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Contributions to Dunkl Theory

Dissertation

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Zusammenfassung

In dieser Arbeit werden neue Beiträge zur Dunkl-Theorie geleistet. Seit Ende der 1980er Jahre entwickelte sich die Dunkl-Theorie als Verallgemeinerung radialer Analysis auf riemannschen symmetrischen Räumen und wird in zwei Zweige unterteilt: die rationale und die trigonometrische Theorie. Die rationale Theorie wurde von C.F. Dunkl eingeführt, während die trigonometrische Theorie durch G. Heckman, E.M. Opdam und I. Cherednik entstand. Mit der Dunkl-Theorie entstand auch eine Theorie von multivariablen speziellen Funktionen, welche unteranderem auf Ideen von I.G. Macdonald zurückgeht. Im Zentrum der Dunkl-Theorie stehen Wurzelsysteme, ihre Spiegelungsgruppen und dazu assoziierte Dunkl- beziehungsweise Cherednik-Operatoren. Im eindimensionalen Fall sind die assoziierten speziellen Funktionen Bessel-Funktionen und Gauß-hypergeometrische Funktionen. In dieser Arbeit werden zunächst für beliebige Wurzelsysteme elliptische Dunkl-Operatoren eingeführt und untersucht, sowie eine multitemporale Wellengleichung studiert. Im Großteil der Arbeit sind Wurzelsysteme vom Typ A und B im Mittelpunkt. Zum Wurzelsystem vom Typ A assoziiert man eine wichtige Klasse orthogonaler Polynome: die Jack-Polynome, welche außerdem auch kombinatorisch betrachtet werden können. Die Einschränkung auf Wurzelsysteme vom Typ A und B kommt von den symmetrischen Kegeln. Symmetrische Kegel sind spezielle riemannsche symmetrische Räume mit einer besonderen Geometrie, assoziiertem Wurzelsystem vom Typ A und Verbindungen zu Objekten zum Wurzelsystem vom Typ B. Basierend auf Ideen und Vermutungen von I.G. Macdonald aus den 1980er Jahren werden Konzepte der radialen Analysis symmetrischer Kegel in die Dunkl-Theorie zu Wurzelsystemen vom Typ A und B verallgemeinert beziehungsweise neu eingeführt. Dies beinhaltet Laplace-Transformationsformeln (unteranderem für Jack-Polynome), eine Hankel-Transformation sowie Zeta-Distributionen und ihre Funktionalgleichungen. Zuletzt können diese Resultate genutzt werden, um Aussagen in der asymptotischen harmonischen Analysis zu beweisen, welche sich insbesondere mit der Konvergenz von sphärischen Funktionen beschäftigen, wenn der Rang des zugrundeliegenden symmetrischen Raumes gegen unendlich geht.

Abstract

This thesis makes new contributions to Dunkl Theory. Since the end of the 1980s, Dunkl theory has developed as a generalization of radial analysis on Riemannian symmetric spaces and is divided into two branches: the rational and the trigonometric theory. The rational theory was developed by C.F. Dunkl, while the trigonometric theory goes back to G. Heckman, E.M. Opdam and I. Cherednik. This includes a theory of multivariate special functions, which was considered before by I.G. Macdonald. In the center of the Dunkl theory are root systems, their reflection groups and associated Dunkl- or Cherednik operators. In the one-dimensional case, the associated special functions are Bessel functions and Gaussian hypergeometric functions. First in this thesis, elliptic Dunkl operators are introduced and examined for arbitrary root systems, and a multitemporal wave equation is studied. The majority of the work focuses on root systems of type A and B. An important class of orthogonal polynomials is associated with the root system of type A: the Jack polynomials, which can also be viewed combinatorially. The restriction to root systems of type A and B comes from the theory of symmetric cones. Symmetric cones are certain Riemannian symmetric spaces with a special geometry, associated root system of type A and connections to objects associated with a root system of type B. Based on ideas and conjectures by I.G. Macdonald from the 1980s, concepts of radial analysis of symmetric cones are generalized or newly introduced into the Dunkl theory for root systems of type A and B. This includes Laplace transform identities (including identities for Jack polynomials), a Hankel transform, as well as zeta distributions and their functional equations. Finally, these results will be used to obtain new results in asymptotic harmonic analysis, which deals, among other things, with the convergence of spherical functions when the rank of the underlying symmetric space tends to infinity.

Acknowledgements

First of all, I would like to thank both my parents and my grandparents for all the support I have received from them over the years. I am also grateful that my wife Tamara has always accompanied me through my studies and doctorate and has supported me unconditionally in everything.
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I would also like to thank all the people at the department of mathematics in Paderborn for the great professional, interdisciplinary and personal exchange. Special thanks go to Alena, Charlene, Daniel, Effie, Jannik, Joachim, Lukas, Tobias, Tomasz and many others.
I would also like to thank the German Research Foundation DFG, who has funded this thesis through the project RO 1264/4-1.
All in all, it was a wonderful time and I would do everything exactly the same again.

Contents

About Dunkl theory and this thesis

History of Dunkl theory

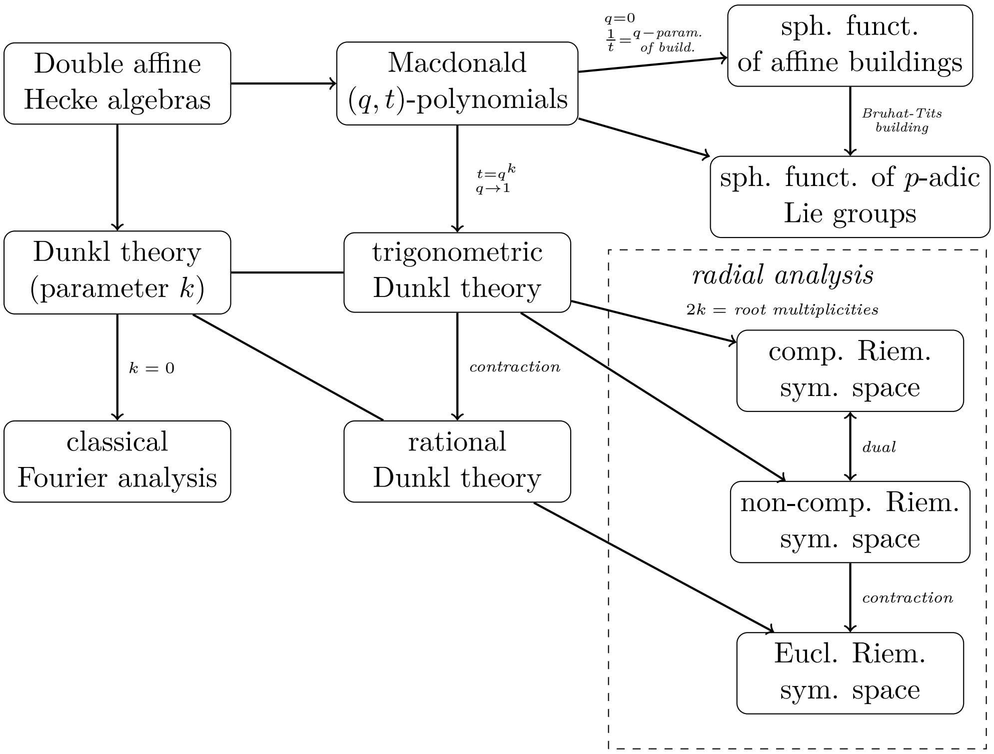
In the late 1980s C.F. Dunkl introduced so-called "rational" Dunkl operators associated with finite reflection groups on a Euclidean space in a series of papers [Dun88, Dun89, Dun90, Dun91, Dun92]. These operators form a class of commuting differential-difference operators and generalize partial derivatives as well as radial parts of invariant differential operators on Riemannian symmetric spaces of Euclidean type. In these papers, the author built up a framework of multivariate special functions related to root systems and a parameter family k of the root system. If the parameters k are chosen as root space multiplicities of a Riemannian symmetric space of Euclidean type, then the associated multivariate special functions reduce to spherical functions. During the last years, rational Dunkl theory was further developed by various people in the areas of special functions with reflection symmetry and harmonic analysis related to root systems. For a general overview of rational Dunkl theory the reader is referred to [Opd93, dJ93, Rö03a, DX14, Ank15].

Parallel to Dunkl in the 1980s, Heckman and Opdam developed the symmetric case of the so-called "trigonometric" Dunkl theory, and in the 1990s, Opdam and Cherednik studied the non-symmetric counterpart. We refer the reader to the articles [HO87, Hec87, Opd89, Hec90, Hec91, HS94, Opd95, Hec97, Opd00]. Even in the trigonometric setting there are certain operators of high relevance, called trigonometric Dunkl operators or Cherednik operators. They form a commuting set of operators generalizing partial derivatives as well as radial parts of invariant differential operator of Riemannian symmetric spaces of the non-compact type. Together, rational and trigonometric Dunkl theory allow to study spherical functions of Riemannian symmetric spaces in a uniform way (no matter whether it is of Euclidean, non-compact or compact type). Thus, Dunkl theory allows to extend spherical functions from a discrete set of parameters depending on the underlying root system to a continuous set of parameters. Moreover, eigenfunctions of the Cherednik operators give rise to a family of orthogonal trigonometric polynomials, called Jacobi polynomials or Heckman-Opdam polynomials, which are of interest in different areas of mathematics. For instance, the root system A_{n-1} leads to Jack polynomials.

Special functions associated with root systems find application in the study of quantum integrable models of Calogero-Moser-Sutherland type in one dimension and random matrix theorey; see [For10] for a background and [AV19] for some probabilistic developments. They have also found increasing interest in the field of integrable probability during the last years, see for instance [BG15].

It is remarkable that Dunkl theory is covered by a more general framework due to Macdonald, cf. [Mac00, Mac03]. He introduced a family of Weyl group invariant orthogonal polynomials depending on two parameters (q, t), later called Macdonald polynomials, with an impressive connection to Dunkl theory and other topics. For instance, in the case of the root systems BC_n they are Koornwinder polynomials. If the parameters are chosen to be $t = q^k$ and q tends to 1, then one obtains the Heckman-Opdam polynomials. The Macdonald polynomials are also related to spherical functions of Gelfand pairs (G, K) associated with Lie groups G of p-adic type. If $q = 0$, then these polynomials are called generalized Hall-Littlewood polynomials and if in addition t consists of parameters associated with an affine building, then the Macdonald polynomials have a simple structure, they are called Macdonald spherical functions of the affine building and are connected to vertex averaging operators on the building, cf. [Par06a, Par06b].

Behind all of this, there is a more general algebraic background given by (double) affine Hecke algebras. They were introduced by Cherednik to prove several conjectures of Macdonald, such as the constant term conjecture. Since these Hecke algebras are not topic of this thesis we do not go into further details and refer the reader to [Che95a, Che05]. Cherednik also introduced a non-symmetric analogue of the Macdonald polynomials such that their Weyl group symmetrization leads to the Macdonald polynomials mentioned before, cf. [Che95b]. The following diagram visualizes the explained connections.



Previous publications

We mention that up to some modifications and extensions, the content of the subsequent chapters are submitted to journals or are already published. This concerns the following chapters and articles:

Chapter 2:
[Bre23] D. Brennecken. Dunkl convolution and elliptic regularity for Dunkl operators. *submitted, preprint: arXiv:2308.07710*, 2023.

Chapter 6:
[BR23] D. Brennecken and M. Rösler. The Dunkl-Laplace transform and Macdonald's hypergeometric series. *Trans. Amer. Math. Soc. 376, 2419–2447*, 2023.

Chapter 7:
[Bre24] D. Brennecken. Hankel transform, K-Bessel functions and zeta distributions in the Dunkl setting. *J. Math. Anal. Appl. 535, 128125*, 2024.

Chapter 9:
[BR24] D. Brennecken and M. Rösler. Limits of Bessel functions for root systems as the rank tends to infinity. *submitted, preprint: arXiv:2401.02515*, 2024.

Concerning the two papers with Margit Rösler, both authors are first authors with equal rights and equal contributed parts to the development of the research questions.

Results of the thesis

First, we start with some general results in rational and trigonometric Dunkl theory. In particular, we shall briefly introduce a minor generalization of the trigonometric Dunkl theory due to Heckman and Opdam. To be more precise, we have a look at situations where the root system does not span the underlying Euclidean space. This is analogous to Lie theory, where one passes from semisimple Lie groups to reductive Lie groups. Later, this will be important when we generalize results from radial analysis on symmetric cones to the Dunkl setting. After this, we have a closer look at the Cherednik kernel, i.e. the joint eigenfunction of the Cherednik operators. We characterize the spectral parameters for which the kernel is a bounded function. For the hypergeometric function, i.e. the Weyl group symmetrization of the Cherednik kernel, this was answered in [NPP14] by giving a generalization of the Helgason-Johnson theorem, which characterizes the bounded spherical functions of a non-compact Riemannian symmetric space. As a result, we can prove a Riemann-Lebesgue lemma for the Cherednik transform.
We further take a closer look at the (rational) Dunkl convolution of distributions with non-compact supports and generalize results from [ØS05], where it was assumed that one of the distributions has compact support. With the results on these convolutions we are able to verify elliptic regularity and hypoellipticity for a certain class of Dunkl operators, called elliptic, in line with results for linear partial differential operators as in [Hö03]. For the particular case of the Dunkl Laplacian, these results were already proven in [MT04]. We will also apply our results to prove a convolution identity for generalized Riesz distributions.
Furthermore, based on our knowledge about generalized convolutions, we study a (rational) Dunkl analogue of the multitemporal wave equation on Riemannian symmetric spaces, cf. [PS93, Hel98, HS99]. The solution of this equation can be used to describe the generalized translation operator. This translation operator is used to define the convolution mentioned before, generalizing the convolution of K-biinvariant functions on a Riemannian symmetric space G/K. It is an object with a wide range of open questions and is only fully understood in rank one. In higher rank it is still open whether this convolution is a continuous L^1-operator, which would lead to a commutative L^1-convolution algebra, generalizing the commutative convolution algebra of K-biinvariant functions on G in the setting of a symmetric space G/K. As a consequence one would obtain L^p-boundedness of generalized translations with important consequences in harmonic analysis associated with root systems. To establish such convolution algebras, one has to find an integral representation for the generalized translation operator, which is equivalent to find a (positive) product formulas for the associated special functions, generalizing the known product formulas of spherical functions. Such results exist in rank one [FJK73], but in higher rank only few examples and partial results have been obtained so far, c.f. [Rö07, Rö10, RR15, Voi15].
For the main part of the thesis, it is important that Dunkl theory for the root system A_{n-1} is in many aspects related to radial analysis on symmetric cones. Indeed, a (simple) symmetric cone of rank n, such as the Lorentz cone or the cone of positive definite $n \times n$ matrices over one of the (skew) fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ is a Riemannian symmetric space $\Omega = G/K$ associated to a reductive Lie group G, with maximal compact subgroup K, whose root system is of type A_{n-1}, see [FK94]. The spherical polynomials of Ω can be identified with Jack polynomials of index $\alpha = 2/d$ as functions in the eigenvalues, where d is the Peirce dimension constant of Ω which only takes finitely many values (except in rank two). For instance, if Ω is the cone of positive definite matrices over $\mathbb{F}$, we have $d = \dim_{\mathbb{R}}(\mathbb{F}) \in \{1, 2, 4\}$. Moreover, via the Harish-Chandra integral formula, the K-spherical functions on the associated Euclidean Jordan algebra V can be identified with Dunkl-type Bessel functions with multiplicity parameter $k = d/2 = 1/\alpha$, c.f. [Rö20] and also [Rö98]. Indeed, we will prove a more general result: if G is a reductive Lie group of the Harish-Chandra class and $K \subseteq G$ is a maximal compact subgroup, then the associated Cartan motion group G_0 of G/K gives a Riemannian symmetric space of Euclidean

type G_0/K, whose spherical function can be identified with Dunkl-type Bessel functions. The Bessel functions govern the radial (i.e. K-biinvariant) analysis on V, such as analysis for functions or measures which depend only on the eigenvalues of their argument. It is therefore natural to ask which of the known results in the radial analysis on symmetric cones have analogues in Dunkl theory, when the multiplicity k is chosen arbitrarily. The technique to obtain such results is to replace the spherical polynomials on the cone by Jack polynomials (symmetric and non-symmetric), and the exponential function on the cone by the Bessel functions of type A (or even the Dunkl kernel). This program was already initiated by I.G. Macdonald in his unpublished manuscript [Mac89] from the 1980s. Since Dunkl theory was developed just around the same time, the author was unable to establish a link to Dunkl theory in [Mac89]. The ideas of Macdonald found applications in [BF97, BF98] within the study of quantum integrable models of Calorgero-Moser-Sutherland type and were also used in [SZ07]. However, most of the results are at a formal level, and matters were not further developed. This may be due to the fact that the analysis of the Dunkl analogue of the Laplace transform, was not well understood for a long time. The transform was introduced in [BF98] and is already contained implicitly in a symmetrized version in [Mac89]. The important observation in [BF98] was that the kernel of the Laplace transform in [Mac89] indeed is a Dunkl Bessel function of type A. But a rigorous foundation of the analysis was given only recently in [Rö20]. This opens up interesting questions and tools to study the interface between Dunkl theory and radial analysis on symmetric cones, which shall be studied in this thesis. For instance, we prove Laplace transform identities for Jack polynomials, the Cherednik kernel and hypergeometric series of Jack polynomials. These identities allow to introduce and study generalized zeta integrals, which satisfy certain characterizing functional equations as in the case of symmetric cones. In particular, important objects that will be introduced are two-variable Bessel functions generalizing the Bessel function and $\mathcal{K}$-Bessel function of a symmetric cone. The study of these special functions is important to verify properties of a generalized Hankel transform and to verify the functional equations of zeta distributions. Our results in the type A Dunkl setting make it possible to study the asymptotic behavior of the Bessel functions of type A and B as the number of variables tends to infinity. This is in accordance with asymptotic harmonic analysis related to Olshanski spherical pairs, cf. [OO97, OO98, Far08]. We generalize results for the asymptotics of the positive definite (Olshanski-)spherical functions of the pairs (G_∞, K_∞) corresponding to the sequences of Gelfand pairs

$$(U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))_{n\in\mathbb{N}},$$
$$((U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F})) \ltimes M_{p_n,q_n}(\mathbb{F}), U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F}))_{n\in\mathbb{N}}$$

with $p_n \geq q_n \to \infty$, where $U_n(\mathbb{F})$ is the unitary group of $\mathbb{F}$, $M_{p,q}(\mathbb{F})$ are the $p \times q$ matrices over $\mathbb{F}$ and $\mathrm{Herm}_n(\mathbb{F})$ are the $n \times n$ hermitian matrices over $\mathbb{F}$. For the latter Gelfand pairs we obtain new results, where only partial results exist in the case $\mathbb{F} = \mathbb{C}$ in [Rab08, Pic90]. Finally, we prove that the Cherednik kernel of type A is the limit of a Cherednik kernel of type BC. This generalizes the results of [RKV13], where an analogous limit transition was proven for the hypergeometric functions, i.e. the Weyl group symmetrizations of the Cherednik kernels. We shall use by analytic extension results and recurrence formulas for the Heckman-Opdam polynomials studied in [Opd95, Sah00a, Sah00b].

Structure of the thesis

First, we collect some basic facts about root systems and their reflection groups in Chapter 1. Then, we give an overview of rational Dunkl theory and known results within this field that are necessary for this thesis. We further introduce the reader to the connection between Dunkl theory and radial analysis on Riemannian symmetric spaces of Euclidean type. In

Chapter 2 we examine the Dunkl convolution of distributions which do not necessarily have compact support, and study its properties. This includes results about the support and generalized singular support of a Dunkl convolution. Afterwards, we introduce elliptic Dunkl operators and prove a theorem on hypoellipicity and elliptic regularity on local Sobolev spaces. Chapter 2 is also contained in the preprint [Bre23]. Chapter 3 deals with a multitemporal wave equation in the Dunkl setting in the spirit of [Hel98], whose solution is closely related to the generalized translation operator. With the results from the previous chapter, we are able to prove that the multitemporal wave equation is well-posed, i.e. for smooth initial data there exists exactly one smooth solution. In Chapter 4 we first introduce the reader to trigonometric Dunkl theory and extend it to the case of non-crystallographic integral root systems, which is motivated by the connection between semisimple and reductive Lie groups. We further explain the connection to Riemannian symmetric spaces of non-compact type and Riemannian symmetric spaces associated with Lie groups of the Harish-Chandra class. We prove a non-symmetric analogue of the Helgason-Johnson theorem in [NPP14] that characterizes the spectral parameters for which the eigenfunction of the Cherednik operators are bounded. As a consequence, we obtain a Riemann-Lebesgue lemma for the Cherednik transform similar to the symmetric analogue contained in [NPP14].

In the second part of the thesis we deal with the generalization of radial analysis on symmetric cones to Dunkl theory for root systems of type A and B. We start in Chapter 5 with a motivation about the connection between radial analysis on symmetric cones and Dunkl theory. Afterwards, we prove Laplace transform identities for Jack polynomials and the Cherednik kernel with techniques using Knop's and Sahi's raising operator and analytic continuation in Chapter 6. Furthermore, we study the convergence properties of Macdonald's hypergeometric series of Jack polynomials and Laplace transformation identities between these series. The chapter also contains a binomial formula for the Cherednik kernel, convolution formulas for generalized Riesz distributions and a Post-Widder inversion formula for the Dunkl-Laplace transform. Most of Chapter 6 is already published in [BR23]. Chapter 7 continues the program of the previous two chapters; it is published in [Bre24]. The chapter deals with generalizations of the Bessel function and the $\mathcal{K}$-Bessel function of a symmetric cone and their characteristic properties. In particular, the new Bessel function is closely related to the Dunkl Bessel function of type B, and defines the kernel of the Hankel transform. The Hankel transform and its properties are examined in detail. The chapter ends with the introduction of zeta integrals and a closely related analytic family of distributions. We prove several functional equations between zeta distributions, where the most characteristic one relates the zeta distributions with their type B Dunkl transform. Finally, we are able to characterize the positive measures within the family of zeta distributions.

The last part of the thesis deals with different limit transitions in Dunkl theory, partially motivated from asymptotic harmonic analysis. The background on asymptotic harmonic analysis and Olshanski spherical pairs is contained in Chapter 8. In Chapter 9 we give a sufficient and necessary condition for sequences of spectral parameters for which the Bessel functions of type A and B converge if the rank of the underlying root systems tends to infinity. As a special case, we are able to characterize the positive definite spherical functions of the pairs $(U_\infty(\mathbb{F}) \ltimes \mathrm{Herm}_\infty(\mathbb{F}), U_\infty(\mathbb{F}))$ and $((U_\infty(\mathbb{F}) \times U_\infty(\mathbb{F})) \ltimes M_{\infty,\infty}(\mathbb{F}), U_\infty(\mathbb{F}) \times U_\infty(\mathbb{F}))$ and show how they can be approximated by positive definite spherical functions of the underlying Gelfand pairs. The main part of Chapter 9 is contained in the preprint [BR24]. Finally, in Chapter 10 we prove that the Cherednik kernel of type A is the limit of the Cherednik kernel of type BC when some of the underlying multiplicities tend to infinity. This extends a known result for the hypergeometric functions, i.e. the Weyl group symmetrization of the Cherednik kernels.

Appendix A is a basis for Chapter 10 and deals with known results on recurrence formulas for Heckman-Opdam polynomials. These recurrence formulas were proven by Sahi in two papers [Sah00a, Sah00b] by different techniques, one for reduced root systems and the other one for the root system BC_n. The appendix gives a uniformization of these two papers for arbitrary crystallographic root systems by using the same language.

Part I

General results in Dunkl theory

CHAPTER 1

Introduction to Dunkl theory

In this chapter, we only provide an overview of the rational Dunkl theory. We explain the current state of research and do not provide any new results in this chapter. For the background on rational Dunkl theory the reader is referred to [Opd93, dJ93, Rö03a, DX14, Ank15]. We further give a brief introduction to root systems and their reflection groups, using the standard reference [Hum90].
The structure of the chapter is as follows. We begin in Section 1 with a brief overview of root systems and their reflection groups. In Section 2, we introduce the reader to the concepts and known results of rational Dunkl theory. We also explain the connection of rational Dunkl theory to radial analysis on Riemannian symmetric spaces of Euclidean type in Section 3.

1.1 Root systems and their reflection groups

Root systems and their finite reflection groups appear in various areas of mathematics, e.g. in (algebraic) combinatorics, geometry, Lie theory, orthogonal polynomials and special functions. Finite reflection groups are finite Coxeter groups. In general, Coxeter groups are a basis for the study of semisimple Lie groups, p-adic groups or buildings. For this work, root systems are a central object, so we will give a brief introduction to root systems to fix the terminology and some notations. Since this is classical theory, we will omit exact citations and instead refer to the standard reference [Hum90].

Basic definitions

Let $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional Euclidean space with norm $|x| := \sqrt{\langle x, x\rangle}$. With a non-zero $\alpha \in \mathfrak{a}$ we associate the orthogonal *reflection* s_α in the hyperplane perpendicular to α, defined by

$$s_\alpha : \mathfrak{a} \to \mathfrak{a}, \quad x \mapsto x - \langle x, \alpha^\vee \rangle \alpha,$$

where $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

Definition 1.1. A non-empty finite subset $R \subseteq \mathfrak{a} \setminus \{0\}$ is called a *root system* if it is invariant under the reflections s_α, for all $\alpha \in R$. Moreover, R is called

(i) *reduced* if $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$.

(ii) *integral* if $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

(iii) *crystallographic* if it is integral and $\operatorname{span}_{\mathbb{R}} R = \mathfrak{a}$.

(iv) *reducible* if it is a disjoint union of root systems, which are orthogonal to each other.

(v) *irreducible* if it is not reducible.

The number $\operatorname{rk} R := \dim(\operatorname{span}_{\mathbb{R}} R)$ is called the *rank* of R. The *(finite) reflection group* associated with a root system R is the group of orthogonal transformations generated by the reflections s_α with $\alpha \in R$, i.e.

$$W := W(R) := \langle s_\alpha, \alpha \in R \rangle_{\text{group}}.$$

If the root system is crystallographic, then W is typically called a *Weyl group*.
We note that each finite group W generated by reflections is the reflection group of a root system. The root system R is uniquely determined up to scaling of the W-orbits inside R. Moreover, the root system is irreducible if and only if the associated reflection group is not the direct product of non-trivial reflection groups.

Example 1.2. Up to isomorphisms of the vector space $\mathfrak{a}$ and rescaling of the orbits, the finite reflection groups with irreducible root systems are classified. To list them, consider the Euclidean space $\mathbb{R}^n$ with the canonical basis $(e_i)_{i=1,\ldots,n}$. The irreducible reduced crystallographic root systems and their Weyl groups are named

$$A_n, B_n/C_n, D_n, E_6, E_7, E_8, F_4, G_2,$$

where the index is the rank of the root system. The root systems E_6, E_7, E_8, F_4, G_2 are called *exceptional* and are constructed in [Hum90]. The infinite families of root systems A_n, B_n, C_n, D_n are called *classical* and can be constructed as follows.

(A_n, $n \geq 1$): The root system consists of the $n(n+1)$ vectors

$$A_n := \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\} \subseteq \mathbb{R}^{n+1}.$$

It is reduced and crystallographic in $\mathbb{R}_0^{n+1} := \{x \in \mathbb{R}^{n+1} \mid x_1 + \ldots + x_{n+1} = 0\}$. The Weyl group is the symmetric group $\mathcal{S}_{n+1}$ on $n+1$ letters, acting by permuting the coordinates. The reflection $s_{e_i - e_j}$ acts by permuting the i-th and j-th component.

(B_n/C_n, $n \geq 2$): These are two root systems, which coincide up to rescaling of orbits. The root systems consist of the $2n^2$ vectors

$$\begin{aligned} B_n &:= \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}, \\ C_n &:= \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}. \end{aligned}$$

They are reduced and crystallographic in $\mathbb{R}^n$. The Weyl group is the hyperoctahedral group $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$, where $\mathbb{Z}_2^n$ acts by changing the signs of the coordinates. The reflection $s_{e_i} = s_{2e_i}$ acts by mapping the i-th coordinate onto its negative.

(D_n, $n \geq 4$): The root system consists of the $2n(n-1)$ vectors

$$D_n := \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\}.$$

It is reduced and crystallographic in $\mathbb{R}^n$. The Weyl group is $\mathcal{S}_n \ltimes \mathbb{Z}_2^{n-1}$, where $\mathbb{Z}_2^{n-1}$ acts by an even number of sign changes.

The irreducible reduced non-crystallographic root systems are named H_3, H_4 and $I_2(m)$ of ranks $3, 4$ and 2, respectively. The root systems H_3 and H_4 are again called *exceptional* and are constructed in [Hum90]. The infinite family $I_2(m)$ can be constructed as follows.

($I_2(m)$, $m = 5, m \geq 7$): The root system consists of the $2m$-th roots of unity

$$I_2(m) := \left\{ e^{i\pi\frac{k}{m}} \mid 0 \leq k \leq 2m-1 \right\} \subseteq \mathbb{C} \cong \mathbb{R}^2.$$

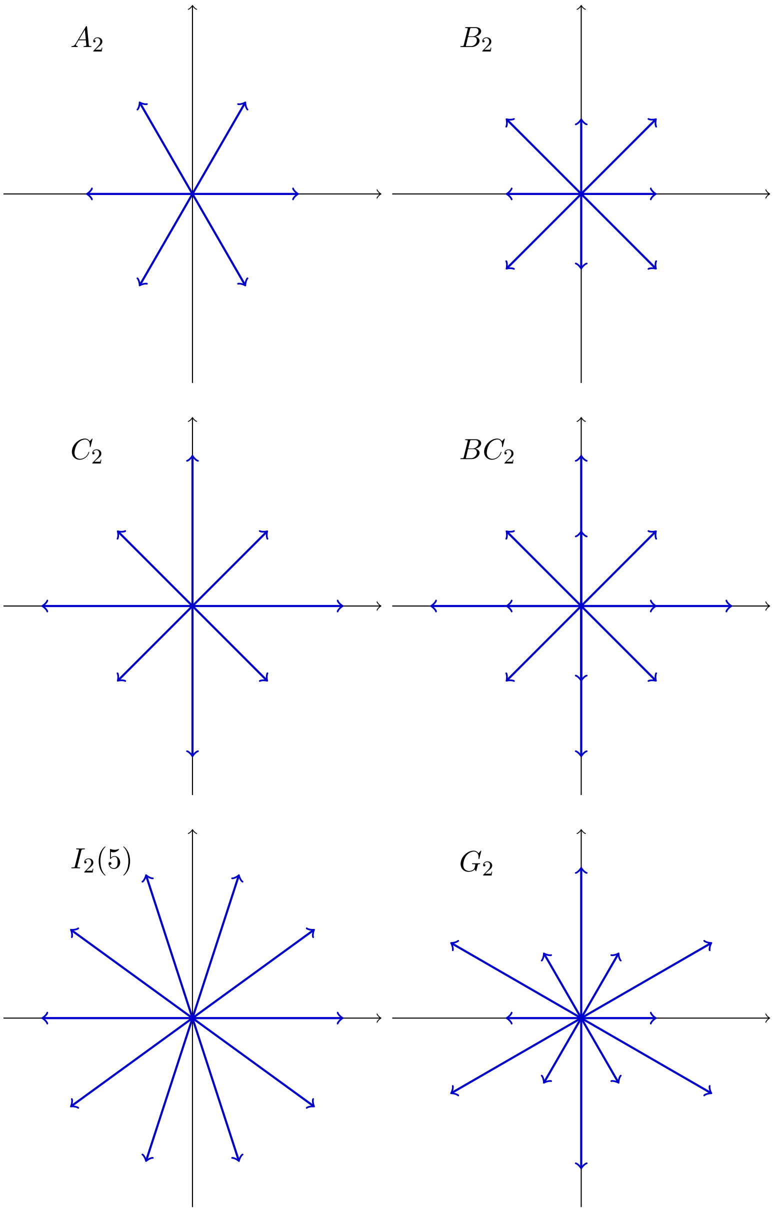
It is reduced, $\operatorname{span}_{\mathbb{R}}(I_2(m)) = \mathbb{C} \cong \mathbb{R}^2$ and the finite reflection group is the dihedral group $\mathcal{D}_{2m} = \mathbb{Z}_2 \ltimes \mathbb{Z}_m$ of order $2m$. In fact, the orbits of $I_2(m)$ cannot be rescaled to obtain a crystallographic root system, except for the case $m = 6$, where we can obtain the exceptional root system G_2.

It is remarkable that there exists exactly one irreducible non-reduced crystallographic root system (up to isomorphisms), called BC_n. It is the union of B_n and C_n.

(BC_n, $n \geq 1$): The root system consists of the $2n(n+1)$ vectors

$$BC_n \coloneqq \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \cup \{\pm e_i, \pm 2e_i \mid 1 \leq i \leq n\} \subseteq \mathbb{R}^n.$$

It is non-reduced and crystallographic in $\mathbb{R}^n$. The Weyl group is $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$.



Positive roots, simple roots and Weyl chambers

Let R be a root system inside the Euclidean space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ with finite reflection group W. The connected components of $\mathfrak{a} \setminus \bigcup_{\alpha \in R} \alpha^\perp$ are open convex cones, called *open Weyl chambers* and the closures are called *closed Weyl chambers*. The group W acts simply transitively on the set of open Weyl chambers. To describe the chambers in terms of the root system, we need the concept of positive and simple roots.

Definition 1.3. Consider two subsets R_+ and Π of the root system R.

(i) Then R_+ is called a *system of positive roots* if R is a disjoint union of R_+ and $R_- \coloneqq -R_+$ such that R_+ and R_- are separated by a hyperplane in $\mathfrak{a}$. The elements of R_- are called *negative roots*.

(ii) Then Π is called a *system of simple roots* or a *basis of* R if Π consists of linearly independent vectors with the following property: for an arbitrary root $\beta = \sum_{\alpha\in\Pi} n_\alpha \alpha \in R$, it holds that either all n_α are non-negative integers or non-positive integers.

There is a one-to-one correspondence between the set of Weyl chambers, positive roots and simple roots which is constructed as follow:

- If C is an open Weyl chamber, then
$$R_+ := \{\alpha \in R \mid \langle x, \alpha\rangle > 0 \text{ for all } x \in C\}$$
is a system of positive roots. Conversely, if R_+ is a system of positive roots, then
$$C := \{x \in \mathfrak{a} \mid \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in R_+\}$$
is an open Weyl chamber, called the *(open) positive Weyl chamber* with respect to R_+. Typically, the positive Weyl chamber is denoted by $\mathfrak{a}_+$ or C_+.
- If Π is a system of simple roots, then
$$R_+ := \{\beta = \textstyle\sum_{\alpha\in\Pi} n_\alpha \alpha \in R \mid n_\alpha \in \mathbb{N}_0 \text{ for all } \alpha \in \Pi\}$$
is a system of positive roots. Conversely, if R_+ is a system of positive roots, the set
$$\Pi := \{\alpha \in R_+ \mid \alpha \text{ is not a sum of elements in } R_+\}$$
is a system of simple roots.

Example 1.4. As an example to visualize the concepts of positive roots, simple roots and Weyl chambers, we look at the root system A_2, as subset of $\mathbb{R}^2 \cong \mathbb{R}^3_0$.

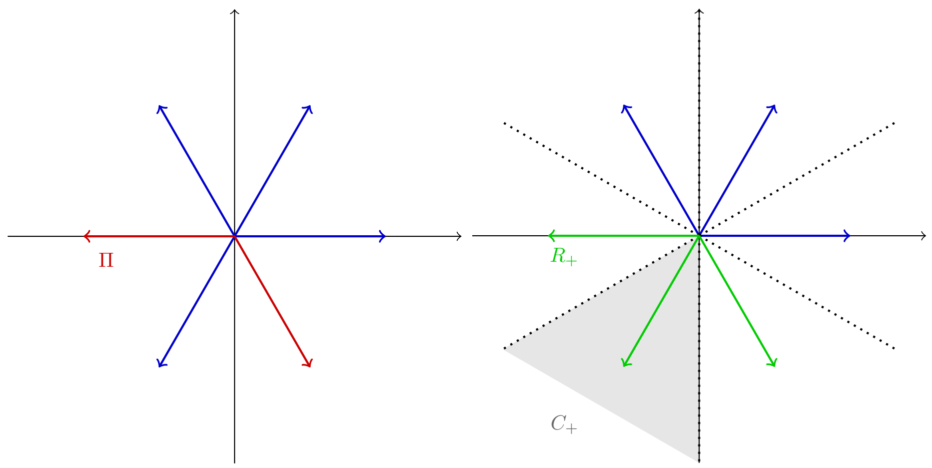


Figure: positive roots, simple roots and positive Weyl chamber of A_2

By fixing a system of positive roots and simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, the reflections $s_i := s_{\alpha_i}$ are called *simple reflections*. They form a set $S = \{s_1, \dots, s_n\}$, which is a minimal generating subset of W. Indeed, the pair (W, S) is a *Coxeter system* in the sense of the subsequent definition.

Definition 1.5. A pair (W, S) consisting of a group W and a finite subset $S \subseteq W$ is called a *Coxeter system* if

(i) S generates W and the elements of S have order 2,

(ii) W is presented with generators S and the relations $(st)^{m_{st}} = 1$, $s, t \in S$, where m_{st} is the order of st in W and the relation does not occur if the order is infinite. This means that W is the quotient of the free group over S by the normal subgroup generated by the upper relations.

In this case, the group W is called a *Coxeter group* with generators S. Each Coxeter system defines a length function by

$$\ell_S : W \to \mathbb{N}_0, \quad \ell_S(w) := \min \{\ell \in \mathbb{N}_0 \mid w = s_1 \cdots s_\ell,\ s_i \in S\}.$$

Coxeter groups have a rich structure theory and even the infinite Coxeter groups are of high relevance, such as affine Weyl groups. For our purpose it is enough to consider finite Coxeter groups, i.e. finite reflection groups W with simple reflections $S = \{s_1, \ldots, s_n\}$ as before. The pair (W, S) has an important element, called the *longest element* $w_0 \in W$, which is uniquely described by the following equivalent properties:

(i) $\ell_S(w) < \ell_S(w_0)$ for all $w_0 \neq w \in W$.

(ii) $\ell_S(w_0) = \#R_+$.

(iii) $w_0\Pi = -\Pi$ and equivalently $w_0R_+ = R_-$.

(iv) $w_0C_+ = -C_+$.

(v) $\ell_S(w_0 s_\alpha) < \ell_S(w_0)$ for all $\alpha \in \Pi$ and equivalentely $\ell_S(w_0 w) < \ell_S(w_0)$ for all $\mathrm{id} \neq w \in W$.

(vi) $\ell_S(w_0 w) = \ell_S(w_0) - \ell_S(w)$ for all $w \in W$.

Since $\ell_S(w) = \ell_S(w^{-1})$, it is immediate that w_0 is an involution, i.e. $w_0^2 = \mathrm{id}$. For instance, if $W = \mathcal{S}_n$ and S consists of the transpositions s_i of the i-th and $(i+1)$-th coordinate, then the longest element w_0 is defined by reversing the order, i.e. $w_0(x_1, \ldots, x_n) = (x_n, \ldots, x_1)$.

1.2 Introduction to rational Dunkl theory

Dunkl operators acting on function spaces

Consider a finite-dimensional Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ with norm $|x| = \sqrt{\langle x, x \rangle}$ and extend the inner product in a complex-bilinear way to the complexification $\mathfrak{a}_\mathbb{C} := \mathbb{C} \otimes \mathfrak{a}$. Let $R \subseteq \mathfrak{a}$ be a reduced root system and $W = W(R)$ the associated finite reflection group. Then W acts on $\mathfrak{a}$ as a group of isometries and therefore on arbitrary functions $f : \Omega \to \mathbb{C}$, defined on W-invariant subsets $\Omega \subseteq \mathfrak{a}$, by the assignment $wf(x) = f(w^{-1}x)$. Moreover, the set of regular elements is

$$\mathfrak{a}_{\mathrm{reg}} := \mathfrak{a} \backslash \bigcup_{\alpha \in R} \alpha^\perp = \{x \in \mathfrak{a} \mid wx \neq x \text{ for all } w \in W \backslash \{\mathrm{id}\}\}.$$

Definition 1.6. Let $k : R \to \mathbb{C}$, $\alpha \mapsto k_\alpha$ be a W-invariant function, called a *multiplicity function* or *multiplicity*. The *(rational) Dunkl operator* associated with (R, k) into the direction $\xi \in \mathfrak{a}$ is the differential-difference operator

$$T_\xi(k) := \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{\langle \alpha, \cdot \rangle} = \partial_\xi + \frac{1}{2} \sum_{\alpha \in R} k_\alpha \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{\langle \alpha, \cdot \rangle},$$

where ∂_ξ is the usual partial derivative into the direction ξ and $R_+ \subseteq R$ is an arbitrary system of positive roots. Sometimes we denote the Dunkl operators by T_ξ, T_ξ^R or $T_\xi^R(k)$, depending on whether (R, k) is clear from the context or not. To be more precise, if $f \in C^1(\Omega)$ for some W-invariant open subset $\Omega \subseteq \mathfrak{a}$, then

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle}.$$

We observe that $\xi \mapsto T_\xi(k)$ is linear, $T_\xi(k)$ does not depend on the length of the roots and $T_\xi(0) = \partial_\xi$ are the usual directional derivatives.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{a}$. As usual, we put for $\beta \in \mathbb{N}_0^n$:

$$\begin{aligned} |\beta| &:= \beta_1 + \ldots + \beta_n, \\ \partial_i &:= \partial_{e_i}, \\ \partial^\beta &:= \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}. \end{aligned}$$

On a W-invariant convex subset of $\mathfrak{a}$, the difference operator occurring in the definition of $T_\xi(k)$ can be represented as

$$\frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle} = \frac{2}{\langle \alpha, \alpha \rangle} \int_0^1 (\partial_\alpha f)(x - t \langle x, \alpha^\vee \rangle \alpha) \, \mathrm{d}t. \tag{1.1}$$

From the version (1.1) of the difference operator we can deduce several properties of the Dunkl operators. In the following we summarize the main properties that are important for our purpose. Consider an open W-invariant subset $\Omega \subseteq \mathfrak{a}$ and $\xi, \eta \in \mathfrak{a}$. The Dunkl operators satisfy:

(i) The operator $T_\xi(k)$ maps $C^m(\Omega)$ into $C^{m-1}(\Omega)$ for $m \in \mathbb{N} \cup \{\infty\}$.

(ii) *Homogeneity*: let $\mathcal{P} = \mathbb{C}[\mathfrak{a}]$ be the space of complex valued polynomial functions on $\mathfrak{a}$. Then, $T_\xi(k)$ is homogeneous of degree -1, i.e.

$$T_\xi(k) : \mathcal{P}_n \to \mathcal{P}_{n-1},$$

where $\mathcal{P}_n \subseteq \mathcal{P}$ is the subspace of homogeneous polynomial functions of degree n.

(iii) *Commutativity*: the Dunkl operators commute, i.e.

$$T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k) \quad \text{on} \quad C^2(\Omega).$$

Thus, we have a well-defined unital morphism of complex algebras

$$\mathcal{P} \to \mathrm{End}(C^\infty(\Omega)), \quad p \mapsto p(T(k)), \ \text{ determined by } \ \langle \cdot, \xi \rangle \mapsto T_\xi(k).$$

The image of this morphism is denoted by $\mathbb{D}(k)$ and is called the *algebra of Dunkl operators*. $\mathbb{D}(0)$ is just the algebra of linear differential operators with constant coefficients.

(iv) *Leibniz rule*: if $f, g \in C^1(\Omega)$, where at least one of them is W-invariant, then

$$T_\xi(k)(f \cdot g) = (T_\xi(k)f) \cdot g + f \cdot (T_\xi(k)g).$$

(v) *Equivariance*: under conjugation with $w \in W$, the Dunkl operators satisfy

$$wT_\xi(k)w^{-1} = T_{w\xi}(k).$$

(vi) *Support*: for $f \in C^1(\Omega)$ we have

$$\mathrm{supp}(T_\xi(k)f) \subseteq W.\mathrm{supp}\, f,$$

where $\mathrm{supp}\, f$ is the support of f. In particular, $C_c^m(\Omega)$ is mapped by $T_\xi(k)$ into $C_c^{m-1}(\Omega)$.

(vii) Let $f \in C^m(\Omega)$ with $m \in \mathbb{N} \cup \{\infty\}$ and let $K \subseteq \Omega$ be a compact convex W-invariant subset. Equation (1.1) leads to the following: For all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m-1$, there exists a constant $C(\beta, k)$ independent of f, Ω and K such that

$$||\partial^\beta T_\xi(k) f||_{\infty,K} \leq C(\beta, k) \max_{|\gamma|=|\beta|+1} ||\partial^\gamma f||_{\infty,K}, \tag{1.2}$$

where $\|f\|_{\infty,K} = \max_{x \in K} |f(x)|$ is the usual supremum norm on K. As a consequence the Dunkl operator $T_\xi(k)$ maps $C^m(\Omega)$ continuously into $C^{m-1}(\Omega)$ (equipped with the usual locally convex topologies).

(viii) Dunkl operators act continuously on the Schwartz space $\mathscr{S}(\mathfrak{a})$ as well as on the spaces $C^\infty(\mathfrak{a})$ and $C_c^\infty(\mathfrak{a})$, each equipped with their usual locally convex space topologies.

Owing to the W-equivariance of Dunkl operators, the algebra of W-invariant Dunkl operators is explicitly given by

$$\mathbb{D}(k)^W = \left\{ p(T(k)) \,\middle|\, p \in \mathcal{P}^W \right\}.$$

Moreover, $\mathbb{D}(k)^W$ acts on $C^\infty(\Omega)^W$ as an algebra of differential operators. To be more precise, for all $p(T(k)) \in \mathbb{D}(k)^W$ there exists a unique linear differential operator $\operatorname{res}(p(T(k)))$ on $\mathfrak{a}_{\mathrm{reg}}$ such that for all $f \in C^\infty(\mathfrak{a})^W$ we have

$$p(T(k))f = \operatorname{res}(p(T(k)))f. \tag{1.3}$$

Example 1.7 (Dunkl-Laplacian)**.** The Dunkl operator Δ_k associated with the quadratic polynomial $p(x) = \langle x, x \rangle$ is called the *Dunkl-Laplacian* and can be written as

$$\Delta_k f(x) = \Delta_{\mathfrak{a}} f(x) + \sum_{\alpha \in R} k_\alpha \left(\frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{|\alpha|^2}{2} \cdot \frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle^2} \right),$$

where $\Delta_{\mathfrak{a}}$ is the usual Laplacian on $\mathfrak{a}$. In particular,

$$L_k := \operatorname{res}(\Delta_k) = \Delta_{\mathfrak{a}} + \sum_{\alpha \in R} k_\alpha \frac{\partial_\alpha}{\langle \alpha, \cdot \rangle}.$$

Example 1.8 (Rank one)**.** The root system is $R = \{\pm 1\} \subseteq \mathbb{R}$ and the Dunkl operators are described by

$$T_1(k) f(x) = f'(x) + k \frac{f(x) - f(-x)}{x},$$

so that the Dunkl Laplacian becomes

$$\Delta_k f(x) = f''(x) + 2k \frac{f'(x)}{x} - k \frac{f(x) - f(-x)}{x^2}.$$

In particular, $L_k = \operatorname{res}(\Delta_k)$ is the *Bessel operator*

$$L_k f(x) = f''(x) + \frac{2k}{x} f'(x).$$

We observe the following: if $k = \frac{n-1}{2}$ with $n \in \mathbb{N}$, then L_k is the $\mathrm{SO}_n(\mathbb{R})$-radial part of the Laplacian $\Delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$ in polar coordinates, i.e. if $f \in C^2(\mathbb{R}^n)$ is radial and $f(x) = F(|x|)$, then

$$\Delta_{\mathbb{R}^n} f(x) = F''(|x|) + \frac{n-1}{|x|} F'(|x|) = L_{\frac{n-1}{2}} F(|x|).$$

Dunkl intertwiner and regular multiplicities

Definition 1.9. On the polynomials $\mathcal{P}$ we define the *generalized Fisher product*

$$[\cdot,\cdot]_k : \mathcal{P} \times \mathcal{P} \to \mathbb{C}, \quad [p,q]_k := (p(T(k))q)(0).$$

In the case $k=0$, this reduces to the usual Fisher product $[p,q]_0 = (p(\partial)q)(0)$. The pairing $[\cdot,\cdot]_k$ defines a W-invariant symmetric bilinear form such that the subspaces $\mathcal{P}_n$, $n \in \mathbb{N}$, are mutually orthogonal. The adjoint of $T_\xi(k)$ with respect to $[\cdot,\cdot]_k$ is multiplication by $\langle \cdot, \xi \rangle$.

If $k \geq 0$ is a non-negative multiplicity, the *Macdonald-Identity*

$$[p,q]_k = \frac{1}{c_k} \int_{\mathfrak{a}} (e^{-\Delta_k/2} p)(x) \cdot (e^{-\Delta_k/2} q)(x) \cdot e^{-|x|^2/2} \omega_k(x) \, \mathrm{d}x$$

holds, where $\mathrm{d}x$ is the Lebesgue measure on $\mathfrak{a}$,

$$\omega_k(x) := \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k_\alpha} = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha} \tag{1.4}$$

is a weight function, and

$$c_k := \int_{\mathfrak{a}} e^{-|x|^2/2} \omega_k(x) \, \mathrm{d}x \tag{1.5}$$

is called the *Macdonald-Mehta constant*.

Remark 1.10. A formula for the constant c_k was conjectured by Macdonald and was proven by Opdam in [Opd89, Opd93] for all irreducible root systems except of H_3, H_4. The proofs of the remaining cases was based on a computer calculation by F. Garvan. Later, Etingof [Eti10] found a uniform proof for all root systems, but under the consideration that the multiplicity is constant. The constant c_k has an explicit form and can be expressed as a product of gamma functions.

Definition 1.11. Let $\mathcal{K}$ be the vector space of multiplicity functions $k : R \to \mathbb{C}$ on R. A multiplicity $k \in \mathcal{K}$ is called *regular* if one of the following equivalent statements hold:

(i) $[\cdot,\cdot]_k$ is non-degenerate. Furthermore, $(p,q) \mapsto [p, \overline{q}]_k$ is an inner product on $\mathcal{P}$.

(ii) $\bigcap_{\xi \in \mathfrak{a}} \ker(T_\xi(k)|_{\mathcal{P}}) = \mathbb{C}$.

(iii) There exists $V_k \in \mathrm{GL}(\mathcal{P})$ such that

$$V_k 1 = 1, \quad V_k \partial_\xi = T_\xi(k) V_k \quad \text{and} \quad V_k(\mathcal{P}_n) \subseteq \mathcal{P}_n$$

for all $\xi \in \mathfrak{a}$ and $n \in \mathbb{N}_0$. The operator V_k is uniquely determined by these properties.

The set of regular multiplicities will be denoted by $\mathcal{K}_{\mathrm{reg}}$ and we have

$$\{k \in \mathcal{K} \mid \operatorname{Re} k \geq 0\} \subseteq \mathcal{K}_{\mathrm{reg}}.$$

For $k \in \mathcal{K}_{\mathrm{reg}}$, the isomorphism V_k is called *Dunkl's intertwining operator*. Furthermore, $\mathcal{K}_{\mathrm{reg}}$ can be described explicitly, see [dJDO94]. One then sees that $\mathcal{K}_{\mathrm{reg}} \subseteq \mathcal{K}$ is dense, open and $\mathcal{K} \backslash \mathcal{K}_{\mathrm{reg}}$ is a countable union of algebraic sets. To be more precise,

$$\mathcal{K} \backslash \mathcal{K}_{\mathrm{reg}} = \bigcup_{n=0}^{\infty} (\mathcal{K}^0 - n \cdot 1_R), \text{ with } \mathcal{K}^0 = \{k \in \mathcal{K} \mid [\pi, \pi]_k = 0\},$$

where 1_R is the multiplicity with constant value 1 and π is the fundamental skew-polynomial

$$\pi(x) = \prod_{\alpha \in R_+} \langle \alpha, x \rangle .$$

We denote the convex hull of a subset $M \subseteq \mathfrak{a}$ by

$$\operatorname{co}(M) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \,\middle|\, m \in \mathbb{N},\, x_i \in M,\, 0 < \lambda_i < 1,\, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

A cornerstone in the study of Dunkl's intertwining operator is the following positivity result due to Rösler.

Theorem 1.12 ([Rö99])**.** *Let $k \geq 0$ be a non-negative multiplicity. Then:*

(i) *the operator V_k is a positivity preserving operator, i.e. if $p \in \mathcal{P}$ is non-negative, then $V_k p$ is non-negative as well.*

(ii) *For all $x \in \mathfrak{a}$ there exists a (unique) probability measure μ_x^k on $\mathfrak{a}$ with support contained in $\operatorname{co}(W.x)$ and*

$$V_k p(x) = \int_{\mathfrak{a}} p \, \mathrm{d}\mu_x^k.$$

Furthermore, Trimèche proved in [Tri01] the following important theorem on the extension of Dunkl's intertwining operator to the space of smooth functions on $\mathfrak{a}$.

Theorem 1.13 ([Tri01])**.** Let $k \geq 0$ be a non-negative multiplicity. If V_k is defined on $C(\mathfrak{a})$ via

$$V_k f(x) = \int_{\mathfrak{a}} f \, \mathrm{d}\mu_x^k,$$

then the following holds:

(i) the intertwiner V_k maps $C(\mathfrak{a})$ into itself so that $\|V_k f\|_{\infty, B_r(0)} \leq \|f\|_{\infty, B_r(0)}$ for all closed balls $B_r(0)$ with radius $r > 0$ around 0.

(ii) the intertwiner V_k is a topological automorphism of $C^\infty(\mathfrak{a})$ satisfying

$$V_k f(0) = f(0) \quad \text{and} \quad V_k \partial_\xi = T_\xi(k) V_k$$

for all $f \in C^\infty(\mathfrak{a})$ and $\xi \in \mathfrak{a}$.

Example 1.14 (Rank one)**.** For $R = \{\pm 1\} \subseteq \mathbb{R}$ one verifies that Dunkl's intertwining operator is given, for $\operatorname{Re} k > 0$, by

$$V_k 1 = 1, \quad V_k(x^{2n}) = \frac{(\frac{1}{2})_n}{(k + \frac{1}{2})_n} x^{2n} \quad \text{and} \quad V_k(x^{2n-1}) = \frac{(\frac{1}{2})_n}{(k + \frac{1}{2})_n} x^{2n-1},$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. In particular, we have

$$\mathcal{K}_{\mathrm{reg}} = \mathbb{C} \setminus \left\{ -\tfrac{1}{2}, -\tfrac{3}{2}, \ldots \right\}.$$

The integral representation of V_k is given by

$$V_k p(x) = \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k)\sqrt{\pi}} \int_{-1}^{1} p(tx)(1-t)^{k-1}(1+t)^k \, \mathrm{d}t.$$

Eigenfunctions

Definition 1.15. For non-negative multiplicities $k \geq 0$, the *Dunkl kernel* associated with (R, k) with spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}$ is defined by

$$E_k(\lambda, \cdot) := V_k(e^{\langle \lambda, \cdot \rangle})$$

and is the unique analytic solution of the eigenvalue problem

$$\begin{cases} T_\xi(k) f = \langle \lambda, \xi \rangle \, f, & \text{for all } \xi \in \mathfrak{a}, \\ \quad f(0) = 1. \end{cases}$$

In fact, by Opdam [Opd93] this eigenvalue problem has a unique analytic solution $E_k(\lambda, \cdot)$ for all $k \in \mathcal{K}_{\text{reg}}$, which extends to an entire map on $\mathfrak{a}_{\mathbb{C}}$. Moreover, the map

$$\mathcal{K}_{\text{reg}} \times \mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}, \quad (k, \lambda, z) \mapsto E_k(\lambda, z)$$

is holomorphic. The *Bessel function* $J_k(\lambda, \cdot)$ associated with (R, k) with spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}$ is by definition

$$J_k(\lambda, z) := \frac{1}{\# W} \sum_{w \in W} E_k(\lambda, wz)$$

and can be characterized as the unique W-invariant analytic solution of

$$\begin{cases} p(T(k)) f = p(\lambda) f, & \text{for all } p \in \mathcal{P}^W, \\ \quad f(0) = 1. \end{cases}$$

Moreover, the Bessel function is also W-invariant in the spectral parameter and holomorphic on the same domain as the Dunkl kernel. In fact, by Opdam [Opd93, Proposition 9.6], $(\mathcal{K} \backslash \mathcal{K}_{\text{reg}}) \times \mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ is precisely the pole set of the Dunkl kernel and the Bessel function, so that the singular multiplicities can be characterized as the poles of these functions.

If $k = 0$, then the Dunkl kernel reduces to the exponential $e^{\langle \lambda, z \rangle}$ and the Bessel function is some kind of generalized hyperbolic cosine. In fact, for $k = 0$ in rank one, the Bessel function is exactly the hyperbolic cosine. The case of general k in rank one is discussed at the end of this section and is the reason why J_k is called a Bessel function. The following theorem shows that the Dunkl kernel behaves like an exponential function in several aspects.

Theorem 1.16 ([dJ93, Rö99])**.** The Dunkl kernel satisfies:

(i) $E_k(\lambda, z) = E_k(z, \lambda)$ for all $\lambda, z \in \mathfrak{a}_{\mathbb{C}}$.

(ii) $E_k(s\lambda, z) = E_k(\lambda, sz)$ for all $\lambda, z \in \mathfrak{a}_{\mathbb{C}}$ and $s \in \mathbb{C}$.

(iii) $E_k(w\lambda, wz) = E_k(\lambda, z)$ for all $\lambda, z \in \mathfrak{a}_{\mathbb{C}}$ and $w \in W$.

If additionally $k \geq 0$, then

(iv) E_k is positive on $\mathfrak{a} \times \mathfrak{a}$.

(v) for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$, $x \in \mathfrak{a}$ and $\alpha \in \mathbb{N}_0^n$

$$|(\tfrac{\partial}{\partial \lambda})^\alpha E_k(\lambda, x)| \leq |x|^{|\alpha|} \, E_k(\operatorname{Re} \lambda, x) \leq |x|^{|\alpha|} \max_{w \in W} e^{\langle \operatorname{Re} \lambda, wx \rangle}.$$

In particular, $|E_k(ix, y)| \leq 1$ for all $x, y \in \mathfrak{a}$.

The same results are still valid if E_k is replaced by the Bessel function J_k.

Remark 1.17 (Product situation)**.** In the cases where R is not irreducible or does not span $\mathfrak{a}$, the Bessel function and Dunkl kernel factorize in the following sense.

(i) If $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is an orthogonal sum and $R = R_1 \sqcup R_2$ with root systems $R_i \subseteq \mathfrak{a}_i$, then each multiplicity function $k : R \to \mathbb{C}$ is uniquely described by its restrictions $k_1 = k|_{R_1}$ and $k_2 = k|_{R_2}$. The Dunkl kernel and Bessel function then factorize as

$$E_k^R(\lambda_1 + \lambda_2, z_1 + z_2) = E_k^{R_1}(\lambda_1, z_1) E_{k_2}^{R_2}(\lambda_2, z_2),$$
$$J_k^R(\lambda_1 + \lambda_2, z_1 + z_2) = J_k^{R_1}(\lambda_1, z_1) J_{k_2}^{R_2}(\lambda_2, z_2),$$

for all $\lambda_i, z_i \in (\mathfrak{a}_i)_\mathbb{C}$.

(ii) Assume that the Euclidean space decomposes into an orthogonal direct sum $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c}$, where $\mathfrak{b}$ is the vector space spanned by R. Then $W(R)$ acts trivial on $\mathfrak{c}$ and

$$E_k^R(\lambda + \mu, z + \zeta) = E_k^R(\lambda, z) e^{\langle \mu, \zeta \rangle},$$
$$J_k^R(\lambda + \mu, z + \zeta) = J_k^R(\lambda, z) e^{\langle \mu, \zeta \rangle},$$

hold for all $\lambda, z \in \mathfrak{b}_\mathbb{C}$ and $\mu, \zeta \in \mathfrak{c}_\mathbb{C}$.

Example 1.18 (rank one)**.** Consider the root system $R = \{\pm 1\} \subseteq \mathbb{R}$. Then $W = \{\pm \mathrm{id}\}$ and $\mathcal{P}^W$ is generated by 1 and $p(x) = x^2$. Hence, $J_k(\lambda, \cdot)$ is the unique even analytic solution of

$$\begin{cases} f''(x) + \frac{2k}{x} f'(x) = \lambda^2 f(x), \\ f(0) = 1. \end{cases}$$

For $k \in \mathbb{C} \setminus \left\{ -\frac{1}{2}, -\frac{3}{2}, \dots \right\}$ this solution is given by

$$J_k(\lambda, z) = j_{k - \frac{1}{2}}(i\lambda z)$$

with the spherical Bessel function

$$j_\alpha(z) = \sum_{k=0}^\infty \frac{\Gamma(\alpha+1)(-1)^k}{\Gamma(\lambda+k+1)k!} \left(\frac{z}{2}\right)^{2k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{izt}\, \mathrm{d}t,$$

where the last equation is true under the consideration $\operatorname{Re}\alpha > \frac{1}{2}$. Especially, for $k = 0, 1$ one has

$$J_0(\lambda, z) = \cosh(\lambda z) \quad \text{and} \quad J_1(\lambda, z) = \frac{\sinh(\lambda z)}{\lambda z}.$$

To compute the Dunkl kernel, recall Example 1.14. This formula for the intertwining operators for $k > 0$ leads to

$$E_k(\lambda, x) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\sqrt{\pi}} \int_{-1}^1 e^{\lambda z t}(1-t)^{k-1}(1+t)^k\, \mathrm{d}t.$$

Simple manipulations on this integral show that

$$E_k(\lambda, x) = j_{k-\frac{1}{2}}(i\lambda z) + \frac{\lambda z}{2k+1} j_{k+\frac{1}{2}}(i\lambda z).$$

Dunkl transform

In the remainder of this section we may assume that the multiplicity k satisfies $\operatorname{Re} k \geq 0$. The Dunkl transform is widely studied in [dJ93, Tri01, dJ06]. If $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ is a complex Radon measure on $\mathfrak{a}$, i.e. the μ_i's are regular Borel measures on $\mathfrak{a}$. We denote the Lebesgue spaces associated with μ by

$$L^p(\mathfrak{a}, \mu) := L^p(\mathfrak{a}, |\mu|) = \bigcap_{i=1}^{4} L^p(\mathfrak{a}, \mu_i), \qquad 1 \leq p \leq \infty \tag{1.6}$$

where $|\mu| = \mu_1 + \mu_2 + \mu_3 + \mu_4$. If μ has a density ω with respect to the Lebesgue measure on $\mathfrak{a}$, then we identify μ with ω and write $L^p(\mathfrak{a}, \omega)$ for the associated Lebesgue spaces. The usual L^p-norms are denoted by

$$\|f\|_{p,\mu} = \begin{cases} \left(\int_{\mathfrak{a}} |f(x)|^p \ \mathrm{d}\,|\mu|\,(x) \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\{c > 0 \mid |f| \leq c \ \text{ a.e. with respect to } \ |\mu|\}, & p = \infty. \end{cases}$$

Definition 1.19. Recall the weight function ω_k and the constant c_k from equations (1.4) and (1.5), respectively. The *Dunkl transform* of a function $f \in L^1(\mathfrak{a}, \omega_k)$ is defined by

$$\mathcal{F}_k f(\xi) := \widehat{f}^k(\xi) := \frac{1}{c_k} \int_{\mathfrak{a}} E_k(-i\xi, x) f(x) \omega_k(x) \, \mathrm{d}x, \quad \xi \in \mathfrak{a}.$$

For $k = 0$ this reduces to the usual Euclidean Fourier transform.

Most of the standard theorems on the Fourier transform can be generalized to the Dunkl transform. For our purposes in this thesis, the subsequent results are of interest.

Theorem 1.20 ([dJ93])**.** With $\psi : \mathfrak{a} \to \mathbb{C}$ we associate the multiplication operator

$$m_\psi : \varphi \mapsto \psi\varphi.$$

For $f, g \in L^1(\mathfrak{a}, \omega_k)$ and $\xi \in \mathfrak{a}$ the Dunkl transform satisfies:

(i) *Equivariance*: $w\mathcal{F}_k = \mathcal{F}_k w$ on $L^1(\mathfrak{a}, \omega_k)$ for all $w \in W$.

(ii) *Riemann-Lebesgue lemma*: $\widehat{f}^k \in C_0(\mathfrak{a})$ with $||\widehat{f}^k||_\infty \leq \frac{1}{c_k} \|f\|_{1,\omega_k}$.

(iii) *Parseval identity*: $\displaystyle\int_{\mathfrak{a}} \widehat{f}^k(x) g(x) \omega_k(x) \, \mathrm{d}x = \int_{\mathfrak{a}} f(x) \widehat{g}^k(x) \omega_k(x) \, \mathrm{d}x.$

(iv) L^1-*Inversion formula*: if $\widehat{f}^k \in L^1(\mathfrak{a}, \omega_k)$, then $f(x) = (\mathcal{F}_k \mathcal{F}_k f)(-x)$ a.e.

(v) If $f \in L^1(\mathfrak{a}, \omega_k) \cap L^2(\mathfrak{a}, \omega_k)$, then $\widehat{f}^k \in L^2(\mathfrak{a}, \omega_k)$.

(vi) *Injectivity*: $\mathcal{F}_k$ is injective on $L^1(\mathfrak{a}, \omega_k)$.

(vii) *Schwartz space automorphism*: $\mathcal{F}_k$ is a topological automorphism of $\mathscr{S}(\mathfrak{a})$.

(viii) *Tempered distribution automorphism*: $\mathcal{F}_k$ defines an automorphism of the tempered distributions $\mathscr{S}'(\mathfrak{a})$ by the assigment $\mathcal{F}_k u := u \circ \mathcal{F}_k$ for $u \in \mathscr{S}'(\mathfrak{a})$.

(ix) *Plancherel theorem*: If $k \geq 0$, then $\mathcal{F}_k$ extends to a unitary operator of $L^2(\mathfrak{a}, \omega_k)$.

(x) If $f \in \mathscr{S}(\mathfrak{a})$, then $T_\xi(k)\mathcal{F}_k = \mathcal{F}_k m_{-i\langle \xi, \cdot \rangle}$ and $m_{i\langle \xi, \cdot \rangle}\mathcal{F}_k = \mathcal{F}_k T_\xi(k)$.

As usual, the inverse Dunkl transform is denoted by

$$f^{\vee k}(x) := \mathcal{F}_k^{-1} f(x) = \mathcal{F}_k f(-x) = \frac{1}{c_k} \int_{\mathfrak{a}} E_k(ix,\xi) f(\xi) \omega_k(\xi) \, \mathrm{d}\xi.$$

Another cornerstone in Fourier analysis is the Paley-Wiener theorem, which has a generalization to the Dunkl setting. There are different versions of the Paley-Wiener theorem, formulated in the subsequent theorem. Let $\mathcal{H}(\mathfrak{a}_{\mathbb{C}})$ be the space of entire functions $f : \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ such that for all $M \in \mathbb{N}_0$ there exist constants $\gamma_M, r > 0$, where r is independent of M, with

$$|f(\xi)| \leq \frac{\gamma_M}{(1+|\xi|)^M} e^{r \cdot |\operatorname{Im} \xi|} \quad \text{for all } \xi \in \mathfrak{a}_{\mathbb{C}}. \tag{1.7}$$

If $S \subseteq \mathfrak{a}$ is a W-invariant convex compact set, then $\mathcal{H}_S(\mathfrak{a}_{\mathbb{C}})$ is defined to be the space of entire functions $f : \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ such that for all $M \in \mathbb{N}_0$ there exists a constant $\gamma_M > 0$ with

$$|f(\xi)| \leq \frac{\gamma_M}{(1+|\xi|)^M} e^{\max_{y \in S} \langle \operatorname{Im} \xi, y \rangle} \quad \text{for all } \xi \in \mathfrak{a}_{\mathbb{C}}. \tag{1.8}$$

Moreover, we define the spaces $\mathbb{H}(\mathfrak{a}_{\mathbb{C}})$ to be the space of entire functions $f : \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ such that there exist constants $M \in \mathbb{N}_0$ and $\gamma_M, r > 0$ satisfying (1.7).

Theorem 1.21 ([dJ06, AAS10])**.** We assume that $k \geq 0$. Let $S \subseteq \mathfrak{a}$ be a W-invariant compact convex neighborhood of 0. Then:

(i) $\mathcal{F}_k$ maps $C_c^\infty(\mathfrak{a})$ onto $\mathcal{H}(\mathfrak{a}_{\mathbb{C}})$.

(ii) If R is integral or $S = B_r(0)$, then $\mathcal{F}_k$ maps the space of smooth functions with support in S onto $\mathcal{H}_S(\mathfrak{a}_{\mathbb{C}})$.

(iii) $\mathcal{F}_k$ maps the space of compactly supported distributions onto $\mathbb{H}(\mathfrak{a}_{\mathbb{C}})$ in the following sense: for a compactly supported distribution u one has

$$\langle \mathcal{F}_k u, f \rangle = \int_{\mathfrak{a}} f(x) g_u(x) \omega_k(x) \, \mathrm{d}x \quad \text{with} \quad g_u = (x \mapsto \langle u, E_k(-ix, \cdot) \rangle) \in \mathbb{H}(\mathfrak{a}_{\mathbb{C}}),$$

such that the map $u \mapsto g_u$ is an isomorphism between the space of compactly supported distributions and $\mathbb{H}(\mathfrak{a}_{\mathbb{C}})$. Moreover, u has its support in $B_r(0)$ if and only if g_u satisfies (1.7).

Part (ii) is proven in [AAS10] for the crystallographic case, but due to the product situation mentioned in Remark 1.17, it is also true for integral root systems, since in this case the Dunkl transform is a tensor product of a (crystallographic) Dunkl transform and an Euclidean Fourier transform. To be more precise, set $\mathfrak{b} := \operatorname{span}_{\mathbb{R}} R$ and $\mathfrak{c} = \mathfrak{b}^\perp$. Let $\mathcal{F}$ be the usual Fourier transform on $\mathfrak{c}$ and $\mathcal{F}_{k,\mathfrak{b}}$ be the Dunkl transform on $\mathfrak{b}$, so that for $f \in L^1(\mathfrak{a}, \omega_k)$:

$$\mathcal{F}_k f = (\mathcal{F} \otimes \mathcal{F}_k^{\mathfrak{b}}) f = (\mathcal{F}^{x_c}(\mathcal{F}_{k,\mathfrak{b}}^{x_b} f(x_b + x_c))) = (\mathcal{F}_{k,\mathfrak{b}}^{x_b}(\mathcal{F}^{x_c} f(x_b + x_c))). \tag{1.9}$$

Generalized translations

In this section let $k \geq 0$ be a non-negative multiplicity.

Definition 1.22. The *generalized translation operator* or *Dunkl translation operator* τ_x^k on $\mathscr{S}(\mathfrak{a})$ is defined by

$$\tau_x^k f = y \mapsto (E_k(ix, \cdot) \widehat{f}^k)^{\vee k}(y).$$

Owing to Trimèche [Tri01], the translation operator can be extended to $f \in C^\infty(\mathfrak{a})$, such that $(x,y) \mapsto \tau_x^k f(y)$ is of class C^∞. To be more precise, $u(x,y) := \tau_x^k f(y)$ is the unique smooth solution of the system

$$\begin{cases} T_\xi(k)^x u = T_\xi(k)^y u, & \text{for all } \xi \in \mathfrak{a}, \\ u(x,0) = f(x), \end{cases} \tag{1.10}$$

where the script x and y denote the relevant variable. In particular, the generalized translation is given in terms of the intertwining operator by

$$\tau_x^k f(y) = V_k^x V_k^y((V_k^{-1} f)(x+y)). \tag{1.11}$$

In particular, if $k=0$, τ_x^0 is the classical translation operator $f \mapsto f(\cdot + x)$.

Many open problems related to the translation operator exist, whose solutions would lead to a progress in Harmonic analysis related to root systems. Let us first summarize the known results from [dJ93, Tri01, Tri02, Rö03b, AAS10, DH19]:

Theorem 1.23. For our purposes, the important properties of generalized translations are:

(i) For $x \in \mathfrak{a}$, the spaces $\mathscr{S}(\mathfrak{a}), C_c^\infty(\mathfrak{a})$ and $C^\infty(\mathfrak{a})$ are invariant under τ_x^k.

(ii) τ_x^k extends to an operator on $L^2(\mathfrak{a}, \omega_k)$ with $||\tau_x^k f||_{2,\omega_k} \leq \|f\|_{2,\omega_k}$.

(iii) $\tau_x^k f(y) = \tau_y^k f(x)$ and $\tau_0^k = \mathrm{id}$ for all $x, y \in \mathfrak{a}$.

(iv) $T_\xi(k)\tau_x^k = \tau_x^k T_\xi(k)$ for all $x, \xi \in \mathfrak{a}$.

(v) ω_k is τ_x^k-invariant, i.e. for $f \in \mathscr{S}(\mathfrak{a})$: $\displaystyle\int_{\mathfrak{a}} (\tau_x^k f)(y)\omega_k(y)\,\mathrm{d}y = \int_{\mathfrak{a}} f(y)\omega_k(y)\,\mathrm{d}y$.

(vi) If $f \in L^2(\mathfrak{a}, \omega_k)$ has support in $B_r(0)$, then $\tau_x^k f$ has support in $W.B_r(-x)$.

(vii) For $x, y \in \mathfrak{a}$, the map $f \mapsto \tau_x^k f(y)$ is a distribution with support in $B_{|x|+|y|}(0)$. If R is integral, then the support is contained in $\mathrm{co}(W.x) + \mathrm{co}(W.y)$.

(viii) If $f \in C^\infty(\mathfrak{a})$ is radial and non-negative, i.e. $f(x) = F(|x|)$ for smooth non-negative F, then $\tau_x^k f(y) \geq 0$ for all $x, y \in \mathfrak{a}$.

(ix) $\tau_x^k(E_k(\lambda, \cdot))(y) = E_k(\lambda, x)E_k(\lambda, y)$.

The situation of integral R in (vii) was proven in [AAS10] for the crystallographic case, but it is also true in the integral case. This can be seen as follows: if R is integral, we put $\mathfrak{b} := \mathrm{span}_{\mathbb{R}}\mathfrak{a}$ and write $\mathfrak{c} = \mathfrak{b}^\perp \leq \mathfrak{a}$ for the orthogonal complement. From the characterization of the Dunkl translation via solutions of (1.10), one obtains for $x = x_b + x_c \in \mathfrak{a}$ with $x_b \in \mathfrak{b}$ and $x_c \in \mathfrak{c}$ that

$$\tau_x^k = \tau_{x_c}^{k,\mathfrak{b}} \otimes S_{x_c}^{\mathfrak{c}} \tag{1.12}$$

as operator on $C^\infty(\mathfrak{a}) = \overline{C^\infty(\mathfrak{b}) \otimes C^\infty(\mathfrak{c})}$, where $\tau_{x_c}^{k,\mathfrak{b}}$ is the Dunkl translation on $\mathfrak{b}$ associated with (R, k) and $S_{x_c}^{\mathfrak{c}}$ is the operator $f \mapsto f(\cdot + x_c)$. Now, the integral case can be reduced to the crystallographic case in [AAS10].

Remark 1.24. It is a big open question if part (viii) is valid for general W-invariant functions (after symmetrization), a fact which is actually only known in rank one. For a non-W-invariant, non-negative function it is known that the translation can be negative. At least, it is open if the distribution $f \mapsto \tau_x^k f(y)$ is of order 0, i.e. whether there exists a complex measure $\mu_{x,y}^k$ such that

$$E_k(\lambda, x)E_k(\lambda, y) = \int_{\mathfrak{a}} E_k(\lambda, \xi)\,\mathrm{d}\mu_{x,y}^k.$$

It is still an open question whether there exists a constant C independent of x such that

$$\|\tau_x f\|_{p,\omega_k} \leq C\,\|f\|_{p,\omega_k}, \quad p \neq 2.$$

In particular, in the case $p = 1$ this would lead to a commutative Banach algebra $(L^1(\mathfrak{a}, \omega_k), *_k)$ with convolution defined in terms of generalized translations.

In the W-invariant case, the positivity of the operator $\sum_{w\in W} \tau_{wx}$ on W-invariant functions is equivalent to the existence of a positive measure $\mu_{x,y}^{k,W}$ such that

$$J_k(\lambda, x)J_k(\lambda, y) = \int_{\mathfrak{a}} J_k(\lambda, \xi)\, \mathrm{d}\mu_{x,y}^{k,W}. \tag{1.13}$$

In the cases where k is related to a symmetric space of Euclidean type (see the next section), this positivity result is true.

1.3 Riemannian symmetric spaces of Euclidean type

This section serves a brief summary of the connection between rational Dunkl theory and radial analysis on Riemannian symmetric spaces, cf. [dJ06, Section 6].
Let G be a connected non-compact semisimple Lie group with finite center, maximal compact subgroup K and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. We consider the associated Cartan motion group $G_0 = K \ltimes \mathfrak{p}$ acting on the Riemannian symmetric space $G_0/K \cong \mathfrak{p}$ of Euclidean type as group of isometries with respect to the Cartan-Killing form. We choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, let Σ be the restricted roots with multiplicities $(m_\alpha)_{\alpha\in\Sigma}$ and consider Σ as a subset of $\mathfrak{a}$ by means of the Killing form. Let W be the associated Weyl group, so that K-invariant functions on $G_0/K \cong \mathfrak{p}$ can be associated with W-invariant functions on $\mathfrak{a}$. Choose a reduced root system $R \subseteq \Sigma$ with Weyl group W and define the multiplicity

$$k_\alpha := \frac{1}{4} \sum_{\beta\in\mathbb{R}\alpha\cap\Sigma} m_\beta, \quad \alpha \in R.$$

Let D be a K-invariant differential operator on $\mathfrak{p}$, i.e. $D = p(\partial)$ with a K-invariant polynomial $p \in \mathbb{C}[\mathfrak{p}]^K$ on $\mathfrak{p}$. Then the spherical functions of the Gelfand pair (G_0, K) are the K-invariant functions on $\mathfrak{p}$ that are eigenfunctions of all K-invariant differential operators on $\mathfrak{p}$. The set of all spherical functions consists of ψ_λ with $\lambda \in \mathfrak{a}_\mathbb{C}$ such that $p(\partial)\psi_\lambda = p(\lambda)\psi_\lambda$ and $\psi_\lambda(0) = 1$. Moreover, $\psi_\lambda = \psi_\mu$ if and only if $\lambda \in W.\mu$. Since K-invariant functions on $\mathfrak{p}$ are in bijection with W-invariant functions on $\mathfrak{a}$, one can assign to each $p \in \mathbb{C}[\mathfrak{p}]^K$ a unique differential operator $\mathrm{rad}(p(\partial))$ on $\mathfrak{a}_{\mathrm{reg}}$, called the radial part, such that

$$(p(\partial)f)|_{\mathfrak{a}_{\mathrm{ref}}} = \mathrm{rad}(p(\partial))(f|_{\mathfrak{a}_{\mathrm{reg}}}).$$

It turns out that $\mathrm{rad}(p(\partial))$ is the restriction of a Dunkl operator associated with (R, k), i.e.

$$\mathrm{rad}(p(\partial)) = \mathrm{res}(p|_{\mathfrak{a}}(T(k))),$$

where res was defined in equation (1.3). In particular, the spherical functions are exactly the Bessel functions associated with (R, k), namely for all $\lambda \in \mathfrak{a}_\mathbb{C}$ and $x \in \mathfrak{a}$

$$\psi_\lambda(x) = J_k(\lambda, x).$$

In particular, the (W-invariant) generalized translations for these pairs (R, k) are positivity preserving, since the product of two Bessel functions (1.13) can be expressed via the product formula for spherical functions.

CHAPTER 2

Dunkl convolution and elliptic regularity

The analysis of linear partial differential operators has a wide range and in particular the theory of elliptic operators has a long history, see for instance [FJ98, Hö3]. Consider a Riemannian manifold M and a linear partial differential operator D on M. If D is an elliptic operator, such as the Laplace-Beltrami operator, it is well known that D is hypoelliptic, i.e.

$$\operatorname{singsupp}(Du) = \operatorname{singsupp} u$$

for any distribution u on M, where singsupp denotes the singular support. Furthermore, for an elliptic operator D of order m, various regularity results about the action on Sobolev spaces are known, such as

$$Du \in H^s_{loc}(M) \text{ if and only if } u \in H^{s+m}_{loc}(M),$$

where $H^s_{loc}(M)$ is the local Sobolev space of order s on M. The aim of this chapter is to describe and prove such results for rational Dunkl operators. To point out why one might expect that this is true, we explain the connection between the analysis of Dunkl operators and radial analysis on Riemannian symmetric spaces of Euclidean type.
Recall the situation from Section 1.3. Consider a connected semisimple Lie group G with finite center, a maximal compact subgroup K and the associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, the Cartan motion group $G_0 := K \ltimes \mathfrak{p}$ gives an associated Riemannian symmetric space of Euclidean type

$$M = \mathfrak{p} \cong G_0/K.$$

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace and $\Sigma \subseteq \mathfrak{a}$ the set of associated restricted roots with Weyl group W. Choose a reduced root system $R \subseteq \Sigma$ and define

$$k_\alpha := \frac{1}{4} \sum_{\beta \in \mathbb{R}\alpha \cap \Sigma} m_\beta, \quad \alpha \in R.$$

Then, the K-radial part of a K-invariant differential operator $p(\partial)$ on M, $p \in \mathbb{C}[\mathfrak{p}]^K$, is given by the restriction of a Dunkl operator associated with (R, k) and $p|_\mathfrak{a}$:

$$\operatorname{rad}(p(\partial)) = \operatorname{res}(p|_\mathfrak{a}(T(k))).$$

For instance, the Laplace-Beltrami operator $\Delta_\mathfrak{p}$ on $\mathfrak{p}$ is K-invariant and the radial part can be written as

$$\operatorname{rad}(\Delta_\mathfrak{p}) = \Delta_\mathfrak{a} + \sum_{\alpha \in R_+} k_\alpha \frac{\partial_\alpha}{\langle \alpha, \cdot \rangle} = \operatorname{res}(\langle T(k), T(k) \rangle),$$

where $\Delta_\mathfrak{a}$ is the usual Laplacian on $\mathfrak{a}$. From this observation, one might expect that elliptic Dunkl operators $q(T)$, i.e. operators $q(T)$ such that the highest order term of q does not vanish on $\mathfrak{a} \setminus \{0\}$, also satisfy some elliptic regularity theorems. One might expect this at least in the W-invariant case, but now for arbitrary parameters $k \geq 0$ and not only for those related to a Riemannian symmetric space of Euclidean type. This seems to be plausible, as ellipticity only depends on the highest order term, which is independent of k. For instance, hypoellipticity of the Dunkl Laplacian was already proven in [MT04]. Important tools are the Dunkl transform and generalized translations, where the latter define the Dunkl convolution $*_k$. Basic ideas for the results and proofs in this chapter are in line with those of classical elliptic regularity such as in [FJ98, Hö03]. However, there are two problems that need to be circumvented. First, the missing (general) Leibniz rule for Dunkl operators, and second by the missing knowledge about the support of generalized translations. The most important property of the Dunkl convolution we are able to prove here states

$$\operatorname{supp}(u *_k v) \subseteq B_r(0) + W.\operatorname{supp} v,$$

for any distributions u, v on $\mathfrak{a}$ such that $\operatorname{supp} u$ is contained in the closed ball $B_r(0)$ of radius r. This behavior of the convolution support is based on an important result of [DH19] on the support of generalized translations of L^2-functions.
The chapter is organized as follows. We start in Section 1 with the discussion on several properties of the Dunkl convolution, which generalizes convolutions of K-invariant functions on $\mathfrak{p} \cong G_0/K$. We study the Dunkl convolution of two distributions and obtain information about the support of a convolution. This extends [ØS05], where one of the distributions was required to have compact support. In Section 2, we introduce a generalized singular support $\operatorname{singsupp}_k u$ of a distribution u, which is defined as the complement of the largest open subset on which u coincides with a function $f\omega$ with $f \in C^\infty(\mathfrak{a})$ and $\omega(x) = \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{k_\alpha}$. This singular support is consistent with the Dunkl setting, and we examine how that singular support behaves under convolution. In Section 3 we give a proof for hypoellipticity of elliptic Dunkl operators, based on the results of the previous sections. To be more precise, for an elliptic Dunkl operator $p(T)$ we prove that

$$W.\operatorname{singsupp}_k(p(T)u) = W.\operatorname{singsupp}_k u,$$

for all distributions u defined on an open W-invariant subset of $\mathfrak{a}$. In Section 4 we prove the following elliptic regularity theorem for an elliptic Dunkl operator $p(T)$ of degree m, stating

$$p(T)u \in H^s_{k,loc}(\Omega) \text{ if and only if } u \in H^{s+m}_{k,loc}(\Omega),$$

where $H^s_{k,loc}(\Omega)$ are generalized local Sobolev spaces on some W-invariant open $\Omega \subseteq \mathfrak{a}$, as introduced in [Tri01, MT04], cf. Section 4.
Finally, as an application we will prove later in this thesis that the Dunkl convolution of Riesz distributions associated with the root system of type A, introduced and studied in [Rö20], exists and that these distributions form a group under Dunkl convolution. This is in line with classical results on Riesz distributions for symmetric cones as in [FK94].

2.1 Dunkl convolution

The Dunkl convolution is already known, and in the literature convolutions of functions and distributions have already been studied, see for instance [ØS05]. Later in this thesis, we will need the convolution of two distributions with non-compact support. To our knowledge, this has not been studied so far; except for the case where one of the distributions is known to have compact support, see for instance [ØS05]. For an open $\Omega \subseteq \mathfrak{a}$, we denote by $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ the spaces of distributions on $\mathfrak{a}$ with support contained in Ω, and compact support contained in Ω, respectively. Both spaces are equipped with the topology of pointwise convergence. As usual, the evaluation of a distribution $u \in \mathcal{D}'(\Omega)$ in $\varphi \in C_c^\infty(\Omega)$ will be denoted by the pairing

$$\langle u, \varphi\rangle := u(\varphi).$$

As usual, we write

$$B_r(x) := \{y \in \mathfrak{a} \mid |x - y| \leq r\}$$

for the closed, and

$$B_r^\circ(x) := \{y \in \mathfrak{a} \mid |x - y| < r\}$$

for the open ball around $x \in \mathfrak{a}$.
To improve readability, we write

$$E = E_k,\ \omega = \omega_k,\ p(T) = p(T(k)),\ V = V_k \ \text{ and } \ \tau = \tau^k$$

for the Dunkl kernel, weight function, Dunkl operators, Dunkl intertwiner and generalized translation. With any locally integrable function $f\colon \Omega \to \mathbb{C}$ we associate a distribution $u_f^k \in \mathcal{D}'(\Omega)$ by the assignment

$$\langle u_f^k, \varphi\rangle := \langle f, \overline{\varphi}\rangle_\omega = \int_\Omega f(x)\varphi(x)\omega(x)\, \mathrm{d}x.$$

This embedding of locally integrable functions into distributions is compatible with the Dunkl setting and differs from the usual embedding, which is the reason to use a superscript k in the notation. In fact, if Ω is W-invariant, the Dunkl operators act continuously on $\mathcal{D}'(\Omega)$ by

$$\langle T_\xi u, \varphi\rangle := -\langle u, T_\xi \varphi\rangle,$$

so that the skew-symmetry of Dunkl operators in $L^2(\mathfrak{a}, \omega)$, cf. [Rö03a], leads to

$$T_\xi u_f^k = u_{T_\xi f}^k.$$

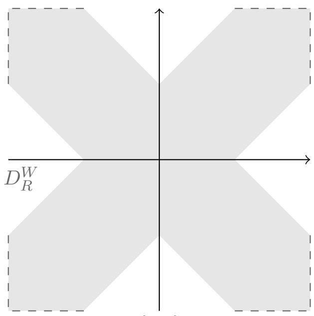
for all $f \in C^1(\Omega)$. Moreover, smooth functions $m \in C^\infty(\Omega)$ act continuously on $\mathcal{D}'(\Omega)$ by multiplication, namely

$$\langle m \cdot u, \varphi\rangle := \langle u, m\varphi\rangle, \text{ so that } m \cdot u_f^k = u_{mf}^k.$$

In order to study the Dunkl convolution of distributions, we introduce the following sets in $\mathfrak{a} \times \mathfrak{a}$. For $r > 0$ we define

$$D_r^W := \bigcup_{w \in W} \{(x, y) \in \mathfrak{a} \times \mathfrak{a} \mid |x + wy| \le r\}.$$

This set is invariant under the canonical action of $W \times W$ on $\mathfrak{a} \times \mathfrak{a}$. In fact, it is a $W \times W$-orbit of a diagonal of width r in $\mathfrak{a} \times \mathfrak{a}$.
In rank one, with $R = \{\pm 1\} \subseteq \mathbb{R}$, we have $W = \{\pm \mathrm{id}\}$ and D_r^W in $\mathbb{R}^2$ is visualized on the right.



$R = \{\pm 1\} \subseteq \mathbb{R}$

Definition 2.1. We call two distributions $u, v \in \mathcal{D}'(\mathfrak{a})$ *W-convolvable* if for each $r > 0$ the intersection $\operatorname{supp}(u \otimes v) \cap D_r^W$ is bounded, i.e. compact.
Here $u \otimes v$ is the usual tensor product of u and v. Note that $\operatorname{supp}(u \otimes v) = \operatorname{supp} u \times \operatorname{supp} v$, so the distributions u, v are W-convolvable in the following cases:

(i) u or v has compact support.

(ii) the supports of u and v are contained in a W-invariant closed convex cone C which is proper, i.e. C does not contain one-dimensional subspaces.

Remark 2.2. We note the following:

(i) $u, v \in \mathcal{D}'(\mathfrak{a})$ are W-convolvable if and only if the restriction of $+ : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ to $(W.\operatorname{supp} u) \times (W.\operatorname{supp} v)$ is a proper map.

(ii) Even in rank one there exist distributions u and v with non-compact support which are W-convolvable. As an example, for $R = \{\pm 1\} \subseteq \mathbb{R}$ and $\delta_x = (\varphi \mapsto \varphi(x))$, consider the distributions

$$u = \sum_{n\in\mathbb{N}} \delta_{2^{2n}} \quad \text{and} \quad v = \sum_{n\in\mathbb{N}} \delta_{2^{2n+1}}$$

with supports $2^{2\mathbb{N}}$ and $2^{2\mathbb{N}+1}$, respectively. Then $D_r^W \cap (2^{2\mathbb{N}} \times 2^{2\mathbb{N}+1})$ is always finite, so that u and v are W-convolvable.

(iii) A non-zero W-invariant proper closed convex cone C does not have to exist. In fact, such a cone exists if and only if R does not span $\mathfrak{a}$. Later in this chapter, the case $\mathfrak{a} = \mathbb{R}^n$, $R = A_{n-1}$ and $C = [0, \infty[^n$ will be of high relevance.

Lemma 2.3. *The Dunkl translation associated with (R, k) defines a continuous linear operator*

$$\tau : C^\infty(\mathfrak{a}) \to C^\infty(\mathfrak{a} \times \mathfrak{a}), \quad \tau\varphi(x, y) = \tau_x\varphi(y) = \tau_y\varphi(x).$$

Moreover, for $\varphi \in C_c^\infty(\mathfrak{a})$ with $\operatorname{supp}\varphi \subseteq B_r(0)$ we have

$$\operatorname{supp}(\tau\varphi) \subseteq D_r^W.$$

Proof. Since τ can be expressed in terms of Dunkl's intertwining operator V and the operator S defined by $Sf(x, y) = f(x + y)$, see (1.11), i.e.

$$\tau = (V \otimes V) \circ S \circ V^{-1},$$

the continuity is a consequence of Theorem 1.13. For $(x, y) \in \mathfrak{a} \times \mathfrak{a}$ we have $(x, y) \in D_r^W$ if and only if $y \in W.B_r(-x)$, so that $\operatorname{supp}(\tau\varphi) \subseteq D_r^W$ holds by Theorem 1.23, proven in [DH19]. ■

Definition 2.4. Assume that $u, v \in \mathcal{D}'(\mathfrak{a})$ are W-convolvable. Choose a cut-off function $\rho \in C^\infty(\mathfrak{a} \times \mathfrak{a})$ with support in an ϵ-neighborhood of $\operatorname{supp} u \times \operatorname{supp} v$ and $\rho \equiv 1$ in a smaller neighborhood. Note that in this case $(\operatorname{supp}\rho) \cap D_r^W$ is still compact for all $r > 0$. Thus, we can define

$$\langle u *_k v, \varphi\rangle := \langle u \otimes v, \rho \cdot \tau\varphi\rangle, \quad \varphi \in C_c^\infty(\mathfrak{a}), \tag{2.1}$$

which does not depend on the particular choice of ρ. It is called the Dunkl convolution of u and v.

This definition was already given in [ØS05] under that assumption that u or v has compact support.

Theorem 2.5. *Consider $u_1, u_2, u, v \in \mathcal{D}'(\mathfrak{a})$, $\lambda \in \mathbb{C}$ and $\xi \in \mathfrak{a}$. Then:*

(i) *If u, v are W-convolvable, then $u *_k v \in \mathcal{D}'(\mathfrak{a})$ and*

$$u *_k v = v *_k u.$$

Moreover, $T_\xi u$ and v are W-convolvable, and so are u and $T_\xi v$, with

$$T_\xi(u *_k v) = (T_\xi u) *_k v = u *_k (T_\xi v).$$

(ii) *If both u_1 and u_2 are W-convolvable with v, then $u_1 + \lambda u_2$ is W-convolvable with v and*

$$(u_1 + \lambda u_2) *_k v = (u_1 *_k v) + \lambda(u_2 *_k v).$$

(iii) *u is W-convolvable with the Dirac distribution $\delta_0 = (\varphi \mapsto \varphi(0))$ and*

$$u *_k \delta_0 = \delta_0 *_k u = u.$$

Proof. Everything is straightforward to verify, we only have to justify the formula for the action of the Dunkl operators on a convolution. For this we choose a more explicit ρ in (2.1), namely

$$\rho(x, y) := \rho_u(x)\rho_v(y), \quad x, y \in \mathfrak{a}$$

with W-invariant ρ_u and ρ_v. Moreover, we choose ρ_u such that it has support in an ϵ-neighborhood of $\operatorname{supp} u$ and $\rho_u \equiv 1$ in a smaller neighborhood. We choose ρ_v in a similar fashion. By Theorem 1.23, the choice of ρ and the Leibniz formula $T_\xi(\chi f) = (\partial_\xi \chi) \cdot f + \chi \cdot (T_\xi f)$ for W-invariant χ we have

$$\begin{aligned}\langle T_\xi(u *_k v), \varphi\rangle &= -\langle u \otimes v, \rho \cdot \tau T_\xi \varphi\rangle = -\langle u \otimes v, \rho \cdot T_\xi^x(\tau\varphi)\rangle \\ &= \langle u \otimes v, (\partial_\xi^x \rho) \cdot (\tau\varphi)\rangle - \langle u \otimes v, T_\xi^x(\rho \cdot \tau\varphi)\rangle \\ &= -\langle u \otimes v, T_\xi^x(\rho \cdot \tau\varphi)\rangle = \langle (T_\xi u) \otimes v, \rho \cdot \tau\varphi\rangle = \langle (T_\xi u) *_k v, \varphi\rangle .\end{aligned}$$

■

Definition 2.6. Similarly to the Euclidean case $k = 0$, we define

$$(f *_k g)(x) := \int_{\mathfrak{a}} (\tau_y f)(-x) g(x) \omega(x) \, \mathrm{d}x,$$

for $f, g \in C^\infty(\mathfrak{a})$, one with compact support, or both $f, g \in \mathscr{S}(\mathfrak{a})$. Moreover, we define

$$(f *_k u)(x) := \langle u(y), (\tau_x f)(-y)\rangle ,$$

for $f \in C^\infty(\mathfrak{a}), u \in \mathcal{D}'(\mathfrak{a})$, one with compact support. Here $u(y)$ means that u acts on functions of the y-variable.

The following properties are straightforward or can be found in [ØS05].

Lemma 2.7. *The Dunkl convolution satisfies:*

(i) *For $f, g \in \mathscr{S}(\mathfrak{a})$ one has $f *_k g \in \mathscr{S}(\mathfrak{a})$ and $(f *_k g)^{\wedge k} = \widehat{f}^k \cdot \widehat{g}^k$.*

(ii) *For $f \in C_c^\infty(\mathfrak{a})$, the map $g \mapsto f *_k g$ is continuous on $C^\infty(\mathfrak{a})$ and satisfies*

$$f *_k g = g *_k f,$$
$$u_f^k *_k u_g^k = u_{f *_k g}^k.$$

(iii) *For $f \in C^\infty(\mathfrak{a})$ and $u \in \mathcal{D}'(\mathfrak{a})$, one with compact support, we have that $f *_k u \in C^\infty(\mathfrak{a})$ and*

$$u_{f *_k u}^k = u_f^k *_k u.$$

Proposition 2.8. *Let $\Omega_1, \Omega_2 \subseteq \mathfrak{a}$ be open W-invariant sets such that $\overline{\Omega_1 \times \Omega_2} \cap D_r^W$ is compact for all $r > 0$. Then the Dunkl convolution defines a sequentially continuous operator*

$$*_k : \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\mathfrak{a}).$$

Proof. This is an immediate consequence of the continuity of the tensor product and Definition 2.1 as ρ can be chosen uniformly for all $(u, v) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$. ■

Corollary 2.9. *The distributions u^k_f, $f \in C_c^\infty(\mathfrak{a})$, form a dense subspace of $\mathcal{D}'(\mathfrak{a})$.*

Proof. Since $\mathcal{E}'(\mathfrak{a})$ is dense in $\mathcal{D}'(\mathfrak{a})$, it suffices to verify its density in $\mathcal{E}'(\mathfrak{a})$. To do so, choose a non-negative $\psi \in C_c^\infty(\mathfrak{a})$ with $\operatorname{supp}\psi \subseteq B_1(0)$ and $\|\psi\|_{L^1(\mathfrak{a},\omega)} = 1$. For $\epsilon > 0$ we put

$$\psi_\epsilon(x) := \frac{1}{\epsilon^\gamma}\psi\left(\frac{x}{\epsilon}\right), \qquad \text{with } \gamma := \dim\,\mathfrak{a} + \frac{1}{2}\sum_{\alpha\in R} k_\alpha.$$

Then $\operatorname{supp}\psi_\epsilon \subseteq B_\epsilon(0)$ and $\|\psi_\epsilon\|_{L^1(\mathfrak{a},\omega)} = 1$. Moreover, $u^k_{\psi_\epsilon}$ tends to δ_0 pointwise since

$$|\langle \delta_0 - u^k_{\psi_\epsilon}, \varphi\rangle| \le \int_{\mathfrak{a}} \psi_\epsilon(x)|\varphi(0) - \varphi(x)|\omega(x)\,\mathrm{d}x \le \|\varphi - \varphi(0)\|_{\infty, B_\epsilon(0)}\,.$$

For $u \in \mathcal{E}'(\mathfrak{a})$ we know from Lemma 2.7 that

$$u *_k \psi_\epsilon \in C^\infty(\mathfrak{a}).$$

Moreover, $\operatorname{supp}(\tau_x\psi_\epsilon) \subseteq W.B_\epsilon(-x)$ and $u \in \mathcal{E}'(\mathfrak{a})$ lead to $u *_k \psi_\epsilon \in C_c^\infty(\mathfrak{a})$ by definition of the convolution. Finally, Lemma 2.7 and Proposition 2.8 leads to

$$u^k_{u *_k \psi_\epsilon} = u *_k u^k_{\psi_\epsilon} \underset{\epsilon\to 0}{\longrightarrow} u *_k \delta_0 = u.$$

■

2.2 (Singular-)support of Dunkl convolutions

As already mentioned, our embedding $f \mapsto u^k_f$ of locally integrable functions into $\mathcal{D}'(\mathfrak{a})$ differs from the usual embedding. Thus, we shall define a specific notion of singular support adapted to this embedding, and hence adapted to the Dunkl setting.

Definition 2.10. For $u \in \mathcal{D}'(\mathfrak{a})$ we define the k-*singular support* of u as the complement of the largest open subset of $\mathfrak{a}$ on which u is of the form u^k_f for some smooth f. To be more precise,

$$\operatorname{singsupp}_k u := \bigcap_{\substack{\Omega\subseteq\mathfrak{a}\\ \text{open}}} \left\{\mathfrak{a}\backslash\Omega \mid u|_\Omega = u^k_f \text{ for some } f \in C^\infty(\Omega)\right\}.$$

It is obvious that

$$\operatorname{singsupp}_k u \subseteq \operatorname{supp} u.$$

Moreover, the usual singular support $\operatorname{singsupp} u = \operatorname{singsupp}_0 u$ differs from the k-singular support only by singular elements, namely

$$\operatorname{singsupp}_k u \cup \mathfrak{a}_{\text{sing}} = \operatorname{singsupp}\, u \cup \mathfrak{a}_{\text{sing}},$$

where $\mathfrak{a}_{\text{sing}} := \bigcup_{\alpha\in R} \alpha^\perp$ is the singular set in $\mathfrak{a}$.

Lemma 2.11. *Let $u, v \in \mathcal{D}'(\mathfrak{a})$ be W-convolvable distributions. Then:*

(i) $\operatorname{supp} u^k_f = \operatorname{supp} f$ *for all locally integrable* $f\colon \mathfrak{a} \to \mathbb{C}$.

(ii) *For distributions $u, v \in \mathcal{D}'(\mathfrak{a})$ with $\operatorname{supp} u \subseteq B_r(0)$, $r > 0$, we have*

$$\operatorname{supp}(u *_k v) \subseteq B_r(0) + W.\operatorname{supp}\, v.$$

(iii) *If R is an integral root system, then*

$$\operatorname{supp}(u *_k v) \subseteq \overline{\operatorname{co}(W.\operatorname{supp} u) + \operatorname{co}(W.\operatorname{supp} v)} = \overline{\operatorname{co}(W.\operatorname{supp} u + W.\operatorname{supp} v)}.$$

PROOF. Part (i) is immediate by definition and the fact that the zero set of ω is a finite union of the hyperplanes.

(ii) By part (i), Lemma 2.7, Proposition 2.8 and Corollary 2.9, it suffices to prove

$$\operatorname{supp}(f *_k g) \subseteq B_r(0) + W.\operatorname{supp} g \tag{2.2}$$

for $f, g \in C_c^\infty(\mathfrak{a})$ with $\operatorname{supp} f \subseteq B_r(0)$. From Theorem 1.23 we obtain supp $\tau_x f \subseteq W.B_r(-x)$, so (2.2) is a consequence of the explicit formula

$$(f *_k g)(y) = \int_{\mathfrak{a}} (\tau_y f)(-x) g(x) \omega(x) \, \mathrm{d}x.$$

(iii) For abbreviation, we write $A := \operatorname{supp} u$ and $B := \operatorname{supp} v$. Consider $\varphi \in C_c^\infty(\mathfrak{a})$ with $\operatorname{supp} \varphi \cap \overline{(\operatorname{co}(W.A) + \operatorname{co}(W.B))} = \emptyset$. By Theorem 1.23 we have $\tau\varphi(x, y) = 0$ for all $(x, y) \in A \times B$ and therefore $\langle u *_k v, \varphi \rangle = 0$.

■

Corollary 2.12. *Assume that R is integral and $C \subseteq \mathfrak{a}$ is a proper W-invariant closed convex cone. Then the space of distributions with support contained in C is a unital, associative and commutative algebra over $\mathbb{C}$ with the Dunkl convolution as multiplication.*

PROOF. By Lemma 2.11 (iii) and the conditions on C, we see that $\operatorname{supp}(u *_k v) \subseteq C$ for all $u, v \in \mathcal{D}'(\mathfrak{a})$ with support contained in C. Theorem 2.5 shows that we have a commutative algebra over $\mathbb{C}$. Moreover, on the Schwartz space $\mathscr{S}(\mathfrak{a})$, the Dunkl convolution is associative. Hence, Lemma 2.7 and Corollary 2.9 show that $*_k$ is associative in general. ■

Theorem 2.13. *Let $u, v \in \mathcal{D}'(\mathfrak{a})$ be W-convolvable distributions. Then:*

(i) *If* $\operatorname{singsupp}_k u \subseteq B_r(0)$,

$$\operatorname{singsupp}_k(u *_k v) \subseteq B_r(0) + W.\operatorname{singsupp}_k v.$$

(ii) *If R is integral,*

$$\operatorname{singsupp}_k(u *_k v) \subseteq \overline{\operatorname{co}(W.\operatorname{singsupp}_k u) + \operatorname{co}(W.\operatorname{singsupp}_k v)}.$$

PROOF. We consider two cases.

(a) First, we may assume that both u and v have compact support. Choose an arbitrary $\epsilon > 0$ and cutoff functions $\chi_u, \chi_v \in C_c^\infty(\mathfrak{a})$ with

$$\begin{aligned} \operatorname{supp} \chi_u &\subseteq \operatorname{singsupp}_k u + B_\epsilon(0), \quad \chi_u \equiv 1 \text{ on } \operatorname{singsupp}_k u, \\ \operatorname{supp} \chi_v &\subseteq \operatorname{singsupp}_k v + B_\epsilon(0), \quad \chi_v \equiv 1 \text{ on } \operatorname{singsupp}_k v. \end{aligned}$$

Hence, we conclude that

$$\operatorname{supp}(\chi_u u) \subseteq \operatorname{singsupp}_k u + B_\epsilon(0), \quad (1 - \chi_u) u = u_f^k, \ f \in C_c^\infty(\mathfrak{a}),$$

$$\operatorname{supp}(\chi_v v) \subseteq \operatorname{singsupp}_k v + B_\epsilon(0), \quad (1-\chi_v)v = u_g^k,\ g \in C_c^\infty(\mathfrak{a}).$$

According to Lemma 2.7 we have

$$\begin{aligned}((1-\chi_v)v) *_k (\chi_u u) &= u_g^k *_k (\chi_u u) = u^k_{g *_k (\chi_u u)},\\ (\chi_v v) *_k ((1-\chi_u)u) &= (\chi_v v) *_k u_f^k = u^k_{f *_k (\chi_v v)},\\ ((1-\chi_v)v) *_k ((1-\chi_u)u) &= u_f^k * u_g^k = u^k_{f *_k g},\end{aligned}$$

and therefore we see that

$$\begin{aligned}\operatorname{singsupp}_k(u *_k v) &= \operatorname{singsupp}_k((\chi_u u) *_k (\chi_v v))\\ &\subseteq \operatorname{supp}((\chi_u u) *_k (\chi_v v)).\end{aligned}$$

Finally, we distinguish between the two situations in the theorem:

(i) In this case, we can conclude that

$$\operatorname{supp}(\chi_u u) \subseteq \operatorname{singsupp}_k u + B_\epsilon(0) \subseteq B_{r+\epsilon}(0)$$

and therefore

$$\operatorname{singsupp}_k(u *_k v) \subseteq B_{r+\epsilon}(0) + W.(\operatorname{singsupp}_k v + B_\epsilon(0))$$

by Theorem 2.11 (ii). Since $\epsilon > 0$ was arbitrary, the claim follows.

(ii) In this case, we conclude from Theorem 2.11 (iii) that

$$\operatorname{singsupp}_k(u *_k v) \subseteq \overline{\operatorname{co}(W.A_\epsilon) + \operatorname{co}(W.B_\epsilon)}$$

with

$$\begin{aligned}A_\epsilon &:= \operatorname{singsupp}_k u + B_\epsilon(0),\\ B_\epsilon &:= \operatorname{singsupp}_k v + B_\epsilon(0).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the claim follows.

(b) For arbitrary $u, v \in \mathcal{D}(\mathfrak{a})$ and $R > 0$, choose a cutoff function $\chi \in C_c^\infty(\mathfrak{a})$ with $\chi \equiv 1$ on $B_{2R}(0)$. Then, from Theorem 2.11 (ii) we obtain

$$u *_k v|_{B_R(0)} = (\chi u) *_k (\chi v)|_{B_R(0)}.$$

Application of step (a) above to the distributions $\chi u, \chi v$ finishes the proof, because $R > 0$ was chosen arbitrarily.

■

2.3 Hypoellipticity of elliptic Dunkl operators

Let $\Omega \subseteq \mathfrak{a}$ be open. In contrast to the usual meaning, we say for $u \in \mathcal{D}'(\Omega)$ and a set M of locally integrable functions that

$$u \in M \quad \text{iff} \quad u = u_f^k \quad \text{for some } f \in M.$$

This means that for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\langle u, \varphi \rangle = \int_\Omega \varphi(x) f(x) \omega(x) \, \mathrm{d}x \quad \text{for some } f \in M.$$

It is important to keep this in mind. In particular, $u \in C^\infty(\mathfrak{a})$ means that u is given (in the usual sense) by $f\omega$ with some $f \in C^\infty(\mathfrak{a})$. Hence, $u \in C^\infty(\mathfrak{a})$ does not mean that u is a smooth function in the usual sense, since ω is in general not smooth along $\mathfrak{a}_{\text{sing}}$. But, as already mentioned, our notion is adapted to the Dunkl setting with the advantage that things are getting similar to the usual theory of elliptic differential operators.
Recall that the Dunkl transform of a tempered distribution $u \in \mathscr{S}'(\mathfrak{a})$ is defined by

$$\langle \widehat{u}^k, f \rangle := \langle \mathcal{F}_k u, f \rangle := \langle u, \widehat{f}^k \rangle, \quad f \in \mathscr{S}(\mathfrak{a}),$$

so that for $f \in \mathscr{S}(\mathfrak{a})$ we have

$$\mathcal{F}_k u_f^k = u_{\mathcal{F}_k f}^k.$$

In this section we are interested in the study of Dunkl operators that are elliptic in the following sense.

Definition 2.14. A Dunkl operator $p(T)$ is called *elliptic of degree* $m \in \mathbb{N}_0$ if $p = \sum_{n=0}^m p_n$ with $p_n \in \mathcal{P}_n$ and $p_m(x) \neq 0$ for all $x \in \mathfrak{a} \setminus \{0\}$.

For $k = 0$, an elliptic Dunkl operator is nothing but an elliptic differential operator with constant coefficients. For instance, the Dunkl Laplacian

$$\Delta_k := \langle T, T \rangle = \Delta_{\mathfrak{a}} + \sum_{\alpha \in R} k_\alpha \left(\frac{\partial_\alpha}{\langle \alpha, \cdot \rangle} - \frac{|\alpha|^2}{2} \frac{1 - s_\alpha}{\langle \alpha, \cdot \rangle^2} \right),$$

where $\Delta_{\mathfrak{a}}$ is the Laplacian on $\mathfrak{a}$, is an elliptic Dunkl operator of degree 2.
As usual, we put

$$\langle x \rangle := \sqrt{1 + |x|^2}$$

as function on $\mathfrak{a}$.

Proposition 2.15. *Consider $f \in L^1(\mathfrak{a}, \omega)$ such that $x \mapsto \langle x \rangle^\ell f(x) \in L^1(\mathfrak{a}, \omega)$ for some fixed $\ell \in \mathbb{N}_0$. Then, $\widehat{f}^k \in C^\ell(\mathfrak{a})$.*

Proof. For $\ell = 0$, this is just the Riemann-Lebesgue Lemma for the Dunkl transform, cf. Theorem 1.20. Otherwise, it is a consequence of standard theorems on differentiable parameter integrals and the estimate

$$|\partial_\xi^\beta E_k(-ix, \xi)| \leq |x|^{|\beta|} \leq \langle x \rangle^\ell, \quad \text{for all } \beta \in \mathbb{N}_0^n, \, |\beta| \leq \ell,$$

where the first inequality holds by Theorem 1.16. ■

Let $a \in C^\infty(\mathfrak{a})$ be a smooth function and $m \in \mathbb{R}$. Assume that for all $\beta \in \mathbb{N}_0^n$ there exists a constant $C_\beta \geq 0$ with

$$|\partial^\beta a(x)| \leq C_\beta \langle x \rangle^{m - |\beta|}. \tag{2.3}$$

Then it is well known that the (distributional) Fourier transform of a has singular support contained in $\{0\}$, cf. [Hö3, Proof of Theorem 7.1.22]. The following lemma is a generalization of this to the case of arbitrary $k \geq 0$.

Lemma 2.16. *Assume that for $a \in C^\infty(\mathfrak{a})$ there exists some $m \in \mathbb{R}$ such that for all $\beta \in \mathbb{N}_0^n$ there exists a constant $C_\beta \geq 0$ with*

$$|T^\beta a(x)| \leq C_\beta \langle x \rangle^{m-|\beta|} \tag{2.4}$$

for all $x \in \mathfrak{a}$. Then we have

$$\operatorname{singsupp}_k(\mathcal{F}_k u_a^k) \subseteq \{0\}.$$

Proof. Note that u_a^k is tempered, as a is at most of polynomial growth. For each $\ell \in \mathbb{N}_0$ there exists $N \in \mathbb{N}$ such that for all $\beta \in \mathbb{N}_0^n$, $|\beta| \geq N$

$$x \mapsto \langle x \rangle^\ell \cdot T^\beta a(x) \in L^1(\mathfrak{a}, \omega_k).$$

With Proposition 2.15 we thus have

$$\mathcal{F}_k(T^\beta a) \in C^\ell(\mathfrak{a}),$$

so that as distributions

$$(ix)^\beta \mathcal{F}_k u_a^k = \mathcal{F}_k(T^\beta u_a^k) \in C^\ell(\mathfrak{a}).$$

Therefore, we conclude

$$(\mathcal{F}_k u_a^k)|_{\mathfrak{a}\setminus\{0\}} \in C^\ell(\mathfrak{a}\setminus\{0\}).$$

But $\ell \in \mathbb{N}_0$ was arbitrary, so $\operatorname{singsupp}_k(\mathcal{F}_k u_a^k) \subseteq \{0\}$. ∎

Assume that $a \in C^\infty(\mathfrak{a})$ satisfies (2.3) and $|a(x)| \geq C\,|x|^m$ for large x and some constant C. Then the reciprocal $\frac{1}{a}$ satisfies (2.3) with $-m$ instead of m and large x. This is an immediate consequence of the quotient rule for partial derivatives. For Dunkl operators there is no general Leibniz rule, thus no quotient rule. To avoid this, we will use the subsequent lemma.

Proposition 2.17. *Let $f = \frac{q}{p} \in C^\infty(\Omega)$ be a rational function on some W-invariant open $\Omega \subseteq \mathfrak{a}$ and $p, q \in \mathcal{P}$. Then, $T_\xi(k)f$ is rational on Ω for all $\xi \in \mathfrak{a}$. To be more precise, there exist finitely many polynomials $\widetilde{q}_i \in \mathcal{P}$ of degree at most $\deg p + \deg q - 1$ and $w_i \in W$, such that for all $x \in \Omega$*

$$T_\xi f(x) = \sum_i \frac{\widetilde{q}_i(x)}{p(x)p(w_i x)}.$$

Proof. Rewriting the difference part of the Dunkl operator, we observe the following

$$\begin{aligned} T_\xi f(x) = &\frac{\partial_\xi q(x) \cdot p(x) - q(x) \cdot \partial_\xi p(x)}{p(x)^2} \\ &+ \sum_{\alpha \in R_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{p(x)p(s_\alpha x)} \frac{q(x)p(s_\alpha x) - q(s_\alpha x)p(x)}{\langle \alpha, x \rangle}. \end{aligned}$$

But the polynomial $q(x)p(s_\alpha x) - q(s_\alpha x)p(x)$ vanishes on $\alpha^\perp$, hence it is divisible by $\langle \alpha, x \rangle$ and the claim holds. ∎

Lemma 2.18. *Let $p \in \mathcal{P}$ be a polynomial of degree $m \in \mathbb{N}_0$. Then:*

(i) $|T^\beta p(x)| \leq C_\beta \langle x \rangle^{m-|\beta|}$ *for all $\beta \in \mathbb{N}_0^n$ and some constant $C_\beta \geq 0$.*

(ii) *If $p = \sum_{j=0}^m p_j$ with p_j homogeneous of degree j and $p_m(x) \neq 0$ for all $x \in \mathfrak{a}\setminus\{0\}$, then there exists some $R > 0$ and $q \in C^\infty(\mathfrak{a})$ with $pq \equiv 1$ on $\mathfrak{a}\setminus B_R(0)$. Moreover,*

$$|T^\beta q(x)| \leq C_\beta \langle x \rangle^{-m-|\beta|} \tag{2.5}$$

for all $\beta \in \mathbb{N}_0^n$ and some constant C_β.

PROOF.

(i) This is obvious, since T_ξ is homogeneous of degree -1 and

$$|x^\alpha| \leq |x|^{|\alpha|} \leq \langle x \rangle^{|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}_0$ with $|\alpha| \leq |\beta|$.

(ii) Since $p_m(x)$ is homogeneous of degree m, there exists some $c > 0$ with $|p_m(x)| \geq c|x|^m$. Therefore,

$$|p(x)| > \tfrac{c}{2}|x|^m \tag{2.6}$$

for all $x \in \mathfrak{a}\backslash B_R(0)$ and some large $R > 0$. Then, choose $q \in C^\infty(\mathfrak{a})$ with $q = \frac{1}{p}$ on $\mathfrak{a}\backslash B_R(0)$. By iteration of Proposition 2.17, we can find for $\beta \in \mathbb{N}_0^n$ finitely many polynomials $\widetilde{q}_i \in \mathcal{P}$ of degree at most

$$\deg \widetilde{q}_i = (2^{|\beta|} - 1) \deg p - |\beta| = (2^{|\beta|} - 1)m - |\beta|$$

and $w_{i,j} \in W, j = 1, \ldots, 2^{|\beta|}$ satisfying

$$T^\beta q(x) = \sum_i \frac{\widetilde{q}_i(x)}{p(w_{i,1}x)p(w_{i,2}x) \cdots p(w_{i,2^{|\beta|}}x)},$$

for all $x \in \mathfrak{a}\backslash B_R(0)$. By part (i) and estimate (2.6), there exists $C'_\beta \geq 0$ with

$$|T^\beta q(x)| \leq C'_\beta \frac{\langle x \rangle^{(2^{|\beta|}-1)m-|\beta|}}{\langle x \rangle^{2^{|\beta|}\cdot m}} = C'_\beta \langle x \rangle^{-m-|\beta|},$$

for all $x \in \mathfrak{a}\backslash B_R(0)$. Finally, as q is continuous, estimate (2.5) follows.

■

The subsequent proof of the theorem on hypoellipticity follows basically classical ideas as in [Hö3, Theorem 7.1.22]. Recall that for any distribution $u \in \mathcal{D}'(\mathfrak{a})$ we have $\delta_0 *_k u = u$.

Theorem 2.19 (Hypoellipticity)**.** *Let $p(T)$ be an elliptic Dunkl operator and $\Omega \subseteq \mathfrak{a}$ a W-invariant open subset. Then for all $u \in \mathcal{D}'(\Omega)$*

$$W.\mathrm{singsupp}_k u = W.\mathrm{singsupp}_k(p(T)u).$$

PROOF. Let m be the degree of p. By Lemma 2.18, we choose $q \in C^\infty(\mathfrak{a})$ with $p(-i\cdot)q \equiv 1$ on $\mathfrak{a}\backslash B_R(0)$ for some large $R > 0$ and such that for all $x \in \mathfrak{a}$ and $\beta \in \mathbb{N}_0^n$

$$|T^\beta q(x)| \leq C_\beta \langle x \rangle^{-m-|\beta|}$$

with some constant $C_\beta \geq 0$. Thus, Lemma 2.16 yields a tempered distribution

$$E := \frac{1}{c_k} \mathcal{F}_k^{-1} u_q^k \in \mathscr{S}'(\mathfrak{a})$$

with k-singular support contained in $\{0\}$, where c_k is the Macdonald-Metha constant from (1.5). We put

$$R := \delta_0 - p(T)E \in \mathscr{S}'(\mathfrak{a}).$$

The Dunkl transform of R is

$$\mathcal{F}_k R = \frac{1}{c_k} u_1^k - \mathcal{F}_k(p(T)E) = \frac{1}{c_k}(u_1^k - u_{p(-i\cdot)q}^k) = u_f^k$$

with $f \in C_c^\infty(\mathfrak{a})$. Therefore, $R = u^k_{\mathcal{F}_k^{-1} f}$ and $\text{singsupp}_k R = \emptyset$.
For $x_0 \in \Omega \backslash W.\text{singsupp}_k(p(T)u)$ choose a cutoff function $\chi \in C_c^\infty(\mathfrak{a})$ such that $\chi \equiv 1$ in a neighborhood of $W.x_0$. We consider χu as a compactly supported distribution on $\mathfrak{a}$ by extending it by 0 outside of Ω. Then

$$\chi u = \delta_0 *_k (\chi u) = (p(T)E + R) *_k (\chi u) = E *_k (p(T)(\chi u)) + R *_k (\chi u). \tag{2.7}$$

Since $\text{singsupp}_k R = \emptyset$ and $\text{singsupp}_k E \subseteq \{0\}$, we can use Theorem 2.13 to obtain

$$\text{singsupp}_k(\chi u) = \text{singsupp}_k(E *_k (p(T)(\chi u))) \subseteq W.\text{singsupp}_k(p(T)(\chi u)).$$

But $\chi \equiv 1$ in a neighborhood of $W.x_0$, so that $p(T)\chi u = p(T)u$ and $\chi u = u$ near $W.x_0$ and therefore $x_0 \notin \text{singsupp}_k u$. From this we have

$$\text{singsupp}_k u \subseteq W.\text{singsupp}_k(p(T)u).$$

Finally, it is obvious that $\text{singsupp}_k(p(T)u) \subseteq W.\text{singsupp}_k u$, which finishes the proof. ■

2.4 Elliptic regularity of Dunkl operators

First, we give a review of Dunkl-type Sobolev spaces and their properties as studied in [MT04].

Definition 2.20. For $s \in \mathbb{R}$ the *Dunkl-type Sobolev space* of order s is defined by

$$H_k^s(\mathfrak{a}) := \left\{ u \in \mathscr{S}'(\mathfrak{a}) \mid \langle x \rangle^s \, \widehat{u}^k \in L^2(\mathfrak{a}, \omega) \right\}.$$

Hence, $u \in H_k^s(\mathfrak{a})$ if and only if $\widehat{u}^k \in \langle x \rangle^{-s} L^2(\mathfrak{a}, \omega)$ and we identify the distribution $\widehat{u}^k$ with the function $f \in \langle x \rangle^{-s} L^2(\mathfrak{a}, \omega)$ such that $\widehat{u}^k = u_f^k$. Under this identification, the inner product on $H_k^s(\mathfrak{a})$ is defined by

$$\langle u, v \rangle_{H_k^s} := \int_{\mathfrak{a}} \langle x \rangle^{2s} \, \widehat{u}^k(x) \overline{\widehat{v}^k(x)} \omega(x) \, \mathrm{d}x.$$

For our purpose, we need the following results from [MT04].

Theorem 2.21. *The Sobolev spaces $H_k^s(\mathfrak{a})$ are Hilbert spaces, satisfying:*

(i) *Let $s \in \mathbb{N}_0$. Then up to identification of u_f^k with f,*

$$H_k^s(\mathfrak{a}) = \left\{ f \in L^2(\mathfrak{a}, \omega) \mid T^\alpha f \in L^2(\mathfrak{a}, \omega) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \leq s \right\}.$$

Moreover, an equivalent norm on $H_k^s(\mathfrak{a})$ is induced by the inner product

$$(f, g) \mapsto \sum_{|\alpha| \leq s} \int_{\mathfrak{a}} (T^\alpha f)(x) \overline{(T^\alpha g)(x)} \omega(x) \, \mathrm{d}x.$$

(ii) *Dunkl operators are continuous linear operators $T_\xi : H_k^s(\mathfrak{a}) \to H_k^{s-1}(\mathfrak{a})$.*

(iii) *For $\psi \in C_c^\infty(\mathfrak{a})$, $u \mapsto \psi u$ is a continuous map from $H_k^s(\mathfrak{a})$ into itself.*

(iv) *$\mathcal{E}'(\mathfrak{a}) \subseteq \bigcup_{s \in \mathbb{R}} H_k^s(\mathfrak{a})$.*

(v) *For $p \in \mathbb{N}$ and $s \in \mathbb{R}$ with $s > p + \frac{n}{2} + \frac{1}{2}\sum_{\alpha\in R} k_\alpha$, the identification $u_f^k \mapsto f$ yields a continuous embedding*

$$H_k^s(\mathfrak{a}) \hookrightarrow C^p(\mathfrak{a}).$$

However, to formulate the theorem on elliptic regularity, we need local Sobolev spaces in the subsequent sense.

Definition 2.22. For a W-invariant open set $\Omega \subseteq \mathfrak{a}$ and $s \in \mathbb{R}$ we define

$$H_{k,loc}^s(\Omega) := \left\{u \in \mathcal{D}'(\Omega) \mid \psi u \in H_k^s(\mathfrak{a}) \text{ for all } \psi \in C_c^\infty(\mathfrak{a})\right\}.$$

Note that $H_k^s(\mathfrak{a}) \subseteq H_k^t(\mathfrak{a})$ for $t \leq s$. Thus $H_{k,loc}^s(\Omega) \subseteq H_{k,loc}^t(\Omega)$ and in particular

$$H_k^0(\mathfrak{a}) = L^2(\mathfrak{a},\omega) \text{ and } H_{k,loc}^0(\Omega) = L_{loc}^2(\Omega,\omega).$$

From [ØS05, Equation (2.13)] we have the following result.

Proposition 2.23. *Let $v \in \mathscr{S}'(\mathfrak{a})$ and $u \in \mathcal{E}'(\mathfrak{a})$ be a tempered and a compactly supported distribution, respectively. Then $\widehat{u}^k \in \mathscr{S}(\mathfrak{a})$, $u *_k v \in \mathscr{S}'(\mathfrak{a})$ and*

$$(u *_k v)^{\wedge k} = \widehat{u}^k \cdot \widehat{v}^k,$$

where $\widehat{u}^k$ again is identified with $f \in \mathscr{S}(\mathfrak{a})$ such that $\widehat{u}^k = u_f^k$.

As an immediate corollary to this proposition, we obtain the following.

Corollary 2.24. *Suppose that $a \in C^\infty(\mathfrak{a})$ satisfies estimate (2.4) for some $m \in \mathbb{R}$. Then for all $v \in H_k^s(\mathfrak{a}) \cap \mathcal{E}'(\mathfrak{a})$ we have*

$$v *_k (\mathcal{F}_k u_a^k) \in H_k^{s-m}(\mathfrak{a}).$$

Theorem 2.25 (Elliptic regularity)**.** *Let $p(T)$ be an elliptic Dunkl operator of degree m. Then for each W-invariant open $\Omega \subseteq \mathfrak{a}$ and $u \in \mathcal{D}'(\Omega)$*

$$p(T)u \in H_{k,loc}^s(\Omega) \text{ if and only if } u \in H_{k,loc}^{s+m}(\Omega).$$

Proof. If $u \in H_{k,loc}^{s+m}(\Omega)$ and $\psi \in C_c^\infty(\Omega)$, choose $\varphi \in C_c^\infty(\Omega)$ with $\varphi|_{W.\mathrm{supp}\,\psi} \equiv 1$, then $\psi p(T)u = \psi p(T)(\varphi u) \in H_k^s(\mathfrak{a})$, as $p(T)(\varphi u) \in H_k^s(\mathfrak{a})$. So we have $p(T)u \in H_{k,loc}^s(\Omega)$.
It remains to prove that $p(T)u \in H_{k,loc}^s(\Omega)$ implies $u \in H_{k,loc}^{s+m}(\Omega)$. Let $\psi \in C_c^\infty(\Omega)$ with $\mathrm{supp}\,\psi \subseteq B_r(0)$ and choose a W-invariant $\chi \in C_c^\infty(\mathfrak{a})$ such that $\chi \equiv 1$ on a neighborhood of $B_r(0)$. Recall the distributions E, R from the proof of Theorem 2.19. Consider ψu and χu as elements of $\mathcal{D}'(\mathfrak{a})$. We obtain similarly to Theorem 2.19, more precisely from (2.7)

$$\psi u = \psi(\chi u) = \psi \cdot (E *_k (p(T)(\chi u))) + \psi \cdot (R *_k (\chi u)). \tag{2.8}$$

We discuss the terms on the right-hand side separately.

(i) As in the proof of Theorem 2.19, we have $R *_k (\chi u) \in C^\infty(\mathfrak{a})$. Thus, $\psi(R *_k (\chi u)) \in C_c^\infty(\mathfrak{a})$ and in particular

$$\psi \cdot (R *_k (\chi u)) \in H_k^t(\mathfrak{a})$$

for all $t \in \mathbb{R}$.

(ii) We claim that

$$p(T)(\chi u) = \chi \cdot p(T)u + v, \tag{2.9}$$

for some $v \in \mathcal{E}'(\mathfrak{a})$ with $v \equiv 0$ in a neighborhood of $B_r(0)$. To see this, consider any $\xi \in \mathfrak{a}$. The W-invariance of χ leads to

$$T_\xi(\chi u) = \chi \cdot (T_\xi u) + (\partial_\xi \chi) \cdot u.$$

Since $\chi \equiv 1$ in a neighborhood of $B_r(0)$, the second summand (and all its images under Dunkl operators) vanishes on this neighborhood. Thus, if we iterate this argument, we obtain the stated equation (2.9).

(iii) Decomposing $p(T)(\chi u)$ according to (2.9), we see that

$$E *_k (p(T)(\chi u)) = E *_k (\chi \cdot p(T)u) + E *_k v.$$

By Theorem 2.13 and $\text{singsupp}_k E \subseteq \{0\}$ we observe

$$\text{singsupp}_k(E *_k v) \subseteq W.\text{singsupp}_k v \subseteq W.\text{supp}\, v \subseteq \mathfrak{a} \backslash B_r(0).$$

Therefore, by $\text{supp}\,\psi \subseteq B_r(0)$, we conclude $\psi(E *_k v) \in C_c^\infty(\mathfrak{a})$ and thus

$$\psi(E *_k v) \in H_k^t(\mathfrak{a})$$

for all $t \in \mathbb{R}$. By assumption, $\chi p(T)u \in H_k^s(\mathfrak{a}) \cap \mathcal{E}'(\mathfrak{a})$, so with Lemma 2.24 we have

$$\psi(E *_k (\chi p(T)u)) + \psi(E *_k v) \in H_k^{s+m}(\mathfrak{a}).$$

Putting things from (i) and (iii) together, we obtain from (2.8) that

$$\psi u \in H_k^{s+m}(\mathfrak{a})$$

for all $\psi \in C_c^\infty(\Omega)$, i.e. $u \in H_{k,loc}^{s+m}(\Omega)$. ■

From Theorem 2.25, we obtain the following corollary.

Corollary 2.26. *Consider some W-invariant open $\Omega \subseteq \mathfrak{a}$, $u \in \mathcal{D}'(\Omega)$ and an elliptic Dunkl operator $p(T)$. Then we have $u \in C^\infty(\Omega)$ in the following cases*

(i) *$p(T)^m u \in L^2_{loc}(\Omega, \omega)$ for all $m \in \mathbb{N}_0$.*

(ii) *$p(T)^m u \in C(\Omega)$ for all $m \in \mathbb{N}_0$.*

(iii) *u is an eigendistribution of $p(T)$.*

Proof.

(i) We recall that $H^0_{k,loc}(\Omega) = L^2_{loc}(\Omega, \omega)$. Furthermore, we note that $p(T)^m$ is elliptic of degree $m \cdot \deg p$. Therefore, by Theorem 2.19, we have that $u \in H_{k,loc}^{m \cdot \deg p}(\Omega)$ for all $m \in \mathbb{N}$, i.e.

$$u \in \bigcap_{s \in \mathbb{R}} H^s_{k,loc}(\Omega) = C^\infty(\Omega).$$

(ii) This is obvious, since $C(\Omega) \subseteq L^2_{loc}(\Omega, \omega)$.

(iii) Assume that $p(T)u = \lambda u$ for $\lambda \in \mathbb{C}$. But $\widetilde{p}(T) := p(T) - \lambda$ is an elliptic Dunkl operator satisfying $\widetilde{p}(T)u = 0$. Thus, by Theorem 2.19 we conclude $u \in C^\infty(\Omega)$.

■

CHAPTER 3

Multitemporal wave equation

The aim of this chapter is to study the multitemporal wave equation in the rational Dunkl setting in line with the results of the papers [PS93, Hel98, HS99, Hel08, STS76]. We prove the uniqueness of smooth solutions if the initial data is smooth. The solution of the multitemporal wave equation will be in close relation to the (W-invariant) generalized translation operator. In [PS93, Hel98, HS99] the authors consider the multitemporal wave equation on a Riemannian symmetric space $X = G/K$ of non-compact type. Let $G = NAK$ be the associated Iwasawa decomposition, denote by $\mathfrak{a}$ the Lie algebra of A inside the Lie algebra $\mathfrak{g}$ of G. The associated Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ acts naturally on $\mathfrak{a}$. Furthermore, let $\mathbb{D}(X)$ be the algebra of G-invariant differential operators on X and denote by $\Gamma : \mathbb{D}(X) \to \mathbb{C}[\mathfrak{a}]^W$ the Harish-Chandra isomorphism of $\mathbb{D}(X)$ onto the W-invariant polynomials on $\mathfrak{a}$. For fixed $f_1, \ldots, f_{\#W}$ one considers the following Cauchy problem for $u \in C^\infty(X \times \mathfrak{a})$:

$$\begin{cases} Du = \Gamma(D)(\partial)u, & \text{for all } D \in \mathbb{D}(X), \\ (p_i(\partial)u)(x, 0) = f_i(x), & \text{for all } 1 \leq i \leq \#W, \end{cases}$$

where $p_1, \ldots, p_{\#W}$ is a basis for the space of W-harmonic polynomials on $\mathfrak{a}$. The equation is called the multitemporal wave equation.
Restricting to K-biinvariant functions, the invariant differential operators $D \in \mathbb{D}(X)$ can be replaced by their radial parts. As in the case of Riemannian symmetric spaces of Euclidean type, these radial parts can be expressed in terms of so-called trigonometric Dunkl operators or Cherednik operators. These operators are introduced in the next chapter. However, some tools are missing in the trigonometric theory, so we consider a flat analogue of the multitemporal wave equation and replace the left-hand side of the equation by W-invariant (rational) Dunkl operators.
The organization of the chapter is the following. In Section 1 we prove the existence of solutions of the multitemporal wave equation in the Dunkl setting and prove the uniqueness of solutions which are compactly supported in the space variable. Then, in Section 2, we deal with the energy inner product, which will be a helpful tool to verify the uniqueness of smooth solutions in Section 3. We further point out the connection between solutions of the multitemporal wave equation and the generalized translation operator.
Let R be a root system inside a Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ with a non-negative multiplicity function $k \geq 0$, associated reflection group W, and a choice of positive roots R_+.
To simplify the notation, we suppress the dependence of the parameter k as in the previous chapter.

3.1 Existence of solutions

Let $\mathcal{P} = \mathbb{C}[\mathfrak{a}]$ be the space of polynomial functions on $\mathfrak{a}$, $\mathcal{P}_n$ the subspace of homogeneous polynomials of degree n and put $\mathcal{P}_+^W := \bigoplus_{n=1}^\infty \mathcal{P}_n^W$ for the subspace of W-invariant polynomials without constant term. A polynomial $p \in \mathcal{P}$ is called *W-harmonic* if $q(\partial)p = 0$ for all $q \in \mathcal{P}_+^W$. We denote by $\mathcal{H}$ the space of all W-harmonics. It is well known that (see for instance [GV88, §5.5, p.215])

$$\mathcal{P} = \mathcal{P} \cdot \mathcal{P}_+^W \oplus \mathcal{H} \quad \text{and} \quad d := \dim_{\mathbb{C}} \mathcal{H} = \#W.$$

Let $1 \equiv p_1, p_2, \ldots, p_d$ be a basis of $\mathcal{H}$ consisting of homogeneous elements with real coefficients. For fixed functions $f_1, \ldots f_d \in C^\infty(\mathfrak{a})$ we study the *multitemporal wave equation*

$$\begin{cases} p(T^x)u = p(\partial_t)u, & \text{for all } p \in \mathcal{P}^W, \\ (p_i(\partial_t)u)(x,0) = f_i(x), & i = 1, \ldots, d. \end{cases} \tag{MW}$$

We are searching for a (unique) solution $u(x,t)$, $u \in C^\infty(\mathfrak{a} \times \mathfrak{a})$. Here the superscript x and subscript t denote the relevant variables and $p(T) = p(T(k))$ is the Dunkl operator associated with the polynomial p and (R,k).

Example 3.1. If $\mathfrak{a} = \mathbb{R}$ and $R = \{\pm 1\}$, then the W-invariant polynomials $\mathcal{P}^W$ are the even polynomials and $\mathcal{H} = \{\mu x + \lambda \mid \lambda, \mu \in \mathbb{C}\}$. Hence, the multitemporal wave equation reduces in the case $k = 0$ to the usual wave equation in $\mathbb{R}$ with the following initial data

$$\begin{cases} \partial_{xx}u = \partial_{tt}u, \\ u(x,0) = f(x), \\ (\partial_t u)(x,0) = g(x). \end{cases}$$

This system has for any initial data $f, g \in C^2(\mathbb{R})$ the unique solution $u(x,t)$ given by d'Alembert's formula

$$u(x,t) = \frac{f(x-t) + f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s.$$

The situation for arbitrary $k \geq 0$ in rank one coincides with the situation of the Dunkl type wave equation in [ØS05].

Later, we make use of the following classical results which can be found in [GV88, §5.5] and [Hel84, Chapter III]. A key object in the following is the fundamental skew-polynomial associated with the root system R, namely

$$\pi(x) := \prod_{\alpha \in R_+} \langle \alpha, x \rangle \,.$$

By [Hel84, Theorem 3.6] the space $\mathcal{H}$ of W-harmonics consists of the derivatives of π

$$\mathcal{H} = \{p(\partial)\pi \mid p \in \mathcal{P}\}\,.$$

Lemma 3.2. *Let $\mathcal{R} = \mathbb{C}(\mathfrak{a})$ be the space of rational functions on $\mathfrak{a}$, equipped with the W-invariant, $\mathcal{R}^W$-valued non-degenerate $\mathcal{R}^W$-bilinear pairing*

$$(f,g)_W := \sum_{w \in W} wf \cdot wg.$$

Consider for $j = 1, \ldots, d$ the functions $q_j \in \mathcal{R}$ that are uniquely defined by the relations $(q_j, p_i)_W = \delta_{ij}$. Then:

(i) $\mathcal{P} = \bigoplus_{i=1}^d \mathcal{P}^W p_i$.

(ii) $\pi q_j \in \mathcal{P}_\mathbb{R} := \mathcal{P} \cap \mathbb{R}[\mathfrak{a}]$.

(iii) $\frac{1}{\pi}\mathcal{P} = \Big\{f \in \mathcal{R} \mid (f,g)_W \in \mathcal{P}^W \textit{ for all } g \in \mathcal{P}\Big\}$.

Moreover, q_j is homogeneous of degree $-\deg p_j$.

Proposition 3.3. *The system* (MW) *has at most one smooth solution u, such that $u(\cdot,t)$ is compactly supported for all t in a neighborhood of* 0.

PROOF. Without loss of generality we may assume that $f_1 = \ldots = f_d = 0$. Let u be a solution of (MW) such that $u(\cdot, t)$ is compactly supported for all $t \in U$, where U is an open neighborhood of 0. Put $\mathcal{F}_k$ for the Dunkl transform that acts on functions of the x-variable. From standard properties of the Dunkl transform (Theorem 1.20) we conclude for $p \in \mathcal{P}^W$ and $t \in U$ the following

$$p(\partial_t)\mathcal{F}_k(u(\cdot,t)) = \mathcal{F}_k(p(\partial_t)u(\cdot,t)) = \mathcal{F}_k(p(T^x)u(\cdot,t)) = p(i\cdot)\mathcal{F}_k(u(\cdot,t)), \tag{3.1}$$

where the first equality is justified by standard theorems on differentiable parameter integrals and the assumptions on the support of u. By Lemma 3.2, there exist for any $q \in \mathcal{P}$ unique polynomials $q_j' \in \mathcal{P}^W$ such that $q = \sum_{j=1}^d q_j' p_j$. Therefore, equation (3.1) yields

$$q(\partial_t)\mathcal{F}_k(u(\cdot,t))(\xi) = \sum_{j=1}^{d} q_j'(i\xi)\mathcal{F}_k(p_j(\partial_t)u(\cdot,t))(\xi).$$

Evaluation at $t = 0$ gives $q(\partial_t)\mathcal{F}_k(u(\cdot,0))(\xi) = 0$. Hence, we can see that $\widetilde{u}(\xi,t) = \mathcal{F}_k(u(\cdot,t))(\xi)$ solves the following initial value problem on $\mathfrak{a} \times U$

$$\begin{cases} p(\partial_t)\widetilde{u}(\xi,t) = p(i\xi)\widetilde{u}(\xi,t), & p \in \mathcal{P}^W, \\ (q(\partial_t)\widetilde{u})(\xi,0) = 0, & q \in \mathcal{P}. \end{cases}.$$

From this differential equation it follows that

$$\widetilde{u}(\xi,t) = \sum_{w \in W} c_w(\xi) e^{i\langle w\xi, t\rangle}$$

with some smooth coefficients c_w, see for instance [Hel84, Chapter 3, Theorem 3.13]. With suitable choices of $q \in \mathcal{P}$, we use the boundary condition to conclude $c_w = 0$ for all $w \in W$, i.e. $\widetilde{u} \equiv 0$ and so $u \equiv 0$. ∎

Definition 3.4. For $t \in \mathfrak{a}$ and $j = 1, \ldots, d$ we consider the functions

$$s_t^{(j)}(\lambda) := \sum_{w \in W} q_j(iw\lambda) e^{i\langle w\lambda, t\rangle}.$$

By Lemma 3.2 we have $h_j := \pi q_j \in \mathcal{P}$. Together with $\pi(w\lambda) = (-1)^{\ell(w)}\pi(\lambda)$, where $\ell(w)$ denotes the length of w, we obtain for $\lambda \in \mathfrak{a}_{\mathbb{C}}$

$$s_t^{(j)}(\lambda) = \frac{1}{\pi(i\lambda)} \sum_{w \in W} (-1)^{\ell(w)} h_j(iw\lambda) e^{i\langle w\lambda, t\rangle}.$$

The sum on the right hand side is holomorphic and skew-symmetric in λ, hence divisible by $\pi(i\lambda)$. Therefore, $s_t^{(j)}$ is an entire function satisfying

$$|s_t^{(j)}(\lambda)| \leq C(1+|\lambda|)^N e^{|t|\cdot|\mathrm{Im}\lambda|}$$

for certain constants $C > 0$ and $N \in \mathbb{N}_0$. By the Paley-Wiener theorem for the Dunkl transform from Theorem 1.21, there exists a tempered distribution $S_t^{(j)} \in \mathcal{S}'(\mathfrak{a})$ with compact support contained in $B_{|t|}(0)$ such that for all $\varphi \in \mathscr{S}(\mathfrak{a})$

$$\langle \mathcal{F}_k S_t^{(j)}, \varphi\rangle = \int_{\mathfrak{a}} \varphi(-x) s_t^{(j)}(x) \omega(x)\, \mathrm{d}x.$$

Recall from Lemma 2.7 that $f *_k u \in C^\infty(\mathfrak{a})$ for $f \in C^\infty(\mathfrak{a})$ and $u \in \mathcal{E}'(\mathfrak{a})$. Moreover, Theorem 2.11 gives that

$$\operatorname{supp}(f *_k u) \subseteq B_r(0) + W.\operatorname{supp} f,$$

whenever $\operatorname{supp} u \subseteq B_r(0)$.

Theorem 3.5. *Let $f_1, \ldots, f_d \in C^\infty(\mathfrak{a})$. Then*

$$u(x,t) := \sum_{j=1}^{d} (f_j *_k S_t^{(j)})(x),$$

defines a smooth solution of (MW) satisfying the finite speed propagation property

$$\operatorname{supp}(u(\cdot,t)) \subseteq \bigcup_{j=1}^{d} (B_{|t|}(0) + W.\operatorname{supp} f_j)$$

In particular, if $f_1, \ldots, f_d$ are compactly supported, then u is the unique solution such that $u(\cdot,t)$ is compactly supported for all t in a neighborhood of 0.

Proof. We first prove that $t \mapsto S_t^{(j)}$ is smooth in the sense that for all $\varphi \in C_c^\infty(\mathfrak{a})$ the map $t \mapsto \langle S_t^{(j)}, \varphi\rangle$ is smooth. In fact, for arbitrary $\varphi \in C_c^\infty(\mathfrak{a})$

$$\langle S_t^{(j)}, \varphi\rangle = \langle \mathcal{F}_k S_t^{(j)}, \mathcal{F}_k \varphi\rangle = \frac{1}{c_k}\int_{\mathfrak{a}} s_t^{(j)}(\lambda)\cdot(\mathcal{F}_k\varphi)(-\lambda)\omega(\lambda)\,\mathrm{d}\lambda.$$

Since for all $\alpha \in \mathbb{N}_0^n$ and any compact $K \subseteq \mathfrak{a}$ we have

$$|\partial_t^\alpha s_t^{(j)}(\lambda)| \le \widetilde{p}(\lambda)$$

for some polynomial $\widetilde{p}$, standard theorems on parameter integrals and $\mathcal{F}_k\varphi \in \mathscr{S}(\mathfrak{a})$ show that $t \mapsto \langle S_t^{(j)}, \varphi\rangle$ is smooth. Moreover, for $p \in \mathcal{P}^W$ we observe

$$\begin{aligned}\langle p(\partial_t)S_t^{(j)}, \varphi\rangle &= \frac{1}{c_k}\int_{\mathfrak{a}} p(i\lambda)s_t^{(j)}(\lambda)\cdot(\mathcal{F}_k\varphi)(-\lambda)\omega(\lambda)\,\mathrm{d}\lambda\\ &= \frac{1}{c_k}\int_{\mathfrak{a}} s_t^{(j)}(\lambda)\cdot(\mathcal{F}_k p(-T)\varphi)(-\lambda)\omega(\lambda)\,\mathrm{d}\lambda\\ &= \langle \mathcal{F}_k S_t^{(j)}, \mathcal{F}_k(p(-T)\varphi)\rangle = \langle S_t^{(j)}, p(-T)\varphi\rangle = \langle p(T)S_t^{(j)}, \varphi\rangle\,.\end{aligned}$$

From the properties of the Dunkl convolution in Theorem 2.5 we obtain $p(T^x)u = p(\partial_t)u$. It remains to prove that u satisfies the correct boundary conditions. To this end, we observe for $\ell = 1, \ldots, d$ that

$$\langle p_\ell(\partial_t)S_t^{(j)}, \varphi\rangle = \frac{1}{c_k}\int_{\mathfrak{a}} \sum_{w\in W} q_j(iw\lambda)p_\ell(iw\lambda)e^{i\langle w\lambda, t\rangle}(\mathcal{F}_k\varphi)(-\lambda)\omega(\lambda)\,\mathrm{d}\lambda.$$

Evaluation at $t = 0$ and the definition of the q_j show that

$$\langle p_\ell(\partial_t)S_t^{(j)}, \varphi\rangle\,|_{t=0} = \frac{\delta_{\ell j}}{c_k}\int_{\mathfrak{a}} \mathcal{F}_k\varphi(-\lambda)\omega(\lambda)\,\mathrm{d}\lambda = \delta_{kj}\cdot\varphi(0),$$

i.e. $p_\ell(\partial_t)S_t^{(j)}|_{t=0} = \delta_{kj}\cdot\delta_0$. Hence, $(p_\ell(\partial_t)u)(x,0) = f_\ell(x)$ holds by Theorem 2.5 and the definition of u. ■

3.2 The energy inner product

Definition 3.6. To $p \in \mathcal{P}$ we associate the $d \times d$ matrix L_p with entries in $\mathcal{P}^W$ defined via Lemma 3.2 by

$$pp_j = \sum_{i=1}^{d} (L_p)_{ij} p_i, \text{ i.e. } (L_p)_{ij} = (q_i, pp_j)_W.$$

Moreover, let $\mathcal{L}_p$ be the $d \times d$ matrix with entries in the algebra of Dunkl operators defined by $(\mathcal{L}_p)_{ij} = (L_p)_{ij}(T)$, i.e. the Dunkl operator associated with $(L_p)_{ij}$.

Lemma 3.7. *The system (MW) is equivalent to the system*

$$\begin{cases} (\mathcal{L}_p^x)^T \mu = p(\partial_t)\mu, & \text{for all } p \in \mathcal{P} \\ \mu(x,0) = F(x) := (f_1(x), \ldots, f_d(x)) \end{cases} \tag{VMW}$$

for $F \in C^\infty(\mathfrak{a}, \mathbb{C}^n)$ and solution $\mu \in C^\infty(\mathfrak{a} \times \mathfrak{a}, \mathbb{C}^d)$. Here the superscript T means transposition. More precisely, if u is a solution of (MW), then $\mu = (p_1(\partial_t)u, \ldots, p_d(\partial_t)u)^T$ is a solution of (VMW). Conversely, if μ is a solution of (VMW), then $u = \mu_1$ is a solution of (MW).

Proof. First, consider a solution μ of (VMW) and put $u = \mu_1$. By $p_1 \equiv 1$ we obtain $(L_{p_\ell})_{i1} = \delta_{i\ell}$, i.e. $(\mathcal{L}_{p_\ell})_{i1} = \delta_{i\ell} \cdot \mathrm{id}$. Thus, we see that

$$p_\ell(\partial_t)u(x,0) = ((p_\ell(\partial_t)\mu)(x,0))_1 = (((\mathcal{L}_{p_\ell}^x)^T\mu)(x,0))_1 = \sum_{i=1}^{n} ((\mathcal{L}_{p_\ell}^x)_{i1}\mu_i)(x,0) = f_\ell(x).$$

Moreover, from Lemma 3.2 for $p \in \mathcal{P}^W$ we have $L_p = p\mathbb{1}_d$, i.e. $\mathcal{L}_p = p(T)\mathbb{1}_d$, so that

$$p(T^x)u = ((p(T^x)\mathbb{1}_d)\mu)_1 = ((\mathcal{L}_p^x)^T\mu)_1 = (p(\partial_t)\mu)_1 = p(\partial_t)u,$$

which means that u is a solution of (MW).
Conversely, let u be a solution of (MW) and put $\mu = (p_1(\partial_t)u, \ldots, p_d(\partial_t)u)$. Then, $\mu(x,0) = F(x)$ and finally

$$(p(\partial_t)\mu)_j = (pp_j)(\partial_t)u = \sum_{i=1}^{d} ((L_p)_{ij}p_i)(\partial_t)u = \sum_{i=1}^{d} (\mathcal{L}_p^x)_{ij}(p_i(\partial_t)u) = ((\mathcal{L}_p^x)^T\mu)_j,$$

i.e. μ is a solution of (VMW). ■

Definition 3.8. Define A to be the $d \times d$ matrix with entries

$$A_{ij} = (\pi q_j, (\pi q_i)(-\cdot))_W = (-1)^{\deg \pi + \deg p_i} \pi^2 (q_j, q_i)_W, \tag{3.2}$$

so that $A_{ij} \in \mathcal{P}^W$ by Lemma 3.2. Define $\mathcal{A}$ to be the associated matrix of Dunkl operators, i.e. $\mathcal{A}_{ij} = A_{ij}(T)$. For fixed $t \in \mathfrak{a}$ and $u, v \in C^\infty(\mathfrak{a} \times \mathfrak{a})$, where either $u(\cdot, t)$ or $v(\cdot, t)$ has compact support, we consider the bilinear form

$$E(u, v; t) := \int_{\mathfrak{a}} (\mu^T \mathcal{A}^x \overline{\nu})(x,t)\omega(x)\, \mathrm{d}x$$

with $\mu = (p_1(\partial_t)u, \ldots, p_d(\partial_t)u)^T$ and $\nu = (p_1(\partial_t)v, \ldots, p_d(\partial_t)v)^T$. Equation (3.2) and the skew-symmetry of Dunkl operators show that the adjoint of $\mathcal{A}_{ij}$ in $L^2(\mathfrak{a}, \omega)$ is $\mathcal{A}_{ji}$. In particular, E is a symmetric bilinear form. We call $E(\cdot, \cdot; t)$ the *Energy inner product* at time t .

Example 3.9. Consider the rank one example, i.e. $\mathfrak{a} = \mathbb{R}$, $R = \{\pm 1\}$ and $W = \{\pm \mathrm{id}\}$. Choose $p_1(x) = 1$ and $p_2(x) = x$ as homogeneous basis for the W-harmonic polynomials. Then $q_1(x) = \frac{1}{2}$ and $q_2(x) = \frac{1}{2x}$ leads to

$$A = \frac{1}{2}\begin{pmatrix} -x^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \frac{1}{2}\begin{pmatrix} -\Delta_k & 0 \\ 0 & 1 \end{pmatrix},$$

where $\Delta_k^2 = T_1(k)^2$ is the Dunkl-Laplacian. Consider $u \in C^\infty(\mathbb{R} \times \mathbb{R})$ such that $u(\cdot, t)$ has compact support for all $t \in \mathfrak{a}$. Then

$$\begin{aligned} E(u, u; t) &= \frac{1}{2}\int_{\mathbb{R}} \left(\partial_t u(x,t) \cdot \overline{\partial_t u(x,t)} + u(x,t)\overline{\Delta_k^x u(x,t)}\right) \omega_k(x)\, \mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(\partial_t u(x,t) \cdot \overline{\partial_t u(x,t)} - u(x,t)\Delta_k^x \overline{u(x,t)}\right) \omega_k(x)\, \mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(|T_1(k)^x u(x,t)|^2 + |\partial_t u(x,t)|^2\right) \omega_k(x)\, \mathrm{d}x. \end{aligned}$$

This coincides with the total energy of u at time t defined in [ØS05, p.13] for solutions of the wave equation of Dunkl operators. In particular, if $k = 0$ this reduces to the usual total energy

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x u(x,t)|^2 + |\partial_t u(x,t)|^2\, \mathrm{d}x.$$

The subsequent theorem can be seen as an analogue of the energy conservation theorem for solutions of the classical wave equation.

Lemma 3.10. *Let $u, v \in C^\infty(\mathfrak{a} \times \mathfrak{a})$ be solutions of (MW). If $v(\cdot, t)$ has compact support for all $t \in \mathfrak{a}$, then $t \mapsto E(u, v; t)$ is constant.*

Proof. Let μ and ν be the solutions of (VMW) associated with u and v, respectively. We divide the proof into two steps.

(i) Consider the $d \times d$ matrix B with entries $B_{ij} := (\pi q_i, \pi q_j)_W$ and let J be the diagonal matrix with entries $J_{ii} = (-1)^{d_i}$, where $d_i = \deg p_i = -\deg q_i$. If $\mathcal{B}$ is the matrix of Dunkl operators associated with B, then

$$(-1)^{\deg \pi} JB = A \quad \text{and} \quad (-1)^{\deg \pi} J\mathcal{B} = \mathcal{A}. \tag{3.3}$$

The entries of B are W-invariant, so that

$$(q_m, \sum_{i=1}^d B_{i\ell} p_i)_W = \sum_{i=1}^d B_{i\ell}(q_m, p_i)_W = B_{mk} = (q_m, \pi^2 q_\ell)_W.$$

Hence, $\sum_{i=1}^d B_{i\ell} p_i = \pi^2 q_\ell$. The matrix L_p has W-invariant entries as well, so

$$(p_k, \sum_{i=1}^d (L_p)_{ji} q_i)_W = \sum_{i=1}^d (L_p)_{ji}(p_k, q_i)_W = (L_p)_{jk} = (q_j, pp_k)_W = (p_k, pq_j)_W.$$

Thus, we obtain $\sum_{i=1}^d (L_p)_{ji} q_i = pq_j$. We claim that this implies that $L_p B$ is symmetric. In fact, with the symmetry of B we observe that

$$\sum_{j=1}^d (L_p B)_{ij} p_j = \sum_{j,\ell=1}^d (L_p)_{i\ell} B_{\ell j} p_j = \pi^2 \sum_{\ell=1}^d (L_p)_{i\ell} q_\ell = \pi^2 pq_i$$

and similarly with the definition of L_p we get

$$\sum_{j=1}^{d}(L_pB)_{ji}p_j = \sum_{j,\ell=1}^{d}(L_p)_{j\ell}B_{\ell i}p_j = \sum_{\ell=1}^{d}B_{\ell i}p_\ell = \pi^2 pq_i.$$

Finally, L_pB has entries in $\mathcal{P}^W$, so Lemma 3.2 (i) implies that $(L_pB)_{ij} = (L_pB)_{ji}$. From the symmetry of L_pB and B we obtain

$$L_pB = BL_p^T \quad \text{and} \quad \mathcal{L}_p\mathcal{B} = \mathcal{B}\mathcal{L}_p^T. \tag{3.4}$$

(ii) Now we prove that $t \mapsto E(u, v; t)$ is constant. Consider the polynomials $p = \xi^* = \langle \xi, \cdot\rangle$ with $\xi \in \mathfrak{a}$. The assumption that μ, ν are solutions of (VMW), together with equations (3.3) and (3.4), now imply

$$\begin{aligned}
&p(\partial_t)E(u, v; t)\\
&\quad= \int_{\mathfrak{a}} \left[((\mathcal{L}_p^x)^T\mu)^T\mathcal{A}^x\overline{\nu} + \mu^T\mathcal{A}^x(\mathcal{L}_p^x)^T\overline{\nu}\right]\omega \,\mathrm{d}x\\
&\quad= (-1)^{\deg\pi}\int_{\mathfrak{a}} \left[((\mathcal{L}_p^x)^T\mu)^TJ\mathcal{B}^x\overline{\nu} + \mu^TJ\mathcal{L}_p^x\mathcal{B}^x\overline{\nu}\right]\omega \,\mathrm{d}x\\
&\quad= (-1)^{\deg\pi}\sum_{i,j=1}^{d}\int_{\mathfrak{a}} \left[(-1)^{d_j}(\mathcal{L}_p^x)_{ij}\mu_i(\mathcal{B}^x\overline{\nu})_j + (-1)^{d_i}\mu_i(\mathcal{L}_p^x)_{ij}(\mathcal{B}^x\overline{\nu})_j\right]\omega \,\mathrm{d}x,
\end{aligned}$$

where the interchange of order of differentation and integration is justified, by v and consequently ν being compactly supported in the x variable. Hence, to verify that $p(\partial_t)E(u, v; t)$ vanishes, it suffices to prove

$$\int_{\mathfrak{a}} \left[(\mathcal{L}_p^x)_{ij}\mu_i \cdot (\mathcal{B}^x\overline{\nu})_j + (-1)^{d_i-d_j}\mu_i(\mathcal{L}_p^x)_{ij}(\mathcal{B}^x\overline{\nu})_j\right]\omega \,\mathrm{d}x = 0.$$

Since p is homogeneous of degree 1 and $pp_j = \sum_{i=1}^{d}(L_p)_{ij}p_i$ holds by definition, we have that $(L_p)_{ij}$ is homogeneous of degree $1 + d_j - d_i$. Finally, the skew-symmetry of Dunkl operators leads to

$$\begin{aligned}
&\int_{\mathfrak{a}} \left[(\mathcal{L}_p^x)_{ij}\mu_i \cdot (\mathcal{B}^x\overline{\nu})_j + (-1)^{d_i-d_j}\mu_i(\mathcal{L}_p^x)_{ij}(\mathcal{B}^x\overline{\nu})_j\right]\omega \,\mathrm{d}x\\
&\qquad= \int_{\mathfrak{a}} \left[(\mathcal{L}_p^x)_{ij}\mu_i \cdot (\mathcal{B}^x\overline{\nu})_j - (\mathcal{L}_p^x)_{ij}\mu_i \cdot (\mathcal{B}^x\overline{\nu})_j\right]\omega \,\mathrm{d}x = 0.
\end{aligned}$$

■

Definition 3.11. For $F, G \in C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$ we define

$$E(F, G) := \int_{\mathfrak{a}} F(x)^T\mathcal{A}\overline{G(x)}\omega(x) \,\mathrm{d}x,$$

which coincides with $E(u, v, 0)$, where u and v are the solutions from Theorem 3.5 with initial data $(p_i(\partial_t)u)(x, 0) = F_i(x)$ and $(p_i(\partial_t)v)(x, 0) = G_i(x)$, respectively. In particular, E is a hermitian bilinear form. Moreover, for $w \in W$ and $\lambda \in \mathfrak{a}$ we define

$$u^w(x, t; \lambda) := e^{i\langle\lambda,t\rangle}E(iw\lambda, x),$$

where E is the Dunkl kernel associated with (R, k). It is obvious, that $u^w(\cdot, \cdot; \lambda)$ is a solution of (MW) with certain initial data. Let $\mu^w(\cdot, \cdot; \lambda)$ be the associated solution of (VMW), i.e.

$$\mu_j^w(x, t; \lambda) = p_j(i\lambda)e^{i\langle\lambda,t\rangle}E(iw\lambda, x).$$

We write $\mathcal{O}(\mathfrak{a}_{\mathbb{C}})$ for the space of entire functions on $\mathfrak{a}_{\mathbb{C}}$. Define

$$\mathcal{E}^w : C_c^\infty(\mathfrak{a}, \mathbb{C}^d) \to \mathcal{O}(\mathfrak{a}_{\mathbb{C}}), \quad F \mapsto \mathcal{E}^w F(\lambda) := \int_{\mathfrak{a}} F(x)^T \mathcal{A}^x \overline{\mu^w(x, 0; \overline{\lambda})} \omega(x) \, \mathrm{d}x.$$

Note that $\mathcal{E}^w F$ in fact defines an entire function by standard theorems on holomorphic parameter integrals.

Lemma 3.12. *For all $w \in W$, $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and $F = (f_1, \dots, f_d) \in C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$*

$$\mathcal{E}^w F(\lambda) = \pi(\lambda)^2 \sum_{\ell=1}^d q_\ell(i\lambda)(\mathcal{F}_k f_\ell)(w\lambda). \tag{3.5}$$

Furthermore, $\mathcal{E}^W F|_{\mathfrak{a}}$ is bounded.

Proof. By Lemma 3.2, we have $\pi q_j \in \mathcal{P}_{\mathbb{R}}$. Therefore, the Paley-Wiener theorem 1.21 shows that the right hand side of (3.5) is an entire function. Thus, it suffices to verify the identity (3.5) for $\lambda \in \mathfrak{a}$. Since p_j has real coefficients, we have

$$\overline{\mu_j^w(x, 0; \overline{\lambda})} = p_j(-i\lambda) E(-iw\lambda, x).$$

Thus, we are able to rewrite $\mathcal{E}^w F$ as follows:

$$\begin{aligned}
\mathcal{E}^w F(\lambda) &= \frac{1}{c_k} \sum_{\ell,j=1}^d A_{\ell j}(-iw\lambda) p_j(-i\lambda) \int_{\mathfrak{a}} f_\ell(x) E(-iw\lambda, x) \omega(x) \, \mathrm{d}x \\
&= \sum_{\ell,j=1}^d (\mathcal{F}_k f_\ell)(w\lambda) A_{\ell j}(-iw\lambda) p_j(-i\lambda).
\end{aligned}$$

With the W-invariance of $A_{\ell j}$ and $\deg p_j = -\deg q_j$ we have

$$\begin{aligned}
\sum_{j=1}^d A_{\ell j}(-iw\lambda) p_j(-i\lambda) &= \sum_{j=1}^d A_{\ell j}(-i\lambda) p_j(-i\lambda) \\
&= \sum_{j=1}^d \sum_{w \in W} (\pi q_j)(-iw\lambda) \cdot (\pi q_\ell)(iw\lambda) \cdot p_j(-i\lambda) \\
&= \sum_{w \in W} (\pi q_\ell)(iw\lambda) \sum_{j=1}^d (\pi q_j)(-iw\lambda) \cdot p_j(-i\lambda) \\
&= (-1)^{\deg \pi} \sum_{w \in W} (\pi q_\ell)(iw\lambda) \sum_{j=1}^d (\pi q_j)(iw\lambda) \cdot p_j(i\lambda) \\
&= (-1)^{\deg \pi} \sum_{j=1}^d p_j(i\lambda) \cdot (\pi q_\ell, \pi q_j)_W(i\lambda) \\
&= (-1)^{\deg \pi} (\pi^2 q_\ell)(i\lambda) = \pi^2(\lambda) q_\ell(i\lambda),
\end{aligned}$$

where the last equality follows immediately by pairing both sides with the q_m via the form $(\cdot, \cdot)_W$. ∎

Definition 3.13. On $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$ we define for $t \in \mathfrak{a}$ the linear map

$$U(t) : C_c^\infty(\mathfrak{a}, \mathbb{C}^d) \to C_c^\infty(\mathfrak{a}, \mathbb{C}^d), \quad F \mapsto \mu_F(\cdot, t),$$

where μ_F is the unique solution (VMW) with $\mu_F(x,0) = F(x)$, such that $\mu_F(\cdot,t)$ is compactly supported for all $t \in \mathfrak{a}$. Note that if $\mu_F(x,t)$ solves (VMW) with initial data F, then $\nu_F(x,t) := \mu_F(x,s+t)$ solves (VMW) with initial data $\mu_F(x,s)$. One verifies

$$U(t)U(s) = U(t+s),$$

i.e. U is a linear transformation group of $C_c^\infty(\mathfrak{a},\mathbb{C}^d)$.

Remark 3.14. In particular, for every $t_0 \in \mathfrak{a}$ and $F \in C_c^\infty(\mathfrak{a},\mathbb{C}^d)$ there exists a unique function $\mu_F^{t_0} := \mu_{U(t_0)^{-1}F} \in C^\infty(\mathfrak{a}\times\mathfrak{a})$, such that $\mu_F^{t_0}(\cdot,t)$ is compactly supported for all $t \in \mathfrak{a}$ and $\mu_F^{t_0}$ solves the system

$$\begin{cases} (\mathcal{L}_p^x)^T\mu = p(\partial_t)\mu, & p \in \mathcal{P}, \\ \mu(x,t_0) = F(x), & i = 1,\dots,d. \end{cases} \tag{3.6}$$

Theorem 3.15. *Consider the measure* $\mathrm{d}\varpi(\lambda) := d\frac{\omega(\lambda)}{\pi(\lambda)^2}\mathrm{d}\lambda$. *The map* $\mathcal{E}^w$ *defines for all* $w \in W$ *a linear operator*

$$\mathcal{E}^w : C_c^\infty(\mathfrak{a},\mathbb{C}^d) \to L^2(\mathfrak{a},\varpi)$$

with dense image. Furthermore, $\mathcal{E}^w$ *has the following properties:*

(i) $E(F,F) = \frac{1}{d}\sum_{w\in W}\|\mathcal{E}^wF\|_{2,\varpi}^2$ *for all* $F \in C_c^\infty(\mathfrak{a},\mathbb{C}^d)$.

(ii) *If* $\mathcal{U}^w(t_0)$ *is the multiplication operator defined by the function* $x \mapsto e^{i\langle wx,t_0\rangle}$, *then on* $C_c^\infty(\mathfrak{a},\mathbb{C}^d)$ *we have*

$$\mathcal{E}^wU(t_0) = \mathcal{U}^w(t_0)\mathcal{E}^w.$$

(iii) *When restricted to* $C_c^\infty(\mathfrak{a},\mathbb{C}^d)^W$, *the operator* $\mathcal{E}^w$ *is injective and for arbitrary* $F = (f_1,\dots,f_d) \in C_c^\infty(\mathfrak{a},\mathbb{C}^d)^W$ *we have*

$$f_j(x) = \mathcal{F}_k\left(\frac{p_j(i\cdot)}{d\,\pi^2}\mathcal{E}^wF\right)(-x). \tag{3.7}$$

Moreover, E *is an inner product on* $C_c^\infty(\mathfrak{a},\mathbb{C}^d)^W$.

(iv) *The closure of* $C_c^\infty(\mathfrak{a},\mathbb{C}^d)^W$ *with respect to the inner product* E *is isometrically isomorphic to* $L^2(\mathfrak{a},\varpi)$, *so that* $\mathcal{E}^w$ *extends to a unitary map between these two spaces.*

(v) *Moreover, we have an injective map*

$$\mathcal{E} = (\mathcal{E}^w)_{w\in W} : C_c^\infty(\mathfrak{a},\mathbb{C}^d) \to \bigoplus_{i=1}^d L^2(\mathfrak{a},\varpi).$$

such that $E(F,F) = \frac{1}{d}\sum_{w\in W}\|\mathcal{E}^wF\|_{2,\varpi}^2$ *and for all* $F = (f_1,\dots,f_d) \in C_c^\infty(\mathfrak{a},\mathbb{C}^d)$

$$f_j(x) = \sum_{w\in W}\mathcal{F}_k\left(\frac{p_j(i\cdot)}{\pi^2}\mathcal{E}^wF\right)(-w^{-1}x).$$

(vi) E *is an inner product on* $C_c^\infty(\mathfrak{a},\mathbb{C}^d)$.

Proof. First we prove that $\mathcal{E}^w$ has a dense image. Since $X := C_c^\infty(\mathfrak{a},\mathbb{C}) \cap L^2(\mathfrak{a},\varpi)$ is dense in $L^2(\mathfrak{a},\varpi)$, it suffices to prove that $g \in X$ vanishes if

$$\int_{\mathfrak{a}}(\mathcal{E}^wF)(\lambda)\overline{g(\lambda)}\,\frac{\omega(\lambda)\mathrm{d}\lambda}{\pi(\lambda)^2} = 0$$

holds for all $F \in C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$. In this case, we use Lemma 3.12 and put $F = (f, 0, \dots, 0)$ with arbitrary $f \in C_c^\infty(\mathfrak{a})$ to obtain

$$\int_{\mathfrak{a}} (\mathcal{F}_k f)(w\lambda)\overline{g(\lambda)}\, \omega(\lambda) \mathrm{d}\lambda = 0.$$

Finally, the Plancherel theorem for the Dunkl transform gives $g \equiv 0$.

(i) From Lemma 3.12 we have

$$\frac{1}{d} \sum_{w \in W} (\mathcal{E}^w F)(\lambda)\overline{(\mathcal{E}^w F)(\lambda)} = \pi(\lambda)^4 \sum_{\ell,j=1}^d q_\ell(i\lambda) q_j(-i\lambda)(\mathcal{F}_k f_\ell)(w\lambda)\overline{(\mathcal{F}_k f_j)(w\lambda)}.$$

Since the measure ϖ is W-invariant, we obtain from the Plancherel Theorem 1.19 for the Dunkl transform

$$\begin{aligned}
\|\mathcal{E}^w F\|_{2,\varpi}^2 &= \sum_{w \in W} \sum_{\ell,j=1}^d \int_{\mathfrak{a}} \pi(\lambda)^2 (\mathcal{F}_k f_\ell)(\lambda)\overline{(\mathcal{F}_k f_j)(\lambda)} \cdot q_\ell(iw\lambda) q_j(-iw\lambda)\omega(\lambda)\, \mathrm{d}\lambda \\
&= \sum_{\ell,j=1}^d \int_{\mathfrak{a}} (\mathcal{F}_k f_\ell)(\lambda)\overline{(\mathcal{F}_k f_j)(\lambda)} \cdot \overline{A_{\ell j}(i\lambda)}\omega(\lambda)\, \mathrm{d}\lambda \\
&= \sum_{\ell,j=1}^d \int_{\mathfrak{a}} (\mathcal{F}_k f_\ell)(\lambda)\overline{(\mathcal{F}_k \mathcal{A}_{\ell j} f_j)(\lambda)}\omega(\lambda)\, \mathrm{d}\lambda \\
&= \sum_{\ell,j=1}^d \int_{\mathfrak{a}} f_\ell(\lambda)\mathcal{A}_{\ell j}\overline{f_j(\lambda)}\omega(\lambda)\, \mathrm{d}\lambda = E(F, F).
\end{aligned}$$

(ii) This is straightforward to verify.

(iii) The proof for (3.7) is the same as for (v) below. In particular, $\mathcal{E}^w$ is injective on $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)^W$ and from $E(F, F) = \|\mathcal{E}^w F\|_{2,\varpi}^2$ we obtain that the hermitian form E is indeed an inner product on $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)^W$.

(iv) This is an immediate conclusion from part (iii).

(v) The formula for $E(F, F)$ follows from part (i). We note that

$$\begin{aligned}
\mathcal{F}_k \left(\frac{p_j(i\cdot)}{\pi^2} \mathcal{E}^w F \right)(-w^{-1}x) &= \frac{1}{c_k} \int_{\mathfrak{a}} p_j(i\lambda)(\mathcal{E}^w F)(\lambda) E(i\lambda, w^{-1}x) \frac{\omega(\lambda)}{\pi(\lambda)^2}\, \mathrm{d}\lambda \\
&= \frac{1}{c_k} \int_{\mathfrak{a}} p_j(iw^{-1}\lambda)(\mathcal{E}^w F)(w^{-1}\lambda) E(i\lambda, x) \frac{\omega(\lambda)}{\pi(\lambda)^2}\, \mathrm{d}\lambda \\
&= \sum_{\ell=1}^d p_j(w^{-1}T) q_\ell(w^{-1}T) f_\ell(x).
\end{aligned}$$

Hence, summation over $w \in W$ yields the stated formula, since

$$\sum_{w \in W} wp_j \cdot wq_\ell = \delta_{j\ell}.$$

(vi) We already noticed that E is a hermitian form. By part (v) E is positive definite, hence an inner product.

■

3.3 Uniqueness of smooth solutions

Lemma 3.16. *Recall the operator $\mathcal{A}$ from Definition 3.8. Consider the Hilbert space $L^2(\mathfrak{a}, \mathbb{C}^d; \omega)$ equipped with the inner product*

$$\langle F, G\rangle_{L^2} = \int_{\mathfrak{a}} \langle F(x), G(x)\rangle\, \omega(x)\, \mathrm{d}x,$$

where $\langle\cdot,\cdot\rangle$ is usual inner product on $\mathbb{C}^d$. Then the subspace

$$\mathcal{X} = \left\{\mathcal{A}F \mid F \in C_c^\infty(\mathfrak{a}, \mathbb{C}^d)\right\}$$

is dense in $L^2(\mathfrak{a}, \mathbb{C}^d; \omega)$.

Proof. Obviously $\mathcal{X}$ is a subspace of $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$, so we only have to verify the density inside $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$. By Theorem 3.15 E is an inner product on $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$. Therefore $E(G, F) = \langle G, \mathcal{A}F\rangle_{L^2}$ vanishes for all $F \in C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$ if and only if $G \equiv 0$, i.e. $\mathcal{X}$ is dense in $C_c^\infty(\mathfrak{a}, \mathbb{C}^d)$. ∎

Theorem 3.17. *For smooth initial data $f_1, \ldots, f_d$ the systems (MW) and (VMW) have unique solutions $u \in C^\infty(\mathfrak{a} \times \mathfrak{a})$ and $\mu \in C^\infty(\mathfrak{a} \times \mathfrak{a}, \mathbb{C}^d)$. In particular, the unique solution of (MW) is given by the formula in Theorem 3.5.*

Proof. By Lemma 3.7 it suffices to prove the theorem for (VMW). Consider a solution μ of (VMW) with initial condition $\mu(\cdot, 0) = 0$. For arbitrary $F \in C_c^\infty(\mathfrak{a})$ and $t_0 \in \mathfrak{a}$ choose $\mu_F^{t_0}$ as stated in Remark 3.14 and consider $u = \mu_1$ and $v = (\mu_F^{t_0})_1$. By Lemma 3.10 we can see that

$$\langle \mu(\cdot, t_0), \overline{\mathcal{A}F}\rangle_{L^2} = E(u, v; t_0) = E(u, v, 0) = 0.$$

By the density property of Lemma 3.16, we have $\langle \mu(\cdot, t_0), F\rangle_{L^2} = 0$ for all $F \in L^2(\mathfrak{a}, \mathbb{C}^d; \omega)$ which are compactly supported. Thus, $\mu(\cdot, t_0) = 0$ which means $\mu \equiv 0$. ∎

Corollary 3.18. *Consider for the system (MW) some initial data $F = (f_1, \ldots, f_d) \in C^\infty(\mathfrak{a}, \mathbb{C}^d)$ and the unique solution $u = u(x, t) \in C^\infty(\mathfrak{a} \times \mathfrak{a})$. Then*

(i) *u is W-invariant in t if and only if $f_2 = \ldots = f_d \equiv 0$.*

(ii) *u is W-invariant in x if and only if F is W-invariant.*

Proof.

(i) Obviously, the equation $p(T^x)u(x, t) = (p(\partial_t)u)(x, t)$ for $p \in \mathcal{P}^W$ is W-invariant in t, i.e. $u(x, t)$ is a solution of (MW) if and only if $u(x, wt)$ solves it for $w \in W$. If $f_2 = \ldots = f_d \equiv 0$, then Theorem 3.17 and Theorem 3.5 show that u is W-invariant in t. We claim that if φ is W-invariant, then $(p_i(\partial)\varphi)(0) = 0$ for $i > 1$. By density of analytic functions in $C^\infty(\mathfrak{a})$ is suffices to check this for analytic φ. Since φ is W-invariant it can be expanded as $\varphi = \sum_{n=0}^\infty \varphi_n$ with W-invariant polynomials φ_n of homogeneous degree n. Then, by symmetry of the Fisher inner product and the definition of harmonic polynomials we have $(\varphi_n(\partial)p_i)(0) = 0$ for $n > 0$. Since p_i has no constant term for $i > 0$, we also have $(\varphi_0(\partial)p_i)(0) = 0$ and therefore
$$(p_i(\partial)\varphi)(0) = \sum_{i=0}^\infty (p_i(\partial)\varphi_n)(0) = \sum_{n=0}^\infty (\varphi_n(\partial)p_i)(0) = 0.$$
If u is a solution of (MW), W-invariant in t, then $f_i(x) = (p_i(\partial_t)u)(x, 0) = 0$, $i > 1$.

(ii) Obviously, for $p \in \mathcal{P}^W$ the equation $p(T^x)u(x,t) = p(\partial_t)u(x,t)$ is W-invariant in x, i.e. $u(x,t)$ is a solution if and only if $u(wx,t)$ is a solution for $w \in W$. Furthermore, the condition $f_i(x) = p_i(\partial_t)u(x,0)$ is W-invariant in the sense that $f_i(wx) = (p_i(\partial_t)u)(wx,0)$. Hence, the uniqueness of the smooth solution finishes the proof of this part.

■

Corollary 3.19. *Let f be a W-invariant smooth function on $\mathfrak{a}$ and let u be the solution of* (MW) *for Cauchy data $F = (f, 0, \dots, 0)$. Consider the W-invariant translation operator*

$$\tau_t^W f(x) = \frac{1}{d} \sum_{w \in W} \tau_t f(wx) = \frac{1}{d} \sum_{w \in W} \tau_{wt} f(x).$$

Then, we have for Dunkl's intertwining operator $V^t = V_k^t$ acting on functions in the t-variable

$$V^t u(x,t) = \tau_t^W f(x) = \tau_x^W f(t).$$

Proof. Put $v(x,t) := (V^t)^{-1} \tau_t^W f$. The translation operator satisfies for all $p \in \mathcal{P}$

$$p(T^t)\tau_t f(x) = p(T^x)\tau_t f(x),$$

so it is immediate that for all $p \in \mathcal{P}^W$ we have

$$p(T^t)\tau_t^W f(x) = p(T^x)\tau_t^W f(x).$$

Hence, by the intertwining relations of V^t, we have for all $p \in \mathcal{P}^W$

$$p(T^x)v(x,t) = p(\partial_t)v(x,t).$$

Furthermore, by $\tau_0 = \mathrm{id}$ and $Vf(0) = f(0)$ we see $v(x,0) = f(x)$. The W-equivariance of V and the definition of τ_t^W show that the function v is W-invariant in t, hence $(p_i(\partial_t)v)(x,0) = 0$ for $i = 2, \dots, d$. Finally, v and u are solutions of (MW) with same initial data, i.e. $u = v$ by Theorem 3.17. ■

Corollary 3.20. *For $f \in C^\infty(\mathfrak{a})$ put $f_i := p_i(\partial)f$ for all $i = 1, \dots, d$. Then, the solution u of* (MW) *satisfies*

$$V^t u(x,t) = \tau_t f(x) = \tau_x f(t).$$

Proof. With the relations $T_\xi \tau_t = \tau_t T_\xi$ and $T_\xi V = V\partial_\xi$ it is straightforward to see that

$$u(x,t) := ((V^t)^{-1}\tau_t f)(x) = V^x((V^{-1}f)(x+t))$$

solves (MW) with data $f_i = p_i(\partial)f$. By uniqueness of smooth solution from Theorem 3.17 the assertion holds. ■

Remark 3.21. One could also study the system

$$\begin{cases} p(T(k)^x)\widetilde{u} = p(T(k')^t)\widetilde{u}, & p \in \mathcal{P}^W \\ (p_i(T(k')^t)\widetilde{u})(x,0) = f_i(x), & i = 1, \dots d \end{cases},$$

where k, k' are multiplicity functions on R, so that $k' = 0$ is the case studied before. Since the distribution $f \mapsto V_k f(x)$ has compact support and $V_k : C^\infty(\mathfrak{a}) \to C^\infty(\mathfrak{a})$ is a topological isomorphism, one has $T_\xi(k)^x V_{k'}^t \widetilde{u} = V_{k'}^t T_\xi(k)^x \widetilde{u}$ for $\widetilde{u} \in C^\infty(\mathfrak{a} \times \mathfrak{a})$. Hence, this system has a unique solution, which is given by $\widetilde{u} = V_{k'}^t u$. Here, u is the solution of the multitemporal wave equation (MW) as studied before with Cauchy data $F = (f_1, \ldots, f_d)$. In particular, the unique solution can be expressed as

$$\widetilde{u}(x,t) = \sum_{j=1}^{d} (f_j *_k S_t^{(j)}(k'))(x),$$

where the compactly supported distribution $S_t^{(j)}(k')$ are defined by

$$[\mathcal{F}_k(S_t^{(j)}(k'))](\lambda) = \sum_{w \in W} q_j(-iw\lambda) E_{k'}(-iw\lambda, t)$$

via the Paley-Wiener theorem for the Dunkl transform. Moreover, in the case $k = k'$ we can characterize the W-invariant translation operator τ^W (with respect to the multiplicity k) as follows: for $f \in C^\infty(\mathfrak{a})^W$ the function $\widetilde{u}(x,t) = \tau_t^W f(x)$ is the unique smooth solution of the system

$$\begin{cases} p(T(k)^x)\widetilde{u} = p(T(k)^t)\widetilde{u}, & p \in \mathcal{P}^W. \\ p_i(T(k)^t)\widetilde{u}(x,0) = 0, & i = 2, \ldots d, \\ \widetilde{u}(x,0) = f(x). \end{cases}$$

Corollary 3.22. *For $f \in C^\infty(\mathfrak{a})$ the function $u(x,t) = \tau_x f(t) = \tau_t f(x)$ is the unique smooth solution of*

$$\begin{cases} p(T(k)^x)u = p(T(k)^t)u, & \text{for all } p \in \mathcal{P}, \\ (p_i(T(k)^t)u)(x,0) = (p_i(T(k))f)(x), & \text{for all } i = 1, \ldots, d. \end{cases}$$

Furthermore, for $f \in C^\infty(\mathfrak{a})^W$ we can characterize $u(x,t) = \tau_x^W f(t) = \tau_t^W f(x)$ as the unique smooth solution of

$$\begin{cases} p(T(k)^x)u = p(T(k)^t)u, & \text{for all } p \in \mathcal{P}^W, \\ (p_i(T(k)^t)u)(x,0) = 0, & \text{for all } i = 2, \ldots, d, \\ u(x,0) = f(x) \end{cases}$$

or as the unique W-invariant smooth solution of

$$\begin{cases} p(T(k)^x)u = p(T(k)^t)u, & \text{for all } p \in \mathcal{P}^W, \\ u(x,0) = f(x). \end{cases}$$

CHAPTER 4

Results in trigonometric Dunkl theory

For the background on trigonometric Dunkl theory, the reader is referred to various articles by Heckman and Opdam like [HO87, Hec87, Opd89, Hec90, Hec91, HS94, Opd95, Hec97, Opd00].
The organization of the chapter is the following. In Section 1 we introduce the reader to trigonometric Dunkl theory for crystallographic root systems. Afterwards, in Section 2, we extend the concepts of trigonometric Dunkl theory to integral root systems in a natural way. The connection to Riemannian symmetric spaces of non-compact type is explained in Section 3, while in Section 4 the connection to Riemannian symmetric spaces associated with reductive Lie groups is explained. Finally, we will prove the generalization of the Helgason-Johnson theorem for the Cherednik kernel in Section 5.

4.1 Introduction to trigonometric Dunkl theory

Trigonometric polynomials and Cherednik operators

As before, consider a finite-dimensional Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ with norm $|x| = \sqrt{\langle x, x\rangle}$ and extend the inner product complex-bilinear to the complexification $\mathfrak{a}_{\mathbb{C}}$. Let $R \subseteq \mathfrak{a}$ be a crystallographic root system, not necessarily reduced, and let $W = W(R)$ be the Weyl group. Fix a system of positive roots $R_+ \subseteq R$, negative roots $R_- = -R_+$ and a corresponding positive open Weyl chamber $\mathfrak{a}_+$. The simple roots associated with R_+ will be denoted by $\alpha_1, \dots, \alpha_n$ and the corresponding simple reflections are $s_i \coloneqq s_{\alpha_i}$. We will write

$$Q \coloneqq \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_n = \operatorname{span}_{\mathbb{Z}} R$$

for the *root lattice*. Inside Q, we fix the cone

$$Q_+ \coloneqq \mathbb{N}_0\alpha_1 \oplus \ldots \oplus \mathbb{N}_0\alpha_n = \operatorname{span}_{\mathbb{N}_0} R.$$

The *weight lattice* P is defined as the dual lattice of the lattice spanned by the coroots, namely

$$P \coloneqq \big\{\lambda \in \mathfrak{a} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\big\} = \mathbb{Z}\omega_1 \oplus \ldots \oplus \mathbb{Z}\omega_n,$$

where $\omega_1, \dots, \omega_n$ are the so-called *fundamental weights*, defined by the condition $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. According to the integrality condition on R we have $R \subseteq Q \subseteq P$. The *dominant weights* are defined as the cone

$$P_+ \coloneqq P \cap \overline{\mathfrak{a}_+} = \mathbb{N}_0\omega_1 \oplus \ldots \oplus \mathbb{N}_0\omega_n.$$

Definition 4.1. We introduce on P_+ the partial order

$$\lambda \le \mu \quad \text{iff} \quad \mu - \lambda \in Q_+,$$

called the *dominance order* on P_+. This order can be extended to P by the assignment

$$\lambda \trianglelefteq \mu \quad \text{iff} \quad \begin{cases} \lambda_+ \le \mu_+, & \text{if } \lambda_+ \neq \mu_+, \\ \mu \le \lambda, & \text{if } \lambda_+ = \mu_+, \end{cases}$$

where λ_+ and μ_+ are the unique dominant weights in the W-orbit of λ and μ, respectively.

The weights P index a basis of the space of *trigonometric polynomials*, which is by definition the algebra

$$\mathcal{T} := \operatorname{span}_{\mathbb{C}}\{e^\lambda \mid \lambda \in P\} \quad \text{for} \quad e^\lambda := e^{\langle \lambda, \cdot \rangle}.$$

The algebra $\mathcal{T}$ is isomorphic to the group algebra $\mathbb{C}[P] = \mathbb{C}[\omega_1, \dots, \omega_n]$ via the isomorphism induced by $\lambda \mapsto e^\lambda$. Under this isomorphism, the usual W-action on $\mathcal{T}$, as an algebra of functions, coincides with the canonical W-action on P, namely $we^\lambda = e^{w\lambda}$.

Definition 4.2. The *Cherednik operator/trigonometric Dunkl operator* associated with (R_+, k) *into direction* $\xi \in \mathfrak{a}$ is defined by

$$D_\xi(R_+, k) := D_\xi(k) := D_\xi := \partial_\xi - \langle \rho(k), \xi \rangle + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{1 - e^{-\alpha}},$$

where $\rho(k)$ is the *Weyl vector*

$$\rho(k) := \rho := \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha.$$

Let $\Omega \subseteq \mathfrak{a}$ be a W-invariant open subset. We summarize the main properties of Cherednik operators that are relevant in this thesis, which can be found in [Opd95, Hec97, Opd00].

(i) $D_\xi(k)$ maps $C^m(\Omega)$ into $C^{m-1}(\Omega)$ and acts continuously on the spaces $C^\infty(\Omega)$ and $C_c^\infty(\Omega)$, equipped with their usual locally convex topologies.

(ii) *Commutativity*: $D_\xi(k)D_\eta(k) = D_\eta(k)D_\xi(k)$

(iii) *Support*: for $f \in C^1(\Omega)$

$$\operatorname{supp}(D_\xi(k)f) \subseteq W.\operatorname{supp} f.$$

(iv) *Leibniz rule*: If $f, g \in C^1(\Omega)$, one of them W-invariant, then

$$D_\xi(k)(fg) = (D_\xi(k)f)g + f(D_\xi(k)g) + \langle \rho(k), \xi \rangle fg.$$

Since the Cherednik operators commute, we associate with $p \in \mathcal{P} = \mathbb{C}[\mathfrak{a}]$ an operator $p(D(k))$ in an obvious manner. A main difference between the rational and the trigonometric Dunkl theory is that Cherednik operators are not W-equivariant, namely $wD_\xi(k)w^{-1} \neq D_{w\xi}(k)$. However, to any W-invariant polynomial $p \in \mathcal{P}^W$ there exists a unique linear partial differential operator $\operatorname{res}(p(D(k))$ on $\mathfrak{a}_{\text{reg}}$ such that

$$p(D(k))f = \operatorname{res}(p(D(k)))f \tag{4.1}$$

for all $f \in C^\infty(\Omega)^W$. Moreover, $p \mapsto \operatorname{res}(p(D(k)))$ defines a morphism of algebras. The operators $\operatorname{res}(p(D(k))$, $p \in \mathcal{P}^W$, generalize the radial part of invariant differential operators on Riemannian symmetric spaces of non-compact type, which will be explained in the end of this chapter.

Cherednik operators on trigonometric polynomials

On $\mathcal{T}$, the Cherednik operators has an upper triangular action. To be more precise, $D_\xi(k)$ acts on monomials e^λ by

$$D_\xi(k)e^\lambda = \langle \widetilde{\lambda}, \xi \rangle e^\lambda + \sum_{\mu \lhd \lambda} c_{\mu\lambda} e^\mu,$$

where $\widetilde{\lambda}$ is given by

$$\widetilde{\lambda} = \lambda - \rho(k) + \sum_{\substack{\alpha \in R_+ \\ \langle \alpha, \lambda \rangle > 0}} k_\alpha \alpha = \lambda + \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \alpha, \lambda \rangle)\alpha, \tag{4.2}$$

with the non-symmetric sign

$$\epsilon(t) = \begin{cases} 1, & t > 0, \\ -1, & t \leq 0. \end{cases}$$

The map $\lambda \mapsto \widetilde{\lambda}$ is injective on $\mathfrak{a}$, since $\mathfrak{a} = \bigsqcup_{w \in W} C_w$ with

$$C_w := \left\{ \lambda \in \mathfrak{a} \,\middle|\, \substack{\langle \alpha, \lambda \rangle > 0 \text{ for } \alpha \in R_+ \cap w R_+ \\ \langle \alpha, \lambda \rangle \leq 0 \text{ for } \alpha \in R_+ \cap w R_-} \right\}$$

and for $\lambda \in C_w$ one has $\widetilde{\lambda} = \lambda + w\rho(k) = w(\lambda_+ + \rho(k)) \in C_w$.

Definition 4.3. For $k \geq 0$, we define on $\mathcal{T}$ the W-invariant inner product

$$\langle f, g \rangle_k := \int_T f(t)\overline{g(t)}\delta_k(t) \, \mathrm{d}t,$$

where $\mathrm{d}t$ is the normalized Haar measure on the torus

$$T := \left\{ i \sum\nolimits_{j=1}^n x_j \alpha_j^\vee \,\middle|\, 0 \leq x_j < 2\pi \right\} \cong i\mathfrak{a}/2\pi i Q^\vee$$

and δ_k is the W-invariant weight function

$$\delta_k = \prod_{\alpha \in R_+} \left| e^{\alpha/2} - e^{-\alpha/2} \right|^{2k_\alpha} = \prod_{\alpha \in R_+} \left| 2 \sinh\left(\tfrac{\langle \alpha, \cdot \rangle}{2} \right) \right|^{2k_\alpha}.$$

For $k \geq 0$, the Cherednik operators are symmetric with respect to $\langle \cdot, \cdot \rangle_k$ on $\mathcal{T}$, i.e. for all $f, g \in \mathcal{T}$

$$\langle D_\xi(k) f, g \rangle_k = \langle f, D_\xi(k) g \rangle_k .$$

The symmetry and the upper triangular action of Cherednik operators on $\mathcal{T}$ lead to the following theorem.

Theorem 4.4 ([Opd95, Sah00a, Sah00b])**.** *The algebra $\mathcal{T}$ has for $k \geq 0$ a unique basis $(E_\lambda(k; \cdot))_{\lambda \in P}$ of polynomials, called non-symmetric Heckman-Opdam polynomials, such that*

(i) $E_\lambda(k; \cdot) = e^\lambda + \sum_{\mu \lhd \lambda} c_{\mu\lambda} e^\mu$ *with* $c_{\mu\lambda} \in \mathbb{C}$.

(ii) $E_\lambda(k; \cdot)$ *is an eigenfunction of all Cherednik operators* $D_\xi(k)$, $\xi \in \mathfrak{a}$.

The second condition can be replaced by the orthogonality condition

(ii') $\langle E_\lambda(k; \cdot), e^\mu \rangle_k = 0$ *for all* $\mu \lhd \lambda$.

Furthermore, $D_\xi(k) E_\lambda(k; \cdot) = \langle \widetilde{\lambda}, \xi \rangle \, E_\lambda(k \cdot)$ *and the coefficients* $c_{\mu\lambda}$ *are non-negative rational functions of* k*, cf. Theorem A.6 in the Appendix A.*

There is a symmetric analog of this theorem. For this, we define for $\lambda \in P_+$ the monomial symmetric function

$$m_\lambda := \sum_{\mu \in W.\lambda} e^\mu.$$

Then the following theorem holds.

Theorem 4.5 ([HO87, HS94, Opd95])**.** *The algebra $\mathcal{T}^W$ has for $k \geq 0$ a unique basis $(P_\lambda(k; \cdot))_{\lambda \in P_+}$ of polynomials, called (symmetric) Heckman-Opdam polynomials/Jacobi polynomials, such that*

(i) $P_\lambda(k; \cdot) = m_\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} m_\mu$ *with* $c_{\mu\lambda} \in \mathbb{C}$.

(ii) $P_\lambda(k;\cdot)$ *is an eigenfunction of the Heckman-Opdam Laplacian*

$$L(k) := \sum_{i=1}^{n} D_{e_i}(k)^2$$

where $e_1,\ldots,e_n$ *is an orthonormal basis of* $\mathfrak{a}$.

The second condition can be replaced by either one of following conditions

(ii') $\langle P_\lambda(k;\cdot), m_\mu\rangle_k = 0$ *for all* $\mu < \lambda$.

(ii") $P_\lambda(k;\cdot)$ *is an eigenfunction of all Cherednik operators* $p(D(k))$ *with* $p \in \mathcal{P}^W$.

Furthermore, $p(D(k))P_\lambda(k;\cdot) = p(\lambda+\rho(k))P_\lambda(k;\cdot)$ *for all* $p \in \mathcal{P}^W$*, the coefficients* $c_{\mu\lambda}$ *are non-negative rational functions of* k *and for all* $\mu \in W.\lambda$ *we have*

$$P_\lambda(k;z) := \frac{\#W\lambda}{\#W}\sum_{w\in W} E_\mu(k;wz).$$

Eigenfunctions of the Cherednik operators

Fix a non-negative multiplicity $k \geq 0$. As in the rational Dunkl setting, we can consider for $\lambda \in \mathfrak{a}_\mathbb{C}$ the joint eigenvalue problem

$$\begin{cases} D_\xi(k)f = \langle\lambda,\xi\rangle\, f, & \text{for all } \xi\in\mathfrak{a},\\ f(0)=1. \end{cases}$$

Owing to the work of Opdam [Opd95], this eigenvalue problem has a unique analytic solution

$$G_k(\lambda,\cdot) : \mathfrak{a} \to \mathbb{C},$$

which is called the *Cherednik kernel* associated with (R_+,k).
Furthermore, the symmetric analogue of the Cherednik kernel is the *(Heckman-Opdam) hypergeometric function* associated with (R,k) and is defined by

$$F_k(\lambda,x) := \frac{1}{\#W}\sum_{w\in W} G_k(\lambda,wx).$$

The hypergeometric function $F_k(\lambda,\cdot)$ is the unique W-invariant analytic solution of the joint eigenvalue problem

$$\begin{cases} p(D(k))f = p(\lambda)f, & \text{for all } p\in\mathcal{P}^W,\\ f(0)=1. \end{cases}$$

In particular, the hypergeometric function $F_k(\lambda,x)$ is W-invariant in the parameter λ.
In the rational Dunkl setting, the existence of the Dunkl kernel was proven first and the Bessel function was defined in terms of the Dunkl kernel. In contrast to this, the existence of the hypergeometric function as a the solution of the joint eigenvalue problem was studied as first, cf. [Hec87, HO87, HS94]. Later, Opdam [Opd95] constructed the Cherednik kernel by

$$G_k(\lambda,\cdot) = D(\lambda,k)F_k(\lambda,\cdot),$$

where $D(\lambda,k)$ is a certain linear differential operator.
We note that G_k depends on the choice of positive roots R_+ and k, while F_k only depends on R and k. Furthermore, due to their defining property, the Cherednik kernel and hypergeometric functions extend the Heckman-Opdam polynomials in the sense of the subsequent lemma.

Lemma 4.6 ([HS94, Opd95]). *Let $k \geq 0$. Then*

(i) $G_k(\lambda', \cdot)$ is a trigonometric polynomial if and only if $\lambda' = \widetilde{\lambda}$ for some $\lambda \in P$. In this case we have

$$G_k(\widetilde{\lambda}, x) = \frac{E_\lambda(k; x)}{E_\lambda(k; 0)}.$$

(ii) $F_k(\lambda', \cdot)$ is a trigonometric polynomial if and only if $w\lambda' = \widetilde{\lambda} = \lambda + \rho(k)$ for some $\lambda \in P_+, w \in W$. In this case we have

$$F_k(\widetilde{\lambda}, x) = F_k(\lambda + \rho, x) = \frac{P_\lambda(k; x)}{P_\lambda(k; 0)}.$$

In the end of the Section 4.3 we will explain that the hypergeometric function generalizes the spherical functions on certain Riemannian symmetric space of non-compact type.

Remark 4.7. According to [KO08, Theorem 13.15] (see also [HO21, Corollary 8.6.5]) the hypergeometric function extends to an holomorphic function

$$\mathcal{K}_{\text{reg}} \times \mathfrak{a}_\mathbb{C} \times (\mathfrak{a} + i\Omega) \to \mathbb{C}, \quad (k, \lambda, z) \mapsto F_k(\lambda, z),$$

where the tubular neighborhood $\mathfrak{a} + i\Omega \subseteq \mathfrak{a}_\mathbb{C}$ is given by

$$\Omega := \{x \in \mathfrak{a} \mid |\langle \alpha, x\rangle| < \pi \text{ for all } \alpha \in R\}.$$

The proof of [Opd95, Theorem 3.15] shows that the same is true for the Cherednik kernel.

Example 4.8 (Rank one). Consider the rank one situation, namely $\mathfrak{a} = \mathbb{R}$, $R = \mathrm{BC}_1 = \{\pm 1, \pm 2\}$, $R_+ = \{1, 2\}$ and $W = \{\pm \mathrm{id}\}$. Let the multiplicity k be given by the values k_1 on ± 1 and k_2 on ± 2. Since the W-invariant polynomials are generated by the polynomial x^2, consider the eigenvalue equation for the Heckman-Opdam Laplacian $L(k) = D_1(R_+, k)^2$. Namely, we are looking for an even analytic function f satisfying for $\lambda \in \mathbb{C}$

$$L(k)f(x) = f''(x) + (k_1 \coth(\tfrac{x}{2}) + k_2 \coth(x))f'(x) - (\tfrac{k_1+2k_2}{2})^2 f(x) = \lambda^2 f(x).$$

The change of variables $y = \frac{1}{2} - \frac{1}{4}(e^x - e^{-x}) = \frac{1}{2}(1 - \cosh(x)) = -\sinh^2(\frac{x}{2})$ and $F(y) = f(x)$ leads to a hypergeometric system

$$y(1-y)F''(y) + (c - (1 + a + b)y)F'(y) - abF(y) = 0, \tag{4.3}$$

where

$$a = \lambda + \rho(k) = \lambda + \frac{k_1 + 2k_2}{2}, \; b = -\lambda + \rho(k) = -\lambda + \frac{k_1 + 2k_2}{2}, \; c = \frac{1}{2} + k_1 + k_2.$$

The system (4.3) has a unique holomorphic solution near 0, the Gauss-hypergeometric function ${}_2F_1\left({a,b \atop c}; y\right)$. So the hypergeometric function associated with (BC_1, k) is given by

$$F_k(\lambda, z) = {}_2F_1\left({\lambda + \frac{k_1+2k_2}{2}, -\lambda + \frac{k_1+2k_2}{2} \atop \frac{1}{2} + k_1 + k_2}; -\sinh^2(\tfrac{z}{2})\right).$$

In [Opd95, page 90] it is proven that the Cherednik kernel is then given by

$$G_k(\lambda, z) = F_k(\lambda, z) + \frac{\sinh(z)}{k_1 + 2k_2 - 2\lambda} \frac{dF_k}{dz}(\lambda, z).$$

In particular, the symmetric Heckman-Opdam polynomial with parameter $n \in P_+ = \mathbb{N}_0$ is given by

$$P_n(k; z) = F_k(n + \rho(k), z) = {}_2F_1\left({n + k_1 + 2k_2, -n \atop \frac{1}{2} + k_1 + k_2}; -\sinh^2(\tfrac{z}{2})\right)$$

which is the Jacobi polynomial of degree n (up to a change of variables and normalization).

4.2 Trigonometric Dunkl theory for integral root systems

In their original work for trigonometric Dunkl theory, Heckman and Opdam only considered the case where the root system spans the Euclidean space. It is quite natural to omit this property, as this is also related to the concept in Lie theory of passing from a semisimple Lie group to a reductive Lie group. In particular, this extension is important for the upcoming results in this thesis, motivated by radial analysis on symmetric cones which are symmetric spaces induced by a reductive Lie group. We introduce trigonometric Dunkl theory for integral root systems inline with the theory of Heckman and Opdam including results for the corresponding Cherednik kernel and hypergeometric function. Afterwards, we will show how our extension of trigonometric Dunkl theory to integral root systems is connected to radial analysis on Riemannian symmetric spaces associated with reductive Lie groups of the Harish-Chandra class.
Let R be an integral root system inside the Euclidean space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ with Weyl group W. We fix a system of positive roots $R_+ \subseteq R$ and a regular multiplicity k. Put

$$\mathfrak{a} = \mathfrak{s} \oplus \mathfrak{c} \quad \text{with} \quad \mathfrak{s} := \operatorname{span}_{\mathbb{R}} R \quad \text{and} \quad \mathfrak{c} := \mathfrak{s}^{\perp}.$$

Then R is a crystallographic root system inside $\mathfrak{s}$ and W acts trivially on $\mathfrak{c}$. Denote by $\pi_{\mathfrak{s}}$ and $\pi_{\mathfrak{c}}$ the orthogonal projections onto $\mathfrak{s}$ and $\mathfrak{c}$, respectively.

Definition 4.9. As in the crystallographic case, we define the *Cherednik operator* associated with (R_+, k) into direction $\xi \in \mathfrak{a}$ by

$$D_\xi(k) := D_\xi(R_+, k) := \partial_\xi - \langle \rho(k), \xi\rangle + \sum_{\alpha\in R_+} k_\alpha \langle \alpha, \xi\rangle \frac{1-s_\alpha}{1-e^{-\alpha}}$$

with $\rho(k) = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \alpha$ as before. In particular, $D_\xi(k) = \partial_\xi$ for all $\xi \in \mathfrak{c}$.

With the same proofs as in the crystallographic case one can verify the following. The Cherednik operators are continuous operators $C^1(\Omega) \to C^0(\Omega)$ for any W-invariant open $\Omega \subseteq \mathfrak{a}$. Furthermore, the Cherednik operators also act continuously on the spaces $C^\infty(\mathfrak{a})$ and $C_c^\infty(\mathfrak{a})$ satisfying $\operatorname{supp}(D_\xi(k)f) \subseteq W.\operatorname{supp} f$ for all $f \in C^\infty(\mathfrak{a})$. This can be seen by the decomposition

$$\frac{1-s_\alpha}{1-e^{-\alpha}} = \frac{\langle\alpha,\cdot\rangle}{1-e^{-\alpha}} \cdot \frac{1-s_\alpha}{\langle\alpha,\cdot\rangle},$$

where the first factor is a multiplication operator with a smooth function and the latter satisfies the assertions as known from the rational Dunkl setting.

Proposition 4.10. *For all $\xi \in \mathfrak{a}$ and $f, g \in C^1(\mathfrak{a})$ we have:*

(i) $D_\xi(k)(f\circ\pi_{\mathfrak{s}}) = (D_{\pi_{\mathfrak{s}}\xi} f)\circ\pi_{\mathfrak{s}}$.

(ii) $D_\xi(k)(f\circ\pi_{\mathfrak{c}}) = ((\partial_{\pi_{\mathfrak{c}}\xi} - \langle\rho(k),\xi\rangle)f)\circ\pi_{\mathfrak{c}}$.

(iii) *If g is W-invariant,* $D_\xi(k)(f\cdot g) = (D_\xi(k)f)\cdot g + f\cdot\partial_\xi g$

PROOF.

(i) Note that $\pi_{\mathfrak{s}}$ is a W-equivariant orthogonal projection, $R \subseteq \mathfrak{s}$ and $\rho(k) \in \mathfrak{s}$. So we have

$$\begin{aligned} D_\xi(f\circ\pi_{\mathfrak{s}})(x) &= (\partial_\xi - \langle\rho(k),\xi\rangle)(f\circ\pi_{\mathfrak{s}})(x) + \sum_{\alpha\in R_+} k_\alpha\langle\alpha,\xi\rangle \frac{f(\pi_{\mathfrak{s}}x) - f(\pi_{\mathfrak{s}}s_\alpha x)}{1-e^{-\langle\alpha,x\rangle}} \\ &= (\partial_{\pi_{\mathfrak{s}}(\xi)} - \langle\rho(k),\pi_{\mathfrak{s}}(\xi)\rangle)f(\pi_{\mathfrak{s}}x) + \sum_{\alpha\in R_+} k_\alpha\langle\alpha,\pi_{\mathfrak{s}}\xi\rangle \frac{f(\pi_{\mathfrak{s}}x) - f(s_\alpha\pi_{\mathfrak{s}}x)}{1-e^{-\langle\alpha,\pi_{\mathfrak{s}}x\rangle}} \\ &= D_{\pi_{\mathfrak{s}}(\xi)} f(\pi_{\mathfrak{s}}x). \end{aligned}$$

(ii) Since W acts trivially on $\mathfrak{c}$ we have $\pi_\mathfrak{c} s_\alpha = s_\alpha \pi_\mathfrak{c} = \pi_\mathfrak{c}$. Hence, part (ii) is immediate by $\partial_\xi(f \circ \pi_\mathfrak{c}) = \partial_{\pi_\mathfrak{c}(\xi)} f \circ \pi_\mathfrak{c}$ as the reflection part of the Cherednik operator vanishes.

(iii) This is a straightforward computation.

■

Lemma 4.11. *The Cherednik operators commute, i.e. $D_\xi(k)D_\eta(k) = D_\eta(k)D_\xi(k)$ as operators $C^2(\Omega) \to C^0(\Omega)$ for all $\xi, \eta \in \mathfrak{a}$ and W-invariant open $\Omega \subseteq \mathfrak{a}$.*

Proof. For simplicity assume that $\Omega = \mathfrak{a}$. Since both $(\xi, \eta) \mapsto D_\xi D_\eta$ and $(\xi, \eta) \mapsto D_\eta D_\xi$ are bilinear, it suffices to consider the following cases:

(i) For $\xi, \eta \in \mathfrak{c}$ we have $D_\xi(k) = \partial_\xi$ and $D_\eta(k) = \partial_\eta$, so this is obvious.

(ii) Consider $\xi \in \mathfrak{c}, \eta \in \mathfrak{s}$, i.e. $D_\xi(k) = \partial_\xi$. Then, for all $\alpha \in R_+$ we obtain from $\langle \alpha, \xi \rangle = 0$

$$\begin{aligned}\partial_\xi \frac{1 - s_\alpha}{1 - e^{-\alpha}} &= \left(\partial_\xi \frac{1}{1 - e^{-\alpha}}\right)(1 - s_\alpha) + \frac{1}{1 - e^{-\alpha}} \partial_\xi (1 - s_\alpha) \\ &= \frac{\partial_\xi - s_\alpha \partial_{s_\alpha \xi}}{1 - e^{-\alpha}} = \frac{\partial_\xi - s_\alpha \partial_\xi}{1 - e^{-\alpha}} = \frac{1 - s_\alpha}{1 - e^{-\alpha}} \partial_\xi\end{aligned}$$

This leads to $D_\xi(k)D_\eta(k) = D_\eta(k)D_\xi(k)$.

(iii) Consider $\xi, \eta \in \mathfrak{s}$. The Cherednik operators have unique decompositions

$$D_\xi D_\eta = \sum_{w \in W} D_{w,\xi,\eta} w, \quad D_\xi D_\eta = \sum_{w \in W} D_{w,\eta,\xi} w, \tag{4.4}$$

where $D_{w,\xi,\eta}$ and $D_{w,\eta,\xi}$ are unique differential operators on $\mathfrak{a}$ with coefficients in the algebra $\mathcal{R}$ generated by the functions $\frac{1}{1-e^{-\alpha}}$ with $\alpha \in R$. By assumption $\eta, \xi \in \mathfrak{s}$, so these differential operators can be expressed as

$$D_{w,\xi,\eta} = \sum_\alpha c_\alpha^{(w)} \partial_\mathfrak{s}^\alpha, \quad D_{w,\eta,\xi} = \sum_\alpha d_\alpha^{(w)} \partial_\mathfrak{s}^\alpha, \tag{4.5}$$

with $c_i^{(w)}$, $d_i^{(w)} \in \mathcal{R}$, where $\partial_\mathfrak{s}^\alpha$ are partial derivatives on $\mathfrak{s}$.
According to Proposition 4.10 (i) we can restrict $D_\xi(k)D_\eta(k)$ and $D_\eta(k)D_\xi(k)$ to $C^2(\mathfrak{s})$. Since R is crystallographic in $\mathfrak{s}$, we have from the classical trigonometric Dunkl theory that $D_\eta(k)D_\xi(k) = D_\xi(k)D_\eta(k)$ on $C^2(\mathfrak{s})$. From this we conclude that $D_{w,\xi,\eta} = D_{w,\eta,\xi}$ on $C^2(\mathfrak{s})$ for all $w \in W$, i.e. $c_i^{(w)} = d_i^{(w)}$ on $\mathfrak{s}$. Finally, $c_i^{(w)}, d_i^{(w)} \in \mathcal{R}$ are constant along the sets $x + \mathfrak{c}$ for all $x \in \mathfrak{s}$, so they coincide on $\mathfrak{a}$. Hence, by (4.4) and (4.5) $D_\xi(k)D_\eta(k) = D_\eta(k)D_\xi(k)$ globally on $C^2(\mathfrak{a})$.

■

Theorem 4.12. *For $\lambda \in \mathfrak{a}_\mathbb{C}$ the eigenvalue problem*

$$\begin{cases} D_\xi(k)f = \langle \lambda, \xi \rangle f, & \textit{for all } \xi \in \mathfrak{a}, \\ f(0) = 1 \end{cases}$$

has a unique analytic solution $f = G_k(\lambda, \cdot)$. This solution extends to a holomorphic function

$$\mathcal{K}_{\text{reg}} \times \mathfrak{a}_\mathbb{C} \times (\mathfrak{a} + i\Omega) \to \mathbb{C}, \quad (k, \lambda, z) \to G_k(\lambda, z)$$

with $\Omega = \langle x \in \mathfrak{a} \mid |\langle \alpha, x\rangle| < \pi$ for all $\alpha \in R\rangle$. This function will also be called the Cherednik kernel associated with (R_+, k) and it can be expressed as

$$G_k(\lambda, z) = e^{\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}} z\rangle} G_k^{\mathfrak{s}}(\pi_{\mathfrak{s}}\lambda, \pi_{\mathfrak{s}} z), \tag{4.6}$$

where $G_k^{\mathfrak{s}}$ is the classical Cherednik kernel associated with (R_+, k) on $\mathfrak{s}$.

Proof. Using Proposition 4.10, it is straightforward to see that G_k defined by (4.6) solves the eigenvalue equation.
Let f be another solution of the eigenvalue problem in a neighborhood of 0. Then by Proposition 4.10, we have for $\eta \in \mathfrak{s}$

$$D_\eta(f \circ \pi_{\mathfrak{s}}) = D_\eta f \circ \pi_{\mathfrak{s}} = \langle \lambda, \eta\rangle\, f \circ \pi_{\mathfrak{s}} = \langle \pi_{\mathfrak{s}}\lambda, \eta\rangle\, f \circ \pi_{\mathfrak{s}},$$

i.e. $f(\pi_{\mathfrak{s}} x) = G_k^{\mathfrak{s}}(\pi_{\mathfrak{s}}\lambda, \pi_{\mathfrak{s}} x)$ by the uniqueness property of the Cherednik kernel on $\mathfrak{s}$.
Again by Proposition 4.10, we have for $\xi \in \mathfrak{c}$

$$\begin{aligned} D_\xi(e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}(\cdot)\rangle} f) &= (\partial_\xi e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}(\cdot)\rangle}) \cdot f + e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}(\cdot)\rangle} \partial_\xi f \\ &= (\langle \pi_{\mathfrak{c}}\lambda, \xi\rangle - \langle \lambda, \xi\rangle) e^{-\langle \pi_{\mathfrak{c}}(\lambda), \pi_{\mathfrak{c}}(\cdot)\rangle} f \\ &= 0. \end{aligned}$$

Hence, $e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}(\cdot)\rangle} f$ is constant in each point into direction $\mathfrak{c}$. Thus, we obtain

$$e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}} x\rangle} f(x) = e^{-\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}\pi_{\mathfrak{s}} x\rangle} f(\pi_{\mathfrak{s}} x) = G_k^{\mathfrak{s}}(\pi_{\mathfrak{s}}\lambda, \pi_{\mathfrak{s}} x).$$

■

Definition 4.13. We define the hypergeometric function associated with (R, k) as

$$F_k(\lambda, z) = \frac{1}{\# W} \sum_{w \in W} G_k(\lambda, wz).$$

Thus, the hypergeometric function can be expressed as

$$F_k(\lambda, z) = e^{\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}} z\rangle} F_k^{\mathfrak{s}}(\pi_{\mathfrak{s}}\lambda, \pi_{\mathfrak{s}} z),$$

with the classical hypergeometric function $F_k^{\mathfrak{s}}$ associated with (R, k) on $\mathfrak{s}$. Hence, F_k is also W-invariant in λ.

Theorem 4.14. *The function $F_k(\lambda, \cdot)$ is the unique W-invariant analytic solution of*

$$\begin{cases} p(D(k))F_k(\lambda, \cdot) = p(\lambda)F_k(\lambda, \cdot), & \text{for all } p \in \mathbb{C}[\mathfrak{a}]^W, \\ F_k(\lambda, 0) = 1. \end{cases}$$

Proof. This can be shown similarly to the proof for the Cherednik kernel. The only crucial part is that for $\xi \in \mathfrak{c}$ the polynomial $\langle \cdot, \xi\rangle \in \mathbb{C}[\mathfrak{a}]^W$ is W-invariant with $\langle D(k), \xi\rangle = D_\xi(k) = \partial_\xi$.
■

Proposition 4.15. *Let $(D_\xi(R_+, k))_{\xi \in \mathfrak{a}}$ be the Cherednik operators associated with (R_+, k). Put $w_0 \coloneqq w_0(R_+)$ for the longest element of W with respect to the choice of R_+. Furthermore, we denote by $G_k(R_+, \lambda, z)$ the Cherednik kernel associated with (R_+, k). Then:*

(i) $wD_\xi(R_+, k)w^{-1} = D_{w\xi}(wR_+, k)$ for all $w \in W$.

(ii) $D_\xi(R_+,k)f^- = -(D_\xi(-R_+,k)f)^-$, *where* $f^-(x) = f(-x)$.

(iii) $G_k(R_+,\lambda,z) = G_k(wR_+,w\lambda,wz)$ *for all* $w \in W$.
Moreover, the hypergeometric function F_k *does not depend on the choice of* R_+.

(iv) $G_k(R_+,\lambda,-z) = G_k(R_+,-w_0\lambda,w_0z)$.

(v) $F_k(\lambda,-z) = F_k(-\lambda,z)$.

PROOF.

(i) Let $\rho(R_+,k)$ be the Weyl vector related to R_+. Then we have

$$w(\partial_\xi - \langle\rho(R_+,k),\xi\rangle)w^{-1} = \partial_{w\xi} - \langle w\rho(R_+,k),w\xi\rangle = \partial_{w\xi} - \langle\rho(wR_+,k),w\xi\rangle\,.$$

Furthermore,

$$\begin{aligned} w\left(\sum_{\alpha\in R_+} k_\alpha\,\langle\alpha,\xi\rangle\,\frac{1-s_\alpha}{1-e^{-\alpha}}\right)w^{-1} &= \sum_{\alpha\in R_+} k_{w\alpha}\,\langle w\alpha,w\xi\rangle\,\frac{1-s_{w\alpha}}{1-e^{-w\alpha}} \\ &= \sum_{\beta\in wR_+} k_\beta\,\langle\beta,w\xi\rangle\,\frac{1-s_\beta}{1-e^{-\beta}}. \end{aligned}$$

Hence, we see that $wD_\xi(R_+,k)w^{-1} = D_{w\xi}(wR_+,k)$.

(ii) We have $(\partial_\xi - \langle\rho(R_+,k),\xi\rangle)f^- = -(\partial_\xi - \langle\rho(-R_+,k),\xi\rangle\,f)^-$. Thus, $k_\alpha = k_{-\alpha}$ leads to

$$\begin{aligned} \sum_{\alpha\in R_+} k_\alpha\,\langle\alpha,\xi\rangle\,\frac{f^- - s_\alpha f^-}{1-e^{-\alpha}} &= \sum_{\alpha\in R_+} k_\alpha\,\langle\alpha,\xi\rangle\left(\frac{f-s_\alpha f}{1-e^{\alpha}}\right)^- \\ &= -\left(\sum_{\beta\in -R_+} k_\beta\,\langle\beta,\xi\rangle\,\frac{1-s_\beta}{1-e^{-\beta}}f\right)^-. \end{aligned}$$

Therefore, the assertion $D_\xi(R_+,k)f^- = -(D_\xi(-R_+,k)f)^-$ follows.

(iii) This is immediate from the eigenvalue equation defining the Cherednik kernel, because the eigenvalue equation $D_\xi(R_+,k)f = \langle\lambda,\xi\rangle\,f$ is equivalent to

$$D_{w\xi}(wR_+,k)(wf) = wD_\xi(R_+,k)f = \langle\lambda,\xi\rangle\,(wf) = \langle w\lambda,w\xi\rangle\,(wf),$$

i.e. $G_k(R_+,\lambda,w^{-1}z) = G_k(wR_+,w\lambda,z)$. Let $F_k(R_+,\lambda,z) := \sum_{w\in W} G_k(R_+,\lambda,wz)$, so that we have for all $w \in W$

$$F_k(R_+,\lambda,z) = F_k(wR_+,w\lambda,z).$$

Since W acts transitively on the set of positive subsystems $R_+ \subseteq R$ and F_k is W-invariant in the λ-variable, we obtain that F_k does not depend on the choice of positive roots.

(iv) From part (ii) we observe that

$$D_\xi(-R_+,k)G_k(R_+,\lambda,\cdot)^- = -\langle\lambda,\xi\rangle\,G_k(R_+,\lambda,\cdot)^-$$

which means $G_k(R_+,\lambda,-z) = G_k(-R_+,-\lambda,z)$ by the defining eigenvalue equation of the Cherednik kernel. The longest element $w_0 \in W$ is characterized by $w_0R_+ = -R_+$, so part (iii) leads to

$$G_k(R_+,\lambda,-z) = G_k(-R_+,-\lambda,z) = G_k(w_0R_+,-\lambda,z) = G_k(R_+,-w_0\lambda,w_0z).$$

(v) This is immediate from the invariance properties of F_k.

■

Proposition 4.16. *For $k \geq 0$, $\lambda \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$ the Cherednik kernel satisfies*

$$0 \leq |G_k(\lambda, x)| \leq G_k(\operatorname{Re}\lambda, x) \leq \sqrt{\#W} e^{\max_{w\in W}\langle \operatorname{Re}\lambda, wx\rangle}.$$

The same is true if G_k is replaced by F_k.

Proof. Owing to [Opd95, Proposition 6.1] and [Sch08] the kernel $G_k^{\mathfrak{s}}$ satisfies

$$0 \leq |G_k^{\mathfrak{s}}(\pi_{\mathfrak{s}}\lambda, \pi_{\mathfrak{s}}x)| = G_k^{\mathfrak{s}}(\operatorname{Re}(\pi_{\mathfrak{s}}\lambda), \pi_{\mathfrak{s}}x) \leq \sqrt{\#W} e^{\max_{w\in W}\langle \operatorname{Re}(\pi_{\mathfrak{s}}\lambda), w\pi_{\mathfrak{s}}x\rangle}. \tag{4.7}$$

Since W acts trivially on $\mathfrak{c}$

$$\left| e^{\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}x\rangle} \right| = e^{\langle \operatorname{Re}(\pi_{\mathfrak{c}}\lambda), \pi_{\mathfrak{c}}x\rangle} = e^{\max_{w\in W}\langle \operatorname{Re}(\pi_{\mathfrak{c}}\lambda), w\pi_{\mathfrak{c}}x\rangle}. \tag{4.8}$$

Therefore, the assertion follows from (4.7), (4.8) and the definition of G_k in (4.6). ■

The following estimate is a generalization of an estimate of the Cherednik kernel as stated in [RKV13, Theorem 3.3].

Theorem 4.17. *Assume that $k \geq 0$. Then for all $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}$ and $x \in \mathfrak{a}$:*

$$|G_k(\lambda + \mu, x)| \leq G_k(\operatorname{Re}\mu, x) \cdot e^{\max_{w\in W}\langle \operatorname{Re}\lambda, wx\rangle}.$$

Moreover, since $G_k(-\rho(k), \cdot) \equiv 1$, one has for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$, $x \in \mathfrak{a}$

$$|G_k(\lambda - \rho(k), x)| \leq e^{\max_{w\in W}\langle \operatorname{Re}\lambda, wx\rangle}.$$

The same is true if G_k is replaced by F_k.

Proof. An inspection of the proof in [RKV13] shows that it can be carried out in the exact same way with $\overline{\mathfrak{a}_+}$ replaced by an arbitrary closed Weyl chamber C. Let $C \subseteq \mathfrak{a}$ be such a closed Weyl chamber and put

$$S := \{\lambda \in \mathfrak{a}_{\mathbb{C}} \mid \operatorname{Re}\lambda \in C\}.$$

For $x \in \mathfrak{a}$ let x_C be the unique element in $C \cap W.x$. Then $\langle w\operatorname{Re}\lambda, x\rangle \leq \langle \operatorname{Re}\lambda, x_C\rangle$ holds for all $w \in W$ and $\lambda \in S$. For $w \in W, x \in \mathfrak{a}$ and $\mu \in C$ we define

$$f(\lambda) := e^{-\langle \lambda, x_C\rangle} \frac{G_k(w\lambda + \mu, x)}{G_k(\mu, x)},$$

so that $f : \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ is holomorphic. According to Proposition 4.16 the following estimate holds

$$|G_k(\lambda, x)| \leq G_k(\operatorname{Re}\lambda, x) \leq \sqrt{\#W} e^{\max_{w\in W}\langle \operatorname{Re}\lambda, wx\rangle}.$$

Hence, we can conclude that

$$|f(\lambda)| \leq \sqrt{\#W} \frac{e^{\langle \mu, x_C\rangle}}{G_k(\mu, x)} \quad \text{for all } \lambda \in S$$
$$|f(i\lambda)| \leq 1 \quad \text{for all } \lambda \in \mathfrak{a}.$$

Proceed as in [RKV13] to conclude with the Phragmén-Lindelöf principle that $|f(\lambda)| \leq 1$ for all $\lambda \in S$. Therefore, we have for all $w \in W, \mu \in C$ and $\lambda \in S$

$$|G_k(w\lambda + \mu, x)| \leq G_k(\mu, x) \cdot e^{\langle \mathrm{Re}\,\lambda, x_C \rangle} = G_k(\mu, x) \cdot e^{\max_{w \in W} \langle w \mathrm{Re}\,\lambda, x \rangle}.$$

Since the Weyl chamber C was arbitrary, the assertion follows for $\lambda \in \mathfrak{a}_\mathbb{C}$ and $\mu \in \mathfrak{a}$. Finally, the inequality $|G_k(\mu + \lambda, x)| \leq G_k(\mathrm{Re}\,\mu + \mathrm{Re}\,\lambda, x)$ for all $\mu, \lambda \in \mathfrak{a}_\mathbb{C}$ finishes the proof. ■

Theorem 4.18. *Assume that $k \geq 0$. Then for any two polynomials $p, q \in \mathcal{P}$ there exists a constant $C > 0$ such that for all $\lambda \in \mathfrak{a}_\mathbb{C}$ and $x \in \mathfrak{a}$*

$$\left| p(\tfrac{\partial}{\partial \lambda}) q(\tfrac{\partial}{\partial x}) G_k(\lambda, x) \right| \leq C(1 + |x|)^{\deg(p)} (1 + |\lambda|)^{\deg(q)} F_k(0, x) e^{\max_{w \in W} \langle w \mathrm{Re}(\lambda), x \rangle}.$$

The same is true for the hypergeometric function F_k.

Proof. This can be proven as Proposition 4.16: Decompose the Cherednik kernel into a product of an exponential and a Cherednik kernel of a crystallographic root system. After that, use the results of [Sch08, Theorem 3.4], where the claimed inequality is proven for the crystallographic Cherednik kernel. ■

Remark 4.19 (Product situation)**.** In the cases where R is not irreducible, the Cherednik kernel and hypergeometric function decompose as follows. Assume that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is an orthogonal sum and $R = R_1 \sqcup R_2$ with integral root systems $R_i \subseteq \mathfrak{a}_i$, and positive roots $R_{i,+} \subseteq R_i$ and $R_+ = R_{1,+} \sqcup R_{2,+}$. Moreover, choose a regular multiplicity $k \geq 0$ on R and consider the restrictions $k_1 = k|_{R_1}$ and $k_2 = k|_{R_2}$. Then we have

$$\begin{aligned} G_k(R_+, \lambda_1 + \lambda_2, x_1 + x_2) &= G_{k_1}(R_{1,+}, \lambda_1, x_1) G_{k_2}(R_{2,+}, \lambda_2, x_2), \\ F_k(R, \lambda_1 + \lambda_2, x_1 + x_2) &= F_{k_1}(R_1, \lambda_1, x_1) F_{k_2}(R_2, \lambda_2, x_2), \end{aligned}$$

for all $\lambda_i \in (\mathfrak{a}_i)_\mathbb{C}$ and $x_i \in \mathfrak{a}_i$.

4.3 Riemannian symmetric spaces of non-compact type

Consider a Riemannian symmetric space $X = G/K$ of non-compact type, where G is a semisimple connected Lie group with finite center and $K \leq G$ a maximal compact subgroup. Then the following connection is well known, cf. [Hec97, Remark 2.3] or [HS94]:
Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Consider an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the associated restricted roots $\Sigma = \Delta(\mathfrak{g}, \mathfrak{a})$ and a positive system of roots Σ_+. Then define the rescaled root system $R = 2\Sigma$ with multiplicity function $k_{2\alpha} = \frac{\dim \mathfrak{g}_\alpha}{2}$, where $\mathfrak{g}_\alpha$ is the root space associated with $\alpha \in \Sigma$. For simplicity we consider the roots as a subset of $\mathfrak{a}$ and not as a subspace of the dual $\mathfrak{a}^*$.

Remark 4.20 (Invariant differential operators and spherical functions)**.** Let $\mathbb{D}(G/K)$ be the algebra of G-invariant differential operators on G/K and consider the Harish-Chandra isomorphism, cf. [GV88, Theorem 2.6.7],

$$\gamma : \mathbb{D}(G/K) \to \mathbb{C}[\mathfrak{a}]^W.$$

Consider for $D \in \mathbb{D}(G/K)$ the K-radial part of D which is defined as the unique differential operator $\mathrm{rad}(D)$ on $\mathfrak{a}_{\mathrm{reg}}$ such that for all K-biinvariant $f \in C^\infty(K \backslash G/K) \cong C^\infty(\mathfrak{a})^W$

$$Df = \mathrm{rad}(D) f.$$

In [HS94, Hec97] the authors prove the following:

$$\{\mathrm{rad}(D) \mid D \in \mathbb{D}(G/K)\} = \{\mathrm{res}(p(D(R_+, k))) \mid p \in \mathbb{C}[\mathfrak{a}]^W\},$$

i.e. the restrictions of W-invariant Cherednik operators (see (4.1)) are precisely the K-radial parts of invariant differential operators of G/K. To be more precise, for $p \in \mathbb{C}[\mathfrak{a}]^W$ we have

$$\mathrm{rad}(\gamma^{-1}(p)) = \mathrm{res}(p(D(R_+, k))). \tag{4.9}$$

For instance, by [Hel84, Chapter II, Section 5], the radial part of the Laplace-Beltrami Δ on G/K is given by

$$\mathrm{rad}(\Delta) = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma_+} \dim(\mathfrak{g}_\alpha) \coth(\langle \alpha, \cdot \rangle) \partial_\alpha$$

and its image under the Harish-Chandra isomorphism is

$$\gamma(\Delta)(x) = \langle x, x \rangle - \langle \rho, \rho \rangle, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \dim(\mathfrak{g}_\alpha)\alpha.$$

In fact, one can verify that the restriction of the Cherednik operator associated with $\gamma(\Delta)$ is given by

$$\mathrm{res}(\gamma(\Delta)(D(R_+, k))) = \Delta_{\mathfrak{a}} + \sum_{\alpha \in R_+} k_\alpha \coth\left(\tfrac{\langle \alpha, \cdot \rangle}{2}\right) \partial_\alpha.$$

The spherical functions φ_λ of the Gelfand pair (G, K) are parametrized by $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and can be described as the unique K-biinvariant smooth functions such that

$$\begin{cases} D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda, & \text{for all } D \in \mathbb{D}(G/K), \\ \varphi_\lambda(e) = 1. \end{cases}$$

By (4.9), the spherical functions of (G, K), considered as functions in $C^\infty(\mathfrak{a})^W \cong C^\infty(K\backslash G/K)$, are given by the hypergeometric function $F_k(R; \cdot, \cdot)$ associated with (R, k) via

$$\varphi_\lambda(x) = F_k(R; \lambda, x). \tag{4.10}$$

4.4 Riemannian symmetric spaces of reductive groups

The aim of this section to verify that the identification (4.10) of hypergeometric functions and spherical functions on a Riemannian symmetric space G/K of non-compact type is still true if G is replaced by a suitable reductive Lie group, i.e. a group of the Harish-Chandra class, and F_k is the hypergeometric function for integral root systems as introduced in the Section 4.2 before. This extension will be important in what follows, because later in this thesis we consider the case where $X = G/K$ is a symmetric cone. We further identify the spherical functions of the associated Riemannian symmetric space G_0/K of Euclidean type, where G_0 is the Cartan motion group of G, as Dunkl type Bessel functions in line with Section 1.3. For the background on Lie groups of the Harish-Chandra class, their structure theory, and their spherical functions we refer the reader to [GV88].

Definition 4.21. A (real) Lie group G lies in the Harish-Chandra class (write $G \in \mathcal{H}$) if it satisfies the following four conditions:

(i) G has a reductive Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{g}$ decomposes as a direct Lie algebra sum

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{s},$$

with $\mathfrak{c}$ abelian and $\mathfrak{s}$ semisimple. In particular, one can conclude that $\mathfrak{c} = \mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is the derived Lie algebra.

(ii) G has finitely many connected components.

(iii) $\mathrm{Ad}(G) \subseteq \mathrm{Ad}_{\mathbb{C}}(G_{\mathbb{C}})$, where $G_{\mathbb{C}} = \langle \exp \mathfrak{g}_{\mathbb{C}} \rangle_{\mathrm{group}}$, where Ad and $\mathrm{Ad}_{\mathbb{C}}$ are the adjoint representations of G and $G_{\mathbb{C}}$ respectively.

(iv) $S := \langle \exp \mathfrak{s} \rangle_{\mathrm{group}} \subseteq G$ has finite center.

Proposition 4.22.

(i) *If G is a connected Lie group, then $G \in \mathcal{H}$ if and only if G satisfy the properties (i)+(iv).*

(ii) *If G is a connected semisimple Lie group, then $G \in \mathcal{H}$ if and only if G has finite center.*

(iii) $G, \widetilde{G} \in \mathcal{H} \Rightarrow G \times \widetilde{G} \in \mathcal{H}$.

PROOF. We prove part (i), since (ii) is an immediate consequence. Since both G and $G_{\mathbb{C}}$ are connected if G is connected, we have

$$\mathrm{Ad}(G) = \langle e^{\mathrm{ad}\mathfrak{g}} \rangle_{\mathrm{group}}, \quad \mathrm{Ad}_{\mathbb{C}}(G_{\mathbb{C}}) = \langle e^{\mathrm{ad}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}}} \rangle_{\mathrm{group}}$$

for $\mathrm{ad}(X)(Y) = [X, Y]$. Since $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and $\mathrm{ad}_{\mathbb{C}}|_{\mathfrak{g}} = \mathrm{ad}$ hold, part (i) is proven. Part (iii) is obvious, because for a direct product we have $(G \times \widetilde{G})_{\mathbb{C}} = G_{\mathbb{C}} \times \widetilde{G}_{\mathbb{C}}$ and $\mathfrak{g} \oplus \tilde{\mathfrak{g}}$ is the Lie algebra of $G \times \widetilde{G}$. ■

The following theorem is about symmetric cones and is one of the reasons why we take a more general look on the Harish-Chandra class. For a background on symmetric cones see for instance [FK94].

Lemma 4.23. *Let Ω be a symmetric cone within an Euclidean Jordan algebra V with unit $e \in \Omega$. In particular, $\Omega = G/K$, where G is the connected component of* id *inside the automorphism group of Ω and K is the (connected) stabilizer of e in G. Then, $K \subseteq G$ is maximal compact and $G \in \mathcal{H}$ lies in the Harish-Chandra class.*

PROOF. Without loss of generality, we may assume that Ω is irreducible by Proposition 4.22. Otherwise we can decompose the Jordan algebra and the symmetric cone via $V = V_1 \oplus V_2$, $\Omega = \Omega_1 + \Omega_2$ which leads to

$$\mathrm{Aut}(\Omega) \cong \mathrm{Aut}(\Omega_1) \times \mathrm{Aut}(\Omega_2),$$

i.e. $G = G_1 \times G_2$. If Ω is irreducible, it is known from the classification of irreducible symmetric cones that Ω is one of the following symmetric cones

V	Ω	$\mathfrak{g}$	$\mathfrak{k}$
$\mathrm{Sym}_n(\mathbb{R})$	$\mathrm{Pos}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{so}_n(\mathbb{R})$
$\mathrm{Herm}_n(\mathbb{C})$	$\mathrm{Pos}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}_n(\mathbb{C})$
$\mathrm{Herm}_n(\mathbb{H})$	$\mathrm{Pos}_n(\mathbb{H})$	$\mathfrak{sl}_n(\mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{su}_n(\mathbb{H})$
$\mathrm{Herm}_3(\mathbb{O})$	$\mathrm{Pos}_3(\mathbb{O})$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{f}_4$
$\mathbb{R} \times \mathbb{R}^{n-1}$	Lor_n	$\mathfrak{o}(1, n-1) \oplus \mathbb{R}$	$\mathfrak{o}(n-1)$

A case by case observation shows that in all cases we have $G \in \mathcal{H}$. ■

Let $G \in \mathcal{H}$ be a connected Lie group with exponential map exp. The Lie algebra decomposes into $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{s}$, where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. By connectedness, G decomposes into

$$G = C \cdot S \quad \text{with} \quad C = \exp \mathfrak{c} \quad \text{and} \quad S = \langle \exp \mathfrak{s} \rangle_{\mathrm{group}}.$$

Let $K \subseteq G$ be a maximal compact subgroup which is connected because of the connectedness of G. From this, one has that $K \cap S \subseteq S$ is a (connected) maximal compact subgroup. In particular, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{s} = (\mathfrak{k} \cap \mathfrak{s}) \oplus (\mathfrak{p} \cap \mathfrak{s})$ is a Cartan decomposition of $\mathfrak{s}$. Moreover, there exists a non-degenerate bilinear form B on $\mathfrak{g}_{\mathbb{C}}$ such that:

(i) B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

(ii) $\mathfrak{k} \perp^B \mathfrak{p}$.

(iii) B is invariant under $\mathrm{Ad}(G)$, $\mathrm{ad}\,\mathfrak{g}$ and the associated Cartan involution θ.

(iv) B is real on $\mathfrak{g} \times \mathfrak{g}$.

(v) $\mathfrak{c} \perp^B \mathfrak{s}$.

Then

$$\langle X, Y \rangle := -B(X, \theta Y) \tag{4.11}$$

defines an inner product on $\mathfrak{g}$ invariant under $\mathrm{Ad}(K)$, θ and with the same orthgonality properties as for B.
The construction of B is the following: on $\mathfrak{s} \times \mathfrak{s}$ it is given by the Cartan-Killing form and on $\mathfrak{c}$ one could choose any non-degenerate form which is negative definite on $\mathfrak{k} \cap \mathfrak{c}$ and positive definite on $\mathfrak{p} \cap \mathfrak{c}$.

Proposition 4.24. *Choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and the canonical maximal abelian subspace $(\mathfrak{a} \cap \mathfrak{s}) \subseteq (\mathfrak{p} \cap \mathfrak{s})$. Let $\Sigma \subseteq \mathfrak{a} \cap \mathfrak{s}$ be the roots of $\mathfrak{s}$ with respect to $\mathfrak{a} \cap \mathfrak{s}$ and let $\mathfrak{s} = \mathfrak{s}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{s}_\alpha$ be the root space decomposition, i.e.*

$$\mathfrak{s}_\alpha = \{X \in \mathfrak{s} \mid \langle \alpha, H \rangle X = [H, X] \text{ for all } H \in \mathfrak{a} \cap \mathfrak{s}\}.$$

Then we have:

(i) *The roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ are precisely Σ.*

(ii) *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ is the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, then*

$$\mathfrak{g}_\alpha = \begin{cases} \mathfrak{s}_\alpha, & \alpha \in \Sigma, \\ \mathfrak{c} \oplus \mathfrak{s}_0, & \alpha = 0. \end{cases}.$$

Proof. We first observe that for $X = X_1 + X_2 \in \mathfrak{g}$ with $X_1 \in \mathfrak{c}$ and $X_2 \in \mathfrak{s}$ we have for $H = H_1 + H_2 \in \mathfrak{a}$ with $H_1 \in (\mathfrak{a} \cap \mathfrak{c})$ and $H_2 \in (\mathfrak{a} \cap \mathfrak{s})$ that $[H, X] = [H_2, X_2]$. From this we deduce that

$$\mathfrak{g}_0 = \mathfrak{c} \oplus \mathfrak{s}_0. \tag{4.12}$$

Assume that $\alpha \in \Sigma \subseteq \mathfrak{a} \cap \mathfrak{s}$. Let $X \in \mathfrak{s}_\alpha$, i.e. $\langle \alpha, H \rangle X = [H, X]$ for all $H \in \mathfrak{a} \cap \mathfrak{s}$. Since $\mathfrak{c}$ is the center of $\mathfrak{g}$ and orthogonal to $\mathfrak{s}$, we observe that indeed for all $H \in \mathfrak{a}$

$$\langle \alpha, H \rangle X = [H, X] \quad \text{for all } H \in \mathfrak{a},$$

which means $X \in \mathfrak{g}_\alpha$. Therefore we have proven

$$\mathfrak{s}_\alpha \subseteq \mathfrak{g}_\alpha, \quad \alpha \in \Sigma. \tag{4.13}$$

Let $\Sigma' \subseteq \mathfrak{a}$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, so (4.13) implies that $\Sigma \subseteq \Sigma'$. Finally, from the root space decomposition

$$\mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma'} \mathfrak{g}_\alpha = \mathfrak{g} = \mathfrak{c} \oplus \mathfrak{s} = (c \oplus \mathfrak{s}_0) \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{s}_\alpha$$

we obtain with (4.12) and (4.13) that $\Sigma = \Sigma'$ and $\mathfrak{g}_\alpha = \mathfrak{s}_\alpha$ for $\alpha \in \Sigma$. ■

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$ be Iwasawa decompositions. To be more precise, we choose a system of positive roots $\Sigma_+ \subseteq \Sigma$ and put $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$, $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. From Proposition 4.24 we observe that in this case

$$\mathfrak{s} = (\mathfrak{k} \cap \mathfrak{s}) \oplus (\mathfrak{a} \cap \mathfrak{s}) \oplus \mathfrak{n} \quad \text{and} \quad S = (K \cap S)(A \cap S)N$$

are Iwasawa decompositions for $\mathfrak{s}$ and S, respectively. We denote by $H^G : G \to \mathfrak{a}$ the Iwasawa projection defined in terms of the Iwasawa decomposition by $H^G(kan) = \log a$, where log is the inverse of the diffeomorphism $\exp : \mathfrak{a} \to A$. For S we define in the same manner the Iwasawa projection $H^S : S \to (\mathfrak{a} \cap \mathfrak{s})$. Moreover, let W be the Weyl group associated with the roots Σ. From $\mathfrak{s} \perp \mathfrak{c}$ and $\Sigma \subseteq \mathfrak{a} \cap \mathfrak{s}$, we obtain that W acts trivially on $\mathfrak{a} \cap \mathfrak{c}$.
With our choice of $\langle \cdot, \cdot \rangle$ on $\mathfrak{c}$ it is well known that the spherical functions of the Gelfand pair (G, K) are given by

$$\varphi_\lambda^G(g) = \int_K e^{-\langle \lambda + \rho, H^G(g^{-1}k) \rangle} \, \mathrm{d}k, \quad \lambda \in \mathfrak{a}_\mathbb{C}, \ g \in G \tag{4.14}$$

with $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \dim(\mathfrak{g}_\alpha)\alpha$ and $\varphi_\lambda = \psi_\mu$ if and only if $\lambda \in W.\mu$, cf. [GV88, Proposition 3.2.1]. Moreover, φ_λ is the unique K-biinvariant smooth function on G with

$$D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda, \quad \varphi_\lambda(e) = 1$$

for all G-invariant differential operators $D \in \mathbb{D}(G/K)$ and the Harish-Chandra isomorphism $\gamma : \mathbb{D}(G/K) \to \mathbb{C}[\mathfrak{a}]^W$. The spherical functions of the Gelfand pair $(S, K \cap S)$ are similarly given by

$$\varphi_\lambda^S(s) = \int_{K \cap S} e^{-\langle \lambda + \rho, H^S(s^{-1}k) \rangle} \, \mathrm{d}k, \quad \lambda \in (\mathfrak{a} \cap \mathfrak{s})_\mathbb{C}, \ s \in G.$$

Lemma 4.25. *Let $\pi_\mathfrak{s}$ and $\pi_\mathfrak{c}$ be the orthogonal projections of $\mathfrak{a}$ onto $\mathfrak{s}$ and $\mathfrak{c}$, respectively. We extend the projections canonically to the complexifications of the spaces. Then the spherical functions of (G, K) and $(S, K \cap S)$, considered as W-invariant functions on $\mathfrak{a}$ and $\mathfrak{a} \cap \mathfrak{s}$, respectively, are related by*

$$\varphi_\lambda^G(x) = e^{\langle \pi_\mathfrak{c}\lambda, \pi_\mathfrak{c}x \rangle} \varphi_{\pi_\mathfrak{s}\lambda}^S(\pi_\mathfrak{s}x)$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}$ and $x \in \mathfrak{a}$.

Proof. For $x = x_1 + x_2 \in \mathfrak{a}$ with $x_1 \in \mathfrak{a} \cap \mathfrak{c}$ and $x_2 \in \mathfrak{a} \cap \mathfrak{s}$ we have

$$\exp x = \exp(x_1)\exp(x_2) = a_1 a_2$$

with $a_1 = \exp x_1 \in A \cap C$ and $a_2 = \exp x_2 \in A \cap S$. The assertion holds by the following:

(i) From $a_1 \in A \cap C$ we obtain $H^G(a^{-1}k) = \log a_1^{-1} + H^G(a_2^{-1}k) = H^G(a_2^{-1}) - x_1$. Thus, with $\rho \in \mathfrak{s} \perp \mathfrak{c}$ we have

$$\varphi_\lambda^G(x) = e^{\langle \lambda + \rho, x_1 \rangle} \varphi_\lambda^G(x_2) = e^{\langle \pi_\mathfrak{c}\lambda, x_1 \rangle} \varphi_\lambda^G(x_2) = e^{\langle \pi_\mathfrak{c}\lambda, \pi_\mathfrak{c}x \rangle} \varphi_\lambda^G(x_2).$$

(ii) Since $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{c}) \oplus (\mathfrak{k} \cap \mathfrak{s})$, the connectedness of K shows that $K = (K \cap C) \cdot (K \cap S)$. Hence, the quotient space $K/(K \cap S)$ can be identified with a subset of $K \cap C$ and by Weyl's integration formula there exists a unique probability measure $\mathrm{d}c$ on $K/(K \cap S)$ such that for all $a \in A \cap S$

$$\int_K e^{-\langle \lambda + \rho, H^G(a^{-1}k) \rangle} \, \mathrm{d}k = \int_{K/(K \cap S)} \int_{K \cap S} e^{-\langle \lambda + \rho, H^G(a^{-1}cs) \rangle} \, \mathrm{d}s \, \mathrm{d}c. \tag{4.15}$$

But for $c \in K \cap C$, $a \in A \cap S$ and $s \in K \cap S$ we observe that $H^G(a^{-1}cs) = H^G(x^{-1}s) = H^S(x^{-1}s)$. Therefore, by equation (4.15) and $\rho \in \mathfrak{s} \perp \mathfrak{c}$ we have for $x = \log a \in \mathfrak{a} \cap \mathfrak{s}$

$$\varphi_\lambda^G(x) = \int_{K\cap S} e^{-\langle \lambda+\rho, H^S(a^{-1}k)\rangle}\, \mathrm{d}s = \int_{K\cap S} e^{-\langle \pi_{\mathfrak{s}}\lambda+\rho, H^S(a^{-1}k)\rangle}\, \mathrm{d}s = \varphi^S_{\pi_{\mathfrak{s}}\lambda}(x).$$

■

The following theorem shows that the generalized hypergeometric function for integral root systems, as introduced in Definition (4.13), generalizes the spherical functions of Riemannian symmetric spaces related to Gelfand pairs of reductive Lie groups from the Harish-Chandra class.

Theorem 4.26. *The spherical functions of (G, K) are related to the hypergeometric function $F_k(R; \cdot, \cdot)$ from Definition 4.13 associated with $R = 2\Sigma \subseteq \mathfrak{a}$ and $k_{2\alpha} = \frac{\dim \mathfrak{g}_\alpha}{2} = \frac{\dim \mathfrak{s}_\alpha}{2}$ by the following formula for all $\lambda \in \mathfrak{a}_\mathbb{C}$, $x \in \mathfrak{a}$*

$$\varphi_\lambda^G(x) = F_k(R; \lambda, x).$$

Proof. By the relation between the spherical functions of Riemannian symmetric spaces of non-compact type and the hypergeometric functions of crystallographic root systems from Remark 4.20, the assertion is an immediate consequence of Lemma 4.25 and the Definition 4.13 of the hypergeometric function. ■

Theorem 4.27. *Let $G_0 := K \ltimes \mathfrak{p}$ and $S_0 := (K \cap S) \ltimes (\mathfrak{p} \cap \mathfrak{s})$ be the Cartan motion groups associated with (G, K) and $(S, K \cap S)$, where K and $K \cap S$ act on $\mathfrak{p}$ and $(\mathfrak{p} \cap \mathfrak{s})$ by the adjoint representation, respectively. Then:*

(i) *The spherical functions of $(S_0, K \cap S)$, considered as W-invariant functions on $\mathfrak{a} \cap \mathfrak{s}$, are*
$$\psi_\lambda^{S_0}(x) = \int_{K\cap S} e^{\langle \lambda, kx\rangle}\, \mathrm{d}k, \quad \lambda \in (\mathfrak{a} \cap \mathfrak{s})_\mathbb{C},$$
such that for any K-invariant differential operator $p(\partial)$ on $\mathfrak{p} \cap \mathfrak{s}$ with $p \in \mathbb{C}[\mathfrak{p} \cap \mathfrak{s}]^{K\cap S}$ we have $p(\partial)\psi_\lambda^{S_0} = p(\lambda)\psi_\lambda^{S_0}$.

(ii) *The spherical functions of (G_0, K), considered as W-invariant function on $\mathfrak{a}$, are*
$$\psi_\lambda^{G_0}(x) = \int_K e^{\langle \lambda, kx\rangle}\, \mathrm{d}k. \quad \lambda \in \mathfrak{a}_\mathbb{C}$$
with $p(\partial)\psi_\lambda^{G_0} = p(\lambda)\psi_\lambda^{G_0}$ for all $p \in \mathbb{C}[\mathfrak{p}]^K$. Furthermore, if $x \in \mathfrak{a}$ and $\lambda \in \mathfrak{a}_\mathbb{C}$ we have
$$\psi_\lambda^{G_0}(x) = e^{\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}x\rangle}\psi^{S_0}_{\pi_{\mathfrak{s}}\lambda}(\pi_{\mathfrak{s}}x).$$

(iii) *Let (R, k) be as in Theorem 4.26 and $J_k(R; \cdot, \cdot)$ the associated Bessel function. Then,*
$$\psi_\lambda^{G_0}(x) = J_k(R; \lambda, x).$$
for all $\lambda \in \mathfrak{a}_\mathbb{C}$ and $x \in \mathfrak{a}$.

Proof.

(i) This can be found in [Hel84, Proposition 4.8].

(ii) The integral representation of the spherical functions has the same proof as in the case where G is semisimple. Consider $x = x_c + x_s$ with $x_c \in \mathfrak{a} \cap \mathfrak{c}$ and $x_s \in \mathfrak{a} \cap \mathfrak{s}$. Then, as for all $k \in K$ we have $kx_c = x_c$ and $\mathfrak{s} \perp \mathfrak{c}$, we observe

$$\psi_\lambda^{G_0}(x) = e^{\langle \lambda, x_c\rangle}\psi_\lambda^{G_0}(x_s) = e^{\langle \pi_{\mathfrak{c}}\lambda, \pi_{\mathfrak{c}}x\rangle}\psi_\lambda^{G_0}(\pi_{\mathfrak{s}}x).$$

Finally, for $x \in \mathfrak{a} \cap \mathfrak{s}$ and $c \in K/(K \cap S) \subseteq K \cap C$ we have $cx = x$, as C is contained in the kernel of the adjoint representation. Thus, as in the proof of Theorem 4.10 we obtain with $\mathfrak{s} \perp \mathfrak{c}$

$$\psi_\lambda^{G_0}(x) = \int_{K/(K\cap S)}\int_{K\cap S} e^{\langle \lambda, kcx\rangle}\,\mathrm{d}k\,\mathrm{d}c = \int_{K\cap S} e^{\langle \pi_{\mathfrak{s}}\lambda, kx\rangle}\,\mathrm{d}k = \psi_{\pi_{\mathfrak{s}}\lambda}^{S_0}(x).$$

(iii) This is obtained from part (ii), the product decomposition of the Bessel function in Remark 1.17 and Section 1.3.

■

Remark 4.28. Lemma 4.25, Theorem 4.10 and Theorem 4.27 are still true if $G \in \mathcal{H}$ is not connected. As a maximal compact subgroup $K \subseteq G$ meets every connected component of G, cf. [GV88, Proposition 2.1.7], the spherical functions of (G, K) and (G_e, K_e) (the connected components of the unit e) can be identified.

4.5 Generalization of the Helgason-Johnson theorem

The classical Helgason-Johnson theorem states the following, cf. [HJ69]. Let $G \in \mathcal{H}$ be a Lie group of the Harish-Chandra class and $K \subseteq G$ a maximal compact subgroup. Then the spherical function φ_λ from (4.14) is bounded if and only if $\lambda \in \mathrm{co}(W.\rho) + i\mathfrak{a}$. In [NPP14, Theorem 4.2] the authors extend the Helgason-Johnson theorem to the hypergeometric functions F_k associated with a crystallographic root system R inside a Euclidean space $\mathfrak{a}$ with multiplicity function $k \geq 0$. To become more precise, they prove that $F_k(\lambda, \cdot)$ is a bounded function on $\mathfrak{a}$ if and only if $\lambda \in \mathrm{co}(W.\rho(k)) + i\mathfrak{a}$. In this section we will generalize the results to the Cherednik kernel of an integral root system in an obvious manner. This will have as a consequence a Riemann-Lebesgue lemma for the Cherednik transform.
To do so, we fix a Euclidean space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ with an integral root system $R \subseteq \mathfrak{a}$, positive roots $R_+ \subseteq R$, Weyl group $W = W(R)$ and a non-negative mulitplicity $k \geq 0$ on R.

Lemma 4.29. *Let R be irreducible and crystallographic. Choose inside R_+ simple roots $\alpha_1, \ldots, \alpha_n \in R_+$ and put $s_1, \ldots, s_n$ for the associated reflections. Put $k_i = k_{\alpha_i} + 2k_{2\alpha_i}$ with $k_{2\alpha} = 0$ in the case that $\alpha \notin R$. Let β be the unique highest short root. Then the Cherednik kernel associated with (R_+, k) satisfies:*

(i) *For all $\lambda \in \mathfrak{a}_{\mathbb{C}}$ with $\langle \beta^\vee, \lambda\rangle \neq 1$, i.e. $s_\beta\lambda \neq \lambda - \beta$, we have*

$$\left(1 + \frac{k_\beta}{1 - \langle \beta^\vee, \lambda\rangle}\right) G_k(\beta + s_\beta\lambda, \cdot) = \left(e^{\langle \beta, \cdot\rangle}s_\beta + \frac{k_\beta}{1 - \langle \beta^\vee, \lambda\rangle}\right) G_k(\lambda, \cdot).$$

(ii) *For all $\lambda \in \mathfrak{a}_{\mathbb{C}}$ with $\langle \alpha_i^\vee, \lambda\rangle \neq 0$, i.e. $s_i\lambda \neq \lambda$, we have*

$$\left(1 + \frac{k_i}{\langle \alpha_i^\vee, \lambda\rangle}\right) G_k(s_i\lambda, \cdot) = \left(s_i + \frac{k_i}{\langle \alpha_i^\vee, \lambda\rangle}\right) G_k(\lambda, \cdot).$$

(iii) *Choose* $\lambda \in \mathfrak{a}_- = -\mathfrak{a}_+$ *and* $w = s_{i_1} \cdots s_{i_m} \in W$ *in a reduced expression. If we put* $\lambda_{(j)} := s_{i_{j+1}} \cdots s_{i_m} \lambda$*, then* $\langle \alpha_{i_j}^\vee, \lambda_{(j)} \rangle < 0$ *and we have for all* $j = 0, \ldots, m-1$*,* $x \in \mathfrak{a}$

$$G_k(\lambda_{(j-1)}, x) \leq \left(1 - \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \lambda_{(j)} \rangle}\right) G_k(\lambda_{(j)}, s_{i_j} x).$$

In particular, for all $\lambda \in \mathfrak{a}_-$ *and* $w \in W$ *there is a constant* $C = C(\lambda, w, k)$ *such that*

$$G_k(w\lambda, x) \leq C \cdot G_k(\lambda, w^{-1} x). \tag{4.16}$$

PROOF. By the same arguments as in Lemma A.3 of the appendix, the Cherednik operators for integral roots systems satisfy the same intertwining relation. Moreover, the assertions (i) and (ii) for the Cherednik kernel can be proven, up to a constant factor, working as in Theorem A.4 for the Heckman-Opdam polynomials. The constant factor is then computed by evaluation at 0.

So it remains to prove part (iii). Consider $\lambda \in \mathfrak{a}_-$ and $w \in W$ with reduced expression $w = s_{i_1} \cdots s_{i_m}$ in terms of the simple reflections $s_1, \ldots, s_n$. Set $\lambda_{(j)} = w'\lambda$ with $w' := s_{i_{j+1}} \cdots s_m$. Since the expression is reduced, the lengths of w' and $s_{i_j} w'$ are $m-j$ and $m-j+1$, respectively. By a standard argument as in [Hum90, Chapter 5] we conclude that $w'^{-1} \alpha_{i_j} \in R_+$ is a positive root, i.e. $0 > \langle w'^{-1} \alpha_{i_j}, \lambda \rangle = \langle \alpha_{i_j}, \lambda_{(j)} \rangle$ and $0 < \langle \alpha_{i_j}, \lambda_{(j-1)} \rangle = -\langle \alpha_{i_j}, \lambda_{(j)} \rangle$. Since the Cherednik kernel is positive, we obtain from part (ii) for all $x \in \mathfrak{a}$

$$\begin{aligned} G_k(\lambda_{(j-1)}, x) &\leq G_k(\lambda_{(j-1)}, x) + \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \lambda_{(j-1)} \rangle} G_k(\lambda_{(j-1)}, s_{i_j} x) \\ &= \left(1 + \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \lambda_{(j-1)} \rangle}\right) G_k(s_{i_j} \lambda_{(j-1)}, s_{i_j} x) = \left(1 - \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \lambda_{(j)} \rangle}\right) G_k(\lambda_{(j)}, s_{i_j} x). \end{aligned}$$

Hence, by induction, the estimate (4.16) is true with the constant

$$C := \prod_{j=1}^{n} \left(1 - \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \lambda_{(j)} \rangle}\right).$$

■

For $x \in \mathfrak{a}$ we abbreviate

$$C(x) := \mathrm{co}(W.x),$$

where $\mathrm{co}(W.x)$ is the convex hull of the orbit $W.x$.

We are now in a position to prove the following generalization of the Helgason-Johnson theorem for the Cherednik kernel of an integral root system.

Theorem 4.30. *Assume that* R *is an integral root system.*

(i) *The Cherednik kernel* $G_k(\lambda, \cdot)$ *is bounded as function on* $\mathfrak{a}$ *if and only if* $\lambda \in C(\rho(k)) + i\mathfrak{a}$*.*

(ii) *There exists* $C_k > 0$ *such that* $|G_k(\lambda, x)| \leq C_k$ *for all* $\lambda \in C(\rho(k)) + i\mathfrak{a}$ *and* $x \in \mathfrak{a}$ *.*

(iii) *For* $\lambda \in \mathfrak{a}_{\mathbb{C}}$*,* $\mu \in C(\rho(k)) + i\mathfrak{a}$ *and* $x \in \mathfrak{a}$ *we have*

$$|G_k(\lambda + \mu, x)| \leq C_k \, e^{\max_{w \in W} \langle \mathrm{Re}\, \lambda, wx \rangle}.$$

PROOF. Part (iii) is an immediate consequence of part (ii) and Theorem 4.17. Therefore, it remains to prove parts (i) and (ii). By Remark 4.19 and (4.6) we assume without loss of

generality that R is irreducible and crystallographic.
First of all, if $G_k(\lambda, \cdot)$ is bounded, then $F_k(\lambda, \cdot)$ is bounded as well. Thus, by [NPP14, Theorem 4.2] we have $\lambda \in C(\rho(k)) + i\mathfrak{a}$.
Conversely, assume that $\lambda \in \Omega := C(\rho(k)) + i\mathfrak{a}$ and fix $x \in \mathfrak{a}$. Since $\lambda \mapsto G_k(\lambda, x)$ is holomorphic and $|G_k(\lambda, x)| \leq G_k(\operatorname{Re}\lambda, x)$, we conclude

$$\max_{\lambda\in\Omega} |G_k(\lambda, x)| = \max_{\lambda\in\partial C(\rho(k))} G_k(\lambda, x).$$

Consider $\mu_1, \mu_2 \in \partial C(\rho(k))$, the line segment $[\mu_1, \mu_2] \subseteq \mathfrak{a}$ joining μ_1 and μ_2, as well as $L^{\mathbb{C}} = \{z\mu_1 + (1-z)\mu_2 \mid z \in \mathbb{C}\}$. Then, the function $\lambda \mapsto |G_k(\lambda, x)|$, restricted to $\Omega \cap \left\{\lambda \in L^{\mathbb{C}} \mid \operatorname{Re}\lambda \in [\mu_1, \mu_2]\right\}$, attains its maximum at the real points on the boundary, namely at μ_1 and μ_2. Since the extreme points of $C(\rho(k))$ are precisely $W.\rho(k)$, we see that

$$\max_{\lambda\in\Omega} |G_k(\lambda, x)| = \max_{\lambda\in W.\rho(k)} G_k(\lambda, x). \tag{4.17}$$

By Theorem 4.29 (iii) and $-\rho(k) \in \mathfrak{a}_-$, there exists a constant $C(w, k) > 0$, independent of x and λ, such that for all $w \in W$ we have

$$|G_k(-w\rho(k), x)| \leq C(w, k) \cdot G_k(-\rho(k), wx) = C(w, k). \tag{4.18}$$

Finally, putting (4.17) and (4.18) together, we deduce that for all $\lambda \in \Omega$ and $x \in \mathfrak{a}$ we have

$$|G_k(\lambda, x)| \leq C_k := \max_{w\in W} C(w, k).$$

■

As a consequence of this theorem, we will give decay results for the Cherednik transform which generalizes the spherical Fourier transform on Riemannian symmetric spaces of non-compact type.

Definition 4.31. The Cherednik transform of a suitable function $f : \mathfrak{a} \to \mathbb{C}$ is defined by

$$\mathcal{H}_k f(\lambda) := \int_{\mathfrak{a}} f(x) G_k(i\lambda, -x)\delta_k(x)\, \mathrm{d}x, \quad \lambda \in \mathfrak{a}$$

with the weight function

$$\delta_k(x) := \prod_{\alpha\in R_+} \left|2\sinh\tfrac{\langle\alpha,x\rangle}{2}\right|^{2k_\alpha}.$$

Notice that due to Proposition 4.15 we could replace in the definition of $\mathcal{H}_k$ the kernel $G_k(i\lambda, -x)$ by $G_k(-iw_0\lambda, w_0x)$, where w_0 is the longest element of W with respect to R_+.
The inverse Cherednik transform is defined by

$$\mathcal{I}_k f(x) := \int_{\mathfrak{a}} f(\lambda) G_k(i\lambda, x)\nu(i\lambda)\, \mathrm{d}\lambda,$$

with the weight function

$$\nu(\lambda) = c \cdot \prod_{\alpha\in R_+} \frac{\Gamma(\langle\lambda, \alpha^\vee\rangle + k_\alpha + \frac{1}{2}k_{\alpha/2})\Gamma(-\langle\lambda, \alpha^\vee\rangle + k_\alpha + \frac{1}{2}k_{\alpha/2} + 1)}{\Gamma(\langle\lambda, \alpha^\vee\rangle + \frac{1}{2}k_{\alpha/2})\Gamma(-\langle\lambda, \alpha^\vee\rangle + \frac{1}{2}k_{\alpha/2} + 1)},$$

where c is a suitable normalization constant and $k_{\alpha/2} = 0$ for $\alpha/2 \notin R$.

Remark 4.32. Recall Theorem 4.26, i.e. the relation between the hypergeometric function and spherical functions of a Riemannian symmetric space G/K associated with a Lie group $G \in \mathcal{H}$ of the Harish-Chandra class. Consider $f \in C_c^\infty(\mathfrak{a})^W$ and the associated unique K-biinvariant function $F \in C_c^\infty(G)$ with $F(e^x) = f(x)$ for all $x \in \mathfrak{a}$. Then, the Cherednik transform of f can be rewritten

$$\mathcal{H}_k f(\lambda) = \int_{\mathfrak{a}} f(x) F_k(i\lambda, -x)\delta_k(x)\, \mathrm{d}x = \int_{\mathfrak{a}} f(x)\varphi_{i\lambda}^G(e^x)\delta_k(x)\, \mathrm{d}x = \int_G F(g)\varphi_{i\lambda}^G(g)\, \mathrm{d}g, \quad (4.19)$$

where the last equality can be found in [GV88, Proposition 2.4.6]. The last integral in (4.19) is the Harish-Chandra transform on G/K, i.e. the spherical Fourier transform associated with (G, K).

The following Theorem can be found in [Opd93, Sch08] for crystallographic roots systems and is easily extended to integral root systems.

Theorem 4.33. *Consider the Paley-Wiener space $\mathcal{H}(\mathfrak{a}_\mathbb{C})$ as defined in (1.7). Furthermore, define a weighted Schwartz space*

$$\mathcal{C}(\mathfrak{a}) := \{f \in C^\infty(\mathfrak{a}) \mid \sigma_{n,\alpha}(f) < \infty \text{ for all } n \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n\},$$

here the seminorms $\sigma_{n,\alpha}$ are defined by

$$\sigma_{n,\alpha}(f) := \sup_{x\in\mathfrak{a}} (1 + |x|)^n e^{-\langle \rho(k), x_+\rangle} |\partial^\alpha f(x)|,$$

where x_+ is the unique element in $W.x \cap \overline{\mathfrak{a}_+}$. The space $\mathcal{C}(\mathfrak{a})$ becomes a Fréchet space with the topology induced by the seminorms $(\sigma_{n,\alpha})_{n\in\mathbb{N}_0,\alpha\in\mathbb{N}_0^n}$. Then the following assertions hold:

(i) *$\mathcal{H}_k : C_c^\infty(\mathfrak{a}) \to \mathcal{H}(\mathfrak{a}_\mathbb{C})$ is a topological isomorphism with inverse $\mathcal{I}_k$. Here the spaces $C_c^\infty(\mathfrak{a})$ and $\mathcal{H}(\mathfrak{a}_\mathbb{C})$ have their natural Fréchet space topologies.*

(ii) *$\mathcal{H}_k : \mathcal{C}(\mathfrak{a}) \to \mathscr{S}(\mathfrak{a})$ is a topological isomorphism with inverse $\mathcal{I}_k$.*

We can now prove the non-symmetric generalization of [NPP14, Corollary 5.1].

Theorem 4.34. *Let C_k be the constant from Corollary 4.30. Then the following generalized Riemann-Lebesgue lemma holds for $f \in L^1(\mathfrak{a}, \delta_k)$:*

(i) $\|\mathcal{H}_k f\|_{\infty,\mathfrak{a}+iC(\rho(k))} \le C_k \|f\|_{1,\delta_k}$.

(ii) *The Cherednik transform $\mathcal{H}_k f$ is continuous on $\mathfrak{a} + iC(\rho(k))$ with*

$$\lim_{\substack{|\lambda|\to\infty \\ \operatorname{Im}\lambda\in C(\rho(k))}} \mathcal{H}_k f(\lambda) = 0. \quad (4.20)$$

(iii) *If R is crystallographic, then the interior of $C(\rho) \subseteq \mathfrak{a}$ is non-empty and the Cherednik transform $\mathcal{H}_k f$ is holomorphic in the interior of $\mathfrak{a} + iC(\rho(k))$.*

Proof. Part (i) is an immediate consequence of Theorem 4.30. Let $f \in L^1(\mathfrak{a}, \delta_k)$. The continuity (and holomorphicity in the crystallographic case) of $\mathcal{H}_k f$ is obtained from Theorem 4.30 and standard theorems on continuous and holomorphic parameter integrals. The decay (4.20) of $\mathcal{H}_k f$ follows for $f \in C_c^\infty(\mathfrak{a})$ from Theorem 4.33. For general $f \in L^1(\mathfrak{a}, \delta_k)$ we obtain (4.20) by part (i) by dominated convergence and the fact that $C_c^\infty(\mathfrak{a}) \subseteq L^1(\mathfrak{a}, \delta_k)$ is a dense subspace. ∎

Part II

Dunkl theory in line with radial analysis on symmetric cones

CHAPTER 5

Radial analysis on symmetric cones and Dunkl theory

In his unpublished manuscript [Mac89] from the 1980ies, Macdonald introduced hypergeometric series in terms of Jack polynomials, which include the hypergeometric functions on symmetric cones as special cases. In this context, he also introduced a generalization of the Laplace transform for radial functions on symmetric cones, but many statements in [Mac89] remained at a formal level. Radial analysis on symmetric cones is closely related to Dunkl theory for root systems of type A, and also Macdonald's concepts have a natural interpretation within Dunkl theory, because the ${}_0F_0$-hypergeometric function, which replaces the exponential kernel in the Macdonald's Laplace transform, is just a Dunkl-Bessel function of type A. The connection of the concepts in [Mac89] to Dunkl theory was already observed by Baker and Forrester in their seminal papers [BF97, BF98] related to the study of Calogero-Moser-Sutherland models.
This chapter is intended to give a brief overview about the connection between radial analysis on symmetric cones and Dunkl theory. For a general background on symmetric cones the reader is referred to [FK94].
For this chapter we fix an irreducible symmetric cone Ω inside a simple Euclidean Jordan algebra V with unit $e \in V$ and inner product $(\cdot|\cdot)$, cf. the proof of Lemma 4.23 for a classification. Then, Ω is identified with the symmetric space G/K, where G is the connected component of id inside the automorphism group of Ω and K is the stabilizer of e inside G. In particular, G is a Lie group of the Harish-Chandra class and $K \subseteq G$ is a maximal compact subgroup.
Let m be the dimension of V, n the rank and d the Pierce-dimension constant. According to the classification of simple Euclidean Jordan algebras and symmetric cones we have the following possibilities:

V	Ω	$\mathfrak{g}$	$\mathfrak{k}$	m	n	d
$\mathrm{Sym}_n(\mathbb{R})$	$\mathrm{Pos}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{so}_n(\mathbb{R})$	$\frac{n(n+1)}{2}$	n	1
$\mathrm{Herm}_n(\mathbb{C})$	$\mathrm{Pos}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}_n(\mathbb{C})$	n^2	n	2
$\mathrm{Herm}_n(\mathbb{H})$	$\mathrm{Pos}_n(\mathbb{H})$	$\mathfrak{sl}_n(\mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{su}_n(\mathbb{H})$	$n(2n-1)$	n	4
$\mathrm{Herm}_3(\mathbb{O})$	$\mathrm{Pos}_3(\mathbb{O})$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{f}_4$	27	3	8
$\mathbb{R} \times \mathbb{R}^{n-1}$	Lor_n	$\mathfrak{o}(1,n-1) \oplus \mathbb{R}$	$\mathfrak{o}(n-1)$	n	2	$n-2$

Let $c_1, \ldots, c_n \in V$ be a fixed Jordan frame. Let $\mathfrak{k} \subseteq \mathfrak{g} \subseteq \mathrm{End}(V)$ be the Lie algebras of K and G, respectively. Then, the map $X \mapsto -X^*$, where X^* is the adjoint of $X \in \mathfrak{g}$, is the Cartan involution on $\mathfrak{g}$ associated with $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. A maximal abelian subspace of $\mathfrak{p}$ is given by $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{L(c_i) \mid i = 1, \ldots, n\}$, where $L(c_i) : V \to V, x \mapsto c_i x$. The associated root system Σ is of type A_{n-1}. Moreover, for any element $x \in V$ there exists an element $k \in K$ and unique (up to permutation) $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, called spectral values, with

$$x = k \sum_{i=1}^{n} \lambda_i c_i.$$

Furthermore, $x \in \Omega$ if and only if $\lambda_1, \ldots, \lambda_n > 0$. The Jordan determinant and Jordan trace of $x \in V$ are defined by

$$\det(x) = \lambda_1 \cdots \lambda_n \quad \text{and} \quad \mathrm{tr}(x) = \lambda_1 + \ldots + \lambda_n. \tag{5.1}$$

It is known that there exists a constant $c > 0$, such that the inner product of V is given by $(x|y) = c\mathrm{tr}(xy)$. By the right choice of c, we have an isometric isomorphim

$$\mathbb{R}^n \to \mathrm{span}_{\mathbb{R}} \{c_1, \dots, c_n\}, \quad e_i \mapsto c_i,$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^n$ is equipped with the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Inside $\mathbb{R}^n$, we consider the root system

$$A_{n-1} = \{\pm(e_i - e_j) \mid 1 \le i < j \le n\}$$

with root multiplicity $k = \frac{d}{2}$.

The first important observation is the following integration formula from [FK94, Theorem VI.2.3].

Theorem 5.1. *There exist a constant $c_0 > 0$ such that for all integrable $f : V \to \mathbb{C}$ we have*

$$\begin{aligned}\int_V f(x)\,\mathrm{d}x &= c_0 \int_K \int_{C_+} f\left(k \sum_{i=1}^n \xi_i c_i\right) \prod_{i<j} |\xi_i - \xi_j|^d \,\mathrm{d}\xi\,\mathrm{d}k \\ &= \frac{c_0}{n!} \int_K f\left(k \sum_{i=1}^n \xi_i c_i\right) \omega^A_{d/2}(\xi)\,\mathrm{d}\xi\,\mathrm{d}k\end{aligned}$$

with $C_+ = \{\xi \in \mathbb{R}^n \mid \xi_1 < \ldots < \xi_n\}$ and the weight function $\omega^A_k(x) = \prod_{\alpha\in A_{n-1}} |\langle\alpha, x\rangle|^k$.

The spherical functions of the Gelfand pair (G, K) can be constructed as K-invariant functions on V as follows. Let $V^{(k)} \subseteq V$ be the eigenspace of the multiplication operator $x \mapsto (c_1 + \ldots + c_k)x$ associated with the eigenvalue 1. It turns out that $V^{(k)}$ is a Euclidean Jordan algebra and denote by $\det^{(k)}$ the associated Jordan determinant. If $P_k : V \to V^k$, $k = 1, \dots, n$ are the orthogonal projections, put $\Delta_k(x) := \det^{(k)}(P_k x)$. Then, the generalized power function is defined for $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ by

$$\Delta_{\mathbf{s}} : V \to \mathbb{C}, \quad \Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1-s_2} \cdots \Delta_{n-1}(x)^{s_{n-1}-s_n} \Delta_n(x)^{s_n} \tag{5.2}$$

and it holds

$$\Delta_{\mathbf{s}+(t,\dots,t)}(x) = \det(x)^t \Delta_{\mathbf{s}}(x). \tag{5.3}$$

The spherical functions of the Gelfand pair (G, K) are characterized by the following theorem.

Lemma 5.2 ([FK94, Theorem XIV.3.1])**.** *The spherical functions of (G, K) are indexed by $\lambda \in \mathbb{C}^n$ as K-invariant functions on V by*

$$\varphi^\Omega_\lambda(x) := \int_K \Delta_{\lambda+\rho}(kx)\,\mathrm{d}k.$$

with $\rho = \frac{d}{4}(1-n, 3-n, \dots, n-1) = \frac{k}{2}\sum_{i<j}(e_j - e_i) \in \mathbb{R}^n$. Moreover, $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = \sigma\mu$ for some $\sigma \in \mathcal{S}_n$.

Theorem 5.3. *Let F^A_k and J^A_k be the hypergeometric function and Bessel function associated with (A_{n-1}, k) on $\mathbb{R}^n$, respectively. For $x \in V$ we denote by $\mathrm{spec}\,x \in \mathbb{R}^n$ the spectral values of x in decreasing order. Then,*

(i) $\int_K e^{(kx|y)}\,\mathrm{d}k = J^A_k(\mathrm{spec}\,x, \mathrm{spec}\,y)$ *for all $x, y \in V$.*

(ii) $\varphi^\Omega_\lambda(a_1c_1 + \ldots + a_nc_n) = F^A_k(\lambda, (\ln a_1, \dots, \ln a_n))$ *for all $a_1, \dots, a_n > 0$.*

(iii) $\varphi^\Omega_\lambda(x) = F_k(\lambda, \log\mathrm{spec}\,x)$ *for all $x \in \Omega$, where $\log y = (\ln y_1, \dots, \ln y_n)$ for $y \in \mathbb{R}^n$.*

Proof. Part (i) follows from Theorem 4.27, see also [Rö20, Remark 3.2]. Part (iii) is a consequence of part (ii). For part (ii) we observe the following. The group G decomposes into reversed Iwasawa decomposition $G = NAK$, cf. [FK94, Theorem VI.3.6] with

$$A = \left\{ P(a) \mid a = \sum_{i=1}^{n} a_i c_i, a_i > 0 \right\}, \quad P(x) = (y \mapsto 2x(xy) - x^2 y).$$

The generalized power functions satisfies by [FK94, Proposition VI.3.10] for $n \in N$ and $k \in K$

$$\Delta_{\mathbf{s}}(nP(a)ke) = \Delta_{\mathbf{s}}(P(a)e) = a_1^{2s_1} \cdots a_n^{2s_n} = e^{\langle \mathbf{s}, (\ln a_1^2, \ldots, \ln a_n^2) \rangle}.$$

The Iwasawa projection $H^G : G \to \mathfrak{a} \cong \mathbb{R}^n$ is given with [FK94, Proposition II.3.4] by

$$H^G(kP(a)n) = H^G(e^{L(\sum_{i=1}^{n} \ln(a_i^2) \cdot c_i)}) = (\ln a_1^2, \ldots, \ln a_n^2)$$

and satisfies

$$\Delta_{\mathbf{s}}(k(a_1^2 c_1 + \ldots + a_n^2 c_n)) = \Delta_{\mathbf{s}}(kP(a)e) = e^{-\langle \mathbf{s}, H^G(P(a)^{-1}k^{-1}) \rangle},$$

which leads immediately with Theorem 4.26 to

$$\varphi_\lambda^\Omega(a_1^2 c_1 + \ldots + a_n^2 c_n) = \int_K e^{-\langle \lambda + \rho, H^G(P(a)^{-1}k) \rangle} \, \mathrm{d}k = F_k^A(\lambda, (\ln a_1^2, \ldots, \ln a_n^2)).$$

■

Proposition 5.4. *The spherical function φ_λ^Ω is a polynomial on V if and only if $\lambda = \sigma(\mathbf{m} - \rho)$ with $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n, \sigma \in \mathcal{S}_n$ and $m_1 \geq \ldots \geq m_n \geq 0$. The functions*

$$\Phi_{\mathbf{m}}^\Omega := \varphi_{\mathbf{m} - \rho}^\Omega$$

are called the spherical polynomials of Ω.

One of the most important integral identities that will be generalized in this part of the thesis is the following from [FK94, Proposition VII.1.2]:

Theorem 5.5. *For $y \in \Omega$ and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $\operatorname{Re} s_j > (j-1)\frac{d}{2}$ for all $j = 1, \ldots, n$*

$$\int_\Omega e^{-(x|y)} \Delta_{\mathbf{s}}(x) \det(x)^{-\frac{m}{n}} \, \mathrm{d}x = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}(y^{-1}),$$

with the Gamma function $\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{m-n}{2}} \prod_{j=1}^{n} \Gamma(s_j - (j-1)\frac{d}{2})$ of the cone. In particular, the identity is still true if $\Delta_{\mathbf{s}}$ is replaced by the spherical functions φ_λ^Ω.

CHAPTER 6

Laplace transform of hypergeometric functions

6.1 Introduction

The Laplace transform is an important tool in various areas of harmonic analysis and forms a cornerstone in the analysis on symmetric cones, see [FK94]. In particular, there are important Laplace transform identities between ${}_pF_q$-hypergeometric functions on a symmetric cone, which are given as expansions with respect to the associated spherical polynomials, c.f. [FK94, Chap.XV]. For cones of positive definite matrices, such hypergeometric series trace back to ideas of Bochner and were studied in detail by Herz [Her55], where they were actually defined recursively by means of the Laplace transform. For important further developments see for instance [Con63, GR87, Kan93]. Multivariable hypergeometric series have found many applications in multivariate statistics [Mui82], but also in number theory and mathematical physics.
Consider a symmetric cone $\Omega = G/K$ inside a simple Euclidean Jordan algebra V of dimension m, rank n and with Peirce constant d which takes only specific integer values. Let $F \in L^1_{\text{loc}}(\Omega)$ be K-invariant, that is of the form $F(x) = f(\text{spec}(x))$, where $\text{spec}(x) \in \mathbb{R}^n_+$ with $\mathbb{R}_+ =]0, \infty[$ denotes the set of eigenvalues of x in decreasing order. Then for $y \in \Omega$, the Laplace transform

$$LF(y) = \int_\Omega e^{-(x|y)} F(x)\, \mathrm{d}x$$

depends only on $\eta = \text{spec}(y) \in \mathbb{R}^n_+$ and can be written by Theorems 5.1 and 5.3 as

$$LF(y) = \text{const} \cdot \int_{\mathbb{R}^n_+} J^A_{d/2}(-\xi, \eta) f(\xi) \omega^A_{d/2}(\xi)\, \mathrm{d}\xi. \tag{6.1}$$

An important observation by Macdonald in [Mac89] was that for $k = \frac{d}{2}$

$$J^A_k(z, w) = \sum_{\lambda \in \Lambda^n_+} \frac{1}{|\lambda|!} \frac{C^\alpha_\lambda(z) C^\alpha_\lambda(w)}{C^\alpha_\lambda(1, \ldots, 1)} = {}_0F^\alpha_0(z, w), \quad \alpha = \frac{1}{k} \in]0, \infty]. \tag{6.2}$$

Here Λ^n_+ denotes the set of partitions with at most n parts and the C^α_λ are the (symmetric) Jack polynomials of index α in C-normalization as in Lemma 6.16 below. See e.g. [Rö20] for some details. On the other hand, it is well-known (c.f. [BF98] and Remark 6.21) that for arbitrary $k \geq 0$, the equation (6.2) holds. Macdonald [Mac89] considered the Laplace transform (6.1) with the Bessel function $e(z, w) = {}_0F^\alpha_0(z, w)$ for arbitrary indices $\alpha > 0$. Many of his calculations were of a formal nature and rested on the following "Conjecture (C)" about the Laplace transform of Jack polynomials: For arbitrary $k \geq 0$, let

$$\mu_0 := k(n-1) \text{ and } \Delta(z) := \prod_{j=1}^n z_j \text{ for } z \in \mathbb{C}^n$$

and write $C_\lambda(z) := C^{1/k}_\lambda(z)$ for abbreviation. Then for all $\lambda \in \Lambda^n_+$, $y \in \mathbb{R}^n_+$ and all $\mu \in \mathbb{C}$ with $\text{Re}\, \mu > \mu_0$,

$$\int_{\mathbb{R}^n_+} J^A_k(-y, x)\, C_\lambda(x) \Delta(x)^{\mu - \mu_0 - 1} \omega^A_k(x)\, \mathrm{d}x \,=\, \Gamma_n(\lambda + \underline{\mu})\, C_\lambda(\tfrac{1}{y})\, \Delta(y)^{-\mu}. \tag{6.3}$$

Here $\Gamma_n(\lambda) = \Gamma_n(\lambda; k)$ is Macdonald's multivariate gamma function defined in formula (6.7) below. In [Rö20], there is a rigorous treatment of the Dunkl-Laplace transform

$$\mathcal{L}f(z) := \int_{\mathbb{R}_+^n} E_k^A(-z, x) f(x)\omega_k(x)\, \mathrm{d}x$$

where compared to Macdonald's version, the Bessel function is replaced by the Dunkl kernel E_k of type A_{n-1} and multiplicity k. This transform was already considered by Baker and Forrester [BF98] and later used in [SZ07], but convergence issues had remained open for a long time. The convergence issues and analytic aspects were studied then in [Rö20]. Formula (6.3) generalizes a Laplace transform identity for spherical polynomials on the symmetric cone Ω, which is in turn a consequence of the following important Laplace transform identity for the generalized power functions Δ_s, $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $\mathrm{Re}\, s_j > \frac{d}{2}(j-1)$ (see [FK94, Chapter VII])

$$\int_\Omega e^{-(y|x)} \Delta_s(x) \det(x)^{-m/n}\, \mathrm{d}x \;=\; \Gamma_\Omega(s)\Delta_s(y^{-1}) \quad \text{for all } y \in \Omega, \tag{6.4}$$

where Γ_Ω is the Gindikin gamma function associated with Ω. Taking K-means in (6.4), one gets the same Laplace transform identity for the spherical functions of Ω as

$$\int_\Omega e^{-(y|x)} \varphi_\lambda^\Omega(x) \det(x)^{-m/n}\, \mathrm{d}x \;=\; \Gamma_\Omega(\lambda)\varphi_\lambda^\Omega(y^{-1}). \tag{6.5}$$

For parameters $\lambda \in \Lambda_+^n$, the spherical function $\varphi_{\lambda-\rho}^\Omega$ are just the spherical polynomials of Ω and are related to the Jack polynomials by

$$\varphi_{\lambda-\rho}^\Omega(x) = \frac{C_\lambda^{2/d}(\mathrm{spec}(x))}{C_\lambda^{2/d}(1, \ldots, 1)}.$$

Rewriting (6.5) by means of identity (6.2), one gets formula (6.3) for the particular multiplicity $k = d/2$. It is well-known that the spherical functions of Ω can be expressed in terms of Heckman-Opdam hypergeometric functions of type A_{n-1} and with multiplicity $k = d/2$, see Theorem 5.3. In the present chapter, we shall establish a generalization of formula (6.5) to the Dunkl setting of type A_{n-1} with arbitrary multiplicity $k \geq 0$. Namely, we obtain in Corollary 6.15 the following Laplace transform identity for Heckman-Opdam hypergeometric functions of type A_{n-1} and, more generally, for the associated Opdam-Cherednik kernel $\mathcal{G}_k(\lambda, x) = G_k^{A+}(\lambda, (\ln x_1, \ldots, \ln x_n))$ with positive roots $A_{n-1}^+ = \{e_j - e_i \mid 1 \leq i < j \leq n\}$: For $\lambda \in \mathbb{C}^n$ with $\mathrm{Re}\,\lambda_i \geq \mu_0$ and $z \in \mathbb{C}^n$ with $\mathrm{Re}\, z_i > 0$,

$$\int_{\mathbb{R}_+^n} E_k(-z, x)\, \mathcal{G}_k(\lambda, x)\, \Delta(x)^{-\mu_0 - 1}\omega_k(x)\, \mathrm{d}x \;=\; \Gamma_n(\lambda + \rho)\, \mathcal{G}_k(\lambda, \tfrac{1}{z}). \tag{6.6}$$

The first step towards the proof of (6.6) will be a rigorous proof of Macdonald's Conjecture (C). More generally, we shall prove Dunkl-Laplace transform identities for the non-symmetric Jack polynomials in the sense of [Opd95, KS97], from which (6.3) then follows by symmetrization. These non-symmetric identities were already stated in [BF98], but the proof given there in terms of Laguerre expansions is involved and not fully carried out. The proof we are presenting here is completely different and very natural; it is based on a reformulation via Dunkl operators and is carried out by induction, using the raising operator of Knop and Sahi [KS97] for the non-symmetric Jack polynomials. The statement for the Cherednik kernel is then obtained via analytic continuation with respect to the spectral variable, and for the hypergeometric function it follows by symmetrization.

Based on the Laplace transform identities for Jack polynomials, we then study hypergeometric series in terms of Jack polynomials of the form

$$ {}_pF_q(\mu;\nu;z,w) := \sum_{\lambda\in\Lambda_+^n} \frac{[\mu_1]_\lambda\cdots[\mu_p]_\lambda}{[\nu_1]_\lambda\cdots[\nu_q]_\lambda}\,\frac{C_\lambda(z)C_\lambda(w)}{|\lambda|!\,C_\lambda(\underline{1})} \quad (\mu\in\mathbb{C}^p,\nu\in\mathbb{C}^q)$$

as well as their non-symmetric analogues, and we establish Laplace transform identities between them. This generalizes known results on symmetric cones and settles several conjectural Laplace transform formulas in [Mac89]. As a further application, we finally prove a Post-Widder inversion theorem for the Dunkl-Laplace transform, which complements a result by Faraut and Gindikin in [FG90] for the Laplace transform on symmetric cones.
The organization of this chapter is as follows: Section 2 provides the necessary background on the type A Dunkl-Laplace transform. In Section 3, we collect results on the symmetric and non-symmetric Jack polynomials which will be relevant in the sequel, and we prove the Dunkl-Laplace transform identities for Jack polynomials. In Section 4 the Laplace transform identities for Jack polynomials are extended to the Opdam-Cherednik kernel and to the hypergeometric function. Section 5 is devoted to the study of Jack-hypergeometric series. In Section 6 we are able to prove that the Dunkl type Riesz distributions from [Rö20] is a group under Dunkl convolution. Finally Section 7 contains a binomial formula for the Cherednik kernel and hypergeometric function, and Section 8 contains the Post-Widder inversion formula in the Dunkl setting.

6.2 The type A Dunkl setting

We consider the root system $A_{n-1} = \{\pm(e_i - e_j) \mid 1 \le i < j \le n\}$ in the Euclidean space $\mathbb{R}^n$ with inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and norm $|x| = \sqrt{\langle x, x\rangle}$, where the e_i denote the standard basis vectors. The inner product $\langle\cdot,\cdot\rangle$ is extended to $\mathbb{C}^n$ in a $\mathbb{C}$-bilinear way. The reflection group generated by A_{n-1} is the symmetric group $\mathcal{S}_n$ on n elements, permuting the coordinates in $\mathbb{R}^n$. To avoid notational overload, we shall always suppress in our notations the dependence on the fixed multiplicity parameter $k \ge 0$ on the root system A_{n-1}. The Dunkl kernel and Bessel function on $\mathbb{R}^n$ associated with (A_{n-1}, k) are denoted by E^A and J^A, respectively. Furthermore, the Dunkl weight function is denoted by

$$\omega^A(x) = \prod_{i\neq j} |x_i - x_j|^k\,.$$

Furthermore, we fix the following notations for $\mu \in \mathbb{C}$, $z \in \mathbb{C}^n$ and $\eta \in \mathbb{N}_0^n$:

$$\begin{aligned}
\mathbb{R}_+^n &:=]0,\infty[^n\\
\underline{\mu} &:= (\mu,\dots,\mu)\\
\Delta(z) &:= z_1\cdots z_n\\
\mu_0 &:= k(n-1),\\
\Gamma_n(\lambda) &:= \prod_{j=1}^n \frac{\Gamma(1+jk)\Gamma(\lambda_j - k(j-1))}{\Gamma(1+k)},\\
\Gamma_n(\mu) &:= \Gamma_n(\underline{\mu}),\\
[\mu]_\eta &:= \prod_{j=1}^n (\mu - k(j-1))_{\eta_j} = \frac{\Gamma_n(\underline{\mu}+\eta)}{\Gamma_n(\underline{\mu})}.
\end{aligned} \tag{6.7}$$

The function Γ_n is *Macdonald's gamma function* and $[\mu]_\eta$ is a *generalized Pochhammer symbol.* Furthermore, we write $\mathcal{P}$ for the polynomial functions on $\mathbb{R}^n$ and $\mathcal{P}_m$ for the subspace of

elements that are homogeneous of degree m.
As already mentioned, Dunkl analysis associated with the root system A_{n-1} generalizes the radial analysis on symmetric cones, which just corresponds to the multiplicity values $k = d/2$, where d is the Peirce constant of the cone. According to equation (6.1), we consider the Dunkl-Laplace transform in the sense of the following definition, first considered in [BF98].

Definition 6.1. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ we define the *Dunkl-Laplace transform* by

$$\mathcal{L}f(z) := \int_{\mathbb{R}^n_+} f(x) E^A(-z, x)\omega(x)\, \mathrm{d}x,$$

if the integral converges for $z \in \mathbb{C}^n$.

For $x \in \mathbb{R}^n_+$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq a$ for some $a \in \mathbb{R}^n$ (which is understood componentwise), the type A Dunkl kernel satisfies the exponential bound

$$|E^A(-z, x)| \leq \exp(-\|x\|_1 \cdot \min_{1\leq i\leq n} a_i), \tag{6.8}$$

see [Rö20]. Here $\|x\|_1 = \sum_{i=1}^n |x_i|$. This estimate, which seems to be exclusive in type A, guarantees good convergence properties of the Laplace integral. In the following, we write for $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$

$$x > a \quad \text{iff} \quad x_i > a \text{ for all } i = 1, \dots, n.$$

We recall the following Lemma from [Rö20].

Lemma 6.2. *Suppose that $f : \mathbb{R}^n_+ \to \mathbb{C}$ is measurable and exponentially bounded according to $|f(x)| \leq Ce^{s\|x\|_1}$ with some constants $C > 0$ and $s \in \mathbb{R}$. Then $\mathcal{L}f(z)$ exists and is holomorphic on $\{z \in \mathbb{C}^n \mid \operatorname{Re} z > s\}$. Moreover, for each polynomial $p \in \mathcal{P}$*

$$p(-T)(\mathcal{L}f) = \mathcal{L}(fp)$$

as functions on $\{\operatorname{Re} z > s\}$.

Let us turn to the trigonometric setting. We fix the positive roots

$$A^+_{n-1} = \{e_j - e_i \mid i < j\},$$

and the associated (trigonometric) Cherednik operators

$$D_\xi := D_\xi(A^+_{n-1}, k) := \partial_\xi - \langle \rho(R_+), \xi\rangle + k \sum_{\alpha\in A^+_{n-1}} \langle \alpha, \xi\rangle \frac{1 - s_\alpha}{1 - e^{-\langle \cdot,\alpha\rangle}},$$

with $\xi \in \mathbb{R}^n$ and the Weyl vector

$$\rho := \rho(R_+) := \rho(R_+, k) := \frac{k}{2} \sum_{\alpha\in A^+_{n-1}} \alpha = -\frac{k}{2}(n - 1, n - 3, \dots, -n + 1). \tag{6.9}$$

Let $G(\lambda, z) = G_k(A^+_{n-1}, \lambda, z)$ be the Cherednik kernel on $\mathbb{R}^n$ associated with (A^+_{n-1}, k), see Theorem 4.12. Similar, we denote by F the hypergeometric functions on $\mathbb{R}^n$ associated with (A_{n-1}, k). According to Theorem 4.12, G and F are holomorphic on the domain $\mathbb{C}^n \times (\mathbb{R}^n + i\Omega)$ with

$$\Omega = \{x \in \mathbb{R}^n \mid |x_i - x_j| < \pi \text{ for all } i < j\}.$$

On $H := \{z \in \mathbb{C}^n \mid \operatorname{Re} z > 0\}$ we define the biholomorphic logarithm

$$\log : H \to (\mathbb{R}^n + i\Omega'), \quad (z_1, \dots, z_n) \mapsto (\ln z_1, \dots, \ln z_n).$$

with $\Omega' := \{x \in \mathbb{R}^n \mid 2|x_i| < \pi \text{ for all } i = 1, \dots, n\} \subseteq \Omega$. In view of Theorem 5.3 we make the following definition.

Definition 6.3. We define the *rational Cherednik kernel* $\mathcal{G}$ and *hypergeometric Function* $\mathcal{F}$ as the holomorphic functions with domain $\mathbb{C}^n \times H$ and

$$\mathcal{G}(\lambda, z) = G(\lambda, \log z), \quad \mathcal{F}(\lambda, z) = F(\lambda, \log z).$$

In particular, in the case where the multiplicity is given by $k = \frac{d}{2}$, where d is the Pierce dimension constant of a symmetric cone Ω of rank n, the functions $(\mathcal{F}(\lambda, \cdot))_{\lambda \in \mathbb{C}^n}$ are precisely the spherical functions of the cone Ω, see Theorem 5.3.

Note that the longest element of $\mathcal{S}_n$ with respect to A_{n-1}^+ is given by

$$\mathbb{C}^n \to \mathbb{C}^n, \quad \lambda \mapsto \lambda^R = (\lambda_n, \ldots, \lambda_1).$$

From the results of Section 4.2 and Theorem 4.30 we obtain the following lemma.

Lemma 6.4. *For $z \in H$, $\lambda, \lambda' \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $x \in \mathbb{R}_+^n$ we have*

(i) $\Delta(z)^\mu \, \mathcal{G}(\lambda, z) = \mathcal{G}(\lambda + \underline{\mu}, z)$.

(ii) $\mathcal{G}(\lambda, \frac{1}{z}) = \mathcal{G}(-\lambda^R, z^R)$ *and* $\mathcal{F}(\lambda, \frac{1}{z}) = \mathcal{F}(-\lambda, z)$, *where* $\frac{1}{z}$ *is understood componentwise.*

(iii) $|\mathcal{G}(\lambda + \lambda', x)| \leq \mathcal{G}(\operatorname{Re} \lambda', x) \cdot \max_{\sigma \in \mathcal{S}_n} x^{\operatorname{Re}(\sigma\lambda)}$.

(iv) *There exists $C = C_k > 0$ such that for all $\widetilde{\lambda} \in \operatorname{co}(\mathcal{S}_n.\rho) + i\mathbb{R}^n$*

$$\mathcal{G}(\lambda + \widetilde{\lambda}, x) \leq C \cdot \max_{\sigma \in \mathcal{S}_n} x^{\operatorname{Re}(\sigma\lambda)}.$$

Parts (i),(iii) and (iv) are still true if $\mathcal{G}$ is replaced by $\mathcal{F}$.

6.3 Jack polynomials and Macdonald's conjecture

We first recall some well-known facts about Jack polynomials from [KS97, For10, Sta89] and show how they are connected to the rational Cherednik kernel and hypergeometric function introduced before. Let $\Lambda_+^n = \{\lambda \in \mathbb{N}_0^n \mid \lambda_1 \geq \ldots \geq \lambda_n\}$ denote the set of *partitions of length at most n*. The dominance order on Λ_+^n is given by

$$\mu \leq_D \lambda \quad \text{iff} \quad |\lambda| = |\mu| \ \text{ and } \ \sum_{j=1}^r \mu_j \leq \sum_{j=1}^r \lambda_j \text{ for all } r = 1, \ldots, n\,,$$

where $|\lambda| = \lambda_1 + \ldots + \lambda_n$. The dominance order is extended from Λ_+^n to $\mathbb{N}_0^n$, the *compositions of length at most n*, as follows: For each composition $\eta \in \mathbb{N}_0^n$ denote by $\eta_+ \in \Lambda_+^n$ the unique element in the $\mathcal{S}_n$-orbit of η. Then the extended dominance order on $\mathbb{N}_0^n$ is defined by

$$\kappa \preceq \eta \quad \text{iff} \quad \begin{cases} \kappa_+ \leq_D \eta_+, & \kappa_+ \neq \eta_+, \\ w_\eta \leq w_\kappa, & \kappa_+ = \eta_+, \end{cases}$$

where $w_\eta \in \mathcal{S}_n$ is the shortest element with $w_\eta \eta_+ = \eta$, and $\leq$ refers to the *Bruhat order* on $\mathcal{S}_n$. Consider the rational Cherednik operators

$$\mathcal{D}_j = \mathcal{D}_j(k) := x_j T_j + k(1-n) + k \sum_{i>j} s_{ij}, \quad j = 1, \ldots, n$$

where the $T_j := T_{e_j}(k)$ are the type A Dunkl operators with multiplicity k, x_j denotes the multiplication operator $(x_j f)(y) := y_j f(y)$ and s_{ij} denotes the reflection in the root $e_i - e_j$, which acts by interchanging x_i and x_j. We remark that our notion differs by a factor k from that in [For10]. This facilitates the handling of the case $k = 0$. The operators $\mathcal{D}_j$ are closely related to the usual Cherednik operators $D_j := D_{e_j}(A_{n-1}^+, k)$. Indeed, consider $f \in C^1(U)$ for some open $U \subseteq \mathbb{R}^n$ and define $g : \exp^{-1}(U) \subseteq \mathbb{R}^n \to \mathbb{C}$ by $g(x) := f(e^x)$, where e^x is understood componentwise. Then a short calculation gives

$$(D_j - \tfrac{k(n-1)}{2})g(x) = (\mathcal{D}_j f)(e^x). \tag{6.10}$$

The operators $\mathcal{D}_j$ are upper triangular with respect to $\preceq$ on the polynomials $\mathcal{P}$. More precisely,

$$\mathcal{D}_j x^\eta \ = \ \overline{\eta}_j x^\eta + \sum_{\kappa \prec \eta} d_{\kappa\eta} x^\kappa$$

with some $d_{\kappa\eta} \in \mathbb{R}$ and

$$\overline{\eta}_j = \eta_j - k\#\left\{i < j \mid \eta_i \geq \eta_j\right\} - k\#\left\{i > j \mid \eta_i > \eta_j\right\}. \tag{6.11}$$

Definition 6.5. The *non-symmetric Jack polynomials* of index $\alpha = 1/k$ with $k \in [0, \infty)$ can be characterized as the unique basis $\left(E_\eta = E_\eta^\alpha\right)_{\eta \in \mathbb{N}_0^n}$ of $\mathcal{P}$ satisfying

(i) $E_\eta(x) = x^\eta + \sum_{\kappa \prec \eta} c_{\kappa\eta} x^\kappa$ with $c_{\kappa\eta} \in \mathbb{C}$,

(ii) $\mathcal{D}_j E_\eta = \overline{\eta}_j E_\eta$ for all $j = 1, \ldots, n$.

By definition, E_η is homogeneous of degree $|\eta|$, and for $k = 0$ we have $E_\eta^\infty(x) = x^\eta$. Following [For10, Equation (12.100)], the *symmetric Jack polynomials* $(P_\lambda = P_\lambda^\alpha)_{\lambda \in \Lambda_+^n}$ are defined by

$$P_\lambda(x) = a_{\lambda,\eta} \cdot \sum_{\sigma \in \mathcal{S}_n} E_\eta(\sigma x)$$

for arbitrary $\eta \in \mathbb{N}_0^n$ with $\eta_+ = \lambda$ and $a_{\lambda,\eta} > 0$ such that the coefficient of x^λ equals 1.

Remark 6.6. Property (ii) in Definition 6.5 and identity (6.10) show that the polynomials E_η are related to the (rational) Cherednik kernel via

$$\frac{E_\eta(x)}{E_\eta(\underline{1})} = G(\overline{\eta} + \tfrac{k(n-1)}{2}\underline{1}, \log x) = \mathcal{G}(\overline{\eta} + \tfrac{k(n-1)}{2}\underline{1}, x), \quad \overline{\eta} = (\overline{\eta}_1, \ldots, \overline{\eta}_n). \tag{6.12}$$

Symmetrization in (6.12) yields a relation between the (extended) hypergeometric function $F = F_k$ and the symmetric Jack polynomials: If $\lambda \in \Lambda_+^n$, then

$$\overline{\lambda} + \frac{k}{2}(n-1) \cdot \underline{1} \ = \ \lambda - \rho \tag{6.13}$$

and therefore we have

$$\frac{P_\lambda(x)}{P_\lambda(\underline{1})} \ = \ F(\lambda - \rho, \log x) = \mathcal{F}(\lambda - \rho, x). \tag{6.14}$$

In particular, we have for $\eta \in \mathbb{N}_0^n$

$$\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \frac{E_\eta(\sigma x)}{E_\eta(\underline{1})} = \frac{P_{\eta_+}(x)}{P_\eta(\underline{1})}. \tag{6.15}$$

The Jack polynomials P_λ satisfy a binomial formula

$$\frac{P_\lambda(\underline{1} + x)}{P_\lambda(\underline{1})} = \sum_{\mu \subseteq \lambda} \binom{\lambda}{\mu} \frac{P_\mu(x)}{P_\mu(\underline{1})}, \tag{6.16}$$

where $\mu \subseteq \lambda$ for $\lambda, \mu \in \Lambda_+^n$ means $\mu_i \leq \eta_i$ for all i, and $\binom{\lambda}{\mu} = \binom{\lambda}{\mu}_k \geq 0$ is a generalized binomial coefficient, the non-negativity is proven in [Sah11].

The following Proposition describes how Jack polynomials are related to Heckman-Opdam polynomials.

Proposition 6.7. *Denote by*

$$\mathbb{R}^n_0 := \{x \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}$$

the subspace of $\mathbb{R}^n$ *spanned by the root system* A_{n-1}. *With respect to the positive roots* $A^+_{n-1} = \{e_j - e_i \mid 1 \le i < j \le n\}$ *the dominant weights are*

$$P_+ = \{\lambda \in \mathbb{R}^n_0 \mid \lambda_j - \lambda_i \in \mathbb{N}_0 \textit{ for all } i < j\}$$

inside the weight lattice

$$P = \{\lambda \in \mathbb{R}^n_0 \mid \lambda_j - \lambda_i \in \mathbb{Z} \textit{ for all } i < j\}.$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^n_0$ *be the orthogonal projection* $\pi(x) = x - \frac{\langle x, \underline{1}\rangle}{n}\underline{1}$. *Then we have:*

(i) $\pi(\mathbb{N}^n_0) = P$. *Furthermore, if* $\sigma_0 \in \mathcal{S}_n$ *is the longest element, i.e.* $\sigma_0 x = (x_n, \ldots, x_1)$, *then* $\pi(\Lambda^n_+) = \sigma_0 P_+ = -P_+$

(ii) *The non-symmetric Heckman-Opdam polynomials* $(E_\lambda(k;\cdot))_{\lambda\in P}$ *are given by the non-symmetric Jack polynomials for* $x \in \mathbb{R}^n_0$ *and* $\eta \in \mathbb{N}^n_0$ *via*

$$E_{\pi(\eta)}(k;x) = E^{1/k}_\eta(e^x).$$

(iii) *The symmetric Heckman-Opdam polynomials* $(P_\lambda(k;\cdot))_{\lambda\in P_+}$ *are given by the symmetric Jack polynomials for* $x \in \mathbb{R}^n_0$ *and* $\lambda \in \Lambda^n_+$ *via*

$$P_{\pi(\sigma_0\lambda)}(k;x) = P^{1/k}_\lambda(e^x).$$

In particular, if $k = d/2$, *where* d *is the Peirce dimension constant of a symmetric cone* Ω, *the spherical polynomials* $(\Phi^\Omega_\lambda)_{\lambda\in\Lambda^n_+}$ *of* Ω *are precisely the symmetric Jack polynomials, i.e.* $\Phi^\Omega_\lambda(x) = P^{2/d}_\lambda(\operatorname{spec} x)$.

Proof.

(i) The relation $\pi(\mathbb{N}^n_0) \subseteq P$ and $\pi(\Lambda^n_+) \subseteq \sigma_0 P_+$ are straightforward computations. If $\mu \in P$, then there exist $\nu_1, \ldots, \nu_{n-1} \in \mathbb{Z}$ with

$$\mu = (\mu_1, \mu_1 + \nu_1, \ldots, \mu_1 + \nu_{n-1}).$$

Hence, we can choose $c \in \mathbb{N}^n_0$ large enough such that $\eta := \mu + c\underline{1} \in \mathbb{N}^n_0$, i.e. $\pi(\eta) = \mu$. Obviously, if $\mu \in P_+$, i.e. $0 \le \nu_1 \le \ldots \le \nu_{n-1}$, then $\sigma_0\mu \in (-P_+) = \sigma_0 P_+$ and $\pi(\sigma_0\eta) = \mu$.

(ii) Recall for $\lambda \in \mathbb{R}^n$

$$\widetilde{\lambda} = \lambda + \tfrac{k}{2} \sum_{\alpha\in A^+_{n-1}} \epsilon(\langle\alpha,\lambda\rangle)\alpha$$

from (4.2) with $\epsilon(t) = 1$ if $t > 0$ and $\epsilon(t) = -1$ if $t \le 0$. Obviously, we have $\widetilde{\pi(\lambda)} = \pi(\widetilde{\lambda})$ and for $\eta \in \mathbb{N}^n_0$

$$\widetilde{\eta}_j = \eta_j + \tfrac{k}{2}\sum_{i<\ell} \epsilon(\eta_\ell - \eta_i)(e_\ell - e_i)_j$$

$$\begin{aligned}&= \eta_j + \tfrac{k}{2}\Big(\#\{i<j \mid \eta_j > \eta_i\} - \#\{i<j \mid \eta_j \le \eta_i\}\\ &\quad + \#\{i>j \mid \eta_j \ge \eta_i\} - \#\{i>j \mid \eta_i > \eta_j\}\Big)\\ &= \eta_j + \tfrac{k}{2}(n-1) - k\#\{i<j \mid \eta_i \ge \eta_j\} - k\#\{i>j \mid \eta_i > \eta_j\}\\ &= \overline{\eta}_j + \tfrac{k}{2}(n-1).\end{aligned}$$

Therefore, Remark 6.6 and Theorem 4.4 lead to

$$\frac{E_{\pi(\eta)}(k;x)}{E_{\pi(\eta)}(k;0)} = G(\widetilde{\pi(\eta)},x) = \mathcal{G}(\widetilde{\eta},e^x) = \mathcal{G}(\overline{\eta}+\tfrac{k}{2}(n-1)\underline{1},e^x) = \frac{E_\eta^{1/k}(e^x)}{E_\eta^{1/k}(\underline{1})}, \quad x\in\mathbb{R}_0^n,$$

showing that there exists a constant $c \neq 0$ with $E_{\pi(\eta)}(k;x) = cE_\eta^{1/k}(e^x)$. In the expansion of $E_{\pi(\eta)}(k;x)$ in terms of $e^\mu, \mu \in P$, the coefficient of $e^{\pi(\eta)}$ equals 1. Thus, it suffices to show that the coefficient of $e^{\pi(\eta)}$ in $E_\eta^{1/k}(e^x)$ also equals 1. Consider the monomial expansion of the homogeneous polynomial $E_\eta^{1/k}$, i.e.

$$E_\eta^{1/k}(e^x) = (e^x)^\eta + \sum_{\substack{\mu\in\mathbb{N}_0^n\setminus\{\eta\}\\ |\mu|\neq|\eta|}} c_{\mu\eta}(e^x)^\mu = e^{\langle\pi\eta,x\rangle} + \sum_{\substack{\mu\in\mathbb{N}_0^n\setminus\{\eta\}\\ |\mu|\neq|\eta|}} c_{\mu\eta}e^{\langle\pi\mu,x\rangle}$$

The projection π is injective on the set $\{\mu\in\mathbb{N}_0^n \mid |\mu| = |\eta|\}$ which leads to the coefficient 1 of $e^{\langle\pi\eta,x\rangle}$ in $E_\eta^{1/k}(e^x)$.

(iii) Let $\lambda \in \Lambda_+^n$. Then $\sigma_0\rho = -\rho$ gives

$$\frac{P_{\pi(\sigma_0\lambda)}(k;x)}{P_{\pi(\sigma_0\lambda)(k;0)}} = F(\pi(\sigma_0\lambda)+\rho,x) = F(\pi\lambda-\rho,x) = \mathcal{F}(\lambda-\rho,e^x) = \frac{P_\lambda^{1/k}(e^x)}{P_\lambda^{1/k}(\underline{1})}.$$

Arguing as in part (ii) leads to $P_{\pi(\sigma_0\lambda)}(k;x) = P_\lambda^{1/k}(e^x)$ for all $x\in\mathbb{R}_0^n$.

■

In the following lemma, we collect some further useful properties of the non-symmetric Jack polynomials $E_\eta = E_\eta^{1/k}$ which can be found in [For10, KS97, Sah98] for $k>0$ and are obvious for $k=0$. Here we consider the Jack polynomials as functions on $\mathbb{C}^n$.

Lemma 6.8. *The non-symmetric Jack polynomials $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ satisfy:*

(i) *For all $p\in\mathbb{N}_0$,*

$$\Delta(z)^p E_\eta(z) = E_{\eta+\underline{p}}(z).$$

By this property, the non-symmetric Jack polynomials uniquely extend to indices $\eta\in\mathbb{Z}^n$.

(ii) *Let $z\in\mathbb{C}^n$ with $z_i \neq 0$ for all $i = 1,\dots,n$. Then*

$$E_\eta\left(\tfrac{1}{z}\right) = E_{-\eta^R}(z^R).$$

(iii) *Let Φ be the so-called raising operator acting on functions $f:\mathbb{C}^n\to\mathbb{C}$ by*

$$\Phi f(z) = z_n f(z_n, z_1, \dots, z_{n-1})$$

and on $\mathbb{N}_0^n$ by

$$\Phi\eta = (\eta_2,\dots,\eta_n,\eta_1+1).$$

Then the non-symmetric Jack polynomials satisfy

$$\Phi E_\eta = E_{\Phi\eta}.$$

According to part (i) this identity extends to all $\eta\in\mathbb{Z}^n$.

(iv) *The coefficients* $c_{\eta\kappa}$ *in the monomial expansion of* $E_\eta, \eta \in N_0^n$*, are non-negative.*

Obviously, part (i),(ii) and (iv) are still correct for symmetric Jack polynomials.

The results in (i) and (ii) can also be deduced from the corresponding formulas of the rational Cherednik kernel in Lemma 6.4. In the following we need polynomial bounds on the values of the Jack polynomials in $\underline{1}$.

Lemma 6.9. *There exists a polynomial* $Q \in \mathcal{P}$ *such that for all* $\eta \in \mathbb{N}_0^n$ *and* $\lambda \in \Lambda_+^n$*,*

$$0 \leq E_\eta(\underline{1}) \leq Q(\eta), \quad 0 \leq P_\lambda(\underline{1}) \leq Q(\lambda).$$

PROOF. By [For10, Prop. 12.3.2],

$$E_\eta(\underline{1}) = \prod_{(i,j)\in\eta} \frac{j + kn - k\ell'(\eta,i,j)}{\eta_i - j + 1 + k\ell(\eta,i,j) + k}$$

with the leg length and coleg length $\ell(\eta,i,j), \ell'(\eta,i,j) \in \{0,\dots,n\}$. Therefore

$$E_\eta(\underline{1}) \leq \prod_{i=1}^{n}\prod_{j=1}^{\eta_i} \frac{j+kn}{\eta_i - j + 1} = \prod_{i=1}^{n} \frac{\Gamma(\eta_i + kn + 1)}{\Gamma(kn+1)\Gamma(\eta_i+1)},$$

which is polynomially bounded in η by Stirling's formula. Similarly (c.f. [For10, Prop. 12.6.2]),

$$P_\lambda(\underline{1}) = \prod_{(i,j)\in\lambda} \frac{j - 1 + kn - kl'(\lambda,i,j)}{\lambda_i - j + kl(\eta,i,j) + k} \leq \prod_{i=1}^{n}\prod_{j=1}^{\lambda_i} \frac{j-1+kn}{\lambda_i - j + k}$$

which is also polynomially bounded in λ. ∎

To formulate the main results of this section, recall from (6.7) Macdonald's gamma function $\Gamma_n(\mu)$ and the generalized Pochhammer symbol $[\mu]_\lambda$ for $\mu \in \mathbb{C}$ and $\lambda \in \Lambda_+^n$. Note that Γ_n differs by the factor $d_n(k)$ from the notion in [Rö20, Mac89], but is in accordance with the notion for the gamma function on symmetric cones. We shall obtain the master theorem as a consequence of the following result, which involves the type A Dunkl operators $T = T(k)$ with multiplicity k.

Theorem 6.10. *Consider the non-symmetric Jack polynomials* $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ *and the symmetric Jack polynomials* $(P_\lambda)_{\lambda\in\Lambda_+^n}$ *of index* $1/k$*. Then for all* $\mu \in \mathbb{C}$ *and all* $x \in \mathbb{R}^n$ *with* $x_i \neq 0$ *for all* $i = 1,\dots,n$*:*

(i) $E_\eta(T)\Delta^{-\mu}(x) = (-1)^{|\eta|}\,[\mu]_{\eta_+} E_\eta(\frac{1}{x})\Delta(x)^{-\mu}$*;*

(ii) $P_\lambda(T)\Delta^{-\mu}(x) = (-1)^{|\lambda|}\,[\mu]_\lambda\, P_\lambda(\frac{1}{x})\Delta(x)^{-\mu}$*.*

For the proof, we need the following lemma.

Lemma 6.11. *The set* $\mathbb{N}_0^n$ *can be recursively constructed from* $0 \in \mathbb{N}_0^n$ *by a chain of the following operations:*

(i) *apply the raising operator* Φ *to* $\eta \in \mathbb{N}_0^n$*,*

(ii) *apply a simple permutation* $s_i = (i, i+1)$ *to* $\eta \in \mathbb{N}_0^n$ *with* $\eta_i < \eta_{i+1}$*.*

Proof. This is easily verified by induction on the weight $|\eta|$. Indeed, assume that all elements of weight at most r are already constructed and take $\eta \in \mathbb{N}_0^n$ with $|\eta| = r + 1$. Consider the maximal index $j = 1, \ldots, n$ with $\eta_j \neq 0$ and $\eta_k = 0$ for $j < k \leq n$. Then

$$\eta = (\eta_1, \ldots, \eta_j, 0, \ldots, 0) = s_j \cdots s_{n-1}(\eta_1, \ldots, \eta_{j-1}, 0, \ldots, 0, \eta_j) = s_j \cdots s_{n-1}\Phi\widehat{\eta}$$

with $\widehat{\eta} = (\eta_j - 1, \eta_1, \ldots, \eta_{j-1}, 0, \ldots, 0)$, which is already constructed by induction hypothesis. ∎

Proof of Theorem 6.10. Part (ii) is obtained from (i) by symmetrization. Part (i) is clear for $\eta = 0$, since $E_0 = 1$. In view of the above observation, it therefore suffices to consider the following two cases:

Case 1. Assume formula (i) is correct for some $\eta \in \mathbb{N}_0^n$ with $\eta_i < \eta_{i+1}$, and consider $E_{s_i\eta}$. According to [For10, Proposition 12.2.1] there exists a constant $d_i^\eta \in \mathbb{R}$ such that

$$E_{s_i\eta} = d_i^\eta E_\eta + s_i E_\eta.$$

The Dunkl operators are $\mathcal{S}_n$-equivariant, i.e. $\sigma T_\xi \sigma^{-1} = T_{\sigma\xi}$, $\sigma \in \mathcal{S}_n$. Hence the symmetry of $\Delta(x)$ leads to

$$\begin{aligned}(s_iE_\eta)(T)\Delta(x)^{-\mu} &= (s_iE_\eta(T)(s_i\Delta)^{-\mu})(x) = E_\eta(T)\Delta^{-\mu}(s_ix) \\ &= (-1)^{|\eta|}\,[\mu]_{\eta_+}E_\eta(\tfrac{1}{s_ix})\Delta(s_ix)^{-\mu} \\ &= (-1)^{|\eta|}\,[\mu]_{\eta_+}(s_iE_\eta)\left(\tfrac{1}{x}\right)\Delta(x)^{-\mu}\end{aligned}$$

As $|s_i\eta| = |\eta|$ and $(s_i\eta)_+ = \eta_+$, the formula follows for $s_i\eta$ by linear combination.

Case 2. Assume that formula (i) is correct for some $\eta \in \mathbb{N}_0^n$, and consider $\Phi\eta$. Using the identity $\Phi E_\eta = E_{\Phi\eta}$ from Lemma 6.8 and the product rule for the Dunkl operators, we calculate

$$\begin{aligned}E_{\Phi\eta}(T)\Delta(x)^{-\mu} &= T_nE_\eta(T_n, T_1, \ldots, T_{n-1})\Delta(x)^{-\mu} \\ &= T_n\Big((-1)^{|\eta|}\,[\mu]_{\eta_+}E_\eta(\tfrac{1}{x_n}, \tfrac{1}{x_1}, \ldots, \tfrac{1}{x_{n-1}})\Delta(x)^{-\mu}\Big) \\ &= (-1)^{|\eta|}[\mu]_{\eta_+}\Big((T_n\,\Delta(x)^{-\mu})E_\eta(\tfrac{1}{x_n}, \tfrac{1}{x_1}, \ldots, \tfrac{1}{x_{n-1}}) \\ &\quad + \Delta(x)^{-\mu}(T_nE_\eta(\tfrac{1}{x_n}, \tfrac{1}{x_1}, \ldots, \tfrac{1}{x_{n-1}}))\Big).\end{aligned} \tag{6.17}$$

As T_n acts on symmetric functions as the partial derivative $\frac{\partial}{\partial x_n}$, we have

$$T_n\Delta(x)^{-\mu} = -\mu\, x_n^{-1}\Delta(x)^{-\mu}\,.$$

Parts (i) and (ii) of Proposition 6.8 show that

$$E_\eta(\tfrac{1}{x_n}, \tfrac{1}{x_1}, \ldots, \tfrac{1}{x_{n-1}}) = \Delta^{-p}(x)E_{\eta^*}(x_{n-1}, \ldots, x_1, x_n)$$

with $\eta^* = -\eta^R + \underline{p}$, where $p \in \mathbb{N}$ is so large that $-\eta^R + \underline{p} \in \mathbb{N}_0^n$. Note further that $\frac{1}{x_n}E_\eta(\frac{1}{x_n}, \frac{1}{x_1}, \ldots, \frac{1}{x_{n-1}}) = E_{\Phi\eta}(\frac{1}{x})$. Thus formula (6.17) reduces to

$$\begin{aligned}&E_{\Phi\eta}(T)\Delta(x)^{-\mu} \\ &= (-1)^{|\eta|}[\mu]_{\eta_+}\Delta(x)^{-\mu}\big(-\mu E_{\Phi\eta}(\tfrac{1}{x}) + T_n(\Delta^{-p}(x)E_{\eta^*}(x_{n-1}, \ldots, x_1, x_n))\big).\end{aligned} \tag{6.18}$$

Again by the product rule for T_n and the fact that T_n commutes with $s_1, \ldots, s_{n-2}$, we further obtain

$$\begin{aligned}
&T_n(\Delta^{-p}(x)E_{\eta^*}(x_{n-1},\ldots,x_1,x_n)) \\
&= -p\Delta(x)^{-p}E_{\eta^*}(x_{n-1},\ldots,x_1,x_n) + \Delta(x)^{-p}(T_nE_{\eta^*}(x_{n-1},\ldots,x_1,x_n)) \\
&= -px_n^{-1}\Delta(x)^{-p}E_{\eta^*}(x_{n-1},\ldots,x_1,x_n) + \Delta(x)^{-p}(T_nE_{\eta^*})(x_{n-1},\ldots,x_1,x_n) \\
&= x_n^{-1}\Delta(x)^{-p}\Big(-pE_{\eta^*}(x_{n-1},\ldots,x_1,x_n) + (x_nT_nE_{\eta^*})(x_{n-1},\ldots,x_1,x_n)\Big).
\end{aligned}$$

As $x_nT_n = \mathcal{D}_n + k(n-1)$, we have

$$x_nT_nE_{\eta^*}(x_{n-1},\ldots,x_1,x_n) = (\overline{\eta^*}_n + k(n-1))E_{\eta^*}(x_{n-1},\ldots,x_1,x_n)$$

with $\overline{\eta^*}_n = \eta_n^* - k\#\{\ell < n \mid \eta_\ell^* \geq \eta_n^*\}$, so that

$$\begin{aligned}
&T_n(\Delta^{-p}(x)E_{\eta^*}(x_{n-1},\ldots,x_1,x_n)) \\
&= \big(-p + (\overline{\eta^*}_n + k(n-1))\big)\Delta(x)^{-p}\tfrac{1}{x_n}E_{\eta^*}(x_{n-1},\ldots,x_1,x_n) \\
&= \big(-p + (\overline{\eta^*}_n + k(n-1))\big)\tfrac{1}{x_n}E_\eta(\tfrac{1}{x_n},\tfrac{1}{x_1},\ldots,\tfrac{1}{x_{n-1}}) \\
&= \big(-p + (\overline{\eta^*}_n + k(n-1))\big)E_{\Phi\eta}(\tfrac{1}{x}).
\end{aligned}$$

Thus equation (6.18) reduces to

$$E_{\Phi\eta}(T)\Delta(x)^{-\mu} = (-1)^{|\eta|}[\mu]_{\eta+}(-\mu - p + \overline{\eta^*}_n + k(n-1))E_{\Phi\eta}(\tfrac{1}{x})\Delta(x)^{-\mu}. \tag{6.19}$$

Let $1 \leq j \leq n$ be minimal such that the j-th entry in η_+ is equal to η_1, i.e.

$$j - 1 = \#\{\ell > 1 \mid \eta_1 < \eta_\ell\}. \tag{6.20}$$

At position j in $(\Phi\eta)_+$ is $\eta_1 + 1$. Thus, by definition of j we have

$$\begin{aligned}
\overline{\eta^*}_n &= (\eta^R + p)_n - k\#\Big\{\ell < n \mid -\eta_\ell^R + p \geq -\eta_n^R + p\Big\} \\
&= p + \eta_1 - k\#\{\ell < n \mid \eta_{n-\ell+1} \leq \eta_1\} \\
&= p + \eta_1 - k\#\{\ell > 1 \mid \eta_\ell \leq \eta_1\} \\
&= p + \eta_1 - k(n-j).
\end{aligned}$$

So finally, since j is the position of $\eta_1 + 1 = (\Phi\eta)_n$ in $(\Phi\eta)_+$ we have that $(\Phi\eta)_+$ is exactly η_+ plus an 1 at position j. Therefore

$$\begin{aligned}
&(-1)^{|\eta|}[\mu]_{\eta+}(-\mu - p + \overline{\eta^*}_n + k(n-1)) \\
&= (-1)^{|\eta|}[\mu]_{\eta+}(-\mu + \eta_1 - k(n-j) + k(n-1)) \\
&= (-1)^{|\eta|+1}[\mu]_{\eta+}(\mu - k(j-1) + (\eta_1 + 1) - 1) \\
&= (-1)^{|\Phi\eta|}[\mu]_{(\Phi\eta)_+}.
\end{aligned}$$

Plugging this into (6.19) we obtain the assertion. ■

Theorem 6.12. *Let $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ and $(P_\lambda)_{\lambda\in\Lambda_+^n}$ be the non-symmetric and symmetric Jack polynomials of index $1/k, k \geq 0$. Then for all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \mu_0$ and $z \in \mathbb{C}^n$ with Re $z > 0$,*

(i) $\displaystyle\int_{\mathbb{R}_+^n} E^A(-z,x)E_\eta(x)\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x = \Gamma_n(\eta_+ + \underline{\mu})E_\eta(\tfrac{1}{z})\Delta(z)^{-\mu};$

(ii) $\displaystyle\int_{\mathbb{R}_+^n} J^A(-z,x)P_\lambda(x)\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x = \Gamma_n(\lambda+\underline{\mu})P_\lambda(\tfrac{1}{z})\Delta(z)^{-\mu}.$

Part (ii) is just Macdonald's [Mac89] Conjecture (C), and part (i) corresponds to formula [BF98, Formula (4.38)] (there is a misprint: the Laguerre polynomial $E_\eta^{(L)}$ has to be replaced by E_η).

Proof. The integrals converge by Lemma 6.2. According to [Rö20],

$$\Delta(z)^{-\mu} = \frac{1}{\Gamma_n(\mu)}\mathcal{L}(\Delta^{\mu-\mu_0-1})(z),$$

and for each polynomial $p \in \mathcal{P}$, by Lemma 6.2,

$$p(-T)\Delta^{-\mu}(z) = \frac{1}{\Gamma_n(\mu)}\mathcal{L}(p\,\Delta^{\mu-\mu_0-1})(z).$$

Now part (i) is immediate from Theorem 6.10 (i) and part (ii) follows by symmetrization. ■

6.4 Laplace transform of the Cherednik kernel

In this section we shall extend the statements of Theorem 6.12 to the rational Cherednik kernel $\mathcal{G}$ and the hypergeometric function $\mathcal{F}$. The extension of Theorem 6.12 will be carried out by analytic extension with respect to the spectral parameter, which is based on the following generalization of the classical Carlson theorem [Tit39, p.186].

Lemma 6.13. *Let $U \subseteq \mathbb{C}^n$ be an open neighborhood of $\{\mathrm{Re}\, z \geq 0\} \subseteq \mathbb{C}^n$ and let $f : U \to \mathbb{C}$ be holomorphic. Put $\|z\|_1 := \sum_{i=1}^n |z_i|$. If f satisfies*

$$f(z) = \mathcal{O}(e^{c\|z\|_1}) \text{ for some } c < \pi \text{ and } f|_{\Lambda_+^n} \equiv 0, \tag{6.21}$$

then $f \equiv 0$.

Proof. We proceed by induction on n. The case $n = 1$ is Carlson's classical theorem. To achieve step $n-1 \to n$, consider for fixed $\lambda \in \Lambda_+^n$ the holomorphic function

$$f_\lambda : U' \to \mathbb{C}, \quad \xi \mapsto f(\xi + \lambda_1, \lambda_2, \ldots, \lambda_n)$$

where $U' \subseteq \mathbb{C}$ is a suitable neighborhood of $\{\mathrm{Re}\,\xi + \lambda_1 \geq 0\} \subseteq \mathbb{C}$. Then $f_\lambda|_{\mathbb{N}_0} \equiv 0$ and

$$f_\lambda(\xi) = \mathcal{O}(e^{c|\xi|})$$

with c as in (6.21). Therefore f_λ vanishes identically by the classical Carlson theorem. From this we conclude that for fixed $\xi \in \mathbb{C}$ with $\mathrm{Re}\,\xi \geq 0$, the function

$$g_\xi : \widetilde{U} \to \mathbb{C}, \quad w \mapsto f(\xi, w)$$

vanishes on Λ_+^{n-1} for some suitable neighborhood $\widetilde{U} \subseteq \mathbb{C}^{n-1}$ of $\{\mathrm{Re}\, w \geq 0\}$. Moreover

$$g_\xi(w) = \mathcal{O}(e^{c\|(\xi,w)\|_1}) = \mathcal{O}(e^{c\|w\|_1}),$$

and by the induction hypothesis we obtain that g_ξ vanishes identically. As ξ was arbitrary, we obtain $f \equiv 0$. ■

We are now in the position to prove the following generalization of Theorem 6.12.

Theorem 6.14. *Recall that* $H = \{z \in \mathbb{C}^n \mid \mathrm{Re}\, z > 0\}$. *Let* $\mu \in \mathbb{C}$ *with* $\mathrm{Re}\,\mu > \mu_0 = k(n-1)$. *Then for all* $z \in H$ *and* $\lambda \in \overline{H} + \mathrm{co}(\mathcal{S}_n.\rho)$

(i) $$\int_{\mathbb{R}_+^n} E^A(-z,x)\,\mathcal{G}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \;=\; \Gamma_n(\lambda+\rho+\underline{\mu})\,\mathcal{G}(\lambda,\tfrac{1}{z})\,\Delta(z)^{-\mu}.$$

(ii) $$\int_{\mathbb{R}_+^n} J^A(-z,x)\,\mathcal{F}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \;=\; \Gamma_n(\lambda+\rho+\underline{\mu})\,\mathcal{F}(\lambda,\tfrac{1}{z})\,\Delta(z)^{-\mu}.$$

In view of Lemma 6.4, the above Theorem can be equivalently reformulated as the following generalization of the Laplace transform identities for spherical functions on a symmetric cones as stated in Theorem 5.5.

Corollary 6.15. *Suppose that* $\lambda \in \underline{\mu_0} + \overline{H} + \mathrm{co}(\mathcal{S}_n.\rho)$. *Then, for all* $z \in H$:

(i) $$\int_{\mathbb{R}_+^n} E^A(-z,x)\,\mathcal{G}(\lambda,x)\,\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x \;=\; \Gamma_n(\lambda+\rho)\,\mathcal{G}(\lambda,\tfrac{1}{z});$$

(ii) $$\int_{\mathbb{R}_+^n} J^A(-z,x)\,\mathcal{F}(\lambda,x)\,\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x \;=\Gamma_n(\lambda+\rho)\,\mathcal{F}(\lambda,\tfrac{1}{z}).$$

Proof of Theorem 6.14. It suffices to check part (i). By Carlson's theorem, we shall prove that

$$\frac{1}{\Gamma_n(\lambda+\rho+\underline{\mu})}\int_{\mathbb{R}_+^n} E^A(-z,x)\,\mathcal{G}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \;=\; \mathcal{G}(\lambda,\tfrac{1}{z})\,\Delta(z)^{-\mu}. \tag{6.22}$$

Note first that (6.22) holds for all $\lambda \in \Lambda_+^n - \rho$ by Theorem 6.12 and equations (6.12), (6.13). The right hand side of (6.22) is holomorphic in (λ, z, μ) on $\mathbb{C}^n \times H \times \mathbb{C}$. Moreover, the left hand side exists and is continuous on $(H + \mathrm{co}(\mathcal{S}_n.\rho)) \times \overline{H} \times \{\mathrm{Re}\,\mu > \mu_0\}$ and holomorphic on the interior. Indeed, $(\lambda,\mu) \mapsto \Gamma_n(\lambda+\rho+\underline{\mu})$ is holomorphic on $(H + \mathrm{co}(\mathcal{S}_n.\rho)) \times \{\mathrm{Re}\,\mu > \mu_0\}$ and continuous on the closure. Furthermore, suppose that $\mathrm{Re}\, z \geq \underline{s}$ for some $s > 0$. Then, by estimate (6.8),

$$\left|E^A(-z,x)\right| \leq E^A(-\mathrm{Re}\, z, x) \;\leq\; e^{-\langle \underline{s}, x\rangle}.$$

Together with Lemma 6.4 we obtain for $x \in \mathbb{R}_+^n$, $\lambda \in H$ and $\lambda' \in \mathrm{co}(\mathcal{S}_n.\rho)$.

$$\left|E^A(-z,x)\,\mathcal{G}(\lambda+\lambda',x)\Delta(x)^{\mu-\mu_0-1}\right| \leq C\cdot e^{-\langle \underline{s},x\rangle}\Delta(x)^{\mathrm{Re}\,\mu-\mu_0-1}\max_{\sigma\in\mathcal{S}_n} x^{\mathrm{Re}(\sigma\lambda)}.$$

Hence the integral on the left hand side of formula (6.22) exists and is (by standard arguments) continuous and holomorphic as stated. It therefore suffices to check (6.22) for $z \in \mathbb{R}^n$ with $z > \underline{1}$, $\mu \in \mathbb{R}$ with $\mu > \mu_0$ and $\lambda \in H$. We want to apply Carlson's Theorem 6.13 with respect to λ. As $z > \underline{1}$, the right hand side of (6.22) is bounded in λ according to Lemma 6.4, and it remains to control the growth of the left hand side. For $\lambda \in H$, define $\eta(\lambda) := (\lceil \mathrm{Re}\,\lambda_1)\rceil, \ldots, \lceil \mathrm{Re}\,\lambda_n)\rceil)_+ \in \Lambda_+^n$. Then for arbitrary $x \in \mathbb{R}_+^n$,

$$\max_{\sigma\in\mathcal{S}_n} x^{\sigma(\mathrm{Re}\,\lambda)} \;\leq\; \max_{\sigma\in\mathcal{S}_n}(\underline{1}+x)^{\sigma(\mathrm{Re}\,\lambda)} \;\leq\; P_{\eta(\lambda)}(\underline{1}+x),$$

because the coefficients of the Jack polynomial $P_{\eta(\lambda)}$ in its monomial expansion are nonnegative with coefficient 1 for $x^{\sigma\eta(\lambda)}$, $\sigma \in \mathcal{S}_n$. Now recall the binomial formula (6.16) for the Jack polynomials as well as the identity

$$\int_{\mathbb{R}_+^n} e^{-\langle \underline{1},x\rangle} P_\kappa(x)\Delta(x)^{\mu-\mu_0-1}\omega(x)\,\mathrm{d}x \;=\; \Gamma_n(\kappa+\underline{\mu})P_\kappa(\underline{1})$$

from [Mac89, (6.18)] (c.f. also [Rö20, Lemma 5.1]). We may therefore estimate

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |E^A(-z,x)\,\mathcal{G}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}|\omega^A(x)\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^n_+} e^{-\langle \underline{1},x\rangle}\,P_{\eta(\lambda)}(\underline{1}+x)\,\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \\
&= \sum_{\kappa\subseteq\eta(\lambda)} \binom{\eta(\lambda)}{\kappa} \int_{\mathbb{R}^n_+} e^{-\langle \underline{1},x\rangle} P_\kappa(x)\,\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \\
&= \sum_{\kappa\subseteq\eta(\lambda)} \binom{\eta(\lambda)}{\kappa} P_\kappa(\underline{1})\Gamma_n(\kappa+\underline{\mu}).
\end{aligned}$$

By monotonicity of the classical gamma function,

$$\Gamma_n(\kappa+\underline{\mu}) \leq \Gamma_n(\eta(\lambda)+\underline{\mu}) \leq \Gamma_n((\operatorname{Re}\lambda)_+ + \underline{1}+\underline{\mu}).$$

Moreover, by Remark 6.9,

$$\sum_{\kappa\subseteq\eta(\lambda)} \binom{\eta(\lambda)}{\kappa} P_\kappa(\underline{1}) \;=\; P_{\eta(\lambda)}(\underline{2}) \;=\; 2^{|\eta(\lambda)|}P_{\eta(\lambda)}(\underline{1}) \;\leq\; 2^{\|\lambda\|_1}\cdot Q(\lambda)$$

with some polynomial $Q\in\mathcal{P}$. Therefore

$$\begin{aligned}
I_{z,\mu}(\lambda) &:= \left|\frac{1}{\Gamma_n(\lambda+\rho+\underline{\mu})}\int_{\mathbb{R}^n_+} E(-z,x)\,\mathcal{G}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}\omega(x)\,\mathrm{d}x\right| \\
&\leq Q(\lambda)\cdot\frac{\Gamma_n((\operatorname{Re}\lambda)_+ + 1+\underline{\mu})}{\left|\Gamma_n(\lambda+\rho+\underline{\mu})\right|}\cdot 2^{\|\lambda\|_1}.
\end{aligned}$$

Then we make the following decomposition

$$\frac{\Gamma_n((\operatorname{Re}\lambda)_+ + 1+\underline{\mu})}{\left|\Gamma_n(\lambda+\rho+\underline{\mu})\right|} = \prod_{j=1}^n \frac{\Gamma(((\operatorname{Re}\lambda)_+)_j+\mu+1-k(j-1))}{|\Gamma(\lambda_j+\mu+\rho(k)_j-k(j-1))|} \;=\; F_1(\lambda)\cdot F_2(\lambda)$$

with the functions

$$\begin{aligned}
F_1(\lambda) &= \prod_{j=1}^n \frac{\Gamma((\operatorname{Re}(\lambda)_+)_j+\mu+1-k(j-1))}{\Gamma(\operatorname{Re}(\lambda)_j)+\mu+1-\frac{k}{2}(n-1))}, \\
F_2(\lambda) &= \prod_{j=1}^n \frac{\left|\lambda_j+\mu-\frac{k}{2}(n-1)\right|\cdot\Gamma(\operatorname{Re}\lambda_j+\mu+1-\frac{k}{2}(n-1))}{\left|\Gamma(\lambda_j+\mu+1-\frac{k}{2}(n-1))\right|}.
\end{aligned}$$

By Stirling's formula, $F_1(\lambda)$ is polynomially bounded, i.e. $F_1(\lambda)=\mathcal{O}(e^{\epsilon\|\lambda\|_1})$ for arbitrary $\epsilon>0$. For F_2, we employ the estimate ([NIS10, Formula 5.6.7])

$$\frac{\Gamma(x)}{|\Gamma(x+iy)|} \leq \sqrt{\cosh(\pi y)} = \mathcal{O}(e^{\frac{\pi}{2}|y|}), \quad x>\tfrac{1}{2}, y\in\mathbb{R},$$

which leads to

$$F_2(\lambda) \;=\; \mathcal{O}(e^{(\epsilon+\frac{\pi}{2})\|\lambda\|_1})$$

with arbitrary $\epsilon>0$. Putting things together, we obtain that $I_{z,\mu}(\lambda)$ satisfies the growth condition of Carlson's Theorem 6.13, which finishes the proof. ■

6.5 Macdonald's hypergeometric series and Laplace transform identities

In the setting of symmetric cones, the Laplace transform establishes important identities between hypergeometric series. Analogous formulas were formally stated by Macdonald [Mac89] for general Jack-hypergeometric series, as consequences of his conjecture (C). With Theorem 6.12 at hand, we shall make these identities precise, and extend them to hypergeometric expansions in terms of non-symmetric Jack polynomials. We start with an appropriate normalization of the symmetric and non-symmetric Jack polynomials.

Lemma 6.16. *Consider the non-symmetric and symmetric Jack polynomials $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ and $(P_\lambda)_{\lambda\in\Lambda_+^n}$ of index $1/k$, respectively. Then:*

(i) *There exist $c_\eta > 0$ for all $\eta\in\mathbb{N}_0^n$ such that the renormalized Jack polynomials $C_\lambda := c_\lambda P_\lambda$ and $L_\eta := c_\eta E_\eta$ satisfy for all $m\in\mathbb{N}_0$*

$$\sum_{\substack{\lambda\in\Lambda_+^n\\|\lambda|=m}} C_\lambda(z) = \sum_{\substack{\eta\in\mathbb{N}_0^n\\|\eta|=m}} L_\eta(z) \;=\; (z_1+\ldots+z_n)^m.$$

(ii) $C_\lambda = \sum_{\eta\in\mathcal{S}_n.\lambda} L_\eta$ *for all* $\lambda\in\Lambda_+^n$.

(iii) $c_\lambda \leq \dfrac{|\lambda|!}{\lambda!}$ *for all* $\lambda\in\Lambda_+^n$ *with* $\lambda! = \lambda_1!\cdots\lambda_n!$.

Proof. We may assume that $k>0$. Part (i) for the symmetric Jack polynomials is well-known (see e.g. [For10, (12.135)]), with

$$c_\lambda = \frac{|\lambda|!}{k^{|\lambda|}d'_\lambda}.$$

Here the constants d'_η for $\eta\in\mathbb{N}_0^n$ are given by

$$d'_\eta = \prod_{(i,j)\in\eta}\Big(\frac{1}{k}(\eta_i-j+1)+\ell(\eta,i,j)\Big) > 0,$$

with the leg length $\ell(\eta,i,j) = \#\{\ell>i \mid j\leq\eta_\ell\leq\eta_i\} + \#\{\ell<i\mid j\leq\eta_\ell+1\leq\eta_i\}$. In particular, for each partition $\lambda\in\Lambda_+^n$ we have

$$c_\lambda = \frac{|\lambda|!}{\prod\limits_{(i,j)\in\lambda}((\lambda_i-j+1)+k\ell(\lambda,i,j))} \leq \frac{|\lambda|!}{\prod\limits_{1\leq j\leq\lambda_i}(\lambda_i-j+1)} = \frac{|\lambda|!}{\lambda!},$$

which is part (iii). From [For10, Proposition 12.6.1] it is further known that

$$P_\lambda = d'_\lambda\sum_{\eta\in\mathcal{S}_n.\lambda}\frac{1}{d'_\eta}E_\eta. \tag{6.23}$$

Hence, we put

$$c_\eta := c_{\eta+}\frac{d'_{\eta+}}{d'_\eta} = \frac{|\eta|!}{k^{|\eta|}\,d'_\eta}$$

for $\eta \in \mathbb{N}_0^n$, and part (i) for the non-symmetric Jack polynomials follows. Finally, part (ii) is immediate from the definition of c_η and relation (6.23). ∎

Recall from Definition 1.9 the generalized Fisher inner product on the space $\mathcal{P}_\mathbb{R} = \mathbb{R}[\mathbb{R}^n]$ of real polynomials on $\mathbb{R}^n$ defined by

$$[p, q] = (p(T)q)(0).$$

Polynomials with different homogeneous degree are orthogonal with respect to this pairing, and $[T_\xi p, q] = [p, \langle \cdot, \xi\rangle\, q]$. This property and the invariance under the action of $\mathcal{S}_n$ show that the Cherednik operators $\mathcal{D}_j$ are symmetric with respect to the Dunkl pairing. In particular, the non-symmetric Jack polynomials $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ form an orthogonal basis of $\mathcal{P}_\mathbb{R}$ with respect to $[\cdot,\cdot]$. More precisely, their renormalizations $L_\eta = c_\eta E_\eta$ satisfy

$$[L_\eta, L_\kappa] = |\eta|!\, L_\eta(\underline{1}) \cdot \delta_{\eta,\kappa} \tag{6.24}$$

which is obtained by combining Lemma 6.16 and [BF98, Formula (2.4)].

Lemma 6.17. *The Dunkl kernel of type A_{n-1} with multiplicity $k \geq 0$ satisfies*

$$E^A(z, w) = \sum_{\eta\in\mathbb{N}_0^n} \frac{L_\eta(z)L_\eta(w)}{|\eta|!\, L_\eta(\underline{1})}.$$

The series converges locally uniformly on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. This is immediate from [Rö98, Lemma 3.1] together with identity (6.24). Alternatively, the stated expansion follows from [For10, Propos. 13.3.4]. ∎

Definition 6.18. Consider the Jack polynomials $(L_\eta)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda)_{\lambda\in\Lambda_+^n}$ of index $\alpha = \frac{1}{k}$, respectively (normalized as above). Following [Mac89], [Kan93] and [BF98], we define for indices $\mu \in \mathbb{C}^p$ and $\nu \in \mathbb{C}^q$ with $p, q \in \mathbb{N}_0$ the *non-symmetric hypergeometric series*

$${}_pK_q(\mu; \nu; z, w) := \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu_1]_{\eta^+}\cdots[\mu_p]_{\eta^+}}{[\nu_1]_{\eta^+}\cdots[\nu_q]_{\eta^+}}\, \frac{L_\eta(z)L_\eta(w)}{|\eta|!\, L_\eta(\underline{1})}$$

as well as the *symmetric hypergeometric series*

$${}_pF_q(\mu; \nu; z, w) := \sum_{\lambda\in\Lambda_+^n} \frac{[\mu_1]_\lambda\cdots[\mu_p]_\lambda}{[\nu_1]_\lambda\cdots[\nu_q]_\lambda}\, \frac{C_\lambda(z)C_\lambda(w)}{|\lambda|!\, C_\lambda(\underline{1})}.$$

More common in the literature are hypergeometric series in one variable which are obtained as functions in z with $w = \underline{1}$. For abbreviation, we write for $\lambda \in \Lambda_+^n$

$$[\mu]_\lambda := [\mu_1]_\lambda\cdots[\mu_p]_\lambda; \quad [\nu]_\lambda := [\nu_1]_\lambda\cdots[\nu_q]_\lambda.$$

Note that for $p = 0$ or $q = 0$, an empty product with value 1 occurs. For those values of k for which the $C_\lambda = C_\lambda^{1/k}$ are the spherical polynomials of a symmetric cone, the convergence properties of ${}_pF_q$-hypergeometric series in one variable are well-known, see [FK94, GR87]. For general $k > 0$, partial results on the domain of convergence of ${}_pF_q$ were obtained in [Kan93]. For some values of p and q, the non-symmetric series ${}_pK_q$ were considered in [BF98]. But to our knowledge, their convergence properties have not been studied so far.

Lemma 6.19. *The non-symmetric and symmetric hypergeometric functions are related by*

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} {}_pK_q(\mu; \nu; \sigma z, w) = {}_pF_q(\mu; \nu; z, w).$$

PROOF. By identity (6.15) and Lemma 6.16 we have

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} {}_pK_q(\mu;\nu;\sigma z,w) = \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu]_{\eta+}}{[\nu]_{\eta+}}\frac{1}{|\eta|!}L_\eta(w)\frac{C_{\eta+}(z)}{C_{\eta+}(\underline{1})}$$
$$= \sum_{\lambda\in\Lambda_+^n} \frac{[\mu]_\lambda}{[\nu]_\lambda}\frac{1}{|\lambda|!}\Big(\sum_{\eta\in\mathcal{S}_n.\lambda} L_\eta(w)\Big)\frac{C_\lambda(z)}{C_\lambda(\underline{1})} = {}_pF_q(\mu;\nu;w,w).$$

■

Theorem 6.20. *Let $\mu\in\mathbb{C}^p$ and $\nu\in\mathbb{C}^q$ with $\nu_i\notin\{0,k,\ldots,k(n-1)\}-\mathbb{N}_0$ for all $i=1,\ldots,n$ (i.e. $[\nu]_\lambda\neq 0$ for all $\lambda\in\Lambda_+^n$).*

(i) *If $p\le q$, the series ${}_pK_q(\mu;\nu;\cdot,\cdot)$ and ${}_pF_q(\mu;\nu;\cdot,\cdot)$ are entire functions.*

(ii) *If $p=q+1$, the series ${}_pK_q(\mu;\nu;\cdot,\cdot)$ and ${}_pF_q(\mu;\nu;\cdot,\cdot)$ are holomorphic on the domain $\{(z,w)\in\mathbb{C}^n\times\mathbb{C}^n : \|z\|_\infty\|w\|_\infty<1\}$.*

Moreover, the hypergeometric series are holomorphic in the parameters (μ,ν) on the domain

$$\{(\mu,\nu)\in\mathbb{C}^p\times\mathbb{C}^q \mid \nu_i\notin\{0,k,\ldots,k(n-1)\}-\mathbb{N}_0 \text{ for all } i=1,\ldots,n\}.$$

PROOF. It suffices to verify the statements for ${}_pK_q$. From Lemma 6.8 we have $|L_\eta(z)|\le L_\eta(|z|)\le L_\eta(\underline{1})\,\|z\|_\infty^{|\eta|}$ and therefore

$$S(\mu,\nu;z,w) := \sum_{\eta\in\mathbb{N}_0^n}\left|\frac{[\mu]_{\eta+}}{[\nu]_{\eta+}}\right|\cdot\left|\frac{L_\eta(z)L_\eta(w)}{|\eta|!\,L_\eta(\underline{1})}\right| \le \sum_{\eta\in\mathbb{N}_0^n}\left|\frac{[\mu]_{\eta+}}{[\nu]_{\eta+}}\right|\cdot\frac{\|z\|_\infty^{|\eta|}\|w\|_\infty^{|\eta|}}{|\eta|!}L_\eta(\underline{1})$$
$$= \sum_{\lambda\in\Lambda_+^n}\left|\frac{[\mu]_\lambda}{[\nu]_\lambda}\right|\cdot\frac{\|z\|_\infty^{|\lambda|}\|w\|_\infty^{|\lambda|}}{|\lambda|!}C_\lambda(\underline{1}),$$

where for the last identity, Lemma 6.16 was used. From Lemmata 6.9 and 6.16 we know that

$$C_\lambda(\underline{1}) = c_\lambda P_\lambda(\underline{1}) \le \frac{|\lambda|!}{\lambda!}Q(\lambda) \tag{6.25}$$

with some polynomial $Q\in\mathcal{P}$. Therefore, we can find to each $\epsilon>1$ a constant $C_\epsilon>0$ such that $Q(\lambda)\le C_\epsilon\,\epsilon^{|\lambda|}$. This gives

$$S(\mu,\nu;z,w)\le C_\epsilon\sum_{\lambda\in\Lambda_+^n}\left|\frac{[\mu]_\lambda}{[\nu]_\lambda}\right|\cdot\frac{\big(\epsilon\|z\|_\infty\|w\|_\infty\big)^{|\lambda|}}{\lambda!}. \tag{6.26}$$

To prove part (i), consider the case $p\le q$. In this case, the quotient

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \frac{\prod_{i=1}^p[\mu_i]_\lambda}{\prod_{i=1}^q[\nu_i]_\lambda}$$

is of polynomial growth in λ. To see this, write

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \prod_{i=1}^p\prod_{j=1}^n\frac{\Gamma(\nu_i-k(j-1))}{\Gamma(\mu_i-k(j-1))}\frac{\Gamma(\mu_i+\lambda_j-k(j-1))}{\Gamma(\nu_i+\lambda_j-k(j-1))}.$$

By Stirling's formula we have, locally uniformly in μ and ν,

$$\frac{\Gamma(\mu_i+\lambda_j-k(j-1))}{\Gamma(\nu_i+\lambda_j-k(j-1))}\sim(\nu_i+\lambda_j-k(j-1))^{\nu_i-\mu_i}\text{ for }\lambda_j\to\infty.$$

Moreover, $|[\nu_i]_\lambda|\geq 1$ for large λ. Thus, for each $\epsilon>1$ there are constant $D>0$ and a compact neighborhood $K\subseteq\mathbb{C}^p\times\mathbb{C}^q$ of (μ,ν), such that

$$\left|\frac{[\mu]_\lambda}{[\nu]_\lambda}\right|\leq D\epsilon^{|\lambda|}\text{ for all }(\mu,\nu)\in K.$$

Hence, for each $\epsilon>1$, we find a constant $C_\epsilon>0$ such that

$$\begin{aligned}S(\mu,\nu;z,w)\;&\leq\;C_\epsilon\sum_{\lambda\in\Lambda_+^n}\frac{(\epsilon\|z\|_\infty\|w\|_\infty)^{|\lambda|}}{\lambda!}\;\leq\;C_\epsilon\sum_{\lambda\in\mathbb{N}_0^n}\frac{(\epsilon\|z\|_\infty\|w\|_\infty)^{|\lambda|}}{\lambda!}\\&\leq\;C_\epsilon\,e^{n\epsilon\|z\|_\infty\|w\|_\infty}.\end{aligned}$$

Therefore the ${}_pK_q$-series is converges locally uniformly on $\mathbb{C}^n\times\mathbb{C}^n$ and also locally uniformly on the stated domain of parameters μ and ν, which proves part (i). For part (ii), observe that for $p=q+1$, we have

$$\frac{[\mu]_\lambda}{[\nu]_\lambda}=\frac{\prod_{i=1}^q[\mu_i]_\lambda}{\prod_{i=1}^q[\nu_i]_\lambda}\cdot[\mu_p]_\lambda\,.$$

As in part (i), the first factor is of polynomial growth. Moreover,

$$\frac{[\mu_p]_\lambda}{\lambda!}=\prod_{j=1}^n\frac{\Gamma(\mu_p-k(j-1)+\lambda_j)}{\Gamma(\lambda_j+1)\Gamma(\mu_p-k(j-1))},$$

which is of polynomial growth as well. Starting from estimate (6.26), we therefore obtain that for each $\epsilon>1$, there is a constant $C_\epsilon>0$ with

$$S(\mu,\nu;z,w)\;\leq\;C_\epsilon\sum_{\lambda\in\Lambda_+^n}(\epsilon\|z\|_\infty\|w\|_\infty)^{|\lambda|}\leq C_\epsilon\,\frac{1}{(1-\epsilon\|z\|_\infty\|w\|_\infty)^n}\,.$$

This yields the claim. ■

Note that part (ii) of this theorem improves, in the case $w=1$, the results of [Kan93].

Remark 6.21. For $p=q=0$, one gets the Dunkl kernel and Bessel function of type A_{n-1}, respectively. Indeed, Lemma 6.17 just says that

$$E^A(z,w)={}_0K_0(z,w),$$

and symmetrization yields

$$J^A(z,w)={}_0F_0(z,w),$$

which was already noted in [BF98].

Remark 6.22. The proof of Theorem 6.20 shows that for $p\leq q$ and arbitrary $\epsilon>1$ there is a constant $C_\epsilon>0$ such that

$$\left|{}_pK_q(\mu;\nu;z,w)\right|\leq\;S(\mu,\nu;z,w)\leq C_\epsilon\,e^{n\epsilon\|z\|_\infty\|w\|_\infty}.\tag{6.27}$$

Taking a closer look at the above proof, we see that for $p<q$ this estimate can be improved. Indeed, consider the quotient

$$\frac{[\mu]_\lambda}{[\nu]_\lambda}=\prod_{j=1}^p\frac{[\mu_j]_\lambda}{[\nu_j]_\lambda}\cdot\prod_{j=p+1}^q\frac{1}{[\nu_j]_\lambda}.$$

By Stirling's formula, the first factor is of polynomial growth, and thus of order $\mathcal{O}({\epsilon_1}^{|\lambda|})$ for arbitrary $\epsilon_1 > 1$, while the second factor is of order $\mathcal{O}({\epsilon_2}^{-|\lambda|})$ for arbitrary $\epsilon_2 > 1$. Under the assumption $p < q$ we therefore obtain the estimate

$$\left|{}_pK_q(\mu;\nu;z,w)\right| \leq S(\mu,\nu;z,w) \leq C_\epsilon\, e^{\epsilon\|z\|_\infty\|w\|_\infty}. \tag{6.28}$$

for arbitrary $\epsilon > 0$, with some constant $C_\epsilon > 0$.

The domain of convergence of the hypergeometric series ${}_pK_q$ and ${}_pF_q$ and their growth estimates (6.27), (6.28) are important to obtain from the Laplace transform identities for Jack polynomials in Theorem 6.12 similar Laplace transform identities for the hypergeometric series.

Theorem 6.23. *Let $\mu \in \mathbb{C}^p$, $\nu \in \mathbb{C}^q$ with $\nu_i \notin \{0, k, \ldots, k(n-1)\} - \mathbb{N}_0$ for all $i = 1, \ldots, n$ and let $\mu' \in \mathbb{C}$ with $\operatorname{Re}\mu' > \mu_0$.*

(i) *If $p < q$, then for all $z, w \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$,*

$$\int_{\mathbb{R}_+^n} E^A(-z,x)\, {}_pK_q(\mu;\nu;w,x)\Delta(x)^{\mu'-\mu_0-1}\omega^A(x)\,\mathrm{d}x$$
$$= \Gamma_n(\mu')\Delta(z)^{-\mu'}\, {}_{p+1}K_q((\mu',\mu);\nu;w,\tfrac{1}{z}).$$

(ii) *If $p = q$, then part (i) is valid under the condition $\|w\|_\infty \cdot \left\|\frac{1}{\operatorname{Re} z}\right\|_\infty < \frac{1}{n}$.*

Moreover, both parts remain valid if ${}_pK_q$ is replaced by ${}_pF_q$.

Proof. To prove part (i), expand ${}_pK_q$ into its defining series. Then (i) is immediate from the Laplace transform identity of Theorem 6.12 by interchanging the order of summation and integration. We have to justify this interchange. Choose $\epsilon > 0$ such that $\|w\|_\infty \cdot \|1/\operatorname{Re} z\|_\infty < \frac{1}{\epsilon}$. Under these conditions the estimates (6.8), (6.27) and (6.28) show that

$$\left|E^A(-x,z)\right| S(\mu,\nu;w,x) \leq C_\epsilon\, e^{-d_\epsilon\|x\|_\infty}$$

with $d_\epsilon = \min_{i=1\ldots n}\operatorname{Re} z_i - \epsilon\|w\|_\infty > 0$. Hence, we can apply the dominated convergence theorem to justify the interchange of summation and integration, so that part (i) is proven since $\epsilon > 0$ was chosen arbitrarily. Part (2) is obtained in the same way, by choosing $\epsilon > 1$ such that $\|w\|_\infty \cdot \|1/\operatorname{Re} z\|_\infty < \frac{1}{n\epsilon}$. By symmetrization we get the same identities for ${}_pF_q$. ■

We continue with an integral representation which was already observed in [Mac89, p.39] for the symmetric case, i.e. for ${}_1F_0$, but only at a formal level and without any statement on convergence.

Corollary 6.24. *Let $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \mu_0$. Then the hypergeometric series ${}_1K_0(\mu;-z,w)$ has an analytic continuation to $D := \{\operatorname{Re} z > 0\} \times \{\operatorname{Re} w > 0\}$ which is given by*

$${}_1K_0(\mu;-z,w) = \frac{\Delta(z)^{-\mu}}{\Gamma_n(\mu)} \int_{\mathbb{R}_+^n} E^A(-\tfrac{1}{z},x)E^A(-w,x)\Delta(x)^{\mu-\mu_0-1}\omega^A(x)\,\mathrm{d}x.$$

By symmetrization, the same formula is valid if one replaces ${}_1K_0$ by ${}_1F_0$ and the Dunkl kernel by the Bessel function.

Proof. Recall that $E^A(z,w) = {}_0K_0(z,w)$. Then, by Theorem 6.23, the stated integral formula holds on a suitable open subset of D. Moreover, estimate (6.8) for the Dunkl kernel shows that the integral exists and defines a holomorphic function on D by standard theorems on holomorphic parameter integrals. Hence, analytic continuation finishes the proof. ■

The following proposition is a generalization of the Euler integral for hypergeometric functions on symmetric cones (cf. [FK94, Proposition XV.1.4]) and can be found as a formal statement in [Mac89, formula (6.21)]. It will be obtained from Kadell's [Kad97] formula,

$$\int_{[0,1]^n} \frac{P_\lambda(x)}{P_\lambda(\underline{1})} \Delta(x)^{\mu-\mu_0-1} \Delta(\underline{1}-x)^{\nu-\mu_0-1} \omega^A(x)\,\mathrm{d}x = \frac{\Gamma_n(\mu)\,\Gamma_n(\nu)}{\Gamma_n(\mu+\nu)} \frac{[\mu]_\lambda}{[\mu+\nu]_\lambda} \tag{6.29}$$

for all $\lambda \in \Lambda_+^n$ and $\mu, \nu \in \mathbb{C}$ with $\operatorname{Re}\mu, \operatorname{Re}\nu > \mu_0$. Furthermore, in [BF98] they generalized (6.29) to the case of non-symmetric Jack polynomials, see also [For10, Formulae (4.4),(12.57)], i.e.

$$\int_{[0,1]^n} \frac{E_\eta(x)}{E_\eta(\underline{1})} \Delta(x)^{\mu-\mu_0-1} \Delta(\underline{1}-x)^{\nu-\mu_0-1} \omega^A(x)\,\mathrm{d}x = \frac{\Gamma_n(\mu)\,\Gamma_n(\nu)}{\Gamma_n(\mu+\nu)} \frac{[\mu]_{\eta_+}}{[\mu+\nu]_{\eta_+}}, \tag{6.30}$$

which would also be a consequence of (6.29) together with $\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \frac{E_\eta(\sigma x)}{E_\eta(\underline{1})} = \frac{P_{\eta_+}(x)}{P_{\eta_+}(\underline{1})}$.

Proposition 6.25. *Consider $p \le q+1$ and $\mu', \nu' \in \mathbb{C}$ with $\operatorname{Re}\mu', Re(\nu'-\mu') > \mu_0$. Moreover, let $\mu \in \mathbb{C}^p$ and $\nu \in \mathbb{C}^q$ with $\nu_i \notin \{0, k, \ldots, k(n-1)\} - \mathbb{N}_0$ for all $i = 1, \ldots, n$. Then, for arbitrary $w \in \mathbb{C}^n$ with the additional condition $\|w\|_\infty < 1$ in the case $p = q+1$, one has*

$$\int_{[0,1]^n} {}_pK_q(\mu;\nu;w,x) \Delta(x)^{\mu'-\mu_0-1} \Delta(\underline{1}-x)^{\nu'-\mu'-\mu_0-1} \omega(x)\,\mathrm{d}x$$
$$= \frac{\Gamma_n(\mu)\,\Gamma_n(\nu-\mu)}{\Gamma_n(\nu)} {}_{p+1}K_{q+1}((\mu',\mu);(\nu',\nu);w,\underline{1}).$$

The same is true for the symmetric hypergeometric series.

Proof. This is immediate from Kadell's integral (6.30) after expanding ${}_pK_q$ into its defining series and changing the order of integration and summation. The latter is justified since the series ${}_pK_q(\mu;\nu;w;\cdot)$ is absolutely bounded on $[0,1]^n$ by the estimates in the proof of Theorem 6.20 for $w \in \mathbb{C}^n$ and $\|w\|_\infty < 1$ if $p = q+1$. ■

In Kadell's integral (6.30) we have

$$\frac{\Gamma_n(\mu)\,\Gamma_n(\nu)}{\Gamma_n(\mu+\nu)} \frac{[\mu]_{\eta_+}}{[\mu+\nu]_{\eta_+}} = \frac{\Gamma_n(\nu)\Gamma_n(\eta_+ + \underline{\mu})}{\Gamma_n(\eta_+ + \underline{\mu} + \underline{\nu})}.$$

Using Carlson's Theorem one can deduce the following generalization of (6.30).

Theorem 6.26. *For all $\lambda \in \overline{H} + \operatorname{co}(\mathcal{S}_n.\rho)$ and $\mu, \nu \in \mathbb{C}$ with $\operatorname{Re}\mu$, $\operatorname{Re}\nu > \mu_0$ we have*

$$\int_{[0,1]^n} \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1} \Delta(\underline{1}-x)^{\nu-\mu_0-1} \omega^A(x)\,\mathrm{d}x = \frac{\Gamma_n(\nu)\Gamma_n(\lambda+\rho+\underline{\mu})}{\Gamma_n(\lambda+\rho+\underline{\mu}+\underline{\nu})}.$$

The same is true for the hypergeometric function.

Proof. By Lemma 6.4 and standard theorems on parameter integrals, both side of the equation are continuous as function in (λ,μ,ν) with domain $(\overline{H} + \operatorname{co}(\mathcal{S}_n.\rho)) \times H \times H$ and holomorphic in the interior. Furthermore, by Kadell's integral (6.30), the equation is true for $\lambda \in \Lambda_+^n - \rho$. By Carlson's Theorem 6.13, it suffices to check the growth conditions of both sides of the equation to obtain the assertion.
The right hand side is polynomially bounded in λ by Stirling's formula. The left hand side

is bounded as function of λ. Indeed, Lemma 6.4 shows that $(x \mapsto \mathcal{G}(\lambda, x))_{\lambda \in \overline{H}+\mathrm{co}(\mathcal{S}_n.\rho)}$ is a uniformly bounded family of functions on $[0,1]^n$. Finally, the conditions of Carlson's Theorem 6.13 are satisfied and the assertions holds. ∎

The following theorem generalizes Proposition XV.1.2. of [FK94] for hypergeometric series on symmetric cones.

Theorem 6.27. *The Jack polynomials and the hypergeometric series have the following properties under the action of the Dunkl operator $\Delta(T)$ associated with the polynomial Δ.*

(i) $\Delta(T)E_\eta = c_\eta E_{\eta-\underline{1}}$ *with some constant* $c_\eta \in \mathbb{R}$. *Moreover,* $c_\eta = 0$ *if* $\eta_i = 0$ *for some* $i \in \{1, \ldots, n\}$.

(ii) $\eta \mapsto c_\eta$ *is* $\mathcal{S}_n$*-invariant.*

(iii) $\Delta(T)L_\eta = d_\eta L_{\eta-\underline{1}}$ *and* $\Delta(T)C_\lambda = d_\lambda C_{\lambda-\underline{1}}$, *where*

$$d_\eta = \begin{cases} \dfrac{|\eta|!}{|\eta-\underline{1}|!}, & \text{if } \eta_i \neq 0 \text{ for all } i = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}.$$

(iv) *If* $p \leq q+1$, *then for all* $w \in \mathbb{C}^n$,

$$\Delta(T)\, {}_pK_q(\mu; \nu; w, \cdot) = \frac{[\mu]_{\underline{1}}}{[\nu]_{\underline{1}}} \Delta(w)\, {}_pK_q(\mu + \underline{1}, \nu + \underline{1}; w, \cdot).$$

The same is true if ${}_pK_q$ *is replaced by* ${}_pF_q$.

Proof.

(i) From the properties of the Dunkl pairing together with Lemma 6.8 (i) we can conclude that for compositions $\eta, \kappa \in \mathbb{N}_0^n$,

$$[\Delta(T)E_\eta, E_\kappa] = [E_\eta, \Delta E_\kappa] = [E_\eta, E_{\kappa+\underline{1}}] = \begin{cases} 0, & \text{if } \eta \neq \kappa + \underline{1}\,; \\ [E_\eta, E_\eta] > 0, & \text{if } \eta = \kappa + \underline{1}\,. \end{cases}$$

Hence, $\Delta(T)E_\eta$ must be a scalar multiple of $E_{\eta-\underline{1}}$ if $\eta_i \geq 1$ for all $i = 1, \ldots, n$ and vanishes otherwise.

(ii) Denote again by $\overline{\eta}_i$ the eigenvalue of E_η under the Cherednik operator $\mathcal{D}_i$. It suffices to show that $c_\eta = c_{s_i\eta}$ if $\eta_i < \eta_{i+1}$. From [For10, Proposition 12.2.1], we then obtain

$$E_{s_i\eta} = d_i^\eta E_\eta + s_i E_\eta \tag{6.31}$$

with the constant

$$d_i^\eta = \frac{k}{\overline{\eta}_{i+1} - \overline{\eta}_i}\,.$$

It is immediate that $\overline{\eta + \underline{1}} = \overline{\eta} + \underline{1}$ and therefore $d_i^{\eta+\underline{1}} = d_i^\eta$. Applying $\Delta(T)$ to equation (6.31), using $d_i^{\eta+\underline{1}} = d_i^\eta$ and the $\mathcal{S}_n$-equivariance of the Dunkl operators, we obtain from part (i) that $c_{s_i\eta} = c_\eta$.

(iii) As L_η is a renormalization of E_η, there is a constant d_η such that $\Delta(T)L_\eta = d_\eta L_{\eta-\underline{1}}$, and $d_\eta = 0$ if $\eta_i = 0$ for some i. Since $\eta \mapsto \frac{|\eta|!}{|\eta-\underline{1}|!}$ is $\mathcal{S}_n$-invariant and $C_\lambda = \sum_{\eta \in \mathcal{S}_n\lambda} L_\eta$, it suffices to verify the stated value of d_η with $|\eta| \geq n$. Recall that T_i acts as ∂_i on symmetric polynomials. We therefore conclude that for $m \in \mathbb{N}_0$ with $m \geq n$,

$$m \cdots (m-n+1) \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta| = m-n}} L_\eta(x) = m \cdots (m-n+1)(x_1 + \ldots + x_n)^{m-n}$$

$$= \Delta(T)(x_1 + \ldots + x_n)^m = \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta| = m}} \Delta(T) L_\eta = \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta| = m}} d_\eta L_{\eta - \underline{1}}.$$

Equating the coefficients proves the stated formula for d_η.

(iv) This is an immediate consequence of part (iii) by expanding the hypergeometric series. One has to perform an index shift $\eta \mapsto \eta + \underline{1}$ after applying $\Delta(T)$ and the identity of part (iii) together with

$$\frac{L_\eta(w)}{L_\eta(\underline{1})} = \Delta(w) \frac{L_{\eta-\underline{1}}(w)}{L_{\eta-\underline{1}}(w)}$$

and $[\theta]_{\eta_+} = [\theta + 1]_{\eta_+ - \underline{1}} [\theta]_{\underline{1}}$.

■

6.6 Convolution of type A Riesz distributions

The *Riesz distribution* $R_\mu \in \mathscr{S}'(\mathbb{R}^n)$ for $\mu > \mu_0 := k(n-1)$, associated with (A_{n-1}, k), is defined as the positive tempered distribution

$$\langle R_\mu, f \rangle := \frac{1}{\Gamma_n(\mu)} \int_{\mathbb{R}_+^n} f(x) \Delta(x)^{\mu - \mu_0 - 1} \omega^A(x) \, \mathrm{d}x.$$

The following results were proven in [Rö20] and generalize results about Riesz distributions on a symmetric cones:

(i) $\mu \mapsto R_\mu$ extends to a (weakly) holomorphic map $\mathbb{C} \to \mathscr{S}'(\mathbb{R}^n)$.

(ii) $\Delta(T) R_\mu = R_{\mu-1}$.

(iii) $\Delta \cdot R_\mu = \prod_{j=1}^n (\mu - k(j-1)) \cdot R_{\mu+1}$.

(iv) $\operatorname{supp} R_\mu \subseteq \overline{\mathbb{R}_+^n}$.

(v) $R_0 = \delta_0$.

(vi) R_μ is a positive measure if and only if μ is contained in the generalized Wallach set

$$W_k := \{0, k, \ldots, k(n-1)\} \cup]k(n-1), \infty[.$$

Riesz distributions on a symmetric cone form a group of tempered distributions under convolution, which is still an open question for Dunkl type Riesz distributions. Indeed, it remained open so far whether two Riesz distributions can be convolved. We shall prove the following theorem based on the results of Chapter 2.

Theorem 6.28. *For $\mu, \nu \in \mathbb{C}$, the Riesz distributions R_μ, R_ν are $\mathcal{S}_n$-convolvable and*

$$R_\mu *_k R_\nu = R_{\mu+\nu}.$$

Definition 6.29. For $s \in \mathbb{R}$ we define $M_s(\mathbb{R}^n_+)$ as the space of complex Radon measures μ on $\overline{\mathbb{R}^n_+}$ such that $x \mapsto e^{-\langle \underline{s}, x\rangle}$ is integrable with respect to the total variation $|\mu|$. We define the Dunkl-Laplace transform on $M_s(\mathbb{R}^n_+)$ by

$$\mathcal{L}\mu(z) := \int_{\mathbb{R}^n_+} E^A(-z,x)\,\mathrm{d}\mu(x), \quad z \in \mathbb{C}^n,\ \mathrm{Re}\, z > s,$$

where $\mathrm{Re}\, z > s$ is understood componentwise. We recall that by (6.8) the Dunkl kernel satisfies for $z \in \mathbb{C}^n$ with $\mathrm{Re}\, z > s$

$$|E^A(-x,z)| \leq e^{-\langle \underline{s}, x\rangle}, \quad x \in \mathbb{R}^n_+.$$

Therefore, $\mathcal{L}\mu$ is a holomorphic function on $\{\mathrm{Re}\, z > s\}$.

Furthermore, in [Rö20], a Dunkl-Laplace transform for tempered distributions u with support contained in $\overline{\mathbb{R}^n_+}$ was defined by

$$\mathcal{L}u(z) := \langle u, \widetilde{E}^A(-z,\cdot)\rangle\,, \quad \mathrm{Re}\, z > 0,$$

where $\widetilde{E}^A(-z,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ and $\widetilde{E}^A(-z,x) = E^A(-z,x)$ for $x \in \mathbb{R}^n_+$. In fact, $\mathcal{L}u$ is holomorphic, does not depend on the extension $\widetilde{E}^A$ of the Dunkl kernel and satisfies

$$\mathcal{L}(e^{-\langle \underline{s},\cdot\rangle}u)(z) = \mathcal{L}u(\underline{s}+z), \quad s > 0. \tag{6.32}$$

In particular, if μ is a tempered distribution of order 0, i.e. a complex Radon measure, with support contained in $\overline{\mathbb{R}^n_+}$, then the two notions of Laplace transform of μ coincide. Moreover, the Dunkl-Laplace transform is injective in the following sense. If $\mathcal{L}u(\underline{s}+iy) = 0$ for some $s > 0$ and all $y \in \mathbb{R}^n$, then $u = 0$.

Lemma 6.30. *Let $\mu, \nu \in M_s(\mathbb{R}^n_+)$. Then we have:*

(i) *$\mu *_k \nu$ exists and $e^{-\langle \underline{s},\cdot\rangle}(\mu *_k \nu) \in \mathscr{S}'(\mathbb{R}^n)$ with support contained in $\overline{\mathbb{R}^n_+}$.*

(ii) *As holomorphic functions on $\{\mathrm{Re}\, z > 0\}$ we have for $z \in \mathbb{C}^n$, $\mathrm{Re}\, z > 0$*

$$\mathcal{L}(e^{-\langle \underline{s},\cdot\rangle}(\mu *_k \nu))(z) = \mathcal{L}\mu(\underline{s}+z)\cdot\mathcal{L}\nu(\underline{s}+z).$$

PROOF.

(i) Since the Dunkl transform is continuous on the Schwartz space, and $|E^A(ix,y)| \leq 1$ for $x, y \in \mathbb{R}^n$, we have by definition of the generalized translation τ_x

$$|\tau_x f(y)| \leq \frac{1}{c_k}||\widehat{f}^k||_{1,\omega^A} \leq \widetilde{C}||\,\langle\cdot\rangle^{\widetilde{N}}\,\widehat{f}^k||_\infty \leq C||\,\langle\cdot\rangle^N\,\partial^\alpha f||_\infty, \tag{6.33}$$

for some constants $C, \widetilde{C} > 0$, $N, \widetilde{N}, N' \in \mathbb{N}$, all independent of f and x, y. Moreover, since $\mathbb{R}\underline{1} = A_{n-1}^\perp$, we use (1.12) to see that

$$\tau_x(e^{-\langle \underline{s},\cdot\rangle}f)(y) = e^{-\langle \underline{s}, x+y\rangle}\cdot\tau_x f(y)$$

holds for all $s \in \mathbb{R}$, $f \in C^\infty(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$. Since $\overline{\mathbb{R}^n_+}$ is a proper $\mathcal{S}_n$-invariant closed convex cone, Corollary 2.12 shows that μ, ν are $\mathcal{S}_n$-convolvable with $\mathrm{supp}(\mu *_k \nu) \subseteq \overline{\mathbb{R}^n_+}$. Finally, we observe that

$$\begin{aligned}\langle e^{-\langle \underline{s},\cdot\rangle}(\mu *_k \nu), \varphi\rangle &= \langle \mu\otimes\nu, \tau(e^{-\langle \underline{s},\cdot\rangle}\varphi)\rangle \\ &= \int_{\overline{\mathbb{R}^n_+}}\int_{\overline{\mathbb{R}^n_+}} e^{-\langle \underline{s}, x+y\rangle}(\tau_x f)(y)\,\mathrm{d}\mu(x)\mathrm{d}\nu(y),\end{aligned}$$

so that $e^{-\langle \underline{s},\cdot\rangle}(\mu *_k \nu) \in \mathscr{S}'(\mathbb{R}^n)$ holds by estimate (6.33).

(ii) This is an immediate consequence of $(\tau_x E(-z,\cdot))(y) = E(-z,x)E(-z,y)$ together with equation (6.32).

■

Proof of Theorem 7.1. By Theorem 2.5 and $\Delta(T)R_\mu = R_{\mu-1}$, we may assume without loss of generality, that $\operatorname{Re}\mu, \operatorname{Re}\nu > \mu_0$. In this case, $R_\mu, R_\nu \in M_s(\mathbb{R}^n_+)$ for all $s > 0$. But, due to [Rö20, Theorem 5.9] we have

$$\mathcal{L}R_\alpha = \Delta^{-\alpha}.$$

Together with Lemma 6.30, this leads to

$$\mathcal{L}(e^{-\langle \underline{s},\cdot\rangle}(R_\mu *_k R_\nu))(z) = \Delta(\underline{s}+z)^\mu \Delta(\underline{s}+z)^\nu = \Delta(\underline{s}+z)^{\mu+\nu} = \mathcal{L}(e^{-\langle \underline{s},\cdot\rangle} R_{\mu+\nu})(z),$$

for all $s > 0$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$. Finally, the injectivity of the Dunkl-Laplace transform finishes the proof. ■

6.7 The binomial formula for the Cherednik kernel

As already mentioned, the Jack polynomials satisfy a generalized binomial formula. By virtue of Carlson's theorem 6.13, we are able to generalize this binomial formula to the Cherednik kernel by analytic continuation.

Definition 6.31 ([Sah98] and [For10, Section 12.5])**.** Consider the non-symmetric Jack polynomials $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ of index $\alpha = \frac{1}{k}, k \geq 0$ with corresponding eigenvalue $\overline{\eta}$ under the Cherednik operators from (6.11) and write $\widetilde{E}_\eta = \frac{E_\eta}{E_\eta(\underline{1})}$. There exists a unique polynomial $E^*_\kappa, \kappa \in \mathbb{N}_0^n$ of degree $|\kappa|$, called a *non-symmetric interpolation Jack polynomial* of index $\frac{1}{k}$, such that

(i) $E^*_\kappa(\overline{\eta}) = 0$ if $\kappa \neq \eta \in \mathbb{N}_0^n$ with $|\eta| \leq |\kappa|$.

(ii) $E^*_\kappa(\overline{\kappa}) \neq 0$.

(iii) the coefficient of x^κ in E^*_κ is 1.

The first condition can be replace by the condition $\kappa \not\subseteq \eta$, i.e. there exists an index i with $\kappa_i > \eta_i$.

Theorem 6.32 ([Sah98, Corollary 1.9])**.** *The non-symmetric Jacks polynomials satisfy the following binomial formula*

$$\widetilde{E}_\eta(\underline{1}+z) = \sum_{\kappa\subseteq\eta} \binom{\eta}{\kappa}_k \widetilde{E}_\kappa(z) = \sum_{\kappa\in\mathbb{N}_0^n} \binom{\eta}{\kappa}_k \widetilde{E}_\kappa(z) \tag{6.34}$$

with the generalized binomial coefficients

$$\binom{\eta}{\kappa}_k = \frac{E^*_\kappa(\overline{\eta})}{E^*_\kappa(\overline{\kappa})}.$$

If $k = 0$, then (6.34) reduces to the usual binomial formula $(1+x)^\eta = \sum_{\kappa\in\mathbb{N}_0^n} \binom{\eta}{\kappa} x^\kappa$.

Definition 6.33. Consider the (rational) Cherednik kernel $\mathcal{G}$ from Definition 6.3 with respect to $k \geq 0$, holomorphic on the domain $\mathbb{C}^n \times \{\operatorname{Re} z > 0\}$. Since $\mathcal{G}(\lambda, \cdot)$ is holomorphic on the polydisc $U = \{z \in \mathbb{C}^n \mid \|\underline{1} - z\|_\infty < 1\}$ it has an convergent series expansion on U of the form

$$\mathcal{G}(\lambda, 1 + z) = \sum_{\kappa \in \mathbb{N}_0^n} a_\kappa(\lambda) \widetilde{E}_\eta(z),$$

with some coefficient functions $a_\eta : \mathbb{C}^n \to \mathbb{C}$.

An immediate consequence of combining Sahi's binomial formula for the non-symmetric Jack polynomials (Theorem 6.32) and equation (6.12) is the following proposition.

Proposition 6.34. *For all partitions $\eta \in \Lambda_+^n$ we have*

$$a_\kappa(\eta - \rho) = \frac{E_\kappa^*(\eta - \rho + \frac{k}{2}(1-n) \cdot \underline{1})}{E_\kappa^*(\overline{\kappa})} = \frac{E_\kappa^*(\overline{\eta})}{E_\kappa^*(\overline{\kappa})}$$

with $\rho = \rho(k)$ as in (6.9).

Theorem 6.35. *The coefficient functions $a_\kappa : \mathbb{C}^n \to \mathbb{C}$ are given as*

(i) $a_\kappa(\lambda) = \frac{1}{k^{|\kappa|} d'_\kappa} \cdot E_\kappa(T) \mathcal{G}(\lambda, x)\Big|_{x=\underline{1}}$, *where d'_κ is defined in the proof of Lemma 6.16.*

(ii) $a_\kappa(\lambda) = \dfrac{E_\kappa^*(\lambda + \frac{k}{2}(1-n) \cdot \underline{1})}{E_\kappa^*(\overline{\kappa})}$

(iii) *The Cherednik kernel satisfies for $z \in \mathbb{C}^n$, $\|z\|_\infty < 1$ the binomial formula*

$$\mathcal{G}(\lambda, \underline{1} + z) = \sum_{\kappa \in \mathbb{N}_0^n} \begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_k \widetilde{E}_\kappa(z).$$

where the generalized binomial coefficients are given by

$$\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_k := \frac{E_\kappa^*(\lambda + \frac{k}{2}(1-n) \cdot \underline{1})}{E_\kappa^*(\overline{\kappa})}.$$

(iv) *The hypergeometric function $\mathcal{F}$ satisfies for $z \in \mathbb{C}^n$, $\|z\|_\infty < 1$ the binomial formula*

$$\mathcal{F}(\lambda, \underline{1} + z) = \sum_{\kappa \in \Lambda_+^n} \begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_k^{\mathcal{S}_n} \widetilde{P}_\kappa(z) \text{ for all } z \in \mathbb{C}^n, \|z\|_\infty < 1,$$

where the generalized symmetric binomial coefficients are given by

$$\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_k^{\mathcal{S}_n} := \sum_{\sigma \in \mathcal{S}_n} \begin{bmatrix} \lambda \\ \sigma\kappa \end{bmatrix}_k$$

and $\widetilde{P}_\kappa$ are the symmetric Jack polynomials of index $\alpha = \frac{1}{k}$ normalized to 1 in $\underline{1}$. For $\lambda = \mu - \rho, \mu \in \Lambda_+^n$ this is the known binomial formula of the symmetric Jack polynomials. In particular,

$$\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_k^{\mathcal{S}_n} = \sum_{\sigma \in \mathcal{S}_n} \frac{E_{\sigma\kappa}^*(\lambda + \frac{k}{2}(1-n) \cdot \underline{1})}{E_{\sigma\kappa}^*(\overline{\sigma\kappa})} = \frac{P_\kappa^*(\lambda + \frac{k}{2}(1-n) \cdot \underline{1})}{P_\kappa^*(\overline{\mu})},$$

where P_κ^* are the symmetric interpolation Jack polynomials, cf. [For10, Section 12.7]. In particular, the invariance of $\mathcal{F}$ leads to an invariance of

$$(\lambda, \kappa) \mapsto \begin{bmatrix} \lambda \\ \kappa \end{bmatrix}^{\mathcal{S}_n}$$

in both arguments λ and κ.

PROOF.

(i) From [BF98, Formula (2.4), Proposition 3.18] we have

$$E_\kappa(T)\widetilde{E}_\mu(x)\Big|_{x=0} = k^{|\kappa|}d'_\kappa.$$

Moreover, the Dunkl operators T_i satisfy for any C^1 function f and $g(x) = f(\underline{1}+x)$ the formula $T_i g(x) = (T_i f)(\underline{1}+x)$. Hence, we have

$$E_\kappa(T)\mathcal{G}(\lambda,\underline{1}+x)\Big|_{x=0} = E_\kappa(T)\mathcal{G}(\lambda,x)\Big|_{x=\underline{1}},$$

i.e. the stated formula holds.

(ii)+(iii) Let $K \subseteq \mathbb{R}^n$ be a compact, convex and $\mathcal{S}_n$-invariant subset. Then, by (1.2) (vii) there exists a constant C (independent of K) such that for all $f \in C^1(\mathbb{R}^n)$

$$\|\partial^\gamma T_j f\|_{\infty,K} \leq C \max_{|\alpha|\leq|\gamma|+1} \|\partial_x^\alpha f\|_{\infty,K},$$

where the index x in ∂_x^α means the derivative with respect to x. Choosing $K = \{\underline{1}\}$, part (i) gives

$$|a_\kappa(\lambda)| \leq C \max_{|\alpha|\leq|\kappa|+1} \left|\partial_x^\alpha \mathcal{G}(\lambda,x)|_{x=\underline{1}}\right|.$$

Hence, if G is the Cherednik kernel associated with $A_{n-1} \subseteq \mathbb{R}^n$, then $\mathcal{G}(\lambda,x) = G(\lambda,\log x)$ and the change of variables $y = \log x$, i.e. $\frac{\partial}{\partial x_i} = \frac{1}{x_i}\frac{\partial}{\partial y_i}$, lead to

$$\partial_x^\gamma \mathcal{G}(\lambda,x) = \sum_{|\alpha|\leq|\gamma|} p_\alpha(\tfrac{1}{x})\partial_y^\alpha G(\lambda,y)\Big|_{y=\log x},$$

where p_α are polynomials independent of λ. Evaluation at $x = \underline{1}$ gives

$$\left|\partial_x^\gamma \mathcal{G}(\lambda,x)\Big|_{x=\underline{1}}\right| \leq \sum_{|\alpha|\leq|\gamma|} |p_\alpha(\underline{1})| \left|\partial_y^\alpha G(\lambda,y)\Big|_{y=\underline{0}}\right|,$$

which is polynomially bounded in λ by Theorem 4.18. Thus, $a_\kappa(\lambda)$ is polynomial bounded in λ as well. Finally, $\lambda \mapsto \begin{bmatrix}\lambda\\ \kappa\end{bmatrix}_k$ is a polynomial and coincides with $a_\kappa(\lambda)$ on $\Lambda_+^n - \rho$ by Proposition 6.34. Therefore, analytic continuation with Carlson's theorem 6.13 gives $a_\kappa(\lambda) = \begin{bmatrix}\lambda\\ \kappa\end{bmatrix}_k$ for all $\lambda \in \mathbb{C}^n$.

(iv) The binomial formula follows from the identities

$$\mathcal{F}(\lambda,x) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \mathcal{G}(\lambda,\sigma x) \text{ and } \widetilde{P}_\lambda(x) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \widetilde{E}_\lambda(\sigma x).$$

Finally, the binomial formula for the symmetric Jack polynomials can be found in [For10, Formula 12.177]).

■

As a corollary from the previous Theorem and [For10, Proposition 12.5.3] we obtain the following evaluation formula.

Corollary 6.36. *For arbitrary $\kappa,\eta \in \mathbb{N}_0^n$ we have*

$$E_\kappa(T)E_\eta(\underline{1}) = E_\kappa^*(\overline{\eta} + \tfrac{k}{2}(1-n)\cdot\underline{1}).$$

This is a generalization of $\partial^\kappa x^\eta\Big|_{x=\underline{1}} = \frac{\eta_1!\cdots\eta_n!}{(\eta_1-\kappa_1)!\cdots(\eta_n-\kappa_n)!}$, *occurring for* $k = 0$.

6.8 A Post-Widder inversion formula for the Dunkl-Laplace transform

As shown in [Rö20], the Dunkl-Laplace transform

$$\mathcal{L}f(z) = \int_{\mathbb{R}^n_+} f(x) E^A(-z,x)\omega^A(x)\,\mathrm{d}x$$

satisfies the following Cauchy inversion theorem: Let $f \in L^1_{loc}(\mathbb{R}^n_+)$ such that $\mathcal{L}f(\underline{s})$ exists for some $s \in \mathbb{R}$ (then $\mathcal{L}f(z)$ also exists for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z = \underline{s}$). Assume further that $y \mapsto \mathcal{L}f(\underline{s}+iy) \in L^1(\mathbb{R}^n,\omega)$. Then f has a continuous representative f_0, and

$$\frac{(-i)^n}{c_k^2}\int_{\operatorname{Re} z=\underline{s}} \mathcal{L}f(z)E^A(x,z)\omega^A(z)\,\mathrm{d}z = \begin{cases} f_0(x), & x \in \mathbb{R}^n_+, \\ 0, & \text{otherwise,}\end{cases}$$

with the constant $c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2}\omega(x)\,\mathrm{d}x$. For the classical Laplace transform

$$Lf(z) = \int_0^\infty f(x)e^{-zx}\,\mathrm{d}x, \quad f \in L^1_{loc}(\mathbb{R}_+),$$

a further well-known inversion theorem is the Post-Widder inversion(see e.g. [WAN01]): Assume that $f \in L^1_{loc}(\mathbb{R}_+)$ has a finite abscissa of convergence and is continuous in $\xi \in \mathbb{R}_+$. Then

$$f(\xi) = \lim_{\nu\to\infty}\frac{(-1)^\nu}{\nu!}\Big(\frac{\nu}{\xi}\Big)^{\nu+1}(Lf)^{(\nu)}\Big(\frac{\nu}{\xi}\Big).$$

In this section, we prove a Post-Widder inversion formula for the Dunkl-Laplace transform, which is the counterpart to a result of Faraut and Gindikin [FG90] in the setting of symmetric cones.

Theorem 6.37 (Post-Widder inversion formula for $\mathcal{L}$)**.** *Let $f:\mathbb{R}^n_+ \to \mathbb{C}$ be measurable and bounded, and suppose that f is continuous at $\xi \in \mathbb{R}^n_+$. Then*

$$f(\xi) = \lim_{\nu\to\infty}\frac{(-1)^{n\nu}}{\Gamma_n(\nu+\mu_0+1)}\Delta\Big(\frac{\nu}{\xi}\Big)^{\nu+\mu_0+1}(\Delta(T)^\nu(\mathcal{L}f))\Big(\frac{\nu}{\xi}\Big).$$

The idea of the proof for this theorem is similar to [FG90]. A fundamental ingredient is Levy's continuity theorem for the Dunkl transform. Let us recall this for the reader's convenience. Denote by $M_b^+(\mathbb{R}^n)$ the space of positive bounded Borel measures on $\mathbb{R}^n$. The Dunkl transform of $\mu \in M_b^+(\mathbb{R}^n)$ (associated with A_{n-1} and multiplicity k) is given by

$$\widehat{\mu}(\xi) = \widehat{\mu}^k(\xi) = \int_{\mathbb{R}^n} E^A(-i\xi,x)\,\mathrm{d}\mu(x), \quad \xi \in \mathbb{R}^n.$$

Note that $\widehat{\mu} \in C_b(\mathbb{R}^n)$, since $|E(-i\xi,x)| \le 1$ for all $\xi, x \in \mathbb{R}^n$. The Dunkl transform is injective on $M_b^+(\mathbb{R}^n)$, see [RV98]. The following is the essential part of Levy's continuity theorem for the Dunkl transform.

Lemma 6.38 ([RV98])**.** *Let $(\mu_\nu)_{\nu\in\mathbb{N}} \subseteq M_b^+(\mathbb{R}^n)$ such that the sequence $(\widehat{\mu}_\nu)_{\nu\in\mathbb{N}}$ converges pointwise to a function $\varphi:\mathbb{R}^n \to \mathbb{C}$ which is continuous at 0. Then there exists a unique $\mu \in M_b^+(\mathbb{R}^n)$ with $\widehat{\mu}_\nu = \varphi$, and $(\mu_\nu)_{\nu\in\mathbb{N}}$ converges to μ weakly.*

Proof of Theorem 6.37. We consider on $\mathbb{R}^n_+$ the functions

$$h_\nu(x) := E^A(-\tfrac{\nu}{\xi},x)\,\Delta(x)^\nu, \quad \nu \in \mathbb{N}.$$

By estimate (6.8), the Laplace transform

$$\mathcal{L}h_\nu(z) = \int_{\mathbb{R}_+^n} E^A(-z,x)E^A(-\tfrac{\nu}{\xi},x)\Delta(x)^\nu\omega(x)\,\mathrm{d}x$$

exists for all $z\in\mathbb{C}^n$ with $\operatorname{Re} z\geq 0$. For such z, put $\nu(z) := \max_i\lceil\|z\|_\infty\,\xi_i\rceil \in\mathbb{N}$. Then, for $\nu>\nu(z)$, we calculate

$$\begin{aligned}\mathcal{L}h_\nu(z) &= \int_{\mathbb{R}_+^n}\Big(\sum_{\eta\in\mathbb{N}_0^n}\frac{L_\eta(-z)L_\eta(x)}{|\eta|!\,L_\eta(\underline{1})}\Big)E^A(-\tfrac{\nu}{\xi},x)\Delta(x)^\nu\omega^A(x)\,\mathrm{d}x\\ &= \sum_{\eta\in\mathbb{N}_0^n}\frac{L_\eta(-z)}{|\eta|!\,L_\eta(\underline{1})}\int_{\mathbb{R}_+^n}L_\eta(x)E^A(-\tfrac{\nu}{\xi},x)\Delta(x)^\nu\omega^A(x)\,\mathrm{d}x\\ &= \sum_{\eta\in\mathbb{N}_0^n}\frac{L_\eta(-z)}{|\eta|!\,L_\eta(\underline{1})}\,\Gamma_n(\eta_+ +\nu+\mu_0+1)\,L_\eta(\tfrac{\xi}{\nu})\,\Delta(\tfrac{\nu}{\xi})^{-\nu-\mu_0-1}.\end{aligned}$$

Here the interchange of the sum and the integral is justified by the dominated convergence theorem, because $|L_\eta(-z)|\leq L_\eta(\|z\|_\infty\cdot\underline{1})$ and therefore

$$\begin{aligned}E^A(-\tfrac{\nu}{\xi},x)\sum_{\eta\in\mathbb{N}_0^n}\frac{|L_\eta(-z)L_\eta(x)|}{|\eta|!\,L_\eta(\underline{1})} &\leq E(-\tfrac{\nu}{\xi},x)E(\|z\|_\infty\cdot\underline{1},x)\\ &= E(-\tfrac{\nu}{\xi}+\|z\|_\infty\cdot\underline{1},x).\end{aligned}$$

This decays exponentially on $\mathbb{R}_+^n$, since $-\nu/\xi+\|z\|_\infty\cdot\underline{1}<0$ by our assumption on ν. Thus for $\nu\geq\nu(z)$,

$$f_\nu(z) := \frac{\Delta(\frac{\nu}{\xi})^{\nu+\mu_0+1}}{\Gamma_n(\nu+\mu_0+1)}\,\mathcal{L}h_\nu(z) = \sum_{\eta\in\mathbb{N}_0^n}c_\nu(\eta)\cdot\frac{L_\eta(-z)L_\eta(\xi)}{|\eta|!\,L_\eta(\underline{1})} \tag{6.35}$$

with the coefficients

$$c_\nu(\eta) = \frac{[\nu+\mu_0+1]_{\eta_+}}{\nu^{|\eta|}} = \prod_{j=1}^n\Big(1+\frac{1+k(n-j)}{\nu}\Big)_{\lambda_j},\qquad \lambda=\eta_+\,.$$

They satisfy

$$\lim_{\nu\to\infty}c_\nu(\eta)=1\quad\text{ for fixed }\ \eta,$$

and it follows that

$$\lim_{\nu\to\infty}f_\nu(z) = \sum_{\eta\in\mathbb{N}_0^n}\frac{L_\eta(-z)L_\eta(\xi)}{|\eta|!\,L_\eta(\underline{1})} = E^A(-z,\xi). \tag{6.36}$$

We still have to justify that the limit $\nu\to\infty$ may be taken inside the sum in (6.35). For this, note that $\nu\mapsto c_\nu(\eta)$ is monotonically decreasing. Hence for $\nu\geq\nu(z)$, the series on the right-hand side of (6.35) is dominated by the convergent series

$$\sum_{\eta\in\mathbb{N}_0^n}c_{\nu(z)}(\eta)\,\frac{L_\eta(\|z\|_\infty\cdot\underline{1})\,L_\eta(\xi)}{|\eta|!\,L_\eta(\underline{1})} = f_{\nu(z)}(-\|z\|_\infty\cdot\underline{1})<\infty,$$

which justifies the above limit. We now consider the measures

$$\mathrm{d}m_\nu(x) := \frac{\Delta(\frac{\nu}{\xi})^{\nu+\mu_0+1}}{\Gamma_n(\nu+\mu_0+1)}\cdot 1_{\mathbb{R}_+^n}(x)E^A(-\tfrac{\nu}{\xi},x)\Delta(x)^\nu\omega^A(x)\,\mathrm{d}x\in M_b^+(\mathbb{R}^n).$$

Owing to Theorem 6.12, m_ν is actually a probability measure on $\mathbb{R}^n$. Formula (6.36), considered for arguments $z \in i\mathbb{R}^n$, shows that the Dunkl transforms satisfy

$$\widehat{m_\nu} \to \widehat{\delta_\xi} \quad \text{pointwise on } \mathbb{R}^n,$$

where δ_ξ denotes the point measure in ξ. Levy's continuity theorem (Lemma 6.38) now implies that $m_\nu \to \delta_\xi$ weakly. Thanks to the Portemanteau theorem ([Kle14]) we even get

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n} g \, \mathrm{d}m_\nu = \int_{\mathbb{R}^n} g \, \mathrm{d}\delta_\xi \; = g(\xi)$$

for all measurable bounded functions $g : \mathbb{R}^n \to \mathbb{C}$ which are continuous at ξ. Now suppose $f : \mathbb{R}^n_+ \to \mathbb{C}$ is measurable, bounded and continuous at ξ. Extend f by zero to $\mathbb{R}^n$. Then

$$\frac{\Delta(\frac{\nu}{\xi})^{\nu+\mu_0+1}}{\Gamma_n(\nu+\mu_0+1)} \int_{\mathbb{R}^n_+} f(x) E^A(-\tfrac{\nu}{\xi}, x) \Delta(x)^\nu \omega^A(x) \, \mathrm{d}x \; = \int_{\mathbb{R}^n} f \, \mathrm{d}m_\nu \; \to f(\xi).$$

But in view of to Lemma 6.2 the integral on the left-hand side can be written as

$$\mathcal{L}(\Delta^\nu f)(\tfrac{\nu}{\xi}) = \; (\Delta(-T))^\nu (\mathcal{L}f)(\tfrac{\nu}{\xi}),$$

which finishes the proof. ∎

CHAPTER 7

Bessel functions, Hankel transform and Zeta integrals

7.1 Introduction

In [Mac89], Macdonald introduced a generalization of the Hankel transform for radial functions on symmetric cones, but many statements remained at a formal level. Radial analysis on symmetric cones is closely related to Dunkl theory for root systems of type A, as mentioned in Chapter 5, and many of Macdonald's concepts have a natural interpretation within Dunkl theory. An important ingredient to study the Hankel transform is the analysis of the generalized Laplace transform and hypergeometric functions as done in [Rö20] and the previous chapter. In this chapter we introduce several types of Bessel functions in the framework of Dunkl theory for root systems of type A serving as analogues of Bessel functions on a symmetric cones. For the setting of general symmetric cones, the Bessel functions and $\mathcal{K}$-Bessel functions of a symmetric cone were first considered in [Her55] for matrix cones, see [Cle88, Dib90, FK94]. A key object in the present chapter is a two-variable hypergeometric series of Jack polynomials C_λ (of n variables and arbitrary index α) which we call a Bessel kernel (see Section 3)

$$\mathscr{J}_\nu(w,z) := {}_0F_1(\nu; w, -z) = \sum_{\lambda\in\Lambda_+^n} \frac{(-1)^{|\lambda|}}{[\nu]_\lambda}\frac{C_\lambda(w)C_\lambda(z)}{|\lambda|!C_\lambda(\underline{1})}, \quad \nu\in\mathbb{C},$$

generalizing the Bessel functions of symmetric cones. For this Bessel kernel we prove integral representations and recurrence formulas in the parameter ν, generalizing these on a symmetric cone. The proof involves the Laplace transform identities for Jack polynomials and hypergeometric series of Jack polynomials as analyzed in the previous chapter. A fundamental observation in this context is that $\mathscr{J}_\nu$ can be identified with a Bessel function from Dunkl theory associated with a root system of type B, cf. [Rö07]. We further define a non-symmetric counterpart of the Bessel kernel, denoted by $\mathscr{E}_\nu$, which is also closely related to the type B Dunkl kernel. This is the point where type B Dunkl theory comes in.
In the analysis on symmetric cones, Bessel functions are important objects used to define the kernel of the Hankel transform. As a generalization, we study a Dunkl-type Hankel transform which is, for Schwartz functions f_0 on $\mathbb{R}^n$, defined in terms of the kernel $\mathscr{E}_\nu$ (see Section 6) by

$$\mathcal{H}_\nu f_0(y) := \frac{1}{\Gamma_n(\nu)}\int_{\mathbb{R}_+^n} \mathscr{E}_\nu(-x,y)f_0(x)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x,$$

where $\Delta(x) = x_1\cdots x_n$, Γ_n is Macdonald's gamma function and

$$\omega^A(x) = \prod_{i<j}|x_i-x_j|^{2k}.$$

This transform was already introduced on a formal level in [BF98] and earlier in [Mac89] in a symmetrized version. We discuss basic analytic properties of the Dunkl type Hankel transform and generalize results of the Hankel transform on symmetric cones. Owing to the relation between $\mathscr{E}_\nu$ and the type B Dunkl kernel, the transform $\mathcal{H}_\nu$ is related to the Dunkl transform $\mathcal{F}^B$ of type B (where the multiplicity depends on the parameter ν) via

$$\mathcal{F}^B f(y) = 2^{-n\nu}\mathcal{H}_\nu f_0(\tfrac{y^2}{4}), \tag{7.1}$$

where f is defined by $f(x) = f_0(x^2)$ and x^2 is understood componentwise. A special case is already known for spaces of rectangular matrices, or more generally for symmetric cones, where the Hankel transform is related to Euclidean Fourier transform, cf. [Her55, FK94, Rub06]. When doing radial analysis on symmetric cones, a useful tool are the $\mathcal{K}$-Bessel functions. Similar as before, $\mathcal{K}$-Bessel functions on a symmetric cone can also be interpreted within the type A Dunkl theory. This suggest to define generalized $\mathcal{K}$-Bessel functions (see Section 5) by the formula

$$\mathcal{K}_\nu(w, z) := \int_{\mathbb{R}_+^n} E^A(-x, w) E^A(-\tfrac{1}{x}, z) \Delta(x)^{\nu-\mu_0-1} \omega^A(x)\, \mathrm{d}x,$$

where E^A is the type A Dunkl kernel. In fact, after symmetrization over the action of the symmetric group and a particular choice of a type A multiplicity parameter, $\mathcal{K}_\nu$ coincides with a symmetrized version of the $\mathcal{K}$-Bessel function of a symmetric cone, as studied in [Cle88, Dib90, FK94]. The generalized $\mathcal{K}$-Bessel function $\mathcal{K}_\nu$ is defined for all $\nu \in \mathbb{C}$, $w, z \in \mathbb{C}^n$ with $\operatorname{Re} w$, $\operatorname{Re} z > 0$ and satisfies

$$\mathcal{K}_\nu(w, z) = \mathcal{K}_{-\nu}(z, w) \text{ and } |\mathcal{K}_\nu(w, z)| \leq \mathcal{K}_{\operatorname{Re}\nu}(\operatorname{Re} w, \operatorname{Re} z).$$

The type A Dunkl operator $\Delta(T^A)$ acts on $\mathcal{K}$-Bessel functions by shifting the parameter ν, see Section 5. We shall obtain important growth conditions on $\mathcal{K}_\nu$. For $\mathcal{K}$-Bessel functions on a symmetric cone it is known that they are eigenfunctions of a system of Bessel operators, cf. [FK94, p. 358]. Similarly, we obtain in the present chapter that the Dunkl-type $\mathcal{K}$-Bessel function is (up to squared variables) an eigenfunction of $\mathbb{Z}_2^n$-invariant type B Dunkl operators which play the role of Bessel operators on symmetric cones.

An important application of $\mathcal{K}$-Bessel functions is the investigation of zeta integrals. On a symmetric cone, zeta integrals depend on a representation of the underlying Jordan algebra, see [FK94, Chapter XVI] and [Cle02]. An important special case is that of matrix cones, where zeta distributions are related to Wishart distributions, see [Mui82, FK94, Rub06]. To motivate our results, let us describe a typical example, studied in [Rub06]: Consider the cone $\Omega = \operatorname{Pos}_n(\mathbb{R})$ of real $n \times n$ positive definite matrices inside the Jordan algebra $V = \operatorname{Sym}_n(\mathbb{R})$ of real symmetric matrices. The map $\Phi(x)\xi := x\xi$ defines a self adjoint representation of V on the Euclidean space $E = \mathbb{R}^{n\times m}$ with associated quadratic representation $Q : E \to V$, $\xi \mapsto \xi\xi^T$. For $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \frac{d}{2}(n-1) - \frac{m}{2n}$, the zeta integral of index α is defined for a Schwartz function $f \in \mathscr{S}(E)$, $E = \mathbb{R}^{n\times m}$ by

$$Z(f; \alpha) = \int_E f(\xi) \det(Q(\xi))^\alpha \,\mathrm{d}\xi, \tag{7.2}$$

cf. [FK94, Chapter XVI]. The zeta integral can be meromorphically extended in the parameter α to the whole complex plane such that the following characteristic functional equation holds

$$\frac{Z(\widehat{f}; \alpha - \frac{m}{2n})}{\Gamma_\Omega(\alpha)} = \pi^{\frac{n}{2}} 4^{n\alpha} \frac{Z(f; -\alpha)}{\Gamma_\Omega(\frac{m}{2n} - \alpha)}.$$

Here Γ_Ω is the Gindikin gamma function of Ω and $\widehat{f}$ denotes the Fourier transform of f. For a general symmetric cone Ω, the zeta distribution

$$\zeta_\alpha(f) := \frac{Z(f; \alpha)}{\Gamma_\Omega(\alpha)}$$

gives rise to an analytic family $\mathbb{C} \to \mathscr{S}'(E)$, $\alpha \mapsto \zeta_\alpha$ of tempered distributions. Moreover, ζ_α is closely related to Riesz distributions on the symmetric cone. Riesz distributions were generalized to the Dunkl setting in [Rö20]. In the present chapter, we shall investigate zeta integrals in this setting. The generalized Hankel transform and the $\mathcal{K}$-Bessel function will play the same essential role as in the setting of symmetric cones and their properties will further reveal that type B Dunkl theory is involved. Our family of tempered distributions is defined by analytic continuation of the following generalized zeta integrals (see Section 7)

$$\langle \zeta_\alpha, f\rangle := \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{R}^n} f(x)\Delta(x^2)^{\alpha-\nu}\omega^B(x)\,\mathrm{d}x,$$

for $\alpha \in \mathbb{C}$ with large real part. The analytic extension in the parameter α to the whole complex plane $\mathbb{C}$ is done by the functional equation

$$\Delta(T^B)^2\zeta_\alpha = 4^n b(\alpha-\nu)\zeta_{\alpha-1},$$

here b is a certain polynomial. This equation is an immediate consequence of a Bernstein identity for type B Dunkl operators. In addition, these zeta distributions satisfy a characteristic property relating them to their Dunkl transform

$$\zeta_\alpha = 2^{n(2\alpha-\nu)}\mathcal{F}^B\zeta_{\nu-\alpha}.$$

Finally, the results on Riesz distributions in [Rö20] make it possible to explicitly determine those zeta distributions which are positive measures. The corresponding parameters $\alpha \in \mathbb{C}$ are contained in a generalized Wallach set.
The chapter is organized as follows: Section 2 deals with a review on some standard facts on radial analysis on symmetric cones, in particular with special functions, and its connection with Dunkl theory related to root systems of type A and B. In Section 3, we collect results on the Bessel kernel and the Bessel functions. These will be employed in Section 4, where the generalized Hankel transform (of type A) and its connection to type B Dunkl theory is studied. Finally, in Section 5 we introduce zeta integrals and study the associated family of tempered distributions.

Fixed notations

We equip the n-dimensional Euclidean space $\mathbb{R}^n$ with the usual inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and extend it naturally to a $\mathbb{C}$-bilinear form on $\mathbb{C}^n$. In $\mathbb{R}^n$ we consider the root systems $R \in \{A, B\}$ with

$$\begin{aligned} A &:= A_{n-1} = \{\pm(e_i - e_j) \mid 1 \le i < j \le n\}, \\ B &:= B_n = \{\pm e_i \mid 1 \le i \le n\} \cup \{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\}, \end{aligned}$$

where $(e_i)_{1\le i\le n}$ is the canonical basis of $\mathbb{R}^n$. The corresponding Weyl groups are

$$W_A := \mathcal{S}_n, \quad W_B := \mathcal{S}_n \ltimes \mathbb{Z}_2^n,$$

where the symmetric group $\mathcal{S}_n$ acts by permutations of coordinates and $\mathbb{Z}_2^n$ acts by sign changes of coordinates. We consider multiplicity functions of the specific form

$$\kappa_A = k, \quad \kappa_B = (k, k'), \tag{7.3}$$

where k is the value on $\pm(e_i \pm e_j)$ and k' is the value on e_i. If $\operatorname{Re}\kappa_R \ge 0$, we consider

$$[p, q]^R := (p(T^R)q)(0), \tag{7.4}$$

the generalized Fisher product from Definition 1.9. The Dunkl kernel and Bessel function on $\mathbb{R}^n$ associated with (R, k_R) are denoted by E^R and J^R, respectively. Recall from Definition 6.3 the (rational) Cherednik kernel and hypergeometric functions $\mathcal{G}$ and $\mathcal{F}$ associated with (A_{n-1}, k), respectively. The Dunkl type weight function is denoted by

$$\omega^R(x) = \omega^R_{\kappa_R}(x) = \prod_{\alpha\in R} |\langle x, \alpha\rangle|^{\kappa_R(\alpha)}. \tag{7.5}$$

By the choice of multiplicities (7.3), the weight functions associated with the root systems A and B are related to each other by the equation

$$\omega^B(x) = \Delta(x^2)^{k'}\omega^A(x^2), \tag{7.6}$$

where x^2 is understood componentwise and

$$\Delta(x) := x_1 \cdots x_n. \tag{7.7}$$

The Dunkl transform associated with (R, k_R) is denoted by $\mathcal{F}^R$ and the associated Macdonald-Mehta constant is denoted by

$$c_R := \int_{\mathbb{R}^n} e^{-|x|^2/2}\omega^R(x)\,\mathrm{d}x. \tag{7.8}$$

Throughout the chapter we will always assume that $k \geq 0$. Recall the (type A) Dunkl-Laplace transform of a locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ is defined on the cone $\mathbb{R}^n_+ =]0,\infty[^n$ by

$$\mathcal{L}f(z) := \int_{\mathbb{R}^n_+} E^A(-x,z)f(x)\omega^A(x)\,\mathrm{d}x, \quad z \in \mathbb{C}^n \tag{7.9}$$

provided the integral exists. For our purpose, the following results on the Dunkl-Laplace transform are of relevance.

Theorem 7.1. *Consider $f \in L^1_{loc}(\mathbb{R}^n_+)$. Then:*

(i) *If $\mathcal{L}f(a)$ exists for $a \in \mathbb{R}^n$, then $\mathcal{L}f$ exists and defines a holmorphic function on the halfspace $H_n(a) := \{z \in \mathbb{C}^n \mid \operatorname{Re} z > a\}$, where $\operatorname{Re} z > a$ is defined componentwise. Moreover, for any polynomial $p \in \mathbb{C}[\mathbb{R}^n]$,*

$$p(-T^A)\mathcal{L}f(z) = \mathcal{L}(pf)(z), \quad z \in H_n(a).$$

(ii) *If $|f(x)| \leq e^{-s\|x\|_1}$ for some $s \in \mathbb{R}$, then $\mathcal{L}f$ exists on $H_n(\underline{s})$.*

(iii) *If $\mathcal{L}f(\underline{s})$ exists for some $s \in \mathbb{R}$ and $y \mapsto \mathcal{L}f(\underline{s}+iy) \in L^1(\mathbb{R}^n, \omega^A(x)\mathrm{d}x)$, then f has a continuous representative f_0 such that*

$$\frac{(-i)^n}{c_A^2}\int_{\operatorname{Re} z = \underline{s}} \mathcal{L}f(z)E^A(x,z)\omega^A(z)\,\mathrm{d}z = \begin{cases} f_0(x) & \text{if } x > 0, \\ 0 & \text{otherwise.}\end{cases}$$

The integral is understood as an n-fold line integral.

7.2 The connection to radial analysis on symmetric cones

This section is intended to review some of the standard facts on radial analysis on symmetric cones as in [FK94]. We will touch only a few aspects of the theory and restrict our attention to the special functions associated with a symmetric cone. The aim is to motivate how special functions of symmetric cones can be interpreted as functions in the Dunkl setting of type A. To become more precise, let $\Omega = G/K$ be an irreducible symmetric cone with associated Euclidean Jordan algebra V of dimension m, rank n and with Peirce dimension constant d. In the remainder of this section we require that $k = \frac{d}{2}$. Let $(Z_\lambda)_{\lambda \in \Lambda^n_+}$ be the spherical polynomials of Ω, normalized such that for all $p \in \mathbb{N}_0$

$$(\operatorname{tr} x)^p = \sum_{\substack{\lambda \in \Lambda^n_+ \\ |\lambda| = p}} Z_\lambda(x),$$

where tr is the Jordan trace. Denote by $\operatorname{spec} x \in \mathbb{R}^n$ the spectrum of $x \in V$. The normalized Jack polynomials $(C_\lambda)_{\lambda\in\Lambda_+^n}$ of index $\alpha = \frac{1}{k}$ coincide with the spherical polynomials, cf. Proposition 5.4, i.e.

$$Z_\lambda(x) = C_\lambda(\operatorname{spec} x). \tag{7.10}$$

More general, by Theorem 5.3, the spherical functions $(\varphi_\lambda^\Omega)_{\lambda\in\mathbb{C}^n}$ are given by the hypergeometric function $\mathcal{F}$

$$\varphi_\lambda^\Omega(x) = \mathcal{F}(\lambda - \rho, \operatorname{spec}, x)$$

with ρ given as in 6.9.

Hypergeometric series

Recall the generalized Pochhammer symbols from (6.7), as well as their dependence on the parameter k. A hypergeometric series associated with Ω is defined for parameter $\mu \in \mathbb{C}^p, \nu \in \mathbb{C}^q$ by the series

$${}_pF_q^\Omega(\mu;\nu;x) := \sum_{\lambda\in\Lambda_+^n} \frac{[\mu]_\lambda}{[\nu]_\lambda|\lambda|!} Z_\lambda(x), \tag{7.11}$$

whenever the series converges. There is no canonical terminology of a two-variable hypergeometric series. To get an idea, let P be the quadratic representation of V and consider for a moment the function

$${}_pF_q^\Omega(\mu;\nu;x,y) := {}_pF_q^\Omega(\mu;\nu;P(\sqrt{x})y).$$

Remark 7.2. By [FK94, Corollary XI.3.2] and (7.10), the K-mean of the two-variable ${}_pF_q^\Omega$ coincides with the hypergeometric series of Jack polynomials introduced in Definition 6.18, i.e.

$$\begin{aligned}\int_K {}_pF_q^\Omega(\mu;\nu;x,ky)\,\mathrm{d}k &= \sum_{\lambda\in\Lambda_+^n} \frac{[\mu]_\lambda}{[\nu]_\lambda\,|\lambda|!}\int_K Z_\lambda(P(\sqrt{x})ky)\,\mathrm{d}k\\ &= \sum_{\lambda\in\Lambda_+^n} \frac{[\mu]_\lambda}{[\nu]_\lambda}\frac{Z_\lambda(x)Z_\lambda(y)}{|\lambda|!Z_\lambda(e)} = {}_pF_q(\mu;\nu;\operatorname{spec} x,\operatorname{spec} y).\end{aligned}$$

By definition K stabilizes the unit element $e \in V$ so that the one-variable hypergeometric series from Definition 6.18 and (7.11) are related by

$${}_pF_q^\Omega(\mu;\nu;x) = {}_pF_q(\mu;\nu;\operatorname{spec} x, \underline{1}).$$

Bessel function and Hankel transform

One of the classical special functions associated with Ω are the ($\mathcal{J}$-)Bessel functions. The Bessel function of index $\nu \in \mathbb{C} \setminus (\{k,\dots,k(n-1)\} - \mathbb{N}_0)$ associated with Ω is defined as the following entire function on the complexification $V_\mathbb{C}$ of V:

$$\mathcal{J}_\nu^\Omega(z) := {}_0F_1^\Omega(\nu;-z) = {}_0F_1(\nu;-\operatorname{spec} z). \tag{7.12}$$

The Bessel function $\mathcal{J}_\nu^\Omega$ defines the kernel of the Hankel transform on Ω by

$$\mathcal{H}_\nu^\Omega f(x) := \frac{1}{\Gamma_\Omega(\nu)}\int_\Omega \mathcal{J}_\nu^\Omega(P(\sqrt{x})y)f(y)\det(y)^{\nu-\frac{m}{n}}\,\mathrm{d}y,$$

where det is the Jordan determinant.

Remark 7.3. In view of [FK94, Theorem VI.2.3] and Remark 7.2, the Hankel transform of a K-invariant function $f : V \to \mathbb{C}$, $f(u) = f_0(\operatorname{spec} u)$ can be rewritten as

$$\mathcal{H}^\Omega_\nu f(x) = \text{const} \cdot \int_{\mathbb{R}^n} f_0(\xi)_0F_1(\xi, \operatorname{spec} x)\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\, \mathrm{d}\xi. \tag{7.13}$$

Therefore, it is natural to ask which results on Bessel functions and the Hankel transform can be generalized to the type A Dunkl setting. It is not our purpose to discuss all associated results, but we will briefly sketch the main theorem that will be generalized to the Dunkl setting.

Consider a self-adjoint representation $\phi : V \to \operatorname{End} E$ of V on some real Euclidean space E of dimension N and let $Q : E \to V$ be the associated quadratic representation. Note that the image of Q is contained in $\overline{\Omega}$, see [FK94, Chapter XVI].

Theorem 7.4. *The Fourier transform of an integrable radial function $f : E \to \mathbb{C}$, i.e. $f(\xi) = F(Q(\xi))$, is given by the Hankel transform of F with index $\nu = \frac{N}{2n}$, namely*

$$\widehat{f}(\eta) = \int_E e^{-i\langle\xi,\eta\rangle_E} f(\xi) d\xi = \text{const} \cdot \mathcal{H}^\Omega_\nu F(\tfrac{Q(\eta)}{4})$$

In [Her55, Mui82, Rub06] the following special case is studied in depth: $V = \operatorname{Sym}_n(\mathbb{R})$ is the space of real symmetric matrices, $\Omega = \operatorname{Pos}_n(\mathbb{R})$ is the cone of positive definite matrices, $E = \mathbb{R}^{n\times m}$ is equipped with the Hilbert-Schmidt inner product, the representation is $\phi(x)\xi = x\xi$, and the associated quadratic form is given by $Q(\xi) = \xi\xi^T$. Motivated by this, we will generalize Theorem 7.4 to the Dunkl setting. The key observation is that a ${}_0F_1$ hypergeometric series coincides with a type B Dunkl-Bessel function (cf. [Rö07, Proposition 4.5]), i.e.

$${}_0F_1(\nu; \tfrac{x^2}{2}, \tfrac{y^2}{2}) = J^B_\kappa(x, y),$$

where the multiplicity κ depends on k and ν. This is how type B Dunkl theory is involved. The Fourier transform in Theorem 7.4 is then replaced by an arbitrary Dunkl transform of type B, Q is replaced by $x \mapsto x^2 = (x_1^2, \dots, x_n^2)$ and the parameter ν related to the type B multiplicity.

K-Bessel functions

The $\mathcal{K}$-Bessel function of Ω with index $\mathbf{s} \in \mathbb{C}$ (in variables $x, y \in \Omega$) is defined by

$$\mathcal{K}^\Omega_{\mathbf{s}}(x, y) := \int_\Omega e^{-(x,u^{-1})-(y,u)}\Delta_{\mathbf{s}}(u)\det(u)^{-\frac{m}{n}}\, \mathrm{d}u, \tag{7.14}$$

where (x, y) is the inner product on V, given by the trace. The $\mathcal{K}$-Bessel functions for arbitrary symmetric cones were first considered in [Cle88] and further studied in [Dun90]. In [FK94, Chapter XVI] the following properties of $\mathcal{K}_{\mathbf{s}}$ are proven.

Theorem 7.5. *The integral $\mathcal{K}_{\mathbf{s}}(x, y)$ converges for all $x, y \in \Omega$ and $\mathbf{s} \in \mathbb{C}^n$. Furthermore*

(i) *$\mathcal{K}_{\mathbf{s}}(x, y)$ is an entire function of $\mathbf{s}$.*

(ii) *$\mathcal{K}_{\mathbf{s}}(y, x) = \mathcal{K}_{-\mathbf{s}^*}(m_0x, m_0y)$, where $\mathbf{s}^* = (s_n, \dots, s_1)$ and $m_0 \in K$ is an involution on V, defined in [FK94, page 127].*

These and additional growth properties of the $\mathcal{K}$-Bessel functions are an important tool to verify functional equations for zeta integrals in the setting of symmetric cones.

Remark 7.6. The formula (4.10), as well as the integration formula [FK94, Theorem VI.2.3] show that averaging $\mathcal{K}$ in both variables over K gives

$$\begin{aligned}\int_{K\times K} &\mathcal{K}^{\Omega}_{\mathbf{s}}(kx,k'y)\,\mathrm{d}(k,k')\\ &= \text{const}\cdot\int_{\mathbb{R}^n_+} J^A(-\operatorname{spec} x,\tfrac{1}{\xi})J^A(-\operatorname{spec} y,\xi)\mathcal{F}(\mathbf{s}-\rho,\xi)\Delta(\xi)^{-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi\\ &= \text{const}\cdot\int_{\mathbb{R}^n_+} J^A(-\operatorname{spec} x,\xi)J^A(-\operatorname{spec} y,\tfrac{1}{\xi})\mathcal{F}(-\mathbf{s}^R-\rho,\xi)\Delta(\xi)^{-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where the last equality follows from a change of variable and properties of the hypergeometric function as stated in Section 5. If the parameter is given by $\mathbf{s}=\underline{\nu}=(\nu,\dots,\nu)$ with $\nu\in\mathbb{C}$, then

$$\int_K \mathcal{K}^{\Omega}_{\underline{\nu}}(kx,y)\,\mathrm{d}k = \text{const}\cdot\int_{\mathbb{R}^n_+} J^A(-\operatorname{spec} x,\frac{1}{\xi})J^A(-\operatorname{spec} y,\xi)\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi. \quad (7.15)$$

This relation between the $\mathcal{K}$-Bessel functions on a symmetric cone and the type A Dunkl theory is the starting point to define $\mathcal{K}$-Bessel functions in the more general type A Dunkl theory for arbitrary parameters k. In particular, we expect that most of the results on the $\mathcal{K}$-Bessel function on a symmetric cone will have an analogue in the Dunkl setting.

7.3 Bessel kernel and K-Bessel function

In this section, we return to the Dunkl setting. We define a Bessel kernel for the root system A_{n-1}, playing the role of the two-variable Bessel functions on a symmetric cone. Furthermore, we introduce Dunkl-type $\mathcal{K}$-Bessel functions for the root system A_{n-1}. The section provides a detailed exposition of this generalized Bessel functions and we investigate their properties in line with know results in the radial analysis on symmetric cones. For instance, we give recurrence relations, integral representations as well as growth estimates.

Definition 7.7. For $\nu\in\mathbb{C}\backslash(\{0,k,\dots,k(n-1)\}-\mathbb{N}_0)$ we define the type A *(non-)symmetric Bessel kernels* as

$$\mathscr{E}_\nu(w,z) := {}_0K_1(\nu;w,-z) = \sum_{\eta\in\mathbb{N}_0^n}\frac{(-1)^{|\eta|}}{[\nu]_{\eta+}}\frac{L_\eta(w)L_\eta(z)}{|\eta|!\,L_\eta(\underline{1})},$$

$$\mathscr{J}_\nu(w,z) := {}_0F_1(\nu;w,-z) = \sum_{\lambda\in\Lambda^n_+}\frac{(-1)^{|\lambda|}}{[\nu]_\lambda}\frac{C_\lambda(w)C_\lambda(z)}{|\lambda|!\,C_\lambda(\underline{1})},$$

which are, by Theorem 6.20, entire functions in the variables w,z, and holomorphic in the parameter ν on the domain $\mathbb{C}\backslash(\{0,k,\dots,k(n-1)\}-\mathbb{N}_0)$. In particular, the Bessel kernels are related by taking $\mathcal{S}_n$-means

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\mathscr{E}_\nu(w,\sigma z) = \mathscr{J}_\nu(w,z).$$

Notice that by equation (7.12) the kernel $\mathscr{J}_\nu$ generalizes the Bessel function of a symmetric cone.

From now on, consider a fixed parameter $\nu\in\mathbb{C}$. The type A multiplicity is

$$\kappa_A = k \geq 0,$$

and the type B multiplicity is

$$\kappa_B = (k, k') \quad \text{with} \quad k' = \nu - \mu_0 - \tfrac{1}{2}, \quad \text{where} \quad \mu_0 := k(n-1). \tag{7.16}$$

Hence, we have a one-to-one correspondence between the pairs (κ_A, ν) and the type B multiplicities κ_B. The following proposition justifies the term "Bessel-kernel" for the functions $\mathscr{J}_\nu$ and $\mathscr{E}_\nu$.

Proposition 7.8. *The type B Dunkl kernel and Bessel function with multiplicity κ_B satisfy*

$${}_0K_1(\nu; \tfrac{w^2}{2}, \tfrac{z^2}{2}) = \frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E^B(w, \tau z) \quad \text{and} \quad {}_0F_1(\nu; \tfrac{w^2}{2}, \tfrac{z^2}{2}) = J^B(w, z).$$

In particular,

$$\mathscr{E}_\nu(\tfrac{w^2}{2}, \tfrac{z^2}{2}) = \frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E^B(iw, \tau z) \quad \text{and} \quad \mathscr{J}_\nu(\tfrac{w^2}{2}, \tfrac{z^2}{2}) = J^B(iw, z).$$

We remark that the statement of Proposition 7.8 for the type B Bessel function was already observed in [Rö07, Proposition 4.5].

Proof. The proof is similar to [Rö07] for the Bessel function. Using [BF98, Proposition 4.18], we obtain the following formula for the Dunkl pairing $[\cdot,\cdot]^B$ of non-symmetric Jack polynomials E_η

$$E_\eta((T_x^B)^2) E_\mu(x^2)\Big|_{x=0} = [E_\eta(x^2), E_\mu(x^2)]^B = \begin{cases} 4^{|\eta|} [\nu]_{\eta+} k^{|\eta|} \frac{d'_\eta e_\eta}{d_\eta}, & \text{if } \eta = \mu, \\ 0, & \text{otherwise,} \end{cases}$$

with certain constants d_η, d'_η, e_η, satisfying $\frac{e_\eta}{d_\eta} = E_\eta(\underline{1})$, cf. [For10, Formula (12.3.3)]. Therefore, by definition of the renormalization L_η of the non-symmetric Jack polynomials (Lemma 6.16),

$$[L_\eta(x^2), L_\mu(x^2)]^B = 4^{|\eta|} \, |\eta|! \, [\nu]_{\eta+} L_\eta(\underline{1}) \cdot \delta_{\eta\mu}, \tag{7.17}$$

where $\delta_{\mu\eta}$ is the Kronecker delta. Since $(L_\eta)_{\eta \in \mathbb{N}_0^n}$ is a homogeneous basis for $\mathbb{C}[\mathbb{R}^n]$, we have an expression

$$\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E^B(w, \tau z) = \sum_{\mu \in \mathbb{N}_0^n} a_\mu(w) L_\mu(z^2),$$

with certain coefficients $a_\mu(w) \in \mathbb{C}$. Finally, the $\mathbb{Z}_2^n$-invariance of $L_\mu(x^2)$, the eigenvalue equation for the Dunkl kernel and (7.17) show that

$$\begin{aligned} L_\eta(w^2) &= \frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} L_\eta((\tau w)^2) E^B(\tau w, z)\Big|_{z=0} = \frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} L_\eta((T_z^B)^2) E^B(\tau w, z)\Big|_{z=0} \\ &= \sum_{\mu \in \mathbb{N}_0^n} a_\mu(w) [L_\eta(z^2), L_\mu(z^2)]^B = a_\eta(w) 4^{|\eta|} \, |\eta|! \, [\nu]_{\eta+} L_\eta(\underline{1}). \end{aligned}$$

■

Lemma 7.9. *Let $w, z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$ and $\operatorname{Re} \nu > \mu_0$. Then*

$$\int_{\mathbb{R}_+^n} E^A(-x, z) \mathscr{E}_\nu(w, x) \Delta(x)^{\nu - \mu_0 - 1} \omega^A(x) \, dx = \Gamma_n(\nu) E^A(w, -\tfrac{1}{z}) \Delta(z)^{-\nu}.$$

The same is true if both E^A and $\mathscr{E}_\nu$ are replaced by J^A and $\mathscr{J}_\nu$, respectively.

PROOF. This is immediate from Remark 6.21 and Theorem 6.23, as

$$ {}_1K_1(\nu;\nu;w,z) = {}_0K_0(w,z) = E^A(w,z). $$

■

Theorem 7.10 (Integral representation). *Let* $\operatorname{Re}\nu > 2\mu_0+1, w\in\mathbb{C}^n$ *and* $x\in\mathbb{R}^n_+$. *Then for all* $s\in\mathbb{R}_+$, *we have*

$$ \mathscr{E}_\nu(w,x)\Delta(x)^{\nu-\mu_0-1} = \frac{\Gamma_n(\nu)}{c_A^2\, i^n}\int_{\operatorname{Re}(\zeta)=\underline{s}} E^A(x,\zeta)E^A(w,-\tfrac{1}{\zeta})\Delta(\zeta)^{-\nu}\omega^A(\zeta)\,\mathrm{d}\zeta. $$

The same is true if both E^A *and* $\mathscr{E}_\nu$ *are replaced by* J^A *and* $\mathscr{J}_\nu$, *respectively.*

PROOF. By the Cauchy-type inversion formula of the Dunkl-Laplace transform (Theorem 7.1 (iii)), it suffices to prove that the right hand side of Lemma 7.9 is integrable as a function of z over $\underline{s}+i\mathbb{R}^n$ for $s>0$ with respect to the measure $\omega^A(x)\mathrm{d}x$. The map $z\mapsto\frac{1}{z}$ is bounded on $\underline{s}+i\mathbb{R}^n$ and so is $z\mapsto E^A(w,-\frac{1}{z})$. Thus, we only have to verify the integrability condition for $\Delta(z)^{-\nu}$. Since $\omega^A(x)=|D(x)|^{2k}$ with $D(x)=\prod_{i<j}(x_i-x_j)$ of degree $(n-1)$ in x_i, the function

$$ y\mapsto \Delta(\underline{s}+y^2)^{-\frac{1}{2}\operatorname{Re}\nu}\omega^A(y) $$

is integrable over $\mathbb{R}^n$ if and only if $\operatorname{Re}\nu > 2\mu_0+1$, since $\sqrt{\underline{s}+y_i^2}\sim|y_i|$ for large $|y_i|$. Hence, the claim follows for $\operatorname{Re}\nu>2\mu_0+1$. ■

Corollary 7.11 (Recurrence formulas). *For* $w,z\in\mathbb{C}^n$ *and* $\nu\in\mathbb{C}$ *with* $\operatorname{Re}\nu>\mu_0$ *we have*

(i) $\Delta(T_z^A)\mathscr{E}_\nu(w,z) = \dfrac{(-1)^n}{[\nu]_{\underline{1}}}\Delta(w)\mathscr{E}_{\nu+1}(w,z)$.

(ii) *If in addition* $\operatorname{Re}z>0$, *then*

$$ \Delta(T_z^A)\Big(\mathscr{E}_\nu(w,z)\Delta(z)^{\nu-\mu_0-1}\Big) = [\nu-1]_{\underline{1}}\,\mathscr{E}_{\nu-1}(w,z)\Delta(z)^{\nu-\mu_0-2}. $$

The results are still true if $\mathscr{E}_\nu$ *is replaced by* $\mathscr{J}_\nu$.

PROOF.

(i) Denoting $f^-(x)=f(-x)$, we have $T_\xi^A f^- = -(T_\xi^A f)^-$. Hence, part (i) is a consequence of Remark 6.21 and Theorem 6.27.

(ii) By analyticity in ν and z, we can assume that $\operatorname{Re}\nu>2\mu_0+2$ and $z=x\in\mathbb{R}^n_+$. By changing the order of differentation and integration, the integral representation of $\mathscr{E}_\nu$ in Theorem 7.10 shows that

$$ \begin{aligned} &\Delta(T_x^A)(\mathscr{E}_\nu(w,x)\Delta(x)^{\nu-\mu_0-1}) \\ &\quad= \frac{\Gamma_n(\nu)}{c_A^2\, i^n}\int_{\operatorname{Re}(\zeta)=\underline{s}} \Delta(\zeta)E^A(x,\zeta)E^A(w,-\tfrac{1}{\zeta})\Delta(\zeta)^{-\nu}\omega^A(\zeta)\,\mathrm{d}\zeta \\ &\quad= [\nu-1]_{\underline{1}}\frac{\Gamma_n(\nu-1)}{c_A^2\, i^n}\int_{\operatorname{Re}(\zeta)=\underline{s}} \Delta(\zeta)E^A(x,\zeta)E^A(w,-\tfrac{1}{\zeta})\Delta(\zeta)^{-(\nu-1)}\omega^A(\zeta)\,\mathrm{d}\zeta \\ &\quad= [\nu-1]_{\underline{1}}\mathscr{E}_{\nu-1}(w,x)\Delta(x)^{\nu-\mu_0-2}. \end{aligned} $$

It remains to justify the change of order of differentation and integration. By [Rö03a, Proposition 2.6] we have

$$\left|\partial_\xi^x E^A(x, \underline{s}+iy)\right| \leq (s+|y|)e^{\langle x,\underline{s}\rangle}$$

which is locally bounded in x. By similar computations as in Theorem 7.10, the condition $\operatorname{Re}\nu > 2\mu_0 + 2$ shows that the function

$$y \mapsto \partial_\xi^x E^A(x, \underline{s}+iy)\Delta(\underline{s}+iy)^{\nu-\mu_0-1}\omega^A(y)$$

is dominated on $\mathbb{R}^n$ by

$$y \mapsto e^{\langle x,\underline{s}\rangle}(s+|y|)\Delta(\underline{s}+iy)^{\nu-\mu_0-1}\omega^A(y)$$

which is locally bounded in x and integrable over $\mathbb{R}^n$. This justifies the change of order of differentation and integration.

■

We now come to the definition of the Dunkl-type $\mathcal{K}$-Bessel function which shares important properties with the $\mathcal{K}$-Bessel functions on symmetric cones.

Definition 7.12. The *Dunkl-type $\mathcal{K}$-Bessel function* of index $\nu \in \mathbb{C}$ is defined by

$$\mathcal{K}_\nu(w,z) := \int_{\mathbb{R}_+^n} E^A(-x,w)E^A(-\tfrac{1}{x},z)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x.$$

Its convergence and further properties of the integral will be investigated in the following theorem.

By taking $\mathcal{S}_n$-means we obtain

$$\frac{1}{n!^2}\sum_{\sigma,\tau\in\mathcal{S}_n}\mathcal{K}_\nu(\sigma w,\tau z) = \int_{\mathbb{R}_+^n} J^A(-x,w)J^A(-\tfrac{1}{x},z)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x$$

which generalizes the averaging property of the $\mathcal{K}$-Bessel function of a symmetric cone as noted in Remark 7.6, equation (7.15).

By a change of variables, the following Lemma is immediate.

Lemma 7.13. *On $\mathbb{R}_+^n$, the $\mathcal{S}_n$-invariant measure*

$$\Delta(x)^{-\mu_0-1}\omega^A(x)\mathrm{d}x \tag{7.18}$$

is invariant under the transformation $x \mapsto \frac{1}{x}$ as well as under $x \mapsto sx$ for $s > 0$.

This invariance will be highly relevant in the subsequent results. The measure (7.18) has to be understood as the Dunkl analogue of the invariant measure of a symmetric cone. In fact, if k is related to a symmetric cone $\Omega = G/K$, then the measure (7.18) is the K-radial part of the G-invariant measure on Ω, cf. [FK94, Theorem VI.2.3].
The subsequent theorem generalizes the results for the $\mathcal{K}$-Bessel functions on a symmetric cone, such as in Theorem 7.5 or [FK94, Chapter XVI, Section 3].

Theorem 7.14. *The Dunkl-type $\mathcal{K}$-Bessel function $\mathcal{K}_\nu$ exists for all $\nu \in \mathbb{C}$ and $w, z \in \mathbb{C}^n$ with $\operatorname{Re} w, \operatorname{Re} z > 0$. Moreover, $\mathcal{K}$ is holomorphic in ν, w, z and has the following properties:*

(i) $\mathcal{K}_\nu(\sigma w, \sigma z) = \mathcal{K}_\nu(w,z)$ *for all $\sigma \in \mathcal{S}_n$.*

(ii) $\mathcal{K}_\nu(w,z) = \mathcal{K}_{-\nu}(z,w)$ *and* $|\mathcal{K}_\nu(w,z)| \leq \mathcal{K}_{\mathrm{Re}\,\nu}(\mathrm{Re}\,w, \mathrm{Re}\,z)$.

(iii) *If* $\nu \in \mathbb{R}$ *and* $x, y \in \mathbb{R}^n_+$, *then*

$$0 < \mathcal{K}_\nu(x,y) \leq \begin{cases} \Gamma_n(\nu)\Delta(x)^{-\nu} & \text{if } \nu > \mu_0 \\ \Gamma_n(-\nu)\Delta(y)^{\nu} & \text{if } \nu < -\mu_0 \end{cases}$$

and $\nu \mapsto \mathcal{K}_\nu(x,y)$ *is convex.*

(iv) *If* $\nu \in \mathbb{R}$, $|\nu| \leq \mu_0$ *and* $x, y \in \mathbb{R}^n_+$, *then for all* $\epsilon > 0$,

$$0 < \mathcal{K}_\nu(x,y) \leq \Gamma_n(\mu_0+1)\Delta(x)^{-\mu_0-1} + \Gamma_n(\mu_0+\epsilon)\Delta(y)^{-\mu_0-\epsilon}.$$

(v) *Recurrence formulas:*

$$\Delta(T^A)\mathcal{K}_\nu(w,\cdot) = (-1)^n \mathcal{K}_{\nu-1}(w,\cdot),$$
$$\Delta(T^A)\mathcal{K}_\nu(\cdot,z) = (-1)^n \mathcal{K}_{\nu+1}(\cdot,z).$$

(vi) *If* $\mathrm{Re}\,\nu < -\mu_0$, *then*

$$\lim_{\epsilon\to 0} \epsilon^{-n\nu}\mathcal{K}_\nu(w,\epsilon z) = \Gamma_n(-\nu)\Delta(z)^{\nu}.$$

If $\mathrm{Re}\,\nu > \mu_0$, *then*

$$\lim_{\epsilon\to 0} \epsilon^{n\nu}\mathcal{K}_\nu(\epsilon w, z) = \Gamma_n(\nu)\Delta(w)^{-\nu}.$$

The proof is similar to the one in the case of symmetric cones as in [FK94, Chapter XVI, Section 3]. See also [Rub06] for the cone of real positive definite symmetric matrices.

Proof. Let $\mathrm{Re}\,\nu > \mu_0$ and $w, z \in \mathbb{C}^n$ with $\mathrm{Re}\,w, \mathrm{Re}\,z > 0$. According to the estimates (6.8),

$$\left| E^A(-x,w)E^A(-\tfrac{1}{x},z)\right| \leq E^A(-x, \mathrm{Re}\,w).$$

Hence, the Laplace transform identities in Theorem 6.12 lead to

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \left| E^A(-x,w)E^A(-\tfrac{1}{x},z)\Delta(x)^{\nu-\mu_0-1}\right| \omega^A(x)\,\mathrm{d}x \\ &\quad \leq \int_{\mathbb{R}^n_+} E^A(-x,\mathrm{Re}\,w)\Delta(x)^{\mathrm{Re}\,\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x = \Gamma_n(\mathrm{Re}\,\nu)\Delta(\mathrm{Re}\,w)^{-\mathrm{Re}\,\nu}. \end{aligned}$$

In particular, $\mathcal{K}_\nu(w,z)$ exists and is holomorphic in $(w,z,\nu) \in \{\mathrm{Re}\,w > 0\} \times \{\mathrm{Re}\,z > 0\} \times \{\mathrm{Re}\,\nu > \mu_0\}$ by standard theorems on holomorphic parameter integrals. Moreover, the stated estimate in (iii) is true in the case $\mathrm{Re}\,\nu > \mu_0$.
Consider the case $\mathrm{Re}\,\nu < -\mu_0$. By the change of variables $\xi \mapsto \frac{1}{\xi}$ in the defining integral for $\mathcal{K}_\nu(w,z)$ and Lemma 7.13 we have $\mathcal{K}_\nu(w,z) = \mathcal{K}_{-\nu}(z,w)$. Thus, $\mathcal{K}_\nu(w,z)$ exists and is holomorphic in $(w,z,\nu) \in \{\mathrm{Re}\,w > 0\} \times \{\mathrm{Re}\,z > 0\} \times \{\mathrm{Re}\,\nu < -\mu_0\}$. Thereby, the estimate in (iii) for $\mathrm{Re}\,\nu < -\mu_0$ is true as well. Finally, for $x \in \mathbb{R}^n_+$, $\nu \mapsto \Delta(x)^{\mathrm{Re}\,\nu-\mu_0-1}$ is convex on $\mathbb{R}$, so that $\mathcal{K}_\nu(w,z)$ exists and is holomorphic in $(w,z,\nu) \in \{\mathrm{Re}\,w > 0\} \times \{\mathrm{Re}\,z > 0\} \times \mathbb{C}$, where $\nu \mapsto \mathcal{K}_\nu(x,y)$ is convex on $\mathbb{R}$ for fixed $x, y \in \mathbb{R}^n_+$. It remains to prove the properties (i), (ii) and (iv)-(vi).

(i) Since $E^A(\sigma w, \sigma z) = E^A(w,z)$, this follows by a change of variables.

(ii) By Lemma 7.13, this is done by the change of variables $x \mapsto \frac{1}{x}$. The estimate is an immediate consequence of equation (6.8).

(iv) To prove the stated estimate, consider an arbitrary $\epsilon > 0$ and split the defining integral for $\mathcal{K}_\nu$ according to

$$\mathbb{R}_+^n = \{\Delta(\xi) < 1\} \sqcup \{\Delta(\xi) > 1\} \sqcup \{\Delta(\xi) = 1\},$$

where the last set has measure zero and need not to be discussed.

(a) Since $E^A(-\frac{1}{\xi}, y) \leq 1$ for $\xi, y \in \mathbb{R}_+^n$ and $\Delta(\xi)^{\nu-\mu_0-1} < 1$ for $\Delta(\xi) > 1$, we obtain

$$\begin{aligned}\int_{\Delta(\xi)>1} & E^A(-\xi, x)E^A(-\tfrac{1}{\xi}, y)\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi \\ &\leq \int_{\mathbb{R}_+^n} E^A(-\xi, x)\omega^A(\xi)\,\mathrm{d}\xi = \Gamma_n(\mu_0+1)\Delta(x)^{-\mu_0-1},\end{aligned}$$

where again Theorem 6.12 was used.

(b) As $\Delta(\xi)^{-\nu-\mu_0-1} < \Delta(\xi)^{\epsilon-1}$ for $\Delta(\xi) > 1$ and $\epsilon > 0$, we obtain

$$\begin{aligned}\int_{\Delta(\xi)<1} & E^A(-\xi, x)E^A(-\tfrac{1}{\xi}, y)\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi \\ &= \int_{\Delta(\xi)>1} E^A(-\tfrac{1}{\xi}, x)E^A(-\xi, y)\Delta(\xi)^{-\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi \\ &\leq \int_{\mathbb{R}_+^n} E^A(-\xi, x)\Delta(\xi)^{\epsilon-1}\omega^A(\xi)\,\mathrm{d}\xi = \Gamma_n(\mu_0+\epsilon)\Delta(x)^{-\mu_0-\epsilon}.\end{aligned}$$

(v) This is a consequence of the eigenvalue equation for E^A.

(vi) It suffices to check the second limit, the first one can then be deduced from (ii). Recall from (6.8), that $\left|E^A(-\frac{\epsilon}{x}, z)\right| \leq 1$ for $\operatorname{Re} z > 0$. By the change of variables $x \mapsto \frac{x}{\epsilon}$, dominated convergence and Theorem 6.12, we just have

$$\begin{aligned}\epsilon^{n\nu}\mathcal{K}_\nu(\epsilon w, z) &= \epsilon^{n\nu}\int_{\mathbb{R}_+^n} E^A(-x, \epsilon w)E^A(-\tfrac{1}{x}, z)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^n} E^A(-x, w)E^A(-\tfrac{\epsilon}{x}, z)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x,\end{aligned}$$

and as ϵ tends to 0, the last integral converges to

$$\int_{\mathbb{R}_+^n} E^A(-x, w)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x = \Gamma_n(\nu)\Delta(w)^{-\nu}.$$

■

Proposition 7.15. *For $w \in \mathbb{C}^n$ with $\operatorname{Re} w > 0$ put $f_w(x) = \mathcal{K}_{\nu-\mu_0-\frac{1}{2}}(x^2, w)$. Then f_w satisfies the eigenvalue equation*

$$\frac{1}{4}(T_i^B)^2 f_w = w_i f_w \quad \text{for all } i = 1, \ldots, n,$$

where the type B multiplicity of the Dunkl operator on the left side is given as in (7.16) by the value k on $\pm e_i \pm e_j$ and $k' = \nu - \mu_0 - \frac{1}{2}$ on $\pm e_i$.

This is a generalization of the Bessel system on a symmetric cone which is solved by the $\mathcal{K}$-Bessel function. See for instance [FK94, Page 358] and [Dib90] for the one-variable case, i.e. $w = \underline{1}$, or [Mö13] for some further eigenvalue equations of the $\mathcal{K}$-Bessel function.

Proof. To improve readability we put $\nu' = \nu + \frac{1}{2}$. The proof will be done by direct computation and is divided into several steps:

(i) First, consider a C^2-function $f(x) = f_0(x^2)$. Then, as noticed in [BF98],

$$\frac{1}{4}(T_i^B)^2 f(x) = x_i^2((T_i^A)^2 f_0)(x^2) + (\nu' - \mu_0)(T_i^A f_0)(x^2) + k\sum_{j\neq i}(T_i^A f_0)(s_{ij}x^2).$$

Hence,

$$\frac{1}{4}(T_{i,x}^B)^2 E^A(-x^2,\xi) = \Big(x_i^2\xi_i^2 - (\nu' - \mu_0)\xi_i\Big)E^A(-x^2,\xi) - k\sum_{j\neq i}\xi_i E^A(-x^2, s_{ij}\xi).$$

(ii) By a change of variables and $E^A(\sigma x, \sigma y) = E^A(x,y)$ for $\sigma \in \mathcal{S}_n$, we have

$$\begin{aligned}\int_{\mathbb{R}_+^n}\sum_{j\neq i}\xi_i E^A(-x^2, s_{ij}\xi)E^A(-\tfrac{1}{\xi}, w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega_k(\xi)\,\mathrm{d}\xi \\ = \int_{\mathbb{R}_+^n} E^A(-x^2,\xi)\left(\sum_{j\neq i}\xi_j E^A(-\tfrac{1}{s_{ij}\xi}, w)\right)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega_k(\xi)\,\mathrm{d}\xi.\end{aligned}$$

(iii) For abbreviation put $g(\xi) = E^A(\xi, w)$, so that

$$\begin{aligned}T_{i,\xi}^A(\xi_i^2 g(-\tfrac{1}{\xi})) &= \frac{\partial g}{\partial \xi_i}(-\tfrac{1}{\xi}) + 2\xi_i g(-\tfrac{1}{\xi}) + k\sum_{j\neq i}\frac{\xi_i^2 g(-\tfrac{1}{\xi}) - \xi_j^2 g(-\tfrac{1}{s_{ij}\xi})}{\xi_i - \xi_j} \\ &= (T_i^A g)(-\tfrac{1}{\xi}) + 2\xi_i g(-\tfrac{1}{\xi_i}) + k\sum_{j\neq i}\left(\frac{\xi_i^2 g(-\tfrac{1}{\xi}) - \xi_j^2 g(-\tfrac{1}{s_{ij}\xi})}{\xi_i - \xi_j}\right. \\ &\quad \left. - \frac{g(-\tfrac{1}{\xi}) - g(-s_{ij}\tfrac{1}{\xi})}{\tfrac{1}{\xi_j} - \tfrac{1}{\xi_i}}\right) \\ &= (w_i + 2\xi_i)g(-\tfrac{1}{\xi}) + k\sum_{j\neq i}\xi_i g(-\tfrac{1}{\xi}) + \xi_j g(-\tfrac{1}{s_{ij}\xi}) \\ &= (w_i + (2+\mu_0)\xi_i)g(-\tfrac{1}{\xi}) + k\sum_{j\neq i}\xi_j g(-\tfrac{1}{s_{ij}\xi}).\end{aligned}$$

Moreover, since T_i^A acts on $\mathcal{S}_n$-invariant functions as a partial derivative,

$$T_{i,\xi}^A\Delta(\xi)^{\nu'-2(\mu_0+1)} = (\nu' - 2(\mu_0+1))\frac{\Delta(\xi)^{\nu'-2(\mu_0+1)}}{\xi_i}.$$

Therefore,

$$\begin{aligned}T_{i,\xi}^A&\left(\xi_i^2 E^A(-\tfrac{1}{\xi}, w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\right) \\ &= \Big[\,(w_i + (\nu' - \mu_0)\xi_i)\,E^A(-\tfrac{1}{\xi}, w) + k\sum_{j\neq i}\xi_j E^A(-\tfrac{1}{s_{ij}\xi}, w)\Big]\Delta(\xi)^{\nu'-2(\mu_0+1)}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\int_{\mathbb{R}_+^n} & x_i^2\xi_i^2 E^A(-x^2,\xi)E^A(-\tfrac{1}{\xi}, w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega^A(\xi)\,\mathrm{d}\xi \\ &= -\int_{\mathbb{R}_+^n}\xi_i^2 T_{i,\xi}^A E^A(-x^2,\xi)E^A(-\tfrac{1}{\xi}, w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega^A(\xi)\,\mathrm{d}\xi \qquad (7.19) \\ &= \int_{\mathbb{R}_+^n} E^A(-x^2,\xi)T_{i,\xi}^A\left(\xi_i^2 E^A(-\tfrac{1}{\xi}, w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\right)\omega^A(\xi)\,\mathrm{d}\xi\end{aligned}$$

$$= \int_{\mathbb{R}^n_+} (w_i + (\nu' - \mu_0)\xi_i)\, E^A(-x^2,\xi)E^A(-\tfrac{1}{\xi},w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega^A(\xi)\,\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^n_+} E^A(-x^2,\xi)\cdot k\sum_{j\neq i}\xi_j E^A(-\tfrac{1}{s_{ij}\xi},w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega^A(\xi)\,\mathrm{d}\xi.$$

Here, equation (7.19) is justified by the skew symmetry of the Dunkl operators on $L^2(\mathbb{R}^n,\omega^A(x)\mathrm{d}x)$ which is based on integration by parts. So it suffices to show that there occur no boundary terms if we integrate by parts which can be seen as follows: As $E^A(-\frac{1}{\xi},w) \longrightarrow 0$ for $\xi \to \partial\mathbb{R}^n_+$ and as $E^A(-x^2,\xi) \longrightarrow 0$ exponetially for $\xi\to\infty$ (estimate 6.8), the boundary terms vanish on $\partial\mathbb{R}^n_+$ and in ∞.

(iv) Putting the things from (1),(2) and (3) together, we obtain

$$\frac{1}{4}(T^B_{i,x})^2\mathcal{K}_{\nu'-\mu_0-1}(x^2,w) = \int_{\mathbb{R}^n_+}\frac{1}{4}(T^B_{i,x})^2E^A(-x^2,\xi)E^A(-\tfrac{1}{\xi},w)\Delta(\xi)^{\nu'-2(\mu_0+1)}\omega(\xi)\,\mathrm{d}\xi$$
$$= w_i\mathcal{K}_{\nu'-\mu_0-1}(x^2,w).$$

■

Up to a change of variables, we are able to compute the Cherednik transform (Definition 4.31) of the $\mathcal{K}$-Bessel function as in the case of symmetric cones in [FK94, Proposition XVI.3.3], where the Cherednik transform is given by the spherical Fourier transform. Recall the rational version of the type A Cherednik kernel $\mathcal{G}$ from Definition 6.3.

Theorem 7.16. *Consider $\nu\in\mathbb{C}$ and $\lambda = \widetilde{\lambda}+\lambda'$ with $\lambda'\in i\mathbb{R}^n + \mathrm{co}(\mathcal{S}_n.\rho)$, satisfying the conditions*

$$\operatorname{Re}\widetilde{\lambda} < -\mu_0, \quad \operatorname{Re}\widetilde{\lambda} < \operatorname{Re}\nu - \mu_0.$$

Then we have

$$\Gamma_n(\rho-\lambda)\Gamma_n(\rho-\lambda+\underline{\nu})\mathcal{G}(\lambda,z) = \int_{\mathbb{R}^n_+}\mathcal{G}(\lambda,\tfrac{1}{x})\mathcal{K}_\nu(z,x)\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^n} G(\lambda,-x)\mathcal{K}_\nu(z,e^x)\delta_k(x)\,\mathrm{d}x,$$

where $\delta_k(x) = \prod_{i\neq j}|2\sinh\frac{x_i-x_j}{2}|^k$.

Proof. The second equality is a consequence of the change of variables $x\mapsto e^x$, so it remains to proof the first part. By the estimates from Lemma 6.4 for the Cherednik kernel and Theorem 7.14 (more precisely by the proof of parts (iii) and (iv)) for the $\mathcal{K}$-Bessel function, we observe that

$$(\xi,x)\mapsto \mathcal{G}(\lambda,\tfrac{1}{x})E^A(x,-\xi)E^A(-z,\tfrac{1}{\xi})\Delta(x)^{-\mu_0-1}\Delta(\xi)^{\nu-\mu_0-1}$$

is integrable over $\mathbb{R}^n_+\times\mathbb{R}^n_+$ with respect to $\omega^A(x)\omega^A(\xi)\mathrm{d}x\mathrm{d}\xi$. Hence, we use Fubini's Theorem, Remark 7.13 and Corollary 6.15 (note that $\mathcal{G}(\lambda,\frac{1}{x}) = \mathcal{G}(-\lambda^R,x)$ and $\lambda\mapsto\Gamma_n(\rho+\lambda)$ is $\mathcal{S}_n$-invariant) to obtain

$$\int_{\mathbb{R}^n_+}\mathcal{G}(\lambda,\tfrac{1}{x})\mathcal{K}_\nu(z,x)\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\mathcal{G}(\lambda,\tfrac{1}{x})E^A(x,-\xi)E^A(-z,\tfrac{1}{\xi})\Delta(x)^{-\mu_0-1}\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\omega^A(x)\,\mathrm{d}x\,\mathrm{d}\xi$$

$$
\begin{aligned}
&= \Gamma_n(\rho-\lambda)\int_{\mathbb{R}^n_+} \mathcal{G}(-\lambda^R, \tfrac{1}{\xi^R})E^A(-z,\tfrac{1}{\xi})\Delta(\xi)^{\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi \\
&= \Gamma_n(\rho-\lambda)\int_{\mathbb{R}^n_+} \mathcal{G}(-\lambda^R, \xi^R)E^A(-z,\xi)\Delta(\xi)^{-\nu-\mu_0-1}\omega^A(\xi)\,\mathrm{d}\xi \\
&= \Gamma_n(\rho-\lambda)\Gamma_n(\rho-\lambda+\underline{\nu})\mathcal{G}(-\lambda^R, \tfrac{1}{x^R}) \\
&= \Gamma_n(\rho-\lambda)\Gamma_n(\rho-\lambda+\underline{\nu})\mathcal{G}(-\lambda^R, \tfrac{1}{x^R}).
\end{aligned}
$$

■

We may also study $\mathcal{K}$-Bessel functions with a multivariate index, similar to their analogues on symmetric cones.

Definition 7.17. For arbitrary $\lambda \in \mathbb{C}^n$ we define the *$\mathcal{K}$-Bessel function*

$$
\mathcal{K}_\lambda(w,z) = \int_{\mathbb{R}^n_+} E^A(-x,w)E^A(-\tfrac{1}{x},z)\mathcal{G}(\lambda-\rho,x)\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x.
$$

The convergence of the integral will be discussed in the next theorem and is in accordance with the previous case $\lambda = \underline{\nu}$ with $\nu \in \mathbb{C}$.

Theorem 7.18. *The map $(\lambda, w, z) \mapsto \mathcal{K}_\lambda(w,z)$ is holomorphic on the domain $\mathbb{C}^n \times \{\operatorname{Re} w > 0\} \times \{\operatorname{Re} z > 0\}$ and satisfies:*

(i) $\mathcal{K}_\lambda(w,z) = \mathcal{K}_{-\lambda^R}(z^R, w^R)$ *and* $|\mathcal{K}_\lambda(w,z)| \le \mathcal{K}_{\operatorname{Re}\lambda}(\operatorname{Re} w, \operatorname{Re} z)$.

(ii) *If $\lambda \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^n_+$, then*

$$
\begin{aligned}
0 < \mathcal{K}_\lambda(x,y) &\le \begin{cases} \Gamma_n(\lambda)\mathcal{G}(\lambda-\rho, \tfrac{1}{x}), & \text{if } \lambda > \mu_0, \\ \Gamma_n(-\lambda^R)\mathcal{G}(-\lambda^R-\rho, \tfrac{1}{x^R}), & \text{if } \lambda < \mu_0. \end{cases} \\
&= \begin{cases} \Gamma_n(\lambda)\mathcal{G}(-\lambda^R-\rho, x^R), & \text{if } \lambda > \mu_0, \\ \Gamma_n(-\lambda^R)\mathcal{G}(\lambda-\rho, x), & \text{if } \lambda < \mu_0. \end{cases}.
\end{aligned}
$$

Moreover $\nu \mapsto \mathcal{K}_{\lambda+\underline{\nu}}$ is a convex function $\mathbb{R} \to \mathbb{R}_+$.

(iii) *Recurrence formulas:*

$$
\begin{aligned}
\Delta(T^A)\mathcal{K}_\lambda(w,\cdot) &= (-1)^n\mathcal{K}_{\lambda-\underline{1}}(w,\cdot), \\
\Delta(T^A)\mathcal{K}_\lambda(\cdot,z) &= (-1)^n\mathcal{K}_{\lambda+\underline{1}}(\cdot,z).
\end{aligned}
$$

Proof. The existence and analyticity of $\mathcal{K}_\nu(w,z)$ can be checked for $\operatorname{Re}\lambda > \mu_0$ and $\operatorname{Re}\lambda < -\mu_0$ exactly as in Theorem 7.14, using the stated properties of the Cherednik kernel in Lemma 6.4. The existence and analyticity in the case $-\mu_0 \le \operatorname{Re}\lambda \le \mu_0$ can be deduced by a convexity property as in Theorem 7.14 which will be proven below.

(i) This is a immediate consequence of Lemma 6.4.

(ii) Suppose that $\lambda > \mu_0$. Similar to the proof of Theorem 7.14 we obtain by Lemma 6.4 that

$$
\begin{aligned}
0 < \mathcal{K}_\lambda(x,y) &\le \int_{\mathbb{R}^n_+} E^A(-\xi,x)\mathcal{G}(\lambda-\rho,x)\Delta(x)^{-\mu_0-1}\omega^A(x)\,\mathrm{d}x \\
&= \Gamma_n(\lambda)\mathcal{G}(\lambda-\rho,\tfrac{1}{x}) = \Gamma_n(\lambda)\mathcal{G}(-\lambda^R-\rho, x^R).
\end{aligned}
$$

By part (i), the case $\lambda < -\mu_0$ reduces to the case $\lambda > \mu_0$. Moreover, in view of Lemma 6.4 we have

$$\mathcal{G}(\lambda + \underline{\nu} - \rho(k), x) = \mathcal{G}(\lambda - \rho(k), x)\Delta(x)^{\nu},$$

so that the convexity in ν is verified just as in Theorem 7.14. Together with part (i), this shows that the integral exists for all $\lambda \in \mathbb{C}^n$ and has the stated analyticity property.

(iii) This is the same argument as in the proof of Theorem 7.14.

■

7.4 The Hankel transform and its connection to type B Dunkl theory

The Hankel transform for the root system A_{n-1} was already introduced in [BF98] and earlier in [Mac89] in a symmetrized version, both at a rather formal level. In this section, we will discuss its analytic aspects and its connection to the type B Dunkl transform. We proceed in a way similar to the one in [Rub06], where the case of the symmetric cone $\Omega = \mathrm{Pos}_n(\mathbb{R})$ is treated. For arbitrary symmetric cones see also [FK94, Chapter XVI, Section 2]. Some of our results are inspired by computations in [Mac89] performed in the case of symmetric functions. As in Remark 7.3 we define the following generalized Hankel transform.

Definition 7.19. For $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > \mu_0$, the *Hankel transform* is defined by

$$\mathcal{H}_\nu f(w) := \frac{1}{\Gamma_n(\nu)} \int_{\mathbb{R}_+^n} f(x)\mathscr{E}_\nu(x,w)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x, \quad w \in \mathbb{C}^n,$$

whenever the integral exists for measurable $f : \mathbb{R}_+^n \to \mathbb{C}$.

Lemma 7.20. *Consider $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > \mu_0$ and put $e_z(x) := E^A(x,-z)$ for $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$. Then:*

(i) *$e_z \in L^2_\nu(\mathbb{R}_+^n) := L^2(\mathbb{R}_+^n, \Delta(x)^{\nu-\mu_0-1}\omega^A(x)\mathrm{d}x)$ for $\nu \in \mathbb{R}$ with $\nu > \mu_0$.*

(ii) *For $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$,*

$$\mathcal{H}_\nu(E^A(\cdot,-z)) = \Delta(z)^{-\nu}E^A(\cdot,-\tfrac{1}{z}).$$

In particular, $\mathcal{H}_\nu$ is involutive on $\mathcal{U} := \operatorname{span}_{\mathbb{C}}\left\{E^A(\cdot,-z) \mid \operatorname{Re} z > 0\right\}$.

(iii) *For $\nu \in \mathbb{R}$, $\nu > \mu_0$ equip $L^2_\nu(\mathbb{R}_+^n)$ with the canonical inner product*

$$\langle f, g\rangle_{L^2_\nu(\mathbb{R}_+^n)} := \int_{\mathbb{R}_+^n} f(x)\overline{g(x)}\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x.$$

Then $\mathcal{U} \subseteq L^2_\nu(\mathbb{R}_+^n)$ is dense. More precisely, the space spanned by the e_z with $z = \underline{s} + iy$ and fixed $s > 0$ is already dense in $L^2_\nu(\mathbb{R}_+^n)$.

Proof. Part (i) is a consequence of the estimates (6.8). Part (ii) is a reformulation of Lemma 7.9. Thus, it remains to prove part (iii). Fix some $s > 0$ and assume that $f \in L^2_\nu(\mathbb{R}_+^n)$ satisfies $\langle e_z, f\rangle_{L^2_\nu(\mathbb{R}_+^n)} = 0$ for all $z = \underline{s} + iy$, $y \in \mathbb{R}^n$. Recall the Dunkl-Laplace transform from (7.9). Then

$$0 = \langle e_{\underline{s}+iy}, f\rangle_{L^2_\nu(\mathbb{R}_+^n)} = \mathcal{L}(\overline{f}\Delta^{\nu-\mu_0-1})(\underline{s}+iy).$$

By injectivity of the Dunkl-Laplace transform (Theorem 7.1 (iii)) we have $f\Delta^{\nu-\mu_0-1} = 0$, i.e. $f = 0$. ■

Theorem 7.21. *For $\nu \in \mathbb{R}$ with $\nu > \mu_0$, the Hankel transform $\mathcal{H}_\nu$ extends uniquely to an involutive isometric isomorphism of $L^2_\nu(\mathbb{R}^n_+)$.*

PROOF. By Lemma 7.20 it suffices to show that $\mathcal{H}_\nu$ is unitary on the space generated by the functions $e_z(x) = E^A(x, -z)$ with $\operatorname{Re} z > 0$. By Corollary 6.24, ${}_1K_0$ has an analytic extension satisfying for $w, z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$

$$\Gamma_n(\nu)\Delta(z)^{-\nu}{}_1K_0(\nu; w, -\tfrac{1}{z}) = \int_{\mathbb{R}^n_+} E^A(-x, z)E^A(-x, w)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x.$$

As $\overline{e_w} = e_{\overline{w}}$ on $\mathbb{R}^n$

$$\langle e_z, e_w\rangle_{L^2_\nu(\mathbb{R}^n_+)} = \Gamma_n(\nu)\Delta(z)^{-\nu}{}_1K_0(\nu; \overline{w}, -\tfrac{1}{z}).$$

Thus, by part (i) of Lemma 7.20 we conclude that

$$\begin{aligned}\langle \mathcal{H}_\nu e_z, \mathcal{H}_\nu e_w\rangle_{L^2_\nu(\mathbb{R}^n_+)} &= \Delta(z)^{-\nu}\Delta(\overline{w})^{-\nu}\left\langle e_{1/z}, e_{1/w}\right\rangle_{L^2_\nu(\mathbb{R}^n_+)}\\ &= \Gamma_n(\nu)\Delta(\overline{w})^{-\nu}{}_1K_0(\nu; \tfrac{1}{\overline{w}}, -z)\\ &= \overline{\Gamma_n(\nu)\Delta(w)^{-\nu}{}_1K_0(\nu; \tfrac{1}{w}, -\overline{z})}\\ &= \overline{\langle e_w, e_z\rangle_{L^2_\nu(\mathbb{R}^n_+)}} = \langle e_z, e_w\rangle_{L^2_\nu(\mathbb{R}^n_+)}.\end{aligned}$$

■

The Hankel transform $\mathcal{H}_\nu f$ of a function $f \in L^2_\nu(\mathbb{R}^n_+)$ can also be uniquely characterized in terms of the Dunkl-Laplace transform with the following lemma.

Lemma 7.22. *Consider $f \in L^2_\nu(\mathbb{R}^n_+)$, so in particular $\mathcal{L}(f\Delta^{\nu-\mu_0-1})(z)$ exists for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$. Moreover, assume that $g : \mathbb{R}^n_+ \to \mathbb{C}$ is measurable and $s > 0$, such that $\mathcal{L}(g\Delta^{\nu-\mu_0-1})(\underline{s})$ exists. Then:*

(i) *For all $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$*

$$\mathcal{L}((\mathcal{H}_\nu f)\Delta^{\nu-\mu_0-1})(z) = \Delta(z)^{-\nu}\mathcal{L}(f\Delta^{\nu-\mu_0-1})(\tfrac{1}{z}).$$

(ii) *If $\mathcal{L}(g\Delta^{\nu-\mu_0-1})(z) = \Delta(z)^{-\nu}\mathcal{L}(f\Delta^{\nu-\mu_0-1})(\frac{1}{z})$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z > s$, then $g \in L^2_\nu(\mathbb{R}^n_+)$, $\mathcal{L}(g\Delta^{\nu-\mu_0-1})(z)$ exists for all $\operatorname{Re} z > 0$ and $g = \mathcal{H}_\nu f$.*

PROOF. For $g = \mathcal{H}_\nu f$ we have by Lemma 7.20 and Theorem 7.21

$$\begin{aligned}\mathcal{L}(g\Delta^{\nu-\mu_0-1})(z) &= \langle e_z, \overline{g}\rangle_{L^2_\nu(\mathbb{R}^n_+)} = \langle \mathcal{H}_\nu e_z, \overline{f}\rangle_{L^2_\nu(\mathbb{R}^n_+)}\\ &= \Delta(z)^{-\nu}\left\langle e_{1/z}, \overline{f}\right\rangle_{L^2_\nu(\mathbb{R}^n_+)} = \Delta(z)^{-\nu}\mathcal{L}(f\Delta^{\nu-\mu_0-1})(\tfrac{1}{z})\end{aligned}$$

which proves (i). From the assumption in part (ii) we conclude for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z > s$

$$\mathcal{L}(g\Delta^{\nu-\mu_0-1})(z) = \mathcal{L}((\mathcal{H}_\nu f)\Delta^{\nu-\mu_0-1})(z).$$

Hence, injectivity of $\mathcal{L}$ leads to the statements in part (ii). ■

To facilitate readability, we will write $L^p(\Omega, h(x)\mathrm{d}x) := L^p(\Omega, |h(x)|\,\mathrm{d}x)$ for measurable $h : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{R}^n$ is a Borel set.
Recall the correspondence between the pairs of type A multiplicities $\kappa_A = k$ together with a parameter $\nu \in \mathbb{C}$, and type B multiplicities κ_B from (7.16), namely

$$(\kappa_A, \nu) = (k, \nu) \longleftrightarrow \kappa_B = (k, k') \text{ with } k' = \nu - \mu_0 - \tfrac{1}{2},\ \mu_0 = k(n-1).$$

Furthermore, assume that $\operatorname{Re}\nu > \mu_0 + \frac{1}{2}$, i.e. $\operatorname{Re}\kappa_B \geq 0$, so that $\mathcal{F}^B$ is an automorphism of $\mathscr{S}(\mathbb{R}^n)$, injective on $L^1(\mathbb{R}^n, \omega^B(x)\mathrm{d}x)$, and extends to an unitary map of $L^2(\mathbb{R}^n, \omega^B(x)\mathrm{d}x)$ in the case $\kappa_B \geq 0$.
The following integral decomposition has to be seen as a Dunkl analogue of the formula [FK94, Proposition XVI.2.1], where we replaced the integration over the Stiefel manifold by summation over the $\mathbb{Z}_2^n$-action. This integral decomposition is quite simple, but plays the same important role as the corresponding integration formula on symmetric cones.

Proposition 7.23 (Integral decomposition). *Consider $f \in L^1(\mathbb{R}^n, \omega^B(x)\mathrm{d}x)$. Then*

$$\int_{\mathbb{R}^n} f(x)\omega^B(x)\,\mathrm{d}x = \frac{1}{2^n}\sum_{\tau\in\mathbb{Z}_2^n}\int_{\mathbb{R}_+^n} f(\tau x^{\frac{1}{2}})\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x,$$

where $x^{\frac{1}{2}}$ has to be understood componentwise. In particular, if f is $\mathbb{Z}_2^n$-invariant

$$\int_{\mathbb{R}^n} f(x)\omega^B(x)\,\mathrm{d}x = \int_{\mathbb{R}_+^n} f(x^{\frac{1}{2}})\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x, \tag{7.20}$$

Proof. Since ω^B is $\mathbb{Z}_2^n$-invariant, we have with (7.6) that

$$\int_{\mathbb{R}^n} f(x)\omega^B(x)\,\mathrm{d}x = \sum_{\tau\in\mathbb{Z}_2^n}\int_{\mathbb{R}_+^n} f(\tau x)\Delta(x^2)^{k'}\omega^A(x^2)\,\mathrm{d}x.$$

Hence, the change of variables $x \leftrightarrow x^{\frac{1}{2}}$ and $\nu - \mu_0 - 1 = k' - \frac{1}{2}$ gives the stated formula. ■

The Hankel transform and the type B Dunkl transform are closely related. The connection is given in the following theorem and generalizes the formula (7.13) in the setting of symmetric cones.

Theorem 7.24. *Recall the Dunkl transform $\mathcal{F}^B$ and the constant c_B from (7.8). Then $c_B = 2^{n\nu}\Gamma_n(\nu)$ and if $f \in L^1(\mathbb{R}^n, \omega^B(x)\mathrm{d}x)$ is $\mathbb{Z}_2^n$-invariant, then the measurable function $f_0 : \mathbb{R}_+^n \to \mathbb{C}$ defined by $f(x) = f_0(x^2)$ satisfies*

$$\mathcal{F}^B f(\xi) = 2^{-n\nu}\mathcal{H}_\nu f_0(\tfrac{\xi^2}{4}), \quad \xi \in \mathbb{R}^n. \tag{7.21}$$

Proof. The computation of c_B can be deduced from Theorem 6.12 and Proposition 7.23

$$\begin{aligned} c_B &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}}\omega^B(x)\,\mathrm{d}x = \int_{\mathbb{R}_+^n} e^{-\frac{1}{2}\langle x,\underline{1}\rangle}\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x \\ &= 2^{n\nu}\int_{\mathbb{R}_+^n} E^A(-x,\underline{1})\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x = 2^{n\nu}\Gamma_n(\nu). \end{aligned}$$

Moreover, the integral formula (7.20) of Proposition 7.23 and Proposition 7.8 show that

$$\mathcal{F}^B f(\xi) = \frac{1}{c_B}\int_{\mathbb{R}_+^n} f_0(x)\mathscr{E}_\nu(x, \tfrac{\xi^2}{4})\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x,$$

so (7.21) is proven. ■

The following lemma is a Dunkl analogue of [Rub06, Theorem 3.1]. More precisely, on the space $\mathbb{R}^{n\times m}$ of rectangular matrices the author in [Rub06] describes the action of the so-called Cayley-Laplacian $\det(\partial^T\partial)$ on radial functions, i.e. functions of the form $f(x) = f_0(xx^T)$. We replace the Cayley-Laplacian by the type B Dunkl operator $\Delta(T^B)^2$ and describe its action on $\mathbb{Z}_2^n$-invariant functions f.

Theorem 7.25. *Let $f \in C^\infty(\mathbb{R}^n)$ be a $\mathbb{Z}_2^n$-invariant function.*

(i) *There exists a smooth function $f_0 \in C^\infty(\mathbb{R}^n)$ with $f(x) = f_0(x^2)$.*

(ii) *The Dunkl operator $\Delta(T^B)^2$ acts according to*

$$\Delta(T^B)^2 f(x) = \mathscr{L}_\nu f_0(x^2), \tag{7.22}$$

where the operator on the right hand side is defined by

$$\mathscr{L}_\nu := 4^n \Delta(x)^{1+\mu_0-\nu}\Delta(T^A)\Delta(x)^{\nu-\mu_0}\Delta(T^A).$$

Here the powers of $\Delta(x)$ are understood as multiplication operator.

Proof.

(i) This was done by Whitney [Whi43] for univariate functions, and the multivariate case is easily reduced to this case.

(ii) By continuity it suffices to check (7.22) on $\mathbb{R}^n \setminus \{\Delta(x) = 0\}$. Since

$$\operatorname{supp}(T_\xi^R f) \subseteq W_R.\operatorname{supp} f \qquad \text{for } R \in \{A, B\}$$

we can assume without loss of generality that $f \in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f \cap \{\Delta(x) = 0\} = \emptyset$, i.e. that $f_0 \in C_c^\infty(\mathbb{R}_+^n)$. Since the Dunkl transform $\mathcal{F}^B$ is injective, we can prove identity (7.22) under the action of $\mathcal{F}^B$. The identities from Theorem 1.20 and Theorem 7.24 show that

$$\begin{aligned}\mathcal{F}^B(\Delta(T^B)^2 f)(\xi) &= \Delta(i\xi)^2 \mathcal{F}^B f(\xi)\\ &= \frac{(-1)^n 2^{-n\nu}}{\Gamma_n(\nu)}\Delta(\xi)^2 \int_{\mathbb{R}_+^n} \mathscr{E}_\nu(\tfrac{\xi^2}{4}, x) f_0(x)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x. \end{aligned}\tag{7.23}$$

The recurrence formulas of Corollary 7.11 give

$$\begin{aligned}(-1)^n\Delta(\xi)^2\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)\Delta(x)^{\nu-\mu_0-1} &= (-4)^n\Delta(\tfrac{\xi^2}{4})\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)\Delta(x)^{\nu-\mu_0-1}\\ &= 4^n\Delta(T_x^A)\Big(\Delta(x)^{\nu-\mu_0}\Delta(T_x^A)\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)\Big).\end{aligned}\tag{7.24}$$

Put $g(x) = \mathscr{L}_\nu f_0(x^2)$. Then we obtain by plugging (7.24) into (7.23) and from the skew symmetry of the Dunkl operators on $C_c^\infty(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n, \omega^A(x)\mathrm{d}x)$ (cf. [dJ93]) that

$$\begin{aligned}\mathcal{F}^B(\Delta(T^B)^2 f)(\xi) &= \frac{(-1)^n 2^{-n\nu}}{\Gamma_n(\nu)}\Delta(\xi)^2 \int_{\mathbb{R}_+^n} \mathscr{E}_\nu(\tfrac{\xi^2}{4}, x) f_0(x)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x\\ &= \frac{4^n 2^{-n\nu}}{\Gamma_n(\nu)}\int_{\mathbb{R}_+^n}(\Delta(T_x^A)(\Delta(x)^{\nu-\mu_0-1}\Delta(T_x^A)\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)))f_0(x)\omega^A(x)\,\mathrm{d}x\\ &= \frac{4^n 2^{-n\nu}}{\Gamma_n(\nu)}\int_{\mathbb{R}_+^n}\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)(\Delta(T_x^A)(\Delta(x)^{\nu-\mu_0-1}\Delta(T_x^A)f_0))(x)\omega^A(x)\,\mathrm{d}x\\ &= \frac{2^{-n\nu}}{\Gamma_n(\nu)}\int_{\mathbb{R}_+^n}\mathscr{E}_\nu(\tfrac{\xi^2}{4}, x)(\mathscr{L}_\nu f_0)(x)\Delta(x)^{\nu-\mu_0-1}\omega^A(x)\,\mathrm{d}x\\ &= 2^{-n\nu}\mathcal{H}_\nu(\mathscr{L}_\nu f_0)(\tfrac{\xi^2}{4}) = \mathcal{F}^B g(\xi),\end{aligned}$$

where in the last line again Theorem 7.24 was used.

■

Example 7.26 (Type B Bernstein identity)**.** Owing to [Rö20, Lemma 5.4] or equivalently by Theorem 6.12 for $\eta = \underline{1}$ one has the following type A Bernstein identity:

$$\Delta(T^A)\Delta(x)^\mu = b(\mu)\Delta(x)^{\mu-1}, \quad \mu \in \mathbb{C},$$

where $b(\mu) = \prod_{j=1}^n (\mu + k(j-1))$. If we choose $f(x) = \Delta(x^2)^\mu$ in Theorem 7.25, we obtain from this type A Bernstein identity that

$$\begin{aligned}\Delta(T^B)^2\Delta(x^2)^\mu &= 4^n[\Delta(\xi)^{1+\mu_0-\nu}\Delta(T^A)\Delta(\xi)^{\nu-\mu_0}\Delta(T^A)\Delta(\xi)^\mu]_{\xi=x^2}\\ &= 4^n b(\mu)[\Delta(\xi)^{1+\mu_0-\nu}\Delta(T^A)\Delta(\xi)^{\nu+\mu-\mu_0-1}]_{\xi=x^2}\\ &= 4^n b(\mu)b(\nu+\mu-\mu_0-1)\Delta(x)^{\mu-1}\\ &=: \mathcal{B}(\mu)\Delta(x^2)^{\mu-1}.\end{aligned}$$

Thus, we have

$$\Delta(T^B)^2\Delta(x^2)^\mu = \mathcal{B}(\mu)\Delta(x^2)^{\mu-1}, \tag{7.25}$$

where $\mathcal{B}$ is the polynomial $\mathcal{B}(\mu) = 4^n \prod_{j=1}^n (\mu + k(j-1))(\mu - \frac{1}{2} + k' + k(j-1))$.
This Bernstein identity was independently proven in [Liu16, Proposition 3.1.2] by direct computation.

7.5 Zeta integrals and zeta distributions in the type B Dunkl setting

Recall the situation from the introduction, namely for $\Omega = \mathrm{Pos}_n(\mathbb{R})$, $V = \mathrm{Sym}_n(\mathbb{R})$, $E = \mathbb{R}^{n\times n}$ and $Q(\xi) = \xi\xi^T, \xi \in \mathbb{R}^{n\times n}$. For $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > \frac{1}{2}(n-1) - \frac{1}{2}$, the *zeta integral* of index α is defined for a Schwartz function $f \in \mathscr{S}(E)$ by

$$Z(f;\alpha) = \int_{\mathbb{R}^{n\times n}} f(\xi)\det(Q(\xi))^\alpha \,\mathrm{d}\xi, \tag{7.26}$$

cf. [FK94, Chapter XVI]. Put $m = \dim_{\mathbb{R}}(\mathrm{Sym}_n(\mathbb{R}))$. Assume that f is radial, i.e. $f(\xi) = F(Q(\xi))$ and additionally that f is invariant under the orthogonal group K which acts on E by right multiplication. Then K is the Stiefel-manifold of E and the integral formula [FK94, Proposition XVI.2.1] shows that the zeta integral (7.26) can be rewritten as

$$\begin{aligned}Z(f;\alpha) &= \frac{\pi^{\frac{n}{2}}}{\Gamma_\Omega(\frac{n}{2})}\int_\Omega F(u)\det(u)^{\alpha+\frac{n}{2}-\frac{m}{n}}\,\mathrm{d}u\\ &= \mathrm{const}\cdot\frac{\pi^{\frac{n}{2}}}{\Gamma_\Omega(\frac{n}{2})}\int_{\mathbb{R}_+^n} F(\mathrm{diag}(x_1,\dots,x_n))\Delta(x)^{\alpha+\frac{n}{2}-\frac{m}{n}}\prod_{i<j}|x_i-x_j|\,\mathrm{d}x\\ &= \mathrm{const}\cdot\frac{\pi^{\frac{n}{2}}}{\Gamma_\Omega(\frac{n}{2})}\int_{\mathbb{R}^n} F_0(x)\Delta(x^2)^{\alpha+\frac{n}{2}-\frac{m}{n}+\frac{1}{2}}\omega^A(x^2)\,\mathrm{d}x\end{aligned}$$

where $F_0(x) = F(\mathrm{diag}(x_1^2,\dots,x_n^2))$. This motivates the subsequent definition of the zeta integral in the Dunkl setting of type A. In view of Proposition 7.23, this can also be interpreted within type B Dunkl theory. To become more precise, consider as before $\kappa_A = k \geq 0$, $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu > \mu_0 + \frac{1}{2}$ and $\kappa_B = (k, k')$ with $k' = \nu - \mu_0 - \frac{1}{2}$.

Definition 7.27. We define the zeta integral of index α, $\mathrm{Re}\,\alpha > \mu_0$ of a Schwartz function $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$\mathcal{Z}(f;\alpha) := \int_{\mathbb{R}^n} f(x)\Delta(x^2)^{\alpha-\nu}\omega^B(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\Delta(x^2)^{\alpha-\mu_0-1}\omega^A(x^2)\,\mathrm{d}x.$$

Obviously, the integral converges absolutely and depends holomorphically on α. Moreover, $\mathcal{Z}(\cdot,\alpha)$ is W_B-invariant, i.e. $\mathcal{Z}(\tau f,\alpha) = \mathcal{Z}(f,\alpha)$ for all $\tau \in W_B$.

The following example will show which kind of functional equation we can expect for zeta integrals in the Dunkl setting and how it can be extended meromorphically in the parameter $\alpha \in \mathbb{C}$.

Example 7.28. Consider the Gauss function

$$g(x) := e^{-|x|^2} = E^A(-\underline{1}, x^2).$$

If $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > \mu_0$, then Theorem 6.12 and Proposition 7.23 show that

$$\mathcal{Z}(g; \alpha) = \int_{\mathbb{R}^n_+} E^A(-\underline{1}, x) \Delta(x)^{\alpha - \mu_0 - 1} \omega^A(x) \, \mathrm{d}x = \Gamma_n(\alpha).$$

Therefore, the zeta integral of g can be meromorphically extended in the parameter $\alpha \in \mathbb{C}$. Moreover, the type B Dunkl transform of g can be computed by Lemma 7.9 and Theorem 7.24 as

$$\mathcal{F}^B g(x) = 2^{-n\nu} g(\tfrac{x}{2}) = 2^{-n\nu} E^A(-\underline{1}, \tfrac{x^2}{4}).$$

Similar, for $\operatorname{Re} \alpha < \operatorname{Re} \nu - \mu_0$ we get

$$\begin{aligned} \mathcal{Z}(\mathcal{F}^B g; \nu - \alpha) &= 2^{-n\nu} \int_{\mathbb{R}^n_+} E^A(-\underline{1}, \tfrac{x}{4}) \Delta(x)^{\nu - \alpha - \mu_0 - 1} \omega^A(x) \, \mathrm{d}x \\ &= 2^{n(\nu - 2\alpha)} \int_{\mathbb{R}^n_+} e^{-\langle x, \underline{1} \rangle} \Delta(x)^{\nu - \alpha - \mu_0 - 1} \omega^A(x) \, \mathrm{d}x \\ &= 2^{n(\nu - 2\alpha)} \Gamma_n(\nu - \alpha) \end{aligned}$$

which can be extended meromorphically in $\alpha \in \mathbb{C}$. We therefore obtain

$$\frac{\mathcal{Z}(g, \alpha)}{\Gamma_n(\alpha)} = 2^{n(2\alpha - \nu)} \frac{\mathcal{Z}(\mathcal{F}^B g, \nu - \alpha)}{\Gamma_n(\nu - \alpha)} = 1. \tag{7.27}$$

We see that both sides of this equation are entire functions in α and that we have found a functional equation between zeta integrals of g and $\mathcal{F}^B g$. We shall see that both the functional equation and the analytic extension are true for arbitrary Schwartz functions.

Proposition 7.29. *Consider a $\mathbb{Z}_2^n$-invariant Schwartz function $f \in \mathscr{S}(\mathbb{R}^n)$. Then there exists a Schwartz function $f_0 \in \mathscr{S}(\mathbb{R}^n)$ such that $f(x) = f_0(x^2)$.*

Proof. Since $f|_{\mathbb{R}^n_+}$ extends to a Schwartz function, we can use [Ste19, Theorem 1.1, $\eta = 0$] to obtain for arbitrary $\alpha, \beta \in \mathbb{N}_0^n$ that

$$\sup_{x \in \mathbb{R}^n_+} \left| x^\alpha \left(\frac{1}{x} \frac{\partial}{\partial x} \right)^\beta f(x) \right| < \infty,$$

where $\left(\frac{1}{x} \frac{\partial}{\partial x} \right)^\beta = \left(\frac{1}{x_1} \frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left(\frac{1}{x_n} \frac{\partial}{\partial x_n} \right)^{\beta_n}$. As stated in Theorem 7.25 (i), we can find a smooth function $g \in C^\infty(\mathbb{R}^n)$ with $f(x) = g(x^2)$. Consider the change of variables $t_i = x_i^2$, i.e. $\frac{1}{x_i} \frac{\partial}{\partial x_i} = 2 \frac{\partial}{\partial t_i}$. The function g then satisfies

$$\sup_{t \in \mathbb{R}^n_+} \left| t^\alpha \left(\frac{\partial}{\partial t} \right)^\beta g(t) \right| = \frac{1}{2^{|\beta|}} \sup_{x \in \mathbb{R}^n_+} \left| x^{2\alpha} \left[\left(\frac{1}{x} \frac{\partial}{\partial x} \right)^\beta f \right](x) \right| < \infty.$$

An application of [JP16, Theorem 4.2] shows that the function $g|_{\mathbb{R}^n_+}$ is the restriction of a Schwartz function $f_0 \in \mathscr{S}(\mathbb{R}^n)$. In particular, $f(x) = f_0(x^2)$. ∎

Our zeta integrals have a close connection to the Riesz distributions in the Dunkl setting of type A, defined for $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > \mu_0$ by

$$\langle R_\alpha, f \rangle := \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{R}^n_+} f(x) \Delta(x)^{\alpha - \mu_0 - 1} \omega^A(x) \, \mathrm{d}x, \quad f \in \mathscr{S}(\mathbb{R}^n).$$

These Riesz distributions were introduced and studied in [Rö20]. They extend to a (weakly) holomorphic map $\mathbb{C} \to \mathscr{S}'(\mathbb{R}^n)$, $\alpha \mapsto R_\alpha$.

Lemma 7.30. *Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then the function*

$$\alpha \mapsto \langle \zeta_\alpha, f \rangle := \frac{\mathcal{Z}(f;\alpha)}{\Gamma_n(\alpha)}$$

extends to an entire function on $\mathbb{C}$. If $f \in \mathscr{S}(\mathbb{R}^n)$ is $\mathbb{Z}_2^n$-invariant with $f(x) = f_0(x^2)$ and $f_0 \in \mathscr{S}(\mathbb{R}^n)$, then this extension is given in terms of the Riesz distributions R_α via

$$\langle \zeta_\alpha, f \rangle = \langle R_\alpha, f_0 \rangle .$$

We call ζ_α a *Dunkl-type zeta distribution* of index $\alpha \in \mathbb{C}$. That ζ_α is in fact a tempered distribution will be proven below in Theorem 7.31.

Proof. By $\mathbb{Z}_2^n$-invariance of the zeta integrals, we may asssume that f is a $\mathbb{Z}_2^n$-invariant function, otherwise we can consider its $\mathbb{Z}_2^n$-mean. By Theorem 7.29 we can find $f_0 \in \mathscr{S}(\mathbb{R}^n)$ such that $f(x) = f_0(x^2)$. For $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \mu_0$, Proposition 7.23 shows that

$$\langle \zeta_\alpha, f \rangle = \langle R_\alpha, f_0 \rangle .$$

Hence, that statement follows from the analytic extension property for the Dunkl-type Riesz distributions $R_\alpha, \alpha \in \mathbb{C}$. ■

Recall that the Dunkl operators T_ξ^R and the Dunkl transform $\mathcal{F}^R$ act continuously on the space of Schwartz functions $\mathscr{S}(\mathbb{R}^n)$, equipped with the usual locally convex topology. Hence, by duality we consider the usual actions on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$ via

$$\langle T_\xi^R u, \cdot \rangle = \langle u, -T_\xi^R \cdot \rangle \qquad \text{and} \qquad \langle \mathcal{F}^R u, \cdot \rangle = \langle u, \mathcal{F}^R \cdot \rangle$$

for $u \in \mathscr{S}'(\mathbb{R}^n)$.

Theorem 7.31. *The functionals ζ_α with $\alpha \in \mathbb{C}$ have the following properties:*

(i) *$\zeta_\alpha \in \mathscr{S}'(\mathbb{R}^n)$ and ζ_α is W_B-invariant.*

(ii) *If $\operatorname{Re}\alpha > \mu_0$, then ζ_α is a positive measure with support $\mathbb{R}^n$.*

(iii) *$\Delta(T^B)^2\zeta_\alpha = 4^n b(\alpha - \nu)\zeta_{\alpha-1}$ with $b(\mu) = \prod_{j=1}^n (z + k(j-1))$.*

(iv) *$\Delta(x^2)\zeta_\alpha = b(\alpha - \mu_0)\zeta_{\alpha+1}$.*

Proof. Part (ii) is obvious. If $\operatorname{Re}\alpha > \mu_0 + 1$, then the skew symmetry of Dunkl operators and the Bernstein identity (7.25) of Example 7.26 show that

$$\mathcal{Z}(\Delta(T^B)^2 f;\alpha) = \mathcal{B}(\alpha - \nu)\mathcal{Z}(f;\alpha - 1).$$

In particular,

$$\langle \zeta_\alpha, \Delta(T^B)^2 f \rangle = \frac{\Gamma_n(\alpha-1)}{\Gamma_n(\alpha)}\mathcal{B}(\alpha-\nu)\,\langle \zeta_{\alpha-1}, f \rangle .$$

By definition, $\mathcal{B}(\alpha - \nu) = 4^n b(\alpha-\nu)b(\alpha-\mu_0-1)$ and moreover

$$b(\alpha - \mu_0 - 1) = \prod_{j=1}^n (\alpha - 1 - k(j-1)) = \frac{\Gamma_n(\alpha)}{\Gamma_n(\alpha-1)}.$$

Hence,

$$\langle \zeta_\alpha, \Delta(T^B)^2 f\rangle = 4^n b(\alpha-\nu)\,\langle \zeta_{\alpha-1}, f\rangle\,,$$

so that analytic extension according to Lemma 7.30 shows that $\zeta_{\alpha-1}$ is a tempered distribution provided ζ_α is so and $b(\alpha-\nu)\neq 0$. The set

$$M = \{\alpha\in\mathbb{C} \mid b(\alpha+k-\nu)\neq 0 \text{ for all } k\in\mathbb{Z}\}$$

is dense in $\mathbb{C}$ and for all $\alpha\in M$ we have $\zeta_\alpha\in\mathscr{S}'(\mathbb{R}^n)$. But, the map $\alpha\mapsto\langle\zeta_\alpha, f\rangle$ is holomorphic for all $f\in\mathscr{S}(\mathbb{R}^n)$ and as a dual of a Fréchet space, $\mathscr{S}'(\mathbb{R}^n)$ is closed under pointwise limits. Hence all ζ_α are tempered. Thus, we have proven parts (i) and (iii). Part (iv) is immediate for $\operatorname{Re}\alpha>\mu_0$ from

$$b(\alpha-\mu_0) = \frac{\Gamma_n(\alpha+1)}{\Gamma_n(\alpha)},$$

and follows by analytic extension in general. ∎

In line with Example 7.28 we next obtain a general functional equation for our zeta distributions. The idea of the proof is the same as for the analogous results in [FK94, Rub06]. The $\mathcal{K}$-Bessel functions and their asymptotic properties will play an essential role.

Theorem 7.32. *For all $\alpha\in\mathbb{C}$, the zeta distributions ζ_α satisfy the functional equation*

$$\zeta_\alpha = 2^{n(2\alpha-\nu)}\mathcal{F}^B\zeta_{\nu-\alpha}. \tag{7.28}$$

Moreover, in view of Theorem 7.24,

$$\langle R_\alpha, f\rangle = 4^{n(\alpha-\nu)}\,\langle R_{\nu-\alpha}, \mathcal{H}_\nu g\rangle\,,\quad f\in\mathscr{S}(\mathbb{R}^n),\ g(x)=f(\tfrac{x}{4}).$$

For the proof of identity (7.28), we start with two lemmata which are needed to outsource some technicalities.

Lemma 7.33. *Consider $g\in\mathscr{S}(\mathbb{R}^n)$ and $\alpha\in\mathbb{C}$.*

(i) *Assume that $\operatorname{Re}\alpha>\mu_0$. Then for arbitrary $\epsilon>0$, the following integrals exist and coincide*

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} g(x)E^A(-\tfrac{1}{s}, x^2+\underline{\epsilon})\Delta(s)^{-\alpha-\mu_0-1}\omega^B(x)\omega^A(s)\,\mathrm{d}x\,\mathrm{d}s\\ &= \Gamma_n(\alpha)\int_{\mathbb{R}^n} g(x)\Delta(x^2+\underline{\epsilon})^{-\alpha}\omega^B(x)\,\mathrm{d}x.\end{aligned}$$

(ii) *For $\operatorname{Re}\alpha<\operatorname{Re}\nu-\mu_0$ and $\epsilon>0$, the following integral and limit exists*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n} g(x)\Delta(x^2+\underline{\epsilon})^{-\alpha}\omega^B(x)\,\mathrm{d}x = \mathcal{Z}(g,\nu-\alpha).$$

Furthermore, the integral on the left hand side exists for arbitrary $\alpha\in\mathbb{C}$ and defines an entire function in α.

Proof.

(i) Assume that we can interchange the order of integration. By Theorem 6.12 and the condition $\operatorname{Re}\alpha>\mu_0$, we obtain

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} |g(x)|E^A(-\tfrac{1}{s}, x^2+\underline{\epsilon})\Delta(s)^{-\operatorname{Re}\alpha-\mu_0-1}|\omega^B(x)|\omega^A(s)\,\mathrm{d}x\,\mathrm{d}s$$

$$= \int_{\mathbb{R}^n} |g(x)| \int_{\mathbb{R}^n_+} E^A(-s, x^2 + \underline{\epsilon})\Delta(s)^{\operatorname{Re}\alpha-\mu_0-1}\omega^A(s)\, \mathrm{d}s\, |\omega^B(x)|\, \mathrm{d}x$$
$$= \Gamma_n(\operatorname{Re}\alpha) \int_{\mathbb{R}^n} |g(x)|\Delta(x^2 + \underline{\epsilon})^{-\operatorname{Re}\alpha}|\omega^B(x)|\, \mathrm{d}x < \infty.$$

The interchange of order of integration is justified by the fact that g is a Schwartz function and $x \mapsto \Delta(x^2 + \epsilon)^{-\alpha}\omega^B(x)$ is continuous and of at most polynomial growth on $\mathbb{R}^n$.

(ii) Since $x \mapsto \Delta(x^2 + \epsilon)^{-\alpha}\omega^B(x)$ is of at most polynomial growth, the integral exists for all $\alpha \in \mathbb{C}$. For $\operatorname{Re}\alpha < \operatorname{Re}\nu - \mu_0$ and $0 < \epsilon < 1$

$$\left|\Delta(x^2 + \underline{\epsilon})^{-\alpha}\right| \leq \begin{cases} \Delta(x^2)^{-\operatorname{Re}\alpha} & \text{if } \operatorname{Re}\alpha > 0 \\ \Delta(x^2 + 1)^{-\operatorname{Re}\alpha} & \text{if } \operatorname{Re}\alpha \leq 0 \end{cases}.$$

Therefore, the claimed limit follows by dominated convergence. Finally, analyticity follows by standard theorems on holomorphic parameter integrals.

■

Lemma 7.34. *Choose $m \in \mathbb{N}_0$ such that $\mu_0 - m < \operatorname{Re}\nu - \mu_0$. Consider $g \in \mathscr{S}(\mathbb{R}^n)$, $\widetilde{g}(x) = g(x)\Delta(x)^m$ and let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \mu_0 - m$.*

(i) *For arbitrary $\epsilon > 0$ the following integrals exist and coincide*

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n} g(x)E^A(-s, \tfrac{x^2}{4})E^A(-\tfrac{1}{s}, \underline{\epsilon})\Delta(s)^{\nu-\alpha-\mu_0-1}\omega^B(x)\omega^A(s)\Delta(x)^m\, \mathrm{d}x\, \mathrm{d}s$$
$$= \epsilon^{n(\nu-\alpha)} \int_{\mathbb{R}^n} g(x)\mathcal{K}_{\alpha-\nu}(\underline{1}, \epsilon\tfrac{x^2}{4})\Delta(x)^m\omega^B(x)\, \mathrm{d}x,$$

where $\mathcal{K}_{\alpha-\nu}$ is the $\mathcal{K}$-Bessel function according to Definition 7.12.

(ii) *The following limit exists*

$$\lim_{\epsilon\to 0} \epsilon^{n(\nu-\alpha)} \int_{\mathbb{R}^n} \widetilde{g}(x)\mathcal{K}_{\alpha-\nu}(\underline{1}, \epsilon\tfrac{x^2}{4})\omega^B(x)\, \mathrm{d}x = \frac{\Gamma_n(\nu-\alpha)}{4^{n\alpha}}\mathcal{Z}(g, \alpha-\nu+m).$$

Moreover, the integral on the left hand side is holomorphic in α on the domain $\{\operatorname{Re}\alpha > \mu_0 - m\}$.

Proof.

(i) It suffices to justify that we can change the order of integration. Everything else follows from the definition of the $\mathcal{K}$-Bessel function and its properties (Theorem 7.14). The change of variables $s \mapsto \epsilon s$ and $\mathcal{K}_{\nu-\alpha}(w, z) = \mathcal{K}_{\alpha-\nu}(z, w)$ lead to

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n} |g(x)|E^A(-s, \tfrac{x^2}{4})E^A(-\tfrac{1}{s}, \underline{\epsilon})\Delta(s)^{\operatorname{Re}(\nu-\alpha)-\mu_0-1}\Delta(x)^m|\omega^B(x)|\omega^A(s)\, \mathrm{d}x\, \mathrm{d}s$$
$$= \epsilon^{n\operatorname{Re}(\nu-\alpha)} \int_{\mathbb{R}^n} |g(x)|\, \mathcal{K}_{\operatorname{Re}(\nu-\alpha)}(\epsilon\tfrac{x^2}{4}, \underline{1})|\omega^B(x)|\, \mathrm{d}x. \tag{7.29}$$

To see that the integral (7.29) is finite, we observe that by Theorem 7.14 (iii)

$$\Delta(x^2)^m\mathcal{K}_{\operatorname{Re}(\alpha-\nu)}(\underline{1}, \epsilon\tfrac{x^2}{4})$$

$$\leq C(\alpha)\begin{cases} 1 & \text{if } \operatorname{Re}\alpha > \operatorname{Re}\nu + \mu_0 \\ 1 + \Delta(x^2)^{-\mu_0-\epsilon'} & \text{if } \operatorname{Re}\nu - \mu_0 \leq \operatorname{Re}\alpha \leq \operatorname{Re}\nu + \mu_0 \\ & \text{and arbitrary } 0 < \epsilon' < 1 \\ \epsilon^{n(\operatorname{Re}\alpha - \operatorname{Re}\nu)}\Delta(x^2)^{\operatorname{Re}\alpha-\nu} & \text{if } \operatorname{Re}\alpha < \operatorname{Re}\nu - \mu_0 \end{cases},$$

with some constant $C(\alpha)$. Hence, under the conditions $\operatorname{Re}\alpha > \mu_0 - m$ and $\mu_0 - m < \operatorname{Re}\nu - \mu_0$, the function

$$x \mapsto \Delta(x^2)^m \mathcal{K}_{\operatorname{Re}(\alpha-\nu)}(\underline{1}, \epsilon\tfrac{x^2}{4})$$

is of polynomial growth (choose $0 < \epsilon' < m - 2\mu_0 - \operatorname{Re}\nu$ in the second case). Therefore the integral (7.29) is finite.

(ii) The same estimates as in part (i), dominated converges, and the asymptotics of Theorem 7.14 (vi) lead to the stated limit.

■

Now we are in a position to prove Theorem 7.32.

Proof of Theorem 7.32. The left hand side of the functional equation (7.28) is given by a positive measure if $\operatorname{Re}\alpha > \mu_0$, while the right hand side is a positive measure if $\operatorname{Re}\alpha < \operatorname{Re}\nu - \mu_0$. So there are possibly no indices α for which both sides are positive measures. To bypass this problem, we replace the argument $f \in \mathscr{S}(\mathbb{R}^n)$ by $\widetilde{f} := \Delta^{2m}(\Delta(T^B)^{2m}f) \in \mathscr{S}(\mathbb{R}^n)$ with $m \in \mathbb{N}_0$ large enough so that $\mu_0 - m < \operatorname{Re}\nu - \mu_0$.

(i) Pick $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \mu_0$. For $s \in \mathbb{R}^n_+$ put $e_s(x) = E^A(-s, x^2)$. The type B Dunkl transform of $e_{1/s}$ is computed by Lemma 7.20 and Theorem 7.24 (ii) as

$$\mathcal{F}^B e_{1/s}(x) = 2^{-n\nu} e_s(\tfrac{x}{2})\Delta(s)^\nu.$$

The Plancherel theorem (cf. [dJ93]) for the Dunkl transform therefore leads to

$$\int_{\mathbb{R}^n} (\mathcal{F}^B\widetilde{f})(x) e_{1/s}(x)\omega^B(x)\,\mathrm{d}x = 2^{-n\nu}\Delta(s)^\nu \int_{\mathbb{R}^n} \widetilde{f}(x)\cdot e_s(\tfrac{x}{2})\omega^B(x)\,\mathrm{d}x. \tag{7.30}$$

Multiplying equation (7.30) with $\Delta(s)^{-\alpha-\mu_0-1}E^A(-\frac{1}{s}, \underline{\epsilon})$, $\epsilon > 0$ and integrating over $\mathbb{R}^n_+$ gives

$$\begin{aligned} &\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} (\mathcal{F}^B\widetilde{f})(x)E^A(-\tfrac{1}{s}, x^2 + \underline{\epsilon})\Delta(s)^{-\alpha-\mu_0-1}\omega^B(x)\omega^A(s)\,\mathrm{d}x\,\mathrm{d}s \\ &= 2^{-n\nu}\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} \widetilde{f}(x)E^A(-s, \tfrac{x^2}{4})E^A(-\tfrac{1}{s}, \underline{\epsilon})\Delta(s)^{\nu-\alpha-\mu_0-1}\omega^B(x)\omega^A(s)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{7.31}$$

By parts (i) of Lemmata 7.33 and 7.34, equation (7.31) reduces to

$$\int_{\mathbb{R}^n} \mathcal{F}^B\widetilde{f}(x)\Delta(x^2 + \underline{\epsilon})^{-\alpha}\omega^B(x)\,\mathrm{d}x = \frac{2^{-n\nu}\epsilon^{n(\nu-\alpha)}}{\Gamma_n(\alpha)}\int_{\mathbb{R}^n} \widetilde{f}(x)\mathcal{K}_{\alpha-\nu}(\underline{1}, \epsilon\tfrac{x^2}{4})\omega^B(x)\,\mathrm{d}x. \tag{7.32}$$

In view of Lemma 7.33 (ii), the left hand side of (7.32) exists for all $\alpha \in \mathbb{C}$ and is holomorphic in α. For $\operatorname{Re}\alpha > \mu_0 - m$ the right hand side of (7.32) exists and depends holomorphically on α by Lemma 7.34 (ii), because

$$\widetilde{f}(x)\mathcal{K}_{\alpha-\nu}(\underline{1}, \epsilon\tfrac{x^2}{4}) = \Delta(T^B)^{2m}f(x)\cdot\Delta(x^2)^m\mathcal{K}_{\alpha-\nu}(\underline{1}, \epsilon\tfrac{x^2}{4})$$

and $\Delta(T^B)^{2m}f \in \mathscr{S}(\mathbb{R}^n)$. Hence, equality (7.32) is true for all $\operatorname{Re}\alpha > \mu_0 - m$.

(ii) Assume that $\alpha \in \mathbb{C}$ is contained in the strip $\mu_0 - m < \operatorname{Re}\alpha < \operatorname{Re}\nu - \mu_0$ which is non-empty by our choice of m. Then, on the one hand, by Lemma 7.33 and Theorem 7.31 there exists a polynomial p (independent of f) with

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{F}^B \widetilde{f}(x)\Delta(x^2+\underline{\epsilon})^{-\alpha}\omega^B(x)\,\mathrm{d}x \underset{\epsilon\to 0}{\longrightarrow} &\int_{\mathbb{R}^n} \mathcal{F}^B \widetilde{f}(x)\Delta(x^2)^{-\alpha}\omega^B(x)\,\mathrm{d}x\\ &= \mathcal{Z}(\mathcal{F}^B\widetilde{f};\nu-\alpha)\\ &= \Gamma_n(\nu-\alpha)\,\langle\zeta_\alpha, \mathcal{F}^B\widetilde{f}\rangle\\ &= \Gamma_n(\nu-\alpha)\,\langle\zeta_\alpha, \Delta(T^B)^{2m}(\Delta^{2m}f)\rangle\\ &= \Gamma_n(\nu-\alpha)\,p(\alpha)\,\langle\zeta_\alpha, f\rangle\,.\end{aligned}$$

On the other hand, Lemma 7.34 and Theorem 7.31 show that there exists a polynomial q (independent of f) with

$$\begin{aligned}&\frac{2^{-n\nu}\epsilon^{n(\nu-\alpha)}}{\Gamma_n(\alpha)}\int_{\mathbb{R}^n}\widetilde{f}(x)\mathcal{K}_{\alpha-\nu}(\underline{1},\epsilon\tfrac{x^2}{4})\omega^B(x)\,\mathrm{d}x\\ &\underset{\epsilon\to 0}{\longrightarrow}\frac{\Gamma_n(\nu-\alpha)}{\Gamma(\alpha)2^{n(\nu+2\alpha}}\int_{\mathbb{R}^n}(\Delta(T^B)^{2m}f)(x)\cdot\Delta(x^2)^{\alpha-\nu+m}\omega^B(x)\,\mathrm{d}x\\ &=\frac{\Gamma_n(\nu-\alpha)}{\Gamma_n(\alpha)2^{n(\nu+2\alpha)}}\mathcal{Z}(\Delta(T^B)^{2m}f;\alpha-\nu+m)\\ &=\frac{\Gamma_n(\nu-\alpha)}{\Gamma_n(\alpha)\Gamma_n(\alpha-\nu+m)2^{n(\nu+2\alpha)}}\,\langle\zeta_{\alpha-\nu+m},\Delta(T^B)^{2m}f\rangle\\ &=\frac{\Gamma_n(\nu-\alpha)}{\Gamma_n(\alpha)\Gamma_n(\alpha-\nu+m)}\,q(\alpha)\;\langle\zeta_\alpha,f\rangle\,.\end{aligned}$$

Therefore, we have found a meromorphic function ρ on $\mathbb{C}$ such that

$$\zeta_\alpha = \rho(\alpha)\mathcal{F}^B\zeta_{\nu-\alpha}.$$

Finally, by equation (7.27) of Example 7.28, we see that $\rho(\alpha) = 2^{n(2\alpha-\nu)}$.

■

7.6 Regularity of the zeta distributions

As for Riesz distributions in [Rö20], one may ask which of the zeta distributions are regular and which are positive measures. The case of zeta distributions reduces to the case of Riesz distributions, where the question was answered in [Rö20]. It turns out that the set of indices $\alpha \in \mathbb{C}$ for which ζ_α is a positive measure only depends on the multiplicity parameter k of $\kappa_B = (k, k')$.

Lemma 7.35. *Consider $\alpha \in \mathbb{C}$.*

(i) *ζ_α is a positive or complex measure if and only if R_α is a positive or complex measure, respectively.*

(ii) *If ζ_α is a complex measure, then $\operatorname{Re}\alpha > \mu_0$ or α is contained in the finite set*

$$[0,\infty[\,\cap\,(\{0,k,\ldots,k(n-1)\} - \mathbb{N}_0).$$

(iii) *ζ_α is a positive measure if and only if α is contained in the generalized Wallach set*

$$W_k = \{0,k,\ldots,k(n-1)\}\cup\,]k(n-1),\infty[.$$

PROOF. If we can show part (i), then the other parts are immediate from [Rö20, Theorem 5.15]. Assume that R_α is a measure μ. Then $\operatorname{supp}\mu \subseteq \overline{\mathbb{R}^n_+}$ (cf. [Rö20]). Let $f \in \mathscr{S}(\mathbb{R}^n)$ and choose $f_0 \in \mathscr{S}(\mathbb{R}^n)$ so that

$$\frac{1}{2^n}\sum_{\tau\in\mathbb{Z}_2^n} f(\tau x) = f_0(x^2).$$

Thus, the W_B-invariance of ζ_α gives

$$\langle \zeta_\alpha, f\rangle = \langle R_\alpha, f_0\rangle = \int_{\overline{\mathbb{R}^n_+}} f_0(x)\,\mathrm{d}\mu(x) = \frac{1}{2^n}\sum_{\tau\in\mathbb{Z}_2^n}\int_{\overline{\mathbb{R}^n_+}} f(\tau\sqrt{x})\,\mathrm{d}\mu(x),$$

i.e. ζ_α is a measure. Conversely, if ζ_α is a measure μ, then for $f \in \mathscr{S}(\mathbb{R}^n)$

$$\langle R_\alpha, f\rangle = \langle \zeta_\alpha, f(x^2)\rangle = \int_{\mathbb{R}^n} f(x^2)\ \mathrm{d}\mu(x),$$

i.e. R_α is a measure. ■

To conclude the study of zeta distributions, we shall explicitly compute ζ_α for $\alpha \in \{0, k, \ldots, k(n-1)\}$, the discrete part of the generalized Wallach set. On the continuous part of the generalized Wallach set, ζ_α is given by the measure

$$\tfrac{1}{\Gamma_n(\alpha)}\Delta(x^2)^{\alpha-\nu}\omega^B(x)\,\mathrm{d}x = \tfrac{1}{\Gamma_n(\alpha)}\Delta(x^2)^{\alpha}\omega^A(x^2)\,\mathrm{d}x.$$

In the following, W_B acts on $\mathscr{S}'(\mathbb{R}^n)$ by $\langle wu, f\rangle = \langle u, w^{-1}f\rangle$ for $u \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{S}(\mathbb{R}^n)$ and $w \in W_B$.

Theorem 7.36. *The zeta distribution on the discrete part of the Wallach set are given as*

(i) $\zeta_0 = \delta_0$.

(ii) *For $r = 1, \ldots, n-1$, ζ_{kr} is a positive measure, namely*

$$\zeta_{kr} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(\zeta_{kn}^{(r)}\otimes\delta_0^{(n-r)})^\sigma,$$

where $\zeta_{kn}^{(r)}$ is the zeta distribution of index kn on $\mathbb{R}^r$ and $\delta_0^{(n-r)}$ is the Dirac measure in $0 \in \mathbb{R}^{n-r}$. Moreover, the support of ζ_{kr} is the stratum

$$\partial_r\mathbb{R}^n := \Big\{x \in \mathbb{R}^n \mid x_{i_j} = 0 \textit{ for a sequence } 1 \le i_1 < \ldots < i_r \le n\Big\}.$$

PROOF.

(i) By Theorem 7.32 and Theorem 7.24 (i) we obtain

$$\begin{aligned}\langle \zeta_0, f\rangle = 2^{-n\nu}\langle \zeta_\nu, \mathcal{F}^B f\rangle &= \frac{1}{2^{n\nu}\Gamma_n(\nu)}\int_{\mathbb{R}^n}\mathcal{F}^B f(x)\omega^B(x)\,\mathrm{d}x \\ &= \mathcal{F}^B\mathcal{F}^B f(0) = f(0) = \langle \delta_0, f\rangle.\end{aligned}$$

(ii) It suffices to verify the stated formula for W_B-invariant $f \in \mathscr{S}(\mathbb{R}^n)$. Hence there is an $\mathcal{S}_n$-invariant $f_0 \in \mathscr{S}(\mathbb{R}^n)$ with $f(x) = f_0(x^2)$. By [Rö20, Theorem 5.11] we have

$$R_{kr} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(R_{kn}^{(r)}\otimes\delta_0^{(n-r)})^\sigma,$$

where $R_{kn}^{(r)}$ is the Dunkl-type Riesz distribution on $\mathbb{R}^r$. Therefore

$$\begin{aligned}\langle \zeta_{kr}, f\rangle = \langle R_{kr}, f_0\rangle &= \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\langle (R_{kr}^{(r)}\otimes\delta_0^{(n-r)})^\sigma, f_0\rangle \\ &= \langle R_{kr}^{(r)}, f_0(\cdot, 0^{n-r})\rangle = \langle \zeta_{kr}^{(r)}, f(\cdot, 0^{n-r})\rangle \\ &= \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\langle (\zeta_{kr}^{(r)}\otimes\delta_0^{(n-r)})^\sigma, f\rangle\,.\end{aligned}$$

Since $r < n$, the index kn is contained in the continuous part of the Wallach set associated with the zeta distributions $\zeta_\alpha^{(r)}$ on $\mathbb{R}^r$. Thus, $\zeta_{kn}^{(r)}$ has support $\mathbb{R}^r$, by Theorem 7.31 (ii). In particular, the support of ζ_{kr} is

$$\bigcup_{\sigma\in\mathcal{S}_n}\sigma(\mathbb{R}^r\times\{0\}^{n-r}) = \partial_r\mathbb{R}^n.$$

■

Part III

Limit transitions and Olshanski pairs

CHAPTER 8

Olshanski spherical pairs

We give a brief summary about Olshanski spherical pairs in analogy to Gelfand pairs. References for this short chapter are [Ols90, OV96, Far08]. An Olshanski spherical pair is the inductive limit of an increasing sequence of Gelfand pairs. This chapter is intended to give a brief overview on the main objects and questions. One of the main question is the following. Which of the spherical functions of an Olshanski spherical pair can be approximated by spherical functions of the underlying Gelfand pairs.

Definition 8.1. A *Gelfand pair* (G, K) consists of a locally compact group G and a compact subgroup $K \subseteq G$ such that one of the following equivalent assertions holds:

(i) The convolution algebra $(C_c(G), *)$ is commutative.

(ii) The convolution algebra $(L^1(G), *)$ is commutative.

(iii) For any irreducible unitary representation $(\pi, \mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, the space of K-fixed vectors $\mathcal{H}^K = \{v \in \mathcal{H} \mid \pi(k)v = v \text{ for all } k \in K\}$ is at most one dimensional.

If (G, K) is a Gelfand pair, it is known that G is an unimodular group.

Definition 8.2. Let (G, K) be a Gelfand pair. Let $\varphi : G \to \mathbb{C}$ be a non-zero K-biinvariant continuous function, i.e. $\varphi(kgk') = \varphi(g)$ for all $g \in G$ and $k, k' \in K$. Then, φ is called *spherical* if it satisfies one of the following equivalent statements

(i) $\varphi(g)\varphi(h) = \int_K \varphi(gkh)\, \mathrm{d}k$ for all $g, h \in G$. Here $\mathrm{d}k$ is the normalized Haar measure on K, that means $\int_K 1\, \mathrm{d}k = 1$.

(ii) $f \mapsto \int_G f(g)\varphi(g^{-1})\, \mathrm{d}g$ is a character of the convolution algebra $(C_c(G), *)$.

(iii) $\varphi(e) = 1$ and for all K-biinvariant $f \in C_c(G)$ there exists $\lambda_f \in \mathbb{C}$ with $f * \varphi = \lambda_f \varphi$.

Remark 8.3. A continuous function $\varphi : G \to \mathbb{C}$ is called *positive definite* if for any choice of $g_1, \ldots, g_n \in \mathbb{C}$ and $c_1, \ldots, c_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(g_i g_j^{-1}) \geq 0.$$

To any such φ, there exists a unitary representation $(\pi, \mathcal{H})$ of G and a *cyclic vector* $\xi \in \mathcal{H}$, that means $\operatorname{span}_{\mathbb{C}} \{\pi(g)\xi \mid g \in G\} \subseteq \mathcal{H}$ is dense, with

$$\varphi(g) = \langle \pi(g)\xi, \xi \rangle . \tag{8.1}$$

The triple $(\pi, \mathcal{H}, \xi)$ is uniquely determined up to unitary equivalence and is called the *Gelfand-Naimark-Siegel representation associated with* φ, short GNS-representation. Conversely, if $(\pi, \mathcal{H})$ is a representation of G and $\xi \in \mathcal{H}$, then (8.1) defines a positive definite function.

The positive definite spherical functions can be characterized by the subsequent theorem.

Lemma 8.4. *Let (G, K) be a Gelfand pair, $P(K\backslash G/K)$ the set of K-biinvariant positive definite functions φ on G and $P_1(K\backslash G/K) = \{\varphi \in P(K\backslash G/K) \mid \varphi(e) = 1\}$. Then, for $\varphi \in P_1(K\backslash G/K)$ the following assertions are equivalent:*

(i) *φ is spherical.*

(ii) *φ is extremal in $P_1(K\backslash G/K)$, i.e. there does not exists $\varphi_1, \varphi_2 \in P_1(K\backslash G/K)$ and $\lambda \in]0, 1[$ with $\varphi = \lambda\varphi_1 + (1 - \lambda)\varphi_2$.*

(iii) *φ is extremal in $P_1(G) = \{\varphi : G \to \mathbb{C} \mid \varphi \text{ positive definite}, \varphi(e) = 1\}$.*

(iv) *the Gelfand-Naimark-Siegel representation associated with φ is irreducible.*

Hence, the set of positive definite spherical functions is in bijection with the set of equivalence classes of irreducible unitary representations $(\pi, \mathcal{H})$ with $\dim_{\mathbb{C}} \mathcal{H}^K = 1$. Such representations are called spherical.

We now come to the definition of an Olshanski spherical pair and the associated spherical functions.

Definition 8.5. Let $(G_n, K_n)_{n\in\mathbb{N}}$ be a *increasing sequence of Gelfand pairs*, i.e. $G_n \subseteq G_{n+1}$ is a closed subgroup, $K_n \subseteq K_{n+1}$ and $K_n = G_n \cap K_{n+1}$ for all $n \in \mathbb{N}$. We put $G := \bigcup_{n\in\mathbb{N}} G_n$ and $K := \bigcup_{n\in\mathbb{N}} K_n$ and equip G with the inductive limit topology, i.e. $U \subseteq G$ is open if and only if $U \cap G_n \subseteq G_n$ is open for all $n \in \mathbb{N}$. Then we call the pair (G, K) an *Olshanski (spherical) pair*.

We note that in general the group G is not locally compact.

Definition 8.6. Consider an Olshanski spherical pair $(G, K) = \lim_{n\to\infty}(G_n, K_n)$. A non-zero K-biinvariant continuous function $\varphi : G \to \mathbb{C}$ is called *spherical* if for all $g, h \in G$

$$\varphi(g)\varphi(h) = \lim_{n\to\infty} \int_{K_n} \varphi(gkh)\, \mathrm{d}_n k,$$

where $\mathrm{d}_n k$ is the normalized Haar measure on K_n.

Similar to Gelfand pairs, we have the following characterizing theorem for spherical functions of an Olshanski spherical pair.

Theorem 8.7. *Let (G, K) be an Olshanski pair. Then for $\varphi \in P_1(K\backslash G/K)$ the following assertions are equivalent:*

(i) *φ is spherical.*

(ii) *φ is extremal in $P_1(K\backslash G/K)$.*

(iii) *φ is extremal in $P_1(G)$.*

(iv) *the Gelfand-Naimark-Siegel representation associated with φ is irreducible.*

Furthermore, for any irreducible unitary representation $(\pi, \mathcal{H})$ it holds that $\mathcal{H}^K$ is at most one dimensional. Hence, the set of positive definite spherical functions is in bijection with the set of equivalence classes of irreducible unitary representations $(\pi, \mathcal{H})$ with $\dim_{\mathbb{C}} \mathcal{H}^K = 1$. Such representations are called spherical.

In general, we can not expect that every spherical function of an Olshanski pair is a limit of spherical functions of the underlying Gelfand pairs. But this will be true if we consider positive definite spherical functions. Let $\mathrm{ex}(K)$ denote the extremal points of a convex compact set K of a topological vector space.

Theorem 8.8. *Assume that (G, K) is an Olshanski spherical pair of an increasing family of Gelfand pairs $(G_n, K_n)_{n\in\mathbb{N}}$ such that G_n is second-countable for all $n \in \mathbb{N}$. Then, for $\varphi \in P_1(K\backslash G/K)$ the following is equivalent:*

(i) *φ is spherical.*

(ii) *there exist a sequence of spherical $\varphi_n \in P_1(K_n\backslash G_n/K_n)$, $n \in \mathbb{N}$ with $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ locally uniformly.*

Proof. In [Ols90, Theorem 22.10] they proved for the spaces

$$P_1(G_n) = \{\varphi \in C(G_n) \mid \text{ positive definite}, \varphi(e) = 1\},$$

and similar $P_1(G)$ the following: $\varphi \in \mathrm{ex}(P_1(G))$ if and only if there exists a sequence $\varphi_n \in \mathrm{ex}(P_1(G_n))$ with $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ locally uniformly. But an observation of the proof shows that this statement is still true if $P_1(G)$ and $P_1(G_n)$ are replaced by $P_1(K\backslash G/K)$ and $P_1(K_n\backslash G_n/K_n)$, respectively. This is done by replacing all vector spaces of functions occurring in the proof of [Ols90, Theorem 22.10] by their analogues of K-biinvariant and K_n-biinvariant functions, respectively. Finally, this is exactly the assertion of Theorem 8.8. ■

Proposition 8.9. *Assume that (G_n, K_n) and (G'_n, K'_n) are increasing families of Gelfand pairs with associated Olshanski pairs (G, K) and (G', K'), respectively. If for any $n \in \mathbb{N}$ there exists $\ell, m \in \mathbb{N}$ with the property*

$$G_n \subseteq G'_\ell,\ G'_n \subseteq G_m \ \text{ as closed subgroups} \tag{8.2}$$

then $(G, K) = (G', K')$ as topological spaces.

Proof. It is immediate that (G, K) and (G', K') consist of the same sets. Let τ be the topology of G and τ' the topology of G'.
By definition of the inductive limit topology we have the following. If $A \subseteq G = G'$ is closed with respect to τ, then by definition $A \cap G_n \subseteq G_n$ is closed for all $n \in \mathbb{N}$. Consider some fixed $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $G'_n \subseteq G_m$ is a closed subset. Therefore, $A \cap G'_n = (A \cap G'_n) \cap G_m \subseteq G'_n \cap G_m = G'_n$ is a closed subset of G'_n. Since $n \in \mathbb{N}$ was arbitrary, we have that A is closed with respect to τ', i.e. $\tau \subseteq \tau'$. The proof for the converse subset relation is the same. ■

CHAPTER 9

Bessel Functions as the rank tends to infinity

The asymptotic analysis of multivariate special functions has a long tradition in infinite dimensional harmonic analysis, tracing back to the work of Olshanski, Vershik, and Kerov, see [VK82, Ols90, OV96]. Of particular interest in this context are the behaviour of spherical representations and the limits of spherical functions of increasing families of Gelfand pairs as specific dimensions tend to infinity. Bessel functions associated with root systems generalize the spherical functions of Riemannian symmetric spaces of Euclidean type, which occur for special values of the multiplicity parameters. There are two classes of particular interest, including applications to β-ensembles in random matrix theory, namely those of type A_{n-1} and type B_n. We refer to [For10] for a general background and to [BGCG22] for some recent developments. In the cases of type A and B, the Bessel functions can be expressed as hypergeometric series involving Jack polynomials, c.f. Remark 6.21 and Proposition 7.8. Bessel functions of type A_{n-1} have a continuous multiplicity parameter $k \geq 0$ and include as special cases the spherical functions of the motion groups $U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F})$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where the unitary group $U_n(\mathbb{F})$ acts by conjugation on the space $\mathrm{Herm}_n(\mathbb{F})$ of Hermitian matrices over $\mathbb{F}$. These cases correspond to $k = \frac{d}{2}$ with $d = \dim_{\mathbb{R}}\mathbb{F} \in \{1, 2, 4\}$. Bessel functions of type B_q have non-negative multiplicity parameters of the form $\kappa = (k', k)$, with k the multiplicity on the roots $\pm(e_i \pm e_j)$ and k' that on the roots $\pm e_i$. They generalize the spherical functions of the motion groups $(U_p(\mathbb{F}) \times U_q(\mathbb{F})) \ltimes M_{p,q}(\mathbb{F})$, with $p \geq q$, where $M_{p,q}(\mathbb{F})$ are the $p \times q$ matrices with entries in $\mathbb{F}$. Here the multiplicities are $k = \frac{d}{2}$, $k' = \frac{d}{2}(p-q+1) - \frac{1}{2}$. In [RV13], the limits of the spherical functions of these motion groups as $p \to \infty$ and the associated Olshanski spherical pairs were studied, where the rank q remained fixed. In the present chapter, we shall study Bessel functions of type A_{n-1} and type B_n with arbitrary positive multiplicities as the rank tends to infinity, in the spirit of the work of Okounkov and Olshanski [OO98, OO06]about Jack polynomials (type A) and multivariate Jacobi polynomials (type BC). See also [Cue18] for a more recent extension of their results. We obtain explicit asymptotic results for Bessel functions of type A and type B with arbitrary positive multiplicities as the rank goes to infinity. In the type A case, our results coincide with these in [AN21], given with a different proof from the geometric settings $2k = 1, 2, 4$ in [OV96] and [Bou07] for the limits of the spherical functions of the Gelfand pairs $(U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))$ as $n \to \infty$. In contrast to [Bou07], whose results in the geometric cases are also weaker than ours (c.f. Remark 9.7) and different from [AN21], we follow the direct approach of [OV96] for $\mathbb{F} = \mathbb{C}$ via spherical expansions of the involved Bessel functions, which are replaced by hypergeometric expansions in terms of Jack polynomials in our general setting. To become more precise, we consider the Bessel functions $J_{A_{n-1}}(i\lambda(n), (x, 0, \ldots, 0))$ with fixed multiplicity $k > 0$ and $x \in \mathbb{R}^r$ for sequences of spectral parameters $\lambda(n) \in \mathbb{R}^n$ as $n \to \infty$. Following [OV96, OO98], we characterize those sequences $(\lambda(n))_{n\in\mathbb{N}}$ for which the associated sequence of Bessel functions converges (locally uniformly), in terms of specific real parameters $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, β, γ with $\gamma \geq 0$. These parameters describe the growth of the so-called Vershik-Kerov sequence $(\lambda(n))$ as $n \to \infty$. In Theorem 9.6, the main result of Section 2, we obtain that for $x \in \mathbb{R}^{(\infty)} = \bigcup_{n=1}^{\infty} \mathbb{R}^n$,

$$\lim_{n\to\infty} J_{A_{n-1}}(i\lambda(n), x) = \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{2k}x_j^2} \prod_{l=1}^{\infty} \frac{e^{-i\alpha_\ell x_j}}{\left(1 - \frac{i\alpha_\ell x_j}{k}\right)^k},$$

where the convergence is locally uniform in a tubular neighborhood of $\mathbb{R}^r$ in $\mathbb{C}^r$ for each $r \in \mathbb{N}$. In the group cases $k = \frac{d}{2}$, the limiting functions are products of Polya functions (in the sense of [Far08]). They coincide with the positive definite Olshanski spherical functions of the spherical pairs $(U_\infty(\mathbb{F}) \ltimes \mathrm{Herm}_\infty(\mathbb{F}), U_\infty(\mathbb{F}))$ which were already determined by Pickrell [Pic91]; see also [OV96], where they occur in the characterization of the ergodic measures of $\mathrm{Herm}_\infty(\mathbb{F})$ with respect to the action of $U_\infty(\mathbb{F})$. In the type B case, we consider Bessel functions $J_{B_n}(\kappa_n, i\lambda(n), (x, 0, \ldots, 0))$ for $n \to \infty$, where the multiplicity is of the form $\kappa_n = (k_n', k)$, i.e. the first multiplicity parameter may also vary with n. This is motivated by the geometric cases. Again we characterize the sequences $(\lambda(n))_{n\in\mathbb{N}}$ for which the associated Bessel functions converge locally uniformly on a tubular neighbordhood of $\mathbb{R}^r$ in $\mathbb{C}^r$ as $n \to \infty$, and we determine the possible limits, which are now given by the functions

$$\varphi_{(\alpha,\beta)}(x) = \prod_{j=1}^{\infty} e^{-\frac{\beta}{4}x_j^2} \prod_{\ell=1}^{\infty} \frac{e^{\frac{\alpha_\ell}{4}x_j^2}}{\left(1 + \frac{\alpha_\ell}{4k}x_j^2\right)^k}, \quad x \in \mathbb{R}^{(\infty)},$$

with real parameters $\beta \geq 0$ and $\alpha_\ell \geq 0$ with $\sum_{\ell=1}^n \alpha_\ell \leq \beta$. It turns out that for $k = \frac{d}{2}$ with $d = 1, 2, 4$, these can be identified with the positive definite Olshanski spherical functions of spherical pairs (G_∞, K_∞), which are obtained as inductive limits of the motion groups $(U_p(\mathbb{F}) \times U_q(\mathbb{F})) \ltimes M_{p,q}(\mathbb{F})$ as both dimension parameters p and q tend to infinity. In the case $\mathbb{F} = \mathbb{C}$, the set of spherical functions were determined in [Pic90].
The organization of this chapter is as follows: The type A case is treated in Section 2. While in this case our results generalize known results in the geometric cases, our results for type B, which are developed in Section 3, seem to be new even in the geometric cases.

9.1 Introduction

For a reduced root system $R \subset \mathbb{R}^n$ we fix a multiplicity function k and the Dunkl kernel $E = E_k$ associated with (R, k). Recall, for each $\lambda \in \mathbb{C}^n$, there exists a compactly supported probability measure μ_λ on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$

$$E(\lambda, x) = \int_{\mathbb{R}^n} e^{\langle \xi, x\rangle} \, \mathrm{d}\mu_\lambda(\xi). \tag{9.1}$$

In particular, $E(\lambda, \cdot)$ is positive-definite on the additive group $\mathbb{R}^n$ if and only if $\lambda \in i\mathbb{R}^n$. The Bessel function associated with (R, k) is denoted by

$$J(\lambda, z) = \frac{1}{\#W} \sum_{w\in W} E(\lambda, wz).$$

And as in the case of the Dunkl kernel, $J(\lambda, \cdot)$ is positive-definite if and only if $\lambda \in i\mathbb{R}^n$. We shall be concerned with the root systems

$$A_{n-1} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\} \subset \mathbb{R}^n,$$
$$B_n = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\} \subset \mathbb{R}^n,$$

where $(e_i)_{1\leq i\leq n}$ denotes the standard basis of $\mathbb{R}^n$. In both cases, the Bessel functions can be written as hypergeometric series in terms of Jack polynomials, cf. Remark 6.21 and Proposition 7.8. For A_{n-1}, the multiplicity function is given by a single parameter $k \geq 0$. We write Λ_+^n for the set of partitions $\kappa = (\kappa_1, \kappa_2, \ldots)$ of length $\ell(\kappa) \leq n$ and denote by $C_\kappa^{(n)}$, $\kappa \in \Lambda_+^n$ the symmetric Jack polynomials in n variables of index $\alpha = \frac{1}{k}$ in C-normalization as in Proposition 6.16, namely

$$\sum_{|\kappa|=m} C_\kappa^{(n)}(z) = (z_1 + \ldots + z_n)^m, \quad m \in \mathbb{N}_0.$$

The Jack polynomials are stable with respect to the number of variables, i.e. for $\kappa \in \Lambda_+^r$ with $r < n$ we have

$$C_\kappa^{(r)}(z_1, \ldots, z_r) = \begin{cases} C_{\kappa'}^{(n)}(z_1, \ldots, z_r, \underline{0}_{n-r}) & \text{if } \kappa' = (\kappa, 0, \ldots), \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

with the notation $\underline{a}_j := (a, \ldots, a) \in \mathbb{C}^j$ for $a \in \mathbb{C}$. See [Sta89, Prop. 2.5] together with [Kan93, formula(16)]. Therefore the Jack polynomials $C_\kappa^{(n)}$ uniquely extend to continuous functions C_κ on $\mathbb{C}^{(\infty)} = \bigcup_{n=1}^\infty \mathbb{C}^n$, equipped with the inductive limit topology. We shall often consider elements from $\mathbb{C}^{(\infty)}$ as sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $x_n \neq 0$ for at most finitely many $n \in \mathbb{N}$, for $\mathbb{R}^{(\infty)}$ accordingly.

9.2 The type A case

The following theorem justifies why we consider type A Bessel functions if the rank tends to infinity.

Theorem 9.1. *The spherical functions of the Gelfand pairs*

$$(G_n, K_n) = (U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))$$

are given as $U_n(\mathbb{F})$-invariant functions on $\mathrm{Herm}_n(\mathbb{F})$ by

$$\varphi_\lambda(X) = \int_{U_n(\mathbb{F})} e^{\mathrm{tr}(kXk^{-1}\cdot \mathrm{diag}(\lambda))}\, \mathrm{d}k, \quad \lambda \in \mathbb{R}^n,$$

where $\mathrm{diag}(\lambda)$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Then:

$$\varphi_\lambda(X) = J^{A_{n-1}}_{\dim_{\mathbb{R}}(\mathbb{F})/2}(\lambda, \mathrm{spec}\, X),$$

where $\mathrm{spec}\, X \in \mathbb{R}^n$ are the spectral values of X in an arbitrary order.

Proof. This is a consequence of Theorem 4.27 as (G_n, K_n) is the Gelfand pair associated with the Cartan motion group of the Lie group $G = \mathrm{GL}_n(\mathbb{F})$ of the Harish-Chandra class with root system of type A_{n-1} and root space dimension $\dim_{\mathbb{R}} \mathbb{F}$. One has only to observe that in this case $\mathfrak{p} = \mathrm{Herm}_n(\mathbb{F})$, $(X, Y) \mapsto \mathrm{tr}(XY^*)$ is the inner product as stated in (4.11), $\lambda \mapsto \mathrm{diag}(\lambda)$ is an isometric isomorphism $\mathbb{R}^n \to \mathfrak{a}$ and $X \mapsto \mathrm{spec}\, X$ induces an identification $K\backslash G_0/K = \mathfrak{p}/K \cong \mathbb{R}^n/\mathcal{S}_n$. ■

The embedding

$$M_{n,n}(\mathbb{F}) \hookrightarrow M_{n+1,n+1}(\mathbb{F}), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

of $n \times n$ matrices into $(n+1) \times (n+1)$ matrices induces an Olshanski spherical pair

$$(G_\infty, K_\infty) = \lim_{n\to\infty} (G_n, K_n) = \lim_{n\to\infty} (U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F})). \tag{9.3}$$

The positive definite spherical functions of (G_∞, K_∞) were completely determined by Pickrell [Pic91, Section 5], see also [OV96] for $\mathbb{F} = \mathbb{C}$, and [Far08, Section 3]. As functions on $\mathrm{Herm}_\infty(\mathbb{F})$, they are given for $d = \dim_{\mathbb{R}} \mathbb{F}$ by

$$\varphi(X) = \prod_{j=1}^\infty e^{i\beta x_j - \frac{\gamma}{d} x_j^2} \prod_{\ell=1}^\infty \frac{e^{-i\alpha_\ell x_j}}{(1 - i\frac{2}{d}\alpha_\ell x_j)^{d/2}},$$

where $\beta, \gamma \in \mathbb{R}, \gamma \geq 0$, $\alpha_\ell \in \mathbb{R}$ with $\sum_{\ell=1}^\infty \alpha_\ell^2 < \infty$, and $(x_1, x_2, \dots) \in \mathbb{R}^{(\infty)}$ are the eigenvalues of X ordered by size and counted according to their multiplicity. The product is invariant under rearrangements of the α_ℓ. For $\mathbb{F} = \mathbb{C}$ it is also noted in [OV96] that the set of positive definite spherical functions is bijectively parametrized by the set

$$\Big\{(\alpha, \beta, \gamma) \mid \beta \in \mathbb{R}, \gamma \geq 0, \alpha = \{\alpha_1, \alpha_2, \dots\} \text{ a multiset with } \alpha_\ell \in \mathbb{R} \text{ and } \sum\nolimits_\ell \alpha_\ell^2 < \infty\Big\}.$$

In [OV96], explicit approximations of the positive definite spherical functions by positive definite spherical functions of the pairs (G_n, K_n) with $n \to \infty$ by use of spherical expansions were obtained in the case $\mathbb{F} = \mathbb{C}$. In [Bou07], this was generalized by completely different methods to $\mathbb{F} = \mathbb{R}, \mathbb{H}$. In the present section, we shall obtain the result of Pickrell and explicit approximations of Olshanski spherical functions as particular cases of a more general asymptotic result for Bessel functions of type A_{n-1} with an arbitrary multiplicity parameter $k > 0$, as already proven in [AN21], but with a very natural proof in line with the geometric cases $k = \frac{1}{2}, 1, 2$.

Let us first turn to the spectral parameters to be considered for $n \to \infty$. Instead of working with multisets, it will be convenient for us to work with sequences (or finite tuples) with a prescribed order of their components. We introduce the following order on $\mathbb{R}$:

$$x \ll y \text{ iff either } |x| < |y| \text{ or } |x| = |y| \text{ and } x \leq y.$$

For instance, the sequence $(3, -3, 2, 1, -1, -1, 0, 0, \dots)$ is decreasing w.r.t. $\ll$.

Definition 9.2. Consider $\lambda(n) \in \mathbb{R}^n$ such that its entries are decreasing with respect to $\ll$. We regard $(\lambda(n))_{n \in \mathbb{N}}$ as a sequence in $\mathbb{R}^{(\infty)} = \bigcup_{n=1}^\infty \mathbb{R}^n$ and call it a *Vershik-Kerov sequence* (VK-sequence for short), if the following limits exist:

$$\begin{aligned} \alpha_i &:= \lim_{n\to\infty} \frac{\lambda(n)_i}{n}, \quad i \in \mathbb{N}, \\ \beta &:= \lim_{n\to\infty} \frac{p_1(\lambda(n))}{n}, \\ \delta &:= \lim_{n\to\infty} \frac{p_2(\lambda(n))}{n^2}, \end{aligned}$$

where

$$p_m(x) := \sum_{i=1}^\infty x_i^m \quad \text{for } m \in \mathbb{N},\ p_0 \equiv 1$$

are the *power sum symmetric functions* on $\mathbb{R}^{(\infty)}$. They generate the algebra of symmetric functions on $\mathbb{R}^{(\infty)}$, i.e. the symmetric polynomial functions in arbitrary many variables.

The definition of a VK-sequence is equivalent to the Olshanski-Vershik condition of [AN21, Definition 2.2], which are slightly weaker than the conditions of [OO98].

Lemma 9.3. *Let $(\lambda(n))_{n\in\mathbb{N}}$ be a VK-sequence with associated parameters $(\alpha_i)_{i\in\mathbb{N}}, \beta, \delta$ as above. Then we have*

(i) *The sequence $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ is square-summable with*

$$\gamma := \delta - \sum_{i=1}^\infty \alpha_i^2 \geq 0.$$

(ii) *If in addition $\lambda(n)_i \geq 0$ for all $i, n \in \mathbb{N}$, then $\gamma = 0$.*

Definition 9.4. Suppose that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK-sequence. Then the triple $\omega = (\alpha, \beta, \gamma)$ with $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ are called the *VK-parameters* of the sequence $(\lambda(n))_{n\in\mathbb{N}}$. Note that the entries of α are also ordered with respect to $\ll$.

Proof of Lemma 9.3.

(i) For fixed $N \in \mathbb{N}$ and all $n \geq N$ one has

$$\sum_{i=1}^{N} \alpha_i^2 \leq \sum_{i=1}^{N} \left(\alpha_i^2 - \frac{\lambda(n)_i^2}{n^2} \right) + \sum_{i=1}^{n} \frac{\lambda(n)_i^2}{n^2}.$$

By definition of α and δ the right-hand side tends to δ as $n \to \infty$. This proves part (i).

(ii) By the ordering of the entries of $\lambda(n)$, we obtain for $N \in \mathbb{N}$ and $n \geq N$ that

$$\begin{aligned} \frac{p_2(\lambda(n))}{n^2} &= \sum_{i=1}^{N-1} \left(\frac{\lambda(n)_i}{n} \right)^2 + \sum_{i=N}^{n} \left(\frac{\lambda(n)_i}{n} \right)^2 \\ &\leq \sum_{i=1}^{N-1} \left(\frac{\lambda(n)_i}{n} \right)^2 + \frac{\lambda(n)_N}{n} \sum_{i=1}^{n} \frac{\lambda(n)_i}{n}, . \end{aligned}$$

Taking the limit $n \to \infty$ on both sides, we obtain that

$$\delta \leq \sum_{i=1}^{N-1} \alpha_i^2 + \alpha_N \beta.$$

As $\lim_{N \to \infty} \alpha_N = 0$, this implies that $\delta \leq \sum_{i=1}^{\infty} \alpha_i^2$ and therefore $\gamma = 0$.

■

We shall throughout fix a strictly positive multiplicity $k > 0$ on A_{n-1} and suppress it in our notation. For sequences $(\lambda(n))_{n \in \mathbb{N}}$ of spectral parameters $\lambda(n) \in \mathbb{R}^n$ with growing dimension n, we are interested in the convergence behaviour of the Bessel functions $J_{A_{n-1}}(i\lambda(n), \cdot)$ as $n \to \infty$. For this, we consider $J_{A_{n-1}}(\lambda, \cdot)$ as a function on $\mathbb{C}^r$ for all $r \leq n$ by

$$J_{A_{n-1}}(\lambda, z) := J_{A_{n-1}}(\lambda, (z, \underline{0}_{n-r})), \quad z \in \mathbb{C}^r. \tag{9.4}$$

For later use, we record the following representation.

Proposition 9.5. *For $\lambda \in \mathbb{C}^n$ and $z \in \mathbb{C}^r$ with $r \leq n$,*

$$J_{A_{n-1}}(\lambda, z) = \sum_{\kappa \in \Lambda_+^r} \frac{C_\kappa(\lambda)[kr]_\kappa}{[kn]_\kappa |\kappa|!} \mathcal{P}_\kappa(z),$$

with the renormalized Jack polynomials

$$\mathcal{P}_\kappa(z) = \frac{C_\kappa(z)}{C_\kappa(\underline{1}_r)}$$

and the generalized Pochhammer symbol

$$[\mu]_\kappa = \prod_{j=1}^{\ell(\kappa)} (\mu - k(j-1))_{\kappa_j}.$$

PROOF. Consider the expansion of $J_{A_{n-1}}$ as Jack hypergeometric function from Remark 6.21, namely for $x \in \mathbb{R}^n$

$$J_{A_{n-1}}(\lambda, x) = {}_0F_0(\lambda, x) = \sum_{\kappa \in \Lambda_+^n} \frac{C_\kappa(\lambda) C_\kappa(x)}{C_\kappa(\underline{1}_n)\, |\kappa|!}$$

From [Kan93, formula (17)] it is known that for all $\kappa \in \Lambda_+^r$,

$$\frac{C_\kappa(\underline{1}_r)}{C_\kappa(\underline{1}_n)} = \frac{[kr]_\kappa}{[kn]_\kappa}.$$

Together with the stability property (9.2), the assertion follows. ∎

We shall prove the following theorem.

Theorem 9.6. *Let $(\lambda(n))_{n\in\mathbb{N}}$ be a sequence of spectral parameters $\lambda(n) \in \mathbb{R}^n$ such that each $\lambda(n)$ is decreasing with respect to $\ll$. Then for fixed multiplicity $k > 0$, the following statements are equivalent.*

(i) *$(\lambda(n))_{n\in\mathbb{N}}$ is a Vershik-Kerov sequence.*

(ii) *The sequence of Bessel functions $\big(J_{A_{n-1}}(i\lambda(n), \cdot)\big)_{n\in\mathbb{N}}$ converges locally uniformly on compact subsets of $\mathbb{R}^{(\infty)}$, i.e. the convergence is locally uniform on each of the spaces $\mathbb{R}^r$, $r \in \mathbb{N}$.*

(iii) *The sequence of Bessel functions $\big(J_{A_{n-1}}(i\lambda(n), \cdot)\big)_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$ against a function which is continuous in 0.*

(iv) *For each fixed multi-index of length r, the corresponding coefficients in the Taylor of expansion of $J_{A_{n-1}}(i\lambda(n), \cdot)$ around $0 \in \mathbb{R}^r$ converge as $n \to \infty$.*

(v) *For all symmetric functions $f : \mathbb{R}^{(\infty)} \to \mathbb{C}$, the limit*

$$\lim_{n\to\infty} \frac{f(\lambda(n))}{n^{\deg f}}$$

exists.

Moreover, in this case one has

$$\lim_{n\to\infty} J_{A_{n-1}}(i\lambda(n), z) = \prod_{j=1}^\infty e^{i\beta z_j - \frac{\gamma}{2k} z_j^2} \prod_{\ell=1}^\infty \frac{e^{-i\alpha_\ell z_j}}{\left(1 - \frac{i\alpha_\ell z_j}{k}\right)^k}, \tag{9.5}$$

locally uniformly in

$$z \in S_{r,k}^\omega := \left\{ z \in \mathbb{C}^r \,\middle|\, \|\mathrm{Im}\, z\|_\infty < \frac{k}{r\,|\alpha_1|} \right\} \tag{9.6}$$

for all $r \in \mathbb{N}$, where (α, β, γ) are the VK-parameters of the VK-sequence $(\lambda(n))_{n\in\mathbb{N}}$, and the product on the right side extends analytically to $\mathbb{R}^r$ for each $r \in \mathbb{N}$.

Remark 9.7. In the geometric case $k = 1$, i.e. for Hermitian matrices over $\mathbb{C}$, this result essentially goes back to [OV96], while in [Bou07], where also $\mathbb{F} = \mathbb{R}$ and $\mathbb{H}$ are considered, only the limit (9.5) is established, by completely different methods and under the additional condition $\gamma = 0$.
The equivalence of (i) and (iii) including the complex domain of convergence was independently proven in [AN21]. But, the authors used a different, probabilistic approach. We feel that the method used here is very natural, which is also suggested by [AN21, Remark 1.15].

Our proof of Theorem 9.6 is inspired by the methods of [OV96], [OO98] and [Far08, Chapter 3]. We start with the following observation, which is already done in [AN21, Proposition 2.3], but for the sake of completeness we give it again, with a clearer proof.

Theorem 9.8. *Assume that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK-sequence with parameters $\omega = (\alpha, \beta, \gamma)$. Then*

$$\lim_{n\to\infty} \frac{p_m(\lambda(n))}{n^m} = \widetilde{p}_m(\omega) := \begin{cases} 1, & m = 0, \\ \beta, & m = 1, \\ \delta = \gamma + \sum_{i=1}^{\infty} \alpha_i^2\,, & m = 2, \\ \sum_{i=1}^{\infty} \alpha_i^m\,, & m \geq 3, \end{cases}$$

where the series in the last case is absolutely convergent. In particular, for each symmetric function f on $\mathbb{R}^{(\infty)}$, the limit

$$\widetilde{f}(\omega) := \lim_{n\to\infty} \frac{f(\lambda(n))}{n^{deg f}}$$

exists.

Proof. We only have to consider the case $m \geq 3$. In view of the ordering of $\lambda(n)$ we have for arbitrary $N \in \mathbb{N}$ that

$$\sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m \leq \Big|\frac{\lambda(n)_N}{n}\Big|^{m-2} \cdot \frac{p_2(\lambda(n))}{n^2}. \tag{9.7}$$

The expression on the right side converges to $\alpha_N^{m-2}\delta$ as $n \to \infty$. As α is square-summable by Lemma 9.3, this implies that for each $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\sum_{i=N}^{\infty} |\alpha_i|^m + \sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m < \epsilon. \tag{9.8}$$

Estimate (9.8) further leads to

$$\begin{aligned} \Big|\frac{p_m(\lambda(n))}{n^m} - p_m(\alpha)\Big| &\leq \sum_{i=N}^{\infty} |\alpha_i|^m + \sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m + \sum_{i=1}^{N-1} \Big|\frac{\lambda(n)_i^m}{n^m} - \alpha_i^m\Big| \\ &\leq \epsilon + \sum_{i=1}^{N-1} \Big|\frac{\lambda(n)_i^m}{n^m} - \alpha_i^m\Big|. \end{aligned}$$

By the definition of a VK-sequence, the last sum tends to zero as $n \to \infty$. As $\epsilon > 0$ was arbitrary, this finishes the proof. ∎

We next consider for $\lambda \in \mathbb{C}^{(\infty)}$ the complex function

$$\Phi(\lambda; z) := \prod_{j=1}^{\infty} \frac{1}{(1 - \lambda_j z)^k},$$

where $\zeta \mapsto \zeta^k$ denotes the principal holomorphic branch of the power function on $\mathbb{C}\backslash]-\infty, 0]$. For fixed λ, the product is finite and $\Phi(\lambda; \cdot)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}$. According to formula (2.9) of [OO98],

$$\Phi(\lambda; z) = \sum_{j=0}^{\infty} g_j(\lambda) z^j \tag{9.9}$$

with

$$g_j(\lambda) = \sum_{i_1\leq \ldots \leq i_j} \frac{(k)_{m_1}(k)_{m_2}\cdots}{m_1!\,m_2!\cdots} \cdot \lambda_{i_1}\cdots\lambda_{i_j}, \tag{9.10}$$

where $m_\ell := \#\{r \in \mathbb{N} : i_r = l\}$ denotes the multiplicity of the number l in the tuple $(i_1, \dots, i_j)$ and $(k)_m = k(k+1)\cdots(k+m-1)$ is the Pochhammer symbol. Moreover, from [OO98, formula (2.8)] and the connection between the C- and P-normalizations of the Jack polynomials according to formula (12.135) of [For10], one calculates that

$$g_j(\lambda) = \frac{(k)_j}{j!} \cdot C_{(j)}(\lambda). \tag{9.11}$$

(For partitions $\kappa = (j)$ with just one part, the Jack polynomials $C_{(j)}$ and $P_{(j)}$ coincide).

Lemma 9.9. *Suppose $\omega = (\alpha, \beta, \gamma)$ are the VK-parameters of a Vershik-Kerov sequence. Then the following hold.*

(i) *The infinite product*

$$\Psi(\omega; z) := e^{k\beta z + \frac{k\gamma}{2} z^2} \prod_{\ell=1}^{\infty} \frac{e^{-k\alpha_\ell z}}{(1-\alpha_\ell z)^k}$$

is holomorphic in $S := \mathbb{C}\backslash (\,]-\infty, -\frac{1}{|\alpha_1|}] \cup [\frac{1}{|\alpha_1|}, \infty[\,)$.
If in addition $\alpha_\ell \geq 0$ for all $l \in \mathbb{N}$, then $\Psi(\omega; \cdot)$ is holomorphic in $\widetilde{S} := \mathbb{C}\backslash [\frac{1}{\alpha_1}, \infty[$.

(ii) *ω is uniquely determined by $\Psi(\omega; \cdot)$.*

Proof.

(i) A power series expansion around $z = 0$ shows that for $|\alpha_\ell z| \leq \delta < 1$,

$$\left| 1 - \frac{e^{-k\alpha_\ell z}}{(1-\alpha_\ell z)^k} \right| \leq C_\delta |\alpha_\ell z|^2$$

with some constant $C_\delta > 0$. Recall that α is decreasing with respect to $\ll$ and square-summable. Hence for fixed $n \in \mathbb{N}$, the product

$$\prod_{\ell=n}^{\infty} \frac{e^{-k\alpha_\ell z}}{(1-\alpha_\ell z)^k}$$

defines a holomorphic function in the open disc $\left\{ z \in \mathbb{C} \,\middle|\, |z| < \frac{1}{|\alpha_n|} \right\}$. Moreover,

$$\prod_{\ell=1}^{n-1} \frac{e^{-k\alpha_\ell z}}{(1-\alpha_\ell z)^k}$$

is holomorphic in S and even in $\widetilde{S}$ if $\alpha_\ell \geq 0$ for all $\ell \in \mathbb{N}$. As $\lim_{l\to\infty} \alpha_\ell = 0$, it follows that $\psi(\omega; \cdot)$ is holomorphic in S or even in $\widetilde{S}$. Unless α is identical zero (which is equivalent to $\alpha_1 = 0$), $\Psi(\omega; \cdot)$ has a singularity in $z = \frac{1}{\alpha_1}$.

(ii) If $\psi(\omega; \cdot)$ is entire, then $\alpha_1 = 0$. Otherwise $\lim_{z\to 1/\alpha_1} |\Psi(\omega; z)| = \infty$. Thus α_1 is uniquely determined by $\Psi(\omega; \cdot)$. Multiplying successively by $(1-\alpha_1 z)^k, \dots$, we further obtain that $\alpha_2, \alpha_3, \dots$ are uniquely determined by $\Psi(\omega; \cdot)$ as well. It is then obvious that also β and γ are uniquely determined by $\Psi(\omega; \cdot)$.

∎

Remark 9.10. A closer observation of the proof of Lemma 9.9 shows that $\Psi(\omega;\cdot)$ is even holomorphic on the domain

$$\mathbb{C}\backslash(]-\infty,\alpha_-^*]\cup[\alpha_+^*,\infty[),$$

where

$$\alpha_-^* := \frac{1}{\max\{\alpha_\ell \mid \ell\in\mathbb{N},\alpha_\ell<0\}} \in [-\infty,0[,$$
$$\alpha_+^* := \frac{1}{\min\{\alpha_\ell \mid \ell\in\mathbb{N},\alpha_\ell>0\}} \in]0,\infty].$$

Proposition 9.11.

(i) *For $\lambda\in\mathbb{C}^{(\infty)}$ with decreasing absolute values and $z\in\mathbb{C}$ with $|z|<\frac{1}{|\lambda_1|}$,*

$$\Phi(\lambda;z)=\exp\Big(k\sum_{m=1}^{\infty}p_m(\lambda)\frac{z^m}{m}\Big). \tag{9.12}$$

(ii) *Moreover, if $(\lambda(n))_{n\in\mathbb{N}}$ is a VK-sequence with parameters $\omega=(\alpha,\beta,\gamma)$, then*

$$\lim_{n\to\infty}\Phi\Big(\frac{\lambda(n)}{n};z\Big)=\Psi(\omega;z),$$

where the convergence is locally uniform in z in $\{z\in\mathbb{C}\mid |z|<1/|\alpha_1|\}$.

PROOF.

(i) The left hand side of (9.12) is obviously holomorphic on the domain $\{z\in\mathbb{C}:|z|<1/|\lambda_1|\}$. By $|p_\lambda(\lambda)|\le r\cdot|\lambda_1|^m$, the right hand side is holomorphic on the same domain. Since both sides of (9.12) have value 1 in $z=0$, it suffices to verify that they have the same logarithmic derivative. Let ln be the principle holomorphic branch of the logarithm in $\mathbb{C}\backslash]-\infty,0]$. Then for $|z|$ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\Phi(\lambda;z)=\sum_{j=0}^{\infty}\frac{k\lambda_j}{1-\lambda_j z}=k\sum_{m=0}^{\infty}p_{m+1}(\lambda)z^m.$$

This is exactly the logarithmic derivative of the right-hand side in (9.12).

(ii) For the second assertion, recall from (9.7) that for $m\ge2$ we may estimate

$$\Big|\frac{p_m(\lambda(n))}{n^m}\Big|\le\Big|\frac{\lambda(n)_1}{n}\Big|^{m-2}\cdot\frac{p_2(\lambda(n))}{n^2}$$

Since the right-hand side converges to $|\alpha_1|^{m-2}\delta$ for $n\to\infty$, the sequence on the left-hand side is uniformly bounded in n. Hence, for all $\epsilon>0$, the series

$$h_n(z)=\sum_{m=1}^{\infty}p_m\Big(\frac{\lambda(n)}{n}\Big)\frac{z^m}{m}$$

converges for $|z|<\frac{1}{|\alpha_1|}-\epsilon$, and the dominated convergence theorem shows that $(h_n)_{n\in\mathbb{N}}$ converges for $n\to\infty$ to $\sum_{m=0}^{\infty}\widetilde{p}_m(\omega)\frac{z^m}{m}$ locally uniformly in $\Big\{z\in\mathbb{C}\mid |z|<\frac{1}{|\alpha_1|}-\epsilon\Big\}$. Thus

$$\lim_{n\to\infty}\Phi\Big(\frac{\lambda(n)}{n};z\Big)=\exp\Big(k\sum_{m=0}^{\infty}\widetilde{p}_m(\omega)\frac{z^m}{m}\Big) \tag{9.13}$$

locally uniformly in the disc $\left\{z \in \mathbb{C} \mid |z| < \frac{1}{|\alpha_1|}\right\}$. Now consider $\Psi(\omega;\cdot)$, which is holomorphic in a neighborhood of 0. Taking the logarithmic derivative as in the proof of [Far08, Prop. 3.12] and recalling Theorem 9.8, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\Psi(\omega;z) = k\left(\beta+\gamma z-\sum_{\ell=1}^{\infty}\Big(\alpha_\ell-\frac{\alpha_\ell}{1-\alpha_\ell z}\Big)\right) = k\sum_{m=0}^{\infty}\widetilde{p}_{m+1}(\omega)z^m.$$

The right-hand side in equation (9.13) has the same logarithmic derivative. Since $\Phi(\frac{\lambda(n)}{n};0) = 1 = \Psi(\omega;0)$, this proves the stated limit.

■

We now consider the asymptotic behaviour of the Bessel functions $J_{A_{n-1}}$ as $n\to\infty$. For $z\in\mathbb{C}^{(\infty)}$, we put

$$\widehat{\Psi}(\omega;z) := \prod_{j=1}^{\infty}\Psi(\omega;z_j),$$

which is actually a finite product.

Theorem 9.12. *Assume that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK-sequence with parameters $\omega=(\alpha,\beta,\gamma)$ and recall $S_{r,k}\subseteq\mathbb{C}^r$ from (9.6). Then for $z\in S^{\omega}_{r,k}$, the Bessel functions of type A_{n-1} with multiplicity $k>0$ satisfy*

$$\lim_{n\to\infty}J_{A_{n-1}}(i\lambda(n),z) = \widehat{\Psi}\Big(\omega;\frac{iz}{k}\Big) = \prod_{j=1}^{\infty}e^{i\beta z_j-\frac{\gamma}{2k}x_j^2}\prod_{\ell=1}^{\infty}\frac{e^{-i\alpha_\ell z_j}}{(1-\frac{i\alpha_\ell}{k}z_j)^k}$$

locally uniformly in $z\in S^{\omega}_{r,k}$ for all $r\in\mathbb{N}$.

Proof. First, we do a rank one reduction as in [AN21, Proposition 6.8]. Recall that there exists a probability measure $\mu_n=\mu_{\lambda(n)}$ such that for all $z\in\mathbb{C}^r$ and $n\geq r$

$$J_{A_{n-1}}(i\lambda(n),z) = \int_{\mathbb{R}^n}e^{i\langle\xi,z\rangle}\,\mathrm{d}\mu_n(\xi).$$

Therefore, the Hölder inequality and Theorem 1.16 (v) lead to

$$\begin{aligned}\big|J_{A_{n-1}}(i\lambda(n),z)\big| &\leq J_{A_{n-1}}(\lambda(n),-\mathrm{Im}\,z)\\ &= \int_{\mathbb{R}^n}e^{-\langle\xi,\mathrm{Im}\,z\rangle}\,\mathrm{d}\mu_n(\xi)\\ &\leq \prod_{j=1}^{r}\Big(\int_{\mathbb{R}^n}e^{-r\xi_j\cdot\mathrm{Im}z_j}\,\mathrm{d}\mu_n(\xi)\Big)^{\frac{1}{r}} = \prod_{j=1}^{r}J_{A_{n-1}}(\lambda(n),-r\mathrm{Im}\,z_j)^{\frac{1}{r}}.\end{aligned}$$

We now divide the prove into two steps.

(i) We claim that the family $(J_{A_{n-1}}(i\lambda(n),\cdot))_{n\in\mathbb{N}}$ is uniformly bounded on compact subsets of $S^{\omega}_{r,k}$. By our rank one reduction, it suffices to prove that $(J_{A_{n-1}}(\lambda(n),\cdot))_{n\in\mathbb{N}}$ is locally uniformly bounded on $I:=\left\{x\in\mathbb{R}\mid |x|<\frac{k}{|\alpha_1|}\right\}$.
In rank one, the Jack polynomials $\mathcal{P}_\kappa,\kappa\in\mathbb{N}_0$ are the monomials, so that Proposition 9.5 becomes for $x\in I$

$$J_{A_{n-1}}(\lambda(n),x)\leq\sum_{\kappa=0}^{\infty}\frac{C_\kappa(\lambda(n))(k)_\kappa}{(kn)_\kappa\kappa!}x^\kappa$$

$$\leq \sum_{\kappa=0}^{\infty} \frac{|C_\kappa(\lambda(n))|\,(kr)_\kappa}{(kn)^\kappa \kappa!}|x|^\kappa$$
$$\leq \sum_{\kappa=0}^{\infty} \frac{\left|C_\kappa\left(\frac{\lambda(n)}{n}\right)\right|(kr)_\kappa}{\kappa!}\left(\frac{|x|}{k}\right)^\kappa = \sum_{\kappa=0}^{\infty} |\alpha_\kappa(\lambda(n))|\left(\frac{|x|}{k}\right)^\kappa, \qquad (*)$$

where the coefficients $\alpha_\kappa(\lambda(n)) \in \mathbb{C}$ are given by (9.9) through

$$\Phi(\tfrac{\lambda(n)}{n}, \tfrac{z}{k}) = \sum_{\kappa=0}^{\infty} \alpha_\kappa(\lambda(n))\left(\frac{z}{k}\right)^\kappa \xrightarrow[n\to\infty]{} \Psi(\omega, \tfrac{z}{k}),$$

where the convergence is locally uniform on $\left\{z \in \mathbb{C} \mid |z| < \frac{k}{|\alpha_1|}\right\}$ by Proposition 9.11. By the Cauchy inequalities for holomorphic functions, the family of series in $(*)$ is locally uniformly bounded in $x \in I$ as

$$|a_\kappa(\lambda(n))| \leq \frac{\sup_{|z|=r}\left|\Phi(\frac{\lambda(n)}{n}, z)\right|}{r^\kappa}$$

holds for all $r < \frac{k}{|\alpha_1|}$.

(ii) Fix $r \in \mathbb{N}$. By part (i), the sequence $\varphi_n := J_{A_{n-1}}(i\lambda(n), \cdot)$ is uniformly bounded on compact subsets of $S^\omega_{r,k}$. Therefore, by Montel's theorem we can find a subsequence converging locally uniformly on $S^\omega_{r,k}$ to a holomorphic function φ. In particular, in a small neighborhood of 0 we can expand φ as follows

$$\varphi(z) = \sum_{\kappa \in \Lambda^r_+} a_\kappa \mathcal{P}_\kappa(z)$$

for some coefficients $a_\kappa \in \mathbb{C}$. By the uniform convergence, the coefficients of the φ_n with respect to the expansion in terms of the Jack polynomials converge to those of φ, that means

$$\lim_{n\to\infty} \frac{i^{|\kappa|} C_\kappa(\lambda(n))[kr]_\kappa}{[kn]_\kappa\,|\kappa|!} = a_\kappa.$$

But, as $[kn]_\kappa \sim (kn)^{|\kappa|}$ for $n \to \infty$, we obtain with Theorem 9.8

$$a_\kappa = \frac{i^{|\kappa|}\widetilde{C}_\kappa(\omega)[kr]_\kappa}{k^{|\kappa|}\,|\kappa|!}.$$

The Cauchy identity for Jack polynomials, see for instance [Sta89, Prop. 2.1], states for $\lambda \in \mathbb{C}^{(\infty)}$ and $z \in \mathbb{C}^r$ with $|z_j|$ small enough that

$$\sum_{\kappa \in \Lambda^r_+} \frac{C_\kappa(\frac{\lambda(n)}{n})[kr]_\kappa}{|\kappa|!}\mathcal{P}_\kappa(\tfrac{iz}{k}) = \prod_{j,\ell} \frac{1}{(1 - i\frac{\lambda(n)_\ell}{n}\frac{z_j}{k})^k} = \prod_{j=1}^{r} \Phi(\tfrac{\lambda(n)}{n}; \tfrac{iz_j}{k}) \xrightarrow[n\to\infty]{} \widehat{\Psi}(\omega, \tfrac{iz}{k}),$$

where the convergence is a consequence of Proposition 9.11. Therefore, as we have $\lim\limits_{n\to\infty} C_\kappa(\frac{\lambda(n)}{n}) = \widetilde{C}_\kappa(\omega)$, we conclude that

$$\varphi(z) = \widehat{\Psi}(\omega; \tfrac{iz}{k})$$

for all $z \in S^\omega_{r,k}$. Finally, using again Montel's theorem we have $\varphi_n(z) \to \varphi(z) = \widehat{\Psi}(\omega; \frac{iz}{k})$ locally uniformly in $z \in S^\omega_{r,k}$ for $n \to \infty$.

∎

Remark 9.13. If the VK-sequence $(\lambda(n))_{n\in\mathbb{N}}$ is non-negative, then (9.5) is true locally uniformly in the domain

$$S_k^\omega := \left\{ z \in \mathbb{C}^{(\infty)} \mid \|\operatorname{Im} z\|_\infty < \tfrac{k}{|\alpha_1|} \right\}.$$

Note that $\lambda(n) \geq 0$ implies that $C_\kappa(\lambda(n)) \geq 0$ due to the non-negative coefficients of C_κ in the monomial expansion. We observe that for $\kappa \in \Lambda_+^r$ we have

$$[kn]_\kappa \geq (k(n-r+1))^{|\kappa|}$$

and therefore with Proposition 9.5 we obtain similar to the proof of Theorem 9.12, but without rank one reduction

$$\begin{aligned}
\left|J_{A_{n-1}}(i\lambda(n), z)\right| &\leq \sum_{\kappa\in\Lambda_+^r} \frac{C_\kappa\left(\frac{\lambda(n))}{n-r+1}\right)[kr]_\kappa}{|\kappa|!} \left|\widetilde{P}_\kappa\left(\tfrac{\operatorname{Im} z}{k}\right)\right| \\
&\leq \sum_{\kappa\in\Lambda_+^r} \frac{C_\kappa\left(\frac{\lambda(n))}{n-r+1}\right)[kr]_\kappa}{|\kappa|!} \widetilde{P}_\kappa\left(\tfrac{(|\operatorname{Im} z_1|,\ldots,|\operatorname{Im} z_r|)}{k}\right) \\
&= \Phi\left(\frac{\lambda(n)}{n-r+1}, \tfrac{(|\operatorname{Im} z_1|,\ldots,|\operatorname{Im} z_r|)}{k}\right) \xrightarrow[n\to\infty]{} \widehat{\Psi}\left(\omega, \tfrac{(|\operatorname{Im} z_1|,\ldots,|\operatorname{Im} z_r|)}{k}\right).
\end{aligned}$$

locally uniformly in $z \in S_k^\omega \cap \mathbb{C}^r$. Proceeding as in the proof of Theorem 9.12 proves the assertion that (9.5) is true as locally uniform limit on S_k^ω.

Remark 9.14. The proof shows that for $z \in \mathbb{C}^r$ with $|z| < \frac{1}{|\alpha_1|}$,

$$\widehat{\Psi}(\omega; z) = \sum_{\kappa\in\Lambda_+^r} \frac{[kr]_\kappa}{|\kappa|!} \widetilde{C}_\kappa(\omega)\mathcal{P}_\kappa(z).$$

Lemma 9.15. *Consider a sequence $(\lambda(n))_{n\in\mathbb{N}}$ such that each $\lambda(n) \in \mathbb{R}^n$ is decreasing with respect to $\ll$. Suppose that the sequence of Bessel functions $J_{A_{n-1}}(i\lambda(n), \cdot)$ converges pointwise on $\mathbb{R}$ to a function which is continuous at 0. Then $(\lambda(n))_{n\in\mathbb{N}}$ is a VK-sequence.*

Proof. Put $\varphi_n(x) := J_{A_{n-1}}(i\lambda(n), x)$, $x \in \mathbb{R}$ and $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$. In view of representation (9.1), there exist compactly supported probability measures μ_n on $\mathbb{R}$ such that

$$\varphi_n(x) = \int_{\mathbb{R}} e^{ix\xi}\, \mathrm{d}\mu_n(\xi)$$

for all $x \in \mathbb{R}$. By Lévy's continuity theorem, there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n \to \mu$ weakly and for all $x \in \mathbb{R}$

$$\varphi(x) = \int_{\mathbb{R}} e^{ix\xi}\, \mathrm{d}\mu(\xi).$$

In particular, the family of measures $\{\mu_n \mid n \in \mathbb{N}\}$ is tight. Recall the functions $g_j(\lambda)$ from (9.10). By Proposition 9.5 and formula (9.11) we have

$$\varphi_n(x) = \sum_{j=0}^{\infty} \frac{C_{(j)}(\lambda(n)) \cdot (k)_j}{(kn)_j \cdot j!}(ix)^j = \sum_{j=0}^{\infty} \frac{g_j(\lambda(n))}{(kn)_j}(ix)^j\,.$$

Hence the moments of the measures μ_n are given by

$$\int_{\mathbb{R}} \xi^j\, \mathrm{d}\mu_n(\xi) = j!\, \frac{g_j(\lambda(n))}{(kn)_j}.$$

We now employ Lemma 5.2 of [OO98]. From the definition of the functions g_j one can find a constant $C > 0$ such that $g_4(\lambda) \leq C g_2(\lambda)^2$ for all $\lambda \in \mathbb{R}^{(\infty)}$, which shows that the quotient

$$\frac{\int_{\mathbb{R}} \xi^4 \, \mathrm{d}\mu_n(\xi)}{\left(\int_{\mathbb{R}} \xi^2 \, \mathrm{d}\mu_n(\xi)\right)^2}$$

is bounded as a function of $n \in \mathbb{N}$. Hence we conclude from Lemma 5.1. of [OO98] that the sequence $\left(\int_{\mathbb{R}} \xi^2 \, \mathrm{d}\mu_n(\xi)\right)_{n\in\mathbb{N}}$ is bounded, which in turn implies that the sequence $\left(\frac{g_2(\lambda(n))}{n^2}\right)_{n\in\mathbb{N}}$ is bounded. As $2g_2 = k^2 p_1^2 + k p_2$, the sequences

$$\left(\frac{|p_1(\lambda(n))|}{n}\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\frac{p_2(\lambda(n))}{n^2}\right)_{n\in\mathbb{N}} \tag{9.14}$$

are bounded as well. Standard compactness arguments and a diagonalization argument imply that $(\lambda(n))_{n\in\mathbb{N}}$ has a subsequence which is Vershik-Kerov. Finally, consider two such subsequences $(\lambda_\ell(n))_{n\in\mathbb{N}}$ with VK-parameters ω_ℓ, $l = 1, 2$. Then by Theorem 9.12 and our assumptions,

$$\varphi(x) = \lim_{n\to\infty} J_{A_{n-1}}(i\lambda_\ell(n), x) = \Psi\left(\omega_\ell; \frac{ix}{k}\right)$$

for all $x \in \mathbb{R}$. Hence $\Psi(\omega_1; \cdot) = \Psi(\omega_2; \cdot)$, and Proposition 9.9 implies that $\omega_1 = \omega_2$. It follows that the full sequence $(\lambda(n))_{n\in\mathbb{N}}$ is Vershik-Kerov. ∎

Putting things together, we are now able to finalize the proof of Theorem 9.6.

Proof of Theorem 9.6. The implication (i) ⇒ (ii) is contained in Theorem 9.12. The implications (iii),(ii)⇒ (i) are just Lemma 9.15, while (ii) ⇒(iii) is obvious. Further, Theorem 9.8 proves the implication (i)⇒ (v). The equivalence of statements (iv) and (v) is obvious from the expansion of Proposition 9.5, because the Jack polynomials span the algebra of symmetric functions. It thus remains to prove the implication (v)⇒ (i). For this, suppose that $(\lambda(n))_{n\in\mathbb{N}}$ is a sequence with each $\lambda(n) \in \mathbb{R}^n$ decreasing with respect to $\ll$, and such that $\lim_{n\to\infty} \frac{f(\lambda(n))}{n^{\deg f}}$ exists for all symmetric functions f. Then in particular, the sequences $\left(\frac{p_1(\lambda(n))}{n}\right)$ and $\left(\frac{p_2(\lambda(n))}{n^2}\right)$ are bounded. Again by a compactness argument, $(\lambda(n))$ has a subsequence which is Vershik-Kerov. Suppose $(\lambda_\ell(n))$, $l = 1, 2$ are two such subsequences with VK-parameters ω_ℓ. Then by Theorem 9.12, the sequences $(J_{A_{n-1}}(i\lambda_1(n), \,.\,))$ and $(J_{A_{n-1}}(i\lambda_2(n), \,.\,))$ converge locally uniformly on $\mathbb{R}^r$ to the same limit, because for each $\kappa \in \Lambda_+^r$, the limit

$$\lim_{n\to\infty} \frac{C_\kappa(\lambda_\ell(n))}{n^{|\kappa|}} = \lim_{n\to\infty} \frac{C_\kappa(\lambda(n))}{n^{|\kappa|}}$$

is independent of l. Arguing further as in the proof of Lemma 9.15, we obtain that $\omega_1 = \omega_2$ and that $(\lambda(n))$ is a VK-sequence.This finishes the proof of the theorem. ∎

We shall now parametrize the possible limit functions in Theorem 9.12. We put

$$\Omega := \left\{(\alpha, \beta, \gamma) \,\middle|\, \beta \in \mathbb{R}, \gamma \geq 0, \alpha = (\alpha_i)_{i\in\mathbb{N}} \text{ with } \alpha_i \in \mathbb{R}, \alpha_{i+1} \ll \alpha_i \,, \textstyle\sum_{i=1}^\infty \alpha_i^2 < \infty\right\}.$$

Note that for $(\alpha, \beta, \gamma) \in \Omega$, either all entries of α are non-zero, or all entries up to finitely many are zero.

Proposition 9.16. *For any element $\omega = (\alpha, \beta, \gamma) \in \Omega$ there exists a VK-sequence $(\lambda(n))_{n\in\mathbb{N}}$ with VK-parameters ω.*

Proof. We divide the proof into several steps.

(i) Assume that $\alpha = 0$. Then for arbitrary $\epsilon > 0$, there exists a sequence $x = (x_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$|x_i| \le \epsilon \quad \text{for all } i \in \mathbb{N}, \ \sum_{i=1}^{\infty} x_i = \beta \quad \text{and} \quad \sum_{i=1}^{\infty} x_i^2 = \gamma. \tag{9.15}$$

To see this, choose $N \in \mathbb{N}$ such that $\left(\frac{6\gamma}{\pi^2 N}\right)^{1/2} \le \epsilon$ and start with the alternating sequence

$$x_i' := \left(\frac{6\gamma}{\pi^2 N}\right)^{1/2} \cdot \frac{(-1)^i}{k+1} \qquad \text{if } k < \frac{i}{N} \le k+1, \, k \in \mathbb{N}_0.$$

It satisfies the first and the third condition of (9.15), and by the Riemann rearrangement theorem, there exists a rearrangement $(x_i)_{i\in\mathbb{N}}$ of $(x_i')_{i\in\mathbb{N}}$ satisfying the second condition as well. For each $m \in \mathbb{N}$ we can therefore find a real sequence $x^{(m)} = (x_i^{(m)})$ and an index $n_m \in \mathbb{N}$ with $n_m \to \infty$ for $m \to \infty$, such that for all $n \ge n_m$,

$$|x_i^{(m)}| \le \frac{1}{m} \quad \text{for all } i \in \mathbb{N}, \quad |\sum_{i=1}^{n} x_i^{(m)} - \beta| \le \frac{1}{m}, \quad |\sum_{i=1}^{n} (x_i^{(m)})^2 - \gamma| \le \frac{1}{m}.$$

We may also assume that $n_{m+1} > n_m$ for all m. Rearranging the entries of each tuple $(x_1^{(m)}, \ldots, x_{n_m}^{(m)})$ according to $\ll$, we thus obtain a sequence $(\lambda(n_m)')_{m\in\mathbb{N}}$ where each $\lambda(n_m)' \in \mathbb{R}^{n_m}$ is decreasing with respect to $\ll$ and satisfies

$$\lim_{m\to\infty} \lambda(n_m)_i' = 0 \quad \text{for all } i \in \mathbb{N},$$
$$\lim_{m\to\infty} \sum_{i=1}^{n_m} \lambda(n_m)_i' = \beta,$$
$$\lim_{m\to\infty} \sum_{i=1}^{n_m} (\lambda(n_m)_i')^2 = \gamma.$$

Finally, put $\lambda(n_m) := n_m \lambda(n_m)'$ and $\lambda(n) := (n\lambda(n_m)', 0, \ldots, 0) \in \mathbb{R}^n$ for $n_m < n < n_{m+1}$. Then $(\lambda(n))_{n\ge n_1}$ is a VK-sequence with parameters $(\alpha = 0, \beta, \gamma)$.

(ii) Assume that α has finitely many non-zero entries and let $m \in \mathbb{N}$ be maximal such that $\alpha_m \neq 0$. Let $(\lambda(n)')_{n\in\mathbb{N}}$ be a VK-sequence with parameters $(0, \beta', \gamma)$, where $\beta' = \beta - \sum_{i=1}^{m} \alpha_i$. For $n > m$, put

$$\lambda(n) := (n\alpha_1, \ldots, n\alpha_m, \lambda(n)_1', \ldots, \lambda(n)_{n-m}').$$

For n large enough, say $n \ge n_0$, the entries of $\lambda(n)$ are decreasing with respect to $\ll$, because $\lim_{n\to\infty} \frac{\lambda(n)_i'}{n} = 0$. Then $(\lambda(n))_{n\ge n_0}$ is Vershik-Kerov with parameters (α, β, γ).

(iii) Assume that all entries of α are non-zero. For $m \in \mathbb{N}$, put $\omega^{(m)} := (\alpha^{(m)}, \beta, \gamma)$, where $\alpha^{(m)} = (\alpha_1, \ldots, \alpha_m, 0, \ldots)$. According to part (ii), there exists a VK-sequence $(\lambda^{(m)}(n))_{n\in\mathbb{N}}$ with VK-parameters $\omega^{(m)}$. By a diagonalization argument we obtain a sequence $\lambda(n_m) := \lambda^{(m)}(n_m)$ with $n_{m+1} > n_m$ satisfying

$$\lim_{m\to\infty} \lambda(n_m)_i = \alpha_i \quad \text{for all } i \in \mathbb{N},$$
$$\lim_{m\to\infty} \sum_{i=1}^{n_m} \lambda(n_m)_i = \beta,$$
$$\lim_{m\to\infty} \sum_{i=1}^{n_m} (\lambda(n_m)_i)^2 = \delta = \gamma + \sum_{i=1}^{\infty} \alpha_i^2.$$

Finally, for $n \in \mathbb{N}$ with $n_m \le n < n_{m+1}$ put $\lambda(n) := (\frac{n}{n_m}\lambda(n_m), 0, \ldots, 0) \in \mathbb{R}^n$. Then $(\lambda(n))_{n\ge n_1}$ is Vershik-Kerov with VK-parameters (α, β, γ).

■

Together with Lemma 9.9, this result shows that the possible limits (for $n \to \infty$) of the Bessel functions $J_{A_{n-1}}(i\lambda(n), x)$ with $x \in \mathbb{R}^r$ and $\lambda(n) \in \mathbb{R}^n$ are exactly all the infinite products $\widehat{\Psi}(\omega; \frac{ix}{k})$, of Theorem 9.12, which are in bijective correspondence with the parameters $\omega \in \Omega$.

Let us finally come back to the Olshanski spherical pair (G_∞, K_∞) as in (9.3). From our results, we obtain the following result of Pickrell [Pic91] mentioned at the beginning of this section.

Corollary 9.17. *The set of positive definite spherical functions of the Olshanski spherical pair* $(G_\infty, K_\infty) = (U_\infty(\mathbb{F}) \ltimes \mathrm{Herm}_\infty(\mathbb{F}), U_\infty(\mathbb{F}))$*, considered as* $U_\infty(\mathbb{F})$*-invariant functions on* $\mathrm{Herm}_\infty(\mathbb{F})$*, is uniquely parametrized by the set* Ω *via*

$$\varphi_\omega(X) = \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{d}x_j^2} \prod_{\ell=1}^{\infty} \frac{e^{-i\alpha_\ell x_j}}{(1 - i\frac{2}{d}\alpha_\ell x_j)^{d/2}}, \quad \omega = (\alpha, \beta, \gamma) \in \Omega,$$

where $x = (x_1, x_2, \ldots)$ *are the eigenvalues of* X*.*
Moreover, a sequence of positive definite spherical functions $(\varphi_{i\lambda(n)})_{n\in\mathbb{N}}$*,* $\lambda(n) \in \mathbb{R}^n$*, of the Gelfand pairs* $(G_n, K_n) = (U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))$ *from Theorem 9.1 converge locally uniformly on* (G_∞, K_∞) *if and only if* $(\lambda(n))_{n\in\mathbb{N}}$ *is (up to permutations) a VK-sequence. And if* ω *are the VK-parameters, then* $\lim_{n\to\infty} \varphi_{\lambda(n)} = \varphi_\omega$*.*

Proof. Let $\varphi : G_\infty \to \mathbb{C}$ be a positive definite spherical functions of (G_∞, K_∞). By Theorem 9.1 and Theorem 8.8 there exists a sequence of positive definite spherical functions $\varphi_n := \varphi_{\lambda(n)} : G_n \to \mathbb{C}$ of (G_n, K_n), $n \in \mathbb{N}$ and $\lambda(n) \in \mathbb{R}^n$, which converges locally uniformly to φ. By Theorem 9.1 any such φ_n is given by a positive Bessel function on $\mathbb{R}^n$ associated with $(A_{n-1}, \dim_{\mathbb{R}} \mathbb{F}/2)$

$$\varphi_n(X) = J_{A_{n-1}}(i\lambda(n), x),$$

where $x = (x_1, \ldots, x_n)$ are the eigenvalues of X. Without loss of generality assume that $\lambda(n)$ is decreasing with respect to $\ll$. By Theorem 9.6 $(\lambda(n))_{n\in\mathbb{N}}$ has to be a VK-sequence and if ω are the VK-parameters of the sequence, then we conclude that $\varphi = \varphi_\omega$.
Conversely, starting with an arbitrary $\omega \in \Omega$ we choose corresponding VK-sequence $(\lambda(n))_{n\in\mathbb{N}}$ by Lemma 9.16. Then, $\varphi_n(X) = J_k^A(i\lambda(n), x)$ are positive definite spherical functions of (G_n, K_n) converging locally uniformly to φ_ω. By Theorem 8.8, φ_ω is a positive definite spherical functions of (G_∞, K_∞). The injectivity of the map $\omega \mapsto \varphi_\omega$ is a consequence Lemma 9.9. ■

9.3 The type B case

Consider the action $U_p(\mathbb{F}) \times U_q(\mathbb{F})$ on $M_{p,q}(\mathbb{F})$, the space of $p \times q$ matrices with entries in $\mathbb{F}$, defined by $(U, V).M = UMV^{-1}$. The following theorem justifies why we consider type B Bessel functions if the rank tends to infinity.

Theorem 9.18. *Let* $p, q \in \mathbb{N}$ *with* $p \geq q$*. The spherical functions of the Gelfand pairs*

$$(G_p, K_q) = ((U_p(\mathbb{F}) \times U_q(\mathbb{F})) \ltimes M_{p,q}(\mathbb{F}), U_p(\mathbb{F}) \times U_q(\mathbb{F}))$$

are given as $(U_p(\mathbb{F}) \times U_q(\mathbb{F}))$*-invariant functions on* $M_{p,q}(\mathbb{F})$ *by*

$$\varphi_\lambda(X) = J^{B_q}_{(k',k)}(\lambda, \operatorname{sing} X), \quad \lambda \in \mathbb{R}^q$$

where $\operatorname{sing} X = \operatorname{spec}(\sqrt{X^*X}) \in \mathbb{R}^q$ *are the singular values of* X *in an arbitrary order and the multiplicity is* $k' = k(p-q+1) - \frac{1}{2}$ *on* $\pm e_i$ *and* $k = \frac{\dim_{\mathbb{R}} \mathbb{F}}{2}$ *on* $\pm e_i \pm e_j$.

PROOF. We refer the reader to [Rö07]. It could also be verified by hand with Theorem 4.27, as the pair (G_p, K_q) is the Cartan motion group associated the semisimple indefinite group $G = U(p, q, \mathbb{F}) \in \mathcal{H}$ of signature (p, q) and $K = U_p(\mathbb{F}) \times U_q(\mathbb{F})$, embedded into G via $(U, V) \mapsto \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. ■

The embedding

$$M_{p,q}(\mathbb{F}) \hookrightarrow M_{p+1,q+1}(\mathbb{F}), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

of $p \times q$ matrices into $(p+1) \times (q+1)$ matrices induces for any sequences $p_n \geq q_n \xrightarrow[n\to\infty]{} \infty$ an Olshanski spherical pair

$$(G_\infty, K_\infty) = \lim_{n\to\infty} (G_n, K_n) = \lim_{n\to\infty} ((U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F})) \ltimes M_{p_n,q_n}(\mathbb{F}), U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F})). \quad (9.16)$$

which is independent among the possible sequences $p_n \geq q_n \xrightarrow[n\to\infty]{} \infty$ by Proposition 8.9. For simplicity we consider $q_n = n$ in the following.
Because of Theorem 9.18 we will consider first the type B Bessel functions. As $n \to \infty$, we shall consider them with the multiplicities $\kappa_n := (k'_n, k)$ with value $k > 0$ on the roots $\pm(e_i \pm e_j)$ and $k'_n \geq 0$ on the roots $\pm e_i$. From Theorem 9.18 its is clear why the multiplicity parameter k_n is allowed to vary with n. With $\nu_n := k'_n + k(n-1) + \frac{1}{2}$ we have

$$J_{B_n}(\kappa_n; \lambda, z) = \sum_{\kappa \in \Lambda_n^+} \frac{1}{4^{|\kappa|} [\nu_n]_\kappa} \frac{C_\kappa(\lambda^2) C_\kappa(z^2)}{|\kappa|!\, C_\kappa(\underline{1}_n)}, \quad (9.17)$$

where the Jack polynomials are of index $1/k$ by Proposition 7.8. Recall the stability property (9.2) of the Jack polynomials. Adopting the notation from (9.4), we therefore have for $\lambda \in \mathbb{C}^n$ and $z \in \mathbb{C}^r$ with $r \leq n$ the representation

$$J_{B_n}(\kappa_n; \lambda, z) := J_{B_n}(\kappa_n; \lambda, (z, \underline{0}_{n-r}) = \sum_{\kappa \in \Lambda_r^+} \frac{C_\kappa(\lambda^2)[kr]_\kappa}{4^{|\kappa|}[kn]_\kappa [\nu_n]_\kappa |\kappa|!} \mathcal{P}_\kappa(z^2). \quad (9.18)$$

Note that for $\lambda \in \mathbb{R}^n$, $J_{B_n}(\kappa_n; \lambda, \cdot)$ is positive definite on $\mathbb{R}^r$ for each $r \leq n$.

Remark 9.19. We note that the restriction onto non-negative multiplicity κ_n is essential as in this case we can characterize by the integral representation (9.1) the spectral parameters for which the Bessel function is positive definite. So we cannot omit the restriction $k'_n \geq 0$, i.e. $\nu_n \geq k(n-1) + \frac{1}{2}$. But, equation (9.17) is still true on a larger set of multiplicities. As the right hand side of (9.17) is holomorphic in the parameter ν_n, and so in k'_n (cf. Theorem 6.20), the equation is true for all regular multiplicities $\kappa_n = (k'_n, k)$ with $k \geq 0$. Note that the Bessel function is holomorphic in κ_n on the set of regular multiplicities, which can be characterized as exactly the pole set of the Bessel function, cf. [Opd93, Proposition 9.6]. Hence, (9.17) is true for all $\kappa_n = (k'_n, k)$ such that $k \geq 0$ and k'_n is not of the form $-kj - \frac{1}{2} - m$, $j \in \{0, \ldots, n-1\}, m \in \mathbb{N}$, i.e. $\nu_n \notin \{0, k, \ldots, k(n-1)\} - \mathbb{N}_0$, which means $[\nu_n]_\kappa \neq 0$ for all $\kappa \in \Lambda_n^+$.

The following counterpart of Theorem 9.6 will be the main result of this section.

Theorem 9.20. *Let* $(\lambda(n))_{n\in\mathbb{N}}$ *be a sequence of spectral parameters* $\lambda(n) \in \mathbb{R}^n$ *non-negative and decreasing. Then the following statements are equivalent.*

(i) $\left(\frac{\lambda(n)^2}{\nu_n}\right)_{n\in\mathbb{N}}$ *is a Vershik-Kerov sequence.*

(ii) *The sequence of Bessel functions* $\big(J_{B_n}(\kappa_n; i\lambda(n), \,.\,)\big)_{n\in\mathbb{N}}$ *converges locally uniformly on each of the spaces* $\mathbb{R}^r$, $r \in \mathbb{N}$.

(iii) *The sequence of Bessel functions* $\big(J_{B_n}(\kappa_n; i\lambda(n), \,.\,)\big)_{n\in\mathbb{N}}$ *converges pointwise on* $\mathbb{R}$ *against a function continuous in* 0.

(iv) *For each fixed multi-index of length* r*, the corresponding coefficients in the Taylor of expansion of* $J_{B_n}(\kappa_n; i\lambda(n), \,.\,)$ *around* $0 \in \mathbb{R}^r$ *converge as* $n \to \infty$.

(v) *For all symmetric functions* $f : \mathbb{R}^{(\infty)} \to \mathbb{C}$*, the limit*

$$\lim_{n\to\infty} \frac{f(\lambda(n)^2)}{(n\nu_n)^{degf}}$$

exists.

In this case, let $\omega = (\alpha, \beta, \gamma)$ *be the VK-parameters of the sequence* $\left(\frac{\lambda(n)^2}{\nu_n}\right)$. *Then* $\gamma = 0$, $\alpha_\ell \geq 0$ *for all* l*, and*

$$\lim_{n\to\infty} J_{B_n}(\kappa_n; i\lambda(n), z) \;=\; \widehat{\Psi}\Big(\omega; -\frac{z^2}{4k}\Big) \;=\; \prod_{j=1}^{\infty} e^{-\frac{\beta z_j^2}{4}} \prod_{\ell=1}^{\infty} \frac{e^{\frac{\alpha_\ell z_j^2}{4}}}{\big(1+\frac{\alpha_\ell z_j^2}{4k}\big)^k},$$

locally uniformly in $z \in \Big\{z \in \mathbb{C}^{(\infty)} \mid \|\mathrm{Im}\, z\| < 2\sqrt{\frac{k}{\alpha_1}}\Big\}$.

Remark 9.21.

(i) It is a consequence of Lemma 9.3 that the VK-parameter γ is 0 in the present situation.

(ii) We do not have any restriction to the asymptotic behavior of ν_n (or k_n'), which grows by the restriction $k_n' \geq 0$ at least linearly. Only the characterization of the spectral parameters, for which the set of Bessel function converges, depends on ν_n. However, the set of limits we are able to obtain by such a limit transition is independent of the asymptotic behavior of ν_n.

(iii) Assume that there exists $C \geq 0$ with $\lim_{n\to\infty} \frac{k_n'}{Cn} = 1$ (with the convention $\frac{0}{0} = 1$ here). Then $\nu_n \sim (C+k)n$ for $n \to \infty$. In particular, $\left(\frac{\lambda(n)^2}{\nu_n}\right)_{n\in\mathbb{N}}$ is Vershik-Kerov with VK parameters $(\alpha, \beta, 0)$ if and only if $\left(\frac{\lambda(n)^2}{n}\right)_{n\in\mathbb{N}}$ is Vershik-Kerov with VK parameters $\big((C+k)\alpha, (C+k)\beta, 0\big)$.

For the proof of Theorem 9.20, we start with the following.

Lemma 9.22. *Let* $(\lambda(n))_{n\in\mathbb{N}}$ *with* $\lambda(n) \in \mathbb{R}^n$ *non-negative and decreasing such that* $\left(\frac{\lambda(n)^2}{\nu_n}\right)$ *is a Vershik-Kerov sequence with VK-parameters* $\omega = (\alpha, \beta, 0)$. *Then for*

$$z \in S_{k,+}^{\omega} := \left\{ z \in \mathbb{C}^{(\infty)} \,\middle|\, \|\mathrm{Im}\, z\|_\infty < 2\sqrt{\frac{k}{\alpha_1}} \right\}$$

we have

$$\lim_{n\to\infty} J_{B_n}(\kappa_n; i\lambda(n), z) \;=\; \widehat{\Psi}\Big(\omega; -\frac{z^2}{4k}\Big).$$

The convergence is uniform on compact subsets of $S_{k,+}^{\omega}$.

PROOF. For $r \in \mathbb{N}$ and $z \in S^{\omega}_{k,+} \cap \mathbb{C}^r$, consider

$$\varphi_n(z) := J_{B_n}(\kappa_n; i\lambda(n), z) = \sum_{\kappa \in \Lambda_r^+} \frac{C_\kappa(\lambda(n)^2)[kr]_\kappa}{[kn]_\kappa[\nu_n]_\kappa|\kappa|!4^{|\kappa|}} \mathcal{P}_\kappa(-z^2). \tag{9.19}$$

As in Remark 9.13 we observe with $\lambda(n) \geq 0$ that

$$|\varphi_n(z)| \leq \sum_{\kappa \in \Lambda_r^+} \frac{C_\kappa\left(\frac{\lambda(n)^2}{(n-r+1)(\nu_n-r+1)}\right)[kr]_\kappa}{|\kappa|!} \mathcal{P}_\kappa\left(\frac{(\operatorname{Im} z_j)^2}{4k}\right) = \prod_{j=1}^r \Phi\left(\frac{\lambda(n)^2}{(n-r+1)(\nu_n-r+1)}, \frac{(\operatorname{Im} z)^2}{4k}\right),$$

where the upper bound converges locally uniformly on $S^{\omega}_{k,+}$ to $\widehat{\Psi}(\omega, \frac{(\operatorname{Im} z)^2}{4k})$. Therefore, $(\varphi_n)_{n\in\mathbb{N}}$ is a family of holomorphic functions that are uniformly bounded on compact subsets of $S^{\omega}_{k,+}$. Arguing with a Montel argument as in Theorem 9.12 shows that on $S^{\omega}_{k,+}$ we have

$$\lim_{n\to\infty} \varphi_n(z) = \widehat{\Psi}(\omega, -\tfrac{z^2}{4k})$$

locally uniformly in $z \in S^{\omega}_{k,+}$, as with $[kn]_\kappa[\nu_n]_\kappa \sim (kn\nu_n)^{|k|}$ for $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{C_\kappa(\lambda(n)^2)}{[kn]_\kappa[\nu_n]_\kappa} = \frac{\widetilde{C_\kappa}(\omega)}{k^{|\kappa|}}.$$

■

Lemma 9.23. *Consider a sequence $(\lambda(n))_{n\in\mathbb{N}}$ with $\lambda(n) \in \mathbb{R}^n$ non-negative and decreasing. Assume that the sequence of Bessel functions $J_{B_n}(i\lambda(n), \cdot)$ converges pointwise on $\mathbb{R}$ to a function which is continuous at 0. Then the sequence $(\frac{\lambda(n)^2}{\nu_n})$ is Vershik-Kerov.*

PROOF. The proof is similar to that of Lemma 9.15. For $x \in \mathbb{R}$, put

$$\varphi_n(x) := J_{B_n}(\kappa_n; i\lambda(n), x) = \int_{\mathbb{R}} e^{ix\xi} \, \mathrm{d}\mu_n(\xi)$$

with certain compactly supported probability measures μ_n on $\mathbb{R}$. By the symmetry properties of J_{B_n}, the measure μ_n is even, hence its odd moments vanish. Let further $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$. Again by Lévy's continuity theorem, there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n \to \mu$ weakly and

$$\varphi(x) = \int_{\mathbb{R}} e^{ix\xi} \, \mathrm{d}\mu(\xi) \quad \text{for all } x \in \mathbb{R}.$$

Further, the family $\{\mu_n : n \in \mathbb{N}\}$ is tight. From (9.18) and formula (9.11) we deduce that

$$\varphi_n(x) = \sum_{j=0}^{\infty} \frac{g_j(\lambda(n)^2)}{4^j(\nu_n)_j(kn)_j} (-x)^{2j}.$$

This shows that the even moments of μ_n are given by

$$\int_{\mathbb{R}} \xi^{2j} \, \mathrm{d}\mu_n(\xi) = (2j)! \, \frac{g_j(\lambda(n)^2)}{4^j(kn)_j(\nu_n)_j}.$$

As in the proof of Lemma 9.15, we deduce that the quotient

$$\frac{\int_{\mathbb{R}} \xi^8 \, \mathrm{d}\mu_n(\xi)}{\left(\int_{\mathbb{R}} \xi^4 \, \mathrm{d}\mu_n(\xi)\right)^2}$$

is bounded in $n \in \mathbb{N}$. Now we conclude from [OO98, Lemma 5.2] (employing the Lemma for the image measure of μ_n under $\xi \mapsto \xi^2$) that the sequence $\left(\int_{\mathbb{R}} \xi^4 \, \mathrm{d}\mu_n(\xi)\right)$ is bounded. As $(\nu_n)_2 \sim \nu_n^2$ and $(kn)_2 \sim (kn)^2$ for $n \to \infty$, it follows that the sequence

$$\Big(g_2\Big(\frac{\lambda(n)^2}{n\nu_n}\Big)\Big)_{n\in\mathbb{N}}$$

is bounded as well. Continuing as in the proof of Lemma 9.15 we obtain that $(\frac{\lambda(n)^2}{\nu_n})$ is a Vershik-Kerov sequence. ■

PROOF OF THEOREM 9.20. From Theorem 9.6 it is clear that the statements (i) and (v) are equivalent. The equivalence of (iv) and (v) follows from expansion (9.19) and the fact that the Jack polynomials span the algebra of symmetric functions. By Lemma 9.23, statement (iii) implies (i). Finally, Lemma 9.22 shows that statement (i) implies statement (ii). ■

We finally want to determine the set of all parameters $\omega = (\alpha, \beta, 0)$ which occur as VK-parameters of a non-negative Vershik-Kerov sequence as in Theorem 9.20. Recall that in the non-negative case, the parameter γ is automatically zero by Lemma 9.3.

Proposition 9.24. *The set Ω_+ of all pairs (α, β) for which there exists a non-negative VK-sequence with parameters $(\alpha, \beta, 0)$ is given by*

$$\Omega_+ = \Big\{(\alpha, \beta) \mid \beta \geq 0,\ \alpha = (\alpha_i)_{i\in\mathbb{N}} \text{ with } \alpha_i \in \mathbb{R},\ \alpha_1 \geq \alpha_2 \geq \ldots \geq 0,\ \textstyle\sum_{i=1}^{\infty} \alpha_i \leq \beta\Big\}.$$

PROOF.

(i) If $(\alpha, \beta, 0)$ are the VK-parameters of a VK-sequence $(\lambda(n))$ with $\lambda(n)_i \geq 0$ for all i, then obviously $\beta \geq 0$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Moreover, for fixed $N \in \mathbb{N}$ and $n \geq N$ we have

$$\sum_{i=1}^{N} \alpha_i \leq \sum_{i=1}^{N}\Big(\alpha_i - \frac{\lambda(n)_i}{n}\Big) + \sum_{i=1}^{n} \frac{\lambda(n)_i}{n}.$$

As $n \to \infty$, the first sum tends to 0 and the second sum tends to β. This proves that $\sum_{i=1}^{\infty} \alpha_i \leq \beta$.

(ii) Conversely, let $(\alpha, \beta) \in \Omega_+$. In order to construct an associated non-negative VK-sequence, we proceed in two steps.

(a) Assume that α has at most finitely many non-zero entries. If $\alpha \neq 0$, let $m \in \mathbb{N}$ be maximal such that $\alpha_i \neq 0$ for $i \leq m$. If $\alpha = 0$, let $m := 0$. Put

$$\beta' := \beta - \sum_{i=1}^{m} \alpha_i \geq 0.$$

For $n > m$, define $\lambda(n) \in \mathbb{R}^n$ by

$$\lambda(n)_i := \begin{cases} n\alpha_i & \text{if } i \leq m \\ \frac{n\beta'}{n-m} & \text{if } m < i \leq n. \end{cases}$$

Note that the entries of $\lambda(n)$ are non-negative and decreasing for n large enough, say $n \geq n_0$. It is now straightforward to verify that $(\lambda(n))_{n\geq n_0}$ is a VK-sequence with parameters $(\alpha, \beta, 0)$.

(b) Assume that all entries of α are strictly positive. Then a diagonalization argument as in the proof of Proposition 9.16 shows that there exists a VK-sequence with parameters $(\alpha, \beta, 0)$.

■

Corollary 9.25. *The set of positive definite spherical functions of the Olshanski spherical pair (G_∞, K_∞), considered as functions on $U_\infty(\mathbb{F}) \times U_\infty(\mathbb{F})$-invariant functions on $M_{\infty,\infty}(\mathbb{F})$ is given by the functions*

$$\varphi_{(\alpha,\beta)}(X) = \prod_{j=1}^{\infty} e^{-\frac{\beta}{4}x_j^2} \prod_{\ell=1}^{\infty} \frac{e^{\frac{\alpha_\ell}{4}x_j^2}}{(1+\frac{\alpha_\ell}{2d}x_j^2)^{d/2}}, \quad (\alpha,\beta) \in \Omega_+,$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$ and $x = (x_1, x_2, \ldots)$ are the singular values of X.
Let (p_n, n) be an increasing sequence with $p_n \geq n$. Then, a sequence of positive definite spherical functions $(\varphi_{i\lambda(n)})_{n\in\mathbb{N}}$, $\lambda(n) \in \mathbb{R}^n$, of the Gelfand pairs $(G_n, K_n) = ((U_{p_n}(\mathbb{F}) \times U_n(\mathbb{F})) \ltimes M_{p_n,n}(\mathbb{F}), U_{p_n}(\mathbb{F}) \times U_n(\mathbb{F}))$ from Theorem 9.18 converge locally uniformly on (G_∞, K_∞) if and only if $(\frac{\lambda(n)^2}{kp_n})_{n\in\mathbb{N}}$ is (up to permutations) a VK-sequence. And if $\omega = (\alpha, \beta, 0)$ are the VK-parameters, then $\lim_{n\to\infty} \varphi_{i\lambda(n)} = \varphi_{(\alpha,\beta)}$.

PROOF. The proof is the same as that of Corollary 9.17 and uses Theorem 9.18 and Theorem 9.20. ■

Remark 9.26. We mention that for $\mathbb{F} = \mathbb{C}$, the first part of this corollary is in accordance with results of [Bou19], where for the semigroup $\mathrm{Herm}_\infty^+(\mathbb{C})$ of infinite dimensional positive definite matrices over $\mathbb{C}$, the positive definite Olshanski spherical functions of $(U_\infty(\mathbb{C}) \ltimes \mathrm{Herm}_\infty^+(\mathbb{C}), U_\infty(\mathbb{C}))$ were determined by semigroup methods and a reduction to the type A case.

CHAPTER 10

Limit transition between root systems of type BC and A

In [RKV13] the authors described a limit transition from the type BC hypergeometric function to the type A hypergeometric function. The limit is taken by the condition that certain BC-multiplicities tend to infinity. The proof is done first for the symmetric Heckman-Opdam polynomials and is then extended to the hypergeometric functions using Montel's theorem and analytic continuation via Carlson's theorem. The polynomial case is done by considering the defining eigenvalue equation of the Heckman-Opdam polynomials. The aim of this chapter is to generalize the results from [RKV13] to the non-symmetric setting. The non-symmetric polynomials setting has a big effort: the Heckman-Opdam polynomials can be constructed recursively by recurrence relations going back to Sahi in the papers [Sah00a, Sah00b] and earlier by Opdam in [Opd95].
The chapter is organized as follows. In the first section, we introduce the reader to the recurrence formulas for Heckman-Opdam polynomials by Sahi. Afterwards, in Section 2, we use these recurrence formulas to prove a limit transition between non-symmetric Heckman-Opdam polynomials of type BC_n and A_{n-1}, the latter can be identified with Jack polynomials. Finally, by analytic continuation, we extend this limit transition to the Cherednik kernels of type BC_n and A_{n-1} in Section 3.

10.1 Sahi's recurrence formulas

We introduce the reader to the recurrence relations of the non-symmetric Heckman-Opdam polynomials proven in [Sah00a]. They were verified under the assumption that the root system is reduced, but the results for the Heckman-Opdam polynomials in [Sah00a] remain true for non-reduced root system without any bigger change. The BC setting was also done in the paper [Sah00b] in the more general BC-Koornwinder setting. In the Appendix A we have verified the recurrence formulas of [Sah00b] for non-reduced root systems in the language of the paper [Sah00a].
Let $R \subseteq \mathbb{R}^n$ be an irreducible crystallographic root system with Weyl group W, weight lattice P, positive roots $R_+ \subseteq R$ with simple roots $\alpha_1, \dots, \alpha_n$ and corresponding simple reflections $s_1, \dots, s_n$. Let $\beta \in R$ be the unique highest short root and define the corresponding affine reflection

$$s_0 := \beta + s_\beta,$$

which is the reflection in the hyperplane $\{x \in \mathbb{R}^n \mid \langle \beta^\vee, x\rangle = 1\}$. The *dual affine Weyl group* is defined by

$$W^{\vee,\text{aff}} := \langle s_0, \dots, s_n\rangle_{\text{group}} = W \ltimes Q$$

and acts transitively on the weight lattice P. Therefore, the action of $W^{\vee,\text{aff}}$ on P induces an action on the trigonometric polynomials $\mathcal{T} := \text{span}_{\mathbb{C}}\{e^\mu \mid \mu \in P\}$. Note, that this action on $\mathcal{T}$ coincides on $W \subseteq W^{\vee,\text{aff}}$ with the usual action $wf(x) = f(w^{-1}x)$ on functions $f : \mathfrak{a} \to \mathbb{C}$. But the actions are not equal on the whole dual affine Weyl group $W^{\vee,\text{aff}}$. Indeed, s_0 acts on trigonometric polynomials f via $s_0 f(x) = e^{\langle \beta, x\rangle} f(s_\beta x)$.
The key tool in this chapter is the following theorem.

Theorem 10.1 ([Sah00a, Sah00b] or Appendix A). *Let $k = (k_\alpha)_{\alpha\in R} \geq 0$ be a non-negative multiplicity function and $(E_\mu(k; \cdot))_{\mu\in P}$ the non-symmetric Heckman-Opdam polynomials asso-*

ciated with (R_+, k). Recall from equation (4.2) the eigenvalue vector $\widetilde{\mu}$ of $E_\mu(k;\cdot)$ under the Cherednik operators. Then the following statements hold:

(i) *For all $\mu \in P$ with $s_i\mu \neq \mu$ we have $\widetilde{s_i\mu} \neq \widetilde{\mu}$ and there exists $d_i(k;\mu) \in \mathbb{R}$ with*

$$d_i(k;\mu)E_{s_i\mu}(k;\cdot) = (s_i + c_i(k;\mu))E_\mu(k;\cdot),$$

where

$$c_i(k;\mu) = \begin{cases} \dfrac{k_\beta}{1-\langle \beta^\vee, \widetilde{\mu}\rangle}, & i = 0,\\ \dfrac{k_{\alpha_i} + 2k_{2\alpha_i}}{\langle \alpha_i^\vee, \widetilde{\mu}\rangle}, & i = 1, \dots, n, \end{cases}$$

with $k_{2\alpha} = 0$ if $2\alpha \notin R$.

(ii) *Let $\mathcal{O} := \{\mu \in P \mid \langle \alpha^\vee, \mu\rangle \in \{0,1\}$ for all $\alpha \in R_+\}$ be the minuscle weights. Then for all $\mu \in P$ there exists a unique element $w_\mu = s_{i_1}\cdots s_{i_m} \in W^{\vee,aff}$ (reduced expression) of minimal length with $\overline{\mu} := w_\mu\mu \in \mathcal{O}$. Then for $\mu_{(j)} = s_{i_{j-1}}\cdots s_{i_1}\overline{\mu}$ we have*

$$E_\mu(k;\cdot) = \begin{cases} e^\mu, & \mu \in \mathcal{O},\\ (s_{i_m} + c_{i_m}(k;\mu_{(m)}))\cdots(s_{i_1} + c_{i_1}(k;\mu_{(1)}))e^{\overline{\mu}}, & \text{otherwise.}\end{cases}$$

Furthermore $c_{i_j}(k;\mu_{(j)}) \geq 0$ for all $j = 1, \dots, m$.

To fix notations, we consider in the following the unique irreducible non-reduced crystallographic root system in $\mathbb{R}^n$ defined by

$$BC_n := \{e_i, 2e_i \mid 1 \leq i \leq n\} \cup \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \subseteq \mathbb{R}^n$$

with Weyl group $W_B = \mathbb{Z}_2^n \ltimes \mathcal{S}_n$. We fix a non-negative multiplicity $\kappa = (k_1, k_2, k_3) \geq 0$, where k_1 is the value on the e_i, k_2 is the value on $2e_i$ and k_3 is the value on $\pm(e_i \pm e_j)$. Furthermore, we consider the positive roots

$$BC_n^+ = \{e_i, 2e_i \mid 1 \leq i \leq n\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$$

with simple roots $\alpha_1, \dots, \alpha_n$ defined by

$$\alpha_i = \begin{cases} e_i - e_{i+1}, & 1 \leq i \leq n-1,\\ e_n, & i = n.\end{cases}$$

The weights and dominant weights are then given by

$$P^{BC} = \mathbb{Z}^n \quad \text{and} \quad P_+^{BC} = \Lambda_+^n = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\},$$

respectively. The positive Weyl chamber is

$$C_+^{BC} := \{\lambda \in \mathbb{R}^n \mid \lambda_1 > \ldots > \lambda_n > 0\}$$

and the Weyl vector is

$$\rho^{BC}(\kappa) = \frac{1}{2}\sum_{i=1}^n (k_1 + 2k_2 + 2k_3(n-i))e_i.$$

The highest short root of BC_n is $\beta := e_1$ and we write

$$\begin{aligned} s_0 &:= \beta + s_\beta &&= x \mapsto (1-x_1, x_2, \dots, x_n),\\ s_i &:= s_{e_i - e_{i+1}} &&= x \mapsto (x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n), \text{ for } 1 \leq i < n,\\ s_n &:= s_{e_n} &&= x \mapsto (x_1, \dots, x_{n-1}, -x_n).\end{aligned}$$

We denote the space of trigonometric polynomials of type BC_n by $\mathcal{T}^{\mathrm{BC}}$. The dual affine Weyl group $W_B^{\vee,\mathrm{aff}} = \langle s_0, \dots, s_n \rangle_{\mathrm{group}}$ acts on $\mathcal{T}^{\mathrm{BC}}$ by the induced action on P^{BC}, namely $we^\lambda := e^{w\lambda}$. Therefore, the action of W_B on $\mathcal{T}^{\mathrm{BC}}$ is the usual one and $s_0 f(x) = e^{x_1} f(-x_1, x_2, \dots, x_n)$ for $f \in \mathcal{T}^{BC}$, $x \in \mathbb{R}^n$.
The type BC recurrence relation for Heckman-Opdam polynomials are summarized in the following Corollary.

Corollary 10.2. *Let $(E_\mu^{\mathrm{BC}}(\kappa;\cdot))_{\mu\in P^{BC}}$ be the non-symmetric Heckman-Opdam polynomials associated with (BC_n^+, κ). Then we have*

(i) *The minuscle weights are $\mathcal{O} = \{0\}$.*

(ii) *For $\mu \in P^{BC} = \mathbb{Z}^n$ with $s_i\mu \neq \mu$ for some $0 \le i \le n$, there exists $d_i = d_i(\kappa;\mu)$ with $d_i E_{s_i\mu}^{\mathrm{BC}}(\kappa;\cdot) = (s_i + c_i(\kappa;\mu)) E_\mu^{\mathrm{BC}}(\kappa;\cdot)$ and*

$$c_i(\kappa;\mu) = \begin{cases} \dfrac{k_1}{1-2\widetilde{\mu}_1}, & i = 0, \\ \frac{k_3}{\widetilde{\mu}_i - \widetilde{\mu}_{i+1}}, & 1 \le i < n, \\ \frac{k_1+2k_2}{2\widetilde{\mu}_n}, & i = n. \end{cases}$$

Hence, $P^{BC} = \mathbb{Z}^n$ can be recursively constructed from 0 by the operations $s_0, \dots, s_n$ and therefore, the Heckman-Opdam polynomials can be constructed recursively from $E_0^{\mathrm{BC}}(\kappa;\cdot) \equiv 1$.

Proof. $\mathcal{O} = \{0\}$ is obvious as for $\mu \in \mathcal{O}$ and $i = 1, \dots, n$ we have $2\mu_i = \langle e_i^\vee, \mu\rangle \in \{0,1\}$. The remaining things are exactly Theorem 10.1. ■

Theorem 10.3. *Let $\trianglelefteq$ be the partial order on $P^{BC} = \mathbb{Z}^n$ defined in Definition 4.1 and $\mu \in \mathbb{Z}^n$. Then there exists a trigonometric polynomial*

$$E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot) = e^\mu + \sum_{\lambda \triangleleft \mu} c_{\lambda\mu}(k_3) e^\mu \in \mathcal{T}^{BC}$$

such that for fixed $k_3 \ge 0$:

(i) $\lim\limits_{\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}} E_\mu^{\mathrm{BC}}(\kappa;\cdot) = E_\mu^{\mathrm{BC}}(\infty;k_3;\cdot)$ *locally uniformly on $\mathbb{C}^n$, including the case $k_2 = 0$ with the convention $k_1/k_2 = \infty$.*

(ii) *If $w_\mu = s_{i_1} \cdots s_{i_m} \in W_B^{\vee,aff}$ is reduced and of minimal length with $w_\mu\mu = 0$, then*

$$E_\mu^{\mathrm{BC}}(\infty;k_3;\cdot) = \begin{cases} 1, & \mu = 0, \\ (s_{i_m} + c_m) \cdots (s_{i_1} + c_1) \cdot 1, & \mu \neq 0, \end{cases} \tag{10.1}$$

where $c_\ell = c_{i_\ell}(\infty; k_3; \mu_{(\ell)}) \ge 0$ is defined via

$$c_\ell(\infty;k_3;x) = \lim_{\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}} c_\ell(\kappa;x) \text{ for all } 0 \le \ell \le n, x \in \mathbb{R}^n$$

with $c_\ell(\kappa;x)$ as in Corollary 10.2. In particular, $E_\mu^{\mathrm{BC}}(\infty,\cdot)$ is non-zero.

(iii) *Let $\epsilon(t) = -1$ if $t \le 0$ and $\epsilon(t) = 1$ if $t > 0$. The limits $c_i = c_i(\infty;k_3;x)$ satisfy for $i = 0, n$*

$$c_i = \begin{cases} -\epsilon(x_1), & i = 0, \\ \epsilon(x_n), & i = n. \end{cases}$$

For $1 \leq i < n$ *we have* $c_i = 0$ *if* $\epsilon(x_i) \neq \epsilon(x_{i+1})$ *and*

$$c_i = \frac{k_3}{x_i - x_{i+1} + \frac{k_3}{2}\left(\sum_{j>i}\epsilon(x_i \pm x_j) + \sum_{j<i}\delta(x_j, x_i) - \sum_{j>i+1}\epsilon(x_{i+1} \pm x_j) - \sum_{j<i+1}\delta(x_j, x_{i+1})\right)},$$

otherwise with $\delta(x_j, x_i) = \epsilon(x_j + x_i) - \epsilon(x_j - x_i)$.

PROOF.

- First of all, we can compute the limit $c_j(\infty; k_3; x) = \lim_{\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}} c_j(\kappa; x)$.
  - As $\widetilde{x}_1 = x_1 + \frac{1}{2}\left((k_1 + 2k_2)\epsilon(x_1) + k_3 \sum_{i>1} \epsilon(x_1 \pm x_i)\right)$ we obtain

    $$\frac{1 - 2\widetilde{x}_1}{k_1} \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} -\epsilon(x_1).$$

    Thus, by $\epsilon(x_1) \in \{\pm 1\}$, we conclude $\lim_{k_1+k_2,\, k_1/k_2\to\infty} c_1(\kappa; x) = -\epsilon(x_1)$.
  - Similar computations show that

    $$c_n(\kappa; x) = \frac{k_1 + 2k_2}{2\widetilde{x}_n} \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} \epsilon(x_n).$$

  - If x satisfies $\epsilon(x_i) \neq \epsilon(x_{i+1})$ for $1 \leq i < n$, i.e. $\epsilon(x_{i+1}) = -\epsilon(x_i)$, then

    $$c_i(\infty; k_3; x) = \frac{k_3}{\widetilde{x}_i - \widetilde{x}_{i+1}} = \frac{k_3}{x_i - x_{i+1} + (k_1 + 2k_2)\epsilon(x_i) + d(k_3, \mu)} \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} 0,$$

    where $d(k_3, x) \in \mathbb{R}$ is independent of k_1, k_2. If conversely $\epsilon(x_i) = \epsilon(x_{i+1})$, then

    $$c_i(\kappa; x) = \frac{k_3}{\widetilde{x}_i - \widetilde{x}_{i+1}} = \frac{k_3}{x_i - x_{i+1} + d(k_3, x)} = c_i(\infty, k_3; x)$$

    is independent of k_1, k_2 and $d(k_3, x)$ is given by

    $$d(k_3, x) = \frac{k_3}{2}\left(\sum_{j>i}\epsilon(x_i \pm x_j) + \sum_{j<i}\delta(x_j, x_i) - \sum_{j>i+1}\epsilon(x_{i+1} \pm x_j) - \sum_{j<i+1}\delta(x_j, x_{i+1})\right).$$

- Let $\mu \in \mathbb{Z}^n$ and $w_\mu = s_{i_1} \cdots s_{i_m}$ given in a reduced expression. From the first part of the proof and Corollary 10.2 the limit

  $$\begin{aligned} E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot) &:= \lim_{k_1+k_2,\, k_1/k_2\to\infty} \begin{cases} 1, & \mu = 0, \\ (s_{i_m} + c_{i_m}(\kappa; \mu_{(m)})) \cdots (s_{i_1} + c_{i_1}(\kappa; \mu_{(1)})) \cdot 1, & \mu \neq 0, \end{cases} \\ &= \lim_{k_1+k_2,\, k_1/k_2\to\infty} E_\mu^{\mathrm{BC}}(\kappa; \cdot) \end{aligned}$$

  exists as locally uniform limit and gives formula (10.1). In particular, $E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot)$ lies in $\mathcal{T}^{BC}$ and is of the form $e^\mu + \sum_{\lambda \trianglelefteq \mu} c_{\lambda\mu}(k_3) e^\mu$ as $E_\mu^{\mathrm{BC}}(\kappa; \cdot)$ is of the form.

■

Corollary 10.4. *For $\mu \in \mathbb{Z}^n$ with $s_i\mu \neq \mu$, there exists $d_i := d_i(k_3;\mu) \in \mathbb{R}$ with*

$$d_i E^{\mathrm{BC}}_{s_i\mu}(\infty;k_3;\cdot) = (s_i + c_i) E^{\mathrm{BC}}_{\mu}(\infty;k_3;\cdot),$$

with $c_i = c_i(k_3;\mu)$.

Proof. From Theorem 10.2 we have a constant $d_i(\kappa;\mu) \in \mathbb{R}$ with

$$d_i(\kappa;\mu) E^{\mathrm{BC}}_{s_i\mu}(\kappa;\cdot) = (s_i + c_i(\kappa;\mu)) E^{\mathrm{BC}}_{\mu}(\kappa;\cdot),$$

i.e. $d_i(\kappa;\mu) = \frac{(s_i+c_i(\kappa;\mu))E^{\mathrm{BC}}_{\mu}(\kappa;0)}{E^{\mathrm{BC}}_{s_i\mu}(\kappa;0)}$. Hence, the claim holds by Theorem 10.3 with

$$d_i = \lim_{\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}} d_i(\kappa;\mu) = \frac{(s_i+c_i)E^{\mathrm{BC}}_{\mu}(\infty;k_3;0)}{E^{\mathrm{BC}}_{s_i\mu}(\infty;k_3;0)}.$$

■

We further recall the recurrence relations for Jack polynomials which can be found in [For10, Proposition 12.2.1, Proposition 12.2.3].

Proposition 10.5. *Consider the non-symmetric Jack polynomials $(E^{Jack}_{\mu}(k;\cdot))_{\mu\in\mathbb{N}_0^n}$ of index $\alpha = \frac{1}{k} \in]0,\infty]$*

(i) $E^{Jack}_{\Phi\mu}(k;\cdot) = \Phi E^{Jack}_{\mu}(k;\cdot)$ with the raising operator Φ defined on $\mu \in \mathbb{N}_0^n$ and functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi\mu = (\mu_2, \dots, \mu_n, \mu_1 + 1),$$
$$\Phi f(x) = x_n f(x_n, x_1, \dots, x_{n-1}).$$

(ii) $E^{Jack}_{s_i\mu}(k;\cdot) = (s_i + \frac{k}{\overline{\mu}_{i+1}-\overline{\mu}_i})E^{Jack}_{\mu}(k;\cdot)$ for $\mu_i < \mu_{i+1}$ and $i = 1,\dots,n-1$ with

$$\overline{\mu}_i = \mu_i - k\#\{j < i \mid \mu_j \geq \mu_i\} - k\#\{j > i \mid \mu_j > \mu_i\}.$$

10.2 Limit transition in the polynomial case

Remark 10.6. Recall from Definition 4.1 the ordering $\trianglelefteq$ on $P^{BC} = \mathbb{Z}^n$ defined by

$$\mu \trianglelefteq \lambda \quad \text{iff} \quad \begin{cases} \lambda \leq \mu, & \lambda_+ = \mu_+, \\ \mu_+ \leq \lambda_+, & \lambda_+ \neq \mu_+, \end{cases}$$

where the λ_+ is the unique element in $W_B.\lambda \cap C_+^{BC}$ and $\leq$ is the dominance order defined by

$$\lambda \leq \mu \quad \text{iff} \quad \mu - \lambda \in Q_+ = \operatorname{span}_{\mathbb{N}_0}(BC_n^+) = \left\{ a_1e_1 + \sum_{i=2}^n (a_i - a_{i+1})e_i \,\middle|\, a_i \in \mathbb{N} \right\}$$
$$\text{i.e. } \sum_{i=1}^p \lambda_i \leq \sum_{i=1}^p \mu_i \text{ for all } p = 1,\dots,n$$

Proposition 10.7. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda \triangleleft \mu$.*

(i) *If $\mu_{i+1} < \mu_i$, then $s_i\lambda \neq \mu$ for all $i = 1,\dots,n-1$.*

(ii) *If $\mu_n \geq 0$, then $s_n\lambda \neq \mu$.*

(iii) *If $\mu_1 \leq 0$, then $s_0\lambda \neq \mu$.*

PROOF.

(i) Assume that $s_i\lambda = \mu$. Then $\lambda \lhd \mu$ is equivalent to $\mu_i < \lambda_i$ and $\mu_i + \mu_{i+1} \leq \lambda_i + \lambda_{i+1}$. But $\mu_i = \lambda_{i+1}$ and $\mu_{i+1} = \lambda_i$, which is a contradiction to $\mu_i > \mu_{i+1}$.

(ii) Assume that $s_n\lambda = \mu$. Then $\lambda \lhd \mu$ is equivalent to $\mu_n < \lambda_n$. But $\mu_n = -\lambda_n$, so $\mu_n \geq 0$ leads to a contradiction.

(iii) Assume that $s_0\lambda = \mu$. Then $\lambda_i = \mu_i$ for $1 < i \leq n$ and $1 - \lambda_1 = \mu_1$. In particular, $\mu_+ \neq \lambda_+$ and therefore $\lambda \lhd \mu$ is equivalent to $\lambda_+ < \mu_+$. By $\mu_1 \leq 0$, there exists a permutation $\sigma \in \mathcal{S}_n$ with

$$\mu_+ = \sigma(-\mu_1, |\mu_2|, \ldots, |\mu_n|) \text{ and } \lambda_+ = \mu_+ + e_i,\ i := \sigma(1).$$

Therefore, $\lambda_+ < \mu_+$ means that $(\mu_+)_i + 1 = (\lambda_+)_i < (\mu_+)_i$. Hence $1 < 0$, a contradiction.

■

Definition 10.8. The following operator on $\mathbb{Z}^n$ plays an important role in the following

$$\widetilde{\Phi} = s_n \cdots s_0 = \eta \mapsto (\eta_2, \ldots, \eta_n, \eta_1 - 1).$$

This operator is related to the raising operator Φ by the equation $\widetilde{\Phi}(-\eta) = -\Phi\eta$. Hence, the operator $\widetilde{\Phi}$ will be important to characterize the limit of the BC_n Heckman-Opdam polynomials as Jack polynomials.

Lemma 10.9. *The limits $(E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot))_{\mu\in\mathbb{Z}^n}$ satisfy the following recurrence relations.*

(i) *If $\mu \in -\mathbb{N}_0^n$, then*

$$E_{\widetilde{\Phi}\mu}^{\mathrm{BC}}(\infty; k_3; \cdot) = (s_n + 1)s_{n-1} \cdots s_1(s_0 + 1)E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot).$$

(ii) *If $\mu = -\eta \in -\mathbb{N}_0^n$ with $\mu_{i+1} < \mu_i$, then*

$$E_{s_i\mu}^{\mathrm{BC}}(\infty; k_3; \cdot) = (s_i + \tfrac{k_3}{\overline{\eta}_{i+1} - \overline{\eta}_i})E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot)$$

with $\overline{\eta}$ as in Proposition 10.5 for $k = k_3$.

(iii) *Let $1 \leq i_1 < .... < i_\ell \leq n$ be the indices with $\mu_{i_j} > 0$ and define*

$$\mu^* = (s_0 \cdots s_{i_\ell - 1}) \cdots (s_0 \cdots s_{i_1 - 1})\mu \in -\mathbb{N}_0^n.$$

Then we have

$$E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot) = [s_{i_1-1} \cdots s_1(s_0 + 1)] \cdots [s_{i_\ell - 1} \cdots s_1(s_0 + 1)]E_{\mu^*}^{\mathrm{BC}}(\infty; k_3; \cdot),$$

with convention $s_j \cdots s_1(s_0 + 1) = 1$ if $j = 0$.

PROOF. The most important argument in this proof is based on the triangular form

$$E_\mu^{\mathrm{BC}}(\infty; k_3; \cdot) = e^\mu + \sum_{\lambda \lhd \mu} c_{\lambda\mu} e^\lambda,$$

from Theorem 10.2.

(i) We will divide this part into several steps.

- Since $\mu_1 \leq 0$, we can conclude that $\epsilon(\mu_1) = -1$ and therefore due to Corollary 10.4

$$d_0 E^{\mathrm{BC}}_{s_0\mu}(\infty, k_3; \cdot) = (s_0 + 1) E^{\mathrm{BC}}_{\mu}(\infty, k_3; \cdot) \tag{10.2}$$

for some constant d_0. According to Proposition 10.7 (part (iii), since $\mu_1 \leq 0$) we can compare the coefficients of $e^{s_0\mu}$: the left hand side of (10.2) has coefficient d_0 and the right hand side 1, i.e. $d_0 = 1$.

- Consider $i = 1, \ldots, n-1$. Then $\mu^* := s_{i-1} \ldots s_0 \mu$ is given by

$$\mu^* = (\mu_2, \ldots, \mu_{i-1}, 1 - \mu_1, \mu_{i+1}, \ldots, \mu_n).$$

Thus, $\epsilon(\mu^*_i) = 1 = -\epsilon(\mu^*_{i+1})$. Proceeding by induction over $i = 1, \ldots, n-1$ this leads by Corollary 10.4 to

$$d_i E^{\mathrm{BC}}_{s_i\mu^*}(\infty, k_3; \cdot) = s_i E^{\mathrm{BC}}_{\mu^*}(\infty, k_3; \cdot) \tag{10.3}$$

for some constant d_i. According to Proposition 10.7 (part (i), since $\mu^*_{i+1} \leq 0 < \mu^*_i$) we can compare the coefficients of $e^{s_i\mu^*}$: the left hand side of (10.3) has coefficient d_i and the right hand side 1, i.e. $d_i = 1$.

- Let $\mu^* = s_{n-1} \cdots s_0 \mu = s_n \Phi\mu$, then $\mu^*_n = 1 - \mu_1$, i.e. $\epsilon(\mu^*_n) = 1$. Hence, Corollary 10.4 gives

$$d_n E^{\mathrm{BC}}_{\widetilde{\Phi}\mu}(\infty, k_3; \cdot) = (s_n + 1) E^{\mathrm{BC}}_{\mu^*}(\infty, k_3; \cdot) \tag{10.4}$$

for some constant d_n. According to Proposition 10.7 (part (ii), since $\mu^*_n \geq 0$) we can compare the coefficients of $e^{s_n\mu^*}$: the left hand side of (10.4) has coefficient d_n and the right hand side 1, i.e. $d_n = 1$.

The assertion follows by combining these three steps.

(ii) Consider $\mu = -\eta \in -\mathbb{N}_0^n$ satisfying $\mu_i > \mu_{i+1}$, i.e. $\eta_i < \eta_{i+1}$.

- We claim that

$$\begin{aligned} \overline{\eta}_{i+1} - \overline{\eta}_i = \mu_i - \mu_{i+1} + \tfrac{k_3}{2}\Big(&\sum_{j>i} \epsilon(\mu_i \pm \mu_j) + \sum_{j<i} \delta(\mu_j, \mu_i) \\ &- \sum_{j>i+1} \epsilon(\mu_{i+1} \pm \mu_j) - \sum_{j<i+1} \delta(\mu_j, \mu_{i+1})\Big). \end{aligned} \tag{10.5}$$

According to $\mu = -\eta \in \mathbb{N}_0^n$ and $\eta_i < \eta_{i+1}$ we have by explicit combinatorical computations

$$\begin{aligned} &\sum_{j>i} \epsilon(\mu_i \pm \mu_j) + \sum_{j<i} \delta(\mu_j, \mu_i) - \sum_{j>i+1} \epsilon(\mu_{i+1} \pm \mu_j) - \sum_{j<i+1} \delta(\mu_j, \mu_{i+1}) \\ &\quad = -(n-i) + \#\{j > i \mid \mu_i > \mu_j\} - \#\{j > i \mid \mu_i \leq \mu_j\} \\ &\qquad - (i-1) + \#\{j < i \mid \mu_j \leq \mu_i\} - \#\{j < i \mid \mu_j > \mu_i\} \\ &\qquad + (n-(i+1)) - \#\{j > i+1 \mid \mu_{i+1} > \mu_j\} - \#\{j > i+1 \mid \mu_{i+1} \leq \mu_j\} \\ &\qquad + i + \#\{j < i+1 \mid \mu_j \leq \mu_{i+1}\} - \#\{j < i+1 \mid \mu_j > \mu_{i+1}\} \\ &\quad = \#\{j > i \mid \eta_j > \eta_i\} + \#\{j < i \mid \eta_j \geq \eta_i\} \\ &\qquad - \#\{j > i+1 \mid \eta_j > \eta_{i+1}\} - \#\{j < i+1 \mid \eta_j \geq \eta_{i+1}\} \\ &\qquad - \#\{j > i \mid \eta_j \leq \eta_i\} + \#\{j > i+1 \mid \eta_j \leq \eta_{i+1}\} \\ &\qquad - \#\{j < i \mid \eta_j < \eta_i\} + \#\{j < i+1 \mid \eta_j < \eta_{i+1}\} \end{aligned}$$

$$= 2 \cdot \Big(\# \{ j > i \mid \eta_j > \eta_i \} + \# \{ j < i \mid \eta_j \geq \eta_i \} \\ - \# \{ j > i+1 \mid \eta_j > \eta_{i+1} \} - \# \{ j < i+1 \mid \eta_j \geq \eta_{i+1} \} \Big).$$

From here it is immediate that equation (10.5) is true.

- By Corollary 10.4 and (10.5) there exists a constant d_i with

$$d_i E^{\mathrm{BC}}_{s_i\mu}(\infty, k_3; \cdot) = (s_i + \tfrac{k_3}{\overline{\eta}_{i+1} - \overline{\eta}_i}) E^{\mathrm{BC}}_{\mu}(\infty, k_3; \cdot). \tag{10.6}$$

According to Proposition 10.7 (part (ii), since $\mu_{i+1} < \mu_i$) we can compare the coefficients of $e^{s_i\mu}$: the left hand side of (10.6) has coefficient d_i and the right hand side 1, i.e. $d_i = 1$.

(iii) Assume that $\mu \in \mathbb{Z}^n$ has exactly ℓ positive entries, namely for $1 \leq i_1 < \ldots < i_\ell \leq n$ with $\mu_{i_j} > 0$. We defined

$$\mu' \coloneqq s_0 \cdots s_{i_1 - 1}\mu = (1 - \mu_{i_1}, \mu_2, \ldots, \mu_{i_1 - 1}, \mu_{i_1+1}, \ldots, \mu_n).$$

In particular, $\mu'_j \leq 0$ for $1 \leq j < i_2$. By the proof of part (i) (to become more precise, the first two steps of the proof of part (i)) we have

$$E^{\mathrm{BC}}_{\mu}(\infty; k_3; \cdot) = E^{\mathrm{BC}}_{s_{i_1-1}\cdots s_0\mu'}(\infty; k_3; \cdot) = s_{i_1-1}\cdots s_1(s_0+1)E^{\mathrm{BC}}_{\mu'}(\infty; k_3; \cdot).$$

Since μ' has precisely $\ell - 1$ we can proceed by induction to obtain the claimed formula.

■

Theorem 10.10. *For $x \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ define $f(x) \coloneqq (f(x_1), \ldots, f(x_n)) \in \mathbb{R}^n$ componentwise. The limits $(E^{\mathrm{BC}}_{\mu}(\infty; k_3; \cdot))_{\mu \in \mathbb{N}_0^n}$ are explicitly given by*

(i) *If $\eta \in \mathbb{N}_0^n$, then $E^{\mathrm{BC}}_{-\eta}(\infty; k_3; x) = 4^{|\eta|} E^{Jack}_{\eta}(k_3; \cosh^2(\frac{x}{2}))$.*

(ii) *If $\mu \in \mathbb{Z}^n \setminus \mathbb{N}_0^n$ and $1 \leq i_1 < \ldots < i_\ell \leq n$ are the indices with $\mu_{i_j} > 0$. Then for*

$$\mu^* = (s_0 \cdots s_{i_\ell - 1}) \cdots (s_0 \cdots s_{i_1-1})\mu = (1 - \mu_{i_\ell}, \ldots, 1 - \mu_{i_1}, \mu') \in -\mathbb{N}_0^n,$$

where $\mu' \in \mathbb{N}_0^{n-\ell}$ is the vector μ with deleted entries μ_{i_j} it holds

$$\begin{aligned} E^{\mathrm{BC}}_{\mu}(\infty; k_3; \cdot) &= (e^{x_{i_1}} + 1) \cdots (e^{x_{i_\ell}} + 1) E^{\mathrm{BC}}_{\mu^*}(\infty; k_3; x_{i_\ell}, \ldots, x_{i_1}, x') \\ &= 4^{|\mu| - \ell}(e^{x_{i_1}} + 1) \cdots (e^{x_{i_\ell}} + 1) E^{Jack}_{-\mu^*}(k_3; \cosh^2(\tfrac{x_{i_\ell}}{2}, \ldots, \tfrac{x_{i_1}}{2}, \tfrac{x'}{2})), \end{aligned}$$

where $x' \in \mathbb{R}^{n-\ell}$ is the vector with deleted entries x_{i_j}. Moreover, E^{BC}_{μ} is $\mathbb{Z}_2$-invariant in the variables x_j with $j \neq i_1, \ldots, i_\ell$.

(iii) *In the situation of part (ii) it holds*

$$\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}^n} E^{\mathrm{BC}}_{\mu}(\infty; k_3; \tau x) = 4^{|\mu| - \frac{\ell}{2}} \prod_{j=1}^{\ell} \cosh^2(\tfrac{x_{i_j}}{2}) \cdot E^{Jack}_{-\mu^*}(k_3; \cosh^2(\tfrac{\sigma_\mu x}{2})),$$

with $\sigma_\mu = (s_1 \cdots s_{i_\ell - 1}) \cdots (s_1 \cdots s_{i_1 - 1})$, i.e. $\sigma_\mu x = (x_{i_\ell}, \ldots, x_{i_1}, x')$. This formula also makes sense by part (i) if $\mu \in -\mathbb{N}_0^n$ with $\mu^* = \mu$ and $\sigma_\mu = 1$.

(iv) *For $\mu \in \mathbb{Z}^n$ it holds for $1 \leq i_1 < \ldots < i_\ell \leq n$ the indices with $\mu_{i_j} > 0$*

$$\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}^n} E_\mu^{\mathrm{BC}}(\infty; k_3; \tau x) = 4^{|\mu| - \frac{\ell}{2}} E_{-\mu^{**}}^{Jack}(k_3; \sigma_\mu^* \cosh^2(\tfrac{x}{2})),$$

where $\mu^{**} = (\mu', -\mu_{i_\ell}, \ldots, -\mu_{i_1})$, and $\sigma_\mu^* = (s_{i_1} \cdots s_{n-1}) \cdots (s_{i_\ell} \cdots s_{n-1})$, i.e. $\sigma_\mu^* x = (x', x_{i_\ell}, \ldots, x_{i_1})$, where x' and μ' are choosen as before, as the vectors with deleted entries of index i_j.

PROOF.

(i) By Proposition 10.5 the family $f_\eta(x) := 4^{|\eta|} E_\eta^{\mathrm{Jack}}(k_3; \cosh^2(\frac{x}{2})), \eta \in \mathbb{N}_0^n$ is uniquely determined by

$$\begin{aligned} f_0(x) &= 1 \\ f_{\Phi\eta}(x) &= 4\cosh^2(\tfrac{x_n}{2}) f_\eta(x_n, x_1, \ldots, x_{n-1}) \\ f_{s_i\eta}(x) &= (s_i + \tfrac{k_3}{\overline{\eta}_{i+1} - \overline{\eta}_i}) f_\eta(x), \text{ if } \eta_i < \eta_{i+1}. \end{aligned}$$

Hence we want to prove that $g_\eta := E_{-\eta}^{\mathrm{BC}}(\infty; k_3; \cdot), \eta \in \mathbb{N}_0^n$ satisfy the same equations. For $\eta = 0$ the equations are immediate. Assume that the equations hold for $\eta \in \mathbb{N}_0^n$, in particular g_η is $\mathbb{Z}_2^n$ invariant.

- s_0 acts on $\mathcal{T}^{BC}$ by $s_0 f(x) = e^{x_1} f(-x_1, x_2, \ldots, x_n)$ and therefore by the induction hypothesis and Lemma 10.9

$$\begin{aligned} g_{\Phi\eta}(x) &= E_{-\Phi\eta}^{\mathrm{BC}}(\infty; k_3; x) = E_{\widetilde{\Phi}(-\eta)}^{\mathrm{BC}}(\infty; k_3; x) \\ &= (s_n + 1) s_{n-1} \cdots s_1 (s_0 + 1) E_{-\eta}^{BC}(\infty; k_3; x) \\ &= (s_n + 1) s_{n-1} \cdots s_1 \left(e^{x_1} g_\eta(-x_1, x_2, \ldots, x_n) + g_\eta(x_1, \ldots, x_n)\right) \\ &= (s_n + 1) s_{n-1} \cdots s_1 (e^{x_1} + 1) g_\eta(x) \\ &= (s_n + 1)(e^{x_n} + 1) g_\eta(x_n, x_1, \ldots, x_{n-1}) \\ &= (e^{-x_n} + 1) g_\eta(-x_n, x_1, \ldots, x_{n-1}) + (e^{x_n} + 1) g_\eta(x_n, x_1, \ldots, x_{n-1}) \\ &= 4\cosh^2(\tfrac{x_n}{2}) g_\eta(x_n, x_1, \ldots, x_{n-1}). \end{aligned}$$

- If $\eta_i < \eta_{i+1}$, then $(-\eta)_{i+1} < (-\eta)_i$ and therefore by the induction hypothesis and Lemma 10.9

$$\begin{aligned} g_{s_i\eta}(x) &= E_{s_i(-\eta)}^{\mathrm{BC}}(\infty; k_3; \cdot) = (s_i + \tfrac{k_3}{\overline{\eta}_{i+1} + \overline{\eta}_i}) E_{-\eta}^{\mathrm{BC}}(\infty; k_3; x) \\ &= (s_i + \tfrac{k_3}{\overline{\eta}_{i+1} + \overline{\eta}_i}) g_\eta(x). \end{aligned}$$

Thus we conclude $g_\eta = f_\eta$ for all $\eta \in \mathbb{N}_0^n$.

(ii) This is immediate from Lemma 10.9.

(iii) Since $E_{\mu^*}^{\mathrm{BC}}(\infty; k_3; \cdot)$ is $\mathbb{Z}_2^n$-invariant, this formula is immediate from parts (i)+(ii) together with $|\mu^*| = |\mu| - \ell$ and

$$\begin{aligned} \tfrac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} (e^{(\tau x)_{i_1}} + 1) \cdots (e^{(\tau x)_{i_\ell}} + 1) &= \tfrac{1}{2^\ell} (e^{x_{i_1}} + e^{-x_{i_1}} + 2) \cdots (e^{x_{i_n}} + e^{-x_{i_n}} + 2) \\ &= 2^\ell \cosh^2(\tfrac{x_{i_1}}{2}) \cdots \cosh^2(\tfrac{x_{i_\ell}}{2}). \end{aligned}$$

(iv) This can be inductively constructed from part (iii), since the Jack polynomials satisfy

$$E^{\text{Jack}}_{\Phi\eta}(k_3;y) = \Phi E^{\text{Jack}}_{\eta}(k_3;y),$$
$$\text{i.e. } E^{\text{Jack}}_{(\eta_2,\dots,\eta_n,\eta_1+1)}(k_3;y) = y_n E^{\text{Jack}}_{\eta}(k_3;y_n,y_1,\dots,y_{n-1}).$$

■

The following result was already proven in [RKV13, Theorem 4.2], but we give an independent proof, based on the results for the non-symmetric setting.

Theorem 10.11. *Let $(P^{Jack}_\lambda(k_3;\cdot))_{\lambda\in\Lambda^n_+}$ be the symmetric Jack polynomials of index $\alpha = \frac{1}{k_3}$. Then for fixed $k_3 \geq 0$ and $\kappa = (k_1,k_2,k_3) \geq 0$ the symmetric Heckman-Opdam polynomials $(P^{BC}_\lambda(\kappa;\cdot))_{\lambda\in\Lambda^n_+}$ satisfy the asymptotic*

$$P^{\text{BC}}_\lambda(\kappa;z) \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 4^{|\lambda|}P^{Jack}_\lambda(k_3;\cosh^2(\tfrac{z}{2})),$$

locally uniformly in $z\in\mathbb{C}^n$.

Proof. Consider the renormalizations

$$\widetilde{P}^{\text{BC}}_\lambda(\kappa;\cdot) := \frac{P^{\text{BC}}_\lambda(\kappa;\cdot)}{P^{\text{BC}}_\lambda(\kappa;0)}, \quad \widetilde{P}^{\text{Jack}}_\lambda(k_3;\cdot) := \frac{P^{\text{Jack}}_\lambda(k_3;\cdot)}{P^{\text{Jack}}_\lambda(k_3;\underline{1})}.$$

(i) For the type BC_n Cherednik kernel G^{BC}_κ and hypergeometric Function F^{BC}_κ one has for $\lambda\in P^{BC}_+ = \Lambda^n_+$ that $\widetilde{-\lambda} = -\lambda-\rho^{BC}(\kappa)$ and therefore

$$G^{\text{BC}}_\kappa(-\lambda-\rho^{BC}(\kappa),\cdot) = \frac{E^{\text{BC}}_{-\lambda}(\kappa;\cdot)}{E^{\text{BC}}_{-\lambda}(\kappa,0)}.$$

The longest element in W_B is $w_0 = -\text{id}$ and therefore the W_B-invariance of the F^{BC}_κ in both arguments leads to

$$\frac{P^{\text{BC}}_\lambda(\kappa;\cdot)}{P^{\text{BC}}_\lambda(\kappa;0)} = F^{\text{BC}}_\kappa(\lambda+\rho^{BC}(\kappa),\cdot) = F^{\text{BC}}_\kappa(-\lambda-\rho^{BC}(\kappa),\cdot)$$
$$= \frac{1}{2^n n!}\sum_{w\in W_B} G^{\text{BC}}_\kappa(-\lambda-\rho^{BC}(\kappa),w\cdot) = \frac{1}{2^n n!}\sum_{w\in W_B}\frac{E^{\text{BC}}_{-\lambda}(\kappa;w\cdot)}{E^{\text{BC}}_{-\lambda}(\kappa,0)}.$$

The Jack polynomials also satisfy by equation (6.15)

$$\frac{P^{\text{Jack}}_\lambda(k_3;\cdot)}{P^{\text{Jack}}_\lambda(k_3;\underline{1})} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\frac{E^{\text{Jack}}_\lambda(k_3;\sigma\cdot)}{E^{\text{Jack}}_\lambda(k_3;\underline{1})}.$$

Since the map $\mathbb{R}^n\to\mathbb{R}^n$, $x\mapsto\cosh^2(\frac{x}{2})$ is W_B-equivariant we obtain from Theorem 10.10 that locally uniformly

$$\widetilde{P}^{\text{BC}}_\lambda(\kappa;x) = \frac{1}{2^n n!}\sum_{w\in W_B}\frac{E^{\text{BC}}_{-\lambda}(\kappa;wx)}{E^{\text{BC}}_{-\lambda}(\kappa,0)} \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 4^{|\lambda|}\widetilde{P}^{\text{Jack}}_\lambda(k_3;\cosh^2(\tfrac{x}{2})).$$

(ii) To prove the limit transition for the non-renormalized polynomials, it suffices to check the limit

$$P_\lambda^{\mathrm{BC}}(\kappa;0) \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 4^{|\lambda|}P_\lambda^{\mathrm{Jack}}(k_3;\underline{1}).$$

Owing to Heckman, the value $P_\lambda^{\mathrm{BC}}(\kappa;0)$ can be expressed in terms of a generalized Harish-Chandra c-function, namely

$$P_\lambda^{\mathrm{BC}}(\kappa;0) = \prod_{\alpha\in BC_n^+} \frac{\Gamma\big(\langle\rho^{BC}(\kappa),\alpha^\vee\rangle + \frac{k_{\alpha/2}}{2}\big)\Gamma\big(\langle\lambda+\rho^{BC}(\kappa),\alpha^\vee\rangle + \frac{k_{\alpha/2}+2k_\alpha}{2}\big)}{\Gamma\big(\langle\rho^{BC}(\kappa),\alpha^\vee\rangle + \frac{k_{\alpha/2}+2k_\alpha}{2}\big)\Gamma\big(\langle\lambda+\rho^{BC}(\kappa),\alpha^\vee\rangle + \frac{k_{\alpha/2}}{2}\big)},$$

see for instance [HO87, HS94]. Denote by d_α the quotient behind the product sign for fixed $\alpha\in BC_n^+$. We will study the asymptotic of the values d_α in a case by case situation under the asymptotic equality

$$\frac{\Gamma(z+w)}{\Gamma(z)} \approx z^w \text{ for } z\to\infty,\ \mathrm{Re}(z)>0.$$

$\alpha = e_i$: We have $e_i^\vee = 2e_i$ and $\rho^{BC}(\kappa) = \frac{1}{2}\sum_{i=1}^n (k_1+2k_2+2(n-i)k_3)e_i$, so for large k_1+k_2 and k_1/k_2:

$$\begin{aligned} d_\alpha &= \frac{\Gamma(k_1+2k_2+2(n-i)k_3)\Gamma(2\lambda_i+2k_1+2k_2+2(n-i)k_3)}{\Gamma(2\lambda_i+k_1+2k_2+2(n-i)k_3)\Gamma(2k_1+2k_2+2(n-i)k_3)} \\ &\approx \left(\frac{2k_1+2k_2+2(n-i)k_3}{k_1+2k_2+2(n-i)k_3}\right)^{2\lambda_i} \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 4^{\lambda_i}. \end{aligned}$$

$\alpha = 2e_i$: We have $(2e_i)^\vee = e_i$, so for large k_1+k_2 and k_1/k_2:

$$\begin{aligned} d_\alpha &= \frac{\Gamma(k_1+k_2+(n-i)k_3)\Gamma(2\lambda_i+k_1+2k_2+(n-i)k_3)}{\Gamma(2\lambda_i+k_1+k_2+(n-i)k_3)\Gamma(k_1+2k_2+(n-i)k_3)} \\ &\approx \left(\frac{k_1+2k_2+(n-i)k_3}{k_1+k_2+(n-i)k_3}\right)^{2\lambda_i} \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 1. \end{aligned}$$

$\alpha = e_i+e_j$: We have $(e_i+e_j)^\vee = e_i+e_j$, so for large k_1+k_2 and k_1/k_2:

$$\begin{aligned} d_\alpha &= \frac{\Gamma(k_1+2k_2+((n-i)+(n-j))k_3)}{\Gamma(\lambda_i+\lambda_j+k_1+2k_2+((n-i)+(n-j))k_3)} \\ &\quad\times \frac{\Gamma(\lambda_i+\lambda_j+k_1+2k_2+(1+(n-i)+(n-j))k_3)}{\Gamma(k_1+2k_2+(1+(n-i)+(n-j))k_3)} \\ &\approx \left(\frac{k_1+2k_2+k_3+(1+(n-i)+(n-j))k_3}{k_1+2k_2+((n-i)+(n-j))k_3}\right)^{\lambda_i+\lambda_j} \xrightarrow[\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}]{} 1. \end{aligned}$$

$\alpha = e_i-e_j$: We have $(e_i-e_j)^\vee = e_i-e_j$, so for large k_1+k_2 and k_1/k_2:

$$\begin{aligned} d_\alpha &= \frac{\Gamma(((n-i)-(n-j))k_3)\Gamma(\lambda_i-\lambda_j-+(1+(n-i)-(n-j))k_3)}{\Gamma(\lambda_i-\lambda_j+((n-i)-(n-j))k_3)\Gamma((1+(n-i)-(n-j))k_3} \\ &= \frac{\Gamma((j-i)k_3)\Gamma(\lambda_i-\lambda_k+k_3+(j-i)k_3)}{\Gamma(\lambda_i-\lambda_k+(j-i)k_3)\Gamma(k_3+(j-i)k_3)}. \end{aligned}$$

Therefore, together with the Pochhammer symbol $(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}$ we have proven that

$$\begin{aligned}
P_\lambda^{\mathrm{BC}}(\kappa, 0) \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} & 4^{|\lambda|} \prod_{i<j} \frac{\Gamma((j-i)k_3)\Gamma(\lambda_i - \lambda_j + k_3 + (j-i)k_3)}{\Gamma(\lambda_i - \lambda_k + (j-i)k_3)\Gamma(k_3 + (j-i)k_3)} \\
&= 4^{|\lambda|} \prod_{i<j} (\lambda_i - \lambda_j + (j-i)k_3)_{k_3} \cdot \prod_{1\le j\le n} \frac{\Gamma(k_3)}{\Gamma(jk_3)} \\
&= 4^{|\lambda|} P_\lambda^{\mathrm{Jack}}(k_3; \underline{1}),
\end{aligned}$$

the last equality can be found for instance in [OO98, Formula (6.4)]

■

10.3 Limit transition of the Cherednik kernels

Let G_κ^{BC} be the Cherednik kernel associated with (BC_n^+, κ). Furthermore, let $\mathcal{G}_{k_3}$ be the rational (type A) Cherednik kernel defined in Definition 6.3 by

$$\mathcal{G}_{k_3}(\lambda, x) = \prod_{i=1}^n x_i^{\langle \lambda, \underline{1}\rangle/n} \cdot G_{k_3}^{\mathrm{A}}(\pi(\lambda), \pi(\log x)), \text{ for all } x > 0.$$

where $G_{k_3}^{\mathrm{A}}$ is the Cherednik kernel of type A_{n-1} on $\mathbb{R}_0^n$ with respect to the positive system $A_{n-1}^+ = \{e_j - e_i \mid i < j\} \subseteq \mathbb{R}_0^n$, $\pi : \mathbb{R}^n \to \mathbb{R}_0^n$ is the orthogonal projection and log is the inverse of $\exp : \mathbb{R}^n \to \mathbb{R}_+^n$.

Theorem 10.12. *For all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$*

$$G_\kappa^{\mathrm{BC}}(-\lambda - \rho^{BC}(\kappa), x) \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} \mathcal{G}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2})),$$

locally uniformly in λ.

Proof.

(i) Define for fixed $x \in \mathbb{R}^n$ the entire function

$$f_\kappa(\lambda) := G_\kappa^{\mathrm{BC}}(-\lambda - \rho^{BC}(\kappa), x).$$

Then, $\{f_\kappa \mid \kappa \ge 0\}$ is a locally uniformly bounded family of entire functions due to Theorem 4.17. In fact, they are bounded by

$$|f_\kappa(\lambda)| \le e^{\max\limits_{w\in W_B} \langle -\mathrm{Re}\,\lambda, wx\rangle}.$$

Hence, by Montel's Theorem each sequence

$$\kappa_n = (k_1^{(n)}, k_2^{(n)}, k_3) \ge 0 \text{ with } k_1^{(n)} + k_2^{(n)}, k_1^{(n)}/k_2^{(n)} \xrightarrow[n\to\infty]{} \infty,$$

has a subsequence $(\kappa_{n_j})_{j\in\mathbb{N}} \subseteq (\kappa_n)_{n\in\mathbb{N}}$, such that there is some entire function f with

$$f_{\kappa_{n_j}} \xrightarrow[j\to\infty]{} f \text{ locally uniformly on } \mathbb{C}^n.$$

(ii) Consider on $S = \{\lambda \in \mathbb{C}^n \mid \operatorname{Re}\lambda_1 < ... < \operatorname{Re}\lambda_n < 0\} = -C_+^{BC} + i\mathbb{R}^n$, where C_+^{BC} is the positive Weyl chamber, the holomorphic functions

$$h := e^{-\langle\lambda, x_+\rangle} f, \quad g := e^{-\langle\lambda, x_+\rangle}\mathcal{G}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2})),$$

where x_+ is the unique element in the W_B-orbit contained in C_+^{BC}. Owing to the geometry of the root system it holds for all $\lambda \in \overline{S}$

$$\max_{w\in W_B} \langle \operatorname{Re}\lambda, x\rangle = \max_{w\in W_B} \langle -\operatorname{Re}\lambda, wx\rangle = \langle \operatorname{Re}\lambda, x\rangle.$$

Hence, h is absolutely bounded by 1 on S. By Theorem 4.17 we also have for $\lambda \in \mathbb{C}^n$

$$\begin{aligned}&\left|\mathcal{G}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2}))\right| \\ &= e^{\langle \operatorname{Re}\lambda, \underline{1}\rangle\langle\log(\cosh^2(\frac{x}{2})),\underline{1}\rangle/n}\left|G_{k_3}^A(\pi(\lambda) - \rho^A(k_3), \pi(\log(\cosh^2(\tfrac{x}{2}))))\right| \\ &\le e^{\langle \operatorname{Re}\lambda, \underline{1}\rangle\langle\log(\cosh^2(\frac{x}{2})),\underline{1}\rangle/n} e^{\max_{\sigma\in\mathcal{S}_n}\langle\pi(\operatorname{Re}\lambda), \sigma\pi(\log(\cosh^2(\frac{x}{2})))\rangle} \\ &= e^{\max\limits_{\sigma\in\mathcal{S}_n}\langle\operatorname{Re}\lambda, \sigma\log(\cosh^2(\frac{x}{2}))\rangle}\end{aligned}$$

Since $\cosh^2(\cdot/2)$ is $\mathbb{Z}_2^n$-invariant we conclude that $(\log(\cosh^2(\frac{x}{2})))_+ = \log(\cosh^2(\frac{x_+}{2}))$. Therefore with $\cosh^2(t/2) \le e^t$ for $t \ge 0$ we have

$$\begin{aligned}|g(\lambda)| &\le e^{-\langle\operatorname{Re}\lambda, x_+\rangle} e^{\max\limits_{\sigma\in\mathcal{S}_n}\langle\operatorname{Re}\lambda, \sigma\log(\cosh^2(\frac{x}{2}))\rangle} \\ &= \prod_{i=1}^n \left(e^{-(x_+)_i}\cosh^2(\tfrac{(x_+)_i}{2})\right)^{\operatorname{Re}\lambda_i} \le 1.\end{aligned}$$

(iii) From part (ii) of the proof, we obtain that g and h are holomorphic on S, such that $|g-h| \le 2$ on S. Moreover, since for $\lambda \in \Lambda_+^n \subseteq S$ it holds $\widetilde{-\lambda} = -\lambda - \rho^{BC}(\kappa)$, we obtain with Theorem 10.10 and Remark 6.6 that

$$\begin{aligned}h(\lambda) &= \lim_{j\to\infty} e^{-\langle\lambda, x_+\rangle} G_{\kappa_{n_j}}^{\mathrm{BC}}(-\lambda - \rho^{BC}(\kappa_{n_j}), x) = e^{-\langle\lambda, x_+\rangle}\frac{E_{-\lambda}^{\mathrm{BC}}(\kappa_{n_j}; x)}{E_{-\lambda}^{\mathrm{BC}}(\kappa_{n_j}; 0)} \\ &= e^{-\langle\lambda, x_+\rangle}\frac{E_\lambda^{\mathrm{Jack}}(k_3; \cosh^2(\frac{x}{2}))}{E_\lambda^{\mathrm{Jack}}(k_3; \underline{1})} = e^{-\langle\lambda, x_+\rangle}\mathcal{G}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2})) = g(\lambda).\end{aligned}$$

Hence, $g-h$ is vanishes on Λ_+^n, is absolutely bounded by 2 on $\overline{S}$ and we can write $\overline{S} = \{\sum_{i=1}^n z_i\omega_i \mid \operatorname{Re} z_i \ge 0\}$, where $\omega_i \in \Lambda_+^n = P_+^{BC}$ are the fundamental weights. Thus, by Carlson's Theorem we have $g \equiv h$ on $\overline{S}$, i.e.

$$\lim_{j\to\infty} G_{\kappa_{n_j}}(\lambda - \rho^{BC}(\kappa_{n_j}), x) = f(\lambda) = g(\lambda) = \mathcal{G}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2})), \quad \lambda \in \overline{S}.$$

Finally, by the identity theorem, the limit is valid on $\mathbb{C}^n$.

(iv) The same argumentation as above shows, that whenever $(\kappa_{n_j})_{j\in\mathbb{N}} \subseteq (\kappa_n)_{n\in\mathbb{N}}$ is any subsequence, such that $f_{\kappa_{n_j}} \xrightarrow[j\to\infty]{} \widetilde{f}$ locally uniformly for some entire function, then it has to be $\widetilde{f} = G_{k_3}(\cdot - \rho^A(k_3), \cosh^2(\frac{x}{2}))$. Thus $f_{\kappa_n} \xrightarrow[n\to\infty]{} G_{k_3}(\cdot - \rho^A(k_3), \cosh^2(\frac{x}{2}))$ locally uniformly.

■

Remark 10.13. Let $F^{\mathrm{A}}_{k_3}$ be the type A hypergeometric function on $\mathbb{R}^n_0$ associated with (A^+_{n-1}, k_3) and $\mathcal{F}_{k_3}$ the rational hypergeometric function on $\mathbb{R}^n$ as defined in Definition 6.3. Averaging in Theorem 10.12 gives

$$\begin{aligned} F^{\mathrm{BC}}_{\kappa}(\lambda, x) \xrightarrow[\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}]{} \prod_{1\le i\le n} (\cosh^2(\tfrac{x}{2}))^{\frac{\langle\lambda,\underline{1}\rangle}{2}} \cdot F^{\mathrm{A}}_{k_3}(\lambda - \rho^A(k_3), \pi(\log(\cosh^2(\tfrac{x}{2})))) \\ = \mathcal{F}_{k_3}(\lambda - \rho^A(k_3), \cosh^2(\tfrac{x}{2})). \end{aligned}$$

The different sign in the ρ-shift, compared to [RKV13, Theorem 5.1] comes from the different choice of positive subsystems in A_{n-1}, i.e. $-\rho^A(k_3)$ here is precisely the term $+\rho^A(k_3)$ in [RKV13]. This results was already proven in [RKV13, Theorem 5.1], but here we have received it from the non-symmetric setting.

Remark 10.14. A limit transition for the B_n root system can be easily obtained from the type BC_n case. Choose for the root system $B_n = BC_n \setminus \{2e_i \mid i = 1, \dots, n\} \subseteq \mathbb{R}^n$ with positive subsystem

$$B^+_n = BC^+_n \cap B_n = \{e_i \mid 1 \le i \le n\} \cup \{e_i \pm e_j \mid 1 \le i < j \le n\}.$$

Then, both root system have the same Weyl group, simple roots, weight lattice and dominant weights. Denote by $\kappa' = (k_1, k_3) \ge 0$ a multiplicity function on B_n, where k_1 is the value on e_i and k_3 is the value on $e_i \pm e_j$. Then the Cherednik operators $D^B_\xi(\kappa')$ and $D^{BC}_\xi(\kappa)$ for the given positive subsystems, $\xi \in \mathbb{R}^n$, are related by

$$D^B_\xi(k_1, k_3) = D^{BC}_\xi(k_1, 0, k_3),$$

and also $\rho^B(k_1, k_3) = \rho^{BC}(k_1, 0, k_3)$. Hence, the non-symmetric and symmetric Heckman-Opdam polynomials, as well as the Cherednik kernels and hypergeometric functions are related by

$$\begin{aligned} E^{\mathrm{B}}_\mu(k_1, k_3; z) &= E^{\mathrm{BC}}_\mu(k_1, 0, k_3; z), \quad \mu \in \mathbb{Z}^n,\ k_1, k_3 \ge 0,\ z \in \mathbb{C}^n; \\ P^{\mathrm{B}}_\lambda(k_1, k_3; z) &= P^{\mathrm{BC}}_\lambda(k_1, 0, k_3; z), \quad \lambda \in \Lambda^n_+,\ k_1, k_3 \ge 0,\ z \in \mathbb{C}^n; \\ G^{\mathrm{B}}_{(k_1,k_3)}(\lambda; x) &= G^{\mathrm{BC}}_{(k_1,0,k_3)}(\lambda, x), \quad \lambda \in \mathbb{C}^n,\ k_1, k_3 \ge 0,\ x \in \mathbb{R}^n; \\ F^{\mathrm{B}}_{(k_1,k_3)}(\lambda; x) &= F^{\mathrm{BC}}_{(k_1,0,k_3)}(\lambda, x), \quad \lambda \in \mathbb{C}^n,\ k_1, k_3 \ge 0,\ x \in \mathbb{R}^n. \end{aligned}$$

Hence, all proven limit transitions are also correct for the root system B_n instead of BC_n in the upper sense, one has only to consider $k_2 = 0$.

Part IV

Appendix

CHAPTER A

Recurrence relations for Heckman-Opdam polynomials

This chapter was intended to verify the results of [Sah00a] for the case of (non-)reduced irreducible crystallographic root system. Since the only non-reduced irreducible crystallographic root systems are these of type BC_n, we only translate the results of [Sah00b, Section 6] in the language of the paper [Sah00a]. To become more precise, we present a uniformized proof for the results of [Sah00a] for all irreducible crystallographic root system.
Let R be a (not necessarily reduced) irreducible crystallographic root system in a Euclidean space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$. Fix a system of positive roots $R_+ \subseteq R$ with associated simple roots $\alpha_1, \dots, \alpha_n$ and a multiplicity function $k = (k_\alpha)_{\alpha \in R}$. Moreover, let W be the Weyl group of R, P the weight lattice with dominant weights P_+ and $s_i = s_{\alpha_i}$ the associated simple reflection. Since R is irreducible, their exists a unique highest short root denoted by β. We define the map

$$s_0 \coloneqq x \mapsto \beta + s_\beta x$$

which is the affine reflection in the hyperplane $\{\langle\beta^\vee, x\rangle = 1\}$. The dual affine Weyl group is the group $W^{\vee,\mathrm{aff}} = W \ltimes Q$ generated by $s_0, \dots, s_n$. The orbit space $W^{\vee,\mathrm{aff}} \backslash P$ has the following set of representatives

$$\mathcal{O} \coloneqq \left\{\lambda \in P \mid \langle\alpha^\vee, \lambda\rangle \in \{0,1\} \text{ for all } \alpha \in R_+\right\},$$

called the *minuscle weights*, cf. [Hum90].

Proposition A.1. *The highest short root β satisfies:*

(i) $\beta \in P_+$.

(ii) *If $\beta \neq \alpha \in R_+$, then $\langle\alpha^\vee, \beta\rangle \in \{0, 1\}$.*

PROOF.

(i) For $\alpha \in R_+$ we have that $s_\alpha\beta = \beta - \langle\alpha^\vee, \beta\rangle\,\alpha$ is a root of the same length as β. But β is the highest short root, so $\langle\alpha^\vee, \beta\rangle \in \mathbb{N}_0$.

(ii) By part (i) we have $\beta \in P_+$, i.e. $\langle\alpha^\vee, \beta\rangle \in \mathbb{N}_0$. Since β is a short root, we have $\langle\beta, \beta\rangle \leq \langle\alpha, \alpha\rangle$ and therefore by the Cauchy-Schwartz inequality

$$\langle\alpha^\vee, \beta\rangle = 2\frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \leq 2\frac{\langle\alpha, \beta\rangle}{\sqrt{\langle\alpha, \alpha\rangle\,\langle\beta, \beta\rangle}} \leq 2. \tag{A.1}$$

Since $\alpha \neq \beta$, one of the following cases occur:

(a) $\alpha = 2\beta$, i.e. $\langle\alpha^\vee, \beta\rangle = \langle(2\beta)^\vee, \beta\rangle = 1$.

(b) $\alpha \notin \mathbb{R}\beta$, then the last inequality in (A.1) is strict, i.e. $\langle\alpha^\vee, \beta\rangle \in \{0, 1\}$.

■

Lemma A.2. *For $i = 0, 1, 2$ put $R^i \coloneqq \{\alpha \in R_+ \mid \langle\alpha^\vee, \beta\rangle = i\}$.*

(i) $R_+ = R^0 \sqcup R^1 \sqcup R^2$ *and* $R^i = \{\alpha \in R \mid \langle \alpha^\vee, \beta \rangle = i\}$.

(ii) *The map*

$$\alpha \mapsto \alpha' := \begin{cases} s_\beta \alpha, & \alpha \in R^0, \\ -s_\beta \alpha, & \alpha \in R^1 \cup R^2 \end{cases}$$

acts trivially on $R^0 \cup R^2$ *and permutes* R^1.

(iii) *If* $\lambda \in P$ *with* $s_j \lambda \neq \lambda$ *for some* $j = 0, \ldots, n$, *then* $s_j \widetilde{\lambda} = \widetilde{s_j \lambda}$ *with*

$$\widetilde{\lambda} = \lambda + \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \alpha^\vee, \lambda \rangle) \alpha,$$

where $\epsilon(t) = -1$ *for* $t \leq 0$ *and* $\epsilon(t) = 1$ *for* $t > 0$.

Proof.

(i) This is immediate from Proposition A.1.

(ii) The assertion is clear for R^0 and R^2. Let $\alpha \in R^1$. Then

$$\langle \alpha'^\vee, \beta \rangle = \langle \alpha^\vee, -s_\beta \beta \rangle = \langle \alpha^\vee, \beta \rangle = 1. \tag{A.2}$$

By part (i) we have $\alpha' \in R^1$. Finally, the injectivity of s_β shows that $\alpha \mapsto \alpha'$ permutes R^1.

(iii) We consider the cases $j > 0$ and $j = 0$.

$j > 0$: Write

$$\widetilde{\lambda} = \lambda + \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle > 0}} k_\alpha \alpha - \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle < 0}} k_\alpha \alpha - \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle = 0}} k_\alpha \alpha.$$

Note that $s_j R_+ = (R_+ \setminus \{\alpha_j, 2\alpha_j\}) \cup \{-\alpha_j, -2\alpha_j\}$ if $2\alpha_j \in R$ or $s_j R_+ = (R_+ \setminus \{\alpha_j\}) \cup \{-\alpha_j\}$ if $2\alpha_j \notin R$. Furthermore, by $s_j \lambda \neq \lambda$ we have $\langle \alpha_j^\vee, \lambda \rangle \neq 0$. Together with

$$\langle \alpha^\vee, \lambda \rangle = 0 \quad \text{iff} \quad \langle (s_j \alpha)^\vee, s_j \lambda \rangle = 0$$

we conclude that

$$s_j \sum_{\substack{\alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle = 0}} k_\alpha \alpha = \sum_{\substack{\alpha \in R_+ \\ \langle \alpha^\vee, s_j \lambda \rangle = 0}} k_\alpha \alpha.$$

Moreover, s_j is an bijection between the following two sets

$$\{\alpha_j, 2\alpha_j \neq \alpha \in R_+ \mid \langle \alpha^\vee, \lambda \rangle > 0\} \leftrightarrow \{\alpha_j, 2\alpha_j \neq \alpha \in R_+ \mid \langle \alpha^\vee, s_j \lambda \rangle > 0\}$$

and therefore

$$s_j \sum_{\substack{\alpha_j \neq \alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle > 0}} k_\alpha \alpha = \sum_{\substack{\alpha_j \neq \alpha \in R_+ \\ \langle \alpha^\vee, s_j \lambda \rangle > 0}} k_\alpha \alpha$$

and similar

$$s_j \sum_{\substack{\alpha_j \neq \alpha \in R_+ \\ \langle \alpha^\vee, \lambda \rangle < 0}} k_\alpha \alpha = \sum_{\substack{\alpha_j \neq \alpha \in R_+ \\ \langle \alpha^\vee, s_j \lambda \rangle < 0}} k_\alpha \alpha.$$

Finally, $\langle (s_j \alpha_j)^\vee, \lambda \rangle = -\langle \alpha_j^\vee, \lambda \rangle$ and $s_j(k_{\alpha_j} \alpha_j) = -k_{\alpha_j} \alpha_j$ show in sum

$$s_j \widetilde{\lambda} = \widetilde{s_j \lambda}.$$

$j = 0$: By definition $k_\alpha = k_{\alpha'}$ and together with part (ii) we obtain

$$s_0\widetilde{\lambda} = \beta + s_\beta\lambda + \frac{1}{2}\sum_{\alpha\in R^1} k_\alpha \epsilon(\langle\alpha^\vee,\lambda\rangle)\alpha - \frac{1}{2}\sum_{\alpha\in R^1\cup R^2} k_\alpha\epsilon(\langle\alpha^\vee,\lambda\rangle)\alpha.$$

Hence, to show $s_0\widetilde{\lambda} = \widetilde{\mu}$ with $\mu = s_0\lambda$ it suffices to show that

$$\epsilon(\langle\alpha^\vee,\mu\rangle) = \begin{cases}\epsilon(\langle\alpha^\vee,\lambda\rangle), & \alpha\in R^0,\\ -\epsilon(\langle\alpha^\vee,\lambda\rangle), & \alpha\in R^1\cup R^2.\end{cases}$$

If $\alpha \in R^0$, i.e. $\langle\alpha^\vee,\beta\rangle = 0$, then

$$\langle\alpha^\vee,\mu\rangle = \langle\alpha^\vee,\beta\rangle + \langle\alpha^\vee,s_0\lambda\rangle = \langle\alpha,\lambda\rangle\,,\ \text{i.e. } \epsilon(\langle\alpha^\vee,\mu\rangle) = \epsilon(\langle\alpha^\vee,\lambda\rangle).$$

If $\alpha\in R^1$, i.e. $\langle\alpha^\vee,\beta\rangle = 1$ then by part (ii)

$$\langle\alpha^\vee,\mu\rangle = \langle\alpha^\vee,\beta\rangle + \langle\alpha^\vee,s_\beta\lambda\rangle = 1 - \langle\alpha'^\vee,\beta\rangle = 1 - \langle\alpha^\vee,\lambda\rangle\,. \tag{A.3}$$

The inner products in (A.3) are all integers, i.e. $\epsilon(\langle\alpha^\vee,\mu\rangle) = -\epsilon(\langle\alpha^\vee,\lambda\rangle)$. Finally, for $\alpha\in R^2$, i.e. $\alpha = \alpha' = \beta$, we have

$$\langle\beta^\vee,\mu\rangle = 2 - \langle\beta^\vee,\lambda\rangle\,.$$

Since $s_0\lambda \neq \lambda$ implies that $\langle\beta^\vee,\lambda\rangle \neq 1$, we see that either $\langle\beta^\vee,\lambda\rangle \geq 2$ or $\langle\beta^\vee,\lambda\rangle \leq 0$. In both cases $\epsilon(\langle\beta^\vee,\lambda\rangle) = -\epsilon(\langle\beta^\vee,\mu\rangle)$ holds.

■

Lemma A.3. *Let $(D_\xi)_{\xi\in\mathfrak{a}}$ be the Cherednik operators associated with (R_+,k), i.e.*

$$D_\xi = \partial_\xi - \langle\rho,\xi\rangle + \sum_{\alpha\in R_+} k_\alpha\,\langle\alpha,\xi\rangle\,\frac{1-s_\alpha}{1-e^{-\langle\alpha,\cdot\rangle}}$$

with $\rho = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha\alpha$. Since $W^{\vee,aff}$ acts on P, it induces an action on the complex unital algebra of trigonometric polynomials

$$\mathcal{T} = \operatorname{span}_{\mathbb{C}}\left\{e^\lambda \mid \lambda\in P\right\},\quad e^\lambda = x\mapsto e^{\langle\lambda,x\rangle}.$$

Then the Cherednik operators satisfy the intertwining relation

(i) *For $j = 1,\dots,n$: $s_jD_\xi - D_{s_j\xi}s_j = -(k_{\alpha_j} + 2k_{2\alpha_j})\,\langle\xi,\alpha_j\rangle$, with $k_{2\alpha} = 0$ if $2\alpha\notin R$*

(ii) *$s_0(D_{s_\beta\xi} + \langle\xi,\beta\rangle) - D_\xi s_0 = -k_\beta\,\langle\xi,\beta\rangle$.*

(iii) *The assertions in (i) and (ii) are still valid if D_ξ is considered as an operator $C^1(\mathfrak{a})\to C(\mathfrak{a})$ under the assumption that s_0 acts on arbitrary function $f:\mathfrak{a}\to\mathbb{C}$ by $s_0f(x) = e^\beta f(s_\beta x)$.*

Proof.

(i) First, since s_j is a simple reflection, we have that

$$s_j(R_+ \setminus \{\alpha_j, 2\alpha_j\}) = R_+ \setminus \{\alpha_j, 2\alpha_j\} \, .$$

Put $k_{2\alpha_j} = 0$ if $2\alpha_j \notin R$. From the relations $s_j \partial_\xi = \partial_{s_j\xi} s_j$, $\langle \alpha, \xi \rangle = \langle s_j\alpha, s_j\xi \rangle$ and $s_j \frac{1-s_\alpha}{1-e^{-\alpha}} = \frac{1-s_{s_j\alpha}}{1-e^{-s_j\alpha}} s_j$ we have that

$$\begin{aligned} s_j D_\xi - D_{s_j\xi} s_j &= (\langle \rho, \xi \rangle - \langle \rho, s_j\xi \rangle) s_j \\ &\quad + k_{\alpha_j} \langle \alpha_j, \xi \rangle \left(\frac{1}{1-e^{-\alpha_j}} + \frac{1}{1-e^{\alpha_j}} \right) (1-s_j)s_j \\ &\quad + 2k_{2\alpha_j} \langle \alpha_j, \xi \rangle \left(\frac{1}{1-e^{-2\alpha_j}} + \frac{1}{1-e^{2\alpha_j}} \right) (1-s_j)s_j \\ &= -(k_{\alpha_j} + 2k_{2\alpha_j}) \langle \alpha_j, \xi \rangle s_j + (k_{\alpha_j} + 2k_{2\alpha_j}) \langle \alpha_j, \xi \rangle (1-s_j)s_j \\ &= -(k_{\alpha_j} + 2k_{\alpha_j}) \langle \alpha_j, \xi \rangle \, . \end{aligned}$$

(ii)+(iii) First, we put $\nabla_\alpha = \frac{1-s_\alpha}{1-e^{-\alpha}}$, so that

$$D_\xi s_0 = \partial_\xi s_0 - \langle \rho, \xi \rangle s_0 + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \nabla_\alpha s_0$$

and

$$D_{s_\beta\xi} = \partial_{s_\beta\xi} - \langle \rho, s_\beta\xi \rangle + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, s_\beta\xi \rangle \nabla_\alpha.$$

Since $s_\beta \nabla_\alpha = \nabla_{s_\beta\alpha} s_\beta$ and $s_0 f = e^\beta \cdot s_\beta f$ we have

$$s_0(D_{s_\beta\xi} + \langle \beta, \xi \rangle) = e^\beta \left(\partial_{s_\beta\xi} - \langle \rho, s_\beta\xi \rangle + \langle \xi, \beta \rangle + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, s_\beta\xi \rangle \nabla_{s_\beta\alpha} \right) s_\beta. \quad \text{(A.4)}$$

Moreover, straightforward computations show

(a) $\partial_\xi s_0 = \partial_\xi(e^\beta \cdot s_\beta) = e^\beta(\langle \beta, \xi \rangle + \partial_{s_\beta\xi}) s_\beta$.

(b) Since we have

$$\rho = \frac{1}{2} \left(\sum_{\alpha \in R^0} k_\alpha \alpha + \sum_{\alpha \in R^1 \cup R^2} k_\alpha \alpha \right)$$

we can use Lemma A.2 to obtain

$$s_\beta \rho = \frac{1}{2} \left(\sum_{\alpha \in R^0} k_\alpha \alpha - \sum_{\alpha \in R^1 \cup R^2} k_\alpha \alpha \right) = \rho - \sum_{\alpha \in R^1 \cup R^2} k_\alpha \alpha.$$

(c) For $\alpha \in R^0$ it holds

$$e^\beta \langle \alpha, s_\beta\xi \rangle \nabla_{s_\beta\alpha} s_\beta = \langle \alpha, \xi \rangle \nabla_\alpha s_0.$$

(d) For $\alpha \in R^2$, i.e. $\alpha = \beta$ it holds

$$e^\beta \langle \beta, s_\beta\xi \rangle \nabla_{s_\beta\beta} s_\beta = \langle \beta, \xi \rangle (\nabla_\beta - 1) s_0 - \langle \beta, \xi \rangle \, .$$

(e) For $\alpha \in R^1$ it holds for $\alpha' = -s_\beta\alpha \in R^1$ by Lemma A.2 that

$$e^\beta \langle \alpha, s_\beta\xi \rangle \nabla_{s_\beta\alpha} s_\beta = -e^\beta \langle \alpha', \xi \rangle \nabla_{-\alpha'} s_\beta = \langle \alpha', \xi \rangle (\nabla_{\alpha'} - 1) s_0.$$

Using these formulas, we obtain in (A.4) term by term

$$\begin{aligned} s_0 D_{s_0\xi} &= \partial_\xi s_0 - \langle \beta, \xi \rangle s_0 - \langle \rho, \xi \rangle s_0 + \langle \beta, \xi \rangle s_0 + \sum_{\alpha \in R^1 \cup R^2} k_\alpha \langle \alpha, \xi \rangle s_0 \\ &\quad + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \nabla_\alpha s_0 - k_\beta \langle \beta, \xi \rangle - \sum_{\alpha \in R^1 \cup R^2} k_\alpha \langle \alpha, \xi \rangle s_0 \\ &= D_\xi s_0 - k_\beta \langle \beta, \xi \rangle , \end{aligned}$$

hence the assertion holds.

■

Theorem A.4. *Let $(E_\lambda)_{\lambda \in P}$ be the non-symmetric Heckman-Opdam polynomials associated with (R_+, k). Define $k_i = k_{\alpha_i} + 2k_{2\alpha_i}$ for $i = 1, \ldots, n$ and $k_0 = k_\beta$. Then:*

(i) $E_\lambda = e^\lambda$ *for all* $\lambda \in \mathcal{O}$.

(ii) $cE_{s_i\lambda} = \left(s_i + \frac{k_i}{\langle \alpha_i^\vee, \widetilde{\lambda} \rangle} \right) E_\lambda$ *for some constant* $c \in \mathbb{R}$ *if* $s_i\lambda \neq \lambda$ *with* $i = 1, \ldots, n$.

(iii) $cE_{s_0\lambda} = \left(s_0 + \frac{k_0}{1 - \langle \beta^\vee, \widetilde{\lambda} \rangle} \right) E_\lambda$ *for some constant* $c \in \mathbb{R}$ *if* $s_0\lambda \neq \lambda$.

Proof.

(i) For $\lambda \in \mathcal{O}$ and $\alpha \in R_+$ we have

$$\frac{e^\lambda - s_\alpha e^\lambda}{1 - e^{-\alpha}} = e^\lambda \frac{1 - e^{-\langle \alpha^\vee, \lambda \rangle \alpha}}{1 - e^{-\alpha}} = \begin{cases} e^\lambda, & \langle \alpha^\vee, \lambda \rangle = 1, \\ 0, & \langle \alpha^\vee, \lambda \rangle = 0. \end{cases}$$

Moreover, from the identity $\widetilde{\lambda} = \lambda - \rho + \sum_{\alpha \in R_+ : \langle \alpha^\vee, \lambda \rangle = 1} k_\alpha \alpha$, we obtain for all $\xi \in \mathfrak{a}$ that $D_\xi e^\lambda = \langle \widetilde{\lambda}, \xi \rangle e^\lambda$. Finally, both e^λ and E_λ have leading coefficient 1 with respect to the ordering $\trianglelefteq$ on P, i.e. part (i) holds by the definition of E_λ.

(ii) Consider $i = 1, \ldots, n$. Let $F := \left(s_i + \frac{k_i}{\langle \alpha_i^\vee, \widetilde{\lambda} \rangle} \right) E_\lambda$. Then Lemma A.3 shows that

$$D_\xi F = \left(s_i D_{s_i \xi} - k_i \langle \xi, \alpha_i \rangle + \frac{k_i}{\langle \alpha_i^\vee, \widetilde{\lambda} \rangle} \right) E_\lambda.$$

The eigenvalue equation for E_λ leads to

$$D_\xi F = \left(\langle s_i \xi, \widetilde{\lambda} \rangle s_i + k_i \frac{\langle \xi, \widetilde{\lambda} \rangle}{\langle \alpha_i^\vee, \widetilde{\lambda} \rangle} - k_i \langle \alpha_i, \xi \rangle \right) E_\lambda.$$

Thus, with $(\xi, \widetilde{\lambda}) - \langle \xi, \alpha_i \rangle \langle \widetilde{\lambda}, \alpha_i^\vee \rangle = \langle s_i \xi, \widetilde{\lambda} \rangle = \langle \xi, s_i \widetilde{\lambda} \rangle$ and Lemma A.2

$$D_\xi F = \langle \xi, s_i \widetilde{\lambda} \rangle F = \langle \xi, \widetilde{s_i \lambda} \rangle F,$$

i.e. F is a scalar multiple of $E_{s_i\lambda}$.

(iii) For $i=0$ we use Lemma A.3 for $F := \left(s_i + \frac{k_i}{\langle \alpha_i^\vee, \widetilde{\lambda}\rangle}\right) E_\lambda$ and the eigenvalue equation for E_λ to obtain that

$$D_\xi F = \left(s_0(D_{s_\beta \xi} + \langle \xi, \beta\rangle) + k_0 \langle \xi, \beta\rangle + \frac{k_0}{1-\langle \beta^\vee, \widetilde{\lambda}\rangle}\right) E_\lambda$$
$$= \left(\langle s_0\xi, \widetilde{\lambda}\rangle\, s_0 + k_0 \frac{\langle \xi, \widetilde{\lambda}\rangle}{1-\langle \beta^\vee, \widetilde{\lambda}\rangle} + k_0 \langle \xi, \beta\rangle\right) E_\lambda.$$

Finally, with $\langle \xi, \widetilde{\lambda}\rangle - \langle \beta^\vee, \widetilde{\lambda}\rangle \langle \xi, \beta\rangle + \langle \xi, \beta\rangle = \langle \xi, s_0\widetilde{\lambda}\rangle$ and Lemma A.2 we have

$$D_\xi F = \langle \xi, s_0\widetilde{\lambda}\rangle\, F = \langle \xi, \widetilde{s_0\lambda}\rangle\, F,$$

i.e. F is a scalar multiple of $E_{s_0\lambda}$.

■

Corollary A.5. *Let $\lambda \in P$. Then choose $w_\lambda \in W^{\vee,aff}$ of minimal length with $\overline{\lambda} = w_\lambda \lambda \in \mathcal{O}$ and reduced expression $w_\lambda = s_{i_1} \cdots s_{i_m}$, $0 \leq i_j \leq n$. Put $\lambda_{(j)} = s_{i_{j-1}} \cdots s_{i_1} \overline{\lambda}$. Then*

$$E_\lambda = (s_{i_m} + c_m) \cdots (s_{i_1} + c_1) e^{\overline{\lambda}},$$

with

$$c_j = \begin{cases} \frac{k_{i_j}}{\langle \alpha_{i_j}^\vee, \widetilde{\lambda_{(j)}}\rangle}, & i_j = 1, \ldots, n, \\ \frac{k_0}{1-\langle \beta^\vee, \widetilde{\lambda_{(j)}}\rangle}, & i_j = 0. \end{cases}$$

Proof. The minimality of w_λ shows that $s_{i_j}\lambda_{(j)} \neq \lambda_{(j)}$. Hence, by Theorem A.4, the stated equation holds up to a scalar multiple. But due to the minimality of w_λ we have $w\overline{\lambda} \neq \lambda$ for all subexpressions w of w_λ^{-1}. So both sides of the claimed formula have coefficient 1 for the term e^λ, i.e. the assertion holds. ■

Theorem A.6. *The numbers c_j in Corollary A.5 are positive and rational in k. In particular, the coefficients in the monomial expansion $E_\lambda = \sum_{\mu \trianglelefteq \lambda} c_{\mu\lambda} E_\mu$ are non-negative.*

Proof. This is the same proof as in [Sah00a] without any adaption in the case of a non-reduced root system. ■

References

[AAS10] B. Amri, J-P. Anker, and M. Sifi. Three results in Dunkl theory. *Colloq. Math. 188, 299–312*, 2010.

[AN21] T. Assiotis and J. Najnudel. The boundary of the orbital beta process. *Mosc. Math. J. 21, 659–694*, 2021.

[Ank15] J-P. Anker. An introduction to Dunkl theory and its analytic aspects. *Sommer School (Banach Converence Center, Bedlewo, Poland), arXiv:1611.08213*, 2015.

[AV19] S. Andraus and M. Voit. Limit theorems for multivariate Bessel processes in the freezing regime. *Stochastic Process. Appl. 129, 4771–4790*, 2019.

[BF97] T.H. Baker and P.J. Forrester. The Calogero-Sutherland model and generalized classical polynomials. *Commun. Math. Phys. 188, 175–216*, 1997.

[BF98] T.H. Baker and P.J. Forrester. Non-Symmetric Jack polynomials and Integral Kernels. *Duke. Math. J. 95, 1–50*, 1998.

[BG15] A. Borodin and V. Gorin. General β-Jacobi corner process and the Gaussian free field. *Comm. Pure Appl. Math. 68, 1774–1844*, 2015.

[BGCG22] F. Benaych-Georges, C. Cuenca, and V. Gorin. Matrix addition and the Dunkl transform at high temperature. *Comm. Math. Phys. 394, 735–795*, 2022.

[Bou07] M. Bouali. Application des théorèmes de Minlos et Poincaré à l' étude asymptotique d'une intégrale orbitale. *Ann. Fac. Sci. Toulouse Math. 16, 49–70*, 2007.

[Bou19] M. Bouali. Olshanski spherical pairs of semigroup type. *Infinite dimensional Analysis, Quantum Probability and Related Topics 22, 1950021*, 2019.

[BR23] D. Brennecken and M. Rösler. The Dunkl-Laplace transform and Macdonald's hypergeometric series. *Trans. Amer. Math. Soc. 376, 2419–2447*, 2023.

[BR24] D. Brennecken and M. Rösler. Limits of Bessel functions for root systems as the rank tends to infinity. *Preprint, arXiv:2401.02515*, 2024.

[Bre23] D. Brennecken. Dunkl convolution and elliptic regularity for Dunkl operators. *Preprint, arXiv:2308.07710*, 2023.

[Bre24] D. Brennecken. Hankel transform, K-Bessel functions and zeta distributions in the Dunkl setting. *J. Math. Anal. Appl. 535, 128125*, 2024.

[Che95a] I. Cherednik. Double affine Hecke algebras and Macdonald's conjecture. *Ann. of Math. 141, 191–216*, 1995.

[Che95b] I. Cherednik. Nonsymmetric Macdonald polynomials. *Int. Math. Res. Not. IMRN 1995, 483–515*, 1995.

[Che05] I. Cherednik. Double affine Hecke algebras. *London. Math. Soc. Lect. Note. Ser., vol. 319, Cambridge Univ. Press*, 2005.

[Cle88] J.L. Clerc. Fonctions K de Bessel pour les algèbres de Jordan. *Harmonic Analysis, P. Eymard, J.P. Pier (Eds.), Lecture Notes in Math. 1359, 122–134*, 1988.

[Cle02] J.L. Clerc. Zeta distributions associated to a representation of a Jordan algebra. *Math. Z. 239, 263–276*, 2002.

[Con63] A.G. Constantine. Some non-central distributions problems in multivariate analysis. *Ann. Math. Statist. 34, 1270–1285*, 1963.

[Cue18] C. Cuenca. Pieri integral formula and asymptotics of Jack unitary characters. *Selecta Math. (N.S.) 24, 2737–2789*, 2018.

[DH19] J. Dziubański and A. Hejna. Hörmander's multiplier theorem for the Dunkl transform. *J. Funct. Anal. 277, 2133–2159*, 2019.

[Dib90] H. Dib. Fonctions de Bessel sur une algèbre de Jordan. *J. Math. Pures Appl. 403–448*, 1990.

[dJ93] M.F.E. de Jeu. The Dunkl transform. *Invent. Math. 133, 147–162*, 1993.

[dJ06] M.F.E. de Jeu. Paley-Wiener Theorems for the Dunkl Transform. *Trans. Amer. Math. Soc. 258, 4225–4250*, 2006.

[dJDO94] M.F.E. de Jeu, C.F. Dunkl, and E. Opdam. Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc. 346, 237–256*, 1994.

[Dun88] C.F. Dunkl. Reflection groups and orthogonal polynomials on the sphere. *Math. Z. 197, 33–60*, 1988.

[Dun89] C.F. Dunkl. Differential-difference operators associated to relflection groups. *Trans. Amer. Math. Soc. 311, 167–183*, 1989.

[Dun90] C.F. Dunkl. Operators commuting with Coxeter group actions on polynomials. In: Invariant Theory and Tableaux, ed. D. Stanton. *IMA Vol. Math. Appl. 19, 107–117*, 1990.

[Dun91] C.F. Dunkl. Integral kernels with reflection group invariance. *Canad. J. Math. 43, 1213–1227*, 1991.

[Dun92] C.F. Dunkl. Hankel transforms associated to finite reflection groups. In: Proc. of. the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications (Tampa, FL, 1991). *Contemp. Math. 138, 123–138*, 1992.

[DX14] C.F. Dunkl and Y. Xu. Orthogonal Polynomials of Several Variables, 2nd ed. *Encyclopedia Math. Appl. 81, Cambridge Univ. Press*, 2014.

[Eti10] P. Etingof. A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups. *Math. Res. Lett. 17, 275–282*, 2010.

[Far08] J. Faraut. Infinite dimensional spherical analysis. With notes by Sho Matsumoto. *COE Lect. Note, 10. Kyushu University, The 21st Century COE Program "DMHF", Fukouka*, 2008.

[FG90] J. Faraut and S. Gindikin. Deux formules d'inversion pour la transformation de Laplace sur un cône symétrique. *C.R. Acad. Sci. Paris Sér. I Math. 310, 5–8*, 1990.

[FJ98] G. Friedlander and M. Joshi. Introduction to The Theory of Distributions, 2nd ed. *Cambridge University Press*, 1998.

[FJK73] M. Flensted-Jensen and T. Koornwinder. The convolution structure for Jacobi function expansions. *Ark. Mat. 11, 245–262*, 1973.

[FK94] J. Faraut and A. Korányi. Analysis on Symmetric Cones. *The Clarendon Press, Oxford University Press, New York*, 1994.

[For10] P.J. Forrester. Log-Gases and Random Matrices. *London Math. Soc. Monogr. Ser., 34 Princeton University Press, Princeton, NJ*, 2010.

[GR87] K.I. Gross and D. St. P. Richards. Special functions of matrix argument. I: Algebraic induction, zonal polynomials, and hypergeometric functions. *Trans. Amer. Math. Soc. 301, 781–811*, 1987.

[GV88] R. Gangolli and V.S. Varadarajan. Harmonic Analysis of Spherical Functions on Real Reductive Groups. *Ergeb. Math. Grenzgeb., 101, Springer Verlag, Berlin*, 1988.

[H03] L. Hörmander. The Analaysis of Linear Partial Differential Operators i. Distribution theory and Fourier analysis. Reprint of the second (1990) edition. *Classics Math., Springer Verlag, Berlin*, 2003.

[Hec87] G. Heckman. Root Systems and Hypergeometric Functions II. *Compos. Math. 64, 353–373*, 1987.

[Hec90] G. Heckman. Hecke algebras and hypergeometric functions. *Invent. Math. 100, 403–417*, 1990.

[Hec91] G. Heckman. An elementary approach to the hypergeometric shift operators of Opdam. *Invent. Math. 103, 341–350*, 1991.

[Hec97] G. Heckman. Dunkl Operators. Séminaire N. Bourbaki 1996/1997. *Astérique 245, Exp. no. 828, 223–246*, 1997.

[Hel84] S. Helgason. Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, Spherical Functions. *Pure Appl. Math. 113, Academic Press*, 1984.

[Hel98] S. Helgason. Integral Geometry and Multitemporal Wave Equations. *Comm. Pure and Appl. Math. 51, 1035–1071*, 1998.

[Hel08] S. Helgason. Geometric Analaysis on Symmetric Spaces (2nd edition). *Math. Surveys Monographs 39, Amer. Math. Soc.*, 2008.

[Her55] C. Herz. Bessel functions of matrix argument. *Ann. of Math. 61, 474–523*, 1955.

[HJ69] S. Helgason and K. Johnson. The bounded spherical functions on symmetric spaces. *Adv. Math. 3, 586–593*, 1969.

[HO87] G. Heckman and E.M. Opdam. Root Systems and Hypergeometric Functions I. *Compos. Math. 64, 329–352*, 1987.

[HO21] G. Heckman and E.M. Opdam. Jacobi polynomials and hypergeometric functions associated with root systems. *In: Encyclopedia of Special Functions, Part II: Multivariable Special Functions, eds. T.H. Koornwinder, J.V. Stokman, Cambridge Univ. Press*, 2021.

[HS94] G. Heckman and H. Schlichtkrull. Harmonic analysis and special functions on symmetric spaces. *Perspect. Math. 16, Academic Press, Inc., San Diego, CA*, 1994.

[HS99] S. Helgason and H. Schlichtkrull. The Paley-Wiener Space of Multitemporal Wave Equation. *Comm. Pure and Appl. Math. 52, 49–52*, 1999.

[Hum90] J.E. Humphreys. Reflection Groups and Coxeter Groups. *Cambridge Stud. Adv. Math. 29. Cambridge University Press, Cambridge*, 1990.

[JP16] S. Jakšić and B. Prangoski. Extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}^d_+)$ by use of the Kernel theorem. *Publ. Inst. Math.(Beograd)(N.S.) Volume 99, 59–65*, 2016.

[Kad97] K.W.J. Kadell. The Selberg-Jack symmetric functions. *Adv. Math. 130, 33–102*, 1997.

[Kan93] J. Kaneko. Selberg integrals and hypergeometric functions associated with Jack polynomials. *SIAM J. Math. Anal. 24, 1986–1110*, 1993.

[Kle14] A. Klenke. Probability. *Springer-Verlag, 2nd ed.*, 2014.

[KO08] B. Krötz and E.M. Opdam. Analysis on the crown domain. *Geom. Funct. Anal. 18, 1326–1421*, 2008.

[KS97] F. Knop and S. Sahi. A recursion and a combinatorial formula for jack polynomials. *Invent. Math. 128 , 9–22*, 1997.

[Liu16] Y. Liu. Dissertation: Riesz distributions and the Laplace transform in the Dunkl setting of type A. *Cornell University*, 2016.

[Mö13] J. Möllers. A geometric quantization of the Kostant-Sekiguchi correspondence for scalar type unitary highest weight representations. *Doc. Math. 18, 785-855*, 2013.

[Mac89] I.G. Macdonald. Hypergeometric Functions 1. *Unpublished Manuscript (1989), available at arXiv:1309.4568v1 from 2013*, 1989.

[Mac00] I.G. Macdonald. Orthogonal polynomials associated with root systems. *Sém. Lothar. Combin. 45, Art. B45a.*, 2000.

[Mac03] I.G. Macdonald. Affine hecke algebras and orthogonal polynomials. *Cambridge Tracts Math. 157, Cambridge Univ. Press, Cambridge*, 2003.

[MT04] H. Mejjaoli and K. Trimèche. Hypoellipticity and hypoanalyticity of the Dunkl Laplacian operator. *Integral Transforms and Special Functions 15, 523–548*, 2004.

[Mui82] R. Muirhead. Aspects of Multivariate Statistical Theory. *John Wiley & Sons, Inc., New York*, 1982.

[NIS10] NIST. Handbook of Mathematical Functions. Eds. F. Olver, D. Lozier, R. Boisvert and C. Clark. *Cambridge University Press, Cambridge*, 2010.

[NPP14] E.K. Narayanan, A. Pasquale, and S. Pusti. Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications. *Adv. Math. vol. 252, 227–259*, 2014.

[Ols90] G. Olshanski. Unitary representations of infinite dimensional paris (g, k) and the formalism of R. Howe. In: Representation of Lie groups and related topics, eds. A. vershik and D. zhelobenko. *Adv. Stud. Contemp. Math., vol. 7, Gordon and Breach, New York, 269–463*, 1990.

[OO97] A. Okounkov and G. Olshanski. shifted Jack polynomials, binomial formula, and applications. *Math. Res. Lett. 4, 69–78*, 1997.

[OO98] A. Okounkov and G. Olshanski. Asymptotics of Jack polynomials as the numver of variables goes to infinity. *Internat. Math. Res. Notices, 641–682*, 1998.

[OO06] A. Okounkov and G. Olshanski. Asymptotics of BC-type orthogonal polynomials as the number of variables goes to infinity. *Contemp. Math. 417, 281–318*, 2006.

[Opd89] E.M. Opdam. Some applications of hypergeometric shift operators. *Invent. Math. 98, 1–18*, 1989.

[Opd93] E.M. Opdam. Dunkl operators, Bessel functions and the discriminant of a finite coxeter group. *Compos. Math. 8, 333–373*, 1993.

[Opd95] E.M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math. 175, 75–121*, 1995.

[Opd00] E.M. Opdam. Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups. With a preface by Toshio Oshima. *MSJ Memoirs 8, Mathematical Society of Japan, Tokyo*, 2000.

[ØS05] B. Ørsted and S. Ben Said. The wave equation for Dunkl operators. *Indag. Mathem. 16, 351–391*, 2005.

[OV96] G. Olshanski and A. Vershik. Ergodic unitarily invariant measures on the spaces of infinite Hermitian matrices. *Amer. Math. Soc. Transl. 175,137–175*, 1996.

[Par06a] J. Parkinson. Buildings and Hecke algebras. *J Algebra 297, 1–49*, 2006.

[Par06b] J. Parkinson. Spherical analysis on affine buildings. *Math. Z. 253, 571–606*, 2006.

[Pic90] D. Pickrell. Separable representation for automorphism groups of infinite symmetric spaces. *J. Funct. Anal. 90, 1-26*, 1990.

[Pic91] D. Pickrell. Mackey analysis of infinite classical motion groups. *Pacific. J. Math. 150, 136–166*, 1991.

[PS93] R.S. Philips and M.M. Shahshahani. Scattering theory for symmetric spaces of noncompact type. *Duke Math. J. 72, 1–29*, 1993.

[Rö98] M. Rösler. Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys. 192, 519–542*, 1998.

[Rö99] M. Rösler. Positivity of Dunkl's intertwining operator. *Duke Math. J. 98, 445–463*, 1999.

[Rö03a] M. Rösler. Dunkl operators: Theory and Applications. *Lecture Notes in Math. 1817, 93–136*, 2003.

[Rö03b] M. Rösler. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc. 355, 110–126*, 2003.

[Rö07] M. Rösler. Bessel convolutions on matrix cones. *Compositio Math. 143, 749–779*, 2007.

[Rö10] M. Rösler. Positive convolution structure for a class of Heckman-Opdam hypergeometric functions of type BC. *J. Funct. Anal. 258, 2779–2800*, 2010.

[Rö20] M. Rösler. Riesz distributions and Laplace transform in the Dunkl setting of type A. *J. Funt. Anal. 278, 108506, 29pp*, 2020.

[Rab08] M. Rabaoui. Asymptotic harmonic analysis on the space of square complex matrices. *J. Lie Theory 18, 645–670*, 2008.

[RKV13] M. Rösler, T. Koornwinder, and M. Voit. Limit transition between hypergeometric functions of type BC and type A. *Compos. Math. 149, 1381–1400*, 2013.

[RR15] H. Remling and M. Rösler. Convolution algebras for Heckman-Opdam polynomials derived from compact Grassmannians. *J. Approx. Theory 197, 30–48*, 2015.

[Rub06] B. Rubin. Riesz potentials and integral geometry in the space of rectangular matrices. *Adv. Math. 205, 549–598*, 2006.

[RV98] M. Rösler and M. Voit. Markov processes related with Dunkl operators. *Adv. Appl. Math. 21, 575–643*, 1998.

[RV13] M. Rösler and M. Voit. Olshanski spherical functions for infinite dimensional motion groups of fixed rank. *J. Lie Theory 23, 899–920*, 2013.

[Sah98] S. Sahi. The binomial formula for nonsymmetric Macdonald polynomials. *Duke Math J. 94, 465–477*, 1998.

[Sah00a] S. Sahi. A new formula for weight mulitplicities and characters. *Duke Math. J. vol 101, 77–85*, 2000.

[Sah00b] S. Sahi. Some properties of Koornwinder polynomials. *Contemp. Math. 254, 395–411*, 2000.

[Sah11] S. Sahi. Binomial coefficients and Littlewood-Richardson coefficients for Jack polynomials. *Int. Math. Res. Not. IMRN, 1597–1612*, 2011.

[Sch08] B. Schapira. Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel. *Geom. Funct. Anal. 18, 222–250*, 2008.

[Sta89] R.P. Stanley. Some combinatorical propoerties of Jack symmetric functions. *Adv. Math. 77, 76–155*, 1989.

[Ste19] K. Stempak. An extension problem for C^∞ functions with symmetries. *J. Approx. Theory 240, 114–125*, 2019.

[STS76] M.A. Semenov-Tian-Shansky. Harmonic analysis on Riemannian symmetric spaces of negative curvature and scattering theory. *Izv. Akad. Nauk. SSSR Ser. Math. 40, 562–592*, 1976.

[SZ07] S. Sahi and G. Zhang. Biorthogonal expansion of non-symmetric Jack functions. *SIGMA Symmetry Integrability Geom. Methods Appl. 3, Paper 106, 9 pp*, 2007.

[Tit39] E.C. Titchmarsh. The Theory of Functions. *2nd ed., Oxford University Press, Oxford*, 1939.

[Tri01] K. Trimèche. The Dunkl Intertwining Operators on Spaces of Functions and Distributions and Integral Representation of its Dual. *Integral Transfrom. Spec. Funct. 12, 349–374*, 2001.

[Tri02] K. Trimèche. Paley-Wiener Theorems for the Dunkl Transform and Dunkl Translation Operators. *Integral Transfrom. Spec. Funct. 13, 17–38*, 2002.

[VK82] A. Vershik and S. Kerov. Characters and factor representations of the infinite unitary group. *Soviet Math. Dokl. 26, 570–574*, 1982.

[Voi15] M. Voit. Product formulas for a two parameter family of Heckman-Opdam hypergeometric functions of type BC. *J. Lie Theory, 9–36*, 2015.

[WAN01] M. Hieber W. Arendt, C. Batty and F. Neubrander. Vector-valued Laplace transforms and Cauchy problems. *Birkhäuser Verlag*, 2001.

[Whi43] H. Whitney. Differentiable even functions. *Duke Math. J. 10, 159–160*, 1943.

List of Symbols

Symbol	Description	Page
$\mathbb{N}_0$	set of non-negative integers	12
$\mathbb{N}$	set of positive integers	12
$\mathbb{Z}$	ring of integers	12
$\mathbb{Q}$	field of rational numbers	12
$\mathbb{R}$	field of real numbers	12
$\mathbb{C}$	field of complex numbers	12
$\mathbb{H}$	skew-field of quaternions	12
$\mathrm{span}_S M$	set of finite linear combinations of M with scalars from S	12
$\bigsqcup, \sqcup$	disjoint union/coproduct	12
$\# M$	cardinality of the set M	12
$\mathbb{Z}_m$	the quotient $\mathbb{Z}/m\mathbb{Z}$	12
$C^m(\Omega)$	complex valued m-times continuously differentiable functions on Ω, $m \in \mathbb{N} \cup \{\infty\}$	12
$C(\Omega), C^0(\Omega)$	complex valued continuous functions on Ω.	12
$C^m(\Omega)$	complex valued m-times continuously differentiable functions on Ω with compact support, $m \in \mathbb{N} \cup \{\infty\}$	12
$C_0(\Omega)$	complex valued continuous functions on Ω vanishing at infinity	12
$(\mathfrak{a}, \langle \cdot, \cdot \rangle)$	Euclidean space	14
$\lvert\cdot\rvert, \lVert\cdot\rVert$	Norm of an Euclidean space	14
s_α	Reflection in hyperplane perpendicular to α	14
$\alpha^\vee$	vector $\frac{2\alpha}{\langle \alpha, \alpha \rangle}$	14
$\mathrm{rk}\, R$	rank of the root system R	14
$W, W(R)$	Reflection group/Weyl group generated by the roots R	14
$A_n, B_n, C_n, D_n, \ldots$	Classification of root systems/finite reflection groups	15
$\mathbb{R}^n_0$	set of vectors $x \in \mathbb{R}^n$ with $x_1 + \ldots + x_n = 0$	15
R_+	system of positive roots	16
R_-	system of negative roots	16
Π	system of simple roots	17
$\mathfrak{a}_+$, C_+	positive Weyl chamber	17
s_i	simple reflection of a reflection group	17
$\mathfrak{a}_\mathbb{C}$	complexification $\mathbb{C} \otimes \mathfrak{a}$ of the real vector space $\mathfrak{a}$	18
wf	group translation of a function f by the element w	18
k, k_α	multiplicity function, i.e. W-invariant function on a root system	18
$T_\xi, T_\xi(k), T_\xi^R$	(rational) Dunkl operators of a root system R and multiplicity k	18
$\lvert\beta\rvert$	weight $\beta_1 + \ldots + \beta_n$ of a multiindex $\beta \in \mathbb{N}_0^n$	19
$\partial_i, \frac{\partial}{\partial x_i}$	partial derivatives	19
∂^β	differential operator $\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$	19
$\mathcal{P}, \mathbb{C}[\mathfrak{a}]$	complex polynomial functions on $\mathfrak{a}$	19
$\mathcal{P}_n$	polynomial functions homogeneous of degree n	19
$p(T(k))$	Dunkl operator associated with the polynomial p	19
$\mathbb{D}(k)$	complex unital algebra of Dunkl operators	19
supp	support of a distribution/function/support	19

Symbol	Description	Page
$M_{p,q}(\mathbb{F})$	space of $p \times q$ matrices over $\mathbb{F}$	141
$\mathbb{R}^{(\infty)}, \mathbb{C}^{(\infty)}$	sequences of real or complex numbers with at most finitely many non-zero entries	143
$\operatorname{spec} X$	spectral values of X	143
$\ll$	a particular order on $\mathbb{R}$	144
$p_m(x)$	power sum symmetric functions, $m \in \mathbb{N}_0$	144
Ω	set of Vershik-Kerov parameters	153
$\operatorname{sing} X$	singular values of X	156
Ω_+	set of positive Vershik-Kerov parameters	159
$W^{\vee,\mathrm{aff}}$	dual affine Weyl group	161

Index