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Dissertation

Factors in graphs

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Abstract

Factors in graphs form a classical area of graph theory. In particular, perfect matchings in cubic graphs and regular factors of regular graphs are well-studied. One major conjecture that is still open was formulated by Fulkerson in 1971 and states that every bridgeless cubic graph has six perfect matchings such that each edge is in exactly two of them. An r-graph, which can be seen as a generalization of a bridgeless cubic graph, is an r-regular graph in which every odd set of vertices is connected to its complement by at least r edges. Similar to the cubic case, Seymour conjectured that every r-graph has $2r$ perfect matchings such that each edge is in exactly two of them. Both conjectures are trivially true for graphs with chromatic index Δ, but widely open in general.

In this thesis we present various new results in the broad area of graph factors; most of them are well-related to the two aforementioned conjectures. In particular we study perfect matchings in r-graphs. The main motivation is to get a better understanding of the structure of graphs that are not Δ-edge-colorable.

Zusammenfassung

Faktoren in Graphen bilden ein klassisches Gebiet der Graphentheorie. Insbesondere perfekte Matchings in kubischen Graphen und reguläre Faktoren von regulären Graphen sind gut erforscht. Eine bedeutende Vermutung, die immer noch offen ist, wurde 1971 von Fulkerson formuliert und besagt, dass jeder brückenlose kubische Graph sechs perfekte Matchings hat, so dass jede Kante in genau zwei von ihnen vorkommt. Ein r-Graph, der als Verallgemeinerung eines brückenlosen kubischen Graphen angesehen werden kann, ist ein r-regulärer Graph, in dem jede ungerade Menge von Knoten mit seinem Komplement durch mindestens r Kanten verbunden ist. Ähnlich wie im kubischen Fall vermutete Seymour, dass jeder r-Graph $2r$ perfekte Matchings hat, so dass jede Kante in genau zwei von ihnen vorkommt. Beide Vermutungen sind trivialerweise wahr für Graphen mit chromatischem Index Δ, aber im Allgemeinen noch weitgehend offen.

In dieser Arbeit stellen wir verschiedene neue Ergebnisse auf dem Gebiet der Graphenfaktoren vor; die meisten stehen in engem Zusammenhang mit den beiden oben genannten Vermutungen. Insbesondere untersuchen wir perfekte Matchings in r-Graphen. Die Hauptmotivation ist, ein besseres Verständnis der Struktur von Graphen die nicht Δ-kantenfärbbar sind, zu erlangen.

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Contents

Chapter 1

Introduction

1.1 Background

A very famous long-standing problem in mathematics has been the 4-Color Conjecture, which was formulated by Guthrie in 1852. He conjectured that the regions of every map in the plane can be colored by four colors such that no two regions sharing a boundary receive the same color. When studying this problem, the notion of a graph turned out to be very useful. In graph theoretical terms, the 4-Color Conjecture states that the vertices of every planar graph can be colored with four colors such that no two adjacent vertices receive the same color. The 4-Color Conjecture bothered mathematicians for more than one century; several proofs and disproofs turned out to be false. Nevertheless, early proofs by Kempe and Tait, despite being incorrect, contained useful new ideas and techniques. In 1880, Tait [83] proved that the 4-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph is 3-edge-colorable. In 1977, the 4-Color Conjecture was finally verified with the help of a computer [6, 7]; now it is a theorem. Moreover, the equivalent formulation of Tait opened the door to some classic areas in graph theory including edge-colorings and factors of graphs.

Vizing proved fundamental results of edge-coloring by showing that $\chi'(G) \leq$

$\Delta(G) + \mu(G)$ for every graph G and $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ if G is simple. The *density* of a graph G, denoted $\Gamma(G)$, is defined by

$$\Gamma(G) = \max\left\{\left\lceil \frac{|E(G[S])|}{\lfloor \frac{1}{2}|S| \rfloor} \right\rceil : S \subseteq V(G), |S| \geq 2\right\}$$

if $|V(G)| \geq 2$ and $\Gamma(G) = 0$ otherwise. In every edge-coloring of G at most $\left\lfloor \frac{1}{2}|S| \right\rfloor$ edges of $G[S]$ can receive the same color for all $S \subseteq V(G)$. As a consequence, $\chi'(G) \geq \Gamma(G)$ and thus, $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$. In the 1970s various authors including Goldberg [26] and Seymour [77] independently conjectured that $\chi'(G) \leq \max\{\Delta(G)+1, \Gamma(G)\}$ for every graph G, which is also known as the Goldberg-Seymour Conjecture. In a very recent breakthrough result, this conjecture was verified by Chen, Jing and Zang [14]. Hence, $\chi'(G) = \max\{\Delta(G), \Gamma(G)\}$ or $\chi'(G) = \max\{\Delta(G)+1, \Gamma(G)\}$ for every graph G. Based on the aforementioned results, it is natural to divide the set of graphs into two classes. A graph G is *class 1* if $\chi'(G) = \Delta(G)$; otherwise G is *class 2*. The density of a given graph can be computed in polynomial time (see for example [17]). Nevertheless, the decision problem whether a given graph G is $\Delta(G)$-edge-colorable is NP-complete, even when reduced to 3-regular graphs [34]. So far, very little is known about the structure of graphs with chromatic index $\Delta + 1$.

First results concerning graph factors were obtained by Petersen and König; these theorems are fundamental results of graph theory in general. In 1891, in his seminal paper “Die Theorie der regulären graphs” [72], Petersen proved that (1) every bridgeless cubic graph has a perfect matching and (2) every $2r$-regular graph can be decomposed into r 2-factors. In 1916, König [49] showed that every regular bipartite graph is class 1. Since then, factors in graphs has been a subject of intensive research. In particular perfect matchings of cubic graphs and, more general, regular factors of regular graphs have been studied. For an excellent overview, the interested reader is referred to [2].

Cubic graphs

A *snarks* is a bridgeless cubic graph that is not 3-edge-colorable. The smallest snark is the well-known Petersen graph, which was discovered in 1898. Snarks play an important role in graph theory, since some notorious hard conjectures are true in general if they are true for bridgeless cubic class 2 graphs. This includes the following two long-standing open conjectures.

Conjecture 1.1.1 (Berge-Fulkerson Conjecture [24]). *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

Conjecture 1.1.2 (Cycle Double Cover Conjecture [78,82]). *Every bridgeless graph has a collection of cycles such that each edge belongs to exactly two of them.*

The Berge-Fulkerson Conjecture was first proposed by Berge, but it was put into print by Fulkerson [24] in 1971 (cf. [77]). The Cycle Double Cover Conjecture was independently proposed by Szekeres [82] in 1973 and Seymour [78] in 1979. As a unifying approach to both conjectures, Jaeger (see [39]) introduced colorings with edges of another graph. For two graphs G and H, an H*-coloring* of G is a mapping $f\colon E(G) \to E(H)$ such that

- if $e_1, e_2 \in E(G)$ are adjacent, then $f(e_1) \neq f(e_2)$,
- for every $v \in V(G)$ there exists a vertex $u \in V(H)$ such that $f(\partial_G(v)) = \partial_H(u)$.

If such a mapping exists, we say H *colors* G. H-colorings have the useful property that the existence of specific substructures in H implies the existence of similar substructures in G. In 1980 Jaeger [37] made the following famous conjecture, which is known as the Petersen Coloring Conjecture.

Conjecture 1.1.3 (Petersen Coloring Conjecture [37]). *The Petersen graph colors every bridgeless cubic graph.*

Since the Petersen graph satisfies the Berge-Fulkerson Conjecture and the Cycle Double Cover Conjecture, these two conjectures are true if the Petersen Coloring Conjecture is true.

All three conjectures are trivially true for bridgeless cubic class 1 graphs but, despite much effort, remain widely open for snarks.

r-graphs

An *r-graph* is an r-regular graph G with $|\partial_G(S)| \geq r$ for every $S \subseteq V(G)$ of odd cardinality. A 3-graph is nothing else than a bridgeless cubic graph; r-graphs can be seen as a generalisation of bridgeless cubic graphs. Note that r-graphs are of even order and might have small edge-cuts separating two sets of even cardinality. The density of an r-regular graph G is at least r; it equals r if and only if G is an r-graph. Hence, r-graphs satisfy the necessary condition for being class 1. Nevertheless, for every $r \geq 3$ there are r-graphs of class 2. By the proof of the Goldberg-Seymour Conjecture, every r-graph has chromatic index either r or $r + 1$. Thus, class 2 r-graphs behave like simple graphs.

Similar to the cubic case, r-graphs are important since some well-known conjectures can be reduced to r-graphs (or are formulated directly for r-graphs). One example is Tutte's 3-Flow Conjecture, which states that every bridgeless graph without 3-edge-cuts has a nowhere-zero 3-flow (see also [12] unsolved problem 48). It is folklore that this conjecture can be reduced to 5-graphs. Furthermore, some conjectures for bridgeless cubic graphs might be true in the more general setting of r-graphs. The following conjecture was proposed by Seymour [77] in 1979 and is known as Seymour's Exact Conjecture.

Conjecture 1.1.4 (Seymour's Exact Conjecture [77])**.** *Every planar r-graph is class 1.*

Note that for $r = 3$ the statement is true by the 4-Color Theorem. Furthermore, Seymour's Exact Conjecture is verified for all $r \in \{4, \dots, 8\}$ by various authors [18, 19, 21, 29]. Also in [77], Seymour generalized the Berge-Fulkerson

Conjecture to r-graphs.

Conjecture 1.1.5 (Generalized Berge-Fulkerson Conjecture [77]). *Every r-graph has $2r$ perfect matching such that each edge is in exactly two of them.*

Similar to the cubic case, the Generalized Berge-Fulkerson Conjecture is trivially true for r-graphs of class 1 but widely open for r-graphs of class 2. Thus, structural properties of snarks and more generally of class 2 r-graphs are of huge interest.

1.2 Outline and contributions of this thesis

In this thesis different problems concerning factors in graphs are considered; most of them are related to the conjectures mentioned above. Our main motivation is to get a better understanding of the structure of graphs with chromatic index $\Delta + 1$. In this section we shortly introduce each topic and summarize the main results.

In Chapter 2 all basic notation concerning graph theory that is used throughout this thesis is defined. In Chapter 3 and 4 we consider some problems about factors in simple graphs. Chapter 5 focuses on regular graphs, whereas in Chapter 6, 7 and 8 r-graphs are under investigation.

Chapter 3: Isolated toughness and component factors

The isolated toughness of a simple graph G, denoted $I(G)$, was first introduced by Yang, Ma and Liu [91] and is defined as follows:

$$I(G) = \min \left\{ \frac{|S|}{iso(G-S)} : S \subseteq V(G), iso(G-S) \geq 2 \right\}$$

if G is not a complete graph and $I(G) = \infty$ otherwise. For $t \in \mathbb{R}$, a simple graph G is *isolated t-tough* if $I(G) \geq t$. The isolated toughness is strongly related to the existence of specific component factors. For instance, in the case of isolated $\frac{1}{n}$-tough graphs the Star Factor Theorem states that for every positive integer n, a simple graph G has a $\{K_{1,i}\colon 1 \leq i \leq n\}$-factor if and only if $iso(G-S) \leq n|S|$ for all $S \subseteq V(G)$ [4, 50]. Similar characterisations are obtained in [92] for isolated $\frac{m}{n}$-tough graphs when n is odd, $n \geq 3$ and $m = 2$. In Chapter 3 we consider the following general problem which was proposed by Kano, Lu and Yu [45] (see also Problem 7.10 in [2]). If n, m are two positive integers and G is a simple graph such that $iso(G-S) \leq \frac{n}{m}|S|$ for all $\emptyset \neq S \subset V(G)$, what factor does G have? We characterize isolated $\frac{m}{n}$-tough graphs in terms of their component factors when $n > m$. This extends the results of [92] and give an answer to the above problem when $n > m$.

Chapter 4: Factors in edge-chromatic critical graphs

A simple graph G is (edge-chromatic) *critical*, if it is class 2 but every proper subgraph has a smaller chromatic index. Clearly, every simple graph contains a critical subgraph. Thus, in order to get a better understanding of the structure of simple class 2 graphs, one attempt is to study critical graphs. One famous conjecture in this field is Vizings's 2-Factor Conjecture [88], which was proposed in 1965 and states that every critical graph has a 2-factor. So far, this conjecture has only been verified for some specific classes of critical graphs such as overfull graphs [28] or critical graphs with large maximum degree in relation to their order [15, 53]. In Chapter 4 we focus on slightly easier statements, which are implied by Vizings's 2-Factor Conjecture. In particular we study the question whether every critical graph has a cycle-factor, which was conjectured to be true in [9]. Our main result in Chapter 4 is that every critical graph with a small number of divalent vertices (compared to its maximum degree) has a cycle-factor.

Chapter 5: Factors intersecting disjoint odd circuits in regular graphs

The Berge-Fulkerson Conjecture has been open for more than 50 years; notwithstanding a solution seems to be far away. Thus, in order to make some progress, weaker conjectures moved into focus. One of them was proposed by Mazzuoccolo [65] in 2013 and states that every bridgeless cubic graph has two perfect matchings such that the complement of their union is bipartite. Clearly, this statement is true if the Berge-Fulkerson Conjecture is true. Very recently, Kardoš, Máčajová and Zerafa [47] proved the following statement, which implies the above conjecture of Mazzuoccolo. If G is a bridgeless cubic graph, $\mathcal{O}$ is a set of pairwise edge-disjoint odd circuits and e is an edge of G, then G has a perfect matching containing e and at least one edge of every element of $\mathcal{O}$. In Chapter 5 we study whether similar statements are true for graphs of higher regularity. Our main results are the following: (1) for every 2-connected $3k$-regular graph G and every set $\mathcal{O}$ of pairwise edge-disjoint odd circuits of G there exists a k-factor F of G such that $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$, and (2) for every 2-connected $4k$-regular graph G and every set $\mathcal{O}$ of pairwise edge-disjoint odd circuits of G there exists a $2k$-factor F of G such that $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$. Furthermore, we show that these results are best possible in some sense.

Chapter 6: Rotation r-graphs

Since the discovery of the Petersen graph, many other non-trivial snarks as well as infinite families of non-trivial snarks (one example are the well-known Flower snarks [35]) were constructed. Hoffmann-Ostenhof and Jatschka [33] introduced a family of highly symmetrical snarks, which they called rotation snarks. Informally a rotation snark is a snark that has a $\frac{2\pi}{3}$-rotation symmetry fixing one vertex and a balanced spanning tree not containing divalent vertices (for a precise definition see [33]). At first glance rotation snarks seem to be very special; nevertheless the Petersen graph as well as the two Loupekine's snarks are rotation snarks (see [33]). Moreover, there are infinitely many cyclically 5-edge-connected rotation snarks, as shown by Máčajová and Škoviera [59]. In Chapter 6 we generalize the notion of rotation snarks to r-graphs of odd regularity and show that every r-graph of odd regularity can be "blown up" to a simple rotation r-graph (which produces many small edge-cuts). As a consequence, some hard long-standing open conjectures including the aforementioned (generalized) Berge-Fulkerson Conjecture and Tutte's 3-Flow Conjecture can be reduced to simple rotation r-graphs. However, our proof heavily relies on the fact that we allow 2-edge-cuts in the definition of rotation r-graphs.

Chapter 7: Pairwise disjoint perfect matchings in r-graphs

Class 2 r-graphs, which exist for every $r \geq 3$, have at most $r-2$ pairwise disjoint perfect matchings. One natural question concerning the structure of r-graphs is the following. What is the maximum number t such that every r-graph has t pairwise disjoint perfect matchings? On one side, every r-graph has a perfect matching [77]. On the other side, snarks are 3-graphs in which every two perfect matchings intersect. In former times the general opinion was that the cubic case is very exclusive. In 1979, Seymour [77] conjectured that when $r \geq 4$ every r-graph has a perfect matching M such that $G - M$ is an $(r-1)$-graph. If true this would imply that every r-graph has $r-2$ pairwise disjoint perfect matchings. It turned out that this is not the case. In 1999, Rizzi [75] constructed r-graphs in which every two perfect matchings intersect for every $r \geq 4$, which completely answers the above question. Nevertheless, all r-graphs with this property that are known so far have a 4-edge-cut. Thus, it is natural to ask whether the situation changes for r-graphs with larger edge-connectivity. Thomassen [84] proposed the problem whether every r-edge-connected r-graph has $r-2$ pairwise disjoint perfect matchings. For $r = 4$ the answer to Thomassen's problem is "no" by Rizzi. As an extension, Mattiolo and Steffen [63] constructed counterexamples when r is a multiple of 4. For the remaining cases, Thomassen's problem is still open. Moreover, there is also very little known about the number of pairwise disjoint perfect matchings in r-graphs whose edge-connectivity is in between 4 and r. Chapter 7 is divided into three parts. In the first part we prove that for every $1 < k < r$ it is NP-complete to decide whether a given r-graph has k pairwise disjoint perfect matchings. In the second part, r-edge-connected r-graph are under investigation. We extend the result of Mattiolo and Steffen to all even integers by constructing r-edge-connected r-graphs without $r-2$ pairwise disjoint perfect matchings when $r \equiv 2$ *(mod* 4). Furthermore, we relate statements on the number of pairwise disjoint perfect matchings in 5-edge-connected 5-graphs to some of the aforementioned conjectures in cubic graphs. In the third part of Chapter 7 r-graphs with arbitrary edge-connectivity are considered. We construct λ-edge-connected r-graphs without $\frac{3}{2}\lambda - 5$ pairwise disjoint perfect matchings, for every even $\lambda \geq 6$ and every $r \geq \lambda$. This result suggests that there might be a relation between the edge-connectivity and the number of pairwise disjoint perfect matchings in r-graphs.

Chapter 8: Complete sets

A set $\mathcal{A}$ of connected r-graphs is r-*complete* if every connected r-graph can be colored by an element of $\mathcal{A}$. As in the cubic case, if there exists an r-complete set in which every element satisfies the generalized Berge-Fulkerson Conjecture, then every r-graph satisfies this conjecture. Thus, one approach to the generalized Berge-Fulkerson Conjecture is to study r-complete sets. The Petersen Coloring Conjecture states that the set whose only element is the Petersen graph is a 3-complete set. Similar to the cubic case, Mazzuoccolo et al. (Problem 4.8 in [68]) asked whether there exists an r-complete set of cardinality 1 for every $r \geq 4$. In Chapter 8 we first prove that for every $r \geq 3$ there is exactly one inclusion-wise minimal r-complete set, which is denoted by $\mathcal{H}_r$. Next, we show that either $\mathcal{H}_3$ consists of the Petersen graph or it is an infinite set. Moreover, we prove that $\mathcal{H}_r$ is an infinite set for every $r \geq 4$, which gives a negative answer to the problem of Mazzuoccolo et al. As a by-product we determine the smallest r-graphs of class 2.

1.3 Publications and preprints

Major parts of this thesis have been already published (or are available as preprints). This thesis is based on the following publications (the numbering is consistent with the bibliography).

[54] Y. Ma, D. Mattiolo, E. Steffen and I. H. Wolf. Pairwise disjoint perfect matchings in r-edge-connected r-regular graphs. *SIAM J. on Discrete Math.*, 37(3):1548-1565, 2023.

[55] Y. Ma, D. Mattiolo, E. Steffen and I. H. Wolf. Sets of r-graphs that color all r-graphs. *arXiv:2305.08619*, submitted 2023.

[56] Y. Ma, D. Mattiolo, E. Steffen and I. H. Wolf. Edge-connectivity and pairwise disjoint perfect matchings in regular graphs. *Combinatorica*, 44(2):429-440, 2024.

[79] E. Steffen and I. H. Wolf. Even factors in edge-chromatic-critical graphs with a small number of divalent vertices. *Graphs and Combinatorics*, 38(104), 2022.

[81] E. Steffen and I. H. Wolf. Rotation r-graphs. *Discrete Mathematics*, 347(8):113457, 2024.

[90] I. H. Wolf. Fractional factors and component factors in graphs with isolated toughness smaller than 1. *arXiv:2312.11095* (to appear in *J. Graph Theory*), submitted 2023.

In addition, during his PhD studies the author was involved in the following publications, which are not part of this thesis.

[25] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, J. Renders and I. H. Wolf. Non-double covered cubic graphs. *arXiv:2402.08538*, submitted 2024.

[80] E. Steffen and I. H. Wolf. Bounds for the chromatic index of signed multigraphs. *Discrete Applied Mathematics*, 337:185-189, 2023.

Furthermore, Chapter 5 is based on an ongoing joined work with J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, J. Renders and L. Toffanetti. The results of this chapter were mainly obtained during two research stays in Italy (one in Verona and one in Modena) and are not published yet.

A short explanation which results are already published can be found at the beginning of each chapter.

Chapter 2

Notation and basic definitions

This chapter is designated to introduce all basic notations concerning graphs that are used in this thesis. We mainly follow the notation used in [11]. For notation that we may have missed, the interested reader is referred to [11].

A graph G is a pair $(V(G), E(G))$ consisting of two disjoint sets $V(G)$, $E(G)$ together with a function ψ_G that maps every element of $E(G)$ to an one- or two-elemental subset of $V(G)$. The elements of $V(G)$ are called *vertices*; the elements of $E(G)$ are called *edges*. A graph is *finite*, if its vertex-set and its edge-set are finite. For an edge e, the elements of $\psi_G(e)$ are called the *end-vertices* of e. An edge with only one end-vertex is a *loop*; two edges with the same set of end-vertices are *parallel*. In this thesis we only consider finite graphs without loops that may have parallel edges. Thus, from now on with the notation "graph" we always mean a finite, loopless graph. A graph without parallel edges is called *simple*.

A graph can be represented by a drawing in the plane, where every vertex is represented by one point (drawn as a circle) and every edge is represented by a line connecting the two points corresponding to its end-vertices. A graph is *planar* if it admits a drawing in the plane such that edges intersect only in the points corresponding to their common end-vertices.

Let G be a graph. The number of vertices of G is the *order* of G. For an edge e with end-vertices u, v, we say e *connects* u and v, and we occasionally denote e by uv. Two edges (respectively, two vertices) are *adjacent* if they share an end-vertex (respectively, if they are connected by an edge). For a vertex v, the set of vertices adjacent to v is denoted by $N_G(v)$; the elements of $N_G(v)$ are called *neighbours* of v. For a subset $X \subseteq V(G)$ we write $N_G(X)$ for $\bigcup_{v \in X} N(v) \setminus X$. A vertex v and an edge e are *incident* if v is an end-vertex of e. For two vertices u, v, the number of edges connecting u and v is denoted by $\mu_G(u, v)$; if $\mu_G(u, v) = 1$, then the edge uv is *simple*. Furthermore, $\mu(G) = \max\{\mu_G(u, v) \colon u, v \in V(G)\}$. The *degree* of a vertex v, denoted $d_G(v)$, is the number of edges incident with v. A vertex of degree 2 is called *divalent*; a vertex of degree 0 is called *isolated*. The set of isolated vertices of G is denoted by $Iso(G)$; we write $iso(G)$ for $|Iso(G)|$. The maximum degree of a vertex of G is denoted by $\Delta(G)$. The graph G is *regular*, if every vertex has the same degree; and G is r*-regular*, if every vertex is of degree r. A 3-regular graph is also called a *cubic* graph.

The *underlying graph* of G is the simple graph H with $V(H) = V(G)$ and $\mu_H(u, v) = 1$ if and only if $\mu_G(u, v) \geq 1$.

A graph H is *isomorphic* to G, denoted by $G \cong H$, if there are two bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = \{u, v\}$ if and only if $\psi_H(\phi(e)) = \{\theta(u), \theta(v)\}$. In this case we call the pair of mappings (θ, ϕ) an *isomorphism* between G and H. In particular, an *automorphism* of a graph is an isomorphism of the graph to itself.

A graph H is a *subgraph* of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $\psi_H = \psi_G|_{E(H)}$. In this case, we say G *contains* H. If $V(H) = V(G)$, then H is *spanning*; if $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is *proper*. Two graphs are *disjoint* (*edge-disjoint*, respectively) if their vertex-sets (edge-sets, respectively) are disjoint. If $H_1, \ldots, H_k$ are pairwise edge-disjoint subgraphs of G such that $E(G) = \bigcup_{i=1}^{k} E(H_i)$, then G can be *decomposed* into $H_1, \ldots, H_k$ and $\{H_1, \ldots, H_k\}$ is a *decomposition* of G.

For two disjoint subsets X, Y of $V(G)$ the set of edges with one end-vertex in X and the other in Y is denoted by $E_G(X, Y)$; the cardinality of $E_G(X, Y)$ is denoted by $e_G(X, Y)$. We write $\partial_G(X)$ for $E_G(X, V(G) \setminus X)$; if X is a non-empty proper subset of $V(G)$ we call $\partial_G(X)$ an *edge-cut* of G. For convenience, if X or Y consist of a single vertex we omit the set-brackets in these notations. For example, we write $E_G(v, Y)$ and $\partial_G(v)$ instead of $E_G(\{v\}, Y)$ and $\partial_G(\{v\})$. A *k-edge-cut* is an edge-cut consisting of k edges; an edge of a 1-edge-cut is called a *bridge*. The graph G is *k-edge-connected* if its order is at least 2 and there is no edge-cut with less than k edges; a 1-edge-connected graph is called a *connected* graph. The *edge-connectivity* of G, denoted $\lambda(G)$, is the maximum number t such that G is t-edge-connected. A component of G is a maximum connected subgraph of G. A set of vertices $X \subseteq V(G)$ is a *vertex-cut* of G if $G - X$ has more components than G; a vertex cut consisting of k elements is a *k-vertex-cut*. The only element of a 1-vertex-cut is called a *cut-vertex*. The graph G is *k-connected* if G is connected, $|V(G)| > k$ and every vertex-cut contains at least k vertices.

For a positive integer r, the graph G is an *r-graph* if G is r-regular and $|\partial_G(X)| \geq r$ for every $X \subseteq V(G)$ of odd cardinality.

A spanning subgraph of G is called a *factor* of G. A factor is a *k-factor*, if every vertex is of degree k. The edge-set of a 1-factor of G is called a *perfect matching*; the edge-set of a 1-regular subgraph of G is called a *matching*. For a set of graphs $\mathcal{G}$, a factor is a *$\mathcal{G}$-factor* if every component is isomorphic to an element of $\mathcal{G}$.

For a subset $X \subseteq V(G)$ the subgraph with vertex-set X and whose edge-set consists of all edges of G having both end-vertices in X is denoted by $G[X]$. We say that X *induces* $G[X]$ and call $G[X]$ an *induced* subgraph. We write $G - X$ for $G[V(G) \setminus X]$ and $G - v$ for $G - X$ if X consists of a single vertex v. Similarly, for a subset $E \subseteq E(G)$ the subgraph with edge-set E and whose vertex-set consists of all vertices of G incident with an edge of E is denoted by

$G[E]$. We say that E *induces* $G[E]$. The subgraph with vertex-set $V(G)$ and edge-set $E(G) \setminus E$ is denoted by $G - E$. When $E = \{e\}$, we write $G - e$ instead of $G - E$.

For a subset $X \subseteq V(G)$, a new graph can be obtained form G as follows: add a new vertex w_X; delete all edges having both end-vertices in X; for every remaining edge e with an end-vertex $w \in X$, change the end-vertex w of e to w_X; delete every vertex in X. The resulting graph is denoted by G/X and we say G/X is obtained from G by *identifying* the vertices in X (to a new vertex w_X). Note that we use the same labels for the edges in G and in G/X, i.e. $E(G/X) \subseteq E(G)$. Furthermore, if we do not explicitly introduce another label, the vertex in $V(G/X) \setminus V(G)$ will always be denoted by w_X. We remark, that in the literature the notation G/X is sometimes also used to denote the graph obtained from G by contracting every edge in $G[X]$, i.e. $G[X]$ needs to be connected. In our notation, G/X is also defined when $G[X]$ is not connected.

For a vertex $v \in V(G)$ and a graph H disjoint from G, a new graph G' can be obtained from G as follows: add H; for every edge $e \in E(G)$ incident to v, replace the end-vertex v of e by a vertex of H; delete v. We say G' is obtained from G by *replacing* v with H. Note that there are many different graphs that can be obtained from G by replacing v with H; all of them have vertex-set $(V(G) \setminus v) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

A *k-edge-coloring* of G is a function $\varphi : E(G) \to \{1, ..., k\}$; the elements of $\{1, ..., k\}$ are called *colors*. A k-edge-coloring φ is *proper*, if no two adjacent edges receive the same color. If a proper k-edge-coloring of G exists, then G is *k-edge-colorable*. The *chromatic index*, denoted $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. If $\chi'(G) = \Delta(G)$, then G is *class 1*; otherwise G is *class 2*.

An *orientation* D of G is a pair of two functions *tail*: $E(G) \to V(G)$ and *head*: $E(G) \to V(G)$ such that for every $e \in E(G)$ the set of end-vertices of e equals $\{tail(e), head(e)\}$. We say e is *directed* from $tail(e)$ towards $head(e)$.

For a vertex $v \in V(G)$ the number of edges directed towards v is the *indegree* of v. For an integer $k \geq 2$ a *nowhere-zero k-flow* is a function $f\colon E(G) \to \{\pm 1, \ldots, \pm(k-1)\}$ together with an orientation $D = (tail, head)$ such that $\sum_{\substack{e \in E(G)\\ tail(e)=v}} f(e) = \sum_{\substack{e' \in E(G)\\ head(e')=v}} f(e')$ for every $v \in V(G)$.

A *cycle* is a graph in which every vertex is of positive even degree. The graph G is *acyclic* if it does not contain a cycle; it is *cyclically k-edge-connected* if $G - E$ has at most one component containing a cycle for every edge-cut $E \subseteq E(G)$ of cardinality less than k. A *tree* is a connected acyclic graph T. A vertex of degree 1 is called a *leaf* of T; the set of leaves of T is denoted by $Leaf(T)$. Every edge incident with a leaf of T is a *pendant edge*.

A *circuit* is a 2-regular connected graph. A circuit is *even* (*odd*, respectively) if its order is even (odd, respectively). A circuit of order k is called a *k-circuit*. Up to isomorphism there is only one k-circuit, which is denoted by C_k. For convenience, we also denote a circuit with vertex-set $\{v_1, \ldots, v_k\}$ and edge-set $\{v_k v_1, v_i v_{i+1}\colon i \in \{1, \ldots, k-1\}\}$ by $v_1 \ldots v_k v_1$.

A *path* is a connected graph P in which exactly two vertices are of degree 1 and every other vertex is of degree 2. The vertices of degree 2 of P are called *inner* vertices. If $d_P(u) = d_P(v) = 1$, then P is a *u, v-path* and u, v are the *ends* of P. Up to isomorphism there is only one path of order k, which is denoted by P_k. For convenience, we also denote a path with vertex-set $\{v_1, \ldots, v_k\}$ and edge-set $\{v_i v_{i+1}\colon i \in \{1, \ldots, k-1\}\}$ by $v_1 \ldots v_k$. For two vertices $u, v \in V(G)$ that belong to the same component of G, the *distance* between u and v is the number of edges of a shortest u, v-path contained in G.

A factor is a *cycle-factor*, if it is a cycle; a factor is a *path-factor*, if every component is a path.

A *complete graph* is a simple graph in which every two vertices are adjacent. Up to isomorphism there is only one complete graph of order n, which is denoted by K_n.

A set $X \subseteq V(G)$ of pairwise non-adjacent vertices is called *stable*. If $V(G)$

can be partitioned into two stable sets A, B, then G is *bipartite* and we call $\{A, B\}$ a *bipartition* of G. If additionally G is simple and u, v are adjacent for every $u \in A$, $v \in B$, then G is a *complete bipartite graph.* A complete bipartite graph whose bipartition contains a set of cardinality 1 is called a *star.* For two integers $0 < n \leq m$, up to isomorphism there is only one complete bipartite graph, denoted by $K_{n,m}$, whose bipartition consists of a set of cardinality n and a set of cardinality m.

In all notations defined above, when it is clear which graph we consider, we will omit the lower index that indicates the graph we are referring to.

Chapter 3

Fractional factors, component factors and isolated vertex conditions

This chapter is based on [90]; all results in Chapter 3 can be found in that preprint.

Recall that the isolated toughness of a graph G, denoted $I(G)$, was first introduced in [91] and is defined as follows:

$$I(G) = \min \left\{ \frac{|S|}{iso(G-S)} : S \subseteq V(G), iso(G-S) \geq 2 \right\}$$

if G is not a complete graph and $I(G) = \infty$ otherwise. For $t \in \mathbb{R}$, a graph G is *isolated t-tough* if $I(G) \geq t$. The isolated toughness is strongly related to the existence of specific component factors. Tutte [85] characterized isolated 1-tough graphs by the existence of component factors as follows.

Theorem 3.0.1 (Tutte [85])**.** *Let G be a simple graph. Then, G has a $\{K_{1,1}, C_i \colon i \geq 3\}$-factor if and only if*

$$iso(G-S) \leq |S| \quad \text{for all } S \subset V(G).$$

This result was extended by Amahashi, Kano [4] and Las Vergnas [50] to isolated $\frac{1}{n}$-tough graphs.

Theorem 3.0.2 (Amahashi, Kano [4], Las Vergnas [50]). *Let G be a simple graph and let $n \geq 2$ be an integer. Then, G has a $\{K_{1,i}\colon 1 \leq i \leq n\}$-factor if and only if*

$$iso(G - S) \leq n|S| \quad \textit{for all } S \subset V(G).$$

Kano, Lu and Yu [45] asked for a general relation between isolated toughness and the existence of component factors.

Problem 3.0.3 (Problem 1 in [45], Problem 7.10 in [2]). *Let G be a simple graph and let n, m be two positive integers. If*

$$iso(G - S) \leq \frac{n}{m}|S| \quad \textit{for all } \emptyset \neq S \subset V(G),$$

what factor does G have?

The same authors [92] gave an answer to Problem 3.0.3 when n is odd, $n \geq 3$ and $m = 2$. Let $\mathcal{T}(3)$ be the set of trees that can be obtained as follows (see [92] for a more detailed definition):

1. start with a tree T in which every vertex has degree 1 or 3,
2. insert a new vertex of degree 2 into every edge of T,
3. add a new pendant edge to every leaf of T.

For every integer $k \geq 2$, let $\mathcal{T}(2k+1)$ be the set of trees that can be obtained as follows (see [92] for a more detailed definition):

1. start with a tree T such that for every $v \in V(T)$
   - $d_{T-Leaf(T)}(v) \in \{1, 3, \dots 2k+1\}$, and
   - $2|\{w\colon w \in Leaf(T) \cap N_T(v)\}| + d_{T-Leaf(T)}(v) \leq 2k+1$,
2. insert a new vertex of degree 2 into every edge of $T - Leaf(T)$,
3. for every $v \in T - Leaf(T)$ with $d_{T-Leaf(T)}(v) = 2l + 1 < 2k + 1$, add $k - l - |\{w\colon w \in Leaf(T) \cap N_T(v)\}|$ new pendant edges to v.

Theorem 3.0.4 (Kano, Lu, Yu [92]). *A simple graph G has a $\{P_2, C_3, P_5, T\colon T \in \mathcal{T}(3)\}$-factor if and only if*

$$iso(G - S) \leq \frac{3}{2}|S| \quad \text{for all } S \subset V(G).$$

Theorem 3.0.5 (Kano, Lu, Yu [92]). *Let $k \geq 2$ be an integer. A simple graph G has a $\{K_{1,i}, T\colon 1 \leq i \leq k, T \in \mathcal{T}(2k+1)\}$-factor if and only if*

$$iso(G - S) \leq \frac{2k+1}{2}|S| \quad \text{for all } S \subset V(G).$$

We extend these results and give an answer to Problem 3.0.3 when $n > m$. The main tool are fractional factors, which are defined as follows. Let $g_1, f_1 : V(G) \to \mathbb{Z}$ and $g_2, f_2 : V(G) \to \mathbb{R}$ be functions with $g_i(w) \leq f_i(w)$ for every $w \in V(G)$ and every $i \in \{1, 2\}$. A factor F of G is a (g_1, f_1)*-factor*, if $g_1(w) \leq d_F(w) \leq f_1(w)$ for every $w \in V(G)$. For a function $h : E(G) \to [0, 1]$ we define $d^h(v) := \sum_{e \in \partial_G(v)} h(e)$. If $g_2(w) \leq d^h(w) \leq f_2(w)$ for every $w \in V(G)$, then h is a *fractional* (g_2, f_2)*-factor* of G. Additionally, if $g_2(w) = a$ and $f_2(w) = b$ for every $w \in V(G)$, then a fractional (g_2, f_2)-factor is called a *fractional* $[a, b]$*-factor*.

Furthermore, for every two integers n, m with $0 < m < n$ let $\mathcal{T}_{\frac{n}{m}}$ be the set of trees T such that

- $iso(T - S) \leq \frac{n}{m}|S|$ for all $S \subset V(T)$, and
- for every $e \in E(T)$ there is a set $S^* \subset V(T)$ with $iso((T - e) - S^*) > \frac{n}{m}|S^*|$.

The following theorem is the main result of this chapter.

Theorem 3.0.6. *Let G be a simple graph and let n, m be integers with $0 < m < n$. Then the following statements are equivalent:*

1) *$iso(G - S) \leq \frac{n}{m}|S|$ for every $S \subset V(G)$.*

2) *G has a fractional $[1, \frac{n}{m}]$-factor.*

3) *G has a fractional $[1, \frac{n}{m}]$-factor with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$.*

4) *G has a $\{C_{2i+1}, T\colon 1 \leq i < \frac{m}{n-m}, T \in \mathcal{T}_{\frac{n}{m}}\}$-factor.*

The remainder of this chapter is structured as follows. In Section 3.1 we give a relation between the isolated toughness and the existence of fractional factors, which proves the equivalence of 1), 2) and 3). In Section 3.2 we prove the equivalence of 1) and 4) by using fractional factors. In Section 3.3 we characterize the trees in $\mathcal{T}_{\frac{n}{m}}$ and deduce further structural properties.

3.1 Isolated vertex conditions and fractional factors

There is a strong relation between the isolated toughness of a graph and the existence of fractional $[1, \frac{n}{m}]$-factors. When $\frac{n}{m}$ is an integer, Ma, Wang and Li [57] obtained the following relation.

Theorem 3.1.1 (Ma, Wang, Li [57]). *Let G be a simple graph and $b > 1$ be an integer. Then*

$$iso(G - S) \leq b|S| \quad \textit{for all } S \subset V(G)$$

if and only if G has a fractional $[1, b]$-factor.

As shown by Yu, Kano and Lu [92], similar results are true for isolated $\frac{2}{n}$-tough graphs, where n is an odd integer with $n \geq 3$.

Theorem 3.1.2 (Kano, Lu, Yu [92]). *Let G be a simple graph and $k \geq 1$ be an integer. Then*

$$iso(G - S) \leq \frac{2k+1}{2}|S| \quad \textit{for all } S \subset V(G)$$

if and only if G has a fractional $[1, \frac{2k+1}{2}]$-factor with values in $\{0, \frac{1}{2}, 1\}$.

It turned out that their proof also works for $\frac{m}{n}$-tough graphs, where n, m are arbitrary integers with $0 < m < n$. By substituting $2k + 1$ with n and 2 with m in the proof of Theorem 3.1.2 given in [92], this result can be extended as follows.

Theorem 3.1.3. *Let G be a simple graph and let n, m be integers with $0 < m < n$. Then*

$$iso(G-S) \leq \frac{n}{m}|S| \quad \text{for all } S \subset V(G) \tag{1}$$

if and only if G has a fractional $[1, \frac{n}{m}]$-factor with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$.

For the sake of completeness, in the remainder of this section we state the proof of [92] (with the substitutions mentioned above) and deduce the equivalence of statements 1), 2) and 3) of Theorem 3.0.6.

The main tool to prove Theorem 3.1.2 (respectively Theorem 3.1.3) is provided by the next theorem. For a function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ and a set $X \subseteq V(G)$, set $f(X) := \sum_{x \in X} f(x)$.

Theorem 3.1.4 (Anstee [5], Heinrich et al. [32])**.** *Let G be a graph and $g, f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ with $0 \leq g(x) < f(x)$ for all $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$g(T) - d_{G-S}(T) \leq f(S) \text{ for all } S \subset V(G),$$

where $T = \{v \in V(G) \setminus S \colon d_{G-S}(v) < g(v)\}$.

Proof of Theorem 3.1.3 (cf. Kano, Lu, Yu [92]). Assume that G satisfies (1). Let G^* denote the graph obtained from G by replacing each edge e of G by m parallel edges $e(1), ..., e(m)$. Then $V(G^*) = V(G)$, and $d_{G^*}(v) = m \cdot d_G(v)$ for every $v \in V(G^*)$. Define two functions $g, f : V(G^*) \to \mathbb{Z}^+ \cup \{0\}$ as

$$g(x) = m \qquad \text{and} \qquad f(x) = n \qquad \text{for all } x \in V(G^*).$$

Then $g < f$, and for any $S \subset V(G^*)$, we have

$$\begin{aligned} T &= \{v \in V(G^*) \setminus S \colon d_{G^*-S}(v) < g(v) = m\} \\ &= \{v \in V(G^*) \setminus S \colon d_{G^*-S}(v) = 0\} \\ &= Iso(G-S). \end{aligned}$$

Thus it follows from the above equality and (1) that

$$\begin{aligned} g(T) - d_{G^*-S}(T) &= m \cdot iso(G-S) - 0 \\ &\leq n|S| = f(S). \end{aligned}$$

Hence by Theorem 3.1.4, G^* has a (g, f)-factor F. Now we construct a fractional $[1, \frac{n}{m}]$-factor $h : E(G) \to \{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$ as follows: for every edge e of G, define $h(e) = \frac{k(e)}{m}$ where $k(e)$ is the number of integers $i \in \{1, ..., m\}$ with $e(i) \in E(F)$. It is easy to see, that h is the desired fractional $[1, \frac{n}{m}]$-factor with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$.

Next assume that G has a fractional $[1, \frac{n}{m}]$-factor h. Let $S \subset V(G)$, and let F be the spanning subgraph of G induced by $\{e \in E(G) \colon h(e) \neq 0\}$. Clearly, the neighbours of each isolated vertex u of $G - S$ are contained in S and $d^h(u) \geq 1$, thus we have

$$\begin{aligned} iso(G-S) &\leq \sum_{e \in E_F(Iso(G-S), S)} h(e) \\ &\leq \sum_{x \in S} d^h(x) \leq \frac{n}{m}|S|. \end{aligned}$$

Hence, $iso(G-S) \leq \frac{n}{m}|S|$, i.e. (1) holds. □

The fact, that h has values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$ is not needed in the second part of the proof of Theorem 3.1.3. As a consequence, we obtain the following corollary:

Corollary 3.1.5. *Let G be a simple graph and let n, m be integers with $0 < m < n$. If G has a fractional $[1, \frac{n}{m}]$-factor, then G has a fractional $[1, \frac{n}{m}]$-factor with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$.*

Therefore, the equivalence of statements 1), 2) and 3) of Theorem 3.0.6 is proved.

3.2 Isolated vertex conditions and component factors

In this section we use Theorem 3.1.3 to prove the following equivalence, which completes the proof of Theorem 3.0.6:

Theorem 3.2.1. *Let G be a simple graph and let n, m be integers with $0 < m < n$. Then*

$$iso(G - S) \leq \frac{n}{m}|S| \quad \textit{for all } S \subset V(G)$$

if and only if G has a $\{C_{2i+1}, T\colon 1 \leq i < \frac{m}{n-m}, T \in \mathcal{T}_{\frac{n}{m}}\}$-factor.

Observe that $\frac{m}{n-m} \leq 1$ if and only if $\frac{n}{m} \geq 2$, and hence, $\{C_{2i+1}, T\colon 1 \leq i < \frac{m}{n-m}, T \in \mathcal{T}_{\frac{n}{m}}\} = \mathcal{T}_{\frac{n}{m}}$ in this case.

For two positive integers n, m we say a graph G satisfies the $\frac{n}{m}$*-isolated-vertex-condition*, if $iso(G - S) \leq \frac{n}{m}|S|$ for all $S \subset V(G)$. In order to prove Theorem 3.2.1 we need the following observation.

Observation 3.2.2. *A simple graph G satisfies the $\frac{n}{m}$-isolated-vertex-condition, if and only if every component of G satisfies the $\frac{n}{m}$-isolated-vertex-condition.*

Proof. If G satisfies the $\frac{n}{m}$-isolated-vertex-condition and C is a component of G, then for every $S \subset V(C)$ we have:

$$iso(C - S) \leq iso(G - S) \leq \frac{n}{m}|S|$$

On the other hand, if G is a graph with components $H_1, ..., H_l$ and every component satisfies the $\frac{n}{m}$-isolated-vertex-condition, then for each $S \subset V(G)$ we have:

$$iso(G - S) = \sum_{i=1}^{l} iso\left(H_i - (S \cap V(H_i))\right) \leq \sum_{i=1}^{l} \frac{n}{m}|S \cap V(H_i))| = \frac{n}{m}|S|.$$

□

For a fractional $[1, b]$-factor h of a graph G and $v \in V(G)$, we call v a *$(+)$-vertex* if $d^h(v) > 1$ and a *$(-)$-vertex* if $d^h(v) = 1$.

Proof of Theorem 3.2.1. First, assume that G has a $\{C_{2i+1}, T\colon 1 \leq i < \frac{m}{n-m}, T \in \mathcal{T}_{\frac{n}{m}}\}$-factor F. Let $H_1, ..., H_l$ be the components of F. Clearly, every component of F satisfies the $\frac{n}{m}$-isolated-vertex-condition and thus, F also does. For every $S \subset V(G)$ each isolated vertex of $G - S$ is also an isolated vertex of $F - S$, and thus $iso(G - S) \leq iso(F - S) \leq \frac{n}{m}|S|$.

Next, assume G satisfies $iso(G - S) \leq \frac{n}{m}|S|$ for all $S \subset V(G)$. Let F be an inclusion-wise minimal factor of G, that also satisfies the $\frac{n}{m}$-isolated-vertex-condition. By Theorem 3.1.3, F has a fractional $[1, \frac{n}{m}]$-factor, whereas every spanning proper subgraph of F does not admit such a fractional factor. In particular, for every $e \in E(F)$, the graph $F - e$ does not have a fractional $[1, \frac{n}{m}]$-factor. In conclusion, the following claim holds:

Claim 1. $h(e) \neq 0$ for every $e \in E(F)$ and every fractional $[1, \frac{n}{m}]$-factor h of F.

We now prove that F is the desired factor.

A closed trail of length k (of F) is a sequence $(v_0, e_0, v_1, e_1, ..., e_{l-1}, v_l)$ of alternately vertices and edges of F with $e_i = v_i v_{i+1}$ for all $i < l$ and $v_0 = v_l$.

Claim 2. F does not contain a closed trail of an even length.

Proof of Claim 2. Suppose F contains a closed trail X of an even length. Let e be an arbitrary edge of X. Now fix a fractional $[1, \frac{n}{m}]$-factor h of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$, such that

(i) $h(e)$ is as small as possible,

(ii) with respect to (i), $\sum_{e' \in E(F)} h(e')$ is as small as possible.

Now suppose, there is an edge $e' \in E(F)$ between two $(+)$-vertices. By Claim 1, the edge e' did not receive the value 0. Thus, reducing $h(e')$ by $\frac{1}{m}$ leads to a new fractional $[1, \frac{n}{m}]$-factor with a smaller sum, which contradicts the choice of h. Therefore, the set of $(+)$-vertices (with respect to h) is stable in F. This implies, that an edge of F received the value 1 if and only if it is incident with a vertex of degree 1 in F. As a consequence, $h(e') < 1$ for every edge e' of X. Now we modify the fractional factor h as follows: add $\frac{1}{m}$ and $-\frac{1}{m}$ alternately to the edges of X such that $-\frac{1}{m}$ is added to e (see Figure 3.1).

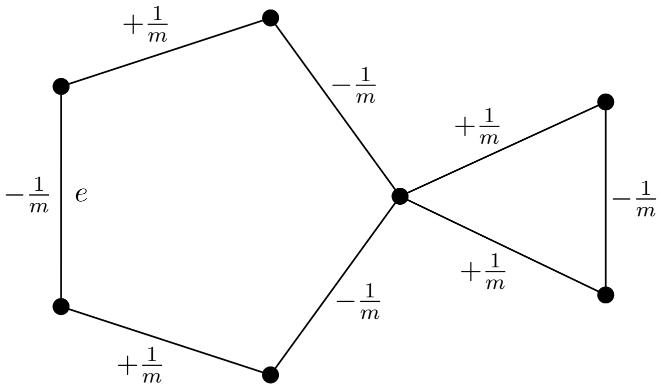


Figure 3.1: The modifying of h if F contains a closed trail of an even length.

Since no edge of X had the value 0 or 1, this leads to a new fractional $[1, \frac{n}{m}]$-factor h' of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$. This contradicts the choice of h, since $h'(e) = h(e) - \frac{1}{m}$. ■

As a consequence the following claims hold:

Claim 3. F does not contain an even circuit.

Claim 4. F does not contain two circuits that share an edge.

Proof of Claim 4. Suppose Claim 4 is false. Then F contains two circuits C, C' such that there common edges induce a path P in F. By Claim 2, the circuits C, C' are odd and thus the graph induced by $(E(C) \cup E(C')) \backslash E(P)$ is an even circuit. This contradicts Claim 3. ■

Claim 5. F does not contain two circuits that share a vertex.

Proof of Claim 5. Suppose F contains two circuits C, C' that share a vertex. By Claim 3, C, C' are odd circuits; by Claim 4, $E(C) \cap E(C') = \emptyset$. Hence, the edgeset $E(C) \cup E(C')$ provides a closed trail of an even length, which contradicts Claim 2. ■

Claim 6. F does not contain two disjoint circuits C, C' and a path P such that $V(C) \cap V(P) = x$ and $V(C') \cap V(P) = y$, where x, y are the ends of P.

Proof of Claim 6. Suppose F contains two disjoint circuits C, C' and a path P with the above properties. Let e be an arbitrary edge of P. Now fix a fractional $[1, \frac{n}{m}]$-factor h of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$, such that

(*i*) $h(e)$ is as small as possible,

(*ii*) with respect to (*i*), $\sum_{e' \in E(F)} h(e')$ is as small as possible.

Again, no two (+)-vertices are adjacent in F. This implies, $h(e') < 1$ for all $e' \in E(C) \cup E(C') \cup E(P)$. Since C and C' are odd by Claim 3, both circuits contain adjacent (−)-vertices. In conclusion, there is a path $P' = (v_1, ..., v_l)$ such that $E(P') \subset E(C) \cup E(C') \cup E(P)$, $e \in E(P')$ and v_1, v_2 are two (−)-vertices of C and v_{l-1}, v_l are two (−)-vertices of C'. Now, add $\frac{1}{m}$ and $-\frac{1}{m}$ alternately to the edges of $P' - \{v_1v_2, v_{l-1}v_l\}$ such that $-\frac{1}{m}$ is added to e. If v_2v_3 or $v_{l-2}v_{l-1}$ received $-\frac{1}{m}$, add $\frac{1}{m}$ to v_1v_2 or $v_{l-1}v_l$, respectively. An example is shown in Figure 3.2.

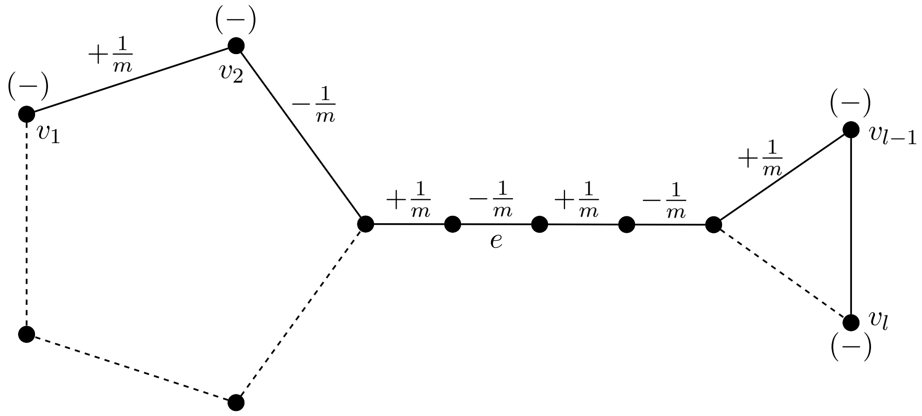


Figure 3.2: The modifying of h if F contains two disjoint circuits connected by a path. The solid edges are the edges of P'.

The resulting function h' has values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$, since no edge of C, C' or P had the value 0 or 1 before. Furthermore, we have $d^{h'}(v) \in \{d^h(v), d^h(v) + \frac{1}{m}\}$ for every $v \in \{v_1, v_2, v_{l-1}, v_l\}$ and $d^{h'}(w) = d^h(w)$ for every other vertex w. Since v_1, v_2, v_{l-1} and v_l are (−)-vertices (with respect to h), h' is a fractional $[1, \frac{n}{m}]$-factor of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$. This contradicts the choice of h, since $h'(e) = h(e) - \frac{1}{m}$. ■

Claim 7. No component of F contains a circuit and a vertex of degree 1.

Proof of Claim 7. Suppose F contains a component with a circuit C and a vertex x with $N_F(x) = \{y\}$. Let $z \in N_F(y) \setminus \{x\}$ be a vertex such that either z and C belong to the same component of $F - y$ or $y, z \in V(C)$. Now, fix a fractional $[1, \frac{n}{m}]$-factor h of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$, such that

(i) $h(yz)$ is as small as possible,

(ii) with respect to (i), $\sum_{e \in E(F)} h(e)$ is as small as possible.

Again, no two (+)-vertices are adjacent in F, which implies that C contains adjacent (−)-vertices. Furthermore, an edge received the value 1 if and only if it is incident with a vertex of degree 1, in particular $h(xy) = 1$ and hence y is a (+)-vertex. By the choice of z, there is a path $P = v_1...v_l$ such that $v_1 = y$, $v_2 = z$ and v_{l-1}, v_l are two (−)-vertices of C. Now, add $\frac{1}{m}$ and $-\frac{1}{m}$ alternately to the edges of $P - v_{l-1}v_l$ such that $-\frac{1}{m}$ is added to yz. If $v_{l-2}v_{l-1}$ received $-\frac{1}{m}$, add $\frac{1}{m}$ to $v_{l-1}v_l$ (see Figure 3.3).

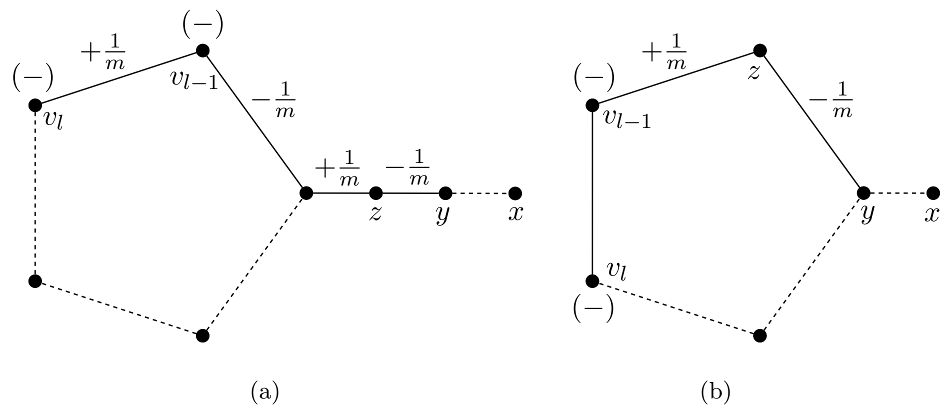


Figure 3.3: The modifying of h if F contains a component with a circuit and a vertex of degree 1 in the cases (a) $y \notin V(C)$ and (b) $y \in V(C)$. The solid edges are the edges of P.

The resulting function is denoted by h'. For each edge $e \in E(P)$ we have

$h(e) > 0$ by Claim 1 and $h(e) < 1$ since P does not contain a vertex of degree 1 in F. In conclusion, h' has values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$. Furthermore we have $d^{h'}(y) = d^h(y) - \frac{1}{m}$, $d^{h'}(v) \in \{d^h(v), d^h(v) + \frac{1}{m}\}$ for every $v \in \{v_{l-1}, v_l\}$ and $d^{h'}(w) = d^h(w)$ for every other vertex w. Since y is a $(+)$-vertex and v_{l-1}, v_l are $(-)$-vertices (with respect to h), h' is a fractional $[1, \frac{n}{m}]$-factor of F with values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$. This contradicts the choice of h, since $h'(yz) = h(yz) - \frac{1}{m}$. ■

By Claims 3-7, each component of F is isomorphic to either an odd circuit or a tree.

Claim 8. If i is a positive integer and C is a component of F isomorphic to C_{2i+1}, then $i < \frac{m}{n-m}$.

Proof of Claim 8. By the choice of F and Observation 3.2.2, no proper subgraph of C satisfies the $\frac{n}{m}$-isolated-vertex-condition. In particular, P_{2i+1} does not satisfy the $\frac{n}{m}$-isolated-vertex-condition. Therefore, $\frac{i+1}{i} > \frac{n}{m}$, which is equivalent to $i < \frac{m}{n-m}$. ■

Claim 9. If T is a component of F that is isomorphic to a tree, then $T \in \mathcal{T}_{\frac{n}{m}}$.

Proof of Claim 9. By Observation 3.2.2, T satisfies the $\frac{n}{m}$-isolated-vertex-condition, whereas no proper subgraph of T satisfies this condition. Hence, $T \in \mathcal{T}_{\frac{n}{m}}$. ■

In conclusion, every component of F is isomorphic to an element of $\{C_{2i+1}, T \colon 1 \leq i < \frac{m}{n-m}, T \in \mathcal{T}_{\frac{n}{m}}\}$ and thus, F is the desired factor. This completes the proof of Theorem 3.2.1. □

3.3 Structural properties of the trees in $\mathcal{T}_{\frac{n}{m}}$

In this section we characterize the trees in $\mathcal{T}_{\frac{n}{m}}$ in terms of their bipartition.

Theorem 3.3.1. *Let n, m be integers with $0 < m < n$ and let T be a tree with bipartition $\{A, B\}$, where $0 < |B| \leq |A|$. Then, the following statements are* *equivalent:*

1) $T \in \mathcal{T}_{\frac{n}{m}}$.

2) *for every* $x \in B$, T *has a fractional* $[1, \frac{n}{m}]$*-factor* h *with values in* $\{\frac{1}{m}, ..., \frac{m-1}{m}, 1\}$ *such that* $d^h(a) = 1$ *for every* $a \in A$, $d^h(b) = \frac{n}{m}$ *for every* $b \in B \setminus \{x\}$ *and* $d^h(x) = \frac{n}{m} + |A| - \frac{n}{m}|B|$.

3) $|A| \leq \frac{n}{m}|B|$ *and for every* $e = xy \in E(T)$: $|V(T_e) \cap A| > \frac{n}{m}|V(T_e) \cap B|$, *where* T_e *is the component of* $T - e$ *that contains the unique vertex in* $\{x, y\} \cap A$.

Proof. 1) $\Rightarrow$ 2). For stars 2) trivially holds. Thus, we assume T is not a star and hence, there is an $u \in Leaf(T - Leaf(T))$. Recall that no fractional $[1, \frac{n}{m}]$-factor of T uses value 0. Let h be a fractional $[1, \frac{n}{m}]$-factor of T with values in $\{\frac{1}{m}, ..., \frac{m-1}{m}, 1\}$, such that

(*i*) $d^h(u)$ is as small as possible,

(*ii*) with respect to (*i*), $\sum_{e \in E(T)} h(e)$ is as small as possible.

Observe that no two (+)-vertices are adjacent and as a consequence, $h(e) = 1$ if and only if e is a pendant edge of T. Furthermore, every vertex adjacent to a leaf of T is a (+)-vertex since T is not isomorphic to K_2.

First, suppose T contains a path $P = uv_1 \dots v_l$ in T such that v_{l-1} is a ($-$)-vertex and $d^h(v_l) < \frac{n}{m}$. Modify h as follows: add $-\frac{1}{m}$ and $\frac{1}{m}$ alternately to the edges of $P - v_{l-1}v_l$ such that $-\frac{1}{m}$ is added to uv_1. If $v_{l-2}v_{l-1}$ received $-\frac{1}{m}$, add $\frac{1}{m}$ to $v_{l-1}v_l$, see Figure 3.4.

Note that v_l is not a leaf, since it is adjacent to a ($-$)-vertex. Hence, no edge of P is a pendant edge of T and thus, every $e \in E(P)$ satisfies $h(e) < 1$. In conclusion, the modification of h, denoted h', has values in $\{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$. Moreover, h' is a fractional $[1, \frac{n}{m}]$-factor of T since $d^h(u) > 1$, $d^h(v_{l-1}) = 1$ and $d^h(v_l) < \frac{n}{m}$. This contradicts the choice of h, since $d^{h'}(u) = d^h(u) - \frac{1}{m}$.

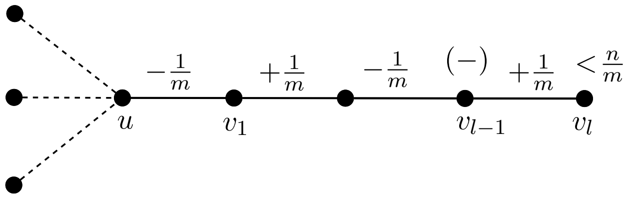


Figure 3.4: The modifying of h if T contains a path $P = uv_1 \dots v_l$ such that v_{l-1} is a $(-)$-vertex and $d^h(v_l) < \frac{n}{m}$. The solid edges belong to P.

The non-existence of such a path implies that the set of $(-)$-vertices is stable and every $v \in V(T) \setminus \{u\}$ that is a $(+)$-vertex satisfies $d^h(v) = \frac{n}{m}$. The former implies that A consists of all $(-)$-vertices and B of all $(+)$-vertices. Hence,

$$|A| = \sum_{a \in A} d^h(a) = \sum_{b \in B} d^h(b) = \frac{n}{m}(|B| - 1) + d^h(u),$$

which implies $d^h(u) = \frac{n}{m} + |A| - \frac{n}{m}|B|$.

Now, let x be an arbitrary $(+)$-vertex, let P be the u,x-path contained in T and let $l = m\left(\frac{n}{m} - d^h(u)\right)$. Note that $|V(P)|$ is odd, since P consists of alternately $(+)$- and $(-)$-vertices. Set $h_0 = h$ and for $i \in \{1, \dots, l\}$ let h_i be the function obtained from h_{i-1} by alternately adding $\frac{1}{m}$ and $-\frac{1}{m}$ to the edges of P such that $\frac{1}{m}$ is added to the edge of P incident with u (see Figure 3.5).

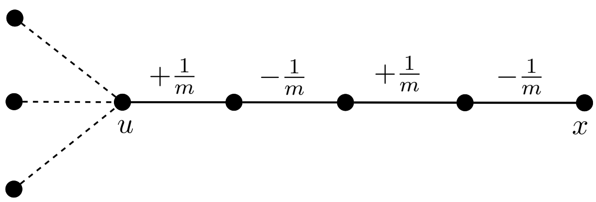


Figure 3.5: The modifying of h_{i-1} to obtain h_i. The solid edges belong to P.

We have $d^{h_l}(u) = d^h(u) + \frac{l}{m} = \frac{n}{m}$ and $d^{h_l}(x) = d^h(x) - \frac{l}{m} = \frac{n}{m} - \frac{l}{m} = d^h(u)$. As a consequence, $d^{h_i}(v) \in [1, \frac{n}{m}]$ for every $i \in \{1, \dots, l\}$ and every $v \in V(T)$. Furthermore, for every $i \in \{1, \dots, l\}$, if h_{i-1} is a fractional $[1, \frac{n}{m}]$-factor that does not use value 0 nor 1 on P, then h_i is a fractional $[1, \frac{n}{m}]$-factor. Thus, h_i also does not use value 0 on P. Moreover, it also does not use value 1 on P, since every edge of P is incident with a $(-)$-vertex (with respect to h_i) that is not a leaf of T. As a consequence, for every $i \in \{1, \ldots, l\}$, h_i only uses values $\frac{1}{m}, \ldots, \frac{m-1}{m}$ on P and therefore, h_l is the desired fractional factor.

2) $\Rightarrow$ 3) By Theorem 3.1.3, T satisfies the $\frac{n}{m}$-isolated-vertex-condition and hence $|A| = iso(T - B) \leq \frac{n}{m}|B|$. Let $e = xy \in E(T)$, where $y \in A$, and let h be a fractional $[1, \frac{n}{m}]$-factor of T with the properties stated in 2) (with pre-described vertex x). Then,

$$\begin{aligned}|V(T_e) \cap A| &= \sum_{v \in V(T_e) \cap A} d^h(v) = h(xy) + \sum_{w \in V(T_e) \cap B} d^h(w) \\ &= h(xy) + \frac{n}{m}|V(T_e) \cap B|,\end{aligned}$$

which proves 3), since $h(xy) > 0$.

3) $\Rightarrow$ 1) For every $e \in E(T)$, statement 3) implies

$$iso\left((T - e) - (V(T_e) \cap B)\right) = |V(T_e) \cap A| > \frac{n}{m}|V(T_e) \cap B|.$$

Thus, by Theorem 3.1.3 it suffice to show that T has a fractional $[1, \frac{n}{m}]$-factor. For every $e \in E(T)$ set

$$h(e) = |V(T_e) \cap A| - \frac{|A|}{|B|}|V(T_e) \cap B|.$$

For every $e \in E(T)$, statement 3) imply

$$h(e) = |V(T_e) \cap A| - \frac{|A|}{|B|}|V(T_e) \cap B| \geq |V(T_e) \cap A| - \frac{n}{m}|V(T_e) \cap B| > 0.$$

By the definition of h, for every $a \in A$ and every $b \in B$ we have

$$\begin{aligned}d^h(a) &= \sum_{e' \in \partial_T(a)} h(e') = (d_T(a) - 1)(|A| - 1) + d_T(a) - \frac{|A|}{|B|}(d_T(a) - 1)|B| \\ &= d_T(a)|A| - |A| + 1 - |A|(d_T(a) - 1) = 1\end{aligned}$$

and

$$d^h(b) = \sum_{e' \in \partial_T(b)} h(e') = |A| - \frac{|A|}{|B|}\left(|B| - 1\right) = \frac{|A|}{|B|}.$$

Note that $1 < \frac{|A|}{|B|} \leq \frac{n}{m}$, since $|A| > |B|$. Furthermore, for every $e = xy \in E(T)$, where $x \in B$, the above calculations imply

$$h(e) = d^h(y) - \sum_{e' \in \partial_T(y) \setminus \{e\}} h(e') \leq 1.$$

In conclusion, h is a fractional $[1, \frac{n}{m}]$-factor of T, which proves $T \in \mathcal{T}_{\frac{n}{m}}$. □

Note that, by the proof of 3) ⇒ 1), every $T \in \mathcal{T}_{\frac{n}{m}}$ has a fractional $[1, \frac{n}{m}]$-factor h such that $d^h(a) = 1$ for every $a \in A$ and $d^h(b) = \frac{|A|}{|B|}$ for every $b \in B$. On the other hand, not every tree with such a factor belongs to $\mathcal{T}_{\frac{n}{m}}$. As the following corollary shows, Theorem 3.3.1 imply some structural properties of trees in $\mathcal{T}_{\frac{n}{m}}$.

Corollary 3.3.2. *Let n, m be integers with $0 < m < n$ and let $T \in \mathcal{T}_{\frac{n}{m}}$ be a tree with bipartition $\{A, B\}$, where $0 < |B| \leq |A|$. Then, the following holds*

- (i) *either $T \cong K_{1,1}$, or $Leaf(T) \subseteq A$,*
- (ii) *$d_T(a) \leq m$ for every $a \in A$,*
- (iii) *$d_T(b) \leq n$ for every $b \in B$,*
- (iv) *$d_T(x) = \lfloor \frac{n}{m} \rfloor + 1$ for every $x \in Leaf(T - Leaf(T))$,*
- (v) *if $n \equiv 1$ (mod m), then either T is a star or $|A| = \frac{n}{m}|B|$ and $|V(T)|$ is a multiple of $n + m$.*

Proof. For stars the statements are trivial. Thus, assume T is not a star and hence, there are two distinct vertices $x_1, x_2 \in Leaf(T - Leaf(T))$. Note that every vertex $v \in Leaf(T - Leaf(T))$ belongs to B, since $h(v) > 1$ for every fractional $[1, \frac{n}{m}]$-factor h of T. Consider two fractional $[1, \frac{n}{m}]$-factors h_1, h_2 of T with the properties stated in statement 2) of Theorem 3.3.1 (with respect to x_1 and x_2, respectively). The existence of h_1 imply $(i), (ii), (iii)$ and $d(x) = \lfloor \frac{n}{m} \rfloor + 1$ for every $x \in Leaf(T - Leaf(T)) \setminus \{x_1\}$. By the existence of h_2 we have $d(x_1) = \lfloor \frac{n}{m} \rfloor + 1$, which proves (iv). Furthermore, if $n \equiv 1$ (mod m), then $\frac{n}{m} = \lfloor \frac{n}{m} \rfloor + \frac{1}{m} \leq d^{h_1}(x_1) \leq \frac{n}{m}$. Hence, $\frac{n}{m} = d^{h_1}(x_1) = \frac{n}{m} + |A| - \frac{n}{m}|B|$, i.e. $|A| = \frac{n}{m}|B|$. Moreover, we observe that $\frac{|B|}{m}$ is an integer, since $\frac{n-1}{m}$ is an integer and $|A| = \frac{n}{m}|B| = \frac{n-1}{m}|B| + \frac{|B|}{m}$. As a consequence, $|V(G)| = |A| + |B| = \frac{n}{m}|B| + |B| = (n+m)\frac{|B|}{m}$, which proves (v). $\square$

By $(i), (iii)$ and (iv), for every $T \in \mathcal{T}_{\frac{n}{1}}$ the set $Leaf(T - Leaf(T))$ is empty, which is equivalent to T being a star. As a consequence, $\mathcal{T}_{\frac{n}{1}} = \{K_{1,i} \colon 1 \leq i \leq n\}$, or equivalently, Theorem 3.0.2 holds.

Chapter 4

Factors in edge-chromatic critical graphs

This chapter (excluding Sections 4.2 and 4.3) is based on [79]. Theorems 4.0.2, 4.0.3 and 4.2.1 are unpublished; all other results in Chapter 4 are published in [79].

Vizing [87] proved the fundamental result on edge-coloring simple graphs by showing that the chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G)+1$. An edge $e \in E(G)$ is *critical*, if $\chi'(G) = \Delta(G)+1$ and $\chi'(G-e) = \Delta(G)$. If G is connected, $\Delta(G) = k$ and all edges of G are critical, then G is k-*critical*. Clearly, every simple graph H with $\chi'(H) = \Delta(H)+1$ contains a $\Delta(H)$-critical subgraph. There had been several conjectures with regard to the order or to (near) perfect matchings of critical graphs, which all turned out to be false, see [9] for a survey. The situation changes when we consider 2-factors. In 1965, Vizing [88] conjectured that every critical graph has a 2-factor. This conjecture has been verified for some specific classes of critical graphs as overfull graphs [28] or critical graphs with large maximum degree in relation to their order [15, 53]. Furthermore, some equivalent formulations or reduction to some classes of critical graphs as e.g. critical graphs of even order are proved in [9, 16]. All these approaches have not yet led to significant progress in answering the question whether critical graphs have a 2-factor.

To gain more insight into structural properties of critical graphs it might be useful to investigate slightly easier statements about factors in critical graphs. If Vizing's 2-factor conjecture is true, then (1) every critical graph has a cycle-factor, (2) every critical graph does not contain an inclusion-wise minimal vertex-cut consisting of an odd number of divalent vertices, and (3) every critical graph has a path-factor. Statements (1) and (3) were conjectured to be true in [9]; Statement (3) is verified in [48]. For Statement (1) note that every bridgeless graph with minimum degree at least 3 has a cycle-factor [23]. Thus, the question whether every critical graph has a cycle-factor is reduced to critical graphs with divalent vertices.

In this chapter, first we prove Statement (1) for critical graphs with a small number of divalent vertices. Next, we show that every vertex-cut (not necessary of odd cardinality) consisting of divalent vertices in a critical graph has a huge cardinality compared to the maximum degree, which partially proves (2). Furthermore, we slightly extend the result of [48]. The following theorems are the main results of Chapter 4:

Theorem 4.0.1. *Let $k \geq 3$ and let G be a k-critical graph. If G has at most $2k-6$ divalent vertices, then G has a cycle-factor.*

Theorem 4.0.2. *Let $k \geq 2$ and let G be a k-critical graph. If A is an inclusion-wise minimal vertex-cut consisting of divalent vertices, then*

$$|A| > \begin{cases} \lceil \frac{1}{4}(k^3 - k^2) \rceil & \text{, if } k \leq 6 \\ \frac{3}{2}(k^2 - k) & \text{, if } k > 6. \end{cases}$$

Theorem 4.0.3. *Let $k \geq 3$ and let G be a k-critical graph. Then, G has a path-factor F with $d_F(v) = 2$ for all $v \in V(G)$ with $d_G(v) = 2$.*

In order to prove the above results, we need some further definitions as well as an observation.

Let φ be a proper k-edge-coloring of a simple graph G and $v \in V(G)$. For a color $i \in \{1,..,k\}$, we say color i is *present* at v if an edge incident to v is colored with color i. Otherwise, color i is *missing* at v. The set of colors present at v is denoted by $\varphi(v)$; the set of colors missing at v is denoted by $\bar{\varphi}(v)$. For two different colors $i, j \in \{1, ..., k\}$, the subgraph induced by the edges that are colored i or j is denoted by $K(i, j)$. Its components are called (i, j)-*Kempe chains* or sometimes just *Kempe chains*. Clearly, a Kempe chain is a path or a circuit. If $\{i, j\} \cap \varphi(v) \neq \emptyset$, then the unique component of $K(i, j)$ that contains v is denoted by $P_v^{\varphi}(i, j)$. We will omit the upper index if this does not cause any ambiguity. A new proper k-edge-coloring, denoted by $\varphi/P_v(i, j)$, can be obtained from φ by interchanging colors i and j in $P_v(i, j)$.

In the proofs of Lemma 4.1.1, 4.1.2 and Theorem 4.2.1 we will use the following basic observation without reference: Let G be a simple graph with a critical edge vw and let φ be a proper $\Delta(G)$-edge-coloring of $G - vw$. If color i is missing at v and j is missing at w, then color i is present at w, color j is present at v and $P_v^{\varphi}(i, j)$ is a v, w-path.

4.1 Cycle-factors

In this section we prove Theorem 4.0.1. In order to do so, we first prove two technical lemmas as well as a theorem.

Lemma 4.1.1. *Let G be a simple graph with $\Delta(G) = k$, $\chi'(G) = k + 1$, and let $A \subseteq V(G)$ be a set of vertices such that*

- *$e_G(A, v) = 1$ for every $v \in N(A)$, and*
- *$N(A) = \{x, y, w_1, ..., w_l\}$ with $l \geq 1$, $d(y) \leq d(x) < k$ and $d(w_i) = 2$ for every $i \in \{1, ..., l\}$.*

If at least one edge in $E_G(A, \{w_1, ..., w_l\})$ is critical, then $l > k(k - d(y)) - d(x) + 1$.

Proof. Let $w \in \{w_1, ..., w_l\}$ be a divalent vertex, let w' be the unique neighbour of w that belongs to A, and let the edge $w'w$ be critical.

Claim 1. There is a proper k-edge-coloring φ of $G - w'w$ such that $\bar{\varphi}(w') = \varphi(w) = \{1\}$ and $1 \in \bar{\varphi}(x)$.

Proof of Claim 1. Since $w'w$ is critical there is a proper k-edge-coloring φ' of $G - w'w$. Furthermore, $\bar{\varphi}'(w') = \varphi'(w) = \{i\}$ for a color $i \in \{1, ..., k\}$. Since $d(x) < k$, there is a color j that is missing at x. If $i = j \neq 1$, then we obtain a coloring with the desired properties by interchanging colors i and 1. If $i \neq j$, then $P_{w'}(i, j)$ is a w', w-path and thus does not contain x. Therefore, the coloring φ'', defined by $\varphi'' = \varphi'/P_{w'}(i, j)$, satisfies $\bar{\varphi}''(w') = \varphi''(w) = \{j\}$ and $j \in \bar{\varphi}''(x)$. Again, if $j \neq 1$, then a coloring with the desired properties can be obtained by interchanging colors j and 1. Thus, the claim is proved. ■

Now fix a proper k-edge-coloring φ of $G - w'w$ with the properties stated in Claim 1. Define a set M as follows:

$$M = \{(h, z, h') : \; z \in N(A) \setminus \{w\}, \{h, h'\} \subseteq \varphi(z), h \neq h', \\ \varphi(e) = h, \text{ where } e \text{ is the unique edge in } E_G(A, z)\}.$$

We prove a lower bound for the number of triples in M, which will be used to obtain the lower bound for l.
For each triple (h, z, h') of M there is a unique Kempe chain P, that contains the two edges incident with z that are colored h and h'. In this case we say P *contains* (h, z, h'). Furthermore, if P is a path and v is an end of P, then we can interpret P as a vertex-list starting with v. This gives an order of the vertices of P and thus an order of the triples contained in P. We define the *first* and the *last* triple contained in P (starting with v) in the natural way. An example is given in Figure 4.1.

If $i \in \{2, ..., k\}$, then the Kempe chain $P_{w'}(1, i)$ is a path with ends w' and w and thus, it contains at least one triple of M. Therefore, we can define a subset M_1 of M as follows: For every $i \in \{2, ...k\}$ let (i_1, z_i, i_2) be the last triple

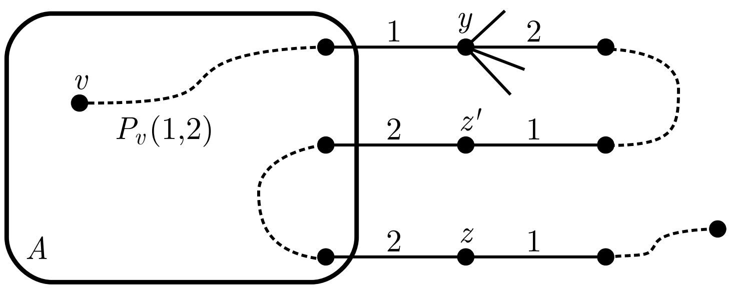


Figure 4.1: $P_v(1,2)$ contains $(1,y,2)$, $(2,z',1)$ and $(2,z,1)$. $(1,y,2)$ is the first and $(2,z,1)$ the last triple contained in $P_v(1,2)$ (starting with v).

contained in $P_{w'}(1,i)$ (starting with w') and set

$$M_1 = \{(i_1, z_i, i_2) : i \in \{2, ...k\}\}.$$

Figure 4.2 shows an example. We note, that $\{i_1, i_2\} = \{1, i\}$ for every $i \in \{2, ..., k\}$ and in particular, x is not in a triple of M_1, since color 1 is missing at x.

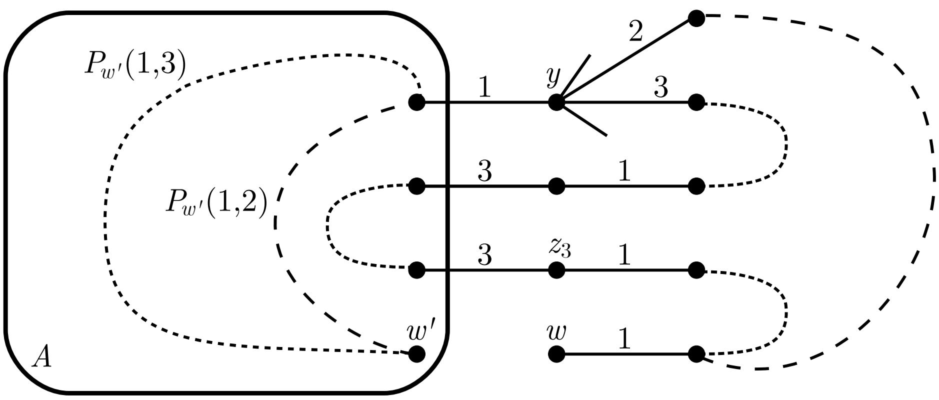


Figure 4.2: The triple $(3,z_3,1)$ is the last triple contained in $P_{w'}(1,3)$; $(1,y,2)$ is the last triple contained in $P_{w'}(1,2)$(starting with w'). Thus, $(3,z_3,1),(1,z_2,2) \in M_1$ where $z_2 = y$.

Claim 2. $|M_1| = k - 1$.

Proof of Claim 2. Let i, i' be two different colors of $\{2, ..., k\}$. Then, $\{i_1, i_2\} = \{1, i\} \neq \{1, i'\} = \{i'_1, i'_2\}$, and hence $(i_1, z_i, i_2) \neq (i'_1, z_{i'}, i'_2)$. ■

Claim 3. Let $i \in \{2, ..., k\}$ and $j \in \bar{\varphi}(z_i)$. Then $P_{z_i}(i_1, j)$ contains a triple of M.

Proof of Claim 3. Suppose, $P_{z_i}(i_1, j)$ does not contain a triple of M. Then, the coloring φ', defined by $\varphi' = \varphi/P_{z_i}(i_1, j)$, satisfies $\bar{\varphi}'(w') = \varphi'(w) = \{1\}$. Since (i_1, z_i, i_2) is the last triple contained in $P_{w'}(1, i)$ (starting with w'), the Kempe chain $P_w^{\varphi'}(i_1, i_2)$ has ends w and z_i. In particular $P_w^{\varphi'}(i_1, i_2)$ is not a w', w-path. We have either $i_1 = 1$ or $i_2 = 1$, a contradiction. See Figure 4.3 for an example. ■

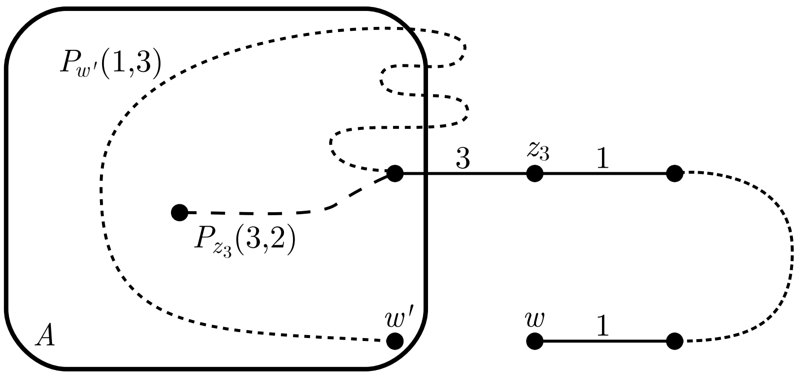


Figure 4.3: The triple $(3, z_3, 1)$ is in M_1. Color 2 is missing at z_3. The Kempe chain $P_{z_3}(3, 2)$ does not contain a triple of M. Interchanging colors 3 and 2 in $P_{z_3}(3, 2)$ produces a contradiction, since $P_w(1, 3)$ is not longer a w', w-path.

Next, define a second subset M_2 of M as follows:

$$M_2 = \{(h, z, h') : i \in \{2, ..., k\}, j \in \bar{\varphi}(z_i), (h, z, h') \text{ is the first triple contained in } P_{z_i}(i_1, j) \text{ (starting with } z_i)\}.$$

An example is given in Figure 4.4.

Claim 4. $M_1 \cap M_2 = \emptyset$.

Proof of Claim 4. We have $z_i \notin \{w', w\}$ for every $i \in \{2, ..., k\}$. Hence, every triple of M_2 is contained in a path with an end that is neither w' nor w, whereas every triple of M_1 is contained in a w', w-path. ■

Claim 5. $|M_2| = |\{(j, z_i) : i \in \{2, ..., k\}, j \in \bar{\varphi}(z_i)\}|$.

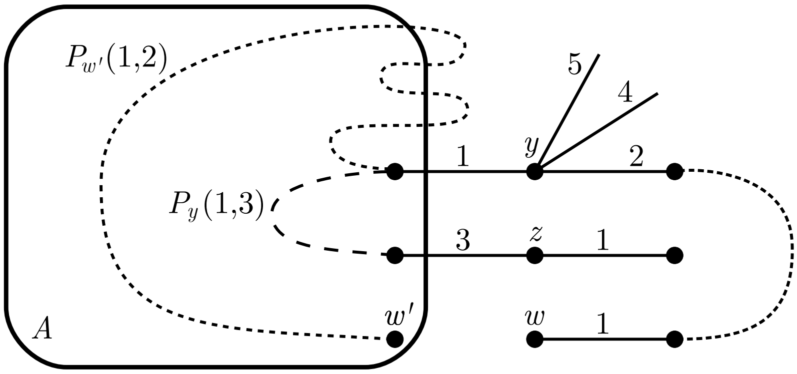


Figure 4.4: The triple $(1, z_2, 2)$ is in M_1, where $z_2 = y$. Color 3 is missing at y. The triple $(3, z, 1)$ is the first triple contained in $P_y(1, 3)$ (starting with y) and thus in M_2.

Proof of Claim 5. Let $i, i' \in \{2, ..., k\}$, $j \in \bar{\varphi}(z_i)$ and $j' \in \bar{\varphi}(z_{i'})$ such that $(j, z_i) \neq (j', z_{i'})$. Then, the paths $P_{z_i}(i_1, j)$ and $P_{z_{i'}}(i'_1, j')$ have at least one different color or a different starting vertex (interpreted as a vertex-list starting with z_i or $z_{i'}$ respectively). Therefore, these two paths have different first triples (in the case $z_i \neq z_{i'}$ we use the fact that both triples are first triples). ■

Claim 6. $|\{(j, z_i) : i \in \{2, ..., k\}, j \in \bar{\varphi}(z_i)\}| \geq (k - d(y))(k - 1)$.

Proof of Claim 6. Since color 1 is missing at x, there is no $i \in \{2, ..., k\}$ with $z_i = x$. If $z_i = z_{i'}$ for two different integers i, i' of $\{2, ..., k\}$, then $z_i = z_{i'} = y$, since every vertex in $N(A) \setminus \{x, y\}$ is divalent. Furthermore, the number of indices $i \in \{2, ..., k\}$ with $z_i = y$ is at most $d(y) - 1$. Vertex y misses $k - d(y)$ colors whereas all other vertices in $N(A)\setminus\{w, x\}$ miss $k-2$ colors. In conclusion:

$$\begin{aligned}&|\{(j, z_i) : i \in \{2, ..., k\}, j \in \bar{\varphi}(z_i)\}|\\ \geq&(k - d(y)) + (k - 1 - (d(y) - 1))(k - 2)\\ =&(k - d(y))(k - 1).\end{aligned}$$

■

We now prove that $l \geq k(k - d(y)) - d(x) + 1$. Since $e_G(A, v) = 1$ for every $v \in N(A)$ and all vertices in $N(A) \setminus \{x, y\}$ are divalent, the inequality $l > |M| - (d(x) - 1) - (d(y) - 1)$ holds. By claims 2, 4, 5 and 6, we have:

$$\begin{aligned} l &> |M| - (d(x) - 1) - (d(y) - 1) \\ &\geq |M_1| + |M_2| - (d(x) - 1) - (d(y) - 1) \\ &\geq k - 1 + (k - d(y))(k - 1) - (d(x) - 1) - (d(y) - 1) \\ &= k(k - d(y)) - d(x) + 1. \end{aligned}$$

□

Lemma 4.1.2. *Let $k > 3$ and let G be a simple graph with $\Delta(G) = k$ and $\chi'(G) = k + 1$. If $E' \subseteq E(G)$ is an inclusion-wise minimal edge-cut consisting of three critical edges, then no edge in E' is incident to a divalent vertex.*

Proof. Let G be a simple graph with $\Delta(G) = k$ and $\chi'(G) = k+1$. Furthermore, let $E' \subseteq E(G)$ be an inclusion-wise minimal edge-cut consisting of three critical edges e_1, e_2 and e_3; let A and B be the components of $G - E'$, and let $e_i = x_i y_i$, where x_i belongs to A and y_i to B. Let G_A be the subgraph induced by $V(A) \cup \{y_1, y_2, y_3\}$, and let G_B be the subgraph induced by $V(B) \cup \{x_1, x_2, x_3\}$. We say a k-edge-coloring φ of G_A or G_B is of

- type 1, if $\varphi(e_1) = \varphi(e_2) = \varphi(e_3)$,
- type 2, if $\varphi(e_1) = \varphi(e_2) \neq \varphi(e_3)$,
- type 3, if $\varphi(e_1) = \varphi(e_3) \neq \varphi(e_2)$,
- type 4, if $\varphi(e_2) = \varphi(e_3) \neq \varphi(e_1)$,
- type 5, if $\varphi(e_1) \neq \varphi(e_2)$, $\varphi(e_1) \neq \varphi(e_3)$, $\varphi(e_2) \neq \varphi(e_3)$.

Suppose to the contrary that there is an edge of E' that is incident to a divalent vertex. We will show that there is a proper k-edge-coloring φ_A of G_A and a proper k-edge-coloring φ_B of G_B such that φ_A and φ_B can be combined to a proper k-edge-coloring of G (by possibly relabeling the colors in one of the colorings), a contradiction.

In order to label the appearing colorings properly, we use the following definition: For an edge $e \in E'$, a k-edge-coloring φ of $G - e$ and a color $i \in \{1, ..., k\}$, let φ_i denote the k-edge-coloring of G obtained from φ by coloring e with i.

Suppose to the contrary that $d(y_1) = 2$. First of all we use the fact that e_1 is critical. Let φ be a proper k-edge-coloring of $G - e_1$ such that w.l.o.g. $\bar{\varphi}(x_1) = \varphi(y_1) = \{1\}$ holds. Thus, for every $i \in \{2, ..., k\}$ the Kempe chain $P_{x_1}(1, i)$ is an x_1, y_1-path. Since $k > 3$, at least two of these paths, say $P_{x_1}(1, 2)$ and $P_{x_1}(1, 3)$, contain w.l.o.g. e_2, which means $\varphi(e_2) = 1$. We first prove that the coloring φ can be used to obtain a proper type 1 and a proper type 2 k-edge-coloring of G_A and a proper type 3, a proper type 4 and a proper type 5 k-edge-coloring of G_B, no matter which color the edge e_3 has received.

Case 1. $\varphi(e_3) = 1$.

In this case, the coloring $\varphi_1 \mid_{E(G_A)}$ is a proper type 1 k-edge-coloring of G_A. On the other hand, $\varphi_2 \mid_{E(G_B)}$ is a proper type 4 k-edge-coloring of G_B. Furthermore, the coloring φ', defined by $\varphi' = \varphi / P_{x_3}(1, 2)$, satisfies $\varphi'(e_3) = 2$, while $\bar{\varphi}'(x_1) = \varphi'(y_1) = \{1\}$ and $\varphi'(e_2) = 1$ still hold. Therefore, $\varphi'_1 \mid_{E(G_A)}$ is a proper type 2 k-edge-coloring of G_A, the coloring $\varphi'_2 \mid_{E(G_B)}$ is a proper type 3 k-edge-coloring of G_B, and $\varphi'_3 \mid_{E(G_B)}$ is a proper type 5 k-edge-coloring of G_B.

Case 2. $\varphi(e_3) \neq 1$.

If $e_3 \in P_{x_1}(1, \varphi(e_3))$, then the coloring φ', defined by $\varphi' = \varphi / P_{x_1}(1, \varphi(e_3))$, satisfies $\bar{\varphi}'(x_1) = \varphi'(y_1) = \{\varphi(e_3)\}$ and $\varphi'(e_2) = \varphi'(e_3) = 1$. Hence, for any $i \in \{2, ..., k\} \setminus \{\varphi(e_3)\}$ the Kempe chain $P^{\varphi'}_{x_1}(i, \varphi(e_3))$ is not an x_1, y_1-path, a contradiction. Therefore, we may assume $e_3 \notin P_{x_1}(1, \varphi(e_3))$, which implies $e_2 \in P_{x_1}(1, \varphi(e_3))$. As a consequence, the coloring φ', defined by $\varphi' = \varphi / P_{x_3}(1, \varphi(e_3))$, satisfies $\bar{\varphi}'(x_1) = \varphi'(y_1) = \{1\}$ and $\varphi'(e_2) = \varphi'(e_3) = 1$. Since $P^{\varphi'}_{x_1}(1, 2)$ and $P^{\varphi'}_{x_1}(1, 3)$ still contain e_2, Case 1 applies.

In both cases there is a proper type 1 and a proper type 2 k-edge-coloring of G_A, and a proper type 3, a proper type 4 and a proper type 5 k-edge-coloring of G_B.
We now use the fact that the edge e_3 is critical as well. Let φ' be a proper k-edge-coloring of $G - e_3$ with $i \in \bar{\varphi}'(x_3)$ and $j \in \bar{\varphi}'(y_3)$. If e_1 and e_2 are colored with the same color, then $\varphi'_j \mid_{E(G_B)}$ is a proper k-edge-coloring of G_B that is of type 1 or 2. On the other hand, if $\varphi'(e_1) \neq \varphi'(e_2)$, then $\varphi'_i \mid_{E(G_A)}$ is a proper type 3, type 4 or type 5 k-edge-coloring of G_A.
In every case there are two proper k-edge-colorings, one of G_A and one of G_B, that are of the same type. This contradicts the fact that G is not k-edge-colorable. □

A graph without a cycle-factor can be characterized as follows (see Theorem 6.2 (p. 221) in [2]).

Theorem 4.1.3 ([2]). *If G is a graph, then G has no cycle-factor, if and only if there is an $X \subset V(G)$ with*

$$\sum_{v \in X} (d(v) - 2) - q(G; X) < 0, \tag{1}$$

where $q(G; X)$ denotes the number of components D of $G - X$ such that $e_G(V(D), X)$ is odd.

We will give a more detailed formulation of Theorem 4.1.3 with regard to a minimal set X that satisfies inequality (1).

Theorem 4.1.4. *If G is a connected graph, then G has no cycle-factor, if and only if there is an $X \subset V(G)$ with the following properties: Let $D_1, \ldots, D_n$ be the components of $G - X$.*

(a) $\sum_{v \in X} (d(v) - 2) - q(G; X) < 0,$

(b) $e_G(V(D_i), v) \leq 1$ *for every* $v \in X$ *and every* $i \in \{1, ..., n\}$,

(c) X *is stable,*

(d) $e_G(V(D_i), X)$ *is odd for every* $i \in \{1, ..., n\}$,

(e) $\sum_{\substack{v \in X \\ d(v) \neq 2}} (d(v) - 3) + \frac{1}{2} \sum_{i=1}^{n} (e_G(V(D_i), X) - 3) < |\{v \in X : d(v) = 2\}|$.

Proof. By Theorem 4.1.3 it suffices to prove one direction. Let G be a connected graph without a cycle-factor. By Theorem 4.1.3, there is a set that satisfies inequality (1). Let $X \subset V(G)$ be the smallest set with $\sum_{v \in X}(d(v) - 2) - q(G; X) < 0$. We show that X satisfies *(b)* - *(e)*.

For each $v \in X$ let $c(v)$ be the number of components D of $G - X$ with $e_G(V(D), X) \equiv 1 \pmod 2$ and $e_G(V(D), v) \geq 1$. We first prove $c(v) = d(v)$ for every $v \in X$, which implies properties *(b)* - *(d)*, since G is connected.

Let $x \in X$ and $X' = X \setminus \{x\}$. By the choice of X, the set X' does not satisfy inequality *(a)*. Furthermore, we observe that $-2|X| + \sum_{v \in X} d(v) - q(G; X)$ is even. As a consequence,

$$\begin{aligned}
0 &\leq \sum_{v \in X'} (d(v) - 2) - q(G; X') \\
&\leq -2|X| + 2 + \sum_{v \in X} d(v) - d(x) - (q(G; X) - c(x)) \\
&= -2|X| + \sum_{v \in X} d(v) - q(G; X) + 2 - d(x) + c(x) \\
&\leq -2 + 2 - d(x) + c(x).
\end{aligned}$$

Thus, $d(x) \leq c(x)$, which implies $d(x) = c(x)$. Therefore, the set X satisfies *(b)* - *(d)*.

Next, by using *(c)* and *(d)* we can transform *(a)* to *(e)* as follows:

$$
\begin{aligned}
& \sum_{v\in X}(d(v)-2) - q(G;X) < 0 \\
\underset{(d)}{\Leftrightarrow}\quad & \sum_{v\in X}(d(v)-2) < n \\
\Leftrightarrow\quad & \frac{1}{2}\sum_{v\in X}(d(v)-2) + \sum_{v\in X}(d(v)-2) < \frac{3}{2}\sum_{i=1}^{n} 1 \\
\underset{(c)}{\Leftrightarrow}\quad & \frac{1}{2}\sum_{i=1}^{n}(e_G(V(D_i),X)) - |X| + \sum_{v\in X}(d(v)-2) < \sum_{i=1}^{n}\frac{3}{2} \\
\Leftrightarrow\quad & \frac{1}{2}\sum_{i=1}^{n}(e_G(V(D_i),X)-3) + \sum_{v\in X}(d(v)-3) < 0 \\
\Leftrightarrow\quad & \frac{1}{2}\sum_{i=1}^{n}(e_G(V(D_i),X)-3) + \sum_{\substack{v\in X\\ d(v)\neq 2}}(d(v)-3) - |\{v\in X : d(v)=2\}| < 0 \\
\Leftrightarrow\quad & \sum_{\substack{v\in X\\ d(v)\neq 2}}(d(v)-3) + \frac{1}{2}\sum_{i=1}^{n}(e_G(V(D_i),X)-3) < |\{v\in X : d(v)=2\}|.
\end{aligned}
$$

□

Proof of Theorem 4.0.1. For $k = 3$ there is nothing to prove. Let $k > 3$. Let G be a k-critical graph without a cycle-factor. Hence, there is a subset $X \subset V(G)$ that satisfies conditions *(a)* -*(e)* of Theorem 4.1.4. We show that X contains more than $2k-6$ divalent vertices.

Let $D_1, ..., D_n$ be the components of $G - X$ and $g : \{D_1, ..., D_n\} \to \mathbb{R}$ with

$$g(D_i) := \sum_{v\in N(V(D_i))} \tfrac{d(v)-2}{d(v)} \text{ for } i \in \{1, ..., n\}.$$

Properties *(a)* -*(d)* imply

$$
\begin{aligned}
\sum_{i=1}^{n} g(D_i) \quad &= \quad \sum_{i=1}^{n}\sum_{v\in N(V(D_i))}\frac{d(v)-2}{d(v)} \underset{(b),(c)}{=} \sum_{v\in X} d(v)-2 \\
&\underset{(a)}{<} \quad q(G;X) \underset{(d)}{=} n.
\end{aligned}
$$

Thus, there is at least one component $D \in \{D_1, ..., D_n\}$ with $g(D) < 1$. Every critical graph does not contain a vertex of degree 1. Therefore, there are at most two vertices in $N(V(D))$ that are not divalent. Moreover, if $N(V(D))$

contains two vertices of degree at least 3, then one of them is of degree 3 and the other is of degree at most 5. Furthermore, since every critical graph is bridgeless, the component D has at least three neighbours in X by *(b)* and *(d)*. In conclusion, we can assume $N(V(D)) = \{x, y, w_1, ..., w_l\}$, where $l \geq 1$, the vertices $w_1, ..., w_l$ are divalent and either $d(y) = 2$, or $d(y) = 3$ and $d(x) \leq 5$. We consider the following two cases:

Case 1. $d(x) < k$.

By condition *(b)* and Lemma 4.1.1 it follows that

$$l > k(k - d(y)) - d(x) + 1 \geq k(k-3) - k + 2 = k(k-4) + 2 \geq 2k - 6.$$

Case 1. $d(x) = k$.

If there are three components adjacent to x such that each has exactly three edges to X, then none of these components is adjacent with a divalent vertex by Lemma 4.1.2. In conclusion, we obtain with property *(b)*

$$\sum_{\substack{i \in \{1,...,n\} \\ x \in N(V(D_i))}} g(D_i) \geq d(x) - 2 + 6\left(\frac{1}{3}\right) = d(x).$$

Since $\sum_{i=1}^{n} g(D_i) < n$, there is another component $D' \in \{D_1, ..., D_n\}$ with $g(D') < 1$, but $x \notin N(D')$. If D' is not adjacent to a vertex of degree k, then $N(V(D'))$ contains at least $2k - 6$ divalent vertices since Case 1 applies. Otherwise X contains at least two vertices of degree k. Since G is bridgeless, property *(d)* implies that $e_G(V(D_i), X) \geq 3$ for every $i \in \{1, ..., n\}$. In conclusion, property *(e)* implies that

$$|\{v \in X : d(v) = 2\}| > 2(k-3) = 2k - 6.$$

If at most two components adjacent to x have exactly three edges to X, then by properties *(b)* - *(d)* there are at least $d(x) - 2$ components such that each has at least five edges to X. Therefore, property *(e)* implies that

$$|\{v \in X : d(v) = 2\}| > d(x) - 3 + (d(x) - 2)\frac{1}{2}(5 - 3) = 2k - 5,$$

and the proof is completed. □

4.2 Vertex-cuts consisting of divalent vertices

In this section we prove Theorem 4.0.2. Lemma 4.1.1 already provides some information about the cardinality of a vertex-cut containing only divalent vertices (in a critical graph). More precisely, there is no k-critical graph with a vertex-cut consisting of $(k-1)^2$ or less divalent vertices. The following theorem, which implies Theorem 4.0.2, improves this bound.

Theorem 4.2.1. *Let G be a simple graph with $\Delta(G) = k$ and $\chi'(G) = k+1$; let $A \subset V(G)$ be a set of divalent vertices. If A is an inclusion-wise minimal vertex-cut and at least one edge in $E_G(A, N(A))$ is critical, then*

$$|A| > \begin{cases} \left\lceil \frac{1}{4}(k^3 - k^2) \right\rceil & \text{, if } k \leq 6 \\ \frac{3}{2}(k^2 - k) & \text{, if } k > 6. \end{cases}$$

Proof. Let $vw \in E(G)$ be a critical edge with $v \in A$ and $w \notin A$; let φ be a proper k-edge-coloring of $G - vw$ with $\varphi(v) = \bar{\varphi}(w) = \{1\}$. We count the number of vertices in $A \setminus \{v\}$ that are incident with a 1-colored edge. The set of such vertices is denoted by M, i.e.

$$M = \{z \colon z \in A \setminus \{v\},\, 1 \in \varphi(z)\}.$$

For each $j \in \{2, ..., k\}$, the Kempe-Chain $P_v(1, j)$ is a v,w-path, thus it contains at least one inner vertex that belongs to M. Some of these paths may contain exactly one inner vertex belonging to M, and we define:

$$M_1 = \{z_j \colon z_j \text{ is the only inner vertex of } P_v(1, j) \text{ that belongs to } A\}.$$

The remaining Kempe-Chains contain at least 3 inner vertices that belong to M, and we define analogously:

$$M_2 = \{z \colon z \in A \setminus M_1,\, z \text{ is an inner vertex of } P_v(1, j) \text{ for one } j \in \{2, ..., k\}\}.$$

Since all vertices in M are divalent, we have $|M_1| \in \{0, ..., k-1\}$, $M_1 \cap M_2 = \emptyset$ and $|M_2| \geq 3((k-1) - |M_1|)$.

Now for every $z_j \in M_1$ and each $i \in \bar{\varphi}(z_j)$, the path $P_{z_j}(1,i)$ also contains at least one inner vertex belonging to M. Otherwise the coloring φ', defined by $\varphi' = \varphi/P_{z_j}(1,i)$, satisfies $\varphi'(v) = \bar{\varphi}'(w) = \{1\}$, but z_j is an endvertex of either $P_v^{\varphi'}(1,j)$ or $P_w^{\varphi'}(1,j)$, a contradiction. Let M_3 be the set of these inner vertices, i.e.

$$M_3 = \{z\colon z_j \in M_1,\, i \in \bar{\varphi}(z_j),\, z \in A,\, z \text{ is an inner vertex of } P_{z_j}(1,i)\}.$$

Each vertex of M_1 misses $k-2$ colors. Furthermore for each vertex $z \in M_3$ there are at most two different pairs (j,i) and (j',i') such that z is an inner vertex of $P_{z_j}(1,i)$ and $P_{z_{j'}}(1,i')$. This happens if and only if $P_{z_j}(1,i) = P_{z_{j'}}(1,i')$. Thus we obtain:

$$|M_3| \geq \left\lceil \frac{|M_1|(k-2)}{2} \right\rceil$$

For every $z_j \in M_1$ and every $i \in \bar{\varphi}(z_j)$ the path $P_{z_j}(1,i)$ has an end that is neither v nor w, and hence $M_3 \cap (M_1 \cup M_2) = \emptyset$. In conclusion $|M|$ is bounded from below as follows:

$$\begin{aligned}
|M| &\geq |M_1| + |M_2| + |M_3| \\
&\geq |M_1| + 3((k-1) - |M_1|) + \left\lceil \frac{|M_1|(k-2)}{2} \right\rceil \\
&= 3k - 3 + \left\lceil \frac{|M_1|(k-6)}{2} \right\rceil \\
&\geq \begin{cases} 3k - 3 + \left\lceil \frac{(k-1)(k-6)}{2} \right\rceil & \text{, if } k \leq 6 \\ 3k-3 & \text{, if } k > 6 \end{cases} \\
&= \begin{cases} \frac{k(k-1)}{2} & \text{, if } k \leq 6 \\ 3(k-1) & \text{, if } k > 6 \end{cases} \quad =: f(k)
\end{aligned}$$

We now choose an arbitrary $h \in \{2, ..., k\}$ and consider the k-edge-coloring φ_h defined by $\varphi_h = \varphi/P_v(1,h)$. This coloring satisfies $\varphi_h(v) = \bar{\varphi}_h(w) = \{h\}$. By using the argumentation above, there are at least $f(k)$ different vertices $z \in A \setminus \{v\}$ with $h \in \varphi_h(z)$. We note that the color swap did not change the colors appearing on a vertex in $A \setminus \{v\}$, i.e. $\varphi(z) = \varphi_h(z)$ for every $z \in A \setminus \{v\}$. Therefore, there are also at least $f(k)$ edges incident to a vertex of $A \setminus \{v\}$ that are colored h in the coloring φ. Since h was arbitrary, there are at least $kf(k)$ different edges incident to a vertex of $A \setminus \{v\}$. By using the fact that every vertex in $A \setminus \{v\}$ is divalent, we finally obtain:

$$|A| > \left\lceil \frac{kf(k)}{2} \right\rceil = \begin{cases} \lceil \frac{1}{4}(k^3 - k^2) \rceil & \text{, if } k \leq 6 \\ \frac{3}{2}(k^2 - k) & \text{, if } k > 6. \end{cases}$$

□

4.3 Path-factors

In this section we prove Theorem 4.0.3. We use the following two results.

Lemma 4.3.1 (Vizing's Adjacency Lemma [89])**.** *Let G be a critical graph and $xy \in E(G)$. Then at least $\Delta(G) - d(y) + 1$ vertices in $N(x) \setminus \{y\}$ have degree $\Delta(G)$.*

For two integers a, b with $0 \leq a \leq b$ and a graph G a factor F of G is an $[a, b]$*-factor* if $a \leq d_F(v) \leq b$ for every $v \in V(G)$. Note that a $[1, 2]$-factor is a factor whose components are paths and circuits. Graphs admitting a $[1, 2]$-factor can be characterized as follows.

Theorem 4.3.2 ([1])**.** *A simple graph G has a $[1, 2]$-factor if and only if*

$$iso(G - S) \leq 2|S| \text{ for all } S \subset V(G).$$

Proof of Theorem 4.0.3. Let G be a critical graph. We first prove that G has a $[1, 2]$-factor F such that every divalent vertex of G is also a divalent vertex in F. Let G^* be the graph obtained from G by splitting each divalent vertex x into two vertices x_1, x_2 of degree 1. That is: delete x; add two new vertices x_1, x_2 of degree 1, where x_1 is adjacent to one former neighbour of x and x_2 is adjacent to the other former neighbour of x. It is easy to see that G has a

$[1,2]$-factor F with $d_F(v) = 2$ for all divalent vertices v of G if and only if G^* has a $[1,2]$-factor. Thus, by Theorem 4.3.2 it is sufficient to show

$$iso(G^* - S) \leq 2|S| \tag{1}$$

for all $S \subset V(G^*)$.

Let S be an arbitrary subset of $V(G^*)$. We use the Discharging-Method to prove (1). Define an initial charge function $ch : Iso(G^* - S) \to \mathbb{R}$ by $ch(u) = 1$ for all $u \in Iso(G^* - S)$. Now ch is modified by moving charge locally around as follows: every $u \in Iso(G^* - S)$ distributes its charge equally among all neighbours. Since each neighbour of a vertex $u \in Iso(G^* - S)$ is in S, this leads to a new charge function $ch' : S \to \mathbb{R}$. Furthermore, since we just move charge around, the following holds:

$$iso(G^* - S) = \sum_{u \in Iso(G^* - S)} ch(u) = \sum_{w \in S} ch'(w).$$

We prove that each $w \in S$ has modified charge at most 2, which implies (1). Let w be an arbitrary vertex of S. If w is of degree 1, then w received at most charge 1 from its neighbour and thus $ch'(w) \leq 2$ obviously holds. If w is adjacent to a vertex v of degree 1, then w is adjacent to v and $\Delta(G) - 1$ vertices of degree $\Delta(G)$ by the construction of G^* and Vizing's Adjacency Lemma. Hence $ch'(w) \leq 1 + \frac{\Delta(G)-1}{\Delta(G)} \leq 2$. Therefore, we may assume, that w and none of its neighbours is of degree 1. As a consequence, each vertex in $\{w\} \cup N(w)$ has the same degree in G^* and G. Let s be the smallest degree of a vertex in $N(w)$ and let n be the number of vertices of degree $\Delta(G)$ in $N(w)$. Clearly, w received charge at most $\frac{1}{\Delta(G)}$ from each neighbour of degree $\Delta(G)$ and at most charge $\frac{1}{s}$ from every other neighbour and thus $ch'(w) \leq \frac{n}{\Delta(G)} + \frac{\Delta(G)-n}{s}$. Furthermore, Vizing's Adjacency Lemma implies $n \geq \Delta(G) - s + 1$. In conclusion:

$$\begin{aligned} ch'(w) &\leq \frac{n}{\Delta(G)} + \frac{\Delta(G) - n}{s} = \left(\frac{1}{\Delta(G)} - \frac{1}{s}\right) n + \frac{\Delta(G)}{s} \\ &\leq \left(\frac{1}{\Delta(G)} - \frac{1}{s}\right) (\Delta(G) - s + 1) + \frac{\Delta(G)}{s} = \frac{\Delta(G) - s + 1}{\Delta(G)} + \frac{s-1}{s} \leq 2. \end{aligned}$$

Thus, G has a $[1, 2]$-factor F with $d_F(v) = 2$ for all divalent vertices v of G. By Vizing's Adjacency Lemma, for $k \geq 3$ every two divalent vertices of a k-critical graph have distance at least 3. Hence, every circuit of F contains two adjacent vertices that are not divalent in G. As a consequence, by deleting an appropriate edge of every circuit, F can be transformed to a path-factor F' of G such that $d_{F'}(v) = 2$ for all $v \in V(G)$ with $d_G(v) = 2$. $\square$

We also note, that a critical graph G has a 2-factor, if and only if G^* has a $[1, 2]$-factor F such that every vertex of degree 1 in F is also a vertex of degree 1 in G^*.

Chapter 5

Factors intersecting disjoint odd circuits in regular graphs

This chapter is based on a joined work with J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, J, Renders and L. Toffanetti, which was mainly carried out during two research stays in Italy (one in Verona and one in Modena). The results in this chapter are not yet published.

Recall that the Berge-Fulkerson Conjecture (Conjecture 1.1.1) states that every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them. If G is a 3-edge-colorable cubic graph, then its edge-set can be partitioned into three perfect matchings. By taking each of these perfect matchings twice we obtain six perfect matchings with the desired property. Thus, Conjecture 1.1.1 reduces to snarks, i.e. bridgeless cubic graphs of class 2. With the help of a computer, the Berge-Fulkerson Conjecture was verified for all snarks of order at most 36 [13], which can be seen as a strong indication that the conjecture might be true in general. Nevertheless, a general solution seems to be far away.

If true, Conjecture 1.1.1 implies that every bridgeless cubic graph has five perfect matchings such that each edge is in at least one of them. This statement was conjectured to be true by Berge (unpublished) and is also known as the Berge Conjecture. At first glance, the Berge-Fulkerson Conjecture seems to be stronger than the Berge Conjecture; but it turned out that they are in fact equivalent (Mazzuoccolo [64]). Despite much effort, both conjectures remain widely open; the Berge-Fulkerson Conjecture is unsolved for over 50 years. Hence, in order to make some progress, weaker statements moved into focus. The following three conjectures are all implied by the Berge-Fulkerson Conjecture and decrease in their strength, i.e. each conjecture is implied by the previous.

Conjecture 5.0.1 (Fan, Raspaud [22])**.** *Every bridgeless cubic graph has three perfect matchings with an empty intersection.*

Conjecture 5.0.2 (Máčajová, Škoviera [60], see also [43])**.** *Every bridgeless cubic graph has two perfect matchings such that their intersection does not contain an edge-cut of odd cardinality.*

Conjecture 5.0.3 (Mazzuoccolo [65])**.** *Every bridgeless cubic graph has two perfect matchings M_1, M_2 such that $G - (M_1 \cup M_2)$ is bipartite.*

Very recently, the weakest of these conjectures (Conjecture 5.0.3) was finally verified by Kardoš, Máčajová and Zerafa [47] by proving the following more general statement.

Theorem 5.0.4 (Kardoš, Máčajová, Zerafa [47])**.** *Let G be a bridgeless cubic graph. Let F be a factor of G such that every vertex is of degree at least 1 in F and let $e \in E(G)$. Then, there exists a perfect matching M of G such that $e \in M$ and $G - (E(F) \cup M)$ is bipartite.*

Theorem 5.0.4 is equivalent to the following statement; in fact in [47] they proved this equivalent version.

Theorem 5.0.5 (Kardoš, Máčajová, Zerafa [47])**.** *Let G be a 2-connected cubic graph. Let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G and let $e \in E(G)$. Then, there exists a 1-factor F of G such that $e \in E(F)$ and $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$.*

It is natural to ask whether similar statements are true for graphs of higher regularity. In this chapter we consider the following problem.

Problem 5.0.6. *Let r, t be integers with $1 \leq t \leq r-2$. Is it true that for every (sufficiently connected) r-regular graph G and every set $\mathcal{O}$ of pairwise edge-disjoint odd circuits of G there is a t-factor F of G such that $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$?*

Note that there are combinations of r and t such that there exist r-regular graphs that do not have a t-factor. Thus, we need to make some connectivity assumptions on G to make sure that G admits a t-factor. Problem 5.0.6 seems to be particularly interesting when t is small. In this chapter we answer some instances of Problem 5.0.6. In particular, we give a positive answer in the cases when $t = \frac{r}{3}$ and when $t = \frac{r}{2}$, where t is even. The following two theorems are our main results.

Theorem 5.0.7. *Let $k \geq 1$ be an integer and let G be a 2-connected $3k$-regular graph. Let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G and let $e \in E(G)$. Then, there exists a k-factor F of G such that $e \in E(F)$ and $E(F) \cap E(O)$ is a non-empty matching of G for every $O \in \mathcal{O}$.*

Theorem 5.0.8. *Let $k \geq 1$ be an integer and let G be a 2-connected $4k$-regular graph. Let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G. Then, there exists a $2k$-factor F of G such that $E(O) \cap E(F) \neq \emptyset$ and $E(O) \cap (E(G) \setminus E(F)) \neq \emptyset$ for every $O \in \mathcal{O}$.*

We furthermore prove that these results are best possible in the sense that (1) the answer to Problem 5.0.6 is negative when $t < \frac{r}{3}$ and (2) for all r, t, the assumption that G has no cut-vertices is a necessary condition for a positive answer to Problem 5.0.6.

5.1 $3k$-regular graphs

In this section we prove Theorem 5.0.7. For that, we define a series of sets of trees $\mathcal{T}^1, \mathcal{T}^2, \ldots$ inductively as follows:

- $\mathcal{T}^1 = \{K_{1,3}\}$
- for every $k > 1$, $\mathcal{T}^k$ consists of all trees that can be obtained as follows:
  1. start with a tree $T \in \mathcal{T}^{k-1}$
  2. add two copies H_1, H_2 of $K_{1,3}$
  3. identify l, l_1 and l_2 to a new vertex, where $l \in Leaf(T)$, $l_1 \in Leaf(H_1)$ and $l_2 \in Leaf(H_2)$.

The only graph in $\mathcal{T}^2$ as well as an element of $\mathcal{T}^3$ is depicted in Figure 5.1. Note that for every positive integer k, every tree of $\mathcal{T}^k$ has exactly $3k$ leaves.

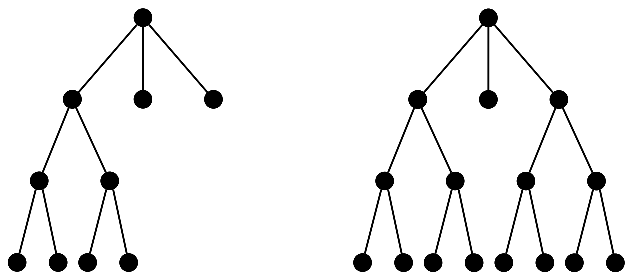


Figure 5.1: The only element of $\mathcal{T}^2$ (left) and an element of $\mathcal{T}^3$ (right).

Moreover, the set $\mathcal{T}^k$ contains a tree such that at most one pendant edge is not adjacent to another pendant edge. Furthermore, we observe the following.

Observation 5.1.1. *Let $T \in \mathcal{T}^k$ and let $M \subset E(T)$ be a matching such that every vertex of $V(T) \setminus Leaf(T)$ is incident to an element of M. Then, M contains exactly k pendant edges of T.*

Proof. We prove the statement by induction on k. For $k = 1$ the statement is trivially true. Next, assume the statement is true for every $k' < k$. Let $T' \in \mathcal{T}^{k-1}$ and let $T \in \mathcal{T}^k$ be obtained from T' by the procedure described in the definition of $\mathcal{T}^k$. Let e be the edge of T that is a pendant edge in T' but not in T, let $e_1, \ldots, e_4$ be the pendant edges of T not belonging to T'. If $e \in M$, then M contains exactly two edges of $\{e_1, \ldots, e_4\}$. Hence, M contains exactly $k - 1 - 1 + 2 = k$ pendant edges of T by induction. If $e \notin M$, then M contains exactly one edge of $\{e_1, \ldots, e_4\}$. Thus, the statement follows again by induction. □

Proof of Theorem 5.0.7. Let $T \in \mathcal{T}^k$ such that at most one pendant edge of T is not adjacent to another pendant edge. First, we transform G to a new graph G' as follows. For every $v \in V(G)$ replace v by a copy T_v of $T - Leaf(T)$ such that (1) every vertex of T_v is of degree 3 and (2) if $e, f \in \partial_G(v)$ belong to the same circuit of $\mathcal{O}$, then e, f remain adjacent in the resulting graph. An example is given in Figure 5.2. Note that (1) is possible since T has exactly

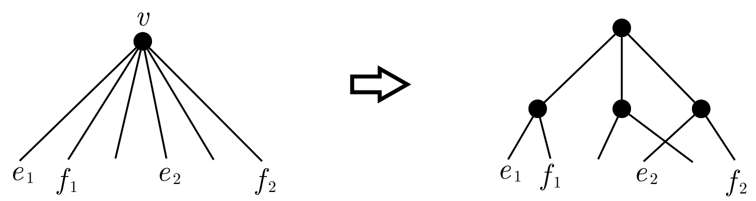


Figure 5.2: The replacement of a vertex v in the proof of Theorem 5.0.7 in the case that $k = 2$ and the edges e_1, f_1 as well as e_2, f_2 belong to the same circuit.

$3k$ leaves; (2) is possible since at most one pendant edge of T is not adjacent to another pendant edge of T. We obtain a cubic graph G' with vertex-set $\bigcup_{v \in V(G)} V(T_v)$ and edge-set $E(G) \cup \bigcup_{v \in V(G)} E(T_v)$. Furthermore, for every $v \in V(G)$, the graph $G' - V(T_v)$ is connected, since G is 2-connected. As a consequence, G' is bridgeless. For every $O \in \mathcal{O}$ the subgraph of G' induced by $E(O)$ is an odd circuit in G', which will be denoted by O'. Let $\mathcal{O}' = \{O' \colon O \in \mathcal{O}\}$. By Theorem 5.0.5, G' has a perfect matching M such that $e \in M$ and $M \cap E(O') \neq \emptyset$ for every $O' \in \mathcal{O}'$. Let F be the subgraph of G induced by the edge set $M \cap E(G)$. By Observation 5.1.1, $|\partial_{G'}(V(T_v)) \cap M| = k$ for every $v \in V(G)$ and hence, F is a k-factor of G. Furthermore, $E(O') \cap M$ is a non-empty matching in G' for every $O' \in \mathcal{O}'$ and therefore, $E(O) \cap E(F)$ is a non-empty matching in G for every $O \in \mathcal{O}$ by the construction of G'. Thus, F has the desired properties. $\square$

Note that if r and t have the same parity, then for every r-regular graph G and every t-factor F of G the graph $G - E(F)$ can be decomposed into 2-factors. Thus, for every $t' \in \{t, t+2, \dots r\}$ the graph G has a t'-factor that contains F. As a consequence, Theorem 5.0.7 implies the following corollary.

Corollary 5.1.2. *Let $k \geq 1$ be an integer and let G be a 2-connected $3k$-regular graph. Let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G and let $e \in E(G)$. Then, for every $t \in \{k, k+2, \dots, 3k\}$ there exists a t-factor F of G such that $e \in E(F)$ and $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$.*

5.2 $4k$-regular graphs

In this section we prove Theorem 5.0.8. Let G be a graph with an orientation D. A circuit C of G is an *oriented* circuit (with respect to D), if for every $v \in V(C)$ exactly one edge of $\partial_G(v) \cap E(C)$ is directed towards v. We first prove the following lemma.

Lemma 5.2.1. *Let k be a positive integer, let G be a 2-connected $4k$-regular graph and let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G. Then, there exists an orientation of G such that*

(i) every vertex has an even in-degree,

(ii) no circuit of $\mathcal{O}$ is an oriented circuit.

Proof. By Petersen, the graph $G - \bigcup_{O \in \mathcal{O}} E(O)$ can be decomposed into circuits, since it only contains vertices of even degree. Thus, G has a decomposition $\mathcal{Q}$ into circuits such that $\mathcal{O} \subseteq \mathcal{Q}$. Let D be an orientation of G such that every circuit of $\mathcal{Q}$ is an oriented circuit. We will change the direction of some edges in order to obtain the desired orientation.

First, transform G into a cubic graph as follows. Let T be a tree such that T has $4k$ leaves, every other vertex is of degree 3 and every pendant edge is adjacent with another pendant edge. Note that T as well as $T - Leaf(T)$ are of even order; if $k = 1$, then $T - Leaf(T)$ is isomorphic to K_2. For every $v \in V(G)$ replace v by a copy T_v of $T - Leaf(T)$ such that (1) every vertex of T_v is of degree 3 and (2) if $e, f \in \partial_G(v)$ belong to the same circuit of $\mathcal{O}$, then e, f remain adjacent in the resulting graph. We obtain a cubic graph G' with $V(G') = \bigcup_{v\in V(G)} V(T_v)$ and $E(G') = E(G) \cup \bigcup_{v\in V(G)} E(T_v)$. Furthermore, G' is bridgeless, since G is 2-connected. For every $O \in \mathcal{O}$ the subgraph of G' induced by $E(O)$ is an odd circuit in G', which will be denoted by O'. Let $\mathcal{O}' = \{O' \colon O \in \mathcal{O}\}$. Hence, by Theorem 5.0.5, G' has a perfect matching M such that $E(O') \cap M \neq \emptyset$ for every $O' \in \mathcal{O}'$.

Now, for every $e \in E(G)$ for which the corresponding edge in G' belongs to M, change the direction of e in D to obtain a new orientation D' of G. For every $v \in V(G)$, the set $M \cap \partial_{G'}((V(T_v))$ is of even cardinality, since T_v is of even order. Hence, for every $v \in V(G)$ we changed the direction of an even number of edges of $\partial_G(v)$. Thus, D' satisfies (i) since in D every vertex has indegree $2k$. Furthermore, for every $O' \in \mathcal{O}'$, the set $E(O') \cap M$ is a non-empty matching in G'. As a consequence, D' satisfies (ii). □

Proof of Theorem 5.0.8. Consider an orientation D of G that satisfies properties (i) and (ii) of Lemma 5.2.1. For every $v \in V(G)$, split v into $2k$ vertices $v_1, \ldots, v_{2k}$ of degree 2 (that is, replace v by a graph H_v consisting of $2k$ isolated vertices $v_1, \ldots, v_{2k}$ such that every vertex of $V(H_v)$ is of degree 2 in the resulting graph). We obtain a 2-regular graph G' with $E(G') = E(G)$ and $V(G') = \bigcup_{v\in V(G)} \{v_1, \ldots, v_{2k}\}$. Since every vertex in G has even indegree (with respect to D), this procedure can be done such that (1) for every $v \in V(G)$ and every $i \in \{1, \ldots, 2k\}$ the two edges incident with v_i in G' are either both directed towards v or both not directed towards v in G and (2) if $O \in \mathcal{O}$, $x \in V(O)$ and $e, f \in E(O) \cap \partial_G(x)$ are such that e, f are either both directed towards x or both not directed towards x, then e, f are adjacent in G'. By (1), the graph G' is bipartite. Hence, it has a perfect matching M. Let F be the subgraph of G induced by the edge set M. Observe that F is a $2k$-factor of G, since M is a perfect matching of G'. Furthermore, for every $O \in \mathcal{O}$ there is a $v_O \in V(O)$ such that the two edges in $\partial_G(v_O) \cap E(O)$ are either both directed towards v_O or both not directed towards v_O, since no circuit of $\mathcal{O}$ is an oriented circuit (with respect to D). Thus, by the construction of G', these two edges are adjacent in G' and hence, M as well as $E(F)$ contain exactly one of these edges. As a consequence, $E(O) \cap E(F) \neq \emptyset$ and $E(O) \cap (E(G) \setminus E(F)) \neq \emptyset$. Thus, F has the desired properties.

□

As is the case of $3k$-regular graphs, we obtain the following corollary.

Corollary 5.2.2. *Let $k \geq 1$ be an integer and let G be a 2-connected $4k$-regular graph. Let $\mathcal{O}$ be a set of pairwise edge-disjoint odd circuits of G. Then, for every $t \in \{2k, 2k+2, \ldots, 4k\}$ there exists a t-factor F of G such that $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$.*

5.3 Problem 5.0.6 when t is small

The smallest t for which we know that Problem 5.0.6 has a positive answer is $t = \lceil \frac{r}{3} \rceil$ (only in the cases $r \equiv 0 \pmod 3$ and $r = 4$). In this section we prove that this is indeed best possible. The following theorem gives a negative answer to Problem 5.0.6 for all $t < \frac{r}{3}$, even when we only consider r-connected r-regular graphs of even order.

Theorem 5.3.1. *For every $r \geq 3$ there is an r-connected r-regular graph G of even order and a set $\mathcal{O}$ of pairwise disjoint odd circuits of G with the property that if F is a t-factor of G with $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$, then $t \geq \frac{r}{3}$.*

Proof. For $i \in \{1, 2, 3\}$ let G_i be a graph isomorphic to $K_{r-2,r}$, where the two partitions are given by $A_i = \{a_1^i, \ldots, a_r^i\}$ and $B_i = \{b_1^i, \ldots, b_{r-2}^i\}$. Construct a new graph G from G_1, G_2, G_3 by adding the edges $a_j^1 a_j^2$, $a_j^1 a_j^3$ and $a_j^2 a_j^3$ for every $j \in \{1, \dots, r\}$. Let $\mathcal{O}$ be the set of pairwise disjoint triangles of G that are induced by the added edges, i.e. $\mathcal{O} = \{a_j^1 a_j^2 a_j^3 a_j^1 \colon j \in \{1, \dots, r\}\}$. The graph G and the set of circuits $\mathcal{O}$ are depicted in Figure 5.3 in the case when $r = 4$. We claim that G and $\mathcal{O}$ have the desired properties. By construction,

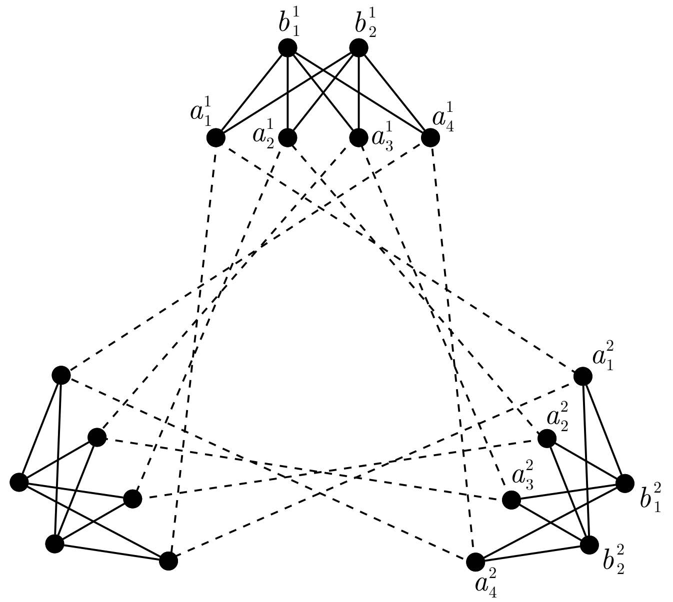


Figure 5.3: The graph G constructed in the proof of Theorem 5.3.1 in the case $r = 4$. The set $\mathcal{O}$ consists of the triangles whose edges are drawn with dashed lines.

G is an r-regular graph of even order. Let $X \subset V(G)$ be a vertex-cut of G. If $G[V(G_i) \setminus X]$ is connected for every $i \in \{1, 2, 3\}$, then X contains at least one vertex of every triangle of $\mathcal{O}$. As a consequence $|X| \geq r$. Thus, we can assume that w.l.o.g. $G[V(G_1) \setminus X]$ is not connected. Hence, $A_1 \subseteq X$ or $B_1 \subseteq X$. In both cases, $|X| \geq r$ since $G - B_1$ does not have a cut-vertex. Therefore, G is r-connected. Next, let F be a t-factor of G with $E(F) \cap E(O) \neq \emptyset$ for every $O \in \mathcal{O}$. By the construction of G, we have $|E(F) \cap \partial_G(B_1 \cup B_2 \cup B_3)| = 3(r-2)t$. Furthermore, $E(F)$ contains at least one edge of every triangle in $\mathcal{O}$. As a consequence, $3(r-2)t + r \leq |E(F)| = t(3r-3)$, which can be transformed to $\frac{r}{3} \leq t$ by a short calculation. $\square$

5.4 Graphs with cut-vertices

In Theorem 5.0.7 and Theorem 5.0.8 we assumed that G is 2-connected. In this section we show that this assumption is necessary, even when we consider Problem 5.0.6 only for graphs admitting a t-factor.

For a set $E = \{u_1v_1, \dots, u_lv_l\}$ of pairwise non-parallel edges having both end-vertices in $V(G)$, we denote by kE the set consisting of k parallel edges connecting u_i and v_i for every $i \in \{1, \dots, l\}$.

Theorem 5.4.1. *Let r, t be integers with $0 < t \leq r-2$. Then, there exists an r-regular graph G and a set $\mathcal{O}$ of pairwise edge-disjoint odd circuits of G such that G has a t-factor but every t-factor is edge-disjoint with at least one element of $\mathcal{O}$.*

Proof. We consider four cases depending on the parity of r and t.

Case 1. r and t are odd.

In this case define $G_{r,t}$ by

$$V(G_{r,t}) = \{u, v, w\} \cup \{x_i, y_i, z_i : i \in \{1, \dots, 3t\}\}$$

and

$$E(G_{r,t}) = \left(\frac{r-t}{2}\right) A \cup B_1 \cup B_2 \cup B_3 \cup \left(\frac{r-1}{2}\right) D_1 \cup \left(\frac{r+1}{2}\right) D_2,$$

where

$$\begin{aligned} A &= \{uv, uw, vw\}, \\ B_1 &= \{ux_i : i \in \{1, \dots, t\}\}, \\ B_2 &= \{vx_i : i \in \{t+1, \dots, 2t\}\}, \\ B_3 &= \{wx_i : i \in \{2t+1, \dots, 3t\}\}, \\ D_1 &= \{x_iy_i, x_iz_i : i \in \{1, \dots, 3t\}\}, \\ D_2 &= \{y_iz_i : i \in \{1, \dots, 3t\}\}. \end{aligned}$$

Set $\mathcal{O} = \{uvwu\}$; an example is shown in Figure 5.4.

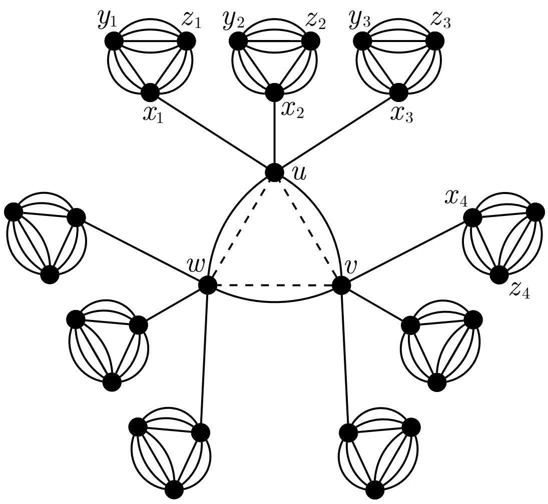


Figure 5.4: The graph $G_{r,t}$ when r and t are odd in the case $r = 7$, $t = 3$. The set $\mathcal{O}$ consists of the triangle whose edges are drawn with dashed lines.

By construction $G_{r,t}$ is r-regular and the edge set $B_1 \cup B_2 \cup B_3 \cup \left(\frac{t-1}{2}\right) D_1 \cup \left(\frac{t+1}{2}\right) D_2$ induces a t-factor. Moreover, every t-factor of $G_{r,t}$ contains the only edge in $\partial(\{x_i, y_i, z_i\})$ for every $i \in \{1, \ldots, 3t\}$ by parity reasons, and hence is edge-disjoint with the triangle $uvwu$.

Case 2. r and t are even.

In this case define $G_{r,t}$ by

$$V(G_{r,t}) = \{v\} \cup \{x_i, y_i, z_i : i \in \{1, \ldots, r\}\}$$

and

$$E(G_{r,t}) = A \cup B \cup \left(\frac{r-2}{2}\right) D_1 \cup \left(\frac{r+2}{2}\right) D_2,$$

where

$$\begin{aligned} A &= \{vx_i : i \in \{1, \ldots, r\}\}, \\ B &= \{x_i x_{i+1} : i \in \{1, \ldots, r-1\}, i \text{ odd}\}, \\ D_1 &= \{x_i y_i, x_i z_i : i \in \{1, \ldots, r\}\}, \\ D_2 &= \{y_i z_i : i \in \{1, \ldots, r\}\}. \end{aligned}$$

Set $\mathcal{O} = \{vx_i x_{i+1} v : i \in \{1, \ldots, r-1\}, i \text{ odd}\}$, see Figure 5.5.

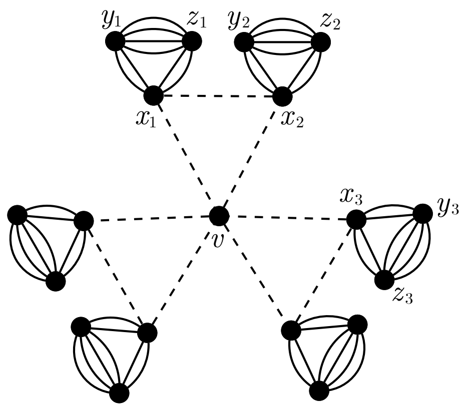


Figure 5.5: The graph $G_{r,t}$ when r and t are even in the case $r = 6$. The set $\mathcal{O}$ consists of the triangles whose edges are drawn with dashed lines.

By construction $G_{r,t}$ is an r-regular graph, which has a t-factor since r and t are even. Let F be a t-factor of $G_{r,t}$. Without loss of generality we assume $vx_1 \notin E(F)$. By parity reasons, $|\partial_F(\{x_1, y_1, z_1\})|$ and $|\partial_F(\{x_2, y_2, z_2\})|$ are even, which implies that F and the triangle vx_1x_2v are edge-disjoint.

Case 3. r is odd and t is even.

In this case define $G_{r,t}$ by

$$V(G_{r,t}) = \{u, v, w\} \cup \{u', v', w'\} \cup \{x_i, y_i, z_i : i \in \{1, \ldots, 6\}\}$$

and

$$E(G_{r,t}) = A \cup (r-2)\, B \cup C \cup \left(\frac{r-1}{2}\right) D_1 \cup \left(\frac{r+1}{2}\right) D_2,$$

where

$$\begin{aligned}
A &= \{uv, uw, vw\},\\
B &= \{uu', vv', ww'\},\\
C &= \{u'x_1, u'x_2, v'x_3, v'x_4, w'x_5, w'x_6\},\\
D_1 &= \{x_iy_i, x_iz_i : i \in \{1, \ldots, 6\}\},\\
D_2 &= \{y_iz_i : i \in \{1, \ldots, 6\}\}.
\end{aligned}$$

Set $\mathcal{O} = \{uvwu\}$. Figure 5.6 shows $G_{r,t}$ in the case $r = 5$.

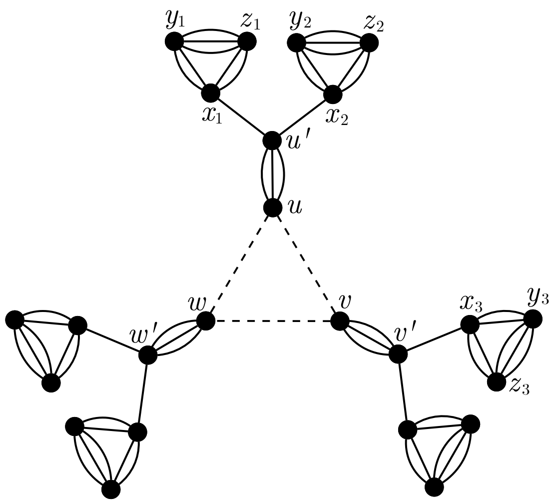


Figure 5.6: The graph $G_{r,t}$ when r is odd and t is even in the case $r = 5$. The set $\mathcal{O}$ consists of the triangle whose edges are drawn with dashed lines.

By construction $G_{r,t}$ is r-regular and the edge set $tB \cup \binom{t}{2} D_1 \cup \binom{t}{2} D_2$ induces a t-factor. Moreover, every t-factor of $G_{r,t}$ does not contain the only edge in $\partial(\{x_i, y_i, z_i\})$ for every $i \in \{1, \ldots, 6\}$ by parity reasons. Hence, it contains every edge of $\partial(\{u, v, w\})$, and thus, it is edge-disjoint with the triangle $uvwu$.

Case 4. r is even and t is odd.

In this case define $G_{r,t}$ by

$$V(G_{r,t}) = \{u, v\} \cup \{w_i, x_i, y_i, z_i : i \in \{1, \ldots, 2r - 4\}\}$$

and

$$E(G_{r,t}) = 2A \cup B \cup C \cup \binom{\frac{r-2}{2}}{} D_1 \cup D_2 \cup \binom{r}{2} D_3,$$

where

$$\begin{aligned}
A &= \{uv\},\\
B &= \{uw_i, vw_{i+(r-2)} : i \in \{1,\dots,r-2\}\},\\
C &= \{w_iw_{i+1} : i \in \{1,\dots,2r-4\}, i \text{ odd}\},\\
D_1 &= \{w_ix_i, w_iy_i : i \in \{1,\dots,2r-4\}\},\\
D_2 &= \{x_iy_i : i \in \{1,\dots,2r-4\}\},\\
D_3 &= \{x_iz_i, y_iz_i : i \in \{1,\dots,2r-4\}\}.
\end{aligned}$$

Set $\mathcal{O} = \{uw_iw_{i+1}u : i \in \{1,\dots,r-2\}, i \text{ odd}\}$, see Figure 5.7 for an example.

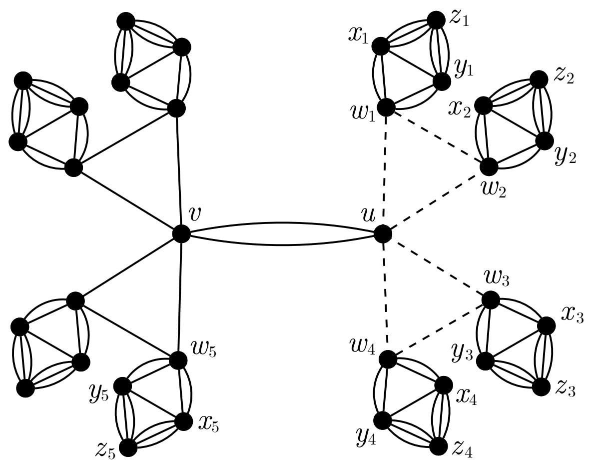


Figure 5.7: The graph $G_{r,t}$ when r is even and t is odd in the case $r = 6$. The set $\mathcal{O}$ consists of the triangles whose edges are drawn with dashed lines.

By construction $G_{r,t}$ is r-regular. Furthermore, the edge set $2A \cup D_1 \cup D_3$ can be partitioned into two disjoint perfect matchings M_1, M_2 of $G_{r,t}$, since it induces a 2-factor that only contains even circuits. By Petersen, $G_{r,t} - M_1 \cup M_2$ has a $(t-1)$-factor, since $r-2$ and $t-1$ are even. A t-factor of $G_{r,t}$ can be obtained by adding either M_1 or M_2 to this $(t-1)$-factor. Next, let F be a arbitrary t-factor of $G_{r,t}$. Without loss of generality we assume $uw_1 \notin E(F)$. By parity reasons, $|\partial_F(\{w_1, x_1, y_1, z_1\})|$ and $|\partial_F(\{w_2, x_2, y_2, z_2\})|$ are even, which implies that F and the triangle uw_1w_2u are edge-disjoint. $\square$

5.5 Concluding remarks and open problems

In addition to solving other instances of Problem 5.0.6, it is also interesting to study this problem in the case that $\mathcal{O}$ consists of pairwise edge-disjoint circuits (of order at least 3), no matter whether they are even or odd. In the cubic case, Kardoš, Máčajová and Zerafa [46] extended their result to arbitrary circuits by proving the following statement. For every cyclically 3-edge-connected cubic graph G and every set $\mathcal{O}$ of pairwise edge-disjoint circuits, there is a perfect matching containing at least one edge of every element of $\mathcal{O}$ (see [46] for more details). In an ongoing work we study possible extensions of this result to graphs of higher regularity.

Another interesting question is whether the statement of Conjecture 5.0.3 (which is true by Theorem 5.0.5) can be extended to r-graphs of higher regularity. More precisely, the following problem seems to be natural. What is the minimum number t such that every r-graph has t perfect matchings whose removal leaves a bipartite graph? Note that if the Generalized Berge-Fulkerson Conjecture (Conjecture 1.1.5) is true, then $2r - 4$ perfect matchings suffices.

Chapter 6

Rotation r-graphs

This chapter is based on [81]; all results of Chapter 6 are published in [81].

A tree is homeomorphically irreducible if it has no vertex of degree 2 and if a graph G has a homeomorphically irreducible spanning tree T, then T is called a hist and G a hist graph. The study of hist graphs has been a very active area of research within graph theory for decades, see for example [3, 31, 36].

Cubic hist graphs then have a spanning tree in which every vertex has either degree 1 or 3. They further have the nice property that their edge-set can be partitioned into the edges of the hist and of an induced cycle on the leaves of the hist. Recall that a snark is a bridgeless cubic graph that is not 3-edge-colorable. Informally, a rotation snark is a snark that has a balanced hist and a $\frac{2\pi}{3}$-rotation symmetry which fixes one vertex. Hoffmann-Ostenhof and Jatschka [33] studied rotation snarks and conjectured that there are infinitely many non-trivial rotation snarks. This conjecture was proved by Máčajová and Škoviera [59] by constructing an infinite family of cyclically 5-edge-connected rotation snarks. It is natural, to ask whether some notoriously difficult conjectures can be proved for rotation snarks. As a first result in this direction, Liu et al. [52] proved that the Berge-Fulkerson Conjecture (Conjecture 1.1.1) is true for the rotation snarks of [59].

We generalize the notion of rotation snarks to r-graphs of odd regularity and show that every r-graph of odd regularity can be "blown up" to a simple rotation r-graph (which produces many small edge-cuts). As a consequence, some hard long-standing open conjectures can be reduced to simple rotation r-graphs. However, our proof heavily relies on the fact that we allow 2-edge-cuts. It would be interesting to study rotation r-graphs with high edge-connectivity.

6.1 Definition of rotation r-graphs

Recall that an *automorphism* of a graph G consists of two bijections $\theta : V(G) \to V(G)$ and $\phi : E(G) \to E(G)$ such that e has end-vertices u, v if and only if $\phi(e)$ has end-vertices $\theta(u), \theta(v)$ for every $e \in E(G)$. Note that in this case, for every two vertices $u, v \in V(G)$ the number of edges between u and v is the same as the number of edges between $\theta(u)$ and $\theta(v)$. On the other hand, every bijection $\alpha\colon V(G) \to V(G)$ with this property can be extended to an automorphism (by appropriately defining a bijection $\beta\colon E(G) \to E(G)$). Thus, an automorphism can be defined alternatively as follow. An *automorphism* of a graph G is a mapping $\alpha\colon V(G) \to V(G)$, such that for every two vertices $u, v \in V(G)$ the number of edges between u and v is the same as the number of edges between $\alpha(u)$ and $\alpha(v)$. In Chapter 6 we will stick with this alternative definition, since it is more convenient for our purposes.

For an automorphism $\alpha\colon V(G) \to V(G)$ and a vertex $v \in V(G)$, the smallest positive integer k such that $\alpha^k(v) = v$ is denoted by $d_\alpha(v)$. An automorphism α of a tree T is *rotational* with respect to a vertex $v \in V(T)$, if $d_\alpha(v) = 1$ and $d_\alpha(u) = d_T(v)$ for every $u \in V(T) \setminus \{v\}$. The unique tree with vertex degrees in $\{1, r\}$ and a vertex x with distance i to every leaf is denoted by T_i^r. Vertex x is unique and it is called the *root* of T_i^r.

Recall that an r-regular graph G is an *r-graph*, if $|\partial(S)| \geq r$ for every $S \subseteq V(G)$ of odd cardinality. An r-regular graph G is a *T_i^r-graph*, if G has a spanning tree T isomorphic to T_i^r. If, additionally, G has an automorphism that is rotational on T (with respect to the root), then G is a *rotation* T_i^r-graph.

Note that G can be embedded in the plane (crossings allowed) such that the embedding has a $\frac{2\pi}{r}$-rotation symmetry fixing the root. A *rotation r-graph* is an r-graph that is a rotation T_i^r-graph for some integer i.

Observation 6.1.1. *Let r, i be positive integers, let G be a T_i^r-graph with corresponding spanning tree T. The order of G is $1+\sum_{j=0}^{i-1} r(r-1)^j$, which is even if and only if r is odd. In particular, if G is an r-graph, then r is odd, $G[Leaf(T)]$ is a cycle and $E(G)$ can be partitioned into $E(T)$ and $E(G[Leaf(T)])$.*

6.2 Main result

Let G be an r-graph and $S \subseteq V(G)$ be of even cardinality. If $|\partial(S)| = 2$, then $N_G(S)$ consists of precisely two vertices, say u, v. Let G' be obtained from G by deleting $G[S] \cup \partial(S)$ and adding the edge uv. We say that G' is obtained from G by a 2-*cut reduction* (of S). The following theorem is the main result of this chapter.

Theorem 6.2.1. *Let r be a positive odd integer. For every r-graph G there is a simple rotation r-graph G', such that G can be obtained from G' by a finite number of 2-cut reductions.*

The following corollary is a direct consequence of Theorem 6.2.1.

Corollary 6.2.2. *Let r be a positive odd integer and let A be a graph-property that is preserved under 2-cut reduction. Every r-graph has property A if and only if every simple rotation r-graph has property A.*

As a consequence, some notoriously difficult conjectures can be reduced to rotation r-graphs.

Corollary 6.2.3. *Let r be a positive odd integer. The following statements are equivalent:*

1. *(generalized Berge-Fulkerson Conjecture [77]) Every r-graph has $2r$ perfect matchings such that each edge is in exactly two of them.*

2. *Every simple rotation r-graph has $2r$ perfect matchings such that each edge is in exactly two of them.*

3. *(generalized Berge Conjecture) Every r-graph has $2r-1$ perfect matchings such that each edge is in at least one of them.*

4. *Every simple rotation r-graph has $2r-1$ perfect matchings such that each edge is in at least one of them.*

Proof. Let G and G' be two r-graphs such that G can be obtained from G' by a 2-cut reduction of a set $S \subset V(G')$. For parity reasons, every perfect matching of G' contains either both or no edges of $\partial(S)$. Hence, each perfect matching of G' can be transformed into a perfect matching of G, which implies the equivalences $(1 \Leftrightarrow 2)$ and $(3 \Leftrightarrow 4)$. The equivalence $(1 \Leftrightarrow 3)$ is proved in [66]. □

As mentioned in the previous chapter, for $r = 3$, statement 3 of Corollary 6.2.3 is usually attributed to Berge and is known as the Berge Conjecture.

Fan and Raspaud [22] conjectured that every 3-graph has three perfect matchings such that every edge is in at most two of them (Conjecture 5.0.1). Equivalent formulations of this conjecture are studied in [42].

Corollary 6.2.4. *Let r be an odd integer and $2 \leq k \leq r-1$. Every r-graph has r perfect matchings, such that each edge is in at most k of them if and only if every simple rotation r-graph has r perfect matchings, such that each edge is in at most k of them.*

In 1954, Tutte [86] stated his seminal conjecture that every bridgeless graph admits a nowhere-zero 5-flow. In 1972, Tutte formulated the no less challenging conjecture that every bridgeless graph without 3-edge-cuts has a nowhere-zero 3-flow. It is a folklore that the 5-Flow Conjecture can be reduced to snarks, whereas the 3-Flow Conjecture is true if and only if it is true for 5-graphs. Admitting a nowhere-zero k-flow is invariant under 2-cut reduction. Hence, we obtain the following consequences of Theorem 6.2.1.

Corollary 6.2.5. *Every snark admits a nowhere-zero 5-flow if and only if every simple rotation snark admits a nowhere-zero 5-flow.*

Corollary 6.2.6. *Every 5-graph admits a nowhere-zero 3-flow if and only if every simple rotation 5-graph admits a nowhere-zero 3-flow.*

6.3 Proof of Theorem 6.2.1

6.3.1 Preliminaries

For the proof of Theorem 6.2.1 we will use the following lemma. The non-trivial direction of the statement is proved by Rizzi in [75] (Lemma 2.3).

Lemma 6.3.1 ([75]). *Let G be an r-regular graph and let $S \subseteq V(G)$ be a set of odd cardinality with $|\partial(S)| = r$. Then, G is an r-graph, if and only if G/S and G/S^c are both r-graphs.*

Let G be an r-graph and T be a spanning tree of G. We need the following two expansions of G and T.

Edge-expansion: Let e be an edge with $e = uv \in E(G) \setminus E(T)$. Let G' be the graph obtained from $G - e$ by adding two new vertices u', v' that are connected by $r - 1$ edges, and adding two edges uu' and vv'. Extend T to a spanning tree T' of G' by adding the vertices u', v' and the edges uu', vv' (see Figure 6.1). For $S = \{u, u', v'\}$ it follows with Lemma 6.3.1 that G' is an r-graph.

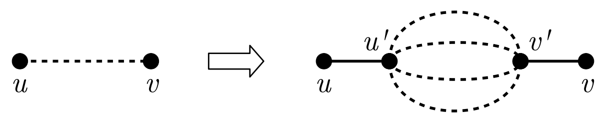


Figure 6.1: An edge-expansion in the case $r = 5$. The solid edges belong to the spanning tree T'.

Leaf-expansion: Let r be odd. Let l be a leaf of T and let u be the neighbor of l in T. Let K be a copy of K_r and let $V(K) = \{l_1, \ldots, l_r\}$. Let G'

be the unique r-regular graph obtained from G by replacing l with K. Without loss of generality we assume $ul_1 \in E(G')$. Extend $T - l$ to a spanning tree T' of G' by adding $V(K)$ and the edges ul_1 and $l_1 l_j$ for $j \in \{2, \dots, r\}$. Vertex l_1 has degree r in T', whereas all other vertices of K are leaves of T'. Furthermore, if l has distance d to a vertex $x \in V(T)$, then the $r-1$ leaves $l_2, \dots, l_r$ of T' have distance $d+1$ to x in T'.

Since K_{r+1} is an r-graph, G' is an r-graph by Lemma 6.3.1. We note that a leaf-expansion of leaf l has the following properties:

(i) In G', no vertex of K is incident with parallel edges.

(ii) Let $S \subseteq V(G)$ be a set of even cardinality with $l \in S$ and $|\partial(S)| = 2$. In the leaf-expansion G', the set $S' = S \setminus \{l\} \cup V(K)$ is of even cardinality and satisfies $|\partial(S')| = 2$. Moreover, the graph obtained from G by a 2-cut reduction of S is the same graph that is obtained from G' by a 2-cut reduction of S'.

An example of leaf-expansion is shown in Figure 6.2.

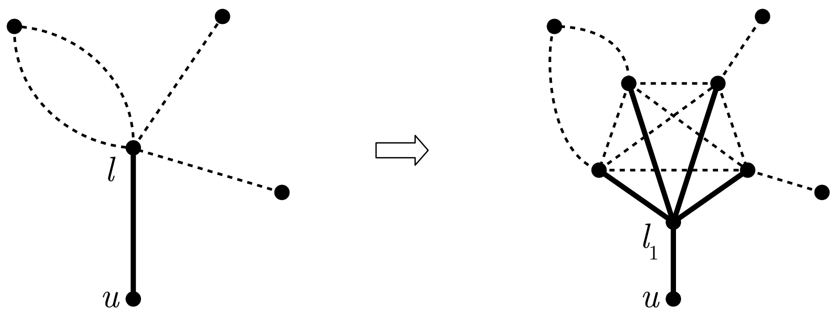


Figure 6.2: An example of a leaf-expansion of the leaf $l \in V(G)$ in the case $r = 5$. The solid edges belong to the spanning trees T and T' respectively.

6.3.2 Construction of G'

Let $r \geq 1$ be an odd integer and G be an r-graph. We will construct G' in two steps.

1. We construct a simple r-graph H with a spanning tree T_H isomorphic to T_i^r for some integer i such that G can be obtained from H by a finite number of 2-cut reductions.

Let T_G be an arbitrary spanning tree of G. Apply an edge-expansion on every edge of $E(G) \setminus E(T_G)$ to obtain an r-graph H_1 with spanning tree T_1. Clearly, G can be obtained from H_1 by 2-cut reductions. Furthermore, $V(G) \subseteq V(H_1)$, every vertex of $V(G)$ has degree r in T_1 and all vertices of $V(H_1) \setminus V(G)$ are leaves of T_1.

Let $x \in V(H_1)$ with $d_{T_1}(x) = r$ and let d be the maximal distance of x to a leaf in T_1. Repeatedly apply leaf-expansions until every leaf has distance $d+1$ to x. Let H_2 be the resulting graph and T_2 be the resulting spanning tree of H_2. By the construction, T_2 is isomorphic to T_{d+1}^r, where x is the root of T_2. By the definition of d, we applied a leaf-expansion of l for every leaf l of T_1. Hence, the graph H_2 is simple by property (i) of leaf-expansions. Furthermore, no expansion of a vertex in $V(G)$ (and degree r in T_1) is applied. As a consequence, property (ii) of leaf-expansions implies that G can be obtained from H_2 by 2-cut reductions. Thus, by setting $H = H_2$ and $T_H = T_2$ we obtain a graph with the desired properties.

2. We construct a simple rotation r-graph G' from which H can be obtained by a 2-cut reduction.

Let $y_1, \dots, y_r$ be the neighbors of x in H. Let R be an arbitrary simple rotation r-graph with a spanning tree T_R isomorphic to T_{d+1}^r. For example, such a graph can be obtained from the rotational T_1^r-graph K_{r+1} by repeatedly applying leaf-expansions. Let x_R be the root of T_R and let α_R be the corresponding rotational automorphism. Label the neighbors of x_R with $z_1, \dots, z_r$ such that $\alpha_R(z_i) = z_{i+1}$ for every $i \in \{1, \dots, r\}$, where the indices are added modulo r.

Take r copies $H^1, \dots, H^r$ of H and $(r-1)^2 - r$ copies $R^1, \dots, R^{(r-1)^2-r}$

of R. In each copy we label the vertices accordingly by using an upper index. For example, if v is a vertex of H, then v^i is the corresponding vertex in H^i. Furthermore, the automorphism of R^i that correspond to α_R will be denoted by α_{R^i}. Delete the root in each of the $(r-1)^2$ copies, i.e. in each copy of H and in each copy of R. The resulting $r(r-1)^2$ vertices of degree $r-1$ are called root-neighbors.

Take a tree T isomorphic to T_2^r with root x_T. The graph $T-x_T$ consists of r pairwise isomorphic components, thus it has a rotation automorphism α_T with respect to x_T. Let $l_1,\dots,l_{r-1}$ be the leaves of one component of $T-x_T$. Clearly, the set of leaves of T is given by $\{\alpha_T^i(l_j)\colon i\in\{0,\dots,r-1\}, j\in\{1,\dots,r-1\}\}$, where $\alpha_T^0 = id_T$.

Connect the $r(r-1)$ leaves of T with the $r(r-1)^2$ root-neighbors by adding $r(r-1)^2$ new edges as follows. For every $i\in\{1,\dots,r\}$ define an ordered list N_i of root-neighbors and an ordered list L_i of leaves of T by

$$N_i := (y_1^i,\dots,y_r^i, z_i^1,\dots,z_i^{(r-1)^2-r}) \quad \text{and} \quad L_i := (\alpha_T^{i-1}(l_1),\dots,\alpha_T^{i-1}(l_{r-1})).$$

The list N_i has $(r-1)^2$ entries, whereas L_i has $r-1$ entries. For each $i\in\{1,\dots,r\}$, connect the first $r-1$ entries of N_i with the first entry of L_i by $r-1$ new edges; connect the second $r-1$ entries of N_i with the second entry of L_i by $r-1$ new edges and so on. The set of new edges is denoted by E and the resulting graph by G'. In Figure 6.3 the construction of G' is shown in the case $r=3$.

Every root-neighbor appears exactly once in the lists $N_1,\dots,N_r$, whereas every leaf of T appears exactly once in the lists $L_1,\dots,L_r$. Consequently, G' is an r-regular simple graph with a spanning tree $T_{G'}$ that is obtained from the union of the trees of each copy of H and R (without its roots) and T by adding the edge set E. Note that $T_{G'}$ is isomorphic to T_{d+3}^r and x_T is the root

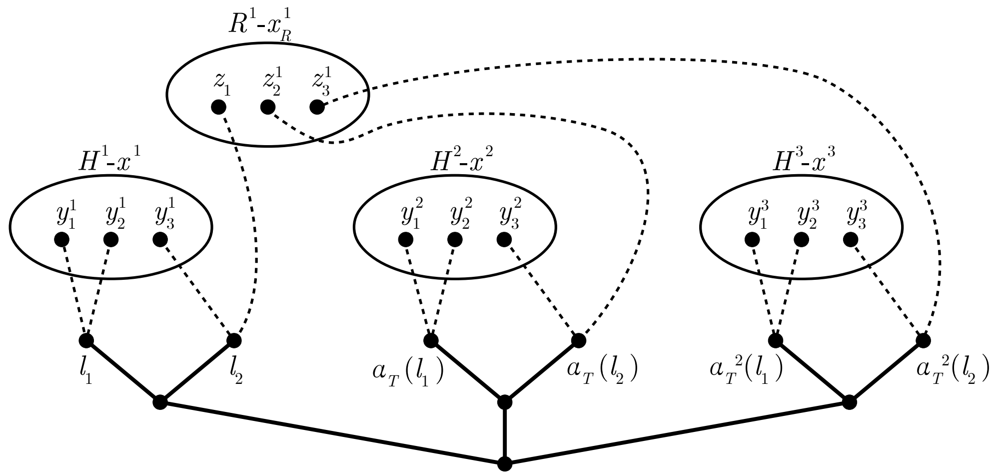


Figure 6.3: The construction of G' in the case $r = 3$. The solid edges belong to T; the dashed edges belong to E.

of $T_{G'}$. Let $\alpha_{G'}\colon V(G') \to V(G')$ be defined as follows:

$$\alpha_{G'}(v) = \begin{cases} \alpha_T(v) & \text{if } v \in V(T), \\ \alpha_{R^i}(v) & \text{if } v \in V(R^i) \setminus \{x_R^i\},\ i \in \{1, \dots, (r-1)^2 - r\}, \\ v^{i+1} & \text{if } v = v^i \in V(H^i) \setminus \{x^i\},\ i \in \{1, \dots, r\} \text{ and the indices} \\ & \text{are added modulo } r. \end{cases}$$

By definition, $\alpha_{G'}$ is an automorphism of $G' - E$ and $T_{G'} - E$ that fixes the root x_T of T and satisfies $d_{\alpha_{G'}}(v) = r$ for every other vertex v of G'. For $i \in \{1, ..., r\}$, if we apply $\alpha_{G'}$ on each element of N_i (or L_i respectively), then we obtain the ordered list N_{i+1} (or L_{i+1} respectively), where the indices are added modulo r. As a consequence, if $uv \in E$, then $\alpha_{G'}(u)\alpha_{G'}(v) \in E$ and hence, $\alpha_{G'}$ is an automorphism of G' and a rotational automorphism of $T_{G'}$.

To see that G' is an r-graph, transform G' as follows: for each $i \in \{1, \dots, r\}$ identify all vertices in $V(H^i) \setminus x^i$ to a vertex $\bar{H}^i$ and for every $j \in \{1, \dots, (r-1)^2 - r\}$ all vertices in $V(R^j) \setminus x_R^j$ to a vertex $\bar{R}^j$ (see Figure 6.4). The resulting graph is an r-regular bipartite graph and therefore, an r-graph. Since every copy of H and of R is an r-graph, it follows by successively application of Lemma 6.3.1 that G' is an r-graph.

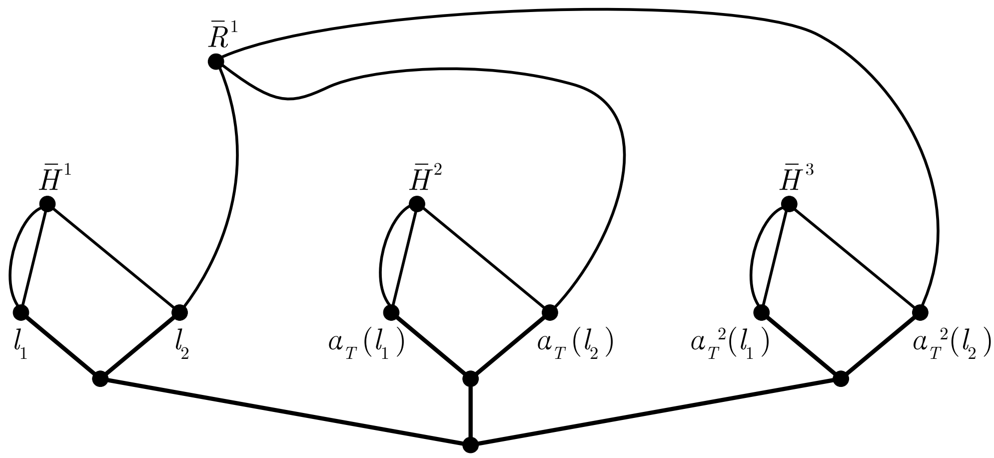


Figure 6.4: The graph constructed from G' in the case $r = 3$.

Last, the set $S \subseteq V(G')$ defined by $S = (V(H^1) \setminus \{x^1\}) \cup \{l_1\}$ is a set of even cardinality, such that $|\partial(S)| = 2$. Applying a 2-cut reduction on $V(G') \setminus S$ transforms G' into the copy H^1 of H. In conclusion, G can be obtained from G' by a finite number of 2-cut reductions, which completes the proof.

6.4 Concluding remarks

The graph G' constructed in the proof of Theorem 6.2.1 has many small edge-cuts. It would be interesting to construct and study highly edge-connected rotation r-graphs. For example, is there an r-edge-connected rotation r-graph of class 2 for every positive odd integer r? In particular, the case $r = 5$ seems to be of interest, as we will see in the next chapter. Furthermore, it might also be possible to prove some of the conjectures mentioned in Corollaries 6.2.3 - 6.2.6 for some families of rotation r-graphs with high edge-connectivity.

Chapter 7

Number of pairwise disjoint perfect matchings in r-graphs

Major parts of Chapter 7 are already published in [56] and [54]. The results of Sections 7.1 and 7.4 appeared in [56]; the results of Section 7.3 appeared in [54]. The results of Section 7.2 are unpublished.

Every r-graph has a perfect matching [77]. On the other hand, class 2 r-graphs, which exist for every $r \geq 3$, have at most $r-2$ pairwise disjoint perfect matchings. A set of k pairwise disjoint perfect matchings of a graph G is called a *k-PDPM*. It is natural to ask "What is the maximum number s such that every r-graph has an s-PDPM?". Rizzi [75] constructed r-graphs in which every two perfect matchings intersect; such r-graphs are called *poorly matchable*. Thus, in general the answer to the above question is $t = 1$. However, every poorly matchable r-graph know so far has a 4-edge-cut. It might be that the situation changes for r-graphs with larger edge-connectivity.

For $1 \leq t \leq r$ let $m(t, r)$ be the maximum number s such that every t-edge-connected r-graph has an s-PDPM. This gives rise to the following problem.

Problem 7.0.1. *Determine $m(t, r)$ for all $r \geq t \geq 1$.*

The function $m(t, r)$ is monotone increasing in t, in other words $m(t, r) \leq m(t', r)$ for $t \leq t'$. In particular we have that $m(t, r) \leq m(r, r)$ for all $t \in$

$\{1, \ldots, r\}$. Clearly, $m(1,1) = 1$, $m(2,2) = 2$, $m(3,3) = 1$ and $m(t,r) \geq 1$ for every $r \geq t \geq 1$. Furthermore, $m(4,r) = 1$ for every $r \geq 4$ by the result of Rizzi [75]. In addition to its exact determination, lower and upper bounds for this parameter are of great interest.

For all $r \geq 3$ and $r \neq 5$, class 2 r-edge-connected r-graphs are known (see [69]). Thus, $m(r,r) \leq r-2$ for these r. Surprisingly, no such graphs seem to be known for $r = 5$, i.e. the following problem seems to be unsolved.

Problem 7.0.2. *Is there any* 5*-edge-connected* 5*-regular class* 2 *graph?*

Note that for planar graphs, the answer to the above question is "no". Guenin [29] proved that all planar 5-graphs are class 1. For general r, Thomassen (Problem 1 of [84]) proposed the following question for the value of $m(r,r)$.

Problem 7.0.3 (Thomassen [84])**.** *For all* $r \geq 3$*, is it true that* $m(r,r) = r-2$*?*

For $r = 4$ the answer is "no" by Rizzi. Furthermore, in [63] it is proved that $m(r-1,r) \leq r-3$ if r is odd and $m(r,r) \leq r-3$ if $r \equiv 0 \mod 4$, which gives a negative answer to Problem 7.0.3 when r is a multiple of 4. It is worth mentioning, that up to now there is no non-trivial lower bound known for $m(t,r)$. Nevertheless, Thomassen [84] conjectured that such bounds exist for sufficiently large r. Precisely, he conjectured that there is an integer r_0 such that there is no poorly matchable r-graph for every $r \geq r_0$.

This chapter is divided into three parts. In Section 7.2 we prove that for every $1 < k < r$ it is NP-complete to decide whether a given r-graph has a k-PDPM. In Section 7.3 we consider r-edge-connected r-graphs. In particular we study the remaining cases of Problem 7.0.3. Our main results in the second part are the following.

Theorem 7.0.4. *If* $r \equiv 2 \mod 4$*, then* $m(r,r) \leq r-3$*.*

Theorem 7.0.5. *If $m(5,5) \geq 2$, then the Fan-Raspaud Conjecture holds. Moreover, if $m(5,5) = 5$, then both the 5-Cycle Double Cover Conjecture and the Berge-Fulkerson Conjecture hold.*

In Section 7.4 we study r-graphs with arbitrary edge-connectivity and obtain an upper bound for $m(t,r)$ that only depends on the edge-connectivity parameter as follows.

Theorem 7.0.6. *For every $l \geq 3$ and $r \geq 2l$, $m(2l,r) \leq 3l - 6$.*

7.1 Preliminaries

In order to prove the main results of this chapter (as well as the main result of Chapter 8) we need some further notation (mainly concerning the Petersen graph) as well as some lemmas, which will be introduced in this section.

A *multiset* $\mathcal{M}$ consists of objects with possible repetitions. We denote by $|\mathcal{M}|$ the number of (not necessary distinct) objects in $\mathcal{M}$. For a positive integer k, we define $k\mathcal{M}$ to be the multiset consisting of k copies of each element of $\mathcal{M}$. Let G be a graph and N a multiset of edges of the complete graph on $V(G)$. The graph $G + N$ is obtained by adding a copy of all edges of N to G. This operation might generate parallel edges. More precisely, if N contains exactly t edges connecting the vertices u and v of G, then $\mu_{G+N}(u,v) = \mu_G(u,v) + t$.

For a multiset $\mathcal{N}$ of perfect matchings of a graph G and an edge $e \in E(G)$, we say that $\mathcal{N}$ *contains* (*avoids*, respectively) e if $e \in \bigcup_{N \in \mathcal{N}} N$ ($e \notin \bigcup_{N \in \mathcal{N}} N$, respectively).

We will frequently use the following simple fact without reference.

Observation 7.1.1. *Let G be a graph with a perfect matching M. For any subset $X \subseteq V(G)$, if $|X|$ is odd, then $|\partial_G(X) \cap M|$ is odd.*

7.1.1 The Petersen graph and its perfect matchings

In the remainder of this chapter as well as in Chapter 8 we make extensive use of the Petersen graph, denoted by $\mathcal{P}$, and of the properties of its perfect matchings. Rizzi [75] observed that every two distinct 1-factors of the Petersen graph have precisely one edge in common, and proved that there is a one-to-one correspondence between edges and pairs of distinct 1-factors in the Petersen graph. Then we have the following proposition immediately.

Proposition 7.1.2. *The Petersen graph has exactly six perfect matchings, and each edge is contained in exactly two of them.*

We fix a drawing of $\mathcal{P}$ as in Figure 7.1 left. With reference to Figure 7.1, we define M_0 to be the perfect matching consisting of all edges u_iv_i, for $i \in \{1, \dots, 5\}$. Moreover, for $i \in \{1, \dots, 5\}$, by Proposition 7.1.2 we let M_i be the only other perfect matching of $\mathcal{P}$ different from M_0 and containing u_iv_i, see Figure 7.1. Let $\mathcal{M}$ be a multiset of perfect matchings of $\mathcal{P}$. We denote by

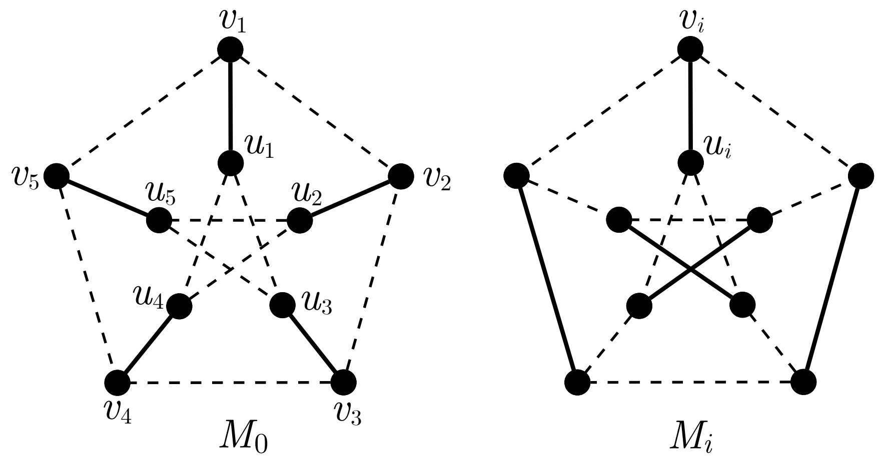


Figure 7.1: The Petersen graph $\mathcal{P}$, and its perfect matchings M_0 and M_i.

$n_{\mathcal{M}}(i)$ the number of copies of M_i appearing in $\mathcal{M}$. We define $\mathcal{P}^{\mathcal{M}}$ to be the graph $\mathcal{P} + \sum_{F \in \mathcal{M}} F$ and we remark that it has the following nice property.

Lemma 7.1.3 ([27]). *For every finite multiset $\mathcal{M}$ of perfect matchings of the Petersen graph $\mathcal{P}$, the graph $\mathcal{P}^{\mathcal{M}}$ is class 2.*

Now, let $\mathcal{N}$ be a multiset of perfect matchings of $\mathcal{P}^{\mathcal{M}}$. Note that each perfect matching N of $\mathcal{N}$ can be interpreted as a perfect matching of $\mathcal{P}$ by caring only about the end-vertices of each edge. If N corresponds to the perfect matching M_i in $\mathcal{P}$, then we say N is of *typ* i in $\mathcal{P}^{\mathcal{M}}$. Moreover, the multiset $\mathcal{N}$ can be interpreted as a multiset of perfect matchings of $\mathcal{P}$, which is denoted by $\mathcal{N}_{\mathcal{P}}$. Note that $|\mathcal{N}_{\mathcal{P}}| = |\mathcal{N}|$. We need the following two lemmas.

Lemma 7.1.4. *Let $\mathcal{M}$ be a multiset of perfect matchings of $\mathcal{P}$. Let $\mathcal{N}$ be a set of pairwise disjoint perfect matchings of $\mathcal{P}^{\mathcal{M}}$. There is at most one $i \in \{0, \dots, 5\}$ such that $n_{\mathcal{N}_{\mathcal{P}}}(i) > n_{\mathcal{M}}(i)$.*

In particular, there is no triple of different vertices u, v, w in $\mathcal{P}^{\mathcal{M}}$, with w adjacent to both v and u, such that $\mathcal{N}$ contains all edges of $E_{\mathcal{P}^{\mathcal{M}}}(\{u, v\}, \{w\})$.

Proof. First, suppose that there are two indices i and j such that $i \neq j$, $n_{\mathcal{N}_{\mathcal{P}}}(i) > n_{\mathcal{M}}(i)$, and $n_{\mathcal{N}_{\mathcal{P}}}(j) > n_{\mathcal{M}}(j)$. Let uv be the edge of $\mathcal{P}$ belonging to both M_i and M_j by Proposition 7.1.2. Since the perfect matchings of $\mathcal{N}$ are pairwise disjoint, at most $\mu_{\mathcal{P}^{\mathcal{M}}}(u, v)$ perfect matchings in $\mathcal{N}$ can contain an edge connecting u and v. This implies $n_{\mathcal{N}_{\mathcal{P}}}(i) + n_{\mathcal{N}_{\mathcal{P}}}(j) \leq \mu_{\mathcal{P}^{\mathcal{M}}}(u, v)$. Then the following contradiction arises.

$$\mu_{\mathcal{P}^{\mathcal{M}}}(u, v) = n_{\mathcal{M}}(i) + n_{\mathcal{M}}(j) + 1 \leq n_{\mathcal{N}_{\mathcal{P}}}(i) + n_{\mathcal{M}}(j) < n_{\mathcal{N}_{\mathcal{P}}}(i) + n_{\mathcal{N}_{\mathcal{P}}}(j).$$

Next, we prove the second part of the lemma. Let u, v be two different vertices both adjacent to the vertex w in $\mathcal{P}^{\mathcal{M}}$. Suppose by contradiction that $\mathcal{N}$ contains all edges of $E_{\mathcal{P}^{\mathcal{M}}}(\{u, v\}, \{w\})$. By Proposition 7.1.2, we may assume without loss of generality that $\{uw\} = M_0 \cap M_1$ and $\{vw\} = M_2 \cap M_3$. Then, since all edges of $E_{\mathcal{P}^{\mathcal{M}}}(\{u, v\}, \{w\})$ are contained in $\mathcal{N}$, we similarly deduce that

- $n_{\mathcal{N}_{\mathcal{P}}}(0) + n_{\mathcal{N}_{\mathcal{P}}}(1) = \mu_{\mathcal{P}^{\mathcal{M}}}(u, w) = n_{\mathcal{M}}(0) + n_{\mathcal{M}}(1) + 1;$
- $n_{\mathcal{N}_{\mathcal{P}}}(2) + n_{\mathcal{N}_{\mathcal{P}}}(3) = \mu_{\mathcal{P}^{\mathcal{M}}}(v, w) = n_{\mathcal{M}}(2) + n_{\mathcal{M}}(3) + 1.$

Then we conclude that there is $s \in \{0, 1\}$ and $t \in \{2, 3\}$, such that $n_{\mathcal{N}_{\mathcal{P}}}(s) > n_{\mathcal{M}}(s)$ and $n_{\mathcal{N}_{\mathcal{P}}}(t) > n_{\mathcal{M}}(t)$, which is impossible. □

Lemma 7.1.5. *Let $\mathcal{M}$ be a multiset of k perfect matchings of $\mathcal{P}$ and let $\mu = \mu(\mathcal{P}^{\mathcal{M}})$. Then, $\lambda(\mathcal{P}^{\mathcal{M}}) = \min\{k+3, 2k+6-2\mu\}$.*

Proof. Note that $\mathcal{P}$ is a 3-graph, and $\mathcal{P}^{\mathcal{M}}$ is a $(k+3)$-graph since every perfect matching of $\mathcal{P}$ intersects each edge-cut that separates two vertex sets of odd cardinality. Let X be a non-empty proper subset of $V(\mathcal{P}^{\mathcal{M}})$ minimizing $|\partial_{\mathcal{P}^{\mathcal{M}}}(X)|$. It implies that $\mathcal{P}^{\mathcal{M}}[X]$ is connected. If $|X|$ is odd, then $|\partial_{\mathcal{P}^{\mathcal{M}}}(X)| \geq k+3$ since $\mathcal{P}^{\mathcal{M}}$ is a $(k+3)$-graph. If $|X|$ is even, then it suffices to consider the cases $|X| \in \{2, 4\}$. Since $\mathcal{P}$ does not contain a circuit of order less than 5, either $\mathcal{P}[X]$ is a path on two or four vertices, or it is isomorphic to $K_{1,3}$. Then $\partial_{\mathcal{P}^{\mathcal{M}}}(X)$ contains at least $k+3-\mu$ edges for each vertex of degree 1 in $\mathcal{P}[X]$, and so we get that $|\partial_{\mathcal{P}^{\mathcal{M}}}(X)| \geq 2(k+3-\mu)$. Consequently, $\lambda(\mathcal{P}^{\mathcal{M}}) \geq \min\{k+3, 2k+6-2\mu\}$. Finally, observe that $|\partial_{\mathcal{P}^{\mathcal{M}}}(\{u,v\})| = 2k+6-2\mu$ for every two vertices u, v with $\mu_{\mathcal{P}^{\mathcal{M}}}(u,v) = \mu$. Thus, the statement follows. □

7.1.2 An useful graph operation

Definition 7.1.6. *Let G and H be two graphs with $u, v \in V(G)$ and $x, y \in V(H)$ such that $\mu_G(u,v) \geq t$ and $\mu_H(x,y) \geq r-t$. Then, $(G,u,v) \oplus_t (H,x,y)$ is the graph obtained from G and H by deleting exactly t edges connecting u and v in G and $r-t$ edges connecting x and y in H, identifying u and x to a new vertex w_{ux}, and identifying v and y to a new vertex w_{vy}, see Figure 7.2.*

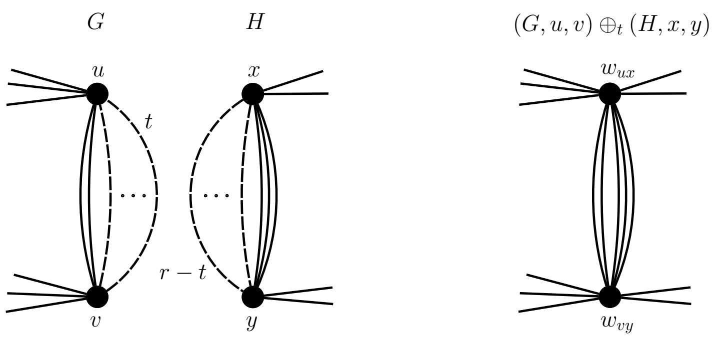


Figure 7.2: The operation of Definition 7.1.6.

Lemma 7.1.7. *Let G and H be two r-graphs with $u, v \in V(G)$ and $x, y \in V(H)$ such that $\mu_G(u, v) \geq t$ and $\mu_H(x, y) \geq r - t$. Then, $G' = (G, u, v) \oplus_t (H, x, y)$ is an r-graph with*

$$\lambda(G') = \min\{\lambda(G), \lambda(H)\}.$$

Proof. By the construction in Definition 7.1.6, $G' = (G, u, v) \oplus_t (H, x, y)$ is r-regular. For every $Y \subseteq V(G) \setminus \{v\}$, we have $|\partial_G(Y)| = |\partial_{G'}((Y \setminus \{u\}) \cup \{w_{ux}\})|$ if $u \in Y$ and $|\partial_G(Y)| = |\partial_{G'}(Y)|$ otherwise. Similarly, for every $Y' \subseteq V(H) \setminus \{y\}$, we have $|\partial_H(Y')| = |\partial_{G'}((Y' \setminus \{x\}) \cup \{w_{ux}\})|$ if $x \in Y'$ and $|\partial_H(Y')| = |\partial_{G'}(Y')|$ otherwise. As a consequence, there is an $X \subseteq V(G')$ with $|\partial_{G'}(X)| = \min\{\lambda(G), \lambda(H)\}$. Thus, it suffices to prove that, for each non-empty proper subset $S \subset V(G')$, we have $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd. Since, for all $Y \subseteq V(G'), \partial_{G'}(Y) = \partial_{G'}(V(G') \setminus Y)$, we just need to consider the two cases when $|S \cap \{w_{ux}, w_{vy}\}| = 0$ and $|S \cap \{w_{ux}, w_{vy}\}| = 1$.

First, assume $|S \cap \{w_{ux}, w_{vy}\}| = 0$. It is clear that $|\partial_{G'}(S)| = |\partial_G(S \cap V(G))| + |\partial_H(S \cap V(H))|$. Note that one of $|S \cap V(G)|$ and $|S \cap V(H)|$ is odd if $|S|$ is odd. So $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd.

Next, we assume $|S \cap \{w_{ux}, w_{vy}\}| = 1$. Without loss of generality, say $w_{ux} \in S$. Note that $|\partial_{G'}(S)| = |\partial_G((S \setminus \{w_{ux}\} \cup \{u\}) \cap V(G))| - t + |\partial_H((S \setminus \{w_{ux}\} \cup \{x\}) \cap V(H))| - (r - t)$. If one of $|(S \setminus \{w_{ux}\} \cup \{u\}) \cap V(G)|$ and $|(S \setminus \{w_{ux}\} \cup \{x\}) \cap V(H)|$ is odd, then the other has the same parity as $|S|$. This implies $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd. Thus, the remaining case is that both $|(S \setminus \{w_{ux}\} \cup \{u\}) \cap V(G)|$ and $|(S \setminus \{w_{ux}\} \cup \{x\}) \cap V(H)|$ are even, and $|S|$ is odd. Since $|(S \setminus \{w_{ux}\}) \cap V(G)|$ is odd in this case and G is an r-graph, we obtain $|\partial_G((S \setminus \{w_{ux}\} \cup \{u\}) \cap V(G))| \geq 2\mu_G(u, v) \geq 2t$. Similarly, $|\partial_H((S \setminus \{w_{ux}\} \cup \{x\}) \cap V(H))| \geq 2\mu_H(x, y) \geq 2(r - t)$. So $|\partial_{G'}(S)| \geq 2t - t + 2(r - t) - (r - t) = r$. This completes the proof. □

Lemma 7.1.8. *Let r, t be two integers with $2 \leq t < r$, let G be an r-graph* *and let $u, v \in V(G)$ such that $\mu_G(u, v) \geq t$. Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of $\mathcal{P}$, let $x, y \in V(\mathcal{P}^{\mathcal{M}})$ such that $\mu_{\mathcal{P}^{\mathcal{M}}}(x, y) \geq r - t$ and let $G' = (G, u, v) \oplus_t (\mathcal{P}^{\mathcal{M}}, x, y)$. If G' has a k-PDPM $\mathcal{N}'$, then G has a k-PDPM $\mathcal{N}$ such that*

(i) $\mathcal{N}$ avoids at least one edge connecting u and v,

(ii) for every $e \in E(G'[V(G) \setminus \{u, v\}])$, if $\mathcal{N}'$ avoids e, then $\mathcal{N}$ avoids e.

Proof. Assume that $\mathcal{N}'$ is a k-PDPM of G'. Every perfect matching of G' contains either zero or exactly two edges of $\partial_{G'}(V(\mathcal{P}^{\mathcal{M}}) \setminus \{x, y\})$, since $|V(\mathcal{P}^{\mathcal{M}}) \setminus \{x, y\}|$ is even. The same holds for $V(G) \setminus \{u, v\}$, since $|V(G) \setminus \{u, v\}|$ is also even. Hence, every perfect matching of G' can be transformed into a perfect matching of G and of $\mathcal{P}^{\mathcal{M}}$ by adding either uv or xy. In particular, $\mathcal{N}'$ can be transformed into a k-PDPM $\mathcal{N}$ of G, which satisfies (ii). Suppose that $\mathcal{N}$ contains all edges connecting u and v, which implies that $\mathcal{N}'$ contains all edges of $\partial_{G'}(V(G))$. As a consequence, $\mathcal{P}^{\mathcal{M}}$ has a k-PDPM that contains all edges of $\partial_{\mathcal{P}^{\mathcal{M}}}(\{x, y\})$. This means that $\mathcal{P}^{\mathcal{M}}$ has a k-PDPM containing all edges incident with y and not with x, a contradiction to Lemma 7.1.4. □

7.2 The complexity of $PDPM(k, r)$

For every $r \geq 3$ the problem whether a given r-regular graph is class 1 is NP-complete, which was shown by Leven and Galil [51]. We extend this result as follows. For every two integers k, r with $1 < k < r$ let $PDPM(k, r)$ be the problem to decide whether a given r-graph has k pairwise disjoint perfect matchings.

Theorem 7.2.1. *For every two integers k, r with $1 < k < r$ the decision problem $PDPM(k, r)$ is NP-complete.*

Proof. First, we prove that for every $r \geq 3$ the decision problem $PDPM(r - 1, r)$ is NP-complete. Leven and Galil [51] proved that for every $r \geq 3$ the problem whether a given r-regular graph is class 1 is NP-complete. Recall that an r-regular graph is an r-graph if and only if its density equals r, where the density of a graph G is defined by

$$\Gamma(G) = \max\left\{\left\lceil \frac{|E(G[S])|}{\lfloor \frac{1}{2}|S| \rfloor} \right\rceil : S \subseteq V(G), |S| \geq 2\right\}$$

if $|V(G)| \geq 2$ and $\Gamma(G) = 0$ otherwise. The density of graph can be computed in polynomial time (see for example [17]) and therefore, it can be decided in polynomial time whether a given r-regular graph is an r-graph. Furthermore, every r-regular graph that is not an r-graph is class 2. Therefore, the result of Leven and Galil implies that for every $r \geq 3$ it is NP-complete to decide whether a given r-graph is class 1, i.e. $PDPM(r-1, r)$ is NP-complete.

Next, we complete the proof by showing that for every two integers k, r with $1 < k < r$, if $PDPM(k, r)$ is NP-complete, then $PDPM(k, r+1)$ is NP-complete. Let G be an r-graph. Every r-graph has a perfect matching [77]; let $M = \{x_1y_1, \ldots, x_sy_s\}$ be a perfect matching of G. For every $i \in \{1, \ldots, s\}$, let H^i be a copy of $\mathcal{P} + (r-2)M_0$. In each copy, the vertices and perfect matchings are labelled accordingly by using an upper index, i.e. the vertex of H^i corresponding to u_1 in $\mathcal{P} + (r-2)M_0$ is labelled as u_1^i. Define graphs $G^0, \ldots, G^s$ inductively as follows:

$$G^0 := G + M,$$
$$G^i := (G^{i-1}, x_i, y_i) \oplus_2 (H^i, u_1^i, v_1^i) \text{ for every } i \in \{1, \ldots, s\}$$

and set $G' = G^s$. Note that $\mu_{G^0}(x_i, y_i) \geq 2$ and $\mu_{H^i}(u_1^i, v_1^i) = r-1$ for every $i \in \{1, \ldots, s\}$ and hence, G' is well-defined. Furthermore, by Lemma 7.1.7 G' is an $(r+1)$-graph, since G^0 and $\mathcal{P}+(r-2)M_0$ are both $(r+1)$-graphs. Lemma 7.1.8 implies, if G' has a k-PDPM, then G_0 has a k-PDPM avoiding every edge of M, i.e. G has a k-PDPM. On the other hand, let $\mathcal{N} = \{N_1, \ldots, N_k\}$ be a k-PDPM of G. Note that $\mu_{G'}(w_{x_iu_1^i}, w_{y_iv_1^i}) = \mu_G(x_i, y_i) - 1$ for every $i \in \{1, \ldots, s\}$, and thus $G - M$ is a subgraph of G'. Hence, we can assume that $N_j \subseteq E(G')$ for every $j \in \{1, \ldots, k-1\}$. For each $i \in \{1, \ldots, s\}$ let $\mathcal{N}^i = \{N_1^i, \ldots, N_k^i\}$ be a k-PDPM of H^i, where N_k^i is is a perfect matching of type 2 in H^i and all other elements of $\mathcal{N}^i$ are of type 0. Furthermore, for every $j \in \{1, \dots, k-1\}$ let $N'_j = N_j \cup \bigcup_{i=1}^{s} N_j^i \setminus \{u_1^i v_1^i\}$ and set $N'_k = \bigcup_{i=1}^{s} N_k^i$. Then, $\{N'_1, \dots, N'_k\}$ is a k-PDPM of G'. As a consequence, G' has a k-PDPM if and only if G has a k-PDPM. Clearly, G' can be obtained from G in polynomial time. Therefore, if $PDPM(k, r)$ is NP-complete, then $PDPM(k, r+1)$ is NP-complete, witch completes the proof. □

Rizzi [75] constructed a poorly matchable $(r + 1)$-graph starting with a poorly matchable r-graph. We remark that the construction of G' from G in the above proof is similar to that construction.

7.3 r-edge-connected r-graphs

In this section we are mainly motivated by the open cases of Problem 7.0.3. First, we prove Theorem 7.0.4, which, together with the results of [63, 75], imply the following corollary.

Corollary 7.3.1. *If $r > 2$ is even, then $m(r, r) \leq r - 3$.*

The graphs that prove Corollary 7.3.1 have a 2-vertex-cut. It is easy to see that for odd r, an r-edge-connected r-graph is 3-vertex-connected (see Observation 7.3.7). This shows that our methods are limited to the case when r is even. Thus, the main motivation for Subsection 7.3.2 is the study of Problems 7.0.1 and 7.0.3 for odd r. We prove that every r-edge-connected r-graph has $k \in \{2, \dots, r-2\}$ pairwise disjoint perfect matchings if and only if every r-edge-connected r-graph has k pairwise disjoint perfect matchings that contain (or that avoid) a fixed edge. For odd r, we prove the stronger statement that every r-edge-connected r-graph has an $(r - 2)$-PDPM if and only if for every r-edge-connected r-graph and every $\lfloor \frac{r}{2} \rfloor$ adjacent edges, there is an $(r - 2)$-PDPM of G containing all $\lfloor \frac{r}{2} \rfloor$ edges. In Subsection 7.3.3 we consider Problem 7.0.1 when $r = t = 5$. For $r \geq 3$, $r \neq 5$, an r-edge-connected r-graph can be constructed by appropriately adding $r-3$ perfect matchings to the Petersen graph. Such graphs are class 2 by Lemma 7.1.3. However, we cannot easily construct a 5-edge-connected 5-regular graph in the same way. Indeed, adding two perfect matchings to $\mathcal{P}$ generates a 4-edge-cut. So far, we have not succeeded in constructing 5-edge-connected 5-regular class 2 graphs. Also, intensive literature research and computer-assisted searches in graph databases did not lead to the desired success. Thus, for $m(5,5)$ we only have the trivial bounds, i.e. $1 \leq m(5,5) \leq 5$. In Subsection 7.3.3 we use the results from Subsection 7.3.2 to prove Theorem 7.0.5. Furthermore, we deduce some properties of a minimum possible 5-edge-connected class 2 5-graph.

7.3.1 Proof of Theorem 7.0.4

In this subsection we construct a $(4k+2)$-edge-connected $(4k+2)$-graph G_k without a $4k$-PDPM for each integer $k \geq 1$. As in [63], we first construct a graph $\mathcal{P}_k$ by adding perfect matchings to the Petersen graph and a graph Q_k by using two copies of $\mathcal{P}_k$. Then, we construct a graph S_k and "replace" some edges of S_k by copies of Q_k to obtain the graph G_k with the desired properties.

The graphs $\mathcal{P}_k$ and Q_k

For each $k \geq 1$, let

$$\mathcal{P}_k = \mathcal{P} + k(M_0 + M_1 + M_2) + (k-1)M_5,$$

as shown in Figure 7.3.

Let $\mathcal{P}_k^1$ and $\mathcal{P}_k^2$ be two distinct copies of $\mathcal{P}_k$. For each $w \in V(\mathcal{P}_k)$, the vertex of $\mathcal{P}_k^1$ ($\mathcal{P}_k^2$, respectively) that corresponds to w is denoted by w^1 (w^2, respectively). Now, we obtain the graph Q_k from $\mathcal{P}_k^1$ and $\mathcal{P}_k^2$ by removing the $2k+1$ parallel edges connecting u_1^i and v_1^i from $\mathcal{P}_k^i$, for each $i \in \{1,2\}$, and identifying u_1^1 and u_1^2 to a new vertex, denoted by u_{Q_k}. Note that the degree of u_{Q_k} in Q_k is $4k+2$. For a graph G containing Q_k as a subgraph, let

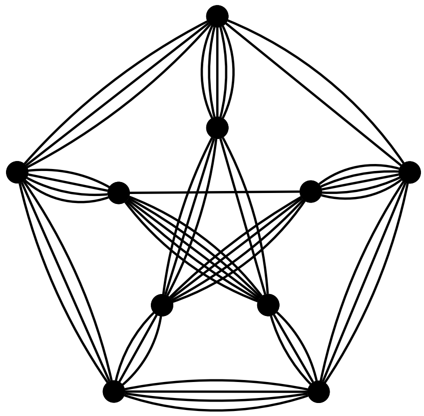


Figure 7.3: The graph $\mathcal{P}_2$.

$E_k^i = E_G(v_1^i, V(G) \setminus V(Q_k))$ for each $i \in \{1, 2\}$. The subgraph Q_2 and the edge sets E_2^1 and E_2^2 are shown in Figure 7.4.

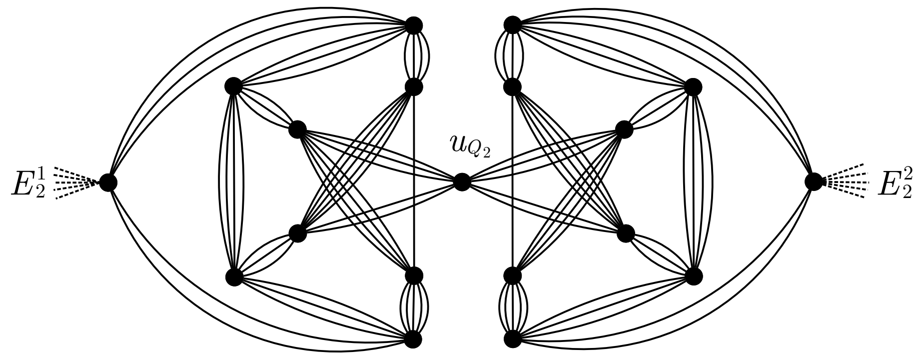


Figure 7.4: The subgraph Q_2 (solid lines) and the edge sets E_2^1 and E_2^2 (dashed lines).

The following lemma is similar to Lemma 2.5 in [63] (Q_k is different), and it can be proved analogously. In order to keep this thesis self-contained, we present the proof here.

Lemma 7.3.2. *Let G be a graph that contains Q_k as an induced subgraph. Let $\mathcal{N} = \{N_1, \dots, N_{4k}\}$ be a set of pairwise disjoint perfect matchings of G and let $N = \bigcup_{i=1}^{4k} N_i$. If $\partial(V(Q_k)) = E_k^1 \cup E_k^2$, then*

$$|E_k^1 \cap N| = |E_k^2 \cap N| = 2k.$$

Proof. Every perfect matching of G intersects $\partial(V(Q_k))$ precisely once since

$|V(Q_k)|$ is odd and $\{v_1^1, v_1^2\}$ is a 2-vertex-cut. It remains to show that E_k^i intersects precisely $2k$ elements of $\mathcal{N}$. Recall that Q_k is constructed by using two copies of $\mathcal{P} + \sum_{M \in \mathcal{M}} M$, where

$$\mathcal{M} = \{\underbrace{M_0, M_1, M_2, \ldots, M_0, M_1, M_2}_{k \text{ times}}, \underbrace{M_5, \ldots, M_5}_{k-1 \text{ times}}\}.$$

We argue by contradiction. Without loss of generality, suppose that $|E_k^1 \cap N| < 2k$, which is equivalent to $|E_k^2 \cap N| > 2k$. Every perfect matching of $\mathcal{N}$ that intersects E_k^1 also intersects the set $E_G(u_{Q_k}, V(\mathcal{P}_k^2))$, and vice versa. Consequently, the existence of $\mathcal{N}$ implies that there is a set $\mathcal{N}'$ of $4k$ pairwise disjoint perfect matchings in $\mathcal{P}_k^1$ such that $\mathcal{N}'$ contains at most $2k-1$ edges of $E_{\mathcal{P}_k^1}(u_1^1, v_1^1)$. Hence, $\mathcal{N}'$ contains all edges of $\partial_{\mathcal{P}_k^1}(u_1^1) \setminus E_{\mathcal{P}_k^1}(u_1^1, v_1^1)$, a contradiction to Lemma 7.1.4. Hence, $|E_k^1 \cap N| = |E_k^2 \cap N| = 2k$. □

The graph S_k

For every $k \geq 1$, let S_k be the graph with vertex-set $\{x_i, y_i, z_i, w : i \in \{1, \ldots, 4k+2\}\}$ and edge-set $A_k \cup kB_k \cup (k+1)D_k \cup (2k+1)(E_k \cup F_k)$ where

$$A_k = \{wz_i : i \in \{1, \ldots, 4k+2\}\},$$

$$B_k = \{z_i x_i, z_i y_i : i \in \{1, \ldots, 4k+2\}\},$$

$$D_k = \{x_i y_i : i \in \{1, \ldots, 4k+2\}\},$$

$$E_k = \{y_i x_{i+1} : i \in \{1, \ldots, 4k+2\}\},$$

$$F_k = \{z_i z_{i+2k+1} : i \in \{1, \ldots, 2k+1\}\},$$

and the indices are added modulo $4k+2$, see Figure 7.5.

Lemma 7.3.3. *For all $k \geq 1$, S_k is $(4k+2)$-edge-connected and $(4k+2)$-regular.*

Proof. By definition, S_k is $(4k+2)$-regular. Let $X \subset V(G)$ be a non-empty set. First, we consider the case that there are two vertices $u, v \in \{x_i, y_i : i \in \{1, \ldots, 4k+2\}\}$ such that X contains exactly one of them. Clearly, there

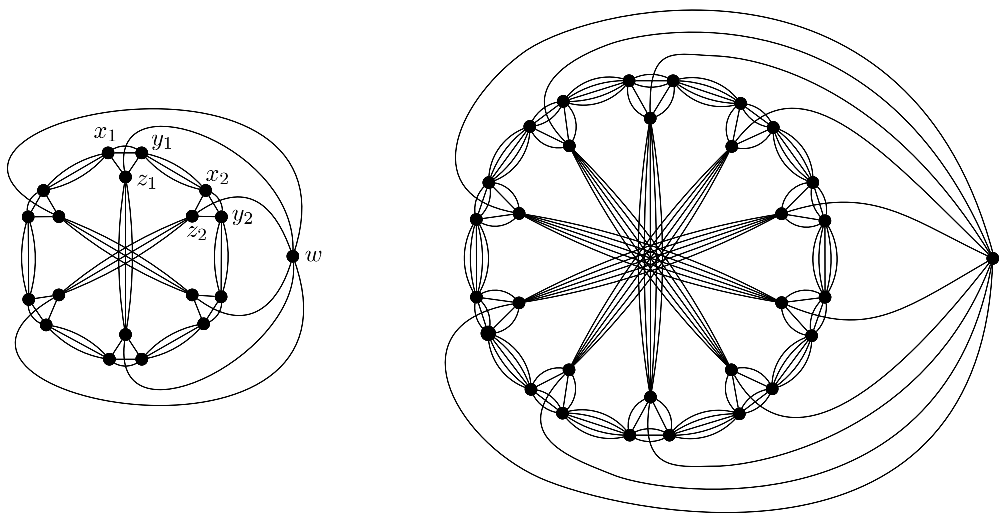


Figure 7.5: The graphs S_1 (left) and S_2 (right).

are $4k+2$ pairwise edge-disjoint u,v-paths in S_k, which only contain edges of $kB_k \cup (k+1)D_k \cup (2k+1)E_k$. Hence, $|\partial_{S_k}(X)| \geq 4k+2$. Therefore, without loss of generality we may assume $\{x_i, y_i : i \in \{1,\ldots,4k+2\}\} \cap X = \emptyset$. Since S_k is $(4k+2)$-regular and $\mu_{S_k}(u,v) \leq 2k+1$ for every $u,v \in V(S_k)$, we have $|X| \notin \{1,2\}$. Hence, X either contains at least three vertices of $\{z_i : i \in \{1,\ldots,4k+2\}\}$ or w and exactly two vertices of $\{z_i : i \in \{1,\ldots,4k+2\}\}$. In the first case, $\partial(X)$ contains at least $6k$ edges of kB_k. In the second case, $\partial(X)$ contains $4k$ edges of A_k and at least $4k$ edges of kB_k, which completes the proof. □

The graph G_k

For every $k \geq 1$, let G_k be the graph obtained from S_k as follows. First, remove all edges of $(2k+1)(E_k \cup F_k)$. Then, for every edge $e = uv \in E_k \cup F_k$, add a copy Q_k^e of Q_k, connect u with the vertex corresponding to v_1^1 by $2k+1$ new parallel edges and connect v with the vertex corresponding to v_2^1 by $2k+1$ new parallel edges, see Figure 7.6.

In order to prove that G_k has the desired properties, we need the following two observations.

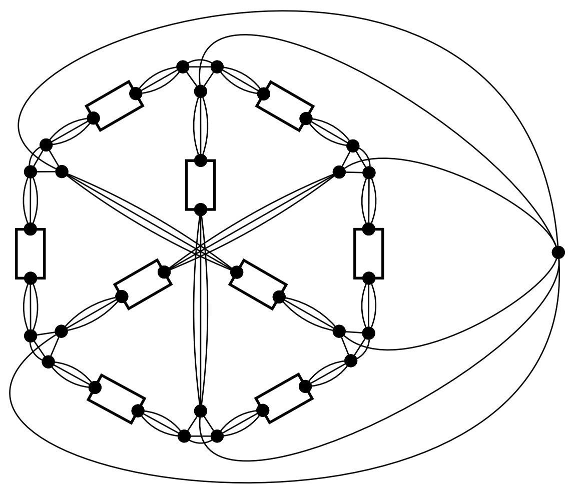


Figure 7.6: The graph G_1, where the boxes are copies of Q_1.

Observation 7.3.4. *Let G be a graph and let $u, v \in V(G)$ with $\mu_G(u, v) = t$. Let H be the graph obtained from G by identifying u and v to a (new) vertex w. If G is $2t$-edge-connected and $2t$-regular, then H is $2t$-edge-connected and $2t$-regular.*

Proof. Assume that G is $2t$-edge-connected and $2t$-regular. Since $\mu_G(u, v) = t$, it follows that $\partial_G(\{u, v\}) = 2t$ and hence, H is $2t$-regular. Let $X \subset V(H)$ be a non-empty set. If $w \in X$, then $|\partial_H(X)| = |\partial_G(X \setminus \{w\} \cup \{u, v\})| \geq 2t$. If $w \notin X$, then $|\partial_H(X)| = |\partial_G(X)| \geq 2t$. □

Observation 7.3.5. *Let G and G' be two disjoint graphs and let $u, v \in V(G)$ and $u', v' \in V(G')$ such that $\mu_G(u, v) = \mu_{G'}(u', v') = t$. Let H be the graph obtained from G and G' as follows. Remove the t parallel edges between u and v and the t parallel edges between u' and v'. Add t parallel edges between u and u' and t parallel edges between v and v'. If G and G' are $2t$-edge-connected and $2t$-regular, then H is $2t$-edge-connected and $2t$-regular.*

Proof. Clearly, H is $2t$-regular. Note that $G - E_G(u, v)$ and $G' - E_{G'}(u', v')$ are t-edge-connected. Let $X \subset V(H)$ be a non-empty set.

Case 1. $X \cap V(G) = V(G)$ or $X \cap V(G') = V(G')$.

Say $V(G) \subseteq X$, then $\partial_H(X)$ contains either (i) $E_H(u,u')$ and $E_H(v,v')$ or (ii) one of $E_H(u,u')$ and $E_H(v,v')$ and a t_1-edge-cut of $G' - E_{G'}(u',v')$ with $t_1 \geq t$ or (iii) a t_2-edge-cut of $G' - E_{G'}(u',v')$ with $t_2 \geq 2t$.

Case 2. $X \cap V(G) \neq V(G)$ and $X \cap V(G') \neq V(G')$.

If $X \cap V(G) \neq \emptyset$ and $X \cap V(G') \neq \emptyset$, then $|\partial_H(X)| \geq |\partial_G(X \cap V(G))| - t + |\partial_{G'}(X \cap V(G'))| - t \geq 2t$. If $X \cap V(G') = \emptyset$, then $|\partial_H(X)| \geq |\partial_G(X)| \geq 2t$. □

Theorem 7.3.6. *For all $k \geq 1$, G_k is a $(4k+2)$-edge-connected $(4k+2)$-graph without $4k$ pairwise disjoint perfect matchings.*

Proof. By construction, $\mu(\mathcal{P}_k) = 2k+1$. As a consequence, $\mathcal{P}_k$ is $(4k+2)$-edge-connected and $(4k+2)$-regular by Lemma 7.1.5. Hence, by Observations 7.3.4 and 7.3.5, the graph $Q_k + (2k+1)\{v_1^1 v_1^2\}$ is $(4k+2)$-edge-connected and $(4k+2)$-regular. Thus, G_k is $(4k+2)$-edge-connected and $(4k+2)$-regular by Observation 7.3.5 again. Furthermore, the order of G_k is $|V(S_k)| + 19|E_k \cup F_k|$, which is even. Suppose to the contrary that G_k has $4k$ pairwise disjoint perfect matchings. Let $N \subseteq E(G_k)$ be the union of them and let $wz_i \in N$. Lemma 7.3.2 implies $|N \cap \partial_{G_k}(\{x_i, y_i, z_i\})| = 6k+1$. On the other hand, every perfect matching contains an odd number of edges of $\partial_{G_k}(\{x_i, y_i, z_i\})$ by Observation 7.1.1. Therefore, $|N \cap \partial_{G_k}(\{x_i, y_i, z_i\})|$ is even, a contradiction. □

Theorem 7.3.6 implies that $m(r,r) \leq r-3$ if $r \equiv 2 \mod 4$. Thus, Theorem 7.0.4 and Corollary 7.3.1 are proved.

7.3.2 Equivalences for statements on the existence of a k-PDPM

The graph G_k from the previous subsection has many 2-vertex-cuts. The following observation shows that such a construction will not apply for the odd case of Problem 7.0.3.

Observation 7.3.7. *For odd $r \geq 3$, every r-edge-connected r-graph is 3-connected.*

Proof. Let G be an r-edge-connected r-graph. Clearly, G is of even order and 2-connected. Suppose that there are two vertices v_1, v_2 such that $G-\{v_1, v_2\}$ is not connected. Then $G-\{v_1, v_2\}$ has exactly two components A and B. Since the order of G is even, A and B are either both of even order or both of odd order. In the first case, $|\partial(V(A))| + |\partial(V(B))| \leq |\partial(v_1)| + |\partial(v_2)| = 2r$. Since A and B are of even order, $|\partial(V(A))|$ and $|\partial(V(B))|$ are both even. Hence, it follows that either $|\partial(V(A))| < r$ or $|\partial(V(B))| < r$ since r is odd. In the second case, $|\partial(V(A) \cup \{v_1\})| + |\partial(V(B) \cup \{v_1\})| = |\partial(v_1)| + |\partial(v_2)| = 2r$. Thus, $|\partial(V(A) \cup \{v_1\})| < r$ or $|\partial(V(B) \cup \{v_1\})| < r$ since A and B are of odd order. Therefore, both cases lead to a contradiction with the assumption that G is r-edge-connected. □

We are going to prove some equivalent statements about the existence of a k-PDPM in r-edge-connected r-graphs.

We recall the following definition from Chapter 2. For a graph G, a vertex $v \in V(G)$ and a graph H disjoint from G, a new graph G' can be obtained from G as follows: add H; for every edge $e \in E(G)$ incident to v, replace the end-vertex v of e by a vertex of H; delete v. We say G' is obtained from G by *replacing* v with H. Note that there are many different graphs that can be obtained from G by replacing v with H; all of them have vertex-set $(V(G) \setminus v) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

In the following we need a special case of the replacing operation.

Definition 7.3.8. *Let G and H be two disjoint r-regular graphs with $u \in V(G)$ and $v \in V(H)$. Let $(G,u)|(H,v)$ be the set of all graphs obtained by replacing the vertex u of G by (H,v), that is, start with G and replace v by $H-u$ such that the resulting graph is r-regular.*

Lemma 7.3.9. *If G and H are two disjoint r-edge-connected r-regular graphs with $u \in V(G)$ and $v \in V(H)$, then every graph in $(G,u)|(H,v)$ is r-regular and r-edge-connected.*

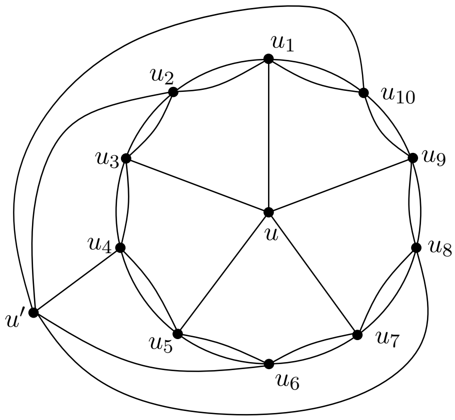
Proof. Suppose to the contrary that there exists a graph $G' \in (G,u)|(H,v)$ with a set $X \subset V(G')$ such that $|\partial_{G'}(X)| \leq r-1$. If $X \subseteq V(G-u)$ or $X \subseteq V(H-v)$, then $|\partial_G(X)| = |\partial_{G'}(X)| \leq r-1$ or $|\partial_H(X)| = |\partial_{G'}(X)| \leq r-1$, a contradiction. Hence, by symmetry, we assume $X \cap V(G-u) = X_1$, $X \cap V(H-v) = X_2$, $X^c \cap V(G-u) = X_3$ and $X^c \cap V(H-v) = X_4$, where $X^c = V(G') - X$ and $X_i \neq \emptyset$ for each $i \in \{1,2,3,4\}$. Since $|\partial_{G'}(X)| \leq r-1$, we have $e_{G'}(X_1, X_3) \leq \lfloor \frac{r-1}{2} \rfloor$ or $e_{G'}(X_2, X_4) \leq \lfloor \frac{r-1}{2} \rfloor$. It implies that $G-u$ or $H-v$ has an edge-cut of cardinality at most $\lfloor \frac{r-1}{2} \rfloor$, which contradicts the assumption that both G and H are r-edge-connected. □

In what follows we show that if every r-edge-connected r-graph has a k-PDPM, then every r-edge-connected r-graph has a k-PDPM containing or avoiding a fixed set of edges.

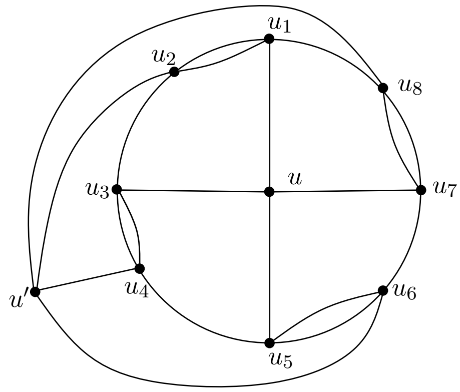
Theorem 7.3.10. *Let $r \geq 4$ and $2 \leq k \leq r-2$. The following statements are equivalent.*
(i) Every r-edge-connected r-graph has a k-PDPM.
(ii) For every r-edge-connected r-graph G and every $e \in E(G)$, there exists a k-PDPM of G containing e.
(iii) For every r-edge-connected r-graph G and every $e \in E(G)$, there exists a k-PDPM of G avoiding e.
(iv) For every r-edge-connected r-graph G, every $v \in V(G)$ and $e \in \partial_G(v)$, there are at least $s = r - \lfloor \frac{r-k}{2} \rfloor - 1$ edges $e_1, \ldots, e_s$ in $\partial_G(v) \setminus \{e\}$ such that, for each $i \in \{1, \ldots, s\}$, there exists a k-PDPM of G containing e_i and e.

Proof. Clearly, each of (ii), (iii) and (iv) implies (i). Thus, it suffices to prove that (i) implies (ii); (i) implies (iii); and (ii) implies (iv).

$(i) \Rightarrow (ii), (iii)$. Assume that statement (i) is true and let G be an r-edge-connected r-graph with an edge vv_1. We use the same construction for both implications. Let $C_{2r} = u_1u_2 \ldots u_{2r}u_1$ be a circuit of order $2r$. Denote $U_o = \{u_i : i \text{ is odd}\}$ and $U_e = \{u_i : i \text{ is even}\}$. We construct a new graph H from C_{2r} as follows. Replace each edge of C_{2r} by $\frac{r-1}{2}$ parallel edges, if r is odd, and replace the edge u_iu_{i+1} (u_ju_{j+1} and $u_{2r}u_1$, respectively) of C_{2r} by $\frac{r}{2}$ ($\frac{r-2}{2}$, respectively) parallel edges for each $u_i \in U_o$ ($u_j \in U_e \setminus \{u_{2r}\}$, respectively), if r is even. Add two new vertices, denoted by u and u', such that u is adjacent to each vertex in U_o and u' is adjacent to each vertex in U_e, see Figure 7.7. Clearly, H is r-regular and r-edge-connected.



(a) $r = 5$

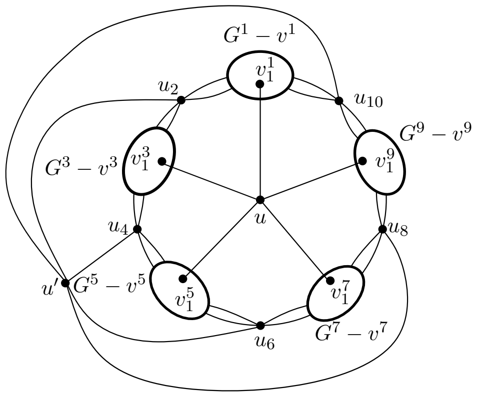


(b) $r = 4$

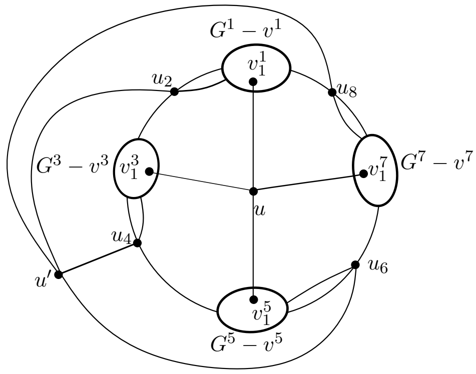
Figure 7.7: Two examples for the graph H obtained from C_{2r} as in the proof of Theorem 7.3.10.

Let $I = \{i : i \in \{1, \ldots, 2r\}, i \text{ is odd}\}$ and for every $i \in I$ let G^i be a copy of G, in which the vertices are labeled accordingly by using an upper index. For example, v^i is the vertex of G^i that corresponds to the vertex v of G. Following the procedure described in Definition 7.3.8, we construct another new graph H' from H by successively replacing each vertex $u_i \in U_o$ of H by (G^i, v^i) such that for each $i \in I$ the vertex v_1^i is adjacent to u (see Figure 7.8). By Lemma 7.3.9, H' is r-regular and r-edge-connected. Note that H' is an r-graph since it is of even order.

In order to prove statements (ii) and (iii) we observe the following. Let M be an arbitrary perfect matching of H' and for every $i \in I$, let $m_i = |\partial_{H'}(V(G^i - v^i)) \cap M|$. The set M contains exactly one edge incident with u and one edge incident with u'. Thus, by the construction of H' we have $\sum_{i \in I} m_i = |M \cap \partial_{H'}(U_e)| = |I|$. Observation 7.1.1 implies $m_i \geq 1$ and hence,



(a) $r = 5$



(b) $r = 4$

Figure 7.8: Two examples for the graph H' obtained from G and H as in the proof of Theorem 7.3.10.

$m_i = 1$ for every $i \in I$. Thus, every perfect matching of H' can be translated into a perfect matching of G^i for each $i \in I$.

Now, by statement (i), H' has a k-PDPM $\mathcal{N}$. Furthermore there are two integers $i, j \in I$ such that $\mathcal{N}$ contains uv_1^i and avoids uv_1^j. By the above observation, the graph G^i has a k-PDPM containing $v^i v_1^i$ and G^j has a k-PDPM avoiding $v^j v_1^j$, which proves statements (ii) and (iii).

$(ii) \Rightarrow (iv)$. Let G be an r-edge-connected r-graph and let $e_1 = vv_1 \in E(G)$. Suppose $|\{e \in \partial_G(v) \setminus \{e_1\}$: there exists a k-PDPM of G containing $e, e_1\}| < s$. As a consequence, $\partial_G(v) \setminus \{e_1\}$ contains at least $t = r-1-(s-1) = \lfloor \frac{r-k}{2} \rfloor + 1$ edges $e_2, \dots, e_{t+1}$, such that for every $j \in \{2, \dots, t+1\}$ there is no k-PDPM of G containing e_1 and e_j. For each $j \in \{2, \dots, t+1\}$ denote $e_j = vv_j$.

Let K_4 be the complete graph of order 4 and let $V(K_4) = \{u_1, u_2, u_3, u_4\}$. We construct a new r-regular graph H from K_4 by replacing each edge of $\{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ by $\frac{r-1}{2}$ parallel edges if r is odd, and replacing each edge of $\{u_1u_2, u_3u_4\}$ ($\{u_2u_3, u_4u_1\}$, respectively) by $\frac{r}{2}$ ($\frac{r-2}{2}$, respectively) parallel edges if r is even, see Figure 7.9. Clearly, H is r-edge-connected.

For each $i \in \{1, 3\}$, let G^i be a copy of G in which the vertices and edges are labeled accordingly by using an upper index and let $V^i = \{v_j^i : j \in \{2, \dots, t+$

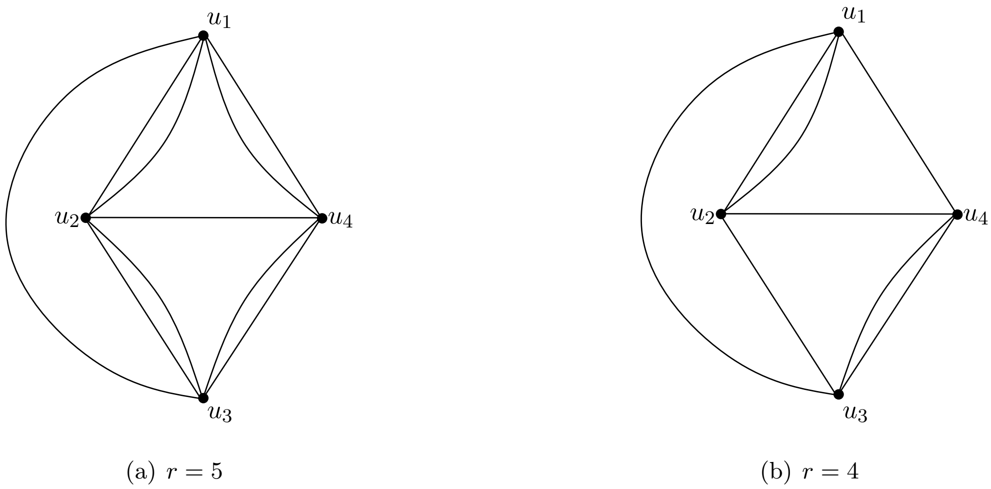


Figure 7.9: Two examples for the graph H obtained from K_4 as in the proof of Theorem 7.3.10.

$1\}\}$. Following the procedure in Definition 7.3.8, we construct another new graph H' from H by successively replacing each vertex $u_i \in \{u_1, u_3\}$ of H by (G^i, v^i) such that v_1^1 is adjacent to v_1^3 and $E_{H'}(u_2, V^1 \cup V^3)$ contains as many edges as possible, see Figure 7.10. The graph H' is r-regular and r-edge-connected by Lemma 7.3.9. By statement (ii), H' has a k-PDPM $\mathcal{N} = \{N_1, \dots, N_k\}$ containing u_2u_4. Clearly, $v_1^1v_1^3$ and u_2u_4 are in the same perfect matching of $\mathcal{N}$ and so each $N_i \in \mathcal{N}$ contains exactly one edge of $\partial_{H'}(V(G^1 - v^1))$ and one edge of $\partial_{H'}(V(G^3 - v^3))$ by Observation 7.1.1. Thus, $N_i \cap E_{H'}(u_2, V^1 \cup V^3) = \emptyset$ for each $i \in \{1, \dots, k\}$. Now we consider the following two cases.

Case 1. r is odd.

Since $t = \lfloor \frac{r-k}{2} \rfloor + 1 \leq \frac{r-1}{2}$, the set $E_{H'}(u_2, V^i)$ contains t edges for each $i \in \{1, 3\}$ by the construction of H'. Note that $N_i \cap E_{H'}(u_2, V^1 \cup V^3) = \emptyset$ for each $i \in \{1, \dots, k\}$. Hence, the k-PDPM $\mathcal{N}$ of H' contains at most $r - 2t = r - 2(\lfloor \frac{r-k}{2} \rfloor + 1) \leq r - 2(\frac{r-k-1}{2} + 1) = k - 1$ edges in $\partial_{H'}(u_2)$, a contradiction.

Case 2. r is even.

Case 2.1. $k = 2$.

Since $t = \lfloor \frac{r-2}{2} \rfloor + 1 = \frac{r}{2}$, the set $E_{H'}(u_2, V^1 \cup V^3)$ contains $2t - 1 = r - 1$

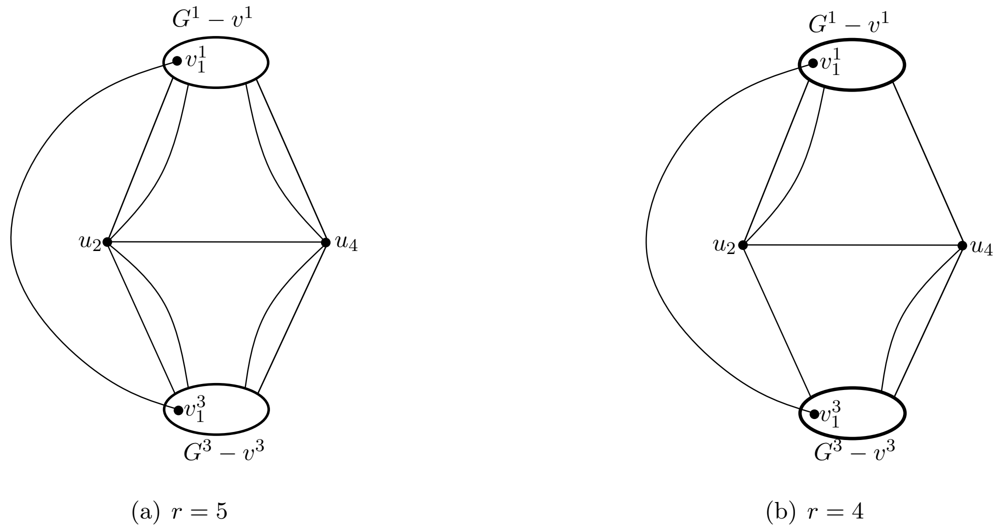


Figure 7.10: Two examples for the graph H' obtained from G and H as in the proof of Theorem 7.3.10.

edges. Hence, the k-PDPM $\mathcal{N}$ of H' contains at most $r - (2t - 1) = 1$ edges in $\partial_{H'}(u_2)$, a contradiction.

Case 2.2. $k > 2$.

Since $t = \lfloor \frac{r-k}{2} \rfloor + 1 \leq \frac{r-4}{2} + 1 = \frac{r}{2} - 1$, we have that $E_{H'}(u_2, V^i)$ contains t edges for each $i \in \{1, 3\}$ by the construction of H'. Hence, the k-PDPM $\mathcal{N}$ of H' contains at most $r - 2t = r - 2(\lfloor \frac{r-k}{2} \rfloor + 1) \leq r - 2(\frac{r-k-1}{2} + 1) = k - 1$ edges in $\partial_{H'}(u_2)$, a contradiction again. □

For the special case $k = r - 2$, we can obtain a stronger result as follows.

Theorem 7.3.11. *Let $k \geq 1$. The following statements are equivalent.*
(i) Every $(2k+1)$-edge-connected $(2k+1)$-graph has a $(2k-1)$-PDPM.
(ii) For every $(2k+1)$-edge-connected $(2k+1)$-graph G and every k edges sharing a common vertex, there exists a $(2k-1)$-PDPM of G containing these k edges.

Proof. It suffices to prove that statement (i) implies statement (ii). Let G be a $(2k+1)$-edge-connected $(2k+1)$-graph and let $v \in V(G)$ be a vertex with

$\partial_G(v) = \{e_i : i \in \{1, \dots, 2k+1\}\}$. We show that there is a $(2k-1)$-PDPM of G that contains the edges $e_1, \dots, e_k$.

Denote $e_i = vv_i$ for each $i \in \{1, \dots, 2k+1\}$. Let G^1 be a copy of G in which the vertices and edges are labeled accordingly by using an upper index. As described in Definition 7.3.8, construct a new graph H from G by replacing v with (G^1, v^1) such that $\partial_H(V(G) \setminus v) = \{v_{2k+1}v^1_{2k+1}\} \cup E_1 \cup E_2$, where $E_1 = \{v_i v^1_{i+k} : i \in \{1, \dots, k\}\}$ and $E_2 = \{v^1_i v_{i+k} : i \in \{1, \dots, k\}\}$. By Lemma 7.3.9, H is $(2k+1)$-edge-connected and $(2k+1)$-regular. Thus, by statement (i) and Theorem 7.3.10 there is a $(2k-1)$-PDPM $\mathcal{N}$ of H avoiding $v_{2k+1}v^1_{2k+1}$. By Observation 7.1.1, every perfect matching of $\mathcal{N}$ contains exactly one edge of $\partial_H(V(G) \setminus \{v\})$ and hence, $\mathcal{N}$ contains either every edge of E_1 or every edge of E_2. In the first case, G has a $(2k-1)$-PDPM that contains $e_1, \dots, e_k$; in the second case, G^1 has a $(2k-1)$-PDPM that contains $e^1_1, \dots, e^1_k$. This proves statement (ii).

□

7.3.3 5-graphs

In this subsection we first related statements on the value of $m(5,5)$ to well-known conjectures for cubic graphs. In particular we prove Theorem 7.0.5. Next, we deduce structural properties of a smallest 5-edge-connected 5-graph of class 2, if such a graph exists.

We recall the following three conjectures.

Conjecture 1.1.1 (Berge-Fulkerson Conjecture [24]). *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

Conjecture 1.1.2 (Cycle Double Cover Conjecture [78,82]). *Every bridgeless graph has a collection of cycles such that each edge belongs to exactly two of them.*

Conjecture 5.0.1 (Fan-Raspaud Conjecture [22]). *Every bridgeless cubic graph has three perfect matchings with an empty intersection.*

A minimum counterexample to Conjecture 1.1.2, if it exists, is a snark. Hence, the Cycle Double Cover is implied by the following stronger conjecture.

Conjecture 7.3.12 (5-Cycle Double Cover Conjecture, see [93]). *Every bridgeless cubic graph has* 5*-cycles such that each edge is in exactly two of them.*

Let G be a cubic graph and let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a multiset of subsets F_i of $E(G)$. For an edge e of G, we denote by $\nu_{\mathcal{F}}(e)$ the number of elements of $\mathcal{F}$ containing e. A *Fan-Raspaud triple*, or *FR-triple*, is a multiset $\mathcal{T}$ of three perfect matchings of G such that $\nu_{\mathcal{T}}(e) \leq 2$ for all $e \in E(G)$. A 5-*cycle double cover*, or 5-CDC, is a multiset $\mathcal{C}$ of five cycles in G such that, for every edge $e \in E(G)$, $\nu_{\mathcal{C}}(e) = 2$. A *Berge-Fulkerson cover*, or *BF-cover*, is a multiset $\mathcal{T}$ of six perfect matchings of G such that $\nu_{\mathcal{T}}(e) = 2$ for all $e \in E(G)$.

Relation to the Fan-Raspaud Conjecture

We show that the Fan-Raspaud Conjecture is true if there is no poorly matchable 5-edge-connected 5-graph. For that we need one result from [44] as well as an equivalent formulation of the Fan-Raspaud Conjecture.

Theorem 7.3.13 (Kaiser and Škrekovski [44]). *Every bridgeless cubic graph has a* 2*-factor that intersects every edge-cut of cardinality* 3 *and* 4*. Moreover, any two adjacent edges can be extended to such a* 2*-factor.*

As proved in [71], the following conjecture is equivalent to the Fan-Raspaud Conjecture.

Conjecture 7.3.14 (Mkrtchyan and Vardanyan [71]). *Let G be a bridgeless cubic graph. For every $e \in E(G)$ and $i \in \{0, 1, 2\}$, there is an FR-triple $\mathcal{T}$ with $\nu_{\mathcal{T}}(e) = i$.*

In the same paper, they also pointed out the following observation but without proof. To keep this thesis self-contained, we present a short proof here.

Observation 7.3.15 (Mkrtchyan and Vardanyan [71]). *A minimum possible counterexample G to Conjecture 7.3.14 with respect to $|V(G)|$ is 3-edge-connected.*

Proof. Suppose that G is a minimum counterexample to Conjecture 7.3.14 with respect to $|V(G)|$. Then, there is $e \in E(G)$ and $i \in \{0, 1, 2\}$ such that no FR-triple $\mathcal{T}$ satisfies $\nu_{\mathcal{T}}(e) = i$. Suppose that there is a set $X \subseteq V(G)$ with $u, v \in X$ and $\partial_G(X) = \{ux, vy\}$. Let $H_1 = G[X] + \{uv\}$ and $H_2 = G - X + \{xy\}$. Notice that both H_1 and H_2 are bridgeless cubic graphs. If $e \in \{ux, vy\}$, since $|V(G)|$ is minimum, there is an FR-triple $\mathcal{T}_1$ of H_1 and an FR-triple $\mathcal{T}_2$ of H_2 such that $\nu_{\mathcal{T}_1}(uv) = \nu_{\mathcal{T}_2}(xy) = i$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ can be used to construct an FR-triple $\mathcal{T}$ of G with $\nu_{\mathcal{T}}(e) = i$, a contradiction. Hence, without loss of generality we may assume $e \in E(H_1)$. Since $|V(G)|$ is minimum, there is an FR-triple $\mathcal{T}_1$ of H_1 and an FR-triple $\mathcal{T}_2$ of H_2 such that $\nu_{\mathcal{T}_1}(e) = i$ and $\nu_{\mathcal{T}_2}(xy) = \nu_{\mathcal{T}_1}(uv)$. Again, $\mathcal{T}_1$ and $\mathcal{T}_2$ can be used to construct an FR-triple $\mathcal{T}$ of G with $\nu_{\mathcal{T}}(e) = i$, a contradiction. □

Theorem 7.3.16. *If $m(5, 5) \geq 2$, then Conjecture 7.3.14 is true.*

Proof. By contradiction, suppose that $m(5, 5) \geq 2$ and Conjecture 7.3.14 is false. Let G be a minimum counterexample to Conjecture 7.3.14 with respect to $|V(G)|$. Then, there is an edge $e = uv$ of G and an $i \in \{0, 1, 2\}$ such that no FR-triple $\mathcal{T}$ satisfies $\nu_{\mathcal{T}}(e) = i$. By Observation 7.3.15, G is 3-edge-connected.

First, we consider the case $i = 0$. By Theorem 7.3.13 there is a 2-factor F of G such that $e \in E(F)$ and F intersects every edge-cut of cardinality 3 and 4. Let $H = G + E(F)$ and let e' be the new edge parallel to e. Since G is 3-edge-connected, the graph H is 5-edge-connected by the choice of F. Since $m(5, 5) \geq 2$, it follows with Theorem 7.3.10 (*iv*) that for each edge

$e_0 \in \partial_H(v) \setminus \{e, e'\}$, there are at least three edges $e_1, e_2, e_3 \in \partial_H(v) \setminus \{e_0\}$ such that for each $j \in \{1, 2, 3\}$ there exists a 2-PDPM containing e_j and e_0. This implies that H has two disjoint perfect matchings N_1 and N_2 such that e and e' are in none of them. In the graph G, let N_1' and N_2' be the perfect matchings corresponding to N_1 and N_2, respectively. Let $N_3 = E(G) \setminus E(F)$. Since N_1 and N_2 are disjoint, every edge of $N_1' \cap N_2'$ belongs to $E(F)$, i.e. $\mathcal{T} = \{N_1', N_2', N_3\}$ is an FR-triple of G. Furthermore $\nu_{\mathcal{T}}(e) = 0$, a contradiction.

Next suppose $i \in \{1, 2\}$. By Theorem 7.3.13 we can choose a 2-factor F of G such that $e \notin E(F)$ and F intersects every edge-cut of cardinality 3 and 4. Again, the graph H defined by $H = G + E(F)$ is 5-edge-connected. Since $m(5,5) \geq 2$, by statements (ii) and (iii) of Theorem 7.3.10, H has two disjoint perfect matchings N_1 and N_2 such that e is in exactly $i-1$ of them. Therefore, $\mathcal{T} = \{N_1', N_2', N_3\}$ is an FR-triple of G with $\nu_{\mathcal{T}}(e) = i$ where N_1' and N_2' are the perfect matchings of G that correspond to N_1 and N_2, respectively, and $N_3 = E(G) \setminus E(F)$. This leads to a contradiction again. □

If $m(5,5) \geq 2$, then in particular every 5-edge-connected 5-graph with an underlying cubic graph has two disjoint perfect matchings. By adjusting Theorem 7.3.10, one can show the following strengthening of Theorem 7.3.16 (for a sketch of the proof, see Appendix A.1).

Theorem 7.3.17. *If every 5-edge-connected 5-graph whose underlying graph is cubic has two disjoint perfect matchings, then Conjecture 7.3.14 is true.*

Relation to the 5-Cycle Double Cover Conjecture

Now we focus on the consequences of the non-existence of 5-edge-connected class 2 5-graphs. Let $k \geq 3$ be an integer. A *k-wheel* W_k is a k-circuit C_k plus one additional vertex w adjacent to all vertices of C_k.

Theorem 7.3.18. *The following statements are equivalent.*

(i) Every 5-edge-connected 5-graph is class 1.

(ii) Every 5-edge-connected 5-graph with an underlying cubic graph is class 1.

Proof. The first statement implies trivially the second one. We prove now the other implication. Let G be a 5-edge-connected 5-graph. For every vertex v of G, let W_5^v be a copy of the graph $W_5 + E(C_5)$. Moreover, let w^v and C_5^v be the vertex and, respectively, the circuit of W_5^v corresponding to w and C_5 in W_5. Following the procedure described in Definition 7.3.8, successively replace every vertex v of G with (W_5^v, w^v) to obtain a new graph H, which is 5-regular and 5-edge-connected. Moreover, its underlying graph is cubic and so H is class 1 by statement (ii). Hence, H has a 5-PDPM, denoted by $\mathcal{N} = \{N_1, \ldots, N_5\}$. Since $|V(C_5^v)|$ is odd, by Observation 7.1.1, we have that, for all $i \in \{1, \ldots, 5\}$, $|N_i \cap \partial_H(V(C_5^v))| = 1$. Hence, the restriction N_i' of N_i to the graph G is a perfect matching of G. Moreover, $\{N_1', \ldots, N_5'\}$ is a 5-PDPM of G. Therefore, G is class 1. □

It is well known that a counterexample of minimum order to Conjecture 7.3.12 is a cyclically 4-edge-connected cubic class 2 graph.

Theorem 7.3.19. *If $m(5,5) = 5$, then Conjecture 7.3.12 is true.*

Proof. Let K be the graph obtained from a 4-wheel by doubling the edges of the outer circuit and of one spoke. Note that K has one vertex of degree 6, which we denote by w, and four vertices of degree 5.

Let G be a minimum counterexample to Conjecture 7.3.12 with respect to $|V(G)|$. Then, G is cubic and cyclically 4-edge-connected. Thus, the graph $2G = G + E(G)$ is 6-edge-connected. For every vertex v of G, let K^v be a copy of K and let w^v be the vertex of K^v corresponding to w in K. Analogously to Definition 7.3.8, let H be the 5-regular graph obtained by replacing each vertex v of $2G$ by (K^v, w^v), in such a way that parallel edges of $2G$ are incident with the same vertex of K^v. Then, H is a 5-edge-connected 5-graph and therefore, it has a 5-PDPM $\mathcal{N} = \{N_1, \ldots, N_5\}$. For every $v \in V(2G)$, there exist exactly three perfect matchings of $\mathcal{N}$, say N_1', N_2', N_3', such that $|N_i' \cap \partial_H(K^v - w^v)| = 2$ for each $i \in \{1,2,3\}$. Hence, for every $j \in \{1, \ldots, 5\}$, the restriction of each

$N_j \in \mathcal{N}$ on G induces a cycle F_j in G. Moreover, we have $\nu_{\mathcal{C}}(e) = 2$ for each $e \in E(G)$, where $\mathcal{C} = \{F_1, \ldots, F_5\}$. So $\mathcal{C}$ is a 5-CDC of G. □

Relation to the Berge-Fulkerson Conjecture

Observation 7.3.20. *If $m(5,5) = 5$, then Conjecture 1.1.1 is true.*

Proof. Assume $m(5,5) = 5$ and suppose that G is a counterexample to the Berge-Fulkerson Conjecture such that the order of G is minimum. Let F be a 2-factor of G. As shown in [58], G is cyclically 5-edge-connected and hence, $G + E(F)$ is 5-edge-connected. Therefore, $G + E(F)$ has five pairwise disjoint perfect matchings. The corresponding five perfect matchings of G and $E(G) \setminus E(F)$ are a BF-cover of G, a contradiction. □

Properties of a minimum possible 5-edge-connected class 2 5-graph

We are going to prove some structural properties of a smallest possible 5-edge-connected class 2 5-graph. Let G be a graph and let $x \in V(G)$ with $|N_G(x)| \geq 2$. A *lifting* (of G) at x is the following operation: Choose two distinct neighbors y and z of x, delete an edge e_1 connecting x with y, delete an edge e_2 connecting x with z and add a new edge e connecting y with z; additionally, if e_1 and e_2 were the only two edges incident with x, then delete the vertex x in the new graph. We say e_1 and e_2 are *lifted to* e.

Theorem 7.3.21 (Mader [61]). *Let G be a finite graph and let $v \in V(G)$ such that $d(v) \geq 4$, $|N(v)| \geq 2$ and $G - v$ is connected. There is a lifting of G at v such that, for every pair of distinct vertices $u, w \in V(G) \setminus \{v\}$, the number of edge-disjoint u, w-paths in the resulting graph equals the number of edge-disjoint u, w-paths in G.*

Statement (ii) of the following theorem is already mentioned in [19] for planar r-graphs without proof.

Theorem 7.3.22. *Let G be a 5-edge-connected class 2 5-graph such that the order of G is as small as possible. The following statements hold.*

(i) Every 5-edge-cut of G is trivial, i.e. if $X \subset V(G)$ and $|\partial(X)| = 5$, then $|X| = 1$ or $|V(G) \setminus X| = 1$.

(ii) Every 3-vertex-cut is trivial, i.e. if $X \subset V(G)$, $|X| = 3$ and $G - X$ is not connected, then one component of $G - X$ is a single vertex.

Proof. (i). The proof follows easily and is left to the reader.

(ii). By contradiction, suppose that $X = \{v_1, v_2, v_3\} \subset V(G)$ is a 3-vertex-cut of G such that none of the components of $G - X$ is a single vertex. By Observation 7.3.7 and the edge-connectivity of G, the graph $G-X$ has at most three components. First, we consider the case that $G - X$ has exactly three components. Denote the vertex-sets of these three components by A, B and C. We have that $|\partial_G(S)| = 5$, for each $S \in \{A, B, C\}$, and so $|A| = |B| = |C| = 1$ by statement (i), a contradiction.

Next, we assume that $G-X$ has exactly two components whose vertex-sets are denoted by A and B. Since G has even order, we may assume $|A|$ is odd and $|B|$ is even. For each $i \in \{1, 2, 3\}$, set $n_i = |\partial_G(B) \cap \partial_G(v_i)|$ and let

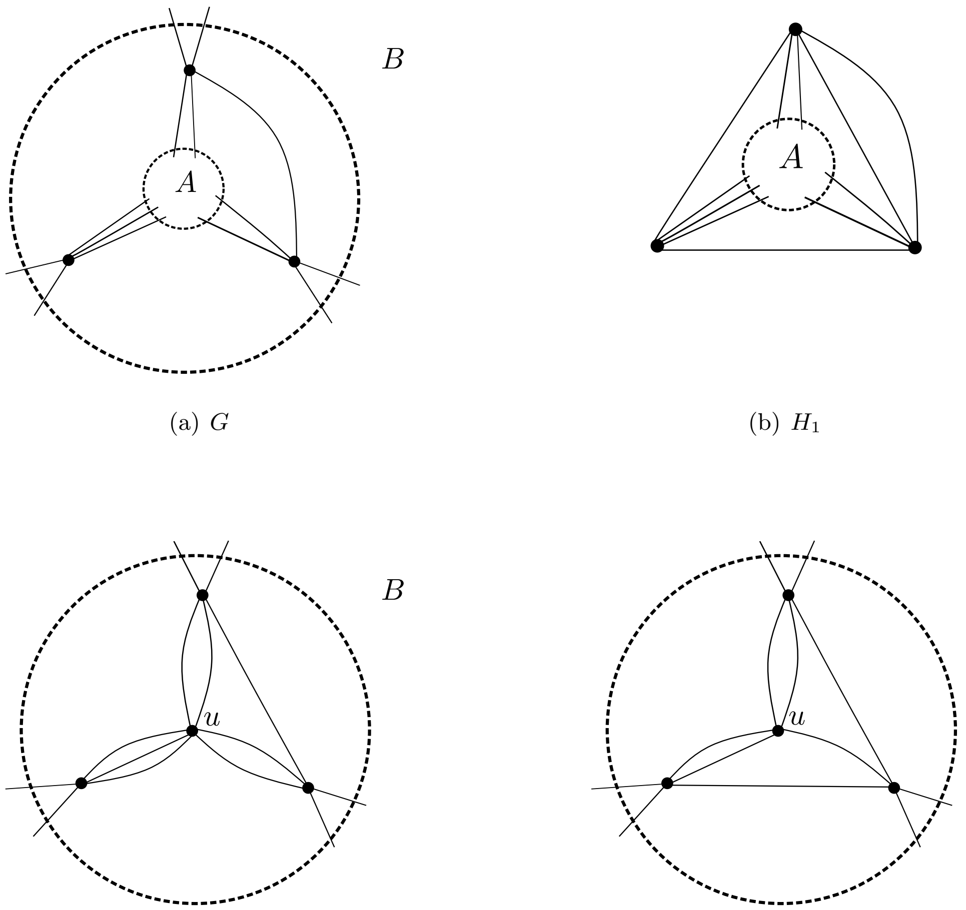
$$a = \frac{1}{2}(n_1 + n_2 - n_3), \quad b = \frac{1}{2}(-n_1 + n_2 + n_3), \quad c = \frac{1}{2}(n_1 - n_2 + n_3).$$

We have that $n_1 + n_2 + n_3 = |\partial_G(B)|$ is even, since $|B|$ is even. Thus, all of a, b, c are integers. Furthermore, $5 \leq |\partial_G(B \cup \{v_3\})| = n_1 + n_2 + (5 - n_3)$ and hence, $a \geq 0$. Analogously, we obtain $b, c \geq 0$. Therefore, we can define a new graph H_1 as follows (see Figure 7.11).

$$H_1 = (G - B) + a\{v_1v_2\} + b\{v_2v_3\} + c\{v_3v_1\}.$$

By the definitions of a, b, c, the graph H_1 is 5-regular. Moreover H_1 is also 5-edge-connected. Indeed, let $Y \subseteq V(H_1)$. We can assume, without loss of generality, that $|Y \cap \{v_1, v_2, v_3\}| \leq 1$ (otherwise, we argue by taking its complement). By the choices of a, b and c, we have $|\partial_{H_1}(Y)| = |\partial_G(Y)| \geq 5$ and so H_1 is 5-edge-connected.

Let H' be the graph obtained from G by identifying all vertices in A to a new vertex u, see Figure 7.11. Then, H' is 5-edge-connected and every vertex is of degree 5 except u. Since $|A|$ is odd, we have that $|\partial_G(A)|$ is odd. Hence, the vertex u has an odd degree of at least 5 in H'. Now, by Theorem 7.3.21, a new 5-edge-connected 5-graph H_2 can be obtained from H' by $\frac{1}{2}(d_{H'}(u) - 5)$ liftings at u, see Figure 7.11.



(a) G (b) H_1

(c) H' (d) H_2

Figure 7.11: An example for the graphs H_1, H' and H_2 obtained from G in the proof of Theorem 7.3.22.

We will refer to the edges of H_2 obtained by a lifting at u as lifting edges and denote the set of all lifting edges by $\mathcal{L}$.

By the minimality of $|V(G)|$, H_1 has a 5-PDPM $\{N_1^1, \ldots, N_5^1\}$ and H_2 has a 5-PDPM $\{N_1^2, \ldots, N_5^2\}$. Since u has at most three neighbors in H_2, every perfect matching of H_2 contains at most one lifting edge. For each $i \in \{1, \ldots, 5\}$, let N_i be the subset of edges of H' defined as follows.

$$N_i = \begin{cases} N_i^2 & \text{if } N_i^2 \cap \mathcal{L} = \emptyset; \\ (N_i^2 \setminus \{e\}) \cup \{e_1, e_2\} & \text{if } N_i^2 \cap \mathcal{L} = \{e\} \text{ and } e_1, e_2 \text{ are the two edges} \\ & \text{lifted to } e. \end{cases}$$

Every perfect matching of H_1 contains either one or three edges of $\partial_{H_1}(A)$ by Observation 7.1.1. Let s_1 be the number of integers $i \in \{1, \ldots, 5\}$ with $|N_i^1 \cap \partial_{H_1}(A)| = 3$, let $s_2 = |\mathcal{L}|$ and let s' be the number of integers $j \in \{1, \ldots, 5\}$ with $|N_j \cap \partial_{H'}(u)| = 3$. We have that $s_2 = s'$. Moreover, we have $\partial_G(A) = 3s_1 + (5 - s_1) = 5 + 2s_2$ and so $s_1 = s_2 = s'$. Note that $\partial_{H_1}(A) = \partial_G(A)$ and recall that H' is obtained from G by identifying all vertices in A to u. As a consequence, the sets of edges $N_1, \ldots, N_5$ of H' and the perfect matchings $N_1^1, \ldots, N_5^1$ of H_1 can be combined to obtain a 5-PDPM of G, a contradiction. □

7.4 r-graphs with arbitrary edge-connectivity

In this section, we prove Theorem 7.0.6, i.e. we show $m(2l, r) \leq 3l - 6$ for every $l \geq 3$ and $r \geq 2l$. Note that this bound only depends on the edge-connectivity parameter. Furthermore, for an r-graph G with a subset $X \subseteq V(G)$, we observe that $|\partial_G(X)| = r \cdot |X| - 2|E(G[X])|$ is even if $|X|$ is even. Therefore, the edge-connectivity of an r-graph is either r or an even number.

7.4.1 Proof of Theorem 7.0.6

Recall that $m(t, r) \leq m(t', r)$ whenever $t \leq t'$. Moreover, $m(4, 5) = 1$ and $m(r, r) \leq r - 2$ for each $r \geq 3$, $r \neq 5$ and thus, $r - 2$ is a trivial upper bound for $m(2l, r)$. As a consequence, Theorem 7.0.6 trivially holds for the case $2l \leq r \leq 3l - 4$, i.e. it suffices to prove Theorem 7.0.6 for the case $r \geq 3l - 3$.

We will construct $2l$-edge-connected r-graphs inductively starting with a $2l$-edge-connected $(3l - 4)$-graph without a $(3l - 5)$-PDPM if $l \geq 4$ and a 6-edge-connected 6-graph without a 4-PDPM if $l = 3$.

For this we first describe the induction step. Then we give the base graphs for the two cases. Finally we deduce the statement of Theorem 7.0.6.

Induction step from r to $r + 1$

Lemma 7.4.1. *Let r, l, k be integers such that $r \geq 3l - 4$, $l \geq 2$ and $2 \leq k \leq r$. If there is an r-graph G such that*

- *$\lambda(G) \geq 2l$,*
- *G has a perfect matching M such that $\mu_G(u, v) \geq l - 1$ for every $uv \in M$,*
- *G has no k-PDPM,*

then there is an $(r + 1)$-graph G' such that

- *$\lambda(G') \geq 2l$,*
- *G' has a perfect matching M' such that $\mu_{G'}(u, v) \geq l - 1$ for every $uv \in M'$,*
- *G' has no k-PDPM.*

Proof. Assume that the order of G is $2s$ and let $M = \{x_1y_1, \dots, x_sy_s\}$. In order to construct G' we define a graph $\mathcal{P}_{(r+1,l)}$ by

$$\mathcal{P}_{(r+1,l)} = \mathcal{P} + \left\lceil \frac{r-l}{2} \right\rceil M_0 + \left\lfloor \frac{r-l}{2} \right\rfloor M_1 + (l-2)M_2.$$

Since G is $2l$-edge-connected, we have $r \geq 2l$. Thus, $\mathcal{P}_{(r+1,l)}$ is well defined. For every $i \in \{1, \dots, s\}$, take a copy $\mathcal{P}^i_{(r+1,l)}$ of $\mathcal{P}_{(r+1,l)}$. In each copy, the vertices and perfect matchings are labelled accordingly by using an upper index, i.e.

the vertex of $\mathcal{P}^i_{(r+1,l)}$ corresponding to u_1 in $\mathcal{P}_{(r+1,l)}$ is labeled as u^i_1. Define graphs $H^0, \dots, H^s$ inductively as follows:

$$H^0 := G + M,$$
$$H^i := (H^{i-1}, x_i, y_i) \oplus_l (\mathcal{P}^i_{(r+1,l)}, u^i_1, v^i_1) \text{ for every } i \in \{1, \dots, s\}.$$

Note that H^0 and $\mathcal{P}_{(r+1,l)}$ are both $(r+1)$-graphs. Furthermore, $\mu_{H^0}(x_i, y_i) \geq l$ for every $i \in \{1, \dots, s\}$ by the choice of M. Recall that $u_1v_1 \in E(\mathcal{P})$ is the unique edge in $M_0 \cap M_1$. Thus, $\mu_{\mathcal{P}_{(r+1,l)}}(u_1, v_1) = (r+1) - l$ by the definition of $\mathcal{P}_{(r+1,l)}$. As a consequence, $H^0, \dots, H^s$ are well defined. Set

$$G' := H^s \qquad \text{and} \qquad M' := \bigcup_{i=1}^{s} M^i_2.$$

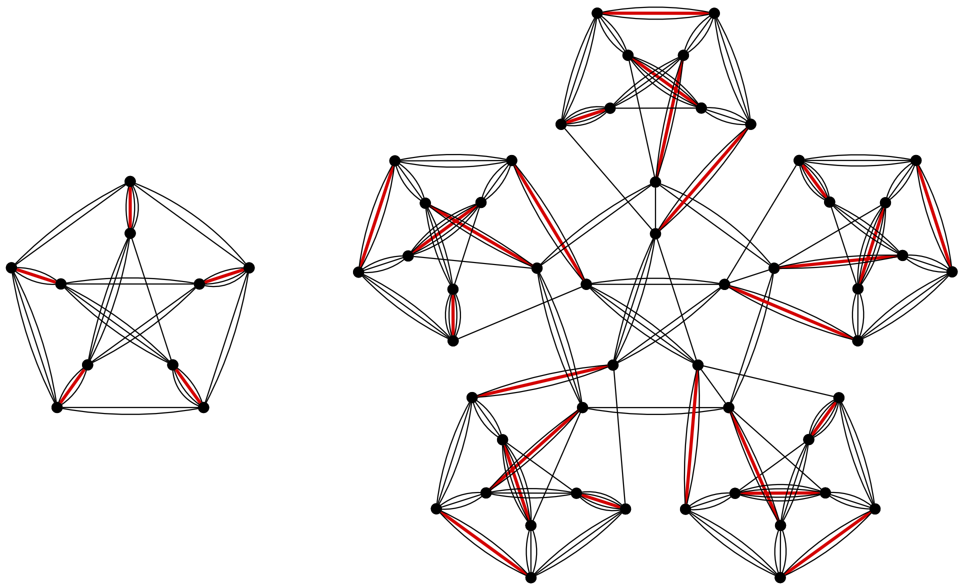


Figure 7.12: The graph $G = \mathcal{P} + 2M_0 + M_1 + M_2 + M_3$ (left) and the graph G' (right) constructed from G in the proof of Lemma 7.4.1. The edges of M and M' respectively are drawn in bold red lines.

An example is given in Figure 7.12. We claim that G' and M' have the desired properties.

The perfect matching M_2 does not contain the edge u_1v_1. Thus, M' is well defined. Furthermore, M' is a perfect matching of G' since M is a perfect matching of G. By the definition of $\mathcal{P}_{(r+1,l)}$, we have $\mu_{G'}(u,v) \geq l-1$ for every $uv \in M'$. Hence, M' has the desired properties.

The graph H^0 is a $2l$-edge-connected $(r+1)$-graph, since G is a $2l$-edge-connected r-graph. Furthermore, $\lfloor \frac{r-l}{2} \rfloor \geq l-2$ since $r \geq 3l-4$. Thus, $r-l+1$ is the maximum number of parallel edges of $\mathcal{P}_{(r+1,l)}$ and hence, $\lambda(\mathcal{P}_{(r+1,l)}) = 2l$ by Lemma 7.1.5. Therefore, for each $i \in \{1,\dots,s\}$, H^i is a $2l$-edge-connected $(r+1)$-graph by Lemma 7.1.7, and so is G'.

Now, suppose that H^s has a k-PDPM $\mathcal{N}^s$. By applying Lemma 7.1.8 with $t = l$ to the $(r+1)$-graph H^s and $\mathcal{N}^s$ we obtain a k-PDPM $\mathcal{N}^{s-1}$ of H^{s-1}, which avoids $x^s y^s$ by property (i). Apply Lemma 7.1.8 to H^{s-1} and $\mathcal{N}^{s-1}$ to obtain a k-PDPM $\mathcal{N}^{s-2}$ of H^{s-2}, which avoids $x^{s-1}y^{s-1}$ by property (i) and $x^s y^s$ by property (ii). By inductively repeating this process, we obtain a k-PDPM of H^0 that avoids every edge of M. This is not possible, since G has no k-PDPM. Therefore, G' has no k-PDPM, which completes the proof. □

We note that the condition $r \geq 3l-4$ is necessary in Lemma 7.4.1 since $\lambda(\mathcal{P}_{(r+1,l)}) < 2l$ if $r < 3l-4$. In view of Lemma 7.4.1, we need to construct suitable base graphs for all $l \geq 3$, which will be done now.

Base graph if $l = 3$.

In order to construct the required base graph G^6, we need the graph G_1 constructed in the Subsection 7.3.1 (see also Figure 7.6). For a precise definition of G_1 (and its vertex-labeling) the reader is referred to that subsection. Every perfect matching of G_1 contains an edge in $\partial_{G_1}(w)$, which is simple. Thus, in order to use Lemma 7.4.1 we need to slightly modify G_1. For any $v \in V(G_1)$, we define a 3-*expansion* to be the operation that splits v into two vertices v' and v'' (edges formerly incident with v will be incident with exactly one of v' and v'') and adds three parallel edges between them.

Let G^6 be the graph obtained from G_1 by applying a 3-expansion to the vertices z_2, z_4, z_6 and w. Let w' and w'' be the new vertices in which w has been split (see Figure 7.13).

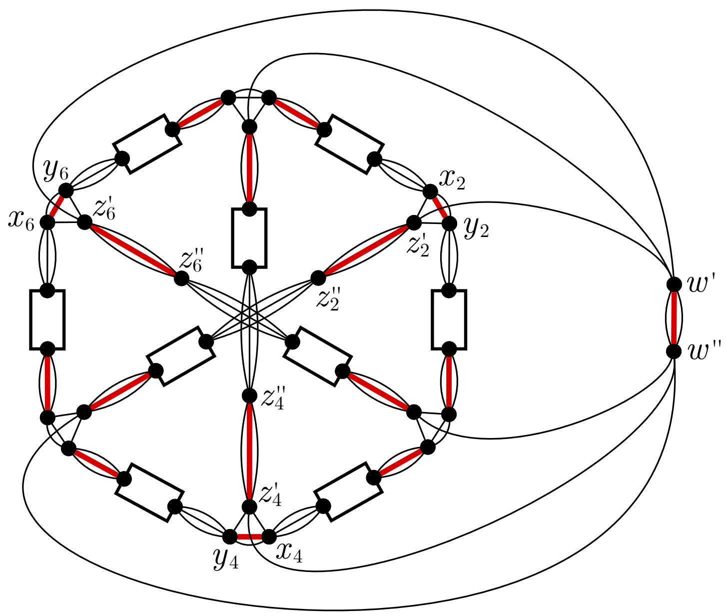


Figure 7.13: The graph G^6, where the boxes a re copies of Q_1. The bold red edges are used to construct M^6 in the proof of Theorem 7.0.6.

It is straightforward that G^6 is still a 6-edge-connected 6-graph. By using similar arguments as in the Proof of Theorem 7.3.6, it follows that G^6 has no 4-PDPM. For the sake of completeness we present a short proof here.

Proposition 7.4.2. *The graph G^6 has no 4-PDPM.*

Proof. In this proof vertex labelings of G^6 are considered with reference to Figure 7.13. Assume by contradiction that G^6 has a 4-PDPM $\mathcal{M} = \{N_1, \dots, N_4\}$. Then, there is $j \in \{1, \dots, 4\}$ such that $\partial_{G^6}(\{w', w''\}) \cap N_j \neq \emptyset$. Let $e \in \partial_{G^6}(\{w', w''\}) \cap N_j$. We can assume without loss of generality that e is incident with z_2'. Let $X = \{x_2, y_2, z_2'\} \subseteq V(G^6)$. Then, from Lemma 7.3.2, we infer that $|\partial_{G^6}(X) \cap N|$ is odd, where $N = \cup_{i=1}^4 N_i$. On the other hand, since X is an odd set, we have that for every $i \in \{i, \dots, 4\}$, $|X \cap N_i|$ is an odd number. Thus, $|X \cap N| = \sum_{i=1}^4 |X \cap N_i|$ must be an even number, a contradiction. $\square$

Base graphs if $l \geq 4$.

Let $l \geq 4$ and consider the following graph

$$G^{3l-4} = \mathcal{P} + (l-2)M_0 + (l-3)M_1 + (l-3)M_2 + M_3.$$

The graph G^8 is shown in the left-hand side of Figure 7.12. By definition, G^{3l-4} is a $(3l-4)$-graph, which is $2l$-edge-connected by Lemma 7.1.5. It is well known, see [27], that G^{3l-4} is of class 2 and hence has no $(3l-5)$-PDPM.

Now we are ready to prove Theorem 7.0.6.

Proof of Theorem 7.0.6. We prove the statement by induction on r. When $l \geq 4$ we choose G^{3l-4} as base graph (defined above) and we consider the perfect matching M_0 of G^{3l-4}.

Recall that G^{3l-4} is a $2l$-edge-connected $(3l-4)$-graph with no $(3l-5)$-PDPM. Furthermore, for all $uv \in M_0$, $\mu_{G^{3l-4}}(u,v) \geq l-1$. Hence the base case is settled. Then, the inductive step follows by Lemma 7.4.1 and the statement is proved.

When $l = 3$, we again argue by induction on r. We choose G^6 as base graph. We have already proved that it is a 6-edge-connected 6-graph without a 4-PDPM. Hence, $m(6,6) \leq 3$.

Let M^6 be the perfect matching of G^6 defined as follows. Consider the matching consisting of the bold red edges depicted in Figure 7.13. Extend this matching to a perfect matching of G^6 by choosing, for every copy of Q_1, the bold red edges depicted in Figure 7.14.

Note that the chosen set of edges is indeed a perfect matching and each edge of such perfect matching has at least one other parallel edge. This means that the condition on the multiplicities of Lemma 7.4.1 is satisfied, i.e. for every edge $uv \in M^6$, $\mu_{G^6}(u,v) \geq 2 = l-1$. Therefore the base step is settled. Again, by Lemma 7.4.1, the inductive step follows. Then Theorem 7.0.6 is proved. □

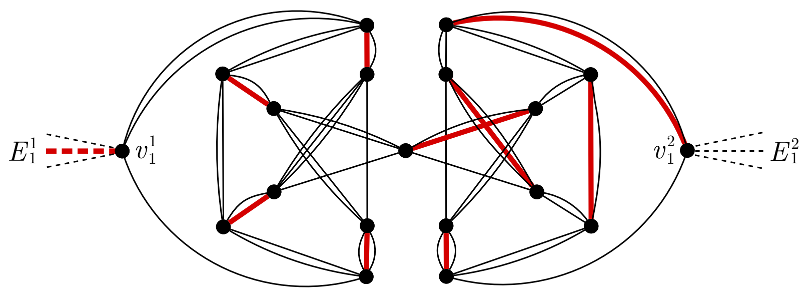


Figure 7.14: The graph Q_1 (solid lines) and the edge sets E_1^1, E_1^2 (dashed lines). The bold red edges are used to construct M^6 in the proof of Theorem 7.0.6.

7.4.2 Concluding remarks

By asking for lower bounds on the parameter $m(t, r)$, one can prove the existence of sets of perfect matchings having specific intersection properties in regular graphs. For example, it can be proved that for $l \geq 5$, if $m(2l, 3l) \geq 2l - 1$, then every bridgeless cubic graph admits a perfect matching cover of cardinality $2l - 1$. As another example, it can be proved that, for $l \geq 3$, if $m(2l, 3l) \geq l$, then every bridgeless cubic graph has l perfect matchings with empty intersection. Both these proofs rely on the properties of the Petersen graph described in Lemma 7.1.4.

We though believe that these lower bounds are quite strong conditions. We believe the following statement to be true.

Conjecture 7.4.3. *For all $l \geq 2$ and $r \geq 2l$, $m(2l, r) \leq l - 1$.*

Note that when $l = 2$, Conjecture 7.4.3 is true by Rizzi [75].

Chapter 8

Complete sets

Chapter 8 is based on [55]; all results in this chapter appeared in that preprint.

As a unifying approach to study some hard conjectures on cubic graphs, Jaeger (see [39]) introduced colorings with edges of another graph. Recall that for two graphs G and H an H*-coloring* of G is a mapping $f\colon E(G) \to E(H)$ such that

- if $e_1, e_2 \in E(G)$ are adjacent, then $f(e_1) \neq f(e_2)$,
- for every $v \in V(G)$ there exists a vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$.

If such a mapping exists, then we write $H \prec G$ and say H *colors* G. A set $\mathcal{A}$ of connected r-graphs such that for every connected r-graph G there is an $H \in \mathcal{A}$ with $H \prec G$ is said to be r*-complete*.

For $r = 3$, Jaeger [37] conjectured that the Petersen graph colors every bridgeless cubic graph (Conjecture 1.1.3), i.e. he conjectured that $\{\mathcal{P}\}$ is a 3-complete set. If true, this conjecture would have far reaching consequences. For instance, it would imply that the Berge-Fulkerson Conjecture (Conjecture 1.1.1) and the 5-Cycle Double Cover Conjecture (Conjecture 7.3.12) are also true. The Petersen Coloring Conjecture is a starting point for research in several directions. Different aspects of it are studied and partial results are proved, see for instance [20, 30, 38, 41, 67, 73, 76].

In this chapter we are mainly motivated by the generalized Berge-Fulkerson Conjecture (Conjecture 1.1.5), which was proposed by Seymour [77] and states that every r-graph has $2r$ perfect matchings such that every edge is in precisely two of them. Analogously to the cubic case, for every $r \geq 3$ if all elements of an r-complete set would satisfy the generalized Berge-Fulkerson Conjecture, then every r-graph would satisfy it. Mazzuoccolo et al. [68] asked whether there exists a connected r-graph H such that $H \prec G$ for every (simple) r-graph G, for all $r \geq 3$. In other words, they asked whether there is an r-complete set of cardinality 1 for every $r \geq 3$.

By definition, any r-graph G of class 1 can be colored with any r-graph H. Indeed, let $N_1, \dots, N_r$ be r pairwise disjoint perfect matchings of G and v a vertex of H with $\partial_H(v) = \{e_1, \dots, e_r\}$. Every edge of N_i of G can be mapped to e_i in H. Hence, the aforementioned questions and conjectures reduce to r-graphs of class 2.

For every $r \geq 3$, let $\mathcal{H}_r$ be an inclusion-wise minimal r-complete set. The following theorem is the main result of this chapter and gives a negative answer to the question of Mazzuoccolo et al. when $r \geq 4$.

Theorem 8.0.1. *Either $\mathcal{H}_3 = \{\mathcal{P}\}$ or $\mathcal{H}_3$ is an infinite set. Moreover, if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set.*

This chapter is organized as follows. In Section 8.1 we characterize $\mathcal{H}_r$ for every $r \geq 3$. The following statement is the main result of that section, which implies that $\mathcal{H}_r$ is unique.

Theorem 8.0.2. *Let $r \geq 3$ and let G be a connected r-graph. The following statements are equivalent.*

1) $G \in \mathcal{H}_r$.

2) The only connected r-graph coloring G is G itself.

3) G cannot be colored by a smaller r-graph.

In Section 8.2 we are going to prove Theorem 8.0.1 by showing that if $\mathcal{H}_r$ has more than one element, then it has infinitely many elements. For $r \geq 1$, let $\mathcal{S}_r$ be the set of the smallest r-graphs of class 2. For example, the only element of $\mathcal{S}_3$ is the Petersen graph. As a partial result in Section 8.2 we determine the set $\mathcal{S}_r$ of the smallest r-graphs of class 2 for each $r \geq 3$, which we think is of interest of its own. We show that $\mathcal{S}_r \subseteq \mathcal{H}_r$.

In Section 8.3 we prove similar results for simple r-graphs. We conclude Chapter 8 with Section 8.4, where we state some open problems.

In this chapter the following observation will frequently be used without reference.

Observation 8.0.3. *Let $r \geq 3$, let G be an r-graph and let $X \subseteq V(G)$. If $|X|$ is even, then $|\partial_G(X)|$ is even. If $|X|$ is odd, then $|\partial_G(X)|$ has the same parity as r.*

8.0.1 Order structure

Jaeger [37] initiated the study of the Petersen Coloring Conjecture in terms of partial ordered sets. DeVos, Nešetřil and Raspaud [20] studied cycle-continuous mappings and asked whether there is an infinite set $\mathcal{G}$ of bridgeless graphs such that every two of them are cycle-continuous incomparable, i.e. there is no cycle-continuous map between any two graphs in $\mathcal{G}$. Šámal [76] gave an affirmative answer to the above question by constructing such an infinite set $\mathcal{G}$ of bridgeless cubic graphs. In fact, he also mentioned that this result can be considered in view of a quasi-order induced by cycle-continuous mappings on the set of bridgeless cubic graphs. That is, this quasi-ordered set contains infinite antichains.

For every integer $r \geq 1$, H-colorings give a quasi-order on the set of r-graphs, which is denoted by $(\mathcal{G}_r, \prec)$. Thus, Theorem 8.0.1 can be restated as follows.

Theorem 8.0.1'. *For $r = 3$, either $\mathcal{H}_3 = \{\mathcal{P}\}$ or $\mathcal{H}_3$ is an infinite antichain* *in* $(\mathcal{G}_3, \prec)$*. For each* $r \geq 4$*,* $\mathcal{H}_r$ *is an infinite antichain in* $(\mathcal{G}_r, \prec)$*.*

8.1 Characterization of $\mathcal{H}_r$

In this section we will characterize $\mathcal{H}_r$. We start with some preliminary technical results. In particular, we introduce a lifting operation for r-graphs.

8.1.1 Substructures and lifting

Some of the following observations appeared also in [68].

Observation 8.1.1. *Let* H *and* G *be graphs and let* f *be an* H*-coloring of* G*.*

(i) $\chi'(G) \leq \chi'(H)$*.*

(ii) If $N_1, \ldots, N_k$ *are* k *pairwise disjoint perfect matchings in* H*, then* $f^{-1}(N_1), \ldots, f^{-1}(N_k)$ *are* k *pairwise disjoint perfect matchings in* G*.*

(iii) If C *is a* 2*-regular subgraph of* H*, then* $f^{-1}(E(C))$ *induces a* 2*-regular subgraph in* G*.*

(iv) If H' *is a* $\{K_{1,1}, C_m \colon m \geq 2\}$*-factor in* H*, then* $f^{-1}(E(H'))$ *induces a* $\{K_{1,1}, C_m \colon m \geq 2\}$*-factor in* G*.*

Proof. Let H' be a subgraph of H and G' be the subgraph of G induced by $f^{-1}(E(H'))$. By the definition of H-coloring, if H' is k-regular (spanning, respectively) then G' is k-regular (spanning, respectively). Then statements $(i), (ii)$ and (iii) can be obtained immediately. In order to show statement (iv), assume that H' is a $\{K_{1,1}, C_m \colon m \geq 2\}$-factor. We decompose H' into a 1-regular subgraph H_1 and a 2-regular subgraph H_2. The sets $f^{-1}(E(H_1))$ and $f^{-1}(E(H_2))$ induce a 1-regular subgraph G_1 and a 2-regular subgraph G_2 of G, respectively. By the definition of H-coloring, G_1 and G_2 are disjoint. This completes the proof. □

We recall the following definition. Let G be a graph and let $x \in V(G)$ with $|N_G(x)| \geq 2$. A *lifting* (of G) at x is the following operation: Choose two distinct neighbors y and z of x, delete an edge e_1 connecting x with y, delete an edge e_2 connecting x with z and add a new edge e connecting y with z; additionally, if e_1 and e_2 were the only two edges incident with x, then delete the vertex x in the new graph. We say e_1 and e_2 are *lifted to* e. Moreover, the new graph is denoted by $G(e_1, e_2)$.

We will make use of the following fact. Let G be a graph, then $|\partial_G(X \cap Y)| + |\partial_G(X \cup Y)| \leq |\partial_G(X)| + |\partial_G(Y)|$ for every $X, Y \subseteq V(G)$.

Lemma 8.1.2. *Let $r \geq 2$ be an integer and let G be a connected graph of order at least 2 with a vertex $x \in V(G)$ such that*

- *$d_G(v) = r$ for all $v \in V(G) \setminus \{x\}$, and*
- *if $|V(G)|$ is even, then $d_G(x) \neq r$, and*
- *$|\partial_G(S)| \geq r$ for every $S \subseteq V(G) \setminus \{x\}$ of odd cardinality.*

Then, for every labeling $\partial_G(x) = \{e_1, \dots, e_{d_G(x)}\}$ there exists an $i \in \mathbb{Z}_{d_G(x)}$ such that $G(e_i, e_{i+1})$ is a connected graph with $|\partial_{G(e_i, e_{i+1})}(S')| \geq r$ for every $S' \subseteq V(G(e_i, e_{i+1})) \setminus \{x\}$ of odd cardinality.

Proof. We argue by contradiction. Let G be a possible counterexample of smallest order, let $d = d_G(x)$, and let $e_i = xy_i$ for every $i \in \{1, \dots, d\}$.

First we show $|N_G(x)| \geq 2$. Suppose that x has just one neighbor x'. Note that $d_G(x') = r$ by our assumptions. If $|V(G)|$ is even, then $d_G(x) \neq r$. As a consequence, the set $S = V(G) \setminus \{x\}$ is a set of odd cardinality with $|\partial_G(S)| = d_G(x) < r$, a contradiction. If $|V(G)|$ is odd, then the set $S = V(G) \setminus \{x, x'\}$ is a set of odd cardinality with $|\partial_G(S)| = r - d_G(x) < r$, a contradiction again. Therefore, $|N_G(x)| \geq 2$.

Hence, we can choose an $i \in \mathbb{Z}_d$ such that $y_i \neq y_{i+1}$ and, if $G - x$ is not connected, then y_i and y_{i+1} belong to different components of $G - x$. Suppose that G has a bridge e. Then, for parity reasons, the component H of $G-e$ not containing x is of odd order, a contradiction since $|\partial_G(V(H))| = 1 < r$. Thus, G is bridgeless and hence, the graph $G(e_i, e_{i+1})$ is connected by the choice of i. As a consequence, there is a set $T \subseteq V(G(e_i, e_{i+1}))\setminus\{x\}$ of odd cardinality with $|\partial_{G(e_i,e_{i+1})}(T)| < r$, since G is a counterexample. Observe that $|\partial_G(T)|$ has the same parity as r, which implies $|\partial_G(T)| = r$ and $y_i, y_{i+1} \in T$. Set $G_1 = G/T$ and label the edges of $\partial_{G_1}(x)$ with the same labels as in G. Then, G_1 and x satisfy the conditions of the statement. Therefore, by the minimality of $|V(G)|$, there is an integer $j \in \mathbb{Z}_d$ such that the graph $G_2 = G_1(e_j, e_{j+1})$ satisfies $|\partial_{G_2}(S)| \geq r$ for every $S \subseteq V(G_2)\setminus\{x\}$ of odd cardinality. Set $G_3 = G(e_j, e_{j+1})$. The graphs G, G_1, G_2 and G_3 are depicted in Figure 8.1.

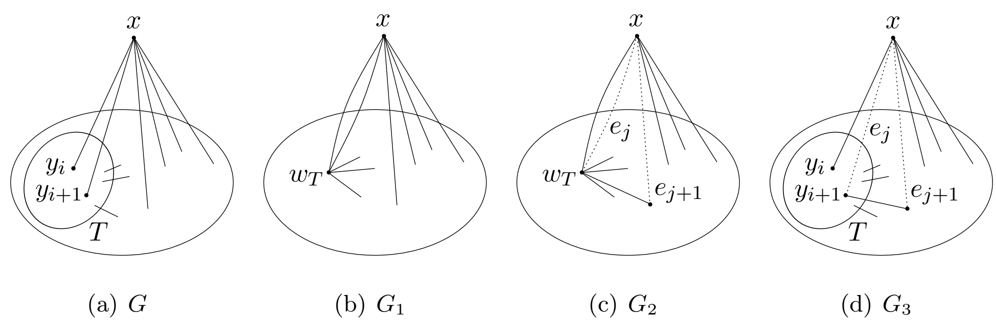


Figure 8.1: An example for the graphs G, G_1, G_2 and G_3.

Note that $V(G) = V(G_3)$ and $V(G_2) \setminus \{w_T\} = V(G_3) \setminus T$. Furthermore, we observe the following:

- for every $X \subseteq T$: $|\partial_G(X)| = |\partial_{G_3}(X)|$,
- for every $X \subseteq V(G_2)\setminus\{w_T\}$: $|\partial_{G_2}(X)| = |\partial_{G_3}(X)|$ and $|\partial_{G_2}(X\cup\{w_T\})| = |\partial_{G_3}(X \cup T)|$.

Now, let $S \subseteq V(G_3) \setminus \{x\}$ be a set of odd cardinality. Set $A = S \cap T$ and $B = S \setminus A$. We consider two cases.

Case 1. $|A|$ is even.

As a consequence, B and $T \setminus A$ are sets of odd cardinality. Therefore, by using the above observations we obtain the following:

$$\begin{aligned}|\partial_{G_3}(S)| = |\partial_{G_3}(S^c)| &\geq |\partial_{G_3}(S^c \cap T)| + |\partial_{G_3}(S^c \cup T)| - |\partial_{G_3}(T)| \\ &= |\partial_{G_3}(T \setminus A)| + |\partial_{G_3}(B)| - |\partial_{G_3}(T)| \\ &= |\partial_{G}(T \setminus A)| + |\partial_{G_2}(B)| - |\partial_{G}(T)| \\ &\geq r + r - r \\ &= r.\end{aligned}$$

Case 2. $|A|$ is odd.

Thus, B is a set of even cardinality, which implies

$$\begin{aligned}|\partial_{G_3}(S)| &\geq |\partial_{G_3}(S \cap T)| + |\partial_{G_3}(S \cup T)| - |\partial_{G_3}(T)| \\ &= |\partial_{G_3}(A)| + |\partial_{G_3}(B \cup T)| - |\partial_{G_3}(T)| \\ &= |\partial_{G}(A)| + |\partial_{G_2}(B \cup \{w_T\})| - |\partial_{G}(T)| \\ &\geq r + r - r \\ &= r.\end{aligned}$$

In any case, we have $|\partial_{G_3}(S)| \geq r$, which implies $|\partial_{G(e_j,e_{j+1})}(S')| \geq r$ for every $S' \subseteq V(G(e_j, e_{j+1})) \setminus \{x\}$ of odd cardinality. This is a contradiction to the assumption that G is a counterexample. □

The previous lemma can be used in r-graphs as follows.

Theorem 8.1.3. *Let $r \geq 2$ be an integer, let G be a connected r-graph and let X be a non-empty proper subset of $V(G)$. If $|X|$ is even, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}|\partial_G(X)|$ lifting operations at w_X. If $|X|$ is odd, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}(|\partial_G(X)| - r)$ lifting operations at w_X.*

Proof. Consider any labeling of $\partial_{G/X}(w_X)$. The statement follows by applying repeatedly Lemma 8.1.2 to G/X at w_X. Note that w_X is removed in the last step when $|X|$ is even. □

Note that the previous lifting operations can be applied such that they preserve embeddings of graphs in surfaces.

8.1.2 The set $\mathcal{H}_r$

Let f be an H-coloring of G. The subgraph of H induced by the edge set $Im(f)$ is denoted by H_f. Observe that H_f also colors G. Furthermore, if H has no two vertices u_1, u_2 with $\partial_H(u_1) = \partial_H(u_2)$, then f induces a mapping $f_V \colon V(G) \to V(H)$, where every $v \in V(G)$ is mapped to the unique vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$. Note that f_V is well defined if H is a connected graph with $|V(H)| > 2$. A vertex of $V(H) \setminus Im(f_V)$ is called *unused*.

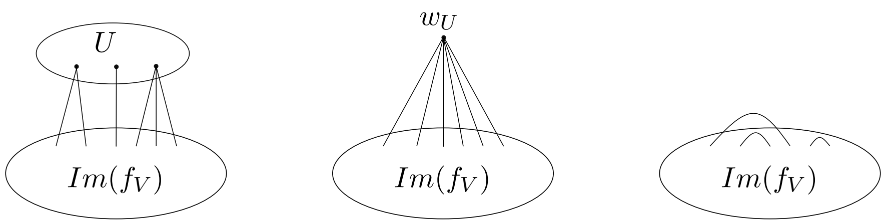
Theorem 8.1.4. *Let $r \geq 3$ and let G be an r-graph of class 2 that cannot be colored by an r-graph of smaller order. If H is a connected r-graph and f is an H-coloring of G, then (f_V, f) is an isomorphism, i.e. $H \cong G$.*

Proof. Let $f \colon E(G) \to E(H)$ be an H-coloring of G. Note, that since G is class 2, H is also class 2 and therefore, f_V is well defined. We first prove three claims.

Claim 1. f is injective.

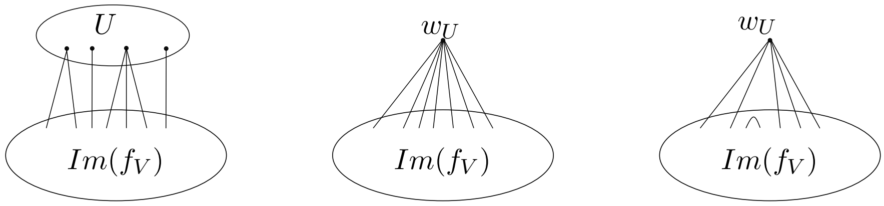
Proof of Claim 1. Suppose to the contrary that f is not injective, which implies $|E(H_f)| < |E(G)|$. If H contains no unused vertices, then $|E(H)| = |E(H_f)| < |E(G)|$, which contradicts the assumption that G cannot be colored by an r-graph of smaller order. Thus, H contains unused vertices; let $U \subseteq V(H)$ be the set of them. Transform the graph H/U into a new r-graph H' as follows. If $|U|$ is even, then apply $\frac{1}{2}\,|\partial_H(U)|$ lifting operations at w_U (see Figure 8.2). If $|U|$ is odd, then apply $\frac{1}{2}\,(|\partial_H(U)| - r)$ lifting operations at w_U (see Figure 8.3). By Theorem 8.1.3, this can be done in such a way that the resulting graph H' is indeed an r-graph.

Note that every edge of $Im(f)$ is incident with at most one vertex of U. Thus, we can define a function $f' \colon E(G) \to E(H/U)$ as follows. For every $e \in E(G)$ let $f'(e)$ be the edge of H/U corresponding to the edge $f(e)$ of H.



(a) H (b) H/U (c) H'

Figure 8.2: An example for the graphs H, H/U and H' when $|U|$ is even.



(a) H (b) H/U (c) H'

Figure 8.3: An example for the graphs H, H/U and H' when $|U|$ is odd.

Observe that f' is an H/U-coloring of G, where w_U is the only unused vertex. Next, define a new mapping $f'' \colon E(G) \to E(H')$ as follows. For every $e \in E(G)$ set

$$f''(e) = \begin{cases} e' & \text{if } f'(e) \text{ is one of the two edges lifted to } e', \\ f'(e) & \text{if } f'(e) \in E(H'). \end{cases}$$

By construction, $f''(\partial_G(v)) = \partial_{H'}(f_V(v))$ for every $v \in V(G)$. Since G and H' are r-regular it follows that f'' is an H'-coloring. Therefore, $H' \prec G$ and hence $|V(H')| \geq |V(G)|$ by our assumptions. This is a contradiction, since

$$|E(H')| \leq |E(H/U)| = |E(H_f)| < |E(G)|.$$

■

Claim 2. f_V is surjective.

Proof of Claim 2. Suppose that H contains unused vertices. Then, there are $v_1, v_2 \in V(G)$ and $e \in E_G(v_1, v_2)$ such that $f(e)$ is incident with exactly one unused vertex in H, since H is connected. Thus, $f(\partial_G(v_1)) = f(\partial_G(v_2))$, which contradicts Claim 1. ■

Claim 3. $|V(H)| = |V(G)|$.

Proof of Claim 3. Since G cannot be colored by an r-graph of smaller order, we have $|V(H)| \geq |V(G)|$. On the other hand, $|V(H)| \leq |V(G)|$ by Claim 2. ■

Claims 1, 2 and 3 imply that f and f_V are bijections. Furthermore, we obtain that $e \in E_G(v_1, v_2)$ if and only if $f(e) \in E_H(f_V(v_1), f_V(v_2))$. Therefore, (f_V, f) is an isomorphism between G and H, i.e. $H \cong G$. □

In [70], Mkrtchyan proved that if a connected 3-graph H colors the Petersen graph $\mathcal{P}$, then $H \cong \mathcal{P}$. The following result is implied by Theorem 8.1.4 together with Observation 8.1.1 (*ii*) and gives a generalization of Mkrtchyan's result in the r-regular case. For every r-graph G let $\pi(G)$ be the largest integer t such that G has t pairwise disjoint perfect matchings.

Corollary 8.1.5. *Let $r \geq 3$ and let G be an r-graph of class 2 such that $\pi(G') > \pi(G)$ for every r-graph G' with $|V(G')| < |V(G)|$. If H is a connected r-graph with $H \prec G$, then $H \cong G$.*

Now we can prove Theorem 8.0.2.

Theorem 8.0.2. *Let $r \geq 3$ and let G be a connected r-graph. The following statements are equivalent.*

1) $G \in \mathcal{H}_r$.

2) The only connected r-graph coloring G is G itself.

3) G cannot be colored by a smaller r-graph.

Proof. 2) $\Longrightarrow$ 1) follows trivially.

1) $\Longrightarrow$ 3). Assume by contradiction that 3) is not true. Then, let H be a smallest r-graph smaller than G such that $H \prec G$. Note that H cannot

be colored by a smaller r-graph because otherwise, since the relation $\prec$ is transitive, G would be colored by an r-graph smaller than H. Hence, $H \in \mathcal{H}_r$ by Theorem 8.1.4. Thus, $\mathcal{H}_r \setminus \{G\}$ is an r-complete set, in contradiction to the inclusion-wise minimality of $\mathcal{H}_r$.

3) $\implies$ 2) follows by Theorem 8.1.4. □

Corollary 8.1.6. *For every $r \geq 3$, there exists only one inclusion-wise minimal r-complete set, i.e. $\mathcal{H}_r$ is unique.*

8.2 Elements of $\mathcal{H}_r$

Let $r \geq 3$ and $k \in \{1, \dots, r\}$ be integers. Let $\mathcal{G}(r,k) = \{G\colon G$ is an r-graph with $\pi(G) = k\}$. Note that $\mathcal{G}(r, r-1) = \emptyset$, since every r-graph with $r-1$ pairwise disjoint perfect matchings is a class 1 graph and thus, it has r pairwise disjoint perfect matchings. If $k \leq r-2$, then the elements of $\mathcal{G}(r,k)$ are class 2 graphs and $\mathcal{G}(r,i) \cap \mathcal{G}(r,j) = \emptyset$, if $1 \leq i \neq j \leq r-2$. We are interested in the subset of $\mathcal{G}(r,k)$ consisting of all such graphs with the smallest order. This set is denoted by $\mathcal{S}(r,k)$. By definition, $\mathcal{S}_r \subseteq \bigcup_{i=1}^{r-2} \mathcal{S}(r,i)$.

By Corollary 8.1.5, $\mathcal{H}_r$ contains the smallest r-graphs of class 2 and the smallest poorly matchable r-graphs, i.e. $\mathcal{S}_r \cup \mathcal{S}(r,1) \subseteq \mathcal{H}_r$. Note that for $r = 3$, we have $\mathcal{S}_r = \mathcal{S}(r,1) = \{\mathcal{P}\}$. The Petersen Coloring Conjecture states that $\mathcal{H}_3 = \{\mathcal{P}\}$. This situation is very exclusive as we show in this section. We first determine the elements of $\mathcal{S}_r$, and show that H_r has more than one element for $r \geq 4$. Then, we show that if $\mathcal{H}_r$ has more than one element, then it has infinitely many elements, which proves Theorem 8.0.1.

In order to prove the results above, we heavily rely on the properties of the Petersen graph $\mathcal{P}$ and its perfect matchings. We will use the labels and notations defined in Section 7.1; for definitions concerning the Petersen graph that are not given here, the reader is referred to Section 7.1.

8.2.1 Smallest r-graphs of class 2

Recall that for every multiset $\mathcal{M}$ of perfect matchings of $\mathcal{P}$, the graph $\mathcal{P}^{\mathcal{M}}$ is class 2 (Lemma 7.1.3). The following theorem extends Lemma 7.1.3 and characterizes the perfect matchings M on $V(\mathcal{P})$ such that $\mathcal{P}+M$ is a class 2 graph.

Theorem 8.2.1. *Let $\mathcal{P}$ be the Petersen graph and H be a 1-regular graph on $V(\mathcal{P})$ with edge-set M. Then $\mathcal{P}+M$ is class 2 if and only if $M \subseteq E(\mathcal{P})$.*

Proof. Lemma 7.1.3 has shown that $M \subseteq E(\mathcal{P})$ is a sufficient condition for $\mathcal{P}+M$ to be class 2. We establish its necessity by way of contradiction. Suppose that there exists an edge $e \in M$, that is not parallel to an edge of $E(\mathcal{P})$. Let $H_1 = \mathcal{P}+M$. Since any two vertices of the Petersen graph are in a 5-circuit, the subgraph $\mathcal{P}$ of H_1 can be decomposed into two 5-circuits, C_5^1 and C_5^2, and a 1-factor H' such that both ends of e belongs to $V(C_5^1)$. Without loss of generality, we assume $C_5^1 = u_1u_2u_3u_4u_5u_1$ with $e = u_2u_5$, as shown in Figure 8.4. Let $H_2 = H_1 - E(H') = \mathcal{P}+M-E(H')$. Note that H_2 is 3-regular

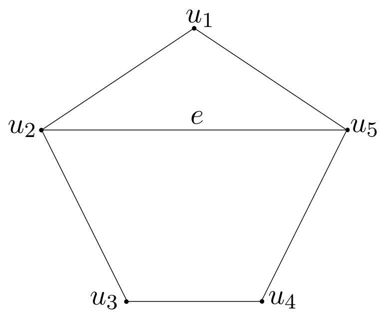


Figure 8.4: The 5-circuit C_5^1 with the edge e.

and contains C_5^1 and C_5^2. If $|\partial_{H_2}(V(C_5^1))| \neq 1$, then H_2 is 2-edge-connected. This implies that H_2 is class 1 since it is not isomorphic to $\mathcal{P}$, as it contains a 4-circuit $u_2u_3u_4u_5u_2$. So, $H_1 = H_2 + E(H')$ is also class 1, a contradiction. Therefore, we may assume $|\partial_{H_2}(V(C_5^1))| = 1$ and set $\partial_{H_2}(V(C_5^1)) = \{e'\}$. The remaining proof is split into two cases. First, if e' is incident with u_1, then M contains an edge incident with u_3 and u_4. Thus, $H_3 = H_1 - N_1$

contains a 3-circuit $u_1u_2u_5u_1$, a 2-circuit $u_3u_4u_3$ and a 5-circuit C_5^2, where $N_1 = (M \setminus \{u_2u_5, u_3u_4\}) \cup \{u_2u_3, u_4u_5\}$. Moreover, there are five edges between $V(C_5^1)$ and $V(C_5^2)$ in H_3, which implies that H_3 is 2-edge-connected. Thus, H_3 is class 1 and so is H_1, a contradiction. Second, if e' is incident with u_3 or u_4, then, without loss of generality, we assume that e' is incident with u_3, and so M contains the edge u_1u_4. Let $N_2 = (M \setminus \{u_1u_4, u_2u_5\}) \cup \{u_1u_2, u_4u_5\}$ and let $H_4 = H_1 - N_2$. There are two adjacent vertices v_1 and v_4 in $\mathcal{P}$ such that $v_i \in N_{\mathcal{P}}(u_i) \setminus V(C_5^1)$ for each $i \in \{1, 4\}$. Then H_4 contains a 4-circuit $u_1u_4v_4v_1u_1$. Moreover, H_4 is 2-edge-connected since there are five edges between $V(C_5^1)$ and $V(C_5^2)$. This implies that H_4 is class 1 and therefore, H_1 is also class 1, a contradiction. □

Theorem 8.2.2. *For all $r \geq 3$, $\mathcal{S}_r = \mathcal{S}(r, r-2) = \{\mathcal{P}^{\mathcal{M}} \colon \mathcal{M}$ is a multiset of $r-3$ perfect matchings of the Petersen graph $\mathcal{P}\}$.*

Proof. For an r-graph G and an odd set $X \subseteq V(G)$, we say the edge-cut $\partial_G(X)$ is tight if it consists of exactly r edges; and it is trivial if $|X| = 1$ or $|X^c| = 1$. We will deduce the statement from the following three claims.

Claim 1. Let $r \geq 3$. If G is a smallest r-graph of class 2, then G has no non-trivial tight edge-cut.

Proof of Claim 1. Suppose that there is an odd set $X \subseteq V(G)$ such that $|\partial_G(X)| = r$ and neither X nor X^c consists of a single vertex. By the minimality of $|V(G)|$, the r-graphs G/X and G/X^c are class 1. As a consequence, G is also class 1, a contradiction. ■

Claim 2. Let $r \geq 3$. If G is a smallest r-graph of class 2, then $|V(G)| = 10$ and G has $r-2$ pairwise disjoint perfect matchings.

Proof of Claim 2. We prove the claim by induction on r. When $r = 3$, the statement follows from the fact that the smallest 3-graph of class 2 is the Petersen graph. Hence, let $r \geq 4$ and assume the statement is true for every $r' < r$.

Let G be a smallest r-graph of class 2. By Lemma 7.1.3, $|V(G)| \leq 10$. Note that every r-graph has a perfect matching [77]. Thus, let M be a perfect matching of G.

If $H = G - M$ is an $(r-1)$-graph, then H is also class 2, since otherwise G would be class 1. Furthermore, we have $|V(G)| = |V(H)| \geq 10$ in this case, which implies $|V(G)| = |V(H)| = 10$. Thus, the statement follows by induction .

Therefore, we may assume that $H = G - M$ is not an $(r-1)$-graph. By the definition and Observation 8.0.3, there is an odd set $X \subseteq V(G)$ such that $|\partial_G(X) \setminus M| \leq r - 3$. Moreover, we have $|\partial_G(X)| \geq r + 2$ by Claim 1. Hence, $|\partial_G(X) \cap M| = |\partial_G(X)| - |\partial_G(X) \setminus M| \geq 5$. Since M is a perfect matching, we conclude that $|V(G)| = 10$. As a consequence, M has cardinality 5 and thus, $|\partial_G(X) \cap M| = 5$ and $|\partial_G(X)| = r + 2$. Let x_1y_1 and x_2y_2 be two different edges of $\partial_G(X) \cap M$, where $x_1, x_2 \in X$. The graph $G' = G - \{x_1y_1, x_2y_2\} + \{x_1x_2, y_1y_2\}$ is still an r-graph. Indeed, for any odd set $Y \subseteq V(G')$ we have $|\partial_{G'}(Y)| \geq |\partial_G(Y)| - 2 \geq r$. Moreover, $|\partial_{G'}(X)| = r$ and hence, G' is class 1 by Claim 1. Let $\mathcal{N}$ be a set of r pairwise disjoint perfect matchings of G' and let N_x and N_y be the perfect matchings containing x_1x_2 and y_1y_2 respectively (note that $N_x \neq N_y$ since otherwise G itself would be class 1). Then $\mathcal{N} \setminus \{N_x, N_y\}$ is a set of $r-2$ pairwise disjoint perfect matchings of G. ■

Claim 3. Let $r \geq 3$. If G is a smallest r-graph of class 2, then there is a set $\mathcal{M}$ of $r-3$ pairwise disjoint perfect matchings of G such that $G - \bigcup_{M \in \mathcal{M}} M \cong \mathcal{P}$.

Proof of Claim 3. We prove the claim by induction on r. When $r = 3$, the statement is trivial since the smallest 3-graph of class 2 is the Petersen graph. Hence, let $r \geq 4$ and assume the statement is true for every $r' < r$.

Let G be a smallest r-graph of class 2. By Claim 2, G is of order 10 and has a set $\mathcal{N}$ of $r-2$ pairwise disjoint perfect matchings. Let $M \in \mathcal{N}$. Then $G - M$ is class 2, since otherwise G would be class 1. If $G - M$ is an $(r-1)$-

graph, then the statement follows by induction. Hence, there exists an odd set $X \subseteq V(G-M)$ with $|\partial_{G-M}(X)| \leq r-3$. Furthermore, $V(G-M) = V(G)$ and $|\partial_G(X) \setminus M| = |\partial_{G-M}(X)|$. By Claim 1 and Claim 2, we have $|\partial_G(X)| \geq r+2$ and $|M| = 5$. As a consequence, we obtain $|\partial_G(X)| = r+2$ and $|\partial_G(X) \cap M| = 5$, which implies $|X| = 5$. Set $H = G - \cup_{N \in \mathcal{N}} N$ and note that H is a 2-factor of G, which contains at least two odd circuits, since otherwise G would be class 1. Every perfect matching of $\mathcal{N}$ contains at least one edge of $\partial_G(X)$ and hence, $|\partial_H(X)| = 0$. Thus, both $H[X]$ and $H[X^c]$ either consists of a 5-circuit or a 3-circuit and a 2-circuit. We consider the following two cases.

Case 1. $H + M$ is a 3-graph.

In this case $H + M \cong \mathcal{P}$, since otherwise $H + M$ is class 1 which would imply that G is also class 1.

Case 2. $H + M$ is not a 3-graph.

Thus, $H + M$ has a bridge, which implies that both $H[X]$ and $H[X^c]$ consists of a 3-circuit and a 2-circuit and $|\partial_{H+M}(V(C) \cup V(C'))| = 1$, where C is the 3-circuit of $H[X]$ and C' is the 2-circuit of $H[X^c]$. As a consequence, there is only one possibility for the structure of $G + M$, which is depicted in Figure 8.5. With respect to the vertex labels in Figure 8.5, set $M' = (M \setminus \{z_1z_4, z_2z_3\}) \cup \{z_1z_2, z_3z_4\}$ and $\mathcal{N}' = (\mathcal{N} \setminus \{M\}) \cup \{M'\}$. Then, $\mathcal{N}'$ is a set of $r-2$ pairwise disjoint perfect matchings of G. Now, consider $\mathcal{N}'$ and M' instead of $\mathcal{N}$ and M, respectively, and repeat the same arguments as above. We deduce that $G - M'$ is an $(r-1)$-graph and the statement follows by induction. ■

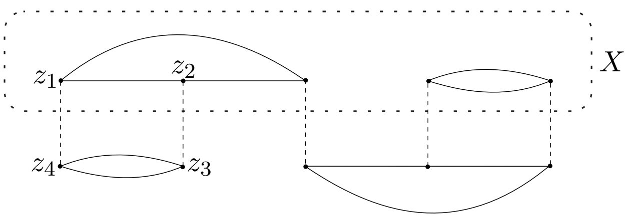


Figure 8.5: The graph $H + M$ in Case 2 of the proof of Claim 3 (Theorem 8.2.2). The dashed edges belong to M.

By Claim 2, we have $\mathcal{S}_r = \mathcal{S}(r, r-2)$. Moreover, by Theorem 8.2.1 and Claim 2, for any multiset $\mathcal{M}$ of $r-3$ perfect matchings of $\mathcal{P}$, the graph $\mathcal{P}^{\mathcal{M}}$ is in $\mathcal{S}_r$. It remains to show that, if $G \in \mathcal{S}_r$, then $G \cong \mathcal{P}^{\mathcal{M}}$ for a suitable multiset $\mathcal{M}$. By Claim 3, there is a set $\mathcal{N}$ of $r-3$ pairwise disjoint perfect matchings of G such that the graph $H = G - \bigcup_{N\in\mathcal{N}} N$ is isomorphic to the Petersen graph. For every $N \in \mathcal{N}$, the graph $H + N$ is class 2, since otherwise G is class 1. Therefore, $G \cong \mathcal{P}^{\mathcal{N}}$ by Theorem 8.2.1. □

8.2.2 Lower bounds for $|\mathcal{S}_r|$

The following lemma is a direct consequence of the fact that the Petersen graph is 3-arc-transitive, see e.g. Corollary 1.8 in [8]. That is, for any two paths of order 4 of $\mathcal{P}$ there is an automorphism of $\mathcal{P}$ which maps one to the other. Recall that the six perfect matchings of the Petersen graph are denoted by $M_1, \dots, M_6$ (see Section 7.1).

Lemma 8.2.3. *Let* $N_1, N_2, N_3 \in \{M_1, \dots, M_6\}$ *and* $g\colon \{N_1, N_2, N_3\} \to \{M_1, \dots, M_6\}$ *be an injective function. There is an automorphism* (θ, ϕ) *of* $\mathcal{P}$ *such that, for all* $i \in \{1, 2, 3\}$*,* $\phi(N_i) = g(N_i)$*.*

Proof. Let N_1, N_2 and N_3 be pairwise different perfect matchings of $\mathcal{P}$. If we prove the statement in this case then the proof is complete.

Note that the unique edge x_1x_2 in $N_1 \cap N_2$ and the unique edge x_3x_4 in $N_1 \cap N_3$ are at distance one, i.e. the subgraph $\mathcal{P}[\{x_1, x_2, x_3, x_4\}]$ is a path T on four vertices. Up to changing names to such vertices, we may assume that $T = x_1x_2x_3x_4$. The same holds for the unique edge y_1y_2 in $g(N_1) \cap g(N_2)$ and the unique edge y_3y_4 in $g(N_1) \cap g(N_3)$. Without loss of generality, we can assume again that $y_1y_2y_3y_4$ is a path on four vertices.

Since $\mathcal{P}$ is 3-arc-transitive there is an automorphism (θ, ϕ) of $\mathcal{P}$ such that, for all $i \in \{1, \dots, 4\}$, $\theta(x_i) = y_i$. Since (θ, ϕ) is an automorphism, $\phi(N_1)$ must be a perfect matching. Moreover, since the only perfect matching of $\mathcal{P}$ containing both y_1y_2 and y_3y_4 is $g(N_1)$ we get $\phi(N_1) = g(N_1)$.

Similarly, $\phi(N_2)$ and $\phi(N_3)$ are perfect matchings of $\mathcal{P}$ different from $\phi(N_1)$, such that $y_1y_2 \in \phi(N_2)$ and $y_3y_4 \in \phi(N_3)$. Then, the only possibility is that $\phi(N_2) = g(N_2)$ and $\phi(N_3) = g(N_3)$. □

We now consider partitions of integers, which are ways of writing an integer as a sum of positive integers, see e.g. [62]. We are interested in partitions of an integer into a fixed number of parts. We allow 0 to be a part of a partition. A *partition* of an integer n into k parts is a multiset of k integers $n_1, \dots, n_k$ with $n_i \geq 0$ for $i \in \{1, \dots, k\}$ such that $n = \sum_{i=1}^{k} n_i$. Two partitions of n are equal if they yield the same multiset, i.e. if they differ only in the order of their elements. For two positive integers $k \leq n$, let $p'(n, k)$ be the number of partitions of n into k parts. Set $p'(0, k) = 1$.

Theorem 8.2.4. *If* $3 \leq r \leq 8$, *then* $|\mathcal{S}_r| = p'(r-3, 6)$, *and if* $r \geq 9$, *then* $|\mathcal{S}_r| > p'(r-3, 6)$.

Proof. By Theorem 8.2.2, any graph $G \in \mathcal{S}_r$ can be expressed as $G = \mathcal{P} + \sum_{i=1}^{6} n_i M_i$. In this case, $n_1, \dots, n_6$ is a partition of $r-3$ into six parts. We say that G *induces* this partition of $r-3$.

Claim 1. Let $r \geq 3$ be an integer and $G, G' \in \mathcal{S}_r$. If $G \cong G'$, then G and G' induce the same partition of $r-3$.

Proof of Claim 1. We can assume that $G = \mathcal{P} + \sum_{j=1}^{6} n_j M_j$ and $G' = \mathcal{P} + \sum_{j=1}^{6} n'_j M_j$. For the subgraph $\mathcal{P}$ of G and G', we label an edge e of $\mathcal{P}$ by the set $\{p, q\}$ if $M_p \cap M_q = \{e\}$, $p \neq q$. Then all possible labels are used and no two edges receive the same label in $\mathcal{P}$.

Since $G \cong G'$, there is an isomorphism between G and G' which maps the labeled edge $\{p, q\}$ of G to a labeled edge $\{i_p, i_q\}$ of G' for each $\{p, q\} \subseteq \{1, \dots, 6\}$. Furthermore, $n_p + n_q = n'_{i_p} + n'_{i_q}$. Thus, $\sum_{j=2}^{6}(n_1 + n_j) = 4n_1 + \sum_{j=1}^{6} n_j = 4n'_{i_1} + \sum_{j=1}^{6} n'_{i_j}$. Since $\sum_{j=1}^{6} n_j = \sum_{j=1}^{6} n'_{i_j} = r-3$, it follows that $n_1 = n'_{i_1}$. With similar arguments, we further obtain that $n_j = n'_{i_j}$ for each $j \in \{1, \dots, 6\}$. ■

Claim 2. If $r \geq 9$, then there are non-isomorphic graphs in $\mathcal{S}_r$ which induce the same partition.

Proof of Claim 2. Let $N_1, \ldots, N_4$ be four pairwise different perfect matchings of $\mathcal{P}$ such that the edge in $N_1 \cap N_2 = \{uv\}$ is adjacent to the edge in $N_3 \cap N_4 = \{uz\}$. There is a fifth perfect matching N_5 of $\mathcal{P}$ such that the unique edge in $N_3 \cap N_5$ is not adjacent to uv.

Let $t \geq 2$ be an integer and consider the $(t+7)$-graphs $G_t^1 = \mathcal{P} + tN_1 + 2N_2 + N_3 + N_4$ and $G_t^2 = \mathcal{P} + tN_1 + 2N_2 + N_3 + N_5$. Note that both G_t^1 and G_t^2 have exactly one pair of vertices connected by $t+3$ edges, i.e. $\mu_{G_t^1}(u,v) = \mu_{G_t^2}(u,v) = t+3$. On one hand, uv is adjacent to uz and $\mu_{G_t^1}(u,z) = 3$. On the other hand, by the choice of N_5, uv is adjacent only to edges xy such that $\mu_{G_t^2}(x,y) \leq 2$. We deduce that $G_t^1 \not\cong G_t^2$. ■

Claim 3. Let $r \leq 8$ and $G, G' \in \mathcal{S}_r$. If G and G' induce the same partition of $r-3$, then $G \cong G'$.

Proof of Claim 3. Assume that $G = \mathcal{P}^{\mathcal{M}} = \mathcal{P} + \sum_{j=1}^{6} n_j M_j$ and $G' = \mathcal{P}^{\mathcal{M}'} = \mathcal{P} + \sum_{j=1}^{6} n'_j M_j$ induce the same partition of $r-3$. Let $\mathcal{M}_0 = \{M_j \colon n_j \neq 0\}$ and $\mathcal{M}'_0 = \{M_j \colon n'_j \neq 0\}$. Then $|\mathcal{M}_0| = |\mathcal{M}'_0|$.

If $|\mathcal{M}_0| \leq 3$, choose a bijection $g \colon \mathcal{M}_0 \to \mathcal{M}'_0$ such that if $g(M_\alpha) = M_\beta$, then $n_\alpha = n'_\beta$. By Lemma 8.2.3, there is an automorphism (θ, ϕ) of $\mathcal{P}$ such that, for each perfect matching $N \in \mathcal{M}_0$, $\phi(N) = g(N)$. It follows that (θ, ϕ') is an isomorphism of $\mathcal{P}^{\mathcal{M}}$ to $\mathcal{P}^{\mathcal{M}'}$, where $\phi'(M_i) = \phi(M_i)$ for each $i \in \{1, \ldots, 6\}$. The only other cases are the following.

- $r - 3 = 4$ with partition $1,1,1,1,0,0$;
- $r - 3 = 5$ with partitions $2,1,1,1,0,0$ or $1,1,1,1,1,0$.

In such cases, we let $\mathcal{M}_1 = \{M_j \colon n_j = 1\}$ and $\mathcal{M}'_1 = \{M_j \colon n'_j = 1\}$. Let $\mathcal{N}_1$ be the set of perfect matchings of $\mathcal{P}$ different from those of $\mathcal{M}_1$ and $\mathcal{N}'_1$ be the set of perfect matchings of $\mathcal{P}$ different from those of $\mathcal{M}'_1$. Then, there is a bijection $g \colon \mathcal{N}_1 \to \mathcal{N}'_1$ such that if $g(M_\alpha) = M_\beta$, then $n_\alpha = n'_\beta$. The proof now, follows as above. Namely, since $|\mathcal{N}_1| = |\mathcal{N}'_1| \leq 3$, by Lemma 8.2.3, there is an automorphism (θ, ϕ) of $\mathcal{P}$ such that, for all $N \in \mathcal{N}_1$, $\phi(N) = g(N)$. Then, (θ, ϕ') is an isomorphism of $\mathcal{P}^{\mathcal{M}}$ to $\mathcal{P}^{\mathcal{M}'}$, where $\phi'(M_i) = \phi(M_i)$ for each $i \in \{1, \ldots, 6\}$. ■

By Claims 1, 2 and 3, the theorem is proved. □

By Theorem 8.2.2 and Corollary 8.1.5 we obtain the following theorem, which implies that $|\mathcal{H}_r| \geq 2$ when $r \geq 4$.

Theorem 8.2.5. *For every $r \geq 3$, $\mathcal{S}(r, r-2) \cup \mathcal{S}(r, 1) \subseteq \mathcal{H}_r$.*

8.2.3 Infinite subsets of $\mathcal{H}_r$

Lemma 8.2.6. *Let $r \geq 3$, let G and H be two connected r-graphs and let f be an H-coloring of G. For any 2-edge-cut $F = \{e_1, e_2\} \subseteq E(G)$, either $|f(F)| = 1$ or $f(F)$ is a 2-edge-cut of H.*

Proof. Let u and v be the endvertices of $f(e_1)$. Suppose by contradiction that $|f(F)| = 2$ but $f(F)$ is not a 2-edge-cut of H. Then, there is a u,v-path T in H avoiding the edges of $f(F)$. Consider the circuit $C = T + f(e_1)$. By Observation 8.1.1 (*iii*), $f^{-1}(E(C))$ is a union of circuits of G. This is a contradiction, since $f^{-1}(E(C))$ contains e_1 but not e_2. □

Let G, H be two graphs, let $f\colon E(G) \to E(H)$, $g\colon V(G) \to V(H)$ and let G' be a subgraph of G. The restriction of f to $E(G')$ is denoted by $f|_{G'}$; the restriction of g to $V(G')$ is denoted by $g|_{G'}$.

Lemma 8.2.7. *Let G and H be two r-graphs, where $r \geq 3$, and let f be an H-coloring of G. Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of $\mathcal{P}$ and let $e_0 \in E(\mathcal{P}^{\mathcal{M}})$. Let G' be an induced subgraph of G isomorphic to $\mathcal{P}^{\mathcal{M}} - e_0$ and H' be the subgraph of H induced by $f(E(G'))$. Then, $(f_V|_{G'}, f|_{G'})$ is an isomorphism between G' and H', i.e. $H' \cong G'$.*

Proof. By the definition of G', we have $|\partial_G(V(G'))| = 2$. Assume that $\partial_G(V(G')) = \{e_1, e_2\}$ and $e_i = w_i z_i$ with $w_i \in V(G')$ for each $i \in \{1, 2\}$.

We first consider the case $f(e_1) = f(e_2)$. Let G^* be the r-graph obtained from G' by adding a new edge e_3 connecting w_1 and w_2. Set $f^*(e) = f(e) = f|_{G'}(e)$ for each $e \in E(G^*) \setminus \{e_3\}$ and $f^*(e_3) = f(e_1) = f(e_2)$. Then f^* is an H-coloring of G^*. Since $G^* \cong \mathcal{P}^{\mathcal{M}}$, we have that (f^*_V, f^*) is an isomorphism between G^* and H by Theorem 8.1.4. Thus $(f_V|_{G'}, f|_{G'})$ is an isomorphism of G' to H' by the definition of f^*.

Now we assume that $f(e_1) \neq f(e_2)$. By Lemma 8.2.6, $\{f(e_1), f(e_2)\}$ is a 2-edge-cut of H. Let X be a subset of $V(H)$ such that $\partial_H(X) = \{f(e_1), f(e_2)\}$. Denote $f(e_i) = x_i y_i$ with $x_i \in X$ for each $i \in \{1, 2\}$. We consider the following two cases.

Case 1. $f_V(V(G')) \subseteq X$ or $f_V(V(G')) \subseteq V(H) \setminus X$.

Without loss of generality, assume that $f_V(V(G')) \subseteq X$. Let G^* be the r-graph obtained from G' by adding a new edge e_3 connecting w_1 and w_2, and H^* be the r-graph obtained from $H[X]$ by adding a new edge e_4 connecting x_1 and x_2. Set $f^*(e) = f(e) = f|_{G'}(e)$ for each $e \in E(G^*) \setminus \{e_3\}$ and $f^*(e_3) = e_4$. Then f^* is an H^*-coloring of G^*. Since $G^* \cong \mathcal{P}^{\mathcal{M}}$, we have that (f^*_V, f^*) is an isomorphism between G^* and H^* by Theorem 8.1.4. Thus $(f_V|_{G'}, f|_{G'})$ is an isomorphism of G' to H' by the definition of f^* and the statement follows.

Case 2. $f_V(V(G')) \cap X \neq \emptyset$ and $f_V(V(G')) \cap (V(H) \setminus X) \neq \emptyset$.

We show that this case does not apply. Let $Z_1 = f_V(V(G')) \cap X$ and $Z_2 = f_V(V(G')) \cap (V(H) \setminus X)$. Observe that $\{f(e_1), f(e_2)\} \subseteq \partial_H(Z_1) \cup \partial_H(Z_2)$. Set $U_1 = X \setminus Z_1$ and $U_2 = (V(H) \setminus X) \setminus Z_2$. Note that U_1 and U_2 might be empty. We construct a new r-graph H_2 from H in two steps. First, if $U_1 = \emptyset$, set $H_1 = H$. Otherwise we can construct an r-graph H_1 starting from H/U_1 by taking suitable lifting operations at w_{U_1} as described in Theorem 8.1.3, namely: if $|U_1|$ is even, then apply $\frac{1}{2}|\partial_H(U_1)|$ lifting operations at w_{U_1}; if $|U_1|$ is odd, then apply $\frac{1}{2}(|\partial_H(U_1)| - r)$ lifting operations at w_{U_1}. Observe that

$U_2 \subset V(H_1)$. Next, if $U_2 = \emptyset$, set $H_2 = H_1$. Otherwise let H_2 be a graph obtained from H_1/U_2 by taking similar lifting operations as described above at the vertex w_{U_2}. An example for the construction of H_2 is given in Figure 8.6.

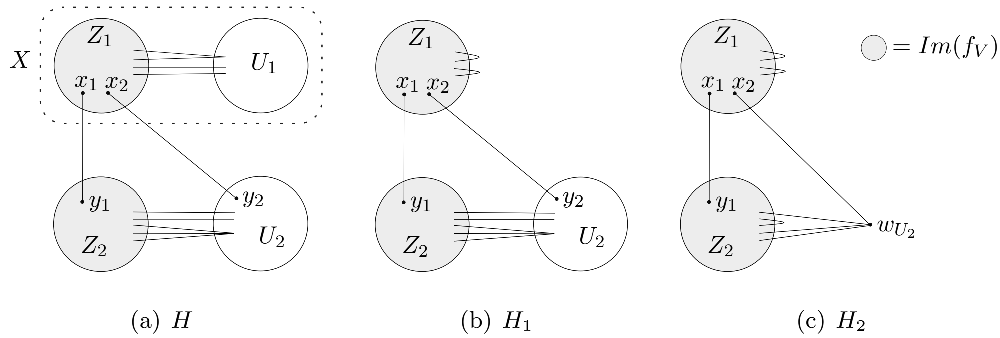


Figure 8.6: An example for the graphs H, H_1 and H_2, when U_1, U_2 are non-empty, U_1 is of even cardinality and U_2 is of odd cardinality.

By Theorem 8.1.3, this can be done such that H_2 is an r-graph. Furthermore, we have

$$|E(H_2)| \leq |E(H') \cup \{f(e_1), f(e_2)\}| \leq |E(G')| + 2.$$

As a consequence, $|V(H_2)| \leq 10$. Thus, H_2 is class 1 since it has a 2-edge-cut and hence, H_2 has r pairwise disjoint perfect matchings. By the construction of H_2, we deduce that H contains r pairwise disjoint sets of edges, denoted by $S_1, \dots, S_r$, such that $|\partial_H(y) \cap S_j| = 1$ for each $y \in f_V(V(G'))$ and each $j \in \{1, \dots, r\}$. Then $f^{-1}(S_1), \dots, f^{-1}(S_r)$ are r pairwise disjoint sets of edges of G such that $|\partial_G(u) \cap f^{-1}(S_j)| = 1$ for each $u \in V(G')$ and each $j \in \{1, \dots, r\}$. This is a contradiction since G' is class 2. □

Let G and G' be two disjoint r-graphs of class 2 with $e \in E(G)$ and $e' \in E(G')$. Denote by $(G, e)|(G', e')$ the set of all graphs obtained from G by replacing the edge e of G by (G', e'), that is, deleting e from G and e' from G', and then adding two edges between $V(G)$ and $V(G')$ such that the resulting graph is regular (see Figure 8.7).

In fact, any graph in $(G, e)|(G', e')$ is an r-graph of class 2. Furthermore,

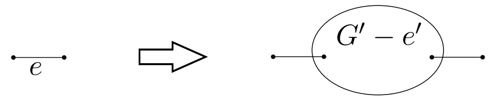


Figure 8.7: A replacement of the edge e by (G', e').

we use $G|(G', e')$ to denote the set of all graphs obtained from G by replacing each edge of G by (G', e').

Theorem 8.2.8. *Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of $\mathcal{P}$, where $r \geq 3$, and let $e_0 \in E(\mathcal{P}^{\mathcal{M}})$. Let G be an r-graph such that $G \ncong \mathcal{P}^{\mathcal{M}}$. If $G \in \mathcal{H}_r$, then $G|(\mathcal{P}^{\mathcal{M}}, e_0) \subset \mathcal{H}_r$.*

Proof. By Theorem 8.0.2, it suffices to prove that any $G^* \in G|(\mathcal{P}^{\mathcal{M}}, e_0)$ cannot be colored by a connected r-graph of smaller order. Let H be a connected r-graph such that G^* has an H-coloring, denoted by f. Label all subgraphs of G^* isomorphic to $\mathcal{P}^{\mathcal{M}} - e_0$ as $G_1, \ldots, G_\ell$, where $\ell = |E(G)|$, and denote by H_i the subgraph of H induced by $f_V(V(G_i))$. Note that $H_i \cong \mathcal{P}^{\mathcal{M}} - e_0$ by Lemma 8.2.7. For each $i \in \{1, \ldots, \ell\}$, we label the two edges of $\partial_{G^*}(V(G_i))$ as e_i^1 and e_i^2, and let $e_i^t = u_i^t v_i^t$ with $v_i^t \notin V(G_i)$ for each $t \in \{1, 2\}$.

Claim 1. $f(\partial_{G^*}(V(G_i)))$ is a 2-edge-cut in H, for every $i \in \{1, \ldots, \ell\}$.

Proof of Claim 1. By Lemma 8.2.6, we suppose to the contrary that there is $i \in \{1, \ldots, \ell\}$ such that $f(e_i^1) = f(e_i^2)$. With $G_i \cong \mathcal{P}^{\mathcal{M}} - e_0$, we have $H \prec \mathcal{P}^{\mathcal{M}}$ by Lemma 8.2.7, and so $H \cong \mathcal{P}^{\mathcal{M}}$ by Theorem 8.1.4. Then, $|f(F)| = 1$ for any 2-edge-cut $F \subset E(G^*)$ by Lemma 8.2.6 since $\mathcal{P}^{\mathcal{M}}$ is 3-edge-connected. Thus, by the construction of G^*, we have $H \prec G$, which implies $H \cong G$ by Theorem 8.1.4. This is a contradiction to the fact that $G \ncong \mathcal{P}^{\mathcal{M}}$. ■

Claim 2. $V(H_i) = V(H_j)$ or $V(H_i) \cap V(H_j) = \emptyset$, for every $i, j \in \{1, \ldots, \ell\}$.

Proof of Claim 2. Assume $V(H_i) \cap V(H_j) \neq \emptyset$. To complete the proof, we shall show $V(H_i) \setminus V(H_j) = \emptyset$ and $V(H_j) \setminus V(H_i) = \emptyset$. Without loss of generality, suppose to the contrary that $V(H_j) \setminus V(H_i) \neq \emptyset$. Note that $f(\partial_{G^*}(V(G_i)))$ is a 2-edge-cut in H by Claim 1. Observe that both H_i and

H_j are isomorphic to $\mathcal{P}^{\mathcal{M}} - e_0$ by Lemma 8.2.7. Thus, at least one edge of $f(\partial_{G^*}(V(G_i)))$ is contained in $E(H_j)$, since H_j is connected. As a consequence, H_j either has a bridge or a 2-edge-cut consisting of two non-adjacent edges, since an r-graph has no cut-vertex. This is not possible. ■

Claim 3. $f_V(z) \notin \bigcup_{i=1}^{\ell} V(H_i)$, for every $z \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$.

Proof of Claim 3. Suppose to the contrary that there is a vertex $z \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$ such that $f_V(z) \in V(H_j)$ for some $j \in \{1, \ldots, \ell\}$. Let e be an edge incident with $f_V(z)$ in H_j. By the construction of G^*, the only edge of $f^{-1}(e) \cap \partial_{G^*}(z)$ is an element of $\partial_{G^*}(V(G_k))$ for some $k \in \{1, \ldots, \ell\}$. Thus, e is in a 2-edge-cut of H by Claim 1, contradicting the fact that $H_j \cong \mathcal{P}^{\mathcal{M}} - e_0$ by Lemma 8.2.7. ■

By Claim 1, $\partial_H(V(H_i)) = f(\partial_{G^*}(V(G_i))) = \{f(e_i^1), f(e_i^2)\}$. Let $f(e_i^t) = x_i^t y_i^t$ with $y_i^t \notin V(H_i)$ for each $t \in \{1, 2\}$.

Claim 4. $\{y_i^1, y_i^2\} \cap V(H_j) = \emptyset$, for every $i, j \in \{1, \ldots, \ell\}$.

Proof of Claim 4. By contradiction, suppose $y_i^t \in V(H_j)$ for some $t \in \{1, 2\}$. Note that $f_V(v_i^t) \in \{y_i^t, x_i^t\}$ and $x_i^t \in V(H_i)$. Thus, $f_V(v_i^t) \in V(H_i) \cup V(H_j)$. This is a contradiction to Claim 3 since $v_i^t \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$ by the construction of G^*. ■

Note that G can be obtained from G^* by deleting all vertices of G_i and adding a new edge edge e_i connecting v_i^1 and v_i^2 for each $i \in \{1, \ldots, \ell\}$. By Claims 2 and 4, $(V(H_i) \cup \{y_i^1, y_i^2\}) \cap V(H_j) = \emptyset$ if $V(H_i) \neq V(H_j)$ for each $i, j \in \{1, \ldots, \ell\}$. Thus, we can construct an r-graph H' from H by deleting all vertices of H_i and adding a new edge g_i connecting y_i^1 and y_i^2 for each $i \in \{1, \ldots, \ell\}$. Note that, for some $i \neq j \in \{1, \ldots, \ell\}$, it might happen that $V(H_i) = V(H_j)$. In such a case, $g_i = g_j$. Define a mapping $f' \colon E(G) \to E(H')$ by letting $f'(e_i) = g_i$, for each $i \in \{1, \ldots, \ell\}$. By Claim 3, $f'_V(z) \in V(H')$ for every vertex $z \in V(G) \subset V(G^*)$. Furthermore, we have $f'(\partial_G(z)) = \partial_{H'}(f'_V(z))$. Since both G and H' are r-graphs, f' is proper. Thus, f' is an H'-coloring of G. Then, (f'_V, f') is an isomorphism between G and H' by Theorem 8.1.4. This implies that $|V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))| = |V(H) \setminus (\bigcup_{i=1}^{\ell} V(H_i))|$, and $V(H_i) \neq V(H_j)$ for any distinct $i, j \in \{1, \dots, \ell\}$ since $f'(e_i) \neq f'(e_j)$. Therefore, $|V(G^*)| = |V(H)|$ by Claims 2 and 3, which completes the proof. □

Thus we now can prove Theorem 8.0.1.

Theorem 8.0.1. *Either $\mathcal{H}_3 = \{\mathcal{P}\}$ or $\mathcal{H}_3$ is an infinite set. Moreover, if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set.*

Proof. If $\mathcal{H}_3 \neq \{\mathcal{P}\}$, then $\mathcal{H}_3$ contains a graph not isomorphic to $\mathcal{P}$. Thus, we can use Theorem 8.2.8 to inductively construct infinitely many graphs belonging to $\mathcal{H}_3$.

By Theorem 8.2.5, $\mathcal{S}(r, 1) \subset \mathcal{H}_r$. Note that the set $\mathcal{S}(r, 1)$ is non-empty (see [75]), and for $r \geq 4$, it does not contain any graph isomorphic to $\mathcal{P}^{\mathcal{M}}$, where $\mathcal{M}$ is any multiset of $r - 3$ perfect matchings of $\mathcal{P}$. Hence, we can use Theorem 8.2.8 to inductively construct infinitely many graphs belonging to $\mathcal{H}_r$. □

8.3 Simple r-graphs

In [68] the authors also asked whether for every $r \geq 4$, there is a connected r-graph coloring all simple r-graph. In this section we answer this question by showing that there is no finite set of connected r-graphs $\mathcal{H}'_r$ such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.

Lemma 8.3.1 ([40]). *Let r be a positive integer, G be an r-graph and $F \subseteq E(G)$. If $|F| \leq r - 1$, then $G - F$ has a 1-factor.*

Recall that, for an r-graph G and an odd set $X \subseteq V(G)$, the edge-cut $\partial_G(X)$ is *tight* if it consists of exactly r edges; and it is *trivial* if $|X| = 1$ or $|X^c| = 1$.

Lemma 8.3.2. *Let $r \geq 3$, let G, H be connected r-graphs and let f be an H-coloring of G. If $F \subseteq E(G)$ is a tight edge-cut in G, then $f(F)$ is a tight edge-cut in H.*

Proof. Since F is a tight edge-cut, we have $|f(F)| \leq r$. Suppose that $|f(F)| < r$. By Lemma 8.3.1, $H - f(F)$ has a perfect matching M. Thus, $f^{-1}(M)$ is a perfect matching of G such that $f^{-1}(M) \cap F = \emptyset$, a contradiction. Therefore, $|f(F)| = r$, and let $H_1, \ldots, H_m$ be the components of $H - f(F)$.

We first claim that the two endvertices of each edge in $f(F)$ are in different components of $H - f(F)$. By contradiction, suppose that there is an edge $xy \in f(F)$ such that x and y are on the same component H' of $H - f(F)$. Let T be an x, y-path contained in H'. Then, $f^{-1}(E(T) \cup \{xy\})$ induces a 2-regular subgraph in G (see Observation 8.1.1 (*iii*)) and intersects F exactly once, a contradiction.

The remaining proof is split into two cases as follows.

Case 1. $H - f(F)$ has a component of odd order.

If $m > 2$, then there is an odd component H' with $|\partial_G(V(H'))| < r$, since $H - f(F)$ has at least two components of odd order, a contradiction. Hence, $H - f(F)$ has exactly two components, which are of odd order and therefore, $f(F)$ is a tight edge-cut in H.

Case 2. Every component of $H - f(F)$ is of even order.

Let $\tilde{H}$ be the graph obtained from H by identifying all vertices in $V(H_i)$ to a new vertex for each $i \in \{1, \ldots, m\}$. Since every component is of even order, $\tilde{H}$ is a connected graph on $|f(F)| = r$ edges in which every vertex is of even degree.

Now, we shall prove that $\tilde{H}$ is bipartite. Suppose by contradiction that $\tilde{H}$ has an odd circuit of order $2t + 1$. This means that there is an odd number of components $H_{i_1}, \ldots, H_{i_{2t+1}}$ in $H - f(F)$ such that, for all $j \in \mathbb{Z}_{2t+1}$ there is an edge $x_j y_{j+1} \in f(F)$ such that $x_j \in V(H_{i_j})$ and $y_{j+1} \in V(H_{i_{j+1}})$. Moreover, for all $j \in \mathbb{Z}_{2t+1}$ there is an x_j, y_j-path T_j contained in the component H_{i_j}, i.e. such that $E(T_j) \cap f(F) = \emptyset$. Consider the circuit C induced by $x_j y_{j+1}$ and all edges of T_j for all $j \in \mathbb{Z}_{2t+1}$. Then $|E(C) \cap f(F)| = 2t + 1$ and $f^{-1}(E(C))$ induces a 2-regular subgraph in G such that $|F \cap f^{-1}(E(C))| = 2t + 1$, a contradiction.

Since $\tilde{H}$ is a bipartite graph, we can assume without loss of generality that there is an $s \in \{1, \ldots, m-1\}$ such that $f(F) = \partial_H(W)$, where $W = V(H_1) \cup \cdots \cup V(H_s)$. Note that $|W|$ is even since every component of $H - f(F)$ has even order. Thus, a perfect matching M of H is such that $|M \cap \partial_H(W)| = |M \cap f(F)|$ is even. But then $|f^{-1}(M) \cap F|$ is even as well, a contradiction. □

Lemma 8.3.3. *Let $r \geq 3$, let G and H be two r-graphs, and let X be a subset of $V(H)$ such that $\partial_H(X)$ is a tight cut and $\chi'(H/X^c) = r$. If $H \prec G$, then $H/X \prec G$.*

Proof. Assume that f is an H-coloring of G. Label the edges of $\partial_H(X)$ as $e_1, \ldots, e_r$. Since $\chi'(H/X^c) = r$, the subset $E(H[X]) \cup \partial_H(X)$ of $E(H)$ can be partitioned into r pairwise disjoint matchings, denoted by $N_1, \ldots, N_r$, such that each edge of $\partial_H(X)$ is contained in exactly one of them. Without loss of generality, we may assume $e_i \in N_i$ for each $i \in \{1, \ldots, r\}$. Note that $E(G) = f^{-1}(E(H)) = f^{-1}(E(H[X^c])) \cup f^{-1}(N_1) \cup \ldots \cup f^{-1}(N_r)$. Moreover, for convenience, every edge and every vertex of H/X is labeled as in H. We define a mapping $f' \colon E(G) \to E(H/X)$ as follows. For every $e \in E(G)$, set

$$f'(e) = \begin{cases} f(e) & \text{if } e \in f^{-1}(E(H[X^c])); \\ e_i & \text{if } e \in f^{-1}(N_i),\ \text{for } i \in \{1, \ldots, r\}. \end{cases}$$

To conclude the proof, we shall show that f' is an H/X-coloring of G. Let v be a vertex of $V(G)$. If $f(\partial_G(v)) = \partial_H(u)$ for some vertex $u \in X^c \subset V(H)$, then $f'(\partial_G(v)) = f(\partial_G(v)) = \partial_H(u) = \partial_{H/X}(u)$ by the definition of f'. If $f(\partial_G(v)) = \partial_H(u)$ for some vertex $u \in X$, then the image under f of each edge of $\partial_G(v)$ is contained in one of $N_1, \ldots, N_r$. Hence, the image under f' of each edge of $\partial_G(v)$ appears once in $\partial_{H/X}(w_X)$. This implies $f'(\partial_G(v)) = \partial_{H/X}(w_X)$. Thus, f' is an H/X-coloring of G. □

A simple graph H is *regularizable* if we can obtain a regular graph from H by replacing each edge of H by a nonempty set of parallel edges. We need the following lemma, which follows from two results of [10] and [74]. The equivalence of the first two statements is shown in [10]; the equivalence of the first and the third statement is shown in [74].

Lemma 8.3.4. *Let G be a simple connected graph which is not bipartite with two partition sets of the same cardinality. The following statements are equivalent:*

- *$iso(G - S) < |S|$, for all $S \subseteq V(G)$.*
- *G is regularizable [10].*
- *for every $v \in V(G)$, both $G - v$ and G have a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor [74].*

Lemma 8.3.5. *Let $r \geq 3$, let G and H be r-graphs, where H is connected, and let $S \subseteq V(G)$ such that $\partial_G(S)$ is a tight cut and $G[S]$ has no $\{K_{1,1}, C_m : m \geq 3\}$-factor. If G has an H-coloring $f \colon E(G) \to E(H)$ and $\partial_H(X) = f(\partial_G(S))$ for an $X \subseteq V(H)$, then H/X or H/X^c is a bipartite graph with two partition sets of the same cardinality.*

Proof. Suppose to the contrary that both H/X and H/X^c are not bipartite graphs with two partition sets of the same cardinality. By Lemma 8.3.2, the edge-cut $\partial_H(X)$ is tight and so both H/X and H/X^c are r-regular. Thus, the underling graphs of H/X and H/X^c are both regularizable and hence, both $H/X - w_X$ and $H/X^c - w_{X^c}$ have a $\{K_{1,1}, C_m : m \geq 3\}$-factor, by Lemma 8.3.4. Let H' be the union of these two factors. Note that H' is a $\{K_{1,1}, C_m : m \geq 3\}$-factor of H, which contains no edge of $\partial_H(X)$. Since $\partial_H(X) = f(\partial_G(S))$ and by Observation 8.1.1 (iv), G has a $\{K_{1,1}, C_m : m \geq 2\}$-factor F, which contains no edge of $\partial_G(S)$. By deleting one edge of every component of F isomorphic to C_2, we obtain a $\{K_{1,1}, C_m : m \geq 3\}$-factor of G, which contradicts the assumption that $G[S]$ has no $\{K_{1,1}, C_m : m \geq 3\}$-factor. □

For an r-regular graph G and a vertex $v \in V(G)$, a *Meredith extension* of G at v is the operation that replaces v by $K_{r-1,r}$ such that the resulting graph is r-regular.

Theorem 8.3.6. *Let $r \geq 3$ and let $\mathcal{H}$ be a set of connected r-graphs such that every $H \in \mathcal{H}$ does not contain a non-trivial tight edge-cut $\partial_H(X)$ such that H/X or H/X^c is class 1. If every connected simple r-graph can be colored by an element of $\mathcal{H}$, then every connected r-graph can be colored by an element of $\mathcal{H}$.*

Proof. Let G be an arbitrary r-graph. By applying a Meredith extension on every vertex of G, we obtain a simple r-regular graph G^e. From the fact that both G and $K_{r,r}$ are r-graphs, we know that G^e is also an r-graph by Lemma 6.3.1. Hence, there is $H \in \mathcal{H}$ such that $H \prec G^e$. Let f be an H-coloring of G^e. Note that for any induced subgraph G' of G^e isomorphic to $K_{r,r-1}$, the edge-cut $\partial_{G^e}(V(G'))$ is tight, and so $f(\partial_{G^e}(V(G')))$ is also tight in H by Lemma 8.3.2. Let $X \subset V(H)$ such that $\partial_H(X) = f(\partial_{G^e}(V(G')))$. Since $K_{r,r-1}$ contains no $\{K_{1,1}, C_m \colon m \geq 3\}$-factor, Lemma 8.3.5 implies that H/X or H/X^c is a bipartite graph with two partition sets of the same cardinality. In particular, H/X or H/X^c is class 1, which implies that X or X^c is a single vertex by the choice of $\mathcal{H}$. Therefore, the edge-cut $\partial_{G^e}(V(G'))$ is mapped to a trivial edge-cut of H under f. Since G' was chosen arbitrarily, we conclude that G also has an H-coloring, which completes the proof. □

Thus, we can deduce our main result for simple graphs as well.

Theorem 8.3.7. *Let $r \geq 3$ and let $\mathcal{H}'_r$ be a set of connected r-graphs such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.*

i) If the Petersen Coloring Conjecture is false, then $\mathcal{H}'_3$ is an infinite set.

ii) If $r \geq 4$, then $\mathcal{H}'_r$ is an infinite set.

Proof. By Lemma 8.3.3 we can identify suitable subsets of vertices of graphs in $\mathcal{H}'_r$ to obtain a set $\mathcal{H}''_r$ of connected r-graphs with the following properties.

- Every connected simple r-graph can be colored by an element of $\mathcal{H}''_r$.

- For every $H \in \mathcal{H}_r''$, there is no non-trivial tight edge-cut $\partial_H(X)$ such that H/X or H/X^c is class 1.

Hence, by Theorem 8.3.6, every connected r-graph can be colored by an element of $\mathcal{H}_r''$. Thus, $\mathcal{H}_r \subset \mathcal{H}_r''$ and hence, $\mathcal{H}_r''$ is an infinite set by Theorem 8.0.1. By the construction of $\mathcal{H}_r''$ we have $|\mathcal{H}_r'| \geq |\mathcal{H}_r''|$, and hence, $\mathcal{H}_r'$ is also an infinite set. □

8.4 Open Problems

Recall that the edge connectivity of an r-graph is equal to r or it is an even number. We have shown that for every $r \geq 3$ and every multiset $\mathcal{M}$ of $r-3$ perfect matchings of the Petersen graph, the graph $\mathcal{P}^{\mathcal{M}}$ belongs to $\mathcal{H}_r$. Thus, for $r \neq 5$, for each possible edge-connectivity t there is a t-edge-connected r-graph in $\mathcal{H}_r$. For $r = 5$, we do not know any 5-edge-connected 5-graph in $\mathcal{H}_r$. However, we know only a finite number of t-edge-connected r-graphs of $\mathcal{H}_r$ if $t \geq 3$.

Problem 8.4.1. *For $r, t \geq 3$, does $\mathcal{H}_r$ contain infinitely many t-edge-connected r-graphs?*

It is also not clear whether $\mathcal{H}_r$ contains elements of $\mathcal{S}(r,k)$ for $k \in \{2, \dots, r-3\}$. So far, these sets are not determined for $k \in \{1, \dots, r-3\}$. Indeed, we even do not know the order of their elements. Let $o(r,k)$ be the order of the graphs of $\mathcal{S}(r,k)$.

Problem 8.4.2. *For all $r \geq 3$ and $k \in \{1, \dots, r-2\}$: Determine $o(r,k)$.*

By our results, $o(r, r-2) = 10$. By results of Rizzi [75], $o(r,1) \leq 2 \times 5^{r-2}$. We conjecture the following to be true.

Conjecture 8.4.3. *For all $r \geq 3$ and $k \in \{2, \dots, r-2\}$: $o(r, k-1) \geq o(r,k)$.*

If Conjecture 8.4.3 would be true, then it would follow with Corollary 8.1.5 that $\mathcal{S}(r,k) \subset \mathcal{H}_r$ for each $k \in \{1, \dots, r-2\}$.

Similar problems arise for simple r-graphs. Let $o_s(r,k)$ be the smallest order of a simple r-graph G with $\pi(G) = k$. Small simple r-graphs of class 2 can be obtained as follows. Consider a perfect matching M of $\mathcal{P}$ and the graph $G = \mathcal{P} + (r-3)M$. Let H be a simple r-graph of smallest order and $v \in V(H)$. Then, H is class 1 and $|V(H)| = r+1$ if r is odd and $|V(H)| = r+2$ if r is even. Now, replace appropriately five vertices of G by $H - v$ (such that the resulting graph is r-regular) to obtain a simple r-graph G'. Since H is class 1 and $\pi(G) = r-2$, we have $\pi(G') = r-2$. Therefore, if r is odd, then $o_s(r, r-2) \leq 5(r+1)$ and if r is even, then $o_s(r, r-2) \leq 5(r+2)$. Furthermore, bounds for $o_s(r,k)$ can be obtained by using Meredith extensions, since if G' is a Meredith extension of an r-graph G, then $\pi(G') = \pi(G)$.

Appendix

A.1: A sketch of the proof of Theorem 7.3.17

In order to prove Theorem 7.3.17 we adjust Theorem 7.3.10 as follows.

Theorem A.1. *The following statements are equivalent.*

(i) Every 5*-edge-connected* 5*-graph with an underlying cubic graph has a 2-PDPM.*

(ii) For every 5*-edge-connected* 5*-graph* G *with an underlying cubic graph and every simple* $e \in E(G)$*, there is a 2-PDPM containing* e*.*

(iii) For every 5*-edge-connected* 5*-graph* G *with an underlying cubic graph and every simple* $e \in E(G)$*, there is a 2-PDPM avoiding* e*.*

(iv) For every 5*-edge-connected* 5*-graph* G *with an underlying cubic graph and every simple* $e \in E(G)$ *and every two parallel edges* e_1, e_2 *adjacent with* e*, there is a 2-PDPM containing* e *and avoiding* e_1, e_2*.*

Proof. Clearly, each of (ii), (iii) and (iv) implies (i). Thus, it suffices to prove that (i) implies (ii); (i) implies (iii); and (ii) implies (iv).

$(i) \Rightarrow (ii), (iii)$. Let G be a 5-edge-connected 5-graph whose underlying graph is cubic and let $e = vv_1$ be a simple edge of G. Let H and H' be the graphs constructed in the part "$(i) \Rightarrow (ii), (iii)$" of the proof of Theorem 7.3.10 by using C_{2r} and r copies of G in the case $r = 5$ (see Figures 7.7 (a) and 7.8 (a)). Clearly, the graph H' can be constructed from H such that every vertex of $V(H') \setminus \{u, u'\}$ has degree 3 in the underlying graph of H'. Let $W = W_5 + E(C_5)$. Now, according to Definition 7.3.8 replace u by (W^1, w^1)

and replace u' by (W^2, w^2), where W^l is a copy of W and w^l is the vertex of W^l corresponding to w, for $l \in \{1,2\}$. The resulting graph, denoted by H'', is a 5-edge-connected 5-graph by Lemma 7.3.9. Since its underlying graph is cubic, H'' has two disjoint perfect matchings N_1, N_2 by statement (i). Let $N = N_1 \cup N_2$ and recall that $I = \{1,3,5,7,9\}$. For every $i \in I$ and $j \in \{1,2\}$, Observation 7.1.1 implies $m_{ij} \in \{1,3\}$, where $m_{ij} = |\partial_{H''}(V(G^i) \setminus \{v^i\}) \cap N_j|$. Furthermore, we have $|\partial_{H''}(V(W^l) \setminus \{w^l\}) \cap N_j| \in \{1,3\}$ for every $l, j \in \{1,2\}$ also by Observation 7.1.1. Thus, $|\partial_{H''}(V(W^l) \setminus \{w^l\}) \cap N| \in \{2,4\}$ for every $l \in \{1,2\}$. As a consequence, there is an integer $i \in I$ such that N does not contain the unique edge in $E_{H''}(v_1^i, V(W^1) \setminus \{w^1\})$. We have $m_{i1} = m_{i2} = 1$. Therefore, G^i has two disjoint perfect matchings such that $v^i v_1^i$ is in none of them, which proves statement (iii). For statement (ii), we consider the following cases.

Case 1. $|N \cap \partial_{H''}(V(W^1) \setminus \{w^1\})| = |N \cap \partial_{H''}(V(W^2) \setminus \{w^2\})| = 2$.

In this case, H' has a 2-PDPM, and hence, statement (ii) follows by the same argumentation as in the proof of Theorem 7.3.10 part "$(i) \Rightarrow (ii), (iii)$".

Case 2. Without loss of generality $|N_1 \cap \partial_{H''}(V(W^1) \setminus \{w^1\})| = 3$.

In this case, there is an integer $i \in I$, say $i = 1$, such that N_1 contains the unique edge in $E_{H''}(v_1^i, V(W^1) \setminus \{w^1\})$ and the unique edge in $E_{H''}(v_1^{i+2}, V(W^1) \setminus \{w^1\})$. The set N_1 contains exactly one edge incident with u_2 and thus, $m_{11} = 1$ or $m_{31} = 1$ by the construction of H''. Therefore, G^1 has two disjoint perfect matchings such that $v^1 v_1^1$ is in one of them or G^3 has two disjoint perfect matchings such that $v^3 v_1^3$ is in one of them, which proves statement (ii).

Case 3. Without loss of generality $|N_1 \cap \partial_{H''}(V(W^2) \setminus \{w^2\})| = 3$.

In this case, there is an integer $i \in I$, say $i = 3$, such that N_1 contains the unique edge in $E_{H''}(u_{i-1}, V(W^2) \setminus \{w^2\})$ and the unique edge in $E_{H''}(u_{i+1}, V(W^2) \setminus \{w^2\})$. As a consequence, N_1 contains the unique edge in $E_{H''}(v_1^3, V(W^1) \setminus \{w^1\})$ and $m_{31} = 1$. Therefore, G^3 has two disjoint perfect matchings such that $v^3v_1^3$ is in one of them, which proves statement (ii).

$(ii) \Rightarrow (iv)$. Let G be a 5-edge-connected 5-graph whose underlying graph is cubic, let $v \in V(G)$ and let $N_G(v) = \{v_1, v_2, v_3\}$. Furthermore, let $\mu_G(v, v_1) = 1$, $\mu_G(v, v_2) = \mu_G(v, v_3) = 2$, let e be the edge connecting v and v_1 and let e_1, e_2 be the two parallel edges connecting v and v_2. We show that there are two disjoint perfect matchings such that their union contains e but neither e_1 nor e_2.

Let G^1 and G^3 be two copies of G in which the vertices and edges are labeled accordingly by using an upper index. Let H be the graph constructed in the part "$(ii) \Rightarrow (iv)$" of the proof of Theorem 7.3.10 by using K_4 in the case $r = 5$, see Figure 7.9 (a). According to Definition 7.3.8, construct a new graph H' from H by replacing u_1 with (G^1, v^1) and replacing u_3 with (G^3, v^3) such that $\mu_{H'}(v_1^1, v_1^3) = 1$ and $\mu_{H'}(v_2^1, u_2) = \mu_{H'}(v_2^3, u_2) = \mu_{H'}(v_3^1, u_4) = \mu_{H'}(v_3^3, u_4) = 2$. The graph H' is 5-edge-connected and 5-regular by Lemma 7.3.9 and its underlying graph is cubic. Therefore, by statement (ii) there are two disjoint perfect matchings N_1, N_2 of H' such that $u_2u_4 \in N_1$. By Observation 7.1.1, we have $v_1^1v_1^3 \in N_1$ and $|\partial_{H'}(V(G^i) \setminus \{v^i\}) \cap N_j| = 1$ for every $i \in \{1, 3\}$ and every $j \in \{1, 2\}$. Furthermore, $N_1 \cup N_2$ either does not contain the two edges connecting v_2^1 and u_2 or does not contain the two edges connecting v_2^3 and u_2. In the first case, G^1 has two disjoint perfect matchings such that their union contains e^1 but neither e_1^1 nor e_2^1; in the second case G^3 has two disjoint perfect matchings such that their union contains e^3 but neither e_1^3 nor e_2^3. This proves statement (iv). □

Theorem 7.3.17 can be proved like Theorem 7.3.16 by using Theorem A.1 instead of Theorem 7.3.10.

List of Figures

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