} for $i \in \{1, 3\}$, but none with C_{F_j} for $j \in \{1, 3\} \setminus \{i\}$.

By symmetries, assume that each of C_{F_4} and C_{F_5} shares a common edge with C_{F_1} but none with C_{F_3}. See Figure 2.5. Therefore, C_{F_3} is edge-disjoint from the negative facial circuits C_{F_1}, C_{F_4}, and C_{F_5}. Furthermore, by Lemma 2.4.1, $C_{F_1} \cup C_{F_4} \cup C_{F_5}$ contains a critically 2-frustrated signed graph. Note that such a critically 2-frustrated signed graph is edge-disjoint from C_{F_3}. Since C_{F_3} is a negative facial circuit (i.e., a critically 1-frustrated signed graph), (G, Σ) is decomposable, a contradiction. □

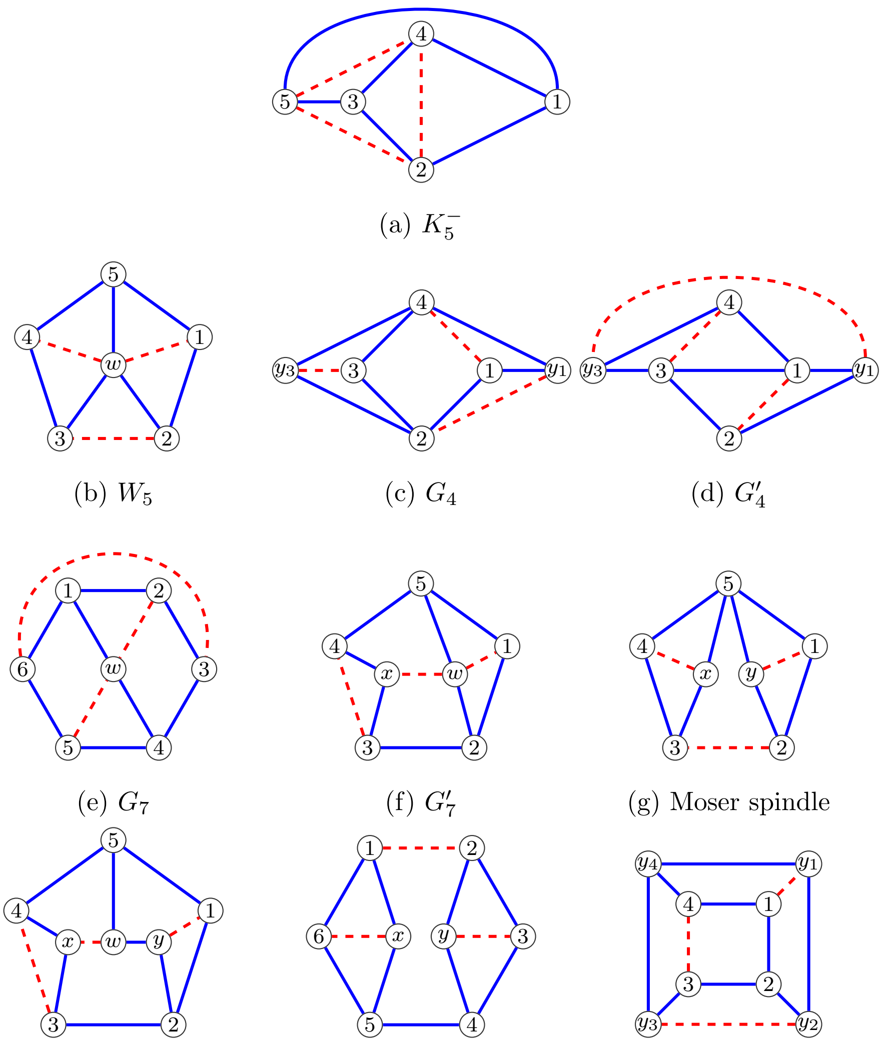
Corollary 2.4.4. *If $(G, \Sigma) \in \mathcal{P}^*(3)$, then (G, Σ) is simple. Moreover, for each minimum signature Σ every facial circuit contains exactly one negative edge.*

Proof. By Proposition 2.2.1, there is no loop in (G, Σ) and no two parallel edges of different sign. If there exists two parallel edges with the same sign, then in some planar embedding of (G, Σ) they induce a positive facial circuit, contradicting Theorem 2.4.3. The moreover part is immediate from the fact that there are six facial circuits. □

Now we are ready to describe the elements of the family $\mathcal{P}^*(3)$.

Theorem 2.4.5. *The family $\mathcal{P}^*(3)$ consists of ten signed graphs, depicted in Figure 2.6.*

Proof. Let $(G, \Sigma) \in \mathcal{P}^*(3)$ with a planar embedding. By Theorem 2.4.3, in (G, Σ) there are six facial circuits all of which are negative. This determines the signature up to a switching. So it remains to classify the underlying graphs G. Let $n = |V(G)|$, $m = |E(G)|$, and $f = |F(G)|$ where $F(G)$ is the set of facial circuits of G. Note that $f = 6$ by Theorem 2.4.3. By Euler's formula, and the fact that $\delta(G) \geq 3$, we have that $n - \frac{3}{2}n + 6 \geq 2$. Hence, every irreducible non-decomposable critically 3-frustrated signed planar graph contains at most



(a) K_5^-

(b) W_5 (c) G_4 (d) G'_4

(e) G_7 (f) G'_7 (g) Moser spindle

(h) G_8 (i) G'_8 (j) Signed cube

Figure 2.6: The family $\mathcal{P}^*(3)$

8 vertices. Note that any simple signed graph on at most four vertices has frustration index at most 2, thus $n \geq 5$. Depending on the values of n we consider four cases. Noting that in each case G has 6 faces, the number of edges is determined by Euler's formula.

- $n = 5,\ m = 9$: The underlying graph is K_5^- as it has only one edge less than K_5. This graph has a unique planar embedding and in (G, Σ) all faces must be negative. In Figure 2.6a one such signature is presented.
- $n = 6,\ m = 10$: Either G consists of one 5-vertex and four 3-vertices or it consists of four 3-vertices and two 4-vertices. In the first case, G is isomorphic to W_5, see Figure 2.6b. In the second case, we consider two subcases: (1) The two 4-vertices are not adjacent. In this case, these two 4-vertices are both adjacent to all the remaining vertices; moreover, there are only two edges induced by the four 3-vertices. See Figure 2.6c. (2) The two 4-vertices are adjacent. In this case, the two 4-vertices share at most two common neighbors. Otherwise a $K_{3,3}$ is forced by just counting degrees, contradicting planarity. The degree conditions then lead to the unique example of Figure 2.6d.
- $n = 7,\ m = 11$: G consists of one 4-vertex and six 3-vertices. We consider the graph G_1 obtained from G by removing the 4-vertex. Note that G_1 consists of two 3-vertices and four 2-vertices, and moreover, G is planar and there is a planar embedding such that the four 2-vertices are in a facial circuit. Then one of the followings must be the case for G_1: (1) It consists of two 4-circuits sharing one edge, see Figure 2.6e; (2) It consists of one 5-circuit sharing one edge with a triangle, see Figure 2.6f; (3) It consists of two triangles connected by an edge, see Figure 2.6g.
- $n = 8,\ m = 12$: There is a total of five cubic 2-connected graphs, see for example [4]. Of these, we have Wagner graph which is not planar, and one obtained from $K_{3,3}$ by blowing up a vertex to a triangle. The other three form the full list of cubic 2-connected simple planar graphs on 8

vertices. They are depicted in Figures 2.6h, 2.6i, and 2.6j.

To complete the proof we need to verify that each graph in the list is 3-critical. That is to say, after removing any edge in any of these graphs, the remaining subgraph has frustration index at most, and hence exactly, 2. To see this we note that each of these 10 graph is 2-edge-connected and each has only six facial circuits all of which are negative. Thus once an edge is removed, we have five facial circuits, one of which (the new one) is positive and the other four are negative. It can then be readily verified that in each case these four negative faces can be covered with 2 edges. Assigning a negative sign to these two then we have an equivalent signature with two negative edges. $\square$

It is interesting to note that the list of signed graphs in Theorem 2.4.5 consists of all planar graphs which are simple, have minimum degree three, and have exactly six faces. Therefore the following holds.

Corollary 2.4.6. *A simple signed planar graph belongs to $\mathcal{P}^*(3)$ if and only if it has minimim degree three and it consists of exactly six negative faces.*

Chapter 3

The families $\mathcal{L}$ and $\mathcal{L}^*$

General results for critically frustrated signed graphs are often difficult to prove, therefore some of them remain as open questions and we conjecture them in Chapter 7. One of our principal questions in this thesis concerns the finiteness of critically k-frustrated signed graphs, for $k \geq 1$. Trivially, if we allow subdivision, then each critical signed graph provides infinitely many new critical signed graphs, therefore, concerning this problem, we focus on the family of irreducible critical signed graphs $\mathcal{L}$, and eventually on the family of irreducible and non-decomposable critical signed graphs $\mathcal{L}^*$. In the previous chapter we saw that $\mathcal{L}(k)$ and $\mathcal{L}^*(k)$ are small for $k \in \{1, 2\}$ but -as was shown in Section 2.4- this may not be the case for each $k \geq 3$.

In this chapter we first provide infinitely large families of k-critical signed graphs for $k \geq 3$. Since these families consist of decomposable graphs, we still believe that $\mathcal{L}^*(k)$ may be finite for each value of k. However, for each $k \geq 3$, $\mathcal{L}^*(k)$ is expected to grow significantly. To prove this, in Section 3.2 we describe a construction which provides non-decomposable and critically k-frustrated signed graphs from two non-decomposable critical signed graphs with smaller frustration index. In Section 3.3 we provide some general results regarding structural properties on signed graphs in $\mathcal{L}^*(k)$. These results are also used in Chapter 4 to prove the finiteness of $\mathcal{L}^*(3)$. Lastly, in Section 3.4, we provide some more structural results for k-critical signed graphs with no

$-K_5$-minor. In particular, we prove a statement about the maximum degree of critical signed graphs which we believe to be true in general.

3.1 Infinite families of k-critical signed graphs

The results of this section have been published in [5].

The main result of this section is the following Theorem:

Theorem 3.1.1. *There exist infinitely many irreducible critically 3-frustrated signed graphs.*

Clearly, given an irreducible critically 3-frustrated signed graph (G, Σ), for each $k \geq 4$ we can obtain a critically k-frustrated signed graph by adding $k-3$ negative loops at one vertex in $V(G)$. Therefore, Theorem 3.1.1 can be easily generalized.

Corollary 3.1.2. *For each positive integer $k \geq 3$, there exist infinitely many irreducible critically k-frustrated signed graphs.*

In order to prove our statements, we first define a set of signed graphs as follows: Let $\hat{G}_0$ be the signed graph obtained from K_4 on vertices x, y, z, w by first assigning negative signs to xw and yz, positive signs to the remaining four edges, and secondly adding a positive edge xw and a negative edge yz. See Figure 3.1. Observe that $\hat{G}_0$ can be decomposed into three negative circuits: xwx (2-circuit), $xyzx$ (3-circuit), and $wyzw$ (3-circuit).

The signed graph $\hat{G}_t$ is built from $\hat{G}_0$ as follows. We first introduce $2t$ points by subdividing the positive edge connecting x and w, and two sets of t points by subdividing each of xz and yw. Then we identify the $2t$ points of the xw-path with the $2t$ points, alternating between the points from xz and wy. See Figure 3.2 for the case of $t = 2$.

Proof of Theorem 3.1.1. We prove this claim by showing that $\hat{G}_t \in \mathcal{L}(3)$ for each $t \geq 1$. Observe that subdivisions of each of the three circuits given in decomposition of $\hat{G}_0$ give a decomposition of $\hat{G}_t$. It implies that $\ell(\hat{G}_t) = 3$. What remains is to show that $\hat{G}_t$ is irreducible and critically 3-frustrated.

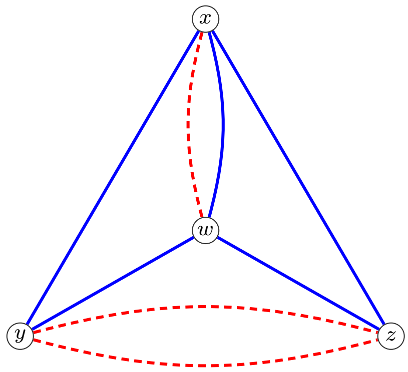


Figure 3.1: $\hat{G}_0$

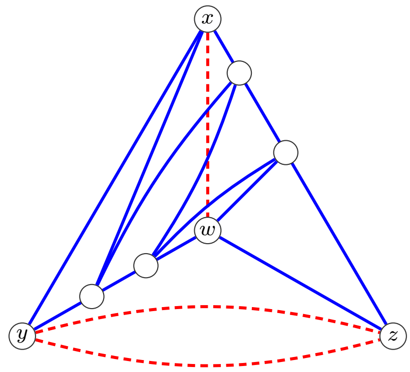


Figure 3.2: $\hat{G}_2$

That $\hat{G}_t$ is irreducible follows from Theorem 2.2.3, that states that in a subdivision of a signed graph there is always a vertex that has only two distinct neighbors. But there is no such vertex in $\hat{G}_t$. Now we provide a sketch of the proof of $\hat{G}_t$ being critically 3-frustrated. First, observe that each edge incident with y (or z) is in an equilibrated edge-cut $\partial(y)$ (respectively, $\partial(z)$). All other edges are the results of subdivisions (and then identifying some vertices). For an edge uv where u is a vertex on the subdivision of xz and v is a vertex on the subdivision of yw, the following six edges form an equilibrated edge-cut: uv, the edge on the xz-path that forms a triangle with uv, the edge on the yw-path that forms a triangle with uv and the three negative edges. $\square$

In fact, we can modify these signed graphs to get an infinite family of irreducible critically 3-frustrated signed planar graphs. For each $\hat{G}_t$, we apply the following modification to get $\hat{G}'_t$. First, by modifying the embedding of Figure 3.2 and putting w on the outside of the xyz-triangle, we may have an embedding with one cross which is the crossing of the edge of the yw-path incident with w and the edge of the xz-path incident with z. Then introduce a new vertex s at this crossing point to get the planar signed graph $\hat{G}'_t$. See Figure 3.3 for a depiction of $\hat{G}'_2$. The only remaining point to verify is that each of the new edges is in an equilibrated edge-cut. Such two edge-cuts are $\partial(\{w, z\})$ and $\partial(\{w, z, s\})$. Therefore, we obtain the following result for planar

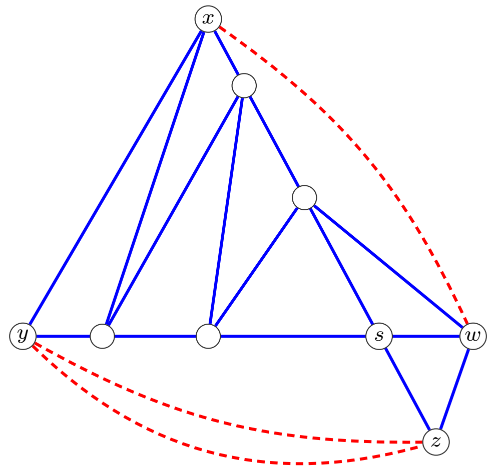


Figure 3.3: $\hat{G}_2'$

graphs.

Theorem 3.1.3. *Given a positive integer $k \geq 3$, there exist infinitely many irreducible critically k-frustrated planar signed graphs.*

3.2 Construction for non-decomposable critically frustrated signed graphs

The results of this section have been published in [5].

In this section, we build signed graphs in $\mathcal{L}^*(k)$ from two given non-decomposable critically frustrated signed graphs, one being k_1-frustrated and the other being k_2-frustrated such that $k = k_1 + k_2 - 1$. For that, we define the following operation.

Definition 3.2.1. *Let (G_1, Σ_1) and (G_2, Σ_2) be two signed graphs, and let xy be a negative edge of (G_1, Σ_1) and uv be a negative edge of (G_2, Σ_2). We define $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ to be the signed graph obtained from the disjoint union of (G_1, Σ_1) and (G_2, Σ_2) by deleting edges xy and uv, and then adding a negative edge xu and a positive edge yv.*

Proposition 3.2.2. *Given integers $k_1, k_2 \geq 2$, let $(G_1, \Sigma_1) \in \mathcal{L}^*(k_1)$ and $(G_2, \Sigma_2) \in \mathcal{L}^*(k_2)$ be two signed graphs such that $|\Sigma_1| = k_1$ and $|\Sigma_2| = k_2$. Let xy be a negative edge of (G_1, Σ_1) and uv be a negative edge of (G_2, Σ_2). Then $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}] \in \mathcal{L}^*(k_1 + k_2 - 1)$.*

Proof. Let Σ be the signature of $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ and note that it has $k_1 + k_2 - 1$ negative edges. We first verify that Σ is a minimum signature by showing that there is no edge-cut with more negative edges than positive ones. Suppose to the contrary that there exists an edge-cut of $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ with more negative edges than positive ones. As Σ_1 (resp. Σ_2) is a minimum signature of (G_1, Σ_1) (resp. (G_2, Σ_2)), such an edge-cut, say $\partial(X)$, must contain the new negative edge xu. The vertices x and y are not separated by $\partial(X)$ because otherwise in the restriction of $\partial(X)$ to (G_1, Σ_1) we would get a contradiction. Similarly, u and v are not separated by $\partial(X)$. Then yv is also an edge of $\partial(X)$. However, in this case in one of the restrictions of $\partial(X)$ to (G_1, Σ_1) and (G_2, Σ_2) we get a contradiction.

Next we show that $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ is critically frustrated. By Theorem 2.1.2, it suffices to prove that each positive edge of $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ belongs to an equilibrated edge-cut. For any positive edge e of (G_1, σ_1), the equilibrated edge-cut of (G_1, Σ_1) containing e is also an equilibrated edge-cut of $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ by replacing xy with xu if needed. The same argument holds for positive edges of (G_2, Σ_2). For the new positive edge yv, $\partial(V(G_1))$ is the required equilibrated edge-cut.

Observe now that $V(H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]) = V(G_1) \cup V(G_2)$ and each vertex in $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ has at least as many neighbors as in the original signed graph. Therefore, by Theorem 2.2.3 and by the fact that both (G_1, Σ_1) and (G_2, Σ_2) are irreducible, it follows that $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ is also irreducible.

It remains to show that $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}]$ is not decomposable. Assume to the contrary that it is and suppose there is a $(r_1, \dots, r_t)$-decomposition $(r_1 + \cdots + r_t = k_1 + k_2 - 1)$ into signed subgraphs $\hat{H}'_1, \dots, \hat{H}'_t$. We may furthermore assume that each $\hat{H}_i'$ is connected. Then they must be 2-connected because a critically frustrated signed graph cannot have a bridge. Thus one of the $\hat{H}_i'$'s, say $\hat{H}_1'$, should contain both xu and yv. Each of the others then should be a subgraph of either (G_1, Σ_1) or (G_2, Σ_2). Without loss of generality, we assume that $\hat{H}_2'$ is a subgraph of (G_2, Σ_2). Let $(H_2, \Sigma) = \hat{H}_2'$, and let (H_1, Σ) be the signed subgraph obtained from putting together all other $\hat{H}_i'$'s (that is $H[(G_1, \Sigma_1)_{xy},\ (G_2, \Sigma_2)_{uv}] - (H_2, \Sigma))$. This gives us an (l_1, l_2)-decomposition where $l_1 = k_1 + k_2 - 1 - r_2$ and $l_2 = r_2$.

Observe that $l_2 \leq k_2 - 1$, because uv is not an edge of the critically l_2-frustrated signed graph (H_2, Σ) which is a subgraph of the critically k_2-frustrated signed graph (G_2, Σ_2). Let (H', Σ_2) be the signed subgraph of (G_2, Σ_2) obtained by removing all edges of H_2 (recall that uv is a negative edge of this signed subgraph). Observe that $\ell(H', \Sigma_2) \leq k_2 - l_2$, but moreover if $\ell(H', \Sigma_2) = k_2 - l_2$ then (G_2, Σ_2) is $(l_2, k_2 - l_2)$-decomposable, a contradiction. Thus $\ell(H', \Sigma_2) \leq k_2 - l_2 - 1$. Thus there exists a switching-equivalent signature Π of Σ_2 such that $|\Pi| = k_2 - l_2 - 1$. Assume Π is obtained by switching at an edge-cut $\partial(X)$, with $X \subseteq V(G_2)$.

We consider two cases based on whether $uv \in \Pi$. If $uv \in \Pi$, then X contains either both of u and v or none of them. We consider a switching at an edge-cut $\partial(X)$, where X is a subset of the vertices of (H_1, Σ). This switching does not change the signs of the edges in the (G_1, Σ_1) part, thus there remain $k_1 - 1$ negative edges in this part, noting that xy is not an edge in $E(H_1 \cap G_1)$. On $\{xu, yv\}$ there would remain one negative edge. And on $(H' - uv, \Pi)$ we have $k_2 - l_2 - 1$ negative edges. Altogether we have $k_1 + k_2 - l_2 - 2$ negative edges in this switching of (H_1, Σ), contradicting the fact that its frustration index is $k_1 + k_2 - l_2 - 1$. If $uv \notin \Pi$, then X contains exactly one of u or v, by symmetry of switching on X or X^c, we may assume $u \in X$. As in the previous case we consider a switching at the edge-cut $\partial(X)$ such that X is a subset of the vertices of (H_1, Σ). Since $u \in X$ and $v \notin X$, both xu and yv are positive edges after this switching. A similar calculation as before then counts the number of negative edges in this switched signed graph to be $k_1 + k_2 - l_2 - 2$, which leads to the same contradiction. $\square$

Observe that, since there exists only one element in $\mathcal{L}^*(2)$ and it is symmetric, the only signed graph in $\mathcal{L}^*(3)$ given by the operation described is the signed graph in Figure 2.6i. However, by combining the signed graphs in $\mathcal{P}^*(3)$ with $-K_4$ in all the possible ways (since not all of such graphs are symmetric), we may obtain a significant number of elements in $\mathcal{L}^*(4)$.

3.3 The family $\mathcal{L}^*$

In order to deal with critically k-frustrated signed graphs, for big values of k, it is necessary to obtain more structural results. In this section we provide such results by studying the relation between non-decomposable k-critical signed graphs and the $k-1$-critical subgraphs.

Proposition 3.3.1. *A non-decomposable critically k-frustrated signed graph has no k-multiedges.*

Proof. Let (G, Σ) be a non-decomposable critically k-frustrated signed graph and assume to the contrary that there exists a k-multiedge E_{xy} in (G, Σ). By Proposition 2.2.1, all the parallel edges in E_{xy} have the same sign. By criticality, there is a minimum signature Σ' under which all of the edges of E_{xy} are negative. Thus $\Sigma' = E_{xy}$. Since (G, Σ') is non-decomposable, by Theorem 2.3.3 it contains a $(K_4, -)$-subdivision which has at least two negative edges that are not parallel, contradicting that $\Sigma' = E_{xy}$. $\square$

Lemma 3.3.2. *Given a signed graph (G, Σ) such that Σ is a minimum signature, if there exist t negative edges sharing a vertex $v \in V(G)$, then $d(v) \geq 2t$.*

The next propositions give a deep insight in the structure of critical signed graphs.

Proposition 3.3.3. *Let $k \geq 2$ be a positive integer and let $(G, \Sigma) \in \mathcal{L}^*(k)$. For each critically $(k-1)$-frustrated subgraph (H, Σ) of (G, Σ), it holds that $G - E(H)$ is a forest.*

Proof. Suppose to the contrary that $G - E(H)$ is not a forest. Therefore, there exists a circuit C edge-disjoint from H. If C is a negative circuit, then, by Proposition 2.2.1, (G, Σ) is decomposable. Hence, we may assume that C is positive. Let e be an edge of C. By criticality, there exists a minimum signature Σ' containing e. Since C is positive, it holds that $|E(C) \cap \Sigma'| \geq 2$. Therefore, there are at most $k-2$ negative edges in (H, Σ'), a contradiction. □

Let Θ be a signed theta-graph. We say that a signed theta-graph is *fully-negative* with respect to a signature Σ if $|E(\Theta) \cap \Sigma| = 3$.

Let $(G, \Sigma) \in \mathcal{L}^*(k)$, and assume that (H, Σ) is a $k-1$-critical subgraph. An H-path P of (G, Σ) is a path such that for each $v \in V(P)$ it holds $d_H(v) \geq 3$ if and only if v is an endpoint of P. In particular, if (H, Σ) is given by subdividing at a signed graph $(H', \Sigma') \in \mathcal{L}^*(k-1)$, then an H-path is the path given by subdividing an edge of H'. Moreover, an H-subpath is a subpath of an H-path. By observing that in a minimum signature each H-path can contain at most one negative edge, and that (H, Σ) contains at least $k-1$ negative edges, the following lemma holds.

Lemma 3.3.4. *Let $(G, \Sigma) \in \mathcal{L}^*(k)$ with $|\Sigma| = k$ and let (H, Σ) be a $k-1$-critical subgraph of (G, Σ). If $|\Sigma \cap E(H)| = k-1$, then any fully-negative theta-graph in (G, Σ) must intersect with at least two H-paths.*

Proposition 3.3.5. *Let $(G, \Sigma) \in \mathcal{L}^*(k)$ and let (H, Σ) be a $k-1$-critical subgraph of (G, Σ). If $v \notin V(H)$, then there exist at most two all-positive internally vertex-disjoint paths which are internally vertex-disjoint from H and connect v to an all-positive H-subpath.*

Proof. Suppose to the contrary that there exist three internally vertex-disjoint paths connecting v to an all-positive H-subpath P, say Q_1, Q_2, Q_3. We denote with v_i the endpoint of Q_i on the H-path, for $i \in \{1, 2, 3\}$. Observe that those endpoints must be three different vertices, since otherwise there would exist a positive circuit edge-disjoint from H, contradicting Proposition 3.3.3. Without loss of generality, we can assume that v_2 is contained in the v_1v_3-path in P. Therefore, $Q_2 \cup Q_1 \cup P$ contains a positive circuit C, and similarly $Q_2 \cup Q_3 \cup P$ contains a positive circuit C'. Furthermore, it holds that $C \cap C' = Q_2$ and $C \cup C'$ is a theta-graph. Let $e \in E(Q_2)$. By criticality, there exists a minimum signature Σ' such that $e \in \Sigma'$. Since $E(Q_2) \cap E(H) = \emptyset$, H contains exactly $k-1$ edges from Σ'. Since $E(C \cap C' \cap H) = \emptyset$, $C \cup C'$ is a fully-negative theta-graph intersecting only one H-path, a contradiction to Lemma 3.3.4.

□

Those results show that each element of $\mathcal{L}^*(k)$ is given by adding a forest to (the subdivision of) a $k-1$-critical signed graph. In particular, Proposition 3.3.5 implies some constraints on the trees of this forest.

In Chapter 4 we use these results to prove that $\mathcal{L}^*(3)$ contains finitely many elements. We expect that such proof could be extended to prove the finiteness of $\mathcal{L}^*(k)$.

3.4 $-K_5$-minor-free signed graphs

The results of this section, except for Proposition 3.4.4 have been published in [5].

A signed graph (H, Π) is a *minor* of (G, Σ) if it is obtained from (G, Σ) by a sequence of the following operations: Deleting vertices or edges, contraction of positive edges, switching.

This section follows from a known result on the frustration index of $-K_5$-minor-free signed graphs:

Theorem 3.4.1 ([27]). *Let (G, Σ) be a $-K_5$-minor-free Eulerian signed graph. The maximum number of edge-disjoint negative circuits in (G, Σ) is equal to the frustration index of (G, Σ).*

A set $\mathcal{C}$ of negative circuits of (G, Σ) is said to be a $\leq_2$-*negative circuit cover* if each edge belongs to at most two circuits of $\mathcal{C}$. If each edge belongs to exactly two circuits in $\mathcal{C}$, then we say that $\mathcal{C}$ is a *negative circuit double cover.*

Let (G, Σ) be a $-K_5$-minor free signed graph. Note that, by doubling each edge with the respective sign, we obtain a new $-K_5$-minor free signed graph which is Eulerian and whose frustration index equals to $2\ell(G, \Sigma)$.

By applying Theorem 3.4.1 to this new graph, we obtain the following result.

Theorem 3.4.2. *Let (G, Σ) be a $-K_5$-minor-free signed graph. Then $\ell(G, \Sigma) = \frac{1}{2}\ |\mathcal{C}|$ where $\mathcal{C}$ is a $\leq_2$-negative circuit cover of (G, Σ).*

We use this theorem to strengthen it using the notion of critically frustrated signed graphs.

Theorem 3.4.3. *Given a $-K_5$-minor-free critically k-frustrated signed graph (G, Σ), there exists a negative circuit double cover $\mathcal{C}$ of (G, Σ) of order $2k$.*

Proof. Let (G, Σ) be a $-K_5$-minor-free signed graph and assume it is critically k-frustrated. By Theorem 3.4.2 there exists a $\leq_2$-negative circuit cover $\mathcal{C}$ of cardinality $2k$. We prove that $\mathcal{C}$ is indeed a negative circuit double cover. That is to say that each edge of G is in two circuits of $\mathcal{C}$. Assume to the contrary, that an edge e is not in two circuits of $\mathcal{C}$, thus it is either in none of them or only in one of them.

First consider the case that e does not belong to any circuit of $\mathcal{C}$. Then $\ell(G - e, \Sigma) \geq k = \frac{1}{2}|\mathcal{C}|$, contradicting criticality of (G, Σ).

Next suppose that e belongs to exactly one circuit of $\mathcal{C}$. By criticality, $\ell(G - e, \Sigma) = 2k - 1$. Hence, by Theorem 3.4.2, each $\leq_2$-negative circuit cover $\mathcal{C}$ of $\ell(G - e, \Sigma)$ is of order at most $2k - 2$. Since e belongs to exactly one circuit of $C \in \mathcal{C}$, the set $\mathcal{C} \setminus \{C\}$ is a $\leq_2$-negative circuit cover of $(G - e, \Sigma)$ with $|\mathcal{C} \setminus \{C\}| = 2k - 1$, a contradiction. □

Such a theorem has strong consequences on the structure of critically k-frustrated signed graphs without $-K_5$-minor.

In Theorem 2.1.2 we showed that each positive edge is contained in an equilibrated edge-cut. This cannot be said for negative edges, since they may be negative in each signature, as it happens with negative loops. For $-K_5$-minor-free signed graphs, we show that loops are actually the only exception.

Proposition 3.4.4. *Let (G, Σ) be a loopless $-K_5$-minor-free signed graph. If (G, Σ) is k-critical, then each edge $e \in E(G)$ belongs to an equilibrated edge-cut.*

Proof. By Theorem 2.1.2, it suffices to show that for each edge e there exists an equivalent signature Σ' such that $e \notin \Sigma'$.

Assume that Σ is a minimum signature, and let $e \in \Sigma$. Since (G, Σ) has no $-K_5$-minors, by Theorem 3.4.3 there exists a negative circuit double cover $\mathcal{C}$ of cardinality $2k$. Note that, since $|\mathcal{C}| = 2k$ and $|\Sigma| = k$, each negative edge is the only negative edge of the two circuits from $\mathcal{C}$ it belongs to. Let $C_1, C_2 \in \mathcal{C}$ be the two circuits containing e. Since (G, Σ) is loopless, there exists another edge $e' \in C_1$, $e' \notin \Sigma$. By criticality, there exists an equivalent minimum signature Σ' such that $e' \in \Sigma'$. If also $e \in \Sigma'$, then $|C_1 \cap \Sigma'| \geq 3$, which is a contradiction. Hence, $e \notin \Sigma'$. □

In Section 2.2 we showed that, if a critical signed graph has a vertex of degree 2, then the graph is given by a subdivision and the vertex can be compressed. Furthermore, it is trivial to observe that critical signed graphs have no vertices of degree 1. Hence, an irreducible k-critical graph has minimum degree 3, and this bound cannot be improved for any value of k -see for example the negative Wheel.

For the maximum degree, things become more interesting. Trivially, $-kC_1$ has maximum degree $2k = 2\ell(-kC_1)$. Our expectation is that this is an upper bound, and that, if circuits share more edges, then the maximum degree should decrease. For $-K_5$-minor-free signed graphs this is true, and it easily follows by the fact that, given a negative circuit double cover, the edges incident with each vertex v belong to at most $2k$ circuits.

Corollary 3.4.5. *Let (G, Σ) be a $-K_5$-minor-free signed graph. If (G, Σ) is* *critically k-frustrated, then* $\Delta(G) \leq 2k$.

Chapter 4

Finiteness of $\mathcal{L}^*(3)$

As we already observed along this work, the number of critically k-frustrated signed graphs seems to significantly increase for increasing values of k.

We previously showed that, for $k \in \{1,2\}$, the families $\mathcal{L}(k)$ of irreducible critically k-frustrated signed graphs are finite and small, but for $k \geq 3$ $\mathcal{L}(k)$ has always infinitely many elements. It remains to understand whether (or under which conditions) $\mathcal{L}^*(k)$ is finite. In this chapter we show that $\mathcal{L}^*(3)$ has finitely many elements. In particular, we prove the following result.

Theorem 4.0.1. *For any $(G,\Sigma) \in \mathcal{L}^*(3)$, $|V(G)| \leq 210$, and $\mathcal{L}^*(3)$ is a finite set.*

4.1 Preliminaries

We recall that, given a graph G, if H_1 and H_2 are subgraphs of G, we may denote with $H_1 \cup H_2$ (resp. $H_1 \cap H_2$) the graph having vertex set $V(H_1) \cup V(H_2)$ (resp. $V(H_1) \cap V(H_2)$) and edge set $E(H_1) \cup E(H_2)$ (resp. $E(H_1) \cap E(H_2)$). A $-K_4$-subdivision is a signed subdivision of K_4 with a signature such that each circuit corresponding to a triangle of K_4 is negative. If $(G,\Sigma) \in \mathcal{L}^*(3)$ and (H,Σ) is a $-K_4$-subdivision in (G,Σ), an H-path is a path in G obtained by subdividing an edge of the $-K_4$. Furthermore, an H-subpath is a subpath of an H-path. We say that two H-paths are *matching* if they do not share any endpoints. Note that a minimum signature of a $-K_4$-subdivision (H, Σ) has exactly two negative edges, and that they are on two matching H-paths. Furthermore, if a signature has three negative edges, then those edges are on three different H-paths whose union induces a circuit (and a triangle in the original $-K_4$).

Since we aim to prove the finiteness of $\mathcal{L}^*(3)$ and we already showed in Theorem 2.4.5 that $\mathcal{P}^*(3)$ is finite and each of its signed graphs has few vertices, from now on we focus on non-planar signed graphs. As we know all of the signed graphs in $\mathcal{P}^*(3)$, it is easy to see that many of our statements are also true for planar signed graphs. In particular, it is trivial that Theorem 4.0.1 is true for planar signed graphs.

Therefore, let $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$ be a signed graph such that $\Sigma = \{e_1^-, e_2^-, e_3^-\}$. Since (G, Σ) is non-decomposable, by Theorem 2.3.3 there exists a $-K_4$-subdivision (H, Σ) in (G, Σ). Furthermore, by criticality we can assume that $\Sigma \cap E(H) = \{e_1^-, e_2^-\}$. A negative circuit F of length three in (G, Σ) is called a *flag* if $E(F) \cap \Sigma = \{e_i^-\}$, for $i \in \{1, 2\}$, and the vertex v of $V(F)$ which is not incident with e_i^- has $d_G(v) = 3$ and it is incident with e_3^-. Note that, since F is negative, e_3^- does not belong to F. Furthermore, observe that F in $(G - e_3^-, \Sigma)$ is the result of a subdivision at one of two parallel edges having different signs. This implies that $E(F) \cap E(H) = \{e_i^-\}$.

Lemma 4.1.1. *Given a signed graph $(G, \Sigma) \in \mathcal{L}^*(3)$ with $\Sigma = \{e_1^-, e_2^-, e_3^-\}$, let (H, Σ) be a $-K_4$-subdivision of (G, Σ) such that $\Sigma \cap E(H) = \{e_1^-, e_2^-\}$. Then at most one of e_1^- and e_2^- belongs to a flag.*

Proof. Suppose to the contrary that each of e_1^- and e_2^- belongs to a flag. First of all, we recall that e_1^- and e_2^- are on two matching H-paths of (H, Σ), say $e_1^- \in E(P_1)$ and $e_2^- \in E(P_2)$. Secondly, note that, by definition, these two flags to which e_1^- and e_2^- belong are edge-disjoint.

Let e be a positive edge on one all-positive H-path, say P_3, of (H, Σ), and let Π be a minimum signature containing e. Note that either $|\Pi \cap E(H)| = 2$ or $|\Pi \cap E(H)| = 3$. If $|\Pi \cap E(H)| = 2$, then, besides e, the other negative edge of $\Pi \cap E(H)$ must be on the H-path matching to P_3, which can neither be P_1 nor P_2 by definition. Since the flags are two edge-disjoint negative circuits of (G, Σ), we need two more negative edges from $\Pi \setminus E(H)$, a contradiction. If $|\Pi \cap E(H)| = 3$, then the three negative edges of (G, Π) cover at most one of the two flags, a contradiction. □

Definition of $(\bar{H}, \Sigma)$

We begin by providing a special $-K_4$-subdivision which has to be contained in each $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$, but we first make some assumptions on the signature Σ. Therefore, let $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$. By criticality, we can assume Σ to be a minimum signature and to satisfy one of the following conditions:

- If G contains parallel edges, then by Proposition 3.3.1 it contains a 2-multiedge. In this case, we can assume that Σ contains these two parallel edges, denoted by e_1^- and e_3^-, and we denote the third edge of Σ by e_2^-. Note that $(G - e_3^-, \Sigma)$ contains a $-K_4$-subdivision.

- If G is simple, there is an edge $e \in E(G)$ such that $(G - e, \Sigma)$ contains a $-K_4$-subdivision. In this case, we assume that Σ contains e and we relabel e with e_3^-. Thus the two remaining negative edges of Σ are in the $-K_4$-subdivision. By Lemma 4.1.1, at most one of two negative edges of $\Sigma \setminus \{e_3^-\}$ belongs to a flag. If there exists one flag containing an edge of $\Sigma \setminus \{e_3^-\}$, we call it e_2^- and the third is denoted by e_1^-. Note that for any $-K_4$-subdivision of $(G - e_3^-, \Sigma)$ that contains e_1^- and e_2^-, only e_2^- can be contained in one flag, since a flag is a subdivision of two parallel edges of different signs in $(G - e_3^-, \Sigma)$.

Definition 4.1.2. *Among all the possible $-K_4$-subdivisions of $(G - e_3^-, \Sigma)$, we choose $(\bar{H}, \Sigma)$ to be a $-K_4$-subdivision of (G, Σ) satisfying the following conditions, in this order of priority:*

(1) The length of the $\bar{H}$-path containing e_1^- is minimized.

(2) The length of the $\bar{H}$-path containing e_2^- is minimized.

(3) $|V(\bar{H})|$ is minimized.

We denote the vertices and the $\bar{H}$-paths as in Figure 4.1. In particular, we denote by P_i, for $i \in \{1, 2, \ldots, 6\}$, the $\bar{H}$-paths of $(\bar{H}, \Sigma)$ and assume that P_i and P_{i+1} are matching paths for $i \in \{1, 3, 5\}$. We assume $e_1^- \in E(P_1)$ and $e_2^- \in E(P_2)$. We recall that e_3^- is the third negative edge under Σ which is in $E(G - E(\bar{H}))$.

For given i and j, we may say there is a path *connecting P_i to P_j* if there is a path Q internally-vertex-disjoint from $\bar{H}$ such that the endpoints of Q are on the $\bar{H}$-paths P_i and P_j, respectively. Note that Q might be an edge. In this case, we assume that such edge is not contained in $E(\bar{H})$.

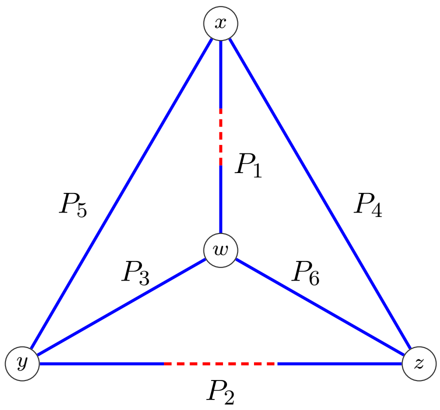


Figure 4.1: Notation on $(\bar{H}, \Sigma)$

From now on, we work with the $-K_4$-subdivision $(\bar{H}, \Sigma)$.

Furthermore, we consider a special forest $G^*_{\bar{H}}$: Let G' be a graph obtained from G by deleting all the edges of $\bar{H}$, all the edges parallel to edges of $\bar{H}$, and vertices v with $d_G(v) = d_{\bar{H}}(v)$. Note that G' is a subgraph of G. By Proposition 3.3.3, G' is a forest. We define the graph $G^*_{\bar{H}}$ from G' as follows: We start from a vertex $v \in V(\bar{H})$ which has n neighbors $u_1, u_2, \ldots, u_n$ in $V(G')$. We first delete v, take n copies of v, namely $v_1, v_2, \ldots, v_n$, and add edges v_iu_i, for $i \in \{1, 2, ..., n\}$. We repeat this process for all the vertices $v \in V(\bar{H})$ with $d_G(v) \neq d_{\bar{H}}(v)$. We observe that $G^*_{\bar{H}}$ consists of trees where each copy of $v \in V(\bar{H}) \cap V(G')$ is a leaf. Furthermore, since (G, Σ) is irreducible, any nonleaf vertex $v \in V(G^*_{\bar{H}})$ satisfies that $d_{G^*_{\bar{H}}}(v) \geq 3$. Note that $V(G) \setminus V(G^*_{\bar{H}}) \subseteq \{x, y, z, w\}$. Therefore, $|V(G^*_{\bar{H}})| \geq |V(G)| - 4$. In the following, with a bit of abuse of notation, we may refer to the vertices of $G^*_{\bar{H}}$ as vertices of G and vice-versa.

4.2 Proof of Theorem 4.0.1

The proof of Theorem 4.0.1 is based on four main results. Since some of these results require long proofs, we first only provide the statements. The proofs are presented later in different sections. Note that some of the results are trivially true when $(G, \Sigma) \in \mathcal{P}^*(3)$, therefore we can give them in general.

The main idea of the proof of Theorem 4.0.1 is to bound the size of the forest $G^*_{\bar{H}}$.

Therefore, the first statement bounds the size of each tree in $G^*_{\bar{H}}$.

Proposition 4.2.1. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$. If T is an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$, then $|V(T)| \leq 8$. If T is a tree of $(G^*_{\bar{H}}, \Sigma)$ containing a negative edge, then $|V(T)| \leq 10$.*

The other results provide a way to count the number of trees in $G^*_{\bar{H}}$.

Proposition 4.2.2. *Let $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$. For each vertex of degree 2 in $\bar{H}$, exactly one of the following conditions holds:*

(i) v is in the tree of $(G^*_{\bar{H}}, \Sigma)$ which contains the negative edge e_3^-;

(ii) v belongs to an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$ whose leaves are on at least two different $\bar{H}$-paths.

(iii) v belongs to an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$ consisting of an edge connecting two internal vertices of P_2 belonging to two different components of $P_2 - \{e_2^-\}$.

Proposition 4.2.3. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$. There exist at most two all-positive pairwise internally vertex-disjoint paths in $(G - E(\bar{H}), \Sigma)$ which are internally vertex-disjoint from $\bar{H}$ and connect the two connected components of $P_2 - \{e_2^-\}$.*

Proposition 4.2.4. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$. There are at most 24 all-positive pairwise internally vertex-disjoint paths in $(G - E(\bar{H}), \Sigma)$ which are internally vertex-disjoint from $\bar{H}$ and connect different $\bar{H}$-paths.*

We can now prove the main result of this chapter.

Proof of Theorem 4.0.1.

As we said at the beginning, it is enough to prove the statements for non-planar signed graphs.

Let $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$ and consider the special $-K_4$-subdivision $(\bar{H}, \Sigma)$ and the forest $(G^*_{\bar{H}}, \Sigma)$. By Proposition 4.2.2, each vertex of degree 2 in $\bar{H}$ belongs to one of three types of trees of $(G^*_{\bar{H}}, \Sigma)$ and we give the number of these trees as follows:

(i) There is a unique tree containing the negative edge e_3^- in (G, Σ);

(ii) By Proposition 4.2.4 there exist at most 24 all-positive trees connecting two different $\bar{H}$-paths. This is because of the observation that each all-positive tree with leaves on different $\bar{H}$-paths provides at least one all-positive path internally vertex-disjoint from $(\bar{H}, \Sigma)$ connecting two $\bar{H}$-paths;

(iii) by Proposition 4.2.3 there are at most two pairwise internally-vertex-disjoint paths connecting the two connected components of $P_2 - \{e_2^-\}$. In particular, by Proposition 4.2.2 such paths consist of single edges.

Since there are at most four vertices which do not belong to any tree of $(G^*_{\bar{H}}, \Sigma)$, and since each tree must have leaves on G, it follows from Proposition 4.2.1 that $|V(G^*)| \leq 1 \times 10 + 24 \times 8 + 2 \times 2 = 206$ and thus $|V(G)| \leq 210$.

That $\mathcal{L}^*(3)$ has finitely many elements set follows from the fact that, by Proposition 3.3.1, (G, Σ) has at most two parallel edges between two adjacent vertices. Thus, it follows that $|E(G)|$ is bounded and consequently, $|\mathcal{L}^*(3)|$ is bounded. $\square$

4.3 Structural properties of $(\bar{H}, \Sigma)$

In order to prove our four statements we first need to show the special properties of $(\bar{H}, \Sigma)$. We actually provide the first results for general $-K_4$-subdivisions, but they are essential for most of the proofs related to $(\bar{H}, \Sigma)$.

The next two lemmas are based on an extension of Proposition 3.3.5 for signed graphs in $\mathcal{L}^*(3)$.

We recall that a fully-negative theta-graph is a theta-graph containing three negative edges. Since each minimum signature of a $-K_4$-subdivision (H, Σ) has two negative edges on two matching H-paths, the following lemma easily follows.

Lemma 4.3.1. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$ with $|\Sigma| = 3$ and let (H, Σ) be a $-K_4$-subdivision of (G, Σ). Assume that $|\Sigma \cap E(H)| = 2$. Then any fully-negative theta-graph in (G, Σ) intersects with at least one pair of matching paths of (H, Σ).*

Lemma 4.3.2. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$ and let (H, Σ) be a $-K_4$-subdivision of (G, Σ). For each $v \in V(G)$, there exist at most two all-positive pairwise internally vertex-disjoint paths in $(G - E(H), \Sigma)$ internally vertex-disjoint from H and connecting v to two non-matching H-paths of (H, Σ).*

Proof. Suppose to the contrary that the statement is not true. Let Q_1, Q_2, Q_3 be three paths as in the statement, such that Q_k intersects H on v_k, for each $k \in \{1, 2, 3\}$, and possibly on v. Let P and P' be the H-paths to whom v_1, v_2, v_3 belong. We note that here v_1, v_2, v_3 may belong to the same H-path. Since P and P' share a common vertex, there exists a pair $i, j \in \{1, 2, 3\}$ such that there is a v_iv_j-path P'' in $P \cup P'$ containing $\{v_1, v_2, v_3\}$. Without loss of generality, we assume that $i = 1$ and $j = 3$. We consider two cases based on whether v belongs to P''.

Case (1) $v \notin V(P'')$.

In this case, $P'' \cup Q_1 \cup Q_2 \cup Q_3$ has a theta-graph Θ as a subgraph. Note that Θ does not intersect any pair of two matching H-paths of (H, Σ) on edges. Furthermore, there exist two circuits C_1 and C_2 such that $C_1 \cup C_2 = \Theta$, $Q_1 \subset C_1$, $Q_3 \subset C_2$, and $C_1 \cap C_2 = Q_2$. We consider the sign of the circuits. First suppose that $E(\Theta) \cap \Sigma = \emptyset$. Let e be an edge of Q_2 and let Σ' be a minimum signature of (G, Σ) containing e. Θ is fully-negative with respect to Σ', contradicting Lemma 4.3.1. Then we may suppose that $|E(\Theta) \cap \Sigma| = 1$. In this case, exactly one of the two circuits C_1 and C_2 is negative, say C_1. Given $e \in E(Q_3)$, let Σ' be a minimum signature of (G, Σ) containing e. Since C_2 is positive and C_1 is negative in (G, Σ), Θ is fully-negative under Σ', contradicting Lemma 4.3.1.

Case (2) $v \in V(P'')$.

By Pigeonhole Principle, there is a $k \in \{1, 3\}$ such that v_2 belongs to the vv_k-path in P'', without loss of generality, say $k = 1$. Let Θ be the theta-graph contained in the vv_1-subpath together with Q_1 and Q_2. If Θ is all-positive under Σ, we get a contradiction by considering an edge in Q_2 and a minimum signature Σ' containing such an edge. If Θ contains a negative edge, then the contradiction is similarly given by considering a minimum signature Σ'' containing an edge $e'' \in E(Q_3)$. □

We can now focus on $(\bar{H}, \Sigma)$ and show some of its structural properties based on our choice. We recall that we denote the $\bar{H}$-paths and the vertices of degree three in $\bar{H}$ as in Figure 4.1. We next prove that P_1 is indeed the edge e_1^-.

Lemma 4.3.3. *The $\bar{H}$-path P_1 consists of exactly one edge, i.e. e_1^-.*

Proof. Let $e_1^- = vu$ and suppose that $|E(P_1)| \geq 2$. Thus, without loss of generality, we may assume that v is contained in the connected component of $P_1 - \{e_1^-\}$ containing x and $d_{\bar{H}}(v) = 2$. As (G, Σ) is irreducible, $d_G(v) \geq 3$. Note that, by Lemma 3.3.2, there exists at least one positive edge e incident with v and such that $e \notin E(\bar{H})$. We first observe that e cannot be parallel to the negative edge e_1^-, or by Proposition 2.2.1 (G, Σ) would be decomposable. Furthermore, we may assume that e is not parallel to the positive edge incident with v on P_1 as well. If not, by the choice of Σ, the negative edge e_1^- should be parallel to e_3^-, and thus $d_G(v) \geq 5$ as (G, Σ) is irreducible. By Proposition 3.3.1, there exists another positive edge e' (under Σ) incident with v and not parallel to any edge of P_1. In this case, we replace e with e'.

Hence, by the connectivity, there exists an all-positive path Q under Σ containing e, which is internally vertex-disjoint from $\bar{H}$, connecting v to another vertex $v' \in V(\bar{H})$. It is because by Proposition 3.3.3 there always exists an all-positive path in a signed tree (with at most one negative edge) whose minimum degree of vertices that are not leaves is 3. Depending on where the endpoint v' of the path Q is, we consider the following two possibilities:

Case 1. $v' \notin V(P_1)$.

In this case, we show that there exists another $-K_4$-subdivision using the path Q where P_1 can be replaced by the vw-subpath of P_1 (which is shorter), which contradicts Condition (1) on the choice of $(\bar{H}, \Sigma)$. Two subcases may occur:

(1) $v' \in V(P_3 \cup P_6 \cup P_2)$. We first argue with the case when v' is either on P_3 or on the connected component of $P_2 - \{e_2^-\}$ containing y. In this case, the degree-3 vertices of the new $-K_4$-subdivision are the following: v, v', z, w. The rest of the case can be easily verified by considering the new $-K_4$-subdivision with v, v', y, w being its degree-3 vertices.

(2) $v' \in V(P_4 \cup P_5)$. The degree-3 vertices of the new $-K_4$-subdivision are the following: v, z, y, w.

In fact, similar arguments may be applied to show that for any internal vertex in the vx-subpath of P_1 or, by symmetry more generally, any internal vertex on P_1, does not belong to an all-positive path internally vertex-disjoint from $\bar{H}$ having one endpoint on a $\bar{H}$-path P_j with $j \in \{2, 3, 4, 5, 6\}$.

Case 2. $v' \in V(P_1)$.

First, we claim that, if an edge not in $E(Q)$ is incident with an internal vertex (if any) of Q, then such an edge belongs to Σ. As otherwise, either

(i) there exist three edge-disjoint paths connecting this internal vertex to P_1, contradicting Lemma 4.3.2, or (ii) there exists an all-positive path connecting v (via this interval vertex) to one of the other $\bar{H}$-paths, contradicting Case 1.

As $|\Sigma| = 3$, this implies that $|E(Q)| \leq 2$. In particular, if $|E(Q)| = 2$, then the internal vertex of Q is of degree 3, it is incident with e_3^- and, furthermore, (G, Σ) is a simple graph.

We denote with u the endpoint of e_3^- different from v, that is $e_3^- = vu$, and consider two cases.

(1) v' is on the xv-subpath of P_1.

As discussed above Q is not an edge parallel to any edge of P_1. Hence, if $|E(Q)| = 1$, then by replacing the vv'-subpath of P_1 with Q, we obtain a new $-K_4$-subdivision with a shorter P_1, contradicting Condition (1) on the choice of $(\bar{H}, \Sigma)$. Hence, we may assume that $|E(Q)| = 2$.

By Condition (1), it holds that the vv'-subpath of P_1 has length at most two, as for otherwise, we can find a shorter P_1 by replacing the vv'-subpath with Q. Furthermore, if the other endpoint of e_3^- is an internal vertex of the vv'-subpath of P_1, then (G, Σ) has to be one of the critical signed graphs in $\mathcal{P}^*(3)$ provided in Theorem 2.4.5, that is either the one in Figure 2.6g (if $x = v'$), or the one in Figure 2.6i (if $x \neq v'$). As we have assumed that $(G, \Sigma) \notin \mathcal{P}^*(3)$, this is a contradiction.

Recalling that e is the edge of Q incident with v, let Σ' be a minimum signature containing e. Since Q together with the vv'-subpath of P_1 induces a positive circuit, the second negative edge of Σ', say e', must be on the vv'-subpath of P_1, and thus the third negative edge of Σ' is on P_2.

We note that as (G, Σ) is non-decomposable, e_3^- is not incident with v.

Assume that e' is incident with v, that is e and e' are adjacent. Then, $d(v) \geq 4$ and thus there exists another positive edge e'' (under both Σ and Σ') incident with v. Therefore, e'' belongs to a path Q' all-positive under Σ internally vertex-disjoint from $\bar{H} \cup Q$ connecting v to a vertex, say v'', of $\bar{H}$. By Case 1, v'' belongs to $V(P_1)$. If v'' belongs to the vx-subpath of P_1, Q' together with P_1 induces a circuit which contains exactly one edge (i.e., e') from Σ' but no edge from Σ, a contradiction. If v'' belongs to the vw-subpath of P_1, noting that $v \neq v''$, then the circuit induced by $Q' \cup P_1$ contains exactly one edge from Σ but no edges from Σ', a contradiction.

Similarly, if e is not adjacent to e', we can repeat the previous argument for the edge incident with the internal vertex of the vv'-subpath of P_1 and reach a contradiction.

(2) v' is on the uw-subpath of P_1.

In this case, first, note that e_1^- is not incident with v', as for otherwise either (1) e_1^- would belong to a flag, which contradicts the choice of e_1^-, or (2) e_1^- would belong to a 2-multiedge with edges of different sign, contradicting the fact that (G, Σ) is non-decomposable (see Proposition 2.2.1). Considering the symmetry between the vertices v and u, by repeating the previous arguments (given for v) for u, we conclude that there exists an all-positive uu'-path Q' internally vertex-disjoint from $\bar{H}$ such that u' belongs to the vx-subpath of P_1, see Figure 4.2.

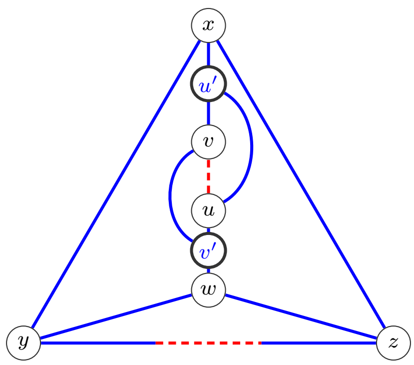


Figure 4.2: The path Q and Q' on P_1

Note that there is a new $-K_4$-subdivision induced by $\{u, u', v, v'\}$ where P_1 is just the negative edge uv, a contradiction.

This completes the proof of the lemma. $\square$

We note that since P_1 is an edge, each path Q connecting to P_1 is also connecting to some other all-positive $\bar{H}$-path P_i. Thus we may view Q to be a connection to P_i rather than P_1, and in this case, we assume from now on that there is no all-positive path connecting P_1 to any other $\bar{H}$-path of $(\bar{H}, \Sigma)$.

In the sequel, we denote the endpoints of e_2^- with y' and z', where y' is the vertex in the connected component of $P_2 - \{e_2^-\}$ containing y. We provide more structural properties of the $\bar{H}$-path P_2 in the following three lemmas.

Lemma 4.3.4. *Each of the all-positive paths in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$ starting from the internal vertices of the yz'-subpath of P_2 (resp., zy'-subpath of P_2) and connecting to $\bar{H} - P_2$, has endpoints on $P_4 \cup P_6$ (resp., on $P_3 \cup P_5$).*

Proof. We prove the statement for the paths starting from internal vertices of the yz'-subpath of P_2. The remaining part holds by symmetry.

Assume to the contrary that there exists a vv'-path Q all-positive in $(G - E(\bar{H}), \Sigma)$ and internally-vertex disjoint from $\bar{H}$ such that v is an internal vertex of the yz'-subpath of P_2, and $v' \notin V(P_4 \cup P_6)$. By Lemma 4.3.3, we can assume $v' \notin P_1$, since the vertices of P_1 are also vertices of other all-positive $\bar{H}$-paths. Therefore, we have $v' \in V(P_3 \cup P_5)$. It holds that there exists a $-K_4$-subdivision (H', Σ) where the degree-3 vertices are v, x, z, w. Here the H'-path containing e_1^- has the same length as the $\bar{H}$-path containing e_1^-, and since $v \neq y$, the length of the H'-path containing e_2^- is smaller than the one in $\bar{H}$, contradicting Condition (2). □

Lemma 4.3.5. *Let y_1 and y_2 be two vertices on $\bar{H}$ in the same connected component of $P_2 - \{e_2^-\}$ and let P be an all-positive y_1y_2-path in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$. There is no negative path in $(G - E(\bar{H}), \Sigma)$ which is internally vertex-disjoint from $\bar{H}$ and connects an internal vertex of P to $\bar{H}$.*

Proof. Suppose to the contrary that there exists a negative path P^- in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$ whose endpoints are w_1 and w_2, where $w_1 \in V(P) \setminus \{y_1, y_2\}$ and $w_2 \in V(\bar{H})$. In particular, this implies that $e_3^- \in P^-$. Let $P_{y_1y_2}$ be the subpath of P_2 connecting y_1 and y_2. We consider the following two cases.

(1) $w_2 \in V(P_2)$.

If $w_2 \in V(P_2)$, then w_2 belongs to the $P_{y_1y_2}$, as for otherwise there exists a fully-negative theta-graph under some minimum signature Σ' intersecting $\bar{H}$ only on P_2, a contradiction. However, if w_2 belongs to $P_{y_1y_2}$, since $w_2 \notin \{y_1, y_2\}$, then (G, Σ) has to be one of the planar critical signed graphs provided in Theorem 2.4.5, that is either the one in Figure 2.6g, or the one in Figure 2.6i. As we have assumed that $(G, \Sigma) \notin \mathcal{P}^*(3)$, this is a contradiction.

(2) $w_2 \notin V(P_2)$.

Without loss of generality, we assume y_1 and y_2 to be two vertices in the connected component of $P_2 - \{e_2^-\}$ containing y. We can also assume y, y_1, and y_2 to be ordered in this way on P_2.

Let e' be an edge of P incident with y_2, and take a minimum signature Σ' containing e'. Since $P_{y_1y_2} \cup P$ induces a positive circuit, one negative edge of (G, Σ') must be on $P_{y_1y_2}$ and the other negative edge must be e_1^-. Let G' be the graph given by removing from $\bar{H} \cup P$ the y_1z-path on P_2, e_1^-, and e'. It is easy to verify that this graph is connected and does not contain any edge from $\Sigma \cup \Sigma'$. Furthermore, by assumption, we have $w_1, w_2 \in V(G')$. Therefore, there is a path P^* in G' connecting w_1 and w_2. It follows that $P^- \cup P^*$ induces a circuit that is negative in (G, Σ) and positive in (G, Σ'), a contradiction.

This completes the proof. □

Lemma 4.3.6. *Each of the all-positive paths in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$ connecting the yy'-subpath of P_2 to the zz'-subpath of P_2 either consists of one edge or it has exactly one internal vertex of degree three in G which is incident with e_3^-.*

Proof. Suppose to the contrary that there exists a path Q internally vertex-disjoint from $\bar{H}$ and connecting the two connected components of $P_2 - \{e_2^-\}$, such that there exists a positive edge $e \notin E(Q)$ in (G, Σ) incident with an internal vertex of Q. By the fact that $\delta(G) \geq 3$, e belongs to a path internally vertex-disjoint from $\bar{H} \cup Q$ connecting Q to a vertex $v \in V(\bar{H})$. If $v \in V(P_2)$, then it contradicts Lemma 4.3.2. Hence, $v \in V(P_j)$, for $j \in \{3, 4, 5, 6\}$. Depending on whether $j \in \{3, 5\}$, or $j \in \{4, 6\}$, we can choose an all-positive path connecting one endpoint of Q (the one on the yy'-subpath when $j \in \{3, 5\}$, the one in the zz'-subpath otherwise) to v which contradicts Lemma 4.3.4.

Noting that $\delta(G) \geq 3$, if Q has an internal vertex, then this internal vertex must be incident with a negative edge. Due to the fact that there are only three negative edges under Σ, Q has at most one such internal vertex. □

4.4 Proof of Proposition 4.2.1

We can now provide the proof of Proposition 4.2.1.

We observe that a similar proof can be given for general $-K_4$-subdivisions of (G, Σ). In this case, we would not have any information on P_1, therefore the bound for the vertices of each tree would be higher. We first recall the statement of the proposition.

Proposition 4.2.1. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$. If T is an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$, then $|V(T)| \leq 8$. If T is a tree of $(G^*_{\bar{H}}, \Sigma)$ containing a negative edge, then $|V(T)| \leq 10$.*

Proof of Proposition 4.2.1.

Let L_T be the set of leaves of a tree T belonging to $G^*_{\bar{H}}$. Note that $L_T \subset V(\bar{H})$. We first observe that $|V(T)| \leq 2|L_T| - 2$. That is because in computing the number of edges in such a tree, each leaf contributes $\frac{1}{2}$ and any other vertex contributes at least $\frac{3}{2}$, i.e., $\frac{1}{2}|L_T| + \frac{3}{2}(|V(T)| - |L_T|) \leq |V(T)| - 1$.

We now prove that $|L_T| \leq 5$, which implies that $|V(T)| \leq 8$. Assume to the contrary that T has at least six leaves. We observe first that for any

$i \in \{1, 2, \ldots, 6\}$, there are at most 2 leaves of T belonging to a $\bar{H}$-path P_i, as for otherwise there exists a vertex in T which is connected to P_i by at least three all-positive edge-disjoint paths contained in $(G - E(\bar{H}), \Sigma)$, contradicting Lemma 4.3.2. Recalling that P_1 consists of only one edge, by Pigeonhole Principle, there exists a path P_k, for $k \in \{2, 3, \ldots, 6\}$, such that there are at least two leaves of T belonging to P_k and by the above observation, we may assume that there are exactly two leaves of T on P_k. Furthermore, if there is another leaf of T belonging to one of the adjacent $\bar{H}$-paths of P_k, then the contradiction follows from Lemma 4.3.2. Therefore, all of the other (at least) four leaves have to be on the $\bar{H}$-path matching to P_k, a contradiction to Lemma 4.3.2.

Assume now that T contains a negative edge. Since $E(T) \cap E(\bar{H}) = \emptyset$, T can only contain one negative edge, i.e. e_3^-. In this case, we consider a minimum signature Σ^* of (G, Σ) such that $e^* \in \Sigma^*$, where $e^* \in E(T)$ and e^* is incident with a leaf v^* of T. By repeating the previous argument, it holds that $T - v^*$ (which is all-positive under Σ^*) has at most 5 leaves. Hence, $|L_T| \leq 6$ and thus $|V(T)| \leq 10$. □

4.5 Proof of Proposition 4.2.2

Proposition 4.2.2. *Let $(G, \Sigma) \in \mathcal{L}^*(3) \setminus \mathcal{P}^*(3)$. For each vertex of degree 2 in $\bar{H}$, exactly one of the following conditions holds:*

(i) v is in the tree of $(G^*_{\bar{H}}, \Sigma)$ which contains the negative edge e_3^-;

(ii) v belongs to an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$ whose leaves are on at least two different $\bar{H}$-paths.

(iii) v belongs to an all-positive tree of $(G^*_{\bar{H}}, \Sigma)$ consisting of an edge connecting two internal vertices of P_2 belonging to two different components of $P_2 - \{e_2^-\}$.

Proof of Proposition 4.2.2.

It is trivial to see that the case (i) is disjoint from cases (ii) and (iii). That case (ii) is disjoint from case (iii) follows from Lemma 4.3.6. Therefore we need to show that at least one of these cases occurs.

Let $v \in V(P_i)$ for $i \in \{2, 3, \ldots, 6\}$ such that $d_{\bar{H}}(v) = 2$. Suppose to the contrary that v does not satisfy any of conditions (i), (ii), and (iii). Noting that (G, Σ) is irreducible, by Theorem 2.2.3 v has at least three neighbors. Hence, there exists an edge $e \notin E(\bar{H})$ incident with v in (G, Σ) and which is not parallel to any edge of $\bar{H}$. We may assume that $e \notin \Sigma$ (i.e., $e \neq e_3^-$), as for otherwise it satisfies condition (i). By Lemma 4.3.2, considering the connectivity of G, if e belongs to a tree in G^* having more than one edge, then condition (ii) is satisfied. Hence, the tree containing e is just an edge. We assume that $e = vv'$ and $v' \in P_i$, as for otherwise condition (ii) is satisfied. Furthermore, note that the vv'-subpath of P_i does not contain e_2^-, or it satisfies condition (iii). Therefore, by Condition (3) on the choice of $\bar{H}$ (i.e., the minimality of $V(\bar{H})$), e is parallel to an edge of P_i, a contradiction. □

4.6 Proof of Proposition 4.2.3

Proposition 4.2.3. *Let* $(G, \Sigma) \in \mathcal{L}^*(3)$. *There exist at most two all-positive pairwise internally vertex-disjoint paths in* $(G - E(\bar{H}), \Sigma)$ *which are internally vertex-disjoint from* $\bar{H}$ *and connect the two connected components of* $P_2 - \{e_2^-\}$.

Proof of Proposition 4.2.3.

Suppose to the contrary that there exist three pairwise internally vertex-disjoint and all-positive paths $Q_1, Q_2,$ and Q_3 in $(G - E(\bar{H}), \Sigma)$, internally vertex-disjoint from $\bar{H}$, and with endpoints on the two different components of $P_2 - \{e_2^-\}$. For each $i \in \{1, 2, 3\}$, we denote by y_i the endpoint of Q_i in the yy'-subpath of P_2 and by z_i the other endpoint of Q_i. Furthermore, we also assume that y, y_1, y_2, y_3, y' are in this order on P_2. We consider two cases based on the order of z_1, z_2, z_3.

- There exist $i, j \in \{1, 2, 3\}$ such that $i < j$ and the vertices z', z_j, z_i, z are in this order on P_2. Since we only consider Q_i and Q_j, without loss of generality, we may assume $i = 1, j = 2$. In this case, Q_1 together with P_2 contains a negative circuit C^-, while $Q_1 \cup Q_2 \cup P_2$ contains an all-positive circuit C^+. Furthermore, it holds that $E(C^- \cap C^+ \cap \bar{H}) = \emptyset$. Let Π be a minimum signature of (G, Σ) such that $\Pi \cap E(Q_1) \neq \emptyset$. Since $E(Q_1) \cap E(\bar{H}) = \emptyset$, $|\Pi \cap E(\bar{H})| = 2$. Observe that $Q_1 \cup Q_2 \cup P_2$ contains a fully-negative theta-graph under Π, contradicting Lemma 4.3.1.

- The vertices $z' z_1 z_2 z_3 z$ are in this order on P_2. In this case, let Π be a minimum signature such that $\Pi \cap E(Q_2) \neq \emptyset$. Note that $Q_1 \cup Q_2 \cup Q_3 \cup P_2 - \{e_2^-\}$ contains a theta-graph, which is fully-negative under Π, a contradiction to Lemma 4.3.1.

$\square$

4.7 Proof of Proposition 4.2.4

Therefore, it remains to prove Proposition 4.2.4, that is:

Proposition 4.2.4. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$. There are at most 24 all-positive pairwise internally vertex-disjoint paths in $(G - E(\bar{H}), \Sigma)$ which are internally vertex-disjoint from $\bar{H}$ and connect different $\bar{H}$-paths.*

Proof of Proposition 4.2.4.

Proposition 4.2.4 is a consequence of many statements which provide bounds for the number of edge-disjoint paths connecting two different $\bar{H}$-paths. In each of these statements we consider special cases depending on the positions of the paths with respect to the $\bar{H}$-paths.

We first prove that the number of all-positive and pairwise internally vertex-disjoint paths between two adjacent $\bar{H}$-paths is bounded.

Proposition 4.7.1. *There are at most two all-positive pairwise internally vertex-disjoint paths in $(G - E(\bar{H}), \Sigma)$ connecting P_i to P_j, for $i, j \in \{2, \ldots, 6\}$ with $i \neq j$, where the $\bar{H}$-paths P_i and P_j are not matching.*

Proof. We first show for any $i, j \in \{3, 4, 5, 6\}$, that there are at most two all-positive pairwise internally vertex-disjoint paths connecting P_i to P_j, where P_i and P_j are not matching. Note that P_i and P_j share a common vertex u. Suppose to the contrary that there are three all-positive and pairwise internally vertex-disjoint paths Q_1, Q_2, and Q_3 connecting P_i to P_j. For $l \in \{1, 2, 3\}$, let v_l and u_l denote the endpoints of Q_l on the path P_i and P_j, respectively. Assume that u, v_1, v_2, and v_3 are ordered on P_i following this ordering and $u, u_{l_1}, u_{l_2}, u_{l_3}$ are ordered on P_j in this order. We consider two possibilities:

- There exists at least one pair $l_k, l_{k'}$ satisfying that $k < k'$ and $l_k < l_{k'}$. In this case, by choosing an edge in Q_{l_k} to be negative, Q_{l_k}, $Q_{l_{k'}}$, $uv_{l_{k'}}$-subpath of P_i and $uu_{l_{k'}}$-subpath of P_j form a fully-negative theta-graph which does not share any edge with the remaining four paths (except P_i and P_j) of $\bar{H}$, a contradiction to Lemma 4.3.1.

- We have $l_1 > l_2 > l_3$, i.e., u, u_3, u_2, u_1 are located on P_j following this order. In this case, Q_1, Q_2, Q_3, the v_1v_3-subpath of P_i and the u_3u_1-subpath of P_j form a fully-negative theta-graph by choosing one edge in Q_2 to be negative, which does not share any edge with the remaining four paths of $\bar{H}$, a contradiction.

It remains to consider the number of all-positive paths connecting P_2 to P_j, for $j \in \{3, 4, 5, 6\}$. By Lemma 4.3.4, each P_j can only be connected to one of the two components of $P_2 - \{e_2^-\}$. Hence, depending on the P_j we can restrict us on one of the two components of $P_2 - \{e_2^-\}$ and repeat a similar argument as in the previous case. □

It remains to show that the number of edge-disjoint all-positive paths in $(G - E(\bar{H}), \Sigma)$ connecting two all-positive matching $\bar{H}$-paths is bounded. To do that, we consider two different situations: (I). There is at least one path of length at least two connecting matching $\bar{H}$-paths; (II). All of the paths connecting matching $\bar{H}$-paths are of length one.

We start with some preparations.

Lemma 4.7.2. *Let $i \in \{3,5\}$, and let Π be a minimum signature of (G,Σ). If $\Pi \cap \Sigma = \emptyset$ and $|\Pi \cap E(P_i \cup P_{i+1})| = 2$, then the underlying graph $\bar{H} - \{\Pi \cup \{e_1^-, e_2^-\}\}$ has two connected components, $G - \{\Pi \cup \{e_1^-, e_2^-\}\}$ is connected, and $\{\Pi \cup \Sigma\}$ is an edge-cut of G.*

Proof. Since $(\bar{H},\Sigma)$ is a $-K_4$-subdivision, it is straightforward to see that $\bar{H} - \{\Pi \cup \{e_1^-, e_2^-\}\}$ consists of two connected components. Suppose that $G - \{\Pi \cup \{e_1^-, e_2^-\}\}$ is not connected, that is to say, $\{\Pi \cup \{e_1^-, e_2^-\}\}$ is an edge-cut. Thus we have an edge-cut in (G,Π) with two positive edges and three negative edges, a contradiction. That $\{\Pi \cup \Sigma\}$ is an edge-cut of G trivially follows from Theorem 2.1.2. □

Lemma 4.7.3. *Let $i \in \{3,5\}$ and $e \in E(P_j \cup P_{j+1})$, with $j \in \{3,5\} \setminus \{i\}$. If there exist two edge-disjoint all-positive paths Q_1 and Q_2 in $(G - E(\bar{H}),\Sigma)$, internally vertex-disjoint from $\bar{H}$, connecting P_i to P_{i+1}, then for each minimum signature Π containing e it holds that $|\Pi \cap E(\bar{H})| = 3$.*

Proof. Suppose to the contrary that there exists a minimum signature Π containing e such that $|\Pi \cap E(\bar{H})| = 2$. As $(\bar{H},\Pi)$ is a $-K_4$-subdivision, the two negative edges of $(\bar{H},\Pi)$ are on P_j and P_{j+1}, say $e \in \Pi \cap E(P_j)$ and $e' \in \Pi \cap E(P_{j+1})$. Note that the third negative edge of Π is not contained in $E(\bar{H})$. Let C be the positive circuit under Σ contained in $P_i \cup P_{i+1} \cup P_j \cup Q_1$ and let C' be the positive circuit under Σ contained in $P_i \cup P_{i+1} \cup P_{j+1} \cup Q_2$. Observe that $e \in E(C)$ and $e' \in E(C')$, but $e \notin E(C')$ and $e' \notin E(C)$. Note that each of C and C' must contain exactly two edges from Π but they share no edges outside $\bar{H}$, a contradiction. □

The next lemma holds generally for P_i and P_{i+1} for $i \in \{3,5\}$. Here we only state it for P_3 and P_4 for the convenience of the statement.

Lemma 4.7.4. *Let Q_1 and Q_2 be all-positive paths in $(G - E(\bar{H}),\Sigma)$ (which might be the same path) internally vertex-disjoint from $\bar{H}$ connecting P_3 to P_4 such that $V(Q_1) \cap V(P_3) = \{v\}$ and $V(Q_2) \cap V(P_4) = \{u\}$. Assume that there exists a minimum signature Σ^a with edges $e_1^a \in E(Q_1)$, e_2^a in the vw-subpath* *of P_3 and e_3^a in the xu-subpath of P_4. Furthermore, assume that there exists a minimum signature Σ^b with edges $e_1^b \in E(Q_2)$, e_2^b in the uz-subpath of P_4 and e_3^b in the yv-subpath of P_3.*

If there exists an all-positive path Q_3 in $(G - E(\bar{H}), \Sigma)$ edge-disjoint from Q_1 and Q_2 connecting P_3 to P_4, internally vertex-disjoint from $\bar{H}$ and such that $E(Q_3) \cap (\Sigma \cup \Sigma^a \cup \Sigma^b) = \emptyset$, then both of $\bar{H} \cup Q_1 \cup Q_2 - \{e_1^-, e_2^-, e_3^a, e_3^b\}$ and $\bar{H} \cup Q_1 \cup Q_2 - \{e_1^-, e_2^-, e_2^a, e_2^b\}$ are connected.

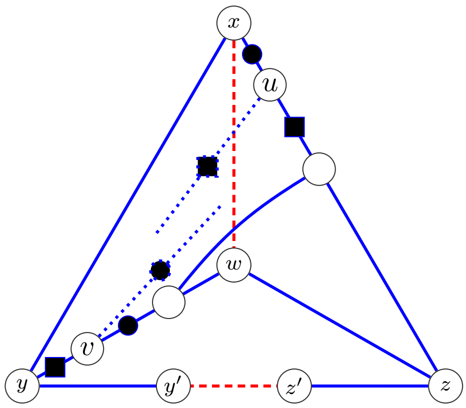


Figure 4.3: The circles and boxes represent the two signatures Σ^a and Σ^b, respectively, of Lemma 4.7.4

Proof. Assume to the contrary that both signatures Σ^a and Σ^b are minimum signatures of (G, Σ) (see Figure 4.3) but one of $\bar{H} \cup Q_1 \cup Q_2 - \{e_1^1, e_2^-, e_3^a, e_3^b\}$ and $\bar{H} \cup Q_1 \cup Q_2 - \{e_1^-, e_2^-, e_2^a, e_2^b\}$ is disconnected. By symmetry, we may restrict ourselves to the case when $\bar{H} \cup Q_1 \cup Q_2 - \{e_1^-, e_2^-, e_3^a, e_3^b\}$ is disconnected. We first prove the following claim.

Claim 4.7.5. *$G - (\Sigma \cup \{e_3^a, e_3^b\})$ is connected.*

Proof of the claim: Assume to the contrary that $G - (\Sigma \cup \{e_3^a, e_3^b\})$ is disconnected. Since no edge-cut of (G, Σ) can have more negative edges than positive ones while $G - (\Sigma \cup \{e_3^a, e_3^b\})$ is indeed disconnected, there exists a proper subset E' of $\Sigma \cup \{e_3^a, e_3^b\}$ such that E' is an edge-cut and thus $G - E'$ is disconnected.

We claim that $E' = \{e_1^-, e_2^-, e_3^a, e_3^b\}$. If not, any proper subset of $\{e_1^-, e_2^-, e_3^a, e_3^b\}$ even together with e_3^- would not disconnect $(\bar{H}, \Sigma)$, a contradiction.

We denote the connected components of $G - \{e_1^-, e_2^-, e_3^a, e_3^b\}$ by O_1 and O_2 such that $\{e_1^a, e_2^a, e_1^b, e_2^b\} \subset E(O_1)$. Let P^- denote the path containing e_3^- which is internally vertex-disjoint from $\bar{H}$, and let $V(P^-) \cap V(\bar{H}) = \{w_1, w_2\}$.

First, observe that w_1 and w_2 are in the same connected component of $G - \{e_1^-, e_2^-, e_3^b, e_3^a\}$ and, moreover, $\{w_1, w_2\} \subset V(O_1)$, as for otherwise, P^- together with $\bar{H}$ would contain a circuit which is negative under Σ but positive under both Σ^a and Σ^b. By Lemma 4.7.2, w_1 and w_2 are in two different connected components of $G - (\Sigma \cup \Sigma^a)$ and in two different components of $G - (\Sigma \cup \Sigma^b)$. Let O_u and O_v denote the connected components of $\bar{H} - (\Sigma^a \cup \Sigma^b)$ containing the vertex u and v, respectively. Observe that one of the connected components of $\bar{H} - (\Sigma \cup \Sigma^b)$ (resp. $\bar{H} - (\Sigma \cup \Sigma^a)$) is entirely contained in O_1, while the other connected component intersects O_1 in O_u (resp., O_v). Therefore, since O_u and O_v are disjoint, then each of them contains one endpoint of P^-. Without loss of generality, we can assume $w_1 \in V(O_u)$ and $w_2 \in V(O_v)$.

Note that by $E(Q_3) \cap (\Sigma \cup \Sigma^a \cup \Sigma^b) = \emptyset$, the endpoints of Q_3 must be on the same connected component of $\bar{H} - (\Sigma \cup \Sigma^a \cup \Sigma^b)$. Hence, either $Q_3 \cup P_3 \cup P_4 \cup P_5$ or $Q_3 \cup P_3 \cup P_4 \cup P_6$ contain a positive circuit C^+ in $\bar{H} - (\Sigma \cup \Sigma^a \cup \Sigma^b)$. Note that in each case, there is a negative circuit C^- containing P^- (thus e_3^-) which is edge-disjoint from C^+. Let e^* be an edge of $E(C^+ \cap (P_5 \cup P_6))$ and let Σ^* be a minimum signature containing e^*. Noting that Q_1 and Q_3 are two all-positive (under Σ) edge-disjoint paths connecting P_3 to P_4, by Lemma 4.7.3, $|\Sigma^* \cap E(\bar{H})| = 3$, and in particular, one edge of Σ^* has to belong to $E(P_1 \cup P_2)$. Since C^+ and C^- are edge-disjoint, it holds that $\Sigma^* \subset E(C^+ \cup C^-)$, but this is a contradiction since C^+ and C^- do not share edges with P_1 and P_2. $\Diamond$

We claim that each of $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma^a \cup \{e_1^-, e_2^-\})$ and $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma^b \cup \{e_1^-, e_2^-\})$ is disconnected. It is enough to show this for $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma^a \cup \{e_1^-, e_2^-\})$. Suppose this is not the case and let P^- be a path containing e_3^- internally vertex-disjoint from $\bar{H} \cup Q_1 \cup Q_2$ and with endpoints $w_1, w_2 \in$

$V(\bar{H} \cup Q_1 \cup Q_2)$. Since $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma^a \cup \{e_1^-, e_2^-\})$ is connected, it contains a w_1w_2-path P^+. By construction, $P^+ \cup P^-$ contains a circuit C such that $E(C^+) \cap \Sigma = \{e_3^-\}$ but $E(C^+) \cap \Sigma^a = \emptyset$, a contradiction.

By Claim 4.7.5, there is a path P^* all-positive under Σ and internally vertex-disjoint from $\bar{H} \cup Q_1 \cup Q_2$ connecting the two connected components of $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma \cup \{e_3^b, e_3^a\})$. Since $\bar{H} - (\Sigma \cup \{e_3^a\})$ is connected, in $\bar{H} \cup P^* - (\Sigma \cup \{e_3^a\})$ there exists a circuit C containing P^* and e_3^b, and observe that C contains no edges of Σ. So the circuit C is positive under Σ and thus C must contain an even number of edges from $\{e_1^a, e_2^a\}$.

Since $\bar{H} \cup Q_1 \cup Q_2 - (\Sigma^a \cup \{e_1^-, e_2^-\})$ is disconnected, up to suppression of vertices of degree 2 in $\bar{H} \cup Q_1 \cup Q_2$, the edges e_3^b, e_1^a, and e_2^a are adjacent (at v) in $\bar{H} \cup Q_1 \cup Q_2$. Thus it is impossible for C to contain all of the three edges e_1^a, e_2^a, and e_3^b at the same time. It follows that C contains no edge from $\{e_1^a, e_2^a\}$. We now consider the signed graph $G - (\Sigma \cup \Sigma^a)$, which by Lemma 4.7.2, has two connected components. Moreover, as C contains no edges of $\Sigma \cup \Sigma^a$, C is contained in one connected component of $G - (\Sigma \cup \Sigma^a)$ which contains e_3^b (since $e_3^b \in E(C)$). However, by assumption on the structure, e_1^b and e_2^b are in the other connected component. Hence, C is negative in (G, Σ^b), contradicting the fact that C is positive under Σ. □

Lemma 4.7.6. *For $i \in \{3, 5\}$, assume that under Σ there is an all-positive path Q connecting P_i to P_{i+1} with endpoints v and u. Let $e \in E(Q)$. If there exists a minimum signature Σ' of (G, Σ) containing e and two more edges e_2' and e_3' which are on the path P_i and P_{i+1} respectively, then following the order in Figure 4.4, one of the two following conditions is satisfied: (1) Both of the two endpoints of e_2' are before v and both of the two endpoints of e_3' are after u; (2) Both of the two endpoints of e_2' are after v and both of the two endpoints of e_3' are before u.*

Given $i \in \{3, 5\}$ and a positive integer n, let $Q_1, ..., Q_n$ be all-positive paths in $(G - E(\bar{H}), \Sigma)$ pairwise vertex-disjoint connecting P_i to P_{i+1} such that for each $j \in \{1, ..., n\}$, $V(Q_j) \cap V(P_i) = \{v_j\}$, $V(Q_j) \cap V(P_{i+1}) = \{u_j\}$, and the vertices $v_1, ..., v_n$ are ordered in this way following the direction of the arrow in Figure 4.4. If the vertices $w_1, ..., w_n$ are also ordered following the direction given in Figure 4.4, then we say that $Q_1, ..., Q_n$ are *aligned* (see Figure 4.5). If two paths connecting P_i and P_{i+1} are internally vertex-disjoint but not aligned, we say that they are *crossing*.

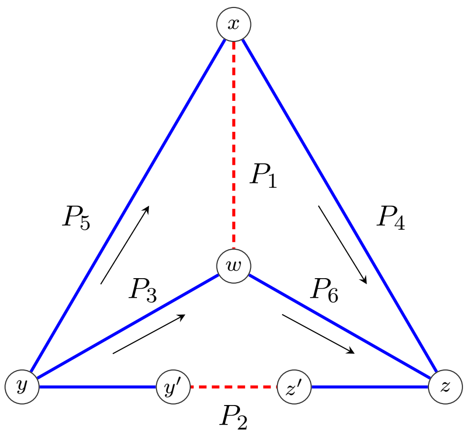


Figure 4.4: Ordering of the vertices

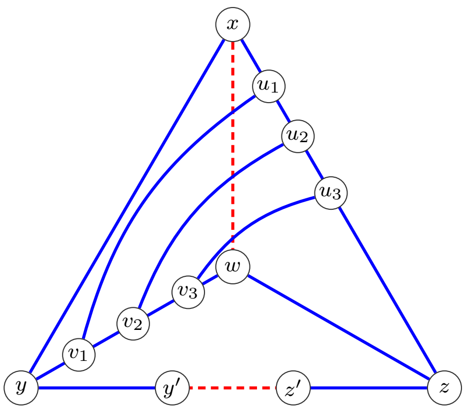


Figure 4.5: An example of three aligned paths

Now we can show that there is at most one all-positive path of length at least 2 connecting two matching $\bar{H}$-paths.

Proposition 4.7.7. *For $i \in \{3, 5\}$, if there is an all-positive path Q, internally vertex-disjoint from $\bar{H}$, of length at least two connecting P_i to P_{i+1} in (G, Σ), then there is no other all-positive path in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H} \cup Q$ connecting P_i to P_{i+1}.*

Proof. Without loss of generality, we assume $i = 3$. Suppose to the contrary that in $(G - E(\bar{H}), \Sigma)$ there exist two all-positive internally vertex-disjoint paths Q_1 and Q_2 connecting P_3 to P_4, such that one of those paths has at least two edges. By symmetry, we can assume $|E(Q_1)| \geq 2$. For $k \in \{1, 2\}$, let $V(Q_k) \cap V(P_3) = \{v_k\}$ and $V(Q_k) \cap V(P_4) = \{u_k\}$. We denote by C_1^+ the positive circuit given by Q_1, Q_2, the $v_1 v_2$-subpath of P_3, and the $u_1 u_2$-subpath of P_4 and note that $E(C_1^+ \cap \bar{H}) \subset E(P_3 \cup P_4)$.

Noting that $|E(Q_1)| \geq 2$, let s be an internal vertex of Q_1. Since (G, Σ) is irreducible, by Proposition 3.3.3, there is a vertex $s' \in V(\bar{H} \cup Q_2)$ such that there exists a path R internally vertex-disjoint from $\bar{H} \cup Q_2$ connecting s to s'. We consider two possibilities based on whether Q_1 and Q_2 are aligned or not.

Case 1. Q_1 and Q_2 are crossing.

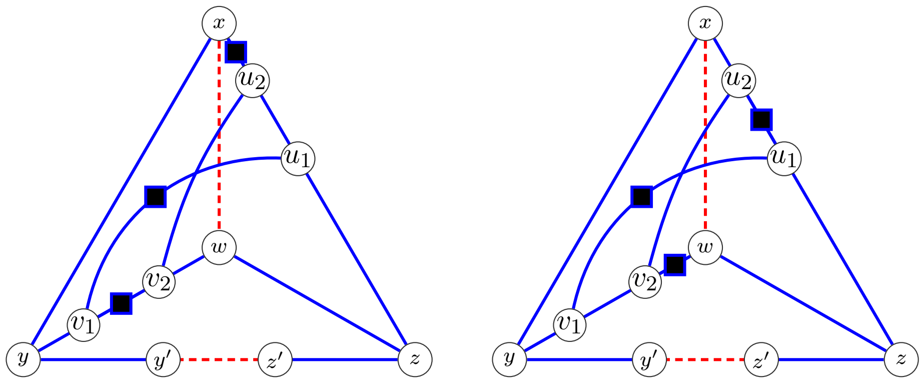


Figure 4.6: Possible signatures of Case 1 indicated by the black boxes

Case 1.1. The vertex s' is not on one of the following paths: The v_1w-subpath of P_3, the xu_1-subpath of P_4, or Q_2.

First, observe that for any minimum signature Σ^* such that $|\Sigma^* \cap E(Q_1)| = 1$, Σ^* has to be one of the two signatures shown in Figure 4.6, since C_1^+ is positive, $|\Sigma^* \cap E(\bar{H})| = 2$, and Lemma 4.7.6 applies. Note that, in both of the cases, v_1 and u_1 belong to different connected components in both $\bar{H} - (\Sigma \cup \Sigma^*)$ and $G - (\Sigma \cup \Sigma^*)$. We denote the two connected components of $G - (\Sigma \cup \Sigma^*)$ by $O_1^{\Sigma^*}$ and $O_2^{\Sigma^*}$ satisfying that $v_1 \in V(O_1^{\Sigma^*})$ and $u_1 \in V(O_2^{\Sigma^*})$. Let s^* be a vertex of $\bar{H}$ which is neither a vertex of the v_1w-subpath of P_3 nor a vertex of the xu_1-subpath of P_4. Note that there is a $j \in \{1, 2\}$ such that for any such minimum signature Σ^* depicted in Figure 4.6, s^* belongs to the connected component $O_j^{\Sigma^*}$. Two cases may occur depending on whether e_3^- belongs to R.

- Assume that $e_3^- \notin E(R)$. In this case, R is an all-positive path in (G, Σ). Let e_1^a be an edge of the v_1s-subpath of Q_1 and let Σ^a be a minimum signature such that $e_1^a \in \Sigma^a$. The vertices u_1 and s' are connected by a path all-positive under both of Σ and Σ^a, thus they are in the same connected component of $G - (\Sigma \cup \Sigma^a)$. Similarly, let e_1^b be an edge of the u_1s-subpath of Q_1 and let Σ^b be a minimum signature such that $e_1^b \in \Sigma^b$. Arguing as before, it follows that v_1 and s' are in the same connected component of $G - (\Sigma \cup \Sigma^b)$. However this is impossible because v_1 and u_1 are always in two different connected components of $G - (\Sigma \cup \Sigma^*)$ where $|\Sigma^* \cap E(Q_1)| = 1$, a contradiction.

- Assume now $e_3^- \in E(R)$. We argue similarly as before. Let e_1^a be an edge of the v_1s-subpath of Q_1 and let Σ^a be a minimum signature containing such an edge. By Lemma 4.7.2, s' and u_1 are on two different connected components of $G - (\Sigma \cup \Sigma^a)$. Let e_1^b be an edge of the u_1s-subpath of Q_1 and let Σ^b be a minimum signature containing such an edge. By Lemma 4.7.2, s' and v_1 belong to different components of $G - (\Sigma \cup \Sigma^b)$. This is not possible since there are only two connected components of $G - (\Sigma \cup \Sigma^*)$, where $|\Sigma^* \cap E(Q_1)| = 1$, containing either v_1 or u_1, a contradiction.

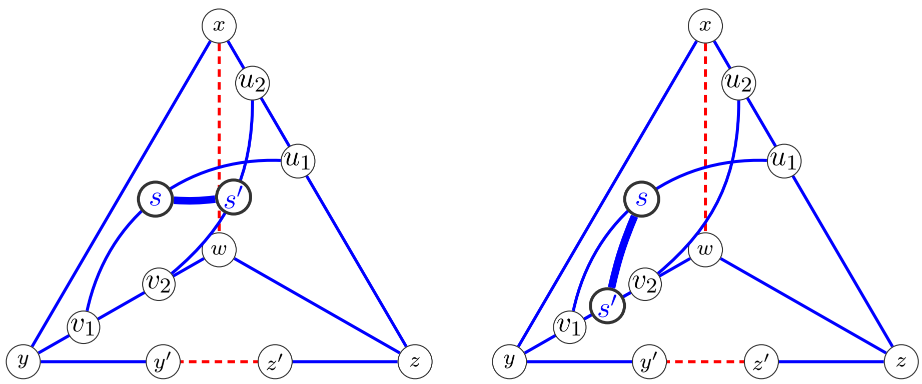


Figure 4.7: Case 1.2

Case 1.2. The vertex s' is on one of the following paths: The v_1w-subpath of P_3, the xu_1-subpath of P_4, or Q_2.

We first observe that if s' is an internal vertex of the v_2w-subpath of P_3, then we can replace Q_1 with another path Q_1' given by the union of R and the su_1-subpath of Q_1. In this case, we obtain two aligned paths, Q_1' and Q_2, which satisfy the conditions of the statement. Hence, the case where s' is an internal vertex of the v_2w-subpath of P_3 will be studied later. Therefore, by symmetries we have two cases shown in Figure 4.7. We consider two cases depending on whether $e_3^- \in E(R_3)$.

- If $e_3^- \notin E(R)$, then we define a minimum signature Σ^a as follows: We choose one edge on R to be negative and denote it by e_1^a, then the other two negative edges have to be on the v_1v_2-subpath of P_3 and on the u_1u_2-subpath of P_4, respectively. However, Σ^a is not switching equivalent to Σ as Q_1 together with $\bar{H} - \{e_1^-, e_2^-\}$ induces two negative circuits under Σ^a but they are positive under Σ, a contradiction.

- If $e_3^- \in E(R)$, then e_3^- belongs to a negative circuit C^- such that $C^- \cap \bar{H}$ is a path S on the v_1w-subpath of P_3. Let e_1^b be an edge of P_5 and note that e_1^b belongs to an all-positive circuit of (G, Σ), denoted by C_b, contained in $P_5 \cup P_3 \cup P_4 \cup Q_1$. In particular, $C_b \cap P_3$ is the yv_1-subpath. Since s' is not in the yv_1-subpath of P_3, $C_b \cap \bar{H} \cap C^- = \emptyset$. Let Σ^b be a minimum signature of (G, Σ) containing e_1^b. By Lemma 4.7.3, $|\Sigma^b \cap E(\bar{H})| = 3$. Therefore, exactly one negative edge of Σ^b, say e_2^b, can be on $P_3 \cup P_4$. Since C_b is a positive circuit and $e_1^b \in E(C_b)$, we have that $e_2^b \in E(C_b)$. By the fact that $E(C_b \cap C^- \cap \bar{H}) = \emptyset$, it follows that C^- contains no edge of Σ^b and thus is positive in (G, Σ^b), a contradiction.

Case 2. Q_1 and Q_2 are aligned.

We recall that C_1^+ is the circuit contained in $Q_1 \cup Q_2 \cup P_3 \cup P_4$. Since Q_1 and Q_2 are aligned, there exists a positive circuit C_2^+ in $\bar{H} \cup Q_1$ such that $C_2^+ \cap C_1^+ = Q_1$. As in the previous case, we discuss two subcases.

Case 2.1. The vertex s' is not an internal vertex of $P_3 \cup P_4$.

There exist two circuits C_1 and C_2 in $\bar{H} \cup Q_1 \cup Q_2 \cup R - \{e_1^-, e_2^-\}$ such that C_1 intersects P_3 but not P_4, C_2 intersects P_4 but not P_3, and both of C_1 and C_2 contain R. Furthermore, for a certain $k \in \{1, 2\}$, it also holds that

$(C_1 \cup C_2) \cap C_k^+ = Q_1$. Note that, for each minimum signature Σ' such that $|\Sigma' \cap E(Q_1 \cup Q_2)| = 1$, each of the other two negative edges of Σ' is on P_3 and P_4, respectively (see Lemma 4.7.6).

- We first assume $e_3^- \notin E(R)$. Here, we just prove the case when $(C_1 \cup C_2) \cap C_1^+ = Q_1$. The case when $(C_1 \cup C_2) \cap C_2^+ = Q_1$ can be proved in the same way, so we leave the details to the reader.

  As $e_3^- \notin E(R)$, C_1 and C_2 are all-positive in (G, Σ). In this case, s' is not an internal vertex of P_2, as for otherwise, we would have a contradiction to Lemmas 4.3.4.

  Let e_1^a be an edge on the $v_1 s$-subpath of Q_1 and let Σ^a be a minimum signature of (G, Σ) such that $e_1^a \in \Sigma^a$. Hence, one more negative edge e_2^a of Σ^a belongs to $P_3 \cap C_1$, and the other one, e_3^a, satisfies that $e_3^a \in E(P_4 \cap C_1^+)$. Similarly, let e_1^b be an edge on the $u_1 s$-subpath of Q_1 and let Σ^b be a minimum signature of (G, Σ) containing e_1^b. The other two negative edges of Σ^b are $e_2^b \in E(P_4 \cap C_2)$ and $e_3^b \in E(P_3 \cap C_1^+)$, see Figure 4.8. It is easy to verify that for the two minimum signatures Σ^a and Σ^b, the conditions of Lemma 4.7.4 on $\bar{H} \cup Q_1$ are all satisfied. Therefore, Lemma 4.7.4 implies that $\bar{H} \cup Q_1 - \{e_1^-, e_2^-, e_3^a, e_3^b\}$ and $\bar{H} \cup Q_1 - \{e_1^-, e_2^-, e_2^a, e_2^b\}$ are connected, a contradiction.

- Assume $e_3^- \in E(R)$. In this case, C_1 and C_2 are negative circuits, as both of them contain the path R.

  First, note that $s' \notin V(P_5 \cup Q_2)$. Or otherwise, given an edge $e^* \in E(P_6)$, a minimum signature Σ^* such that $e^* \in \Sigma^*$ provides a contradiction. Similarly, $s' \notin V(P_6)$, as for otherwise the contradiction is given by taking a minimum signature where an edge on P_5 is negative.

  Furthermore, if s' belongs to the yy'-subpath of P_2, then the contradiction follows by considering a minimum signature where an edge of Q_2 is taken to be negative. Hence, together with the condition that $s' \notin V(P_3 \cup P_4)$,

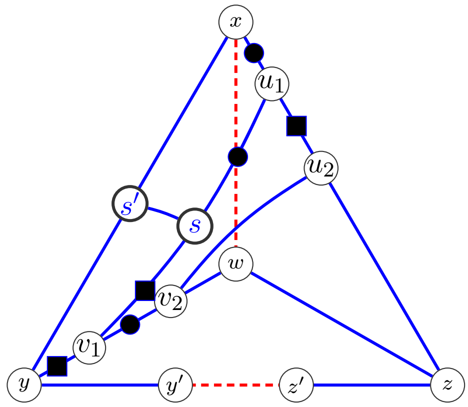


Figure 4.8: Case 2.1, with $s' \in V(P_4)$. The boxes represent Σ^a and the circles represent Σ^b

it remains to consider the case when s' is an internal vertex of the $y'z$-subpath of P_2.

First, observe that $z \neq z'$. By Lemma 3.3.2 and the fact that $d_{\bar{H}}(z') = 2$, there exists a positive edge under Σ incident with z' which is not in $E(\bar{H})$. By connectivity, this edge belongs to a path R' all-positive under Σ, which is internally vertex-disjoint from $\bar{H}$, connecting z' to another vertex of $V(\bar{H})$, say s''. Furthermore, by Lemma 4.3.4, $s'' \in V(P_3 \cup P_5 \cup P_2)$.

We now show that s'' is not on the zz'-subpath of P_2. Assume the contrary and in this case, R' cannot belong to a multi-edge, as for otherwise e_3^- is parallel to e_1^- by the choice of Σ, contradicting the fact that R contains e_3^-. Moreover, R' is not an edge connecting two vertices of the $z'z$-subpath of P_2 which are in distance more than 1, as it would contradict the minimality of $\bar{H}$. Thus R' has an internal vertex of degree at least 3. If this vertex is incident with an edge positive under Σ, then by Lemma 4.3.2, it can only connect this internal vertex of R' to an internal vertex of P_1, a contradiction to Lemma 4.3.3. If this vertex is incident with an edge negative under Σ, then the contradiction follows from Lemma 4.3.5. Therefore R cannot consist of more than one edge and has to be a parallel edge, a contradiction.

Moreover, we claim that s'' is neither on C_2^+ nor on the yy'-subpath of P_2. Otherwise, given a minimum signature Σ^* such that an edge of Q_2 is chosen to be negative, it can be observed that $G - (\Sigma \cup \Sigma^*)$ is not disconnected, contradicting Lemma 4.7.2. Brought together, we have that $s'' \in V(P_3) \backslash V(C_2^+)$. This implies that there exists a negative circuit C^- in $R' \cup P_2 \cup P_3$ such that $E(C_2 \cap C^- \cap \bar{H}) = \emptyset$.

Let $e_1' \in P_5$ and let Σ' be a minimum signature containing e_1'. By Lemma 4.7.3, exactly one of the other edges of Σ', say e_2', belongs to $E(P_3 \cup P_4) \cap E(C_2^+)$. Since C_2 is a negative circuit under Σ, $E(C_2 \cap \bar{H}) \subset E(P_2 \cup P_4)$, and by the fact that $E(C_2 \cap P_4) \cap E(C_2^+ \cap P_4) = \emptyset$, the third negative edge of Σ' is on the $s'z$-subpath of P_2. So $e_2' \in E(C_2^+ \cap P_4)$. Hence, C^- is a positive circuit under Σ', a contradiction.

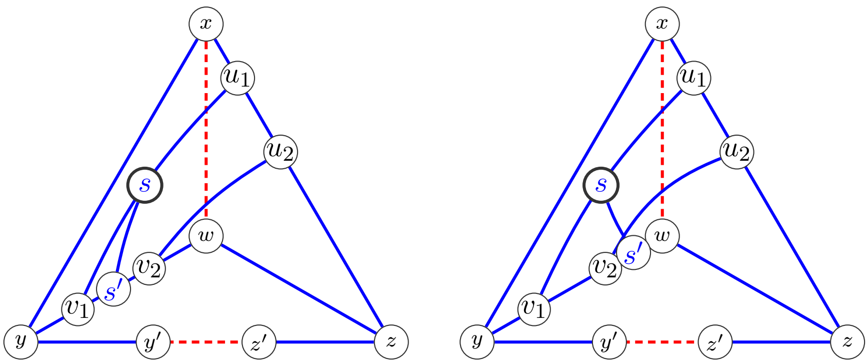


Figure 4.9: Case 2.2

Case 2.2. The vertex s' is an internal vertex of $P_3 \cup P_4$.

Without loss of generality, we assume $s' \in V(P_3)$. There are two possibilities depicted in Figure 4.9.

We first prove that R is a path all-positive under Σ. If not, first assume that s' is on the v_1w-subpath of P_3. We consider $e^* \in E(P_5)$ and note that e^* is contained in the positive circuit C_2^+. By Lemma 4.7.3, each minimum signature Σ^* containing e^* has three negative edges on $\bar{H}$. Hence, (G, Σ^*)

has exactly one negative edge on $P_3 \cup P_4$. Consider the negative circuit C^- in $R \cup Q_1 \cup P_3$. Since $E(C^-) \cap E(C_2^+) \cap E(\bar{H}) = \emptyset$, we have either $|E(P_3) \cap \Sigma^*| = 2$ or $|E(P_3 \cup P_4) \cap \Sigma^*| = 2$, each of which is a contradiction. Similarly, if s' is on the yv_1-subpath of P_3, then we get a similar contradiction by taking a minimum signature Σ' such that $\Sigma' \cap E(P_6) \neq \emptyset$.

Therefore, we can assume that R is a path all-positive under Σ. We first assume that s' is not on the v_2w-subpath of P_3. Note that both the sv_1-path and ss'-path are all-positive and internally vertex-disjoint from $\bar{H}$, and thus we may switch the role of v_1 and s'. By symmetry, we only consider the case when s' is on the v_1v_2-subpath of P_3.

We now define two minimum signatures Σ^a and Σ^b as follows: For Σ^a, we choose one edge e_1^a on the sv_1-subpath of Q_1 to be negative, then the second negative edge e_2^a must be on the v_1s'-subpath of P_3 and the third negative edge e_3^a must be on u_1x-subpath of P_4. Similarly, for Σ^b, we choose one edge e_1^b on R to be negative, and the second and third edges e_2^b, e_3^b have to be on the v_1s'-subpath of P_3 and on the u_1u_2-subpath of P_4, respectively.

Considering a minimum signature containing an edge e_1^p from the su_1-subpath of Q_1, by the assumption on the structure of the graph, there are only two possibilities: Either we have a minimum signature Σ_1^p containing e_1^p and two negative edges e_2^p and e_3^p, where e_2^p is from the u_1x-subpath of P_4 and e_3^p from the $s'v_2$-subpath of P_3; or we have a minimum signature Σ_2^p containing e_1^p and two negative edges $e_2'^p$ and $e_3'^p$, $e_2'^p$ from the u_1u_2-subpath of P_4 and $e_3'^p$ from the yv_1-subpath of P_3, respectively. In each case, we may apply Lemma 4.7.4 to the minimum signatures Σ_1^p and Σ^b (similarly, Σ_2^p and Σ^a) and obtain a contradiction on the connectivity.

The case where s' is on the v_2w-subpath of P_3 is proved similarly by applying Lemma 4.7.4 on the two following minimum signtures: Σ^a, containing an edge of the v_1s-subpath of Q_1, and therefore one more edge on the xu_1-subpath of P_4, and the other on the v_1v_2-subpath of P_3; Σ^b, containing one edge of the su_1-subpath of Q_1, and therefore one more edge on the yv_1-subpath of P_3, and the other on the u_1u_2-subpath of P_4. This completes the proof of the proposition. □

It remains to consider the case when the all-positive matching $\bar{H}$-paths are only connected by paths of length one (i.e. edges).

Lemma 4.7.8. *For $i \in \{3,5\}$, if e_1, e_2, and e_3 are three positive edges in $(G - E(\bar{H}), \Sigma)$ connecting P_i to P_{i+1}, then at least two of them are aligned.*

Proof. By symmetry, we assume $i = 3$. Suppose to the contrary that there exist three crossing edges $e_j = v_ju_j$, for $j \in \{1,2,3\}$, such that the vertices y, v_1, v_2, v_3, w are labeled in this order on P_3, see Figure 4.10.

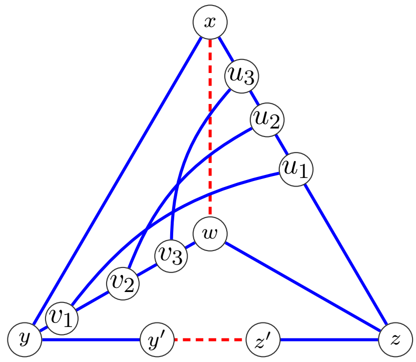


Figure 4.10: Three crossing edges connecting P_3 to P_4

By criticality, there exists a minimum signature Σ' such that $e_2 \in \Sigma'$. Since there are two positive circuits (under Σ) contained in $P_3 \cup P_4 \cup \{e_1, e_2, e_3\}$ sharing only e_2, the two remaining negative edges of Σ' belong to the v_1v_3-subpath of P_3, and to the u_1u_3-subpath of P_4, respectively. Hence, the circuit given by $e_3 \cup P_3 \cup P_4 \cup P_5$ and containing $\{e_3\}$ has exactly one negative edge from Σ' but no negative edge from Σ, a contradiction. □

We define an *Z-Path* to be a path in (G, Σ) such that its vertices are internal vertices of $P_i \cup P_{i+1}$, for a certain $i \in \{3,5\}$, and each edge is either (1) an edge parallel to an edge of a $\bar{H}$-path P_i or P_{i+1}, or (2) an edge whose endpoints are on P_i and P_{i+1}, respectively. Moreover, there are at least two edges of the type (2), and if we label the vertices of the path with an increasing order, the vertices on P_i (resp., on P_{i+1}) will have a consecutive order. In particular, by referring to the direction of the arrows in Figure 4.4, the vertices either will have an increasing order on both P_i and P_{i+1}, or a decreasing order on both P_i and P_{i+1} (see e.g. Figure 4.11).

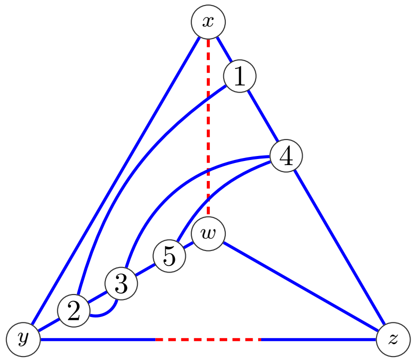


Figure 4.11: An example of Z-Path

Lemma 4.7.9. *For $i \in \{3, 5\}$, if there are three aligned positive edges in $(G - E(\bar{H}), \Sigma)$ connecting P_i to P_{i+1}, then at least one of them belongs to a Z-path.*

Proof. By symmetry, we only provide the proof for the case when $i = 3$. Assume that there exist three positive aligned edges connecting P_3 to P_4, say e_1, e_2, and e_3. For $j \in \{1, 2, 3\}$, let the endpoints of e_j on P_3 and P_4 be v_j and u_j, respectively. Let C_1^+ be the positive circuit contained in $P_3 \cup P_4 \cup \{e_1, e_2\}$ and C_2^+ be the positive circuit contained in $P_3 \cup P_4 \cup \{e_2, e_3\}$. Note that there exists a minimum signature Σ' such that $e_2 \in \Sigma'$. Since C_1^+ and C_2^+ are both positive circuits and they have no common edge on $\bar{H}$, $C_1^+ \cup C_2^+$ is a fully-negative theta-graph with respect to Σ'. Hence, without loss of generality, we may assume $e_1^* \in E(P_3 \cap C_1^+) \cap \Sigma'$ and $e_2^* \in E(P_4 \cap C_2^+) \cap \Sigma'$. By Lemma 4.7.2, $\bar{H} - \{\Sigma' \cup \{e_1^-, e_2^-\}\}$ has two connected components O_1 and O_2, and $G - (\Sigma' \cup \{e_1^-, e_2^-\})$ is connected. We can assume that O_1 is the component containing P_5.

We now consider the edge e^* as follows: If $d_G(u_2) \geq 4$, we choose e^* to be an edge incident with u_2 and not in $E(\bar{H}) \cup \{e_2\}$. Observe that, if e_2 is adjacent to e_2^*, then by Lemma 3.3.2 it always hold that $d_G(u_2) \geq 4$. Similarly, if $d_G(u_2) = 3$, e_2 and e_2^* are not adjacent, then there exists an edge $f = u_2\tilde{u}$ such that $f \in E(P_4) \cap E(C_2^+)$ and $f \neq e_2^*$, and moreover, $d_G(\tilde{u}) \geq 3$ as (G, Σ) is irreducible. In this case, we define e^* to be an edge incident with $\tilde{u}$ and not in $E(\bar{H})$.

Let u' be a vertex of e^* on P_4. Note that it holds $u' \in \{\tilde{u}, u_2\}$ by definition.

Next, we define a path Q as follows: If e^* is not an edge parallel to an edge of $\bar{H}$, we define Q to be a the path in $(G - E(\bar{H}), \Sigma)$, internally vertex-disjoint from $\bar{H}$, containing e^*, and connecting u' to another vertex of $\bar{H}$. Similarly, if e^* is an edge parallel to an edge of $\bar{H}$, we denote $e^* = u'u^*$. Note that in this case we can assume $u' = u_2$, or there would exist an edge incident with u' and not parallel to any edge of $\bar{H}$, and we could replace e^* with such an edge. Since (G, Σ) is irreducible, there exists an edge e^{**} incident with u^* and not parallel to any edge of $\bar{H}$. In this case, we define Q' to a the path in $(G - E(\bar{H}), \Sigma)$, internally vertex-disjoint from $\bar{H}$, containing e^{**}, and connecting u^* to another vertex of $\bar{H}$, and we define $Q = Q' \cup e^*$. In both cases, we denote with u'' the endpoint of Q different from u'.

We first claim that Q is all-positive under Σ. We assume to the contrary that $e_3^- \in E(Q)$. Since by Lemma 4.7.2 $G - (\Sigma \cup \Sigma')$ is disconnected, we have that $u'' \in V(O_2)$. Then, we define Π to be a minimum signature of (G, Σ) containing e_1. Observe that $e_1 \cup e_2 \cup P_5 \cup P_3 \cup P_4$ contains a fully-negative theta-graph Θ under Π. Therefore, $u'' \in V(\Theta)$, as for otherwise, there would exist a circuit containing Q which is negative under Σ but positive under Π, a contradiction. Therefore, it holds that $u'' \in V(\Theta \cap O_2)$. In particular, it follows that u'' is on the v_1v_2-subpath of P_3. By taking another minimum signature Π' such that $\Pi' \cap E(P_5) \neq \emptyset$, and considering the negative circuit contained in $Q \cup e_2 \cup P_3 \cup P_4$, the contradiction follows.

Next, we prove that u'' is not on P_4. Suppose to the contrary that $u'' \in V(P_4)$. We consider three cases:

1. First, assume that $|E(Q)| \geq 2$ and Q does not contain an edge parallel to an edge of P_4, that is Q is internally vertex-disjoint from $\bar{H}$. Since (G, Σ) is irreducible, there exists an edge $f^* \notin E(Q)$ incident with u^*. By connectivity, let Q' be the path that contains f^* and is internally vertex-disjoint from $\bar{H}$, connecting u' to another vertex of $\bar{H}$. By Lemma 4.3.2, Q' can only connect u' to a vertex of P_3. Note that $|E(Q')| \geq 2$ and $e_2 \notin E(Q')$, contradicting Proposition 4.7.7.

2. Assume now $|E(Q)| = 1$. Then by the minimality of $V(\bar{H})$ (Condition (3)), Q belongs to a 2-multiedge, a contradiction to the choice of Q.

3. Similarly, if $|E(Q)| \geq 2$ and Q contains an edge e^* parallel to an edge of P_4, then by definition $Q - \{e^*\}$ does not consist of an edge parallel to an edge of $\bar{H}$. Hence, either $|E(Q - \{e^*\})| = 1$, and we get the same contradiction as in Case 2., or $|E(Q - \{e^*\})| \geq 2$, and we get the contradiction from Case 1.

We next claim that $u'' \in V(P_3)$. Suppose to the contrary that $u'' \in V(P_j)$, with $j \neq 3$. Since $G - (\Sigma \cup \Sigma')$ is disconnected and Q is all-positive, u'' and u_2 belong to the same component. Therefore, u'' is either on P_5 or on the yy'-subpath of P_2. Let Σ'' be a minimum signature containing e_1. By Lemma 4.7.2, $G - (\Sigma \cup \Sigma'')$ consists of two connected components, and moreover, by assumption on the structure of the signed graph, u'' and u' are now in different components of $G - (\Sigma \cup \Sigma'')$. However, it is impossible since they are connected by Q, which is an all-positive path in both (G, Σ) and (G, Σ'').

Hence, $u'' \in V(3)$. Furthermore, u'' belongs to the same component as u_2 in $G - (\Sigma \cup \Sigma')$, i.e. it is an internal vertex of the yv_2-subpath of P_3.

By the above claims, it follows that $Q \cup P_3 \cup P_4 \cup e_2$ contains an all-positive circuit under Σ. By Proposition 4.7.7, Q is either just an edge $u'u'' := f'$, or it consists of two edges, one parallel to an edge of $\bar{H}$, and the other (which we also denote by f') with endpoints on P_3 and P_4. In the latter case, two subcases may occur: If f' and e_2 are aligned, then we have a Z-path. If f' and e_2 are crossing, then we get a contradiction by choosing a minimum signature where the edge in $E(Q)\backslash\{f'\}$ is negative and considering the circuits in $Q \cup e_2 \cup P_3 \cup P_4$.

Hence, we can assume that $|E(Q)| = 1$ and Q and e_2 are crossing.

Recalling that $e_1^* \in E(P_3 \cap C_1^+) \cap \Sigma'$, we have that either $d_G(v_2) \geq 4$, or there exists an edge $t = v_2\tilde{v} \in E(C_1 \cap P_3)$ and $d(\tilde{v}) \geq 3$. We therefore choose an edge t' incident to a vertex $v' \in \{v_2, \tilde{v}\}$ with the same conditions used to choose e^*. Similarly, we define a path S starting on v' following the rules given for Q. We choose a path S by following the conditions given for Q, so that S contains t' and connects v' to a vertex $v'' \in V(\bar{H})$. By repeating the previous arguments, we obtain that S consists of an edge and $v'' \in V(P_4)$, and we have the structure as in Figure 4.12.

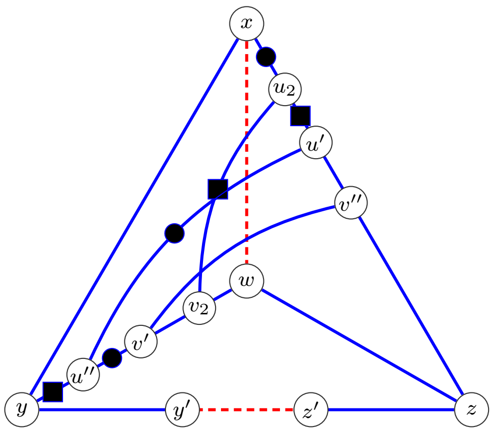


Figure 4.12: Configuration of Lemma 4.7.9: The boxes represent Σ^f, the circles represent $\Sigma^{f'}$

Observe that $u'' \neq v'$, since they are on two different components of $G - (\Sigma^- \cup \Sigma')$. For the same reason, it also holds that $u' \neq v''$. Furthermore, if $u' = u_2$ or $v' = v_2$, then there exists a Z-path and the statement follows. Therefore, we assume $u' \neq u_2$ and $v' \neq v_2$. We consider the edge $f = u_2u'$. In a minimum signature Σ^f containing f, e_2 must be negative and a third negative edge, say $\tilde{f}$, of Σ^f must be on the yu''-subpath of P_3. Similarly, recalling

$f' = u'u''$, for a minimum signature $\Sigma^{f'}$ such that $f' \in \Sigma^{f'}$, the two other negative edges will be one on the $u''v'$-subpath of P_3 and the other, say f^{**}, on the xu_2-subpath of P_4. Observe that $\bar{H} \cup f' \cup e_2 - \{\tilde{f}, f^{**}, e_1^-, e_2^-\}$ is not connected, contradicting Lemma 4.7.4. □

Lemma 4.7.10. *There exists no Z-Path in* (G, Σ)*.*

Proof. To prove this lemma we argue by contradiction and we first provide some structural results, and second, we proceed with a case analysis where we show that, in each case, there exist three edge-disjoint negative circuits.

Suppose that there exists a Z-Path Z in (G, Σ). Without loss of generality, we can assume Z to be maximal and to intersect with P_3 and P_4 (and we denote P_5 and P_6 as in Figure 4.4). We denote the vertices in $V(Z) \cap V(P_3)$ by $v_1, v_2, \ldots, v_t$ and the vertices in $V(Z) \cap V(P_4)$ by $u_1, u_2, \ldots, u_{t'}$, in this order, following the direction given in Figure 4.4. In the following, we only consider the case when the start vertex and the end vertex of Z are on two different $\bar{H}$-paths. The case when both of its end vertices are on the same $\bar{H}$-path can be proved similarly. Furthermore, by symmetry, we can assume that Z starts at u_1 and ends at v_t.

In the proof we are going to use our Z-path as a path connecting u_1 to v_t in $(G - E(\bar{H}), \Sigma)$, and not intersecting on edges any of the other paths which we are going to consider. Thus, the presence of edges parallel to edges of $\bar{H}$ has no influence on our proof, and therefore, to make the notation easier, we also assume that there are no edges parallel to edges of $\bar{H}$.

Let $e_1^Z = u_1v_1$ and let Σ^Z be a minimum signature containing e_1^Z. By the assumptions on the structure, the other two negative edges of Σ^Z are on the u_1u_2-subpath of P_4 and on the yv_1-subpath of P_3, say $e_2^Z \in P_4$ and $e_3^Z \in P_3$.

We now define the edge f' in the following way: If $d_G(u_1) \geq 4$, we define f' to be an edge incident with u_1 and not in $E(\bar{H}) \cup \{e_1^Z\}$. If $d_G(u_1) = 3$, then by Lemma 3.3.2 e_1^Z and e_2^Z are not adjacent; hence there exists an edge $f = u_1u'$ on the u_1u_2-subpath of P_4. Noting that $d(u') \geq 3$, we define f' as the edge $f' \notin E(\bar{H})$ incident with u'.

Claim 4.7.11. *f' does not belong to any negative path in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$.*

Proof of the claim: By connectivity, f' belongs to a path Q internally vertex-disjoint from $\bar{H}$ and connecting P_4 to a vertex $u'' \in V(\bar{H})$. Suppose to the contrary that such path is negative.

By Lemma 4.7.2, $\Sigma \cup \Sigma^Z$ is an edge-cut. It implies that u'' has to be in the same component as v_1 in $G - (\Sigma \cup \Sigma^Z)$. It follows that u'' can neither be on P_5, nor on the yy'-subpath of P_2, nor on the xu_1-subpath of P_4. Furthermore, if u'' is on the yv_1-subpath of P_3, it is easy to see that a minimum signature containing an edge of P_6 would provide a contradiction. Similarly, if u'' is on the v_1w-subpath of P_3, on the u_1z-subpath of P_4, or on P_6, a contradiction easily follows by taking a minimum signature containing an edge on P_5.

Therefore, u'' is on the zz'-subpath of P_2. Let C^- be the circuit contained in $Q \cup P_2 \cup P_4$. By construction, C^- is negative under Σ.

Observe now that, since $z \neq z'$, there exists a positive edge incident with z' and belonging to a path Q' all-positive under Σ, internally vertex-disjoint from $\bar{H}$ and connecting $\bar{H}$ to a vertex $u''' \in V(\bar{H})$. By the choice of $(\bar{H}, \Sigma)$ and by Lemma 4.3.4, u''' is either on P_3, or on the zz'-subpath of P_2. In the first case, Q' forms a circuit C' together with P_2 and P_3, and the contradiction follows by taking a minimum signature such that one edge of P_5 is negative. Therefore, suppose that u''' is on the zz'-subpath. If Q' consists of an edge, by the choice of $\bar{H}$ it has to belong to a multiedge, therefore e_3^- is an edge parallel to e_1^-, contradicting the fact that it belongs to Q. Therefore, there exists at least an internal vertex of Q'. By Lemma 4.3.5, e_3^- is not incident with that vertex, while by Lemma 4.3.4 and Lemma 4.3.2, no positive edge can be incident with such a vertex, a contradiction. $\diamondsuit$

Hence, by the claim we can define Q to be an all-positive path in $(G - E(\bar{H}), \Sigma)$ containing f', internally vertex-disjoint from $\bar{H}$ and connecting a vertex $u' \in V(P_4)$ to a vertex $u'' \in V(\bar{H})$, where f' is incident with u' and, depending on the degree of u_1 (and hence on the choice of f'), it may be

$u' = u_1$.

Note that, by Lemma 4.7.2 and since Q is all-positive, we know that u'' is contained in the same component as u_1 in $G - (\Sigma \cup \Sigma^Z)$. Furthermore, it holds that u'' is not an internal vertex of P_3: If $u' = u_1$, this follows from the fact that Z is maximized; if $u' \neq u_1$, assuming u'' to be an internal vertex of P_3 provides the same construction and contradiction as in Lemma 4.7.9 (by considering the path Q, e_1^Z, and the edge v_1u_2 on Z).

We also claim that u'' is not an internal vertex of P_4. If not, by the assumptions on Z and by Condition (3) on $|V(\bar{H})|$, we have that either u'' is in the u_1u_2-subpath of P_4, or $|E(Q)| \geq 2$.

In the first case, we get a contradiction by taking a minimum signature containing an edge of Q.

Hence, consider the latter case and suppose that $|E(Q)| \geq 2$, that is, there exists an internal vertex of Q, say $\tilde{u}$. Since $d_G(\tilde{u}) \geq 3$, there exists a path in $G - E(Q)$ internally vertex-disjoint from $\bar{H}$ which connects $\tilde{u}$ to $\bar{H}$. By Claim 4.7.11 such path is all-positive and by Lemma 4.3.2 it has a vertex on P_3. This implies that there exists a path connecting P_3 and P_4 of length at least two. Since such path is internally vertex-disjoint from the edges of Z, this contradicts Proposition 4.7.7. Note that, if we would allow the existence of edges parallel to edges of $\bar{H}$ in Z, we could repeat the same argument by considering the fact that Z is maximized.

Hence, this implies that $u'' \in V(P_5 \cup P_2)$.

Claim 4.7.12. *The edge e_1^Z is adjacent to e_2^Z. Therefore, $u' = u_1$.*

Proof of the claim: Suppose to the contrary that e_1^Z is not adjacent to e_2^Z. That is we have an edge $f = u_1u'$. Note that f belongs to a circuit C contained in $Q \cup P_4 \cup P_5 \cup P_2$ which is all-positive under Σ. Let Σ^f be a minimum signature such that $f \in \Sigma^f$. By assumption on the structure of the graph, one edge of Z has to be negative in (G, Σ^f), therefore the other negative edge has to be on P_3. It follows that f is the only negative edge contained in C, a contradiction.

Hence, by Lemma 3.3.2, $d_G(u_1) \geq 4$ and, by the choice of f', it holds that $u_1 = u'$. $\Diamond$

By symmetry, we can prove that there exist an all-positive path Q' in $(G - E(\bar{H}), \Sigma)$ connecting v_t to a vertex $v' \in V(P_6 \cup P_2)$ which is internally vertex-disjoint from $\bar{H}$. In particular, if $v' \in V(P_2)$, then v' is a vertex on the zz'-subpath of P_2.

Observe that Q and Q' are both positive under Σ and Σ^Z, and they belong to two different components of $G - (\Sigma \cup \Sigma^Z)$, therefore Q and Q' are disjoint.

Recall that by Lemma 4.3.3 we have $P_1 = e_1^-$ and by Lemma 4.7.3, given an edge $e \in P_5 \cup P_6$, each minimum signature Σ' containing e has exactly three negative edges on $\bar{H}$ and one of these negative edges is either on P_1 or on P_2. In particular, as $P_1 = e_1^-$ by Lemma 4.3.3, if $e_1^- \in \Sigma'$, then $\Sigma \cap \Sigma' = \{e_1^-\}$. This implies that (G, Σ') can be obtained from (G, Σ) by switching at an edge-cut $\partial(X)$ equilibrated under Σ which contains e_3^- and such that $d(X) = 4$. Furthermore, in this case the following holds.

Claim 4.7.13. *If there exists a minimum signature Σ_5 (respectively, Σ_6) containing e_1^- and one edge $e_5 \in E(P_5)$ (resp., $e_6 \in E(P_6)$), then e_2^- is incident with y (resp., with z).*

Proof of the claim: By symmetry, we only prove the claim assuming that there exists a signature Σ_5 such that $e_5 \in \Sigma_5$, for a certain edge $e_5 \in E(P_5)$.

Suppose to the contrary that e_2^- is not incident with y. Note that, if (G, Σ) has parallel edges, then by assumption we have that e_1^- and e_3^- are parallel, that is to say, they form a circuit which is positive in (G, Σ) but negative in (G, Σ_5), a contradiction. Hence, G is a simple graph.

Since $y' \neq y$ and e_2^- is negative in (G, Σ), there exists a positive edge $f \notin E(\bar{H})$ incident with y'. In particular, there exists a vertex $y'' \in V(\bar{H})$ and an all-positive $y'y''$-path P^* which is internally vertex-disjoint from $\bar{H}$ such that f belongs to $E(P^*)$. By Lemma 4.3.4, $y'' \notin V(P_3 \cup P_5)$. Furthermore, if y'' belongs to the connected component of $P_2 - \{e_2^-\}$ containing z, or to $P_4 \cup P_6$, then $G - (\Sigma \cup \Sigma_5)$ is connected, a contradiction. Therefore, y'' is a vertex of the yy'-subpath of P_2.

Note now that P^* cannot consist of one edge, as for otherwise since G is simple, we would find a smaller $-K_4$-subdivision with respect to condition (2), a contradiction. Hence, $|E(P^*)| \geq 2$. Let w' be an internal vertex of P^* and let $\tilde{f}$ be an edge not in P^* incident with w'. By Lemma 4.3.5, $\tilde{f}$ is positive, and there is a vertex $w'' \in V(\bar{H})$ such that $\tilde{f}$ is in the all-positive $w'w''$-path which is internally vertex-disjoint from $\bar{H}$. By Lemma 4.3.2, $w'' \notin V(P_2)$, but the existence of the all-positive $y'w''$-path contradicts the argument discussed above. $\diamondsuit$

In the following, we proceed with a case analysis considering which signatures occur on the edges of $P_5 \cup P_6$. We prove that, in each case, there exist three edge-disjoint negative circuits and therefore, the graph cannot be in $\mathcal{L}^*(3)$.

We have three possible cases:

(i) For each $j \in \{5, 6\}$, there exists a minimum signature Σ'_j such that $\Sigma'_j \cap E(P_j) \neq \emptyset$ and $e_1^- \in \Sigma'_j$.

(ii) For exactly one value of $j \in \{5, 6\}$, there exists a minimum signature Σ' such that $\Sigma' \cap E(P_j) \neq \emptyset$ and $e_1^- \in \Sigma'$.

(iii) For each $j \in \{5, 6\}$, there exists no minimum signature Σ' such that $\Sigma' \cap E(P_j) \neq \emptyset$ and $e_1^- \in \Sigma'$.

Case (i) For each $j \in \{5, 6\}$, there exists a minimum signature Σ'_j such that $\Sigma'_j \cap E(P_j) \neq \emptyset$ and $e_1^- \in \Sigma'_j$.

Let $e_5 \in E(P_5)$ and let Σ'_5 be a minimum signature containing both of e_5 and e_1^-. Thus the third negative edge of Σ'_5, denoted by e', is on the yv_1-path on P_3. Note that, by Claim 4.7.13, e_2^- is incident with y. Hence, (G, Σ'_5) is obtained from (G, Σ) by switching at an edge-cut $\partial(X')$ such that $\partial(X') = \{e_2^-, e_3^-, e', e_5\}$. Let $X'_{\bar{H}} = X' \cap V(\bar{H})$, then we have $X'_{\bar{H}} \subset V(P_5 \cup P_3)$.

Similarly, let $e_6 \in E(P_6)$ and let Σ''_6 be a minimum signature containing both of e_6 and e_1^-. Noting that e_2^- is incident with z by Claim 4.7.13, by repeating the previous argument, there exists an edge-cut $\partial(X'')$ with $X''_{\bar{H}} = X'' \cap V(\bar{H}) \subset V(P_4 \cup P_6)$ which contains e_3^-.

Considering the fact that $e_3^- \in \partial(X') \cup \partial(X'')$, we conclude that there exists a path P^- internally vertex-disjoint from $\bar{H}$ which connects $X'_{\bar{H}}$ to $X''_{\bar{H}}$ and contains e_3^-.

We recall that Q is the all-positive path in (G, Σ) connecting u_1 to u'', where $u'' \in V(P_2 \cup P_5)$, which is internally vertex-disjoint from $\bar{H}$. In fact, since $y = y'$, the yy'-path (on P_2) is of length 0 and thus $u'' \in V(P_5)$. We note that in this case, Q together with P_5 and the xu_1-path on P_4 induces a positive circuit C^+. Since C^+ contains neither e_1^- nor e', it does not contain e_5 either. In particular, this implies that none of its edges is contained in $G[X'_{\bar{H}}]$.

We now prove that the endpoint of P^- in $X'_{\bar{H}}$ is y. Suppose that this is not the case. Let C' be the positive circuit contained in the subgraph $P_2 \cup P^- \cup G[X'_{\bar{H}}] \cup G[X''_{\bar{H}}]$. Thus, there is an edge $\tilde{e} \in E(C') \cap E(\bar{H}) \setminus \{e_2^-\}$. Let $\tilde{\Sigma}$ be a minimum signature containing $\tilde{e}$. We note that $|\tilde{\Sigma} \cap E(C')| = 2$. If $\tilde{\Sigma} \cap E(P_5) \neq \emptyset$, then $|\tilde{\Sigma} \cap E(\bar{H})| = 3$, which is not possible because of the positive circuit induced by $V(Q) \cup V(P_4) \cup V(P_5)$. If $\tilde{\Sigma} \cap E(P_5) = \emptyset$, then the two negative edges of $\tilde{\Sigma} \cap E(\bar{H})$ are on $P_3 \cap G(X'_{\bar{H}})$ and on the xu_1-subpath of P_4. Therefore the third edge of $\tilde{\Sigma}$ has to be on Q, contradicting the fact that $|\tilde{\Sigma} \cap E(C')| = 2$.

By symmetry, by using the all-positive path Q' (internally edge-disjoint from $\bar{H}$) connecting v_t to a vertex $v' \in V(P_6)$, we conclude that the other endpoint of P^- in $X''_{\bar{H}}$ is z.

We finish the proof of this case by showing the existence of three edge-disjoint negative circuits as follows.

(1) e_1^- together with the xu_1-subpath of P_4, Z, and the $v_t w$-subpath of P_3.

(2) e_2^- together with the yu''-subpath of P_5, Q, and the $u_1 z$-subpath of P_4.

(3) e_3^- together with P^-, the yv_t-subpath of P_3, Q', and the $v'z$-subpath of P_6.

Case (ii) For exactly one value of $j \in \{5, 6\}$, there is no signature Σ' such that $\Sigma' \cap E(P_j) \neq \emptyset$ and $e_1^- \in \Sigma'$.

Without loss of generality, we assume that there exist $e_5 \in E(P_5)$ and a minimum signature Σ' such that e_5 and e_1^- belong to Σ'. Hence, for each edge $e_6 \in P_6$, any minimum signature Σ'' containing e_6 must have a negative edge in P_2, say $\tilde{e}_2^-$. We choose e_6 to be the edge of $E(P_6)$ incident with w. Note that $\Sigma'' \subset E(P_2 \cup P_3 \cup P_6)$.

By repeating the same argument as in Case (i), we note that (G, Σ') is obtained from (G, Σ) by switching at an edge-cut $\partial(X')$ such that $e_3^- \in \partial(X')$ and $d(X') = 4$. Again, by Claim 4.7.13, e_2^- is incident with y. In particular, there exists a path P^- internally vertex-disjoint from $\bar{H}$ such that $e_3^- \in E(P^-)$ and it has one vertex of $X' \cap V(\bar{H})$ as an endpoint. Note that this implies that e_3^- is not parallel to e_1^- and thus, by the choice of $(\bar{H}, \Sigma)$, (G, Σ) is a simple signed graph.

Recall that Q' is an all-positive path connecting v_t to $v' \in V(P_6 \cup P_2)$, which is internally edge-disjoint from $\bar{H}$. We have two subcases to consider:

Subcase (ii.a) $\tilde{e}_2^- = e_2^-$.

In this subcase, we know that (G, Σ'') is obtained from (G, Σ) by switching at an edge-cut $\partial(X'')$ such that $e_1^-, e_3^- \in \partial(X'')$ and $d(X'') = 4$. It follows that P^- must have the other endpoint in $X'' \cap V(\bar{H})$. Furthermore, since both e_6 and e_1^- are incident with w, $X'' \cap V(\bar{H}) \subset V(P_3)$ and thus the endpoint of P^- (in X'') is in $V(P_3)$.

We now consider the path Q'. We claim that $v' \in V(P_6)$. If not, $v' \in V(P_2) \setminus \{z\}$ (see Figure 4.13) and there is an edge on P_2 of the $v'z$-subpath of P_2 which does not belong to any minimum signature (we leave the details to the reader).

Furthermore, as in the previous case it is easy to see that no endpoint of P^- can be an internal vertex of P_5, hence, the three edge-disjoint negative circuits are as follows:

(1) e_1^- together with the xu_1-subpath of P_4, Z, Q' and the $v'w$-subpath of

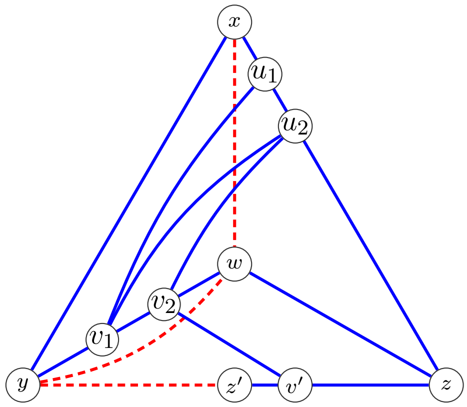


Figure 4.13: Case (ii.a), when $v' \in V(P_2)$, and P^- is represented as a negative edge connecting y to w

P_6.

(2) e_2^- together with P_2, the yu''-subpath of P_5, Q, the u_1z-subpath of P_4.

(3) e_3^- together with P^- and P_3.

Subcase (ii.b) $\tilde{e}_2^- \neq e_2^-$. In this case, as $\tilde{e}_2^- \in E(P_2)$, $|E(P_2)| \geq 2$. Noting that e_2^- is incident with y, we consider z'. As $d(z') \geq 3$, there is an all-positive path Q'' in $(G - E(\bar{H}), \Sigma)$ internally vertex-disjoint from $\bar{H}$ connecting z' to a vertex v'' of $V(\bar{H})$. Furthermore, since Q'' cannot consists of one edge connecting P_2 to P_2 (by Condition (2) and by the fact that (G, Σ) is simple), by Lemmas 4.3.2 and 4.3.5, we can assume $v'' \notin V(P_2)$.

By Lemma 4.3.4, $v'' \in V(P_3 \cup P_5)$. In fact, if $v'' \in V(P_5)$, noting that $\Sigma'' \cap E(P_5) = \emptyset$, then Q'', e_2^-, and the yv''-subpath of P_5 together form a negative circuit (under Σ) which contains no edges from Σ'', a contradiction. Let $\Sigma'' = \{\tilde{e}_2^-, e_6, e''\}$ and note that e'' belongs to the v_tw-subpath of P_3. Thus we know that the all-positive path Q'', together with P_3, P_6, and P_2, forms a positive circuit, denoted by C_1, and thus C_1 contains exactly the two negative edges $e_6, \tilde{e}_2^-$ from Σ''. Similarly, the all-positive path Q', together with P_3, P_6, and possibly P_2, forms a positive circuit C_2 and the circuit C_2 contains exactly two negative edges e'', e_6 from Σ''. Since v_t (one endpoint of Q') and v'' (one endpoint of Q'') are not in the same connected component of $P_3 - e''$, we know that $v_t \neq v''$ and similarly, $z' \neq v'$ (where z' is the other endpoint of the path Q'' and v' is the other endpoint of Q'). Moreover, v' must be on the path P_6, as for otherwise, any minimum signature Σ''' containing a negative edge of the $v'z$-subpath of P_2 must satisfy that $|\Sigma''' \cap E(\bar{H})| = 3$, $|\Sigma''' \cap E(P_6)| = 1$, and $|\Sigma''' \cap E(P_2)| = 1$, which is impossible.

Let e_1^* be the edge in Q'' incident with z' and let Σ^* be a minimum signature containing e_1^*. We now aim to prove that $e_1^- \in \Sigma^*$. We suppose to the contrary that $e_1^- \notin \Sigma^*$ and thus $\Sigma^* \cap E(P_2) = \emptyset$. Furthermore, since $|\Sigma^* \cap E(\bar{H})| = 2$, by Lemma 4.7.3 no edge of P_6 can be in Σ^*. Since the second negative edge e_2^* of Σ^* must be on the circuit C_1, $|\Sigma^* \cap E(P_3)| = 1$. Together with the fact that $|\Sigma^* \cap (E(P_2) \cup E(P_6))| = 0$, it is a contradiction to the existence of the positive circuit C_2. Hence, we conclude that e_2^* belongs to the $z'z$-subpath of P_2 and $\Sigma^* = \{e_1^*, e_2^*, e_1^-\}$.

It follows that (G, Σ^*) is obtained from (G, Σ) by switching at an edge-cut $\partial(X'')$ such that $e_3^-, e_2^-, e_1^*, e_2^* \in \partial(X'')$ and $d(X'') = 4$. Similar to Case (i), we can prove that the endpoint of P^- in X' is y and we omit the details. Therefore, in this case, P^- has one endpoint y and the other endpoint being on the $z'z$-subpath of P_2, say z''.

This also implies that $v'' = w$, as for otherwise we would get a contradiction by considering a minimum signature containing an edge of the $v''w$-subpath of P_3.

Therefore, the three edge-disjoint negative circuits are as follows:

(1) e_1^- together with the xu_1-path on P_4, Z, Q' and the $v'w$-subpath of P_6.

(2) e_2^- together with the yv_t-subpath of P_3, Q', the $v'w$-subpath of P_6, and Q''.

(3) e_3^- together with the yu''-subpath of P_5, Q, the u_1z-subpath of P_4, the $z''z$-subpath of P_2, and P^-.

Case (iii) For each $j \in \{5, 6\}$, there exists no signature Σ' such that $\Sigma' \cap P_j \neq \emptyset$

and $e_1^- \in \Sigma'$.

In this case, every signature containing an edge of $E(P_5)$ (or $E(P_6)$) does not contain e_1^- as a negative edge. By Lemma 4.7.3, any such signature contains an edge of $E(P_2)$.

Note that u'' (one endpoint of Q) must be on P_2, as for otherwise, there would exist an edge e' on P_5 which is not contained in the positive circuit induced by $Q \cup P_5 \cup P_4$. Since each minimum signature containing e' has one edge on the xu_1-subpath of P_4, the circuit contained in $Q \cup P_5 \cup P_4$ would be negative under such a signature, a contradiction. By symmetry, v' (one endpoint of Q') must be on P_2.

Claim 4.7.14. *For each $j \in \{5, 6\}$, there exists a minimum signature Σ'_j such that $\Sigma'_j \cap P_j \neq \emptyset$ and $e_2^- \in \Sigma'_j$.*

Proof of the claim: By symmetry, we only prove the claim for $j = 5$. Let $e_5 \in E(P_5)$, and Σ' be a minimum signature containing e_5. If $e_2^- \in \Sigma'$, then the claim trivially follows, so we assume that $\Sigma' \cap E(P_2) = \{\tilde{e}_2^-\}$ and $\tilde{e}_2^- \neq e_2^-$. Moreover, the third negative edge e' of Σ' is on the xu_1-subpath of P_4.

We first show that $\tilde{e}_2^-$ is on the yy'-subpath of P_2, which implies $y \neq y'$. This claim is trivial for the case $z' = z$. Therefore, assume $z' \neq z$ and $d_{\bar{H}}(z') = 2$. By the exact same argument as shown in Case (ii.b), there is an all-positive path (internally edge-disjoint from $\bar{H}$) in (G, Σ) connecting z' to a vertex on P_3. This path, together with P_3, P_6, and P_2, induces a positive circuit. Since this positive circuit (under Σ) contains neither e_5 nor e', it cannot contain $\tilde{e}_2^-$. Therefore, $\tilde{e}_2^-$ is on the yy'-subpath of P_2.

We now claim that $y' \notin V(Q)$. The reason for that is that the circuit C_Q induced by $Q \cup P_2 \cup P_5 \cup P_4$ is positive (under Σ) and it contains both e_5 and e'. Thus, it cannot contain $\tilde{e}_2^-$ which is an edge on the yy'-subpath of P_2.

Since $y' \neq y$, and $d(y') \geq 3$, there exists an all-positive path (under Σ) internally vertex-disjoint from $\bar{H}$ connecting y' to an internal vertex, say s, of the xu_1-subpath on P_4. Note that this $y's$-path is not Q.

We now consider a minimum signature Σ'' such that the edge of the xs-

subpath of P_4 incident with x is negative. As such an edge is contained in C_Q, and C_Q is positive under Σ, $|\Sigma'' \cap E(P_5)| = 1$.

Thus the third negative edge of Σ'' must be on $P_2 - E(C_Q)$. Furthermore, such an edge cannot be on the $u''y'$-subpath of P_2 (nor on the zz'-subpath of P_2), since otherwise the circuit induced by the $y's$-path internally-vertex-disjoint from $\bar{H}$ together with the yy'-subpath of P_2, P_5, and the xs-subpath of P_4 (or the path internally vertex-disjoint from $\bar{H}$ connecting z' to P_3, together with P_3, P_6, and the zz'-subpath of P_2) would induce a circuit which is positive under Σ, and negative under Σ'', a contradiction. Therefore $e_2^- \in \Sigma''$ and Σ'' is the required signature. $\diamondsuit$

Hence, for $j \in \{5, 6\}$, let Σ_j be a minimum signature such that $\Sigma_j \cap P_j \neq \emptyset$ and $e_2^- \in \Sigma_j$. As in the previous cases, (G, Σ_j) is obtained from (G, Σ) by switching at an edge-cut X_j such that $e_3^- \in \partial(X_j)$ and $d(X_j) = 4$. Therefore, there exists a path P^- internally vertex-disjoint from $\bar{H}$ such that $e_3^- \in P^-$ and it has one endpoint in $X_5 \cap V(\bar{H})$, say s, and another endpoint in $X_6 \cap V(\bar{H})$, say s'. It is easy to check $s = x$ and $s' = w$. If not, we get a contradiction by taking one edge on the sx-path (or on the $s'w$-path) and arguing with the two negative circuits of (G, Σ) contained in $P^- \cup \bar{H} - (P_1 \cup P_2)$.

Hence, we obtain the following three edge-disjoint negative circuits:

(1) e_1^- together with P_4 and P_6.

(2) e_2^- together with $u''v'$-subpath of P_2, Q, Z, and Q'.

(3) e_3^- together with P^-, P_5, and P_3.

This completes the proof of the lemma. $\square$

By Lemmas 4.7.9 and 4.7.10, the following corollary holds.

Corollary 4.7.15. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$ and consider the special $-K_4$-subdivision $(\bar{H}, \Sigma)$. For $i \in \{3, 5\}$, there are at most two aligned positive edges in $(G - E(\bar{H}), \Sigma)$ connecting P_i to P_{i+1}.*

We will next make use of the following statement of Erdős–Szekeres Theorem:

Theorem 4.7.16 ([12])**.** *Let r, s be positive integers. Any sequence of distinct real numbers with length at least $(r-1)(s-1)+1$ contains either a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s.*

In particular, it implies that any sequence of $(n-1)^2-1$ integers contains a monotone (increasing or decreasing) subsequence of length at least n. Applying Lemma 4.7.8 and Corollary 4.7.15, we obtain the next result.

Corollary 4.7.17. *Let $(G, \Sigma) \in \mathcal{L}^*(3)$ and consider the special $-K_4$-subdivision $(\bar{H}, \Sigma)$. For $i \in \{3, 5\}$, there are at most four positive edges connecting P_i to P_{i+1}.*

Proof. Suppose to the contrary that there are five positive edges $e_i = v_i u_i$ for $i \in \{1, 2, \ldots, 5\}$ connecting P_3 to P_4. We may also assume that $v_1, \ldots, v_5$ are ordered on the path P_3 following this order and let $u_{j_1}, \ldots, u_{j_5}$ denote the corresponding ordered sequence on the path P_4 (as shown in Figure 4.4). By Theorem 4.7.16, in any sequence $j_1 j_2 j_3 j_4 j_5$, there is a triple (k_1, k_2, k_3) with $k_1 < k_2 < k_3$ such that either $j_{k_1} > j_{k_2} > j_{k_3}$ or $j_{k_1} < j_{k_2} < j_{k_3}$, but the first one contradicting Lemma 4.7.8 and the second one contradicting Corollary 4.7.15. □

To conclude the proof of Proposition 4.2.4, following the arguments of Lemma 4.3.3, there is no path connecting P_1 to any other $\bar{H}$-path; by Proposition 4.7.1, there are at most two all-positive pairwise internally vertex-disjoint paths connecting two adjacent $\bar{H}$-paths; by Proposition 4.7.7 and Corollary 4.7.17, there are at most four all-positive pairwise internally vertex-disjoint paths connecting two all-positive matching $\bar{H}$-paths.

Therefore, Proposition 4.2.4 follows. □

Chapter 5

The families $\mathcal{S}$ and $\mathcal{S}^*$

In the previous chapters we saw how critically frustrated signed graphs may be decomposable into more critically frustrated signed subgraphs, and that studying these "basic" structures also provides information on the signed graphs given by combining them. A further natural question is "how much non-decomposable" a critically frustrated signed graph can be. Consider the two signed Petersen graphs (P, Σ_1) and (P, Σ_2) in Figure 2.1. Both are 3-critical and by Proposition 2.2.1, both of them are non-decomposable. However, (P, Σ_2) has also the property that each 2-critical subgraph is non-decomposable. This is not true for (P, Σ_1) since it contains two edge-disjoint negative circuits.

The chapter is organized as follows: In Section 5.1 we study the set of k-critical signed graphs having a non-decomposable j-critical subgraph for each $j \in \{2, ..., k\}$. We also define the family $\mathcal{S}^*$ consisting of critically frustrated signed graphs such that each critical subgraph is non-decomposable. We prove that this family can also be described as the family of critically frustrated signed graphs which contain no pair of edge-disjoint negative circuits. Additionally, we provide a characterization for its elements. In Section 5.2, we build k-critical signed graphs belonging to $\mathcal{S}^*$ for each $k \geq 3$. Lastly, in Section 5.3, we completely describe the family of 3-critical signed graphs in $\mathcal{S}^*$.

5.1 A characterization of $\mathcal{S}^*$

The results of this section have been published in [6].

For $k \geq 1$ let $\mathcal{S}(k)$ be the set of k-critical irreducible signed graphs (G, Σ) with the property that (G, Σ) contains a non-decomposable j-critical subgraph (H, Γ) for every $j \in \{1, \ldots, k\}$. Let $\mathcal{S} = \bigcup_{i=1}^{\infty} \mathcal{S}(i)$. Analogously, let $\mathcal{S}^*(k)$ be the set of k-critical irreducible signed graphs (G, Σ) with the property that every j-critical subgraph (H, Γ) is non-decomposable for every $j \in \{1, \ldots, k\}$. Let $\mathcal{S}^* = \bigcup_{i=1}^{\infty} \mathcal{S}^*(i)$.

Proposition 5.1.1. *Let (G, Σ) be a critical irreducible signed graph. Then $(G, \Sigma) \in \mathcal{S}^*$ if and only if (G, Σ) contains no edge-disjoint negative circuits.*

Proof. If $(G, \Sigma) \notin \mathcal{S}^*$, then it has a t-critical subgraph (H, Γ) which can be decomposed into two critical subgraphs (H_1, Γ_1) and (H_2, Γ_2). Since $E(H_1) \cap E(H_2) = \emptyset$ and each of them contains negative circuits, it follows that (G, Σ) contains two edge-disjoint negative circuits.

If (G, Σ) contains two edge-disjoint negative circuits, then it contains a decomposable 2-critical subgraph. Hence, $(G, \Sigma) \notin \mathcal{S}^*$. $\square$

For $i \in \{1, 2\}$ let (H_i, Γ_i) be a signed graph. Let $v_i w_i \in E(H_i) \setminus \Gamma_i$. The *2-edge-sum* $(H_1, \Gamma_1) \oplus_2 (H_2, \Gamma_2)$ is obtained from $(H_1 - v_1 w_1, \Gamma_1)$ and $(H_2 - v_2 w_2, \Gamma_2)$ by adding the positive edges $v_1 v_2$ and $w_1 w_2$.

Let $u_i \in V(H_i)$ be a vertex of degree 3 with neighbors x_i, y_i, z_i such that all edges incident to u_i are positive. The *3-edge-sum* $(H_1, \Gamma_1) \oplus_3 (H_2, \Gamma_2)$ is obtained from $(H_1 - u_1, \Gamma_1)$ and $(H_2 - u_2, \Gamma_2)$ by adding the positive edges $x_1 x_2, y_1 y_2$, and $z_1 z_2$.

Proposition 5.1.2. *For $i \in \{2, 3\}$ let (G, Σ) be the i-edge-sum of an unbalanced signed graph (H_1, Γ_1) and a 2-edge-connected balanced signed graph (H_2, Γ_2).*

1. *(G, Σ) contains no edge-disjoint negative circuits if and only if (H_1, Γ_1) contains no edge-disjoint negative circuits.*

2. *Let (G,Σ) be a k-critical signed graph. If $(G,\Sigma) = (H_1,\Gamma_1) \oplus_2 (H_2,\Gamma_2)$, then (H_2,Γ_2) is a balanced circuit. If $(G,\Sigma) = (H_1,\Gamma_1) \oplus_3 (H_2,\Gamma_2)$, then (H_2,Γ_2) is a balanced theta-graph.*

Proof. 1. For $i = 3$ we prove that (G,Σ) contains two edge-disjoint negative circuits if and only if (H_1,Γ_1) contains two edge-disjoint negative circuits. The case $i = 2$ can be proved analogously.

Let C_1, C_2 be two edge-disjoint negative circuits in (G,Σ). None of them can be a subgraph of $G[H_2 - v_2]$. If they are both subgraphs of $G[H_1 - v_1]$, then the statement follows. Thus one of them, say C_1, contains two edges of $\{x_1x_2, y_1y_2, z_1z_2\}$, say x_1x_2, z_1z_2. We can assume that every (x_2,z_2)-path P in $(G[H_2 - v_2],\Sigma)$ is all-positive. Thus, there is a negative circuit C_2' in (H_1,Γ_1) which is edge-disjoint from C_1, which is also a subgraph of (H_1,Γ_1).

If C_1, C_2 are edge-disjoint negative circuits in (H_1,Γ_1), then at most one of them contains v_1. Since (H_2,Γ_2) is balanced, this circuit can be extended to a negative circuit in (G,Σ). Thus, (G,Σ) contains two edge-disjoint negative circuits.

2. Assume now that (G,Σ) is a critically k-frustrated signed graph such that $(G,\Sigma) = (H_1,\Gamma_1) \oplus_i (H_2,\Gamma_2)$, $i \in \{2,3\}$. Let $V_2 = V(G) \cap V(H_2)$ and $V_1 = V(G) \setminus V_2$. Furthermore, since (H_2,Γ_2) is balanced, we can assume the signed graph induced by $(G[V_2],\Sigma)$ to be all-positive. If (G,Σ) contains an edge-cut $\partial_G(U)$ equilibrated under Σ with more than one edge of $G[V_2]$, then there is an edge-cut in (G,Σ) which is a subset of $(E(G[V_1]) \cap \partial_G(U)) \cup \{v_1v_2, w_1w_2\}$ if $i = 2$, and of $(E(G[V_1]) \cap \partial_G(U)) \cup \{x_1x_2, y_1y_2, z_1z_2\}$ if $i = 3$ and which has more negative than positive edges; a contradiction to Lemma 1.3.5. Thus, $G[V_2]$ is a tree with at most i leaves and the statements follow.

□

Proposition 5.1.3. $\mathcal{S}(1) = \mathcal{S}^*(1) = \mathcal{L}(1) = \{-C_1\}$, $\mathcal{S}(2) = \mathcal{S}^*(2) = \mathcal{L}^*(2) = \{-K_4\}$, *and for all $k \geq 3$: $\emptyset \neq \mathcal{S}^*(k) \subset \mathcal{S}(k)$. In particular, $\mathcal{S}^* \subset \mathcal{S}$.*

Proof. The statement for $k = 1$ is trivial. For $k = 2$, note that $-K_4$ is the only non-decomposable 2-critical signed graph (see also Theorem 2.3.2).

Clearly, $\mathcal{S}^*(k) \subseteq \mathcal{S}(k)$. We show that these families are not empty and that there are signed graphs in $\mathcal{S}(k)$ which are not in $\mathcal{S}^*(k)$, for $k \geq 3$. Reed [26] proved that for every $s \geq 2$ there exists an s-frustrated signed graph (G_s, Σ_s) which does not contain any edge-disjoint negative circuits. By Proposition 2.1.1, (G_s, Σ_s) contains an m-critical subgraph (H, Γ) for each $m \in \{1, \dots, s\}$. The edge sets of any two negative circuits of (H, Γ) have a non-empty intersection. Thus, $(H, \Gamma) \in \mathcal{S}^*(m)$ by Proposition 5.1.1 and $\mathcal{S}^*(k) \neq \emptyset$ for each $k \geq 1$.

Consider the antibalanced odd wheel $-W_{2t+1}$ for $t \geq 2$. It holds $\ell(-W_{2t+1}) = t+1$ and every $(t+1)$-signature contains precisely one edge e_o of the outer circuit C_{2t+1} and t spokes. It is easy to see that when removing one negative spoke and one positive spoke which are edges of an induced triangle in $-W_{2t+1}$, then the resulting signed graph is t-critical and a subdivision of $-W_{2t-1}$. On the other hand, $-W_{2t+1} - e_o$ contains a t-critical subgraph which is a subdivision of $-tC_1$. Thus, $-W_{2k+1} \in \mathcal{S}(k+1) - \mathcal{S}^*(k+1)$ for every $k \geq 2$. $\square$

Signed graphs which do not contain two edge-disjoint negative circuits are characterized by Lu et al. [22], where the case $k = 2$ is proved in [29, 32]. Let $\hat{G}$ be a contraction of a graph G and let $x \in V(G)$. Then $\hat{x}$ denotes the vertex of $\hat{G}$ which x is contracted into.

Theorem 5.1.4 ([22]). *Let (G, Σ) be a 2-connected k-frustrated ($k \geq 2$) signed graph and $\Sigma = \{x_1y_1, \dots, x_ky_k\}$ be a k-signature. Then the following statements are equivalent.*

1. *(G, Σ) contains no edge-disjoint negative circuits.*
2. *$G - \Sigma$ is contractible to a 2-connected graph containing no edge-disjoint $(\hat{x}_i, \hat{y}_i)$-path and $(\hat{x}_j, \hat{y}_j)$-path for any $i \neq j$.*
3. *The graph G can be contracted to a cubic graph $\hat{G}$ such that either $\hat{G} - \{\hat{x}_1\hat{y}_1, \dots, \hat{x}_k\hat{y}_k\}$ is a circuit C_1 with vertex set $\{\hat{x}_1, \dots, \hat{x}_k, \hat{y}_1, \dots, \hat{y}_k\}$ or it can be obtained from a 2-connected plane cubic graph by selecting a facial circuit C_2 and inserting vertices $\hat{x}_1, \dots, \hat{x}_k, \hat{y}_1, \dots, \hat{y}_k$ on the edges*

of C_2 in such a way that for every 2-element set $\{i, j\} \subseteq \{1, \ldots, k\}$, the vertices $\hat{x}_i, \hat{x}_j, \hat{y}_i, \hat{y}_j$ are around the circuit C_1 or C_2 in this cyclic order.

We now can prove the main theorem which characterizes $\mathcal{S}^*$ as a specific family of projective planar signed cubic graphs.

Theorem 5.1.5. *Let $k \geq 1$ and (G, Σ) be an irreducible k-critical signed graph and $\Sigma = \{x_1y_1, \ldots, x_ky_k\}$. Then $(G, \Sigma) \in \mathcal{S}^*$ if and only if*

1. *$(G, \Sigma) \in \{-C_1, -K_4\}$ or*
2. *(G, Σ) is obtained from a 2-connected plane cubic graph H by selecting a facial circuit C_H and inserting vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$ on the edges of C_H in such a way that for every pair $\{i, j\} \subseteq \{1, \ldots, k\}$, the vertices x_i, x_j, y_i, y_j are around the circuit in this cyclic order.*

Furthermore, for $k \geq 3$: If $(G, \Sigma) \in \mathcal{S}^*(k)$ is irreducible, then G is a cyclically 4-edge connected projective-planar cubic graph.

Proof. Let $(G, \Sigma) \in \mathcal{S}^*(k)$. If $k \in \{1, 2\}$, then $(G, \Sigma) \in \{-C_1, -K_4\}$ by Proposition 5.1.3.

By Proposition 5.1.1, (G, Σ) does not contain two edge-disjoint negative circuits. Thus, by Theorem 5.1.4, G can be contracted to a cubic graph $\hat{G}$ such that either $\hat{G} - \{\hat{x}_1\hat{y}_1, \ldots, \hat{x}_k\hat{y}_k\}$ is a circuit C with vertex set $\{\hat{x}_1, \ldots, \hat{x}_k, \hat{y}_1, \ldots, \hat{y}_k\}$ or $\hat{G}$ can be obtained from a 2-connected plane cubic graph H by selecting a facial circuit C_H and inserting vertices $\hat{x}_1, \ldots, \hat{x}_k, \hat{y}_1, \ldots, \hat{y}_k$ on the edges of C_H in such a way that for every 2-element set $\{i, j\} \subseteq \{1, \ldots, k\}$, the vertices $\hat{x}_i, \hat{x}_j, \hat{y}_i, \hat{y}_j$ are around the circuit C or C_H in this cyclic order.

Let $\hat{\Sigma} = \{\hat{x}_1\hat{y}_1, \ldots, \hat{x}_k\hat{y}_k\}$ be the corresponding k-signature on $\hat{G}$. We can assume $k \geq 3$ and thus, $\hat{G} - \hat{\Sigma}$ is a subdivision of a cubic graph, i.e. the second case of the above statement applies. We show that $(G, \Sigma) = (\hat{G}, \hat{\Sigma})$.

Let $X = \{x_1, \ldots, x_k, y_1, \ldots, y_k\} \subseteq V(G)$ and $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k, \hat{y}_1, \ldots, \hat{y}_k\} \subseteq V(\hat{G})$.

Suppose to the contrary that there is $\hat{s} \in V(\hat{G})$, which is the result of a contraction of a subgraph $G[S]$ of G with $s \in S$ and $|S| > 1$. Since $\hat{X} \subseteq V(\hat{G})$

and $\hat{\Sigma} \subseteq E(\hat{G})$, it follows that $|X \cap S| \leq 1$. Furthermore, $\Sigma \cap E(G[S]) = \emptyset$, $|\partial_G(S)| = 3$ and at least two edges of $\partial_G(S)$ are positive under Σ. If all three edges are positive (i.e. $|X \cap S| = 0$), then the statement follows from the fact that (G, Σ) is irreducible and Proposition 5.1.2.

Now, let $|X \cap S| = 1$, say $x_1 \in S$. If there is an edge-cut $\partial_G(U)$ equilibrated in (G, Σ) that contains more than one edge of $G[S]$, then there is an edge-cut in (G, Σ) which is a subset of $(E(G[V(G) \setminus S]) \cap \partial_G(U)) \cup \partial_G(S)$ and which has more negative than positive edges; a contradiction to Lemma 1.3.5. Thus, $G[S]$ is a tree with at most two leaves. Since (G, Σ) is irreducible it follows that $S = \{x_1\}$ and there is nothing to contract.

Hence, $(G, \Sigma) = (\hat{G}, \hat{\Sigma})$ and (G, Σ) is obtained from a 2-connected plane cubic graph H by selecting a facial circuit C_H and inserting vertices $x_1, \dots, x_k$, $y_1, \dots, y_k$ on the edges of C_H in such a way that for every pair $\{i, j\} \subseteq \{1, \dots, k\}$, the vertices x_i, x_j, y_i, y_j are around the circuit in this cyclic order. Clearly, G has an embedding into the projective plane. Furthermore, if $k \geq 3$, then the vertices $x_1, x_2, x_3, y_1, y_2, y_3$ are the six trivalent vertices of a subdivision of a $K_{3,3}$. Hence, G is not planar.

It remains to prove that G is cyclically 4-edge-connected. By the same arguments as above, (G, Σ) has no non-trivial 3-edge-cut. Suppose to the contrary that G has a 2-edge-cut $\partial_G(U)$. Then $\partial_G(U) \subseteq E(H)$ and $\partial_G(U) \cap \Sigma = \emptyset$. If $\partial_G(U) \not\subseteq E(C_H)$, then $\partial_G(U)$ is an edge-cut in H. By Proposition 5.1.2, one component of $G - \partial_G(U)$ is a path, contradicting the fact that G is cubic. If $\partial_G(U) \subseteq E(C)$, then one part of $C - \partial_G(U)$ does not contain any vertex of X. But then, we deduce a contradiction again with Proposition 5.1.2. Thus, G is cyclically 4-edge-connected. $\square$

Figure 5.1 shows the signed Petersen graph (P, Σ_2) embedded into the projective plane. The planar graph to start with is K_4.

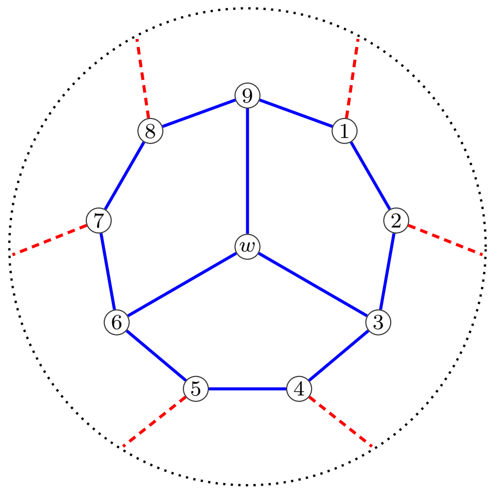


Figure 5.1: (P, Σ_2) embedded into the projective plane; Σ_2 is indicated by the dotted lines

5.2 Families of signed graphs in $\mathcal{S}^*$

The results of this section have been published in [6].

As we already observed, all of the critical subgraphs of the Escher walls described by Reed [26] belong to $\mathcal{S}^*$. Thus, $\mathcal{S}^*(k) \neq \emptyset$ for every $k \geq 1$. However, those graphs are subdivisions of cubic graphs, whose structural properties are not that obvious. In this section we construct signed cubic graphs $(E_k, \Sigma_k) \in \mathcal{S}^*(k)$ for every $k \geq 3$. Let $\mathcal{W} = \{(E_k, \Sigma_k) \colon k \geq 3)\}$ be this family. We first give the construction of the elements of $\mathcal{W}$ and then prove that its elements belong to $\mathcal{S}^*$.

Construction of the elements of $\mathcal{W}$

A *row* R^i of length k is the graph obtained from two distinct paths $P^i = v^i_1, \ldots, v^i_{2k+1}$ and $Q^i = w^i_1, \ldots, w^i_{2k+1}$ with vertex set $V(R^i) = V(P^i) \cup V(Q^i)$ and edge set $E(R^i) = E(P^i) \cup E(Q^i) \cup \{v^i_{2j+1} w^i_{2j+1} \colon j \in \{0, \ldots, k\}\}$. The edges of the path P_i are said to be *horizontal*, while the other edges are said to be *vertical*. The circuits of length 6 in R_i are called *bricks*. Furthermore, note that a row of length k has exactly k bricks.

We say that we *stick* two rows R^i and R^j when we identify the path Q^j with P^i so that either $w_n^j = v_{n-1}^i$, for $n \in \{2, \ldots, 2k+1\}$, or $w_n^j = v_{n+1}^i$, for $n \in \{1, \ldots, 2k\}$.

The construction is given for the even and the odd case separately.

If $k = 2t$ is an even positive integer, then (E_k, Σ_k) is defined as follows:

We first define an all-positive *even wall* $(W_e(k), \emptyset)$ of size k. Let $(R^t, \emptyset)$ be a signed row of length $k-1$. For $j \in \{1, \ldots, t-1\}$ we stick -sequentially- two rows of length $k-j-1$, $(R^{t-j}, \emptyset)$ and $(\check{R}^{t-j}, \emptyset)$, one on the top and one on the bottom so that the first vertex of the path is identified with the second vertex of row $(R^{t-j+1}, \emptyset)$ and $(\check{R}^{t-j+1}, \emptyset)$ respectively. If a row R^i or $\check{R}^i$ has more than $4i$ vertical edges, we remove all the vertical edges except for the first $2i$ edges and the last $2i$ edges. We relabel with x_i, for $i \in \{1, \ldots, k\}$, the first vertex of each path, from the top to the bottom, and y_i, for $i \in \{1, \ldots, k\}$, the last vertex of each path, from the bottom to the top, as in Figure 5.2. The signed graph (E_k, Σ_k) is given by adding the set $\Sigma_k = \{x_i y_i : i \in \{1, \ldots, k\}\}$ to the wall $W_e(k)$, i.e. $E_k = W_e(k) + \Sigma_k$. Observe that a wall of size k can be constructed from a wall of size $k-2$ by adding two bricks to each row, two more rows and possibly one or two vertical edges to some rows and then shifting the vertices x_i, y_i, for $i \in \{1, \ldots, k\}$.

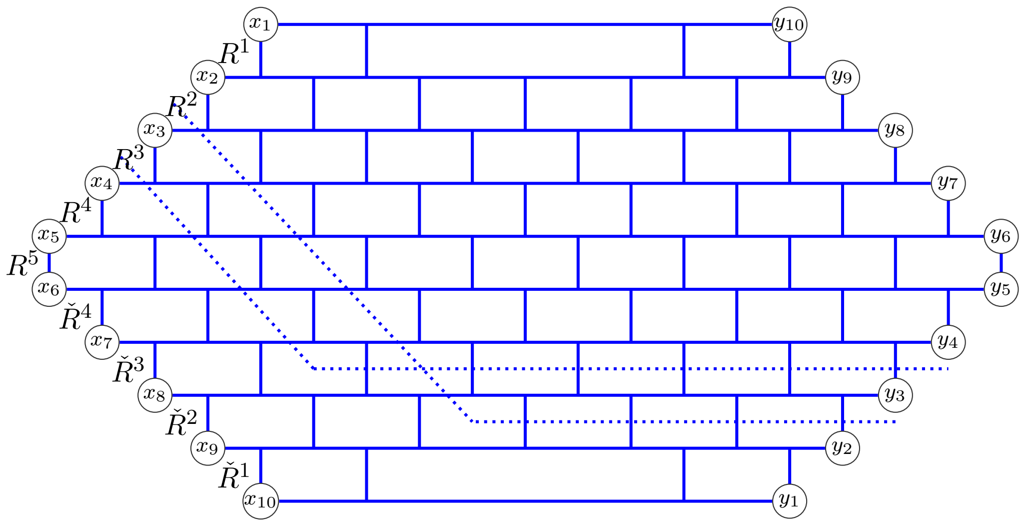


Figure 5.2: The wall of size 10 with two edge-cuts of cardinality 10

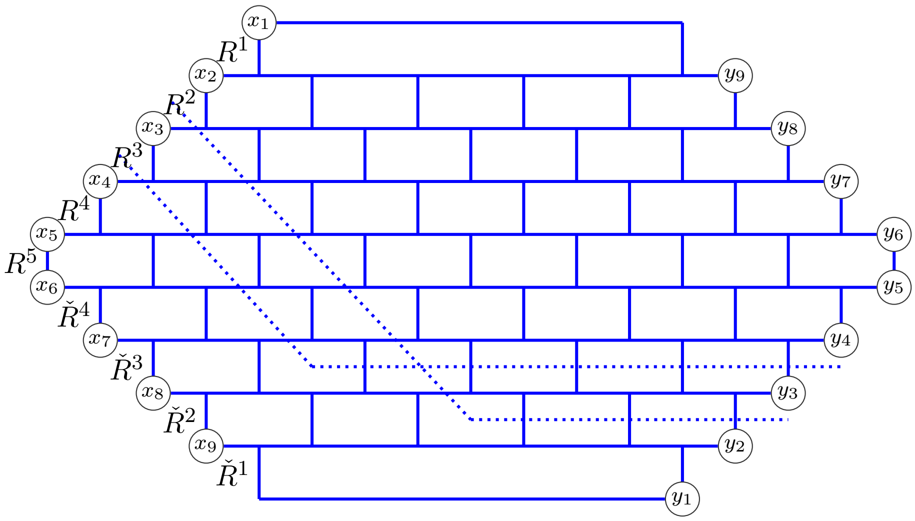


Figure 5.3: The wall of size 9 constructed from the wall of size 7 and with two edge-cuts of cardinality 9

If $k = 2t + 1$ is an odd positive integer, then (E_k, Σ_k) is defined as follows:

We first define an all-positive *odd wall* $(W_o(k), \emptyset)$ of size k. Let $(R^{t+1}, \emptyset)$ be a signed row of length $2t$. For $j \in \{1, \ldots, t\}$ we stick -sequentially- two rows $(R^{t+1-j}, \emptyset)$ and $(\check{R}^{t+1-j}, \emptyset)$ of length $2t + 1 - j$, the first on the top and the second on the bottom, so that the first vertex of the path of the new rows is identified with the second vertex of row $(R^{t-j+2}, \emptyset)$ and $(\check{R}^{t-j+2}, \emptyset)$ respectively. If a row R^i or $\check{R}^i$ has more than $4i - 2$ vertical edges, we remove all the vertical edges except for the first $2i - 1$ edges and the last $2i - 1$ edges. We relabel with x_i, for $i \in \{1, \ldots, k + 1\}$, the first vertex of each path, from the top to the bottom, and y_i, for $i \in \{1, \ldots, k + 1\}$, the last vertex of each path, from the bottom to the top. Lastly, we suppress the divalent vertices x_{k+1} and y_{k+1} (see Fig. 5.3). The signed graph (E_k, Σ_k) is given by adding the set $\Sigma_k = \{x_i y_i \colon i \in \{1, \ldots, k\}\}$ to the wall $W_o(k)$, i.e. $E_k = W_o(k) + \Sigma_k$. In the following, we denote by the boundary B of (E_k, Σ_k) the boundary of the outer faces of embeddings of $W_e(k)$ or of $W_o(k)$ as shown in Figures 5.2 and 5.3 for the cases $k \in \{9, 10\}$.

Theorem 5.2.1. *For each positive integer $k \geq 3$, $(E_k, \Sigma_k) \in \mathcal{S}^*(k)$.*

Proof. First, we prove that (E_k, Σ_k) is k-frustrated, and then that it is also critical. From this together with Theorem 5.1.5 it follows that $(E_k, \Sigma_k) \in \mathcal{S}^*(k)$. For the first step we show that for each $U \subseteq V(E_k)$, it holds $d^-(U) \leq d^+(U)$. In particular, it suffices to prove this property for edge-cuts containing exactly two edges of B. To see this, observe the following.

Define $X = \{x_i \colon i \in \{1, \dots, k\}\}$ and $Y = \{y_i \colon i \in \{1, \dots, k\}\}$. Note that, if $d^-(U) > 0$, then neither $X \cup Y \subseteq U$ nor $X \cup Y \subseteq V(G - U)$. In particular, we can assume that all of the connected components of U contain at least one vertex from $X \cup Y$. Consider the vertices of $X \cup Y$ with the cyclic order $x_1, ..., x_n, y_1, ..., y_n$, that is the order they have in B. Intervals of this set belong to U, that is, an even number of edges of B belongs to $\partial(U)$. Hence, $\partial(U)$ can be seen as the result of the symmetric difference of n edge-cuts $\partial(U_1), ..., \partial(U_n)$ such that $\partial(U_i)$, for $i \in \{1, ..., n\}$, contains exactly two edges of B, and $\partial(U_i) \cap \partial(U_j) \cap (E(E_k) \setminus \Sigma_k) = \emptyset$ for each $i, j \in \{1, ..., n\}$, $i \neq j$. It follows that $d^-(U) \leq \sum_{i=1}^{n} d^-(U_i) \leq \sum_{i=1}^{n} d^+(U_i) = d^+(U)$.

Thus, we assume $\partial(U)$ to be an edge-cut such that $|\partial(U) \cap E(B)| = 2$.

In the following, we say that an edge-cut *crosses* a brick if the edge-cut contains at least one edge of the brick.

If $Y \subseteq U$ then $d^-(U) = |X \cap V(G - U)|$. We can assume $X \cap V(G - U) = \{x_i \colon i \in \{1, \dots, s\}\}$. Hence, by construction, $\partial(U)$ crosses at least $s - 1$ bricks inside the wall, that is, it contains at least s positive edges and $d^-(U) \leq d^+(U)$. The same holds if $X \subseteq U$. Thus, we can assume that there exist i', j' such that $x_{i'}$, $y_{j'+1} \in U$ and $x_{i'+1}$, $y_{j'} \notin U$. We study separately the even and the odd case.

Let $k = 2t$. First, assume that $i' \leq t$ and $j' \geq t$. We can also assume that the edge-cut contains the vertical edges $v_1^{i'} w_1^{i'}$ and $v_{2(k-j')+1}^{k-j'} w_{2(k-j')+1}^{k-j'}$, that is the vertical edges of the shortest paths between $x_{i'}$ and $x_{i'+1}$ and between $y_{j'+1}$ and $y_{j'}$. In particular, by symmetry, we can assume that the edge-cut goes from row R^i, where $i = i'$, to row R^j, where $j = k - j'$, with $i \leq j$. Hence, it holds that $(\{x_s : s \in \{1, ..., i\}\} \cup \{y_r : r \in \{k - j + 1, ..., k\}\}) \subseteq U$, and no other element of X or Y is in U. In particular, $x_s y_s \in \partial(U)$ if and only if $s \in \{1, ..., i\} \cup \{k-j+1, ..., k\}$. It implies that $d^-(U) = i+j$. We aim to prove that $d^+(U) \geq i+j$.

Observe that at least $j-i$ horizontal edges have to belong to the edge-cut in order to go from one row to the other, so the number of vertical edges contained in the edge-cut has to be at least $i+j-(j-i) = 2i$. But after we crossed the bricks using the horizontal edges we still need to cross as many vertical edges as the number of vertical edges contained in row R^i, that is either $4i$ or $i+t$. In both cases it holds $d^+(U) \geq i+j$. Assume now that $i' \leq t$ and $j' < t$. This edge-cut crosses the graph from row R^i, with $i = i'$, to row $\check{R}^j$, with $j = j'$. Again, we can assume, by symmetry, $i \leq j$. Since $x_s \in U$ if and only if $s \in \{1, ..., i\}$, and $y_r \in U$ if and only if $r \in \{j+1, ..., 2t\}$, it holds that $x_s y_s \in \partial(U)$ if and only if $s \in \{1, ..., i\} \cup \{j+1, ..., 2t\}$. Hence, $d^-(U) = i+2t-j$. Furthermore, the number of horizontal edges in the edge-cut is at least $2t-j-i$. We aim to prove that at least $i+2t-j-(2t-j-i) = 2i$ vertical edges belong to the edge-cut.

Observe that, by taking two horizontal edges in the edge-cut, we can also "move" laterally by one brick without using any vertical edge. In particular, by taking n horizontal edges, we can move laterally by $\lfloor \frac{n}{2} \rfloor$ bricks. That is, if we assume that row R^i has $i+t$ vertical edges and since there are $j-i$ more vertical edges in $\check{R}^j$ to cross, the number of vertical edges of the edge-cut is at least $i+t+(j-i) - \left\lfloor \frac{2t-j-i}{2} \right\rfloor \geq j + \frac{j+i}{2} \geq 2i$. Note that, after we crossed the wall vertically, by construction, we always have at least $2i$ edges to take. This implies that, also if row R^i has $4i < i+t$ vertical edges, by construction $2i$ of them belong to the edge-cut. Therefore, it holds $d^+(U) \geq d^-(U)$.

Let $k = 2t+1$. We first consider the case where $i' \leq t+1$ and $j' \geq t+1$. This edge-cut goes from row R^i, $i = i'$, to row R^j, $j = k+1-j'$. As in the previous case, by assuming $i \leq j$, we have that $x_s y_s \in \partial(U)$ if and only if $s \in \{1, ..., i\} \cup \{k-j+2, ..., k\}$ and it follows that $d^-(U) = i+j-1$. Since the edge-cut has to contain at least $j-i$ horizontal edges, it remains to show that it also contains at least $i+j-1-(j-i)=2i-1$ vertical edges. As for the even case, we can assume that these edges are all of the edges belonging to row R^i, so they are either $4i-2$ or $i+t \geq 2i-1$.

Assume now $i' \leq t+1$ and $j' \leq t+1$, $i' \leq j'$. The edge-cut "goes" from row R^i, $i=i'$, to row $\check{R}^j$, $j=j'$. Repeating the same argument as in the previous cases, we have $d^-(U)=i+2t+1-j$, and there are at least $2t+2-i-j$ horizontal edges belonging to the edge-cut. We claim that there are at least $i+2t+1-j-(2t+2-i-j)=2i-1$ vertical edges in the edge-cut. Indeed, by arguing as in the even case, since row R^i has length $t+i-1$, after we "moved" laterally we still need to cross $t+i+(j-i)-\left\lfloor\frac{2t+2-i-j}{2}\right\rfloor \geq j-1+\frac{i+j}{2} \geq 2i-1$. Since by construction we always have the first and the last $2i-1$ vertical edges in a row R^i, it implies that, also when the row has $4i-2<t+i$ vertical edges, there are still at least $2i-1$ vertical positive edges belonging to the edge-cut. As a consequence, it always holds that $d^+(U) \geq d^-(U)$.

It remains to prove that each edge belongs to an equilibrated edge-cut. For the horizontal edges not belonging to B, this is trivial. Similarly, for the edges of B belonging to the path from x_1 to y_k or to the path from x_k to y_1.

For the other edges, it can be observed that the previous considerations about edge-cuts provide equality by taking $i'=j'$. In particular, for each row R^i we can take the first $2i$ ($2i-1$ for the odd case) vertical edges, and then $k-2i$ ($k-2i+1$) horizontal edges. Since the horizontal edges allow the edge-cut to reach the middle of the graph, by symmetry it can be easily observed that this edge-cuts exist for all of the edges. □

Note that this is not the only family of critical subgraphs of the Escher walls. In the following we provide one more family $\mathcal{W}'=\{(E'_k,\Sigma'_k)\colon k= 2t+1 \text{ and } t\geq 1\}$ for the odd case.

Let $k=2t+1$. The corresponding walls $W'(k)$ are defined as follows: Define rows $(R^t,\emptyset)$ and $(R^{t'},\emptyset)$ two rows of length, respectively, $2t$ and $2t-1$. We stick a new row $(R^{t-j},\emptyset)$ to $(R^{t-j+1},\emptyset)$, of length $t+j$, for $j\in\{1,\ldots,t-1\}$. Similarly, we stick a new row $(R^{t-j'},\emptyset)$ to $(R^{t-j+1'},\emptyset)$, of length $t+j-1$, for

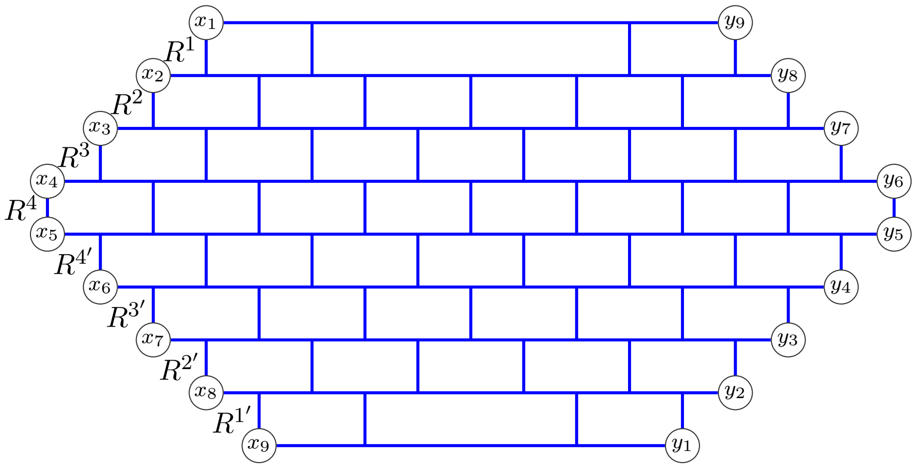


Figure 5.4: One more critical subgraph of the Escher wall of size 9

$j \in \{1, \ldots, t-1\}$. Define x_i, for $i \in \{1, \ldots, k\}$ the first vertex of each path, from the top to the bottom, and y_i, $i \in \{1, \ldots, k\}$ the last vertex of each path, from the bottom to the top, as in Figure 5.4.

As in the previous cases, (E'_k, Σ'_k) is obtained by adding $\Sigma_k = \{x_i y_i \colon i \in \{1, \ldots, k\}\}$ to $W'(k)$. That the elements of $\mathcal{W}'$ belong to $\mathcal{S}^*$ can be proved as in the previous case. Furthermore, $\mathcal{W}$ and $\mathcal{W}'$ are different. To see this observe, for example, that (E'_3, Σ'_3) is the signed Petersen graph. This is not the case when we consider (E_3, Σ_3).

5.3 The family of $\mathcal{S}^*(3)$

The results of this section have been published in [5].

Given the strong conditions on the structure of graphs in $\mathcal{S}^*$ described in Theorem 5.1.5, we do expect that for each $k \geq 1$ the family $\mathcal{S}^*(k)$ has finitely many members. To address this problem, in this section we first provide some general results for signed graphs in $\mathcal{S}^*$, and we then focus on $\mathcal{S}^*(3)$. In particular, we show that $\mathcal{S}^*(3)$ consists of exactly two elements, which are few compared to the cardinality of $\mathcal{L}^*(3)$ (since $\mathcal{L}^*(3)$ contains at least all of the planar graphs described in Section 2.4).

In the following, we strongly refer to Theorem 5.1.5. In particular, let

$(G,\Sigma) \in \mathcal{S}^*(k)$ with $\Sigma = \{x_1y_1,...,x_ky_k\}$ being a minimum signature. An embedding of (G,Σ) into the projective plane as described in Theorem 5.1.5 is called a *canonical projective-planar embedding* of (G,Σ). Where the choice of such signature is clear from the context, we denote the subgraph $G-\Sigma$ of G by G'. Moreover, G' will always be considered together with its planar embedding that is implied from this theorem. The facial circuit of the outer face of this plane graph G' will be denoted by C_O. One may observe that in G' the vertices $x_1,\dots,x_k,y_1,\dots,y_k$ are all of degree 2 and they appear on C_O in this cyclic order.

Given vertices u and v of C_O, by A_{uv} we denote the path on C_O connecting u to v which is in clockwise direction starting at u and ending at v. When referring to a face of G' we do not consider the outer face. Thus a face F of G' is also a face of (G,Σ) in the projective plane embedding of it from which G' is obtained. The boundary of this face F, which must be a circuit, will be denoted by C_F.

A face of G' is said to be *internal* if its boundary shares no edge with C_O. We note that, since G' is subcubic, the boundary of an internal face does not intersect C_O at a vertex either. In particular, the boundary of a face F which is not internal shares at least two vertices with C_O. We classify such faces depending on how many of those common vertices are in the set $\mathcal{R} = \{x_1,...,x_k,y_1,...,y_k\}$. More precisely, a face F is said to be an *i-face* of G' if C_F contains i elements from the set $\mathcal{R}$. Two faces F_1 and F_2 are said to be *adjacent on the boundary* if $V(C_{F_1}\cap C_{F_2}\cap C_O)\neq\emptyset$.

A face F of G' is called a *bridge face* if the subgraph induced by $C_F\cap C_O$ is disconnected. See Figure 5.5 for an example. Note that curves represent paths that might contain more vertices.

Note that in (G,Σ), each edge-cut with negative edges contains at least two edges of C_O. Furthermore, based on the cyclic order of the elements of $\mathcal{R}$ on C_O, we have the following facts.

Lemma 5.3.1. *Let (G,Σ) be a canonically projective-planar embedded signed*

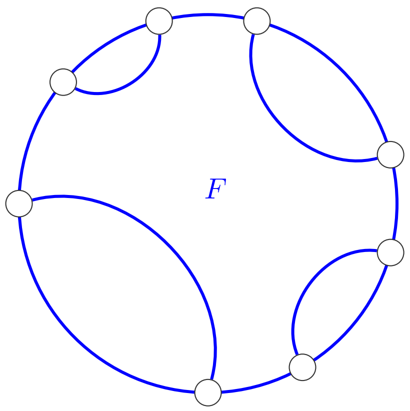


Figure 5.5: A bridge-face in G'

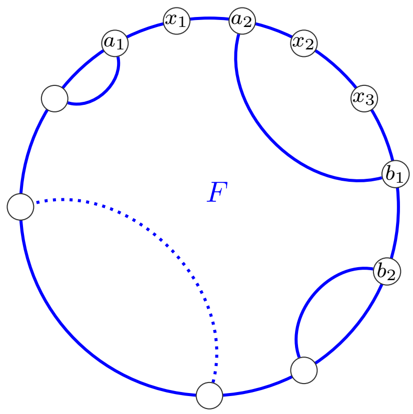


Figure 5.6: Proposition 5.3.3

graph in $\mathcal{S}^*(k)$ for $k \geq 2$. If an edge-cut $\partial_G(X)$ contains exactly two edges e_1 and e_2 of C_O, then $d^-_{(G,\Sigma)}(X) = \min\{|V(A_1) \cap \mathcal{R}|, |V(A_2) \cap \mathcal{R}|\}$ where A_1 and A_2 are the two connected components of $C_O - \{e_1, e_2\}$.

Lemma 5.3.2. *Let (G, Σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(k)$ for $k \geq 3$. Assume F to be a bridge-face and let $A_{a_1a_2}$ and $A_{b_1b_2}$ be two connected components of $C_F \cap C_O$ such that $A_{a_2b_1}$ is a connected component in $C_O - E(C_F)$. Then $|V(A_{a_2b_1}) \cap \mathcal{R}| \in \{2, 2k-2\}$.*

Proof. Let e_1 (resp. e_2) be the edge in $A_{a_1a_2}$ (resp. $A_{b_1b_2}$) that has a_2 (resp. b_1) as an endpoint. Let G'' be the connected component of $G' - \{e_1, e_2\}$ containing a_2 (and b_1).

We first show that $|V(A_{a_2b_1}) \cap \mathcal{R}| \notin \{3, 4, \dots, 2k-3\}$. Otherwise, the edge-cut $\partial_G(V(G''))$ must contain at least three negative edges, but it has only two positive edges. This contradicts the fact that Σ is a minimum signature.

Next we show that $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$ is not possible either. That $|V(A_{a_2b_1}) \cap \mathcal{R}| \geq 2k-1$ is not possible follows similarly. Suppose to the contrary that $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$. In $C_F - \{e_1, e_2\}$, there is a path connecting a_2 to b_1 and let e be an edge of this path. By criticality, there exists an equilibrated edge-cut $\partial(X)$ containing e. Since each equilibrated edge-cut of (G, Σ) contains at least two (positive) edges from C_O and noting that e is also a positive edge, we have $d^+(X) \geq 3$, and hence $d^-(X) \geq 3$. Moreover, by the choice of e, at least one of the edges of $A_{a_2b_1}$, say e', is in $\partial(X)$. Thus in total, at least two edges of G'' are in $\partial(X)$. We now consider the following two edge-cuts: $E_1 = \partial(X) \setminus E(G'') \cup \{e_1\}$ and $E_2 = \partial(X) \setminus E(G'') \cup \{e_2\}$. Since $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$, it follows that one of these two edge-cuts, say E_1, has the same set of negative edges as $\partial(X)$. However, E_1 has fewer positive edges than $\partial(X)$, contradicting the minimality of Σ. □

Proposition 5.3.3. *Let (G, Σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(k)$ for $k \geq 3$. Then the following statements hold:*

1. *Every bridge-face of G' is a 0-face.*

2. *For $i \geq 3$ there is no i-face in G'.*

Proof. 1 Let F be a bridge-face of G' and assume that $C_F \cap C_O$ consists of t connected components (thus $t \geq 2$). Let $A_{a_1a_2}$ and $A_{b_1b_2}$ be two connected components of $C_F \cap C_O$ such that $A_{a_2b_1}$ is a connected component in $C_O - E(C_F)$. By Lemma 5.3.2, $|V(A_{a_2b_1}) \cap \mathcal{R}| \in \{2, 2k-2\}$. Toward a contradiction and without loss of generality, assume that $x_1 \in \mathcal{R} \cap V(A_{a_1a_2})$ and $x_2, x_3 \in \mathcal{R} \cap V(A_{a_2b_1})$, depicted in Figure 5.6.

We claim that each connected component of $C_O \cap C_F$ contains exactly two vertices from $\mathcal{R}$. If not, then one of them contains $2k-2$ vertices from $\mathcal{R}$. In this case, since $|\mathcal{R}| = 2k$, there is only one other component in $C_O \cap C_F$. Furthermore, this component must contain the other two vertices of $\mathcal{R}$. This in turn implies that F is a 0-face.

Let e_1 be the edge on $A_{a_1x_1}$ incident with x_1 and let e_2 be the edge on $A_{b_1b_2}$ incident with b_1. Then the set $\{e_1, e_2, x_1y_1, x_2y_2, x_3y_3\}$ is an edge-cut consisting of two positive edges and three negative edges, contradicting the fact that Σ is a minimum signature.

2 Suppose to the contrary that F is an i-face of G' for $i \geq 3$. By Claim 1, we know that F is not a bridge-face. Therefore, by the symmetry of labeling, we assume that $x_1, x_2, x_3 \in V(C_F) \cap \mathcal{R}$. Let $e_1 = vx_1, e_2 = x_3u \in E(C_F \cap C_O)$ such that $v \notin V(A_{x_1x_2})$ and $u \notin V(A_{x_2x_3})$. Then the edge set

$\{e_1, e_2, x_1y_1, x_2y_2, x_3y_3\}$ is an edge-cut that contains three negative edges but only two positive edges, a contradiction. □

From now on, we focus on the family $\mathcal{S}^*(3)$. We give some structural properties of signed graphs in $\mathcal{S}^*(3)$ in the following lemmas.

Lemma 5.3.4. *Let (G, Σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. Then each face of G' is either a bridge-face or an i-face for $i \in \{1, 2\}$.*

Proof. By Proposition 5.3.3 2, if F is an i-face of G', then $i \in \{0, 1, 2\}$. Thus, it remains to show that there are no internal faces and that every 0-face is a bridge-face.

For the first claim, assume to the contrary that there exists an internal face F of G'. Note that each equilibrated edge-cut containing one edge of C_F must have at least two (positive) edges from C_F and two (positive) edges from C_O, hence a minimum of four positive edges. However, there are only three negative edges in (G, Σ), contradicting the fact that each equilibrated edge-cut has the same number of positive and negative edges.

For the second claim, assume F to be a 0-face of G' which is not a bridge-face. As C_F shares at least one edge with the outer facial circuit C_O of G', there is a face F' such that $C_{F'}$ shares a common vertex with both C_F and C_O. Assume that F' is an i-face for $i \in \{0, 1, 2\}$. Let e_0 be a (positive) edge in the path $C_F \cap C_{F'}$. Let $\partial(X)$ be the equilibrated edge-cut containing e_0. Recall that any equilibrated edge-cut must contain at least two edges of C_O. As $(G, \Sigma) \in \mathcal{S}^*(3)$, $\partial(X)$ contains exactly two edges of C_O. Furthermore, one of these two edges belongs to $E(C_O \cap C_F)$ while the other is in $E(C_O \cap C_{F'})$. To complete the proof, it suffices to show that $|X \cap \mathcal{R}| \neq 3$, which would contradict the fact that $\partial(X)$ is an equilibrated edge-cut. If F' is not a bridge-face, then by Proposition 5.3.3 there are at most two elements of $\mathcal{R}$ in $C_{F'}$, and consequently at most two elements of $\mathcal{R}$ in X (i.e., $|X \cap \mathcal{R}| \leq 2$). If F' is a bridge-face, then by Lemma 5.3.2 the number of elements of $\mathcal{R}$ in each connected component of $C_O - E(C_{F'})$ is either 2 or 4. As either all of the vertices of a connected component of $C_O - E(C_{F'})$ are contained in X or none of them is in X, $|X \cap \mathcal{R}|$ has to be an even number and clearly $|X \cap \mathcal{R}| \neq 3$. $\square$

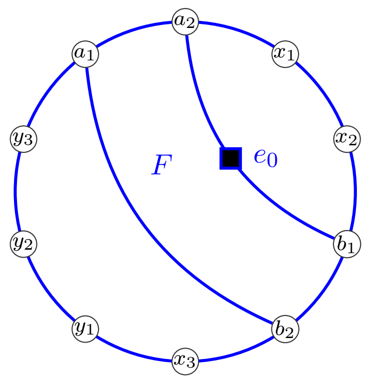


Figure 5.7: Case in Lemma 5.3.5

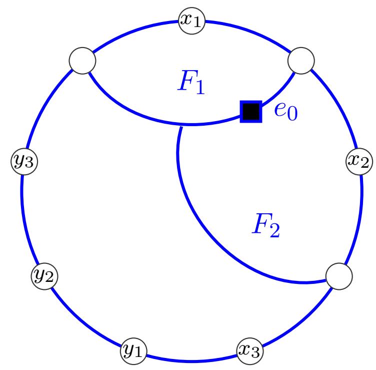


Figure 5.8: Case in Lemma 5.3.6

Lemma 5.3.5. *Let (G, Σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. If F is a bridge-face of G', then $C_F \cap C_O$ consists of exactly three connected components. In particular, there is at most one bridge-face.*

Proof. As $(G, \Sigma) \in \mathcal{S}^*(3)$ we have $|\mathcal{R}| = 6$. As F is a bridge-face, $C_F \cap C_O$ has at least two connected components, and, by Lemma 5.3.2, it has at most three components. It remains to show that $C_F \cap C_O$ does not have exactly two components. Suppose to the contrary that $C_F \cap C_O$ has exactly two components, say $A_{a_1a_2}$ and $A_{b_1b_2}$. Then one of $A_{a_2b_1}$ or $A_{b_2a_1}$, say $A_{a_2b_1}$ without loss of generality, has two elements from $\mathcal{R}$, and the other, $A_{b_2a_1}$ in this case, has four elements from $\mathcal{R}$. See Figure 5.7 for a depiction.

Let e_0 be an edge on the a_2b_1-path of C_F which is internally vertex-disjoint from C_O. Let $\partial(X)$ be an equilibrated edge-cut containing e_0. As $\partial(X)$ must contain two (positive) edges, say e_1 and e_2, of C_O, it has to be an edge-cut of size 6 and hence e_0, e_1, and e_2 are the only positive edges of it. Thus one of e_1 or

e_2, say e_1, is on $A_{a_2b_1}$ and the other i.e. e_2, is on $A_{a_1a_2} \cup A_{b_1b_2}$. Noting that each bridge-face is a 0-face by Proposition 5.3.3 1 and $A_{a_2b_1}$ contains two elements from $\mathcal{R}$, X has at most two vertices of $\mathcal{R}$ and, therefore, $\partial(X)$ contains at most two negative edges, contradicting the fact that it is an equilibrated edge-cut.

Finally, by Lemma 5.3.2, as each of the connected components of $C_O - E(C_F)$ must contain at least two or four elements of $\mathcal{R}$, and since there are three connected components, each of them contains exactly two elements of $\mathcal{R}$ and thus there is no other bridge-face. $\square$

Lemma 5.3.6. *Let (G, Σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. Let F_1 and F_2 be an i_1-face and an i_2-face of G', respectively. If F_1 is adjacent to F_2 on the boundary, then either (i) $i_1 + i_2 \geq 3$ or (ii) one of F_1 and F_2 is a bridge-face.*

Proof. Assume that neither F_1 nor F_2 is a bridge-face. By Lemma 5.3.4 $i_1+i_2 \geq 2$, thus it remains to prove that $i_1 + i_2 \neq 2$.

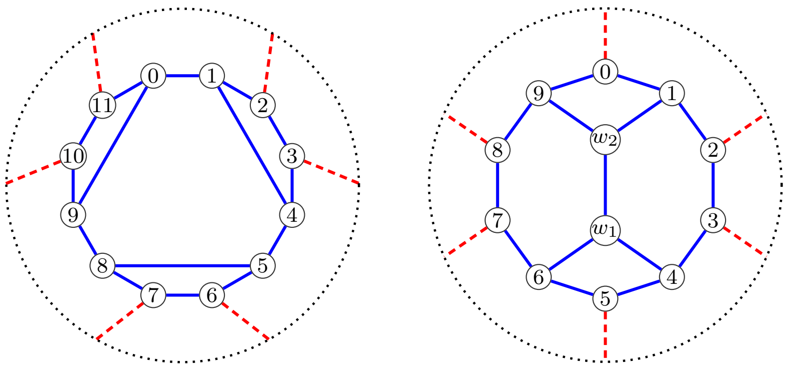
Assume to the contrary that $i_1 + i_2 = 2$. By Lemma 5.3.4, $i_1 = 1_2 = 1$. Let e_0 be an edge on the path $C_{F_1} \cap C_{F_2}$. See Figure 5.8. Each equilibrated edge-cut containing the edge e_0 must have two more (positive) edges of C_O, say e_1 and e_2. It follows as before that e_1 is on $C_{F_1} \cap C_O$ and e_2 is on $C_{F_2} \cap C_O$. Since $i_1 + i_2 = 2$, a similar argument implies that X can contain at most two vertices from $\mathcal{R}$, leading to a contradiction with $\partial(X)$ being an equilibrated edge-cut. $\square$

We are now ready to give the full description of $\mathcal{S}^*(3)$.

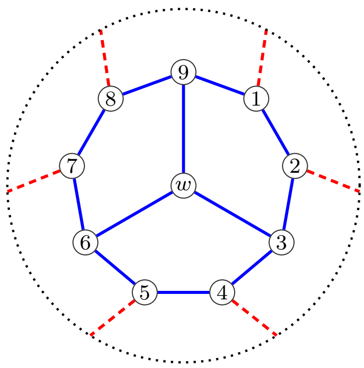
Theorem 5.3.7. *The family $\mathcal{S}^*(3)$ consists of two signed graphs, depicted in Figure 5.9.*

Proof. We consider the following three cases:

- G' has a bridge-face F. By Lemma 5.3.5, F is the only bridge-face of G' and $C_O \cap C_F$ consists of three components each of which has exactly two elements from $\mathcal{R}$. Furthermore, it follows from Lemma 5.3.6 that



(a) (G_1, Σ_1) with two embeddings



(b) (G_2, Σ_2)

Figure 5.9: $\mathcal{S}^*(3)$

the vertices of $\mathcal{R}$ are the only vertices on each of these components, as for otherwise a vertex not in $\mathcal{R}$ would result in an i_1-face and an i_2-face with $i_1 + i_2 \leq 2$. This leads to the projective planar graph of Figure 5.9a (left).

- G' has at least one 1-face (and no bridge-face). Let F_1 be a 1-face of G'. As G' has no bridge-face, by Lemma 5.3.4, it has no 0-face. Furthermore, by Lemma 5.3.6, each of the two faces adjacent to F_1 on the boundary are 2-faces. As there are only six vertices in $\mathcal{R}$, and as there is no internal face by Lemma 5.3.4, there is only one remaining face. Furthermore, this face is a 1-face. Let G'' be the graph obtained from G' by suppressing all vertices of $\mathcal{R}$. Note that G'' is cubic and planar.

  It then follows from Euler's formula that $|V(G'')| - \frac{3}{2}|V(G'')| + 5 = 2$, i.e., $|V(G'')| = 6$. But there are only two cubic graphs on 6 vertices: $K_{3,3}$ and the 3-prism. As $K_{3,3}$ is not planar, G'' is the 3-prism. As each 1-face of G' is incident with two 2-faces of G', both of the triangles of G'' correspond to faces of the same type in G'. More precisely, either each one corresponds to a 1-face or each one corresponds to a 2-face. The former case leads to the projective planar graph of Figure 5.9a (right). In the latter case we consider the middle edge of the 3-thread in G' and we observe that this edge cannot be in an equilibrated edge-cut.

- Each face of G' is a 2-face. Hence G' has exactly three 2-faces. Similar to the previous case we consider the graph G'' obtained from G' by replacing each thread with an edge. It follows from Euler's formula that G'' has four vertices and hence, since it is cubic, it must be K_4. Thus (G, Σ) is the signed graph in Figure 5.9b.

We note that the two signed graphs in Figure 5.9a are switching-isomorphic and thus up to switching $\mathcal{S}^*(3)$ consists of two signed graphs. □

Observe that the signed graph (G_2, Σ_2) of Figure 5.9b is a signed Petersen graph.

Note that, even though the classes $\mathcal{S}^*(3)$ and $\mathcal{P}^*(3)$ are fully described in this work, the full description of the class $\mathcal{L}^*(3)$ is far from clear. In particular, $\mathcal{L}^*(3) \setminus (\mathcal{S}^*(3) \cup \mathcal{P}^*(3)) \neq \emptyset$. Two examples of such signed graphs are given in Figure 5.10.

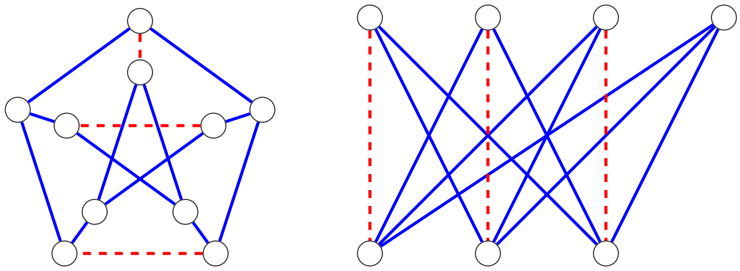


Figure 5.10: Examples in $\mathcal{L}^*(3)$ neither in $\mathcal{S}^*(3)$ nor in $\mathcal{P}^*(3)$

Chapter 6

Signed graph coloring

Graph coloring is one of the most popular topics in graph theory, not only because of the various applications it can provide, but also because of the numerous ways it can be extended. As a consequence, the question whether and how colorings can be extended to signed graphs spontaneously arises. There are different approaches to answer this question. We consider colorings of signed graphs which are defined by assigning colors to its vertices.

We recall that, in order to make notation simpler for the reader, in this chapter we consider a signature of a graph G to be a map $\sigma : E(G) \to \{+1, -1\}$. Moreover, given a signed graph (G, σ), we denote with E_σ^- the set of negative edges of (G, σ).

Let (G, σ) be a signed graph and S be a set of colors. A *coloring* of (G, σ) is a map $c : V(G) \longrightarrow S$. A coloring c is *proper* if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. If (G, σ) admits a proper coloring with elements from S, we say that (G, σ) is S-colorable.

While the coloring-condition for positive edges remains unchanged with respect to the unsigned case, the condition on a negative edge $e = vw$ requires that $c(v) \neq -c(w)$. It implies that $-s \in S$ for each $s \in S$. At this point, the choice of the elements of S has strong consequences on the colorings. Therefore, two cases have to be distinguished: When s is a non-self-inverse element, that is $s \neq -s$, and when s is a self-inverse element, and so $s = -s$.

The objective of this chapter is to define a coloring and a corresponding chromatic number for a signed graph (G, σ) which gives a minimum color coded partition of the vertex set and which does not depend on the number of self-inverse colors which are allowed for coloring. To achieve this we first discuss colorings where the number of self-inverse elements is fixed. This somewhat technical part is performed in Section 6.1, where we prove a Brooks'-type theorem for these kinds of coloring, and we determine the chromatic spectrum of signed graphs and the chromatic polynomial. These results are used in Section 6.2, where we show that the symset chromatic number (which is defined later in this subsection) describes the minimum partition of the signed graph into independent sets and antibalanced non-bipartite subgraphs. It follows that this parameter gives a lower bound on the number of pairwise vertex-disjoint negative circuits of a signed graph. We further give an upper bound for the symset-chromatic number for a specific family of signed graphs and show that circular coloring of signed graphs [20, 25] is also covered by our approach and that all these colorings can also be formalized as DP-coloring.

Note that, if a vertex v of (G, σ) is incident with a positive loop, then (G, σ) does not have a proper coloring. If v is incident to a negative loop, then it has to be colored with a non-self-inverse color, which in some sense counteracts our aforementioned objective. For these reasons, in this chapter we consider multigraphs without loops.

6.1 The symset t-chromatic number

The results of this section, except for subsection 6.1.3 have been published in [7].

We recall that a symmetric set $S_{2k}^t = \{0_1, ..., 0_t, \pm 1, \pm k\}$ is a set containing t self-inverse elements and $2k$ non-self-inverse elements. Furthermore, given a signed graph (G, σ) and a positive integer t, we define the symset t-chromatic number as the minimum $\lambda_t = t + 2k$ for which (G, σ) admits an S_{2k}^t-coloring.

As we said in the introduction, self-inverse elements "cancel" the effect of the signature, therefore each signed graph (G, σ) has a $S_0^{\chi(G)}$-coloring. For this reason, given a signed graph (G, σ), we always assume $t \leq \chi(G)$.

An S_{2k}^t-coloring of (G, σ) provides some information on the structure of (G, σ).

Proposition 6.1.1. *If a signed graph (G, σ) admits a proper S_{2k}^t-coloring c, then c induces a partition of $V(G)$ such that $c^{-1}(0)$ is an independent set in G for every self-inverse color 0, and $(G[c^{-1}(\pm s)], \sigma)$ is an antibalanced subgraph of (G, σ) for every pair of non-self-inverse color $\pm s$.*

We first prove upper bounds for the symset t-chromatic number in terms of the chromatic number of the underlying graph. The cases $t \leq 1$ and $t = 2$ have been proved in [23] and [18], respectively.

Theorem 6.1.2. *Let G be a graph with chromatic number k. Then for every $t \in \{0, \dots, k\}$: $\chi_{sym}^t(G, \sigma) \leq 2k - t$. Furthermore, $\chi_{sym}^t(\pm G) = 2k - t$ and there are simple signed graphs (H, σ_H) such that $\chi(H) = k$ and $\chi_{sym}^t(H, \sigma_H) = 2k - t$.*

Proof. Let c be a k-coloring of G with colors from $\{0_1, \dots 0_t, s_{t+1}, \dots, s_k\}$. C can be extended to a coloring of $\pm G$ with colors from $\{0_1, \dots 0_t, \pm s_{t+1}, \dots, \pm s_k\}$. Hence, $\chi_{sym}^t(\pm G) \leq 2k - t$.

If $t = k$, then $\chi_{sym}^t(\pm G) = 2k - t$, since $\chi(G) = k$.

Let $t < k$ and suppose to the contrary, that $\chi_{sym}^t(\pm G) < 2k - t$. Then, there is a coloring with elements $\{0_1, \dots 0_t, \pm s_{t+1}, \dots, \pm s_l\}$ and $l < k$. If necessary by switching there is a $2l - t$ coloring of (G, σ) which only uses colors $\{0_1, \dots 0_t, s_{t+1}, \dots, s_l\}$. This is also an l-coloring of G, a contradiction. Hence, $\chi_{sym}^t(\pm G) = 2k - t$ and $\chi_{sym}^t(G, \sigma) \leq 2k - t$, since (G, σ) is a subgraph of $\pm G$.

Let G_k be the Turan graph on $k(k - t + 1)$ vertices which is the complete k-partite graph with k independent sets of cardinality $k - t + 1$. Thus, G_k contains $k - t + 1$ pairwise disjoint copies $H_1, \dots, H_{k-t+1}$ of K_k. Let σ be a signature on G_k with $E_\sigma^- = \bigcup_{i=2}^{k-t+1} E(H_i)$. Clearly, $\chi(G_k) = k$ and therefore, $\chi_{sym}^t(G_k, \sigma) \leq 2k - t$.

If $t = k$, then $\chi^t_{sym}(G_k, \sigma) = \chi(G_k) = k(= 2k - k)$. Let $t \in \{0, \dots, k-1\}$ and suppose to the contrary that $\chi^t_{sym}(G_k, \sigma) < 2k - t$, say (G, σ) is colored with colors from $\{0_1, \dots, 0_t, \pm s_{t+1} \cdots \pm s_l\}$ with $l < k$.

Then, at least $k - t$ vertices of H_1 are colored with pairwise different colors from $\{\pm s_{t+1} \cdots \pm s_l\}$. Furthermore, for each $i \in \{2, \dots, k-t\}$ each all-negative copy H_i of K_k contains at least two vertices of the same color of $\{\pm s_{t+1} \cdots \pm s_l\}$. Since for all $2 \le i < j \le k - t + 1$ each vertex of H_i is connected by a positive edge to every vertex of H_j, it follows that for all $i \neq j$, the multiple used colors in H_i are different from the multiple used colors in H_j. Thus, at least $2(k - t)$ pairwise different non-self-inverse colors are needed; $k - t$ for the all-negative copies of K_k and $k - t$ for the coloring of H_1. But we have only $2(l - t)$ non-self-inverse colors and $l < k$, a contradiction. Thus, $\chi^t_{sym}(G_k, \sigma) = 2k - t$. □

6.1.1 Brooks' type theorem for the symset t-chromatic number

We next prove a Brooks' type theorem for the symset t-chromatic number, which implies Brooks' Theorem for unsigned graphs.

Observe that the symset t-chromatic number has the same parity as t. Furthermore, if $t = \chi(G) - l$, then Theorem 6.1.2 can be reformulated as $\chi^t_{sym}(G, \sigma) \le \chi(G) + l$. By parity we obtain equality in the following statement.

Proposition 6.1.3. *Let (G, σ) be a signed graph. If $t = \chi(G) - 1$, then $\chi^t_{sym}(G, \sigma) = \chi(G) + 1$.*

If G is a graph with $\chi(G) = \Delta(G) = t + 1$, then $\chi^t_{sym}(G, \sigma) = \Delta(G) + 1$ by Proposition 6.1.3. The following Brooks' type statement is the main result of this section.

Theorem 6.1.4. *Let G be a connected graph and $t \in \{0, \dots, \chi(G)\}$.*
If $\Delta(G) - t$ is odd, then $\chi^t_{sym}(G, \sigma) \le \Delta(G) + 1$.
If $\Delta(G) - t$ is even, then $\chi^t_{sym}(G, \sigma) = \Delta(G) + 2$ or $\chi^t_{sym}(G, \sigma) \le \Delta(G)$. Furthermore, $\chi^t_{sym}(G, \sigma) = \Delta(G) + 2$ if and only if

- *G is a complete graph and $t = \chi(G) - 1 (= \Delta(G))$ or*

- *(G, σ) is a balanced complete graph or*
- *(G, σ) is a balanced odd circuit or*
- *(G, σ) is an unbalanced even circuit and $t = 0$ or*
- *(G, σ) is an unbalanced odd circuit and $t = 2$.*

We prove the statement by formulating several propositions, some of which might be of interest on their own.

Proposition 6.1.5. *Let K_n be the complete graph on $n \geq 3$ vertices and let $t \in \{0, \dots, n\}$.*
If $\Delta(K_n) - t$ is odd, then $\chi^t_{sym}(K_n, \sigma) \leq \Delta(K_n) + 1$.
If $\Delta(K_n) - t$ is even, then $\chi^t_{sym}(K_n, \sigma) = \Delta(K_n) + 2$ or $\chi^t_{sym}(K_n, \sigma) \leq \Delta(K_n)$.
Furthermore, $\chi^t_{sym}(K_n, \sigma) = \Delta(K_n) + 2$ if and only if (K_n, σ) is equivalent to $(K_n, +)$ or $t = n - 1$.

Proof. If $t = n$ or $t = n - 2$, then $\chi^t_{sym}(K_n, \sigma) = \chi(K_n) = n = \Delta(K_n) + 1$. If $t = n - 1$, then by Proposition 6.1.2, $\chi^t_{sym}(K_n, \sigma) = \chi(K_n) + 1 = n + 1 = \Delta(K_n) + 2$.

Let $t \leq n-3$ and $\chi^t_{sym}(K_n, \sigma) = t+2k$. Hence, $k \geq 1$. First we consider the case when (K_n, σ) is not balanced. Then it contains an induced antibalanced circuit C_3 of length 3, which can be colored with one pair of non-self-inverse colors. Thus, $(K_n - V(C_3), \sigma)$ can be colored with at most $n - 3$ pairwise different colors. Taking the parity into account it follows that if $\Delta(K_n) - t$ is even, then $\chi^t_{sym}(K_n, \sigma) \leq n - 1 = \Delta(K_n)$, and if $\Delta(K_n) - t$ is odd, then $\chi^t_{sym}(K_n, \sigma) \leq n = \Delta(K_n) + 1$.

If σ is equivalent to $+$, then any coloring of $(K_n, +)$ needs n pairwise different colors. Thus, $\Delta(K_n) - t$ is even if and only if $\chi^t_{sym}(K_n, +) = n + 1 = \Delta(K_n) + 2$. $\square$

Proposition 6.1.6. *For each circuit C_n on n vertices:*

- *If $t \in \{1, 3\}$, then $\chi^t_{sym}(C_n, \sigma) = 3$.*

- *If* $t \in \{0, 2\}$, *then* $\chi^t_{sym}(C_n, \sigma) \in \{2, 4\}$, *and* $\chi^t_{sym}(C_n, \sigma) = 4$ *if and only if*
  - (C_n, σ) *is a balanced odd circuit or*
  - (C_n, σ) *is an unbalanced even circuit and* $t = 0$ *or*
  - (C_n, σ) *is an unbalanced odd circuit and* $t = 2$.

Proof. Since we assume that $t \leq \chi(C_n)$, it follows that $t \in \{0, 1, 2, 3\}$, where $t = 3$ only applies if n is odd, and there it holds $\chi^3_{sym}(C_n, \sigma) = \chi(C_n) = 3$. Furthermore, it is easy to check that $\chi^1_{sym}(C_n, \sigma) = 3$. The statements for $t = 2$ follow with Proposition 6.1.3. It is easy to see that $\chi^0_{sym}(C_n, \sigma) \leq 4$ and $\chi^0_{sym}(C_n, \sigma) = 2$ if and only if n is even and C_n is balanced or n is odd and C_n is unbalanced. □

The following statement is a standard lemma for coloring.

Lemma 6.1.7. *The vertices of a connected graph* G *can be ordered in a sequence* $x_1, x_2, ..., x_n$ *so that* x_n *is any preassigned vertex of* G *and for each* $i < n$ *the vertex* x_i *has a neighbor among* $x_{i+1}, ..., x_n$.

Lemma 6.1.8. *Let* (G, σ) *be a simple connected signed graph. If* G *is not regular, then*

$$\chi^t_{sym}(G, \sigma) \leq \begin{cases} \Delta(G) + 1, & \text{if } \Delta(G) - t \text{ is odd} \\ \Delta(G), & \text{if } \Delta(G) - t \text{ is even.} \end{cases}$$

Proof. Let v be a vertex having degree $d_G(v) \leq \Delta - 1$. By Lemma 6.1.7, there exists an ordering of the vertices $x_1, ..., x_n$ such that $x_n = v$ and for each $i < n$ the vertex x_i has neighbors among $x_{i+1}, ..., x_n$. We follow this order to color the vertices by using the greedy algorithm. We can first use the t self-inverse colors, and then add pairs of non-self-inverse colors when it is necessary.

If $\Delta - t = 2n$ is even, then we use exactly n non-self-inverse colors $\pm s$. Each vertex x_i, $i < n$, has at most $\Delta - 1$ neighbors which have been colored previously. Since it also holds $d(x_n) \leq \Delta - 1$, the graph has an S^t_{2k}-coloring, with $t + 2k = \Delta$.

If $\Delta - t$ is odd, then the result follows similarly.

□

For the proof of Theorem 6.1.4 we also use the following lemma.

Lemma 6.1.9 ([21]). *Let G be a 2-connected graph with $\Delta(G) \geq 3$ other than a complete graph. Then G contains a pair of vertices a and b at distance 2 such that the graph $G - \{a, b\}$ is connected.*

Proof of Theorem 6.1.4.

Propositions 6.1.5 and 6.1.6 imply that the statement is true for complete graphs and circuits. By Lemma 6.1.8 it suffices to prove it for non-complete regular graphs with maximum vertex degree at least 3. We can also assume that the graph is connected.

Let (G, σ) be a signed graph of order n and $0 \leq t \leq \chi(G)$. If $\Delta(G) - t$ is odd, then (G, σ) can be colored greedily with $\Delta(G) + 1$ colors. Hence, we focus on the case where $\Delta(G) - t$ is even. We show that the graph has an S^t_{2k}-coloring with $t + 2k \leq \Delta(G)$.

Assume that (G, σ) is 2-connected. By Lemma 6.1.9 there are two non-adjacent vertices a and b which have a common neighbor x and $G - \{a, b\}$ is connected. By possible switching we can assume that ax and bx both are positive. Order the vertices of G as in Lemma 6.1.7 so that $x_1 = a$, $x_2 = b$ and $x_n = x$. The vertices x_1 and x_2 can receive the same color since they are not adjacent. The vertices x_3,...,x_{n-1} can be colored greedily. Indeed each vertex has at most $\Delta(G) - 1$ neighbors which are already colored. Since two neighbors of x have the same color, there is an element of S^t_{2k} which is not used in the neighborhood of x.

Assume now that (G, σ) is not 2-connected, that is, there exists a edge-cut vertex v.

Let H_1, H_2,...,H_k be the components of $G - v$. For each $i \in \{1, ..., k\}$, the subgraph $H'_i = H_i \cup v$ is not regular and $d_{H'_i}(v) < \Delta(H'_i)$. Thus, it can be colored by $\Delta(G)$ colors by Lemma 6.1.8. By relabeling we can always suppose that v is colored with the same element in each graph, so the entire graph is also S_{2k}^t-colorable, with $t + 2k = \Delta(G)$. □

Corollary 6.1.10 (Brooks' Theorem [3])**.** *Let G be a connected graph. If G is neither complete nor an odd circuit, then $\chi(G) \leq \Delta(G)$.*

Proof. By induction we get $\chi(G) \leq \Delta(G) + 1$. Assume that $\chi(G) = \Delta(G) + 1$. For $t = \Delta(G)$ it follows by Proposition 6.1.3 that $\chi_{sym}^t(G, +) = \Delta(G) + 2$. Hence, by Theorem 6.1.4, G is a complete graph or it is an odd circuit. □

As a simple consequence of Theorem 6.1.2 and Corollary 6.1.10 we obtain the following statement on the signed extension of a graph.

Corollary 6.1.11. *Let G be a connected graph. If G is a complete graph or an odd circuit, then $\chi_{sym}^t(\pm G) = \Delta(\pm G) + 2 - t$. Otherwise $\chi_{sym}^t(\pm G) \leq \Delta(\pm G) - t$.*

6.1.2 Symset t-chromatic spectrum

Let G be a graph and $\Sigma(G)$ be the set of its non-equivalent signatures. The *symset t-chromatic spectrum* of G is the set $\Sigma_{\chi_{sym}^t}(G) := \{\chi_{sym}^t(G, \sigma) : \sigma \in \Sigma(G)\}$. We define $m_{\chi_{sym}^t}(G) = min\Sigma_{\chi_{sym}^t}(G)$, and $M_{\chi_{sym}^t}(G) = max\Sigma_{\chi_{sym}^t}(G)$.

Since $|S_{2k}^t|$ has the same parity as t, it follows that the t-chromatic spectrum contains only values of the same parity.

Questions on the t-chromatic spectrum of a signed graph for $t \in \{0, 1, 2\}$ were first studied in [19]. There, it is shown that $\Sigma_{\chi_{sym}^0}(G) \cup \Sigma_{\chi_{sym}^1}(G)$ and $\Sigma_{\chi_{sym}^1}(G) \cup \Sigma_{\chi_{sym}^2}(G)$ are intervals of integers.

Observe that, if $t = \chi(G)$, then it follows that $\Sigma_{\chi_{sym}^t}(G) = \{t\}$. Hence, we assume $t \leq \chi(G) - 1$.

Proposition 6.1.12. *Let G be a graph and t a positive integer. Then $m_{\chi_{sym}^t}(G) = t + 2$.*

Proof. Consider the signed graph $(G, -)$ and the coloring $c : V(G) \to S_2^t$ with $c(v) = \ 1$ for each $v \in V(G)$. This coloring is proper and uses $t + 2$ colors.

Since $t < \chi(G)$, there exists no signature σ' such that $\chi^t_{sym}(G, \sigma') = t$, so $m_{\chi^t_{sym}}(G) = t + 2$. □

Lemma 6.1.13. *Let (G, σ) be a λ_t-chromatic graph, with $\lambda_t = t + 2k$. Then $\chi^t_{sym}(G - v, \sigma) \in \{t + 2k, t + 2k - 2\}$.*

Proof. Suppose that there exists a vertex v such that $\chi^t_{sym}(G-v, \sigma) \leq t+2k-4$. The coloring can be easily extended to (G, σ) by adding at most two colors, so $\chi^t_{sym}(G, \sigma) \leq t + 2k - 2$, which is a contradiction. □

We say that a signed graph (G, σ) is *critical λ_t-chromatic* if $\chi^t_{sym} = \lambda_t$ and for each vertex $v \in V(G)$, it holds that $\chi^t_{sym}(G - v, \sigma) < \lambda_t$.

By Lemma 6.1.13 it follows that in a critical λ_t-chromatic signed graph (G, σ) it holds that $\chi^t_{sym}(G - v, \sigma) = \lambda_t - 2$ for each $v \in V(G)$. In particular, the following statement holds:

Theorem 6.1.14. *If (G, σ) is a λ_t-chromatic graph, with $\lambda_t = t + 2k$, then (G, σ) has a critical λ^i_t-chromatic subgraph for each $\lambda^i_t = t + 2i$, $i \in \{1, ..., k\}$.*

Proof. First, we step-wise remove vertices v such that the removal of v does not decrease the symset t-chromatic number. The remaining subgraph (G', σ) is λ_t-critical.

Second, we remove another vertex w from G'. Lemma 6.1.13 implies that this graph has t-chromatic number $t + 2k - 2$. By proceeding as before, we find a critical subgraph with the same t-chromatic number. This process can be iterated until we obtain a λ^i_t-critical graph, for each $i \in \{1, ..., k\}$. □

Theorem 6.1.15. *Let G be a graph, then $\Sigma_{\chi^t_{sym}}(G) = \{m_{\chi^t_{sym}(G)} = t + 2, t + 4, ..., t + 2k = M_{\chi^t_{sym}}(G)\}$.*

Proof. Let (G, σ) be a signed graph with a signature σ such that $\chi^t_{sym}(G, \sigma) = M_{\chi^t_{sym}}(G) = t + 2k$. By Theorem 6.1.14, we know that for each value of $\lambda^i_t = t + 2i$, where $i \in \{1, ..., k\}$, (G, σ) has a λ^i_t-chromatic subgraph (H, τ). Our aim is to prove that the signature τ can be extended to a signature τ' in G such that $\chi^t_{sym}(G, \tau') = t + 2i$.

Let $c : V(H) \to S_{2i}^t$ the λ_t^i-coloring of (H, τ). For each edge $uv \in E(G)$ we define τ' in the following way:
If $u, v \in V(H)$, $\tau(uv) = \tau'(uv)$.
If $u, v \notin V(H)$ or $v \in V(H)$ and $u \notin V(H)$ and v is colored with 1, $\tau'(uv) = -1$.
If $v \in V(H)$ and $u \notin V(H)$ and v is not colored with 1, $\tau'(uv) = +1$.

By defining $c' : V(G) \to S_{2i}^t$ as $c'(v) = c(v)$ if $v \in V(H)$ and $c'(v) = 1$ if $v \notin V(H)$ we obtain a proper S_{2i}^t coloring, so the statement follows. □

6.1.3 The t-chromatic polynomial

In this section, given a graph G and an edge $e \in E(G)$, we denote with G/e the graph resulting from the contraction of e.

Given a graph G, the *chromatic polynomial* $P_G(\lambda)$ of G is the function describing how many different proper colorings of G can be provided by using λ colors. In order to see that $P_G(\lambda)$ is a polynomial, observe the following. Let $e \in E(G)$ and assume $e = vw$. $P_G(\lambda)$ is equal to $P_{G-e}(\lambda)$ minus all colorings where v and w receive the same color, that is $P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda)$. This is called the *deletion-contraction* formula. This formula can be iteratively applied until all edges are removed, that is, we have sums and differences of chromatic polynomial of subgraphs H_i's consisting of n_i vertices, which can be trivially colored in λ^{n_i} different ways.

In the signed case this approach requires to be adjusted. Indeed, in a signed graph only positive edges can be contracted. As a consequence, after repeatedly applying the deletion-contraction formula, the remaining graphs consist of vertices having negative loops, eventually. In particular, negative loops are not colorable with self-inverse-elements, hence the number of self-inverse colors strongly impacts the chromatic polynomial.

Zaslavsky introduced a first approach to study signed chromatic polynomials in [33], which was further developed by Cheng et al. [10]. Here, we aim to extend their definition to our approach for coloring.

Let (G, σ) be a signed graph with n vertices. We define the *t-chromatic* *polynomial* $P^t_{(G,\sigma)}(\lambda)$ to be the number of S^t_{2k}-colorings of (G,σ), with $\lambda = t+2k$.

Proposition 6.1.16. *Let (G,σ) be a signed graph with n vertices. Then $P^t_{(G,\sigma)}(\lambda)$ is a monic polynomial in n.*

Proof. As in the unsigned case, (G,σ) is decomposable through the deletion-contraction formula into subgraphs (H,τ) such that $P^t_{(H,\tau)}(\lambda)$ is easy to compute. Since all of the positive edges can be deleted and contracted, and all of the negative edges which are not loops can be switched into positive edges, it follows that the remaining signed graphs consist of vertices and negative loops. We denote these graphs $H^{m_i}_{n_i}$, where m_i is the number of disjoint negative loops and n_i the number of vertices. Vertices without loops can be colored with λ different colors, while those with negative loops are colorable in $\lambda - t$ ways. Hence $P^t_{H^{m_i}_{n_i}}(\lambda) = \lambda^{n_i - m_i}(\lambda - t)^{m_i}$, and $P^t(G,\sigma)$ is given by sums and differences of those polynomials.

Note that the only subgraph with $n = |V(G)|$ vertices is the one given only by deleting all edges, hence it has no negative loops. Since $P^t_{H^0_n} = \lambda^n$, the polynomial is monic in n. $\square$

As mentioned in Section 1.4, Zaslavsky [33] separately considers the 0-free chromatic polynomial and the chromatic polynomial (where 0 is allowed). Furthermore, he shows the relation between them. In our terms, this is the relation between the 0-chromatic polynomial and the 1-chromatic polynomial. We extend this relation to all possible values of t.

Proposition 6.1.17. *Let (G,σ) be a signed graph, and let λ, t, k be positive integers such that $\lambda = t + 2k$. Then it holds that*

$$P^{t+1}_{(G,\sigma)}(\lambda+1) = \sum_{R \subseteq V} P^t_{(G[R],\sigma)}(\lambda),$$

where R's are all possible sets of vertices such that R^c is an independent set.

Proof. Let c be an S^{t+1}_{2k}-coloring. The set T of all elements colored by 0_{t+1} is an independent set. Hence, c can be given by a S^t_{2k}-coloring c' of the subgraph induced by T^c.

Similarly, an S^t_{2k}-coloring of a subgraph $(G[R], \sigma)$ is extendable to a S^{t+1}_{2k}-coloring of (G, σ) by assigning a new self-inverse element 0_{t+1} to all the vertices in the independent set R^c. $\square$

Proposition 6.1.18. *Let (G, σ) be a signed graph and let t, r, k, λ be positive integers such that $t \geq r$ and $\lambda = t + 2k$. It holds*

$$P^t_{(G,\sigma)}(\lambda) = \sum_{R \subseteq V} r! P^{t-r}_{(G[R],\sigma)}(\lambda - r),$$

where R's are all the possible sets of vertices such that R^c are different r-partitions.

Proof. Let c be an S^t_{2k}-coloring of (G, σ) and let R^c be a r-partition such that $R^c = \bigcup_{i=t-r+1}^{t} T_i$, where T_i's are the independent sets induced by $c^{-1}(0_i)$. Each S^{t-r}_{2k}-coloring c' of $(G[R], \sigma)$ can be extended in $r!$ different ways to a S^t_{2k}-coloring of (G, σ) such that the same partition is used.

Similarly, let c' be a S^{t-r}_{2k}-coloring of $(G[R], \sigma)$, with R such that R^c is a r-partition. c' can be extended to an S^t_{2k} coloring of (G, σ) in $r!$ different ways such that the same partition is induced by $\bigcup_{i=t-r+1}^{t} c^{-1}(0_i)$. $\square$

The relation between the t-chromatic polynomial and the 0-chromatic polynomial follows consequentially.

Corollary 6.1.19. *Let (G, σ) be a signed graph, t, k, λ positive integers such that $\lambda = t + 2k$. Then, it holds*

$$P^t_{(G,\sigma)}(\lambda) = \sum_{R \subseteq V} t! P^0_{(G[R],\sigma)}(\lambda - t),$$

where R's are all the possible sets of vertices such that R^c is a different t-partition.

6.2 The symset chromatic number

The results of this section have been published in [7].

In this section we skip the constraint on the set of colors given by fixing the value of t and we focus on the symset chromatic number. We recall that the symset chromatic number of a signed graph (G, σ) is defined as

$$\chi_{sym} = \max_t \min\{\chi^t_{sym}(G, \sigma) : 0 \leq \chi(G)\}.$$

In the following, the set of vertices induced by a self-inverse color is called a *self-inverse color class*. Similarly, a set of vertices induced by a pair of non-self-inverse colors is called a *non-self-inverse color class*. Clearly, a self-inverse color class is an independent set of the graph, while a non-self-inverse color class induces an antibalanced subgraph. If the color classes are induced by a λ_t-coloring of (G, σ) and $\lambda_t = \chi_{sym}(G, \sigma)$, then any non-self-inverse color class induces a non-bipartite subgraph of G, see Proposition 6.1.1.

6.2.1 The chromatic spectrum and structural implications

As we did for the symset t-chromatic number, we can define the symset chromatic spectrum. Let G be a graph and $\Sigma(G)$ be the set of its non-equivalent signature. The *symset chromatic spectrum* of G is the set $\Sigma_{\chi_{sym}}(G) := \{\chi_{sym}(G, \sigma) : \sigma \in \Sigma(G)\}$.

We start with proving that the symset chromatic spectrum is an interval of integers. Observe that the symset chromatic spectrum of signed graphs without edges is trivially $\{1\}$, therefore in the following we assume G to have at least one edge.

Theorem 6.2.1. *The symset chromatic spectrum of a graph G is the interval $\Sigma_{\chi_{sym}}(G) = \{2, \ldots, \chi(G)\}$.*

Proof. We prove the theorem by induction on the order of the graph. If G is the K_2, then the statement is trivial. Let us remark that this is obviously true for bipartite graphs.

Let $v \in V(G)$ and $G' = G - v$. By induction hypothesis $\Sigma_{\chi_{sym}}(G') = \{2, \ldots, \chi(G')\}$. Let $i \in \{2, \ldots, \chi(G') - 1\}$ and σ'_i be a signature of G' such that $\chi_{sym}(G', \sigma'_i) = i$. Since $i < \chi(G')$ it follows that $i = t + 2k$ and $k \geq 1$. Let c'

be an S^t_{2k}-coloring of (G', σ'_i). We also assume, by switching, that c' does not use the negative colors. It implies that all of the edges connecting vertices in the same non-self-inverse color class are negative.

Extend σ'_i to a signature σ_i of G as follows. Let $vw \in E(G)$. If $c'(w)$ is self-inverse, then let $\sigma_i(vw) = +1$ and $\sigma_i(vw) = -1$ for otherwise. Since v is connected to a non-self-inverse color class by negative edges only, it can be colored with the same color. Thus, $\chi_{sym}(G, \sigma_i) \leq i$. It cannot be smaller, since we would otherwise get a symset-coloring of (G', σ'_i) with less than i colors. Thus, $\chi_{sym}(G, \sigma_i) = i$ and therefore, $\{2, \dots, \chi(G') - 1\} \subseteq \Sigma_{\chi_{sym}}(G)$. Since $\chi_{sym}(G, +) = \chi(G)$, if $\chi(G') = \chi(G)$ the statement follows.

Assume now that $\chi(G') = \chi(G) - 1$. We now define a signature σ of G such that $\chi_{sym}(G, \sigma) = \chi(G) - 1$.

Let S_1 and S_2 be two of the $\chi(G')$ self-inverse color classes induced by the all-positive signature of G'. Define σ' in the following way: For each $e = wz$, $\sigma'(e) = -1$ if $\{w, z\} \subseteq S_1 \cup S_2$ and $\sigma'(e) = +1$ otherwise. (G', σ') can be colored with a pair of non-self-inverse colors instead of two self-inverse colors. By extending σ' to σ as before, we obtain a $S_2^{\chi(G)-3}$-coloring, so the statement follows. $\square$

The symset chromatic number gives some information on the circuits in the underlying graph G and on the frustration index.

Theorem 6.2.2. *Let (G, σ) be a signed graph and let $t, k \geq 0$ be positive integers. If $\chi_{sym}(G, \sigma) = t + 2k$, then G has at least k pairwise vertex-disjoint odd circuits, which are unbalanced in (G, σ). In particular, $k \leq \ell(G, \sigma)$.*

Proof. For $k = 0$ there is nothing to prove. So assume $k \geq 1$. Let c be an S^t_{2k}-coloring of (G, σ) and let S be a non-self-inverse color class. Since t is maximized, it follows that $\chi(S) > 2$. Hence, S is not bipartite. Thus, it contains an odd and therefore unbalanced circuit. Since this is true for every subgraph induced by a non-self-inverse color class, the statement follows. $\square$

Furthermore, the bound regarding the frustration index is sharp: The complete graph K_6 together with the signature in Fig. 6.1 has frustration index 2 and a minimum coloring requires two non-self-inverse elements.

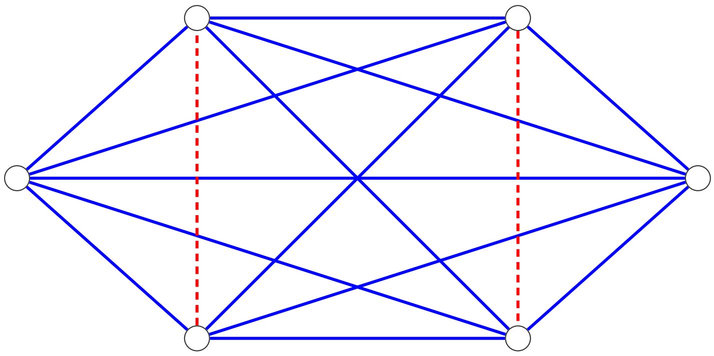


Figure 6.1: A graph with frustration index 2 and $\chi_{sym} = \chi^0_{sym} = 4$

Let (G, σ) be a signed graph with $\chi_{sym}(G, \sigma) = \lambda_t = t + 2k$ (t maximum) and let c be a λ_t-coloring of (G, σ). Let $0_1, \ldots, 0_t$ be the self-inverse colors and $\pm s_1, \ldots, \pm s_k$ be the non-self-inverse colors. Let $I_p = \bigcup_{j=1}^{p} c^{-1}(0_{i_j})$ be the union of p self-inverse color classes and $S_q = \bigcup_{j=1}^{q} c^{-1}(\pm s_{i_j})$ be the union of q non-self-inverse color classes, and $(H_{p,q}, \sigma_{p,q}) = (G[I_p \cup S_q], \sigma)$.

Theorem 6.2.3. *Let (G, σ) be a signed graph with $\chi_{sym}(G, \sigma) = \lambda_t = t + 2k$ (t maximum) and let c be a λ_t-coloring of (G, σ). Then $\chi_{sym}(H_{p,q}, \sigma_{p,q})) = \chi^p_{sym}(H_{p,q}, \sigma_{p,q})) = p + 2q$, for each $p \in \{0, \ldots, t\}$ and $q \in \{0, \ldots, k\}$.*

Proof. By the coloring c of (G, σ) we have that $\chi_{sym}(H_{p,q}, \sigma_{p,q})) \leq p + 2q$. However, if there would be a better coloring with less colors or one with the same number of colors but more self-inverse colors, then there would be a better coloring for (G, σ), a contradiction. □

For $(p, q) = (t, 0)$ and $(p, q) = (0, k)$ we obtain the following corollary.

Corollary 6.2.4. *Let (G, σ) be a signed graph with $\chi_{sym}(G, \sigma) = \lambda_t = t+2k$ (t maximum) and let c be a λ_t-coloring of (G, σ). Then (G, σ) can be partitioned into two induced subgraphs (H_1, σ_1) and (H_2, σ_2), such that $\chi_{sym}(H_1, \sigma_1) = t = \chi(H_1)$ and $\chi_{sym}(H_2, \sigma_2) = 2k = \chi^0_{sym}(H_2, \sigma_2)$.*

We conclude with the following structural statement.

Theorem 6.2.5. *Let (G, σ) be a signed graph. Then $\chi_{sym}(G, \sigma) = \lambda_t = t + 2k$ if and only if (G, σ) can be partitioned into t' independent sets and k' non-bipartite antibalanced subgraphs with $t' + 2k' = t + 2k$ minimum.*

Proof. Clearly, each self-inverse color class induces an independent set in (G, σ) and each non-self-inverse color class an antibalanced subgraph (H, γ). Since t is maximum it follows that (H, γ) is not bipartite.

On the other side, if (G, σ) has a partition into t' independent sets and k' non-bipartite antibalanced subgraphs, then it has a $S^{t'}_{2k'}$-coloring. Let s be the maximum number such that there is $S^{s}_{2k_s}$-coloring. Then for $s = t$ and $k = k_s$ we have the desired λ_t-coloring. □

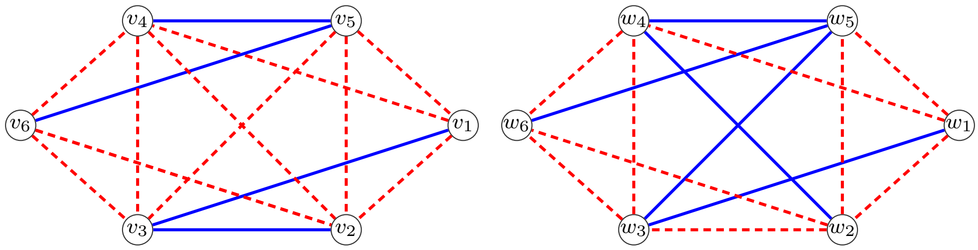
Upper bounds for the symset chromatic number

A natural expectation may be that, when more non-self-inverse colors are required, the difference between the symset chromatic number and the chromatic number of the underlying graph increases. In particular, one may expect that, given a signed graph (G, σ) such that $\chi_{sym}(G, \sigma) = t + 2k$, then $\chi_{sym}(G, \sigma) \leq \chi(G) - k$. Surprisingly this is not true. That is, we may require more non-self-inverse elements in order to save only one self-inverse element. In the following, we provide an example.

Let (G', σ_1) and (G'', σ_2) be the signed graphs in Figure 6.2. We also refer to the vertices as in the Figure 6.2. The underlying graph contains the complete graph K_5 and two independent vertices, therefore the chromatic number is $\chi(G') = \chi(G'') = 5$.

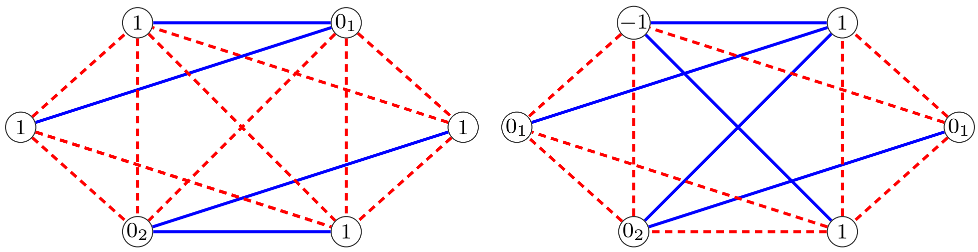
Observe now that none of the two signed graphs is S^1_2-colorable. Otherwise, there would exist an independent set such that its removal makes the signed graph antibalanced. Since each independent set consists of either one vertex or the two vertices of degree 4, this cannot happen. As a consequence, the colorings represented in Figure 6.3 are minimum for both signed graphs.

Claim 6.2.6. *If c is a S^2_2-coloring of (G', σ_1), then $c(v_1)$ and $c(v_6)$ are non-self-inverse elements.*



(a) (G', σ_1) (b) (G'', σ_2)

Figure 6.2: The signed graphs (G', σ_1) and (G'', σ_2)



(a) (G', σ_1) (b) (G'', σ_2)

Figure 6.3: A S_2^2-coloring of (G, σ_1) and (G, σ_2)

Proof. Suppose that there exists a S_2^2-coloring c' such that at least one vertex among v_1 and v_6 is colored by a self-inverse element, say 0_1. Since 0_1 cannot be assigned to any other vertex, we can also assume $c'(v_1) = c'(v_6) = 0_1$. Observe now that $(G - \{v_1, v_6\}, \sigma)$ is switching equivalent to $(K_4, +)$. Since 0_1 has already been used, c' provides a $(1, 2)$-coloring for $(K_4, +)$, and it contradicts Proposition 6.1.5. $\square$

Claim 6.2.7. *If c is a S_2^2-coloring of (G'', σ_2), then $c(w_1)$ and $c(w_6)$ are self-inverse elements.*

Proof. As before, suppose that there exists a S_2^2-coloring c' such that at least one vertex among w_1 and w_6 is colored by ± 1. This implies that there exist two vertices w_i and w_j, where $i, j \in \{2, ..., 5\}$, such that $(G'' - \{w_i, w_j\}, \sigma)$ is antibalanced. By a simple verification it can be seen that this never happens, so c' does not exist. $\square$

Consider now the graph (H, σ) defined in the following way:

- $V(H) = V(G') \cup V(G'')$.
- $E(H) = E(G') \cup E(G'') \cup \{w_1 v_i, \text{ for } i = 2, ..., 5\} \cup \{w_6 v_i, \text{ for } i = 2, ..., 5\}$.
- $\sigma_{|_{G'}} = \sigma_1$, $\sigma_{|_{G''}} = \sigma_2$, $\sigma(w_1v_2) = \sigma(w_1v_3) = \sigma(w_6v_4) = \sigma(w_6v_5) = -1$ and all other edges are positive.

A 5-partition of H in independent sets is given by the sets $V_1 = \{v_1, v_6, w_1, w_6\}$ and $V_i = \{v_i w_i\}$ for $i \in \{2, ..., 5\}$. Since H contains two graphs with chromatic number 5, it holds $\chi(H) = 5$. In addition, the subgraph induced by $\{v_i, w_i, \text{ for } i \in \{1, 2, 3\}\}$ and the one induced by $\{v_i, w_i, \text{ for } i \in \{4, 5, 6\}\}$ are both antibalanced, so $\chi^0_{sym}(H, \sigma) = 0 + 4$.

We aim to prove that $\chi_{sym}(H, \sigma) = \chi^0_{sym}(H, \sigma)$. Since (H, σ) has subgraphs with symset chromatic number $2+2$, it is enough to prove that $\chi^2_{sym}(H, \sigma) > 4$. In particular, we show that by allowing only one non-self inverse element, we need at least three more colors.

Suppose that c is an S^2_2-coloring of (H, σ). This coloring is also an S^2_2-coloring for (G'', σ_2), so it holds $c(w_1) = c(w_6) = 0_i$, for $i = 1, 2$. We can assume $c(w_1) = c(w_6) = 0_1$. Similarly, c is a S^2_2-coloring of (G', σ_1), so 0_1 and 0_2 are not assigned to v_1 or v_6. It implies that $c(v_i) = 0_1$ for a certain $i \in \{2, ..., 5\}$. By construction, it means that there exists i such that v_i and w_1 are neighbors and $c(v_i) = c(w_1) = 0_1$, so c cannot be a proper coloring.

A weaker conjecture relies on Brook Theorem. Note that by definition, $\chi_{sym}(G, \sigma) \leq \chi(G)$ and therefore, Brooks' Theorem can easily be extended to the symset chromatic number. Indeed, if $\chi_{sym}(G, \sigma) \neq \chi(G)$, then $\chi_{sym}(G, \sigma) \leq \Delta(G) - 1$ unless G is complete or an odd circuit. We expect that this bound can be improved for the symset chromatic number.

Conjecture 6.2.8. *If (G, σ) is a signed graph with $\chi_{sym}(G, \sigma) = t + 2k$, then $\chi_{sym}(G, \sigma) \leq \Delta - k + 1$.*

We prove this statement for a specific case.

Theorem 6.2.9. *Let (G, σ) be a signed graph and $\chi_{sym}(G, \sigma) = \lambda_t = t + 2k < \chi(G)$. If there exists a λ_t-coloring with a non-self-inverse color class of cardinality 3, then $\chi_{sym}(G, \sigma) \leq \Delta(G) - k + 1$.*

Proof. Among all λ_t-colorings of (G, σ) which have a non-self-inverse color class with precisely three vertices we choose a coloring c with maximum number of vertices in the union of the self-inverse color classes. Furthermore, we then choose the non-self-inverse color classes $S_1, \ldots, S_k$, such that $|S_1|$ is maximum, according to the choice of $S_1, \ldots, S_i$ choose S_{i+1} such that $|S_{i+1}|$ is maximum. Since every non-self-inverse color class has at least three vertices we can assume that $S_k = T$ contains three vertices.

Clearly, $G[T]$ is a triangle. Note that by the choice of c every vertex of T is connected to each self-inverse color class by an edge and to each non-self-inverse color class by a positive and a negative edge. This implies that each vertex $v \in T$ has $d_G(v) \geq t + 2k$.

We show that there is a vertex $v \in T$ with $d_G(v) \geq t + 3k - 1$.

Let $S = c^{-1}(\pm s)$ be a non-self-inverse color class. Then c induces an S_4^0-coloring of $(G[S \cup T], \sigma)$, and we can assume that all edges of $G[S]$ and $G[T]$ are negative. Furthermore, each vertex of T has degree at least 4 in $G[T \cup S]$ and $d_{G[S \cup T]}(T) \geq 6$. We show that there are more than six edges between T and S. Let $V(T) = \{v_1, v_2, v_3\}$ and we assume that $d_{G[T \cup S]}(v_1) \leq d_{G[T \cup S]}(v_2) \leq d_{G[T \cup S]}(v_3)$.

Claim 6.2.10. $d_{G[S \cup T]}(T) \geq 9$.

Proof of the claim:

Suppose to the contrary that the claim is not true. Then $d_{G[T \cup S]}(v_1) = 4$ and $8 \leq d_{G[T \cup S]}(v_2) + d_{G[T \cup S]}(v_3) \leq 10$. Thus, $d_{G[T \cup S]}(v_2) \leq 5$. Let $\{w_1, w_2\}$ be the neighbors of v_1 in S. We assume that $v_1 w_1$ is negative and $v_1 w_2$ is positive.

Suppose that w_2 is not a neighbor of v_i, $i \in \{2, 3\}$. Then w_2 and v_i can be colored with one self-inverse color, v_j $(j \neq 1, i)$ can be colored with another self-inverse color and v_1 with color s, since it is connected by a negative edge to its second neighbor in S. Thus, w_2 is also a neighbor with v_2 and v_3. By switching at T we deduce that w_1, w_2 are both neighbors of v_2 and of v_3.

Let $d_{G[T\cup S]}(v_2) = 5$. Hence, $d_{G[T\cup S]}(v_3) = 5$. Let w_3 be the third neighbor of v_2 in S. By possible switching at T we can assume that v_2w_3 is negative. If $G[\{v_3, w_1, w_2\}]$ is bipartite, then we color it with two (new) self-inverse colors and v_1, v_2 with color s to obtain an S_2^2-coloring of $G[T \cup S]$, a contradiction.

Thus, $w_1w_2 \in E(G)$ (indeed in E_σ^-) and $G[\{v_3, w_1, w_2\}]$ is a triangle. Furthermore, $G[\{v_1, w_1, w_2\}]$ is a balanced triangle. Suppose that $G[\{v_2, w_1, w_2\}]$ is anti-balanced, then v_2 can be colored with $\pm s$ and v_1, v_3 with two self-inverse colors to obtain an S_2^2-coloring of $G[T \cup S]$, a contradiction. By possible switching we analogously argue for $G[\{v_3, w_1, w_2\}]$ and hence, $G[\{v_i, w_1, w_2\}]$ is a balanced triangle for each $i \in \{1, 2, 3\}$. Since w_1w_2 is negative, precisely one of the remaining two edges is positive. If one of w_1, w_2, say w_1 is incident to three positive edges v_1w_1, v_2w_1, v_3w_1, then we color w_1 and w_4 -the third neighbor of v_3 in S- with two self-inverse colors and the remaining vertices with color s to obtain an S_2^2-coloring of $G[T \cup S]$, a contradiction.

Hence, $G[T \cup \{w_1, w_2\}]$ is a complete signed subgraph (H_5, σ_5) of (G, σ). Clearly, all edges within two vertices of T are negative and all edges between two vertices of S are negative. We discuss the following distribution of positive and negative edges: v_1w_2, v_2w_2, v_3w_1 are positive and all other edges in (H_5, σ_5) are negative. Furthermore, we can assume that v_2w_3 is negative (see Figure 6.4). The argumentation for other distributions is similar.

If $w_3 = w_4$ and v_3w_3 is negative or $w_3 \neq w_4$, then we color v_3 and w_2 with two self-inverse colors and the remaining vertices with color s to obtain the desired contradiction.

(*) If $w_3 = w_4$ and v_3w_3 is positive, then we color v_3, w_2 with two self-inverse colors and the remaining vertices with color s to obtain an S_2^2-coloring of $G[T \cup S]$, which is the desired contradiction and finishes the proof of this case.

It remains to consider the case when $d_{G[T\cup S]}(v_2) = 4$. We analogously deduce that (G,σ) contains (H_5,σ_5). If $d_{G[T\cup S]}(v_3) = 4$, then w_1, w_2 is a bipartite edge-cut in $G[T\cup S]$ and we easily get an S_2^2-coloring of $G[T\cup S]$. If $d_{G[T\cup S]}(v_3) = 5$, we similarly argue as above by discussing the edge v_3w_3 instead of v_2w_3. If $d_{G[T\cup S]}(v_3) = 6$, then we may assume that v_3w_3 is negative. However, the coloring given in **(*)** works here as well and the proof of the claim is finished.

$\diamondsuit$

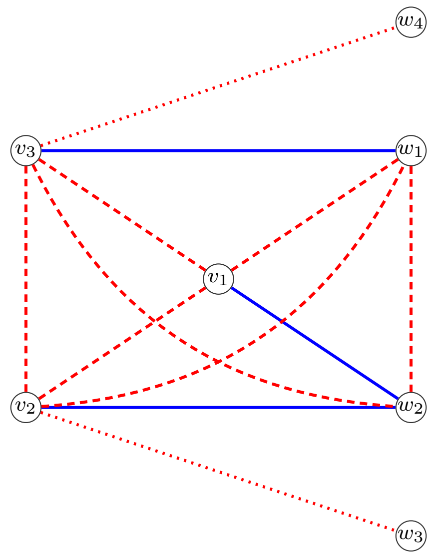


Figure 6.4: The graph $(G[T\cup S],\sigma)$, with dotted edges negative and small dotted edges undefined

Since $(G[T],\sigma)$ is connected to t self-inverse color classes and $k-1$ non-self-inverse color classes, it holds that $d_G(T) \geq 3t + 9(k-1)$. Seeing that T only contains three vertices, each of degree 2 in $G[T]$, it follows that there exists $v \in T$ such that $d_G(v) \geq t + 3(k-1) + 2 = t + 3k - 1$.

$\square$

6.2.2 Concluding remarks on variants of coloring parameters of signed graphs

Circular coloring

Circular coloring is a well studied refinement of ordinary coloring of graphs. Here the set of colors is provided with a (circular) metric. Kang and Steffen [20] used elements of cyclic groups as colors for their definition of (k, d)-coloring of a signed graph (G, σ). For positive integers k, d with $k \geq 2d$, a (k, d)-*coloring* of a signed graph (G, σ) is a map $c : V(G) \to \mathbb{Z}_k$ such that for each edge $e = vw$, $|c(v) - \sigma(e)c(w)| \geq d \mod k$. Hence, this coloring is a specific $S^1_{2k'}$-coloring if $k = 2k' + 1$, and a specific $S^2_{2(k'-1)}$-coloring if $k = 2k'$.

Naserasr, Wang and Zhu [25] generalized circular coloring of graphs to signed graphs as follows. For $i, j \in \{0, 1, \ldots, p-1\}$, the modulo-$p$ distance between i and j is $d_{(\bmod p)}(i, j) = \min\{|i-j|, p - |i-j|\}$. For an even integer p, the antipodal color of $x \in \{0, 1, \ldots, p-1\}$ is $\overline{x} = x + \frac{p}{2} \mod p$.

Let p be an even integer and $q \leq \frac{p}{2}$ be a positive integer. A (p, q)-coloring of a signed graph (G, σ) is a mapping $f \colon V(G) \to \{0, 1, \ldots, p-1\}$ such that for each positive edge xy, $d_{(\bmod p)}(f(x), f(y)) \geq q$, and for each negative edge xy, $d_{(\bmod p)}(f(x), \overline{f(y)}) \geq q$. Now it is easy to see that this defines a specific S^0_p-coloring of (G, σ).

DP-coloring

In this part we show that coloring of signed graphs with elements from a symmetric set can be described as a special *DP-coloring*. The *DP*-coloring was introduced for graphs by Dvořák and Postle [11] under the name correspondence coloring. We follow Bernshteĭn, Kostochka, and Pron [2] and consider multigraphs.

Let G be a multigraph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is the graph with vertex set $\bigcup_{v \in V(G)} L(v)$ satisfying the following conditions:

1. $H[L(v)]$ is an independent set for each $v \in V(G)$.

2. For any two distinct vertices v, w of G the set of edges between $L(v)$ and $L(w)$ is the union of $\mu_G(v, w)$ (possible empty) matchings, where $\mu_G(v, w)$ denotes the number of edges between v and w in G.

An (L, H)-*coloring* of G is an independent transversal T of cardinality $|V(G)|$ in H, i.e. for each vertex $v \in V(G)$ exactly one vertex of $L(v)$ belongs to T and $H[T]$ is edgeless. We also say that G is (L, H)-*colorable*.

Let t, k be positive integers and $L_{2k}^t = \{s_1, \dots, s_t, r_0, \dots, r_{2k-1}\}$. An (L, H_{2k}^t)-*cover* of a signed multigraph is a cover of (G, σ) with $L(v) = L_{2k}^t$ for each vertex $v \in V(G)$ and H_{2k}^t satisfies the following conditions:

1. $H_{2k}^t[L(v)]$ is an independent set for each $v \in V(G)$.

2. If there is no edge between u and w, then $E_{H_{2k}^t}(L(u), L(w)) = \emptyset$.

3. For each edge e between u and w we associate a perfect matching M_e of $E_{H_{2k}^t}(L(u), L(w))$ with the property that, if e is a positive edge, then $M_e = \{((q, u), (q, w)) \colon q \in L_{2k}^t\}$ and if e is a negative edge, then M_e is a perfect matching of $E_{H_{2k}^t}(L(u), L(w))$ which consists of the edges $((s_i, u), (s_i, w))$ for each $i \in \{1, \dots, t\}$ and $((r_j, u), (r_{j+k}, w))$ for each $j \in \{0, \dots, 2k-1\}$, where the indices are added mod $2k$.

It is easy to see that a signed graph is S_{2k}^t-colorable if and only if it is (L, H_{2k}^t)-colorable. The associated chromatic numbers are to be defined accordingly.

If we consider coloring of signed graphs we can restrict ourselves to multigraphs with edge multiplicity at most 2, since more than one positive and one negative edge between two vertices do not have any effect on the coloring properties of the multigraph. That is, if we consider (L, H_{2k}^t)-cover of a signed extension $\pm G$ of a graph G, then $H_{2k}^t[E_H(L(u), L(w)]$ is a 2-regular multigraph whose components are digons and circuits of length 4, for any two adjacent vertices u, v of G.

However, the DP-coloring approach allows further flexibility and generalizations. For instance, DP-coloring is considered in the more general context of gain graphs in a short note of Slilaty [30], where the corresponding chromatic polynomials are defined.

Chapter 7

Conclusion and future work

The main aim of this work is to reach a deeper understanding of signed graphs. In particular, since the unbalanced part of a signed graph is the main source of the differences between the signed and the unsigned case, we focused on a measurement of unbalance, that is on the frustration index, and on the signed graphs critical with respect to it.

This problem is also of interest since it generalizes problems existing in the unsigned case: Given a graph G, in the signed graph $(G, E(G))$ containing negative circuits is equivalent to containing odd circuits in G. Therefore, problems related to the presence of odd circuits or to the bipartite subgraphs (e.g. the Max-Cut problem) are covered by our approach.

As a first step, we give a characterization of critically k-frustrated signed graphs, for each value of $k \geq 1$, and we show some families of critical graphs.

Secondly, we define decomposition and subdivision, which are two basic operations to deal with critical graphs, since they both allow us to focus on a smaller set of graphs.

In particular, for each value of k, there exist infinitely many critically k-frustrated signed graphs which are the result of a subdivision. Similarly, if we allow decomposable graphs, for each $k \geq 3$, there exist infinitely many critically k-frustrated signed graphs.

It comes as a natural question whether the set $\mathcal{L}^*(k)$ of the critically

k-frustrated signed graphs which are non-decomposable and irreducible has finitely many members.

For $k \leq 3$, we show that the answer to this question is yes. However, while it is easy to characterize $\mathcal{L}^*(1)$ and $\mathcal{L}^*(2)$, the family $\mathcal{L}^*(3)$ is much more complicated and it contains significantly more elements than the previous ones. We still believe that the same answer holds for $k \geq 4$.

Conjecture 7.0.1. *For each positive integer k, $\mathcal{L}^*(k)$ has finitely many elements.*

Relaxations of this conjecture regard some families of their own interest.

First of all, the family $\mathcal{S}^*(k)$ of critically k-frustrated signed graphs where each critical subgraph is also non-decomposable, that is the signed graphs where there are no two edge-disjoint negative circuits. In this thesis this family is characterized and it is precisely described for $k \leq 3$. While $\mathcal{L}^*(3)$ has many elements, $\mathcal{S}^*(3)$ is still small and just consists of two graphs. Therefore, we conjecture the following.

Conjecture 7.0.2. *For each positive integer k, $\mathcal{S}^*(k)$ has finitely many elements.*

Another interesting family is that of the critical, non-decomposable, and irreducible planar signed graphs $\mathcal{P}^*(k)$. As for $\mathcal{S}^*(k)$, they can be precisely described for $k \leq 3$. For $k \in \{1, 2\}$, it holds that $\mathcal{S}^*(k) = \mathcal{P}^*(k)$, while $\mathcal{P}^*(3)$ consists of more signed graphs that $\mathcal{S}^*(3)$, namely it has ten elements. For the planar case, our conjecture is a bit different, but it still implies the finiteness of $\mathcal{P}^*(k)$.

Conjecture 7.0.3. *Let (G, Σ) be a planar graph which is critically k-frustrated and non-decomposable. Then (G, Σ) consists of exactly $2k$ faces and all of the faces are negative.*

One last conjecture which we strongly believe to be true regards a structural property.

Conjecture 7.0.4. *Let (G, Σ) be a critically k-frustrated signed graph, then $\Delta(G) \leq 2k$.*

Note that, on one side, the bound is reached by the graph consisting of k negative loops such that all of them share one vertex, that is a signed graph which is "extremely" decomposable. On the other side, signed graphs which are "extremely" non-decomposable, like the elements of $\mathcal{S}^*$, have maximum degree 3. Therefore we expect that all of the other k-critical signed graphs (G, Σ) have $4 \leq \Delta(G) \leq 2k - 1$.

We prove this conjecture for graphs without $-K_5$-minors, but the remaining cases are still open.

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