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Spectra of Higher Rank Locally Symmetric Spaces

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Abstract

The aim of this thesis is to contribute to the spectral geometry of higher rank locally symmetric spaces. The theory for rank one spaces is well developed but in the higher rank case much less is known. Therefore, the main interest is the higher rank case.

There are two different sets of operators that we are considering: First, the algebra of invariant differential operators on the locally symmetric space. This is the proper replacement for the Laplace operator in the higher rank setting and encodes the quantum mechanics of the manifold. Secondly, the classical dynamics are described by the geodesic flow in the rank one case. In higher rank this is replaced by the Weyl chamber flow.

We prove a quantum-classical correspondence between the spectra of these two sets of operators for compact locally symmetric space. This is used to determine the location of the classical Ruelle-Taylor resonances and to prove a Weyl law as well as a spectral gap.

In the non-compact setting we concentrate on the quantum spectrum. We prove that there are no principal L^2-eigenvalues under some dynamical condition, i.e. that there are no tempered spherical representations occurring discretely in $L^2(\Gamma\backslash G)$.

Concerning the non-tempered part of the spectrum, we relate its extent to the growth rate of the fundamental group in the case where the universal cover is a product of rank one symmetric spaces. In particular, we obtain that the space is tempered if the growth rate is small enough.

Zusammenfassung

Das Ziel dieser Arbeit ist die Erweiterung der Spektralgeometrie von lokal symmetrischen Räumen. Die Theorie für Räume vom Rang eins ist gut entwickelt, aber im Fall höheren Rangs ist deutlich weniger bekannt. Daher gilt das Hauptinteresse dem Fall höheren Rangs.

Es gibt zwei verschiedene Gruppen von Operatoren, die wir hier betrachten: Erstens, die Algebra der invarianten Differentialoperatoren auf dem lokal symmetrischen Raum. Dies ist der geeignete Ersatz für den Laplace-Operator im höheren Rang und kodiert die Quantenmechanik der Mannigfaltigkeit. Zweitens wird die klassische Dynamik im Rang eins durch den geodätischen Fluss beschrieben. In höherem Rang wird dieser durch den Weyl-Kammer-Fluss ersetzt.

Wir beweisen eine Quanten-Klassische-Korrespondenz zwischen den Spektren dieser beiden Gruppen von Operatoren für kompakte lokal symmetrische Räume. Dies wird verwendet, um die Lage der klassischen Ruelle-Taylor-Resonanzen zu bestimmen und ein Weyl-Gesetz sowie eine spektrale Lücke zu beweisen.

Im nicht-kompakten Fall konzentrieren wir uns auf das Quantenspektrum. Wir beweisen, dass es unter bestimmten dynamischen Bedingungen keine temperierten L^2-Eigenwerte gibt, d.h. es gibt keine temperierten sphärischen Darstellungen, die diskret in $L^2(\Gamma\backslash G)$ auftreten.

Was den nicht temperierten Teil des Spektrums betrifft, setzen wir seine Ausdehnung mit der Wachstumsrate der Fundamentalgruppe in Beziehung. Diesen Zusammenhang erhalten wir in dem Fall, dass die universelle Überlagerung ein Produkt von symmetrischen Räumen vom Rang eins ist. Insbesondere erhalten wir, dass der Raum temperiert ist, wenn die Wachstumsrate niedrig genug ist.

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Preface

This thesis is the product of my work as a PhD student at the Institute for Mathematics at Paderborn University. It has been supervised by Prof. Dr. Tobias Weich. The main part of this thesis consists of three papers:

- Paper I: *Higher rank quantum-classical correspondence.*
  arXiv:2103.05667 This preprint is co-authored with Joachim Hilgert and Tobias Weich. It has been accepted for publication in *Analysis and PDE.*
- Paper II: *Absence of principal eigenvalues for higher rank locally symmetric spaces.*
  arXiv:2205.03167 This preprint is co-authored with Tobias Weich. It has been accepted for publication in *Communications in Mathematical Physics.*
- Paper III: *Temperedness of locally symmetric spaces: The product case.*
  arXiv:2304.09573 This preprint is also co-authored with Tobias Weich. It has been submitted and is currently under review.

All three papers are self-contained and include their own introductions and bibliographies. They do not differ from the published versions up to minor changes.

In addition to the published articles I included additional material that I worked on during my time as a PhD student but that were not included in the final versions of the papers. More precisely, Chapter 3 contains some preliminary discussion on G-invariant differential operators. Sections I.6 and I.7 contain alternative proofs for statements included in Paper I. Furthermore, Chapter II contains an introduction to compactifications on symmetric spaces which is only touched shortly in Paper II.

In all three papers the authors contributed in equal parts to the development of the research question and to the proof strategy.

I worked out the proofs and wrote the manuscripts. This process was accompanied by regular joint blackboard discussions among the authors.

All authors contributed in equal parts to the proof checking and the proofreading of the manuscripts.

Contents

Projects

1. Introduction

The mathematical field of spectral geometry concerns the interplay between the geometry of a manifold and spectra of differential operators defined by the structure of this manifold. The most prominent example of such an operator is the *Laplace-Beltrami operator* on a Riemannian manifold. A classical occurrence of the relation between the geometry and the spectrum is that the spectrum of the Laplace-Beltrami operator on a compact manifold is discrete, i.e. it consists of a discrete set of eigenvalues with finite multiplicities. Even more is true: The asymptotics of the number $N(T)$ of eigenvalues less than T is precisely described by the Weyl law:

$$\lim_{T\to\infty} \frac{N(T)}{T^{d/2}} = \frac{\omega_d}{(2\pi)^d} \operatorname{vol}(M),$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ and d is the dimension of the manifold M.

Another important example of a differential operator that is of a different flavor is the *geodesic flow* ϕ_t respectively its generator the *geodesic vector field* X. It is defined as follows: If (x, v) is a vector in the unit tangent bundle SM of the Riemannian manifold M, then there is a unique geodesic $\gamma_v : (-\varepsilon, \varepsilon) \to M$ such that $\gamma_v(0) = x$ and $\dot{\gamma}_v(0) = v$. Then $\phi_t(x, v) = (\gamma_v(t), \dot{\gamma}_v(t))$. If M has negative sectional curvature, then the spectrum of X on $L^2(SM)$ is equal to $i\mathbb{R}$. Nevertheless one can associate a discrete spectrum to the geodesic vector field by continuing the resolvent of X meromorphically on suitable Hilbert spaces. This leads to the notion of *Ruelle resonances.* The goal of the present thesis was to study the spectra of generalized versions of the Laplacian and the geodesic vector field on certain types of manifolds M with lots of symmetries.

One class of examples is given by hyperbolic surfaces with finite topology, i.e. geometrically finite surfaces of constant curvature -1. This example is the simplest case and serves as motivation for the questions dealt with in this thesis. Let us first describe the two different kinds of spectra in this case.

1.1. Ruelle resonances

On the dynamical side we have the Ruelle resonances of the geodesic flow. These can be defined as follows. For $\lambda \in \mathbb{C}$ we first define the space of *resonant states*

$$\operatorname{Res}(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(SM) \mid (X + \lambda)u = 0\},$$

where $\mathcal{D}'_{E_u^*}(SM)$ are the distributions on SM with wavefront set contained in the unstable bundle E_u^*. Then the set of Ruelle resonances can be defined as

$$\sigma_{\mathrm{Rue}}(X) \coloneqq \{\lambda \in \mathbb{C} \mid \mathrm{Res}(\lambda) \neq 0\}.$$

Note that this definition is only valid for compact hyperbolic surfaces but there are different approaches that can be extended to non-compact settings (see [BW22]).

For the location of the resonances we have the following theorem.

Theorem 1.1 (see [DFG15])**.** *$\sigma_{\mathrm{Rue}}(X)$ is discrete and if $\lambda \in \sigma_{\mathrm{Rue}}(X)$ then either $\mathrm{Im}\,\lambda = 0$ or $\mathrm{Re}\,\lambda \in -\frac{1}{2} - \mathbb{N}_0$.*

In particular, the theorem establishes a band structure with bands at the lines where the real part is $-\frac{1}{2} - \mathbb{N}_0$. Moreover, the discreteness together with the band structure imply the existence of a spectral gap: There is $\varepsilon > 0$ such that

$$\sigma_{\mathrm{Rue}}(X) \cap \{\mathrm{Re} > -\varepsilon\} = \{0\}.$$

The existence of a spectral gap is strongly related to mixing properties of the geodesic flow such as decay of correlations. On hyperbolic surfaces exponential decay of correlations has been shown by Moore [Moo87] which by its own means provides a spectral gap. For Weyl chamber flows, which are the higher rank analogues of geodesic flows, the interplay between a spectral gap and exponential decay of correlations is more subtle. Exponential decay of correlations has been shown by Katok and Spatzier [KS94]. However the existence of a spectral gap does not follow immediately due to the definition of the resonances in the higher rank setting [BGHW20]. We will prove the existence of a spectral gap in this more general setting and also give a precise resonance-free region (see Theorem I.5.1).

1.2. Laplace spectrum

The Laplace spectrum on a hyperbolic surface M of finite topology highly depends on the geometry of M. If we stick to the setting of compact surfaces, the spectrum is discrete and satisfies a Weyl law as described above. However, the Laplace spectrum can be studied on a general hyperbolic surface. In the non-compact setting it is more elaborate: If M is non-compact and geometrically finite, the interval $[1/4, \infty[$ is always contained in the spectrum and the spectrum below $1/4$ consists of finitely many eigenvalues (see [Bor16] for an overview).

In the special case of the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$ the only eigenvalue below $1/4$ is 0 corresponding to the constant functions. However there are infinitely many eigenvalues embedded in the continuous spectrum $[1/4, \infty[$ satisfying a Weyl asymptotic. These Maass wave forms are studied in the theory of automorphic forms and are of fundamental importance in number theory.

For a general finite area geometrically finite hyperbolic surface the situation of embedded eigenvalues is not understood completely. There is a conjecture by Phillips and Sarnak that embedded eigenvalues do not exist for a generic surface. Furthermore, it is conjectured that there are infinitely many embedded eigenvalues if and only if the fundamental group is arithmetic (see [PS85]).

If we proceed to infinite area surfaces, then the aspect of embedded eigenvalues becomes clearer. Here we have the following classical theorem.

Theorem 1.2 (see [Pat75])**.** *For a geometrically finite hyperbolic surface of infinite area there are no L^2-eigenvalues for the Laplace-Beltrami operator in the continuous spectrum* $[1/4, \infty[$.

This theorem will be generalized in Project II to higher rank locally symmetric spaces.

On the other side of the spectrum one is interested in the first eigenvalue or more precisely in the bottom λ_0 of the Laplace spectrum. Clearly, if M has finite area, then $\lambda_0 = 0$ as the constant functions are harmonic L^2-functions. If M has infinite area, λ_0 is related to the growth rate of the fundamental group as follows: The universal cover of the hyperbolic surface is the hyperbolic plane $\mathbb{H}$ on which the fundamental group $\Gamma \leq PSL_2(\mathbb{R})$ acts freely. The critical exponent δ of Γ is defined by

$$\delta := \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma x_0, x_0) < R\}, \quad x_0 \in \mathbb{H} \text{ arbitrary},$$

which is also equal to the Hausdorff dimension of the limit set of Γ. It is related to the bottom of the Laplace spectrum λ_0 by the following theorem.

Theorem 1.3 (see [Els73, Pat76])**.**

$$\lambda_0 = \begin{cases} 1/4 & : \delta \leq 1/2 \\ \delta(1-\delta) & : \delta \geq 1/2 \end{cases}.$$

In particular, if $\delta \leq 1/2$ the spectrum of Δ equals $[1/4, \infty[$ and is therefore equal to the Laplace spectrum on the universal cover $\mathbb{H}$. In this case we call M tempered as all $PSL_2(\mathbb{R})$-representations occurring in $L^2(\Gamma\backslash PSL_2(\mathbb{R}))$ are then tempered. In Project III we give criteria for temperedness of manifolds which have a product of rank one spaces as universal cover.

1.3. Quantum-classical correspondence

In physics there is a general principle that states that the large scale behavior of a system described by quantum mechanics has to agree with the classical description. In our setting this means that there should be a strong relation between the Laplace operator (quantum side) and the geodesic flow (classical side). A manifestation of this principle is the Selberg trace formula which holds for finite area hyperbolic surfaces [Sel56]. Using this formula McKean [McK72] and Müller [Mül92] showed that the Laplace spectrum and the length spectrum (i.e. the length of primitive closed geodesics on M) determine each other in the compact case and in the finite area case, respectively. (The compact case is originally due to Huber [Hub59] by a different method.) With this theorem in mind one also expects a correspondence between the Laplace spectrum and the Ruelle resonance spectrum in the case where both spectra are defined. In particular in the case of compact hyperbolic surfaces we have the following theorem.

Theorem 1.4 (see [DFG15, GHW18]). *For a compact hyperbolic surface M the Ruelle resonances for the geodesic flow on SM are*

(i) $\lambda = s-1-m$, $m \in \mathbb{N}_0, \operatorname{Re} s \in [0,1]$, $s \neq 0,1$, *with multiplicity* $\dim\ker(\Delta - s(1-s))$ *if* $s \neq 1/2$ *and* $2\dim\ker(\Delta - 1/4)$ *if* $s = 1/2$. *Moreover, there is an explicit relation between Ruelle resonant states and eigenfunctions of the Laplacian.*

(ii) $-n$, $n \in \mathbb{N}_0$, *with multiplicity* 1 *if* $n = 0$ *and* $n^2|\chi(M)| + 2$ *if* $n \neq 0$ *where* $\chi(M)$ *is the Euler characteristic of* M*.*

Note that Theorem 1.1 is achieved by this quantum-classical correspondence in combination with the positivity of Δ. We will follow the same strategy in Section I.6 and establish a generalization of such a quantum-classical correspondence on higher rank locally symmetric spaces in Project I.

1.4. Locally symmetric spaces

The previous descriptions of the different spectra on hyperbolic surfaces are well-known for some time. Clearly there are generalizations of many results to more general settings. For example one could take a look at manifolds of higher dimension or relaxing the condition of hyperbolicity. The direction we are taking is the following. By the uniformization theorem the universal cover of a hyperbolic surface is the hyperbolic plane $\mathbb{H}$. The group of Deck transformations Γ is isomorphic to the fundamental group of the hyperbolic surface. Γ is a discrete torsion-free subgroup of the orientation-preserving isometries on $\mathbb{H}$ and the hyperbolic surface is $\Gamma\backslash\mathbb{H}$. The group of orientation-preserving isometries on $\mathbb{H}$ is $PSL_2(\mathbb{R})$ which acts transitively on $\mathbb{H}$ with stabilizer of a base point conjugated to $PSO(2)$. Hence, the hyperbolic surface is isomorphic to the biquotient $\Gamma\backslash PSL_2(\mathbb{R})/PSO(2)$. The way we want to generalize the above mentioned results is to extend our knowledge to manifolds with a symmetric space G/K as a universal cover, i.e. to biquotients $\Gamma\backslash G/K$ for other real semisimple Lie groups G of finite center with maximal compact subgroup K and discrete torsion-free subgroups $\Gamma \leq G$. The resulting manifolds $\Gamma\backslash G/K$ are called *locally symmetric spaces* and are the main objects of our study.

1.5. Quantum and classical operators

The two operators – the Laplacian and the geodesic flow – that define the two spectra are defined on a locally symmetric space merely by the property that they are Riemannian manifolds. However, one observes that the corresponding operators on the symmetric space G/K are invariant by the action of G. Indeed, they are defined by means of the metric and (the identity component of) G is the isometry group of G/K. Hence, they descend to the locally symmetric space $\Gamma\backslash G/K$ and there they coincide with the operators directly defined by the metric. Not only these operators descend to their local versions but so do any G-invariant ones. In particular, each element of the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K descends to $\Gamma\backslash G/K$. This algebra is central in all three projects contained in this thesis so that we included a preliminary discussion of its properties in Chapter 3. The algebra $\mathbb{D}(G/K)$ always contains the Laplace-Beltrami operator, but in general it is generated by multiple algebraically independent operators with the number of generators equal to the rank of the symmetric space. For a better understanding of the relation between spectra and geometry it is more fruitful to consider a joint spectrum of $\mathbb{D}(G/K)$ instead of Δ alone (see Proposition III.3.6 for multiple equivalent definitions).

For the geodesic flow the construction is a little bit more involved. Recall that it is defined on the sphere bundle of the manifold which is given by $\Gamma\backslash PSL_2(\mathbb{R}) = \Gamma\backslash SL_2(\mathbb{R})/\{\pm 1\}$ for a hyperbolic surface $\Gamma\backslash SL_2(\mathbb{R})/SO(2)$. The geodesic flow is then obtained by right multiplication by $a_t = \operatorname{diag}(e^{t/2}, e^{-t/2})$. Note that the set $\{a_t \mid t \in \mathbb{R}\}$ is precisely the group A in the Iwasawa decomposition

$$SL_2(\mathbb{R}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \{a_t \mid t \in \mathbb{R}\} SO(2) = NAK$$

and $\{\pm 1\}$ is precisely the subgroup M of $K = SO(2)$ commuting with all a_t. This leads to the following definition generalizing the dynamical action.

Definition 1.5. The right action of A on $\Gamma\backslash G/M$ is called *Weyl chamber action* where $G = NAK$ is an Iwasawa decomposition and M is the centralizer of A in K.

[BGHW20] provides a resonance spectrum for this action for compact locally symmetric spaces (see Section I.2.1 and I.3). This spectrum is then called *Ruelle-Taylor resonance spectrum* as it is defined using the notion of the Taylor spectrum for commuting operators.

This thesis is concerned with these spectra and their connection to each other as well as the relation to the geometry especially in the case of higher rank, i.e. if $\dim A \geq 2$. We will summarize the results in the next chapter.

2. Summary of the publications

In this chapter we will summarize the results of the three projects.

Project I: Quantum-classical correspondence

In this project we determine the location of certain Ruelle-Taylor resonances for the Weyl chamber action, i.e. we generalize Theorem 1.4 to the higher rank setting. As in the surface case this is achieved by proving a quantum-classical correspondence, i.e. a 1:1-correspondence between horocycle invariant Ruelle-Taylor resonant states and joint eigenfunctions of the algebra of invariant differential operators on G/K. The description of the quantum spectrum is due to [DKV79] in this case and leads to a Weyl-lower bound on an appropriate counting function for the Ruelle-Taylor resonances. Furthermore, we establish a spectral gap which is uniform in Γ if G/K is irreducible of higher rank. In contrast to the rank one case this does not follow from the discreteness of the spectrum. We rather have to use Kazdhan's Property (T) to prove its existence. The size of the gap is made explicit by L^p-bounds for elementary spherical functions.

In addition to the published article, we give an alternative proof for the obstructions on the location of the resonances that avoids the abstract theory of unitary representations connected to spherical functions (Section I.6). This line of arguments is more in the spirit of the rank one case where one obtains the location of the resonances from the positivity of the Laplacian together with the quantum-classical correspondence. We also added an alternative proof for the uniform spectral gap that does not use an explicit description of Kazhdan's Property (T) (Section I.7).

Project II: Absence of principal eigenvalues

As described above (see Theorem 1.2) given a geometrically finite hyperbolic surface of infinite volume it is a classical result of Patterson that the positive Laplace-Beltrami operator has no L^2-eigenvalues $\geq 1/4$. In this project we prove a generalization of this result for the joint L^2-eigenvalues of the algebra of commuting differential operators on Riemannian locally symmetric spaces $\Gamma\backslash G/K$ of higher rank. We derive dynamical assumptions on the Γ-action on the geodesic and the Satake compactifications of the globally symmetric space G/K which imply the absence of the corresponding principal eigenvalues. A large class of examples fulfilling these assumptions are the non-compact quotients by Anosov subgroups. To get a more complete picture of the compactifications we included a preliminary discussion of the geodesic and the Satake compactifications (Sections II.2 and II.3).

Project III: Temperedness of local product spaces

Theorem 1.3 establishes a connection between the growth rate of the fundamental group Γ and the bottom of the Laplace spectrum as well as the temperedness of the surface. The growth rate of Γ is measured by the translation distance in the universal cover G/K which equals the size of the A-component in the $K\overline{A^+}K$-decomposition of the isometry group G. In the higher rank setting where $\dim A \geq 2$ there are different ways to measure this size. For example if G is a product $G_1 \times G_2$ then one can measure the growth in the two directions determined by the factors. In this project we show that the quantum spectrum is related to the growth rate of Γ in the two directions similar to Theorem 1.3 precisely in the case where G/K is a product of rank one spaces. We also obtain a condition for the temperedness of the space and we can show that this condition is satisfied for a large class of Γ.

3. Preliminaries

Let us shortly fix the notation for this preliminary discussion of the algebra of invariant differential operators. G is a real semisimple non-compact Lie group with finite center and K is a maximal compact subgroup. There is a Cartan involution θ on G such that K is the set of fixed points of θ. The Lie algebra $\mathfrak{g}$ splits into the ±1-eigenspaces of θ. The $+1$-eigenspace is the Lie algebra $\mathfrak{k}$ of K and we call the -1-eigenspace $\mathfrak{p}$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In $\mathfrak{p}$ we choose a maximal abelian subalgebra $\mathfrak{a}$. The action on $\mathfrak{g}$ of this algebra splits into joint eigenspaces $\mathfrak{g}_\alpha \coloneqq \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{a}\}$ where $\alpha \in \mathfrak{a}^*$. The set of roots Σ is the collection of $\alpha \in \mathfrak{a}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$. We choose a positive set of roots $\Sigma^+ \subseteq \Sigma$ and define $\mathfrak{n} \coloneqq \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ as well as $\rho \coloneqq \frac{1}{2}\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\cdot\alpha$. We then have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ which also holds on the group level with the maximal compact subgroup K and the corresponding analytic subgroups A and N. The group W acting on $\mathfrak{a}^*$ generated by the reflections along the roots $\alpha \in \Sigma$ is called Weyl group.

3.1. Invariant differential operators

In this section we introduce one of the main objects of this thesis, namely the algebra $\mathbb{D}(G/K)$ of *G-invariant differential operators* on G/K, i.e. differential operators commuting with the left regular representation L_g for elements $g \in G$ where $L_g f(x) \coloneqq f(g^{-1}x)$. This algebra can be identified with a set of polynomials by the following theorem.

Theorem 3.1.1 (Harish-Chandra isomorphism, see [Hel84, II-Thm. 5.17])**.** *There is an algebra isomorphism*

$$\mathrm{HC}\colon \mathbb{D}(G/K) \to \mathit{Poly}(\mathfrak{a}^*_\mathbb{C})^W$$

from the G-invariant differential operators $\mathbb{D}(G/K)$ to the Weyl group invariant polynomials on $\mathfrak{a}^*_\mathbb{C}$. We write $\chi_\lambda(D)$ instead of $\mathrm{HC}(D)(\lambda)$.

The construction is as follows: We represent a differential operator D in $\mathbb{D}(G/K)$ as an element X in $\mathcal{U}(\mathfrak{g})^K$, the K-invariant elements in the universal enveloping algebra of $\mathfrak{g}$. The element X is unique modulo $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}_\mathbb{C}$. We consider the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Using the Poincaré-Birkhoff-Witt theorem we can define the projection

$$\delta \colon \mathcal{U}(\mathfrak{g})^K \subseteq \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{a})$$

with kernel $\mathfrak{n}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{k}_{\mathbb{C}}$. Furthermore we define the algebra isomorphism

$$\eta \colon \mathcal{U}(\mathfrak{a}) \to \mathcal{U}(\mathfrak{a}) \quad \text{by} \quad \mathfrak{a} \ni X \mapsto X + \rho(X).$$

Then

$$\mathrm{HC}(D) := (\eta \circ \delta)(X),$$

where we identify $Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$ with the Weyl group invariants in the symmetric algebra $S(\mathfrak{a}) = \mathcal{U}(\mathfrak{a})$. To see that HC is a homomorphism we only need to see that δ is a homomorphism. For $X, Y \in \mathcal{U}(\mathfrak{g})^K$ we have

$$XY - \delta(X)\delta(Y) = \delta(X)(Y - \delta(Y)) + (X - \delta(X))Y.$$

By definition of δ,

$$X - \delta(X), Y - \delta(Y) \in \mathfrak{n}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{k}_{\mathbb{C}}.$$

Since Y is K-invariant we infer $(X - \delta(X))Y \in \ker\delta$ and since $\mathfrak{a}$ normalizes $\mathfrak{n}$ we also have $\delta(X)(Y - \delta(Y)) \in \ker\delta$. Hence, HC is a homomorphism.

To see that its image consists of W-invariant polynomials we consider the function

$$e_{\lambda,kM}(gK) := e^{-(\lambda+\rho)H(g^{-1}k)}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, kM \in K/M, g \in G,$$

where $H \colon G \to \mathfrak{a}$ is defined by $g \in Ke^{H(g)}N$. Clearly, $e_{\lambda,eM}$ is a left N-invariant function on G/K. Hence by construction,

$$De_{\lambda,eM} = \delta(X)(\lambda + \rho)e_{\lambda,eM} = \eta(\delta(X))(\lambda)e_{\lambda,eM} = \chi_\lambda(D)e_{\lambda,eM}.$$

As $e_{\lambda,kM} = L_k e_{\lambda,eM}$ the same is true if we replace eM by kM. In particular the elementary spherical function

$$\phi_\lambda(g) := \int_K e_{\lambda,kM}(gK)\, dk = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\, dk$$

satisfies $D\phi_\lambda = \chi_\lambda(D)\phi_\lambda$. It follows from [Hel84, Ch. II Thm. 5.16] (which is essentially an application of several integral formulas) that $\phi_\lambda = \phi_{w\lambda}$ for all $w \in W$. This shows that $\mathrm{HC}(D)$ is W-invariant. Note that if we choose two different positive systems Σ^+ in Σ, the resulting HC differ by the action of a Weyl group element. By the W-invariance we get that HC does not depend on this choice.

In order to prove injectivity and surjectivity we use the following symmetrization map:

$$\lambda \colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \quad X_1 \cdots X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}.$$

By construction this is well defined and $\mathrm{Ad}(G)$-invariant. Bijectivity follows from the following observation which is easily derived from commutator relations in the universal enveloping algebra:

$$\lambda(X_1 \cdots X_n) - X_1 \cdots X_n \in \mathcal{U}^{n-1}(\mathfrak{g}), \quad X_i \in \mathfrak{g}, \tag{3.1}$$

where the latter $X_1 \cdots X_n$ is understood as an element of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}^k(\mathfrak{g})$ is the subspace of $\mathcal{U}(\mathfrak{g})$ of elements of degree $\leq k$.

Now injectivity of λ follows easily. Let $X \in \mathcal{U}(\mathfrak{g})$ of degree n (i.e. $X \in \mathcal{U}^n(\mathfrak{g}) \setminus \mathcal{U}^{n-1}(\mathfrak{g})$) with $\lambda(X) = 0$. Then by (3.1) $X \in \mathcal{U}^{n-1}(\mathfrak{g})$ contradicting the choice of X.

To prove surjectivity of λ we proceed by induction. Without loss of generality let

$$X \in X_1 \cdots X_n + \mathcal{U}^{n-1}(\mathfrak{g}) \subseteq \mathcal{U}(\mathfrak{g}).$$

Then by (3.1) $X - \lambda(X_1 \cdots X_n) \in \mathcal{U}^{n-1}(\mathfrak{g})$ and by induction we can find $X' \in S(\mathfrak{g})$ such that $\lambda(X') = X - \lambda(X_1 \cdots X_n)$. Hence, X is contained in the image of λ.

Since λ is $\mathrm{Ad}(G)$-invariant it is clear that every differential operator $D \in \mathbb{D}(G/K)$ can be represented as an image of $S(\mathfrak{g})^K$ under λ. The following lemma shows that elements from $S(\mathfrak{p})^K$ are sufficient. The proof is similar to the above.

Lemma 3.1.2 ([Hel84, Ch. II Cor. 4.8])**.**

$$\mathcal{U}(\mathfrak{g})^K = (\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}_{\mathbb{C}}) \oplus \lambda(S(\mathfrak{p})^K)$$

In particular, $\lambda\colon S(\mathfrak{p})^K \to \mathbb{D}(G/K)$ *is a bijection that preserves the degree.*

As before we identify $S(\mathfrak{a})^W$ with the space of Weyl group invariant polynomials on $\mathfrak{a}_{\mathbb{C}}^*$. Similarly, we identify $S(\mathfrak{p})^K$ with K-invariant polynomials on $\mathfrak{p}_{\mathbb{C}}^*$. In addition let us identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}^*$ with $\mathfrak{p}_{\mathbb{C}}$ with respect to the Killing form. Then we have a map $Poly(\mathfrak{p}_{\mathbb{C}}) \to Poly(\mathfrak{a}_{\mathbb{C}})$ by restriction. On the level of symmetric algebras it is the projection on $S(\mathfrak{a})$ with respect to the decomposition $S(\mathfrak{p}) = S(\mathfrak{a}) \oplus S(\mathfrak{p})\mathfrak{q}$ where $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. This map is injective when we restrict it to K-invariant polynomials on $\mathfrak{p}_{\mathbb{C}}$ since $\mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$ and its image is contained in $Poly(\mathfrak{a}_{\mathbb{C}})^W$. To see that it is also surjective let $p \in Poly(\mathfrak{a}_{\mathbb{C}})$. By the injectivity there is only one possible way to define the preimage $\tilde{p}$. Namely, for $X \in \mathfrak{p}$ we must have $\tilde{p}(X) = \tilde{p}(\mathrm{Ad}(k)X)$ for every $k \in K$. But since $\mathfrak{p} = \mathrm{Ad}(K)\mathfrak{a}$ this already defines $\tilde{p}\colon \mathfrak{p}_{\mathbb{C}} \to \mathbb{C}$. ($\tilde{p}$ is well-defined by [Kna02, Lemma 7.38].) The following lemma completes the proof of the surjectivity.

Lemma 3.1.3. *$\tilde{p}$ is a polynomial on $\mathfrak{p}_{\mathbb{C}}$ of the same degree as p.*

Proof. See [Hel84, Ch. II Thm. 5.8] for smoothness. Then decompose p into homogeneous summands and conclude by using the fact that a smooth homogeneous function is a polynomial. □

The last step for the proof of the surjectivity in Theorem 3.1.1 follows a similar strategy as the proof of the bijectivity of the symmetrization mapping λ including the following statement that is similar to (3.1).

Lemma 3.1.4. *For* $q \in S(\mathfrak{p})^K$ *we have* $\deg(\eta(\delta(\lambda(q))) - \overline{q}) \leq \deg(q) - 1$ *where* $\overline{q}$ *denotes the restriction of* q *to* $\mathfrak{a}_{\mathbb{C}}$.

Proof. Without loss of generality we can assume that q is homogeneous of degree $d > 0$. For $X \in \mathfrak{q}$ we find $Z \in \mathfrak{n}$ such that

$$X = Z - \theta Z = 2Z - (Z + \theta Z) \in \mathfrak{n} \oplus \mathfrak{k}.$$

Therefore, $q - \overline{q} \in \mathfrak{n}S^{d-1}(\mathfrak{g}) + S^{d-1}(\mathfrak{g})\mathfrak{k}$ (where $S^{d-1}(\mathfrak{g})$ is the subspace of $S(\mathfrak{g})$ of elements of degree $\leq d-1$) and also $\lambda(q) - \overline{q} \in \mathfrak{n}_{\mathbb{C}}\mathcal{U}^{d-1}(\mathfrak{g}) + \mathcal{U}^{d-1}(\mathfrak{g})\mathfrak{k}_{\mathbb{C}}+$ terms of lower order. It follows that $\delta(\lambda(q)) - \overline{q}$ has degree $< d$ and therefore the same holds for $\eta(\delta(\lambda(q))) - \overline{q}$ as η does not change the highest order term. □

Now we can prove the injectivity of HC. Let $D \in \mathbb{D}(G/K)$ be represented by $X \in \mathcal{U}(\mathfrak{g})^K$ with $\eta(\delta(X)) = 0$. By Lemma 3.1.2 $X = \lambda(p) + Y$ with $p \in S(\mathfrak{p})^K$ and $Y \in \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}_{\mathbb{C}}$. But this decomposition implies that $\eta(\delta(\lambda(p))) = 0$ so by Lemma 3.1.4 $\deg \overline{p} \leq \deg p - 1$. This is a contradiction unless $\deg(p) = -\infty$, i.e. $p = 0$.

For the surjectivity we define the pseudo inverse

$$\mathrm{Op}\colon S(\mathfrak{a})^W \to \mathbb{D}(G/K), \quad p \mapsto \lambda(\tilde{p}).$$

By Lemma 3.1.2 and 3.1.3 this is well-defined and bijective. It is not the inverse of HC but by Lemma 3.1.4 it satisfies that $p - \mathrm{HC}(\mathrm{Op}(p))$ is of lower degree than p for $p \in S(\mathfrak{a})^W$. Using this we can easily prove by induction that HC is surjective: By the induction hypothesis there is $D \in \mathbb{D}(G/K)$ such that $\mathrm{HC}(D) = p - \mathrm{HC}(\mathrm{Op}(p))$. Then $\mathrm{HC}(D + \mathrm{Op}(p)) = p$. This completes the proof of Theorem 3.1.1.

Remark 3.1.5. Note that by [Hel84, Ch. II Thm. 4.9] $\mathrm{Op}(p)$ can be expressed as a differential operator as follows. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{p}$ so that $\widetilde{p} = \sum_\alpha a_\alpha X^\alpha$ for some $a_\alpha \in \mathbb{C}$. For $f \in C^\infty(G/K)$ and $g \in G$ we then have

$$\lambda(\widetilde{p})f(gK) = \left(\sum_\alpha a_\alpha \partial^\alpha\right) f\left(g \exp\left(\sum_{i=1}^r t_i X_i\right) K\right)\Bigg|_{t_i=0}.$$

If $X_1', \ldots, X_r'$ is the dual basis of $X_1, \ldots, X_r$, then $\widetilde{p}(t_1 X_1' + \cdots + t_r X_r') = \sum_\alpha a_\alpha t^\alpha$. Therefore,

$$\lambda(\widetilde{p})f(gK) = \left(\widetilde{p}\left(\frac{\partial}{\partial t_1}X_1' + \cdots + \frac{\partial}{\partial t_r}X_r'\right)\right) f\left(g \exp\left(\sum_{i=1}^r t_i X_i\right) K\right)\Bigg|_{t_i=0}.$$

This construction is carried out for the case of $SL_n(\mathbb{R})$ in the next section.

3.2. Invariant differential operators for $SL_n(\mathbb{R})$

In this section we want to take a look at the invariant differential operators in the special case of $G = SL_n(\mathbb{R})$. We choose $\mathfrak{a} = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \mid \sum \lambda_i = 0\}$ and identify $\mathfrak{a}^*$ with $\mathfrak{a}$ via $\langle X, Y\rangle = \mathrm{Tr}(X \cdot Y)$. The root system of restricted roots Σ is given by $\{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)) = \lambda_i$. The Weyl group W is the symmetric group S_n and acts on $\mathfrak{a}$ and $\mathfrak{a}^*$ by permuting the diagonal entries. This root system is of type A_{n-1} and it is well known that the algebra of W-invariant polynomials is generated by the following homogeneous algebraically independent polynomials (see [Hum90, Section 3.12]):

$$p_i(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)) = \lambda_1^i + \cdots + \lambda_n^i, \quad i = 2, 3, \ldots n.$$

In order to find the invariant differential operators for $SL_n(\mathbb{R})/SO(n)$ we use the surjective map Op as defined in Section 3.1. First we need to extend the polynomials p_i to K-invariant polynomials $\tilde{p}_i$ on $\mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid X^T = X, \mathrm{Tr}(X) = 0\}$. As described in Section 3.1 $\tilde{p}_i$ is determined by $\tilde{p}_i(kHk^{-1}) = \tilde{p}_i(\mathrm{Ad}(k)H) = p_i(H)$ for $H \in \mathfrak{a}$. We observe that $p_i(H) = \mathrm{Tr}(H^i)$ for $H \in \mathfrak{a}$. This description allows us to extend p_i easily:

$$\tilde{p}_i(kHk^{-1}) = \mathrm{Tr}(H^i) = \mathrm{Tr}(kH^ik^{-1}) = \mathrm{Tr}((kHk^{-1})^i).$$

Hence, $\tilde{p}_i(X) = \mathrm{Tr}(X^i)$ for $X \in \mathfrak{p}$.

Example 3.2.1. For $n = 3$ let $X = \begin{pmatrix} a & x & z \\ x & b & y \\ z & y & c \end{pmatrix} \in \mathfrak{p}$. Then we have

$$\tilde{p}_2(X) = a^2 + b^2 + c^2 + 2x^2 + 2y^2 + 2z^2$$

and

$$\tilde{p}_3(X) = a^3 + b^3 + c^3 - 3cx^2 - 3ay^2 - 3bz^2 + 6xyz.$$

The next step is to express $\tilde{p}_i$ with respect to a basis, i.e. as an element of $S(\mathfrak{p})^K$ via the isomorphism $S(\mathfrak{p}) \simeq Poly(\mathfrak{p}^*)$. We introduce the following matrices.

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then clearly

$$\begin{aligned} \langle X, H_1\rangle &= a - b, & \langle X, E_1\rangle &= 2x, \\ \langle X, H_2\rangle &= b - c & \langle X, E_2\rangle &= 2y, \\ &= a + 2b, & \langle X, E_3\rangle &= 2z. \end{aligned}$$

Hence,

$$\begin{aligned} a &= \frac{1}{3}(2H_1 + H_2) & x &= \frac{1}{2}E_1 \\ b &= \frac{1}{3}(-H_1 + H_2) & y &= \frac{1}{2}E_2 \\ c &= \frac{1}{3}(-H_1 - 2H_2) & z &= \frac{1}{2}E_3. \end{aligned}$$

We obtain

$$\widetilde{p}_2 = \frac{2}{3}(H_1^2 + H_1H_2 + H_2^2) + \frac{1}{2}(E_1^2 + E_2^2 + E_3^2)$$

and

$$\begin{aligned} \widetilde{p}_3 = &\frac{1}{9}(2H_1^3 + 3H_1^2H_2 - 3H_1H_2^2 - 2H_2^3) \\ &+ \frac{1}{4}(E_1^2(H_1 + 2H_2) + E_2^2(-2H_1 - H_2) + E_3^2(H_1 - H_2)) + \frac{3}{4}E_1E_2E_3. \end{aligned} \qquad \square$$

Let us now determine the operators $\mathrm{Op}(\widetilde{p}_i)$ acting on $f \in C^\infty(SL_n(\mathbb{R})/SO(n))$. The greatest obstacle is that there is no nice orthonormal basis and hence either the basis or the dual basis is hard to work with. Therefore we simply choose the basis coming from the simple roots, i.e. let $H_i = \mathrm{diag}(0, \dots, 0, 1, -1, 0, \dots, 0)$, $i = 1, \dots, n-1$, where the 1 is the i-th diagonal entry. Then one calculates that the dual basis is given by $H_i' = \mathrm{diag}(1, \dots, 1, 0, \dots, 0) - \frac{i}{n}I$. By Remark 3.1.5

$$\mathrm{Op}(\widetilde{p}_k)f(gSO(n)) = \mathrm{Tr}\left(\left(\frac{\partial}{\partial X}\right)^k\right) f\left(g\exp(X)SO(n)\right)\Big|_{t_i = x_{ij} = 0}$$

where

$$X = \begin{pmatrix} t_1 & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{12} & t_2 - t_1 & x_{23} & \cdots & x_{2n} \\ x_{13} & x_{23} & t_3 - t_2 & \cdots & x_{3n} \\ & \vdots & & \ddots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \cdots & -t_{n-1} \end{pmatrix}$$

and

$$\frac{\partial}{\partial X} = \begin{pmatrix} \sum_{i=1}^{n-1} \partial_{t_i} - \sum_{i=1}^{n-1} \frac{i}{n}\partial_{t_i} & \frac{1}{2}\partial_{x_{12}} & \cdots & \frac{1}{2}\partial_{x_{1n}} \\ \frac{1}{2}\partial_{x_{12}} & \sum_{i=2}^{n-1} \partial_{t_i} - \sum_{i=1}^{n-1} \frac{i}{n}\partial_{t_i} & & \vdots \\ \vdots & & \ddots & \\ \frac{1}{2}\partial_{x_{1n}} & \cdots & & -\sum_{i=1}^{n-1} \frac{i}{n}\partial_{t_i} \end{pmatrix}.$$

Since this expression is quite cumbersome one can also take the detour over $GL_n(\mathbb{R})$ where one has a nice orthonormal basis. This is done in [BCH21]. They obtain a different generating set of invariant differential operators given by the Maass-Selberg operators δ_i which are defined for $f \in C^\infty(SL_n(\mathbb{R})/SO(n))$ by

$$\delta_i f(gSO(n)) = \operatorname{Tr}\left(\left(\frac{\partial}{\partial X}\right)^i\right)\Bigg|_{X=0} f\left(g\exp\left(X - \frac{1}{n}\operatorname{Tr}(X)I_n\right)SO(n)\right),$$

where

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial X} = \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \cdots & \frac{\partial}{2\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{2\partial x_{1n}} & \cdots & \frac{\partial}{\partial x_{nn}} \end{pmatrix}.$$

Example 3.2.2. Let us calculate the image of $\operatorname{Op}(p_k)$ under HC for $G = SL_3(\mathbb{R})$. We already know from Example 3.2.1 how $\widetilde{p}_k$ looks like as an element in $S(\mathfrak{p})$. Let us begin with p_2. From the expression $\widetilde{p}_2 = \frac{2}{3}(H_1^2 + H_1H_2 + H_2^2) + \frac{1}{2}(E_1^2 + E_2^2 + E_3^2)$ we see that $\lambda(\widetilde{p}_2) = \widetilde{p}_2$. We also observe that this is the Laplace operator in $\mathbb{D}(SL_3(\mathbb{R})/SO(3))$ since it coincides with the Casimir operator up to an element in $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$. Hence we expect $\operatorname{HC}(\operatorname{Op}(p_2)) = p_2 - \|\rho\|^2 = p_2 - 2$. Let us calculate this explicitly.

Let

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $K_i = N_i - N_i^T \in \mathfrak{k}$. Then $E_i = 2N_i - K_i$ and hence we calculate in $\mathcal{U}(\mathfrak{g})$ mod $\mathfrak{n}_\mathbb{C}\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{k}_\mathbb{C}$:

$$E_i^2 = -2K_iN_i = -2N_iK_i + [-K_i, 2N_i] = 2[N_i^T, N_i] = -2H_i$$

where $H_3 = H_1 + H_2$. By the definition of δ in Section 3.1 we have

$$\delta(\operatorname{Op}(p_2)) = \frac{2}{3}(H_1^2 + H_1H_2 + H_2^2) - 2H_1 - 2H_2.$$

To obtain $\operatorname{HC}(\operatorname{Op}(p_2))$ we have to apply η, i.e. we have to replace H_i by $H_i + \rho(H_i) = H_i + 1$, $i = 1, 2$. This results in

$$\begin{aligned} \operatorname{HC}(\operatorname{Op}(p_2)) &= \frac{2}{3}((H_1+1)^2 + (H_1+1)(H_2+1) + (H_2+1)^2) - 2(H_1+1) - 2(H_2+1) \\ &= p_2 - 2 \end{aligned}$$

as expected.

For p_3 the calculations become more involved since we are now dealing with elements of degree 3. Recall that

$$\begin{aligned} \widetilde{p}_3 = &\frac{1}{9}(2H_1^3 + 3H_1^2H_2 - 3H_1H_2^2 - 2H_2^3) \\ &+ \frac{1}{4}(E_1^2(H_1 + 2H_2) + E_2^2(-2H_1 - H_2) + E_3^2(H_1 - H_2)) + \frac{3}{4}E_1E_2E_3. \end{aligned}$$

First of all observe that applying λ to $\widetilde{p}_3$ only affects the last part $E_1E_2E_3$ since E_i commutes with the attached linear combination $H_i^\perp$ of H_1 and H_2. This is due to the fact that E_i is an element of the direct sum $\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}$ where $\alpha_i = \langle H_i, \cdot\rangle \in \Sigma^+$ and $H_i \perp H_i^\perp$. Again we calculate in $\mathcal{U}(\mathfrak{g})$ mod $\mathfrak{n}_\mathbb{C}\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{k}_\mathbb{C}$:

$$\begin{aligned}E_iH_i^\perp E_i &= (2N_i - K_i)H_i^\perp(2N_i - K_i) = -K_iH_i^\perp 2N_i = -K_i 2N_iH_i^\perp \\ &= -2N_iK_iH_i^\perp - 2[K_i, N_i]H_i^\perp = 2[N_i^T, N_i]H_i^\perp = -2H_iH_i^\perp.\end{aligned}$$

Hence,

$$\begin{aligned}\delta(\mathrm{Op}(p_3)) &= p_3 - \frac{1}{2}\sum_{i=1,2,3} H_iH_i^\perp + \frac{3}{4}\delta(\lambda(E_1E_2E_3)) \\ &= p_3 - H_1^2 + H_2^2 + \frac{3}{4}\delta(\lambda(E_1E_2E_3)).\end{aligned}$$

For the last part we observe that $\delta(E_iE_jE_k) = \delta(E_iE_kE_j)$ since the commutator bracket of two elements in $\mathfrak{p}$ is contained in $\mathfrak{k}$. Now,

$$E_iE_1E_j = E_1E_iE_j + [E_i, E_1]E_j = E_1E_iE_j + E_j[E_i, E_1] + [[E_i, E_1], E_j]$$

and therefore

$$\delta(E_iE_1E_j) = \delta(E_1E_iE_j) + \delta([[E_i, E_1], E_j]) = \delta(E_1E_jE_i) + \delta(E_1[E_i, E_j]) + \delta([[E_i, E_1], E_j]).$$

We conclude that

$$\begin{aligned}\delta(\lambda(E_1E_2E_3)) &= \frac{1}{3}\delta(E_1E_2E_3) + \frac{1}{3}\delta(E_2E_1E_3) + \frac{1}{3}\delta(E_3E_1E_2) \\ &= \delta(E_1E_2E_3) + \frac{1}{3}\delta([[E_2, E_1], E_3]) + \frac{1}{3}\delta([[E_3, E_1], E_2]) \\ &= \delta(E_1E_2E_3) + \frac{1}{3}\delta([-K_3, E_3]) + \frac{1}{3}\delta([-K_2, E_2]) \\ &= \delta(E_1E_2E_3) - \frac{2}{3}H_3 - \frac{2}{3}H_2.\end{aligned}$$

We are left with computing $\delta(E_1E_2E_3)$:

$$\begin{aligned}E_1E_2E_3 &= -K_1(2N_2 - K_2)(2N_3) = [-K_1, 2N_2](2N_3) + (-K_1)[-K_2, 2N_3] \\ &= [[-K_1, 2N_2], 2N_3] + [-K_1, [-K_2, 2N_3]] = [-2N_3, 2N_3] + [-K_1, -2N_1] \\ &= 2H_1 \mod \mathfrak{n}_\mathbb{C}\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{k}_\mathbb{C}.\end{aligned}$$

All in all,

$$\begin{aligned}\delta(\mathrm{Op}(p_3)) &= p_3 - H_1^2 + H_2^2 + \frac{3}{4}\left(2H_1 - \frac{2}{3}H_2 - \frac{2}{3}H_3\right) \\ &= \frac{1}{9}(2H_1^3 + 3H_1^2H_2 - 3H_1H_2^2 - 2H_2^3) - H_1^2 + H_2^2 + H_1 - H_2.\end{aligned}$$

As we did for p_2 we apply η to obtain the image under HC:

$$\begin{aligned}\mathrm{HC}(\mathrm{Op}(p_3)) &= \frac{1}{9}(2(H_1+1)^3 + 3(H_1+1)^2(H_2+1) - 3(H_1+1)(H_2+1)^2 - 2(H_2+1)^3)\\ &\quad - (H_1+1)^2 + (H_2+1)^2 + (H_1+1) - (H_2+1)\\ &= \frac{1}{9}(2H_1^3 + 3H_1^2H_2 - 3H_1H_2^2 - 2H_2^3) + H_1^2 - H_2^2 + H_1 - H_2\\ &\quad - H_1^2 + H_2^2 - 2H_1 + 2H_2\\ &\quad + H_1 - H_2\\ &= p_3.\end{aligned}$$

Actually, this computation could have been avoided by Proposition I.6.5. Namely, $\mathrm{HC}(\mathrm{Op}(p_3)) = p_3 +$ lower order terms where the degree of the lower order terms are odd as well. Hence, $\mathrm{HC}(\mathrm{Op}(p_3)) - p_3 \in S(\mathfrak{a})^W$ is homogeneous of degree one and therefore vanishes since there is no W-invariant element in $\mathfrak{a}$ except 0. $\square$

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Projects

I. Higher rank quantum-classical correspondence

Outline

Published Paper

Abstract

For a compact Riemannian locally symmetric space $\Gamma\backslash G/K$ of arbitrary rank we determine the location of certain Ruelle-Taylor resonances for the Weyl chamber action. We provide a Weyl-lower bound on an appropriate counting function for the Ruelle-Taylor resonances and establish a spectral gap which is uniform in Γ if G/K is irreducible of higher rank. This is achieved by proving a quantum-classical correspondence, i.e. a 1:1-correspondence between horocyclically invariant Ruelle-Taylor resonant states and joint eigenfunctions of the algebra of invariant differential operators on G/K.

I.1. Introduction

Ruelle resonances for an Anosov flow provide a fundamental spectral invariant that does not only reflect many important dynamical properties of the flow but also geometric and topological properties of the underlying manifold. Very recently the concept of resonances was extended to higher rank $\mathbb{R}^n$-Anosov actions and led to the notion of *Ruelle-Taylor*[1] *resonances* which were shown to be a discrete subset $\sigma_{\mathrm{RT}} \subset \mathbb{C}^n$ [BGHW20]. It was furthermore shown in [BGHW20] that the leading resonances (i.e. those with vanishing real part) are related to mixing properties of the considered Anosov action. In particular, it was shown that if the action is weakly mixing in an arbitrary direction of the abelian group $\mathbb{R}^n$, then $0 \in \mathbb{C}^n$ is the only leading resonance. Furthermore, the resonant states at zero give rise to equilibrium measures that share properties of SRB measures of Anosov flows.

Apart from the leading resonances the spectrum of Ruelle-Taylor resonances has so far not been studied if $n \geq 2$. In particular, when $n \geq 2$, it was not known whether there are other resonances than the resonance at zero. Neither was it known whether there is a spectral gap, i.e. whether the real parts of the resonances are bounded away from zero. In this article we shed some light on these questions by examining the Ruelle-Taylor resonances for the class of Weyl chamber flows via harmonic analysis.

Let us briefly introduce the setting: Let G be a real connected non-compact semisimple Lie group with finite center and Iwasawa decomposition $G = KAN$. Let $\mathfrak{a}$ be the Lie

[1] They were named Ruelle-Taylor resonances because the notion of the Taylor spectrum for commuting operators is a crucial ingredient of their definition.

algebra of A and M the centralizer of A in K. Then A is isomorphic to $\mathbb{R}^n$ where n is the real rank of G and acts on G/M from the right. Hence A also acts on the compact manifold $\mathcal{M} := \Gamma\backslash G/M$, where $\Gamma \leq G$ is a cocompact torsion-free lattice. It can be easily seen that this action is an Anosov action with hyperbolic splitting $T\mathcal{M} = E_0 \oplus E_s \oplus E_u$ which can be described explicitly in terms of associated vector bundles (see Section I.2.1 for a general definition of Anosov actions and Proposition I.3.1 for the description of the hyperbolic splitting for Weyl chamber flows). Furthermore, if $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots with simple system Π and positive system Σ^+ then the positive Weyl chamber is given by $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Pi\}$.

The *Ruelle-Taylor resonances* of this Anosov action are defined as follows: For $H \in \mathfrak{a}$ let X_H be the vector field on $\mathcal{M}$ defined by the right A-action. Then

$$\sigma_{\mathrm{RT}} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \setminus \{0\} \colon (X_H + \lambda(H))u = 0 \,\forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(M)$ is the set of distributions with wavefront set contained in the annihilator $E_u^* \subseteq T^*\mathcal{M}$ of $E_0 \oplus E_u$. The distributions $u \in \mathcal{D}'_{E_u^*}(M)$ satisfying $(X_H + \lambda(H))u = 0$ for all $H \in \mathfrak{a}$ are called *resonant states* of λ and the dimension of the space of all such distributions is called the multiplicity $m(\lambda)$ of the resonance λ. It has been shown in [BGHW20] that $\sigma_{\mathrm{RT}} \subset \mathfrak{a}_{\mathbb{C}}^*$ is discrete and that all resonances have finite multiplicity. It also follows from that work that the real part of the resonances are located in a certain cone $\overline{{}_{-}\mathfrak{a}^*} \subset \mathfrak{a}^*$ which is the negative dual cone of the positive Weyl chamber $\mathfrak{a}_+$ (see Section I.2.2 for a precise definition).

In this article we will prove that there is a bijection between a certain subset of the Ruelle-Taylor resonant states and certain joint eigenfunctions of the invariant differential operators on the locally symmetric space $\Gamma\backslash G/K$. Before explaining this correspondence in more detail we state two results on the spectrum of Ruelle-Taylor resonances that we can conclude from the correspondence.

The first result says that for any Weyl chamber flow there exist infinitely many Ruelle-Taylor resonances by providing a Weyl-lower bound on an appropriate counting function.

Theorem I.1.1. *Let ρ be the half sum of the positive restricted roots, W the Weyl group (see Section I.2.2 for a precise definition), and for $t > 0$ let*

$$N(t) := \sum_{\lambda \in \sigma_{\mathrm{RT}}, \mathrm{Re}(\lambda) = -\rho, \|\mathrm{Im}(\lambda)\| \leq t} m(\lambda).$$

Then for $d := \dim(G/K)$

$$N(t) \geq |W| \mathrm{Vol}(\Gamma\backslash G/K) \left(2\sqrt{\pi}\right)^{-d} \frac{1}{\Gamma(d/2+1)} t^d + \mathcal{O}(t^{d-1}).$$

More generally, let $\Omega \subseteq \mathfrak{a}^*$ be open and bounded such that $\partial\Omega$ has finite $(n-1)$-dimensional Hausdorff measure. Then

$$\sum_{\lambda \in \sigma_{\mathrm{RT}}, \mathrm{Re}(\lambda) = -\rho, \mathrm{Im}(\lambda) \in t\Omega} m(\lambda) \geq |W| \mathrm{Vol}(\Gamma\backslash G/K) \, (2\pi)^{-d} \, \mathrm{Vol}(\mathrm{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1}).$$

The second result guarantees a uniform spectral gap.

Theorem I.1.2. *Let G be a real semisimple Lie group with finite center, then for any cocompact torsion-free discrete subgroup $\Gamma \subset G$ there is a neighborhood $\mathcal{G} \subset \mathfrak{a}^*$ of 0 such that*

$$\sigma_{\mathrm{RT}} \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{0\}.$$

If G furthermore has Kazhdan's property (T) (e.g. if G is simple of higher rank), then the spectral gap $\mathcal{G}$ can be taken uniformly in Γ and only depends on the group G.

Let us now explain in some detail the spectral correspondence that is the key to the above results:

We define the space of *first band resonant states* as those resonant states that are in addition horocyclically invariant

$$\mathrm{Res}^0_X(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M})\colon (X_H + \lambda(H))u = 0 \text{ and } \mathcal{X}u = 0\, \forall H \in \mathfrak{a}, \mathcal{X} \in C^\infty(\mathcal{M}, E_u)\}$$

and we call a Ruelle-Taylor resonance a *first band resonance* iff $\mathrm{Res}^0_X(\lambda) \neq 0$. By working with horocycle operators and vector valued Ruelle-Taylor resonances we will be able to show that all resonances with real part in a certain neighborhood of zero in $\mathfrak{a}^*$ are always first band resonances (see Proposition I.3.7). As the Weyl chamber flow is generated by mutually commuting Hamilton flows, we consider the set of Ruelle-Taylor resonances as a *classical spectrum.*

Let us briefly describe the *quantum* side: In the rank one case the quantization of the geodesic flow is given by the Laplacian on G/K. In the higher rank case we have to consider the algebra of G-invariant differential operators on G/K which we denote by $\mathbb{D}(G/K)$. As an abstract algebra this is a polynomial algebra with n algebraically independent operators, among them the Laplace operator. These operators descend to $\Gamma\backslash G/K$ and we can define the joint eigenspace

$${}^\Gamma E_\lambda = \{f \in C^\infty(\Gamma\backslash G/K) \mid Df = \chi_\lambda(D)f \quad \forall D \in \mathbb{D}(G/K)\}$$

where χ_λ is a character of $\mathbb{D}(G/K)$ parametrized by $\lambda \in \mathfrak{a}^*_\mathbb{C}/W$ with the Weyl group W. Here χ_ρ is the trivial character (see Section I.2.4). Let σ_Q denote the corresponding *quantum spectrum* $\{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid {}^\Gamma E_\lambda \neq \{0\}\}$.

We have the following correspondence between the classical first band resonant states and the joint quantum eigenspace:

Theorem I.1.3. *Let $\lambda \in \mathfrak{a}^*_\mathbb{C}$ be outside the exceptional set $\mathcal{A} := \{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid \frac{2\langle\lambda+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in -\mathbb{N}_{>0}$ for some $\alpha \in \Sigma^+\}$. Then there is a bijection between the finite dimensional vector spaces*

$$\pi_* : \mathrm{Res}^0_X(\lambda) \to {}^\Gamma E_{-\lambda-\rho}$$

where π_* is the push-forward of distributions along the canonical projection $\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Using this 1:1-correspondence we can then use results about the the quantum spectrum to obtain obstructions and existence results on the Ruelle-Taylor resonances. Notably we use results of Duistermaat-Kolk-Varadarajan [DKV79] on the spectrum σ_Q but we also deduce refined information on the quantum spectrum. Here we use L^p-bounds for spherical functions obtained from asymptotic expansions [vdBS87] and L^p-bounds for matrix coefficients based on work by Cowling and Oh [Cow79, Oh02]. Theorem I.1.1 and Theorem I.1.2 as stated above give only a rough version of the information on the Ruelle-Taylor resonances that we can actually obtain. As the full results require some further notation we refrain from stating them in the introduction and refer to Theorem I.5.1. We also refer to Figure I.6 for a visualization of the structure of first band resonances for the case of $G = SL(3, \mathbb{R})$.

Methods and related results:

The key ingredient to the quantum-classical correspondence is that we can in a first step relate the horocyclically invariant first band resonant states with distributional vectors in some principal series representations. Then we can apply the Poisson transform of [KKM$^+$78] to get a bijection onto the quantum eigenspace ${}^\Gamma E_{-\lambda-\rho}$. The prototype of such a quantum-classical correspondence has been first established by Dyatlov, Faure and Guillarmou [DFG15] in the case of manifolds of constant curvature or in other words for the rank one group $G = SO(n, 1)$. Certain central ideas have however already been present for $G = SO(2, 1)$ in the works of Flaminio-Forni and Cosentino [FF03, Cos05]. In the rank one setting there exist several generalizations of the quantum classical correspondence of [DFG15] e.g. to convex cocompact manifolds of constant curvature [GHW18, Had20], general compact locally symmetric spaces of rank one [GHW21] and vector bundles [KW20, KW21].

Besides the correspondence between the classical Ruelle resonant states and the quantum Laplace eigenvalues there are several other approaches in the literature establishing exact relations between the Laplace spectrum and the geodesic flow. One approach is to relate the Laplace spectrum to divisors of zeta functions. Such relations have been obtained for rank one locally symmetric spaces on various levels of generality by Bunke, Olbrich, Patterson and Perry ($G = SO(n, 1)$, Γ convex cocompact: [BO97, BO99, PP01], G real rank one, Γ cocompact [BO95]).

A third approach to an exact quantum-classical correspondence is to relate the Laplace spectrum to a transfer operator which represents a time discretized dynamics of the geodesic flow. This type of correspondence was notably studied for hyperbolic surfaces with cusps (see [LZ01, BLZ15, BP23] for results for $G = SL(2, \mathbb{R})$ and Γ discrete subgroups of increasing generality). We refer in particular to the expository article [PZ20] and the introduction of [BP23] for a current state of the art of these techniques. A very first step towards generalizations of this approach to higher rank has been recently achieved in [Poh20] for the Weyl chamber flow on products of Schottky surfaces by the construction of symbolic dynamics and transfer operators.

Note that in [DFG15] not only the first band of Ruelle resonances was related to the Laplace spectrum but a complete band structure has been established and the higher bands could be related to the Laplace spectrum on divergence free symmetric tensors. In the present article we do not study the higher bands. This will presumably be a very hard question for general semisimple groups G (note that in [DFG15] it was crucial at several points that for $G = SO(n,1)$, $N \cong \mathbb{R}^{n-1}$ is abelian). However it might be tractable for some concrete groups with simple enough root spaces such as $G = SL(3,\mathbb{R})$. For geodesic flows the phenomenon of such a band structure is quite universal and known in the case of compact locally symmetric spaces of rank one [KW21] but also for geodesic flows on manifolds of pinched negative curvature [FT13, GC21, FT21].

As mentioned above an important application of Ruelle resonances for Anosov flows are mixing results. More precisely, the existence of a spectral gap in addition with resolvent estimates imply mixing of the flow. For Weyl chamber flows this relation of gaps and mixing rates is not yet established but conjectured to be true. From this perspective Theorem I.1.2 is related to the work of Katok and Spatzier [KS94] who showed exponential mixing for the Weyl chamber action in every direction of the closure of the positive Weyl chamber if G has Property (T). However it is not known whether their result remains true if the Property (T) assumption is dropped. Our result above (Theorem I.1.2) ensures a Γ-dependent gap in any case but as mentioned above the precise relation to mixing rates is not yet established.

Finally, Weyl laws for Ruelle resonances of geodesic flows can also be established in variable curvature (or more generally contact Anosov flows) in various settings [FS11, DDZ14, FT17]. In particular, in the very recent article [FT21] by Faure and Tsujii the Weyl law also follows because a “first band” of resonances can be related to a quantum operator. The methods in their work are however completely different and are based on microlocal analysis rather then global harmonic analysis.

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I.2. Preliminaries

I.2.1. Ruelle-Taylor resonances for higher rank Anosov actions

In this section we recall the main properties of Ruelle-Taylor resonances for higher rank Anosov actions from [BGHW20]. Let $\mathcal{M}$ be a compact Riemannian manifold, $A \simeq \mathbb{R}^n$ be an abelian group and let $\tau\colon A \to \mathrm{Diffeo}(\mathcal{M})$ be a smooth locally free group action. If $\mathfrak{a} := \mathrm{Lie}(A)$ we define the generating map

$$X\colon \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad H \mapsto X_H := \frac{d}{dt}\Big|_{t=0} \tau(\exp(tH)).$$

Note that $[X_{H_1}, X_{H_2}] = 0$ for $H_i \in \mathfrak{a}$. For $H \in \mathfrak{a}$ we denote by $\varphi_t^{X_H}$ the flow of the vector field X_H. The action is called *Anosov* if there exists $H \in \mathfrak{a}$ and a continuous $\varphi_t^{X_H}$-invariant splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s,$$

where $E_0 := \operatorname{span}\{X_H : H \in \mathfrak{a}\}$ is of dimension n because the action is locally free and there exist $C > 0$, $\nu > 0$ such that for each $x \in \mathcal{M}$

$$\begin{aligned} \forall w \in E_s(x), t \geq 0 : \quad & \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu t}\|w\|, \\ \forall w \in E_u(x), t \leq 0 : \quad & \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu |t|}\|w\|, \end{aligned}$$

where the norm on $T\mathcal{M}$ is given by the Riemannian metric on $\mathcal{M}$. Such an $H \in \mathfrak{a}$ is called *transversally hyperbolic*. We call the set

$$\mathcal{W} := \{H' \in \mathfrak{a} \mid H' \text{ is transversally hyperbolic with the same splitting as } H\}$$

the *positive Weyl chamber containing* H.

Let $E \to \mathcal{M}$ be the complexification of a Euclidean bundle over $\mathcal{M}$ and denote by $\operatorname{Diff}^1(\mathcal{M}, E)$ the set of first order differential operators with smooth coefficients acting on sections of E. Then a linear map $\mathbf{X} : \mathfrak{a} \to \operatorname{Diff}^1(\mathcal{M}, E)$ such that $\mathbf{X}_{H_1}\mathbf{X}_{H_2} = \mathbf{X}_{H_2}\mathbf{X}_{H_1}$ for all $H_i \in \mathfrak{a}$ is called an *admissible lift* of the generic map X if

$$\mathbf{X}_H(fs) = (X_H f)s + f\mathbf{X}_H s \tag{I.1}$$

for $s \in C^\infty(\mathcal{M}, E)$, $f \in C^\infty(\mathcal{M})$, and $H \in \mathfrak{a}$.

For a fixed positive Weyl chamber $\mathcal{W}$ the set of *Ruelle-Taylor resonances* can be defined as

$$\sigma_{\mathrm{RT}}(\mathbf{X}) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \setminus \{0\} : (\mathbf{X}_H + \lambda(H))u = 0 \,\forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(\mathcal{M}, E)$ is the set of distributional sections of the bundle E with wavefront set contained in E_u^*. Here E_u^* is defined as the annihilator of $E_0 \oplus E_u$ in $T^*\mathcal{M}$. The vector space of *Ruelle-Taylor resonant states* for a resonance $\lambda \in \sigma_{\mathrm{RT}}(\mathbf{X})$ is defined by

$$\operatorname{Res}_{\mathbf{X}}(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \mid (\mathbf{X}_H + \lambda(H))u = 0 \,\forall H \in \mathfrak{a}\}.$$

Remark I.2.1. The original definition of Ruelle-Taylor resonances and resonant states is stated via Koszul complexes (see [BGHW20, Section 3]). More precisely, λ is a resonance iff the corresponding Koszul complex is not exact and the resonant states are the cohomologies of this complex. The space of resonant states that we are considering is just the 0th cohomology. However, it turns out that the Koszul complex is not exact iff the 0th cohomology is non-vanishing, i.e. the two notions coincide (see [BGHW20, Theorem 4]).

It is known that the resonances have the following properties.

Proposition I.2.2 (see [BGHW20, Theorems 1 and 4]). *$\sigma_{\mathrm{RT}}(\mathbf{X})$ is a discrete subset of $\mathfrak{a}^*_{\mathbb{C}}$ contained in*

$$\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(H)) \leq C_{L^2}(H) \quad \forall H \in \mathcal{W}\}$$

with $C_{L^2}(H) = \inf\{C > 0 \mid \|e^{-t\mathbf{X}_H}\|_{L^2\to L^2} \leq e^{Ct} \quad \forall t > 0\}$ where $e^{-t\mathbf{X}_H}\colon L^2(\mathcal{M}, E) \to L^2(\mathcal{M}, E)$ is the semigroup with generator $-\mathbf{X}_H$. Moreover, for each $\lambda \in \sigma_{\mathrm{RT}}(\mathbf{X})$ the space $\mathrm{Res}_{\mathbf{X}}(\lambda)$ of resonant states is finite dimensional.

I.2.2. Semisimple Lie groups

In this section we fix the notation for the present article. Let G be a real semisimple connected non-compact Lie group with finite center and Iwasawa decomposition $G = KAN$. Furthermore, let $M := Z_K(A)$ be the centralizer of A in K and $G = KAN_-$ the opposite Iwasawa decomposition. We denote by $\mathfrak{g}, \mathfrak{a}, \mathfrak{n}, \mathfrak{n}_-, \mathfrak{k}, \mathfrak{m}$ the corresponding Lie algebras. For $g \in G$ let $H(g)$ be the logarithm of the A-component in the Iwasawa decomposition. We have a K-invariant inner product on $\mathfrak{g}$ that is induced by the Killing form and the Cartan involution. We have the orthogonal Bruhat decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ into root spaces $\mathfrak{g}_\alpha$ with respect to the $\mathfrak{a}$-action via the adjoint action ad. Here $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots. Denote by W the Weyl group of the root system of restricted roots. Let n be the real rank of G and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ (resp. Σ^+) the simple (resp. positive) system in Σ determined by the choice of the Iwasawa decomposition. Let $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ and $\rho := \frac{1}{2}\Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$. Denote by w_0 the longest Weyl group element, i.e. the unique element in W mapping Π to $-\Pi$. Let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Pi\}$ the positive Weyl chamber and $\mathfrak{a}^*_+$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{a}$. We denote by ${}_+\mathfrak{a}^*$ the dual cone $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) > 0 \,\forall H \in \overline{\mathfrak{a}_+} \setminus \{0\}\}$ and by $\overline{{}_+\mathfrak{a}^*}$ its closure $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) \geq 0 \,\forall H \in \mathfrak{a}_+\} = \mathbb{R}_{\geq 0}\Pi$. Hence, if ω_j is the dual basis of α_j then $\overline{{}_+\mathfrak{a}^*} = \{\lambda \in \mathfrak{a}^* \mid \langle\lambda, \omega_j\rangle \geq 0 \,\forall j = 1, \dots, n\}$. Furthermore, we denote $\overline{{}_-\mathfrak{a}^*} := -\overline{{}_+\mathfrak{a}^*}$. If $\overline{A^+} := \exp(\overline{\mathfrak{a}_+})$, then we have the Cartan decomposition $G = K\overline{A^+}K$.

Example I.2.3. If $G = SL_n(\mathbb{R})$, then we choose $K = SO(n)$, A as the set of diagonal matrices of positive entries with determinant 1, and N as the set of upper triangular matrices with 1's on the diagonal. $\mathfrak{a}$ is the abelian Lie algebra of diagonal matrices and the set of restricted roots is $\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i(\lambda)$ is the i-th diagonal entry of λ. The positive system corresponding to the Iwasawa decomposition is $\Sigma^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ with simple system $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}$. The positive Weyl chamber is $\mathfrak{a}_+ = \{\mathrm{diag}(\lambda_1, \dots, \lambda_n) \mid \lambda_1 > \cdots > \lambda_n\}$ and the dual cone is $\overline{{}_+\mathfrak{a}} = \{\mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{a} \mid \lambda_1 + \cdots + \lambda_k \geq 0 \,\forall k\}$. The Weyl group is the symmetric group S_n acting by permutation of the diagonal entries.

I.2.3. Principal series representations

The concept of a principal series representation is an important tool in representation theory of semisimple Lie groups. It can be described using different pictures. We start

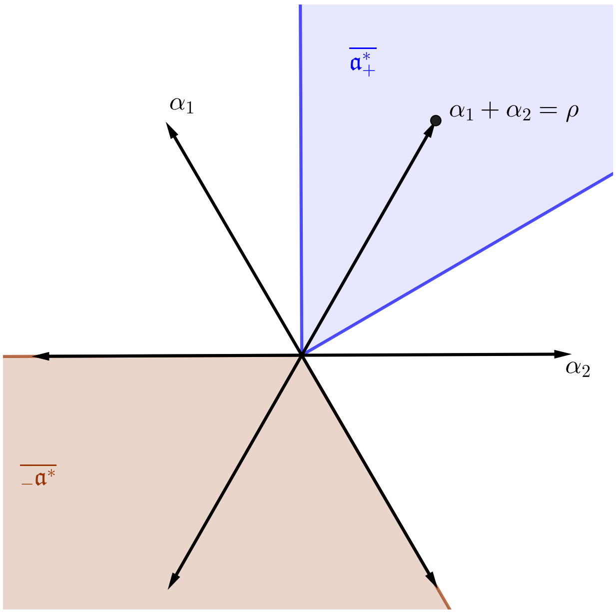

Figure I.1.: The root system for the special case $G = SL_3(\mathbb{R})$: There are three positive roots $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. As all root spaces are one dimensional the special element $\rho = \frac{1}{2}\Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$ equals $\alpha_1 + \alpha_2$.

with the *induced picture*: Pick $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and (τ, V_τ) an irreducible unitary representation of M. We define

$$V^{\tau,\lambda} := \left\{ f\colon G \to V_\tau \text{ cont.}: \begin{array}{l} f(gman) = e^{-(\lambda+\rho)\log a}\tau(m)^{-1}f(g) \\ \text{for all } g \in G, m \in M, a \in A, n \in N \end{array} \right\}$$

endowed with the norm $\|f\|^2 = \int_K \|f(k)\|^2 dk$ where dk is the normalized Haar measure on K. Recall that ρ is the half sum of positive roots. The group G acts on $V^{\tau,\lambda}$ by the left regular representation. The completion $H^{\tau,\lambda}$ of $V^{\tau,\lambda}$ with respect to the norm is called *induced picture of the (non-unitary) principal series representation* with respect to (τ, λ). We also write $\pi_{\tau,\lambda}$ for this representation. If τ is the trivial representation then we write H^λ and π_λ and call it the *spherical principal series* with respect to λ. Note that for equivalent irreducible unitary representations τ_1, τ_2 of M the corresponding principal series representations are equivalent as representations as well. In particular, the Weyl group W acts on the unitary dual of M by $w\tau(m) = \tau(w^{-1}mw)$ where $w \in W$ is given by a representative in the normalizer of A in K and therefore $H^{\lambda,w\tau}$ is well-defined up to equivalence.

A different way to view the principal series representation is the so-called *compact picture*. Although we don't need this description we want to introduce it in order to give a larger overview of these representation. It is given by restricting the function $f\colon G \to V_\tau$ to K, i.e. a dense subspace is given by

$$\{f\colon K \to V_\tau \text{ cont.} \mid f(km) = \tau(m)^{-1}f(k), k \in K, m \in M\}$$

with the same norm as above. In this picture the G-action is given by

$$\pi_{\tau,\lambda}(g)f(k) = e^{-(\lambda+\rho)H(g^{-1}k)}f(k_{KAN}(g^{-1}k)), \quad g \in G, k \in K,$$

where k_{KAN} is the K-component in the Iwasawa decomposition $G = KAN$. Furthermore, recall from section I.2.2 that $H(g) \in \mathfrak{a}$ was defined as the logarithm of the Iwasawa A component.

For the example $G = PSL_2(\mathbb{R})$ the compact picture allows us to describe this representation explicitly without using the Iwasawa decomposition: Since $K = PSO(2) \simeq S^1 \subseteq \mathbb{R}^2$ the representation $H^{1,\lambda\alpha} = H^{\lambda\alpha}$, $\lambda \in \mathbb{C}$, is given by $L^2(S^1)$ with the action $\pi_\lambda(g)f(\omega) = \|g^{-1}\omega\|^{-2\lambda-1}f(g^{-1}\omega/\|g^{-1}\omega\|)$.

I.2.4. Invariant differential operators

Let $\mathbb{D}(G/K)$ be the algebra of *G-invariant differential operators* on G/K, i.e. differential operators commuting with the left translation by elements $g \in G$. Then we have an algebra isomorphism $\mathrm{HC}\colon \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ from $\mathbb{D}(G/K)$ to the W-invariant complex polynomials on $\mathfrak{a}^*$ which is called *Harish-Chandra homomorphism* (see [Hel84, Ch. II Theorem 5.18]). For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let χ_λ be the character of $\mathbb{D}(G/K)$ defined by $\chi_\lambda(D) \coloneqq \mathrm{HC}(D)(\lambda)$. Obviously, $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$. Furthermore, the χ_λ exhaust all characters of $\mathbb{D}(G/K)$ (see [Hel84, Ch. III Lemma 3.11]). We define the space of joint eigenfunctions

$$E_\lambda \coloneqq \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D)f \quad \forall D \in \mathbb{D}(G/K)\}.$$

We will only work with the subspace of functions of moderate growth

$$E_\lambda^* \coloneqq \{f \in E_\lambda \mid \exists c \in \mathbb{R} : |f(kaK)| \leq Ce^{c\|\log a\|} \quad \forall k \in K, a \in A\}.$$

Note that E_λ and E_λ^* are G-invariant.

I.2.5. Poisson transform

The representation of G on E_λ^* can be described via the *Poisson transform*: If $(H^{\tau,\lambda})^{-\infty}$ denotes the distributional vectors in the principal series, then the Poisson transform $\mathcal{P}_\lambda$ maps $(H^{-\lambda})^{-\infty}$ into E_λ^* G-equivariantly. It is given by

$$\mathcal{P}_\lambda f(xK) = \int_K f(k)e^{-(\lambda+\rho)H(x^{-1}k)}dk$$

if f is a sufficiently regular function in the compact picture of the principal series. If f is given in the induced picture, then $\mathcal{P}_\lambda f(xK)$ simply is $\int_K f(xk)dk$. Since K/M can be seen as the boundary of G/K at infinity, the Poisson transform produces a joint eigenfunction for a given boundary value (see [vdBS87] for more details).

It is important to know for which values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the Poisson transform is a bijection. By [vdBS87, Theorem 12.2] we have that $\mathcal{P}_\lambda \colon (H^{-\lambda})^{-\infty} \to E_\lambda^*$ is a bijection if

$$-\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbb{N}_{>0} \quad \text{for all} \quad \alpha \in \Sigma^+. \tag{I.2}$$

In particular, $\mathcal{P}_\lambda$ is a bijection if $\operatorname{Re} \lambda \in \overline{\mathfrak{a}_+^*}$.

I.2.6. L^p-bounds for elementary spherical functions

One can show that in each joint eigenspace E_λ there is a unique left K-invariant function which has the value 1 at the identity (see [Hel84, Ch. IV Corollary 2.3]). We denote the corresponding bi-K-invariant function on G by ϕ_λ and call it *elementary spherical function*. Therefore, $\phi_\lambda = \phi_\mu$ iff $\lambda = w\mu$ for some $w \in W$. It is given by the Poisson transform of the constant function with value 1 in the compact picture, i.e. $\phi_\lambda(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)} dk$.

The aim of this section is to establish the following proposition (see Figure I.2 for a visualization) that will be needed to obtain a spectral gap in Theorem I.4.10.

Proposition I.2.4. *Let $p \in [2, \infty[$. Then the elementary spherical function ϕ_λ is in $L^{p+\varepsilon}(G)$ (where the L^p-space is defined via a Haar measure on G) for every $\varepsilon > 0$ iff* $\operatorname{Re} \lambda \in (1 - 2p^{-1}) \operatorname{conv}(W\rho)$ *where* $\operatorname{conv}(W\rho)$ *is the convex hull of the finite set $W\rho$.*

Proof. First of all note that we only have to consider $\operatorname{Re} \lambda \in \overline{\mathfrak{a}_+^*}$ since $\phi_\lambda = \phi_\mu$ iff $\lambda = w\mu$ for some $w \in W$. In this case $\operatorname{Re} \lambda \in (1 - 2p^{-1}) \operatorname{conv}(W\rho)$ is equivalent to $\operatorname{Re} \lambda \in (1 - 2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$ (see [Hel84, Ch. IV Lemma 8.3]).

With this remark, one implication of the proposition is a straight forward consequence of standard estimates for elementary spherical functions: Suppose that $\operatorname{Re} \lambda \in \overline{\mathfrak{a}_+^*}$ and $\operatorname{Re} \lambda \in (1 - 2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$. Then we have the following bound on ϕ_λ (see [Kna86, Ch. VII Prop. 7.15]):

$$|\phi_\lambda(a)| \leq C e^{(\operatorname{Re} \lambda - \rho)(\log a)} (1 + \rho(\log a))^d, \quad a \in A^+$$

where C and d are constants ≥ 0. By the integral formula for $G = K\overline{A^+}K$ (see [Hel84, Ch. I Theorem 5.8]) and the bi-K-invariance of ϕ_λ we have

$$\begin{aligned}
\int_G |\phi_\lambda(g)|^{p+\varepsilon} dg &= \int_{\mathfrak{a}_+} |\phi_\lambda(\exp H)|^{p+\varepsilon} \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H))^{m_\alpha} dH \\
&\leq \int_{\mathfrak{a}_+} (C e^{(\operatorname{Re} \lambda - \rho)H} (1 + \rho(H))^d)^{p+\varepsilon} e^{2\rho(H)} dH
\end{aligned}$$

for a suitable Lebesgue measure on $\mathfrak{a}$. Because of $\operatorname{Re} \lambda \in (1 - 2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$ we have

$$(p + \varepsilon)(\operatorname{Re} \lambda - \rho)(H) \leq -(2 + 2\varepsilon p^{-1})\rho(H).$$

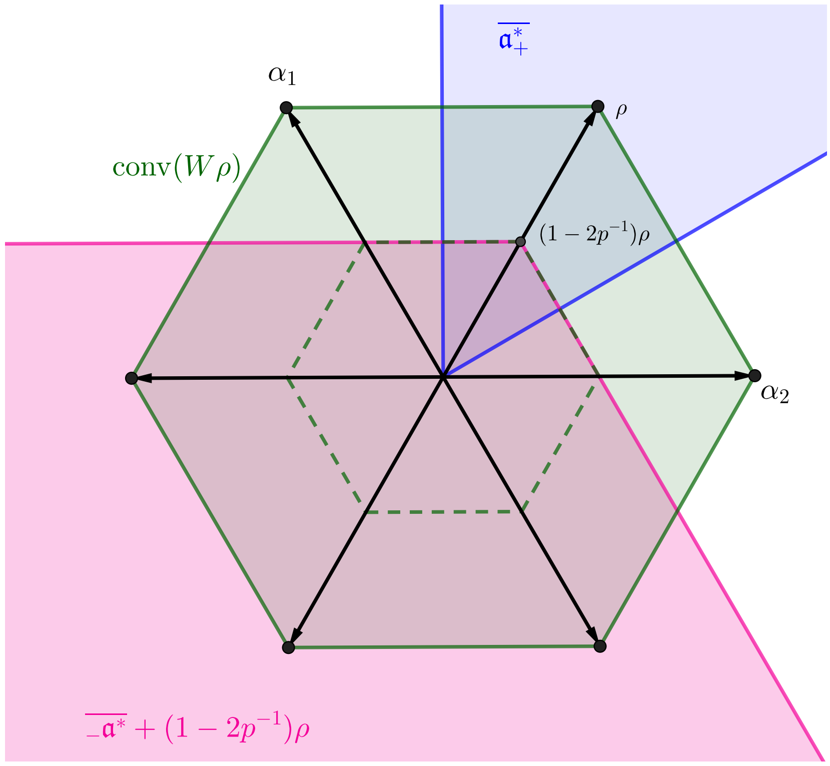

Figure I.2.: Visualization of the regions appearing in Proposition I.2.4 for the case $G = SL_3(\mathbb{R})$: The green dashed region is the boundary of $(1-2p^{-1})\,\mathrm{conv}(W\rho)$. Its intersection with the positive Weyl chamber $\overline{\mathfrak{a}_+^*}$ (blue cone) equals $(1 - 2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$ intersected with $\overline{\mathfrak{a}_+^*}$.

Hence,

$$\int_G |\phi_\lambda(g)|^{p+\varepsilon} dg \leq C^{p+\varepsilon} \int_{\mathfrak{a}_+} (1+\rho(H))^{d(p+\varepsilon)} e^{-2\varepsilon p^{-1}\rho(H)} dH$$

and we see that the latter is indeed finite by coordinizing $\mathfrak{a}_+$ by $x_j \leftrightarrow \alpha_j(H)$ with $x_j > 0$. Then dH is a multiple of dx and $\rho(H) = \sum x_j \rho_j$ with $\rho_j > 0$. Therefore $\phi_\lambda \in L^{p+\varepsilon}(G)$.

The opposite implication will be proved by combining the proof of [Kna86, Theorem 8.48] with [vdBS87]: According to [vdBS87, Corollary 16.2] the elementary spherical function ϕ_λ has a converging expansion

$$\phi_\lambda(\exp H) = \sum_{\xi \in X(\lambda)} p_\xi(\lambda, H) e^{\xi(H)}, \quad H \in \mathfrak{a}_+, \tag{I.3}$$

where $X(\lambda) = \{w\lambda - \rho - \mu \mid w \in W, \mu \in \mathbb{N}_0\Pi\}$ and the $p_\xi(\lambda, \cdot)$ are polynomials of degree $\leq |W|$. The series converges absolutely on $\mathfrak{a}_+$ and uniformly on each subchamber $\{H \in \mathfrak{a}_+ \mid \alpha_i(H) \geq \varepsilon_i > 0\}$. The main ingredient of the proof of Proposition I.2.4 is the fact that (see [vdBS87, Theorem 10.1])

$$p_{\lambda-\rho}(\lambda, \cdot) \neq 0. \tag{I.4}$$

Now, if $\phi_\lambda \in L^{p+\epsilon}(G)$, the proof of [Kna86, Theorem 8.48] shows that $\mathrm{Re}\langle \lambda - (1 - 2(p + \epsilon)^{-1})\rho, \omega_j\rangle < 0$. Hence $\mathrm{Re}\,\lambda - (1 - 2p^{-1})\rho \in \overline{{}_{-}\mathfrak{a}^*}$. □

I.2.7. Positive definite functions and unitary representations

In this section we recall the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations. Recall first that a continuous function $f\colon G \to \mathbb{C}$ is called *positive semidefinite* if the matrix $(f(x_i^{-1}x_j))_{i,j}$ for all $x_1, \ldots, x_k \in G$ is positive semidefinite. If f is positive semidefinite, then f is bounded by $f(1)$ and one has $f(x^{-1}) = \overline{f(x)}$. Moreover, we can define a unitary representation π_f associated to f in the following way: If R denotes the right regular representation of G, then π_f is the completion of the space spanned by $R(x)f$ with respect to the inner product defined by $\langle R(x)f, R(y)f\rangle \coloneqq f(y^{-1}x)$ which is positive definite. G acts unitarily on this space by the right regular representation. If $f(g) = \langle \pi(g)v, v\rangle$ is a matrix coefficient of a unitary representation π, then f is positive semidefinite and π_f is contained in π.

Secondly, recall that a unitary representation is called *spherical* if it contains a non-zero K-invariant vector. Denote by $\widehat{G}_{\mathrm{sph}}$ the subset of the unitary dual consisting of spherical representations. We then have a 1:1-correspondence between positive semidefinite elementary spherical functions and $\widehat{G}_{\mathrm{sph}}$ given by $\phi_\lambda \mapsto \pi_{\phi_\lambda}$ (see [Hel84, Ch. IV Theorem 3.7]). The preimage of an irreducible unitary spherical representation π with normalized K-invariant vector v_K is given by $g \mapsto \langle \pi(g)v_K, v_K\rangle$. If the set $\widehat{G}_{\mathrm{sph}}$ is endowed with the Fell topology (see [BdlHV08, Appendix F.2]) and we use the topology of convergence on compact sets on the set of elementary spherical functions, then the above correspondence is a homeomorphism as is easily seen from the definitions.

I.2.8. Associated vector bundles

In order to define the Weyl chamber flow not only on the base manifold but also on vector bundles we recall the definition of the associated vector bundle $\mathcal{V}_\tau$ over a homogeneous space G/M for a unitary finite dimensional representation (τ, V_τ) of M. Its total space is given by $\mathcal{V}_\tau = G \times_\tau V_\tau = (G \times V_\tau)/_\sim$ where $(gm, v) \sim (g, \tau(m)v)$ with $g \in G$, $m \in M$ and $v \in V_\tau$. The equivalence classes are denoted by $[g, v]$ and the projection to G/M is $[g, v] \mapsto gM$. A section s of this bundle can be identified with a function $\overline{s}\colon G \to V_\tau$ satisfying $\overline{s}(gm) = \tau(m)^{-1}\overline{s}(g)$. We will use this identification throughout this article. We also have a G-action on $\mathcal{V}_\tau$ defined by $g[g', v] \coloneqq [gg', v]$. Therefore, we also have the left regular action on smooth sections of $\mathcal{V}_\tau$:

$$(gs)(g'M) \coloneqq g(s(g^{-1}g'M)), \quad s \in C^\infty(G/M, \mathcal{V}_\tau).$$

Identifying s with $\overline{s}$ this actions reads $\overline{gs}(g') = \overline{s}(g^{-1}g')$.

A special case of an associated vector bundle is the tangent bundle $T(G/M)$. Namely, $T(G/M) = G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$. Hence, vector fields $\mathfrak{X}$ can be identified with smooth functions $\overline{\mathfrak{X}}\colon G \to \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ satisfying $\overline{\mathfrak{X}}(gm) = \mathrm{Ad}(m)^{-1}\overline{\mathfrak{X}}(g)$. Therefore, we have a canonical connection ∇ on $\mathcal{V}_\tau$ given by

$$\overline{\nabla_{\mathfrak{X}} s}(g) = \frac{d}{dt}\Big|_{t=0} \overline{s}(g\exp(t\overline{\mathfrak{X}}(g))),$$

where s is a smooth section identified with a $\overline{s}\colon G \to V_\tau$ and $\mathfrak{X}$ is a vector field of G/M identified with $\overline{\mathfrak{X}}$ as above. This connection will be used to lift the Weyl chamber flow to $\mathcal{V}_\tau$.

I.3. Ruelle-Taylor resonances for the Weyl chamber action

We keep the notation from Section I.2.2. Let Γ be a discrete, torsion-free, cocompact subgroup of G. Then the biquotient $\mathcal{M} = \Gamma\backslash G/M$ is a smooth compact Riemannian manifold where the Riemannian structure is induced by the inner product on $\mathfrak{g}$. More precisely, the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is given by quotient $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-))$ and the norm of some $\Gamma[g, Y], g \in G, Y \in \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ is given by the norm of $Y \in \mathfrak{g}$. We have a well-defined right A-action on $\mathcal{M}$:

$$(\Gamma g M)a := \Gamma g a M, \quad a \in A, g \in G.$$

Therefore we have an $\mathfrak{a}$-action by smooth vector fields

$${}_\Gamma X\colon \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad {}_\Gamma X_H f(\Gamma g M) = \frac{d}{dt}\Big|_{t=0} f(\Gamma g e^{tH} M)$$

which we call *Weyl chamber action.*

For later use we denote by $X\colon \mathfrak{a} \to \mathrm{Diff}^1(G/M)$ the corresponding action on G/M.

Proposition I.3.1. *The A-action on $\mathcal{M}$ is Anosov. More precisely, each $H \in \mathfrak{a}_+$ is transversally hyperbolic with the splitting $E_0 = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a})$, $E_s = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$, and $E_u = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-))$. Moreover, for fixed $H_0 \in \mathfrak{a}_+$ the dynamically defined positive Weyl chamber*

$$\mathcal{W} = \{H \in \mathfrak{a} \mid H \text{ is transversally hyperbolic with the same splitting as } H_0\}$$

equals $\mathfrak{a}_+$. Hence the two notions of positive Weyl chambers agree.

Proof. Pick $\Gamma[g, X_\alpha] \in \Gamma\backslash(G \times_M \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and assume that X_α is in the root space $\mathfrak{g}_\alpha$. Then we calculate

$$d\varphi_t^{\Gamma X_H}(\Gamma g M)\Gamma[g, X_\alpha] = \frac{d}{ds}\Big|_{s=0} \varphi_t^{\Gamma X_H}(\Gamma g e^{sX_\alpha} M) = \frac{d}{ds}\Big|_{s=0} \Gamma e^{sX_\alpha} e^{tH} M =$$
$$= \frac{d}{ds}\Big|_{s=0} \Gamma g e^{tH} e^{s\,\mathrm{Ad}(e^{-tH})X_\alpha} M = \Gamma[g e^{tH}, \mathrm{Ad}(e^{-tH})X_\alpha] = \Gamma[g e^{tH}, e^{-t\alpha(H)} X_\alpha]$$

Hence we have exponential decay if $\alpha \in \Sigma^+$ and exponential growth if $\alpha \in -\Sigma^+$. The general statement is obtained from the observation that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta \perp \mathfrak{a}$ for $\alpha \neq \beta \neq 0$ in Σ. ∎

I.3.1. Lifted Weyl chamber action

In order to define horocycle operators we generalize the Weyl chamber action to associated vector bundles. Let (τ, V_τ) be a finite-dimensional unitary representation of M, that is a complexification of an orthogonal representation. Then we have defined the associated vector bundle $\mathcal{V}_\tau = G \times_\tau V_\tau$ over G/M (see Section I.2.8).

The quotient bundle $\Gamma\backslash\mathcal{V}_\tau$ is the complexification of a Euclidean vector bundle over $\mathcal{M}$, where the Euclidean structure is induced by the inner product on V_τ. We identify smooth sections s of this bundle with smooth functions $\overline{s}\colon G \to V_\tau$ with $\overline{s}(\gamma g m) = \tau(m^{-1})\overline{s}(g)$ for all $\gamma \in \Gamma$, $g \in G$, and $m \in M$.

The canonical connection ∇ descends to a connection

$$
{}_\Gamma\nabla\colon C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes T^*\mathcal{M})
$$

and we have the following formula:

$$
\overline{{}_\Gamma\nabla s(\mathfrak{X})}(g) \coloneqq \overline{{}_\Gamma\nabla_{\mathfrak{X}} s}(g) = \left.\frac{d}{dt}\right|_{t=0} \overline{s}(g \exp(t\overline{\mathfrak{X}}(g))), \tag{I.5}
$$

where s is a smooth section identified as above and $\mathfrak{X}$ is a vector field of $\mathcal{M}$ identified with a smooth function $\overline{\mathfrak{X}}\colon G \to \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ which is left Γ-invariant and right M-equivariant.

Definition I.3.2. The *lifted Weyl chamber action* is defined as

$$
{}_\Gamma\mathbf{X}^\tau\colon \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), \quad {}_\Gamma\mathbf{X}^\tau_H \coloneqq {}_\Gamma\nabla_{\mathfrak{X}_H},
$$

where $\mathfrak{X}_H$ is the vector field identified with the constant mapping $G \to \mathfrak{a} \subseteq \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$, $g \mapsto H$.

The fact that ${}_\Gamma\nabla$ is a covariant derivative implies that ${}_\Gamma\mathbf{X}^\tau$ is an admissible lift of the Weyl chamber action in the sense of Equation (I.1).

For later use we denote by $\mathbf{X}^\tau\colon \mathfrak{a} \to \mathrm{Diff}^1(G/M, \mathcal{V}_\tau)$ the corresponding action on G/M.

We can find a non-trivial tube domain in $\mathfrak{a}^*_\mathbb{C}$ which is independent of τ and contains all Ruelle-Taylor resonances for the lifted Weyl chamber action.

Proposition I.3.3. *The set of Ruelle-Taylor resonances $\sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau)$ is contained in $\overline{{}_-\mathfrak{a}^*} + i\mathfrak{a}^*$.*

Proof. By Proposition I.2.2 we have

$$
\sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau) \subseteq \{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid \mathrm{Re}(\lambda(H)) \le C^\tau_{L^2}(H) \quad \forall H \in \mathfrak{a}_+\}.
$$

Hence, it remains to show that $C^\tau_{L^2}(H) \coloneqq \inf\{C > 0 \mid \|e^{-t{}_\Gamma\mathbf{X}^\tau_H}\|_{L^2\to L^2} \le e^{Ct} \quad \forall t > 0\} = 0$ for all $H \in \mathfrak{a}_+$. We show the stronger statement that $e^{-t{}_\Gamma\mathbf{X}^\tau_H}$ is unitary.

Since M commutes with A we have a well defined action of A on $\Gamma\backslash\mathcal{V}_\tau$. It is given by $(\Gamma[g,v])a = \Gamma[ga,v]$. This action gives rise to an A-action on sections of the bundle $\Gamma\backslash\mathcal{V}_\tau$ defined via $(af)(x) = f(xa)a^{-1}$ with $f \in C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau)$, $x \in \mathcal{M}$ and $a \in A$. If we identify f with a equivariant function $\overline{f}\colon G \to V_\tau$, then $\overline{(af)}(g) = \overline{f}(ga)$. Let $d\Gamma g$ be the normalized right G-invariant Radon measure on $\Gamma\backslash G$. Then the L^2-norm of f is given by $\|f\|_{L^2}^2 = \int_{\Gamma\backslash G} \|\overline{f}(g)\|_{V_\tau}^2 d\Gamma g$ and it follows that the A-action continued to $L^2(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau)$ is unitary. By definition $e^{-t_\Gamma \mathbf{X}_H^\tau} f = \exp(-tH)f$ for $f \in L^2(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau)$ and therefore $e^{-t_\Gamma \mathbf{X}_H^\tau}$ is unitary. □

I.3.2. First band resonances and horocycle operators

In analogy to the rank one setting we make the following definition (see [KW21, Definition 2.11] and [GHW21, Definition 3.1] in the scalar case).

Definition I.3.4. We call $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau)$ a *first band resonance* and write $\lambda \in \sigma^0_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau)$ if the vector space

$$\mathrm{Res}^0_{{}_\Gamma\mathbf{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma\mathbf{X}^\tau}(\lambda) \mid {}_\Gamma\nabla_{\mathfrak{X}} u = 0 \,\forall \mathfrak{X} \in C^\infty(\mathcal{M}, E_u)\}$$

of *first band resonant states* is non-trivial.

The goal of this section is to prove that in a certain neighborhood of 0 in $\mathfrak{a}^*_{\mathbb{C}}$ each Ruelle-Taylor resonance is a first band resonance and $\mathrm{Res}^0_{{}_\Gamma\mathbf{X}^\tau}(\lambda) = \mathrm{Res}_{{}_\Gamma\mathbf{X}^\tau}(\lambda)$. This will be done by introducing so called horocycle operators as follows.

Recall that $T\mathcal{M} = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and the bundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$ decomposes as $\bigoplus_{\alpha \in \Sigma^+} \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$, and similarly for $\mathfrak{n}_-$. Therefore, the cotangent bundle $T^*\mathcal{M}$ is the Whitney sum $\Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{a}^*) \oplus \bigoplus_{\alpha \in \Sigma} \Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{g}^*_\alpha)$. Let us denote the coadjoint action of M on the complexification of $\mathfrak{g}^*_\alpha$ by τ_α. Note that τ_α is unitary with respect to the inner product induced by the Killing form and the Cartan involution. We can now define

$$\mathrm{pr}_\alpha\colon (T^*\mathcal{M})_{\mathbb{C}} \to \Gamma\backslash\mathcal{V}_{\tau_\alpha}$$

by fiber-wise restriction to the subbundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$. This induces a map

$$\widetilde{\mathrm{pr}}_\alpha\colon C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}}) \to C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Definition I.3.5. If ${}_\Gamma\nabla^{\mathbb{C}}\colon C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}})$ denotes the complexification of the canonical connection ${}_\Gamma\nabla$, then the *horocycle operator* $\mathcal{U}_\alpha$ for $\alpha \in \Sigma$ is defined as the composition

$$\mathcal{U}_\alpha \coloneqq \widetilde{\mathrm{pr}}_\alpha \circ {}_\Gamma\nabla^{\mathbb{C}}\colon C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M},\Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Note that we have the explicit formula

$$\overline{\mathcal{U}_\alpha s}(g)(Y) = \frac{d}{dt}\Big|_{t=0} \overline{s}(g\exp(tY)), \quad s \in C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), Y \in \mathfrak{g}_\alpha, \tag{I.6}$$

if we again use the identification of sections of some associated vector bundle with left Γ-invariant and right M-equivariant functions indicated by $\bar{\cdot}$ and the identification $V_\tau \otimes \mathfrak{g}_\alpha^* \simeq \mathrm{Hom}(\mathfrak{g}_\alpha, V_\tau)$.

We should point out that the space of first band resonant states can be rewritten with the horocycle operators as

$$\mathrm{Res}^0_{\Gamma\mathbf{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{\Gamma\mathbf{X}^\tau}(\lambda) \mid \mathcal{U}_{-\alpha}u = 0\ \forall \alpha \in \Sigma^+\}. \tag{I.7}$$

Note that in the case of constant curvature manifolds (i.e. the real hyperbolic case $G = PSO(n,1)$ of rank 1) there is only one positive root and our definition reduces to the original one due to Dyatlov and Zworski (see [DFG15, p. 931]). Furthermore, our definition extends the definition of the horocycle operators for arbitrary G of rank one (see [KW21]).

The horocycle operators fulfill the following important commutation relation.

Lemma I.3.6.

$$\forall H \in \mathfrak{a}\colon \quad {}_\Gamma\mathbf{X}_H^{\tau\otimes\tau_\alpha}\mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma\mathbf{X}_H^\tau = \alpha(H)\mathcal{U}_\alpha.$$

Proof. Using the formulas (I.5) and (I.6) we obtain

$$\overline{({}_\Gamma\mathbf{X}_H^{\tau\otimes\tau_\alpha}\mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma\mathbf{X}_H^\tau)s}(g)(Y) = \\ \frac{d}{dt_1}\Big|_{t_1=0}\frac{d}{dt_2}\Big|_{t_2=0} \overline{s}(g\exp(t_1H)\exp(t_2Y)) - \overline{s}(g\exp(t_1Y)\exp(t_2H))$$

and the latter equals

$$\frac{d}{dt}\Big|_{t=0} \overline{s}(g\exp(t[H,Y])).$$

Since $[H,Y] = \alpha(H)Y$ for $Y \in \mathfrak{g}_\alpha$ the claim follows. □

We can now prove the main result of this section.

Proposition I.3.7. *The horocycle operators can be extended continuously as linear operators to distributional sections, i.e.*

$$\mathcal{U}_\alpha\colon \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Moreover, for $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau)$ *the horocycle operator* $\mathcal{U}_{-\alpha}$ *maps*

$$\mathrm{Res}_{\Gamma\mathbf{X}^\tau}(\lambda) \quad \textit{into} \quad \mathrm{Res}_{\Gamma\mathbf{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha).$$

In particular, each $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}^\tau)$ *with* $\mathrm{Re}\,\lambda \in \bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha)$ *is a first band resonance and* $\mathrm{Res}_{\Gamma\mathbf{X}^\tau}(\lambda) = \mathrm{Res}^0_{\Gamma\mathbf{X}^\tau}(\lambda)$ *holds.*

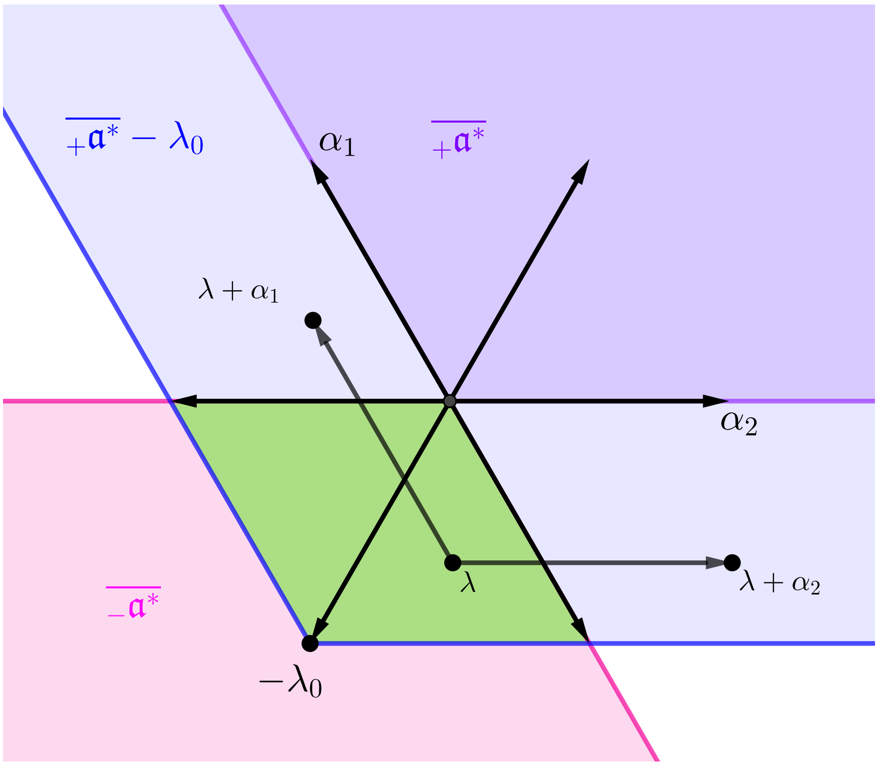

Figure I.3.: For $G = SL_3(\mathbb{R})$ the green region depicts the real part of the region where every resonance is a first band resonance (see Proposition I.3.7).

Proof. Since the horocycle operators are differential operators, we obtain a continuation to distributional sections and Lemma I.3.6 still holds. Let $u \in \mathrm{Res}_{{}_\Gamma\mathbf{X}^\tau}(\lambda)$, i.e. $u \in \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$ with $\mathrm{WF}(u) \subseteq E_u^*$ and ${}_\Gamma\mathbf{X}_H^\tau u = -\lambda(H)u$. Since differential operators do not increase the wavefront set, we have $\mathrm{WF}(\mathcal{U}_{-\alpha}u) \subseteq E_u^*$. Furthermore,

$$ {}_\Gamma\mathbf{X}_H^{\tau\otimes\tau_{-\alpha}}\mathcal{U}_{-\alpha}u = -\alpha(H)\mathcal{U}_{-\alpha}u + \mathcal{U}_{-\alpha}{}_\Gamma\mathbf{X}_H^\tau u = -(\lambda+\alpha)(H)\mathcal{U}_{-\alpha}u $$

by Lemma I.3.6. Hence $\mathcal{U}_{-\alpha}u \in \mathrm{Res}_{{}_\Gamma\mathbf{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha)$.

For the 'in particular' part recall that $\mathrm{Res}_{{}_\Gamma\mathbf{X}^{\tau'}}(\lambda') = 0$ for each unitary representation τ' of M and $\mathrm{Re}(\lambda') \notin \overline{{}_-\mathfrak{a}^*}$ (see Proposition I.3.3) and $\mathrm{Res}^0_{{}_\Gamma\mathbf{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma\mathbf{X}^\tau}(\lambda) \mid \mathcal{U}_{-\alpha}u = 0 \;\forall \alpha \in \Sigma^+\}$. □

Note that $\bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha) = \overline{{}_-\mathfrak{a}^*} \cap ({}_+\mathfrak{a}^* - \lambda_0)$, where $\lambda_0 = \sum_{\alpha\in\Pi}\alpha$. Indeed, let $\lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha \in \mathfrak{a}^*$. Then $\lambda \in \overline{{}_-\mathfrak{a}^*}$ iff $c_\alpha \le 0$ for all $\alpha \in \Pi$, $\lambda \in \overline{{}_-\mathfrak{a}^*} - \alpha$ iff $c_\alpha \le -1$ and $c_\beta \le 0$ for all $\beta \in \Pi \setminus \{\alpha\}$, and $\lambda \in {}_+\mathfrak{a}^*$ iff $c_\alpha > 0$ for all $\alpha \in \Pi$. Combining these statements implies the claim.

I.3.3. First band resonant states and principal series representation

In this section we identify first band resonances states with certain Γ-invariant vectors in a corresponding principal series representation. The proof follows the line of arguments given in [KW21, Section 2] in the rank one case. This will allow us to apply the Poisson transform and obtain a quantum-classical correspondence.

By analogy to [KW21, Definition 2.1] we define

$$\mathcal{R}(\lambda) := \left\{ s \in \mathcal{D}'(G/M, \mathcal{V}_\tau) \colon \begin{array}{ll} (\mathbf{X}_H^\tau + \lambda(H))s = 0 & \forall H \in \mathfrak{a} \\ \nabla_{\mathfrak{X}_-} s = 0 & \forall \mathfrak{X}_- \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-) \end{array} \right\}.$$

The following lemma allows us to first study the representation of G in $\mathcal{R}(\lambda)$ and take Γ-invariants afterwards.

Lemma I.3.8. *The space $\mathrm{Res}^0_{\Gamma\mathbf{X}^\tau}(\lambda)$ is isomorphic to the space of Γ-invariants of $\mathcal{R}(\lambda)$, where the isomorphism is defined by considering Γ-invariant sections as sections of the bundle $\Gamma\backslash\mathcal{V}_\tau$.*

Proof. The only part to observe is that each $s \in \mathcal{R}(\lambda)$ automatically has $\mathrm{WF}(s) \subseteq G \times_{\mathrm{Ad}|_M} \mathfrak{n}^*$. This holds because $G \times_{\mathrm{Ad}^*|_M} \mathfrak{n}^*$ is the joint characteristic set of $\mathbf{X}_H^\tau$ and $\mathfrak{X}_-$ (see [KW21, Lemma 2.5] for details). □

We will now show that the smooth sections in $\mathcal{R}(\lambda)$ correspond to smooth vectors in the principal series representation for the opposite Iwasawa decomposition.

Lemma I.3.9. *The smooth sections $\mathcal{R}(\lambda) \cap C^\infty(G/M, \mathcal{V}_\tau)$ in $\mathcal{R}(\lambda)$ can be identified G-equivariantly with*

$$W = \{\overline{s} \colon G \to V_\tau \text{ smooth} \mid \overline{s}(gman_-) = e^{-\lambda \log a}\tau(m)^{-1}\overline{s}(g), m \in M, a \in A, n_- \in N_-\}.$$

The identification is obtained by considering sections in $s \in \mathcal{R}(\lambda)$ as right M-equivariant functions $\overline{s} : G \to V_\tau$.

Proof. The M-equivariance is clear so it remains to show the transformation properties under A and N_-. The property $(\mathbf{X}_H^\tau + \lambda(H))s = 0$ amounts to $\frac{d}{dt}\big|_{t=0} \overline{s}(ge^{tH}) = -\lambda(H)\overline{s}(g)$ for every $g \in G$ and $H \in \mathfrak{a}$. Hence, the function $\varphi(t) = \overline{s}(ge^{tH})$ satisfies

$$\varphi'(r) = \frac{d}{dt}\bigg|_{t=0} \varphi(ge^{rH}e^{tH}) = -\lambda(H)\overline{s}(ge^{rH}) = -\lambda(H)\varphi(r).$$

Therefore, $\overline{s}(ge^{tH}) = \varphi(t) = e^{-t\lambda(H)}\overline{s}(g)$. This proves the right A-equivariance.

For the N_--invariance, let $Y \in \mathfrak{n}_-$ and consider $\varphi(t) = \overline{s}(ge^{tY})$. For $r \in \mathbb{R}$ let $g_r = ge^{rY} \in G$. Since $[g_r, Y] \in G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-$ is in the fiber over $g_r M \in G/M$, there is a smooth section $\mathfrak{X}_r \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-)$ such that $\mathfrak{X}_r(g_r M) = [g_r, Y]$. In particular, the corresponding right M-equivariant function $\overline{\mathfrak{X}_r}\colon G \to \mathfrak{n}_-$ satisfies $\overline{\mathfrak{X}_r}(g_r) = Y$. It follows that

$$0 = \overline{\nabla_{\mathfrak{X}_r} s}(g_r) = \frac{d}{dt}\bigg|_{t=0} \overline{s}(g_r e^{t\overline{\mathfrak{X}_r}(g_r)}) = \frac{d}{dt}\bigg|_{t=0} \overline{s}(ge^{rY}e^{tY}) = \varphi'(r).$$

Hence, φ is constant. This completes the proof. □

Note that the space W from Lemma I.3.9 is already very close to the definition of the induced picture of the principal series representation (see Section I.2.3). The only difference is that in W we have a right invariance w.r.t. N_- instead of N. This can be easily fixed using a conjugation with the longest Weyl group element and leads to the main result of this section:

Proposition I.3.10. *With the longest Weyl group element w_0 (see Section I.2.2) we have an isomorphism*

$$\mathrm{Res}^0_{\Gamma\mathbf{X}^\tau}(\lambda) \to {}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$$

where ${}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$ denotes the Γ-invariant distributional vectors in the principal series representation $\pi_{w_0\tau, w_0(\lambda+\rho)}$.

Proof. Pick $k_0 \in K$ normalizing $\mathfrak{a}$ such that the action of $\mathrm{Ad}(k_0)$ on $\mathfrak{a}$ is the longest Weyl group element w_0. We consider the map $I\overline{s}(g) \coloneqq \overline{s}(gk_0)$. Then I commutes with the left action by G and one calculates that

$$I\overline{s}(gman) = e^{-(w_0\lambda)\log a}(w_0\tau)(m)^{-1} I\overline{s}(g), \quad g \in G, m \in M, a \in A, n \in N.$$

Hence, we have an intertwiner between W and smooth vectors in $H^{w_0\tau, w_0(\lambda+\rho)}$ which extends to distributional sections. By Lemma I.3.9 we conclude that

$$\mathcal{R}(\lambda) \simeq \left(H^{w_0\tau, w_0(\lambda+\rho)}\right)^{-\infty}$$

as G-representations. Taking Γ-invariants and using Lemma I.3.8 completes the proof. □

I.3.4. Quantum-classical correspondence

In the previous section we identified the first band resonant states $\mathrm{Res}^0_{\Gamma\mathbf{X}^\tau}(\lambda)$ with Γ-invariant distributional vectors in the principal series $(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$. If we restrict ourselves to the scalar case $\tau = 1$, then the Poisson transform $\mathcal{P}_{-w_0(\lambda+\rho)}$ defines a map from ${}^\Gamma(H^{w_0(\lambda+\rho)})^{-\infty}$ to ${}^\Gamma E_{-w_0(\lambda+\rho)}$, as $\mathcal{P}_{-w_0(\lambda+\rho)}$ provides a G-equivariant map $(H^{w_0(\lambda+\rho)})^{-\infty}$ to $E_{-w_0(\lambda+\rho)}$ (see Section I.2.5). Hence, we can identify eigendistributions of the classical motion with quantum states and we call this identification *quantum-classical correspondence.* More precisely, we have the following result, which immediately gives Theorem I.1.3.

Proposition I.3.11. *If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies $\frac{2\langle\lambda+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin -\mathbb{N}_{>0}$ for all $\alpha \in \Sigma^+$, then we have a bijection*

$$\operatorname{Res}^0_{\Gamma\mathbf{X}}(\lambda) \to {}^\Gamma E_{-w_0(\lambda+\rho)} = {}^\Gamma E_{-(\lambda+\rho)}.$$

In particular, $\lambda \in \sigma^0_{\mathrm{RT}}(\Gamma\mathbf{X})$ if and only if ${}^\Gamma E_{-(\lambda+\rho)} \neq 0$. Furthermore, the isomorphism is given by the push-forward π_* of distributions along the canonical projection ${}_\Gamma\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Proof. In view of Section I.2.5 the Poisson transform is a bijection from $(H^{w_0(\lambda+\rho)})^{-\infty} \to E^*_{-\lambda-\rho}$. Restricted to Γ-invariant distributional vectors it is still injective with image ${}^\Gamma E_{-\lambda-\rho}$ since Γ is cocompact and therefore ${}^\Gamma E_{-\lambda-\rho} = {}^\Gamma E^*_{-\lambda-\rho}$.

It remains to show that the isomorphism is the push-forward along the canonical projection. To this end let $s \in \mathcal{R}(\lambda)$ be smooth and $\pi\colon G/M \to G/K$ the canonical projection. Then the isomorphism $\mathcal{R}(\lambda) \to (H^{w_0(\lambda+\rho)})^{-\infty}$ carries s to $\tilde{s}\colon G \to \mathbb{C}, \tilde{s}(g) = s(gk_0)$ where $k_0 \in K$ is as in the proof of Proposition I.3.10. It follows that

$$\mathcal{P}_{-w_0(\lambda+\rho)}\tilde{s}(gK) = \int_K \tilde{s}(gk)dk = \int_K s(gkk_0)dk = \int_K s(gk)dk$$

since K is unimodular. On the other hand, for $f \in C_c^\infty(G/K)$ we have

$$(\pi_* s)(f) = s(f\circ\pi) = \int_{G/M} s(gM)f(gK)dgM = \int_{G/K}\left(\int_{K/M} s(gkM)dkM\right) f(gK)dgK$$

if we normalize the Haar measure on M and choose compatible invariant measures on G/K and K/M. Hence, $\pi_* s = \mathcal{P}_{-w_0(\lambda+\rho)}\tilde{s}$ for $s \in \mathcal{R}(\lambda) \cap C^\infty(G/M)$. Using the density of smooth compactly supported functions in $\mathcal{R}(\lambda)$ [KW21, Corollary 2.9] we obtain the equality for the whole space $\mathcal{R}(\lambda)$. As before we now restrict to Γ-invariant distributions identified with distributions on $\Gamma\backslash G/M$ and $\Gamma\backslash G/K$ to complete the proof. ☐

I.4. Quantum spectrum

In this section we analyze the quantum spectrum of the locally symmetric space $\Gamma\backslash G/K$. Recall the definition of the joint eigenspace

$$E_\lambda = \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D)f \quad \forall\, D \in \mathbb{D}(G/K)\}$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For the definition of χ_λ see Section I.2.2. Since $D \in \mathbb{D}(G/K)$ is G-invariant, it descends to a differential operator ${}_\Gamma D$ on the locally symmetric space $\Gamma\backslash G/K$. Therefore, the left Γ-invariant functions of E_λ (denoted by ${}^\Gamma E_\lambda$) can be identified with joint eigenfunctions on $\Gamma\backslash G/K$ for each ${}_\Gamma D$:

$${}^\Gamma E_\lambda = \{f \in C^\infty(\Gamma\backslash G/K) \mid {}_\Gamma Df = \chi_\lambda(D)f \quad \forall\, D \in \mathbb{D}(G/K)\}.$$

This leads to the following definition.

Definition I.4.1. The *quantum spectrum* of $\Gamma\backslash G/K$ is defined as

$$\sigma_Q \coloneqq \sigma_Q(\Gamma\backslash G/K) \coloneqq \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid {}^\Gamma E_\lambda \neq 0\}.$$

Using the quantum-classical correspondence and the Weyl law from [DKV79] we can now prove Theorem I.1.1.

Proof of Theorem I.1.1. From [DKV79, Theorem 8.9] we have for each set $\Omega \subset \mathfrak{a}^*$ as in Theorem I.1.1

$$\sum_{\lambda\in\sigma_Q\cap i\mathfrak{a}^*, \operatorname{Im}\lambda\in t\Omega} \dim({}^\Gamma E_\lambda)|W\lambda|^{-1} = \operatorname{Vol}(\Gamma\backslash G/K)\,(2\pi)^{-d}\operatorname{Vol}(\operatorname{Ad}(K)\Omega)t^d + \mathcal{O}(t^{d-1}),$$

where $\operatorname{Vol}(\Gamma\backslash G/K)$ is the volume of the compact Riemannian manifold $\Gamma\backslash G/K$ with Riemannian structure induced by the Killing form and $\operatorname{Vol}(\operatorname{Ad}(K)\Omega)$ is the volume of the set $\operatorname{Ad}(K)\Omega \subseteq \operatorname{Ad}(K)\mathfrak{a}$ with respect to the Killing form restricted to $\operatorname{Ad}(K)\mathfrak{a} = \mathfrak{p}$. Replacing Ω by $\Omega \setminus \bigcup_{\alpha\in\Sigma^+}\alpha^\perp$ we deduce $\sum_{\lambda\in\sigma_Q\cap i\mathfrak{a}^*, \operatorname{Im}\lambda\in t\Omega\cap\bigcup\alpha^\perp}\dim({}^\Gamma E_\lambda) = \mathcal{O}(t^{d-1})$ since $\operatorname{Vol}(\operatorname{Ad}(K)\alpha^\perp) = 0$. Therefore,

$$\sum_{\lambda\in\sigma_Q\cap i\mathfrak{a}^*, \operatorname{Im}\lambda\in t\Omega} \dim({}^\Gamma E_\lambda) = |W|\operatorname{Vol}(\Gamma\backslash G/K)\,(2\pi)^{-d}\operatorname{Vol}(\operatorname{Ad}(K)\Omega)t^d + \mathcal{O}(t^{d-1})$$

since W acts freely on the Weyl chambers. To complete the proof we observe that $\sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}) \supseteq \sigma^0_{\mathrm{RT}}({}_\Gamma\mathbf{X})$ and $m(\lambda) \geq \dim(\operatorname{Res}^0_{{}_\Gamma\mathbf{X}}(\lambda)) = \dim({}^\Gamma E_{-\lambda-\rho})$ for $\lambda \in i\mathfrak{a}^*$. $\square$

As $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$ it is obvious that σ_Q is W-invariant. The following properties of σ_Q were derived by Duistermaat-Kolk-Varadarajan [DKV79]. We include the proof for the convenience of the reader.

Proposition I.4.2 (see [DKV79, Prop. 2.4, Prop. 3.4, Cor. 3.5])**.** *If $\lambda \in \sigma_Q$, then the corresponding spherical function ϕ_λ is positive semidefinite. Moreover, there is some $w \in W$ such that $w\lambda = -\overline{\lambda}$ and $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$. In particular, $\langle \operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = 0$ and $\|\operatorname{Re}\lambda\| \leq \|\rho\|$.*

Proof. Pick $u \in {}^\Gamma E_\lambda$, regarded as a right K-invariant element of $L^2(\Gamma\backslash G)$, normalized such that $\langle u, u\rangle_{L^2(\Gamma\backslash G)} = 1$. With the right regular representation R on $L^2(\Gamma\backslash G)$ define $\Phi(g) \coloneqq \langle R(g)u, u\rangle$. Being a matrix coefficient the function Φ is positive semidefinite. We will show that Φ is the elementary spherical function ϕ_λ. By right K-invariance of u and unitarity of R we get that Φ is K-biinvariant. $\Phi(1) = 1$ is obvious. Smoothness follows from the fact that u is smooth. Furthermore, $D\Phi(g) = \langle R(g)Du, u\rangle = \chi_\lambda(D)\Phi(g)$ by left invariance of D. We conclude that Φ is the elementary spherical function for χ_λ, i.e. $\Phi = \phi_\lambda$.

Since ϕ_λ is positive semidefinite we have $\phi_\lambda(g) = \overline{\phi_\lambda(g^{-1})}$ by definition of positive definiteness and $\overline{\phi_\lambda(g^{-1})} = \phi_{-\overline{\lambda}}(g)$ by the integral representation (see Section I.2.7). Therefore $\phi_\lambda = \phi_{-\overline{\lambda}}$ implying that $w\lambda = -\overline{\lambda}$ for some $w \in W$. It easily follows that

$$\langle \operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = \langle w\operatorname{Re}\lambda, w\operatorname{Im}\lambda\rangle = \langle -\operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = 0.$$

Moreover, ϕ_λ is bounded which holds iff $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$ (see [Hel84, Ch. IV Theorem 8.1]). Since $\{\mu \in \mathfrak{a}^* \mid \|\mu\| \leq \|\rho\|\}$ is convex and contains $W\rho$, the last assertion follows. □

Remark I.4.3. In the rank one case Proposition I.4.2 implies for $\lambda \in \sigma_Q$ that $\lambda \in \mathfrak{a}^*$ with $\|\lambda\| \leq \|\rho\|$ or $\lambda \in i\mathfrak{a}^*$. In this particular case, this can be obtained not only from Proposition I.4.2 but also from the positivity of the Laplacian on $\Gamma\backslash G/K$. In the higher rank setting the algebra $\mathbb{D}(G/K)$ contains more operators, more precisely it is a polynomial algebra in n variables. Using the properties of the Harish-Chandra isomorphism HC one can obtain that $-\overline{\lambda} \in W\lambda$ from the self/skew-adjointness of the operators in $\mathbb{D}(G/K)$.

Remark I.4.4. Proposition I.4.2 implies the following obstructions for $\lambda \in \mathfrak{a}^*_\mathbb{C}$ to be in σ_Q.

(i) If $\operatorname{Re}\lambda = 0$, then we get no obstructions on $\operatorname{Im}\lambda$ since $w\lambda = -\overline{\lambda}$ is satisfied with $w = 1$.

(ii) If $\operatorname{Re}\lambda \neq 0$, then $\operatorname{Im}\lambda$ is singular, i.e. $\operatorname{Im}\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$, since $\operatorname{Im}\lambda$ non-singular implies $w = 1$ as W acts simply transitively on open Weyl chambers.

(iii) If $\operatorname{Re}\lambda$ is regular, i.e. $\langle \operatorname{Re}\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Sigma$, we denote by $\tilde{w}_0$ the unique Weyl group element mapping the Weyl chamber containing $\operatorname{Re}\lambda$ to its negative. Then we have $\lambda \in \operatorname{Eig}_{-1}(\tilde{w}_0) + i\operatorname{Eig}_{+1}(\tilde{w}_0) \subseteq \mathfrak{a}^*_\mathbb{C}$ where $\operatorname{Eig}_{\pm 1}$ denotes the eigenspace for ± 1. If -1 is contained in W, then $\operatorname{Im}\lambda = 0$. In particular, this is true in the rank one case but need not hold in general as is seen below.

Let us calculate $\dim \operatorname{Eig}_{+1}(w_0) = \dim \operatorname{Eig}_{+1}(\tilde{w}_0)$ in order to control the amount of freedom for $\operatorname{Im}\lambda$. Let $d_\pm := \dim \operatorname{Eig}_{\pm 1}(w_0)$. Then $n = d_+ + d_-$ and $\operatorname{Tr}(w_0) = d_+ - d_-$. Choosing the basis Π we observe $\operatorname{Tr}(w_0) = -\#\{\alpha \in \Pi \mid w_0\alpha = -\alpha\} \leq 0$. Thus, $d_\pm = \frac{1}{2}(n \pm \operatorname{Tr}(w_0))$ so that $d_+ \leq \frac{n}{2}$. We obtain the following traces and dimensions for the irreducible root systems from the classification.

Type	A_n, n even	A_n, n odd	B_n	C_n	D_n, n even	D_n, n odd	E_6	E_7	E_8	F_4	G_2
$-\operatorname{Tr}(w_0)$	0	1	n	n	n	$n-2$	2	7	8	4	2
d_+	$n/2$	$(n-1)/2$	0	0	0	1	2	0	0	0	0

Example I.4.5. For $G = SL_n(\mathbb{R})$ an element $\lambda \in \mathfrak{a}^* \simeq \mathfrak{a}$ is regular iff the diagonal entries are pairwise distinct. For $\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \sigma_Q$ with $\operatorname{Re}\lambda \in \overline{\mathfrak{a}^*_+}$ satisfies $\operatorname{Re}\lambda_k = -\operatorname{Re}\lambda_{n+1-k}$ and $\operatorname{Im}\lambda_k = \operatorname{Im}\lambda_{n+1-k}$ for all k since the longest Weyl group element is the permutation $(1 \leftrightarrow n)(2 \leftrightarrow n-1)\cdots$.

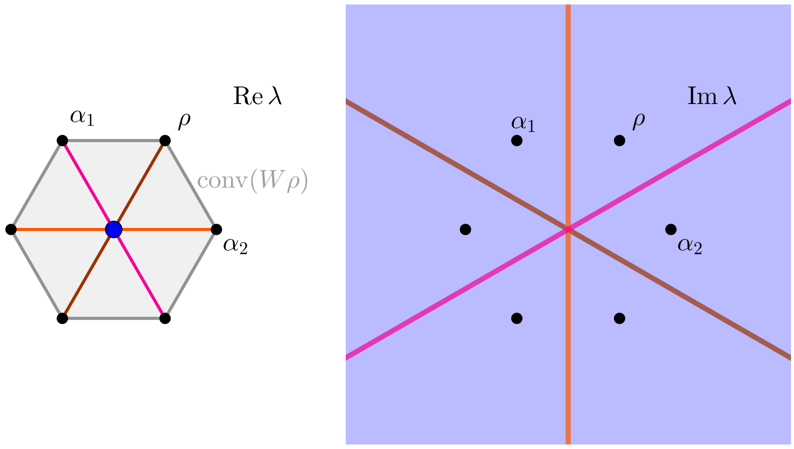

Figure I.4.: Situation for $SL_3(\mathbb{R})$ as obtained from Remark I.4.4: If $\lambda \in \sigma_Q$ then $\operatorname{Re}\lambda$ is either equal to zero (blue dot in the left picture) or lies on one of the purple, orange or brown lines depicted on the left. Furthermore $\operatorname{Im}\lambda$ has to lie in the respective region depicted on the right, i.e. if $\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda$ can take any value (blue shaded plane), if $\operatorname{Re}\lambda$ lies on the orange line, then $\operatorname{Im}\lambda$ has to lie on the orange line and so on.

More specifically, for $G = SL_3(\mathbb{R})$ the only Weyl group elements with eigenvalue -1 are the reflections at hyperplanes perpendicular to the roots. Hence, $\lambda \in \sigma_Q$ implies $\operatorname{Re}\lambda \in [-1,1]\alpha$ and $\operatorname{Im}\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$ or $\lambda \in i\mathfrak{a}^*$. The obstructions on λ to be in σ_Q described by Remark I.4.4 are less concrete and are visualized in Figure I.4.

Let us formulate the condition that ϕ_λ is positive semidefinite in a different way.

Proposition I.4.6. *ϕ_λ is positive semidefinite if and only if the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible for some $w \in W$. Equivalently, $H^{-w\overline{\lambda}}$ has a unitarizable irreducible spherical quotient.*

Proof. By Casselman's embedding theorem π_{ϕ_λ} is a subrepresentation of $H^{\tau,\nu}$ for some $\tau \in \widehat{M}$ and $\nu \in \mathfrak{a}^*_\mathbb{C}$ (see e.g. [Kna86, Theorem 8.37]). More precisely, the $(\mathfrak{g}, K)$-module of K-finite vectors are equivalent. Since the only principal series representations containing K-invariant vectors are the spherical ones, we obtain $\tau = 1$. Since infinitesimally equivalent admissible representations of G have the same set of K-finite matrix coefficients (see [Kna86, Corollary 8.8]), we conclude $\phi_\lambda = \phi_\nu$, i.e. $w\lambda = \nu$.

Conversely assume that the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible. Again by [Kna86, Corollary 8.8] the matrix coefficient $\phi_{w\lambda} = \phi_\lambda$ of $H^{w\lambda}$ is a matrix coefficient of the unitary representation obtained by the unitary structure as well. Hence, ϕ_λ is positive semidefinite. Transition to the dual representation implies the second equivalence. □

Remark I.4.7. Although the unitary dual is classified for many groups, it is difficult to deduce which elementary spherical functions are positive semidefinite. This is due to the fact that most classifications are not obtained in terms of quotients of the spherical principal series but use different descriptions of admissible representations. However, for rank one groups everything is classified (see [Hel84, p.484]): If α denotes the unique reduced root in Σ^+, then ϕ_λ is positive semidefinite iff $\lambda \in i\mathfrak{a}^*$ or $\lambda \in \mathfrak{a}^*$ and $|\langle \lambda, \alpha \rangle| \leq \langle \rho, \alpha \rangle$ for $2\alpha \notin \Sigma$ (i.e. in the real hyperbolic case) and $|\langle \lambda, \alpha \rangle| \leq (m_\alpha/2 + 1)\langle \alpha, \alpha \rangle$ for $2\alpha \in \Sigma$ or $\lambda = \pm\rho$.

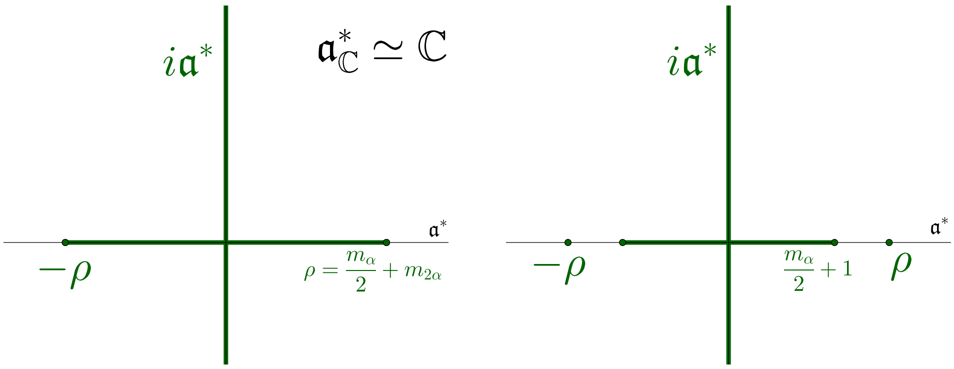

Figure I.5.: Spherical dual in the rank one case. The picture on the left describes the real and complex hyperbolic case $m_{2\alpha} \leq 1$. The picture on the right describes the quaternionic case $m_{2\alpha} \geq 2$. In the latter case note that there is a spectral gap separating ρ.

I.4.1. Property (T)

In this section we review some facts about Kazhdan's Property (T) which will lead to a more precise description of the location of σ_Q. Recall that a locally compact group has *Property (T)* iff the trivial representation is an isolated point in the unitary dual of the group with respect to the Fell topology (see [BdlHV08] for a general reference). It is well known that each real simple Lie group of real rank ≥ 2 has Property (T) (see [BdlHV08, Theorem 1.6.1]). Since the mapping $\lambda \mapsto \phi_\lambda$ is continuous and the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations is a homeomorphism (see Section I.2.7), we obtain that in some neighbourhood of ρ no elementary spherical function is positive semidefinite. We will use a more quantitative description introduced by Oh [Oh02, Section 7.1]. Therefore, we denote by $p_K(G)$ the smallest real number such that the K-finite matrix coefficients of π are in $L^q(G)$ for any $q > p_K(G)$ and nontrivial $\pi \in \widehat{G}$.

Remark I.4.8. (i) Since each matrix coefficient of $\pi \in \widehat{G}$ is bounded, it is contained in L^q for each $q > p$ if it is in L^p. Hence,

$$p_K(G) = \inf\{p \mid \text{all } K\text{-finite matrix coefficients of } \pi \text{ are in } L^p(G) \quad \forall \pi \in \widehat{G} \setminus \{1\}\}.$$

(ii) $p_K(G) \geq 2$.

(iii) By [Cow79] together with [Oh02] we have $p_K(G) < \infty$ iff G has Property (T).

In many examples one knows the number $p_K(G)$ explicitly or at least upper bounds.

Example I.4.9 (see [Oh02, Section 7]). (i) $p_K(SL_n(k)) = 2(n-1)$ for $n \geq 3$ and $k = \mathbb{R}, \mathbb{C}$.

(ii) $p_K(Sp_{2n}(\mathbb{R})) = 2n$ for $n \geq 2$.

(iii) $p_K(G)$ is bounded above by an explicit value for split classical groups of higher rank.

We can now prove the following theorems.

Theorem I.4.10. *Let G be a non-compact real semisimple Lie group with finite center and $\Gamma \leq G$ a discrete, cocompact, torsion-free subgroup. Then*

$$\operatorname{Re}\sigma_Q(\Gamma\backslash G/K) \subseteq (1 - 2p_K(G)^{-1})\operatorname{conv}(W\rho) \cup W\rho.$$

Proof. Let $\lambda \in \sigma_Q(\Gamma\backslash G/K)$. By Proposition I.4.2 ϕ_λ is positive semidefinite so that the irreducible unitary representation π_{ϕ_λ} is defined (see Section I.2.7). ϕ_λ is a matrix coefficient of this representation. By the definition of $p_K(G)$ we have $\phi_\lambda \in L^{p_K(G)+\epsilon}(G)$ for all $\epsilon > 0$ or π_{ϕ_λ} is the trivial representation. By Proposition I.2.4 we get $\operatorname{Re}\lambda \in (1 - 2p_K(G)^{-1})\operatorname{conv}(W\rho)$ in the first case. The latter case occurs iff $\phi_\lambda \equiv 1$, i.e. $\lambda \in W\rho$. □

Theorem I.4.11. *Let G be a non-compact real semisimple Lie group with finite center and $\Gamma \leq G$ a discrete, cocompact, torsion-free subgroup. Then there is a neighborhood $\mathcal{G}$ of ρ in $\mathfrak{a}^*$ such that*

$$\sigma_Q(\Gamma\backslash G/K) \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{\rho\}.$$

Proof. Without loss of generality we assume that G has trivial center, otherwise replace G by $G/Z(G)$. Then G is a product of simple Lie groups $G_1, \ldots, G_l$ such that $G_1, \ldots, G_k$, $k \leq l$, are of rank one. With the obvious notation let $\lambda = (\lambda_1, \ldots, \lambda_l) \in (\mathfrak{a}_1)^*_{\mathbb{C}} \oplus \cdots \oplus (\mathfrak{a}_l)^*_{\mathbb{C}}$ be in σ_Q. By Proposition I.4.2 we have $w\lambda = -\overline{\lambda}$ for some $w \in W$. Since the Weyl group W is the product of the Weyl groups $\lambda_i \in \mathfrak{a}_i^*$ are real for $i \leq k$ if $\operatorname{Re}\lambda_i \neq 0$. The elementary spherical function ϕ_λ is the product of elementary spherical functions $\phi_{\lambda_i}^{G_i}$ for the factors G_i. Again by Proposition I.4.2 we know that ϕ_λ is positive semidefinite and therefore each $\phi_{\lambda_i}^{G_i}$ is positive semidefinite. The same line of arguments as in the proof of Theorem I.4.10 imply that $\operatorname{Re}\lambda_i \in (1 - 2p_K(G_i)^{-1})\operatorname{conv}(W_i\rho_i) \cup W_i\rho_i$ for $i > k$. Since G_i, $i > k$, have Property (T) we conclude that there is a neighborhood U of ρ in $\mathfrak{a}^*$ such that

$$\sigma_Q \cap (U \times i\mathfrak{a}^*) \subseteq \mathfrak{a}_1^* \times \cdots \times \mathfrak{a}_k^* \times \{\rho_{k+1}\} \times \cdots \times \{\rho_l\}.$$

Discreteness of σ_Q implies the theorem. □

I.5. Main Theorem

In this section we present the main theorem of the article and deduce Theorem I.1.2 from it.

Theorem I.5.1. *Let G be a non-compact real semisimple Lie group with finite center and $\Gamma \leq G$ a discrete, cocompact, torsion-free subgroup. Define $\mathcal{A} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \frac{2\langle\lambda+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in -\mathbb{N}_{>0}$ for some $\alpha \in \Sigma^+\}$, $\mathcal{B} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid w\lambda = -\overline{\lambda}$ for some $w \in W\}$, and $\mathcal{F} := \{\lambda \in \mathfrak{a}^* \mid \lambda + \alpha \notin \overline{{}_-\mathfrak{a}^*}$ for all $\alpha \in \Pi\}$. Then we have the following inclusions*

$$\sigma_{\mathrm{RT}}({}_\Gamma\mathbf{X}) \cap (\mathcal{F} \times i\mathfrak{a}^*) \subseteq \sigma_{\mathrm{RT}}^0({}_\Gamma\mathbf{X})$$

and

$$\begin{aligned}\sigma_{\mathrm{RT}}^0({}_\Gamma\mathbf{X}) \cap (\mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{A}) &\subseteq -\sigma_Q(\Gamma\backslash G/K) - \rho \\ &\subseteq \mathcal{B} \cap (((1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho) \cup W\rho) + i\mathfrak{a}^*) - \rho.\end{aligned}$$

Proof. This is immediate from Propositions I.3.7, I.3.11, and I.4.2 and Theorem I.4.10. □

Proof of Theorem I.1.2. It follows from Theorem I.5.1 that the neighborhood can be chosen as $(\mathfrak{a}_+^* - \rho) \cap \mathcal{F} \cap (-\mathcal{G} - \rho)$ where $\mathcal{G}$ is obtained by Theorem I.4.11. If G has Property (T), then $p_K(G)$ is finite and $\mathcal{G}$ can be replaced by the complement of the Γ-independent set $(1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho)$. □

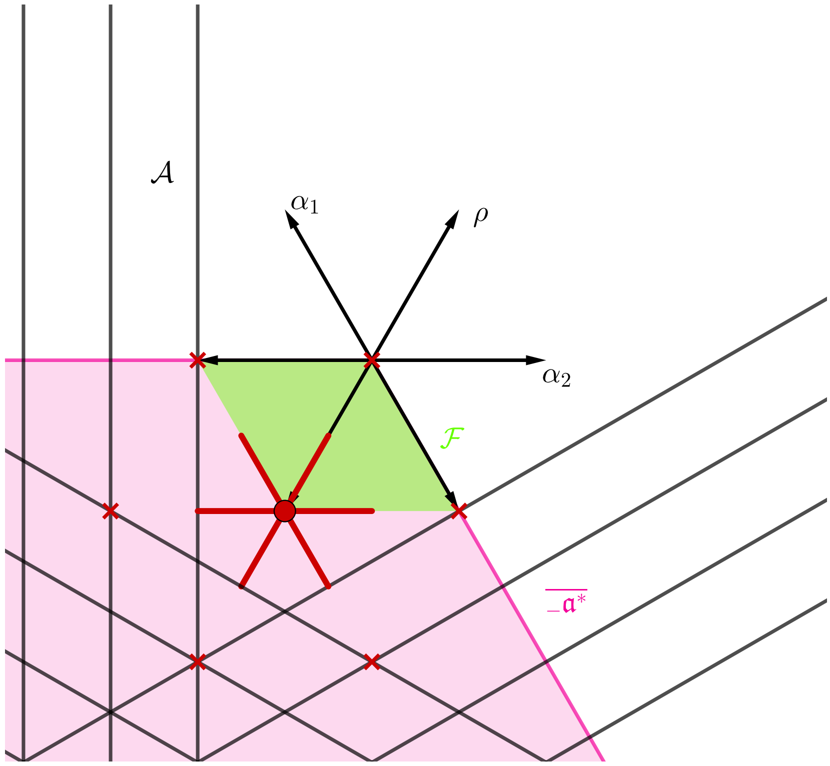

Figure I.6.: Visualization of the real part of $\mathfrak{a}^*_{\mathbb{C}}$ for $G = SL_3(\mathbb{R})$: The pink region is the region where Ruelle-Taylor resonances can a priori be located in view of the results of [BGHW20]. The red points and lines depict the region $(\mathcal{B} \cap \frac{1}{2}\operatorname{conv}(W\rho) \cup W\rho) - \rho$, i.e. the region where first band resonances can occur. The green shaded region illustrates the real parts in which only first band resonances can occur. Further first band resonances might occur inside the exceptional set $\mathcal{A}$ depicted by the black lines.

Supplementary Material

I.6. Alternative proof of Proposition I.4.2

In this section we give an alternative proof of Proposition I.4.2 that does not use the abstract theory of spherical functions and representations. It is inspired by the rank one case where the positivity of the Laplacian gives an exact characterization for the location of the spectrum. Let us recall that fact: If $\lambda \in \sigma_Q$ then there exists $f \in C^\infty(\Gamma\backslash G/K)$ such that ${}_\Gamma Df = \chi_\lambda(D)f$ for all $D \in \mathbb{D}(G/K)$. In particular, $\Delta \in \mathbb{D}(G/K)$ since the action of G on G/K is by isometries. The Laplace operator is a positive self-adjoint operator on $L^2(\Gamma\backslash G/K)$ and therefore it has non-negative eigenvalues. As $C^\infty(\Gamma\backslash G/K) \subseteq L^2(\Gamma\backslash G/K)$ for cocompact Γ we find that $\chi_\lambda(\Delta) \geq 0$. By [Hel84, Ch. II Cor. 5.20] we have

$$\chi_\lambda(\Delta) = -\|\operatorname{Re}\lambda\|^2 + \|\operatorname{Im}\lambda\|^2 + \|\rho\|^2 - 2i\langle \operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle.$$

In the rank one case where $\dim_\mathbb{R} \mathfrak{a} = 1$ the fact that $\chi_\lambda(\Delta)$ is real implies $\operatorname{Re}\lambda = 0$ or $\operatorname{Im}\lambda = 0$. Additionally, the positivity implies $\|\operatorname{Re}\lambda\| \leq \rho$ where equality is attained for $\lambda = \pm\rho$ and the eigenfunctions are the constant functions.

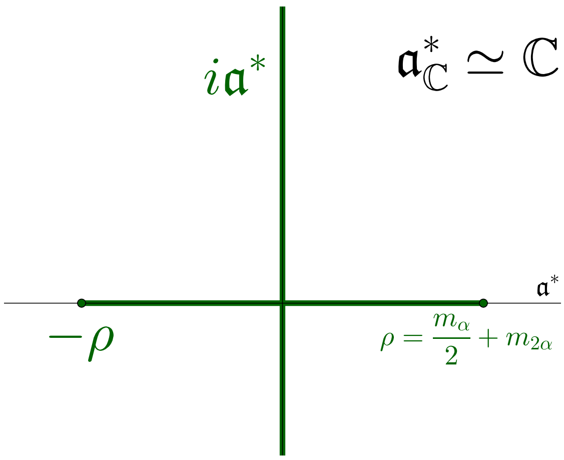

Figure I.7.: σ_Q in the rank one case

In order to prove Proposition I.4.2 and in particular that $-\overline{\lambda} \in W\lambda$ for $\lambda \in \sigma_Q$ we need the following lemma.

Lemma I.6.1 (see [Hel84, Lemma III-3.11])**.** *The algebra of Weyl group invariant polynomials $Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$ separates the points of $\mathfrak{a}_{\mathbb{C}}^*/W$, i.e. for $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}^*$*

$$p(\lambda) = p(\mu) \quad \forall\, p \in Poly(\mathfrak{a}_{\mathbb{C}}^*)^W \iff \exists w \in W\colon w\lambda = \mu.$$

Proof. Pick λ and μ in $\mathfrak{a}_{\mathbb{C}}^*$ such that $p(\lambda) = p(\mu)$ for all $p \in Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$ and assume $\lambda \notin W \cdot \mu$. Since W is finite we can pick $0 \leq \tilde{f} \in C(\mathfrak{a}_{\mathbb{C}}^*)$ such that $\tilde{f}(\lambda) = 1$ and $\tilde{f}(w\mu) = 0$ for all $w \in W$. Define $f = |W|^{-1} \sum_{w \in W} w \cdot \tilde{f}$. Then $f \in C(\mathfrak{a}_{\mathbb{C}}^*)$ is W-invariant with $f(\lambda) \geq 1$ and $f(\mu) = 0$. We use the Weierstrass approximation theorem to uniformly approximate f by polynomials $p_n \in Poly(\mathfrak{a}_{\mathbb{C}}^*)$ on some compact set containing $W \cdot \lambda$ and $W \cdot \mu$. By construction

$$f(\lambda) = |W|^{-1} \sum_{w \in W} f(w\lambda) = \lim_n |W|^{-1} \sum_{w \in W} p_n(w\lambda) = \lim_n |W|^{-1} \sum_{w \in W} (w \cdot p_n)(\lambda).$$

Due to the fact that $|W|^{-1} \sum_{w \in W} w \cdot p_n$ is W-invariant we infer that $f(\lambda) = f(\mu)$ contradicting $1 \leq f(\lambda) = f(\mu) = 0$. □

Let us first prove the special case where $-1 \in W$. This is the case if the root system is of type $B_n, C_n, D_n (n \text{ even}), E_7, E_8, F_4$, or G_2.

Proposition I.6.2. *Assume $-1 \in W$. Then for $\lambda \in \sigma_Q$ there is $w \in W$ such that $w\lambda = -\overline{\lambda}$. If $\operatorname{Re}\lambda$ is regular then we have $\operatorname{Im}\lambda = 0$.*

Proof. Let $p_1, \dots, p_n$ be algebraically independent homogeneous generators of $S(\mathfrak{a})^W$ with real coefficients if p_i is represented by a basis of $\mathfrak{a}$. Assume that $\deg(p_i) \leq \deg(p_j)$ for $i \leq j$ and define $d_i := \deg(p_i)$. Since $-1 \in W$ we have $p_i(\lambda) = p_i(-\lambda) = (-1)^{d_i} p_i(\lambda)$ and therefore d_i is even for all i. Let Op be the composition of the extension map $S(\mathfrak{a})^W \to S(\mathfrak{p})^K \subseteq S(\mathfrak{g})^K$ and the symmetrization $\lambda\colon S(\mathfrak{g})^K \to \mathcal{U}(\mathfrak{g})^K$ as defined in Section I.2.4. By construction we have $\mathrm{HC}(\mathrm{Op}(p_i)) =: \tilde{p}_i = p_i +$ lower order terms where by construction the lower order terms have real coefficients as well (see Section I.2.4). We also have $\mathrm{Op}(p_i)^* = (-1)^{d_i} \mathrm{Op}(p_i)$ denoting the L^2-adjoint by *. Indeed, to obtain $\mathrm{Op}(p_i)^*$ we have to take $X \in \mathcal{U}(\mathfrak{g})^K$ representing $\mathrm{Op}(p_i)$ and take its adjoint as an operator on G and let it act on G/K. But every element in $\mathfrak{g}$ is skew-adjoint acting on $L^2(G)$ so that we have to reverse the order of X and multiply by $(-1)^{d_i}$. The construction of Op includes summing over all permutations and hence $\mathrm{Op}(p_i)^* = (-1)^{d_i} \mathrm{Op}(p_i)$ follows. We already observed that d_i is even for every i. Therefore $\mathrm{Op}(p_i)$ is symmetric.

Now we can use the argument as in the rank one case. For $\lambda \in \sigma_Q$ there is $f \in C^\infty(\Gamma\backslash G/K)$ such that $Df = \chi_\lambda(D) f$ for every $D \in \mathbb{D}(G/K)$. In particular, f is an eigenfunction for $\mathrm{Op}(p_i)$ in $L^2(\Gamma\backslash G/K)$. Since $\mathrm{Op}(p_i)$ is symmetric we must have $\chi_\lambda(\mathrm{Op}(p_i)) = \tilde{p}_i(\lambda) \in \mathbb{R}$. Hence, $\overline{\tilde{p}_i(\lambda)} = \tilde{p}_i(\lambda)$. The former equals $\tilde{p}_i(\overline{\lambda})$ since $\tilde{p}_i$ has real coefficients. Since the p_i and hence also $\tilde{p}_i$ generate $Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$ we deduce $p(\lambda) = p(\overline{\lambda})$ for all $p \in Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$. By Lemma I.6.1 there is $w \in W$ such that $w\lambda = \overline{\lambda}$. Note that in the present setting where $-1 \in W$ this is equivalent to saying $w\lambda = -\overline{\lambda}$ for some w which is the general statement we obtained.

If in addition $\operatorname{Re}\lambda$ is regular then $w = 1$ as the Weyl group acts freely on the open Weyl chambers. In this case we clearly have $\operatorname{Im}\lambda = 0$ since $w \operatorname{Im}\lambda = -\operatorname{Im}\lambda$. □

In the previous proposition we were able to use the assumption that $-1 \in W$ to conclude that all operators considered are symmetric. For the general case we need the following lemma to deal with the non-symmetric operators.

Lemma I.6.3 (see [Hel84, II-Lemma 5.21])**.** *If we identify* $Poly(\mathfrak{a}_{\mathbb{C}}^*)^W \simeq S(\mathfrak{a})^W$ *with the set of invariant differential operators* $\mathbb{D}(A)$ *on* A*, then* HC *is a* $*$*-homomorphism, i.e.*

$$\mathrm{HC}(D^*) = \mathrm{HC}(D)^* \quad \forall\, D \in \mathbb{D}(G/K),$$

where $\cdot^*$ *denotes the adjoint with respect to the corresponding invariant measures.*

Proof. Let D be represented by $X \in \mathcal{U}(\mathfrak{g})^K$ and $f \in C_c^\infty(G/K)$ be real valued and left K-invariant. We consider the integral transform

$$F_f(g) = e^{\rho(H(g^{-1}))} \int_N f(ngK)\, dn.$$

By observing that the map $\eta \colon \mathcal{U}(\mathfrak{a}) \to \mathcal{U}(\mathfrak{a}), X \mapsto X + \rho(X)$ corresponds to the automorphism $D \mapsto e^{-\rho} D e^{\rho}$ of $\mathbb{D}(A)$ it follows from the definitions that

$$\mathrm{HC}(D)F_f(a) = F_{Df}(a) \tag{I.8}$$

as $e^{\rho(H(\,\cdot^{-1}))}F_f$ is left N-invariant and right K-invariant.

Recall that we defined the elementary spherical function

$$\phi_\lambda(g) = \int_K e^{(-\lambda-\rho)H(g^{-1}k)}\, dk$$

which satisfies $D\phi_\lambda = \chi_\lambda(D)\phi_\lambda$. Using this we calculate

$$\begin{aligned}
\mathrm{HC}(D^*)(\lambda) \int_G \phi_\lambda(g) f(gK)\, dg &= \int_G (D^*\phi_\lambda)(g) f(gK)\, dg = \int_G \phi_\lambda(g)(Df)(gK)\, dg \\
&= \int_G \int_K e^{(-\lambda-\rho)H(g^{-1}k)}\, dk (Df)(gK)\, dg \\
&= \int_G e^{(-\lambda-\rho)H(g^{-1})} \int_K (Df)(kgK)\, dk\, dg \\
&= \int_G e^{(-\lambda-\rho)H(g^{-1})} (Df)(gK)\, dg.
\end{aligned}$$

The integral formula for the Iwasawa decomposition (see [Hel84, I-Prop. 5.1]) allows us to reduce to an integral over A:

$$\int_G e^{(-\lambda-\rho)H(g^{-1})}(Df)(gK)\,dg = \int_N\int_A\int_K e^{(-\lambda-\rho)H(a^{-1})}(Df)(naK)e^{-2\rho(\log a)}\,dk\,da\,dn$$
$$= \int_A e^{\lambda\log a}F_{Df}(a)\,da.$$

By (I.8) this equals

$$\int_A e^{\lambda\log a}\,\mathrm{HC}(D)F_f(a)\,da = \int_A (\mathrm{HC}(D)^* e^{\lambda\log a})F_f(a)\,da$$
$$= \mathrm{HC}(D)^*(\lambda)\int_A e^{\lambda\log a}F_f(a)\,da.$$

In the last line $\mathrm{HC}(D)^* \in \mathbb{D}(A)$ is again seen as a polynomial on $\mathfrak{a}^*_{\mathbb{C}}$. Now the same calculation as above (with $D = 1$) shows

$$\int_A e^{\lambda\log a}F_f(a)\,da = \int_G \phi_\lambda(g)f(gK)\,dg.$$

This shows that $\mathrm{HC}(D^*) = \mathrm{HC}(D)^*$. □

Remark I.6.4. Let us explain the meaning of $\mathrm{HC}(D)^*$ as a polynomial on $\mathfrak{a}^*_{\mathbb{C}}$. To do so let $p = aH_1\cdots H_n \in \mathcal{U}(a) = S(\mathfrak{a}) \simeq \mathbb{D}(A)$ be a monomial where $a \in \mathbb{C}$ and $H_i \in \mathfrak{a}$. Since the H_i are skew symmetric, $(aH_1\cdots H_n)^* = (-H_n)\cdots(-H_1)\overline{a} = \overline{a}(-1)^n H_1\cdots H_n$. Evaluating this polynomial at λ we get

$$p^*(\lambda) = \overline{a}(-\lambda(H_1))\cdots(-\lambda(H_n)) = \overline{a(-\overline{\lambda}(H_1))\cdots(-\overline{\lambda}(H_n))} = \overline{p(-\overline{\lambda})}.$$

In particular, $\chi_\lambda(D^*) = \mathrm{HC}(D^*)(\lambda) = \mathrm{HC}(D)^*(\lambda) = \overline{\mathrm{HC}(D)(-\overline{\lambda})} = \overline{\chi_{-\overline{\lambda}}(D)}$.

Proposition I.6.5. *Let $p \in S(\mathfrak{a})^W$ be homogeneous with real coefficients if p is represented by a basis of $\mathfrak{a}$. Let $d := \deg(p)$ and Op as defined in Section I.2.4. Then*

$$\chi_{-\lambda}(\mathrm{Op}(p)) = (-1)^d\chi_\lambda(\mathrm{Op}(p)).$$

In particular, $\mathrm{HC}(\mathrm{Op}(p)) = p +$ lower order terms where the degree of the lower order terms have the same parity as d.

Proof. By construction we have $\mathrm{HC}(\mathrm{Op}(p)) =: \tilde{p} = p +$ lower order terms where by construction the lower order terms have real coefficients as well (see Section I.2.4). As in the proof of Proposition I.6.2 we also have $\mathrm{Op}(p)^* = (-1)^d\,\mathrm{Op}(p)$ denoting the adjoint by *. Therefore $\mathrm{Op}(p_i)$ is symmetric if d is even and skew-symmetric if d is odd.

Applying Lemma I.6.3 to this relation gives

$$\tilde{p}(-\lambda) = \overline{\tilde{p}(-\overline{\lambda})} = \overline{\chi_{-\overline{\lambda}}(\mathrm{Op}(p))} = \chi_\lambda(\mathrm{Op}(p)^*) = \chi_\lambda((-1)^d\,\mathrm{Op}(p)) = (-1)^d\tilde{p}(\lambda). \quad □$$

With these preliminaries we can now prove the general version of Proposition I.6.2. It follows the same line of arguments except that not all operators are symmetric. To deal with this we use Proposition I.6.5.

Proposition I.6.6. *Let $\lambda \in \sigma_Q$. Then there is $w \in W$ such that $w\lambda = -\overline{\lambda}$.*

Proof. Let $p_1, \dots, p_n$ be algebraically independent homogeneous generators of $S(\mathfrak{a})^W$ with real coefficients if p_i is represented by a basis of $\mathfrak{a}$. Assume that $\deg(p_i) \leq \deg(p_j)$ for $i \leq j$ and define $d_i := \deg(p_i)$. Applying Proposition I.6.5 yields $\chi_{-\lambda}(\mathrm{Op}(p_i)) = (-1)^{d_i}\chi_\lambda(\mathrm{Op}(p_i))$.

Now we can use the same argument as in Proposition I.6.2. For $\lambda \in \sigma_Q$ there is $f \in C^\infty(\Gamma\backslash G/K)$ such that $Df = \chi_\lambda(D)f$ for every $D \in \mathbb{D}(G/K)$. In particular, f is an eigenfunction for $\mathrm{Op}(p_i)$ in $L^2(\Gamma\backslash G/K)$. Since $\mathrm{Op}(p_i)$ is symmetric (resp. skew-symmetric) we must have $\chi_\lambda(\mathrm{Op}(p_i)) =: \tilde{p}_i(\lambda) \in i^{d_i}\mathbb{R}$. Hence, $\tilde{p}_i(\lambda) = (-1)^{d_i}\overline{\tilde{p}_i(\lambda)} = \tilde{p}_i(-\overline{\lambda})$. Since the p_i and hence also $\tilde{p}_i$ generate $Poly(\mathfrak{a}_\mathbb{C}^*)^W$ we deduce $p(\lambda) = p(-\overline{\lambda})$ for all $p \in Poly(\mathfrak{a}_\mathbb{C}^*)^W$. By Lemma I.6.1 there is $w \in W$ such that $w\lambda = -\overline{\lambda}$. □

I.7. Alternative proof of Theorem I.4.11

In this section we will give an alternative proof of Theorem I.4.11. This proof will not give an explicit description of the spectral gap as in Theorem I.4.10 since it merely uses the definition of Kazdhan's Property (T) instead of the L^p-bounds for the matrix coefficients. Let us begin with a review of the Fell topology.

Definition I.7.1 (see [BdlHV08, Definition F.2.1 and Proposition F.2.4])**.** The *Fell topology* on the unitary dual $\widehat{G}$ of G is given as follows: A basis for the family of neighbourhoods of $\pi \in \widehat{G}$ is given by the sets $W(\pi, v_1, \dots, v_n, Q, \varepsilon)$ for $v_i \in \mathcal{H}_\pi$, $\|v_i\| = 1$, $Q \subseteq G$ compact, and $\varepsilon > 0$ where

$$W(\pi, v_1, \dots, v_n, Q, \varepsilon) := \{\sigma \in \widehat{G} \mid \exists w_i, \|w_i\| = 1\colon |\langle \pi(x)v_i, v_i\rangle - \langle \sigma(x)w_i, w_i\rangle| \leq \varepsilon \forall x \in Q\}.$$

Note that in [BdlHV08] they do not work with unit vectors. However, a short calculation shows that the topologies coincide.

Example I.7.2. (i) If $G = A$ is abelian then $\widehat{A}$ are the characters $\chi\colon A \to S^1$ and the matrix coefficients are the characters as well. As one easily sees, the Fell topology is then the topology of uniform convergence on compact sets. More specifically, for $A = \mathbb{R}$ the map $\mathbb{R} \to \widehat{\mathbb{R}}, \xi \to e^{i\xi\cdot}$ is a homeomorphsim.

(ii) If $G = K$ is compact then $\widehat{K}$ is discrete. Indeed, if $\pi \in \widehat{K}$ and $(\pi_j)_{j\in J}$ is a net converging to π and ϕ_j (resp. ϕ) are diagonal matrix coefficients of π_j (resp. π) then $\phi_j \to \phi$ uniform on K. This implies that $\int_K \phi(x)\overline{\phi_j(x)}\, dx \to \int_K |\phi(x)|^2\, dx \neq 0$. Therefore, $\int_K \phi(x)\overline{\phi_j(x)}\, dx \neq 0$ for all $j \geq j_0$ for some $j_0 \in J$. By Schur orthogonality we infer $\pi \simeq \pi_j$, $j \geq j_0$, so that $\widehat{K}$ is discrete.

In many examples where the unitary dual is parametrized by some topological space the Fell topology turns this parametrization into a homeomorphism. For example if G is a compact linear connected semisimple Lie group, by the theorem of the highest weight the unitary dual is parametrized by dominant analytically integral functionals λ on $\mathfrak{h}_\mathbb{C}$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (see [Kna86, Thm. 4.28]). This set is a lattice in $\mathfrak{h}_\mathbb{C}^*$, in particular it is discrete. Hence by the above example the parametrization is a homeomorphism.

A second example where this principle holds is the following. If G is a nilpotent connected simply connected Lie group, then the orbit method by Kirillov [Kir62] yields a bijection between $\widehat{G}$ and the set of coadjoint orbits. It turns out that this bijection turns into a homeomorphism if we endow $\widehat{G}$ with the Fell topology and $\mathfrak{g}^*/G$ with the quotient topology.

Let us prove a similar result for $\widehat{G}_{\mathrm{sph}}$ in our usual setting.

Proposition I.7.3. *If we use the induced topology on $\widehat{G}_{sph}$ (also called Fell topology) and the topology of uniform convergence on compact sets on the set $\mathfrak{P}$ of positive semidefinite spherical functions, then the correspondence $\pi_\phi \mapsto \phi$ (see Section I.2.7) is a homeomorphism.*

Proof. Let $\phi_n \to \phi \in \mathfrak{P}$ converge uniformly on compact sets. We have to show that $\pi_{\phi_n} \in W(\pi_\phi, v_1, \dots, v_m, Q, \varepsilon)$ for n large enough. By density it suffices to consider $v_i \in \operatorname{span} R(G)\phi$. Let $v_i = \sum_j \alpha_{ij} R(g_{ij})\phi$ and define $w_i^n := \sum \alpha_{ij} R(g_{ij})\phi_n$. Then we have $\langle \pi_\phi(g) v_i, v_i\rangle = \sum |\alpha_{ij}|^2 \phi(g_{ij}^{-1} g g_{ij})$ and $\langle \pi_{\phi_n}(g) w_i^n, w_i^n\rangle = \sum |\alpha_{ij}|^2 \phi_n(g_{ij}^{-1} g g_{ij})$. Since ϕ_n converges uniformly on the compact set $\bigcup_{i,j} g_{ij}^{-1} Q g_{ij}$ we get that

$$\pi_{\phi_n} \in W(\pi_\phi, v_1, \dots, v_m, Q, \varepsilon)$$

for almost every n.

For the opposite direction suppose $\pi_{\phi_n} \to \pi_\phi$ in the Fell topology. We have to show that $\phi_n \to \phi$ on an arbitrary compact set Q. Let v_K be a K-invariant unit vector for π_ϕ. By definition of the Fell topology there exists $w^n \in \mathcal{H}_{\phi_n}$ with norm 1 such that $|\langle \pi_\phi(x) v_K, v_K\rangle - \langle \pi_{\phi_n}(x) w^n, w^n\rangle| \leq \varepsilon$ for all $x \in Q \cup K$.

We define $\overline{w}^n := \int_k \pi_{\phi_n}(k) w^n dk$ and calculate

$$\|\pi_{\phi_n}(k) w^n - w^n\|^2 = 2(1 - \operatorname{Re}\langle \pi_{\phi_n}(x) w^n, w^n\rangle) \leq 2\varepsilon.$$

Hence $\|\overline{w}^n - w^n\| \leq \sqrt{2\varepsilon}$ and $\overline{w}^n \neq 0$ for $\varepsilon < \frac{1}{2}$. It follows that $\frac{\overline{w}^n}{\|\overline{w}^n\|}$ is a K-invariant unit vector for π_{ϕ_n} and therefore $\phi_n(x) = \langle \pi_{\phi_n}(x)\overline{w}^n, \overline{w}^n\rangle \, \|\overline{w}^n\|^2$. Now we can estimate for $x \in Q$:

$$\begin{aligned}
|\phi(x) - \phi_n(x)| &= |\langle \pi_\phi(x) v_K, v_K\rangle - \langle \pi_{\phi_n}(x)\overline{w}^n, \overline{w}^n\rangle \, \|\overline{w}^n\|^2 \,| \\
&\leq \varepsilon + |\langle \pi_{\phi_n}(x) w^n, w^n\rangle - \langle \pi_{\phi_n}(x)\overline{w}^n, \overline{w}^n\rangle \, \|\overline{w}^n\|^2 \,| \\
&\leq \varepsilon + \|\overline{w}^n\|^2 \, |1 - \|\overline{w}^n\| \,| + \|\overline{w}^n - w^n\| \, \|w^n\| + \|\overline{w}^n - w^n\| \, \|\overline{w}^n\| \,.
\end{aligned}$$

This completes the proof since $\|\overline{w}^n - w^n\| \leq \sqrt{2\varepsilon}$ and therefore $|1 - \|\overline{w}^n\|\,| \leq \sqrt{2\varepsilon}$. □

The following definition will be used to deduce a weaker form of Theorem I.4.10. We will only obtain a neighborhood of $W\rho$ where no quantum spectrum is present instead of explicit structure of the spectral gap.

Definition I.7.4 (see [BdlHV08, Theorem 1.2.5])**.** A locally compact group has *Kazhdan's Property (T)* if the trivial representation is an isolated point in the unitary dual of the group with respect to the Fell topology.

Example I.7.5. (i) Each compact group has Property (T) since $\widehat{G}$ is discrete.

(ii) $\mathbb{R}^n, \mathbb{Z}^n, SL_2(\mathbb{R}), SO(n,1), SU(n,1)$ do not have Property (T).

(iii) Every real connected simple Lie group with real rank ≥ 2 has Property (T) (see [BdlHV08, Thm. 1.6.1])

(iv) The rank one groups $Sp(n,1)$ and F_4 have Property (T).

Recall that by Proposition I.4.2 if $\lambda \in \sigma_Q(\Gamma\backslash G/K)$ then ϕ_λ is positive semidefinite. Hence, the representation $\pi_{\phi_\lambda} \in \widehat{G}_{\mathrm{sph}}$ is defined. Therefore if G has Property (T) then by Proposition I.7.3 and since $\lambda \mapsto \phi_\lambda$ is continuous there is a neighborhood U of ρ in $\mathfrak{a}_\mathbb{C}^*$ such that

$$\sigma_Q(\Gamma\backslash G/K) \cap U = \{\rho\}.$$

However, a priori the quantum spectrum could contain spectral parameters λ with real parts arbitrarily close to ρ. This is possible since the imaginary parts can be big such that $\lambda \notin U$.

In order to rule out this behavior we use the reduction to real infinitesimal character (see [Kna86, Thm. 16.10]). This requires us to identify the representation π_{ϕ_λ} as a quotient of the principal series representation. We will assume that G is a linear connected semisimple group for the rest of this section.

Theorem I.7.6 (Casselman embedding theorem, see [Kna86, Theorem 8.37])**.** *Let π be an irreducible unitary representation of G. Then π is infinitesimally equivalent to a subrepresentation of some nonunitary principal series representation $H^{\tau,\nu}, \tau \in \widehat{M}, \nu \in \mathfrak{a}_\mathbb{C}^*$. More precisely, if $\nu - \rho$ is a leading exponent of π, then π is infinitesimally equivalent to a subrepresentation of $H^{w_0\tau, w_0\nu}$ for some $\tau \in \widehat{M}$.*

Let us recall the definition of a leading exponent. Let π be an irreducible unitary representation and E_1, E_2 orthogonal projections onto two K-types U_1, U_2 of π. Then the spherical function $F(x) = E_1\pi(x)E_2$ has values in $\mathrm{Hom}(U_2, U_1)$, is an eigenfunction of the center of the universal enveloping algebra, and hence has an expansion of the form

$$F(\exp H) = \sum_\nu F_{\nu-\rho}(\exp H)$$

with

$$F_{\nu-\rho}(\exp H) = e^{-\rho(H)} \sum_{|q|\leq q_0} c_{\nu,q}\alpha_1(H)^{q_1}\cdots\alpha_n(H)^{q_n}e^{\nu(H)}.$$

Here, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is the simple system, $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, and q is a multiindex. $\nu - \rho$ is called an *exponent of* F if $F_{\nu-\rho} \neq 0$. $\nu - \rho$ is called *exponent of* π if $\nu - \rho$ is an exponent of F for some K-types U_1, U_2. An exponent $\nu - \rho$ is called *leading exponent* if the only exponent of the form $\nu - \rho + \sum_{\alpha\in\Pi} c_\alpha\alpha$ with $c_\alpha \in \mathbb{N}_0$ is $\nu - \rho$. The set of leading exponents is finite and non-empty. Moreover, if $\nu - \rho$ is an exponent, then there is a leading exponent of the form $\nu - \rho + \sum_{\alpha\in\Pi} c_\alpha\alpha$ (see [Kna86, Ch. VII.8] for details).

Let us return to the representation π_{ϕ_λ} where ϕ_λ is positive semidefinite. Since $\phi_\lambda = \phi_{w\lambda}$ for $w \in W$ we can assume $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$.

By Theorem I.7.6 π_{ϕ_λ} is infinitesimally equivalent to a subrepresentation of $H^{w_0\tau, w_0\nu}$ where $\nu - \rho$ is a leading exponent of π_{ϕ_λ}. By definition of the principal series the restriction to K is equivalent to $\operatorname{Ind}_M^K \tau$. With the Frobenius reciprocity theorem (see [Kna86, Theorem 1.14]) we observe that the trivial representation of K is contained in $H^{w_0\tau, w_0\nu}$ iff τ is the trivial representation on M. Since π_{ϕ_λ} is a spherical representation we obtain that π_{ϕ_λ} is infinitesimally equivalent to the irreducible subrepresentation of $H^{w_0\nu}$ containing the K-trivial representation.

We now determine ν. The elementary spherical function ϕ_λ is the matrix coefficient for the K-invariant vector and by [Kna86, Cor. 8.8] this is also the K-invariant matrix coefficient of $H^{w_0\nu}$. By definition of the principal series representation $H^{w_0\nu}$ its K-invariant matrix coefficient is $\phi_{w_0\nu} = \phi_\nu$. Therefore, $w\lambda = \nu$ for some $w \in W$. On the other hand, the elementary spherical function ϕ_λ is the spherical function of π_{ϕ_λ} for the trivial K-type. By Equation (I.4) we find that $\lambda - \rho$ is an exponent of ϕ_λ and therefore also of π_{ϕ_λ}. Hence, the leading exponent $\nu - \rho$ can be assumed to be of the form $\lambda - \rho + \sum_{\alpha\in\Pi} c_\alpha\alpha$, $c_\alpha \in \mathbb{N}_0$.

We obtain $w\lambda = \lambda + \sum c_\alpha\alpha$. This is only possible if $w\lambda = \lambda$ and $c_\alpha = 0$. Indeed, $\operatorname{Re}\langle\lambda - w\lambda, \rho\rangle = \operatorname{Re}\langle\lambda, \rho - w\rho\rangle \geq 0$ since $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$. Therefore, $\langle\sum_\alpha c_\alpha\alpha, \rho\rangle \leq 0$ and hence $c_\alpha = 0$ for all α.

We summarize the above discussion in the following proposition.

Proposition I.7.7. *Let* $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ *and suppose that* ϕ_λ *is positive semidefinite. Then* π_{ϕ_λ} *is infinitesimally equivalent to the subrepresentation of* $H^{w_0\lambda}$ *that is generated by the* K*-invariant vector.*

If one considers dual representations we get that π_{ϕ_λ} is an irreducible unitary spherical quotient of the dual representation of $H^{w_0\lambda}$, i.e. of $H^{-w_0\overline{\lambda}}$.

Since we are interested in the neighborhood of ρ let us restrict to $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$. In this case, $w_0\lambda = -\overline{\lambda}$ by Remark I.4.4. In particular, π_{ϕ_λ} is an irreducible quotient of H^λ. But by [Kna86, Thm. 7.24 on p.214] there is a unique irreducible quotient of H^λ if $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$ which is called J^λ. As π_{ϕ_λ} is unitary, J^λ is infinitesimally unitary. On the other hand if J^λ is infinitesimally unitary, the diagonal matrix coefficients are positive semidefinite. In particular, ϕ_λ is positive semidefinite, π_{ϕ_λ} is defined and infinitesimally equivalent to J^λ.

We will now describe how to reduce the question whether J^λ is infinitesimally unitary to the case where λ is real.

Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be non-real with $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$ and $w_0\lambda = -\overline{\lambda}$. Set $F := \{\alpha \in \Sigma^+ \mid \alpha \perp \operatorname{Im}\lambda\}$. We have $\emptyset \subsetneq F \subsetneq \Sigma^+$ by Chevalley's lemma (see [Hum90, Thm. 1.12]) and $\operatorname{Im}\lambda \neq 0$. Define $\tilde{\Sigma}^+ := \{\alpha \in \Sigma \mid \langle \alpha, \operatorname{Im}\lambda\rangle > 0\} \cup F$. Then $\tilde{\Sigma}^+$ is a positive system in Σ. Indeed, $\tilde{\Sigma}^+ \cup -\tilde{\Sigma}^+ = \Sigma$ and if $\alpha, \beta \in \tilde{\Sigma}^+$ with $\alpha+\beta \in \Sigma$ then $\alpha+\beta \in \tilde{\Sigma}^+$ if (at least) one root is not in F. If both roots α and β are contained in F, then $\alpha+\beta \in F$ since Σ^+ is a positive system.

The positive system $\tilde{\Sigma}^+$ defines a different minimal parabolic subgroup $\tilde{P} = MA\tilde{N}$. It is contained in a (non-minimal) parabolic subgroup $P_1 = M_1A_1N_1$ with the property that $\operatorname{Im}\lambda$ restricted to the $\mathfrak{a}$-part $\mathfrak{a}_{M_1}$ of M_1 vanishes, i.e. $\lambda|_{\mathfrak{a}_{M_1}}$ is real. Here, M_1 is generated by M and $\mathfrak{g}_\alpha$ with $\alpha \in \pm F$, A_1 is the analytic subgroup of $\mathfrak{a}_1 = \bigcap_{\alpha\in F} \ker\alpha$, and N_1 is the analytic subgroup of $\mathfrak{n}_1 = \bigoplus_{\alpha \in \tilde{\Sigma}^+ \setminus F} \mathfrak{g}_\alpha$. The Lie algebra $\mathfrak{m}_1$ of M_1 is given by $\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_1^\perp \oplus \bigoplus_{\alpha \in \pm F} \mathfrak{g}_\alpha$. Note that $\tilde{P} \neq P_1$ since $\mathfrak{g}_{-\alpha}$ is not in the Lie algebra of $\tilde{P}$ but in P_1 for $\alpha \in F$. Let $A_{M_1} = A \cap M_1 = \exp \mathfrak{a}_1^\perp$ and $N_{M_1} = \tilde{N} \cap M_1 = \exp \bigoplus_{\alpha\in F} \mathfrak{g}_\alpha = N \cap M_1$. Then we have $P_{M_1} := \tilde{P} \cap M_1 = MA_{M_1}N_{M_1}$ and P_{M_1} is the minimal parabolic subgroup for the positive system for (M_1, A_{M_1}) given by F. Then we have indeed $\operatorname{Im}\lambda|_{\mathfrak{a}_{M_1}} = 0$. In fact, if we identify $\mathfrak{a}$ and $\mathfrak{a}^*$ by the Killing form of G, then $\mathfrak{a}_{M_1} = \mathfrak{a}_1^\perp$ is identified with $\langle F\rangle$ which by definition is orthogonal to $\operatorname{Im}\lambda$ (see Example I.7.12 for the case $G = SL_3(\mathbb{R})$).

We want to show that $\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}$ is contained in the positive Weyl chamber for F as a positive system in (M_1, A_{M_1}). Here we have to use the Killing form B_1 of M_1 to identify $\mathfrak{a}_{M_1}$ with its dual and to measure angles. Let $\alpha \in F$ and define $\overline{\alpha} = \alpha|_{\mathfrak{a}_{M_1}}$ and $A_{\overline{\alpha}} \in \mathfrak{a}_{M_1}$ by $B_1(H, A_{\overline{\alpha}}) = \alpha(H)$, $H \in \mathfrak{a}_{M_1}$. Similarly define $A_\alpha \in \mathfrak{a}$ by $B(A_\alpha, H) = \alpha(H)$, $H \in \mathfrak{a}$, where B is the Killing form of G. By [Kna02, Proposition 6.52] we have

$$[X_\alpha, \theta X_\alpha] = B(X_\alpha, \theta X_\alpha) A_\alpha = B_1(X_\alpha, \theta X_\alpha) A_{\overline{\alpha}}$$

for $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in F$. Therefore, $A_{\overline{\alpha}} = c_\alpha A_\alpha$ with $c_\alpha > 0$. Now we can calculate

$$B_1(\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}, \overline{\alpha}) = \operatorname{Re}\lambda(A_{\overline{\alpha}}) = c_\alpha \operatorname{Re}\lambda(A_\alpha) = c_\alpha B(\operatorname{Re}\lambda, \alpha) > 0.$$

We conclude that $\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}$ is in the positive Weyl chamber. Therefore $J^{\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}}$ as a representation of M_1 is defined and we have the following theorem.

Theorem I.7.8 (Reduction to real character, [Kna86, Theorem 16.10])**.** *J^λ is infinitesimally unitary if and only if $J^{\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}}$ is infinitesimally unitary.*

Remark I.7.9. The theorem implies that only the direction of $\operatorname{Im}\lambda$ is important for J^λ to be infinitesimally unitary. More precisely, $J^{\operatorname{Re}\lambda + i\operatorname{Im}\lambda}$ is infinitesimally unitary if and only if $J^{\operatorname{Re}\lambda + ci\operatorname{Im}\lambda}$ is infinitesimally unitary for every $c > 0$. Moreover, $J^{\operatorname{Re}\lambda}$ is infinitesimally unitary in this case. Indeed, J^λ being infinitesimally unitary is equivalent to saying that ϕ_λ is positive semidefinite. $\lambda \mapsto \phi_\lambda$ is continuous and positive semidefiniteness is closed in the topology of convergence on compact sets. Therefore, letting $c \to 0$ implies the claim.

We obtain the following theorem on the quantum spectrum.

Theorem I.7.10. *Suppose G has Property (T). Then there is a neighbourhood U of ρ in $\mathfrak{a}^*$ (independent of Γ) such that*

$$\sigma_Q(\Gamma\backslash G/K) \cap (U \times i\mathfrak{a}^*) = \{\rho\}.$$

Proof. We already observed that for $\lambda \in \sigma_Q(\Gamma\backslash G/K)$ with $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$ the quotient J^λ is infinitesimally unitary and that there is a neighborhood U of ρ in $\mathfrak{a}_\mathbb{C}^*$ such that no quotient J^λ for $\lambda \in U$ is infinitesimally unitary. But now Theorem I.7.8 shows that this is also the case if $\lambda \in U + i\mathfrak{a}^*$. Projecting U onto the real part completes the proof. □

Remark I.7.11. Note that the spectral gap obtained by Theorem I.7.10 is only coming from the definition of Kazhdan's Property (T) where only the existence of some neighborhood is required. Hence we do not have control about its size or its shape. In contrast to that Theorem I.4.10 gives an explicit region where the spectrum is located and the size of this region is controlled by the L^p-boundedness of matrix coefficients of irreducible unitary representations.

Let us carry out the reduction to real characters in the example of $SL_3(\mathbb{R})$.

Example I.7.12. Let $G = SL(3,\mathbb{R})$ and $\lambda = c\rho + id(\alpha_1 - \alpha_2)$ with $c, d > 0$. Then $F = \{\alpha_1 + \alpha_2\}$ and $\tilde{\Sigma}^+ = \{\alpha_1, \alpha_1 + \alpha_2, -\alpha_2\}$. We have $\mathfrak{a}_1 = \mathbb{R}\operatorname{diag}(1,-2,1)$ and $\mathfrak{a}_{M_1} = \mathbb{R}\operatorname{diag}(1,0,-1)$. Furthermore $\tilde{\rho} = \alpha_1$ and $\rho_{M_1} = \frac{1}{2}\sum_{\alpha\in F} m_\alpha \alpha|_{\mathfrak{a}_{M_1}} = \frac{1}{2}\rho|_{\mathfrak{a}_{M_1}}$. We have $\tilde{\rho}|_{\mathfrak{a}_{M_1}}(\operatorname{diag}(1,0,-1)) = 1 = \rho_{M_1}(\operatorname{diag}(1,0,-1))$ so that $\rho_{M_1} = \tilde{\rho}|_{\mathfrak{a}_{M_1}}$. For the convex hulls we get $\operatorname{conv}(W\rho)|_{\mathfrak{a}_{M_1}} = 2\operatorname{conv}(W_{M_1}\rho_{M_1})$. By Theorem I.7.8 J^λ is infinitesimally unitary iff $J^{\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}}}$ is infinitesimally unitary. Hence the a priori bound given by $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$ (i.e. $c \leq 1$) is improved to $\operatorname{Re}\lambda|_{\mathfrak{a}_{M_1}} \in \operatorname{conv}(W_{M_1}\rho_{M_1}) = \frac{1}{2}\operatorname{conv}(W\rho)|_{\mathfrak{a}_{M_1}}$ (i.e. $c \leq \frac{1}{2}$) if $d > 0$ (see Figure I.8 for a visualization).

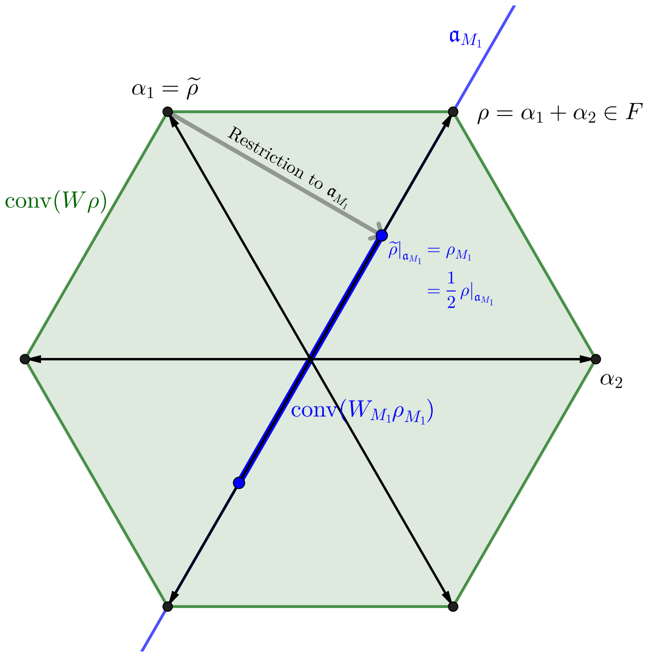

Figure I.8.: Visualization of the reduction to real character for $G = SL_3(\mathbb{R})$.

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II. Absence of principal eigenvalues for higher rank locally symmetric spaces

Outline

Preliminary Material

Before we delve into the paper, let us first discuss some fundamental concepts of compactifications that will play a central role later on.

II.1. Introduction to compactifications

For a topological Hausdorff space X it is often useful to know how it can be compactified. Let us first define the notion of a compactification.

Definition II.1.1. For a topological Hausdorff space X a *compactification* of X is a compact Hausdorff space $\overline{X}$ together with a topological embedding $\iota\colon X \to \overline{X}$ with open dense image.

Clearly, if X is already a compact Hausdorff space, and $\iota\colon X \to \overline{X}$ is a compactification, then $\iota(X)$ is also compact and therefore closed. By assumption $\iota(X)$ is dense so that $\iota(X) = \overline{X}$ and thus X and $\overline{X}$ are homeomorphic. Hence, it is only interesting to speak about compactifications if X is non-compact.

Example II.1.2. If X is locally compact non-compact Hausdorff space, then one can always construct the *Alexandroff* or *one-point compactification* as follows. Let ∞ be any element not contained in X and set $\overline{X} := X^* := X \cup \{\infty\}$. We define the topology on X^* by taking the open sets of X as open sets in X^* as well as all subsets of X^* which are complements in X^* of compact sets of X. This indeed defines a topology on X^*. Obviously, $\emptyset$ and X^* are open. Intersections of two open sets in X^* are again open, since $U \cap X^* \setminus K = U \cap X \setminus K$ is open where $U \subseteq X$ is open and $K \subseteq X$ is compact and hence closed and $(X^* \setminus K_1) \cap (X^* \setminus K_2) = X^* \setminus (K_1 \cup K_2)$ is a complement of a compact set where $K_i \subseteq X$ are compact. If $K_i \subseteq X$ is compact and $U_j \subseteq X$ is open then

$$\bigcup_i (X^* \setminus K_i) \cup \bigcup_j U_j = X^* \setminus \left(\bigcap_i K_i \cap X \setminus \bigcup_j U_j \right)$$

is the complement of a compact subset hence open. Therefore, arbitrary unions of open sets are again open. We conclude that X^* is a topological space.

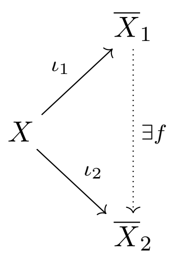
Moreover, $X \to X^*$ is a topological embedding, X is open in X^*, and as X is non-compact it has dense image.

In order to see that X^* is Hausdorff we have to show that a point $x \in X$ and ∞ are separated by open sets. Since we assumed X to be locally compact there is a compact neighborhood K of x. Let $U \subseteq K$ open with $x \in U$. Then U and $X^* \setminus K$ are open and disjoint sets containing x and ∞ respectively proving the Hausdorff property.

Furthermore, X^* is compact. For an open cover $(U_i)_i$ of X^* there is i_0 such that $\infty \in U_{i_0}$ and therefore $U_{i_0} = X^* \setminus K$ for a compact set $K \subseteq X$. Then K is covered by $(U_i \cap X)_{i \neq i_0}$ and hence there is a finite subcover of K which together with $X^* \setminus K$ covers all of X^*. This shows compactness.

The example of the one-point compactification is a very basic construction of a compactification. In general there are many different compactifications of a locally compact space. To compare different compactifications we make the following definition.

Definition II.1.3. We say that a compactification $\iota_1 \colon X \to \overline{X}_1$ *dominates* a compactification $\iota_2 \colon X \to \overline{X}_2$ if there is a continuous map $f \colon \overline{X}_1 \to \overline{X}_2$ such that $\iota_2 = f \circ \iota_1$.



Note that the map f is unique since $\iota_1(X)$ is dense in $\overline{X}_1$ and surjective since

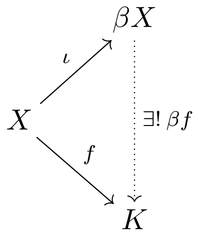
$$f\left(\overline{X}_1\right) = f\left(\overline{\iota_1(X)}\right) \supseteq \overline{f(\iota_1(X))} = \overline{\iota_2(X)} = \overline{X}_2.$$

Proposition II.1.4. *Let X be a locally compact non-compact Hausdorff space. Then every compactification $\overline{X}$ of X dominates the one-point compactification X^*.*

Proof. Define $f \colon \overline{X} \to X^*$ by $f(\iota(x)) = x$, $x \in X$, and $f(x) = \infty$ if $x \notin \iota(X)$. It remains to show that f is continuous. If $U \subseteq X$ is open, then $f^{-1}(U) = \iota(U)$ is open in $\iota(X)$ as ι is a topological embedding and therefore open in $\overline{X}$ as $\iota(X)$ is open in $\overline{X}$. If $K \subseteq X$ is compact then $f^{-1}(X^* \setminus K) = \overline{X} \setminus \iota(K)$ which is open since $\iota(K)$ is compact and therefore closed. This proves that f is continuous. □

The previous proposition shows that X^* is a final object in the category of compactifications of X. There also exists a initial object in this category which is called *Stone-Čech compactification* βX.

Proposition II.1.5 (see [Wal74])**.** *Let X be a locally compact Hausdorff space. Then there is a compactification βX of X with the following universal property: For every compact Hausdorff space K and every continuous map $f\colon X \to K$ there is a unique continuous map $\beta f\colon \beta X \to K$ with $f = \beta f \circ \iota$. In particular, βX dominates every other compactification of X.*



While the one-point compactification is too small to resemble the different ways to diverge to infinity, the Stone-Čech compactification is too unwieldy to work with. Therefore we introduce two compactifications suitable for our setting of symmetric spaces.

II.2. Geodesic compactification

Let X be an n-dimensional *Hadamard manifold*, i.e. a complete simply connected and non-positively curved Riemannian manifold. For these manifolds we have the famous Cartan-Hadamard theorem.

Theorem II.2.1 (see [BC64])**.** *X is diffeomorphic to the Euclidean plane $\mathbb{R}^n$. More precisely, at any point $p \in X$ the exponential mapping $\exp_p\colon T_pX \to X$ is a diffeomorphism.*

This theorem allows us to define a compactification of X by compactifying $\mathbb{R}^n$.

Definition II.2.2. The *geodesic compactification* $X \cup X(\infty)$ of X is the Hausdorff space $\{v \in T_pX \mid \|v\| \leq 1\}$ together with the embedding

$$\iota\colon X \to X \cup X(\infty), \quad x \mapsto \frac{\exp_p^{-1}(x)}{1 + \|\exp_p^{-1}(x)\|}.$$

Clearly, ι is an embedding with open dense image since $v \mapsto \frac{v}{1+\|v\|}$ is a diffeomorphism between T_pX and the unit disc in T_pX. Hence, $X \cup X(\infty)$ is a compactification of X.

To see that this compactification is independent of the base point p, i.e. different choices of p give rise to compactifications that dominate each other, we make the following definition.

Definition II.2.3 (see [BJ06, Section I.2.2])**.** Two (unit speed) geodesics γ_1, γ_2 are equivalent if $\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty$.

Proposition II.2.4 (see [BJ06, Prop. I.2.3])**.** *The factor space {all geodesics}/ ∼ can be canonically identified with the unit sphere in the tangent space T_pX at any base point p.*

Proof. For any unit vector $v \in T_pX$ there is a unique geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Hence, the set of all geodesics through p can be identified with the unit sphere in T_pX. This yields an injective map from the unit sphere in T_pX to the set of all geodesics modulo equivalence by [BC64, Ch. 9.5. Cor. 2]. For a geodesic γ let $x_n \coloneqq \gamma(n)$. Define γ_n to be the unique geodesic with $\gamma_n(0) = p$ and $\gamma_n(t_n) = x_n$ for some $t_n \geq 0$. Since the unit sphere in T_pX is compact, there is a subsequence such that $\dot\gamma_{n_k}(0)$ converges. Therefore, γ_{n_k} converges to a geodesic γ_∞ with $\gamma_\infty(0) = p$ uniformly for t in compact subsets of $\mathbb{R}$. Let $t \geq 0$. Since $t_n = d(p, x_n)$ and $n = d(x_n, \gamma(0))$ we have by the reverse triangle inequality $|t_n - n| \leq d(p, \gamma(0))$. In particular, $t_n \to \infty$. Pick n large such that $t \leq t_n$. Then by convexity of the function $t \mapsto d(\gamma_n(t), \gamma(t))$ (see [BO69]):

$$d(\gamma_n(t), \gamma(t)) \leq \max(d(p, \gamma(0)), d(\gamma_n(t_n), \gamma(t_n))).$$

We also have $d(\gamma_n(t_n), \gamma(t_n)) = d(\gamma(n), \gamma(t_n)) = |t_n - n| \leq d(p, \gamma(0))$ and therefore $d(\gamma_n(t), \gamma(t)) \leq d(p, \gamma(0))$. Letting $n \to \infty$ it follows $d(\gamma_\infty(t), \gamma(t)) \leq d(p, \gamma(0))$. Hence, γ and γ_∞ are equivalent. This completes the proof. □

The proposition shows that the boundary $X(\infty) = \{\text{all geodesics}\}/\sim$ is independent of the base point p. Also the convergence of an unbounded sequence $(x_n)_n$ is independent of the base point p. To see this let γ be a geodesic with $\gamma(0) = p$. Then $x_n \to [\gamma]$ (in the compactification associated with p) if and only if $\exp_p^{-1}(x_n)/(1 + \|\exp_p^{-1}(x_n)\|) \to \dot\gamma(0)$. The geodesic starting in p through x_n is $\gamma_n(t) = \exp_p\left(t\frac{\exp_p^{-1}(x_n)}{\|\exp_p^{-1}(x_n)\|}\right)$ so that $x_n \to [\gamma]$ if and only if $\gamma_n \to \gamma$ uniformly on compact sets. For a different base point p' let γ'_n be the geodesic starting in p' through x_n and γ' the geodesic with $\gamma'(0) = p'$ and $\gamma' \in [\gamma]$. Let us show that $\gamma'_n \to \gamma'$: Let $\varepsilon > 0$ and $R > 0$. Then there is $N \in \mathbb{N}$ such that $d(\gamma_n(t/\varepsilon), \gamma(t/\varepsilon)) < 1$ for all $n \geq N$ and $t \leq R$. By enlarging N we can assume that $d(x_n, p') \geq R/\varepsilon$ for $n \geq N$. Then by convexity

$$\begin{aligned} d(\gamma'_n(t), \gamma(t)) &\leq \varepsilon d(\gamma'_n(t/\varepsilon), \gamma'(t/\varepsilon)) \\ &\leq \varepsilon(d(\gamma'_n(t/\varepsilon), \gamma_n(t/\varepsilon)) + d(\gamma_n(t/\varepsilon), \gamma(t/\varepsilon)) + d(\gamma(t/\varepsilon), \gamma'(t/\varepsilon))). \end{aligned}$$

Again by convexity the first part is bounded by $\max\{d(\gamma'_n(0)), \gamma_n(0)), d(\gamma'_n(t'_n), \gamma_n(t'_n))\}$ where $t'_n = d(x_n, p')$. As in the proof of Proposition II.2.4 this is bounded by $d(p, p')$. The last part is bounded since γ and γ' are equivalent. All in all, $d(\gamma'_n(t), \gamma'(t)) \leq C\varepsilon$ for all $n \geq N$ and $t \leq R$ with a constant C independent of ε and R. This shows that $\gamma_n \to \gamma$ if and only if $\gamma'_n \to \gamma'$ and therefore also the topology of $X \cup X(\infty)$ is independent of the base point.

Let us finish the discussion about the topology by giving a fundamental system of neighborhoods. In the closed unit disc of T_pX a neighborhood of a point v_∞ on the boundary contains a set of the form

$$\{v \in T_pX \mid r < \|v\| \leq 1, \sphericalangle(v, v_\infty) < \varepsilon\}$$

where $r \nearrow 1$ and $\varepsilon \searrow 0$. Therefore the intersection with the interior X of $X \cup X(\infty)$ of a fundamental system of neighborhoods of $[\gamma]$ with $\gamma(0) = p$ is given by

$$\{\tilde{\gamma}(t) \in X \mid \tilde{\gamma} \text{ geodesic with } \tilde{\gamma}(0) = p, t > R, \sphericalangle(\dot{\gamma}(0)), \dot{\tilde{\gamma}}(0)) < \varepsilon\}$$

with $R \nearrow \infty$ and $\varepsilon \searrow 0$.

For a compactification it is an important question which continuous maps can be extended to the compactification. Recall that for the Stone-Čech compactification every continuous map into a compact space can be continued whereas only the functions with a unique limit can be continued to the one-point compactification[1]. For the geodesic compactification we have the following proposition.

Proposition II.2.5. *Every isometry $\varphi\colon X \to X$ extends (uniquely) to a homeomorphism $X \cup X(\infty) \to X \cup X(\infty)$.*

Proof. We define $\varphi([\gamma]) = [\varphi \circ \gamma]$ and have to check continuity. Since $X \subseteq X \cup X(\infty)$ is open and dense it is sufficient to show that $\varphi(x_n) \to \varphi(x)$ for $x_n \to x$ with $x_n \in X$ and $x = [\gamma] \in X(\infty)$. We already observed that $x_n \to [\gamma]$ if and only if $\gamma_n \to \gamma$ (uniformly on compact sets) where γ_n is the geodesic from p to x_n. But clearly $\varphi \circ \gamma_n \to \varphi \circ \gamma$ and $\varphi \circ \gamma$ is a geodesic from the base point $\varphi(p)$ to $\varphi(x_n)$. Hence $\varphi(x_n) \to [\varphi \circ \gamma]$. $\square$

Let us return to the setting where X is a symmetric space G/K. Then X is a Hadamard manifold with canonical base point $x_0 = eK$ and therefore the geodesic compactification is defined. The tangent space $T_{x_0}X$ at x_0 can be identified with $\mathfrak{p}$ and the Riemannian exponential map $\exp_{x_0}$ at x_0 coincides with the exponential map $\mathfrak{p} \to G/K, Y \mapsto \exp(Y)K$. In particular, a fundamental system of neighborhoods of $[\gamma]$ where $\gamma(t) = \exp(tY_0)$ for a unit vector $Y_0 \in \mathfrak{p}$ is given by

$$\{\exp(tY)K \mid t > R, Y \in \mathfrak{p} \text{ normalized}, \sphericalangle(Y, Y_0) < \varepsilon\}$$

with $R \nearrow \infty$ and $\varepsilon \searrow 0$.

Since G acts on G/K by isometries, Proposition II.2.5 shows that this action extends to an action on $G/K \cup (G/K)(\infty)$. Hence, the geodesic compactification is a so-called *G-compactification.*

[1]More precisely, for $\varphi\colon X \to Y$ in order to be continuously extendable to X^* there has to exist $y_0 \in Y$ such that for every neighborhood U of y_0 the set $\varphi^{-1}(Y \setminus U)$ is compact.

II.3. Satake compactifications

While the one-point compactification and the Stone-Čech compactification can be constructed for any locally compact Hausdorff space, the geodesic compactification is only defined for Hadamard manifolds. However the definition of none of the three compactifications is specifically designed for Riemannian symmetric spaces. There are many different compactifiactions for symmetric spaces, each of them having other properties. The Satake compactifications are the most useful compactifications for us as they match up with the asymptotic expansions for the eigenfunctions of $\mathbb{D}(G/K)$ that we will use in the paper. We introduce these compactifications in this section.

The definition uses irreducible faithful projective representations, hence let τ be such a representation of G, i.e. $\tau\colon G \to PSL(n, \mathbb{C})$ is a homomorphism that is injective and there are no proper invariant projective subspaces. Note that an irreducible representation of $\mathfrak{g}$ on some $\mathbb{C}^n$ lifts to an irreducible linear representation of the universal cover $\tilde{G}$. If Z is a central subgroup of $\tilde{G}$, then by Schur's lemma Z acts as scalar operators. Hence we obtain a projective representation of $\tilde{G}/Z$ and thus of G. On the other hand, if τ is an irreducible projective representation of G, then the derived representation $d\tau\colon \mathfrak{g} \to \mathfrak{sl}_n\mathbb{C}$ is also irreducible. Thus we have a correspondence of irreducible representations of $\mathfrak{g}$ and irreducible projective representations of G. This correspondence also preserves faithfulness if the center of G is trivial (see [GJT98, Prop. 4.6]).

For an irreducible projective representation τ by Weyl's unitary trick there is an inner product on $\mathbb{C}^n$ such that $\tau(\theta(g)) = (\tau(g)^{-1})^*$ where θ is the Cartan involution. In particular, $\tau(H)$ is Hermitian for $H \in \mathfrak{a}$ and therefore $\mathbb{C}^n = \bigoplus_{\mu \in \mathfrak{a}^*} V_\mu$ with $V_\mu = \{v \mid \tau(H)v = \mu(H)v \ \forall H \in \mathfrak{a}\}$. $\mu \in \mathfrak{a}^*$ is called *weight* if $V_\mu \neq 0$. The choice of the positive system Σ^+ defines a highest weight μ_τ in the sense that every other weight is of the form $\mu_\tau - \sum_{\alpha \in \Pi} c_\alpha \alpha$ with $c_\alpha \in \mathbb{N}_0$.

Remark II.3.1. The irreducible finite dimensional representations of $\mathfrak{g}$ (and therefore also the irreducible faithful projective representations of G) are parametrized by the integral dominant functionals on a Cartan subalgebra of $\mathfrak{g}$ by the theorem of the highest weight [Hum72, Ch. VI]. In particular, there are infinitely many of them.

Using the irreducible faithful representation τ we can define a corresponding Satake compactification.

Definition II.3.2. Let $\mathcal{H}_n$ be the real vector space of Hermitian $n \times n$ matrices and $\mathbb{P}(\mathcal{H}_n)$ the associated projective space. The *Satake compactification* $\overline{X}_\tau$ associated to an irreducible faithful representation τ is the closure of the range of the map

$$i_\tau\colon G/K \to \mathbb{P}(\mathcal{H}_n), \quad gK \mapsto [\tau(g)\tau(g)^*].$$

Note that i_τ is well-defined since we assumed that $\tau(\theta(g)) = (\tau(g)^{-1})^*$ and thus $\tau(K) \subseteq PSU(n)$. This implies $\tau(k)\tau(k)^* = 1 \in PSL_n(\mathbb{C})$.

The projective space $\mathbb{P}(\mathcal{H}_n)$ is compact and hence $\overline{X}_\tau$ is also compact. In order to see that $\overline{X}_\tau$ is a compactification we have to show that i_τ is an embedding with open image. The space $PSL_n(\mathbb{C})/PSU(n)$ can be identified with positive definite Hermitian matrices of determinant one via $gPSU(n) \mapsto gg^*$. Because of the restriction on the determinant $PSL_n(\mathbb{C})/PSU(n) \to \mathbb{P}(\mathcal{H}_n)$ is injective and clearly an embedding with open image.

For $H \in \mathfrak{p}$ it holds $\tau(\exp H)\tau(\exp H)^* = \exp(2\tau(H))$. Since the derivative $\tau\colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$ is injective we infer that i_τ is an embedding.

$\overline{X}_\tau$ is also a G-compactification: The action of G extends to an action on $\overline{X}_\tau$ and even on $\mathbb{P}(\mathcal{H}_n)$ by $g.[A] = [\tau(g)A\tau(g)^*]$.

Let us analyze the convergence of sequences in $\overline{X}_\tau$. Since $G = K\overline{A^+}K$ and K is compact we first take a look at sequences $x_j = \exp(H_j)K \in X$ with $H_j \in \overline{\mathfrak{a}_+}$. As described above the representation space $\mathbb{C}^n$ decomposes into weight spaces V_μ for the weights μ. Choosing a basis according to the weight spaces $\tau(e^H)$ is a diagonal matrix $\operatorname{diag}(e^{\mu_1(H)}, \dots, e^{\mu_n(H)})$ where $\mu_1, \dots, \mu_n$ are the weights listed with multiplicity. Then $i_\tau(e^H K) = [\operatorname{diag}(e^{2\mu_1(H)}, \dots, e^{2\mu_n(H)})]$. For the sequence $H_j \in \overline{\mathfrak{a}_+}$ there are different ways to diverge to ∞. More precisely, H_j can drift away from all walls of the Weyl chamber or has a bounded distance from some walls. The walls $\mathfrak{a}_I$ are given by subsets $I \subsetneq \Pi$ where $\mathfrak{a}_I = \bigcap_{\alpha\in I} \ker\alpha$. Let $J = \{\alpha \in \Pi \mid \limsup_j \alpha(H_j) < \infty\}$. By extracting a subsequence we can assume that $\lim \alpha(H_j)$ exists and is finite for $\alpha \in J$ and $\alpha(H_j) \to \infty$ for $\alpha \notin J$. Recall that $\mu_i = \mu_\tau - \sum_{\alpha\in\Pi} c_{\alpha,i}\alpha$ and assume that $\mu_1 = \mu_\tau$. Then

$$\begin{aligned} i_\tau(e^{H_j}K) &= [\operatorname{diag}(1, e^{2(\mu_2-\mu_\tau)H_j}, \dots, e^{2(\mu_n-\mu_\tau)H_j})] \\ &= [\operatorname{diag}(1, e^{-2\sum c_{\alpha,2}\alpha(H_j)}, \dots, e^{-2\sum c_{\alpha,n}\alpha(H_j)})]. \end{aligned}$$

It now depends on the $c_{\alpha,i}$ whether the entries converge to 0 or not.

To order the weights in a useful way we define the *support* $\operatorname{supp}(\mu_i)$ of the weight μ_i as $\operatorname{supp}(\mu_i) = \{\alpha \in \Pi \mid c_{\alpha,i} > 0\}$. With this definition a weight μ_i satisfies $\operatorname{supp}(\mu_i) \not\subseteq J$ iff $e^{-2\sum c_{\alpha,i}\alpha(H_j)} \to 0$. Let us order the weights such that for $i = 1, \dots, k$ we have $\operatorname{supp}(\mu_i) \subseteq J$. Note that $\operatorname{supp}(\mu_1) = \emptyset$ so that this does not contradict the assumption $\mu_1 = \mu_\tau$. Then we have

$$\begin{aligned} i_\tau(e^{H_j}K) &= [\operatorname{diag}(1, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_2)} c_{\alpha,2}\alpha(H_j)}, \dots, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_n)} c_{\alpha,n}\alpha(H_j)})] \\ &\to [\operatorname{diag}(1, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_2)} c_{\alpha,2}\lim_j \alpha(H_j)}, \dots, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_k)} c_{\alpha,k}\lim_j \alpha(H_j)}, 0, \dots, 0)]. \end{aligned}$$

Note that the limit only depends on $\lim_j \alpha(H_j)$ for $\alpha \in \bigcup_{\operatorname{supp}(\mu_i)\subseteq J} \operatorname{supp}(\mu_i) =: I \subseteq J$ and not on the limits for $\alpha \in J \setminus I$.

We now want to characterize the limit. Recall that $\mathfrak{a}_I = \bigcap_{\alpha\in I} \ker\alpha$. We define $\mathfrak{a}^I$ to be the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$ (see Figure II.3 for a visualization). By elementary linear algebra I is a basis of $(\mathfrak{a}^I)^*$ and thus there is $H_\infty \in \mathfrak{a}^I$ with $\alpha(H_\infty) = \lim_j \alpha(H_j)$ for all $\alpha \in I$. We conclude

$$i_\tau(e^{H_j}K) \to [\operatorname{diag}(1, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_2)} c_{\alpha,2}\alpha(H_\infty)}, \dots, e^{-2\sum_{\alpha\in\operatorname{supp}(\mu_k)} c_{\alpha,k}\alpha(H_\infty)}, 0, \dots, 0)].$$

In particular, we have a map

$$\begin{aligned} i_I\colon \overline{\mathfrak{a}^I_+} &= \{H \in \mathfrak{a}^I \mid \alpha(H) \geq 0 \,\forall \alpha \in I\} \to \overline{X}_\tau \\ H &\mapsto [\mathrm{diag}(1, e^{-2\sum_{\alpha\in\mathrm{supp}(\mu_2)} c_{\alpha,2}\alpha(H)}, \dots, e^{-2\sum_{\alpha\in\mathrm{supp}(\mu_k)} c_{\alpha,k}\alpha(H)}, 0, \dots, 0)]. \end{aligned}$$

By [BJ06, Lemma I.4.21/22] i_I is an embedding and the images of i_I for different I are disjoint.

To identify the different subsets I that can occur here we define the following notion.

Definition II.3.3. A subset $I \subseteq \Pi$ is called *μ_τ-connected* if $I \cup \{\mu_\tau\}$ cannot be decomposed into to non-empty orthogonal subsets.

The following proposition connects the support of a weight with the notion of μ_τ-connectedness.

Proposition II.3.4 (see [Sat60, Lemma 5])**.** *The μ_τ-connected subsets are precisely the supports of the weights of τ.*

Clearly, the union of μ_τ-connected subsets is μ_τ-connected and therefore for $J \subsetneq \Pi$ the set $I = \bigcup_{\mathrm{supp}(\mu_i)\subseteq J} \mathrm{supp}(\mu_i)$ is the largest μ_τ-connected subset contained in J.

Let us summarize the results in the following proposition.

Proposition II.3.5 (see [BJ06, Prop. I.4.23])**.** *A sequence $e^{H_j}K \in G/K$ for an unbounded sequence $H_j \in \overline{\mathfrak{a}_+}$ converges in $\overline{X}_\tau$ iff for a μ_τ-connected subset I the limit $\lim_j \alpha(H_j)$ exists and is finite for all $\alpha \in I$ and for every larger μ_τ-connected subset I' there is $\alpha \in I' \setminus I$ such that $\alpha(H_j) \to \infty$. If $H_\infty \in \overline{\mathfrak{a}^I_+}$ is the element with $\alpha(H_\infty) = \lim_j \alpha(H_j)$ for all $\alpha \in I$ then $i_\tau(e^{H_j}K) \to i_I(H_\infty)$. Hence,*

$$\overline{i_\tau(e^{\overline{\mathfrak{a}_+}}K)} = i_\tau(e^{\overline{\mathfrak{a}_+}}K) \cup \bigcup_{I\subsetneq\Pi \ \mu_\tau\text{-connected}} i_I\left(\overline{\mathfrak{a}^I_+}\right).$$

Example II.3.6. Let $G = PSL_n(\mathbb{C})$ and $\tau = id$ be the defining representation. $\mathfrak{a}$ is the set of diagonal matrices with real entries and trace 0. The representation space $V = \mathbb{C}^n$ has the weight space decomposition $V = \bigoplus V_{\mu_i}$ where $\mu_i(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_i$ and $V_{\mu_i} = \mathbb{C}e_i$. The usual choice of positivity gives the simple roots $\Pi = \{\alpha_i = \mu_i - \mu_{i+1} \mid i = 1, \dots, n-1\}$. This choice determines the highest weight to be μ_1 and $\mathrm{supp}(\mu_i) = \{\alpha_1, \dots, \alpha_{i-1}\}$. For $J \subseteq \Pi$ the set $I \subseteq J$ defined above is $\{\alpha_1, \dots, \alpha_i\}$ such that $\alpha_{i+1} \notin J$. If $H = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \overline{\mathfrak{a}_+}$ then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Hence for a sequence $H_j \in \overline{\mathfrak{a}_+}$ let k be the biggest index such that $\lambda_{k+1,j} - \lambda_{1,j}$ stays bounded for $j \to \infty$. Such an index k exists since $\sum_i \lambda_{i,j} = 0$ for all j. In particular, we can assume that $\lambda_{i,j} - \lambda_{1,j} \to d_i$ for $j \to \infty$ and $i \leq k+1$ and $\lambda_{i,j} - \lambda_{1,j} \to -\infty$ for $i > k+1$. It follows that

$$i_\tau(e^{H_j}PSU(n)) \to [\mathrm{diag}(1, e^{2d_2}, \dots, e^{2d_{k+1}}, 0, \dots, 0)].$$

In this example $I = \{\alpha_1, \ldots, \alpha_k\}$ but it could be the case that $I \subsetneq J$, i.e. $\limsup_j \lambda_{i,j} - \lambda_{i+1,j} < \infty$ for some $i > k+1$.

The boundary components of $i_\tau(e^{\overline{\mathfrak{a}_+}}K)$ are parametrized by the μ_1-connected subsets $I = \{\alpha_1, \ldots, \alpha_k\}$, $k = 1, \ldots, n-1$. Here $\mathfrak{a}_I = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{a} \mid \lambda_1 = \cdots = \lambda_{k+1}\}$ and $\mathfrak{a}^I = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{a} \mid \lambda_{k+2} = \cdots = \lambda_n = 0\}$. The embedding i_I is given by

$$i_I(\mathrm{diag}(\lambda_1, \ldots, \lambda_{k+1}, 0, \ldots, 0)) = [\mathrm{diag}(e^{2\lambda_1}, \ldots, e^{2\lambda_{k+1}}, 0, \ldots, 0)]. \qquad \square$$

Let us describe the orbit structure of $\overline{X}_\tau$. Denote by x_I the image of 0 under the embedding i_I for a μ_τ-connected subset $I \subsetneq \Pi$. Then we have the following statement.

Proposition II.3.7 (see [BJ06, Prop. I.4.27])**.** *The G-orbits in $\overline{X}_\tau$ are parametrized by the μ_τ-connected subsets $I \subsetneq \Pi$. More precisely,*

$$\overline{X}_\tau = X \cup \bigcup_{I \subsetneq \Pi \ \mu_\tau\text{-connected}} \mathcal{O}_I \tag{II.1}$$

where $\mathcal{O}_I$ is the orbit Gx_I through x_I.

In order to determine the orbits in $\overline{X}_\tau$ as homogeneous spaces we have to introduce parabolic subgroups (see Section II.5.4 for an example).

Definition II.3.8. For $I \subseteq \Pi$ recall that $\mathfrak{a}_I = \bigcap_{\alpha \in I} \ker \alpha$ and $\mathfrak{a}^I$ is the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$. We further define $\mathfrak{n}_I = \bigoplus_{\alpha \in \Sigma^+ \setminus \langle I \rangle} \mathfrak{g}_\alpha$ and $\mathfrak{m}_I = \mathfrak{m} \oplus \mathfrak{a}^I \oplus \bigoplus_{\alpha \in \langle I \rangle} \mathfrak{g}_\alpha$. We define the corresponding subgroups $A_I = \exp \mathfrak{a}_I$, $N_I = \exp \mathfrak{n}_I$ and $M_I = M \langle \exp \mathfrak{m}_I \rangle$. Then the subgroup $P_I = M_I A_I N_I$ is the *standard parabolic subgroup* for the subset I.

We also need the notion of the μ_τ-saturation.

Definition II.3.9. For a subset $I \subsetneq \Pi$ we define the *μ_τ-saturation* of I as the union of I and the simple roots orthogonal to $I \cup \{\mu_\tau\}$.

Now we can identify the orbits $\mathcal{O}_I$ as homogeneous spaces.

Proposition II.3.10 (see [BJ06, Prop. I.4.40])**.** *Let $I \subsetneq \Pi$ be μ_τ-connected and J its μ_τ-saturation. Then the stabilizer of x_I is $N_J A_J M_{J \setminus I}(K \cap M_I)$. In particular, the orbit $\mathcal{O}_I$ is the homogeneous space*

$$G/(N_J A_J M_{J \setminus I}(K \cap M_I)). \tag{II.2}$$

Example II.3.11. For $G = PSL_n\mathbb{C}$ and $\tau = id$ we have the μ_τ-connected subsets $I_k = \{\alpha_1, \ldots, \alpha_k\}$. Clearly, the μ_τ-saturation of I_k is

$$J_k = \{\alpha_1, \ldots, \alpha_k\} \cup \{\alpha_{k+2}, \ldots, \alpha_{n-1}\} = \Pi \setminus \{\alpha_{k+1}\}.$$

Therefore,

$$\mathfrak{a}_{J_k} = \left\{ \begin{pmatrix} \lambda id_{k+1} & \\ & \lambda' id_{n-(k+1)} \end{pmatrix} \in \mathfrak{sl}_n\mathbb{C} \colon \lambda, \lambda' \in \mathbb{R} \right\},$$
$$\mathfrak{n}_{J_k} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_n\mathbb{C} \colon B \in M_{(k+1)\times(n-(k+1))}(\mathbb{C}) \right\},$$
$$\mathfrak{m}_{I_k} = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in \mathfrak{sl}_n\mathbb{C} \colon \operatorname{Tr} A \in i\mathbb{R}, D = \operatorname{diag}(iz_{k+2}, \dots, iz_n), z_j \in \mathbb{R} \right\}, \text{ and}$$
$$\mathfrak{m}_{J_k \smallsetminus I_k} = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in \mathfrak{sl}_n\mathbb{C} \colon \operatorname{Tr} D \in i\mathbb{R}, A = \operatorname{diag}(iz_1, \dots, iz_{k+1}), z_j \in \mathbb{R} \right\}.$$

Thus on the group level

$$A_{J_k} = \left\{ \begin{pmatrix} \lambda id_{k+1} & \\ & \lambda' id_{n-(k+1)} \end{pmatrix} \in PSL_n\mathbb{C} \colon \lambda, \lambda' > 0 \right\},$$
$$N_{J_k} = \left\{ \begin{pmatrix} id_{k+1} & B \\ 0 & id_{n-(k+1)} \end{pmatrix} \in PSL_n\mathbb{C} \colon B \in M_{(k+1)\times(n-(k+1))}(\mathbb{C}) \right\},$$
$$M_{I_k} = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in PSL_n\mathbb{C} \colon D = \operatorname{diag}(\xi_{k+2}, \dots, \xi_n), |\det A| = |\xi_j| = 1 \right\}, \text{ and}$$
$$M_{J_k \smallsetminus I_k} = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in PSL_n\mathbb{C} \colon A = \operatorname{diag}(\xi_1, \dots, \xi_{k+1}), |\det D| = |\xi_j| = 1 \right\}.$$

Since $K = PSU(n)$ we have $\operatorname{Stab}(x_I) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in PSL_n\mathbb{C} \colon A \in \mathbb{R}U(k+1) \right\}$.

This can be seen by direct calculation, too. x_{I_k} is the element $[\operatorname{diag}(1, \dots, 1, 0, \dots, 0)]$ in $\mathbb{P}(\mathcal{H}_n)$. Hence,

$$g.x_{I_k} = \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} id_{k+1} & \\ & 0 \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \right] = \left[\begin{pmatrix} AA^* & AC^* \\ CA^* & CC^* \end{pmatrix} \right]$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in PSL_n\mathbb{C}$. Thus, $g \in \operatorname{Stab}(x_{I_k})$ if and only if $AA^* = a \cdot id_{k+1}$, $a \in \mathbb{R}$, and $C = 0$.

To determine the orbits as subsets of $\mathbb{P}(\mathcal{H}_n)$ note that $g.x_{I_k}$ is the line through a positive semidefinite Hermitian matrix of (complex) rank $k+1$. On the other hand, if A is a rank $k+1$ positive semidefinite Hermitian matrix, then A can be unitarily diagonalized, i.e. there is $U \in SU(n)$ such that U^*AU is a diagonal matrix with $k+1$ positive entries. Taking the square root of the entries and normalizing the determinant shows that

$$\mathcal{O}_{I_k} = PSL_n(\mathbb{C}).x_{I_k} = \{[A] \in \mathbb{P}(\mathcal{H}_n) \mid A \geq 0, \operatorname{rk} A = k+1\}.$$

Now it is clear by taking the image of A that this orbit is a fiber bundle over the Grassmannian $\operatorname{Gr}_{k+1}(\mathbb{C}^n) = \{V \subseteq \mathbb{C}^n \mid V \text{ is a subspace of dimension } k+1\}$. Note that

$\mathrm{Gr}_{k+1}(\mathbb{C}^n)$ is the homogeneous space $PSL_n\mathbb{C}/P_{J_k}$ where

$$P_{J_k} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in PSL_n\mathbb{C} \colon A \in GL_{k+1}(\mathbb{C}), D \in GL_{n-(k+1)}(\mathbb{C}) \right\}.$$

Let us determine the fiber over the canonical base point $\mathbb{C}^{k+1} \times \{0\}^{n-(k+1)}$, i.e. let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{H}_n$ with image $\mathbb{C}^{k+1} \times \{0\}^{n-(k+1)}$. It follows easily that $C = D = 0$ and by Hermiticity also $B = 0$. Hence the fiber is

$$\left\{ \left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right] \in \mathbb{P}(\mathcal{H}_n) \colon A > 0 \right\} \simeq \{[A] \in \mathbb{P}(\mathcal{H}_{k+1}) \mid A > 0\} \simeq PSL_{k+1}(\mathbb{C})/PSU(k+1)$$

the space of positive definite matrices of size $k+1$ and determinant 1. Note that this can also be described by the parabolic subgroups. Indeed,

$$PSL_{k+1}(\mathbb{C})/PSU(k+1) \simeq M_{I_k}/(M_{I_k} \cap PSU(n)). \qquad \square$$

Similar statements for the orbit structure can be made in the general case as well. Recall that $\mathcal{O}_I \simeq G/(N_J A_J M_{J\smallsetminus I}(K \cap M_I))$ for a μ_τ-connected subset I with μ_τ-saturation J. Clearly we have $M_{I_1} \subseteq M_{I_2}$ for $I_1 \subseteq I_2$. It follows that $M_{J\smallsetminus I}(K \cap M_I)$ is contained in M_J and hence $\mathrm{Stab}(x_I) \subseteq N_J A_J M_J = P_J$. We infer that $\mathcal{O}_I$ is a fiber bundle over the flag variety G/P_J with fiber

$$P_J/(N_J A_J M_{J\smallsetminus I}(K \cap M_I)) \simeq M_J/(M_{J\smallsetminus I}(M_I \cap K)) \simeq M_I/(M_I \cap K)$$

(see [BJ06, Cor. I.4.41] for details).

We now deal with the comparison of $\overline{X}_\tau$ for different representations τ. Note that the orbit structure (II.1),(II.2) and the related notions of μ_τ-connectedness and μ_τ-saturation does not depend on the actual representation τ but rather on μ_τ and more precisely only on $\theta_\tau = \{\alpha \in \Pi \mid \langle \alpha, \mu_\tau \rangle \neq 0\}$. This leads to the following proposition.

Proposition II.3.12 (see [BJ06, Prop. I.4.35])**.** *There are only finitely many non-homeomorphic Satake compactifications. Two Satake compactifications $\overline{X}_\tau$ and $\overline{X}_{\tau'}$ are homeomorphic if and only if $\theta_\tau = \theta_{\tau'}$.*

Furthermore, if $\theta_\tau \subseteq \theta_{\tau'}$ then $\overline{X}_{\tau'}$ dominates $\overline{X}_\tau$. In particular if $\mu_\tau \in \mathfrak{a}_+^*$, i.e. $\theta_\tau = \Pi$, then $\overline{X}_\tau$ dominates every other Satake compactification and therefore this compactification is called maximal Satake compactification $\overline{X}^{\max}$.

Let us describe the neighborhoods for the maximal Satake compactification and compare them with non-maximal ones. We first observe that every subset $I \subsetneq \Pi$ is μ_τ-connected. Hence, $\overline{X}^{\max} = X \cup \bigcup_{I \subsetneq \Pi} \mathcal{O}_I$. For $H_\infty \in \overline{\mathfrak{a}_+^I}$ an unbounded sequence $e^{H_j}K \in X$ with $H_j \in \overline{\mathfrak{a}_+}$ converges to $\exp(H_\infty)x_I$ if and only if $\alpha(H_j) \to \infty$ for $\alpha \notin I$ and $\alpha(H_j) \to \alpha(H_\infty)$ for $\alpha \in I$.

If we consider a sequence $k_j e^{H_j} K \in X$ with $k_j \in K$ then this converges to $ke^{H_\infty} x_I$ if and only if additionally $k_j \to k \mod M_I \cap K$ since $\mathrm{Stab}(x_I) \cap K = M_I \cap K$. In particular, (the intersection with the interior X of $\overline{X}^{\max}$ of) a fundamental system of neighborhoods of $ke^{H_\infty} x_I$ in $\overline{X}^{\max}$ is given by

$$V(K \cap M_I) \exp\{H \in \overline{\mathfrak{a}_+} \mid |\alpha(H) - \alpha(H_\infty)| < \varepsilon, \alpha \in I, \alpha(H) > R, \alpha \notin I\}K,$$

where V is a fundamental system of neighborhoods of k in K, $\varepsilon \searrow 0$, $R \nearrow \infty$.

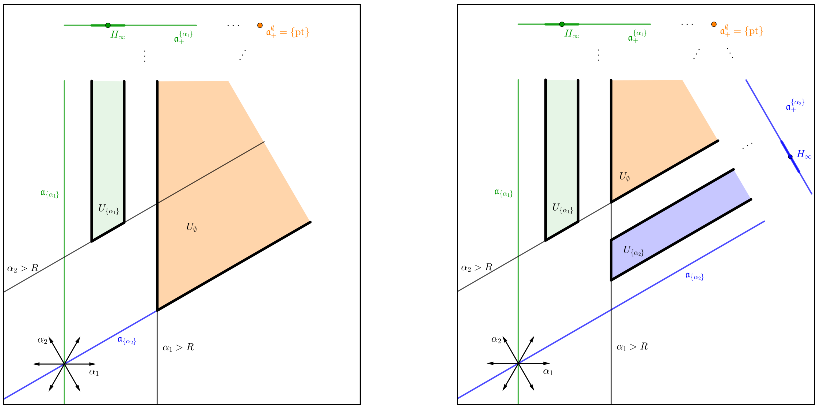


Figure II.1.: The intersection of $e^{\mathfrak{a}_+}K$ with a fundamental system of neighborhoods in $\overline{X}_{id}$ (left) and $\overline{X}^{\max}$ (right) for $G = PSL_3\mathbb{C}$.

Let us conclude the section by comparing the geodesic and the maximal Satake compactification.

Proposition II.3.13. *Let $\overline{X}$ be a G-compactification of X that is dominated by the geodesic compactification $X \cup X(\infty)$ and by the maximal Satake compactification $\overline{X}^{\max}$. If the rank of X is bigger than 1, then $\overline{X}$ is the one-point compactification X^*.*

Proof. Let $i_1 \colon \overline{X}^{\max} \to \overline{X}$ and $i_2 \colon X \cup X(\infty) \to \overline{X}$ be the continuous maps realizing the domination. Recall that $X(\infty)$ can be identified with the unit sphere $\mathfrak{p}_\infty$ in $\mathfrak{p}$. Let $H \in \mathfrak{a}$ normalized and $k \in K$. Then $ke^{nH}K \in X$ converges to $\mathrm{Ad}(k)H$ in $X \cup X(\infty)$. On the other hand $ke^{nH}K$ converges to $kx_\emptyset$ in $\overline{X}^{\max}$ if $H \in \mathfrak{a}_+$. Therefore, $i_1(kx_\emptyset) = i_2(\mathrm{Ad}(k)H)$ for $H \in \mathfrak{a}_+$ normalized. By continuity, this also holds for normalized vectors $H \in \overline{\mathfrak{a}_+}$. Now for $I \subsetneq \Pi$ let $H_n \in \overline{\mathfrak{a}_+}$ be a sequence such that $e^{H_n}K$ converges to x_I. Then we have

$$i_1(kx_I) = \lim_n ke^{H_n}K = i_2(\mathrm{Ad}(k)H) = i_1(kx_\emptyset)$$

for some $H \in \overline{\mathfrak{a}_+}$ normalized. Note that if $k \in K \cap M_I$ then $i_2(\mathrm{Ad}(k)H) = i_1(kx_\emptyset) = i_1(kx_I) = i_1(x_I) = i_1(x_\emptyset)$. By the assumption of higher rank the different $K \cap M_I$ for $I \subsetneq \Pi$ generate K. Hence $i_1(kx_I) = i_2(\mathrm{Ad}(k)H)$ for every $k \in K$, $I \subsetneq \Pi$, and $H \in \overline{\mathfrak{a}_+}$ normalized. This completes the proof. □

Published Paper

Abstract

Given a geometrically finite hyperbolic surface of infinite volume it is a classical result of Patterson that the positive Laplace-Beltrami operator has no L^2-eigenvalues $\geq 1/4$. In this article we prove a generalization of this result for the joint L^2-eigenvalues of the algebra of commuting differential operators on Riemannian locally symmetric spaces $\Gamma\backslash G/K$ of higher rank. We derive dynamical assumptions on the Γ-action on the geodesic and the Satake compactifications which imply the absence of the corresponding principal eigenvalues. A large class of examples fulfilling these assumptions are the non-compact quotients by Anosov subgroups.

II.4. Introduction

Let $\mathbb{H} = SL(2,\mathbb{R})/SO(2)$ be the hyperbolic plane equipped with the Riemannian metric of constant negative curvature and $\Gamma \subset SL(2,\mathbb{R})$ a discrete torsion-free subgroup. Then $\Gamma\backslash\mathbb{H}$ is a Riemannian surface of constant negative curvature and the relations between the geometry of $\Gamma\backslash\mathbb{H}$, the group theoretic properties of Γ, the dynamical properties of the Γ-action on $\mathbb{H}$ or its compactification, and the spectrum of the positive Laplace-Beltrami operator Δ have been intensively studied over several decades. Let us focus on the discrete L^2-spectrum of the Laplace-Beltrami operator, i.e. those $\mu \in \mathbb{R}$ such that $(\Delta - \mu)f = 0$ for some $f \in L^2(\Gamma\backslash\mathbb{H})$, $f \neq 0$. If $\Gamma \subset SL(2,\mathbb{R})$ is cocompact, then $\mu_0 = 0$ is always an eigenvalue corresponding to the constant function and Weyl's law for the elliptic selfadjoint operator Δ implies that there is a discrete set of infinitely many eigenvalues $0 = \mu_0 < \mu_1 \leq \ldots$ of finite multiplicity. From a representation theoretic perspective there is a clear distinction between $\mu_i \in]0, 1/4[$ and $\mu_i \geq 1/4$. The former correspond to complementary series representations and the latter to principal series representations occurring in $L^2(\Gamma\backslash SL(2,\mathbb{R}))$. We call the eigenvalues accordingly principal eigenvalues (if $\mu_i \geq 1/4$) and complementary or exceptional eigenvalues (if $\mu_i \in]0, 1/4[$). Merely by discreteness of the spectrum we know that there are at most finitely many complementary eigenvalues and infinitely many principal eigenvalues.

If we pass to non-compact $\Gamma\backslash\mathbb{H}$, the situation becomes more intricate: For the modular surface $SL(2,\mathbb{Z})\backslash\mathbb{H}$, which is non-compact but of finite volume, it is well known that there are no complementary eigenvalues but still infinitely many principal eigenvalues obeying a Weyl asymptotic. In general the question of existence of principal eigenvalues on finite volume hyperbolic surfaces is wide open. A long standing conjecture by Phillips and Sarnak [PS85] states that for a generic lattice $\Gamma \subset SL(2,\mathbb{R})$ there should be no principal eigenvalues.

If we pass to hyperbolic surfaces of infinite volume the situation is much better understood. A classical theorem by Patterson [Pat75] states that if $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \infty$ and $\Gamma \subset SL(2,\mathbb{R})$ is geometrically finite, then there are no principal eigenvalues. The result has later been generalized to real hyperbolic spaces of higher dimensions by Lax and Phillips [LP82]. Even if we are not aware of a reference, it seems folklore that the statement holds for general rank one locally symmetric spaces.

In this article we are interested in a generalization of Patterson's theorem to higher rank locally symmetric spaces:

Let us briefly[2] introduce the setting: Let $X = G/K$ be a Riemannian symmetric space of non-compact type and $\Gamma \subset G$ a discrete torsion-free subgroup. We will be interested in the L^2-spectrum of the locally symmetric space $\Gamma\backslash X$. As for hyperbolic surfaces the Laplace-Beltrami operator is a canonical geometric differential operator whose spectral theory can be studied. If the symmetric space is of higher rank, there are however further G-invariant differential operators on X that descend to differential operators on $\Gamma\backslash X$. It is from many perspectives more desirable to study the spectral theory of the whole algebra of invariant differential operators $\mathbb{D}(G/K)$ instead of just the spectrum of the Laplacian. In order to introduce the definition of the joint spectrum of $\mathbb{D}(G/K)$ we recall that $\mathbb{D}(G/K)$ is a commutative algebra generated by $r \geq 1$ algebraically independent differential operators and r equals the rank of the symmetric space X. After a choice of generating differential operators a joint eigenvalue of these commuting differential operators would be given by an element in $\mathbb{C}^r$. A more intrinsic way of defining the spectrum which does not require to choose any generators, is provided by the Harish-Chandra isomorphism. This is an algebra isomorphism $\mathrm{HC} : \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ between the invariant differential operators and the complex-valued Weyl group invariant polynomials on the dual of $\mathfrak{a} = \mathrm{Lie}(A)$, where A is the abelian subgroup of G in the Iwasawa decomposition $G = KAN$. If we fix $\lambda \in \mathfrak{a}^* = \mathfrak{a} \otimes \mathbb{C}$ and compose the Harish-Chandra isomorphism with the evaluation of the polynomial at λ we obtain a character $\chi_\lambda := \mathrm{ev}_\lambda \circ \mathrm{HC} : \mathbb{D}(G/K) \to \mathbb{C}$. With this notation we call $\lambda \in \mathfrak{a}^*_\mathbb{C}$ a joint L^2-eigenvalue on $\Gamma\backslash X$ if there exists $f \in L^2(\Gamma\backslash X)$ such that for all $D \in \mathbb{D}(G/K)$:

$$Df = \chi_\lambda(D)f.$$

As for the hyperbolic surfaces we can distinguish two kinds of L^2-eigenvalues: The purely imaginary joint eigenvalues $\lambda \in i\mathfrak{a}^*$ correspond to principal series representations and we call them *principal joint L^2-eigenvalues*. The remaining eigenvalues are called *complementary* or *exceptional* eigenvalues. These two kind of eigenvalues are not only distinguished by representation theory, but they also behave differently from the point of view of spectral theory: In their seminal paper [DKV79], Duistermaat, Kolk and Varadarajan consider the case of cocompact discrete subgroups $\Gamma \subset G$. They prove that

[2]A more detailed description of the setting will be provided in Section II.5.1.

there exist infinitely many principal joint eigenvalues and their asymptotic growth is precisely described by a Weyl law with a remainder term. They furthermore prove an upper bound on the number of complementary eigenvalues whose growth rate is strictly inferior than the Weyl asymptotic of the principal eigenvalues. There are thus much less complementary than principal eigenvalues.

The most prominent non-compact higher rank locally symmetric space is without doubt $\Gamma\backslash X = SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)$. By [Mül07] (and in a more general setting by [LV07]) it is known that there are infinitely many joint L^2-eigenvalues. Assuming the generalized Ramanujan conjecture which implies the absence of complementary eigenvalues (see e.g. [BB13]), we would get infinitely many principal joint L^2-eigenvalues. If one replaces the full modular group by a congruence subgroup $\Gamma(n)$ of level $n \geq 3$, the existence of infinitely many principal joint L^2-eigenvalues has been shown by Lapid and Müller [LM09]. More precisely, there is a Weyl law for the principal joint eigenvalues and the number of complementary eigenvalues are shown to be bounded by a function of lower order growth.

In the recent article [EO22] Edwards and Oh give examples and conditions on the discrete subgroup Γ which imply that the complementary eigenvalues are not only of lower quantity but that they are indeed absent. The main example are selfjoinings of convex-cocompact subgroups in $PSO(n,1)$, but they conjecture that this holds for every Anosov subgroup.

In this article we are interested in conditions on the group Γ which imply the absence of principal eigenvalues. In order to state our main theorem, recall the definition of a wandering point: If Γ acts continuously on a topological space T, then a point $t \in T$ is called wandering, if there exists a neighborhood $U \subset T$ of t such that $\{\gamma \in \Gamma : \gamma U \cap U \neq \emptyset\}$ is finite. The collection of all wandering points is called the wandering set $\mathrm{w}(\Gamma, T)$.

We can now state our main theorem.

Theorem II.4.1. *Let $X = G/K$ be a Riemannian symmetric space of non-compact type and $\Gamma \subset G$ a discrete torsion-free subgroup. Let $\overline{X}$ be the geodesic or the maximal Satake compactification (see Sections II.5.3 and II.5.4) and let $\mathrm{w}(\Gamma, \overline{X})$ be the wandering set for the action of Γ on $\overline{X}$. If $\mathrm{w}(\Gamma, \overline{X}) \cap \partial\overline{X} \neq \emptyset$, then there are no principal joint L^2-eigenvalues on $\Gamma\backslash X$.*

Let us compare our theorem to the classical result of Patterson: First of all, for $\mathbb{H}$ the geodesic compactification and the Satake compactification coincide. Furthermore, if $\Gamma \subset SL(2,\mathbb{R})$ is geometrically finite, then it is well known that the following are equivalent:

(i) $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \infty$.

(ii) The limit set of Γ is not the whole boundary $\Lambda(\Gamma) \neq \partial\mathbb{H}$.

(iii) There is a non-empty open set of discontinuity $\Omega(\Gamma) \subset \partial\mathbb{H}$ on which Γ acts properly discontinuously.

The last point immediately implies the existence of a wandering point of the Γ action on $\overline{\mathbb{H}}$. In this sense our theorem boils down to the classical result of Patterson. Also the higher dimensional result of Lax-Phillips on $\mathbb{H}^n$ is easily recovered from our main theorem: If $\Gamma \subset PSO(1,n)$ is geometrically finite and $\Gamma\backslash\mathbb{H}^n$ of infinite volume, then at least one non-compact end has to be a funnel or a cusp of non-maximal rank, and the existence of such a non-compact end directly implies the wandering condition of Theorem II.4.1.

As discrete subgroups on higher rank semisimple Lie groups are known to be constrained by strong rigidity results, it is a valid question whether there are interesting examples in higher rank which fulfill the wandering condition of Theorem II.4.1. We address this question in Section II.8 and we will see that all images of Anosov representations fulfill our condition. This is a consequence of recent results on compactifications of Anosov symmetric spaces [KL18, GKW15] that are modeled on the Satake compactification.

A further natural question is, whether one can also in the higher rank setting obtain the result by the assumption of infinite volume of the locally symmetric spaces instead of the dynamical assumption on the group action used in our theorem. We do not know a definitive answer. However, it should be noted, that there is so far no good notion of a geometrically finite group Γ in higher rank. Without the assumption of geometric finiteness, to our best knowledge even for $SL(2,\mathbb{R})$ it is unknown if infinite volume implies the absence of principal eigenvalues.

Outline of the proof and the article. Let $f \in C^\infty(X)$ be the Γ-invariant lift of a joint eigenfunction for $\mathbb{D}(X)$ that is in $L^2(\Gamma\backslash X)$. The proof of Theorem II.4.1 relies on the analysis of the asymptotic behavior of f towards the boundary of the compactification at infinity. For the result on the geodesic compactification it suffices to study the asymptotics of f into the regular directions. In order to obtain the result on the Satake compactification we are required to also analyze the behavior in singular directions along the different boundary strata of the Weyl chambers.

In a first step we show that f satisfies a certain growth condition called *moderate growth*. This is done by elliptic regularity combined with coarse estimates on the injectivity radius (see Section II.6).

The knowledge of moderate growth then allows us (see Section II.7) to use asymptotic expansion results for f by van den Ban-Schlichtkrull [vdBS87, vdBS89]. For the asymptotics into the regular directions, i.e. in the interior of the positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, it follows from [vdBS87] that the leading term for the expansion of $f(k\exp(tH)K)$ with $k \in K$ and $H \in \mathfrak{a}^+$ is

$$\sum_{w\in W} p_w(k) e^{(w\lambda-\rho)(tH)} \quad \text{as} \quad t \to \infty,$$

where W is the Weyl group, ρ the usual half sum of roots and $\lambda \in i\mathfrak{a}^*$ a regular spectral parameter (for singular spectral parameters the formula becomes slightly more complicated but is still tractable). The wandering condition of Γ acting on the geodesic compactification $X \cup X(\infty)$ yields a neighborhood U in $X \cup X(\infty)$ of some point in $X(\infty)$ such that $f \in L^2(U \cap X)$. Combining this with the expansion and the description of such neighborhoods U implies that all the boundary values p_w vanish on an open subset of K. This implies, again by [vdBS87], that $f = 0$.

The result for the Satake compactification follows the same strategy but involves more complicated expansions from [vdBS89] that describe the asymptotic behavior into the singular directions along the different boundary strata of the Weyl chamber (see Section II.7.2).

Finally, in Section II.8 we provide some examples of higher rank locally symmetric spaces that fulfill the wandering condition of Theorem II.4.1. In particular, we show that all quotients by Anosov subgroups fulfill the assumption.

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II.5. Preliminaries

II.5.1. Symmetric spaces

In this section we fix the notation for the present article. Let G be a real semisimple non-compact Lie group with finite center and with Iwasawa decomposition $G = KAN$. Furthermore, let $M := Z_K(A)$ be the centralizer of A in K. We denote by $\mathfrak{g}, \mathfrak{a}, \mathfrak{n}, \mathfrak{k}, \mathfrak{m}$ the corresponding Lie algebras. We have a K-invariant inner product on $\mathfrak{g}$ that is induced by the Killing form and the Cartan involution. We further have the orthogonal Bruhat decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ into root spaces $\mathfrak{g}_\alpha$ with respect to the $\mathfrak{a}$-action via the adjoint action ad, i.e. $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y \ \forall H \in \mathfrak{a}\}$. Here $\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\} \subseteq \mathfrak{a}^*$ is the set of restricted roots. Denote by W the Weyl group of the root system of restricted roots. Let n be the real rank of G and Π (resp. Σ^+) the simple (resp. positive) system in Σ determined by the choice of the Iwasawa decomposition. Let $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ and $\rho := \frac{1}{2}\Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$. Let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Pi\}$ denote the positive Weyl chamber. If $\overline{A^+} := \exp(\overline{\mathfrak{a}_+})$, then we have the Cartan decomposition $G = K\overline{A^+}K$. The main object of our study is the symmetric space $X = G/K$ of non-compact type. On X with a natural G-invariant measure dx we have the integral formula

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_+} f(k \exp(H)) \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha} dH dk. \tag{II.3}$$

(see [Hel84, Ch. I Theorem 5.8]).

Example II.5.1. If $G = SL_n(\mathbb{R})$, then we choose $K = SO(n)$, A as the set of diagonal matrices of positive entries with determinant 1, and N as the set of upper triangular matrices with 1's on the diagonal. $\mathfrak{a}$ is the abelian Lie algebra of diagonal matrices and the set of restricted roots is $\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i(\lambda)$ is the i-th diagonal entry of λ. The positive system corresponding to the Iwasawa decomposition is $\Sigma^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ with simple system $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}$. The positive Weyl chamber is $\mathfrak{a}_+ = \{\mathrm{diag}(\lambda_1, \dots, \lambda_n) \mid \lambda_1 > \dots > \lambda_n\}$ and the Weyl group is the symmetric group S_n acting by permutation of the diagonal entries.

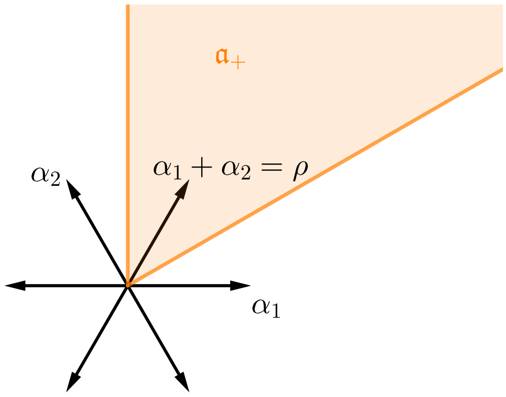

Figure II.2.: The root system for the special case $G = SL_3(\mathbb{R})$: There are three positive roots $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. As all root spaces are one dimensional the special element $\rho = \frac{1}{2}\Sigma_{\alpha \in \Sigma^+} m_\alpha \alpha$ equals $\alpha_1 + \alpha_2$.

II.5.2. Invariant differential operators

Let $\mathbb{D}(G/K)$ be the algebra of *G-invariant differential operators* on G/K, i.e. differential operators commuting with the left translation by elements $g \in G$. Then we have an algebra isomorphism $\mathrm{HC}\colon \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ from $\mathbb{D}(G/K)$ to the W-invariant complex polynomials on $\mathfrak{a}^*$ which is called the *Harish-Chandra homomorphism* (see [Hel84, Ch. II Theorem 5.18]). For $\lambda \in \mathfrak{a}^*_\mathbb{C}$ let χ_λ be the character of $\mathbb{D}(G/K)$ defined by $\chi_\lambda(D) \coloneqq \mathrm{HC}(D)(\lambda)$. Obviously, $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$. Furthermore, the χ_λ exhaust all characters of $\mathbb{D}(G/K)$ (see [Hel84, Ch. III Lemma 3.11]). We define the space of joint eigenfunctions

$$E_\lambda \coloneqq \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D)f \quad \forall D \in \mathbb{D}(G/K)\}.$$

Note that E_λ is G-invariant.

Example II.5.2. For $G = SL_n(\mathbb{R})$ the algebra $\mathrm{Poly}(\mathfrak{a}^*_\mathbb{C})^W$ is generated by $n-1$ elements $p_2, \dots, p_n$. Let us identify $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}^*_\mathbb{C}$ via $\lambda \leftrightarrow \mathrm{Tr}(\lambda\, \cdot)$. Then $p_i(\lambda) = \lambda_1^i + \dots + \lambda_n^i = \mathrm{Tr}(\lambda^i)$ where $\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{a}_\mathbb{C}$. Clearly, these polynomials are invariant under permutations of the diagonal entries and it can be shown that they are algebraically independent and generate $\mathrm{Poly}(\mathfrak{a}_{\mathbb{C}}^*)^W$ (see [Hum90]). $\mathbb{D}(G/K)$ is then generated by the preimages of p_i under HC. Up to lower order terms the resulting invariant differential operators are given by the Maass-Selberg operators δ_i which are defined for $f \in C^\infty(G/K) = C^\infty(SL_n(\mathbb{R})/SO(n))$ by

$$\delta_i f(gK) = \mathrm{Tr}\left(\left(\frac{\partial}{\partial X}\right)^i\right)\Bigg|_{X=0} f\left(g \exp\left(X - \frac{1}{n}\mathrm{Tr}(X) I_n\right) K\right),$$

where

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial X} = \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \cdots & \frac{\partial}{2\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{2\partial x_{1n}} & \cdots & \frac{\partial}{\partial x_{nn}} \end{pmatrix}$$

(see [BCH21]).

Now, let $\Gamma \leq G$ be a torsion-free discrete subgroup. Since $D \in \mathbb{D}(G/K)$ is G-invariant, it descends to a differential operator ${}_\Gamma D$ on the locally symmetric space $\Gamma\backslash G/K$. Therefore, the left Γ-invariant functions of E_λ (denoted by ${}^\Gamma E_\lambda$) can be identified with joint eigenfunctions on $\Gamma\backslash G/K$ for each ${}_\Gamma D$:

$${}^\Gamma E_\lambda = \{f \in C^\infty(\Gamma\backslash G/K) \mid {}_\Gamma D f = \chi_\lambda(D) f \quad \forall\, D \in \mathbb{D}(G/K)\}.$$

The goal is to show that $L^2(\Gamma\backslash G/K) \cap {}^\Gamma E_\lambda = \{0\}$ for $\lambda \in i\mathfrak{a}^*$ and certain discrete subgroups Γ. Then

$$\sigma(\Gamma\backslash X) \coloneqq \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid L^2(\Gamma\backslash G/K) \cap {}^\Gamma E_\lambda \neq \{0\}\}$$

has the property that the set of principal eigenvalues $\sigma(\Gamma\backslash X) \cap i\mathfrak{a}^*$ is empty.

II.5.3. Geodesic compactification

In this section we recall the notion of the geodesic compactification of a simply connected and non-positively curved Riemannian manifold X. A classical reference for this topic is [Ebe96]. In the sequel also the Satake compactification will be crucial thus we provide detailed references to [BJ06] which treats both types of compactifications.

Definition II.5.3 ([BJ06, Section I.2.2])**.** Two (unit speed) geodesics γ_1, γ_2 are equivalent if $\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty$. The space $X(\infty)$ is the factor space of all geodesics modulo this equivalence relation. The union $X \cup X(\infty)$ is called *geodesic compactification*. The topology on $X \cup X(\infty)$ is given as follows: For $[\gamma] \in X(\infty)$ the intersection with X of a fundamental system of neighborhoods is given by $C(\gamma, \varepsilon, R) = C(\gamma, \varepsilon) \smallsetminus B(R)$ where

$$C(\gamma, \epsilon) = \{x \in X \mid \text{the angle between } \gamma \text{ and the geodesic from } x_0 \text{ to } x \text{ is less than } \varepsilon\}$$

and $B(R)$ is the ball of radius R centered at some base point $x_0 \in X$. This topology is Hausdorff and compact.

The space $X(\infty)$ can be canonically identified with the unit sphere in the tangent space at the base point $x_0 \in X$. If $\exp\colon T_{x_0}X \to X$ is the (Riemannian) exponential map at x_0, then a representative of the equivalence class of geodesics corresponding to a unit vector $Y \in T_{x_0}X$ is given by the geodesic $t \mapsto \exp(tY)$. This identification yields the neighborhoods $C(Y_0, \varepsilon, R) = \{\exp tY \mid t > R, \|Y\| = 1, |\cos^{-1}(\langle Y, Y_0\rangle)| < \varepsilon\}$ where $Y_0 \in T_{x_0}X$ is normalized. More precisely, if γ is the geodesic $t \mapsto \exp(tY_0)$ then $C(\gamma, \varepsilon, R) = C(Y_0, \varepsilon, R)$.

Let us return to the setting where $X = G/K$ is a symmetric space of non-compact type, then X is simply connected and non-positively curved. Hence, the geodesic compactification of X is defined and we have the following proposition.

Proposition II.5.4 ([BJ06, Proposition I.2.5])**.** *The action of G on X extends to a continuous action on $X \cup X(\infty)$.*

II.5.4. Maximal Satake compactification

In this section we introduce a different compactification for a Riemannian symmetric space $X = G/K$ the so called maximal Satake compactification. Before entering the technicalities let us give some heuristics: Recall that the Cartan decomposition allows to write $G = K\overline{\exp \mathfrak{a}_+}K$ and since K is compact the "way" in which a point in G/K tends to infinity can be described in $\mathfrak{a}_+$. Recall that the particular simplicity of a rank one locally symmetric space stems from the fact that $\mathfrak{a}_+$ is just a half line (geometrically it corresponds to the distance from the origin of the symmetric space) and there is only one "way" to tend towards infinity. In the higher rank case $\mathfrak{a}_+$ is a higher dimensional simplicial cone bounded by the hyperplanes $\ker \alpha \subset \mathfrak{a}$ for $\alpha \in \Pi$ and the Satake compactifications will "detect" if a sequence tends to infinity inside the cone, while staying at bounded distance to a certain number of chamber walls $\ker \alpha$ for some subset $\alpha \in I \subsetneq \Pi$.

In order to describe the precise structure of the Satake compactification we need to introduce the following notion of standard parabolic subgroups:

For $I \subsetneq \Pi$ let $\mathfrak{a}_I := \bigcap_{\alpha \in I} \ker \alpha$, $\mathfrak{a}^I := \mathfrak{a}_I^\perp$, $\mathfrak{n}_I := \bigoplus_{\alpha \in \Sigma^+ \smallsetminus \langle I \rangle} \mathfrak{g}_\alpha$ and $\mathfrak{m}_I := \mathfrak{m} \oplus \mathfrak{a}^I \oplus \bigoplus_{\alpha \in \langle I \rangle} \mathfrak{g}_\alpha$. Define the subgroups $A_I := \exp \mathfrak{a}_I$, $N_I := \exp \mathfrak{n}_I$ and $M_I := M\langle \exp \mathfrak{m}_I \rangle$. Then $P_I := M_I A_I N_I$ is the standard parabolic subgroup for the subset I. We furthermore introduce the notation $\mathfrak{a}^I_+ := \{H \in \mathfrak{a}^I \mid \alpha(H) > 0 \; \forall \alpha \in I\}$ and $\mathfrak{a}_{I,+} := \{H \in \mathfrak{a}_I \mid \alpha(H) > 0 \; \forall \alpha \in \Pi \smallsetminus I\}$.

Example II.5.5. For $G = SL_n(\mathbb{R})$ the set of simple roots is $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\}$. Let $I = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ be a proper subset of Π. Then $\mathfrak{a}_I = \{\operatorname{diag}(\lambda_1, \ldots, \lambda_n) \mid \lambda_{i_j} = \lambda_{i_j+1}\}$ and $\mathfrak{a}^I = \bigoplus_j \{\operatorname{diag}(0, \ldots, \lambda_{i_j}, -\lambda_{i_j+1}, \ldots, 0)\}$. Note that $\mathfrak{a}^I = \operatorname{span} \alpha_{i_j}$ if one identifies $\mathfrak{a}$ and $\mathfrak{a}^*$ (see Figure II.3 for an illustration). Hence, $\mathfrak{a}^I$ consists of blocks where a single block is a copy of the $\mathfrak{a}$-part of $SL_m(\mathbb{R})$. Each block corresponds to a root in $\Pi \smallsetminus I$. More precisely, if $\alpha_i \in \Pi \smallsetminus I$ then a block ends in row i. Note that the m_i can very well be equal to 1. In this case there is simply a zero at this point on the

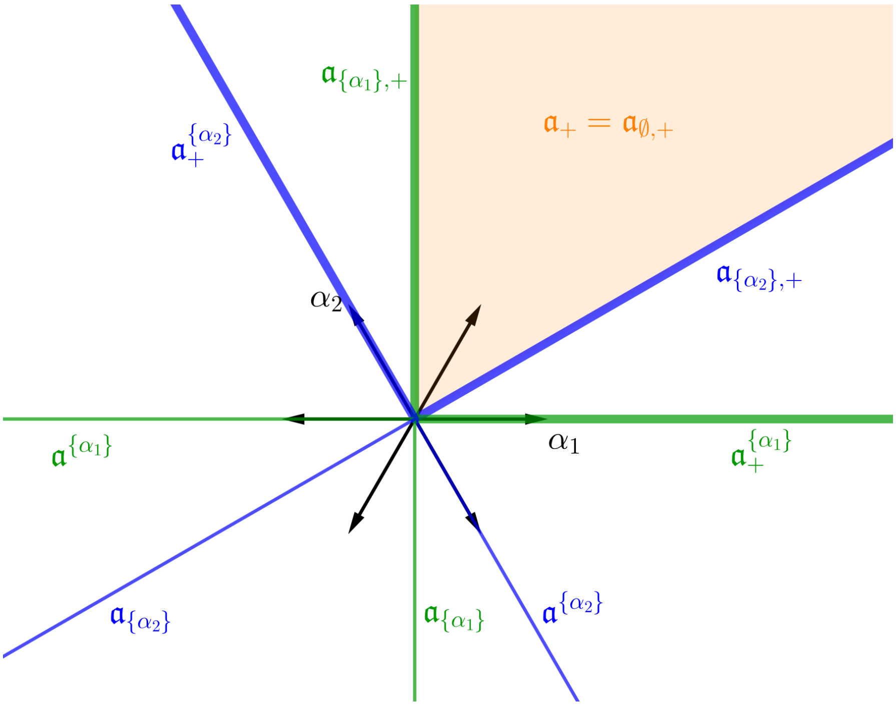

Figure II.3.: The various cones and subspaces in $\mathfrak{a}$ corresponding to subsets of Π for $G = SL_n(\mathbb{R})$. $\mathfrak{a}_\emptyset$ is all of $\mathfrak{a}$ and $\mathfrak{a}^\emptyset$ is the origin.

diagonal. $\mathfrak{m}_I$ adds the corresponding root spaces, so $\mathfrak{m}_I$ is isomorphic to direct sum of different $\mathfrak{sl}_m(\mathbb{R})$.

$$\mathfrak{m}_I = \begin{pmatrix} \mathfrak{sl}_{m_1}(\mathbb{R}) & & \\ & \ddots & \\ & & \mathfrak{sl}_{m_{n-1-k}}(\mathbb{R}) \end{pmatrix}$$

where the bottom rows of the blocks correspond to the index of the roots in $\Pi \setminus I$. $\mathfrak{n}_I$ is the Lie algebra that contains of the upper-triangular matrices with non-zero entries in the positions that are not in the blocks of $\mathfrak{m}_I$. On the group level $A_I = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in A \mid \lambda_{i_j} = \lambda_{i_j+1}\}$ and N_I is the same as $\mathfrak{n}_I$ but with 1's on the diagonal. For M_I one has to multiply by $M = \{\mathrm{diag}(\pm 1, \ldots, \pm 1)\}$ so that M_I consists of block diagonal matrices where each block has determinant ± 1 under the condition that the whole matrix has determinant 1. It follows that the standard parabolic subgroups P_I are the sets of block upper-triangular matrices:

$$\begin{bmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \ddots & \\ 0 & & & \boxed{*} \end{bmatrix}$$

The maximal Satake compactification $\overline{X}^{\max}$ is the G-compactification of X (i.e. a compact Hausdorff space containing X as an open dense subset such that the G-action extends continuously from X to the compactification) with the orbit structure $\overline{X}^{\max} = X \cup \bigcup_{I \subsetneq \Pi} \mathcal{O}_I$. For the orbit $\mathcal{O}_I$ we can choose a base point $x_I \in \mathcal{O}_I$ with $\operatorname{Stab}(x_I) = N_I A_I (M_I \cap K)$. The topology can be described as follows: Since $G = K\overline{A^+}K$ and K is compact, it suffices to consider sequences $\exp H_n$, $H_n \in \overline{\mathfrak{a}_+}$. Such a sequence by definition converges iff $\alpha(H_n)$ converges in $\mathbb{R} \cup \{\infty\}$ for all $\alpha \in \Pi$. If this is the case, to determine the limit, let $I = \{\alpha \in \Pi \mid \lim \alpha(H_n) < \infty\}$ and $H_\infty \in \overline{\mathfrak{a}_+^I}$ such that $\alpha(H_\infty) = \lim \alpha(H_n)$ for $\alpha \in I$. Then $\exp H_n \to \exp(H_\infty) x_I$.

The intersection with X of a fundamental system of neighborhoods of $k \exp(H_\infty) x_I$ with $k \in K, H_\infty \in \overline{\mathfrak{a}_+^I}$ is given by

$$V(K \cap M_I) \exp\{H \in \overline{\mathfrak{a}_+} \mid |\alpha(H) - \alpha(H_\infty)| < \varepsilon, \alpha \in I, \alpha(H) > R, \alpha \notin I\} x_0,$$

where V is a fundamental system of neighborhoods of k in K, $\varepsilon \searrow 0$, $R \nearrow \infty$.

Note that usually one defines the Satake compactification in a different way (see e.g. [BJ06, Ch. I.4]). Namely, let $\tau \colon G \to PSL(n, \mathbb{C})$ be an irreducible faithful projective representation such that $\tau(K) \subseteq PSU(n)$. The closure in the projective space of Hermitian matrices of the image of the embedding of X given by $gK \mapsto \mathbb{R}(\tau(g)\tau(g)^*)$ is then called Satake compactification. It only depends on the highest weight χ_τ of τ. If χ_τ is contained in the interior of the Weyl chamber, then this compactification is isomorphic to the maximal Satake compactification defined above. It is maximal in the sense that it dominates every other Satake compactification $\overline{X}^S$ (i.e. there is a continuous G-equivariant map $\overline{X}^{\max} \to \overline{X}^S$). Since we only need the description of neighborhoods and the orbit structure we chose to introduce $\overline{X}^{\max}$ this way.

II.6. Moderate growth

In this section we show that on a locally symmetric space each joint eigenfunction which is L^2 satisfies a growth condition in the following sense.

Definition II.6.1. (i) A function $f \colon X \to \mathbb{C}$ is called *function of moderate growth* if there exist $r \in \mathbb{R}, C > 0$ such that

$$|f(x)| \leq C e^{r d(x, x_0)}$$

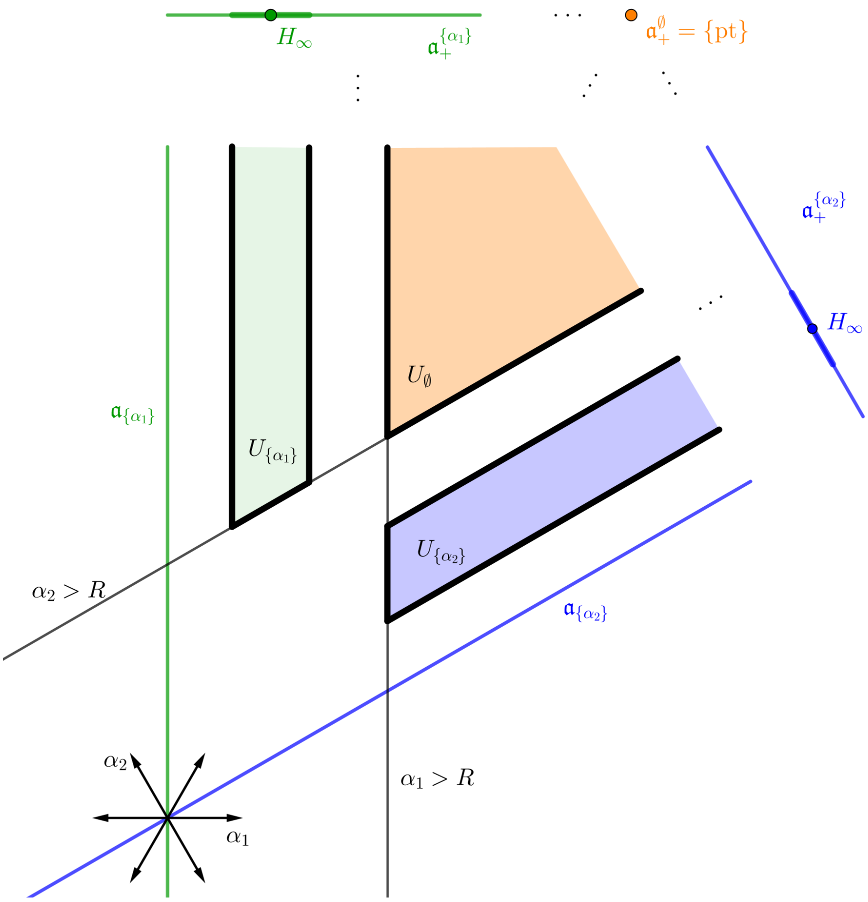

Figure II.4.: The compactification of $\mathfrak{a}_+$ for $G = SL_3(\mathbb{R})$ is obtained by gluing $\overline{\mathfrak{a}_+^{\{\alpha_1\}}}, \overline{\mathfrak{a}_+^{\{\alpha_2\}}}$ and $\overline{\mathfrak{a}_+^{\emptyset}}$ to the boundary of $\overline{\mathfrak{a}_+}$. The sets U_I for $I = \{\alpha_1\}, \{\alpha_2\}, \emptyset$ are the intersection of $\mathfrak{a}_+$ with a fundamental neighborhood of $\exp(H_\infty)x_I$.

for all $x \in X$.

(ii) For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the space E_λ^* is the space of joint eigenfunction with moderate growth, i.e.

$$E_\lambda^* = \{f \in E_\lambda \mid f \text{ has moderate growth}\}.$$

Let $\Gamma \leq G$ be a torsion-free discrete subgroup.

Theorem II.6.2. *Let $f \in {}^\Gamma E_\lambda \cap L^2(\Gamma\backslash G/K)$. Then f (considered as a Γ-invariant function on X) has moderate growth.*

The proof uses Sobolev embedding and the following estimate on the injectivity radius.

Proposition II.6.3 (see [CGT82, Thm. 4.3]). *Let (M, g) be a complete Riemannian manifold such that the sectional curvature K_M satisfies $K_M \leq K$ for constant $K \in \mathbb{R}$. Let $0 < r < \pi/4\sqrt{K}$ if $K > 0$ and $r \in (0, \infty)$ if $K \leq 0$. Then the injectivity radius* $\operatorname{inj}(p)$ *at p satisfies*

$$\operatorname{inj}(p) \geq \frac{r\mathrm{Vol}(B_M(p,r))}{\mathrm{Vol}(B_M(p,r)) + \mathrm{Vol}_{T_pM}(B_{T_pM}(0,2r))},$$

where $\mathrm{Vol}_{T_pM}(B_{T_pM}(0,2r))$ *denotes the volume of the ball of radius $2r$ in T_pM, where both the volume and the distance function are defined using the metric $g^* := exp_p^* g$, i.e. the pull-back of the metric g to T_pM via the exponential map.*

For $M = \Gamma\backslash G/K$ we obtain that the injectivity radius decreases at most exponentially.

Proposition II.6.4. *There are constants $C, s > 0$ such that*

$$\operatorname{inj}_{\Gamma\backslash G/K}(\Gamma x) \geq C^{-1} e^{-sd(x,eK)}$$

for every $x \in G/K$.

Proof. Since $\Gamma\backslash G/K$ is of non-positive curvature we can apply the above proposition for every $r > 0$. Note that $\exp\colon T_pM \to M$ is the universal cover of M and therefore $\mathrm{Vol}_{T_{\Gamma x}M}(B_{T_{\Gamma x}M}(0,2r)) = \mathrm{Vol}_{G/K}(B_{G/K}(x,2r)) = \mathrm{Vol}_{G/K}(B_{G/K}(x_0,2r)) \leq Ce^{sr}$ for some constants C, s independent of x, where x_0 is the base point eK of G/K. Hence,

$$\begin{aligned}\operatorname{inj}(\Gamma x) &\geq r(1 + \mathrm{Vol}_{T_{\Gamma x}M}(B_{T_{\Gamma x}M}(0,2r))/\mathrm{Vol}(B_M(\Gamma x, r)))^{-1} \\ &\geq r(1 + Ce^{sr}/\mathrm{Vol}(B_M(\Gamma x, r)))^{-1}.\end{aligned}$$

For $r = 1 + d(x, x_0)$ we have $B_M(\Gamma x, r) \supseteq B_M(\Gamma x_0, 1)$ and therefore

$$\operatorname{inj}(\Gamma x) \geq (1 + d(x,x_0))(1 + Ce^{s(1+d(x,x_0))}/\mathrm{Vol}(B_M(\Gamma x_0, 1))^{-1} \geq (1 + C' e^{sd(x,x_0)})^{-1}.$$

This finishes the proof. □

Note that this estimate isn't sharp. Indeed, the growth rate s that we obtain in the proof is independent of Γ and only depends on the volume growth in G/K.

Let $m = \dim X$. We need the following well-known lemma on the geodesic balls in G/K.

Lemma II.6.5. *Fix $r > 0$. There is a constant C such that for every $x \in G/K$ and $\varepsilon > 0$ there is a finite set $A \subseteq B(x, r)$ such that $\bigcup_{a\in A} B(a, \varepsilon) \supseteq B(x, r)$ and $\#A \leq C\varepsilon^{-m}$.*

Proof. Let $a_1 = x$ and choose inductively $a_{i+1} \in B(x,r) \smallsetminus \bigcup_{j=0}^{i} B(a_j, \varepsilon)$ if the latter is non-empty. This yields a finite set $A = \{a_1, \ldots, a_N\}$ (since $\overline{B(x,r)}$ is compact) such that $B(x, r + \varepsilon) \supseteq \bigcup_{j=0}^{N} B(a_j, \varepsilon) \supseteq B(x, r)$ and $B(a_j, \varepsilon/2)$ are pairwise disjoint. It follows that $\mathrm{Vol}(B(x, r + \varepsilon)) \geq \sum_i \mathrm{Vol}(B(a_i, \varepsilon/2) = \#A \cdot \mathrm{Vol}(B(x, \varepsilon/2))$ and therefore $\#A \leq \frac{C}{\mathrm{Vol}(B(x,\varepsilon/2))}$. The lemma follows from the fact that the volume is independent from the center and decreases like ε^m as $\varepsilon \to 0$. □

We can now combine Proposition II.6.4 with Sobolev embedding to prove Theorem II.6.2.

Proof of Theorem II.6.2. Since $B(x_0, 1)$ is relatively compact, there exists a constant C such that

$$\sup_{x\in B(x_0,1)} |f(x)| \le C\|(\Delta+1)^{m/4+\varepsilon} f\|_{L^2(B(x_0,1))} = C(\chi_\lambda(\Delta)+1)^{m/4+\varepsilon}\|f\|_{L^2(B(x_0,1))}$$

by ellipticity of the Laplace operator Δ on G/K and the Sobolev embedding

$$H^{m/2+\varepsilon}(B(x_0,1)) \hookrightarrow C(B(x_0,1)).$$

By G-invariance of Δ and d the same holds true for x_0 replaced by an arbitrary point $x \in X$. In particular,

$$|f(x)| \le C(\lambda)\|f\|_{L^2(B(x,1))}.$$

By Proposition II.6.4 there are constants $C, s > 0$ independent of x such that

$$\mathrm{inj}_{\Gamma\backslash G/K}(\Gamma y) \ge C^{-1} e^{-sd(x,eK)}$$

for every $y \in B(x, 1)$. Let $\varepsilon(x) := \frac{1}{C} e^{-sd(x,eK)}$. Then there is a finite set $A(x) \subseteq B(x, 1)$ such that $\bigcup_{a\in A(x)} B(a, \varepsilon(x))$ covers $B(x, 1)$ and $\#A(x) \le C'\varepsilon(x)^{-m}$ by Lemma II.6.5. Hence,

$$\|f\|^2_{L^2(B(x,1))} \le \sum_{a\in A(x)} \|f\|^2_{L^2(B(a,\varepsilon(x)))}$$

Since $\mathrm{inj}_{\Gamma\backslash G/K}(\Gamma a) \ge \varepsilon(x)$ we have $\|f\|_{L^2(B(a,\varepsilon(x)))} \le \|f\|_{L^2(\Gamma\backslash G/K)}$ for $a \in A(x)$. Therefore,

$$\begin{aligned} |f(x)| &\le C(\lambda)\|f\|_{L^2(\Gamma\backslash G/K)}\sqrt{\#A(x)} \\ &\le C(\lambda)\|f\|_{L^2(\Gamma\backslash G/K)} C'^{1/2}\varepsilon(x)^{-m/2} \\ &= C(\lambda)\|f\|_{L^2(\Gamma\backslash G/K)} C'^{1/2} C^{m/2} e^{msd(x,x_0)/2}. \end{aligned}$$ □

Remark II.6.6. In the case of locally symmetric spaces of finite volume there is a different argument showing Theorem II.6.2: If we lift f to a function on G which we also call f, then there is smooth compactly supported function α on G such that $f = f * \alpha$ (see [HC66, Theorem 1]). Then one easily shows that $|f(\Gamma x)| \le C\|f\|_{L^1(\Gamma\backslash G/K)} e^{sd(x,x_0)}$ using simple estimates for lattice point counting. Since $L^2 \subseteq L^1$ for spaces of finite volume, we can deduce moderate growth for f. Unfortunately, this argument does not work for infinite volume locally symmetric spaces since a pointwise bound including the L^2-norm of f would need much better counting estimates.

II.7. Absence of imaginary values in the L^2-spectrum

We introduce the space of smooth vectors in E_λ. It is precisely the space of joint eigenfunctions with smooth boundary values (see [vdBS87]).

Definition II.7.1.

$$E_\lambda^\infty = \{f \in E_\lambda \mid \exists r \; \forall u \in \mathcal{U}(\mathfrak{g}) \; \exists C_u > 0 \colon |(uf)(x)| \leq C_u e^{rd(x,x_0)}\}$$

II.7.1. Geodesic compactification

In this section we want to prove the following theorem.

Theorem II.7.2. *Let $f \in E_\lambda^*$, $\lambda \in i\mathfrak{a}^*$, such that f is square-integrable on $C(Y_0, \varepsilon, R)$ for some ε, R, Y_0 (see Section II.5.3). Then $f = 0$.*

Let $X(\lambda) := \{w\lambda - \rho - \mu \mid w \in W, \mu \in \mathbb{N}_0\Pi\}$ (see Figure II.5 for a visualization in example of $SL(3,\mathbb{R})$). We will use the following asymptotic expansion for functions in E_λ^∞.

Theorem II.7.3 ([vdBS87, Thm 3.5])**.** *For each $f \in E_\lambda^\infty$, $g \in G$, and $\xi \in X(\lambda)$ there is a unique polynomial $p_{\lambda,\xi}(f,g)$ on $\mathfrak{a}$ which is smooth in g such that*

$$f(g\exp(tH)) \sim \sum_{\xi \in X(\lambda)} p_{\lambda,\xi}(f,g,tH)e^{t\xi(H)}, \quad t \to \infty,$$

at every $H_0 \in \mathfrak{a}_+$, i.e. for every N there exist a neighborhood U of H_0 in $\mathfrak{a}_+$, a neighborhood V of g in G, $\varepsilon > 0$, $C > 0$ such that

$$\left| f(y\exp(tH)) - \sum_{\operatorname{Re}\xi(H_0) \geq -N} p_{\lambda,\xi}(f,y,tH)e^{t\xi(H)} \right| \leq Ce^{(-N-\varepsilon)t}$$

for all $y \in V, H \in U$, $t \geq 0$.

Remark II.7.4. The uniformity in x is not stated in [vdBS87] but it follows from (6.18) therein.

Example II.7.5. In the case where G/K is the upper half plane $\mathbb{H}$ a simplified version of this theorem can be stated as follows. Suppose $f \in E_{s-1/2}^\infty$, i.e. $f \in C^\infty(\mathbb{H})$ with $\Delta f = s(1-s)f$ and the derivatives of f satisfy some uniform pointwise exponential bounds. We lift f to a function (also called f) on the sphere bundle $S\mathbb{H}$ which is constant on the fibers. Denote by ϕ_t the geodesic flow. Then if $s \notin \frac{1}{2}\mathbb{Z}$

$$(\phi_t)_* f(x) \sim e^{-ts}\left(\sum_{n=0}^\infty p_n^+(x)e^{-nt}\right) + e^{-t(1-s)}\left(\sum_{n=0}^\infty p_n^-(x)e^{-nt}\right)$$

with $p_n^\pm$ being smooth. If $s \in \frac{1}{2}\mathbb{Z}$ the functions $p_n^\pm$ can be polynomials of degree one in t.

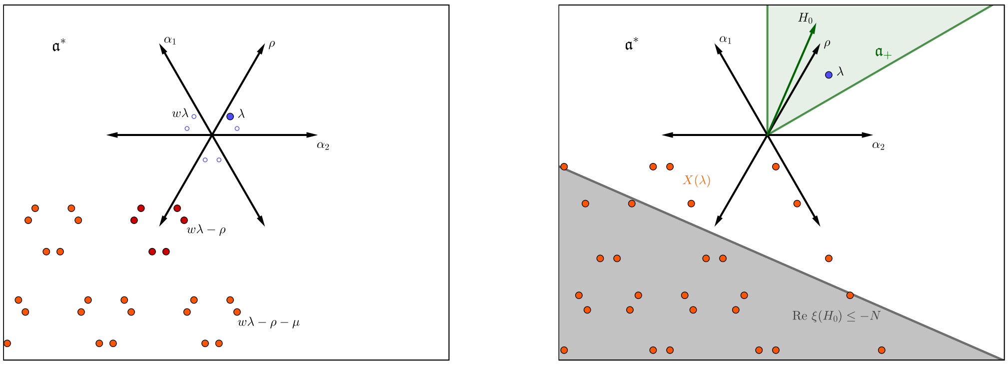

Figure II.5.: Real part of the exponents of the asymptotic expansion in Theorem II.7.3 for $G = SL(3,\mathbb{R})$.

Proof of Theorem II.7.2. First, we will consider the case $f \in E_\lambda^\infty$. By continuity there is a unit vector $H_0 \in \mathfrak{a}_+$, a neighborhood U of H_0 in the unit sphere of $\mathfrak{a}$, and an open set V in K such that

$$\Omega = \left\{ k\exp(H) \colon k \in V, \frac{H}{\|H\|} \in U, \|H\| > R \right\} \subseteq C(Y_0, \varepsilon, R).$$

Let $N = \rho(H_0)$ such that without loss of generality

$$|f(k\exp(H)) - \sum_{w\in W} p_{\lambda, w\lambda-\rho}(f,k,H)e^{(w\lambda-\rho)(H)}| \leq Ce^{(-\rho(H_0)-\varepsilon)\|H\|} \tag{II.4}$$

for all $k \in V$, $\frac{H}{\|H\|} \in U$.

We use the integral formula (II.3) and observe that

$$\int_{(R,\infty)U} e^{-2(\rho(H_0)+\varepsilon)\|H\|} \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha} dH$$
$$\leq \int_{(R,\infty)U} e^{-2(\rho(H_0)+\varepsilon)\|H\|} e^{2\rho(H)} dH \leq \int_{(R,\infty)U} e^{2(\rho(\frac{H}{\|H\|}-H_0)-\varepsilon)\|H\|} dH$$

which is finite after shrinking U such that $\rho(\frac{H}{\|H\|} - H_0) < \varepsilon$ for $H \in U$. Consequently, the right hand side of (II.4) and therefore also the left hand side of (II.4) is square integrable on Ω.

Since f is L^2 and the approximation (II.4) holds,

$$\left| \sum_{w\in W} p_{\lambda,w\lambda-\rho}(f,k,H)e^{(w\lambda-\rho)(H)} \right|^2 \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha}$$

is integrable on $V \times (R,\infty)U$. Hence,

$$\left| \sum_{w\in W} p_{\lambda,w\lambda-\rho}(f,k,H)e^{(w\lambda-\rho)(H)} \right|^2 \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha}$$

is integrable on $(R,\infty)U$ for almost every $k \in V$. Since $\sinh(x) \geq e^x/4$ for $x \geq \frac{1}{2}\log 2$,

$$\left|\sum_{w\in W} p_{\lambda,w\lambda-\rho}(f,k,H)e^{(w\lambda-\rho)(H)}\right|^2 e^{2\rho(H)} = \left|\sum_{w\in W} p_{\lambda,w\lambda-\rho}(f,k,H)e^{w\lambda(H)}\right|^2$$

is integrable on $(R,\infty)U$ for R large enough. This is only possible if $p_{\lambda,w\lambda-\rho}(f,k)$ vanishes on $\mathfrak{a}$ for every $w \in W$ by [Kna86, Lemma 8.50]. Since $p_{\lambda,w\lambda-\rho}(f,\bullet)$ is smooth it vanishes identically on V.

We now show that it also vanishes on VAN. For $n \in N$ [vdBS87, Lemma 8.7] states for $f \in E_\lambda^\infty$

$$p_{\lambda,\xi}(f,n) = \sum_{\mu\in\mathbb{N}_0\Pi,\xi+\mu\in X(\lambda)} p_{\lambda,\xi+\mu}(f_\mu,e), \quad \xi \in X(\lambda),$$

where $f_\mu \in L(\mathcal{U}(\mathfrak{g}))f$ (where L is the left regular representation) are specific joint eigenfunctions obtained by the Taylor expansion of f in the direction of n and $f_0 = f$. For $\xi = w\lambda - \rho$ the only summand comes from $\mu = 0$ since $\lambda \in i\mathfrak{a}^*$ and $X(\lambda) = \{w\lambda - \rho - \mu \mid w \in W, \mu \in \mathbb{N}_0\Pi\}$. In particular, $p_{\lambda,w\lambda-\rho}(f,n) = p_{\lambda,w\lambda-\rho}(f,e)$.

To deal with $a \in A$ we use [vdBS87, Lemma 8.5]:

$$p_{\lambda,\xi}(f,a,H) = a^\xi p_{\lambda,\xi}(f,e,H+\log a), \quad f \in E_\lambda^\infty, \xi \in X(\lambda), H \in \mathfrak{a},$$

where as usual $a^\xi = e^{\xi(\log a)}$.

Let us return to the situation that we achieved earlier, where $p_{\lambda,w\lambda-\rho}(f,k,H) = 0$ for every $k \in V$ and $H \in \mathfrak{a}$. But then

$$\begin{aligned} p_{\lambda,w\lambda-\rho}(f,kan,H) &= p_{\lambda,w\lambda-\rho}(L_{(ka)^{-1}}f,n,H) = p_{\lambda,w\lambda-\rho}(L_{(ka)^{-1}}f,e,H) \\ &= p_{\lambda,w\lambda-\rho}(L_{k^{-1}}f,a,H) = a^{w\lambda-\rho}p_{\lambda,w\lambda-\rho}(L_{k^{-1}}f,e,H) \\ &= a^{w\lambda-\rho}p_{\lambda,w\lambda-\rho}(f,k,H) = 0 \end{aligned}$$

for every $k \in V, a \in A, n \in N$ and $w \in W$. Hence, $p_{\lambda,w\lambda-\rho}(f,x) = 0$ if x is contained in the open set VAN. This is exactly the assumption of [vdBS89, Theorem 4.1] in the case $I = I_\lambda$, i.e. f is an eigenfunction for the whole algebra $\mathbb{D}(G/K)$ and is not only annihilated by an ideal of finite codimension. Note that in this case $X(I) = X(\lambda)$. We infer $f = 0$.

It remains to show that the statement also holds for $f \in E_\lambda^*$.

Since $C(Y_0,\varepsilon,R)$ is a fundamental system of neighborhoods of Y_0 in the geodesic compactification and G acts continuously on $X \cup X(\infty)$, there is a neighborhood V of e in G and ε', R' such that $V^{-1}C(Y_0,\varepsilon',R') \subseteq C(Y_0,\varepsilon,R)$. Let φ_n be an approximate identity on G with $\operatorname{supp}\varphi_n \subseteq V$, i.e. $\varphi_n \in C_c^\infty(G)$ is non-negative with $\int_G \varphi_n(g)dg = 1$ and $\operatorname{supp}(\varphi_n)$ shrinks to $\{e\}$. We consider $(\varphi_n * f)(x) = \int_G \varphi_n(g)f(g^{-1}x)dg$. Obviously, $\varphi_n * f \in E_\lambda^\infty$ since $L_xR_y(\varphi_n * f) = (L_x\varphi_n) * (R_yf), x,y \in G$.

Combining the already established case $f \in E_\lambda^\infty$ with Lemma II.7.6 below we infer that $\varphi_n * f = 0$ for all n and therefore $f = 0$. This completes the proof. □

Lemma II.7.6. *$\varphi_n * f$ is square-integrable on $C(Y_0, \varepsilon', R')$.*

Proof of Lemma II.7.6. Abbreviate $C' = C(Y_0, \varepsilon', R')$ and $C = C(Y_0, \varepsilon', R')$. It suffices to show that

$$\left|\int_{C'} h(x)(\varphi_n * f)(x)dx\right| \leq B\|h\|_{L^2(C')}$$

for $h \in C_c(C')$ with a constant B independent of h.

Let us write $|h(x)\varphi_n(g)f(g^{-1}x)| = (|h|^2(x)\varphi_n(g))^{1/2}(|f|^2(g^{-1}x)\varphi_n(g))^{1/2}$ and use the Cauchy-Schwarz inequality of $L^2(V \times C')$ to obtain

$$\begin{aligned}\left|\int_{C'} h(x)(\varphi_n * f)(x)dx\right| &\leq \int_V \int_{C'} |h(x)\varphi_n(g)f(g^{-1}x)|dxdg \\ &\leq \left(\int_V \int_{C'} |h|^2(x)\varphi_n(g)dxdg \int_V \int_{C'} |f|^2(g^{-1}x)\varphi_n(g)dxdg\right)^{1/2} \\ &\leq \|h\|_{L^2(C')} \left(\int_V \int_C |f|^2(x)\varphi_n(g)dxdg\right)^{1/2} \\ &= \|h\|_{L^2(C')}\|f\|_{L^2(C)}\end{aligned}$$

where we used $V^{-1}C' \subseteq C$ in the last inequality. This finishes the proof. □

II.7.2. Maximal Satake compactification

In this section we prove a statement analogous to Theorem II.7.2 for the maximal Satake compactification. First of all we remark that each neighborhood of an element in the orbit $Gx_\emptyset \subseteq \overline{X}^{\max}$ contains a neighborhood $C(Y_0, \varepsilon, R)$. Hence, we have the following proposition.

Proposition II.7.7. *Let $f \in E^*_\lambda$, $\lambda \in i\mathfrak{a}^*$, such that f is square-integrable in some neighborhood of an element in $Gx_\emptyset \subseteq \overline{X}^{\max}$. Then $f = 0$.*

The goal is to prove this statement for general neighborhoods in $\overline{X}^{\max}$.

Theorem II.7.8. *Let $f \in E^*_\lambda$, $\lambda \in i\mathfrak{a}^*$, such that f is square-integrable in some neighborhood of an element $x_\infty \in \partial X^{\max}$. Then $f = 0$.*

Proof. By the same reasoning as in the proof of Theorem II.7.2 we can assume $f \in E^\infty_\lambda$. Moreover, we can assume that $x_\infty = k\exp(H_\infty)x_I$ with $k \in K$ and $H_\infty \in \mathfrak{a}^I_+$ (instead of $H_\infty \in \overline{\mathfrak{a}^I_+}$) since every neighborhood of $k\exp(H_\infty)x_I$ contains an element $k'\exp(H'_\infty)x_I$ with $H'_\infty \in \mathfrak{a}^I_+$.

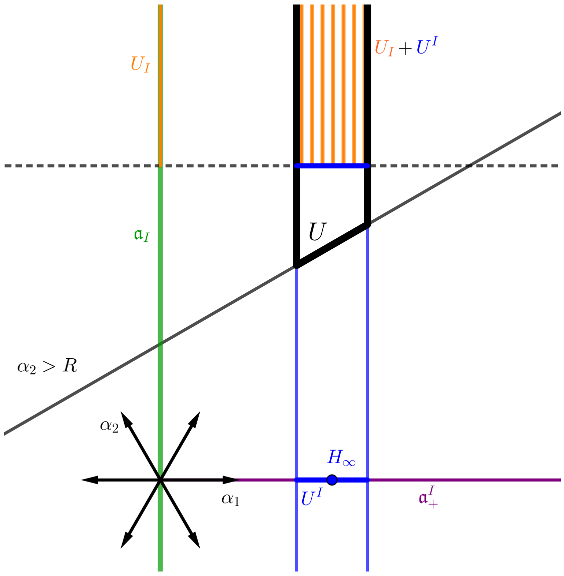

Figure II.6.: Decomposition of U for $G = SL(3, \mathbb{R})$ and $I = \{\alpha_1\}$.

Let $\Omega = V \exp(U) x_0 \subseteq X$ with a k-neighborhood V in K and

$$U := \{H \in \overline{\mathfrak{a}_+} \mid |\alpha(H) - \alpha(H_\infty)| < \varepsilon, \alpha \in I, \alpha(H) > R, \alpha \notin I\},$$

so that Ω is contained in the intersection of a neighborhood of x_∞ with the interior of $\overline{X}^{\max}$. Define $U^I := \{H^I \in \mathfrak{a}^I \mid |\alpha(H^I) - \alpha(H_\infty)| < \varepsilon, \alpha \in I\}$ which is a bounded open set in $\mathfrak{a}^I$ since the set of linear forms I restricted to $\mathfrak{a}^I$ is linear independent. Without loss of generality $U^I \subseteq \mathfrak{a}_+^I$ has positive distance to the boundaries. Let $U_I := \{H_I \in \mathfrak{a}_I \mid \alpha(H_I) > C, \alpha \in \Pi \setminus I\} \subseteq \mathfrak{a}_{I,+}$ so that $U_I + U^I \subseteq U$ for C large enough.

As in Theorem II.7.2 we use the integral formula (II.3) to obtain

$$\int_{U \subseteq \mathfrak{a}_+} |f|^2(k \exp(H)) \prod \sinh(\alpha(H))^{m_\alpha} dH < \infty$$

for almost every $k \in V$. Therefore,

$$\int_{U_I \subseteq \mathfrak{a}_{I,+}} |f|^2(k \exp(H^I) \exp(H_I)) \prod \sinh(\alpha(H_I + H^I))^{m_\alpha} dH_I < \infty$$

for almost every $k \in V$ and $H^I \in U^I \subseteq \mathfrak{a}^I$ (with suitable Lebesgue measures on $\mathfrak{a}_I$ and $\mathfrak{a}^I$).

The property that $U^I \subseteq \mathfrak{a}_+^I$ has positive distance to the boundaries implies that $\alpha(H_I + H^I) > \varepsilon$ and hence

$$\prod_{\alpha \in \Sigma^+} \sinh(\alpha(H_I + H^I))^{m_\alpha} \geq C e^{2\rho(H_I)}, \quad H_I \in U_I, H^I \in U^I.$$

Therefore, $|f|^2(k\exp(H^I)\exp(H_I))e^{2\rho(H_I)}$ is integrable on U_I.

Similarly to the proof of Theorem II.7.2 we use an asymptotic expansion for f, but this time we have to consider asymptotics along the boundary of the positive Weyl chamber instead of regular directions.

Theorem II.7.9 ([vdBS89, Thm 1.5]). *There exists a finite set $S(\lambda, I) \subseteq \mathfrak{a}_I^*$ such that for each $f \in E_\lambda^\infty$, $g \in G$, and $\xi \in X(\lambda, I) = S(\lambda, I) - \mathbb{N}_0\Pi|_{\mathfrak{a}_I}$ there is a unique polynomial $p_{I,\xi}(f, g)$ on $\mathfrak{a}_I$ which is smooth in g such that*

$$f(g\exp(tH_0)) \sim \sum_{\xi \in X(\lambda, I)} p_{I,\xi}(f, g, tH_0)e^{t\xi(H_0)}, \quad t \to \infty,$$

at every $H_0 \in \mathfrak{a}_{I,+}$, i.e. for every N there exist a neighborhood U of H_0 in $\mathfrak{a}_{I,+}$, a neighborhood V of g in G, $\varepsilon > 0$, $C > 0$ such that

$$\left| f(y\exp(tH)) - \sum_{\operatorname{Re}\xi(H_0) \geq -N} p_{I,\xi}(f, y, tH)e^{t\xi(H)} \right| \leq Ce^{(-N-\varepsilon)t}$$

for all $y \in V, H \in U$, $t \geq 0$.

Remark II.7.10. The uniformity in x is not stated in [vdBS89] but it follows from Proposition 1.3 therein.

Let $H_0 \in \mathfrak{a}_{I,+}$, $\|H_0\| = 1$. After shrinking we can assume that

$$\left| f(k\exp(H^I)\exp(H_I)) - \sum_{\operatorname{Re}\xi(H_0) \geq -\rho(H_0)} p_{I,\xi}(f, k\exp(H^I), H_I)e^{\xi(H_I)} \right| \leq Ce^{(-\rho(H_0)-\varepsilon)\|H_I\|}$$

for all $k \in V, H^I \in U^I$, and $\frac{H_I}{\|H_I\|}$ in some neighborhood $\tilde{U}_I$ of H_0 in $\mathfrak{a}_{I,+}$ such that $(R', \infty)\tilde{U}_I \subseteq U_I$.

The error term $e^{(-\rho(H_0)-\varepsilon)\|H_I\|}$ satisfies

$$e^{2(-\rho(H_0)-\varepsilon)\|H_I\|}e^{2\rho(H_I)} = e^{2(\rho(-H_0+\frac{H_I}{\|H_I\|})-\varepsilon)\|H_I\|} \leq e^{-\varepsilon\|H_I\|}$$

if $\tilde{U}_I$ is sufficiently small. Since $e^{-\varepsilon\|H_I\|}$ is integrable on $(R', \infty)\tilde{U}_I$ the same is true for

$$\left| \sum_{\operatorname{Re}\xi(H_0) \geq -\rho(H_0)} p_{I,\xi}(f, k\exp(H^I), H_I)e^{(\xi+\rho)(H_I)} \right|^2.$$

Using [Kna86, Lemma 8.50] we obtain that $p_{I,\xi}(f, k\exp(H^I), H_I) = 0$ if $\operatorname{Re}(\xi+\rho)(H_I) \geq 0$ for almost every $k \in V$ and $H^I \in U^I$. Since $p_{I,\xi}(f, \bullet, H_I)$ is smooth, this holds for every $k \in V$ and $H^I \in U^I$.

By [vdBS89, Corollary 2.5] the mapping $M_I \ni m \mapsto p_{I,\xi}(f, xm, H_I)$, $x \in G$, is real analytic. Therefore, $\mathfrak{a}^I \ni H^I \mapsto p_{I,\xi}(f, k \exp H^I, H_I)$ is real analytic as well and vanishes on the open set U^I for $k \in V$, $\mathrm{Re}(\xi + \rho)(H_I) \geq 0$. Hence it vanishes on $\mathfrak{a}^I$ identically.

In a last step of the proof we show that the vanishing of $p_{I,\xi}(f, k)$ for $\mathrm{Re}(\xi+\rho)(H_I) \geq 0$ implies that the expansion coefficients $p_{\lambda,\eta}(f, k)$ from Theorem II.7.3 vanish for all $\eta \in W\lambda - \rho$ and $k \in V$. For this purpose we use the following expansion for the polynomial $p_{I,\xi}$.

Proposition II.7.11 ([vdBS89, Theorem 3.1])**.** *Let $f \in E_\lambda^\infty, g \in G$, and $\xi \in X(I, \lambda)$.*

(i) For every $H_I \in \mathfrak{a}_{I,+}$ and $H^I \in \mathfrak{a}_+^I$ the following asymptotic expansion holds:

$$p_{I,\xi}(f, g \exp(tH^I), H_I) \sim \sum_{\eta \in w\lambda - \rho - \mathbb{N}_0\Pi, \eta|_{\mathfrak{a}_I} = \xi} p_{\lambda,\eta}(f, g, H_I + tH^I) e^{t\eta(H^I)}.$$

(ii) For all $\eta = w\lambda - \rho - \mathbb{N}_0\Pi$ with $\eta|_{\mathfrak{a}_I} \notin X(\lambda, I)$ we have $p_{\lambda,\eta}(f, g) = 0$.

Let $\eta = w\lambda - \rho$, $w \in W$, and $k \in V, H_I \in U_I$. If $\eta|_{\mathfrak{a}_I} \notin X(I, \lambda)$, then $p_{\lambda,\eta}(f, k) = 0$. If $\eta|_{\mathfrak{a}_I} = \xi \in X(I, \lambda)$, then $\mathrm{Re}(\xi + \rho)(H_I) = \mathrm{Re}\, w\lambda(H_I) = 0 \geq 0$. Therefore, $p_{I,\xi}(f, k \exp H^I, H_I) = 0$ for all $H^I \in \mathfrak{a}^I$ by the previous paragraph. It follows that the asymptotic expansion has every coefficient vanishing (see [vdBS87, Lemma 3.2]), in particular $p_{\lambda,\eta}(f, k, H_I + tH^I) = 0$, $H_I \in U_I$, $H^I \in \mathfrak{a}^I$. Since $p_{\lambda,\eta}(f, k)$ is a polynomial, this implies $p_{\lambda,\eta}(f, k) = 0$. Hence in both cases $p_{\lambda,w\lambda-\rho}(f, k) = 0$ for $k \in V$. The remainder of the proof proceeds the same way as the proof of Theorem II.7.2. □

II.7.3. Proof of Theorem II.4.1

Let $\overline{X}$ be one of the compactifications $X \cup X(\infty)$ or $\overline{X}^{\max}$.

Recall that the *wandering set* $\mathrm{w}(\Gamma, \overline{X})$ is defined to be the points $x \in \overline{X}$ for which there is a neighborhood U of x such that $\gamma U \cap U \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$. Clearly, $\mathrm{w}(\Gamma, \overline{X})$ is open, Γ-invariant and contains X. Theorem II.4.1 is a simple consequence of Theorem II.6.2 combined with Theorem II.7.2, respectively II.7.8.

Proof of Theorem II.4.1. Let $x \in \mathrm{w}(\Gamma, \overline{X}) \cap \partial\overline{X}$. Hence, there is an open subset U of $\overline{X}$ containing x such that $\{\gamma \mid \gamma U \cap U \neq \emptyset\}$ contains $N \in \mathbb{N}$ elements. Let $\lambda \in i\mathfrak{a}^*$ and $f \in L^2(\Gamma\backslash X)$ a joint eigenfunction of $\mathbb{D}(X)$ for the character χ_λ. Let $\overline{f} \in E_\lambda$ be Γ-invariant lift of f to X. Then

$$\begin{aligned}\|\overline{f}\|^2_{L^2(U)} &= \int_U |\overline{f}|^2 = \int_{\Gamma\backslash X} \sum_{\gamma\in\Gamma} 1_U(\gamma y)|\overline{f}|^2(\gamma y) d(\Gamma y) = \int_{\Gamma\backslash X} \#\{\gamma \mid \gamma y \in U\}|f|^2(\Gamma y) d(\Gamma y)\\ &\leq N\|f\|^2_{L^2(\Gamma\backslash X)} < \infty.\end{aligned}$$

Hence, f is L^2 on U and f is of moderate growth by Theorem II.6.2. Using Theorem II.7.2 or II.7.8 finishes the proof. □

II.8. Examples

In this section we discuss three classes of examples that satisfy the wandering condition of Theorem II.4.1. As mentioned in the introduction the condition is satisfied for geometrically finite discrete subgroups of $PSO(n,1)$ of infinite covolume.

Products

The most basic example is the case of products. Let $X = X_1 \times X_2$ be the product of two symmetric spaces of non-compact type where $X_i = G_i/K_i$. Let $\Gamma \leq G_1 \times G_2$ be a discrete torsion-free subgroup that is the product of two discrete torsion-free subgroups $\Gamma_i \leq G_i$. Then it is clear that the spectral theory of $\Gamma\backslash X$ is completely determined by the spectral theory of the two factors. In particular, since the algebra $\mathbb{D}(G/K)$ is generated by $\mathbb{D}(G_i/K_i)$, $i = 1,2$, there are no principal joint eigenvalues if the same holds for one of the factors. The same statement can be obtained by Theorem II.4.1 using the maximal Satake compactification. Indeed, by [BJ06, Prop. I.4.35] it holds that the maximal Satake compactification of X is the product of the maximal Satake compactifications of X_i, i.e. $\overline{X}^{\max} = \overline{X_1}^{\max} \times \overline{X_2}^{\max}$. Then it is clear from the definition of the wandering set that $\mathrm{w}(\Gamma, \overline{X}^{\max}) = \mathrm{w}(\Gamma_1, \overline{X_1}^{\max}) \times \mathrm{w}(\Gamma_2, \overline{X_2}^{\max})$. Hence, the wandering condition $\mathrm{w}(\Gamma, \overline{X}^{\max}) \cap \partial\overline{X}^{\max} \neq \emptyset$ is fulfilled if and only if it is fulfilled for one of the actions $\Gamma_i \curvearrowright \overline{X_i}^{\max}$.

Selfjoinings

A more interesting class of examples is given by selfjoinings of locally symmetric spaces. These are given as follows. As above let $X = X_1 \times X_2$ be the product of two symmetric spaces of non-compact type where $X_i = G_i/K_i$. Now, let Υ be a discrete group and $\rho_i \colon \Upsilon \to G_i$, $i = 1,2$, two representations into real semisimple non-compact Lie groups with finite center. We assume that ρ_1 has finite kernel and discrete image. We want to consider the subgroup Γ of $G_1 \times G_2$ given by $\Gamma = \{(\rho_1(\sigma), \rho_2(\sigma)) \colon \sigma \in \Upsilon\}$ which is discrete. We assume that Γ is torsion-free (e.g. if Υ is torsion-free). In contrast to the previous example the locally symmetric space $\Gamma\backslash X$ is not a product of two locally symmetric spaces anymore, so also the spectral theory cannot be reduced to some lower rank factors. However, we can exploit that the globally symmetric space is still a product and consider the maximal Satake compactification which is given by $\overline{X}^{\max} = \overline{X_1}^{\max} \times \overline{X_2}^{\max}$. Since $\rho_1(\Upsilon)$ is discrete, it acts properly discontinuously on X_1. Hence every point of X_1 is wandering for the action of $\rho_1(\Upsilon)$. It follows easily that $X_1 \times \overline{X_2}^{\max}$ is contained in the wandering set $\mathrm{w}(\Gamma, \overline{X}^{\max})$ of the action $\Gamma \curvearrowright \overline{X}^{\max}$. Therefore, the wandering condition is fulfilled. Indeed, $\mathrm{w}(\Gamma, \overline{X}^{\max}) \cap \partial\overline{X}^{\max} \supseteq X_1 \times \partial\overline{X_2}^{\max} \neq \emptyset$.

Anosov subgroups

The result of Lax and Phillips [LP82] is in particular true if we consider a (non-cocompact) convex-cocompact subgroup of $PSO(n,1)$. Anosov subgroups as introduced by Labourie [Lab06] for surface groups and generalized to arbitrary word hyperbolic groups by Guichard and Wienhard [GW12] generalize convex-cocompact subgroups to higher rank symmetric spaces. For such Γ we have the following proposition.

Proposition II.8.1. *Let Γ be a torsion-free Anosov subgroup that is not a cocompact lattice in a rank one Lie group. Then the wandering condition $\mathrm{w}(\Gamma, \overline{X}^{\max}) \cap \partial\overline{X}^{\max} \neq \emptyset$ is fulfilled.*

Proof. By [KL18] (and [GKW15] for a specific maximal parabolic subgroup) every locally symmetric space arising from an Anosov subgroup admits a compactification modeled on the maximal Satake compactification $\overline{X}^{\max}$, i.e. there is $X \subseteq \Omega \subseteq \overline{X}^{\max}$ open such that Γ acts properly discontinuously and cocompactly on Ω. Since Γ does not act cocompactly on X, we have $\Omega \cap \partial\overline{X}^{\max} \neq \emptyset$. As every point in a region of discontinuity is wandering by definition we have $\Omega \subseteq \mathrm{w}(\Gamma, \overline{X}^{\max})$, and in particular the wandering condition is fulfilled. $\square$

Combining the above proposition with Theorem II.4.1 we obtain the following corollary.

Corollary II.8.2. *Let Γ be a torsion-free Anosov subgroup that is not a cocompact lattice in a rank one Lie group. Then there are no principal joint L^2-eigenvalues on $\Gamma\backslash X$.*

It is worth mentioning that selfjoinings of two representations into $PSO(n,1)$ yield Anosov subgroups if and only if one of the images of the representations is convex-cocompact. One can thus easily construct non-trivial examples which are not Anosov subgroups but fulfill the wandering condition of Theorem II.4.1. This is again parallel to Patterson's result that holds beyond the convex-cocompact case for hyperbolic surfaces admitting cusps and at least one funnel.

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III. Temperedness of locally symmetric spaces: The product case

Outline

Published Paper

Abstract

Let $X = X_1 \times X_2$ be a product of two rank one symmetric spaces of non-compact type and Γ a torsion-free discrete subgroup in $G_1 \times G_2$. We show that the spectrum of $\Gamma\backslash X$ is related to the asymptotic growth of Γ in the two directions defined by the two factors. We obtain that $L^2(\Gamma\backslash G)$ is tempered for a large class of Γ.

III.1. Introduction

If one considers a geometrically finite hyperbolic surface $M = \Gamma\backslash\mathbb{H}$ it is a very classical theorem that the smallest eigenvalue of the Laplace-Beltrami operator Δ is related to the growth rate of Γ. More precisely,

$$\inf \sigma(\Delta) = \begin{cases} 1/4 & : \delta_\Gamma < 1/2 \\ 1/4 - (\delta_\Gamma - 1/2)^2 & : \delta_\Gamma \geq 1/2, \end{cases}$$

where δ_Γ is the critical exponent of the discrete subgroup $\Gamma \subseteq SL_2(\mathbb{R})$

$$\delta_\Gamma := \inf \left\{ s \in \mathbb{R} \colon \sum_{\gamma\in\Gamma} e^{-sd(\gamma x_0, x_0)} < \infty \right\}, \quad x_0 \in \mathbb{H}.$$

This theorem is due to Elstrodt [Els73a, Els73b, Els74] and Patterson [Pat76].

A decade later it has been extended to real hyperbolic manifolds of arbitrary dimension by Sullivan [Sul87] and then to general locally symmetric spaces of rank one by Corlette [Cor90].

We are interested in analog statements for higher rank locally symmetric spaces. To state the theorems let us shortly introduce the setting (see Section III.2.1). Let X be a symmetric space of non-compact type, i.e. $X = G/K$ where G is a real connected semisimple non-compact Lie group with finite center and K is a maximal compact subgroup. G admits a Cartan decomposition $G = K\exp(\overline{\mathfrak{a}_+})K$. Hence for every $g \in G$ there is $\mu_+(g) \in \overline{\mathfrak{a}_+}$ such that $g \in K\exp(\mu_+(g))K$. $\mu(g)$ can be thought of a higher dimensional distance $d(gK, eK)$.

In this setting the bottom of the spectrum of the Laplace-Beltrami operator Δ can be estimated using δ_Γ as well [Web08, Leu04]. Note that in the definition of δ_Γ the term $d(\gamma K, eK)$ is $\|\mu_+(\gamma)\|$. Hence, one only considers the norm of $\mu_+(\gamma)$ but there are different ways to measure the growth rate of γ or $\mu_+(\gamma)$. This is exploited by Anker and Zhang [AZ22] to determine $\inf \sigma(\Delta)$ to an exact value.

However, the spectral theory of $\Gamma\backslash G/K$ is more involved than in the rank one case and is not completely determined by Δ: There is a whole algebra of natural differential operators on $\Gamma\backslash G/K$ that come from the algebra of G-invariant differential operators $\mathbb{D}(G/K)$ on G/K. In the easiest higher rank example $G/K = (G_1 \times G_2)/(K_1 \times K_2) = (G_1/K_1) \times (G_2/K_2)$ of two rank one symmetric spaces this algebra is generated by the two Laplacians acting on the respective factors. In this case we could just consider the Laplace operators on the two factors G_1/K_1 and G_2/K_2 which generate $\mathbb{D}((G_1 \times G_2)/(K_1 \times K_2))$. However, in general there are no canonical generators for $\mathbb{D}(G/K)$. This is the reason why in the higher rank setting it is more natural to work with the whole algebra instead of a generating set.

The importance of this algebra can be seen by considering the representation $L^2(\Gamma\backslash G)$ where G acts by right translation. In the rank one case (where $\mathbb{D}(G/K) = \mathbb{C}[\Delta]$) $L^2(\Gamma\backslash G)$ is tempered (see Definition III.3.9) if $\sigma(\Delta) \subseteq [\|\rho\|^2, \infty[$. In the higher rank case this is not true anymore but an analogous statement can be formulated in terms of $\mathbb{D}(G/K)$ (see Proposition III.3.10). This requires to define a joint spectrum $\widetilde{\sigma}(\Gamma\backslash G/K)$ for $\mathbb{D}(G/K)$ on $L^2(\Gamma\backslash G/K)$. There are different ways to define this spectrum: On the one hand we can use the representation theoretical decomposition of $L^2(\Gamma\backslash G)$ and consider the support of the corresponding measure (see Section III.3.1). On the other hand we can define a joint spectrum for a finite generating set of $\mathbb{D}(G/K)$ using approximate eigenvectors (see Section III.3.2). This definition is more in the spirit of usual spectral theory. In fact both definitions coincide and it holds:

$$\widetilde{\sigma}(\Gamma\backslash G/K) = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \chi_\lambda(D) \in \sigma({}_\Gamma D) \quad \forall D \in \mathbb{D}(G/K)\} \tag{III.1}$$

where χ_λ are the characters of $\mathbb{D}(G/K)$ parametrized by $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (see Proposition III.3.6).

As a first result we prove that

$$i\mathfrak{a}^* \subseteq \widetilde{\sigma}(\Gamma\backslash G/K) \tag{III.2}$$

if $\Gamma\backslash G/K$ has infinite injectivity radius (see Proposition III.3.7).

The above mentioned connection between this spectrum and temperedness of $L^2(\Gamma\backslash G)$ is given by the following fact.

Fact III.1.1 (Proposition III.3.10). *If $\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq i\mathfrak{a}^*$ then $L^2(\Gamma\backslash G)$ is tempered.*

Until recently, it was completely unknown which conditions on Γ (similar to $\delta_\Gamma \leq \|\rho\|$) imply temperedness of $L^2(\Gamma\backslash G)$ even for the example of $G = G_1 \times G_2$ with G_i of rank one. Then Edwards and Oh [EO22] showed temperedness for Anosov subgroups if the growth indicator function ψ_Γ is bounded by ρ (see Section III.4.4 for the definition). This statement is in the same spirit as the original theorems by Patterson, Sullivan, and Corlette, but it only holds for Anosov subgroups for minimal parabolics which are a higher rank analog of convex cocompact subgroups and its proof uses rather different methods including estimates on mixing rates from [ELO20]. The main example where they verify the condition $\psi_\Gamma \leq \rho$ is precisely the product situation $G = G_1 \times G_2$ with G_i of rank one and Γ is an Anosov subgroup.

In this work we present a different proof for the temperedness of $L^2(\Gamma\backslash(G_1 \times G_2))$ that is closer to the original proofs in the rank one case and does not use any mixing results. Moreover, we need not to assume that Γ is Anosov.

Theorem (Theorem III.4.9)**.** Let G_1 and G_2 be of rank one and $\Gamma \leq G_1 \times G_2$ discrete and torsion-free. Let

$$\delta_1 = \sup_{R>0} \inf \left\{ s \in \mathbb{R} \colon \sum_{\gamma\in\Gamma, \|\mu_+(\gamma_2)\|\leq R} e^{-s\|\mu_+(\gamma_1)\|} < \infty \right\}$$

and define δ_2 in the same way. Then

$$\widetilde{\sigma}(\Gamma\backslash(G_1 \times G_2)/(K_1 \times K_2)) \subseteq \{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid \|\operatorname{Re}(\lambda_i)\| \leq \max(0, \delta_i - \|\rho_i\|)\}.$$

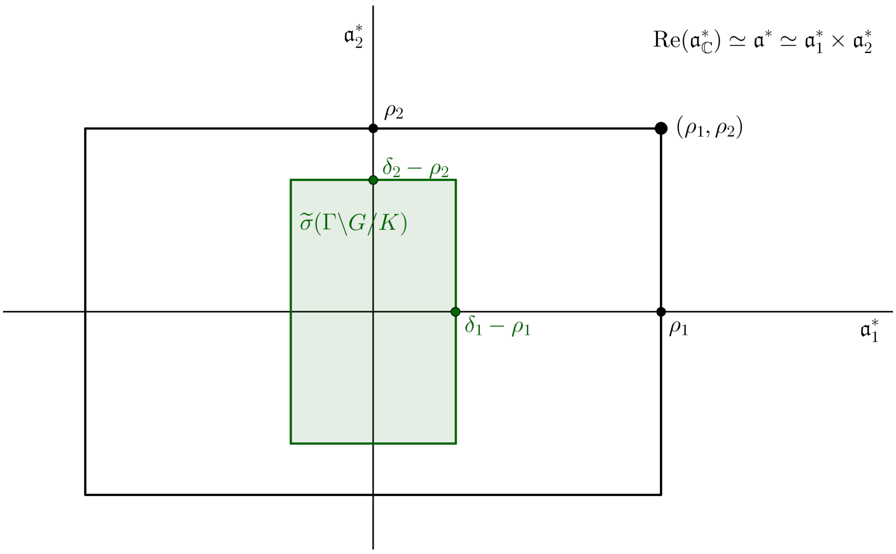

Figure III.1.: $\widetilde{\sigma}(\Gamma\backslash(G_1 \times G_2)/(K_1 \times K_2))$ for two rank one groups G_i

For the proof we consider the Laplace operators on the two factors and use (III.1) to bound $\widetilde{\sigma}$. For these operators the proof is similar to the proofs of Patterson and Corlette, i.e. we obtain information about the spectrum by considering the resolvent kernel on the globally symmetric space G/K and get the local version by averaging over Γ. Analyzing the region of convergence of this averaging process leads to the theorem.

We obtain the following corollary.

Corollary (Corollary III.4.10)**.** If $\delta_1 \leq \|\rho_1\|$ and $\delta_2 \leq \|\rho_2\|$ then $L^2(\Gamma\backslash(G_1 \times G_2))$ is tempered.

An important example is a selfjoining: Let $\pi_i\colon G_1 \times G_2 \to G_i$ be the projection on one factor. Suppose that $\pi_i|_\Gamma$, $i = 1, 2$, both have finite kernel and discrete image. Then $\delta_1 = \delta_2 = -\infty$ and hence $L^2(\Gamma\backslash(G_1 \times G_2))$ is tempered. Any Anosov subgroup with respect to the minimal parabolic subgroup in $G_1 \times G_2$ satisfies this assumption, but also satisfies additional assumptions, e.g. Γ is word hyperbolic and $\|\mu_+(\pi_i(\gamma))\|$ is comparable to the word length of $\gamma \in \Gamma$ [Lab06, GW12]. Therefore we generalize this part of [EO22]. In contrast, [EO22] also provide statements on the connection between temperedness and growth behavior of the Anosov subgroup Γ for more general (globally) symmetric spaces G/K which are not products of rank one symmetric spaces. To extend our work to this more general setting one needs growth estimates for the kernel of the resolvent for suitable generators of the algebra $\mathbb{D}(G/K)$ which so far only seem to be known for the Laplace operator (see [AJ99]).

Outline of the article

In Section III.2 we recall some preliminaries about the symmetric space, spherical functions, the spherical dual, and the Fourier-Helgason transform. After that we define the Plancherel spectrum (see Section III.3.1) and the joint spectrum (see Section III.3.2) and show that they coincide (see Proposition III.3.6). We also prove (III.2) in Proposition III.3.7. In Section III.3.4 we show the connection between $\widetilde{\sigma}(\Gamma\backslash G/K)$ and the temperedness of $L^2(\Gamma\backslash G)$. We suppose that the statements might be considered as folklore among experts in spectral theory of higher rank symmetric spaces, but as the literature on spectral theory of locally symmetric spaces of higher rank and infinite volume is very sparse we provide precise statements with complete proofs in this section. In Section III.4 we prove Corollary III.4.10. To do so we first recall the averaging procedure (see Lemma III.4.2) and reprove the rank one result by [Cor90] in a form that we need later (see Lemma III.4.8). We conclude this article by comparing the quantities δ_i with the growth indicator function ψ_Γ (see Section III.4.4).

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III.2. Preliminaries

III.2.1. Setting

In this section we introduce the notation in the general higher rank setting and only restrict to product spaces once it becomes necessary in order to emphasize clearly what the missing knowledge for the general higher rank setting is. Let G be a real connected semisimple non-compact Lie group with finite center and with Iwasawa decomposition $G = KAN$. We denote by $\mathfrak{g}, \mathfrak{a}, \mathfrak{n}, \mathfrak{k}$ the corresponding Lie algebras. For $g \in G$ let $H(g)$ be the logarithm of the A-component in the Iwasawa decomposition KAN. We have a K-invariant inner product on $\mathfrak{g}$ that is induced by the Killing form and the Cartan involution. We further have the orthogonal Bruhat decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ into root spaces $\mathfrak{g}_\alpha$ with respect to the $\mathfrak{a}$-action via the adjoint action ad. Here $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots. Denote by W the Weyl group of the root system of restricted roots. Let n be the real rank of G and Π (resp. Σ^+) the simple (resp. positive) system in Σ determined by the choice of the Iwasawa decomposition. Let $m_\alpha := \dim_\mathbb{R} \mathfrak{g}_\alpha$ and $\rho := \frac{1}{2}\Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$. Let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Pi\}$ denote the positive Weyl chamber and $\mathfrak{a}_+^*$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{a}$. If $\overline{A^+} := \exp(\overline{\mathfrak{a}_+})$, then we have the Cartan decomposition $G = K\overline{A^+}K$. For $g \in G$ we define $\mu_+(g) \in \overline{\mathfrak{a}_+}$ by $g \in K \exp(\mu_+(g))K$. The main object of our study is the symmetric space $X = G/K$ of non-compact type.

Let $\mathbb{D}(G/K)$ be the algebra of *G-invariant differential operators* on G/K, i.e. differential operators commuting with the left translation by elements $g \in G$. Then we have an algebra isomorphism $\mathrm{HC}\colon \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ from $\mathbb{D}(G/K)$ to the W-invariant complex polynomials on $\mathfrak{a}^*$ which is called the *Harish-Chandra homomorphism* (see [Hel84, Ch. II Thm. 5.18]). For $\lambda \in \mathfrak{a}^*_\mathbb{C}$ let χ_λ be the character of $\mathbb{D}(G/K)$ defined by $\chi_\lambda(D) := \mathrm{HC}(D)(\lambda)$. Obviously, $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$. Furthermore, the χ_λ exhaust all characters of $\mathbb{D}(G/K)$ (see [Hel84, Ch. III Lemma 3.11]). We define the space of joint eigenfunctions

$$E_\lambda := \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D)f \quad \forall D \in \mathbb{D}(G/K)\}.$$

Note that E_λ is G-invariant.

For example the (positive) Laplace operator Δ is contained in $\mathbb{D}(G/K)$ and $\chi_\lambda(\Delta) = -\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle$.

III.2.2. Spherical functions

One can show that in each joint eigenspace E_λ there is a unique left K-invariant function which has the value 1 at the identity (see [Hel84, Ch. IV Corollary 2.3]). We denote the corresponding bi-K-invariant function on G by ϕ_λ and call it *elementary spherical function*. Therefore, $\phi_\lambda = \phi_\mu$ iff $\lambda = w\mu$ for some $w \in W$. It is given by $\phi_\lambda(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}dk$. Note that we differ from the notation in [Hel84] by a factor of i: $\phi_\lambda^{\mathrm{Hel}} = \phi_{i\lambda}$.

III.2.3. Functions of positive type and unitary representations

In this section we recall the correspondence between elementary spherical functions of positive type and irreducible unitary spherical representations. Recall first that a continuous function $f\colon G \to \mathbb{C}$ is called *of positive type* if the matrix $(f(x_i^{-1}x_j))_{i,j}$ for all $x_1,\dots,x_k \in G$ is positive semidefinite. If f is of positive type, then one has $f(x^{-1}) = \overline{f(x)}$ and $|f(g)| \leq f(1)$. Moreover, we can define a unitary representation π_f associated to f in the following way: If R denotes the right regular representation of G, then π_f is the completion of the space spanned by $R(x)f$ with respect to the inner product defined by $\langle R(x)f, R(y)f\rangle \coloneqq f(y^{-1}x)$ which is positive definite. G acts unitarily on this space by the right regular representation. If $f(g) = \langle \pi(g)v, v\rangle$ is a matrix coefficient of a unitary representation π, then f is of positive type and π_f is contained in π.

Secondly, recall that a unitary representation is called *spherical* if it contains a non-zero K-invariant vector. Denote by $\widehat{G}_{\mathrm{sph}}$ the subset of the unitary dual consisting of spherical representations. We then have a 1:1-correspondence between elementary spherical functions of positive type and $\widehat{G}_{\mathrm{sph}}$ given by $\phi_\lambda \mapsto \pi_{\phi_\lambda}$ (see [Hel84, Ch. IV Thm. 3.7]). The preimage of an irreducible unitary spherical representation π with normalized K-invariant vector v_K is given by $g \mapsto \langle \pi(g)v_K, v_K\rangle$.

III.2.4. Harish-Chandra's c-Function

Definition III.2.1. We define the Harish-Chandra **c**-function for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re}\lambda \in \mathfrak{a}_+^*$ as the absolutely convergent integral

$$\mathbf{c}(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})} d\overline{n},$$

where $d\overline{n}$ is normalized such that $\mathbf{c}(\rho) = 1$. It is given by the product formula

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{2^{-\langle\lambda,\alpha_0\rangle}\Gamma(\langle\lambda,\alpha_0\rangle)}{\Gamma(\frac{1}{4}m_\alpha + \frac{1}{2} + \frac{1}{2}\langle\lambda,\alpha_0\rangle)\Gamma(\frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha} + \frac{1}{2}\langle\lambda,\alpha_0\rangle)}$$

where $\Sigma_0^+ = \Sigma^+ \setminus \frac{1}{2}\Sigma^+$, $\alpha_0 = \alpha/\langle\alpha,\alpha\rangle$, and the constant c_0 is determined by $\mathbf{c}(\rho) = 1$.

III.2.5. The Fourier-Helgason transform

For a sufficiently nice function $f\colon G/K \to \mathbb{C}$ we define the Fourier-Helgason transform of f by

$$\mathcal{F}f(\lambda, kM) = \int_{G/K} f(gK) e^{(\lambda-\rho)H(g^{-1}k)} d(gK).$$

Let $e_{\lambda,kM}(gK) = e^{-(\lambda+\rho)H(g^{-1}k)}$. Then we have $De_{\lambda,kM} = \chi_\lambda(D)e_{\lambda,kM}$ by [Hel84, Ch. II Lemma 5.15] for every $D \in \mathbb{D}(G/K)$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $k \in K$, and $g \in G$. Therefore,

$$\begin{aligned}\mathcal{F}(Df)(\lambda, kM) &= \int_{G/K} Df(gK)\overline{e_{-\overline{\lambda},kM}(gK)}d(gK) = \int_{G/K} f(gK)\overline{D^* e_{-\overline{\lambda},kM}(gK)}d(gK)\\ &= \int_{G/K} f(gK)\overline{\chi_{-\overline{\lambda}}(D^*)e_{-\overline{\lambda},kM}(gK)}\, d(gK) = \overline{\chi_{-\overline{\lambda}}(D^*)}\mathcal{F}f(\lambda, kM).\end{aligned}$$

By [Hel84, Lemma 5.21 and Cor. 5.3] $\chi_\lambda(D^*) = \overline{\chi_{-\overline{\lambda}}(D)}$ so that we have the following lemma.

Lemma III.2.2. *The Fourier-Helgason transform satisfies*

$$\mathcal{F}(Df)(\lambda, kM) = \chi_\lambda(D)\mathcal{F}f(\lambda, kM)$$

for every $D \in \mathbb{D}(G/K)$.

Theorem III.2.3 ([Hel89, Ch. III Thm. 1.5])**.** *The Fourier-Helgason transform is an isometry between $L^2(G/K)$ and $L^2(i\mathfrak{a}_+^* \times K/M, |\mathbf{c}(\lambda)|^{-2}d\lambda d(kM))$. Moreover,*

$$\langle f, g\rangle_{L^2(G/K)} = |W|^{-1}\int_{i\mathfrak{a}^*\times K/M} \mathcal{F}f(\lambda, kM)\overline{\mathcal{F}g(\lambda, kM)}|\mathbf{c}(\lambda)|^{-2}d\lambda d(kM)$$

In particular, Lemma III.3.2 implies that $\sigma(D) = \operatorname{essran}[i\mathfrak{a}_+^* \to \mathbb{C}, \lambda \mapsto \chi_\lambda(D)]$ with respect to the measure $|\mathbf{c}(\lambda)|^{-2}d\lambda d(kM)$. Since $\chi_\lambda(D)$ is polynomial and $|\mathbf{c}(\lambda)|^{-2} > 0$ for $\lambda \in i\mathfrak{a}_+^*$ we find that the spectrum of D is the closure of $\{\chi_\lambda(D) \mid \lambda \in i\mathfrak{a}_+^*\}$. As $\chi_\lambda(D)$ is W-invariant this coincides with the closure of $\{\chi_\lambda(D) \mid \lambda \in i\mathfrak{a}^*\}$.

III.3. Spectra for locally symmetric spaces

In this section we recall different types of spectra for the algebra $\mathbb{D}(G/K)$ on a locally symmetric space.

Let $\Gamma \leq G$ be a torsion-free discrete subgroup.

III.3.1. Plancherel spectrum

We want establish a spectrum for the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators. Let us start with the spectrum that is obtained from decomposing the representation $L^2(\Gamma\backslash G)$.

Theorem III.3.1 (see e.g. [BdlHV08, Thm. F.5.3])**.** *Let π be a unitary representation of G. Then there exists a standard Borel space Z, a probability measure μ on Z, and a measurable field of irreducible unitary representations $(\pi_z, \mathcal{H}_z)$ such that π is unitarily equivalent to the direct integral $\int_Z^\oplus \pi_z d\mu(z)$.*

According to the previous theorem let $L^2(\Gamma\backslash G)$ be the direct integral $\int_Z^\oplus \pi_z d\mu(z)$. We denote by Z_{sph} the subset $\{z \in Z \mid \pi_z \text{ is spherical}\}$ of Z where *spherical* means that the representation has a non-zero K-invariant vector. We note that projection $P\colon L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)^K$ onto the K-invariant vectors is given by $\int_K R(k)dk$ where R is the representation of G on $L^2(\Gamma\backslash G)$. Hence, there is a measurable vector field $z \mapsto v_z^K$

such that $v_z^K \in \mathcal{H}_z^K$ is of norm 1 if $\mathcal{H}_z^K \neq 0$. In particular, Z_{sph} is measurable. For $z \in Z_{\text{sph}}$ the representation π_z is unitary, irreducible, and spherical. By Section III.2.3 $\pi_z \simeq \pi_{\phi_{\lambda_z}}$ for some $\lambda_z \in \mathfrak{a}_{\mathbb{C}}^*$ such that ϕ_{λ_z} is of positive type.

Recall the definition of the essential range for a measurable function $f\colon (Z,\mu) \to Y$ from a probability space into a second countable topological space Y:

$$\operatorname{essran} f \coloneqq \{y \in Y \mid \forall U \subseteq Y \text{ open}, y \in U\colon \mu(f^{-1}(U)) > 0\}.$$

By definition $\operatorname{essran} f$ equals the support of the pushforward measure $f_*\mu$ and for $A \subseteq Y$ closed $\operatorname{essran} f \subseteq A$ if and only if $f(z) \in A$ for μ-a.e. $z \in Z$ which we can see as follows: Clearly, if $\mu(\{f(z) \notin A\}) = 0$, then $\operatorname{essran} f \cap Y \setminus A = \emptyset$. Hence, $\operatorname{essran} f \subseteq A$. Conversely, if $\operatorname{essran} f \cap Y \setminus A = \emptyset$ then for every $a \in Y \setminus A$ we find an open neighborhood N_a of a with $\mu(f^{-1}(N_a)) = 0$. Since Y is second countable $Y \setminus A$ can be covered by countably many N_a. Thus $\mu(f^{-1}(Y \setminus A)) \leq \sum \mu(f^{-1}(N_a)) = 0$. Therefore, $f(z) \in A$ for μ-a.e. $z \in Z$.

The following lemma motivates the definition of the Plancherel spectrum.

Lemma III.3.2. *Let $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}_z d\mu(z)$ be the direct integral of the field $(\mathcal{H}_z)_{z\in Z}$ of Hilbert spaces over the σ-finite measure space (Z,μ). Let $T = \int_Z^{\oplus} T_z d\mu(z)$ be the direct integral of the field of operators $(T_z)_{z\in Z}$ such that $T(z) = f(z) id_{\mathcal{H}_z}$ for a measurable function f where the domain of T is $\{\int_Z^{\oplus} y_z\, d\mu(z) \in \mathcal{H} \mid \int_Z^{\oplus} |f(z)|^2 \|y_z\|^2\, d\mu(z) < \infty\}$. Then*

$$\sigma(T) = \operatorname{essran} f = \{y \in \mathbb{C} \mid \forall \varepsilon > 0\colon \mu(f^{-1}(B_\varepsilon(y))) > 0\}.$$

Proof. If $\lambda \notin \operatorname{essran} f$ then there is $\varepsilon > 0$ such that $|f(z) - \lambda| \geq \varepsilon$ for a.e. $z \in Z$. Hence, $\int_Z \frac{1}{f(z)-\lambda} id_{\mathcal{H}_z} d\mu(z)$ is bounded operator with operator norm $\leq 1/\varepsilon$ inverting $T - \lambda$. Therefore, $\lambda \notin \sigma(T)$.

Conversely, let $\lambda \in \operatorname{essran} f$ and $\varepsilon > 0$. Then $A_\varepsilon \coloneqq \{z \in Z \mid |f(z) - \lambda| < \varepsilon\}$ has positive measure and there is a unit vector $y_\varepsilon = \int_Z^{\oplus} y_{\varepsilon,z} d\mu(z) \in \mathcal{H}$ such that $y_{\varepsilon,z} = 0$ for $z \notin A_\varepsilon$. It follows that

$$\|(T-\lambda)y_\varepsilon\|^2 = \left\| \int_Z^{\oplus} (f(z) - \lambda) y_{\varepsilon,z} d\mu(z) \right\|^2 = \int_{A_\varepsilon} |f(z) - \lambda|^2 \|y_{\varepsilon,z}\|^2 d\mu(z) \leq \varepsilon^2.$$

Consequently, $T - \lambda$ cannot be invertible. □

For a locally symmetric space $\Gamma\backslash G/K$ we define

$$\widetilde{\sigma}(\Gamma\backslash G/K) \coloneqq \operatorname{essran}[z \mapsto \lambda_z] \subseteq \mathfrak{a}_{\mathbb{C}}^*/W.$$

Note that $\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W \mid \phi_\lambda \text{ is of positive type}\}$. In particular, since functions of positive type are bounded $\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq \operatorname{conv}(W\rho)$ (see [Hel84, Ch. IV Thm. 8.1]). Furthermore, $\phi_\lambda = \phi_{-\overline{\lambda}}$ so that $\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W \mid -\overline{\lambda} \in W\lambda\}$ (see e.g. [HWW21, Sec. 4]).

III.3.2. The joint spectrum

In this section we describe a different kind of spectrum for $\mathbb{D}(G/K)$ that takes the action of the operators into account instead of the representation theoretical decomposition (see [Sch12, Ch. 5.2.2]).

Definition III.3.3 (see [Sch12, Prop. 5.27])**.** Let T_1 and T_2 be (not necessarily bounded) normal operators on a Hilbert space $\mathcal{H}$. We say that T_1 and T_2 strongly commute if their spectral measures E_{T_1} and E_{T_2} commute.

For strongly commuting normal operators we can define the following joint spectrum.

Definition III.3.4 (see [Sch12, Prop. 5.24])**.** Let $T = \{T_1, \ldots, T_n\}$ be a family of pairwise strongly commuting operators on a Hilbert space $\mathcal{H}$. We define $\sigma_j(T)$ to be the set of all $s \in \mathbb{C}^n$ such that there is a sequence $(x_k)_{k\in\mathbb{N}}$ of unit vectors in $\bigcap_{i=1}^n \operatorname{dom}(T_i) \subseteq \mathcal{H}$ satisfying

$$\lim_{k\to\infty} (T_i - s_i)x_k = 0$$

for all $i = 1, \ldots, n$. We call the sequence (x_k) joint approximate eigenvector.

Clearly, every joint approximate eigenvector is an approximate eigenvector for T_i. Hence, $s_i \in \sigma(T_i)$ for $s \in \sigma_j(T_i)$ and (see [Sch12, Prop. 5.24(ii)]):

$$\sigma_j(T) \subseteq \sigma(T_1) \times \cdots \times \sigma(T_n).$$

Let us come back to the invariant differential operators on a locally symmetric space. By definition $D \in \mathbb{D}(G/K)$ is G-invariant and therefore it maps Γ-invariant elements in $C^\infty(G/K)$ into itself. Since ${}^\Gamma C^\infty(G/K) \simeq C^\infty(\Gamma\backslash G/K)$ we obtain a differential operator ${}_\Gamma D$ on $\Gamma\backslash G/K$. Using the direct integral decomposition it is easy to see that ${}_\Gamma D$ is a normal operator on $L^2(\Gamma\backslash G/K)$ for $D \in \mathbb{D}(G/K)$ (with domain $\{f \in L^2(\Gamma\backslash G/K) \mid {}_\Gamma Df \in L^2(\Gamma\backslash G/K)\}$). Furthermore, the spectral measure is given by

$$E_{{}_\Gamma D}(M) \int_{Z_{\mathrm{sph}}}^{\oplus} f_z \, d\mu(z) = \int_{\{z \mid \chi_{\lambda_z}(D) \in M\}}^{\oplus} f_z \, d\mu(z).$$

We obtain that ${}_\Gamma D_1$ and ${}_\Gamma D_2$ strongly commute for $D_1, D_2 \in \mathbb{D}(G/K)$ and hence we can define the joint spectrum for any finite family $\{{}_\Gamma D_1, \ldots, {}_\Gamma D_n\}$.

III.3.3. Comparison of spectra

In this section we want to see that the Plancherel spectrum and the joint spectrum coincide. In order to achieve this we need the following lemma.

Lemma III.3.5. *Let $p_1, \dots, p_n \in Poly(\mathfrak{a}_\mathbb{C}^*)^W$ be non-constant complex Weyl group invariant homogeneous polynomials of degree d_i on $\mathfrak{a}_\mathbb{C}^*$ that separate the points on $\mathfrak{a}_\mathbb{C}^*/W$. Then $\mathfrak{a}_\mathbb{C}^*/W \to \mathbb{C}^n$, $\lambda \mod W \mapsto (p_1(\lambda), \dots, p_n(\lambda))$ is a topological embedding.*

Proof. By definition the mapping $\Phi\colon \lambda \mod W \mapsto (p_1(\lambda), \dots, p_n(\lambda))$ is injective and continuous. It remains to show that Φ^{-1} is continuous, i.e. for $\lambda_n \in \mathfrak{a}_\mathbb{C}^*$ with $\Phi(\lambda_n) \to \Phi(\lambda_0)$ we have $\lambda_n \mod W \to \lambda_0 \mod W$. Since the polynomials p_i are homogeneous it is clear that $\Phi(0) = 0$ and 0 is not contained in $\Phi(\{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \|\lambda\| = 1\}/W)$. By compactness

$$\|\Phi(\{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \|\lambda\| = 1\}/W)\|_\infty \geq c > 0$$

where we use the maximum norm on $\mathbb{C}^n$. Now for $\|\lambda\| \geq 1$:

$$\begin{aligned}\|\Phi(\lambda \mod W)\|_\infty &= \max |p_i(\lambda)| = \max \|\lambda\|^{d_i} |p_i(\lambda/\|\lambda\|)| \\ &\geq \|\lambda\|^{\min d_i} \max |p_i(\lambda/\|\lambda\|)| \geq c\|\lambda\|^2.\end{aligned}$$

For $\Phi(\lambda_n) \to \Phi(\lambda_0)$ it follows that $\|\lambda_n\|$ is bounded: Indeed if $\limsup \|\lambda_n\| = \infty$ then $\infty = \limsup c\|\lambda_n\|^2 \leq \limsup \|\Phi(\lambda_n)\|_\infty \leq \|\Phi(\lambda_0)\|_\infty + 1$. Therefore, λ_n is contained in the bounded set $B = \{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \|\lambda\| \leq r\}$. But now $\Phi|_{B/W}\colon B/W \to \mathbb{C}^n$ is injective and continuous and since B/W is compact it is a topological embedding. As $\lambda_n, \lambda_0 \in B$ we infer $\lambda_n \mod W \to \lambda_0 \mod W$ and the lemma is proved. □

As before let $L^2(\Gamma\backslash G) = \int_Z^\oplus \pi_z d\mu(z)$. It is clear that $L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash G)^K = \int_{Z_\mathrm{sph}}^\oplus \mathcal{H}_z^K d\mu(z)$. For $z \in Z_\mathrm{sph}$ the representation π_z is unitary, irreducible, and spherical. By Section III.2.3 $\pi_z \simeq \pi_{\phi_{\lambda_z}}$ for some $\lambda_z \in \mathfrak{a}_\mathbb{C}^*/W$ such that ϕ_{λ_z} is of positive type. This reflects that $\widetilde{\sigma}(\Gamma\backslash G/K)$ is the set of spectral parameters λ occurring in $L^2(\Gamma\backslash G/K)$. By definition of $\pi_{\phi_{\lambda_z}}$ the differential operator $D \in \mathbb{D}(G/K)$ acts by $\chi_{\lambda_z}(D)$ on $\mathcal{H}_z^K$.

We now aim to show the following proposition.

Proposition III.3.6. *Let $D_1, \dots, D_n$ be a generating set for $\mathbb{D}(G/K)$ consisting of symmetric operators such that their Harish-Chandra polynomials* $\mathrm{HC}(D_i)$ *are homogeneous. Then the following sets coincide:*

(i) $\widetilde{\sigma}(\Gamma\backslash G/K)$

(ii) $\{\lambda \mid \forall\, D \in \mathbb{D}(G/K)\colon \chi_\lambda(D) \in \sigma({}_\Gamma D)\}$

(iii) $\{\lambda \mid \forall\, p \in Poly(\mathfrak{a}_\mathbb{C}^*)^W\colon p(\lambda) \in \mathrm{essran}[z \mapsto p(\lambda_z)]\}$

(iv) $\{\lambda \mid \forall\, p \in \mathbb{C}[x_1, \dots, x_n]\colon p(\chi_\lambda(D_1), \dots, \chi_\lambda(D_n)) \in \sigma({}_\Gamma p(D_1, \dots, D_n))\}$

(v) $\{\lambda \mid (\chi_\lambda(D_1), \dots, \chi_\lambda(D_n)) \in \sigma_j({}_\Gamma D_1, \dots, {}_\Gamma D_n)\}$

(vi) $\{\lambda \mid {}_\Gamma \sum_{i=1}^n (D_i - \chi_\lambda(D_i))^*(D_i - \chi_\lambda(D_i)) \text{ is not invertible}\}$

Proof. (ii),(iii) and (iv) coincide by the Harish-Chandra isomorphism and Lemma III.3.2 and contain $\widetilde{\sigma}(\Gamma\backslash G/K)$ by continuity of the polynomials $p \in Poly(\mathfrak{a}_{\mathbb{C}}^*)^W$. Taking $p = \sum_{i=1}^n (x_i - \overline{\chi_\lambda(D_i)})(x_i - \chi_\lambda(D_i))$ we see that (iv) is contained in (vi). To see that (vi) is contained in (v) we observe that an approximate eigenvector for the spectral value 0 for ${}_\Gamma \sum_{i=1}^n (D_i - \chi_\lambda(D_i))^*(D_i - \chi_\lambda(D_i))$ is an joint approximate eigenvector for all the ${}_\Gamma D_i$ as

$$\sum_{i=1}^n \|({}_\Gamma D_i - \chi_\lambda(D_i))f\|^2 = \langle {}_\Gamma \sum_{i=1}^n (D_i - \chi_\lambda(D_i))^*(D_i - \chi_\lambda(D_i))f, f\rangle.$$

It remains to show that (v) is contained in $\widetilde{\sigma}(\Gamma\backslash G/K)$. Let $f_n = \int_{Z_{\mathrm{sph}}}^{\oplus} f_{n,z}\, d\mu(z)$ be a joint approximate eigenvector for ${}_\Gamma D_1, \ldots, {}_\Gamma D_n$ and $A_\varepsilon := \{z \mid \sum_{i=1}^n |\chi_{\lambda_z}(D_i) - \chi_\lambda(D_i)|^2 < \varepsilon\}$. Then

$$\begin{aligned} 0 \leftarrow \sum \|({}_\Gamma D_i - \chi_\lambda(D_i))f_n\|^2 &= \int_{Z_{\mathrm{sph}}} \sum_{i=1}^n |\chi_{\lambda_z}(D_i) - \chi_\lambda(D_i)|^2 \|f_{n,z}\|^2 \, d\mu(z) \\ &\geq \int_{Z_{\mathrm{sph}}\backslash A_\varepsilon} \varepsilon \|f_{n,z}\|^2 \, d\mu(z) \end{aligned}$$

but the last expression equals ε if $\mu(A_\varepsilon) = 0$. Hence, A_ε has positive measure for all $\varepsilon > 0$. By Lemma III.3.5 the preimage of a neighborhood in $\mathfrak{a}_{\mathbb{C}}^*/W$ of λ under $z \mapsto \lambda_z$ contains A_ε for some $\varepsilon > 0$ and therefore has positive measure as well. It follows $\lambda \in \widetilde{\sigma}(\Gamma\backslash G/K)$. This completes the proof. □

We now prove that $\widetilde{\sigma}(\Gamma\backslash G/K)$ contains $i\mathfrak{a}^*$ if the injectivity radius is infinite.

Proposition III.3.7. *Suppose that the injectivity radius of $\Gamma\backslash G/K$ is infinite, i.e. for every compact set $C \subseteq G/K$ there is $g \in G$ such that $G/K \to \Gamma\backslash G/K$ restricted to gC is injective. Then $i\mathfrak{a}^* \subseteq \widetilde{\sigma}(\Gamma\backslash G/K)$. In particular, $[\|\rho\|^2, \infty[\subseteq \sigma({}_\Gamma\Delta)$.*

Proof. The proof follows the same idea as [EO22, Prop. 8.4]. Let $\lambda \in i\mathfrak{a}^* = \widetilde{\sigma}(G/K)$. We choose a generating set $D_1, \ldots, D_n$ for $\mathbb{D}(G/K)$ consisting of symmetric operators such that $\mathrm{HC}(D_i)$ are homogeneous. Let $D_{n+1} = (\Delta - \|\rho\|^2)^k$ for k large such that the order of D_{n+1} is bigger than all the orders of $D_1, \ldots, D_n$. Denote the elliptic operator $\sum_{i=1}^{n+1} (D_i - \chi_\lambda(D_i))^*(D_i - \chi_\lambda(D_i))$ by D. By Proposition III.3.6 there exists $(f_n)_n \subset L^2(G/K)$ with $\|f_n\|_{L^2(G/K)} = 1$ and $Df_n \to 0$. Since D is elliptic and positive it is essentially self-adjoint on $C_c^\infty(G/K)$. In particular, we can assume that $f_n \in C_c^\infty(G/K)$. We can now find $g_n \in G$ such that $g_n \operatorname{supp} f_n$ injects into $\Gamma\backslash G/K$. Define $\widetilde{f}_n(\Gamma x) = f_n(g_n^{-1}x)$ for $x \in g_n K_n$ and $\widetilde{f}_n(\Gamma x) = 0$ else. By construction this is well-defined and $\|\widetilde{f}_n\|_{L^2(\Gamma\backslash G/K)} = \|f_n\|_{L^2(G/K)}$. Moreover, $\|{}_\Gamma D\widetilde{f}_n\|_{L^2(\Gamma\backslash G/K)} = \|Df_n\|_{L^2(G/K)} \to 0$. This shows $\lambda \in \widetilde{\sigma}(\Gamma\backslash G/K)$. The 'in particular' part follows from Proposition III.3.6 (ii) and $\chi_\lambda(\Delta) = -\langle\lambda, \lambda\rangle + \|\rho\|^2$. □

Remark III.3.8. The assumption in Proposition III.3.7 is satisfied for the following examples:

(i) If $G = SL_2(\mathbb{R})$ and Γ is geometrically finite, then infinite injectivity radius is equivalent to infinite volume which is again equivalent to saying that $\Gamma\backslash\mathbb{H}$ has at least one funnel.

(ii) If G is simple of real rank at least 2, then a discrete subgroup $\Gamma\backslash G/K$ has infinite injectivity radius iff Γ has infinite covolume by [FG23].

(iii) If $\Gamma \leq G$ is an Anosov subgroup, then $\Gamma\backslash G/K$ has infinite injectivity radius [EO22, Proposition 8.3].

III.3.4. Temperedness of $L^2(\Gamma\backslash G)$

We want to obtain a connection between the spectrum and temperedness of $L^2(\Gamma\backslash G)$. Let us recall the definition of a tempered representation.

Definition III.3.9 (see e.g. [CHH88])**.** A unitary representation $(\pi, \mathcal{H}_\pi)$ is called *tempered* if one of the following equivalent conditions is satisfied:

(i) π is weakly contained in $L^2(G)$, i.e. any diagonal matrix coefficients of π can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of $L^2(G)$.

(ii) for any $\varepsilon > 0$ the representation π is strongly $L^{2+\varepsilon}$ where π is called strongly L^p if there is a dense subspace D of $\mathcal{H}_\pi$ so that for any vectors $v, w \in D$ the matrix coefficient $g \mapsto \langle \pi(g)v, w\rangle$ lies in $L^p(G)$.

To characterize temperedness of $L^2(\Gamma\backslash G)$ we will use the direct integral decomposition (see Section III.3.1).

We will prove the following statement.

Proposition III.3.10. *Suppose that* $\operatorname{Re}\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq \frac{p-2}{p}\operatorname{conv}(W\rho)$ *for some* $p \in [2,\infty[$*. Then* $L^2(\Gamma\backslash G)$ *is strongly* $L^{p+\varepsilon}$*. In particular, if* $\widetilde{\sigma}(\Gamma\backslash G/K) \subseteq i\mathfrak{a}^*/W$ *then* $L^2(\Gamma\backslash G)$ *is tempered.*

Proof. Let $\varepsilon > 0$ and $f_1, f_2 \in C_c(\Gamma\backslash G)$ non-negative. We have to show that

$$\int_G |\langle R(g)f_1, f_2\rangle|^{p+\varepsilon} dg$$

is finite. Obviously, $\langle R(g)f_1, f_2\rangle = \int_{\Gamma\backslash G} f_1(\Gamma hg)f_2(\Gamma h)d\Gamma h$ is bounded by $\langle R(g)F_1, F_2\rangle$ where $F_i(\Gamma h) = \max_{k\in K}|f_i(\Gamma hk)|$. Hence, it is sufficent to show $\int_G |\langle R(g)f_1, f_2\rangle|^{2+\varepsilon}dg < \infty$ for K-invariant f_1, f_2. We decompose f_i in the direct integral decomposition as $f_i = \int_Z^\oplus f_{i,z}d\mu(z)$. Since we assumed f_i to be K-invariant we know that $f_{i,z} \in \mathcal{H}_z^K$ for μ-a.e. $z \in Z$. It follows that we have to integrate only over Z_{sph}.

For $z \in Z_{\text{sph}}$ the representation π_z is unitary, irreducible, and spherical. By Section III.2.3 $\pi_z \simeq \pi_{\phi_{\lambda_z}}$ for some $\lambda_z \in \mathfrak{a}_{\mathbb{C}}^*$ such that ϕ_{λ_z} is of positive type. We also have $\langle \pi_z(g) f_{1,z}, f_{2,z}\rangle = \phi_{\lambda_z}(g) \cdot \langle f_{1,z}, f_{2,z}\rangle$. By assumption, $\lambda_z \in \frac{p-2}{p}\operatorname{conv}(W\rho)$ for a.e. $z \in Z_{\text{sph}}$. This implies that $\phi_{\lambda_z} \in L^{p+\varepsilon}(G)$ by [HWW21, Prop. 2.4] and even $\int_G |\phi_{\lambda_z}|^{p+\varepsilon}\, dg \le C_{\varepsilon,p}$ for μ-a.e. $z \in Z_{\text{sph}}$ with $C_{\varepsilon,p}$ independent of z.

Now we estimate

$$\begin{aligned}\int_G |\langle R(g) f_1, f_2\rangle|^{p+\varepsilon} dg &\le \int_G \left(\int_{Z_{\text{sph}}} |\langle \pi_z(g) f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{p+\varepsilon} dg \\ &= \int_G \left(\int_{Z_{\text{sph}}} |\phi_{\lambda_z}(g)\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{p+\varepsilon} dg.\end{aligned}$$

Using Hölder's inequality we find that

$$\begin{aligned}\int_{Z_{\text{sph}}} |\phi_{\lambda_z}(g)\langle f_{1,z}, f_{2,z}\rangle| d\mu(z) &= \int_{Z_{\text{sph}}} |\phi_{\lambda_z}(g)||\langle f_{1,z}, f_{2,z}\rangle|^{\frac{1}{p+\varepsilon}} |\langle f_{1,z}, f_{2,z}\rangle|^{1/q} d\mu(z) \\ &\le \left(\int_{Z_{\text{sph}}} |\phi_{\lambda_z}(g)|^{p+\varepsilon}|\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{\frac{1}{p+\varepsilon}} \cdot \left(\int_{Z_{\text{sph}}} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{1/q}.\end{aligned}$$

where $\frac{1}{p+\varepsilon} + \frac{1}{q} = 1$.

Therefore,

$$\begin{aligned}\int_G |\langle R(g) f_1, f_2\rangle|^{p+\varepsilon} dg \le \int_G \int_{Z_{\text{sph}}} &|\phi_{\lambda_z}(g)|^{p+\varepsilon} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z) \\ &\cdot \left(\int_{Z_{\text{sph}}} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{\frac{p+\varepsilon}{q}} dg.\end{aligned}$$

Using $\int_G |\phi_{\lambda_z}|^{p+\varepsilon}\, dg \le C_{\varepsilon,p}$ it follows

$$\begin{aligned}\int_G |\langle R(g) f_1, f_2\rangle|^{p+\varepsilon} dg &\le C_{\varepsilon,p} \int_{Z_{\text{sph}}} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z) \cdot \left(\int_{Z_{\text{sph}}} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{\frac{p+\varepsilon}{q}} \\ &\le C_{\varepsilon,p} \left(\int_{Z_{\text{sph}}} |\langle f_{1,z}, f_{2,z}\rangle| d\mu(z)\right)^{p+\varepsilon} \\ &\le C_{\varepsilon,p} \left(\int_{Z_{\text{sph}}} \|f_{1,z}\|^2 d\mu(z) \int_{Z_{\text{sph}}} \|f_{2,z}\|^2 d\mu(z)\right)^{p+\varepsilon/2} \\ &\le C_{\varepsilon,p} \|f_1\|^{p+\varepsilon}_{L^2(\Gamma\backslash G)} \|f_2\|^{p+\varepsilon}_{L^2(\Gamma\backslash G)} < \infty.\end{aligned}$$

This completes the proof. □

III.4. The spectrum for quotients of products of rank one space

III.4.1. The resolvent kernel on a locally symmetric space

In this subsection we determine the Schwartz kernel of the resolvent on a locally symmetric space in terms of its Schwartz kernel on the global space G/K. To do this we need the following well-known lemma.

Lemma III.4.1. *The averaging map* $\alpha\colon C_c^\infty(G/K) \to C_c^\infty(\Gamma\backslash G/K)$ *defined by*

$$\alpha f(\Gamma x) = \sum_{\gamma\in\Gamma} f(\gamma x), \quad x \in G/K,$$

is surjective.

Let us recall that for $D \in \mathbb{D}(G/K)$ we defined the differential operator ${}_\Gamma D$ acting on $L^2(\Gamma\backslash G/K)$. The following lemma tells us how the Schwartz kernel of ${}_\Gamma D^{-1}$ can be expressed provided D is invertible.

Lemma III.4.2. *Let* $D \in \mathbb{D}(G/K)$ *and suppose that* D *is invertible as an unbounded operator* $L^2(G/K) \to L^2(G/K)$. *Let* $K_{D^{-1}} \in \mathcal{D}'(G/K \times G/K)$ *be the Schwartz kernel of* D^{-1}. *Suppose further that* ${}_\Gamma D\colon L^2(\Gamma\backslash G/K) \to L^2(\Gamma\backslash G/K)$ *is invertible. Then the Schwartz kernel* $K_{{}_\Gamma D^{-1}} \in \mathcal{D}'(\Gamma\backslash G/K \times \Gamma\backslash G/K)$ *of* ${}_\Gamma D^{-1}$ *is given by*

$$K_{{}_\Gamma D^{-1}}(\varphi \otimes \psi) = \sum_{\gamma\in\Gamma} K_{D^{-1}}(L_\gamma \tilde\varphi \otimes \tilde\psi),$$

where $\tilde\varphi$ *(and* $\tilde\psi$*) are preimages of* φ *(resp.* ψ*) under the surjective map* $\alpha\colon C_c^\infty(G/K) \to C_c^\infty(\Gamma\backslash G/K)$. *By slight abuse of notation we write*

$$K_{{}_\Gamma D^{-1}}(\Gamma x, \Gamma y) = \sum_{\gamma\in\Gamma} K_{D^{-1}}(x, \gamma y).$$

Proof. First of all note that D and therefore D^{-1} is G-invariant, hence $K_D(\tilde\varphi \otimes \tilde\psi) = K_D(L_g\tilde\varphi \otimes L_g\tilde\psi)$ for all $g \in G$ and $\tilde\varphi, \tilde\psi \in C_c^\infty(G/K)$. Let $\varphi = \alpha\tilde\varphi, \psi = \alpha\tilde\psi \in C_c^\infty(\Gamma\backslash G/K)$. By definition of $K_{{}_\Gamma D^{-1}}$ we have

$$K_{{}_\Gamma D^{-1}}(({}_\Gamma D\varphi) \otimes \psi) = \int_{\Gamma\backslash G/K} \varphi(\Gamma x)\psi(\Gamma x)\, d\Gamma x.$$

On the other hand ${}_\Gamma D\varphi = \alpha(D\tilde\varphi)$ by G-invariance of D so that we can choose $D\tilde\varphi$ as $\widetilde{{}_\Gamma D\phi}$. Therefore we have to show

$$\sum_{\gamma\in\Gamma} K_{D^{-1}}(L_\gamma D\tilde\varphi \otimes \tilde\psi) = \int_{\Gamma\backslash G/K} \varphi(\Gamma x)\psi(\Gamma x)\, d\Gamma x.$$

The left hand side equals

$$\sum_{\gamma\in\Gamma} K_{D^{-1}}(DL_\gamma\tilde{\varphi}\otimes\tilde{\psi}) = \sum_{\gamma\in\Gamma}\int_{G/K} L_\gamma\tilde{\varphi}(x)\tilde{\psi}(x)\,dx$$

again by G-invariance of D and the definition of $K_{D^{-1}}$. Now we can use the definition of the measure of $\Gamma\backslash G/K$ to conclude

$$\sum_{\gamma\in\Gamma}\int_{G/K} L_\gamma\tilde{\varphi}(x)\tilde{\psi}(x)\,dx = \sum_{\gamma\in\Gamma}\int_{\Gamma\backslash G/K}\sum_{\gamma'\in\Gamma}\tilde{\varphi}(\gamma x)\tilde{\psi}(\gamma' x)\,d\Gamma x = \int_{\Gamma\backslash G/K}\varphi(\Gamma x)\psi(\Gamma x)\,d\Gamma x.$$

This shows the lemma. □

III.4.2. Spectrum of the Laplacian in a general locally symmetric space of rank one

In this section we recall the connection between the bottom of the Laplace spectrum on the locally symmetric space $\Gamma\backslash G/K$ of rank one and the critical exponent of Γ which is due to Elstrodt [Els73a, Els73b, Els74] and Patterson [Pat76] for $G = SL_2(\mathbb{R})$, Sullivan [Sul87] for $G = SO_0(n,1)$, and Corlette [Cor90] for general G of rank one. In the higher rank setting this was generalized by Leuzinger [Leu04], Weber [Web08], and Anker and Zhang [AZ22].

Definition III.4.3. We define the abscissa of convergence/critical exponent for Γ as

$$\delta_\Gamma := \inf\left\{ s\in\mathbb{R}\colon \sum_{\gamma\in\Gamma} e^{-s\|\mu_+(\gamma)\|} < \infty \right\}.$$

Let us recall the theorem for the bottom of the spectrum on a locally symmetric space of rank one and its proof as we will use it later in the proof of Theorem III.4.9.

Proposition III.4.4. *Let G/K be a symmetric space of rank one and Γ a torsion-free discrete subgroup. Then*

$$\sigma({}_\Gamma\Delta) \subseteq \begin{cases} [\|\rho\|^2,\infty[& : \delta_\Gamma < \|\rho\| \\ [\|\rho\|^2 - (\delta_\Gamma - \|\rho\|)^2,\infty[& : \delta_\Gamma \geq \|\rho\|. \end{cases}$$

The main ingredient for the proof of Proposition III.4.4 is the Green function which is the resolvent kernel $K_{(\Delta-z)^{-1}}$ for the Laplacian Δ. It is well-known that $K_{(\Delta-z)^{-1}}$ is a smooth function away from the diagonal. By the G-invariance of Δ we have $K_{(\Delta-z)^{-1}}(gx,gy) = K_{(\Delta-z)^{-1}}(x,y)$ and therefore $K_{(\Delta-z)^{-1}}(x,y)$ only depends on the value $\mu_+(x^{-1}y)\in\mathfrak{a}$. This allows us to see $K_{(\Delta-z)^{-1}}$ as a function on A which has the following global bounds:

Theorem III.4.5 ([AJ99, Thm. 4.2.2]).

(i) For every $z < b < \|\rho\|^2$ there is a constant $C_{z,b} > 0$ such that

$$K_{(\Delta - z)^{-1}}(e^H) \le C_{z,b} e^{-(\sqrt{\|\rho\|^2 - b} + \|\rho\|)\|H\|}$$

for all $H \in \mathfrak{a}$ away from the origin.

(ii) For every $z < \|\rho\|^2$ there is a constant C_z such that

$$K_{(\Delta - z)^{-1}}(e^H) \le C_z \begin{cases} \|H\|^{2-\dim(G/K)} & : \dim(G/K) > 2 \\ \log(1/\|H\|) & : \dim(G/K) = 2 \end{cases}$$

for all $H \in \mathfrak{a}$ near the origin.

Remark III.4.6. In addition to the bounds on $K_{(\Delta - z)^{-1}}$ from Theorem III.4.5 we will use the following general estimates:

$$|K_{(\Delta - z)^{-1}}| \le K_{(\Delta - \operatorname{Re} z)^{-1}}$$

which is positive. Moreover,

$$K_{(\Delta - z)^{-1}} \le K_{(\Delta - z')^{-1}} \quad \text{for} \quad z \le z' < \|\rho\|^2.$$

These estimates can been seen e.g. by writing $(\Delta - z)^{-1}$ in terms of the Laplace transform.

We use Stone's formula in order to decide whether the kernel given by the averaging construction of Lemma III.4.2 defines a bounded inverse on $L^2(\Gamma \backslash G/K)$.

Proposition III.4.7 (see e.g. [Sch12, Prop. 5.14]). *Let A be a self-adjoint operator and P_I the spectral projector of A for a Borel subset $I \subseteq \mathbb{R}$. Then*

$$\frac{1}{2}(P_{[a,b]} + P_{]a,b[}) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_a^b (A - (z + i\varepsilon))^{-1} - (A - (z - i\varepsilon))^{-1} dz.$$

Here the limit as $\varepsilon \to 0$ is understood as a strong limit.

The advantage of Stone's formula is that the occurring inverted operators are well-defined by the self-adjointness of A. Hence we can merely consider the Schwartz kernel without having to wonder whether this kernel defines a bounded operator on L^2.

Proof of Prop. III.4.4. According to Proposition III.4.7 we have to determine for which $b < \|\rho\|^2$:

$$\int_0^b ({}_\Gamma\Delta - (z + i\varepsilon))^{-1} - ({}_\Gamma\Delta - (z - i\varepsilon))^{-1}\, dz \to 0 \tag{III.3}$$

in the strong sense as $\varepsilon \to 0$. As in Lemma III.4.2 denote the Schwartz kernel of $({}_\Gamma D - (z \pm i\varepsilon))^{-1}$ by $K_{({}_\Gamma D-(z\pm i\varepsilon))^{-1}}$. Then we need to see that

$$\int_0^b (K_{({}_\Gamma\Delta-(z+i\varepsilon))^{-1}} - K_{({}_\Gamma\Delta-(z-i\varepsilon))^{-1}})(\varphi \otimes \psi)dz \to 0 \tag{III.4}$$

as $\varepsilon \to 0$ for every $\varphi, \psi \in C_c^\infty(\Gamma\backslash G/K)$ for certain $b < \|\rho\|^2$. Let $\tilde{\varphi}$ (resp. $\tilde{\psi}$) be a preimage of φ (resp. ψ) under the map α. Then the expression in (III.4) equals

$$\int_0^b \sum_{\gamma\in\Gamma} (K_{(\Delta-(z+i\varepsilon))^{-1}} - K_{(\Delta-(z-i\varepsilon))^{-1}})(L_\gamma\tilde{\varphi} \otimes \tilde{\psi})\, dz \tag{III.5}$$

by Lemma III.4.2 since Δ is symmetric and therefore ${}_\Gamma\Delta - (z \pm i\varepsilon)$ is invertible.

The following slightly more general lemma shows that (III.3) holds for $b < \|\rho\|^2 - (\max\{0, \delta_\Gamma - \|\rho\|\})^2$ and hence $\sigma({}_\Gamma\Delta) \cap (-\infty, \|\rho\|^2 - (\max\{0, \delta_\Gamma - \|\rho\|\})^2) = \emptyset$. □

Lemma III.4.8. *Let D be a multiset whose underlying set is a discrete subset of a rank one Lie group G and*

$$\delta_D := \inf\left\{ s \in \mathbb{R} \colon \sum_{\gamma\in D} e^{-s\|\mu_+(\gamma)\|} < \infty \right\}.$$

For $b < \|\rho\|^2 - (\max\{0, \delta_D - \|\rho\|\})^2$ it holds that

$$\int_0^b \sum_{\gamma\in D} (K_{(\Delta-(z+i\varepsilon))^{-1}} - K_{(\Delta-(z-i\varepsilon))^{-1}})(L_\gamma\tilde{\varphi} \otimes \tilde{\psi})\, dz \to 0$$

as $\varepsilon \to 0$ for every $\tilde{\varphi}, \tilde{\psi} \in C_c^\infty(G/K)$.

Proof. Since the supports of $\tilde{\varphi}$ and $\tilde{\psi}$ are compact there are only finitely many $\gamma \in \Gamma$ such that $\mathrm{supp}(L_\gamma\tilde{\varphi} \otimes \psi)$ intersects the diagonal in $G/K \times G/K$ non-trivially. For these finitely many $\gamma \in \Gamma$ the term converges to 0 as $\Delta - z$ is invertible on $L^2(G/K)$ for $z < \|\rho\|^2$ and therefore $(\Delta - (z \pm i\varepsilon))^{-1} \to (\Delta - z)^{-1}$.

For the other γ we use that $K_{(\Delta-z)^{-1}}$ is a smooth function away from the diagonal and the estimates from Remark III.4.6.

$$\left|\int_0^b \sum_\gamma (K_{(\Delta-(z+i\varepsilon))^{-1}} - K_{(\Delta-(z-i\varepsilon))^{-1}})(L_\gamma\tilde{\varphi}\otimes\tilde{\psi})\,dz\right|$$
$$\leq \sup_{0\leq z\leq b} b\sum_\gamma \left|(K_{(\Delta-(z+i\varepsilon))^{-1}} - K_{(\Delta-(z-i\varepsilon))^{-1}})(L_\gamma\tilde{\varphi}\otimes\tilde{\psi})\right|$$
$$\leq \sup_{0\leq z\leq b} b\sum_\gamma \int_{G/K}\int_{G/K}\left|(K_{(\Delta-(z+i\varepsilon))^{-1}}(x,y) - K_{(\Delta-(z-i\varepsilon))^{-1}}(x,y))(\tilde{\varphi}(\gamma^{-1}x)\tilde{\psi}(y))\right|\,dx\,dy$$
$$\leq \sup_{0\leq z\leq b} 2b\sum_\gamma \int_{G/K}\int_{G/K}\left|K_{(\Delta-z)^{-1}}(\gamma x,y)\tilde{\varphi}(x)\tilde{\psi}(y)\right|\,dx\,dy$$
$$\leq 2b\sum_\gamma \int_{G/K}\int_{G/K}\left|K_{(\Delta-b)^{-1}}(\gamma x,y)\tilde{\varphi}(x)\tilde{\psi}(y)\right|\,dx\,dy$$

Since the Green function only depends on $\mu_+(y^{-1}\gamma x)$ this can be estimated by a constant times

$$\sup_{x,y\in C}\sum_\gamma |K_{(\Delta-b)^{-1}}(e^{\mu_+(y^{-1}\gamma x)})|$$

where $C\subseteq G$ is compact. Now we use Theorem III.4.5 to see that this is bounded for any $\nu>0$ by

$$C_\nu \sup_{x,y\in C}\sum_\gamma e^{-(\sqrt{\|\rho\|^2-b}+\|\rho\|-\nu)\|\mu_+(y^{-1}\gamma x)\|}. \tag{III.6}$$

By the triangle inequality

$$\|\mu_+(\gamma)\|\leq\|\mu_+(y)\|+\|\mu_+(x)\|+\|\mu_+(x^{-1}\gamma y)\|$$

so that (III.6) is bounded by

$$C_\nu \sup_{x,y\in C} e^{(\sqrt{\|\rho\|^2-b}+\|\rho\|-\nu)(\|\mu_+(y)\|+\|\mu_+(x)\|)}\sum_\gamma e^{-(\sqrt{\|\rho\|^2-b}+\|\rho\|-\nu)\|\mu_+(\gamma)\|}.$$

This is finite (for small ν) if $\sqrt{\|\rho\|^2-b}+\|\rho\|>\delta_\Gamma$, i.e. $b<\|\rho\|^2-(\max\{0,\delta_D-\|\rho\|\})^2$. This estimate allows us to use Lebesgue's dominated convergence theorem to conclude the lemma. □

Note that in Lemma III.4.8 D is not assumed to be a group. We will use this general statement in the proof of Proposition III.4.9.

III.4.3. Product of rank one spaces

Let $X=X_1\times X_2=(G_1\times G_2)/(K_1\times K_2)$ be the product of two rank one symmetric spaces and $\Gamma\subseteq G_1\times G_2$ discrete and torsion-free. In order to determine $\widetilde{\sigma}(\Gamma\backslash G/K)$ in this case we bound the spectrum of the Laplacian acting on one factor and then use Proposition III.3.6.

Theorem III.4.9. *Let Δ_1 be the Laplacian $\Delta \otimes id$ on $L^2(X_1 \times X_2) = L^2(X_1) \otimes L^2(X_2)$ acting on the first factor. Let*

$$\delta_1 = \sup_{R>0} \inf \left\{ s \in \mathbb{R}\colon \sum_{\gamma \in \Gamma, \|\mu_+(\gamma_2)\| \leq R} e^{-s\|\mu_+(\gamma_1)\|} < \infty \right\}.$$

Then

$$\begin{aligned} \widetilde{\sigma}({}_\Gamma\Delta_1) &:= \{\lambda \in \mathfrak{a}_\mathbb{C}^*/W \mid \chi_\lambda(\Delta_1) \in \sigma({}_\Gamma\Delta_1)\} \\ &\subseteq \{\lambda \in \mathfrak{a}_\mathbb{C}^*/W \mid \|\operatorname{Re}(\lambda_1)\| \leq \max(0, \delta_1 - \|\rho_1\|)\}. \end{aligned}$$

Proof. Since the Schwartz kernel of the identity is the Dirac distribution $\delta_{x_2=y_2}$ on the diagonal in $X_2 \times X_2$, the Schwartz kernel of $(\Delta_1 - z)^{-1}$ is

$$K_{(\Delta_1 - z)^{-1}}((x_1, x_2), (y_1, y_2)) = K_{(\Delta - z)^{-1}}(x_1, y_1)\delta_{x_2=y_2}(x_2, y_2)$$

for $z \notin [\|\rho_1\|^2, \infty[$. Therefore, if $({}_\Gamma\Delta_1 - z)$ is invertible the kernel of $({}_\Gamma\Delta_1 - z)^{-1}$ is

$$K_{({}_\Gamma\Delta_1 - z)^{-1}}(\Gamma(x_1, x_2), \Gamma(y_1, y_2)) = \sum_{\gamma \in \Gamma} K_{(\Delta - z)^{-1}}(\gamma_1 x_1, y_1)\delta_{x_2=y_2}(\gamma_2 x_2, y_2)$$

by Lemma III.4.2. According to Proposition III.4.7 we have to determine for which $b < \|\rho_1\|^2$:

$$\int_0^b ({}_\Gamma\Delta_1 - (z + i\varepsilon))^{-1} - ({}_\Gamma\Delta_1 - (z - i\varepsilon))^{-1} \, dz \to 0$$

in the strong sense as $\varepsilon \to 0$. As in Lemma III.4.2 denote the Schwartz kernel of $({}_\Gamma D - z)^{-1}$ by $K_{({}_\Gamma D - z)^{-1}}$. Then we need to see for which $b < \|\rho_1\|^2$

$$\int_0^b (K_{({}_\Gamma\Delta_1 - (z+i\varepsilon))^{-1}} - K_{({}_\Gamma\Delta_1 - (z-i\varepsilon))^{-1}})(\varphi \otimes \psi) dz \to 0 \tag{III.7}$$

as $\varepsilon \to 0$ for every $\varphi, \psi \in C_c^\infty(\Gamma\backslash G/K)$. Let $\tilde{\varphi}$ (resp. $\tilde{\psi}$) be a preimage of φ (resp. ψ) under the map α. Then the expression in (III.7) equals

$$\int_0^b \sum_{\gamma \in \Gamma} (K_{(\Delta_1 - (z+i\varepsilon))^{-1}} - K_{(\Delta_1 - (z-i\varepsilon))^{-1}})(L_\gamma \tilde{\varphi} \otimes \tilde{\psi}) \, dz$$

by Lemma III.4.2. Without loss of generality we can assume that $\tilde{\varphi} = \tilde{\varphi}_1 \otimes \tilde{\varphi}_2 \in C_c^\infty(X_1) \otimes C_c^\infty(X_2) \subseteq C_c^\infty(X_1 \times X_2)$ and in the same way for $\tilde{\psi}$. Then (III.7) reduces to

$$\int_0^b \sum_{\gamma \in \Gamma} \left(K_{(\Delta - (z+i\varepsilon))^{-1}} - K_{(\Delta - (z-i\varepsilon))^{-1}})(L_{\gamma_1}\tilde{\varphi}_1 \otimes \tilde{\psi}_1) \right) (\delta_{x_2=y_2}(L_{\gamma_2}\tilde{\varphi}_2 \otimes \tilde{\psi}_2)) \, dz$$

The latter part of the integrand is $\int_{X_2} \tilde{\varphi}_2(\gamma_2^{-1}x)\tilde{\psi}_2(x)\,dx$ which vanishes if γ_2 is large depending on $\tilde{\varphi}_2$ and $\tilde{\psi}_2$. More precisely, this is the case if

$$\|\mu_+(\gamma_2)\| > 2 \max_{x\in\operatorname{supp}\tilde{\varphi}_2} d(x, eK_2) + \max_{\substack{x\in\operatorname{supp}\tilde{\varphi}_2\\ y\in\operatorname{supp}\tilde{\psi}_2}} d(x,y) =: R.$$

Indeed, $d(x,\gamma_2^{-1}x) \geq d(\gamma_2K_2, eK_2) - 2d(x,eK_2) > \max_{\substack{x\in\operatorname{supp}\tilde{\varphi}_2\\ y\in\operatorname{supp}\tilde{\psi}_2}} d(x,y)$ so that $x \in \operatorname{supp}\tilde{\psi}_2$ excludes $\gamma_2^{-1}x \in \operatorname{supp}\tilde{\varphi}_2$.

Let $\Gamma_R \coloneqq \{\gamma \in \Gamma \mid \|\mu_+(\gamma_2)\| \leq R\}$. It follows that (III.7) is bounded by a constant times

$$\int_0^b \sum_{\gamma\in\Gamma_R} (K_{(\Delta-(z+i\varepsilon))^{-1}} - K_{(\Delta-(z-i\varepsilon))^{-1}})(L_{\gamma_1}\tilde{\varphi}_1 \otimes \tilde{\psi}_1)\,dz$$

Now Lemma III.4.8 yields that this vanishes as $\varepsilon \to 0$ as long as

$$b < \|\rho_1\|^2 - (\max\{0, \delta_{\operatorname{pr}_1(\Gamma_R)} - \|\rho_1\|\})^2$$

where $\operatorname{pr}_1(\Gamma_R)$ is the multiset of $\gamma_1 \in G$ with multiplicity $\#\{(\gamma_1',\gamma_2') \in \Gamma_R \mid \gamma_1 = \gamma_1'\}$. In order to get (III.7) for every φ,ψ the above condition on b has to hold for every $R > 0$, i.e. $b < \|\rho_1\|^2 - (\max\{0,\delta_1 - \|\rho_1\|\})^2$. We infer that

$$\sigma({}_\Gamma\Delta_1) \subseteq \begin{cases} [\|\rho_1\|^2, \infty[& : \delta_1 \leq \|\rho_1\| \\ [\|\rho_1\|^2 - (\delta_1 - \|\rho_1\|)^2, \infty[& : \delta_1 \geq \|\rho_1\|. \end{cases}$$

Reformulating this statement in terms of $\widetilde{\sigma}$ we obtain the stated result. □

Obviously, Theorem III.4.9 is also true if we consider the Laplacian on the second factor with the critical exponent

$$\delta_2 = \sup_{R>0} \inf \left\{ s \in \mathbb{R} \colon \sum_{\gamma\in\Gamma, \|\mu_+(\gamma_1)\|\leq R} e^{-s\|\mu_+(\gamma_2)\|} < \infty \right\}.$$

Using this, Proposition III.3.6, and Proposition III.3.10 we obtain the following corollary giving us temperedness of $L^2(\Gamma\backslash G)$ in dependence of δ_1 and δ_2.

Corollary III.4.10. *If $\delta_1 \leq \|\rho_1\|$ and $\delta_2 \leq \|\rho_2\|$, then $L^2(\Gamma\backslash G)$ is tempered.*

Example III.4.11. (i) Let Γ be a product $\Gamma_1 \times \Gamma_2$ where each $\Gamma_i \leq G_i$ is discrete and torsion-free. Then it is clear that $\delta_i = \delta_{\Gamma_i}$. Hence, we obtain the expected results in this product situation.

(ii) Let Γ be a selfjoining: both projections $\pi_i \colon G_1 \times G_2 \to G_i$ onto one factor restricted to Γ have finite kernel and discrete image. Then the set of $\gamma \in \Gamma$ where $\|\mu_+(\pi_i(\gamma))\| \leq R$ is finite. Therefore $\delta_i = -\infty$ and $L^2(\Gamma\backslash G)$ is tempered.

(iii) Let $\Gamma \leq G_1 \times G_2$ be an Anosov subgroup with respect to the minimal parabolic subgroup, i.e. Γ is a selfjoining such that $\pi_i|_\Gamma$ are convex-cocompact representations. In particular, $L^2(\Gamma\backslash G)$ is tempered.

III.4.4. Growth indicator function

In this section we will take a look at the limit cone and the growth indicator function ψ_Γ introduced by Quint [Qui02] and compare it with δ_1.

Definition III.4.12. The limit cone $\mathcal{L}_\Gamma$ of Γ is defined as the asymptotic cone of $\mu_+(\Gamma)$, i.e.

$$\mathcal{L}_\Gamma = \{\lim t_n \mu_+(\gamma_n) \mid t_n \to 0, \gamma_n \in \Gamma\}.$$

For Γ Zariski dense, $\mathcal{L}_\Gamma$ is a convex cone with non-empty interior [Ben97]. From this definition we obtain the following proposition.

Proposition III.4.13. *Let Γ be a torsion-free discrete subgroup of $G = G_1 \times G_2$ where G_i are of real rank one. If $\mathcal{L}_\Gamma \subseteq \mathfrak{a}_+ \cup \{0\}$, then $L^2(\Gamma\backslash G)$ is tempered.*

Proof. In view of Corollary III.4.10 it is sufficient to show that $\delta_i = -\infty$. Suppose there are infinitely many $\gamma_n \in \Gamma$ pairwise distinct such that $\|\mu_+(\gamma_{n,2})\| \leq R$. By discreteness $\|\mu_+(\gamma_{n,1})\| \to \infty$. Hence we can choose $t_n \coloneqq 1/\|\mu_+(\gamma_{n,1})\|$. Then $t_n\mu_+(\gamma_n)$ converges to $(H_1, 0)$ where $H_1 \in \mathfrak{a}_{1,+}$ is normalized contradicting $\mathcal{L}_\Gamma \subseteq \mathfrak{a}_+ \cup \{0\}$. Therefore, there are only finitely many $\gamma \in \Gamma$ with bounded second component and hence $\delta_1 = -\infty$. The same argument works for δ_2. □

For $\Gamma \leq G$ discrete and Zariski dense let $\psi_\Gamma \colon \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ be defined by

$$\psi_\Gamma(H) \coloneqq \|H\| \inf_{H \in \mathcal{C}} \inf\{s \in \mathbb{R} \mid \sum_{\gamma \in \Gamma, \mu_+(\gamma) \in \mathcal{C}} e^{-s\|\mu_+(\gamma)\|} < \infty\}$$

where the infimum runs over all open cones $\mathcal{C}$ containing H and $\|\cdot\|$ is a Weyl group invariant norm on $\mathfrak{a}$. For $H = 0$ let $\psi_\Gamma(0) = 0$. Note that ψ_Γ is positive homogeneous of degree 1. In general we have the upper bound $\psi_\Gamma \leq 2\rho$. By [Qui02] we know that $\psi_\Gamma \geq 0$ on $\mathcal{L}_\Gamma$, $\psi_\Gamma > 0$ on the interior of $\mathcal{L}_\Gamma$ and $\psi_\Gamma = -\infty$ outside $\mathcal{L}_\Gamma$. Moreover, ψ_Γ is concave and upper-semicontinuous.

Let us compare δ_1 to ψ_Γ in the situation $G = G_1 \times G_2$ where G_i is of real rank one. Let $H_i \in \mathfrak{a}_{i,+}$ of norm 1 and consider the maximum norm on $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2$. In this situation it is clear that $\delta_1 \leq \psi_\Gamma(H_1, 0)$ since every cone $\mathcal{C}$ containing $(H_1, 0)$ contains the strip $\mathfrak{a}_{1,+} \times \{H \in \mathfrak{a}_{2,+} \mid \|H\| \leq R\}$ outside a large enough compact set.

Note that if $\psi_\Gamma \leq \rho$ then by the above comparison this condition implies $\delta_i \leq \|\rho_i\|$ which is enough to obtain:

Corollary III.4.14. *Let $X = X_1 \times X_2 = (G_1 \times G_2)/(K_1 \times K_2)$ be the product of two rank one symmetric spaces and $\Gamma \leq G_1 \times G_2$ discrete and torsion-free. If $\psi_\Gamma \leq \rho$ then $L^2(\Gamma\backslash G)$ is tempered.*

Note that this is precisely the result of [EO22] without the assumption that Γ is the image of an Anosov representation with respect to a minimal parabolic subgroup.

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