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Dissertation

Subsets of finite general linear groups

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“Ain’t no mountain high enough”
— Ashford, Simpson

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Zusammenfassung

Die Hauptresultate dieser Arbeit tragen zur extremalen und algebraischen Kombinatorik bei. Im Kontext der extremalen Kombinatorik ist eines der berühmtesten Ergebnisse das Erdős-Ko-Rado (EKR) Theorem, das die Frage beantwortet, wie groß eine Familie von sich paarweise schneidenden Mengen sein kann. Seitdem Erdős, Ko und Rado ihr Ergebnis in den 1960er Jahren veröffentlicht haben, wurden EKR-Probleme untersucht, die sich aus vielen verschiedenen Objekten und Definitionen von schneidend ergeben. In dieser Arbeit untersuchen wir EKR-Probleme in der endlichen allgemeinen linearen Gruppe $\mathrm{GL}(n, q)$, welche aus allen invertierbaren $n \times n$ Matrizen mit Einträgen im endlichen Körper $\mathbb{F}_q$ besteht. Wir liefern obere Schranken für die Größe verschiedener sich schneidender Mengen in $\mathrm{GL}(n, q)$ und geben eine Charakterisierung der sich schneidenden Mengen maximaler Größe.

Im Kontext der algebraischen Kombinatorik beschäftigen wir uns mit transitiven Teilmengen einer Permutationsgruppe, welche den Begriff der transitiven Untergruppe verallgemeinern. Wir liefern strukturelle Ergebnisse über Teilmengen von $\mathrm{GL}(n, q)$, die transitiv auf fahnenartigen Strukturen wirken. Mithilfe der Theorie der Assoziationsschemata zeigen wir, dass diese transitiven Mengen Delsarte T-Designs im Assoziationsschema von $\mathrm{GL}(n, q)$ sind. Dies verallgemeinert ein gruppentheoretisches Resultat von Perin über Untergruppen von $\mathrm{GL}(n, q)$, die transitiv auf Unterräumen über endlichen Körpern wirken.

Unser Ansatz sich schneidende und transitive Mengen zu untersuchen, verwendet die Theorie der Assoziationsschemata und die Darstellungstheorie von $\mathrm{GL}(n, q)$.

Viele der erzielten Ergebnisse können als q-Analoga der für die symmetrische Gruppe bekannten Resultate interpretiert werden.

Abstract

The main results of this thesis contribute to extremal and algebraic combinatorics. In the context of extremal combinatorics, one of the most famous results is the Erdős-Ko-Rado (EKR) theorem, which answers the question of how large a family of pairwise intersecting sets can be. Ever since Erdős, Ko, and Rado published their result in the 1960s, EKR problems arising from many different objects and notions of intersection have been investigated. In this thesis, we study EKR problems in the finite general linear group $\mathrm{GL}(n, q)$, the group consisting of all invertible $n \times n$ matrices with entries in the finite field with q elements. We provide upper bounds for the size of different intersecting sets in $\mathrm{GL}(n, q)$ and give a characterisation of the intersecting sets of maximal size.

In the context of algebraic combinatorics, we deal with transitive subsets of a permutation group, which generalise the notion of a transitive subgroup. We provide structural results on subsets of $\mathrm{GL}(n, q)$ acting transitively on flag-like structures. Using the theory of association schemes, we show that these transitive sets are Delsarte T-designs in the association scheme of $\mathrm{GL}(n, q)$. This generalises a group-theoretical result of Perin on subgroups of $\mathrm{GL}(n, q)$ acting transitively on subspaces over finite fields.

Our approach for studying both intersecting and transitive sets of $\mathrm{GL}(n, q)$ uses the theory of association schemes and the representation theory of $\mathrm{GL}(n, q)$.

Many of the results obtained can be interpreted as q-analogs of those known for the symmetric group.

Contents

Introduction

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This thesis sits at the intersection of the mathematical disciplines of *extremal combinatorics*, *algebraic combinatorics*, and *representation theory.* The main original contributions of this thesis can be divided into contributions in *extremal combinatorics* on the one hand, and on *algebraic combinatorics* on the other. The first is given by Erdős-Ko-Rado (EKR) results in the finite general linear group $\mathrm{GL}(n, q)$ and provides q-analogs of previous results on the symmetric group. The latter provides combinatorial interpretations of certain Delstarte T-designs in the association scheme of $\mathrm{GL}(n, q)$ as transitive sets on flag-like structures. The main tools for these results come from representation theory.

Extremal combinatorics

Extremal combinatorics addresses questions about the largest or smallest possible size of a collection of objects with given properties. Within this framework, the so-called *intersection problems* are of particular significance. In what follows, we call a subset of $\{1, 2, \ldots, n\}$ of size k a *k-set* of $\{1, 2, \ldots, n\}$. The classical *intersection problem* deals with the following question.

How large can a family of k-sets of $\{1, 2, \ldots, n\}$ be such that every two members of this family have non-empty intersection? (❀)

Families with the property given in (❀) are called *intersecting.* Figure 1 illustrates two intersecting families for $n = 4$ and $k = 2$. The collections of k-sets containing a fixed element of $\{1, 2, \ldots, n\}$ give the *canonical examples* of such intersecting families. When such examples satisfy the further condition that no two sets intersect other than at the fixed element, the families are known as *sunflowers.*

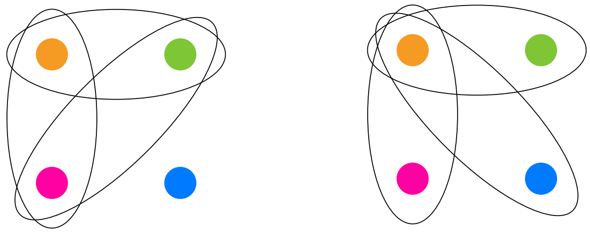


Figure 1: Examples of intersecting families of maximal size.

If $n < 2k$ every family of k-sets of $\{1, 2, \ldots, n\}$ is intersecting, so it is necessary to assume $n \geq 2k$ to avoid triviality. In the non-trivial case, the answer to (❀) was given by Erdős, Ko, and Rado [EKR61] and states that the size of such a family is at most $\binom{n-1}{k-1}$, and, for $n > 2k$, equality holds if and only if the family is a canonical example.

Although this result was published only in 1961, it had already been proved more than 20 years earlier by Erdős, Ko, and Rado. However, according to Erdős, "at that time there was relatively less interest in combinatorics" [Erd87]. Nowadays, combinatorics is recognised as one of the pillars of communication and network theory, showing just how much this perception has changed.

In general, whenever a set of objects and a notion of intersection among these objects is given, an intersection problem can be formulated as follows.

Problem 1. Find an upper bound on the number of pairwise intersecting objects.

In fact we are not only interested in finding upper bounds but also in the structure of the extremal cases, which motivates the following problem.

Problem 2. Characterise the intersecting families of maximal size.

In honour of the authors of [EKR61], these two problems are also known as *EKR problems* or *EKR-type problems*, and the corresponding results as *EKR results*.

For example, a different notion of intersection in the classical case of sets is the *t-intersection*. Specifically, two k-sets of $\{1, 2, \ldots, n\}$ are *t-intersecting* if their intersection has size at least t. With this notion of intersection, the *canonical examples* are the families of k-sets containing a fixed t-set of $\{1, 2, \ldots, n\}$. For n sufficiently large compared to t, Erdős, Ko, and Rado also solved this EKR-type problem in [EKR61]. They showed that the size of a t-intersecting family of k-sets of $\{1, 2, \ldots, n\}$ is at most $\binom{n-t}{k-t}$ and, in the case of equality, the family is a canonical example.

The classical EKR problem has a straightforward formulation, its solution is the expected one, and the extremal cases are given by the canonical examples. This usually happens for EKR problems in general, so that we could say that the slogan of EKR-type problems is that for n large enough, "the expected answer is the right one, and, usually, the 'only one' ".

Despite the simplicity of the formulation and the "obviousness" of the answer, EKR results have had a significant impact in extremal combinatorics.

EKR problems arising from many different objects and notions of intersection have been investigated, with objects such as subspaces or flags, see [GM16] for a survey. In this thesis, we are interested in EKR problems on more structured objects, namely groups. In this context Ellis, Friedgut, and Pilpel [EFP11] proved a remarkable EKR-type result for the symmetric group S_n, building on important earlier work [DF77], [CK03], [LM04], [GM09]. Here the notion of intersection considered is the following: two permutations $\pi, \sigma \in S_n$ are *t-intersecting* if they permute t distinct elements in the same way. The *canonical examples* are cosets of the stabilisers of t-tuples of t distinct elements of $\{1, 2, \ldots, n\}$. It was shown in [EFP11] that for each fixed t and all sufficiently large n, the size of a t-intersecting set in S_n is at most $(n-t)!$ and, in case of equality, the set is a canonical example, which solves the Problems 1 and 2.

Instead of a pointwise notion of intersection, one can consider a setwise notion of intersection in the symmetric group. Two permutations $\pi, \sigma \in S_n$ are *t-set-intersecting* if there exits a t-set S of $\{1, 2, \ldots, n\}$ such that $\pi(S) = \sigma(S)$. In this case, the *canonical examples* are given by cosets of the stabilisers of t-sets of $\{1, 2, \ldots, n\}$. Ellis [Ell12] proved that for each fixed t and all sufficiently large n, the size of a t-set-intersecting set in S_n is at most $t!(n-t)!$ and, in case of equality, the set is a canonical example, which solves the Problems 1 and 2.

In this thesis we study q-analogs of these EKR-type problems for the symmetric group. More precisely, we investigate the finite general linear group $\mathrm{GL}(n, q)$, consisting of all invertible $n \times n$ matrices with entries in the finite field $\mathbb{F}_q$, and consider different notions of intersection, see Chapter 5.

The notions of intersection in $\mathrm{GL}(n, q)$ that we consider are *t-intersection* and *t-space-intersection.* Two elements of $\mathrm{GL}(n, q)$ are *t-intersecting* if they coincide as maps on t linearly independent vectors of $\mathbb{F}_q^n$. Here the *canonical examples* are cosets of the stabilisers of t-tuples of linearly independent vectors of $\mathbb{F}_q^n$.

Two elements of $\mathrm{GL}(n, q)$ are *t-space-intersecting* if they coincide as maps on a t-dimensional subspace of $\mathbb{F}_q^n$. With this notion of intersection, the *canonical examples* are cosets of the stabilisers of t-dimensional subspaces of $\mathbb{F}_q^n$.

It is well known, see [AA14] or [AM15] for example, that the size of a 1-intersecting set in $\mathrm{GL}(n, q)$ is bounded by the size of a canonical example. Moreover, in [MR23], it was proved that if Y is a 1-intersecting set of maximal size in $\mathrm{GL}(2, q)$, then the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of nonzero vectors.

Meagher and Spiga [MS11] proved that the size of a 1-space-intersecting set is bounded by the size of a stabiliser of a 1-dimensional subspace of $\mathbb{F}_q^n$.

Original contributions of this thesis - Part 1

For each of the notions of intersection in $\mathrm{GL}(n, q)$ that we consider (t-intersection and t-space-intersection), we solve the corresponding EKR problems for *all* positive integers t and all n sufficiently large compared to t, providing q-analogs of the results mentioned for the symmetric group. This solves Problem 1, partially solves Problem 2, and greatly improves the previously known results from [MR23], dealing only with the case $t = 1$ and $n = 2$, respectively.

In the case of t-intersecting and t-space-intersecting, we prove the following two theorems in Chapter 5.

Theorem 1. *Let $Y \subseteq \mathrm{GL}(n, q)$ be t-intersecting. If n is sufficiently large compared to t, then the size of Y is at most the size of the canonical example and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of t-tuples of linearly independent vectors of $\mathbb{F}_q^n$.*

Theorem 2. *Let $Y \subseteq GL(n, q)$ be t-space-intersecting. If n is sufficiently large compared to t, then the size of Y is at most the size of the canonical example and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of t-dimensional subspaces of $\mathbb{F}_q^n$.*

The results from Theorems 1 and 2 on the different notions of intersection in $\mathrm{GL}(n, q)$ were published in

[ES23] A. Ernst, K.-U. Schmidt, *Intersection theorems for finite general linear groups*, Math. Proc. Cambridge Philos. Soc., **175** 2023, no. 1, 129–160.

Both authors of the paper *Intersection theorems for finite general linear groups* are first authors with equal rights and equally contributed to the development of the research questions.

It is worth mentioning that, after a first version of this paper was made publicly available (arXiv, May 2022), Ellis, Kindler, and Lifshitz [EKL23] (arXiv, August 2022) independently proved a result in the context of forbidden intersection problems that is more general than our result stated in Theorem 1. Moreover, they also solved the characterisation issue of Problem 2 for t-intersecting sets. Their methods are completely different from ours, and in particular they do not use the representation theory of $\mathrm{GL}(n, q)$.

In the case of t-space-intersecting sets in $\mathrm{GL}(n, q)$ only the partial solution of Problem 2 from Theorem 2 is known. The complete characterisation problem was only solved for $t = 1$ (see [MS11], [MS14], and [Spi19]) but it remains open for all $t \geq 2$.

Moreover, we obtain results for *cross-intersecting* subsets in $\mathrm{GL}(n, q)$. A pair of subsets (Y, Z) of $\mathrm{GL}(n, q)$ is t-cross-intersecting and t-space-cross-intersecting if all pairs in $Y \times Z$ are t-intersecting and t-space-intersecting, respectively. The corresponding upper bounds on $|Y||Z|$ are contained in Chapter 5 and also published in [ES23].

Our methods to prove these EKR-type results heavily rely on the representation theory of $\mathrm{GL}(n, q)$ and on *association schemes* arising from $\mathrm{GL}(n, q)$, which belong to the field of *algebraic combinatorics*.

Algebraic Combinatorics

In the words of Bannai and Ito [BI84], the "very fundamental, perhaps the most important, [theory] in algebraic combinatorics" is the theory of *association schemes*.

The theory of *association schemes* is much richer than finite group theory, which is why it is often referred to as "a group theory without groups" (see [BI84], for example). An *association scheme* consists of a finite set X together with relations on $X \times X$ that satisfy certain regularity and symmetry properties.

In his monumental PhD thesis [Del73], Delsarte demonstrated the profound significance of association schemes by using them to provide a unifying framework for *error-correcting codes* and *combinatorial designs*, which he generalised to so-called *D-cliques* and *T-designs*.

While classical combinatorial t-designs are defined as collections of sets with structural properties to ensure regularities in their intersection behaviour, T-designs, which include classical combinatorial t-designs as a special case, are defined purely algebraically. The definition of a T-design is given in the context of an association scheme. Here, for any given subset Y of the association scheme, it is defined an associated *dual distribution*, which is a tuple obtained from the characteristic vector of Y after some algebraic manipulations involving the matrix algebra associated with the association scheme. Now Y forms a T-design if its dual distribution has zeroes in certain entries specified by T. More context and precise definitions are provided in Chapter 2.

Despite their algebraic definition, it turns out that certain T-designs often have nice combinatorial interpretations, as shown in [Del73] and [Sta86], for example. In the words of Delsarte [Del73] this "motivates [...] the 'conjecture' being that T-designs will often have interesting properties". This leads to the following problem.

Problem 3. Find a combinatorial interpretation for a T-design in a given association scheme.

Moreover, one can often interpret T-designs as approximations of the underlying set, see [Sta86]. It is therefore interesting to study the following problem.

Problem 4. Prove the existence and give constructions of small Delsarte T-designs in association schemes.

In this thesis, several contributions to algebraic combinatorics emerge from the study of T-designs in the association scheme of the finite general linear group $\mathrm{GL}(n, q)$. Specifically, we provide a combinatorial characterisation of Delsarte T-designs in $\mathrm{GL}(n, q)$ and show the existence of small such designs, to mention only some of our results in this context.

These findings on T-designs in $\mathrm{GL}(n, q)$ complement existing work in the field, such as the classical characterisation of T-designs in the association scheme of the symmetric group by Martin and Sagan [MS06], where certain T-designs are described as *sets* (not necessarily subgroups) acting *transitively* on set partitions.

In [MS06], the authors generalise the concept of a transitive group action to subsets as follows. Let G be a group acting transitively on a finite set Ω. A subset Y of G is transitive on Ω if the number of elements in Y that map one element of Ω to another is the same for any two elements in Ω.

In the context of the symmetric group S_n, for an integer partition $\sigma = (\sigma_1, \sigma_2, \dots)$ of n, Ω consists of σ*-partitions*, which are ordered partitions of $\{1, 2, \dots, n\}$ into subsets of size $\sigma_1, \sigma_2, \dots$.

Martin and Sagan [MS06] show that a subset of the symmetric group is transitive on σ-partitions if and only if it is a T-design (where T consists of all partitions of n that are different from (n) and that *dominate* σ). A detailed exposition of these results is provided in Section 3.4. The characterisation of Martin and Sagan in the classical setting of the symmetric group does not only solve Problem 3 in the association scheme of the symmetric group, but also characterises algebraically the very combinatorial object of a transitive set. It is thanks to this characterisation that they generalise the famous Livingstone-Wagner theorem [LW65] on t-homogeneous subgroups of the symmetric group to subsets and σ-partitions.

Moreover, they give a construction and show the existence of small such designs, solving Problem 4 for the association scheme of the symmetric group.

In this thesis we study q-analog problems, replacing the symmetric group with the finite general linear group $\mathrm{GL}(n, q)$. In the q-analog setting we take Ω to be the set of flag-like structures, which can be seen as q-analogs of σ-partitions. More precisely, for an integer partition σ of n, a σ-flag is a sequence of subspaces $(V_1, V_2, \ldots)$ of $\mathbb{F}_q^n$ such that $\{0\} = V_0 \leq V_1 \leq V_2 \leq \ldots$ satisfying $\dim(V_i/V_{i-1}) = \sigma_i$ for each $i \geq 1$. In analogy with the classical case, we prove that certain Delsarte T-designs are also transitive sets, in this case, on flags.

Original contributions of this thesis - Part 2

The contribution of this thesis to algebraic combinatorics and group theory include structural characterisations and existence results of arbitrarily small transitive sets and T-designs in $\mathrm{GL}(n, q)$, respectively. One of the characterisation results for T-designs in $\mathrm{GL}(n, q)$ that we obtain is the following, see also Section 6.2.

Theorem 3. *For an integer partition σ of n, a subset of $\mathrm{GL}(n, q)$ is transitive on σ-flags if and only if it is a T-design.*

Our characterisation solves Problem 3 in the association scheme of $\mathrm{GL}(n, q)$ and not only provides a combinatorial interpretation for Delsarte T-designs in this scheme but also gives a structural algebraic characterisation of transitive subsets of $\mathrm{GL}(n, q)$. In particular, using the latter characterisation, we generalise a theorem of Perin [Per72] on subgroups of $\mathrm{GL}(n, q)$ acting transitively on t-dimensional subspaces to subsets and flags.

Our main results solve Problem 3 more generally: we consider subsets of $\mathrm{GL}(n, q)$ that are transitive on generalisations of t-dimensional subspaces and of bases of t-dimensional subspaces of $\mathbb{F}_q^n$, see Section 6.2.

Furthermore, in Section 6.5, we give a recursive construction and show the existence of small such T-designs, solving Problem 4 in the association scheme arising from $\mathrm{GL}(n, q)$. This is interesting because we are able to show that also in this q-analog setting, such designs somehow approximate $\mathrm{GL}(n, q)$.

Finally, as a byproduct of the aforementioned results, we obtain results on *codes* in $\mathrm{GL}(n, q)$ associated with the *rank distance*. The *rank distance* d_r between two matrices $x, y \in \mathrm{GL}(n, q)$ is the rank of their difference, namely $d_r(x, y) = \mathrm{rk}(x - y)$. A *code* in $\mathrm{GL}(n, q)$ with minimum (rank) distance d is a subset Y of $\mathrm{GL}(n, q)$ such that the minimum of the distances of two distinct elements of Y is d. As a byproduct of Theorem 6.4.4, we provide a sharp upper bound on the size of such a code and also give a construction, see Corollary 6.6.6 and Proposition 6.6.9. It is worth noticing that our bound is strictly lower than the Singleton bound for general rank-metric codes. Moreover, the bound provides a perfect q-analog of the well known bound for permutation codes [BCD79].

Our contributions to algebraic combinatorics are contained in Chapter 6 and were published in

[ES24] A. Ernst, K.-U. Schmidt, *Transitivity in finite general linear groups*, Math. Z., **307** 2024, no. 3.

Both authors of the paper *Transitivity in finite general linear groups* are first authors with equal rights and equally contributed to the development of the research questions.

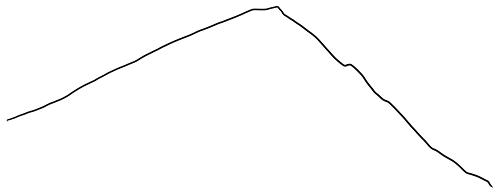
Outline of the thesis

The thesis is organised as follows. The first part consists of the Chapters 1 and 2. In Chapter 1 we summarise without proofs the representation theory of finite groups and the connection to the ring of symmetric functions, where we especially focus on the symmetric group. Then, in Chapter 2 we provide the background on the theory of association schemes required to understand the results and proofs presented later on.

The second part, Chapter 3, deals with the association scheme arising from the symmetric group. It gives an overview of known Erdős-Ko-Rado results, permutation codes, and transitive sets of permutations.

In the third part, including Chapters 4, 5, and 6, we study the q-analog problems of Chapter 3. More precisely, Chapter 4 provides necessary background such as representation theory and association schemes of the finite general linear group. In Chapter 5 our main results on EKR theorems in the finite general linear group are stated and proven. Moreover, we provide a collection of conjectures and open problems of interest for future research. Finally, in Chapter 6, we look more closely at different Delsarte T-designs in the association scheme arising from $\mathrm{GL}(n, q)$, give combinatorial interpretations, show (non-)existence results, and establish connections with orthogonal polynomials. We close this chapter with a collection of some related open problems.

1 Representation Theory



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In this chapter we briefly summarise without proofs the fundamental concepts of the representation theory of finite groups and give an overview of the properties that we need. Unless stated otherwise, the following definitions and statements are taken from [Sag01, Chapter 1]. We refer the reader to [Sag01] and [BI84] for more details.

In the first section we provide an overview of all necessary objects from representation theory for this thesis. In Section 1.2 we put the focus on the representation theory of the symmetric group and characterise its irreducible characters. Finally, in Section 1.3, we connect the irreducible characters of the symmetric group to the ring of *symmetric functions*.

1.1 Definitions and basic properties

From now on, let G be a finite group, and let V and W be nonzero finite dimensional vector spaces over the complex numbers $\mathbb{C}$. By $\mathrm{GL}(V)$ we denote the *complex general linear group* of the vector space V and by $\mathrm{GL}(n,\mathbb{C})$ the general linear group of $\mathbb{C}^n$, that is the group of all invertible $n \times n$ matrices over $\mathbb{C}$. Moreover, the symmetric group on n symbols is denoted by S_n.

We recall that a group G with identity element e *acts* on a set X if $gh.x = g.(h.x)$ and $e.x = x$ holds for all $x \in X$ and all group elements $g, h \in G$. In this thesis, *conjugation* is one of the most important group actions.

Example 1.1.1. The group G acts by conjugation on itself, that means $g.h = ghg^{-1}$ for all group elements $g, h \in G$. Every orbit of this action is called a *conjugacy class* of G. We call two elements g and h of G *conjugate* if they are contained in the same conjugacy class.

For determining the conjugacy classes of the symmetric group S_n we recall the *cycle type* of a permutation π of S_n: every permutation π can be uniquely written as a product of disjoint cycles $\pi = \sigma_1\sigma_2\cdots\sigma_k$, where the cycles σ_i are arranged in nonincreasing order regarding to their cycle length. Then the *cycle type* of π is a tuple given by the lengths of the cycles σ_i, namely $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|)$. The conjugacy classes of the symmetric group are characterised in the following way.

Example 1.1.2 ([DF91, Prop. 4.11]). Two permutations are conjugate if and only if they have the same cycle type.

In the following we denote by $\mathbb{C}X$ the inner product space of all complex-valued functions defined on a finite set X, namely $\mathbb{C}X = \{f | f\colon X \to \mathbb{C}\}$. The addition and scalar multiplication are given by

$$(f+g)(x) = f(x) + g(x), \quad (cf)(x) = c \cdot f(x)$$

and the inner product by

$$\langle f, g\rangle = \frac{1}{|X|}\sum_{y\in X} f(y)\overline{g(y)}$$

for all $f, g \in \mathbb{C}X$, $x \in X$, and $c \in \mathbb{C}$, where $\overline{g(y)}$ is the complex conjugate of $g(y)$. We note that we can interpret $\mathbb{C}X$ as the vector space consisting of all formal sums $\sum_{x\in X} c_x x$, where $c_x \in \mathbb{C}$.

Definition 1.1.3. A *representation* ψ of a group G is a group homomorphism

$$\Psi\colon G \to \mathrm{GL}(V).$$

The *degree* $\deg(\Psi)$ of the representation Ψ is the dimension of the underlying vector space V, namely $\deg(\Psi) = \dim_{\mathbb{C}}(V)$.

Example 1.1.4. The *trivial representation* of a group G sends every $g \in G$ to the (1×1)-matrix (1) and is a representation of degree one.

Example 1.1.5. Let G be a finite group acting on the finite set X. Then $\Xi\colon G \to \mathrm{GL}(\mathbb{C}X)$, given by

$$\Xi(g)(f(x)) = f(g^{-1}.x).$$

for all $g \in G$ and $x \in X$, is a representation of G that is called the *permutation representation* on X and it holds that $\deg(\Xi) = |X|$. In the case $X = G$ and G acts on itself via $g.h = gh$, the arising permutation representation is called the *left regular representation*.

Definition 1.1.6. Two representations $\Psi\colon G \to \mathrm{GL}(V)$ and $\Psi'\colon G \to \mathrm{GL}(W)$ are *equivalent* if there exists a vector space isomorphism $T\colon V \to W$ satisfying $\Psi'(g) = T\Psi(g)T^{-1}$ for all $g \in G$. In that case we write $\Psi \sim \Psi'$.

The definition of equivalent representations allows us to interpret representations in a different way. For this, we consider a representation $\Psi\colon G \to \mathrm{GL}(V)$ of G that has degree n. For a fixed basis $B = \{b_1, b_2, \dots, b_n\}$ of V we define by $T\colon V \to \mathbb{C}^n$, $v = \sum_{i=1}^n c_i b_i \mapsto (c_1, c_2, \dots, c_n)^T$ a vector space isomorphism. This yields a so-called *matrix representation* $\Psi'\colon G \to \mathrm{GL}(n, \mathbb{C})$ which is given by $\Psi'(g) = T\Psi(g)T^{-1}$ and is thus equivalent to Ψ. Consequently, we can think of Ψ and Ψ' to be the same representations.

Sometimes it can be useful to describe representations in terms of *modules*.

Definition 1.1.7. The space V is a *G-module*, if there exists a group action of G on V

$$G \times V \to V, \quad (g, v) \mapsto g.v$$

satisfying $g.(v + cw) = g.v + c(g.w)$ for all $g \in G$, $v, w \in V$ and all scalars $c \in \mathbb{C}$.

We will see that representations of a finite group are uniquely determined by their *characters*.

Definition 1.1.8. Let $\Psi\colon G \to \mathrm{GL}(V)$ be a representation of a group G. Then the *character* ψ of Ψ is given by

$$\psi\colon G \to \mathbb{C}, \quad \psi(g) = \mathrm{Tr}(\Psi(g)).$$

We note that the degree of a representation Ψ equals the evaluation of the corresponding character ψ at the identity element e of G, that is $\deg(\Psi) = \psi(e)$.

Example 1.1.9. The *trivial character* 1_G of a finite group G, which corresponds to the trivial representation from Example 1.1.4, sends every element of G to 1, that is $1_G(g) = 1$ for all $g \in G$.

Example 1.1.10. We consider any degree one representation $\Psi\colon G \to \mathbb{C}$. Then the character ψ of Ψ satisfies $\psi(gh) = \psi(g)\psi(h)$. These types of characters are called *linear characters* of G.

Example 1.1.11. Let Ξ be the permutation representation of G on the finite set X from Example 1.1.5. Then the character ξ of Ξ is called the *permutation character* of G on the set X and is given by

$$\xi\colon G \to \mathbb{C}, \quad \xi(g) = \#\{y \in X : g.y = y\}.$$

The permutation character plays a crucial role in this thesis. We study certain permutation characters of the finite general linear group in Section 4.3.

Since the trace is invariant under conjugation, characters have the following property.

Lemma 1.1.12. *Let ψ be a character corresponding to a representation of a finite group G. Then, for all $g, h \in G$ we have*

$$\psi(g) = \psi(h^{-1}gh).$$

This Lemma implies that characters are constant on the conjugacy classes of the underlying group. Consequently we can write $\psi(C)$ instead of $\psi(g)$, where C is the conjugacy class containing $g \in G$. Characters are an example of so-called *class functions*.

Definition 1.1.13. A function $f \in \mathbb{C}G$ is a *class function* if f is constant on the conjugacy classes of G, more precisely if, for all $g, h \in G$, the function f satisfies

$$f(g) = f(h^{-1}gh).$$

In fact, the set of class functions is a vector space, and we will see that the *irreducible* characters form an orthonormal basis of this space.

The following result gives the characterisation of equivalent representations in terms of their characters.

Theorem 1.1.14. *Two representations $\Phi \colon G \to \mathrm{GL}(V)$ and $\Psi \colon G \to \mathrm{GL}(W)$ with characters φ and ψ, respectively, are equivalent if and only if $\varphi = \psi$.*

A well known task in representation theory is to decompose a given representation or character into so called *irreducible* constituents. The definition of *irreducibility* requires some preliminary preparation.

Definition 1.1.15. Let $\Phi \colon G \to \mathrm{GL}(V)$ be a representation of G. A subspace $U \subseteq V$ is a *G-invariant subspace* of V if, for all $g \in G$, it holds that $\Phi(g)(U) \subseteq U$.

We note that for a given representation $\Phi \colon G \to \mathrm{GL}(V)$ the trivial subspaces $\{0\}$ and V are G-invariant.

Definition 1.1.16. The *(external) direct sum* $\Phi \oplus \Psi$ of two representations Φ and Ψ of G on V and W, respectively, is defined by

$$\Phi \oplus \Psi \colon G \to \mathrm{GL}(V \oplus W), \quad (\Phi \oplus \Psi)(g)(v, w) = (\Phi(g)(v), \Psi(g)(w))$$

for all $g \in G$ and all $(v, w) \in V \oplus W$.

Lemma 1.1.17. *Let Φ and Ψ be representations of G with characters φ and ψ, respectively. Then the character of $\Phi \oplus \Psi$ is $\varphi + \psi$.*

Lemma 1.1.18. *Let $\Phi\colon G \to \mathrm{GL}(V)$ be a representation and let $U \leq V$ be a G-invariant subspace of V. Then the restriction $\Phi|_U\colon G \mapsto \mathrm{GL}(U)$ given by*

$$(\Phi|_U)(g)(u) = \Phi(g)(u) \in U,$$

for all $g \in G$ and $u \in U$, is a representation of G.

Lemma 1.1.19. *Let $\Phi\colon G \to \mathrm{GL}(V)$ be a representation and let $V = U_1 \oplus U_2$, where U_1 and U_2 are G-invariant subspaces of V. Then the representations Φ and $\Phi|_{U_1} \oplus \Phi|_{U_2}$ are equivalent.*

From Lemma 1.1.19 it follows that if the vector space corresponding to a given representation has two G-invariant subspaces, then the representation consists of the sum of at least two subrepresentations. This observation motivates the following definition of *irreducibility*.

Definition 1.1.20. A nonzero representation $\Phi\colon G \to \mathrm{GL}(V)$ is *irreducible* if V does not contain any nontrivial G-invariant subspace. The character of an irreducible representation is called an *irreducible character*. A representation is *completely reducible* if it is a direct sum of irreducible representations.

Example 1.1.21. The trivial representation and more general every representation of degree 1 is irreducible because the only complex subspaces of $\mathbb{C}$ are $\{0\}$ and $\mathbb{C}$.

Lemma 1.1.22. *Let Φ be a representation of G.*

(i) If Φ is equivalent to an irreducible representation, then Φ is irreducible as well.

(ii) If Φ is equivalent to a completely reducible representation, then Φ is completely reducible as well.

Theorem 1.1.23 (Maschke). *Every representation of a finite group is completely reducible.*

Definition 1.1.24. The *character table* of G is a table whose rows are indexed by the irreducible characters, whose columns are indexed by the conjugacy classes of G, and whose entry corresponding to the character χ and the conjugacy class C is the evaluation $\chi(C)$.

For example, the character table of the symmetric group S_3 is given in Table 1.1.

Table 1.1: The character table of S_3.

	$C_{(1^3)}$	$C_{(2,1)}$	$C_{(3)}$
$\chi^{(3)}$	1	1	1
$\chi^{(2,1)}$	2	0	-1
$\chi^{(1,1,1)}$	1	-1	1

In Section 1.2 we explain why not only the conjugacy classes but also the irreducible characters in Table 1.1 are indexed by partitions.

Definition 1.1.25. Let $\langle \cdot, \cdot \rangle$ be an inner product on the vector space V. A representation $\Phi\colon G \to \mathrm{GL}(V)$ is *unitary* if, for all $g \in G$ and all $v, w \in V$ it holds that

$$\langle \Phi(g)(v), \Phi(g)(w) \rangle = \langle v, w \rangle,$$

In other words, Φ is a unitary representation, if $\Phi(g)$ is unitary with respect to $\langle \cdot, \cdot \rangle$ for all $g \in G$.

Lemma 1.1.26. *Every representation of a finite group is equivalent to a unitary representation.*

Theorem 1.1.27 (Schur's Lemma). *Let $\Phi\colon G \to \mathrm{GL}(V)$ and $\Psi\colon G \to \mathrm{GL}(W)$ be representations of G and let $T\colon V \to W$ be a homomorphism satisfying $T\Phi(g) = \Psi(g)T$ for all $g \in G$. Then T is invertible or $T = 0$. Moreover, if $\Phi = \Psi$, then T is a multiple of the identity.*

Henceforth, we write $m\Phi = \Phi \oplus \ldots \oplus \Phi$ for the direct sum of the representation Φ of G taken m times. Due to Maschke's Theorem 1.1.23 and the fact that G only has finitely many inequivalent irreducible representations, which will be part of Lemma 1.1.31, Φ is equivalent to a sum of irreducible representations

$$\Phi \sim m_1\Phi^1 \oplus m_2\Phi^2 \oplus \ldots \oplus m_n\Phi^n, \tag{1.1}$$

where $\Phi^1, \Phi^2, \ldots, \Phi^n$ are the inequivalent irreducible representations of G, and $m_i = 0$ means that the representation Φ^i does not occur in the decomposition.

Definition 1.1.28. The nonnegative integer m_i in the decomposition (1.1) of Φ is called *multiplicity* of Φ. If $m_i > 0$, then Φ^i is an *irreducible constituent* of Φ.

In the following we determine the multiplicities in the decomposition of a given representation and character, respectively.

Since the characters of a finite group G are elements of the space $\mathbb{C}G$, we recall from the beginning of this section that the inner product of two characters φ and ψ of G is given by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

In fact, the irreducible characters of a group are orthonormal with respect to this inner product.

Lemma 1.1.29 (First orthogonality relations). *Let φ and ψ be two irreducible characters of G. Then we have*

$$\langle \varphi, \psi \rangle = \begin{cases} 1 & \textit{for } \varphi = \psi, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 1.1.30. *The irreducible characters of a finite group form an orthonormal basis of the space of class functions.*

Lemma 1.1.31. *The number of inequivalent irreducible representations of a finite group G equals the number of conjugacy classes of G. In particular, this number is finite.*

Theorem 1.1.32. *Let Φ be a representation of G, with corresponding character φ. Assume that Φ is equivalent to $m_1\Phi^1 \oplus m_2\Phi^2 \oplus \ldots \oplus m_n\Phi^n$, where the m_i are nonnegative and $\Phi^1, \Phi^2, \ldots, \Phi^n$ denote the inequivalent irreducible representations of G with characters $\varphi^1, \varphi^2, \ldots, \varphi^n$, respectively. Then we have*

(i) $\varphi = \sum_i m_i \varphi^i$,

(ii) $m_i = \langle \varphi, \varphi^i \rangle$.

From Theorem 1.1.32 we find that the decomposition of a representation of G into irreducible constituents is unique up to equivalence. Moreover, the representation is determined up to equivalence by its character.

Additionally, Theorem 1.1.32 together with the first orthogonality relations from Lemma 1.1.29 imply a characterisation of irreducible representations in terms of their characters.

Corollary 1.1.33. *A representation of a finite group is irreducible if and only if its character φ satisfies*

$$\langle \varphi, \varphi \rangle = 1.$$

Lemma 1.1.18 implies that we can sometimes restrict a given representation $\Psi\colon G \to \mathrm{GL}(V)$ to a subspace $U \leq V$ and obtain again a representation. It is also possible to restrict a given representation of G to a subgroup of G. Since, in this thesis, we almost exclusively work with restricted characters instead of representations, we state the following definitions and properties in terms of characters.

Definition 1.1.34. The *restricted character* $\mathrm{Res}_H^G(\varphi)$ of a given character φ of G to a subgroup $H \leq G$, is defined as

$$\mathrm{Res}_H^G(\varphi)\colon H \to \mathbb{C}, \quad h \mapsto \varphi(h).$$

Lemma 1.1.35. *If φ is a character of G, then the restriction $\mathrm{Res}_H^G(\varphi)$ to a subgroup H of G is a character of H.*

It is worth mentioning that the concept of irreducibility is not preserved when restricting a character to a subgroup.

We can also do the opposite and *induce* a character from a subgroup of G to G itself in order to obtain a character of G.

Definition 1.1.36. Let φ be a character of a subgroup H of G. Then the *induced character* $\mathrm{Ind}_H^G(\varphi)$ is given by

$$\mathrm{Ind}_H^G(\varphi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G\colon \\ x^{-1}gx \in H}} \varphi(x^{-1}gx).$$

Lemma 1.1.37. *Let φ be a character of $H \leq G$. Then the induced character $\mathrm{Ind}_H^G(\varphi)$ is a character of G.*

Like for a restricted character, the concept of irreducibility is not preserved when inducing a character.

Theorem 1.1.38 (Frobenius reciprocity). *Let H be a subgroup of G and let φ and ψ be characters of H and G, respectively. Then we have*

$$\langle \mathrm{Ind}_H^G(\varphi), \psi \rangle_G = \langle \varphi, \mathrm{Res}_H^G(\psi) \rangle_H,$$

where $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle_H$ are the usual inner products on $\mathbb{C}G$ and $\mathbb{C}H$, respectively.

The Frobenius reciprocity can be useful to decompose a character.

Example 1.1.39. Let $E = \{e\}$ be the trivial subgroup of G and 1_E the trivial character of E. Using the Frobenius reciprocity and Theorem 1.1.32 we can determine the multiplicities of an irreducible character φ of G in the decomposition of the induced character $\operatorname{Ind}_E^G(1_E)$, namely

$$\langle \operatorname{Ind}_E^G(1_E), \varphi \rangle_G = \langle 1_E, \operatorname{Res}_E^G(\varphi) \rangle_E = \varphi(e).$$

Consequently, the decomposition of $\operatorname{Ind}_E^G(1_E)$ is given by

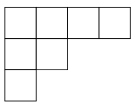
$$\operatorname{Ind}_E^G(1_E) = \sum_{\varphi} \varphi(e)\varphi,$$

where the sum is taken over all irreducible characters of G.

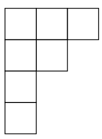
1.2 Symmetric groups

In this section we give an overview of the representation theory of the symmetric group. We determine the irreducible characters, the *Specht characters*, and discuss the decomposition of a certain permutation character of the symmetric group. For more details on the representation theory of the symmetric group we refer to [Sag01].

An (integer) *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers that sum up to n and that satisfy $\lambda_1 \geq \lambda_2 \geq \ldots$. The λ_i are called *parts* of the partition λ. If λ is a partition of n we write $\lambda \vdash n$. The *size* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ and its *length* $\ell(\lambda)$ is the largest index i such that $\lambda_i > 0$. Instead of writing $(\lambda_1, \lambda_2, \ldots)$, we often use $(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$.
We can illustrate an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with a *Ferrers diagram*, which is an array of $|\lambda|$ boxes with left-justified rows and top-justified columns such that row i contains exactly λ_i boxes.
For example the Ferrers diagram of the partition $(4, 2, 1)$ is



For each partition λ there exists the *conjugate* partition λ' whose parts are the number of boxes in the columns of λ. For example the conjugate of the partition $(4, 2, 1)$ is the partition $(3, 2, 1, 1)$



The *dominance order* is a partial order on the set of partitions. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be two partitions of an integer n. Then we say that λ *dominates* μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \text{for each } k \geq 1. \tag{1.2}$$

For example $(4, 2)$ dominates $(4, 1, 1)$ but neither $(3, 3)$ dominates $(4, 1, 1)$ nor does $(4, 1, 1)$ dominate $(3, 3)$. Figure 1.1 illustrates the dominance order in terms of Ferrers diagrams.

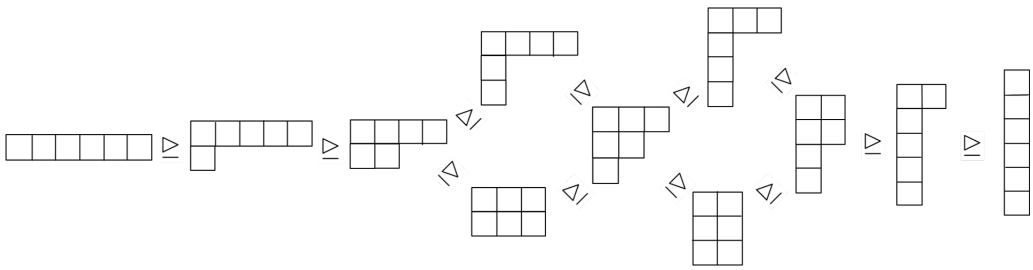


Figure 1.1: The dominance order for the partitions of 6.

For a partition λ, we obtain a λ-*tableau* by filling the boxes of the Ferrers diagram of λ with the integers $1, 2, \ldots, |\lambda|$, each integer appearing exactly once. For example

1	4	6	2
7	3		
5			

is a $(4, 2, 1)$-tableau.

A λ-*tabloid* is an ordered partition of the set $\{1, 2, \ldots, n\}$ into subsets of cardinality $\lambda_1, \lambda_2, \ldots$. We note that the number of different $(\lambda_1, \lambda_2, \ldots, \lambda_k)$-tabloids is $n!/(\lambda_1!\lambda_2! \cdots \lambda_k!)$. Moreover, the symmetric group acts naturally on the set of λ-tabloids.

We use the permutation character of the symmetric group on tabloids to obtain all the irreducible characters of this group. Recall, from Example 1.1.11, that the evaluation of the permutation character ξ^μ of S_n on μ-tabloids at $\pi \in S_n$ is

$$\xi^\mu(\pi) = \#\{\mu\text{-tabloids fixed by } \pi\}.$$

Remark 1.2.1. Let H be the stabiliser of a μ-tabloid T and 1_H the trivial character of H. Then, for a permutation π, we have the following

$$\begin{aligned}
\mathrm{Ind}_H^{S_n}(1_H)(\pi) &= \frac{1}{|H|} \sum_{\substack{\sigma \in S_n\colon \\ \sigma^{-1}\pi\sigma \in H}} 1_H(\sigma^{-1}\pi\sigma) \\
&= \frac{1}{|H|} \#\{\sigma \in S_n \colon \sigma^{-1}\pi\sigma(T) = T\} \\
&= \frac{1}{|H|} \#\{\sigma \in S_n \colon \pi\sigma(T) = \sigma(T)\}.
\end{aligned}$$

We fix permutations $\tau_1, \tau_2, \ldots, \tau_\ell$ of S_n such that $S_n = \dot{\cup}_i \tau_i H$, where $\dot{\cup}$ denotes the disjoint union. Then we get

$$\begin{aligned}\operatorname{Ind}_H^{S_n}(1_H)(\pi) &= \#\{i \in \{1,2,\ldots,\ell\} \colon \pi\tau_i(T) = \tau_i(T)\}\\ &= \#\{\mu\text{-tabloids fixed by } \pi\}\\ &= \xi^\mu(\pi).\end{aligned}$$

Consequently, the permutation character ξ^μ is equal to the induced character $\operatorname{Ind}_H^{S_n}(1_H)$.

Definition 1.2.2. Let λ and μ be partitions. A *generalised tableau of shape λ and content μ* is an array obtained by filling the boxes of a Ferrers diagram with positive integers i, such that i occurs exactly μ_i times.

For example

1	2	3
2	1	
4		

is a generalised tableau of shape $(3, 2, 1)$ and content $(2, 2, 1, 1)$.

Definition 1.2.3. A *semistandard tableau* is a generalised tableau whose rows are weakly and whose columns are strictly increasing.

For example

1	1	2
2	3	
4		

is a semistandard $(3, 2, 1)$ tableau with content $(2, 2, 1, 1)$.

Definition 1.2.4. The *Kostka number $K_{\lambda\mu}$* is the number of semistandard λ-tableaux with content μ.

Theorem 1.2.5 ([Mac15]). *The Kostka numbers $K_{\lambda\mu}$ satisfy*

$$K_{\lambda\lambda} = 1 \text{ and } K_{\lambda\mu} \neq 0 \Rightarrow \lambda \trianglerighteq \mu \tag{1.3}$$

Table 1.2 provides some values of the Kostka numbers.

In the following we introduce the irreducible characters of the symmetric group in an indirect way, which later allows us to emphasise some analogies with the irreducible characters of the finite general linear group. It is well known (see [Sag01, Chapter 2],

Table 1.2: The Kostka numbers $K_{\lambda\mu}$ for λ and μ being partitions of 3.

	(3)	$(2,1)$	$(1,1,1)$
(3)	1	1	1
$(2,1)$	0	1	2
$(1,1,1)$	0	0	1

for example) that using the permutation characters on tabloids, we get all irreducible characters of the symmetric group by a recursive construction.

The *first* permutation character $\xi^{(n)}$ of S_n is irreducible because it is the trivial character. We name it $\chi^{(n)}$. The *next* irreducible character arises from $\xi^{(n-1,1)}$, since $\xi^{(n-1,1)}$ can be decomposed into irreducible characters containing copies of $\chi^{(n)}$ and a single copy of the new irreducible character $\chi^{(n-1,1)}$. We can continue that way and obtain all irreducible characters of S_n. More precisely, we define the characters χ^λ recursively in the following way

$$\chi^\mu = \xi^\mu - \sum_{\lambda \rhd \mu} K_{\lambda\mu}\chi^\lambda,$$

where the $K_{\lambda\mu}$ are the Kostka numbers.

Furthermore, it is well known (see [Sag01, Chapter 2] for example), that for every partition λ of n, these characters χ^λ, called *Specht characters*, are irreducible characters of the symmetric group S_n. Moreover, all irreducible characters of S_n are given by χ^λ with λ being a partition of n.

The parametrisation of the irreducible characters of the symmetric group we use is standard and consistent with the literature.

Moreover, the decomposition of the permutation character ξ^μ is as follows.

Theorem 1.2.6 (Young's rule). *Let μ be a partition of n and let ξ^μ be the permutation character of S_n on μ-tabloids. The decomposition of ξ^μ into irreducibles χ^λ is given by*

$$\xi^\mu = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu}\chi^\lambda,$$

where the $K_{\lambda\mu}$ denote the Kostka numbers.

Remark 1.2.7. We note that our approach of introducing the irreducible characters of the symmetric group does not explain why they are irreducible and it does make Young's rule look like an obvious consequence. We point out that, in fact, proving the irreducibility and Young's rule is a highly non-trivial task.

Using the character table of S_3 from Table 1.1 and the Kostka numbers from Table 1.2 we get the following illustration of Young's rule in terms of a matrix multiplication.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

From Young's rule together with the use of linear algebra it follows that we can express the irreducible characters in terms of the permutation characters on tabloids.

Theorem 1.2.8. *Let χ^λ be the irreducible character of S_n corresponding to the partition λ. There are integers $H_{\mu\lambda}$ satisfying*

$$\chi^\lambda = \sum_\mu H_{\mu\lambda} \xi^\mu, \tag{1.4}$$

where ξ^μ is the permutation character of S_n on μ-tabloids, and

$$H_{\lambda\lambda} = 1 \text{ and } H_{\mu\lambda} \neq 0 \Rightarrow \mu \trianglerighteq \lambda. \tag{1.5}$$

We note that from (1.4) together with the definition of the permutation character ξ^μ, it follows that the irreducible characters of the symmetric group are real-valued.

1.3 The ring of symmetric functions

In this section we give an overview of the connections between representations of the symmetric group and the ring of symmetric functions. In fact there exists a bijection between symmetric functions and class functions of symmetric groups. We follow [Sag01, Chapter 4] and, unless stated otherwise, all results stated in this section can be found there. We also refer to [Mac15, I] for a more algebraic background on symmetric functions.

Definition 1.3.1. Let $x_1, x_2, \dots$ be infinitely many variables and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. Then the *monomial symmetric function* m_λ corresponding to λ is given by

$$m_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell},$$

where the sum is taken over all distinct monomials having exponents $\lambda_1, \lambda_2, \dots, \lambda_\ell$.

For example,

$$m_{(3,1)} = x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3 + \dots.$$

We define the complex vector space Λ^n as the span of all monomial symmetric functions of degree n, more precisely

$$\Lambda^n = \langle m_\lambda \colon \lambda \vdash n \rangle_{\mathbb{C}}. \tag{1.6}$$

In fact the monomial symmetric functions m_λ, where λ is a partition of n, are linearly independent.

Proposition 1.3.2. *The set of monomial symmetric functions $\{m_\lambda \colon \lambda \vdash n\}$ forms a basis of Λ^n. Consequently the dimension of Λ^n is equal to the number of partitions of n.*

Definition 1.3.3. The *ring of symmetric functions* Λ is

$$\Lambda = \bigoplus_{n \geq 0} \Lambda^n. \tag{1.7}$$

Since the decomposition in (1.7) is direct, Λ is a graded ring.

There is an action of the symmetric group on the functions in Λ, namely, for $f \in \Lambda$ and $\pi \in S_n$,

$$\pi f(x_1, x_2, x_3, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \dots), \tag{1.8}$$

where we set $\pi(i) = i$ for all $i > n$.

Since the monomial symmetric functions are invariant under the action (1.8), it is justified to call them *symmetric*.

Besides the monomial symmetric functions there are other functions which are invariant under the action (1.8).

Definition 1.3.4. The *nth power sum symmetric function* p_n is given by $p_n = m_{(n)}$. And for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ the *power sum symmetric function* p_λ corresponding to λ is $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$.

Theorem 1.3.5. *The set of power sum symmetric functions $\{p_\lambda \colon \lambda \vdash n\}$ forms a basis of Λ^n.*

In addition to the monomial symmetric functions and the power sum symmetric functions, other bases for Λ^n are provided by the *elementary symmetric functions*, the *complete homogeneous symmetric functions*, and the *Schur functions*. The latter are important in this thesis. For more background on all the other mentioned symmetric functions we refer to [Sag01, Section 4.4] or [Mac15, Ch. I.2].

In order to define the *Schur functions* we use a combinatorial approach. To do so, we need the notion of a *composition*, that is much like a partition.

Definition 1.3.6. A *composition* of a nonnegative integer n is a sequence $\mu = (\mu_1, \mu_2, \ldots)$ of nonnegative integers that sum up to n. The length $\ell(\mu)$ of a composition is defined as it is for partitions, and we often write $(\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$ instead of $(\mu_1, \mu_2, \ldots)$.

For example $(2, 4, 1)$ is a composition of 7 but not a partition. For a composition μ, the definition of a semistandard λ-tableau with content μ is as expected.

Definition 1.3.7. For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, the corresponding monomial weight x^μ is defined by

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \ldots x_\ell^{\mu_\ell}.$$

Let T be a semistandard λ-tableau with content μ. Then we define a monomial x^T, representing the *weight* of T by

$$x^T = x^\mu.$$

For example, if T is equal to

1	1	4
3	4	

,

then $x^T = x_1^2 x_3 x_4^2$.

Definition 1.3.8. The *Schur function* s_λ associated with the partition λ is defined by

$$s_\lambda = \sum_T x^T,$$

where the sum is taken over all semistandard λ-tableaux T.

Even though we defined the Kostka numbers $K_{\lambda\mu}$ only for partitions (see Definition 1.2.4), the definition can be generalised for arbitrary compositions μ. For more details we refer to [Sag01]. Then, for any partition λ of n, we have the following identity

$$s_\lambda = \sum_\mu K_{\lambda\mu} x^\mu,$$

where the sum is taken over all compositions μ of n. From this it follows that the Schur functions are indeed symmetric functions.

Using Equation (1.3), we can write the Schur functions in terms of the monomial symmetric functions.

Lemma 1.3.9. *It holds that*

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_\mu,$$

where the sum is taken over all partitions μ being dominated by λ.

From that and $K_{\lambda\lambda} = 1$, we get the following.

Theorem 1.3.10. *The set* $\{s_\lambda \colon \lambda \vdash n\}$ *of Schur functions forms a basis of* Λ^n*.*

Moreover, from Lemma 1.3.9, it follows that the transition matrix of the change of basis from the Schur functions to the monomial symmetric functions is given by the Kostka numbers. Two other interesting transition matrices are coming from the change of basis from the power sum symmetric functions to the monomial symmetric functions, and from the Schur functions to the power sum symmetric functions.

Theorem 1.3.11. *Let* ξ^μ_λ *be the permutation character of* S_n *on* μ*-tabloids evaluated on the conjugacy class corresponding to* λ*. Then*

$$p_\lambda = \sum_{\mu \trianglerighteq \lambda} \xi^\mu_\lambda m_\mu.$$

Consequently, the transition matrix of power sum symmetric functions and monomial symmetric functions arises from the permutation character on tabloids.

Theorem 1.3.12. *Let* χ^λ_μ *be the irreducible character of* S_n *associated with the partition* λ *and evaluated on the conjugacy class* C_μ *corresponding to* μ*. Then*

$$s_\lambda = \frac{1}{n!} \sum_{\mu \vdash n} |C_\mu| \chi^\lambda_\mu p_\mu.$$

Consequently, the scaled character table of the symmetric group gives the transition matrix of Schur functions and power sum symmetric functions.

In the following we focus more on the connection between Schur functions and irreducible characters of the symmetric group. In fact there exists a bijection between Λ^n and the space of class functions of the symmetric group S_n.

We define an inner product on Λ^n by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

and sesquilinear extension, which is sufficient since the Schur functions form a basis of Λ^n.

Definition 1.3.13. Let R^n denote the space of class functions of S_n. The *characteristic map* ch^n is defined by

$$\mathrm{ch}^n \colon R^n \to \Lambda^n, \quad \mathrm{ch}^n(\chi^\lambda) = s_\lambda,$$

where χ^λ and s_λ are the irreducible characters of S_n and the Schur functions, respectively.

We emphasise that it is sufficient to define the map ch^n in terms of the irreducible characters χ^λ since they form a basis of R^n.

Lemma 1.3.14. *The characteristic map* ch^n *is an isometry between* R^n *and* Λ^n*.*

Now, for $R = \bigoplus_n R^n$ and $\Lambda = \bigoplus_n \Lambda^n$, Theorem 1.3.15 shows that $\mathrm{ch} = \bigoplus_n \mathrm{ch}^n$ is an isomorphism of algebras. The tensor product $\otimes$ of two class functions ϕ and ψ of S_n and S_m, respectively, is a class function of $S_n \times S_m$ and is by definition

$$\phi \otimes \psi \colon S_n \times S_m \mapsto \mathbb{C}, \quad (\phi \otimes \psi)(\pi, \sigma) = \phi(\pi) \cdot \psi(\sigma).$$

The generalisation of an induced class function is defined in exactly the same way as an induced character (see Definition 1.1.36). We define a product on class functions ϕ and ψ of S_n and S_m, respectively, in the following way

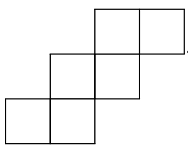
$$\phi \odot \psi = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\phi \otimes \psi). \tag{1.9}$$

Theorem 1.3.15. *The map* $\mathrm{ch} \colon R \to \Lambda$ *is an isomorphism of algebras.*

With the help of the characteristic map ch we can decompose products of irreducible characters. For this we need the notion of a *skew-tableau.*

Definition 1.3.16. Let n and m be nonnegative integers and let λ and μ be partitions of m and $n+m$, respectively. The partition λ is *contained* in μ and we write $\lambda \subseteq \mu$ if the Ferrers diagram of λ is contained in the Ferrers diagram of μ, more precisely if $\lambda_i \leq \mu_i$ for all i. This partial order is called *containment.* For partitions $\lambda \subseteq \mu$, the *skew diagram* μ/λ is the set difference of μ and λ and contains exactly n cells.

For example, for $\lambda = (2,1)$ and $\mu = (4,3,2)$, we have $\lambda \subseteq \mu$ and the skew diagram μ/λ looks as follows

.

We note that $\subseteq$ is a partial order on the set of all partitions.

The definitions of a *skew tableau* and a *semistandard* skew-tableau are as expected.

Definition 1.3.17. Let T be a skew tableau of shape μ/λ. The *word* $w(T)$ derived from T is a sequence obtained by reading the symbols in T from right to left in successive rows, starting with the top row. The word $w(T) = i_1 i_2 \dots i_M$ in the symbols $1, 2, \dots m$ is a *lattice permutation* if for any $1 \leq r \leq M$ and any positive integer $1 \leq \ell \leq m-1$, the number of occurrences of ℓ's in $i_1 i_2 \dots i_r$ is at least as large as the number of occurrences of $(\ell+1)$'s.

As an example, the word associated with the following skew tableau

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ 2 & 3 & & \end{array}$$

is 112132 and is a lattice permutation.

Theorem 1.3.18 (Littlewood-Richardson Rule). *Let χ^λ and χ^ν be irreducible characters of S_n and S_m, respectively. Then the product $\chi^\lambda \odot \chi^\nu$ decomposes into irreducible constituents as follows*

$$\chi^\lambda \odot \chi^\nu = \sum_\mu c^\mu_{\lambda\nu} \chi^\mu,$$

where the Littlewood-Richardson coefficient $c^\mu_{\lambda\nu}$ *is the number of semistandard skew-tableaux T of shape μ/λ and content ν such that the word $w(T)$ is a lattice permutation. Since the Littlewood-Richardson coefficients are nonnegative integers, the product $\chi^\lambda \odot \chi^\nu$ is indeed a character of S_{n+m}.*

We note that $c^\mu_{\lambda\nu} = 0$ unless $|\mu| = |\lambda| + |\nu|$ and $\lambda, \nu \subseteq \mu$.

In the case of $\nu = (m)$ the irreducible character χ^ν is just the trivial character of S_m and the Littlewood-Richardson rule reduces to the so-called *Pieri's rule*.

Corollary 1.3.19 (Pieri's rule). *Let χ^λ and $\chi^{(m)}$ be irreducible characters of S_n and S_m, respectively. Then $\chi^\lambda \odot \chi^{(m)}$ decomposes into irreducible constituents as follows*

$$\chi^\lambda \odot \chi^{(m)} = \sum_\mu \chi^\mu$$

where μ runs through all partitions whose Ferrers diagram is obtained from λ by adding m boxes, no two of which are in the same column.

2 Association schemes

Dinara, 1831m

In this chapter we introduce and collect some notions about *association schemes*.

The theory of *association schemes*, in the words of Bannai and Ito [BI84], "is very fundamental, perhaps the most important, in algebraic combinatorics". *Association schemes* provide a unified and underlying framework for studying coding and design theory. One of the highlights of the theory is its versatility in applying a wide range of algebraic tools, such as eigenvalue techniques and representation theory, as well as graph theory and linear programming. For this reason, *association schemes* are themselves a powerful tool to solve (extremal) combinatorial problems.

It is indeed often possible to interpret combinatorial objects as subsets within certain *association schemes*, which then allows to utilise algebraic or linear programming methods to uncover new characterisations and properties of these combinatorial objects.

First introduced in the 1930s in the context of statistics, *association schemes* were formally recognised as an independent research subject through the work of Bose and Shimamoto in [BS52] in the 1950s. However, it was not until the 1970s that Delsarte demonstrated the theory's profound significance. In his monumental thesis [Del73], Delsarte used *association schemes* to unify the study of combinatorial designs and error-correcting codes, revealing them as two facets in the same broader theoretical framework. This establishes association schemes as an object of fundamental importance in both studies[1].

We provide a brief introduction to the theory of *association schemes* in Section 2.1, including definitions, examples and some basic properties. In Section 2.2 we characterise subsets of *association schemes* in terms of their *inner* and *dual distributions* leading to so-called D*-cliques* and T*-designs*. In Sections 2.3 and 2.4 we collect methods for

[1]The impact is also visible in the fact that Delsarte's thesis has been cited over 2000 times according to Google Scholar as of 2024

studying the sizes of D-*cliques* in association schemes. This includes a graph-theoretical approach involving the *Hoffman bound* and its weighted version, as well as the *linear programming method* introduced by Delsarte. Moreover, we use these methods to analyse special D-*cliques* associated with *intersecting families* of sets.

2.1 Definitions, examples, and properties

In this section we introduce *association schemes* and summarise some of their properties, with a particular focus on the *conjugacy class scheme*. For more details and for proofs we refer to [Del73], [BI84], and [GM16].

From now on, unless stated otherwise, let X be a finite set with at least two elements.

Definition 2.1.1. An *association scheme* with n classes is a pair $(X, \{R_0, R_1, \dots, R_n\})$, where $R_0, R_1, \dots, R_n$ are non-empty relations on $X \times X$ with the following properties.

(A1) The relations $R_0, R_1, \dots, R_n$ partition $X \times X$ and $R_0 = \{(x, x) \colon x \in X\}$.

(A2) For all i, we have $R_i^T \in \{R_0, R_1, \dots, R_n\}$, where $R_i^T = \{(y, x) \colon (x, y) \in R_i\}$.

(A3) For every $x, y \in X$ with $(x, y) \in R_k$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant only depending on i, j, k, more precisely

$$\#\{z \in X \colon (x, z) \in R_i \text{ and } (z, y) \in R_j\} = p_{ij}^k$$

for all i, j, k.

(A4) For all i, j, k, we have $p_{ij}^k = p_{ji}^k$.

The constants p_{ij}^k are called *intersection numbers*. If we have $R_i^T = R_i$ for all i, then the association scheme is *symmetric*. For an $x \in R_i$, we define the *valency* v_i of the relation R_i by

$$v_i = \#\{y \in X \colon (x, y) \in R_i\}.$$

We note that the valencies v_i are well-defined because, for a pair (i, j) such that $R_i^T = R_j$, we have

$$\#\{y \in X \colon (x, y) \in R_i\} = \#\{y \in X \colon (x, y) \in R_i, (y, x) \in R_j\} = p_{ij}^0,$$

so that $v_i = p_{ij}^0$.

Before focusing on properties of association schemes, we give some examples.

Example 2.1.2 (Johnson scheme). Let k and n be nonnegative integers satisfying $n \geq 2k$, and let X be the set of all subsets of $[n] = \{1, 2, \dots, n\}$ having k elements. Hence we have $|X| = \binom{n}{k}$. Let R_i be given by

$$R_i = \{(x, y) \in X \times X \colon |x \cap y| = k - i\}.$$

Then $(X, \{R_0, R_1, \dots, R_k\})$ defines a symmetric association scheme with k classes. This association scheme is called the *Johnson scheme* and is denoted by $J(k, n)$. From a counting argument it follows that the valencies v_i are given by

$$v_i = \binom{k}{k-i}\binom{n-k}{i} = \binom{k}{i}\binom{n-k}{i}.$$

We can extend the concept of classical combinatorics of sets to combinatorics of vector spaces over the finite field $\mathbb{F}_q$. It is well known that the first can be seen as the limiting case $q \to 1$ of the latter, where sets are replaced by vector spaces over $\mathbb{F}_q$ and cardinality is replaced by dimension. The combinatorics of vector spaces over $\mathbb{F}_q$ is typically referred to as *q-analogs* of the classical cases.

The q-analog of the usual binomial coefficient is the *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and is the number of k-dimensional subspaces of $\mathbb{F}_q^n$, namely

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}. \tag{2.1}$$

In terms of the *q-factorial* of a nonnegative integer m given by

$$[m]_q! = [m]_q[m-1]_q \cdots [1]_q \quad \text{with } [\ell]_q = \frac{q^\ell - 1}{q - 1}, \tag{2.2}$$

it holds that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Example 2.1.3 (q-Johnson scheme). Let k and n be nonnegative integers with $n \geq 2k$. Let X be the set of all k-dimensional subspaces of $\mathbb{F}_q^n$ and let

$$R_i = \{(x, y) \in X \times X \colon \dim(x \cap y) = k - i\}.$$

Then $(X, \{R_0, R_1, \dots, R_k\})$ is a symmetric association scheme with k classes, which is called the *q-Johnson scheme* or *Grassmann scheme* and it is denoted by $J_q(k, n)$. The valencies v_i are given by

$$v_i = q^{i^2} \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} n-k \\ i \end{bmatrix}_q.$$

Another class of association schemes arises from finite groups, namely the *conjugacy class schemes*, which plays a crucial role in this thesis.

Example 2.1.4 (Conjugacy class scheme). Let G be a finite group and let all its conjugacy classes be $C_0 = \{e\}, C_1, \dots, C_n$. We define the relations by

$$R_i = \{(g,h) \in G \times G \colon g^{-1}h \in C_i\}.$$

Then $(G, \{R_0, R_1, \dots, R_n\})$ is an association scheme with n classes that is called *conjugacy class scheme* arising from the finite group G. For the valencies v_i, we have $v_i = |C_i|$.

The two groups, which are of great importance in this thesis, are the symmetric group and its q-analog the finite general linear group. In the Chapters 3, 5, and 6 we interpret certain combinatorial objects (transitive and intersecting sets) of the symmetric group and of the finite general linear group as subsets of the corresponding conjugacy class schemes arising from these groups. This allows us to describe further (algebraic) properties of these objects. Due to the fact that we are working almost exclusively with the conjugacy class scheme, we will repeatedly refer to properties of this association scheme in the remainder of this chapter.

Even though there exist groups for which the corresponding conjugacy class scheme is symmetric, like the symmetric group, whose conjugacy classes are closed under inversion, the conjugacy class scheme is not necessarily symmetric. In certain finite general linear groups, for example, not all the conjugacy classes are closed under inversion and thus the associated conjugacy class scheme is not symmetric. The fact that the conjugacy class scheme arising from the finite general linear groups are not necessarily symmetric plays a crucial role in the thesis. However, given an association scheme, it is possible to construct a symmetric one arising from it, as we will see in Lemma 2.1.21.

Remark 2.1.5. There is a strong connection between association schemes and graph theory. We can view an association scheme $(X, \{R_0, R_1, \dots, R_n\})$ as a union of (not necessarily undirected) graphs $\Gamma_i = (X, R_i)$, whose vertices are given by X and whose edges are given by the relations R_i. If the association scheme is symmetric, then the graphs Γ_i are *regular graphs* for each i.

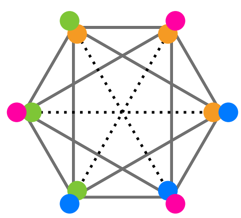


Figure 2.1: The Johnson scheme $J(2,4)$.

Example 2.1.6 (Johnson scheme). Figure 2.1 shows the graph corresponding to the Johnson scheme $J(2,4)$. The dotted lines correspond to the relation R_2, the nondotted ones to the relation R_1, and we omitted to draw the relation R_0.

Given an association scheme, we can bring matrices into play. To do this, we first introduce some notation.

Given a field $\mathbb{K}$ and two finite sets X and Y, we write $\mathbb{K}(X,Y)$ for the set of all $|X| \times |Y|$ matrices whose entries are in $\mathbb{K}$ and whose rows and columns are indexed by X and Y, respectively. For a matrix $A \in \mathbb{K}(X,Y)$, the entry corresponding to $x \in X$ and $y \in Y$ is denoted by $A(x,y)$. If $Y = |1|$, then we write $\mathbb{K}(X)$ instead of $\mathbb{K}(X,Y)$ for the set of all column vectors indexed by X and having entries in $\mathbb{K}$. For the x-entry of $a \in \mathbb{K}(X)$, where $x \in X$, we write $a(x)$.

For a relation R on $X \times X$, the *adjacency matrix* $A \in \mathbb{R}(X,X)$ of R is given by

$$A(x,y) = \begin{cases} 1 & \text{for } (x,y) \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Using adjacency matrices we can give the following equivalent definition of an association scheme.

Definition 2.1.7. Let $R_0, R_1, \ldots, R_n$ be non-empty relations on $X \times X$ with corresponding adjacency matrices $A_0, A_1, \ldots, A_n$. Then $(X, \{R_0, R_1, \ldots, R_n\})$ is an association scheme if the following properties hold.

(A1') We have $\sum_{i=0}^{n} A_i = J$ and $A_0 = I$, where $J, I \in \mathbb{C}(X,X)$ denote the all-ones and the identity matrix, respectively.

(A2') For all i, we have $A_i^T \in \{A_0, A_1, \ldots, A_n\}$.

(A3') For all i,j, there exist integers p_{ij}^k such that $A_iA_j = \sum_{k=0}^{n} p_{ij}^k A_k$.

(A4') For all i,j,k, we have $p_{ij}^k = p_{ji}^k$.

Remark 2.1.8. The adjacency matrices of an association scheme have exactly v_i ones in each row and each column, where v_i denotes the valency.

Example 2.1.9 (Conjugacy class scheme). Given the conjugacy class scheme arising from a finite group G with conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_n$, the corresponding adjacency matrices $A_i \in \mathbb{C}(G,G)$ are given by

$$A_i(g,h) = \begin{cases} 1 & \text{for } g^{-1}h \in C_i, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we often denote the set of relations $\{R_0, R_1, \ldots, R_n\}$ of an association scheme by $\mathcal{R}$. Let $(X, \mathcal{R})$ be an association scheme with n classes and corresponding adjacency matrices $A_0, A_1, \ldots, A_n$. Then the vector space $\langle A_0, A_1, \ldots, A_n \rangle_{\mathbb{C}}$ is a matrix algebra because of $(A3')$ and together with $(A4')$ it follows that this algebra is commutative. Moreover, from $(A1')$ it follows that this vector space is $(n+1)$-dimensional. This brings us to the following definition.

Definition 2.1.10. For an association scheme $(X, \mathcal{R})$ whose adjacency matrices are $A_0, A_1, \ldots, A_n$, the commutative $(n+1)$-dimensional matrix algebra $\langle A_0, A_1, \ldots, A_n \rangle_{\mathbb{C}}$ is called the *Bose-Mesner algebra* of $(X, \mathcal{R})$.

From property $(A2')$ together with the fact that all entries of the adjacency matrix A_i are 0 or 1 it also follows that the A_i's and thus all elements of the Bose-Mesner algebra are normal. Using linear algebra, it follows that the Bose-Mesner algebra is simultaneously unitary diagonalisable. Thus it is possible to construct a second basis of the Bose-Mesner algebra that consists only of pairwise orthogonal idempotent matrices.

Theorem 2.1.11. *The Bose-Mesner algebra of an association scheme $(X, \mathcal{R})$ with n classes has a unique basis of pairwise orthogonal idempotent matrices $E_0, E_1, \ldots, E_n$ that are Hermitian and satisfy*

$$\sum_{k=0}^{n} E_k = I \quad \text{and} \quad \frac{1}{|X|} J \in \{E_0, E_1, \ldots, E_n\},$$

where I and J are the identity and the all-ones matrix, respectively.

From now on, we will assume that $E_0 = \frac{1}{|X|} J$. Since the matrices E_k of the second basis of the Bose-Mesner algebra are idempotent, it follows that they are positive semi-definite.

Definition 2.1.12. The *multiplicities* m_k of an association scheme are the ranks of the matrices E_k from Theorem 2.1.11, that is $m_k = \operatorname{rk}(E_k)$.

The beauty of the conjugacy class scheme lies in its ability to involve representation theory.

Example 2.1.13. The pairwise orthogonal idempotent matrices E_k corresponding to the conjugacy class scheme arising from the finite group G are indexed by the irreducible characters $\chi^0, \chi^1, \ldots, \chi^n$ of G and are given by

$$E_k(g, h) = \frac{\chi^k(e)}{|G|} \chi^k(g^{-1}h), \tag{2.3}$$

where e denotes the identity element of G. Thus, the multiplicities m_k are

$$m_k = \mathrm{rk}(E_k) = \frac{\chi^k(e)}{|G|} \sum_{g \in G} \chi^k(g^{-1}g) = \chi^k(e)^2.$$

There are certain connections between the two bases of an association scheme, which we collect in the following remark.

Remark 2.1.14. The two bases of the Bose-Mesner algebra of an association scheme $(X, \{R_0, R_1, \ldots, R_n\})$ are dual in a certain way, which is illustrated in Table 2.1. There, $\circ$ denotes the *Hadamard product*, that is the component-wise multiplication of two matrices and δ_{ij} denotes the Kronecker-delta.

Table 2.1: Summary of properties of the bases matrices A_i and E_k of the Bose-Mesner algebra corresponding to the association scheme $(X, \{R_0, R_1, \ldots, R_n\})$.

A_i	E_k
$A_0 = I$	$E_0 = \frac{1}{\lvert X\rvert} J$
$\sum_{i=0}^{n} A_i = J$	$\sum_{k=0}^{n} E_k = I$
$A_i \circ A_j = \delta_{ij} A_i$	$E_k E_\ell = \delta_{k\ell} E_k$
entries are 0 or 1	eigenvalues are 0 or 1

From Theorem 2.1.11, we have that $\sum_{k=0}^{n} E_k = I$ and that the E_k are pairwise orthogonal. Consequently, we get the following.

Lemma 2.1.15. *The matrices E_k of an association scheme $(X, \{R_0, R_1, \ldots, R_n\})$ satisfy that*

$$\mathbb{C}X = \bigoplus_{k=0}^{n} \mathrm{colsp}(E_k),$$

where $\mathrm{colsp}(E_k)$ *denotes the column span of E_k over the complex numbers.*

There exist unique complex numbers $P_i(k)$ and $Q_k(i)$ such that

$$A_i = \sum_{k=0}^{n} P_i(k) E_k, \tag{2.4}$$

$$E_k = \frac{1}{|X|} \sum_{i=0}^{n} Q_k(i) A_i. \tag{2.5}$$

We will see at a later stage in this thesis that the change of basis is of great importance in the theory of association schemes. The numbers $P_i(k)$ and $Q_k(i)$ are called *eigenvalues* and *dual eigenvalues* of the corresponding association scheme, respectively. Using the properties of the two bases, we obtain the following values of the (dual) eigenvalues.

Lemma 2.1.16. *Let* $(X, \{R_0, R_1, \dots, R_n\})$ *be an association scheme having valencies* v_i *and multiplicities* m_k*. Then the the eigenvalues* $P_i(k)$ *and dual eigenvalues* $Q_k(i)$ *of the scheme have the following values for every* i, k*:*

(i) $P_0(k) = 1$,

(ii) $Q_0(i) = 1$,

(iii) $P_i(0) = v_i$,

(iv) $Q_k(0) = m_k$.

Moreover, using the equations (2.4) and (2.5) we can deduce the following identities.

Lemma 2.1.17. *Let* $P_i(k)$ *and* $Q_k(i)$ *denote the eigenvalues and dual eigenvalues, respectively, of the association scheme* $(X, \{R_0, R_1, \dots, R_n\})$ *having valencies* v_i *and multiplicities* m_k*. Then we have*

(i) $\frac{1}{|X|}\sum_{k=0}^{n} P_i(k)Q_k(j) = \delta_{ij}$,

(ii) $\frac{1}{|X|}\sum_{i=0}^{n} Q_k(i)P_i(l) = \delta_{kl}$,

(iii) $m_k\overline{P_i(k)} = v_iQ_k(i)$,

where δ_{ij} *denotes the Kronecker-delta.*

Furthermore, we have the following estimates on the absolute value of the (dual) eigenvalues.

Lemma 2.1.18. *Let* $P_i(k)$ *and* $Q_k(i)$ *denote the eigenvalues and dual eigenvalues, respectively, of the association scheme* $(X, \{R_0, R_1, \dots, R_n\})$ *having valencies* v_i *and multiplicities* m_k*. Then, for all* i, k*, we have*

$$|P_i(k)| \leq v_i \quad \textit{and} \quad |Q_k(i)| \leq m_k.$$

Calling the numbers $P_i(k)$ and $Q_k(i)$ eigenvalues and dual eigenvalues, respectively, is justified because, for all i, k, we have

$$A_iE_k = P_i(k)E_k, \tag{2.6}$$

$$A_i \circ E_k = \frac{1}{|X|}Q_k(i)A_i, \tag{2.7}$$

where $\circ$ denotes again the Hadamard product. We emphasise that from (2.6) we find that the columns of the pairwise orthogonal idempotent matrices E_k are the common eigenvectors of the adjacency matrices A_i.

We note that for a symmetric association scheme the adjacency matrices are symmetric and thus their eigenvalues $P_i(k)$ and, due to Lemma 2.1.17, also the dual eigenvalues $Q_k(i)$ are real-valued. In general these numbers are not necessarily real-valued.

Example 2.1.19. The eigenvalues $P_i(j)$ of the Johnson scheme $J(k,n)$ from Example 2.1.2 are given by

$$P_i(j) = \sum_{r=i}^{k} (-1)^{r-i+j} \binom{r}{i} \binom{n-2r}{k-r} \binom{n-r-j}{r-j}.$$

Example 2.1.20. We focus again on the conjugacy class scheme of the finite group G whose conjugacy classes and irreducible characters are $C_0, C_1, \dots, C_n$ and $\chi^0, \chi^1, \dots, \chi^n$, respectively. Then the eigenvalues of this scheme are

$$P_i(k) = \frac{|C_i|}{\chi^k(e)} \overline{\chi_i^k},$$

and the dual eigenvalues are

$$Q_k(i) = \chi^k(e) \chi_i^k,$$

where χ_i^k denotes the irreducible character χ^k evaluated on the conjugacy class C_i. Together with (2.4) and (2.5), respectively, we have

$$A_i = \sum_{k=0}^{n} \frac{|C_i|}{\chi^k(e)} \overline{\chi_i^k} E_k, \tag{2.8}$$

$$E_k = \frac{\chi^k(e)}{|G|} \sum_{i=0}^{n} \chi_i^k A_i. \tag{2.9}$$

Sometimes, it is mandatory to have a symmetric association scheme. Given a non-symmetric association scheme, there is a method to construct a symmetric one from it.

Lemma 2.1.21 (Symmetrisation). *Let $(X, \{R_0, R_1, \dots, R_n\})$ be an association scheme and define*

$$\mathcal{R} = \{R_i \cup R_i^{-1} : i \in \{0, 1, \dots, n\}\}.$$

Then the pair $(X, \mathcal{R})$ is a symmetric association scheme. In this case we call $(X, \mathcal{R})$ the symmetrisation *or the* symmetric closure *of $(X, \{R_0, R_1, \dots, R_n\})$.*

2.2 Subsets of association schemes

One of the reasons why association schemes are such a powerful tool is because it is often possible to embed interesting combinatorial objects as subsets in certain association schemes and to characterise them in terms of their *inner* or *dual distribution*. Delsarte named these subsets *cliques* and *designs* and introduced them in his thesis [Del73].

In the following we write $[n]$ for the set $\{1, 2, \dots, n\}$ and we call a subset of size k of $[n]$ a *k-subset* or *k-set* for short. Moreover, if not stated otherwise, let $(X, \{R_0, R_1, \dots, R_n\})$ be an association scheme with adjacency matrices $A_0, A_1, \dots, A_n$.

Definition 2.2.1. For a subset Y of X, the *inner distribution* of Y is the tuple $(a_0, a_1, \dots, a_n)$ given by

$$a_i = \frac{|(Y \times Y) \cap R_i|}{|Y|}. \tag{2.10}$$

Since R_0 is the set of diagonal elements, we have $a_0 = 1$. We note that the numbers a_i are nonnegative.

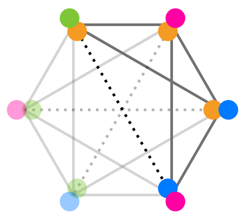


Figure 2.2: A subset of the Johnson scheme $J(2,4)$ consisting of four 2-subsets.

Example 2.2.2 (Johnson scheme $J(2,4)$). The inner distribution (a_0, a_1, a_2) of the given subset in Figure 2.2 satisfies

$$a_0 = 1, \quad a_1 = 5/2, \quad a_2 = 1/2.$$

Let $\mathbb{1}_Y \in \mathbb{C}(X)$ denote the *characteristic vector* of a subset Y of X, that is

$$\mathbb{1}_Y(x) = \begin{cases} 1 & \text{for } x \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Since we can describe an association scheme in terms of its adjacency matrices A_i, we find that the definition (2.10) of the inner distribution of a subset Y of X is equivalent to the following

$$a_i = \frac{1}{|Y|} \mathbb{1}_Y^T A_i \mathbb{1}_Y. \tag{2.11}$$

Certain subsets of an association scheme, known as *cliques*, are defined in terms of their inner distribution and are of particular interest.

Definition 2.2.3. Let D be a subset of $[n]$. A subset Y of X with inner distribution $(a_0, a_1, \dots, a_n)$ is called a *D-clique* if $a_i = 0$ for all $i \in [n] \setminus D$.

In other words, if Y is a D-clique, then the set $D \cup \{0\}$ contains all *permitted* relations that occur among the elements of Y. We are interested in finding upper bounds on the size of D-cliques. One tool to do so is the *linear programming method* by Delsarte, which we discuss in Section 2.4.

Example 2.2.4. Let $(X, \{R_0, R_1, \ldots, R_n\})$ be the Johnson scheme $J(k, n)$ from Example 2.1.2 and take $D = \{1, 2, \ldots, k-1\}$. Then a D-clique of X consists only of k-subsets x, y of $[n]$ such that $x \cap y \neq \varnothing$.

The D-cliques from Example 2.2.4 are called *intersecting sets* or *Erdős-Ko-Rado sets* and will play a crucial role in this thesis. We will give more details on these *intersecting sets* in Section 2.3.2 and will later on discuss *intersecting sets* in other association schemes, namely in the conjugacy class schemes arising from the symmetric group and the finite general linear group in Section 3.2 and Chapter 5, respectively.

The *dual* of a D-clique in an association scheme is a T-*design*. In order to introduce a T-*design*, we first need the dual object of the inner distribution, which is the *dual distribution*.

Definition 2.2.5. The *dual distribution* of a subset Y of X is a tuple $(b_0, b_1, \ldots, b_n)$ given by

$$b_k = \sum_{i=0}^{n} Q_k(i) a_i, \tag{2.12}$$

where the $Q_k(i)$ are the dual eigenvalues of the association scheme.

From (2.11) and (2.5) it follows that the definition (2.12) is equivalent to

$$b_k = \frac{|X|}{|Y|} \mathbb{1}_Y^T E_k \mathbb{1}_Y, \tag{2.13}$$

where the matrices E_k form the pairwise orthogonal idempotent basis from Theorem 2.1.11 of the underlying association scheme.

Since $E_0 = \frac{1}{|X|} J$ it follows that $b_0 = |Y|$. Moreover, since the E_k are positive semi-definite (see Theorem 2.1.11), it follows from (2.13) that the entries of the dual distribution are real and nonnegative.

Table 2.2 gives an overview of the properties of the inner and dual distribution, respectively.

Example 2.2.6. Let $\chi^0, \chi^1, \ldots, \chi^n$ be all irreducible characters of the finite group G and let Y be a subset of G. Then, due to Example 2.1.20, the dual distribution $(b_0, b_1, \ldots, b_n)$ of Y is given by

$$b_k = \frac{\chi^k(e)}{|Y|} \sum_{g,h \in Y} \chi^k(g^{-1}h).$$

Table 2.2: Properties of the inner and dual distribution of a subset Y of X.

inner distribution (a_i)	**dual distribution** (b_k)
$a_i = \frac{1}{\lvert Y\rvert}\mathbb{1}_Y^T A_i \mathbb{1}_Y$	$b_k = \frac{\lvert X\rvert}{\lvert Y\rvert}\mathbb{1}_Y^T E_k \mathbb{1}_Y$
$a_i = \frac{1}{\lvert X\rvert}\sum_{k=0}^{n} P_i(k) b_k$	$b_k = \sum_{i=0}^{n} Q_k(i) a_i$
$a_i \geq 0$	$b_k \geq 0$
$a_0 = 1$	$b_0 = \lvert Y\rvert$
$\sum_{i=0}^{n} a_i = \lvert Y\rvert$	$\sum_{k=0}^{n} b_k = \lvert X\rvert$

Since all entries of the dual distribution are nonnegative, we are particularly interested in the extremal case, namely when some entries are equal to zero. This motivates the definition of a *T-design*.

Definition 2.2.7. Let T be a subset of $[n]$. A subset Y of X with dual distribution $(b_0, b_1, \dots, b_n)$ is a *Delsarte T-design* (or *T-design* for short), if $b_k = 0$ for all $k \in T$.

Remark 2.2.8. Since the matrices E_k of the association scheme $(X, \{R_0, R_1, \dots, R_n\})$ are pairwise orthogonal we conclude that a subset Y of X is a T-design if and only if the characteristic vector $\mathbb{1}_Y$ of Y is orthogonal to the column space of E_k for all $k \in T$.

Ever since Delsarte introduced T-designs in his thesis [Del73], they have been of interest in algebraic combinatorics. We emphasise that the definition of a T-design is algebraic and without any clear combinatorial interpretation. However, it turns out that T-designs in association schemes often have nice combinatorial interpretations, which motivated Delsarte to "the 'conjecture' being that T-designs will often have interesting properties" [Del73]. This conjecture was proved for several classical association schemes, see for example [Del76] or [Sta86]. In Section 3.4 and in Chapter 6 we show that this conjecture is also true for the conjugacy class scheme arising from the symmetric group and from the finite general linear group, respectively.

It is well known that for many association schemes Delsarte T-designs approximate in some sense the underlying set, see for example [Sta86]. Consequently, we are interested in the existence of small T-designs.

For example, in the Johnson scheme certain T-designs are classical combinatorial *t-(n, k, λ) designs*.

Definition 2.2.9. For positive integers $n \geq k$ and $t \in [k]$, we consider the set S of all k-subsets of $[n]$. An element of S is called a *block*. A subset Y of S is a combinatorial t-(n, k, λ) *design* if every t-subset of $[n]$ is contained in exactly λ blocks of Y.

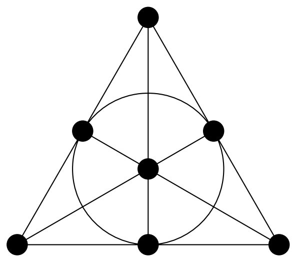


Figure 2.3: Fano plane

Example 2.2.10. The *Fano plane*, illustrated in Figure 2.3, is a 2-$(7, 3, 1)$ design. The set X consists of all 3-subsets of $[7]$. The blocks of the Fano plane are given by the seven lines, where the inner circle also indicates a line.

The existence of combinatorial t-(n, k, λ) designs for given t and n sufficiently large was proven by Teirlinck [Tei87].

Delsarte proved [Del73] that the notions of $\{1, 2, \ldots, t\}$-designs and combinatorial t-(n, k, λ) designs coincide in the Johnson scheme.

Theorem 2.2.11. *Let $(X, \{R_0, R_1, \ldots, R_k\})$ be the Johnson scheme $J(k, n)$ from Example 2.1.2. Additionally, let t be an integer with $1 \leq t \leq k$, and let Y be a subset of X. Then Y is a T-design with $T = \{1, 2, \ldots, t\}$ if and only if Y is a t-(n, k, λ) design with $\lambda = |Y|\binom{k}{t}/\binom{n}{t}$.*

From this theorem it follows that the Fano plane is a $\{1, 2\}$-design in the Johnson scheme $J(3, 7)$.

To obtain a q-analog of a combinatorial t-(n, k, λ) design we replace again sets by subspaces and obtain a *subspace design*.

Definition 2.2.12. For positive integers $n \geq k$ and $t \in [k]$, let X be the set of all k-dimensional subspaces of $\mathbb{F}_q^n$. An element of X is called a *block*. A subset Y of X is a t-$(n, k, \lambda)_q$ *subspace design* if every t-dimensional subspace of $\mathbb{F}_q^n$ is contained in exactly λ blocks of Y.

We have the following combinatorial interpretation for a $\{1, 2, \ldots, t\}$-design in the q-Johnson scheme, see for example [Del76] or [Sta86].

Theorem 2.2.13. *Let $(X, \{R_0, R_1, \ldots, R_k\})$ be the q-Johnson scheme $J_q(k, n)$ from Example 2.1.3. Moreover, let t be an integer with $1 \leq t \leq k$ and let Y be a subset* *of X. Then Y is a T-design with $T = \{1, 2, \ldots, t\}$ if and only if Y is a t-$(n, k, \lambda)_q$ subspace design with $\lambda = |Y| \left[{k \atop t}\right]_q / \left[{n \atop t}\right]_q$.*

In Section 3.4 and in Chapter 6 it turns out that certain T-designs in the conjugacy class scheme arising from the symmetric group and from the finite general linear group, respectively, are so-called *transitive sets* in the corresponding group.

2.3 Hoffman bounds

In this thesis, we are interested in intersecting sets, not only in the classical case from Example 2.2.4, but also within symmetric and finite general linear groups. Since these sets arise as special D-cliques, our focus is on finding upper bounds for their sizes. One approach to obtain these bounds involves exploiting the connection between association schemes and graphs, as observed in Remark 2.1.5, and utilising graph theory tools such as the classical *Hoffman bound* that will be introduced in this section.

Definition 2.3.1. An *(undirected) graph* $\Gamma = (X, E)$ consists of a finite set of *vertices* X and a set of *edges* E, where the latter is a subset of $X \times X$ satisfying $(x, x) \notin E$ for all $x \in X$, and $(x, y) \in E$ if and only if $(y, x) \in E$. The *degree* of a vertex $x \in X$ is the number of $y \in X$ such that $(x, y) \in E$. A graph is *d-regular* if all vertices have degree d.

The *adjacency matrix* $A \in \mathbb{R}(X, X)$ of a graph $\Gamma = (X, E)$ is given by

$$A(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since the adjacency matrix $A \in \mathbb{R}(X, X)$ of an undirected graph is symmetric it has an orthonormal system of $|X|$ eigenvectors that form a basis of $\mathbb{R}(X)$. Moreover, all eigenvalues of A are real and we refer to them as the eigenvalues of the corresponding graph. We note that, if the graph is d-regular, then d is an eigenvalue with the all-ones vector as a corresponding eigenvector. A subset Y of X is called *independent* if for all $x, y \in Y$ there is no edge between x and y in Γ, that means $(x, y) \notin E$ for all $x, y \in Y$.

2.3.1 The classical Hoffman bound

The *Hoffman bound* [Hae21], also known as *ratio bound*, gives an upper bound on the size of an independent set of a regular graph in terms of its minimal eigenvalue. Moreover, it gives a partial characterisation of the extremal case.

Theorem 2.3.2. *(Hoffman bound) Let $\Gamma = (X, E)$ be a d-regular graph with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_\ell$ and corresponding eigenvectors $v_0, v_1, \ldots, v_\ell$, where v_0 is the all-ones vector. If $Y \subseteq X$ is an independent set in Γ, then*

$$\frac{|Y|}{|X|} \leq \frac{|\lambda_{\min}|}{d + |\lambda_{\min}|},$$

where $\lambda_{\min} = \min_{k \neq 0} \lambda_k$. In case of equality we have

$$\mathbb{1}_Y \in \langle \{v_0\} \cup \{v_k \colon \lambda_k = \lambda_{\min}\} \rangle.$$

In the next subsection we apply the Hoffman bound to prove an upper bound on the size of intersecting families of sets from Example 2.2.4. A corresponding theorem on subsets Y, Z of a graph having no edges in between is as follows.

Theorem 2.3.3. *Let $\Gamma = (X, E)$ be a d-regular graph with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_\ell$ and corresponding eigenvectors $v_0, v_1, \ldots, v_\ell$, where v_0 is the all-ones vector. If Y and Z are subsets of X such that there are no edges between Y and Z in Γ, then*

$$\sqrt{\frac{|Y|}{|X|} \frac{|Z|}{|X|}} \leq \frac{\lambda_{\max}}{d + \lambda_{\max}},$$

where $\lambda_{\max} = \max_{k \neq 0} |\lambda_k|$. In case of equality we have

$$\mathbb{1}_Y, \mathbb{1}_Z \in \langle \{v_0\} \cup \{v_k \colon |\lambda_k| = \lambda_{\max}\} \rangle.$$

2.3.2 Application: classical Erdős-Ko-Rado theorems

One of the most famous questions in extremal set theory is the following.

$$\text{How large can a family of } k\text{-sets of } \{1, 2, \ldots, n\} \text{ be such that every two members of this family have non-empty intersection?} \tag{Qu1}$$

Figure 2.4 illustrates two families with the property given in (Qu1) for $n = 4$ and $k = 2$.

The question (Qu1) was first answered by Erdős, Ko, and Rado in [EKR61] in 1961. They also characterised the extremal case for n sufficiently large compared to k.

Theorem 2.3.4 (Erdős-Ko-Rado). *For all fixed k and $n \geq 2k$, the size of a family of k-sets of $[n]$ such that every two members of this family have non-empty intersection is at most $\binom{n-1}{k-1}$. Moreover, for $n > 2k$, equality holds if and only if there is one element of $[n]$ that is contained in all members of the family.*

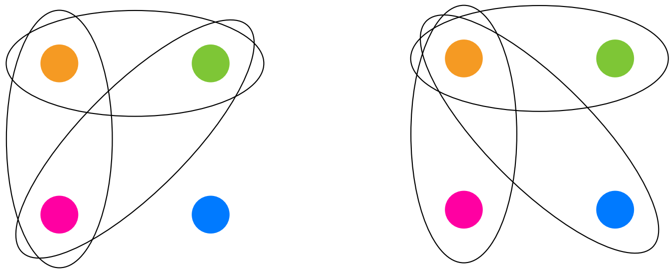


Figure 2.4: Examples of intersecting families of 2-sets of maximal size.

The original proof of the Erdős-Ko-Rado theorem is working with so-called *shiftings*, which are operations on set systems. However, we can use association schemes in order to get an elegant proof that uses algebraic combinatorics (c.f. [GM16]) and applies the Hoffman bound from Theorem 2.3.2. To do so, recall from Example 2.2.4 that, taking the Johnson scheme $J(k,n)$, an intersecting family of k-sets is a D-clique in $J(k,n)$, where $D = \{1,2,\ldots,k-1\}$.

Example 2.3.5. Let $(X, \{R_0, R_1, \ldots, R_k\})$ be the Johnson scheme from Example 2.1.2. Let Γ be the graph with vertex set X and two edges x and y are adjacent if and only if $|x \cap y| = 0$. Then Γ has $\binom{n}{k}$ vertices and is regular of degree $\binom{n-k}{k}$. Moreover, any intersecting set Y of X is an independent set in this graph because, for all elements x and y of an intersecting set, it holds that $|x \cap y| \geq 1$. We obtain the eigenvalues of this graph by applying Example 2.1.19 for $i = k$, namely

$$(-1)^j \binom{n-k-j}{k-j}.$$

Consequently the smallest eigenvalue of Γ is

$$-\binom{n-k-1}{k-1}.$$

Applying the Hoffman bound from Theorem 2.3.2 gives an upper bound on the size of an independent set in Γ and thus on the size of any intersecting subset Y of X, namely

$$|Y| \leq \binom{n}{k} \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

Since the 1960's, questions of the flavour of (Qu1) have been investigated for many different objects and notions of intersection with objects such as subspaces or flags, see [GM16] for a survey.

Another notion of intersection, known as *cross-intersection*, arises from set systems. Two families Y and Z of k-subsets of $[n]$ are *cross-intersecting* if, for all $y \in Y$ and $z \in Z$, the intersection of y and z is nonempty. Kleitmann [Kle68] gave an upper bound on the size of cross-intersecting sets, which was later improved by Pyber [Pyb86] and then by Matsumoto and Tokushige [MT89].

Theorem 2.3.6. *Let $n \geq 2k$ and let Y and Z be cross-intersecting families of k-sets of $[n]$. Then we have*

$$\sqrt{|Y||Z|} \leq \binom{n-1}{k-1}.$$

Moreover, for $n > 2k$, equality holds if and only if $Y = Z$ and there exists one element of $[n]$ that is contained in all members of Y.

The proof of the upper bound works similarly as the proof of the upper bound in Example 2.3.5. Using the same graph Γ whose second largest eigenvalue in absolute value is

$$\binom{n-k-1}{k-1}$$

and applying Theorem 2.3.3 gives the upper bound.

Another notion of intersection arises as a strengthened version from the classical notion of intersection and is called *t-intersection*.

Definition 2.3.7. Two k-sets x, y of $[n]$ are *t-intersecting* if $|x \cap y| \geq t$. A family of k-sets Y of $[n]$ is *t-intersecting* if all pairs in $Y \times Y$ are t-intersecting.

The corresponding question to (Qu1) for t-intersecting families then is the following.

How large can a family of k-sets of $\{1, 2, \ldots, n\}$ be such that every two members of this family are t-intersecting? (Qu2)

The answer to this question (Qu2) was first given in [EKR61] for n sufficiently large compared to t and k. The authors also characterise the extremal case.

Theorem 2.3.8. *Let $t \leq k$. For n sufficiently large compared to t and k, the size of a t-intersecting family of k-sets of $[n]$ is bounded by $\binom{n-t}{k-t}$ and, in case of equality, there exist t distinct elements of $[n]$ being contained in each member of the family.*

In fact, Frankl [Fra78] and later Wilson [Wil84] obtained exact bounds on the value of n for which Theorem 2.3.8 holds. More precisely, Frankl proved that the theorem is valid for all $n \geq (t+1)(k-t+1)$ when $t \geq 15$, and Wilson subsequently extended this result to all t.

There also exists a strengthened version of Theorem 2.3.6, that deals with *t-cross-intersecting sets.*

Definition 2.3.9. Two families Y and Z of k-subsets of $[n]$ are *t-cross-intersecting*, if all pairs in $Y \times Z$ are t-intersecting.

An upper bound for $|Y||Z|$ is, for example, given in [Tok10] and an improved version in [FLST14].

Theorem 2.3.10 ([FLST14]). *Let $n > k \geq t \geq 14$ be nonnegative integers with $n \geq (t+1)k$ and let Y and Z be t-cross-intersecting families of k-sets of $[n]$. Then we have*

$$\sqrt{|Y||Z|} \leq \binom{n-t}{k-t},$$

and, in case of equality, $Y = Z$ and there exist t elements of $[n]$ that are contained in all members of Y.

In order to give an algebraic proof of the upper bound in Theorem 2.3.8, we apply a *weighted version* of the Hoffman bound in the next section.

2.3.3 The weighted version of the Hoffman bound

The following weighted version of the Hoffman bound generalises Theorems 2.3.2 and 2.3.3, and was stated and proven by Ellis, Friedgut, and Pilpel in [EFP11].

Theorem 2.3.11 (Weighted version of the Hoffman bound). *Let $\Gamma = (X, E)$ be a graph on n vertices. Suppose that $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ are regular spanning subgraphs of Γ, all having $\{v_0, v_1, \ldots, v_{n-1}\}$ as an orthonormal system of eigenvectors with v_0 being the all-ones vector. Let $P_i(k)$ be the eigenvalue of v_k in Γ_i. Let $w_0, w_1, \ldots, w_r \in \mathbb{R}$ and write $P(k) = \sum_{i=0}^{r} w_i P_i(k)$.*

(i) If $Y \subseteq X$ is an independent set in Γ, then

$$\frac{|Y|}{|X|} \leq \frac{|P_{\min}|}{P(0) + |P_{\min}|},$$

where $P_{\min} = \min_{k \neq 0} P(k)$. In case of equality we have

$$\mathbb{1}_Y \in \langle \{v_0\} \cup \{v_k \colon P(k) = P_{\min}\} \rangle.$$

(ii) If $Y, Z \subseteq X$ are such that there are no edges between Y and Z in Γ, then

$$\sqrt{\frac{|Y|}{|X|}\frac{|Z|}{|X|}} \leq \frac{P_{\max}}{P(0)+P_{\max}},$$

where $P_{\max} = \max_{k \neq 0} |P(k)|$. In case of equality we have

$$\mathbb{1}_Y, \mathbb{1}_Z \in \langle \{v_0\} \cup \{v_k \colon |P(k)| = P_{\max}\}\rangle.$$

This theorem plays a crucial role in the proofs of Ellis, Friedgut, and Pilpel [EFP11] on the size and characterisation of *t-intersecting* and *t-cross-intersecting* sets of the symmetric group. We summarise these results in Section 3.2. Moreover, the proofs of our main results in Chapter 5 also heavily rely on this weighted version of the Hoffman bound.

2.3.4 Application: t-intersecting families of sets

In the following we apply the weighted version of the Hoffman bound to give an algebraic proof of the upper bound on the size of t-intersecting families of sets as stated in Theorem 2.3.8. For this, we make heavy use of association schemes.

To do so, let $A_0, A_1, \ldots, A_k$ denote the adjacency matrices of the Johnson scheme $J(k,n)$. Let $t \leq k$. For every i with $k-t+1 \leq i \leq k$, let Γ_i be the graph corresponding to the adjacency matrix A_i. Then each graph Γ_i is $\binom{k}{i}\binom{n-k}{i}$-regular. Moreover, a t-intersecting family of sets is an independent set in every graph Γ_i. Let Γ be the graph given by the adjacency matrix

$$\sum_{i=k-t+1}^{k} A_i.$$

We apply Theorem 2.3.11 to the graph Γ and the regular spanning subgraphs Γ_i.

Recall from (2.6) that every vector in the column space of E_j is an eigenvector of A_i with eigenvalue $P_i(j)$. We wish to construct some weight $w \in \mathbb{R}^t$ such that the minimum value over all j of

$$\sum_{i=k-t+1}^{k} w_i P_i(j) \tag{2.14}$$

equals

$$\eta = -\frac{1}{\binom{n}{k}/\binom{n-t}{k-t} - 1} \tag{2.15}$$

and such that w is normalised in the sense that (2.14) equals 1 if $j = 0$. This guarantees that Theorem 2.3.11 gives the upper bound on the size of a t-intersecting family of sets from Theorem 2.3.8.

In the construction of the weight w we make use of a *t-Steiner system.*

Remark 2.3.12. A *t-Steiner system* is a combinatorial t-$(n, k, 1)$ design. It was proved by Keevash [Kee14] (see also [GKLO23]) that t-Steiner systems exist for all $t \leq k$ and all sufficiently large n provided some natural divisibility conditions are satisfied.

Let Z be a t-Steiner system. According to the definition of a combinatorial t-$(n, k, 1)$ design, each t-set of $[n]$ is contained in exactly one member of Z, which implies that $|x \cap y| \leq t - 1$ for all distinct $x, y \in Z$. Consequently, the inner distribution $(a_0, a_1, \ldots, a_k)$ of Z satisfies

$$a_i = 0 \qquad \text{for all } 1 \leq i \leq k - t. \tag{2.16}$$

Due to Theorem 2.2.11, Z is a $\{1, 2, \ldots, t\}$-design of size $\binom{n}{k}/\binom{n-t}{k-t}$ in the Johnson scheme $J(k, n)$. Thus the dual distribution $(b_0, b_1, \ldots, b_k)$ of Y satisfies

$$b_j = 0 \qquad \text{for all } 1 \leq j \leq t. \tag{2.17}$$

With the help of a t-Steiner system and the theory of association schemes we are in a position to construct some $w \in \mathbb{R}^t$ such that the minimum value of (2.14) equals η from (2.15).

Lemma 2.3.13. *Let n, k and t be positive integers such that $t \leq k \leq n/2$ and such that a t-Steiner system Z of size $\binom{n}{k}/\binom{n-t}{k-t}$ exists. Let $(a_0, a_1, \ldots, a_k)$ and $(b_0, b_1, \ldots, b_k)$ be the inner and dual distribution of Z, respectively, and let $v_i = \binom{k}{i}\binom{n-k}{i}$ for all $i \in \{0, 1, \ldots, k\}$. Then, for*

$$w_i = \frac{a_i}{v_i(|Z| - 1)},$$

we have

$$\sum_{i=k-t+1}^{k} w_i P_i(j) \begin{cases} = 1 & \text{for } j = 0 \\ = \eta & \text{for } 1 \leq j \leq t \\ > \eta & \text{otherwise,} \end{cases}$$

where η is given by (2.15)*.*

Proof: Using the identity $P_i(j) = \frac{v_i}{m_j} Q_j(i)$ from Lemma 2.1.17 we have

$$\sum_{i=k-t+1}^{k} w_i P_i(j) = \frac{1}{m_j} \sum_{i=k-t+1}^{k} w_i v_i Q_j(i).$$

For $j = 0$, from (2.12), and (2.16) we obtain

$$\frac{1}{m_0} \sum_{i=k-t+1}^{k} w_i v_i Q_0(i) = \frac{1}{|Z| - 1}(b_0 - 1) = 1.$$

Since, according to (2.17), we have

$$\sum_{i=0}^{k} Q_j(i)a_i = 0 \quad \text{for all } 1 \le j \le t,$$

it follows with (2.16) that

$$\sum_{i=k-t+1}^{k} Q_j(i)a_i = -Q_j(0)a_0 = -m_j \quad \text{for all } 1 \le j \le t.$$

Consequently, for all j satisfying $1 \le j \le t$, it follows that

$$\frac{1}{m_j}\sum_{i=k-t+1}^{k} w_i v_i Q_j(i) = \frac{1}{(|Z|-1)}\frac{1}{m_j}\sum_{i=k-t+1}^{k} a_i Q_j(i) = -\frac{1}{|Z|-1} = \eta.$$

For $j > t$, by using (2.16) once again, we obtain

$$\begin{aligned}\frac{1}{m_j}\sum_{i=k-t+1}^{k} w_i v_i Q_j(i) &= \frac{1}{m_j(|Z|-1)}\sum_{i=k-t+1}^{k} a_i Q_j(i) = \frac{1}{m_j(|Z|-1)}(b_j - 1)\\ &\ge -\frac{1}{m_j(|Z|-1)} > -\frac{1}{|Z|-1} > \eta,\end{aligned}$$

where we have used that $b_j \ge 0$ for all j, and $m_j = \binom{n}{j} - \binom{n}{j-1} > 1$ for all $j \le k \le n/2$.
□

Now, we have all the necessary ingredients to provide an algebraic proof of the upper bound on the size of a t-intersecting family of sets in Example 2.3.14 and to follow the strategy outlined at the beginning of this section.

Example 2.3.14. Let $J(k,n)$ be the Johnson scheme with the adjacency matrices $A_0, A_1, \ldots, A_k$ and pairwise orthogonal idempotent matrices $E_0, E_1, \ldots, E_k$. As explained at the beginning of this section, we apply the weighted version of the Hoffman bound, Theorem 2.3.11, to the graph Γ with adjacency matrix

$$\sum_{i=k-t+1}^{k} A_i$$

and the $\binom{k}{i}/\binom{n-k}{i}$-regular spanning subgraphs Γ_i given by the adjacency matrices A_i for every i with $k-t+1 \le i \le k$. Now, each t-intersecting family of k-sets of $[n]$ is an independent set in this graph Γ. Let $P_i(j)$ be the eigenvalue of A_i corresponding to E_j and let $w \in \mathbb{R}^{t+1}$ be given as in Lemma 2.3.13. Then Lemma 2.3.13 implies that

$$\min_{j \ne 0} \sum_{i=k-t+1}^{k} w_i P_i(j) = \eta.$$

Applying Theorem 2.3.11 gives the upper bound from Theorem 2.3.8 on the size of a t-intersecting family of k-sets of $[n]$ if n is sufficiently large compared to t and k.

This application of the weighted version of the Hoffman bound also gives a partial characterisation of the extremal cases. Namely, the characteristic vector of a t-intersecting family of sets of maximal size is spanned by the eigenvectors, given by the columns of E_j with $0 \leq j \leq t$. However, more work has to be done to obtain the characterisation of the extremal case from Theorem 2.3.8.

2.4 Linear Programming

One of the reasons why association schemes are such a powerful tool is because, using the fact that the dual distribution is a linear transformation of the inner distribution of a subset, see (2.12), we are in a position to use linear programming to derive upper or lower bounds on the size of subsets of an association scheme. This idea goes back to Delsarte [Del73]. First, we present some preliminaries from linear programming before placing it in the context of association schemes.

2.4.1 A brief overview of linear programming

In what follows, we give a brief summary of some basic concepts of linear programming that we need in the remainder of this thesis. For more background and details, see for example [Van20].

Definition 2.4.1. Let $M \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$. Then the *primal linear program* (LP) for this data is given by

$$\begin{aligned} \max_{x\in\mathbb{R}^m} \quad & c^T x \\ \text{subject to} \quad & x_i \geq 0 \text{ for all } i, \\ & Mx \geq -b. \end{aligned} \tag{LP}$$

The mapping $\mathbb{R}^m \to \mathbb{R}, x \mapsto c^T x$ is called the *objective function* of the linear program (LP). If a vector $x \in \mathbb{R}^m$ satisfies the constraints in (LP), then x is a *feasible solution.* The linear program (LP) is *bounded* if $c^T x$ is bounded for all feasible solutions. Otherwise (LP) is *unbounded.* If the linear program (LP) is bounded, then a feasible solution x^* of (LP) that satisfies $c^T x \leq c^T x^*$ for all feasible solutions x, is called *optimal solution.*

For every primal linear program there exists a so-called *dual program*.

Definition 2.4.2. Let M, b, and c be as given in (LP). Then the *dual linear program* (DLP) corresponding to (LP) is given by

$$\begin{aligned} \min_{y \in \mathbb{R}^n} \quad & y^T b \\ \text{subject to} \quad & y_i \geq 0 \quad \text{for all } i, \\ & y^T M \leq -c^T. \end{aligned} \tag{DLP}$$

The definitions of an *objective function*, a *feasible solution*, *boundedness*, and an *optimal solution* are analogous to the ones for the primal linear program (LP).

Remark 2.4.3. Comparing variables and inequalities in the primal and corresponding dual linear program gives the following.

$$\begin{aligned} \#\text{ variables of (LP)} &= \#\text{ inequalities of (DLP)} \\ \#\text{ inequalities of (LP)} &= \#\text{ variables of (DLP)} \end{aligned}$$

In fact there is a stronger connection between the primal and dual linear program.

Theorem 2.4.4 (Weak duality). *Let x and y be feasible solutions of* (LP) *and* (DLP)*, respectively. Then we have*

$$c^T x \leq y^T b.$$

This implies especially that every feasible solution of the dual linear program (DLP) *gives an upper on the objective function of the primal linear program's* (LP) *optimal solution.*

2.4.2 Application: cliques in association schemes

In this section we apply linear programming in the context of association schemes to obtain upper bounds on the size of cliques. This idea goes back to Delsarte [Del73].

Let $(X, \{R_0, R_1, \dots, R_n\})$ be a symmetric association scheme, $D \subseteq [n]$, and let Y be a D-clique of X with inner distribution $(a_0, a_1, \dots, a_n)$ and dual distribution

$(b_0, b_1, \dots, b_n)$. Then it holds that (see Section 2.2)

$$\sum_{i=0}^{n} a_i = |Y|$$

$$a_0 = 1$$

$$a_i \geq 0 \quad \text{for all } i \in \{0, 1, \dots, n\}$$

$$a_i = 0 \quad \text{for all } i \in [n] \setminus D$$

$$b_k \geq 0 \quad \text{for all } k \in \{0, 1, \dots, n\}$$

This motivates the following linear program for D-cliques.

Let $Q_k(i)$ and m_k be the dual eigenvalues and multiplicities, respectively, of the association scheme $(X, \{R_0, R_1, \dots, R_n\})$. The primal linear program for cliques is given by

$$\begin{aligned} \max_{x_i \in \mathbb{R}} \quad & \sum_{i \in D \cup \{0\}} x_i \\ \text{subject to} \quad & x_0 = 1 \\ & x_i \geq 0 \text{ for all } i \in D, \\ & \sum_{i \in D} Q_k(i) x_i \geq -m_k \text{ for all } k \in \{1, 2, \dots, n\}, \end{aligned} \tag{2.18}$$

Applying results from linear programming we obtain the following.

Theorem 2.4.5 ([Del73]). *Let $(X, \{R_0, R_1, \dots, R_n\})$ be a symmetric association scheme with dual eigenvalues $Q_k(i)$. Let D be a subset of $[n]$. Then the linear program* (2.18) *has at least one feasible solution and is bounded. Let $LP(D)$ be the maximum value of its objective function. If $Y \subseteq X$ is a D-clique, then its inner distribution $(a_0, a_1, \dots, a_n)$ is a feasible solution of this linear program and*

$$|Y| \leq LP(D). \tag{2.19}$$

We call the bound (2.19) *linear programming bound* for cliques.

We note that, for the linear program (2.18), the dual eigenvalues have to be real-valued. That is why we assumed the association scheme to be symmetric. In the remainder of this thesis we almost exclusively work with the conjugacy class scheme of the finite general linear group, whose (dual) eigenvalues are not necessarily real-valued. For the conjugacy class scheme, we can modify the constraints of the linear program, so that we are still able to apply the linear programming method.

Let $(G, \{R_0, R_1, \ldots, R_n\})$ be the conjugacy class scheme arising from the finite group G having dual eigenvalues $Q_k(i)$ and multiplicities m_k. For $D \subseteq [n]$, the linear program for D-cliques in G is given by

$$\begin{aligned} \max_{x_i \in \mathbb{R}} \quad & \sum_{i \in D \cup \{0\}} x_i \\ \text{subject to} \quad & x_0 = 1, \\ & x_i \geq 0 \text{ for all } i \in D, \\ & x_i = 0 \text{ for } i \in [n] \setminus D, \\ \sum_{i \in D} \operatorname{Re}(Q_k(i)) x_i \geq -m_k \quad \text{and} \quad & \sum_{i \in D} \operatorname{Im}(Q_k(i)) x_i = 0 \text{ for all } k \in \{1, 2, \ldots, n\}, \end{aligned} \tag{2.20}$$

where the last constraint is due to the fact that all entries of the dual distribution of a subset of G are real and nonnegative, see Section 2.2. Moreover, using $x_0 = 1$ and $Q_0(i) = 1$, we note that the last constraint of (2.20) is equivalent to

$$\sum_{i \in D \cup \{0\}} Q_k(i) x_i \in \mathbb{R}_{\geq 0} \quad \text{for all } k \in \{0, 1, \ldots, n\}. \tag{2.21}$$

Then the corresponding result to Theorem 2.4.5 in the non-symmetric case is as follows.

Theorem 2.4.6. *Let $(G, \{R_0, R_1, \ldots, R_n\})$ be the conjugacy class scheme arising from the finite group G with dual eigenvalues $Q_k(i)$. Let D be a subset of $[n]$. Then the linear program (2.20) has at least one feasible solution and is bounded. Let $LP(D)$ be the maximum of its objective function. If $Y \subseteq G$ is a D-clique, then its inner distribution $(a_0, a_1, \ldots, a_n)$ is a feasible solution to this linear program and*

$$|Y| \leq LP(D). \tag{2.22}$$

Also in this case, we call the bound (2.22) *linear programming bound* for cliques.

Proof: A feasible solution for the linear program is given by $x_0 = 1$ and $x_i = 0$ for all $i \in D$, where we use that the multiplicities m_k are nonnegative integers. To prove that the program is bounded, let us assume that $(x_i)_{i \in D \cup \{0\}}$ is a feasible solution. Then, since $P_0(k) = 1$, and by using Lemma 2.1.17 we have, for $i \in D \cup \{0\}$,

$$\begin{aligned} &\sum_{k=0}^{n} (v_i - P_i(k)) \sum_{j \in D \cup \{0\}} Q_k(j) x_j \\ &= v_i \sum_{j \in D \cup \{0\}} x_j \sum_{k=0}^{n} P_0(k) Q_k(j) - \sum_{j \in D \cup \{0\}} x_j \sum_{k=0}^{n} P_i(k) Q_k(j) \\ &= |G|(v_i - x_i). \end{aligned} \tag{2.23}$$

Taking the real parts on both sides of (2.23) and making use of (2.21), it follows that

$$\sum_{k=0}^{n}(v_i - \mathrm{Re}(P_i(k))) \sum_{j\in D\cup\{0\}} Q_k(j)x_j = |G|(v_i - x_i) \quad \text{for all } i \in D \cup \{0\}. \tag{2.24}$$

Since, according to Lemma 2.1.18, $|P_i(k)| \leq v_i$ for all i, k, we especially have $\mathrm{Re}(P_i(k)) \leq v_i$ for all $i \in D \cup \{0\}$ and all $k \in \{0, 1, \ldots, n\}$. Together with (2.21) we can deduce from (2.24) that

$$0 \leq |G|(v_i - x_i) \quad \text{for all } i \in D \cup \{0\}$$

and thus

$$x_i \leq v_i \quad \text{for all } i \in D \cup \{0\},$$

which implies that the linear program is bounded.

Now, let Y be a D-clique in G with inner distribution $(a_0, a_1, \ldots, a_n)$. Then we have $a_0 = 1$, $a_i \geq 0$ for all i, $a_i = 0$ for all $i \in [n] \setminus D$, and since the dual distribution of Y is real-valued and nonnegative, we also have

$$\sum_{i\in D\cup\{0\}} Q_k(i)a_i \in \mathbb{R}_{\geq 0}.$$

Consequently, the inner distribution of a D-clique gives a feasible solution for the linear program, which proves the second statement. □

A nice application of the linear programming method is the so-called *Clique-Coclique bound* [Del73, Thm. 3.9] whose proof makes use of the dual linear program. The dual linear program corresponding to (2.20) is given by the following

$$\begin{aligned} \min_{\substack{y_0\in\mathbb{R},\\ y^1,y^2,y^3\in\mathbb{R}^n}} \quad & y_0 + \sum_{k=1}^{n} y_k^1 m_k \\ \text{subject to} \quad & y_0 = 1, \\ & y_k^1, y_k^2, y_k^3 \geq 0 \text{ for all } k \in [n], \\ \sum_{k=1}^{n} y_k^1 \mathrm{Re}(Q_k(i)) + \sum_{k=1}^{n} y_k^2 \mathrm{Im}(Q_k(i)) - & \sum_{k=1}^{n} y_k^3 \mathrm{Im}(Q_k(i)) \leq -1 \text{ for all } i \in D, \end{aligned} \tag{2.25}$$

Theorem 2.4.7 (Clique-coclique bound). *Let $(X, \{R_0, R_1, \ldots, R_n\})$ be an association scheme and, for $D \subset [n]$, let $Y \subseteq X$ be a D-clique and $Z \subseteq X$ an $([n] \setminus D)$-clique. Then we have*

$$|Y| \cdot |Z| \leq |X|.$$

The proof for the clique-coclique bound in the case of symmetric association schemes can be found in [Del73]. The general case follows by using the same arguments like in [Del73]. For the sake of completeness, we provide the proof for not necessarily symmetric association schemes.

Proof: Let $(a_0, a_1, \dots, a_n)$ and $(c_0, c_1, \dots, c_n)$ be the inner distributions of the D-clique Y and the $([n] \setminus D)$-clique Z, respectively. Then we define $(z_0, z_1, \dots, z_n)$ by

$$z_k = \frac{1}{|Z|} \frac{1}{m_k} \sum_{j=0}^{n} Q_k(j) c_j \quad \text{for all } k \in \{0, 1, \dots, n\},$$

where $Q_k(j)$ and m_k denote the dual eigenvalues and the multiplicities of the association scheme, respectively. Since $\sum_j c_j Q_k(j)$ is the dual distribution of Z, it turns out that $z_k \geq 0$ for all $k \in \{0, 1, \dots, n\}$. Moreover, since $Q_0(j) = 1$ and $m_0 = 1$, we have that $z_0 = 1$. Additionally, by taking Lemma 2.1.17 into account, for $i \in \{0, 1, \dots, n\}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n} z_k \operatorname{Re}(Q_k(i)) - \sum_{k=0}^{n} z_k \operatorname{Im}(Q_k(i)) &= \sum_{k=0}^{n} z_k \overline{Q_k}(i) \\
&= \frac{1}{|Z|} \sum_{j=0}^{n} c_j \sum_{k=0}^{n} \frac{1}{m_k} \overline{Q_k}(i) Q_k(j) \\
&= \frac{1}{|Z|} \sum_{j=0}^{n} c_j \sum_{k=0}^{n} \frac{1}{v_i} P_i(k) Q_k(j) \\
&= \frac{|X|}{|Z|} \frac{1}{v_i} \sum_{j=0}^{n} c_j \delta_{ij} \\
&= \frac{|X|}{|Z|} \frac{1}{v_i} c_i. \qquad (2.26)
\end{aligned}$$

Since Z is a $([n] \setminus D)$-clique, it follows that

$$\sum_{k=0}^{n} z_k \operatorname{Re}(Q_k(i)) - \sum_{k=0}^{n} z_k \operatorname{Im}(Q_k(i)) = 0 \quad \text{for all } i \in D$$

or equivalently

$$\sum_{k=1}^{n} z_k \operatorname{Re}(Q_k(i)) - \sum_{k=1}^{n} z_k \operatorname{Im}(Q_k(i)) = -1 \quad \text{for all } i \in D.$$

Consequently, taking $y_0 = z_0$, $y^1 = y^3 = (z_2, z_3, \dots, z_n)^T$, and $y^2 = 0 \in \mathbb{R}^n$ gives a feasible solution for the dual linear program (2.25).

We have that $(a_i)_{i \in D \cup \{0\}}$ is a feasible solution for the primal linear program (2.20). The weak duality theorem, Theorem 2.4.4, together with (2.26) for $i = 0$ imply that

$$|Y| = \sum_{i \in D \cup \{0\}} a_i \leq y_0 + \sum_{k=1}^{n} y_k^1 m_k = y_0 + \sum_{k=1}^{n} z_k Q_k(0) = \frac{|X|}{|Z|},$$

which completes the proof. □

To give an example, we apply the clique-coclique bound to give another proof of the upper bound for t-intersecting families of k-sets of $[n]$ from Theorem 2.3.8.

Example 2.4.8. Let n, k and t be positive integers such that $t \leq k \leq n/2$ and such that a t-Steiner system Z of size $\binom{n}{k}/\binom{n-t}{k-t}$ exists. Let Y be a t-intersecting set. Then Y is a D-clique with $D = \{i\colon 1 \leq i \leq k-t\}$ in the Johnson scheme $J(k,n)$. From Remark 2.3.12 it follows that Z is a $([k] \setminus D)$-clique. Applying the clique-coclique bound from Theorem 2.4.7, gives

$$|Y|\frac{\binom{n}{k}}{\binom{n-t}{k-t}} \leq \binom{n}{k}.$$

And thus the upper bound from Theorem 2.3.8 on the size of Y.

3 Subsets of the symmetric group

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In this chapter we survey results on combinatorial subsets of the symmetric group, specifically *intersecting sets*, *codes* and *transitive sets*. These sets arise as cliques and designs, respectively, in the conjugacy class scheme of the symmetric group.

This chapter is structured in four main sections. Initially, we summarise essential properties of the conjugacy class scheme arising from the symmetric group. Subsequently, our focus shifts to the *Erdős-Ko-Rado* theorems for permutations, which study *intersecting sets* of permutations. We then examine the converse scenarios, specifically *permutation codes*, which represent the "counterparts" to intersecting sets. The final section provides an algebraic characterisation of *transitive sets* of permutations.

In Chapters 5 and 6, containing the original parts of this thesis, we discuss q-analogs of the combinatorial objects introduced in this chapter.

3.1 The association scheme of the symmetric group

In this section we collect some properties of the conjugacy class scheme arising from the symmetric group by using its representation theory. See Section 1.2 for the necessary background on the representation theory of the symmetric group.

Since the cycle type of any permutation in the symmetric group S_n corresponds uniquely to one partition of n (see Example 1.1.2), there exists a one-to-one correspondence between the conjugacy classes of S_n and the partitions of n. Consequently, we can index the conjugacy classes of S_n, and thus also the adjacency matrices of the conjugacy class scheme arising from the symmetric group, with the partitions of n (see Example 2.1.9).

From Section 1.2 we find that also the irreducible characters of the symmetric group S_n are indexed by partitions of n. Henceforth, if not stated otherwise, we denote by χ^λ the irreducible character of S_n corresponding to the partition λ of n. Moreover, we recall that the trivial character of S_n is indexed by the partition (n).

Example 2.1.13 implies that the pairwise orthogonal matrices $E_\lambda \in \mathbb{C}(S_n, S_n)$ of the Bose-Mesner algebra arising from S_n are given by

$$E_\lambda(\pi, \tau) = \frac{\chi^\lambda(e)}{n!}\chi^\lambda(\pi^{-1}\tau).$$

Since both the conjugacy classes and the irreducible characters of the symmetric group S_n are indexed by partitions λ and μ of n, we write $P_\mu(\lambda)$ and $Q_\lambda(\mu)$ for the eigenvalues and dual eigenvalues. Consequently, from Example 2.1.20, it follows that the eigenvalues and dual eigenvalues of the conjugacy class scheme arising from the symmetric group S_n are given by

$$P_\mu(\lambda) = \frac{|C_\mu|}{\chi^\lambda(e)}\chi^\lambda_\mu, \qquad Q_\lambda(\mu) = \chi^\lambda(e)\chi^\lambda_\mu, \tag{3.1}$$

where λ and μ are partitions of n and where we have used the fact that the irreducible characters of the symmetric group are real-valued.

Due to (3.1), and since the trivial character and the trivial conjugacy class of S_n are indexed by (n) and (1^n), respectively, Lemma 2.1.16 specialises as follows.

Corollary 3.1.1. *For all partitions λ and μ of n, the eigenvalues $P_\mu(\lambda)$ and the dual eigenvalues $Q_\lambda(\mu)$ of the conjugacy class scheme arising from the symmetric group S_n have the following values:*

(i) $P_{(1^n)}(\lambda) = 1$,

(ii) $Q_{(n)}(\mu) = 1$,

(iii) $P_\mu((n)) = |C_\mu|$,

(iv) $Q_\lambda((1^n)) = \chi^\lambda(e)^2$.

The dual distribution $(b_\lambda)_{\lambda \vdash n}$ of a subset Y of S_n is indexed by the partitions of n and, from Example 2.2.6, we find that

$$b_\lambda = \frac{\chi^\lambda(e)}{|Y|}\sum_{\pi,\tau \in Y} \chi^\lambda(\pi^{-1}\tau).$$

Since the trivial character of S_n corresponds to the partition (n), we obtain

$$b_{(n)} = \frac{\chi^{(n)}(e)}{|Y|}\sum_{\pi,\tau \in Y} \chi^{(n)}(\pi^{-1}\tau) = \frac{1}{|Y|}|Y|^2 = |Y|.$$

Consequently, $b_{(n)}$ plays the role of b_0 in Section 2.2.

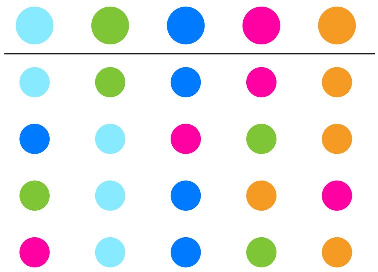


Figure 3.1: An intersecting subset of S_5 consisting of four permutations. The second to fifth rows each provide a permutation of the five colours in the first. Taking any two of these permutations there exists at least one colour on which they are equal.

3.2 Erdős-Ko-Rado theorems

In Section 2.3.2 we studied intersecting families of k-sets. Erdős, Ko, and Rado answered the question of how large such a family can be and also characterised the extremal case, see Theorem 2.3.4. Since the 1960s, many similar problems for different objects and different notions of intersection have been studied, see [GM16], for example. In this section we focus first on one such problem, arising in the context of the symmetric group. We collect, on the one hand, upper bounds on the sizes of these sets, and on the other hand, characterisations of the extremal cases.

Definition 3.2.1. Two permutations $\pi, \sigma \in S_n$ are *intersecting* if there exists an integer $i \in [n]$ such that $\sigma(i) = \pi(i)$. A subset Y of the symmetric group S_n is *intersecting* if all pairs in $Y \times Y$ are intersecting.

Equivalently, a subset $Y \subseteq S_n$ is intersecting if for every two permutations $\pi, \sigma \in Y$ the product $\pi^{-1}\sigma$ has at least one fixed point in $[n]$. Translating this into the language of association schemes, an intersecting set of permutations is a D-clique in the conjugacy class scheme arising from the symmetric group on n elements, where $D = \{(\mu_1, \mu_2, \dots) \vdash n \colon \exists i \text{ with } \mu_i = 1\}$.

Figure 3.1 illustrates one example of an intersecting set of permutations.

As in the classical case, we are interested in finding an upper bound on the size of intersecting sets of permutations. Deza and Frankl [DF77] were first to prove this upper bound.

Theorem 3.2.2 ([DF77]). *Let Y be an intersecting subset of S_n, then $|Y| \leq (n-1)!$.*

By now, many different proofs of this result exist in the literature. For example, we can apply the clique-coclique bound from Theorem 2.4.7, or the Hoffman bound from

Theorem 2.3.2, to a *Cayley graph* generated by the set of fixed-point-free permutations. For more details on this see [GM16, Chapter 14], for example.

The characterisation of the extremal case was conjectured by Deza and Frankl in 1977 [DF77] and proved nearly 30 years later by Cameron and Ku [CK03], and independently by Larose and Malventuo [LM04].

Theorem 3.2.3 ([CK03], [LM04]). *If an intersecting set in S_n meets the bound in Theorem 3.2.2, then it is a coset of the stabiliser of a point in $[n]$.*

An algebraic approach to the proof of this result is the classical Hoffman bound. For more details we refer to [GM16]. Similar to the classical case in Section 2.3.2, we can ask for a strengthened version of this result, which is in terms of *t-intersecting* permutations.

Definition 3.2.4. Let t be a positive integer. Two permutations $\pi, \sigma \in S_n$ are *t-intersecting* if there exist t distinct elements $i_1, i_2, \dots, i_t$ in $[n]$ such that $\pi(i_\ell) = \sigma(i_\ell)$ for all $\ell \in \{1, 2, \dots, t\}$. A subset Y of the symmetric group S_n is *t-intersecting* if all pairs in $Y \times Y$ are t-intersecting.

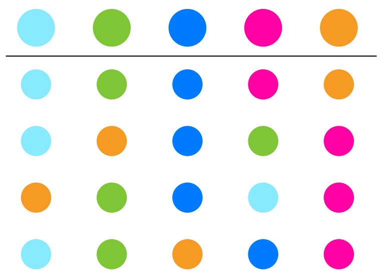


Figure 3.2: A 2-intersecting set of S_5 consisting of four permutations. The second to fifth rows each provide a permutation of the five colours of the first. For any two of these permutations there exist at least two colours on which they are equal.

In other words, a subset Y of the symmetric group S_n is t-intersecting if, for all permutations π, σ in Y, the product $\pi^{-1}\sigma$ has at least t fixed points in $[n]$. We note that similarly to the 1-intersecting case, we can interpret a t-intersecting set of permutations as a clique in the conjugacy class scheme arising from the symmetric group. Figure 3.2 illustrates an example of a 2-intersecting set of S_5.

Every coset of the stabiliser of t distinct elements of $[n]$ is a t-intersecting set of the symmetric group of size $(n-t)!$. These cosets play a crucial role.

Theorem 3.2.5 ([EFP11]). *For t fixed and n sufficiently large compared to t, a t-intersecting set Y in S_n has size at most $(n-t)!$.*

We note that the bound in Theorem 3.2.5 is sharp.

However, it is not possible to prove Theorem 3.2.5 using the classical Hoffman bound for the *Cayley graph* generated by all permutations having less than t fixed points (see [EFP11]). Instead Ellis, Friedgut, and Pilpel [EFP11] apply the weighted version of the Hoffman bound from Theorem 2.3.11. In [EFP11], the regular spanning subgraphs from Theorem 2.3.11 are given by the adjacency matrices corresponding to certain conjugacy classes which do not fix t distinct elements of $[n]$. Then, the eigenvalues of these subgraphs are given by $P_\lambda(\mu)$ from (3.1). Since the adjacency matrices of the conjugacy class scheme all have common eigenspaces, the eigenvalues of a weighted sum of these are weighted sums of the $P_\lambda(\mu)$. Manipulating the weights in the desired way and applying the weighted version of the Hoffman bound then gives the upper bound on the size of t-intersecting sets from Theorem 3.2.5. However, showing the existence of the desired weights is not trivial and involves a lot of representation theory. For example, one difficulty is given by the fact that the irreducible characters χ^λ and $\chi^{\lambda'}$ of S_n are equal in absolute value.

Ellis, Friedgut and Pilpel also obtain a partial characterisation result by using the decomposition of the permutation character on tabloids.

Theorem 3.2.6 ([EFP11]). *Let Y be a t-intersecting set in S_n whose size meets the bound in Theorem 3.2.5. If n is sufficiently large compared to t, then the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of t distinct elements of $[n]$.*

The main tasks in applying the weighted version of the Hoffman bound are choosing the right adjacency matrices, finding appropriate weights and controlling the smallest eigenvalue of the weighted linear combination. In Chapter 5, in a q-analog setting, we apply the weighted version of the Hoffman bound for *t-intersecting sets* in the finite general linear group and we use a similar strategy there.

The full characterisation of the extremal case of t-intersecting sets of permutations is as follows.

Theorem 3.2.7 ([EFP11]). *Let Y be a t-intersecting set in S_n whose size meets the bound in Theorem 3.2.5. If n is sufficiently large compared to t, then Y is a coset of the stabiliser of t distinct elements of $[n]$.*

This full characterisation result follows from the following stability result proved by Ellis.

Theorem 3.2.8 ([Ell11]). *Let Y be a t-intersecting set in S_n. For n sufficiently large compared to t, it holds that if Y is not contained in a coset of the stabiliser of t distinct elements of $[n]$, then $|Y| \leq \left(1 - \frac{1}{e} + o(1)\right)(n-t)!$.*

In 2024, Keller, Lifshitz, Minzer, and Sheinfeld [KLMS24] obtained a linear bound on the size of n compared to t for the validity of the Theorems 3.2.5 and 3.2.7.

Theorem 3.2.9 ([KLMS24]). *There exists a constant c such that for all $n \geq ct$, Theorems 3.2.5 and 3.2.7 hold.*

As in the classical case, also *t-cross-intersecting* subsets of the symmetric group have been studied.

Definition 3.2.10. Two subsets Y and Z of S_n are *t-cross-intersecting* if every pair in $Y \times Z$ is t-intersecting.

The result for t-cross intersecting sets of permutations is as follows.

Theorem 3.2.11 ([EFP11]). *Let t be a positive integer and let Y and Z be t-cross-intersecting subsets of the symmetric group S_n. If n is sufficiently large compared to t, then $|Y| \cdot |Z| \leq ((n-t)!)^2$, and equality holds if and only if $Y = Z$ and Y is a coset of the stabiliser of t distinct points of $[n]$.*

The bound on t-cross-intersecting sets comes as a byproduct from the proof of Theorem 3.2.5.

A more general notion of t-intersection is the *t-set-intersection* of permutations.

Definition 3.2.12. Two permutations $\pi, \sigma \in S_n$ are *t-set-intersecting* if there exists a t-subset I of $[n]$ such that $\sigma(I) = \pi(I)$. A subset Y of S_n is *t-set-intersecting* if every pair in $Y \times Y$ is t-set-intersecting.

Again every coset of the stabiliser of a t-set of $[n]$ is t-set-intersecting and, for n sufficiently large compared to t, it turns out that these are the only t-set-intersecting sets of maximal size.

Theorem 3.2.13 ([Ell12]). *For t fixed and n sufficiently large compared to t, a t-set-intersecting set Y in S_n has size at most $t!(n-t)!$ and, in case of equality, Y is a coset of the stabiliser of a t-set of $[n]$.*

In order to prove the result on t-set-intersecting sets in [Ell12], Ellis applied the weighted version of the Hoffman bound from Theorem 2.3.11.

In Chapter 5 we study q-analog settings, namely *t-intersecting*, *t-cross-intersecting*, *t-space-intersecting*, and *t-space-cross-intersecting* sets in the finite general linear group.

3.3 Permutation codes

In the previous section we studied t-intersecting sets of permutations. More precisely we studied subsets $Y \subseteq S_n$ having the property that $\pi^{-1}\sigma$ has at least t fixed points for every $\pi, \sigma \in Y$. In this section we study the "counterpart", namely subsets $Y \subseteq S_n$ such that $\pi^{-1}\sigma$ has at most t fixed points.

Definition 3.3.1. For a positive integer d, a subset Y of the symmetric group S_n is a *d-code* if for all $\pi, \sigma \in Y$ the permutation $\pi^{-1}\sigma$ has at most $n - d$ fixed points.

These codes are also known as *permutation codes*.

Remark 3.3.2. It is common to define a d-code of permutations in terms of the *Hamming distance* d_H, which is given by

$$d_H(\pi, \sigma) = \#\{i \in [n] \colon \pi(i) \neq \sigma(i)\},$$

where $\pi, \sigma \in S_n$. Then a subset $Y \subseteq S_n$ is a d-code if $d_H(\pi, \sigma) \geq d$ for all $\pi, \sigma \in Y$.

The following upper bound on the size of a d-code in S_n was obtained by [BCD79]. We also refer to [Tar99] for a proof involving linear programming and the representation theory of S_n.

Theorem 3.3.3. *Let $Y \subseteq S_n$ be a d-code, then $|Y| \leq n(n-1)\cdots d$.*

3.4 Transitive sets

In this section we study *transitive sets* (not groups) of permutations. It turns out that they are certain T-designs in the conjugacy class scheme arising from the symmetric group.

Before giving the definition of a *transitive subset*, we recall the special case of a *t-homogeneous* subgroup of permutations. A subgroup of the symmetric group S_n is *t-homogeneous* if it acts transitively on the t-subsets of $[n]$. Livingstone and Wagner proved the following famous result on t-homogeneous subgroups.

Theorem 3.4.1 ([LW65]). *If a subgroup G of S_n is t-homogeneous for some t satisfying $1 \leq t \leq \frac{n}{2}$ then G is also $(t-1)$-homogeneous.*

Martin and Sagan [MS06] generalised this theorem in two ways. They replaced subgroups of S_n by subsets and replaced t-subsets of $[n]$ by tabloids of $[n]$, which are basically set partitions of $[n]$.

Definition 3.4.2. For a partition σ of n, a subset Y of S_n is *transitive* on the set of σ-tabloids if there is a constant r such that the following holds: for all σ-tabloids S, T, there are exactly r elements $\pi \in Y$ such that $\pi(S) = T$.

Figure 3.3 illustrates an example of a subset of S_5 that is transitive on $(4, 1)$-tabloids.

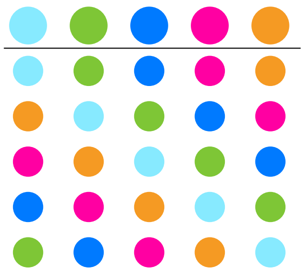


Figure 3.3: A subset of S_5 that is transitive on $(4, 1)$-tabloids. The second to fifth rows each provide a permutation of the five colours of the first. For any two colours there is exactly one permutation that maps the one colour to the other.

We note that if Y is a subgroup of S_n, then the notion of a transitive subset and that of a transitive subgroup coincide.

We are able to determine the constant r in the definition of a subset Y of permutations being transitive on σ-tabloids by double counting the set

$$A = \{(\pi, \sigma) \in Y \times S_n \colon \pi(T) = \sigma(T)\},$$

where T is a σ-tabloid and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ is a partition of n. On the one hand, we have $|A| = |Y|\sigma_1! \cdots \sigma_k!$ and, on the other hand, it is $|A| = |S_n| \cdot r$, from which we conclude that

$$r = \frac{\sigma_1!\sigma_2! \cdots \sigma_k!}{n!}|Y|. \tag{3.2}$$

One of the main results in [MS06] is the following.

Theorem 3.4.3 ([MS06]). *Let σ be a partition of n. A subset Y of the symmetric group S_n is transitive on σ-tabloids if and only if the dual distribution $(b_\lambda)_\lambda$ of Y satisfies*

$$b_\lambda = 0 \quad \text{for all } \sigma \trianglelefteq \lambda \triangleleft (n).$$

On the one hand, this theorem gives a combinatorial interpretation for T-designs, where $T = \{\lambda \colon \sigma \trianglelefteq \lambda \triangleleft (n)\}$, in the conjugacy class scheme arising from the symmetric group. On the other hand, it provides an algebraic notion for the combinatorial object of a transitive set of permutations.

As a consequence Martin and Sagan obtained the following result.

Corollary 3.4.4 ([MS06]). *Let Y be a subset of S_n that is transitive on σ-tabloids. Then G is also transitive on τ-tabloids for all τ satisfying $\tau \trianglerighteq \sigma$.*

From this result the Livingstone-Wagner Theorem 3.4.1 follows as a corollary by noticing that a t-homogeneous subgroup is a subgroup acting transitively on $(n-t,t)$-tabloids and taking $\sigma = (n-t,t)$ and $\tau = (n-t+1, t-1)$.

In [MS06], the proofs of the Theorems 3.4.3 and 3.4.4 make use of the conjugacy class scheme arising from the symmetric group. Thus these proofs heavily use the representation theory of the symmetric group and especially the decomposition of the permutation character on σ-tabloids.

Considering (3.2), we note that a subset Y of the symmetric group S_n is transitive on σ-tabloids if and only if the following equation holds

$$\frac{1}{|Y|}\sum_{\pi\in Y} \mathbb{1}_{\pi(S)=T} = \frac{1}{|S_n|}\sum_{\pi\in S_n} \mathbb{1}_{\pi(S)=T} \qquad \text{for all } \sigma\text{-tabloids } S, T.$$

As a consequence we can understand transitive subsets of permutations as subsets that locally approximate the symmetric group. Thus, we are interested in determining whether small subsets of this type exist.

The existence of small transitive subsets of the symmetric group is another main result in [MS06]. Its proof is based on a recursive construction.

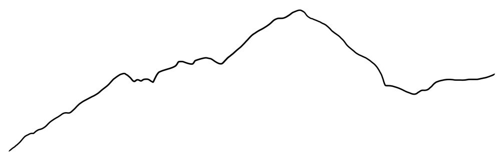
Lemma 3.4.5 ([MS06]). *Taking a combinatorial t-(n,k,λ) design, a subset of S_k that is transitive on $(k-t,1^t)$-tabloids, and a subset of S_{n-k} that is transitive on $(n-k-t,1^t)$-tabloids implies the existence of a subset of S_n that is transitive on $(n-t,1^t)$-tabloids.*

Together with the existence result of Teirlinck [Tei87], which gives the existence of combinatorial t-(n,k,λ) designs for given t and n sufficiently large, Martin and Sagan obtained the following asymptotic existence result of arbitrary small transitive subsets with regard to the size of the symmetric group.

Theorem 3.4.6 ([MS06]). *Let $\sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_k$ be a nonincreasing sequence of positive integers and let $\varepsilon > 0$. Then for all sufficiently large $\sigma_1 \geq \sigma_2$ there exists a subset Y of the symmetric group S_n, where $n = \sigma_1 + \sigma_2 + \ldots, +\sigma_k$, that is transitive on $(\sigma_1, \sigma_2, \ldots, \sigma_k)$-tabloids satisfying $|Y|/n! < \varepsilon$.*

In Chapter 6 we study q-analog problems and replace the symmetric group S_n by the finite general linear group $\mathrm{GL}(n,q)$.

4 Finite general linear groups



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From now on we focus on q-analog settings of those we collected for the symmetric group in Chapter 3. Consequently, we work with the finite general linear group $\mathrm{GL}(n, \mathbb{F}_q)$. In the following we write $\mathrm{GL}(n, q)$ instead of $\mathrm{GL}(n, \mathbb{F}_q)$. The purpose of this chapter is to collect all the properties needed for understanding the main results in this thesis. First, we recall already known results, including conjugacy classes and irreducible characters of $\mathrm{GL}(n, q)$ in Sections 4.1 and 4.2, respectively. Then, in Section 4.3, we give new results and decompose certain permutation and permutation-like characters of $\mathrm{GL}(n, q)$ into their irreducible constituents.

4.1 Conjugacy classes

In this section we study the indexing of the conjugacy classes of the finite general linear group. Moreover, we find appropriate representatives for each conjugacy class. For this, we follow [Mac15, Ch. IV.2].

Let Par be the set of integer partitions, where we denote the unique partition of 0 by $\varnothing$. Moreover, we write Φ for the set of all monic irreducible polynomials in $\mathbb{F}_q[X]$ that are distinct from X.

For every matrix $g \in \mathrm{GL}(n, q)$ we can define a multiplication of the polynomial ring $\mathbb{F}_q[X]$ on the finite dimensional vector space $\mathbb{F}_q^n$ by

$$\mathbb{F}_q[X] \times \mathbb{F}_q^n \to \mathbb{F}_q^n, \quad (f(X), v) \mapsto f(g)v.$$

This makes $\mathbb{F}_q^n$ an $\mathbb{F}_q[X]$-module that we denote by V_g. We observe that two matrices g and h in $\mathrm{GL}(n, q)$ are conjugate if and only if the corresponding $\mathbb{F}_q[X]$-modules V_g

and V_h are isomorphic. Thus we can write V_c instead of V_g, where c is the conjugacy class of g in $\mathrm{GL}(n,q)$.

We note that V_c is a finitely generated module over the principal ideal domain $\mathbb{F}_q[X]$. Hence, the *structure theorem for finitely generated modules over a principal ideal domain* (see [DF91, Ch. 12, Thm. 6], for example), gives that V_c is isomorphic to a sum of cyclic modules. More precisely, there exists a unique partition valued function $\underline{\lambda}_c \colon \Phi \to \mathrm{Par}$ such that

$$V_c \cong \bigoplus_{f\in\Phi} \bigoplus_{i\geq 1} \mathbb{F}_q[X]/(f^{\underline{\lambda}_c(f)_i}) \tag{4.1}$$

as $\mathbb{F}_q[X]$-modules. Comparing dimensions on both sides gives

$$\begin{aligned} n = \dim_{\mathbb{F}_q} V_c &= \sum_{f\in\Phi}\sum_{i\geq 1} \dim_{\mathbb{F}_q}(\mathbb{F}_q[X]/(f^{\underline{\lambda}_c(f)_i})) \\ &= \sum_{f\in\Phi}\sum_{i\geq 1} \deg(f)\underline{\lambda}_c(f)_i \\ &= \sum_{f\in\Phi} \deg(f)|\underline{\lambda}_c(f)|. \end{aligned}$$

Moreover, for a given partition valued function $\underline{\lambda} \colon \Phi \to \mathrm{Par}$, there exists a unique conjugacy class c of $\mathrm{GL}(n,q)$ such that $\underline{\lambda} = \underline{\lambda}_c$ if and only if $\sum_{f\in\Phi} \deg(f)|\underline{\lambda}(f)| = n$.

In the following we omit the index and write $\underline{\lambda}$ instead of $\underline{\lambda}_c$ if it is clear from context, which conjugacy class we consider.

Definition 4.1.1. The *size* of $\underline{\lambda} \colon \Phi \to \mathrm{Par}$ is $\|\underline{\lambda}\| = \sum_{f\in\Phi} \deg(f)|\underline{\lambda}(f)|$. And we put $\Lambda_n = \{\underline{\lambda} \in \Phi \colon \|\underline{\lambda}\| = n\}$.

Using the isomorphism (4.1) we can determine a representative, the so-called *Jordan canonical form*, for each conjugacy class of $\mathrm{GL}(n,q)$:

We recall that the *companion matrix* of a polynomial $f \in \Phi$ with $f = X^d + f_{d-1}X^{d-1} + \cdots + f_1X + f_0$ is

$$C(f) = \begin{bmatrix} & & & & -f_0 \\ 1 & & & & -f_1 \\ & 1 & & & -f_2 \\ & & \ddots & & \vdots \\ & & & 1 & -f_{d-1} \end{bmatrix} \in \mathbb{F}_q^{d\times d}.$$

(where blanks are filled with zeros). For $f \in \Phi$ of degree d and a positive integer k, we write

$$C(f,k) = \begin{bmatrix} C(f) & I & & & \\ & C(f) & I & & \\ & & \ddots & \ddots & \\ & & & \ddots & I \\ & & & & C(f) \end{bmatrix} \in \mathbb{F}_q^{kd \times kd},$$

and for $f \in \Phi$ and $\mu \in \mathrm{Par}$, we define $C(f,\mu)$ to be the block diagonal matrix of order $|\mu| \cdot \deg(f)$ with blocks $C(f,\mu_1), C(f,\mu_2), \dots$. Finally, with every $\underline{\mu} \in \Lambda_n$ we associate the block diagonal matrix $R_{\underline{\mu}}$ of order n whose blocks are $C(f,\underline{\mu}(f))$, where f ranges through the support of $\underline{\mu}$. Then every element g of $\mathrm{GL}(n,q)$ is conjugate to exactly one matrix $R_{\underline{\mu}}$ for $\underline{\mu} \in \Lambda_n$, which is called the *Jordan canonical form* of g. We denote by $C_{\underline{\mu}}$ the conjugacy class containing $R_{\underline{\mu}}$. We note that $C_{X-1\mapsto(1^n)}$ is the conjugacy class containing the identity. In the following, we denote the identity matrix of $\mathrm{GL}(n,q)$ by 1.

An explicit expression for the size of a conjugacy class of $\mathrm{GL}(n,q)$ was first obtained by Stanley [Sta12].

Theorem 4.1.2 ([Sta12]). *For each $\underline{\sigma}\colon \Phi \to \mathrm{Par}$ with $\|\underline{\sigma}\| = n$ we have*

$$\frac{|\mathrm{GL}(n,q)|}{|C_{\underline{\sigma}}|} = \prod_{f\in\Phi} \prod_{i=1}^{|\underline{\sigma}(f)|} \prod_{j=1}^{\mathrm{m}_i(\underline{\sigma}(f))} q^{|f|\,\mathrm{s}_i(\underline{\sigma}(f)')}(1-q^{-|f|j}),$$

where $|f| = \deg(f)$, $\mathrm{m}_i(\sigma) = \#\{j \geq 1\colon \sigma_j = i\}$, and $\mathrm{s}_i(\sigma) = \sum_{j=1}^{i} \sigma_j$ for every partition σ.

Example 4.1.3. We consider the finite general linear group $\mathrm{GL}(3,2)$. The irreducible polynomials in $\mathbb{F}_2[X] \setminus \{X\}$ of degree less than or equal to 3 are

$$f_1 = X - 1,$$

$$f_2 = X^2 + X + 1,$$

$$f_3 = X^3 + X^2 + 1,$$

$$\tilde{f}_3 = X^3 + X + 1.$$

There are 6 partition valued functions whose size is equal to 3, namely

$$f_1 \mapsto (1,1,1), \quad f_1 \mapsto (2,1) \quad f_1 \mapsto (3)$$

$$f_3 \mapsto (1), \quad \tilde{f}_3 \mapsto (1)$$

$$\underline{\lambda}\colon f_1 \mapsto (1), f_2 \mapsto (1).$$

Table 4.1 gives an overview of the 6 conjugacy classes of $\mathrm{GL}(3,2)$. In Section 4.2 we also provide the conjugacy classes of $\mathrm{GL}(4,2)$.

Table 4.1: **The conjugacy classes of** $\mathrm{GL}(3,2)$**.**

Indexing	**Representative**	**Size**
$f_1 \mapsto (1,1,1)$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	1
$f_1 \mapsto (2,1)$	$\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$	21
$f_1 \mapsto (3)$	$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$	42
$f_3 \mapsto (1)$	$\begin{bmatrix}0&0&1\\1&0&0\\0&1&1\end{bmatrix}$	24
$\tilde{f}_3 \mapsto (1)$	$\begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix}$	24
$\underline{\lambda}$	$\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix}$	56

4.2 Representation theory

The representation theory of the finite general linear group $\mathrm{GL}(n,q)$ plays a crucial role in this thesis. In the Chapters 5 and 6, containing the original results, we study subsets of the finite general linear group that we embed in the association scheme arising from this group. Recall from Chapter 2 that we can express the eigenvalues and dual eigenvalues of the conjugacy class scheme arising from any finite group in terms of its irreducible characters. To gain better insights into this, we recall the necessary background on the representation theory of the finite general linear group in this section. Since we almost exclusively work with characters instead of representations or modules, we present the following definitions and properties only in terms of characters. For more background, we refer the reader to [Mac15, Chapter IV] and [Jam86].

The complete set of (complex) irreducible characters has been obtained by Green [Gre55]. The complex irreducible representations were obtained by Gelfand [Gel70] and the irreducible representations over fields of nondefining characteristic were obtained by James [Jam86]. Our approach to obtain the irreducible characters of the finite general linear group follows [Jam86] and is similar to the method we used to obtain the irreducible characters of the symmetric group in Section 1.2. More precisely, similarly to the symmetric group, we construct characters $\xi^{f\mapsto\mu}$ of $\mathrm{GL}(n,q)$ such that the *first* one $\xi^{f\mapsto(n)}$ is irreducible, the *second* one can be written as copies of the first $\xi^{f\mapsto(n)}$ plus a single copy of the new irreducible character, and so on. However, in contrast to the symmetric group, the constructed characters are not necessarily permutation characters, and we cannot obtain all irreducible characters of $\mathrm{GL}(n,q)$ this way.

The construction of the characters of $\mathrm{GL}(n,q)$ which gives us all irreducible characters, relies heavily on *parabolic induction*, which is the induction of characters from *parabolic subgroups* to $\mathrm{GL}(n,q)$.

We recall from Definition 1.3.6 that a composition of the positive integer n is a sequence of nonnegative integers that sum up to n.

Definition 4.2.1. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a composition of n. The *parabolic subgroup* P_λ of $\mathrm{GL}(n,q)$ is the subgroup of $\mathrm{GL}(n,q)$ consisting of block upper-triangular matrices with block sizes $\lambda_1, \lambda_2, \dots, \lambda_k$, namely

$$P_\lambda = \left\{ \begin{bmatrix} A_1 & * & \cdots & * \\ & A_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & A_k \end{bmatrix} : A_i \in \mathrm{GL}(\lambda_i, q) \right\}. \tag{4.2}$$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a composition of n and let $\pi_i \colon P_\lambda \to \mathrm{GL}(\lambda_i, q)$ be the projection of the i-th block of the diagonal, so that

$$\pi_i \colon \begin{bmatrix} A_1 & * & \cdots & * \\ & A_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & A_k \end{bmatrix} \mapsto A_i. \tag{4.3}$$

We note that, for class functions ϕ_i of $\mathrm{GL}(\lambda_i, q)$, the product

$$\prod_i (\phi_i \circ \pi_i)$$

is a class function of the parabolic subgroup P_λ.

Definition 4.2.2. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a composition of n and let ϕ_i be class functions of $\mathrm{GL}(\lambda_i, q)$ for all $1 \le i \le k$. We define the *parabolic induction* as the product $\phi_1 \odot \phi_2 \odot \cdots \odot \phi_k$, which is the induction of the class function $\prod_{i=1}^k (\phi_i \circ \pi_i)$ from P_λ to $\mathrm{GL}(n,q)$, that is

$$\bigodot_{i=1}^{k} \phi_i = \mathrm{Ind}_{P_\lambda}^{\mathrm{GL}(n,q)} \left(\prod_{i=1}^{k} (\phi_i \circ \pi_i) \right), \tag{4.4}$$

where π_i denotes again the projection given in (4.3).

In the following we summarise James' construction [Jam86] of the irreducible characters of $\mathrm{GL}(n,q)$. Unlike in [Jam86], we do not use the terms of modules, since we will almost exclusively work with characters in this thesis.

Similar to the conjugacy classes, the irreducible characters of the finite general linear group $\mathrm{GL}(n,q)$ are indexed by the partition-valued functions Λ_n. We write $\chi^{\underline{\lambda}}$ for the irreducible character associated with $\underline{\lambda} \in \Lambda_n$. Moreover, we write $\chi^{f\mapsto\lambda}$ for $\chi^{\underline{\lambda}}$ if $\underline{\lambda}$ is only supported on $f \in \Phi$ with $\underline{\lambda}(f) = \lambda$. The characters $\chi^{f\mapsto\lambda}$ are called *primary* irreducible characters of $\mathrm{GL}(n,q)$.

In this section, if not stated otherwise, let $f \in \Phi$ be an irreducible polynomial in $\mathbb{F}_q[X] \setminus \{X\}$ of degree d and let k be a positive integer. There are five steps to obtain all irreducible characters of $\mathrm{GL}(n,q)$:

(I): In order to construct the primary irreducible characters James starts with the *cuspidal characters* $\chi^{f\mapsto(1)}$ of $\mathrm{GL}(d,q)$. In [Jam86, p. 241 and Thm. 3.6] these characters are denoted by ψ_s and the corresponding module by $M_F(s,(1))$. The cuspidal characters are irreducible and zero on all conjugacy classes except on those corresponding to the partition valued functions which are only supported on one polynomial. The existence of these cuspidal characters was proved by Green [Gre55]. The degree of the cuspidal character $\chi^{f\mapsto(1)}$ is given by [Jam86, p. 242]

$$\chi^{f\mapsto(1)}(1) = (q-1)(q^2-1)\cdots(q^{d-1}-1).$$

(II): Using the cuspidal characters of $\mathrm{GL}(d,q)$ James constructs characters $\xi^{f\mapsto(1^k)}$ of $\mathrm{GL}(dk,q)$ by parabolic induction, namely

$$\xi^{f\mapsto(1^k)} = \chi^{f\mapsto(1)} \odot \chi^{f\mapsto(1)} \odot \ldots \odot \chi^{f\mapsto(1)},$$

where $\chi^{f\mapsto(1)}$ is a cuspidal character of $\mathrm{GL}(d,q)$ and there are exactly k copies of $\chi^{f\mapsto(1)}$ on the right hand side. In [Jam86, Def. 4.2], the module corresponding to $\xi^{f\mapsto(1^k)}$ is denoted by $M_F(s,(1^k))$.

(III): For a partition μ of k, James defines the characters $\xi^{f\mapsto\mu}$ of $\mathrm{GL}(dk,q)$ based on $\xi^{f\mapsto(1^k)}$. He defines a map F_μ on the set of characters of $\mathrm{GL}(dk,q)$ such that $F_\mu(\xi^{f\mapsto(1^k)}) = \xi^{f\mapsto\mu}$. In fact, the notion is consistent because $F_{(1^k)}(\xi^{f\mapsto(1^k)}) = \xi^{f\mapsto(1^k)}$. James denotes the modules associated with $\xi^{f\mapsto\mu}$ by $M_F(s,\mu)$, see [Jam86, Sec. 6]. Writing $\mu = (\mu_1,\mu_2,\ldots,\mu_k)$, the characters $\xi^{f\mapsto\mu}$ satisfy [Jam86, (6.2)]

$$\xi^{f\mapsto\mu} = \bigodot_{i=1}^{k} \xi^{f\mapsto(\mu_i)}. \tag{4.5}$$

It holds that the characters $\xi^{f\mapsto(k)}$ are irreducible in $\mathrm{GL}(dk,q)$ ([Jam86, Lemma 6.3]).

(IV): Moreover, in some way, the character $\xi^{f\mapsto\mu}$ plays the same role as the permutation character of the symmetric group on μ-tabloids. Similarly as in Section 1.2, we obtain the primary irreducible characters of the finite general linear group from the characters $\xi^{f\mapsto\mu}$. Since, for every positive integer k, the character $\xi^{f\mapsto(k)}$ is irreducible in $\mathrm{GL}(dk,q)$ we name this character $\chi^{f\mapsto(k)}$. Then the decomposition of $\xi^{f\mapsto(k-1,1)}$ into irreducible characters consists of copies of $\chi^{f\mapsto(k)}$ and a single copy of the new irreducible character $\chi^{f\mapsto(k-1,1)}$. We define the primary characters $\chi^{f\mapsto\mu}$ of $\mathrm{GL}(dk,q)$ by

$$\chi^{f\mapsto\mu} = \xi^{f\mapsto\mu} - \sum_{\lambda\rhd\mu} K_{\lambda\mu}\chi^{f\mapsto\lambda},$$

where the $K_{\lambda\mu}$ denote the Kostka numbers from Definition 1.2.4. The characters $\chi^{f\mapsto\mu}$ are irreducible characters of $\mathrm{GL}(dk,q)$, where μ is a partition of k, and it holds [Jam86, (7.19)]

$$\xi^{f\mapsto\mu} = \sum_{\lambda\trianglerighteq\mu} K_{\lambda\mu}\chi^{f\mapsto\lambda}. \tag{4.6}$$

Consequently, comparing this with Theorem 1.2.6, we observe that the character $\xi^{f\mapsto\mu}$ is similar to the permutation character of the symmetric group on μ-tabloids.

(V): Using this approach, we find all primary irreducible characters of the finite general linear group. We obtain the remaining irreducible ones by *gluing* the primary ones together, namely for every $\underline{\lambda} \in \Lambda_n$ the character $\chi^{\underline{\lambda}}$ given by

$$\chi^{\underline{\lambda}} = \bigodot_{f\in\Phi} \chi^{f\mapsto\underline{\lambda}(f)} \tag{4.7}$$

is an irreducible character of $\mathrm{GL}(n,q)$. Moreover, these are all irreducible characters of the finite general linear group $\mathrm{GL}(n,q)$ [Jam86, p.267f].

We note that we are using the indexing of [Jam86] for the irreducible characters of $\mathrm{GL}(n,q)$. In contrast in [Mac15] for example, λ is replaced by the conjugate λ'.

Example 4.2.3. The character table of $\mathrm{GL}(3,2)$ is given in Table 4.2. In Section 4.2 we also provide the character table of $\mathrm{GL}(4,2)$.

In the following, we focus on some identities in the spirit of (4.6). From (4.6) and by using linear algebra, it follows that there exist integers $H_{\mu\lambda}$ satisfying

$$\chi^{f\mapsto\lambda} = \sum_{\mu} H_{\mu\lambda}\xi^{f\mapsto\mu} \tag{4.8}$$

and

$$H_{\lambda\lambda} = 1 \text{ and } H_{\mu\lambda} \neq 0 \Rightarrow \mu \trianglerighteq \lambda \tag{4.9}$$

(see[Mac15, I.6, Ex.4], for example).

Table 4.2: **The character table of** GL(3, 2) (see [Gor22], for example), where we use the notation from Example 4.1.3. In this table $A = \zeta_7 + \zeta_7^2 + \zeta_7^4$, where $\zeta_7 = \exp(2\pi i/7)$ is a 7-th root of unity and $\overline{A}$ denotes the complex conjugate of A.

	$f_1 \mapsto (1,1,1)$	$f_1 \mapsto (2,1)$	$f_1 \mapsto (3)$	$f_3 \mapsto (1)$	$\tilde{f}_3 \mapsto (1)$	$\underline{\lambda}$
$f_1 \mapsto (3)$	1	1	1	1	1	1
$f_1 \mapsto (2,1)$	6	2	0	−1	−1	0
$f_1 \mapsto (1,1,1)$	8	0	0	1	1	−1
$f_3 \mapsto (1)$	3	−1	1	A	$\overline{A}$	0
$\tilde{f}_3 \mapsto (1)$	3	−1	1	$\overline{A}$	A	0
$\underline{\lambda}$	7	−1	−1	0	0	1

Now, for $\underline{\mu} \in \Lambda_n$, we define the characters

$$\xi^{\underline{\mu}} = \bigodot_{f\in\Phi} \xi^{f\mapsto\underline{\mu}(f)}. \tag{4.10}$$

In what follows we decompose $\xi^{\underline{\mu}}$ into irreducible characters $\chi^{\underline{\lambda}}$. In order to do so, we introduce the *shape* of a partition valued function.

Definition 4.2.4. The *shape* of $\underline{\lambda} \in \Lambda_n$ is the mapping $s\colon \Phi \to \mathbb{Z}$ given by $s(f) = |\underline{\lambda}(f)|$ for each $f \in \Phi$. If two partition valued functions $\underline{\lambda}, \underline{\mu} \in \Lambda_n$ have the same shape, we write $\underline{\lambda} \sim \underline{\mu}$.

We note that $\sim$ is an equivalence relation on Λ_n.

For example $\underline{\lambda}, \underline{\sigma} \in \Lambda_5$ given by $\underline{\lambda}(X-1) = (4,1)$ and $\underline{\sigma}(X-1) = (3,1,1)$ have the same shape.

Definition 4.2.5. For $\underline{\lambda}, \underline{\mu} \in \Lambda_n$ with $\underline{\lambda} \sim \underline{\mu}$, we define

$$K_{\underline{\lambda}\underline{\mu}} = \prod_{f\in\Phi} K_{\underline{\lambda}(f)\underline{\mu}(f)},$$

$$H_{\underline{\mu}\underline{\lambda}} = \prod_{f\in\Phi} H_{\underline{\mu}(f)\underline{\lambda}(f)}.$$

Then, from (4.6) and (4.8), it follows that

$$\xi^{\underline{\mu}} = \sum_{\underline{\lambda} \sim \underline{\mu}} K_{\underline{\lambda}\underline{\mu}} \chi^{\underline{\lambda}} \quad \text{for each } \underline{\mu} \in \Lambda_n, \tag{4.11}$$

$$\chi^{\underline{\lambda}} = \sum_{\underline{\mu} \sim \underline{\lambda}} H_{\underline{\mu}\underline{\lambda}} \xi^{\underline{\mu}} \quad \text{for each } \underline{\lambda} \in \Lambda_n. \tag{4.12}$$

Similarly as for the symmetric group (see [Sag01, Thm. 3.10.1], for example), we can calculate the degree of an irreducible character of the finite general linear group by using the *hook lengths*. In the following, we will often write $|f|$ to denote the degree of $f \in \Phi$.

Lemma 4.2.6 ([Gre55, Thm. 14]). *We have*

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \prod_{i=1}^{n} (q^i - 1) = \prod_{f \in \Phi} \frac{1}{q^{|f| b(\underline{\lambda}(f))}} \prod_{(i,j) \in \underline{\lambda}(f)} (q^{|f| h_{i,j}(\underline{\lambda}(f))} - 1), \tag{4.13}$$

where for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$,

$$b(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$$

and $h_{i,j}(\lambda)$ is the hook length *of λ at (i,j), namely*

$$h_{i,j}(\lambda) = \lambda_i + \lambda'_j - i - j + 1$$

and the corresponding product over (i,j) is over all boxes of the Ferrers diagram of $\underline{\lambda}(f)$.

The only characters of the finite general linear group that we need explicitly are the primary irreducible characters $\chi^{f \mapsto (n)}$, where $f \in \Phi$ is a polynomial of degree one. For this, let α be a generator of the multiplicative group $\mathbb{F}_q^*$ of $\mathbb{F}_q$, let $\omega = \exp(2\pi\sqrt{-1}/(q-1))$ be a complex root of unity, and let θ be the linear character of $\mathbb{F}_q^*$ given by

$$\theta\colon \mathbb{F}_q^* \to \mathbb{C}, \quad \theta(\alpha^i) = \omega^i. \tag{4.14}$$

Note that, for all i, the characters $\chi^{X-\alpha^i \mapsto (n)}$ have degree one. Moreover, they are given in terms of the linear character θ.

Lemma 4.2.7 ([Gre55]). *For all $g \in \mathrm{GL}(n,q)$, we have*

$$\chi^{X-\alpha^i \mapsto (n)}(g) = \theta(\det(g)^i).$$

In particular $\chi^{X-1 \mapsto (n)}$ is the trivial character.

Similarly as for the symmetric group, also in the case of the finite general linear group we find an isomorphism between irreducible characters and symmetric functions Λ, from Section 1.3.

Lemma 4.2.8 ([Mac15]). *For every polynomial $f \in \Phi$, the algebra generated by $\{\chi^{f\mapsto\lambda}\colon \lambda \in \mathrm{Par}\}$ with multiplication $\odot$ is isomorphic to the algebra of symmetric functions Λ, with $\chi^{f\mapsto\lambda}$ being send to the Schur function s_λ.*

From this Lemma it follows that we are in a position to decompose the product $\chi^{f\mapsto\lambda} \odot \chi^{f\mapsto\nu}$ into its irreducible constituents. We state the Littlewood-Richardson rule in terms of irreducible characters of the finite general linear group, like we did in Theorem 1.3.18 for the irreducible characters of the symmetric group.

Lemma 4.2.9. *For all $f \in \Phi$ and for every two irreducible characters $\chi^{f\mapsto\lambda}$ and $\chi^{f\mapsto\nu}$ of $\mathrm{GL}(n,q)$ and $\mathrm{GL}(m,q)$, respectively, the product decomposes as follows*

$$\chi^{f\mapsto\lambda} \odot \chi^{f\mapsto\nu} = \sum_{\mu\in\mathrm{Par}} c^{\mu}_{\lambda\nu}\chi^{f\mapsto\mu},$$

where $c^{\mu}_{\lambda\nu}$ are the Littlewood-Richardson coefficients from Theorem 1.3.18.

Recall that we have $c^{\mu}_{\lambda\nu} = 0$ unless $|\mu| = |\lambda| + |\nu|$ and $\lambda, \nu \subseteq \mu$.

Moreover, from Pieri's rule we obtain the following.

Lemma 4.2.10. *Let $\chi^{X-\alpha^i\mapsto\kappa}$ and $\chi^{X-\alpha^i\mapsto(m)}$ be irreducible characters of $\mathrm{GL}(|\kappa|,q)$ and $\mathrm{GL}(m,q)$ respectively. Then the decomposition of the product is given by*

$$\chi^{X-\alpha^i\mapsto\kappa} \odot \chi^{X-\alpha^i\mapsto(m)} = \sum_{\lambda} \chi^{X-\alpha^i\mapsto\lambda},$$

where λ runs through all partitions whose Young diagram is obtained from that of κ by adding m boxes, no two of which in the same column.

In what follows, we obtain a decomposition of the product of any two irreducible characters of the finite general linear group and not only of primary ones. By Λ^c we denote the set of all mappings $\underline{\lambda}\colon \Phi \mapsto \mathrm{Par}$ of finite support (with $\varnothing$ being the zero element in Par). We define for $\underline{\lambda}, \underline{\mu}, \underline{\nu} \in \Lambda^c$, the following generalisation of the Littlewood-Richardson coefficients

$$c^{\underline{\mu}}_{\underline{\lambda}\underline{\nu}} = \prod_{f\in\Phi} c^{\underline{\mu}(f)}_{\underline{\lambda}(f)\underline{\nu}(f)}.$$

We note that $c^{\underline{\mu}}_{\underline{\lambda}\underline{\nu}} = 0$ unless $\|\underline{\mu}\| = \|\underline{\lambda}\| + \|\underline{\nu}\|$ and $\underline{\lambda}, \underline{\nu} \subseteq \underline{\mu}$, where $\underline{\lambda} \subseteq \underline{\mu}$ means $\underline{\lambda}(f) \subseteq \underline{\mu}(f)$ for all $f \in \Phi$. With this notion we get the following as a consequence of Lemma 4.2.9.

Lemma 4.2.11. *For all* $\underline{\lambda}, \underline{\nu} \in \Lambda^c$ *we have*

$$\chi^{\underline{\lambda}} \odot \chi^{\underline{\nu}} = \sum_{\underline{\mu} \in \Lambda} c^{\underline{\mu}}_{\underline{\lambda\nu}} \chi^{\underline{\mu}}.$$

Remark 4.2.12. Let $\underline{\lambda}, \underline{\mu} \in \Lambda^c$ such that $\underline{\lambda}(f) \subseteq \underline{\mu}(f)$ for all $f \in \Phi$. Then there exists $\underline{\nu} \in \Lambda$ such that $c^{\underline{\mu}}_{\underline{\lambda\nu}} > 0$.

Lemma 4.2.13. *For each* $f \in \Phi$ *and each partition* $\mu = (\mu_1, \mu_2, \ldots)$*, we have*

$$\bigodot_{i \geq 1} \chi^{f \mapsto (\mu_i)} = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} \chi^{f \mapsto \lambda},$$

where λ *ranges over the partitions of* $|\mu|$*, and where the* $K_{\lambda\mu}$ *denote the Kostka numbers.*

Example: The character table of $\mathrm{GL}(4,2)$

In this section we provide representatives of the conjugacy classes and the character table of $\mathrm{GL}(4,2)$. The irreducible polynomials in $\mathbb{F}_2[X] \setminus \{X\}$ of degree less than or equal to 4 are

$$\begin{aligned}
f_1 &= X - 1 \\
f_2 &= X^2 + X + 1 \\
f_3 &= X^3 + X^2 + 1 \\
\tilde{f}_3 &= X^3 + X + 1 \\
f_4 &= X^4 + X + 1 \\
\tilde{f}_4 &= X^4 + X^3 + 1 \\
\hat{f}_4 &= X^4 + X^3 + X^2 + X + 1.
\end{aligned}$$

There are 14 partition valued functions whose size is equal to 4, namely

$$\begin{aligned}
&f_1 \mapsto (1^4), \quad f_1 \mapsto (2,1,1) \quad f_1 \mapsto (2,2), \quad f_1 \mapsto (3,1), \quad f_1 \mapsto (4), \\
&f_2 \mapsto (1^2), \quad f_2 \mapsto (2), \\
&f_4 \mapsto (1), \quad \tilde{f}_4 \mapsto (1), \quad \hat{f}_4 \mapsto (1)
\end{aligned}$$

and

$$\underline{\lambda}_1(f) = \begin{cases} (1^2) & \text{for } f = f_1 \\ (1) & \text{for } f = f_2 \end{cases} \qquad \underline{\lambda}_2(f) = \begin{cases} (2) & \text{for } f = f_1 \\ (1) & \text{for } f = f_2 \end{cases}$$

$$\underline{\lambda}_3(f) = \begin{cases} (1) & \text{for } f = f_1 \\ (1) & \text{for } f = \tilde{f}_3 \end{cases} \qquad \underline{\lambda}_4(f) = \begin{cases} (1) & \text{for} f = f_1 \\ (1) & \text{for } f = f_3 \end{cases}$$

Table 4.3 gives an overview of the 14 conjugacy classes of GL$(4, 2)$.

Table 4.3: **The conjugacy classes of** GL$(4,2)$**.**

Indexing	Representative	Size
$f_1 \mapsto (1^4)$	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	1
$f_1 \mapsto (2,1,1)$	$\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	105
$f_1 \mapsto (2,2)$	$\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$	210
$f_1 \mapsto (3,1)$	$\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	1260
$f_1 \mapsto (4)$	$\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$	2520
$f_2 \mapsto (1^2)$	$\begin{bmatrix}0&1&0&0\\1&1&0&0\\0&0&0&1\\0&0&1&1\end{bmatrix}$	112
$f_2 \mapsto (2)$	$\begin{bmatrix}0&1&1&0\\1&1&0&1\\0&0&0&1\\0&0&1&1\end{bmatrix}$	1680

Indexing	Representative	Size
$f_4 \mapsto (1)$	$\begin{bmatrix}0&0&0&1\\1&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$	1344
$\tilde{f}_4 \mapsto (1)$	$\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&1\end{bmatrix}$	1344
$\hat{f}_4 \mapsto (1)$	$\begin{bmatrix}0&0&0&1\\1&0&0&1\\0&1&0&1\\0&0&1&1\end{bmatrix}$	1344
$\underline{\lambda}_1$	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&1\end{bmatrix}$	1120
$\underline{\lambda}_2$	$\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&1\end{bmatrix}$	3360
$\underline{\lambda}_3$	$\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&1\\0&0&1&0\end{bmatrix}$	2280
$\underline{\lambda}_4$	$\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&1\end{bmatrix}$	2280

A possibility to compute character tables of finite general linear groups is provided by the computer algebra system GAP [GAP24]. However, rather than indexing the characters in the same way as we do, GAP stores them in ascending order of degree. If the characters are not uniquely defined by their degree, some additional computations have to be done to determine which character it is in our indexing. To do so, one can use [Gor22], which employs Green's approach to the representation theory of GL(n, q). Using this method, we computed the character table of GL$(4, 2)$, which is shown in Table 4.4.

Table 4.4: **The character table of** $\mathrm{GL}(4,2)$. In this table $A = -\zeta - \zeta^2 - \zeta^4 - \zeta^8$, where $\zeta = \exp(\frac{2\pi i}{15})$ is a 15-th root of unity and $B = \xi^3 + \xi^5 + \xi^6$, where $\xi = \exp(\frac{2\pi i}{7})$ is a 7-th root of unity. By $\bar{A}$ and $\bar{B}$ we denote the complex conjugate of A and B, respectively.

	$f_1 \mapsto (1^4)$	$f_1 \mapsto (2,1^2)$	$f_1 \mapsto (2^2)$	$f_1 \mapsto (3,1)$	$f_1 \mapsto (4)$	$f_2 \mapsto (1^2)$	$f_2 \mapsto (2)$	$f_4 \mapsto (1)$	$\tilde{f}_4 \mapsto (1)$	$\hat{f}_4 \mapsto (1)$	$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$
$f_1 \mapsto (4)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$f_1 \mapsto (3,1)$	14	6	2	2	0	−1	−1	−1	−1	−1	2	0	0	0
$f_1 \mapsto (2,2)$	20	4	4	0	0	5	1	0	0	0	−1	1	−1	−1
$f_1 \mapsto (2,1,1)$	56	8	0	0	0	−4	0	1	1	1	−1	−1	0	0
$f_1 \mapsto (1^4)$	64	0	0	0	0	4	0	−1	−1	−1	−2	0	1	1
$f_2 \mapsto (1^2)$	28	−4	4	0	0	1	1	1	1	−2	1	−1	0	0
$f_2 \mapsto (2)$	7	−1	3	−1	1	4	0	−1	−1	2	1	−1	0	0
$f_4 \mapsto (1)$	21	−3	1	1	−1	−3	1	A	$\bar{A}$	1	0	0	0	0
$\tilde{f}_4 \mapsto (1)$	21	−3	1	1	−1	−3	1	$\bar{A}$	A	1	0	0	0	0
$\hat{f}_4 \mapsto (1)$	21	−3	1	1	−1	6	−2	1	1	1	0	0	0	0
$\underline{\lambda}_1$	70	−2	2	−2	0	−5	−1	0	0	0	1	1	0	0
$\underline{\lambda}_2$	35	3	−5	−1	−1	5	1	0	0	0	2	0	0	0
$\underline{\lambda}_3$	45	−3	−3	1	1	0	0	0	0	0	0	0	B	$\bar{B}$
$\underline{\lambda}_4$	45	−3	−3	1	1	0	0	0	0	0	0	0	$\bar{B}$	B

4.3 Permutation characters

This section serves as a *collection* of certain permutation, or permutation-like characters of the finite general linear group and their decompositions into irreducible constituents. These results were published in [ES23] and [ES24], respectively. The proofs of the main results in the Chapters 5 and 6 heavily rely on these characters and their decompositions.

In the following $\chi^{\underline{\lambda}}$ denotes the irreducible character of $\mathrm{GL}(\|\underline{\lambda}\|, q)$ corresponding to $\underline{\lambda} \in \Lambda^c$. Recall from Section 4.2 that $\xi^{f\mapsto\mu}$, where $f \in \Phi$ and μ being a partition, is the character defined by James. First, we explain that $\xi^{X-1\mapsto(n-t,t)}$ is the permutation character of $\mathrm{GL}(n,q)$ on the set of t-dimensional subspaces of $\mathbb{F}_q^n$ (where $\mathrm{GL}(n,q)$ acts naturally on this set). From (4.5) we find

$$\xi^{X-1\mapsto(n-t,t)} = \xi^{X-1\mapsto(n-t)} \odot \xi^{X-1\mapsto(t)}.$$

Let π_1 and π_2 be the projections onto the diagonal blocks of order t and $(n-t)$, respectively, as given in (4.3). Since $\xi^{X-1\mapsto(n-t)}$ and $\xi^{X-1\mapsto(t)}$ are the trivial characters $1_{\mathrm{GL}(n-t,q)}$ of $\mathrm{GL}(n-t,q)$ and $1_{\mathrm{GL}(t,q)}$ of $\mathrm{GL}(t,q)$, respectively, for $g \in \mathrm{GL}(n,q)$, we obtain

$$\begin{aligned}\xi^{X-1\mapsto(n-t,t)}(g) &= \frac{1}{|P_{(t,n-t)}|} \sum_{\substack{x\in\mathrm{GL}(n,q)\\ x^{-1}gx\in P_{(t,n-t)}}} 1_{\mathrm{GL}(t,q)}(\pi_1(x^{-1}gx)) \cdot 1_{\mathrm{GL}(n-t,q)}(\pi_1(x^{-1}gx))\\ &= \frac{1}{|P_{(t,n-t)}|} \sum_{\substack{x\in\mathrm{GL}(n,q)\\ x^{-1}gx\in P_{(t,n-t)}}} 1_{P_{(t,n-t)}}(x^{-1}gx)\\ &= \mathrm{Ind}_{P_{(t,n-t)}}^{\mathrm{GL}(n,q)}(1_{P_{(t,n-t)}})(g),\end{aligned}$$

The same arguments as those in Remark 1.2.1 imply that $\xi^{X-1\mapsto(n-t,t)}$ is the permutation character of $\mathrm{GL}(n,q)$ on the set of t-dimensional subspaces of $\mathbb{F}_q^n$. Thus, from (4.6), we obtain the following decomposition of this permutation character.

Lemma 4.3.1. *Let $t \le n/2$. Then the permutation character ξ of $\mathrm{GL}(n,q)$ on the t-dimensional subspaces of $\mathbb{F}_q^n$ decomposes as follows*

$$\xi = \sum_{s=0}^{t} \chi^{X-1\mapsto(n-s,s)}.$$

Now we consider a *permutation-like* character $\zeta^{(t,i)}$ of $\mathrm{GL}(n,q)$, which is related to the permutation character on t-tuples of linearly independent vectors of $\mathbb{F}_q^n$.

For $t \leq n$, let $H_{n,t} \leq \mathrm{GL}(n,q)$ be the stabiliser of a fixed t-tuple of linearly independent elements of $\mathbb{F}_q^n$. We define $\zeta^{(t,i)}$ to be the character obtained by inducing the linear character

$$\begin{aligned} H_{n,t} &\to \mathbb{C} \\ g &\mapsto \theta(\det(g)^i) \end{aligned} \tag{4.15}$$

where θ is the linear character of $\mathbb{F}_q^*$ that was defined in (4.14). Then $\zeta^{(t,0)}$ is the permutation character of $\mathrm{GL}(n,q)$ on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$ (where the action of $\mathrm{GL}(n,q)$ on the set of these t-tuples is the one induced by the natural action on the components). These characters are related to each other in the following way.

Lemma 4.3.2. *For each $g \in \mathrm{GL}(n,q)$ we have*

$$\zeta^{(t,i)}(g) = \theta(\det(g)^i)\zeta^{(t,0)}(g).$$

Proof: Since similar matrices have the same determinant, we find from the definition of an induced character, for $g \in \mathrm{GL}(n,q)$, that

$$\begin{aligned} \zeta^{(t,i)}(g) &= \frac{1}{|H_{n,t}|} \sum_{\substack{x \in \mathrm{GL}(n,q) \\ xgx^{-1} \in H_{n,t}}} \theta(\det(xgx^{-1})^i) \\ &= \frac{1}{|H_{n,t}|} \sum_{\substack{x \in \mathrm{GL}(n,q) \\ xgx^{-1} \in H_{n,t}}} \theta(\det(g)^i) \\ &= \theta(\det(g)^i)\zeta^{(t,0)}(g). \end{aligned}$$

□

The following decomposition of $\zeta^{(t,i)}$ into irreducible characters of $\mathrm{GL}(n,q)$ plays a crucial role to obtain the main results in Chapter 5.

Lemma 4.3.3. *We have*

$$\zeta^{(t,i)} = \sum_{\underline{\lambda} \in \Lambda_n} m_{i,\underline{\lambda}} \chi^{\underline{\lambda}},$$

where $m_{i,\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}(X - \alpha^i)_1 \geq n - t$.

Proof: We may choose $H_{n,t}$ to be

$$H_{n,t} = \left\{ \begin{bmatrix} I & * \\ & g \end{bmatrix} : g \in \mathrm{GL}(n-t,q) \right\},$$

so that $H_{n,t}$ is a subgroup of the parabolic subgroup $P_{(t,n-t)}$ given in (4.2). Let π_1 and π_2 be the projections onto the diagonal blocks of order t and $n-t$, respectively, as given in (4.3). Using Lemma 4.2.7, the character (4.15) can be written as

$$(1 \circ \pi_1)(\chi^{X-\alpha^i)\mapsto(n-t)} \circ \pi_2), \tag{4.16}$$

where 1 is the trivial character of the trivial subgroup of $\mathrm{GL}(t,q)$. From Example 1.1.39 we have that 1 induces on $\mathrm{GL}(t,q)$ to the character

$$\sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)\chi^{\underline{\kappa}}.$$

Since $P_{(t,n-t)}/H_{n,t} \cong \mathrm{GL}(t,q)$, it follows that (4.16) induces on $P_{(t,n-t)}$ to the character

$$\sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)(\chi^{\underline{\kappa}} \circ \pi_1)(\chi^{X-\alpha^i\mapsto(n-t)} \circ \pi_2).$$

Hence, by transitivity of induction, we have

$$\zeta^{(t,i)} = \sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)(\chi^{\underline{\kappa}} \odot \chi^{X-\alpha^i\mapsto(n-t)}).$$

From Lemma 4.2.10 we find that

$$\chi^{X-\alpha^i\mapsto\kappa} \odot \chi^{X-\alpha^i\mapsto(n-t)} = \sum_{\lambda} \chi^{X-\alpha^i\mapsto\lambda},$$

where λ runs through all partitions whose Young diagram is obtained from that of κ by adding $n-t$ boxes, no two of which in the same column. Then, the statement of the lemma follows by using (4.7). □

The last character that we study in this section is the permutation character of $\mathrm{GL}(n,q)$ on so-called *α-flags*.

For $q \neq 2$, let $\Sigma_{n,q}$ be set of all pairs $(\rho,\mathcal{I})$, where ρ is a composition of n and $\mathcal{I}$ is a subset of $\{1,2,\dots,\ell(\rho)\}$, namely

$$\Sigma_{n,q} = \{(\rho,\mathcal{I})\colon \rho \text{ is a composition of } n,\ \mathcal{I} \subseteq \{1,2,\dots,\ell(\rho)\}\}, \tag{4.17}$$

and, if $q=2$, we insist that $\rho_i > 1$ for each $i \notin \mathcal{I}$

Definition 4.3.4. For a composition ρ of n, a *ρ-flag* is a tuple of subspaces $(V_1,V_2,\dots,V_{\ell(\rho)})$ of $\mathbb{F}_q^n$ such that

$$\{0\} = V_0 \le V_1 \le \ldots \le V_{\ell(\rho)} = \mathbb{F}_q^n$$

and $\dim(V_i/V_{i-1}) = \rho_i$ for each $i \in \{1,2,\dots,\ell(\rho)\}$.

Definition 4.3.5. Let $\alpha = (\rho,\mathcal{I})$ be an element of $\Sigma_{n,q}$ with $\mathcal{I} = \{i_1,i_2,\dots,i_k\}$. We define a *signed α-flag*, or *α-flag* for short, to be a pair (F,B), where $F = (V_1,V_2,\dots,V_{\ell(\rho)})$ is a ρ-flag and $B = (B_1,B_2,\dots,B_k)$ is a tuple of ordered bases of $V_{i_1}/V_{i_1-1}, V_{i_2}/V_{i_2-1},\dots,V_{i_k}/V_{i_k-1}$ with $V_0 = \{0\}$.

A $((t, n-t), \varnothing)$-flag, for example, is essentially a t-dimensional subspace of $\mathbb{F}_q^n$. And a $((t, n-t), \{1\})$-flag is essentially a t-tuple of linearly independent elements of $\mathbb{F}_q^n$.

In order to give the decomposition of the permutation character of $\mathrm{GL}(n,q)$ on the set of α-flags with $\alpha \in \Sigma_{n,t}$, where the action of $\mathrm{GL}(n,q)$ on the set of α-flags is the one induced by the natural action on the components, we need to introduce some more notions.

Definition 4.3.6. For each $(\rho, \mathcal{I}) \in \Sigma_{n,q}$ we associate a pair of partitions (σ, τ), called the *type* of $(\rho, \mathcal{I})$, where σ is the partition whose parts are those ρ_i with $i \in \mathcal{I}$ and τ is the partition whose parts are those ρ_i with $i \notin \mathcal{I}$.

For example $((25123), \{2,3,5\})$ has type $((531), (22))$.

Definition 4.3.7. We define the *type* of $\underline{\lambda} \in \Lambda_n$ to be a pair of partitions (κ, λ), where $\lambda = \underline{\lambda}(X-1)$ and κ has $|\underline{\lambda}(f)|$ parts of size $|f|$ as f ranges through $\Phi \setminus \{X-1\}$. The type of $\underline{\lambda} \in \Lambda_n$ is denoted by $\mathrm{type}(\underline{\lambda})$.

For example, for $q = 3$, the type of $\underline{\lambda} \in \Lambda_{20}$ given by

$$X - 1 \mapsto (31), \quad X+1 \mapsto (33) \quad X^2+1 \mapsto (2), \quad X^2+X+1 \mapsto (21)$$

equals $((2^5 1^6), (31))$.

Note that, if (κ, λ) is the type of $\underline{\lambda} \in \Lambda$, then $|\kappa| + |\lambda| = n$. And note that the unique irreducible character $\chi^{\underline{\lambda}}$ of $\mathrm{GL}(n,q)$ with $\mathrm{type}(\underline{\lambda}) = (\varnothing, (n))$ is the trivial character of $\mathrm{GL}(n,q)$.

In order to define a partial order on the pairs of partitions, recall from (1.2) the dominance order $\trianglelefteq$ on partitions. There we defined the dominance order only for partitions of the same size. However, this definition can naturally be extended to the set of all partitions Par.

Another partial order on Par is given by the *refinement*. A partition $\mu = (\mu_1, \ldots, \mu_k)$ *refines* a partition λ if $|\mu| \le |\lambda|$ and the parts of λ can be partitioned to produce the parts of $(\mu_1, \ldots, \mu_k, 1^{|\lambda|-|\mu|})$. For example $(3,2,1)$ refines $(7,4,2,2)$.

A partial order on the pairs of partitions is given by the following.

Definition 4.3.8. Let (ν, μ) and (κ, λ) be pairs of partitions. We write $(\nu, \mu) \preceq (\kappa, \lambda)$ if κ refines ν and $\mu \trianglelefteq \lambda$. And we write $(\nu, \mu) \prec (\kappa, \lambda)$ if $(\nu, \mu) \preceq (\kappa, \lambda)$ and $(\nu, \mu) \neq (\kappa, \lambda)$.

The decomposition of the permutation character ξ of $\mathrm{GL}(n,q)$ on α-flags with $\alpha \in \Sigma_{n,t}$ is as follows.

Lemma 4.3.9. *Let $\alpha \in \Sigma_{n,q}$, let ξ be the permutation character of* $\mathrm{GL}(n,q)$ *on the set of α-flags, and let*

$$\xi = \sum_{\underline{\lambda}\in\Lambda_n} m_{\underline{\lambda}} \chi^{\underline{\lambda}}$$

be the decomposition of ξ into irreducible characters. Then we have

$$m_{\underline{\lambda}} \neq 0 \quad \Leftrightarrow \quad \mathrm{type}(\alpha) \preceq \mathrm{type}(\underline{\lambda}).$$

Proof: For $\alpha \in \Sigma_{n,q}$ we write $\alpha = (\rho, \mathcal{I})$, where $\rho = (\rho_1, \rho_2, \dots, \rho_k)$. We define a subgroup H of the parabolic subgroup P_ρ and thus of $\mathrm{GL}(n,q)$ by

$$H = \left\{ \begin{bmatrix} A_1 & * & \cdots & * \\ & A_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & A_k \end{bmatrix} : A_i \in \mathrm{GL}(\rho_i, q),\ A_i = I_{\rho_i} \text{ if } i \in \mathcal{I}. \right\},$$

where I_{ρ_i} denotes the $(\rho_i \times \rho_i)$-identity matrix. Then H is the stabiliser of an α-flag. Using similar arguments like in Remark 1.2.1 we have that the induced character of 1_H to $\mathrm{GL}(n,q)$ is the permutation character of $\mathrm{GL}(n,q)$ on the set of α-flags

$$\xi = \mathrm{Ind}_H^{\mathrm{GL}(n,q)}(1_H).$$

Now we first induce 1_H to the parabolic subgroup P_ρ. For $1 \leq i \leq k$, let $\pi_i \colon P_\rho \to \mathrm{GL}(\rho_i, q)$ denote the projections given in Section 4.2. Hence

$$1_H = \prod_{i=1}^{k} (1_i \circ \pi_i),$$

where 1_i is the trivial character on the trivial subgroup of $\mathrm{GL}_{(\rho_i,q)}$ for $i \in \mathcal{I}$ and 1_i is the trivial character of $\mathrm{GL}(\rho_i, q)$ for $i \in \mathcal{J}$, where $\mathcal{J}$ is the complement of $\mathcal{I}$ in $\{1, 2, \dots, k\}$. We have

$$P_\rho / H \cong \prod_{i\in\mathcal{I}} \mathrm{GL}(\rho_i, q),$$

as a direct product. From Example 1.1.39 it follows that, for each $i \in \mathcal{I}$, we have

$$\mathrm{Ind}_{E_i}^{\mathrm{GL}(\rho_i,q)}(1_i) = \sum_{\underline{\kappa}\in\Lambda_{\rho_i}} \chi^{\underline{\kappa}}(1) \chi^{\underline{\kappa}},$$

where E_i denotes the trivial subgroup of $\mathrm{GL}(\rho_i, q)$. Hence we obtain

$$\mathrm{Ind}_H^{P_\rho}(1_H) = \left(\prod_{i\in\mathcal{J}} (1_i \circ \pi_i) \right) \left(\prod_{i\in\mathcal{I}} \sum_{\underline{\kappa}\in\Lambda_{\rho_i}} \chi^{\underline{\kappa}}(1)(\chi^{\underline{\kappa}} \circ \pi_i) \right).$$

By the transitivity of induction, ξ is obtained by inducing $\mathrm{Ind}_H^{P_\rho}(1_H)$ to $\mathrm{GL}(n,q)$. To determine the irreducible constituents of ξ, it is now enough to determine the irreducible constituents of the induced characters

$$\phi_1 \odot \phi_2 \odot \cdots \odot \phi_k \tag{4.18}$$

where ϕ_i is an irreducible character of $\mathrm{GL}(\rho_i,q)$ for $i \in \mathcal{I}$ and ϕ_i is the trivial character of $\mathrm{GL}(\rho_i,q)$ for $i \in \mathcal{J}$. Since the product of characters is commutative, we may now assume without loss of generality that $\mathcal{I} = \{1,2\dots,r\}$, where $r = |\mathcal{I}|$. We put $\sigma = (\rho_1,\rho_2,\dots,\rho_r)$ and $\tau = (\rho_{r+1},\rho_{r+2},\dots,\rho_k)$ (so that (σ,τ) is the type of $(\rho,\mathcal{I})$). Now consider the parabolic subgroup $P = P_{\rho_1,\dots,\rho_r,|\tau|}$. We have

$$P/P_\rho \cong \mathrm{GL}(|\tau|,q)/P_\tau$$

and hence by Lemma 4.2.13 the character (4.18) induces on P to

$$\sum_{\nu \trianglerighteq \tau} K_{\nu\tau}\chi^{X-1\mapsto\nu} \odot \phi_1 \odot \phi_2 \odot \cdots \odot \phi_r.$$

To determine the irreducible constituents of ξ, it is now enough to determine the irreducible constituents of the induced characters

$$\phi_0 \odot \phi_1 \odot \cdots \odot \phi_r, \tag{4.19}$$

where ϕ_0 is the irreducible character of $\mathrm{GL}(|\tau|,q)$ corresponding to $X-1 \mapsto \nu$, where ν is a partition with $\nu \trianglerighteq \tau$, and ϕ_i is an irreducible character of $\mathrm{GL}(\rho_i,q)$ for $1 \le i \le r$.

To prove the forward direction of the lemma, assume that $\chi^{\underline{\lambda}}$ is a constituent of some induced character of the form (4.19). Let (κ,λ) be the type of $\underline{\lambda}$ and let $(\kappa^{(i)},\lambda^{(i)})$ be the type of the element of Λ_{ρ_i} indexing ϕ_i. Then Lemma 4.2.11 implies that the parts of κ are exactly the parts of $\kappa^{(1)},\kappa^{(2)},\dots,\kappa^{(r)}$. Since ϕ_i is a character of $\mathrm{GL}(\rho_i,q)$, we find that $\kappa^{(i)}$ refines (ρ_i) and hence κ refines σ. By assumption there is some partition ν with $|\nu| = |\tau|$ such that $\nu \trianglerighteq \tau$, which by Lemma 4.2.11 satisfies $\nu \subseteq \lambda$, where $\subseteq$ denotes the containment order from Definition 1.3.16. Hence we have $\lambda \trianglerighteq \tau$, which proves the forward direction of the lemma.

To prove the reverse direction, let $\underline{\lambda} \in \Lambda_n$ be such that its type (κ,λ) satisfies $(\sigma,\tau) \preceq (\kappa,\lambda)$. Then κ is a refinement of σ and $\tau \trianglelefteq \lambda$. It is readily verified that there exists a partition ν with $|\nu| = |\tau|$ such that $\nu \trianglerighteq \tau$ and $\nu \subseteq \lambda$. Let $\underline{\lambda}_0 \in \Lambda_{|\tau|}$ be given by $X-1 \mapsto \nu$. Since κ is a refinement of σ, there is a chain of partition-valued functions

$$\underline{\lambda}_0 \subseteq \underline{\lambda}_1 \subseteq \dots \subseteq \underline{\lambda}_r = \underline{\lambda}$$

with the property $\|\underline{\lambda}_i\| - \|\underline{\lambda}_{i-1}\| = \rho_i$ for all $i \in \{1,2\dots,r\}$. By Remark 4.2.12, we can choose $\underline{\delta}_i \in \Lambda_{\rho_i}$ such that

$$c^{\underline{\lambda}_i}_{\underline{\lambda}_{i-1},\underline{\delta}_i} > 0 \text{ for each } i \in \{1,2\dots,r\}.$$

Now we take $\phi_0 = \chi^{\underline{\lambda}_0}$ and $\phi_i = \chi^{\underline{\delta}_i}$ for each $i \in \{1, 2 \dots, r\}$. Then

$$\phi_0 \odot \phi_1 \odot \cdots \odot \phi_r,$$

has $\chi^{\underline{\lambda}_i}$ as an irreducible constituent for each $i \in \{1, 2, \dots, r\}$. Hence $\chi^{\underline{\lambda}}$ is an irreducible constituent of $\phi_0 \odot \phi_1 \odot \cdots \odot \phi_r$, which completes the proof. □

4.4 Association schemes

In this section we recall the most important objects of the conjugacy class scheme arising from the finite general linear group by taking into account the results on the conjugacy classes and representation theory from the Sections 4.1 and 4.2. Moreover, we study the symmetrisation of the conjugacy class scheme arising from $\mathrm{GL}(n, q)$.

4.4.1 The conjugacy class scheme of the finite general linear group

Since the conjugacy classes of $\mathrm{GL}(n, q)$ are indexed by partition-valued functions $\underline{\mu} \in \Lambda_n$, the adjacency matrices $A_{\underline{\mu}} \in \mathbb{C}(\mathrm{GL}(n, q), \mathrm{GL}(n, q))$ from Example 2.1.9 are indexed by Λ_n as well and are given by

$$A_{\underline{\mu}}(x, y) = \begin{cases} 1 & \text{for } x^{-1}y \in C_{\underline{\mu}}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the irreducible characters $\chi^{\underline{\lambda}}$ of $\mathrm{GL}(n, q)$ are also indexed by Λ_n, the matrices $E_{\underline{\lambda}} \in \mathbb{C}(\mathrm{GL}(n, q), \mathrm{GL}(n, q))$ from Example 2.1.13, are indexed by Λ_n and are given by

$$E_{\underline{\lambda}}(x, y) = \frac{\chi^{\underline{\lambda}}(1)}{|\mathrm{GL}(n, q)|} \chi^{\underline{\lambda}}(x^{-1}y). \tag{4.20}$$

Especially, $E_{X-1 \mapsto (n)}$ plays the role of E_0 in Example 2.1.13 because the trivial character of $\mathrm{GL}(n, q)$ is indexed by $X - 1 \mapsto (n)$. From Lemma 2.1.15 we have the following decomposition

$$\mathbb{C}(\mathrm{GL}(n, q)) = \bigoplus_{\underline{\lambda} \in \Lambda_n} \mathrm{colsp}(E_{\underline{\lambda}}).$$

Moreover, due to Example 2.2.6, the dual distribution $(b_{\underline{\lambda}})_{\underline{\lambda} \in \Lambda_n}$ of a subset $Y \subseteq \mathrm{GL}(n, q)$ is given by

$$b_{\underline{\lambda}} = \frac{\chi^{\underline{\lambda}}(1)}{|Y|} \sum_{x, y \in Y} \chi^{\underline{\lambda}}(x^{-1}y). \tag{4.21}$$

We recall, that the entries of the dual distribution are real and nonnegative. Moreover, we note that, since $X-1 \mapsto (n)$ corresponds to the trivial character of $\mathrm{GL}(n,q)$, it follows that

$$b_{X-1\mapsto(n)} = \frac{\chi^{X-1\mapsto(n)}(1)}{|Y|} \sum_{x,y\in Y} \chi^{X-1\mapsto(n)}(x^{-1}y) = \frac{1}{|Y|}|Y|^2 = |Y|.$$

Thus, $b_{X-1\mapsto(n)}$ plays the role of b_0 from Section 2.2.

4.4.2 The symmetrisation of the conjugacy class scheme of the finite general linear group

Since the conjugacy class scheme arising from the finite general linear group $\mathrm{GL}(n,q)$ is not necessarily symmetric, in this section, we focus on the symmetrisation of this scheme.

In what follows, we describe the inverse set $C_{\underline{\sigma}}^{-1} = \{g^{-1} \colon g \in C_{\underline{\sigma}}\}$ of the conjugacy class $C_{\underline{\sigma}}$ of $\mathrm{GL}(n,q)$. To do so, we recall that for a given polynomial $f \in \Phi$, its *reciprocal polynomial* $f^* \in \Phi$ is the monic polynomial whose roots in the algebraic closure of $\mathbb{F}_q$ are exactly the inverses of the roots of f. This brings us to the following definition.

Definition 4.4.1. Let $\underline{\lambda} \in \Lambda_n$. Then we define $\underline{\lambda}^*$ to be the partition valued function in Λ_n given by $\underline{\lambda}^*(f) = \underline{\lambda}(f^*)$ for all $f \in \Phi$.

Lemma 4.4.2. *For $\underline{\sigma}, \underline{\lambda} \in \Lambda_n$ we have*

(i) $C_{\underline{\sigma}^*} = C_{\underline{\sigma}}^{-1}$,

(ii) $\chi^{\underline{\lambda}^*} = \overline{\chi}^{\underline{\lambda}}$,

(iii) $\chi_{\underline{\sigma}}^{\underline{\lambda}^*} = \chi_{\underline{\sigma}^*}^{\underline{\lambda}}$.

Proof: Property (i) follows from linear algebra, (ii) is essentially [Jam86, (7.32)], and (i) together with (ii) imply (iii). □

We define a subset Ω_n of Λ_n to be a set that contains $\underline{\lambda} \in \Lambda_n$ if $\underline{\lambda} = \underline{\lambda}^*$ and that contains exactly one of $\underline{\lambda}$ or $\underline{\lambda}^*$ otherwise for all $\underline{\lambda} \in \Lambda_n$. Then, for $\underline{\sigma} \in \Omega_n$ we define $D_{\underline{\sigma}} = C_{\underline{\sigma}} \cup C_{\underline{\sigma}^*}$. Note that $D_{\underline{\sigma}}$ is exactly the union of $C_{\underline{\sigma}}$ and $C_{\underline{\sigma}}^{-1}$, which follows from Lemma 4.4.2. From Lemma 2.1.21 together with Example 2.1.4 we can deduce that $(\mathrm{GL}(n,q), \{R_{\underline{\sigma}} \colon \underline{\sigma} \in \Omega_n\})$ with $R_{\underline{\sigma}} = \{(g,h) \colon g^{-1}h \in D_{\underline{\sigma}}\}$ is the symmetric closure of the conjugacy class scheme arising from the finite general linear group $\mathrm{GL}(n,q)$. The adjacency matrices $B_{\underline{\sigma}}$ of this symmetric association scheme are given by

$$B_{\underline{\sigma}} = \begin{cases} A_{\underline{\sigma}} & \text{for } \underline{\sigma} = \underline{\sigma}^* \\ A_{\underline{\sigma}} + A_{\underline{\sigma}^*} & \text{otherwise,} \end{cases}$$

where the $A_{\underline{\sigma}}$ are the adjacency matrices of the conjugacy class scheme arising from $\mathrm{GL}(n, q)$. For this symmetric association scheme, we can express the eigenvalues again in terms of (not necessarily irreducible) characters of $\mathrm{GL}(n, q)$. To do so, we define the character $\psi^{\underline{\lambda}}$ for each $\underline{\lambda} \in \Omega_n$ by

$$\psi^{\underline{\lambda}} = \begin{cases} \chi^{\underline{\lambda}} & \text{for } \underline{\lambda} = \underline{\lambda}^* \\ \chi^{\underline{\lambda}} + \chi^{\underline{\lambda}^*} & \text{otherwise.} \end{cases} \tag{4.22}$$

Then $\psi^{\underline{\lambda}}$ is constant on $D_{\underline{\sigma}}$ for all $\underline{\lambda}, \underline{\sigma} \in \Omega_n$, which follows from Lemma 4.4.2. For $\underline{\lambda}, \underline{\sigma} \in \Omega_n$ this justifies to write

$$\psi^{\underline{\lambda}}_{\underline{\sigma}} = \psi^{\underline{\lambda}}(g) \text{ for an arbitrary element } g \in D_{\underline{\sigma}}. \tag{4.23}$$

We note that the characters $\psi^{\underline{\lambda}}$ are real-valued for all $\underline{\lambda} \in \Omega_n$. Now, for $\underline{\lambda} \in \Omega_n$ we write

$$F_{\underline{\lambda}} = \begin{cases} E_{\underline{\lambda}} & \text{for } \underline{\lambda} = \underline{\lambda}^* \\ E_{\underline{\lambda}} + E_{\underline{\lambda}^*} & \text{otherwise,} \end{cases} \tag{4.24}$$

where the $E_{\underline{\lambda}}$ are the pairwise orthogonal idempotent matrices of the conjugacy class scheme arising from $\mathrm{GL}(n, q)$. We note that all entries of $F_{\underline{\lambda}}$ are real-valued. By $V_{\underline{\lambda}}$ we denote the column span of $F_{\underline{\lambda}}$ over the reals. And for $\underline{\sigma}, \underline{\lambda} \in \Omega_n$, we write

$$P(\underline{\lambda}, \underline{\sigma}) = \frac{|D_{\underline{\sigma}}|}{\psi^{\underline{\lambda}}(1)} \psi^{\underline{\lambda}}_{\underline{\sigma}}. \tag{4.25}$$

The real-valued numbers $P(\underline{\lambda}, \underline{\sigma})$ are in fact the eigenvalues of the symmetric closure of the conjugacy class scheme arising from $\mathrm{GL}(n, q)$, which is shown in the following lemma.

Lemma 4.4.3. *We have an orthogonal direct sum decomposition of* $\mathbb{R}(\mathrm{GL}(n, q))$ *in terms of* $V_{\underline{\lambda}}$, *namely*

$$\mathbb{R}(\mathrm{GL}(n, q)) = \bigoplus_{\underline{\lambda} \in \Omega_n} V_{\underline{\lambda}}.$$

Moreover, for all $\underline{\sigma}, \underline{\lambda} \in \Omega_n$, *the elements in* $V_{\underline{\lambda}}$ *are precisely the eigenvectors of* $B_{\underline{\sigma}}$ *with corresponding eigenvalue* $P(\underline{\lambda}, \underline{\sigma})$.

Proof: The matrices $F_{\underline{\lambda}} \in \mathbb{R}(\mathrm{GL}(n,q),\mathrm{GL}(n,q))$ are pairwise orthogonal and idempotent because the $E_{\underline{\lambda}}$ are. Consequently, the $V_{\underline{\lambda}}$ are orthogonal and the rank of $F_{\underline{\lambda}}$ is equal to the trace of $F_{\underline{\lambda}}$. The latter together with (2.9) imply that

$$\mathrm{rk}(F_{\underline{\lambda}}) = \begin{cases} \chi^{\underline{\lambda}}(1)^2 & \text{for } \underline{\lambda} = \underline{\lambda}^* \\ \chi^{\underline{\lambda}}(1)^2 + \chi^{\underline{\lambda}^*}(1)^2 & \text{otherwise.} \end{cases}$$

Hence we have

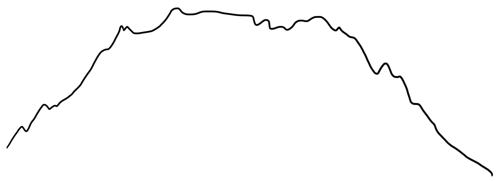
$$\dim\left(\bigoplus_{\underline{\lambda}\in\Omega_n} V_{\underline{\lambda}}\right) = \sum_{\underline{\lambda}\in\Omega_n} \dim V_{\underline{\lambda}} = \sum_{\underline{\lambda}\in\Omega_n} \mathrm{rk}(F_{\underline{\lambda}}) = \sum_{\underline{\lambda}\in\Lambda_n} \chi^{\underline{\lambda}}(1)^2 = |\,\mathrm{GL}(n,q)|,$$

where the last equation is a well known fact from representation theory of finite groups (see [Sag01, Prop. 1.10.1], for example). This proves the first statement. Using Lemma 4.4.2 together with the identity (2.8) it is readily verified that

$$B_{\underline{\sigma}} = \sum_{\underline{\lambda}\in\Omega_n} P(\underline{\lambda},\underline{\sigma}) F_{\underline{\lambda}}$$

for all $\underline{\sigma} \in \Omega_n$. Since the $F_{\underline{\lambda}}$ are pairwise orthogonal we obtain the second statement.□

5 Erdős-Ko-Rado theorems for finite general linear groups



Zugspitze, 2962m

In this chapter we study q-analogous problems of those in Section 3.2 for the symmetric groups. Here, we replace the symmetric group S_n by the finite general linear group $\mathrm{GL}(n, q)$. We study *pointwise*, *spacewise t-intersecting*, and *t-cross-intersecting* sets in $\mathrm{GL}(n, q)$. More precisely, we establish upper bounds on their sizes and partially characterise the extremal cases.

The chapter is organised as follows. In Section 5.1 we introduce the different notions of intersection, namely *t-intersection*, *t-cross-intersection*, *t-space-intersection*, and *t-space-cross-intersection* in finite general linear groups. Moreover, we collect already known results and state our main theorems. In Section 5.2 we prepare some key steps for the proofs of the main results. The Sections 5.3 and 5.4 contain the main arguments of our proofs of the pointwise intersection theorems, Theorems 5.1.4 and 5.1.6, and of the spacewise (cross-)intersection theorems, Theorems 5.1.9 and 5.1.11. In Section 5.5 we collect some open problems and conjectures arising from the study of *pointwise intersecting* and *spacewise intersecting sets* in finite general linear groups.

The results presented in this chapter were published in [ES23].

5.1 Introduction and main results

In the remainder of this chapter, we fix a prime power q. Henceforth, for a positive integer t, a *t-space* of $\mathbb{F}_q^n$ is a t-dimensional subspace of $\mathbb{F}_q^n$.

Pointwise intersecting sets

Definition 5.1.1. Two elements $x, y \in \mathrm{GL}(n, q)$ are *t-intersecting* if there exist t linearly independent vectors $v_1, v_2, \ldots, v_t$ in $\mathbb{F}_q^n$ such that $xv_i = yv_i$ for all i. A subset Y of $\mathrm{GL}(n, q)$ is *t-intersecting* if all pairs in $Y \times Y$ are t-intersecting.

Equivalently, a subset Y of $\mathrm{GL}(n, q)$ is t-intersecting if $\mathrm{rk}(x - y) \leq n - t$ holds for all $x, y \in Y$. The *canonical examples* for t-intersecting sets in $\mathrm{GL}(n, q)$ are given by so-called *t-cosets*.

Example 5.1.2. A coset of the stabiliser of a t-tuple of linearly independent vectors of $\mathbb{F}_q^n$ is given by

$$\{x \in \mathrm{GL}(n, q) \colon xv_i = w_i\},$$

for some t-tuples $(v_1, v_2, \ldots, v_t)$ and $(w_1, w_2, \ldots, w_t)$ of linearly independent vectors in $\mathbb{F}_q^n$. We call such a set *t-coset*. Every t-coset is t-intersecting and its size is given by

$$\prod_{i=t}^{n-1} (q^n - q^i). \tag{5.1}$$

Given a t-intersecting set, we can construct another one.

Example 5.1.3. Let $Y \subseteq \mathrm{GL}(n, q)$ be t-intersecting and let $Y^T = \{y^T \colon y \in Y\}$ be the set of all transposed matrices of Y. Then Y^T is t-intersecting as well because, for all $x^T, y^T \in Y^T$, we have

$$\mathrm{rk}(x^T - y^T) = \mathrm{rk}(x - y) \leq n - t.$$

From Example 5.1.3 it follows that the t-cosets are not the only t-intersecting sets of the size given in (5.1). As before, we are interested in finding an upper bound on the size of a t-intersecting set in $\mathrm{GL}(n, q)$ and in a characterisation of the extremal case.

It is well known (see [AA14] or [AM15], for example) that the size of a 1-intersecting set in $\mathrm{GL}(n, q)$ is bounded by $\prod_{i=1}^{n-1} (q^n - q^i)$. This follows from the existence of a *Singer cycle* in $\mathrm{GL}(n, q)$ and a simple application of the clique-coclique bound from Theorem 2.4.7.

Additionally Meagher and Razafimahatratra [MR23] proved that the characteristic vector of a 1-intersecting set of size $q^2 - q$ in $\mathrm{GL}(2, q)$ is spanned by the characteristic vectors of 1-cosets. We obtain a result for all t and n such that n is sufficiently large compared to t.

Theorem 5.1.4 (Pointwise t-intersection). *Let t be a positive integer and let Y be a t-intersecting set in* $\mathrm{GL}(n, q)$*. If n is sufficiently large compared to t, then*

$$|Y| \leq \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of t-cosets.

After a first version of our paper [ES23] containing Theorem 5.1.4 was publicly available (arXiv, May 2022) Ellis, Kindler, and Lifshitz [EKL23] (arXiv, August 2022) independently proved a slightly more general result than Theorem 5.1.4. However their methods are completely different, in particular they make no use of the representation theory of $\mathrm{GL}(n, q)$, which is one of the main tools in our approach.

Moreover, we obtain a result for *t-cross-intersecting* sets.

Definition 5.1.5. Two subsets Y and Z of $\mathrm{GL}(n, q)$ are *t-cross-intersecting* if all pairs in $Y \times Z$ are t-intersecting.

Theorem 5.1.6 (Pointwise t-cross-intersection). *Let t be a positive integer and let Y and Z be t-cross-intersecting sets in* $\mathrm{GL}(n, q)$*. If n is sufficiently large compared to t, then*

$$\sqrt{|Y| \cdot |Z|} \leq \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, the characteristic vectors of Y and Z are spanned by the characteristic vectors of t-cosets.

The Theorems 5.1.4 and 5.1.6 can be seen as q-analogs of the corresponding results for the symmetric group [EFP11] from Section 3.2.

Spacewise intersecting sets

We can also q-analogise the setwise intersection result from Theorem 3.2.13.

Definition 5.1.7. Two elements $x, y \in \mathrm{GL}(n, q)$ are *t-space-intersecting* if there exists a t-space U of $\mathbb{F}_q^n$ such that $xU = yU$. A subset Y of $\mathrm{GL}(n, q)$ is *t-space-intersecting* if all pairs in $Y \times Y$ are t-space intersecting.

It is natural to state the definition of a t-space-intersecting set in terms of the projective general linear group $\mathrm{PGL}(n, q)$. For consistency, in this thesis, we write the results in terms of the general linear group. However the results of $\mathrm{GL}(n, q)$ and of $\mathrm{PGL}(n, q)$ can be easily translated into each other.

The *canonical examples* of t-space-intersecting sets are given by cosets of stabilisers of t-spaces.

Example 5.1.8. Every coset of the stabiliser of a t-space of $\mathbb{F}_q^n$ is t-space-intersecting and its size is given by

$$\left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]. \tag{5.2}$$

The set of all transposed matrices of a t-space-intersecting set is t-space-intersecting as well. Moreover we note that the set of all transposed matrices of the stabiliser of a t-space is the stabiliser of an $(n-t)$-space and gives another example of a t-space-intersecting set of the size given in (5.2).

It was shown in [MS11] that the size of a 1-space-intersecting set in $\mathrm{GL}(n,q)$ is at most the product given in (5.2) for $t = 1$, which follows again from the existence of a *Singer cyclic subgroup* and the application of the clique-coclique bound from Theorem 2.4.7. We obtain a result for arbitrary t and all sufficiently large n.

Theorem 5.1.9 (Spacewise t-intersection). *Let t be a positive integer and let Y be a t-space-intersecting set in* $\mathrm{GL}(n,q)$. *If n is sufficiently large compared to t, then*

$$|Y| \le \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of t-spaces of $\mathbb{F}_q^n$.

Again, we obtain a corresponding result on cross-intersecting subsets of $\mathrm{GL}(n,q)$.

Definition 5.1.10. Two subsets Y and Z of $\mathrm{GL}(n,q)$ are *t-space-cross-intersecting* if every pair in $Y \times Z$ is t-space-intersecting.

Theorem 5.1.11 (Spacewise t-cross-intersection). *Let t be a positive integer and let Y and Z be t-space-cross-intersecting sets in* $\mathrm{GL}(n,q)$. *If n is sufficiently large compared to t, then*

$$\sqrt{|Y| \cdot |Z|} \le \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, the characteristic vectors of Y and Z are spanned by the characteristic vectors of cosets of stabilisers of t-spaces of $\mathbb{F}_q^n$.

5.2 Preparations for the proofs of the pointwise and spacewise intersection theorems

In this section we provide some of the key ingredients for the proofs of our main results in Section 5.1. First, we explain the proof strategy. Then we study properties of a matrix arising from the character table of $\mathrm{GL}(n,q)$. The last part of this section then deals with rather technical estimates on certain conjugacy class sizes and character degrees.

5.2.1 Proof strategy

In order to explain the strategy of the proof of our main theorems on t-intersecting sets, we first recall the weighted version of the Hoffman bound from Theorem 2.3.11.

Theorem. *Let $\Gamma = (X, E)$ be a graph on n vertices. Suppose that $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ are regular spanning subgraphs of Γ, all having $\{v_0, v_1, \ldots, v_{n-1}\}$ as an orthonormal system of eigenvectors with v_0 being the all-ones vector. Let $P_i(k)$ be the eigenvalue of v_k in Γ_i. Let $w_0, w_1, \ldots, w_r \in \mathbb{R}$ and write $P(k) = \sum_{i=0}^{r} w_i P_i(k)$.*

(i) If $Y \subseteq X$ is an independent set in Γ, then

$$\frac{|Y|}{|X|} \le \frac{|P_{\min}|}{P(0) + |P_{\min}|},$$

where $P_{\min} = \min_{k \neq 0} P(k)$. In case of equality we have

$$\mathbb{1}_Y \in \langle \{v_0\} \cup \{v_k \colon P(k) = P_{\min}\} \rangle.$$

(ii) If $Y, Z \subseteq X$ are such that there are no edges between Y and Z in Γ, then

$$\sqrt{\frac{|Y|}{|X|}\frac{|Z|}{|X|}} \le \frac{P_{\max}}{P(0) + P_{\max}},$$

where $P_{\max} = \max_{k \neq 0} |P(k)|$. In case of equality we have

$$\mathbb{1}_Y, \mathbb{1}_Z \in \langle \{v_0\} \cup \{v_k \colon |P(k)| = P_{\max}\} \rangle.$$

In what follows we explain how we apply the weighted version of the Hoffman bound to prove the pointwise intersection theorems from Theorems 5.1.4 and 5.1.6. The strategy for the spacewise intersecting cases, Theorems 5.1.9 and 5.1.11, is similar. In the remainder of this chapter we will use the notations from Section 4.4.2.

Definition 5.2.1. An element $x \in \mathrm{GL}(n,q)$ is called a *t-derangement* if there is no t-tuple of linearly independent elements of $\mathbb{F}_q^n$ that is fixed by x.

Equivalently $x \in \mathrm{GL}(n,q)$ is a t-derangement if $\mathrm{rk}(x - I) > n - t$. Recall from Section 4.4.2 that $D_{\underline{\sigma}}$ is the union of the conjugacy classes $C_{\underline{\sigma}}$ and $C_{\underline{\sigma}}^{-1}$. We make the following observation.

Observation 5.2.2. Either all elements of $D_{\underline{\sigma}}$ are t-derangements or none of them.

We apply Theorem 2.3.11 to a graph that we construct from the symmetrisation of the conjugacy class scheme of the finite general linear group $\mathrm{GL}(n,q)$. For the construction of the graph, we wish to establish a set of partition valued functions $\Sigma \subseteq \Omega_n$ such that $D_{\underline{\sigma}}$ consists of t-derangements for all $\underline{\sigma} \in \Sigma$. This ensures that a t-intersecting set is an independent set in the graph Γ given by the adjacency matrix $\sum_{\underline{\sigma}\in\Sigma} B_{\underline{\sigma}}$. Then we apply the weighted version of the Hoffman bound from Theorem 2.3.11, to the graph Γ and the $|D_{\underline{\sigma}}|$-regular spanning subgraphs $\Gamma_{\underline{\sigma}}$ having adjacency matrix $B_{\underline{\sigma}}$ for $\underline{\sigma} \in \Sigma$. Recall from Section 2.1 that $\mathbb{R}(\Sigma)$ denotes the set of column vectors indexed by Σ and having entries in $\mathbb{R}$. We wish to construct some weight $w \in \mathbb{R}(\Sigma)$ such that both the minimum value and the negative of the second-largest absolute value over all $\underline{\lambda} \in \Omega_n$ of

$$\sum_{\underline{\sigma}\in\Sigma} \omega(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) \tag{5.3}$$

equals

$$\eta = -\frac{1}{(q^n-1)(q^n-q)\cdots(q^n-q^{t-1})-1} \tag{5.4}$$

and such that w is normalised in the sense that (5.3) equals 1 if $\underline{\lambda} \in \Omega_n$ is given by $X - 1 \mapsto (n)$. This will ensure that Theorem 2.3.11 will give the bounds of Theorems 5.1.4 and 5.1.6.

5.2.2 A special invertible matrix

In the following we focus on identifying the relevant conjugacy classes of $\mathrm{GL}(n,q)$ whose elements are t-derangements and do not fix a t-space, respectively.

Definition 5.2.3. An element of $\mathrm{GL}(n,q)$ is *regular elliptic* if its characteristic polynomial is irreducible over $\mathbb{F}_q$.

Lemma 5.2.4 ([LRS14]). *Each regular elliptic element of* $\mathrm{GL}(n,q)$ *fixes no proper nontrivial subspace of* $\mathbb{F}_q^n$.

This Lemma implies that the regular elliptic elements in $\mathrm{GL}(n,q)$ play the role of an n-cycle in the symmetric group S_n.

We note that, for each polynomial $f \in \Phi$ of degree d, its companion matrix C_f satisfies $\det(C_f) = (-1)^d f(0)$. Moreover it is well known [HM92] that given an element $a \in \mathbb{F}_q^*$ there exists an irreducible polynomial $f \in \mathbb{F}_q[X]$ of degree d such that $f(0) = a$. Consequently we can always find a polynomial in Φ with prescribed degree and prescribed nonzero determinant of its companion matrix. Moreover, we note that, for every $f \in \Phi$, we have $f(0)f^*(0) = 1$, where f^* is the reciprocal polynomial associated with f from Section 4.4.2, and thus we have $\det(C_f)\det(C_{f^*}) = 1$, where again C_f and C_{f^*} are the companion matrices of f and f^*, respectively.

From now on let α be a fixed generator of $\mathbb{F}_q^*$. For all integers ℓ, j with $0 \le \ell < n$ and $0 \le j \le q-2$, we fix an irreducible polynomial $h_{\ell,j} \in \Phi$ of degree $n - \ell$ such that its companion matrix has determinant α^j and such that $h_{\ell,j}^* = h_{\ell,-j}$. We define

$$\Sigma_{\ell,j} = \{\underline{\sigma} \in \Lambda_n \colon \underline{\sigma}(h_{\ell,j}) = (1)\}$$

and

$$\Sigma_\ell = \bigcup_{j=0}^{q-2} \Sigma_{\ell,j} \quad \text{and} \quad \Sigma_{\le t} = \bigcup_{\ell=0}^{t} \Sigma_\ell.$$

We note that for each $\underline{\sigma} \in \Sigma_{\le t-1}$, the conjugacy class $C_{\underline{\sigma}}$ consists only of elements that do not fix a t-space of $\mathbb{F}_q^n$. Moreover, for each $\underline{\sigma} \in \Sigma_t$ except those $q-1$ exceptions $\underline{\sigma} \in \Sigma_t$ satisfying $\underline{\sigma}(X-1) = (1^t)$, the conjugacy class $C_{\underline{\sigma}}$ consists of elements that do not fix a t-space pointwise. Recall from Section 1.2 that the i-th part of a partition λ is denoted by λ_i. For integers $k \le n$ we now define

$$\Pi_{k,i} = \{\underline{\lambda} \in \Lambda_n \colon \underline{\lambda}(X - \alpha^i)_1 = n - k\},$$

and

$$\Pi_k = \bigcup_{i=0}^{q-2} \Pi_{k,i} \quad \text{and} \quad \Pi_{\le t} = \bigcup_{k=0}^{t} \Pi_k.$$

We note that, for $k < n/2$, we have $|\Pi_{k,i}| = |\Sigma_{k,i}|$ and $|\Omega_n \cap \Pi_{k,i}| = |\Omega_n \cap \Sigma_{k,i}|$ for all i. We define the matrix $Q \in \mathbb{R}(\Omega_n, \Omega_n)$ by

$$Q(\underline{\lambda}, \underline{\sigma}) = \psi^{\underline{\lambda}}_{\underline{\sigma}} \quad \text{for each } \underline{\lambda}, \underline{\sigma} \in \Omega_n,$$

where $\psi^{\underline{\lambda}}$ is the character of $\mathrm{GL}(n,q)$ that was defined in (4.22). Let Q_t denote the restriction of Q to $\mathbb{R}(\Omega_n \cap \Pi_{\le t}, \Omega_n \cap \Sigma_{\le t})$. Then Q_t is a square matrix as well. A key step in our proof is the following.

Proposition 5.2.5. *For $n > 2t$, the matrix Q_t has full rank and is independent of n.*

In the remainder of this section we prove this proposition. In order to do so, we define the matrix $R \in \mathbb{C}(\Lambda_n, \Lambda_n)$ by

$$R(\underline{\lambda}, \underline{\sigma}) = \chi^{\underline{\lambda}}_{\underline{\sigma}} \quad \text{for each } \underline{\lambda}, \underline{\sigma} \in \Lambda_n,$$

where $\chi^{\underline{\lambda}}_{\underline{\sigma}}$ denotes the irreducible character of $\mathrm{GL}(n,q)$ corresponding to $\underline{\lambda}$ evaluated on the conjugacy class $C_{\underline{\sigma}}$. Let R_t denote the restriction of R to $\mathbb{C}(\Pi_{\le t}, \Sigma_{\le t})$. We prove a counterpart of Proposition 5.2.5 for the matrix R_t.

Proposition 5.2.6. *For $n > 2t$, the matrix R_t has full rank and is independent of n.*

Note that Proposition 5.2.5 follows from Proposition 5.2.6, since Q_t is obtained from R_t by applying elementary row operations, then deleting some rows, and then deleting duplicate columns.

To prove Proposition 5.2.6 we define a matrix $S \in \mathbb{C}(\Lambda_n, \Lambda_n)$ by

$$S(\underline{\mu}, \underline{\sigma}) = \xi^{\underline{\mu}}_{\underline{\sigma}} \quad \text{for each } \underline{\mu}, \underline{\sigma} \in \Lambda_n, \tag{5.5}$$

where $\xi^{\underline{\mu}}$ is the character of $\mathrm{GL}(n, q)$ introduced in Section 4.2. Let S_t be the restriction of S to $\mathbb{C}(\Pi_{\le t}, \Sigma_{\le t})$. We define $T \in \mathbb{C}(\Lambda_n, \Lambda_n)$ to be given by

$$T(\underline{\mu}, \underline{\lambda}) = \begin{cases} K_{\underline{\lambda}\underline{\mu}}, & \text{for } \underline{\lambda} \sim \underline{\mu}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sim$ is the equivalence relation on Λ_n introduced in Definition 4.2.4, and the numbers $K_{\underline{\lambda}\underline{\mu}}$ are products of Kostka numbers introduced in Section 4.2. Let T_t be the restriction of T to $\mathbb{C}(\Pi_{\le t}, \Sigma_{\le t})$. We first prove the following.

Lemma 5.2.7.

(i) We have $S = TR$ and T has full rank.

(ii) For $n > 2t$, we have $S_t = T_t R_t$ and T_t has full rank and is independent of n.

Proof: From (4.11) we have $S = TR$ and the matrix T is block diagonal, where the blocks are induced by the equivalence classes under $\sim$. Each diagonal block corresponds to one equivalence class. If $s\colon \Phi \to \mathbb{Z}$ is the shape of such an equivalence class, then the corresponding block can be written as a Kronecker product, namely

$$\bigotimes_{f \in \Phi} K^{(s(f))},$$

where $K^{(m)} \in \mathbb{C}(\mathrm{Par}_m, \mathrm{Par}_m)$ is a Kostka matrix given by $K^{(m)}(\mu, \lambda) = K_{\lambda\mu}$ with the convention $K^{(0)} = (1)$ and Par_m is the set of partitions of m. It follows from (1.3) that the Kostka matrices are invertible. Hence T is a block-diagonal matrix whose blocks are Kronecker products of matrices of full rank and so T itself has full rank. This proves (i).

From (1.3) we find that $S_t = T_t R_t$. Note that T_t is still block diagonal with one diagonal block for each equivalence class of Λ_n under $\sim$ whose shape $s\colon \Phi \to \mathbb{Z}$ satisfies $s(X - \alpha^i) \ge n - t$ for some i. The corresponding block can be written as

$$\tilde{K}^{(s(X-\alpha^i))} \otimes \bigotimes_{f \in \Phi \setminus \{X - \alpha^i\}} K^{(s(X-\alpha^i))}, \tag{5.6}$$

where $\tilde{K}^{(s(X-\alpha^i))}$ is the matrix $K^{(s(X-\alpha^i))}$ restricted to partitions λ of $s(X-\alpha^i)$ satisfying

$$\lambda \trianglerighteq (n-t, 1^{s(X-\alpha^i)-(n-t)}).$$

From (1.3) it follows that, after a suitable ordering of rows and columns, all matrices occurring in the Kronecker product (5.6) are upper-triangular with ones on the diagonal. Again T_t is a block-diagonal matrix whose blocks are Kronecker products of matrices of full rank and so T_t itself has full rank.

From the proof of [EFP11, Thm.20] we know that $\tilde{K}^{(s(X-\alpha^i))}$ is independent of n. Moreover, all other matrices occurring in the Kronecker product (5.6) are also independent of n. Hence T_t itself is also independent of n. This proves (ii). □

In the following, we show that also the matrix S_t has full rank. Recall that, for a composition λ, P_λ denotes the parabolic subgroup of $\mathrm{GL}(|\lambda|, q)$ given in (4.2). We start with the following lemma.

Lemma 5.2.8. *Let m and n be positive integers satisfying $m < n$ and let ϕ and ψ be class functions of $\mathrm{GL}(m,q)$ and $\mathrm{GL}(n,q)$, respectively. Let $\pi_1\colon P_{(m,n)} \to \mathrm{GL}(m,q)$ and $\pi_2\colon P_{(m,n)} \to \mathrm{GL}(n,q)$ be the natural projections onto the corresponding diagonal blocks. Let $g \in P_{(m,n)}$ be such that $\pi_2(g)$ is regular elliptic. Then we have*

$$(\phi \odot \psi)(g) = \phi(\pi_1(g))\, \psi(\pi_2(g)).$$

Proof: From the definition of parabolic induction from (4.4) we have

$$(\phi \odot \psi)(g) = \frac{1}{|P_{(m,n)}|} \sum_{\substack{x \in \mathrm{GL}(m+n,q),\\ xgx^{-1} \in P_{(m,n)}}} \phi(\pi_1(xgx^{-1}))\psi(\pi_2(xgx^{-1})). \tag{5.7}$$

Since $\pi_2(g)$ is regular elliptic and $m < n$, we find from Lemma 5.2.4 that g stabilises a unique m-dimensional subspace U of $\mathbb{F}_q^{m+n}$. Hence the number of $x \in \mathrm{GL}(m+n,q)$ such that $xgx^{-1} \in P_{(m,n)}$ is the number of ordered bases $\{u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_n\}$ of $\mathbb{F}_q^{m+n}$ such that $\{u_1, u_2, \ldots, u_m\}$ spans U. This number equals $|P_{(m,n)}|$. Since $xgx^{-1} \in P_{(m,n)}$ for each $x \in P_{(m,n)}$, we conclude that

$$\{x \in \mathrm{GL}(m+n,q) \colon xgx^{-1} \in P_{(m,n)}\} = P_{(m,n)}.$$

Since $\pi_i(xgx^{-1})$ is conjugate to $\pi_i(g)$ for each $i \in \{1,2\}$ and each $x \in P_{(m,n)}$, the statement of the lemma follows from (5.7). □

We use Lemma 5.2.8 to prove the following result on the structure of the matrix S.

Lemma 5.2.9. *Let k,ℓ be integers satisfying $0 \le k,\ell < \frac{n}{2}$ and let $\underline{\mu} \in \Pi_{k,i}$ and $\underline{\sigma} \in \Sigma_{\ell,j}$. If $k > \ell$, then we have $\xi^{\underline{\mu}}_{\underline{\sigma}} = 0$. For $k \le \ell$, let ν be the partition obtained from $\underline{\mu}(X-\alpha^i)$ by replacing the part $n-k$ by $\ell-k$ and define $\underline{\nu},\underline{\tau} \in \Lambda_\ell$ by*

$$\underline{\nu}(f) = \begin{cases} \nu & \text{for } f = X-\alpha^i \\ \underline{\mu}(f) & \text{otherwise.} \end{cases} \quad \text{and} \quad \underline{\tau}(f) = \begin{cases} \varnothing & \text{for } f = h_{\ell,j} \\ \underline{\sigma}(f) & \text{otherwise.} \end{cases}$$

If $k \le \ell$, then we have $\xi^{\underline{\mu}}_{\underline{\sigma}} = \xi^{\underline{\nu}}_{\underline{\tau}}\omega^{ij}$.

Proof: Let $g \in C_{\underline{\sigma}}$. We define $\underline{\kappa} \in \Lambda_k$ by

$$\underline{\kappa}(f) = \begin{cases} (\underline{\mu}(X-\alpha^i)_2, \underline{\mu}(X-\alpha^i)_3, \ldots) & \text{for } f = X-\alpha^i \\ \underline{\mu}(f) & \text{otherwise.} \end{cases}$$

Then by (4.5) and (4.10) we have

$$\xi^{\underline{\mu}} = \xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i \mapsto (n-k)}. \tag{5.8}$$

For $\xi^{\underline{\mu}}(g)$ to be nonzero, g must be conjugate to an element of the parabolic subgroup $P_{(k,n-k)}$. Each such element fixes a k-dimensional subspace of $\mathbb{F}_q^n$. If $k > \ell$, then by Lemma 5.2.4, g fixes no k-dimensional subspace of $\mathbb{F}_q^n$ and hence $\xi^{\underline{\mu}}(g) = 0$.

Henceforth we assume that $k \le \ell$. We shall frequently use $\xi^{f\mapsto(m)} = \chi^{f\mapsto(m)}$, which follows from (4.6) and (1.3). Since $k \le \ell$ we have

$$\xi^{\underline{\nu}} = \xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i \mapsto (\ell-k)}. \tag{5.9}$$

Write

$$E = \bigcup_{\substack{\underline{\rho} \in \Lambda_{n-k} \\ \underline{\rho}(h_{\ell,j}) = (1)}} C_{\underline{\rho}}.$$

We claim that

$$\xi^{X-\alpha^i \mapsto (n-k)}(e) = (\xi^{X-\alpha^i\mapsto(\ell-k)} \odot \xi^{X-\alpha^i \mapsto (n-\ell)})(e) \quad \text{for each } e \in E. \tag{5.10}$$

Indeed, each $e \in E$ is conjugate to an element of $P_{(\ell-k,n-\ell)}$ with blocks $e_1 \in \mathrm{GL}(\ell-k,q)$ and $e_2 \in \mathrm{GL}(n-\ell,q)$ on the main diagonal, where e_2 is regular elliptic. Hence we find from Lemma 4.2.7 that, for each $e \in E$, the left hand side of (5.10) equals

$$\begin{aligned}\theta(\det(e)^i) &= \theta(\det(e_1)^i) \cdot \theta(\det(e_2)^i) \\ &= \xi^{X-\alpha^i\mapsto(\ell-k)}(e_1) \cdot \xi^{X-\alpha^i\mapsto(n-k)}(e_2),\end{aligned}$$

which by Lemma 5.2.8 equals the right hand side of (5.10). From (5.8) we have

$$\xi^{\underline{\mu}}(g) = \frac{1}{|P_{(k,n-k)}|} \sum_{\substack{x\in \mathrm{GL}(n,q)\\ xgx^{-1}\in P_{(k,n-k)}}} \xi^{\underline{\kappa}}(\pi_1(xgx^{-1}))\xi^{X-\alpha^i\mapsto(n-k)}(\pi_2(xgx^{-1})),$$

where $\pi_1 \colon P_{(k,n-k)} \to \mathrm{GL}(k,q)$ and $\pi_2 \colon P_{(k,n-k)} \to \mathrm{GL}(n-k,q)$ are the natural projections onto the diagonal blocks. Since $k,\ell < \frac{n}{2}$, Lemma 5.2.4 implies that each $\pi_2(xgx^{-1})$ occurring in the summation is forced to lie inside E. Hence by subsequent applications of (5.8), (5.10), and (5.9), we then find that

$$\begin{aligned}\xi^{\underline{\mu}}(g) &= (\xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i\mapsto(n-k)})(g)\\ &= (\xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i\mapsto(\ell-k)} \odot \xi^{X-\alpha^i\mapsto(n-\ell)})(g)\\ &= (\xi^{\underline{\nu}} \odot \xi^{X-\alpha^i\mapsto(n-\ell)})(g).\end{aligned}$$

Without loss of generality, we may assume that $g \in P_{(\ell,n-\ell)}$ and that the diagonal blocks of g are g_1 and g_2, where $g_1 \in C_{\underline{\tau}}$ and g_2 is the companion matrix of $h_{\ell,j}$. Since g_2 is regular elliptic, we may apply Lemma 5.2.8 once more to obtain

$$\xi^{\underline{\mu}}(g) = \xi^{\underline{\nu}}(g_1)\xi^{X-\alpha^i\mapsto(n-\ell)}(g_2).$$

Since $g_1 \in C_{\underline{\tau}}$, we have $\xi^{\underline{\nu}}(g_1) = \xi^{\underline{\nu}}_{\underline{\tau}}$, and since g_2 is the companion matrix of $h_{\ell,j}$, we find from Lemma 4.2.7 that

$$\xi^{X-\alpha^i\mapsto(n-\ell)}(g_2) = \theta(\det(g_2)^i) = \omega^{ij}.$$

Hence we obtain $\xi^{\underline{\mu}}(g) = \xi^{\underline{\nu}}_{\underline{\tau}}\omega^{ij}$, as required. □

We can now prove the required property of the matrix S_t.

Lemma 5.2.10. *For $n > 2t$, the matrix S_t has full rank and is independent of n.*

Proof: To indicate dependence on n, write $S^{(n)}$ for the matrix S given in (5.5) and $S^{(n)}_t$ for the corresponding restricted matrix S_t. Let $n > 2t$. From Lemma 5.2.9 we find that all entries in $S^{(n)}_t$ are independent of n, which proves the second statement of the lemma.

To show that $S^{(n)}_t$ is invertible, we view $S^{(n)}_t$ as a block matrix, where the blocks are indexed by Π_k and Σ_ℓ for $k,\ell \in \{0,1,\dots,t\}$. Let $B_{k,\ell}$ be the block corresponding to Π_k and Σ_ℓ. Lemma 5.2.9 implies that $B_{k,\ell}$ is zero for $k > \ell$ and, for $0 \le k \le \ell$, the block $B_{k,k}$ is the Kronecker product of $S^{(k)}$ and the Vandermonde matrix $(\omega^{ij})_{0\le i,j\le q-2}$. Since the character table of irreducible characters of every finite group is invertible, Lemma 5.2.7 implies that $S^{(k)}$ is invertible and so $B_{k,k}$ is invertible. Hence $S^{(n)}_t$ is block upper-triangular and all diagonal blocks are invertible. Therefore $S^{(n)}_t$ itself is invertible. □

Finally, by combining Lemmas 5.2.7 and 5.2.10, we obtain a proof of Proposition 5.2.6.

5.2.3 Estimates on some conjugacy class sizes and character degrees

In this section, we provide bounds on the size of certain conjugacy classes and degrees of certain irreducible characters of $\mathrm{GL}(n,q)$. These are used in the proofs of the upcoming Lemmas 5.3.2 and 5.4.2, which play a crucial role in the proofs of our main theorems.

Lemma 5.2.11. *Let n and t be positive integers satisfying $n > 2t$ and let $\underline{\sigma} \in \Sigma_{\le t}$. Then we have*

$$\frac{|\mathrm{GL}(n,q)|}{|C_{\underline{\sigma}}|} \le q^{t^5} q^n.$$

Proof: From Theorem 4.1.2, with the same notation as in Theorem 4.1.2, we find that

$$\frac{|\mathrm{GL}(n,q)|}{|C_{\underline{\sigma}}|} \le \prod_{f\in\Phi} \prod_{i=1}^{|\underline{\sigma}(f)|} q^{|f| s_i(\underline{\sigma}(f)') m_i(\underline{\sigma}(f))}. \tag{5.11}$$

Since $\underline{\sigma} \in \Sigma_{\le t}$ and $t < \frac{n}{2}$, there is exactly one polynomial $h \in \Phi$ of degree at least $n-t$ in the support of $\underline{\sigma}$. This polynomial must satisfy $\underline{\sigma}(h) = (1)$ and the corresponding factor in (5.11) is at most q^n. There are at most t other polynomials in the support of $\underline{\sigma}$. Each such polynomial f has degree at most t and satisfies $|\underline{\sigma}(f)| \le t$ and hence the corresponding factor in (5.11) has a crude upper bound of q^{t^4}. As there are at most t such factors, the proof is completed. □

Lemma 5.2.12. *Let t be a positive integer. Then there is a constant δ_t such that, for all sufficiently large n and for all $\underline{\lambda} \in \Lambda_n \setminus \Pi_{\le t}$, we have*

$$\chi^{\underline{\lambda}}(1) \ge \delta_t q^{n(t+1)}.$$

Proof: Let $\underline{\lambda} \in \Lambda_n \setminus \Pi_{\le t}$. From Proposition 5.2.13(ii), stated and proved at the end of this section, with $x = \frac{1}{2^i}$, we find that

$$\frac{\prod_{i=1}^n (q^i - 1)}{q^{\frac{1}{2}n(n+1)}} = \prod_{i=1}^n \left(1 - \frac{1}{q^i}\right) \ge \prod_{i=1}^n \left(1 - \frac{1}{2^i}\right) \ge \prod_{i=1}^n 4^{-\frac{1}{2^i}} \ge \prod_{i=1}^\infty 4^{-\frac{1}{2^i}} = \frac{1}{4}.$$

Using this estimation in the q-analog of the hook-length formula (4.13) gives

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \le 4q^{N(\underline{\lambda}) - M(\underline{\lambda}) - \frac{1}{2}n(n+1)}, \tag{5.12}$$

where

$$N(\underline{\lambda}) = \sum_{f\in\Phi} |f| \sum_{(i,j)\in\underline{\lambda}(f)} h_{i,j}(\underline{\lambda}(f)),$$

$$M(\underline{\lambda}) = \sum_{f\in\Phi} |f| b(\underline{\lambda}(f))$$

and b and $h_{i,j}$ are as defined in Lemma 4.2.6. Note that for each partition λ, we have

$$\sum_{(i,j)\in\lambda} h_{i,j}(\lambda) \le \sum_{k=1}^{|\lambda|} k = \frac{1}{2}|\lambda|(|\lambda|+1). \tag{5.13}$$

First we assume that there exists a polynomial $h \in \Phi$ such that $|h| = 1$ and $\underline{\lambda}(h)'_1 \ge n-t$. In this case we have

$$M(\underline{\lambda}) \ge b(\underline{\lambda}(h)) \ge \sum_{k=1}^{n-t}(i-1)\underline{\lambda}(h)_k \ge \sum_{k=1}^{n-t-1} k = \frac{1}{2}(n-t)(n-t-1)$$

and from (5.13) together with $|\underline{\lambda}(f)| \ne n$ for all $f \in \Phi$ and $\sum_{f\in\Phi}|f||\underline{\lambda}(f)| = n$ we find that

$$\begin{aligned} N(\underline{\lambda}) &\le \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|(|\underline{\lambda}(f)|+1) \\ &\le \frac{n+1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)| \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

Therefore (5.12) implies that

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \le 4q^{-\frac{1}{2}(n-t)(n-t-1)},$$

so that we have $\chi^{\underline{\lambda}}(1) \ge q^{n(t+1)}$ for all sufficiently large n by very crude estimates.

Hence we can assume that $\underline{\lambda}(h)'_1 \le n-t-1$ and, since $\underline{\lambda} \notin \Pi_{\le t}$, that $\underline{\lambda}(f)_1 \le n-t-1$ for all $f \in \Phi$ satisfying $|f| = 1$.

In what follows we distinguish between two cases. In the first case we assume that $|\underline{\lambda}(f)| \le n-t-1$ for all $f \in \Phi$ satisfying $|f| = 1$. Let ℓ be the maximum of $|\underline{\lambda}(f)|$ over all $f \in \Phi$ with $|f| = 1$, hence $\ell \le n-t-1$. By (5.13) we have

$$\begin{aligned} N(\underline{\lambda}) &\le \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|(|\underline{\lambda}(f)|+1) \\ &= \frac{n}{2} + \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|^2, \end{aligned}$$

where the last equation holds since $\sum_{f\in\Phi}|f||\underline{\lambda}(f)|=n$. If $\ell\le\frac{n}{2}$, then we have $|\underline{\lambda}(f)|\le\frac{n}{2}$ for all $f\in\Phi$ and so

$$N(\underline{\lambda})\le\frac{n}{2}+\frac{n}{4}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|=\frac{n^2}{4}+\frac{n}{2}.$$

From (5.12) and the trivial bound $M(\underline{\lambda})\ge 0$, we find that

$$\frac{1}{\chi^{\underline{\lambda}}(1)}\le 4q^{-\frac{n^2}{4}},$$

so that we have $\chi^{\underline{\lambda}}(1)\ge q^{n(t+1)}$ for all sufficiently large n, again by very crude estimates. If $\ell>\frac{n}{2}$, and g denoting the polynomial in Φ satisfying $|g|=1$ and $|\underline{\lambda}(g)|=\ell$, then, by (5.13)

$$\begin{aligned}N(\underline{\lambda})&\le\frac{1}{2}(\sum_{f\in\Phi}|f||\underline{\lambda}(f)|+\sum_{f\in\Phi}|f||\underline{\lambda}(f)|^2)\\&\le\frac{1}{2}(n+\ell^2+(\sum_{\substack{f\in\Phi\\f\ne g}}|f||\underline{\lambda}(f)|)^2)\\&=\frac{1}{2}(n+\ell^2+(n-\ell)^2)\\&\le\frac{1}{2}(n+(n-t-1)^2+(t+1)^2)\\&=\frac{n^2+n}{2}-n(t+1)+(t+1)^2,\end{aligned}$$

where we have used that $x^2+(n-x)^2$ is increasing for $x\ge\frac{n}{2}$. Hence in this case we obtain $\chi^{\underline{\lambda}}(1)\ge\frac{1}{4}q^{n(t+1)-(t+1)^2}$ by (5.12) together with the trivial estimate $M(\underline{\lambda})\ge 0$.

In the remaining case we assume that there exists $h\in\Phi$ such that $|h|=1$ and $|\underline{\lambda}(h)|\ge n-t$. Recall that we also assume that $\underline{\lambda}(h)_1\le n-t-1$ and $\underline{\lambda}(h)'_1\le n-t-1$. Since $N(\underline{\lambda})$ depends only on the hook lengths of $\underline{\lambda}(f)$ for $f\in\Phi$, we may replace $\underline{\lambda}(h)$ by its conjugate $\underline{\lambda}(h)'$. Assuming that n is sufficiently large, namely $n\ge(t+2)^2$, we have $\underline{\lambda}(h)_1\ge t+2$ or $\underline{\lambda}(h)'_1\ge t+2$ and we assume without loss of generality that $\underline{\lambda}(h)_1\ge t+2$. Write $\underline{\lambda}(h)_1=n-r$, so that our assumptions imply $t+1\le r\le n-t-2$. Then, writing $s=|\underline{\lambda}(h)|$, there exist nonnegative integers c_j satisfying

$$\sum_{j=1}^{n-r}h_{1j}(\underline{\lambda}(h))=\sum_{j=1}^{n-r}(j+c_j),\qquad\text{where}\qquad\sum_{j=1}^{n-r}c_j=s-(n-r).$$

Hence

$$\sum_{j=1}^{n-r}h_{1j}(\underline{\lambda}(h))=\binom{n-r+1}{2}+(s-n+r).$$

Application of (5.13) with $\lambda = (\underline{\lambda}(h)_2, \underline{\lambda}(h)_3, \ldots)$ gives

$$\begin{aligned}\sum_{(i,j)\in\underline{\lambda}(h)} h_{i,j}(\underline{\lambda}(h)) &\leq \binom{s-n+r+1}{2} + \binom{n-r+1}{2} + (s-n+r)\\ &= \frac{s^2}{2} + \frac{3s}{2} + n^2 - sn - n + r(r-(2n-s-1))\\ &\leq \frac{s^2}{2} + \frac{3s}{2} + n^2 - sn - n + (t+1)((t+1)-(2n-s-1)),\end{aligned}$$

since the term depending on r is maximised for $r = t+1$ over the interval $[t+1, n-t-2]$. This last expression equals

$$\frac{s}{2} + \frac{1}{2}s(s-2(n-t-2)) + n^2 - n + (t+1)((t+1)-(2n-1)).$$

The second summand is increasing for $s \geq n-t$ and so is at most $\frac{1}{2}n(n-2(n-t-2))$. Hence we obtain

$$\sum_{(i,j)\in\underline{\lambda}(h)} h_{i,j}(\underline{\lambda}(h)) \leq \frac{s}{2} + \frac{n^2}{2} - n(t+1) + (t+1)(t+2).$$

Invoking (5.13) once more, we obtain

$$N(\underline{\lambda}) \leq \sum_{(i,j)\in\underline{\lambda}(h)} h_{ij} + \frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|(|\underline{\lambda}(h)|+1).$$

We have

$$\frac{s}{2} + \frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)| = \frac{1}{2}\sum_{f\in\Phi} |f||\underline{\lambda}(f)| = \frac{n}{2}$$

and

$$\frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|^2 \leq \frac{1}{2}\left(\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|\right)^2 \leq \frac{t^2}{2}.$$

Collecting all terms, we find that

$$N(\underline{\lambda}) \leq \frac{n(n+1)}{2} - n(t+1) + (t+1)(t+2) + \frac{t^2}{2}.$$

From (5.12) we then obtain

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \leq 4q^{-n(t+1)+(t+1)(t+2)+\frac{1}{2}t^2},$$

which completes the proof. □

In the proof of Lemma 5.2.12 we used the following technical results.

Proposition 5.2.13.

(i) Let s be an integer with $n-t \leq s \leq n$. Then, for n sufficiently large and all $r \in [t+1, n-t-2]$, we have

$$r(r-(2n-s-1)) \leq (t+1)((t+1)-(2n-s-1)).$$

(ii) For all x with $0 \leq x \leq \frac{1}{2}$, we have

$$1-x \geq 4^{-x}.$$

Proof: Let $g(r) = r(r-(2n-s-1))$. We find that g is decreasing on $(-\infty, \frac{2n-s-1}{2})$ and increasing on $(\frac{2n-s-1}{2}, \infty)$. And since $\frac{2n-s-1}{2} \in [t+1, n-t-2]$ it is sufficient to prove $g(t+1) \geq g(n-(t+2))$, which implies (i). From the identity $g(2n-s-r) = g(r) - 2n + s - 2r$ and the estimate $g(n-(t+2)) \leq g(2n-s-(t+2))$ we find

$$\begin{aligned} g(t+1) - g(n-(t+2)) &\geq g(t+1) - g(2n-s-(t+2)) \\ &= g(t+1) - (g(t+2) - 2n + s - 2(t+2)). \end{aligned}$$

Calculating and using the assumption $s \leq n$ gives that

$$g(t+1) - g(n-(t+2)) \geq 2n > 0,$$

which establishes (i). To prove (ii), let f be given by $f(x) = 1-x-4^{-x}$. Computing the first derivative of f and using elementary calculus gives that f is only monotonically increasing on $\left(-\infty, \frac{\log(\log(4))}{\log(4)}\right]$. Since f is continuous and $f(0) = f(\frac{1}{2}) = 0$, it follows that $f(x) \geq 0$ for all $0 \leq x \leq \frac{1}{2}$, which implies (ii). □

5.3 Proofs of the pointwise intersection theorems

In this section, we prove the pointwise intersection results from Theorem 5.1.4 and Theorem 5.1.6 following the strategy described in Section 5.2.1.

In order to determine the appropriate weight w that appears in (5.3), recall the definition of the eigenvalues $P(\underline{\lambda}, \underline{\sigma})$ given in (4.25) and the definition of the prescribed extremal eigenvalue η from (5.4). We obtain the following existence result.

Proposition 5.3.1. *Let n and t be positive integers satisfying $n > 2t$. Then there exists $\omega \in \mathbb{R}(\Omega_n \cap \Sigma_{\leq t})$ such that $\omega(\underline{\sigma}) = 0$ for $\underline{\sigma}(X-1) = (1^t)$ and*

$$\sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\leq t}} w(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) = \begin{cases} 1 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{0,0} \\ \eta & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{k,0} \text{ and } 1 \leq k \leq t \\ 0 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{k,i} \text{ and } 0 \leq k \leq t \text{ and } 1 \leq i \leq q-2 \end{cases} \tag{5.14}$$

and

$$|w(\underline{\sigma})| \le \frac{\gamma_t}{|D_{\underline{\sigma}}|} \qquad \text{for all } \underline{\sigma} \in \Omega_n \cap \Sigma_{\le t} \tag{5.15}$$

for some constant γ_t depending only on t.

Proof: From Proposition 5.2.5 we know that Q_t has full rank. In view of (4.25) there exists a unique $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\le t})$ satisfying (5.14).

We now show that $w(\underline{\sigma}) = 0$ for the $\lfloor q/2 \rfloor + 1$ elements $\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}$ satisfying $\underline{\sigma}(1) = (1^t)$. Without loss of generality we may assume that Ω_n contains $X - \alpha^i$ and $h_{t,j}$ for all $i, j \in \{0, 1, \dots, \lfloor q/2 \rfloor\}$. Accordingly we define $\underline{\sigma}_j \in \Sigma_{t,j}$ by $\underline{\sigma}_j(1) = (1^t)$ for $j = 0, 1, \dots, \lfloor q/2 \rfloor$. Recall the definition of the character $\zeta^{(t,i)}$ from Section 4.3 and write $\zeta^{(t,i)}_{\underline{\sigma}}$ for this character evaluated on the conjugacy class $C_{\underline{\sigma}}$. We evaluate the sum

$$S_i = \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) |D_{\underline{\sigma}}| \, (\zeta^{(t,i)}_{\underline{\sigma}} + \zeta^{(t,-i)}_{\underline{\sigma}}) \tag{5.16}$$

in two ways. Since $\zeta^{(t,0)}$ is the permutation character on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$, we find by Lemma 4.3.2 that the summand in (5.16) is nonzero only when the elements of $C_{\underline{\sigma}}$ fix a t-tuple of linearly independent elements of $\mathbb{F}_q^n$, hence only when $\underline{\sigma} = \underline{\sigma}_j$ for some j. By the definition of $\underline{\sigma}_j$, each element in $C_{\underline{\sigma}_j}$ has determinant α^j. Hence by applying Lemma 4.3.2 twice we obtain

$$\zeta^{(t,i)}_{\underline{\sigma}_j} = \omega^{ij} \zeta^{(t,0)}_{\underline{\sigma}_j} = \omega^{ij} \zeta^{(t,0)}_{\underline{\sigma}_0}$$

and therefore

$$S_i = 2\zeta^{(t,0)}_{\underline{\sigma}_0} \sum_{j=0}^{\lfloor q/2 \rfloor} w(\underline{\sigma}_j) \, |D_{\underline{\sigma}_j}| \, \cos\left(\frac{2\pi i j}{q-1}\right). \tag{5.17}$$

On the other hand, since $\zeta^{(t,i)} + \zeta^{(t,-i)}$ is a real-valued class function, we find from Lemma 4.4.2 that it is a linear combination of $\psi^{\underline{\lambda}}$ for $\underline{\lambda} \in \Omega_n$. Hence by Lemma 4.3.3 there exist numbers $n_{i,\underline{\lambda}}$ such that

$$\zeta^{(t,i)}_{\underline{\sigma}} + \zeta^{(t,-i)}_{\underline{\sigma}} = \sum_{\substack{\underline{\lambda} \in \Omega_n \\ \underline{\lambda}(X-\alpha^i)_1 \ge n-t}} n_{i,\underline{\lambda}} \, \psi^{\underline{\lambda}}_{\underline{\sigma}}$$

and hence

$$S_i = \sum_{\substack{\underline{\lambda} \in \Omega_n \\ \underline{\lambda}(X-\alpha^i)_1 \ge n-t}} n_{i,\underline{\lambda}} \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) |D_{\underline{\sigma}}| \, \psi^{\underline{\lambda}}_{\underline{\sigma}}. \tag{5.18}$$

Since (5.14) holds, we conclude that $S_i = 0$ for each i satisfying $1 \le i \le \lfloor q/2 \rfloor$. Since $\zeta^{(t,0)}$ is a permutation character, it contains the trivial character with multiplicity

1 (this can be seen by Frobenius reciprocity, for example). Hence we have $n_{0,\underline{\lambda}} = 2$ for $\underline{\lambda} \in \Omega_n$ satisfying $\underline{\lambda}(X-1) = (n)$. We therefore find from (5.18) and (5.14) that

$$S_0 = 2 + \eta \sum_{\substack{\underline{\lambda} \in \Omega_n \\ n-t \le \underline{\lambda}(X-1)_1 < n}} n_{0,\underline{\lambda}}\, \psi^{\underline{\lambda}}(1) = 2 + 2\eta(\zeta^{(t,0)}(1) - 1).$$

Since $\zeta^{(t,0)}(1)$ equals the number of t-tuples of linearly independent elements of $\mathbb{F}_q^n$, we have

$$\zeta^{(t,0)}(1) = (q^n - 1)(q^n - q)\cdots(q^n - q^{t-1}). \tag{5.19}$$

Therefore $S_0 = 0$ and so $S_i = 0$ for each i satisfying $0 \le i \le \lfloor q/2 \rfloor$. Since each element of $C_{\underline{\sigma}_0}$ fixes a t-tuple of linearly independent elements of $\mathbb{F}_q^n$, we have $\zeta_{\underline{\sigma}_0}^{(t,0)} \neq 0$. Thus (5.17) implies

$$\sum_{j=0}^{\lfloor q/2 \rfloor} w(\underline{\sigma}_j)\, |D_{\underline{\sigma}_j}|\, \cos\left(\frac{2\pi ij}{q-1}\right) = 0 \quad \text{for each } i \text{ satisfying } 0 \le i \le \lfloor q/2 \rfloor$$

and, using that $(\omega^{ij})_{0 \le i,j < q-1}$ is a Vandermonde matrix, it follows that this in turn implies that $w(\underline{\sigma}_j) = 0$ for all j satisfying $0 \le j \le \lfloor q/2 \rfloor$, as required.
Now, for each $\underline{\lambda} \in \Omega_n$ satisfying $n - t \le \underline{\lambda}(X-1)_1 < n$, we find from Lemma 4.3.3 that

$$|\eta|\, \psi^{\underline{\lambda}}(1) \le |\eta|\, (\zeta^{(t,0)}(1) - 1) = 1,$$

using (5.19). Since $\psi^{\underline{\lambda}}(1) = \chi^{\underline{\lambda}}(1) = 1$ for $\underline{\lambda} \in \Pi_{0,0}$, we conclude from (5.14) that

$$\left| \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) |D_{\underline{\sigma}}|\, \psi^{\underline{\lambda}}_{\underline{\sigma}} \right| \le 1 \quad \text{for each } \underline{\lambda} \in \Omega_n \cap \Pi_{\le t}.$$

By Proposition 5.2.5 all entries of Q_t (which are precisely the values of $\psi^{\underline{\lambda}}_{\underline{\sigma}}$ occurring in the sum) are independent of n and so are uniformly bounded by some value only depending on t. The same also holds for the inverse of Q_t, which establishes (5.15).□

In what follows we treat the remaining eigenvalues.

Lemma 5.3.2. *Let n and t be positive integers with $n > 2t$ and let $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\le t})$ be such that*

$$|w(\underline{\sigma})| \le \frac{\gamma_t}{|D_{\underline{\sigma}}|} \quad \textit{for all } \underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}$$

for some constant γ_t depending only on t. Then

$$\left| \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) \right| < |\eta| \quad \textit{for all } \underline{\lambda} \in \Omega_n \setminus \Pi_{\le t},$$

provided that n is sufficiently large compared to t.

For the proof of Lemma 5.3.2, we recall from Section 1.1 that the inner product of class functions χ and ψ of $\mathrm{GL}(n,q)$ is given by

$$\langle \chi, \psi \rangle = \frac{1}{|\mathrm{GL}(n,q)|} \sum_{g \in \mathrm{GL}(n,q)} \chi(g)\overline{\psi(g)}. \tag{5.20}$$

Proof (of Lemma 5.3.2): By the definition (4.25) of $P(\underline{\lambda},\underline{\sigma})$ and (4.23) we have

$$P(\underline{\lambda},\underline{\sigma}) = \frac{|\mathrm{GL}(n,q)|}{\psi^{\underline{\lambda}}(1)} \langle \psi^{\underline{\lambda}}, 1_{D_{\underline{\sigma}}} \rangle. \tag{5.21}$$

Since $\chi^{\underline{\lambda}}$ is irreducible, we have $\langle \psi^{\underline{\lambda}}, \psi^{\underline{\lambda}} \rangle = 1$ or 2 and therefore we obtain, by an application of the Cauchy-Schwarz inequality,

$$|\langle \psi^{\underline{\lambda}}, 1_{D_{\underline{\sigma}}} \rangle| \le \sqrt{2\,\langle 1_{D_{\underline{\sigma}}}, 1_{D_{\underline{\sigma}}} \rangle} = \sqrt{\frac{2|D_{\underline{\sigma}}|}{|\mathrm{GL}(n,q)|}}.$$

From (5.21) and our hypothesis on w we then find that

$$\begin{aligned}
\left| \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) P(\underline{\lambda},\underline{\sigma}) \right| &\le \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} |w(\underline{\sigma})|\, |P(\underline{\lambda},\underline{\sigma})| \\
&\le \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} \frac{\gamma_t}{|D_{\underline{\sigma}}|} \frac{|\mathrm{GL}(n,q)|}{\psi^{\underline{\lambda}}(1)} \sqrt{\frac{2|D_{\underline{\sigma}}|}{|\mathrm{GL}(n,q)|}} \\
&\le \frac{\gamma_t\, |\Sigma_{\le t}|}{\psi^{\underline{\lambda}}(1)} \max_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} \sqrt{\frac{2|\mathrm{GL}(n,q)|}{|D_{\underline{\sigma}}|}} \\
&\le \frac{\gamma_t\, |\Sigma_{\le t}|}{\chi^{\underline{\lambda}}(1)} \max_{\underline{\sigma} \in \Sigma_{\le t}} \sqrt{\frac{2|\mathrm{GL}(n,q)|}{|C_{\underline{\sigma}}|}}.
\end{aligned}$$

Note that $|\Sigma_{\le t}|$ is independent of n. Using Lemmas 5.2.11 and 5.2.12 we find that there is a constant γ_t', depending only on t, such that

$$\left| \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}} w(\underline{\sigma}) P(\underline{\lambda},\underline{\sigma}) \right| \le \frac{\gamma_t'}{q^{n/2}} \frac{1}{q^{nt}}$$

for all $\underline{\lambda} \in \Omega_n \setminus \Pi_{\le t}$ and all sufficiently large n. The right hand side is certainly strictly smaller than $1/q^{nt}$ for all sufficiently large n and the proof is completed by noting that $|\eta| > 1/q^{nt}$. □

Recall from Section 4.4.2 that $V_{\underline{\lambda}}$ is the column span of $F_{\underline{\lambda}}$ over the reals. We define

$$U_t = \sum_{\substack{\underline{\lambda} \in \Omega_n \\ \underline{\lambda}(X-1)_1 \ge n-t}} V_{\underline{\lambda}}.$$

We note that, due to Lemma 4.4.3, this is in fact a direct sum. We obtain the following.

Theorem 5.3.3. *Let t be a positive integer. Then, for all sufficiently large n, the following holds.*

(i) Every t-intersecting set Y in $\mathrm{GL}(n,q)$ satisfies

$$|Y| \le \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, we have $\mathbb{1}_Y \in U_t$.

(ii) Every pair of t-cross-intersecting sets Y, Z in $\mathrm{GL}(n,q)$ satisfies

$$\sqrt{|Y|\cdot|Z|} \le \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, we have $\mathbb{1}_Y, \mathbb{1}_Z \in U_t$.

Proof: As explained at the beginning of Section 5.2.1, we apply Theorem 2.3.11 to the graph with adjacency matrix

$$\sum_{\substack{\underline{\sigma}\in\Omega_n\cap\Sigma_{\le t}\\ \underline{\sigma}(X-1)\ne(1^t)}} B_{\underline{\sigma}}$$

and the $|D_{\underline{\sigma}}|$-regular spanning subgraphs $\Gamma_{\underline{\sigma}}$ with adjacency matrix $B_{\underline{\sigma}}$ for those $\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t}$ satisfying $\sigma(X-1) \ne (1^t)$. Since none of the elements in $D_{\underline{\sigma}}$ for such $\underline{\sigma}$ fix a t-space pointwise, every t-intersecting set in $\mathrm{GL}(n,q)$ is an independent set in this graph. Recall from Lemma 4.4.3 that every element of $V_{\underline{\lambda}}$ is an eigenvector of $B_{\underline{\sigma}}$ with eigenvalue $P(\underline{\lambda},\underline{\sigma})$. Let $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\le t})$ be the vector given by Proposition 5.3.1 and write

$$P(\underline{\lambda}) = \sum_{\substack{\underline{\sigma}\in\Omega_n\cap\Sigma_{\le t}\\ \underline{\sigma}(X-1)\ne(1^t)}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma}).$$

Proposition 5.3.1 and Lemma 5.3.2 imply that, for all sufficiently large n, we have

$$P(\underline{\lambda}) = \begin{cases} 1 & \text{for } \underline{\lambda}(X-1)_1 = n \\ \eta & \text{for } n-t \le \underline{\lambda}(X-1)_1 < n \end{cases}$$

and $|P(\underline{\lambda})| < |\eta|$ for $\underline{\lambda}(X-1)_1 < n-t$. Hence, writing $\underline{\lambda}_0$ for $X-1 \mapsto (n)$, we have $P(\underline{\lambda}_0) = 1$ and

$$\eta = \min_{\underline{\lambda}\ne\underline{\lambda}_0} P(\underline{\lambda}) \quad \text{and} \quad |\eta| = \max_{\underline{\lambda}\ne\underline{\lambda}_0} |P(\underline{\lambda})|.$$

Then the required result follows from Theorem 2.3.11 and the decomposition of $\mathbb{R}(\mathrm{GL}(n,q))$ given in Lemma 4.4.3. □

Our proofs of Theorems 5.1.4 and 5.1.6 are completed by the following result.

Theorem 5.3.4. *U_t is spanned by the characteristic vectors of t-cosets.*

Proof: Let $\mathcal{A}_t$ be the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$. Define the incidence matrix $M_t \in \mathbb{C}(\mathrm{GL}(n,q), \mathcal{A}_t \times \mathcal{A}_t)$ of elements of $\mathrm{GL}(n,q)$ versus t-cosets by

$$M_t(x,(u,v)) = \begin{cases} 1 & \text{for } xu = v \\ 0 & \text{otherwise,} \end{cases}$$

so that the columns of M_t are precisely the characteristic vectors of the t-cosets. Let $\zeta^t = \zeta^{(t,0)}$ be the permutation character of $\mathrm{GL}(n,q)$ on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$ from Section 4.3. We define $C_t \in \mathbb{C}(\mathrm{GL}(n,q), \mathrm{GL}(n,q))$ by

$$C_t(x,y) = \zeta^t(x^{-1}y).$$

Denoting by $\mathbb{1}_{xu=v}$ the indicator of the event that $x \in \mathrm{GL}(n,q)$ maps u to v, we have

$$\begin{aligned} (M_t M_t^T)(x,y) &= \sum_{u,v} M_t(x,(u,v)) M_t(y,(u,v)) \\ &= \sum_{u,v} \mathbb{1}_{xu=v} \mathbb{1}_{yu=v} \\ &= \sum_{u} \mathbb{1}_{xu=yu} \\ &= \sum_{u} \mathbb{1}_{x^{-1}yu=u} \\ &= \zeta^t(x^{-1}y) = C_t(x,y). \end{aligned}$$

Hence we have $C_t = M_t M_t^T$ and so the column span of C_t equals the column span of M_t or equivalently the span of the characteristic vectors of the t-cosets.

From Lemma 4.3.3 we have

$$\zeta^t = \sum_{\substack{\underline{\lambda} \in \Lambda_n \\ \underline{\lambda}(X-1)_1 \geq n-t}} m_{\underline{\lambda}} \chi^{\underline{\lambda}}$$

for some integers $m_{\underline{\lambda}}$ satisfying $m_{\underline{\lambda}} \neq 0$ for each $\underline{\lambda}$ occurring in the summation. Since ζ^t is real-valued, we find from Lemma 4.4.2 that $m_{\underline{\lambda}^*} = m_{\underline{\lambda}}$ and therefore we have

$$\zeta^t = \sum_{\substack{\underline{\lambda} \in \Omega_n \\ \underline{\lambda}(X-1)_1 \geq n-t}} m_{\underline{\lambda}} \psi^{\underline{\lambda}}. \tag{5.22}$$

Lemma 4.2.6 implies that $\chi^{\underline{\lambda}^*}(1) = \chi^{\underline{\lambda}}(1)$. We therefore obtain from (4.24) and Example 2.1.13 that

$$F_{\underline{\lambda}}(x,y) = \frac{\chi^{\underline{\lambda}}(1)}{|\mathrm{GL}(n,q)|}\psi^{\underline{\lambda}}(x^{-1}y)$$

and thus we find from (5.22) that

$$C_t = |\mathrm{GL}(n,q)| \sum_{\substack{\underline{\lambda}\in\Omega_n \\ \underline{\lambda}(X-1)_1 \geq n-t}} \frac{m_{\underline{\lambda}}}{\chi^{\underline{\lambda}}(1)} F_{\underline{\lambda}} \tag{5.23}$$

Hence the column span of C_t is contained in U_t. Conversely, let v be a column of $F_{\underline{\kappa}}$ for some $\kappa \in \Omega_n$ satisfying $\underline{\kappa}(X-1)_1 \geq n-t$. Since $F_{\underline{\lambda}}$ is idempotent, we have $F_{\underline{\lambda}}v = v$ for $\underline{\kappa} = \underline{\lambda}$ and Lemma 4.4.3 implies $F_{\underline{\lambda}}v = 0$ for $\underline{\kappa} \neq \underline{\lambda}$. Hence from (5.23), we find that

$$C_t v = |\mathrm{GL}(n,q)| \frac{m_{\underline{\kappa}}}{\chi^{\underline{\kappa}}(1)} v,$$

and, since $m_{\underline{\kappa}} \neq 0$, we conclude that v is in the column span of C_t. This completes the proof. □

5.4 Proofs of the spacewise intersection theorems

The proofs of the spacewise intersection and spacewise cross-intersection results from Theorems 5.1.9 and 5.1.11, follow along similar lines as those in Section 5.3.

Recall from Section 4.3, that the character $\xi^{X-1\mapsto(n-t,t)}$ is the permutation character of $\mathrm{GL}(n,q)$ on the set of t-spaces of $\mathbb{F}_q^n$. From Lemma 4.3.1 we obtain the decomposition of this permutation character, namely

$$\xi^{X-1\mapsto(n-t,t)} = \sum_{s=0}^{t} \chi^{X-1\mapsto(n-s,s)}. \tag{5.24}$$

Recall from (2.1) that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ gives the number of k-dimensional subspaces of $\mathbb{F}_q^n$. Then we have

$$\xi^{X-1\mapsto(n-t,t)}(1) = \begin{bmatrix} n \\ t \end{bmatrix}_q, \tag{5.25}$$

and so (5.24) implies that

$$\chi^{X-1\mapsto(n-s,s)}(1) = \begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q. \tag{5.26}$$

Also note that $\psi^{X-1\mapsto\lambda} = \chi^{X-1\mapsto\lambda}$ for all partitions λ. Throughout this section, we define

$$\varepsilon = -\frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q - 1},$$

which will be our prescribed extremal eigenvalue.

We begin with the following counterpart of Proposition 5.3.1.

Proposition 5.4.1. *Let n and t be positive integers satisfying $n > 2t$. Then there exists $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\le t-1})$ such that*

$$\sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t-1}} w(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) = \begin{cases} 1 & \text{for } \underline{\lambda}(X-1) = (n) \\ \varepsilon & \text{for } \underline{\lambda}(X-1) = (n-s, s) \text{ with } 1 \le s \le t \\ 0 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{\le t-1}, \text{ where} \\ & \quad \underline{\lambda}(X-1) \ne (n-s, s) \text{ with } 0 \le s \le t-1 \end{cases} \tag{5.27}$$

and

$$|w(\underline{\sigma})| \le \frac{\gamma_t}{|D_{\underline{\sigma}}|} \quad \text{for all } \underline{\sigma} \in \Omega_n \cap \Sigma_{\le t-1} \tag{5.28}$$

for some constant γ_t depending only on t.

Proof: From Proposition 5.2.5 we know that Q_{t-1} has full rank. In view of (4.25) there exists a unique $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\le t-1})$ satisfying (5.27) except for $\underline{\lambda}$ of the form $\underline{\lambda}(X-1) = (n-t, t)$.

Next, we show that (5.27) also holds when $\underline{\lambda}(X-1) = (n-t, t)$. By Lemma 5.2.9 we have $\xi_{\underline{\sigma}}^{X-1\mapsto(n-t,t)} = 0$ for each $\underline{\sigma} \in \Sigma_{\le t-1}$. Hence, by using (5.24), we have

$$\begin{aligned} 0 &= \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t-1}} w(\underline{\sigma}) |D_{\underline{\sigma}}| \xi_{\underline{\sigma}}^{X-1\mapsto(n-t,t)} \\ &= \sum_{s=0}^{t} \sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\le t-1}} w(\underline{\sigma}) |D_{\underline{\sigma}}| \chi_{\underline{\sigma}}^{X-1\mapsto(n-s,s)}. \end{aligned} \tag{5.29}$$

Since (5.27) holds with the only exception $\underline{\lambda}(X-1) = (n-t, t)$, the inner sum equals 1 for $s = 0$ and $\varepsilon \chi^{X-1\mapsto(n-s,s)}(1)$ for each s satisfying $1 \le s \le t-1$. Assuming that this is true also for $s = t$ and using (5.26), the right hand side of (5.29) is indeed

$$1 + \varepsilon \sum_{s=1}^{t} \left(\begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q \right) = 1 + \varepsilon \left(\begin{bmatrix} n \\ t \end{bmatrix}_q - 1 \right) = 0.$$

Hence (5.27) also holds when $\underline{\lambda}(X-1) = (n-t, t)$.

It remains to prove (5.28). For each s satisfying $1 \le s \le t$, we find from (5.27) that

$$|\varepsilon| \, \chi^{X-1\mapsto(n-s,s)}(1) \le |\varepsilon| \, (\xi^{X-1\mapsto(n-t,t)}(1) - 1) = 1,$$

using (5.25). Since $\chi^{X-1\mapsto(n)}(1)=1$, we conclude from (5.27) that

$$\Bigg|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\le t-1}} w(\underline{\sigma})|D_{\underline{\sigma}}|\,\psi^{\underline{\lambda}}_{\underline{\sigma}}\Bigg|\le 1\quad\text{for each } \underline{\lambda}\in\Omega_n\cap\Pi_{\le t-1}.$$

By Proposition 5.2.5 all entries of Q_{t-1} are independent of n and so are uniformly bounded by some value only depending on t. The same also holds for the inverse of Q_t, which establishes (5.28). □

The bound (5.28) and Lemma 5.3.2 ensure that the right hand side of (5.27) is small in modulus for each $\underline{\lambda}\in\Omega_n\setminus\Pi_t$. It therefore remains to deal with the case that $\underline{\lambda}\in\Omega_n\cap\Pi_t$ except for $\underline{\lambda}\in\Omega_n$ given by $\underline{\lambda}(X-1)=(n-t,t)$, which is the subject of the following lemma.

Lemma 5.4.2. *Let $w\in\mathbb{R}(\Omega_n\cap\Sigma_{\le t-1})$ be given in Proposition 5.4.1 (so that $n>2t$). Then, for all $\underline{\lambda}\in\Omega_n\cap\Pi_t$ with $\underline{\lambda}(X-1)\ne(n-t,t)$, we have*

$$\Bigg|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\le t-1}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma})\Bigg|<|\varepsilon|,$$

provided that n is sufficiently large compared to t.

Proof: By slight abuse of notation, we view w as an element of $\mathbb{R}(\mathrm{GL}(n,q))$ by setting $w(x)=0$ if $x\notin\Omega_n\cap\Sigma_{\le t-1}$ and $w(x)=w(\underline{\sigma})$ if $x\in\Omega_n\cap\Sigma_{\le t-1}$ and $x\in D_{\underline{\sigma}}$. Recalling the inner product on class functions of $\mathrm{GL}(n,q)$ from (5.20), the statement of the lemma is equivalent to

$$\frac{|\mathrm{GL}(n,q)|}{\psi^{\underline{\lambda}}(1)}\,|\langle w,\psi^{\underline{\lambda}}\rangle|<|\varepsilon| \tag{5.30}$$

for all $\underline{\lambda}\in\Omega_n\cap\Pi_t$ with $\underline{\lambda}(X-1)\ne(n-t,t)$, provided that n is sufficiently large compared to t.

We pick $\underline{\lambda}\in\Omega_n\cap\Pi_t$ such that $\underline{\lambda}(X-1)\ne(n-t,t)$. Then $\underline{\lambda}(X-\alpha^i)_1=n-t$ for some i. First assume that $|\underline{\lambda}(X-1)|\ne n$. Denoting by $\mathrm{Re}(x)$ the real part of a complex number x, we find from Lemma 4.4.2 and (4.12) that

$$\frac{1}{2}|\langle w,\psi^{\underline{\lambda}}\rangle|\le|\mathrm{Re}(\langle w,\chi^{\underline{\lambda}}\rangle)|=\Bigg|\sum_{\underline{\mu}\sim\underline{\lambda}} H_{\underline{\mu}\underline{\lambda}}\,\mathrm{Re}(\langle w,\xi^{\underline{\mu}}\rangle)\Bigg|.$$

Lemma 5.2.9 implies that $\xi^{\underline{\mu}}_{\underline{\sigma}}=0$ for each $\underline{\mu}\notin\Pi_{\le t-1}$ and each $\underline{\sigma}\in\Sigma_{\le t-1}$. For $\underline{\mu}\in\Lambda_n$, we have

$$\mathrm{Re}(\langle w,\xi^{\underline{\mu}}\rangle)=\sum_{\underline{\kappa}\sim\underline{\mu}} K_{\underline{\kappa}\underline{\mu}}\,\mathrm{Re}(\langle w,\chi^{\underline{\kappa}}\rangle).$$

By (1.3), the summation can be taken over all $\underline{\kappa}$ such that $\underline{\kappa}(X-\alpha^i) \trianglerighteq \underline{\mu}(X-\alpha^i)$. Hence, if $\underline{\mu} \in \Pi_{\le t-1}$, then $\underline{\kappa} \in \Pi_{\le t-1}$. By the assumed properties of w given in Proposition 5.4.1, we have $\langle w, \psi^{\underline{\kappa}} \rangle = 0$ for each $\underline{\kappa} \in \Omega_n \cap \Pi_{\le t-1}$ satisfying $|\underline{\kappa}(X-1)| \neq n$. Since $|\underline{\lambda}(X-1)| \neq n$ we conclude that $\langle w, \psi^{\underline{\lambda}} \rangle = 0$.

Now assume that $|\underline{\lambda}(X-1)| = n$ and write $\underline{\lambda}(X-1) = \lambda$. From (4.8) and (4.9) we have

$$\langle w, \psi^{X-1\mapsto\lambda} \rangle = \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} \langle w, \xi^{X-1\mapsto\mu} \rangle,$$

since by Lemma 5.2.9 in the case $\mu_1 = n-t$ we have $\xi^{X-1\mapsto\mu}_{\underline{\sigma}} = 0$ for each $\underline{\sigma} \in \Sigma_{\le t-1}$. From (4.6) and (1.3) we find that

$$\begin{aligned} \langle w, \psi^{X-1\mapsto\lambda} \rangle &= \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} \sum_{\kappa \trianglerighteq \mu} K_{\kappa\mu} \, \langle w, \psi^{X-1\mapsto\kappa} \rangle \\ &= \frac{1}{|\mathrm{GL}(n,q)|} \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} + \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} \sum_{(n) \triangleright \kappa \trianglerighteq \mu} K_{\kappa\mu} \, \langle w, \psi^{X-1\mapsto\kappa} \rangle, \end{aligned} \tag{5.31}$$

using that $|\mathrm{GL}(n,q)| \langle w, \psi^{X-1\mapsto(n)} \rangle = 1$ by the assumed properties of w given in Proposition 5.4.1 and $K_{(n)\mu} = 1$ for each partition μ of n. We first show that the first sum is zero. We have

$$\sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} = \sum_{\mu \trianglerighteq \lambda} K_{(n)\mu} H_{\mu\lambda} - \sum_{\mu \trianglerighteq \lambda} K_{(n-t,t)\mu} H_{\mu\lambda}, \tag{5.32}$$

using that $\lambda_1 = n-t$ and that, for each partition μ of n, we have

$$K_{(n-t,t)\mu} = \begin{cases} 1 & \text{for } \mu_1 = n-t \\ 0 & \text{for } \mu_1 > n-t. \end{cases}$$

It holds that

$$\sum_{\mu \trianglerighteq \lambda} K_{\kappa\mu} H_{\mu\lambda} = \delta_{\kappa\lambda}. \tag{5.33}$$

Since λ is neither (n) nor $(n-t,t)$, we conclude that (5.32) equals zero. Hence (5.31) becomes

$$\langle w, \psi^{X-1\mapsto\lambda} \rangle = \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} \sum_{(n) \triangleright \kappa \trianglerighteq \mu} K_{\kappa\mu} \, \langle w, \psi^{X-1\mapsto\kappa} \rangle. \tag{5.34}$$

By the assumed properties of w given in Proposition 5.4.1, the inner summand is nonzero only when $\kappa = (n-s,s)$ for some s satisfying $1 \le s \le t-1$. In particular, for κ of this form, Proposition 5.4.1 and (5.26) give

$$|\mathrm{GL}(n,q)| \, |\langle w, \psi^{X-1\mapsto\kappa} \rangle| = \frac{\begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q - 1} \le \frac{\begin{bmatrix} n \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q} = \frac{q^t - 1}{q^{n-t+1} - 1} \le \frac{q^{2t-1}}{q^n}.$$

By Lemma 5.2.7 the Kostka numbers $K_{\kappa\mu}$ occurring in (5.34) are independent of n and it follows from (5.33) that the numbers $H_{\mu\lambda}$ occurring in (5.34) are also independent of n. Moreover the number of summands in (5.34) is also independent of n. From Lemma 5.2.12, we have $\psi^{X-1\mapsto\lambda}(1) \geq \delta_{t-1}\, q^{nt}$ for some constant δ_{t-1} only depending on t. Hence there is a constant c_t, depending only on t, such that

$$\frac{|\mathrm{GL}(n,q)|}{\psi^{X-1\mapsto\lambda}(1)}\,|\langle w, \psi^{X-1\mapsto\lambda}\rangle| \leq \frac{c_t}{q^{n(t+1)}}.$$

Since $|\varepsilon| > 1/q^{nt}$, this shows that (5.30) holds provided that n is sufficiently large compared to t. □

Recall from Section 4.4.2 that $V_{\underline{\lambda}}$ is the column span over the reals of $F_{\underline{\lambda}}$. We define

$$W_t = \sum_{\substack{\underline{\lambda}\in\Omega_n \\ \underline{\lambda}(X-1)\trianglerighteq(n-t,t)}} V_{\underline{\lambda}}.$$

We obtain the following.

Theorem 5.4.3. *Let t be a positive integer. Then, for all sufficiently large n, the following holds.*

(i) Every t-space-intersecting set Y in $\mathrm{GL}(n,q)$ satisfies

$$|Y| \leq \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, we have $\mathbb{1}_Y \in W_t$.

(ii) Every pair of t-space-cross-intersecting sets Y, Z in $\mathrm{GL}(n,q)$ satisfies

$$\sqrt{|Y|\cdot|Z|} \leq \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, we have $\mathbb{1}_Y, \mathbb{1}_Z \in W_t$.

Proof: We apply Theorem 2.3.11 to the graph with adjacency matrix

$$\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leq t-1}} B_{\underline{\sigma}}$$

and the $|D_{\underline{\sigma}}|$-regular spanning subgraphs with adjacency matrices $B_{\underline{\sigma}}$ for those $\underline{\sigma} \in \Omega_n \cap \Sigma_{\leq t-1}$. Every t-space-intersecting set in $\mathrm{GL}(n,q)$ is an independent set in this graph. Let $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leq t-1})$ be given by Proposition 5.4.1 and write

$$P(\underline{\lambda}) = \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leq t-1}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma}).$$

Proposition 5.4.1 and Lemmas 5.3.2 and 5.4.2 imply that, for all sufficiently large n, we have

$$P(\underline{\lambda}) = \begin{cases} 1 & \text{for } \underline{\lambda}(X-1) = (n) \\ \varepsilon & \text{for } \underline{\lambda}(X-1) = (n-s,s) \text{ with } 1 \le s \le t \end{cases}$$

and $|P(\underline{\lambda})| < |\varepsilon|$ for $\underline{\lambda}(X-1) \neq (n-s,s)$ with some s satisfying $0 \le s \le t$. Hence, writing $\underline{\lambda}_0$ for $X-1 \mapsto (n)$, we have $P(\underline{\lambda}_0) = 1$ and

$$\varepsilon = \min_{\underline{\lambda} \neq \underline{\lambda}_0} P(\underline{\lambda}) \quad \text{and} \quad |\varepsilon| = \max_{\underline{\lambda} \neq \underline{\lambda}_0} |P(\underline{\lambda})|.$$

Then the required result follows from Theorem 2.3.11 and the decomposition of $\mathbb{R}(\mathrm{GL}(n,q))$ given in Lemma 4.4.3. □

The proof of Theorems 5.1.9 and 5.1.11 is completed by the following result.

Theorem 5.4.4. *W_t is spanned by the characteristic vectors of cosets of stabilisers of t-spaces of $\mathbb{F}_q^n$.*

Proof: The proof is almost identical to that of Theorem 5.3.4 with $\mathcal{A}_t$ replaced by the set of t-spaces of $\mathbb{F}_q^n$ and ζ^t replaced by the permutation character $\xi^{X-1\mapsto(n-t,t)}$ of $\mathrm{GL}(n,q)$ on the set of t-spaces of $\mathbb{F}_q^n$ and the decomposition of ζ^t replaced by the decomposition of $\xi^{X-1\mapsto(n-t,t)}$ given in (5.24). □

5.5 Open Problems

In Theorem 5.1.9 it is shown that the characteristic vector of a t-space-intersecting set of maximal size is spanned by the characteristic vectors of cosets of stabilisers of t-spaces. This only partially characterises the extremal case.

In [MS11] Meagher and Spiga conjectured that the only 1-space-intersecting sets in $\mathrm{GL}(n,q)$ of maximal size are cosets of stabilisers of 1-spaces or cosets of stabilisers of $(n-1)$-spaces. In the same paper [MS11] they proved the conjecture for $n = 2$. In [MS14] the same authors proved the conjecture for $n = 3$ and in [Spi19] it was proven by Spiga for $n \ge 4$. Thus, in [ES23], we made the following conjectures about t-space-(cross-)intersecting sets.

Conjecture 5.5.1. Let t be a positive integer and let Y be a t-space-intersecting set in $\mathrm{GL}(n,q)$ of maximal size, meaning that its size meets the bound in Theorem 5.1.9. If n is sufficiently large compared to t, then Y is a coset of the stabiliser of a t-space or a coset of the stabiliser of an $(n-t)$-space.

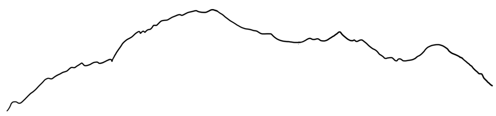
Conjecture 5.5.2. Let t be a positive integer and let Y and Z be t-space-cross-intersecting subsets of $\mathrm{GL}(n,q)$ of maximal size, meaning that their sizes meet the bound in Theorem 5.1.11. If n is sufficiently large compared to t, then $Y = Z$ and Y is a coset of the stabiliser of a t-space or a coset of the stabiliser of an $(n-t)$-space.

Our approach to prove the Theorems 5.1.4, 5.1.6, 5.1.9, and 5.1.11, heavily relies on estimating certain character values and sizes of the conjugacy classes of the finite general linear group. Since our estimates are very rough, the question arises as to how small n can be compared to t.

As pointed out in Section 3.2, for the symmetric group Keller, Lifshitz, Minzer, and Sheinfeld [KLMS24] proved that there exists a constant c with the following property. For all $n > ct$ the results on t-intersecting sets of permutations in [EFP11] (see also Section 3.2) hold. This motivates the following open problem for our q-analog setting.

Open Problem 5.5.3. Find sharp bounds on the sizes of n in the Theorems 5.1.4, 5.1.6, 5.1.9, and 5.1.11.

6 Transitive subsets of finite general linear groups



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This chapter involves the study of q-analog problems of the ones for the symmetric group that we collected in Section 3.4. Here we replace the symmetric group S_n by the finite general linear group $\mathrm{GL}(n, q)$. We study certain designs in the conjugacy class scheme of $\mathrm{GL}(n, q)$ and prove that these designs are transitive sets in $\mathrm{GL}(n, q)$.

The results presented in this chapter were published in [ES24].

6.1 Introduction

Our starting point is the following q-analog of the Livingstone-Wagner theorem from Theorem 3.4.1.

Theorem 6.1.1 ([Per72]). *Let $G \leq \mathrm{GL}(n, q)$ be a subgroup that is transitive on t-dimensional subspaces of $\mathbb{F}_q^n$ for some integer t satisfying $1 \leq t \leq n/2$. Then G is also transitive on $(t-1)$-dimensional subspaces of $\mathbb{F}_q^n$.*

We generalise this result in two ways. Instead of subgroups of $\mathrm{GL}(n, q)$ we study subsets of $\mathrm{GL}(n, q)$ that act *transitively.* In general the notion of a *transitive* subset is given as follows (see also Section 3.4 for the definition of a transitive subset of permutations).

Definition 6.1.2. Let Ω be a set on which the group G acts. We say that a subset Y of G is *transitive* on Ω if there is a constant r such that the following holds: for all $a, b \in \Omega$, there are exactly r elements $g \in Y$ such that $ga = b$. If $r = 1$, then we call Y *sharply* transitive on Ω.

If Y is a subgroup of the group G, then this notion coincides with that of a transitive group action.

Recall from Section 4.3 the Definitions (4.17) and 4.3.5 of the set $\Sigma_{n,q}$ and of an α-flag, with $\alpha \in \Sigma_{n,q}$, respectively.

Our second generalisation of Theorem 6.1.1 replaces subspaces by α-flags, which are generalisations of subspaces of $\mathbb{F}_q^n$ and bases of t-dimensional subspaces of $\mathbb{F}_q^n$.

For a composition σ of n, recall from Definition 4.3.4 the notion of a σ-flag of $\mathbb{F}_q^n$. The following is an example of the results we obtain in this chapter.

Theorem 6.1.3. *Let σ be a composition of n and let $Y \subseteq \mathrm{GL}(n,q)$ be transitive on the set of σ-flags. Then Y is also transitive on the set of τ-flags for all compositions τ satisfying $\tilde{\tau} \trianglerighteq \tilde{\sigma}$, where $\tilde{\sigma}$ and $\tilde{\tau}$ are the partitions obtained from σ and τ, respectively, by rearranging the parts.*

This theorem can be seen as a q-analog of Corollary 3.4.4. In fact we obtain a more general result, namely a characterisation of subsets of $\mathrm{GL}(n,q)$ acting transitively on the set of α-flags.

This chapter is organised as follows. In Section 6.2 we characterise subsets of $\mathrm{GL}(n,q)$ acting transitively on the set of α-flags in terms of T-designs in the corresponding association scheme. This gives us structural results for transitive subsets of $\mathrm{GL}(n,q)$ leading to results like Theorem 6.1.3. In Section 6.3 we study transitive subgroups of $\mathrm{GL}(n,q)$. In Section 6.4 we study the connection of transitive sets and so-called (σ,τ)-*cliques*, where the latter turn out to be cliques in the corresponding association scheme. This study allows us to establish non-existence results for sharply transitive sets in $\mathrm{GL}(n,q)$ for certain cases. Then, in Section 6.5, for all fixed t, we show the existence of small nontrivial subsets of $\mathrm{GL}(n,q)$ that are transitive on the set of t-tuples of linearly independent vectors of $\mathbb{F}_q^n$. This also shows the existence of small nontrivial subsets of $\mathrm{GL}(n,q)$ that are transitive on α-flags. In Section 6.6 we discuss connections between transitive subsets and cliques in $\mathrm{GL}(n,q)$ on the one hand and certain orthogonal polynomials, namely the *Al-Salam-Carlitz* polynomials, on the other hand.

6.2 Designs in finite general linear groups

In this section, we characterise transitive sets in $\mathrm{GL}(n,q)$ in terms of Delsarte T-designs in the conjugacy class scheme arising from $\mathrm{GL}(n,q)$.

First, we provide an example of subsets of $\mathrm{GL}(n,q)$ that are sharply transitive on $((1,n-1),\mathcal{I})$-flags for $\mathcal{I}=\{1\}$ and $\mathcal{I}=\varnothing$, using a representation over a finite field.

Remark 6.2.1. Although we have only introduced representations over the complex numbers in Chapter 1, it is also possible to define representations over finite fields. The definition is as expected from Definition 1.1.3. For an example of a representation over $\mathbb{F}_q$, we take the companion matrix C of an irreducible polynomial over $\mathbb{F}_q$ of degree n. Then $\mathbb{F}_q[C]$ is a representation of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Example 6.2.2. Let $\mathbb{F}_q[C]$ be given as in Remark 6.2.1. Then the multiplicative group $\mathbb{F}_q[C]^*$ of $\mathbb{F}_q[C]$ is sharply transitive on $\mathbb{F}_q^n \setminus \{0\}$. Hence $\mathbb{F}_q[C]^*$ is sharply transitive on $((1, n-1), \{1\})$-flags. Of course $\mathbb{F}_q[C]^*$ is a cyclic subgroup of $\mathrm{GL}(n, q)$, known as the *Singer cycle*. Moreover, $\mathbb{F}_q[C]^*$ contains a cyclic subgroup of order $(q^n - 1)/(q-1)$ that is sharply transitive on the one-dimensional subspaces of $\mathbb{F}_q^n$. Hence this subgroup is sharply transitive on $((1, n-1), \varnothing)$-flags.

Recall from Definition 4.3.6 the type of an $\alpha \in \Sigma_{n,q}$. The set of all possible types of elements in $\Sigma_{n,q}$ is denoted by $\Theta_{n,q}$, namely

$$\Theta_{n,q} = \{\mathrm{type}(\alpha) \colon \alpha \in \Sigma_{n,q}\}.$$

Hence $\Theta_{n,q}$ contains all pairs of partitions (σ, τ) such that $|\sigma| + |\tau| = n$ and all parts of τ are strictly larger than 1 for $q = 2$.

For a partition valued function $\underline{\lambda} \in \Lambda_n$, recall the type of $\underline{\lambda}$ from Definition 4.3.7. Moreover, we recall from Definition 4.3.8 that the partial order $\preceq$ is an order on pairs of partitions, with reverse refinement in the first and dominance in the second coordinate.

The following result gives a combinatorial interpretation of a Delsarte T-design in the finite general linear group, or more precisely in the conjugacy class scheme arising from the finite general linear group.

Theorem 6.2.3. *Let Y be a subset of $\mathrm{GL}(n, q)$ with dual distribution $(b_{\underline{\lambda}})$ and let $\alpha \in \Sigma_{n,q}$. Then Y is transitive on the set of α-flags if and only if*

$$b_{\underline{\lambda}} = 0 \quad \textit{for all } \underline{\lambda} \in \Lambda_n \textit{ satisfying } \mathrm{type}(\alpha) \preceq \mathrm{type}(\underline{\lambda}) \prec (\varnothing, (n)).$$

In other words, a subset of $\mathrm{GL}(n, q)$ is transitive on the set of α-flags if and only if it is a T-design with $T = \{\underline{\lambda} \in \Lambda_n \colon \mathrm{type}(\alpha) \preceq \mathrm{type}(\underline{\lambda}) \prec (\varnothing, (n))\}$. Thus, from Theorem 6.2.3 we obtain two characterisations. On the one hand we have an algebraic characterisation for transitive sets, on the other hand a combinatorial interpretation for the very algebraic object of a Delsarte T-design.

Before proving Theorem 6.2.3 we discuss some of its consequences. The first one is that transitivity on α-flags only depends on the type of α.

Corollary 6.2.4. *Let $\alpha, \beta \in \Sigma_{n,q}$ be of the same type and let Y be a subset of $\mathrm{GL}(n,q)$. Then Y is transitive on the set of α-flags if and only if Y is transitive on the set of β-flags.*

Corollary 6.2.4 motivates the following definition.

Definition 6.2.5. For $(\sigma,\tau) \in \Theta_{n,q}$, a subset Y of $\mathrm{GL}(n,q)$ is (σ,τ)-*transitive* if Y is transitive on the set of α-flags for some $\alpha \in \Sigma_{n,q}$ of type (σ,τ).

We note that in Example 6.2.2 we have (σ,τ)-transitive sets for (σ,τ) equal to $((1),(n-1))$ and $(\varnothing,(n-1,1))$. We may now restate Theorem 6.2.3 as follows.

Corollary 6.2.6. *Let Y be a subset of $\mathrm{GL}(n,q)$ with dual distribution $(b_{\underline{\lambda}})$ and let $(\sigma,\tau) \in \Theta_{n,q}$. Then Y is (σ,τ)-transitive if and only if*

$$b_{\underline{\lambda}} = 0 \qquad \text{for all } \underline{\lambda} \in \Lambda_n \text{ satisfying } (\sigma,\tau) \preceq \mathrm{type}(\underline{\lambda}) \prec (\varnothing,(n)).$$

For every partition τ of n, a $(\varnothing,\tau)$-transitive set is just a subset of $\mathrm{GL}(n,q)$ that is transitive on τ-flags. In this case, Corollary 6.2.6 specialises to the following perfect q-analog of the corresponding result for the symmetric group from Theorem 3.4.3.

Corollary 6.2.7. *Let Y be a subset of $\mathrm{GL}(n,q)$ with dual distribution $(b_{\underline{\lambda}})$ and let τ be a partition of n. Then Y is $(\varnothing,\tau)$-transitive if and only if*

$$b_{\underline{\lambda}} = 0 \qquad \text{for all } \underline{\lambda} \in \Lambda_n \text{ satisfying } \tau \trianglelefteq \underline{\lambda}(X-1) \triangleleft (n).$$

Another immediate consequence of Theorem 6.2.3 is the following.

Corollary 6.2.8. *Let Y be a subset of $\mathrm{GL}(n,q)$ and suppose that Y is (σ,τ)-transitive for some $(\sigma,\tau) \in \Theta_{n,q}$. Then Y is also $(\hat\sigma,\hat\tau)$-transitive for all $(\hat\sigma,\hat\tau) \in \Theta_{n,q}$ satisfying $(\sigma,\tau) \preceq (\hat\sigma,\hat\tau)$.*

From this corollary Theorem 6.1.3 arises as a special case.

In the remainder of this section we give the proof of Theorem 6.2.3. A key ingredient is the decomposition of the permutation character of $\mathrm{GL}(n,q)$ on the set of α-flags from Lemma 4.3.9.

For $\alpha \in \Sigma_{n,q}$, let $\mathcal{F}_\alpha$ be the set of α-flags and define $M_\alpha \in \mathbb{C}(\mathrm{GL}(n,q), \mathcal{F}_\alpha \times \mathcal{F}_\alpha)$ to be the incidence matrix of elements of $\mathrm{GL}(n,q)$ versus left cosets of stabilisers of α-flags by

$$M_\alpha(g,(u,v)) = \begin{cases} 1 & \text{for } gu = v \\ 0 & \text{otherwise.} \end{cases}$$

From now on, we use the notation from Section 4.4.1 for the conjugacy class scheme of the finite general linear group $\mathrm{GL}(n,q)$. Recall from that section, for example, that the pairwise orthogonal idempotent matrices of this scheme are denoted by $E_{\underline{\lambda}}$.

For $(\sigma,\tau)\in\Theta_{n,q}$, we define

$$U_{(\sigma,\tau)} = \sum_{\substack{\underline{\lambda}\in\Lambda_n \\ (\sigma,\tau)\preceq\mathrm{type}(\underline{\lambda})}} \mathrm{colsp}(E_{\underline{\lambda}}), \tag{6.1}$$

where $\mathrm{colsp}(E_{\underline{\lambda}})$ is the column space of the matrix $E_{\underline{\lambda}}$. We note that the sum given in (6.1) is, in fact, direct because the matrices $E_{\underline{\lambda}}$ of an association scheme are pairwise orthogonal.

Corollary 6.2.9. *The column space of M_α equals $U_{\mathrm{type}(\alpha)}$.*

The proof of this Corollary is almost identical to the proof of Theorem 5.3.4. But since we are not working with the symmetrisation of the conjugacy class scheme arising from $\mathrm{GL}(n,q)$, we give the proof here for the sake of completeness. However, it is less detailed.

Proof: Let ξ be the permutation character of $\mathrm{GL}(n,q)$ on the set of α-flags and define $C_\alpha\in(\mathbb{C}(\mathrm{GL}(n,q)),\mathbb{C}(\mathrm{GL}(n,q))$ by $C_\alpha(x,y)=\xi(x^{-1}y)$. Then it follows that

$$M_\alpha M_\alpha^T = C_\alpha.$$

Hence the column space of C_α equals the column space of M_α. From Lemma 4.3.9 and (4.20) we obtain

$$C_\alpha = |\,\mathrm{GL}(n,q)| \sum_{\substack{\underline{\lambda}\in\Lambda_n \\ \mathrm{type}(\alpha)\preceq\mathrm{type}(\underline{\lambda})}} \frac{m_{\underline{\lambda}}}{\chi^{\underline{\lambda}}(1)} E_{\underline{\lambda}}. \tag{6.2}$$

Hence the column space of C_α is contained in $U_{\mathrm{type}(\alpha)}$. Conversely, let v be a column of $E_{\underline{\kappa}}$ for some $\underline{\kappa}\in\Lambda_n$ satisfying $\mathrm{type}(\underline{\kappa})\succeq\mathrm{type}(\alpha)$. Since the $E_{\underline{\lambda}}$ are pairwise orthogonal and idempotent, from (6.2) we find that

$$C_\alpha v = |\,\mathrm{GL}(n,q)|\frac{m_{\underline{\kappa}}}{\chi^{\underline{\kappa}}(1)} v,$$

and, since $m_{\underline{\kappa}}\neq 0$, we conclude that v is in the column space of C_α, as required. □

Now, we are in a position to complete the proof of Theorem 6.2.3.

Proof (of Theorem 6.2.3): Note that Y is transitive on α-flags if and only if

$$\frac{1}{|Y|}M_\alpha^T \mathbb{1}_Y = \frac{1}{|\,\mathrm{GL}(n,q)|}M_\alpha^T \mathbb{1}_{\mathrm{GL}(n,q)}, \tag{6.3}$$

hence, if and only if

$$\mathbb{1}_Y - \frac{|Y|}{|\operatorname{GL}(n,q)|}\mathbb{1}_{\operatorname{GL}(n,q)}$$

is orthogonal to the column space of M_α. In view of the orthogonal decomposition of this space given in Corollary 6.2.9 and the fact that $V_{X-1\mapsto(n)}$ is spanned by $\mathbb{1}_{\operatorname{GL}(n,q)}$, we conclude that Y is transitive on the set of α-flags if and only if $\mathbb{1}_Y$ is orthogonal to $V_{\underline{\lambda}}$ for each $\underline{\lambda} \in \Lambda_n$ satisfying type$(\alpha) \preceq$ type$(\underline{\lambda}) \prec (\varnothing, (n))$. From Remark 2.2.8 it follows that this is equivalent to the statement of the theorem. □

Remark 6.2.10. From (6.3) it follows that a subset Y of $\operatorname{GL}(n,q)$ is transitive on α-flags if and only if

$$\frac{1}{|Y|}\sum_{x\in Y}\mathbb{1}_{xF=F'} = \frac{1}{|\operatorname{GL}(n,q)|}\sum_{x\in \operatorname{GL}(n,q)}\mathbb{1}_{xF=F'}$$

holds for all α-flags F and F' of $\mathbb{F}_q^n$, where $\mathbb{1}_{xF=F'}$ is the indicator of the event that x maps F to F'. As a consequence, we can understand a subset of $\operatorname{GL}(n,q)$ that is transitive on α-flags as a set that locally approximates $\operatorname{GL}(n,q)$.

6.3 Transitive subgroups

In this section we classify subgroups G of $\operatorname{GL}(n,q)$ that are (σ,τ)-transitive. These results are essentially known. If G is $((1),(n-1))$-transitive or $(\varnothing,(n-1,1))$-transitive or $((1^2),\varnothing)$ if $q=2$, then G is transitive on 1-spaces of $\mathbb{F}_q^n$. Such subgroups have been classified by Hering [Her74], see also [GGP23, Table 3.1] for a nice summary. However, as we always have examples of sharply (σ,τ)-transitive subgroups in these cases (see Example 6.2.2), we shall henceforth assume that (σ,τ) is different from $((1),(n-1))$ and $(\varnothing,(n-1,1))$ and also different from $((1^2),\varnothing)$ if $q=2$.

In what follows $\operatorname{SL}(n,q)$ denotes the *finite special linear group* consisting of all $n\times n$ matrices with entries in the finite field $\mathbb{F}_q$ having determinant 1. Moreover, by $\Gamma\mathrm{L}(n,q)$ we denote the *finite general semilinear group* which is consisting of all invertible semilinear transformations $\mathbb{F}_q^n \to \mathbb{F}_q^n$.

First we consider the case $n\geq 4$. By Corollary 6.2.8, G is also $(\varnothing,(n-2,2))$-transitive, namely transitive on 2-spaces of $\mathbb{F}_q^n$. Kantor [Kan73] proved that G is either doubly transitive on 1-spaces of $\mathbb{F}_q^n$ or $G\cong \Gamma\mathrm{L}(1,2^5)$ as a subgroup of $\operatorname{GL}(5,2)$, which acts sharply transitive on 2-spaces of $\mathbb{F}_2^5$. Cameron and Kantor [CK79] proved that, if G is doubly transitive on 1-spaces of $\mathbb{F}_q^n$, then G either contains $\operatorname{SL}(n,q)$, in which case G is $((n-1),(1))$-transitive, or $G\cong A_7$ as a subgroup of $\operatorname{GL}(4,2)$. In fact it is computationally readily verified that the latter example is sharply $((31),\varnothing)$-transitive.

Next we consider the case $n=3$. Then by Corollary 6.2.8, G is also $(\varnothing,(1^3))$-transitive when $q>2$ or $((1^3),\varnothing)$-transitive when $q=2$. That is, G is transitive on

(1^3)-flags in $\mathbb{F}_q^3$, typically just called *flags* in the literature. Kantor [Kan87] proved that G either contains $\mathrm{SL}(n,q)$ or G acts sharply transitive on flags in $\mathbb{F}_q^3$. Higman and McLaughlin [HM61] showed that in the latter case the only possibility is $G \cong \Gamma\mathrm{L}(1,2^3)$ as a subgroup of $\mathrm{GL}(3,2)$.

Now we consider the case $n = 2$. Then only the case that G is $((1^2),\varnothing)$-transitive and $q > 2$ is left. The number of $((1^2),\varnothing)$-flags is $(q^2-1)(q-1)$ and the order of G must be a multiple of this number. Since $|\mathrm{GL}(2,q)| = (q^2-1)(q-1)q$, the index of G in $\mathrm{GL}(2,q)$ must therefore be a divisor of q. Noting that G is transitive on the 1-spaces of $\mathbb{F}_q^2$, an inspection of [GGP23, Thm. 3.1] reveals that the only possible cases are $G \cong \Gamma\mathrm{L}(1,3^2)$ inside $\mathrm{GL}(2,3)$ or q is one of the numbers $5, 7, 9, 11, 19, 23, 29, 59$ and a computer verification reveals that only $\mathrm{GL}(2,3)$ and $\mathrm{GL}(2,5)$ contain subgroups G in question. In the former case we have $G \cong \Gamma\mathrm{L}(1,3^2)$ and in the latter case G is unique up to conjugation. In both cases G is sharply $((1^2),\varnothing)$-transitive.

We summarise these results in the following theorem.

Theorem 6.3.1. *Suppose that G is a (σ,τ)-transitive nontrivial proper subgroup of* $\mathrm{GL}(n,q)$ *and (σ,τ) is different from $((1),(n-1))$ and $(\varnothing,(n-1,1))$ and also different from $((1^2),\varnothing)$ if $q = 2$. Then one of the following holds:*

(1) $q > 2$ *and* $G \geq \mathrm{SL}(n,q)$ *and G is $((n-1),(1))$-transitive.*

(2) $(n,q) = (2,3)$ *and* $G \cong \Gamma\mathrm{L}(1,3^2)$ *is sharply $((1^2),\varnothing)$-transitive.*

(3) $(n,q) = (2,5)$ *and G has order* 96 *and is sharply $((1^2),\varnothing)$-transitive.*

(4) $(n,q) = (3,2)$ *and* $G \cong \Gamma\mathrm{L}(1,2^3)$ *and G is sharply $((1^3),\varnothing)$-transitive.*

(5) $(n,q) = (4,2)$ *and* $G \cong A_7$ *is sharply $((3,1),\varnothing)$-transitive.*

(6) $(n,q) = (5,2)$ *and* $G \cong \Gamma\mathrm{L}(1,2^5)$ *is sharply $(\varnothing,(3,2))$-transitive.*

It should be noted that there exist groups acting transitively on flags in $\mathbb{F}_8^3$, namely $\Gamma\mathrm{L}(1,2^9)$ and a subgroup of index 7 [HM61]. These groups however are not subgroups of $\mathrm{GL}(3,8)$, but rather are subgroups of $\Gamma\mathrm{L}(3,8)$.

6.4 Transitive sets and cliques

In this section we consider cliques in $\mathrm{GL}(n,q)$ and discuss their relationship to transitivity in $\mathrm{GL}(n,q)$.

Definition 6.4.1. Let $(\sigma,\tau) \in \Theta_{n,q}$. A subset Y of $\mathrm{GL}(n,q)$ is a (σ,τ)-*clique* if, for all distinct $x, y \in Y$, there is no α-flag with $\mathrm{type}(\alpha) = (\sigma,\tau)$ fixed by $x^{-1}y$.

Definition 6.4.2. For $\underline{\mu} \in \Lambda$ we define the *conjugate* $\underline{\mu}' \in \Lambda$ to be the mapping $\underline{\mu}' : \Phi \to \mathrm{Par}$ given by $\underline{\mu}'(f) = \underline{\mu}(f)'$.

Note that, if $\mathrm{type}(\underline{\mu}) = (\nu,\mu)$, then we have $\mathrm{type}(\underline{\mu}') = (\nu,\mu')$.

Theorem 6.4.3. *Let Y be a subset of* $\mathrm{GL}(n,q)$ *with inner distribution* $(a_{\underline{\mu}})$ *and let* $(\sigma,\tau)\in\Theta_{n,q}$*. Then Y is a (σ,τ)-clique if and only if*

$$a_{\underline{\mu}}=0 \qquad \textit{for all } \underline{\mu}\in\Lambda_n \textit{ satisfying } (\tau,\sigma)\preceq \mathrm{type}(\underline{\mu}')\prec(\varnothing,(n)).$$

Proof: We fix $\underline{\mu}\in\Lambda_n$ and we note that, for $\alpha\in\Sigma_{n,q}$, either all elements in $C_{\underline{\mu}}$ fix an α-flag or none of the elements in $C_{\underline{\mu}}$. We show that the elements in $C_{\underline{\mu}}$ fix an α-flag with $\mathrm{type}(\alpha)=(\sigma,\tau)$ if and only if $(\tau,\sigma)\preceq(\nu,\mu)$, where (ν,μ) is the type of $\underline{\mu}'$.

First we assume that $(\tau,\sigma)\preceq(\nu,\mu)$, namely $\sigma\trianglelefteq\mu$ and ν refines τ. Since $\sigma\trianglelefteq\mu$, rearranging rows and columns of the Jordan canonical form of $C_{\underline{\mu}}$ shows that $C_{\underline{\mu}}$ contains a block upper-triangular matrix whose diagonal blocks are $I_{\sigma_1},I_{\sigma_2},\dots$ followed by $|\mu|-|\sigma|$ blocks of order 1 followed by blocks of order $\nu_1,\nu_2,\dots$. Since ν refines τ, this matrix fixes an α-flag with $\mathrm{type}(\alpha)=(\sigma,\tau)$.

Now let $g\in C_{\underline{\mu}}$ be in Jordan canonical form and assume that g fixes an α-flag with $\mathrm{type}(\alpha)=(\sigma,\tau)$. By [LRS14, Proposition 4.4] the companion matrix of an irreducible polynomial in $\mathbb{F}_q[X]$ of degree d does not fix a proper subspace of $\mathbb{F}_q^d$. Hence ν must refine τ. Also note that g has μ_i Jordan blocks with eigenvalue 1 of order at least i and each such Jordan block of order i fixes a β-flag with $\mathrm{type}(\beta)=((1^i),\varnothing)$. Hence g must have at least

$$\sigma_i-\sum_{j=1}^{i-1}(\mu_j-\sigma_j)$$

Jordan blocks with eigenvalue 1 of order at least i. The latter statement is equivalent to $\sigma\trianglelefteq\mu$. □

This theorem shows that a (σ,τ)-clique is a D-clique in the association scheme of $\mathrm{GL}(n,q)$ with $D=\{\underline{\mu}\in\Lambda_n\colon\ \mathrm{type}(\underline{\mu}')\not\succeq(\tau,\sigma)\}$. Moreover, by comparing this with Corollary 6.2.6 we note again that the concepts of a clique and a design are dual.

In the following we establish relationships between (σ,τ)-cliques and (σ,τ)-transitive sets in $\mathrm{GL}(n,q)$.

Theorem 6.4.4. *Let Y be a subset of* $\mathrm{GL}(n,q)$*, let* $(\sigma,\tau)\in\Theta_{n,q}$*, and let H be the stabiliser of an α-flag with* $\mathrm{type}(\alpha)=(\sigma,\tau)$*.*

(1) If Y is a (σ,τ)-clique, then $|Y|\le|\mathrm{GL}(n,q)|/|H|$ with equality if and only if Y is (σ,τ)-transitive.

(2) If Y is (σ,τ)-transitive, then $|Y|\ge|\mathrm{GL}(n,q)|/|H|$ with equality if and only if Y is a (σ,τ)-clique.

In both cases, equality implies that Y is sharply (σ,τ)-transitive.

Proof: Since, for each $(x, y) \in H \times Y$, there is a unique $g \in \mathrm{GL}(n, q)$ such that $gx = y$, we have

$$\sum_{g \in \mathrm{GL}(n,q)} |Y \cap gH| = |Y| \cdot |H|. \tag{6.4}$$

The quotient of any two distinct elements in $Y \cap gH$ fixes an α-flag of type (σ, τ). Hence, if Y is a (σ, τ)-clique, then each summand on the left hand side of (6.4) is at most 1, which gives the bound in (1). If H is the stabiliser of the α-flag F, then gH contains precisely all elements of $\mathrm{GL}(n, q)$ mapping F to gF. Hence, if Y is (σ, τ)-transitive, then each summand on the left hand side of (6.4) must be at least 1, which gives the bound in (2). In both cases, equality occurs if and only if $|Y \cap gH'| = 1$ for each $g \in G$ and the stabiliser H' of each α-flag of type (σ, τ). By the same reasoning as above, this establishes the characterisations of equality. □

Another way to approach Theorem 6.4.4 involves the clique-coclique bound from Theorem 2.4.7 and a condition on designs and antidesigns [Roo82, Corollary 3.3] for the conjugacy class scheme of $\mathrm{GL}(n, q)$. The latter was proved in [Roo82] only for the case of symmetric association schemes, but it also holds in general.

Note that, if H is the stabiliser of an α-flag with $\mathrm{type}(\alpha) = (\sigma, \tau)$, then an elementary counting argument gives

$$\frac{|\mathrm{GL}(n,q)|}{|H|} = \frac{[n]_q!}{\left(\prod_{i \geq 1} [\sigma_i]_q!\right)\left(\prod_{i \geq 1} [\tau_i]_q!\right)} \prod_{i \geq 1} \prod_{j=0}^{\sigma_i - 1} (q^{\sigma_i} - q^j),$$

where, for a nonnegative integer m, $[m]_q!$ denotes the q-factorial from (2.2).

In view of Theorems 6.4.3 and 6.4.4 the existence of sharply (σ, τ)-transitive subsets of $\mathrm{GL}(n, q)$ can be ruled out by linear programming. From (2.20) in Section 2.4.2, it follows that the linear-programming (LP) bound for (σ, τ)-cliques is the maximum of

$$\sum_{\underline{\mu} \in \Lambda_n} a_{\underline{\mu}}$$

subject to the constraints

$$a_{\underline{\mu}} \geq 0 \quad \text{for all } \underline{\mu} \in \Lambda_n,$$

$$\sum_{\underline{\mu} \in \Lambda_n} \mathrm{Im}(\chi^{\underline{\lambda}}_{\underline{\mu}})\, a_{\underline{\mu}} = 0 \quad \text{and} \quad \sum_{\underline{\mu} \in \Lambda_n} \mathrm{Re}(\chi^{\underline{\lambda}}_{\underline{\mu}})\, a_{\underline{\mu}} \geq 0 \quad \text{for all } \underline{\lambda} \in \Lambda_n,$$

$$a_{\underline{\mu}} = 0 \quad \text{for all } \underline{\mu} \in \Lambda_n \text{ satisfying } (\tau, \sigma) \preceq \mathrm{type}(\underline{\mu}') \prec (\varnothing, (n)).$$

We have determined the LP bound for (σ, τ)-cliques in $\mathrm{GL}(n, 2)$ for $n \in \{2, 3, 4, 5\}$. The LP bound coincides with the bound of Theorem 6.4.4 (i) except for those pairs (σ, τ) shown in Table 6.1. Consequently no sharply (σ, τ)-transitive subsets of $\mathrm{GL}(n, q)$ can exist in these cases.

Table 6.1: Bounds for cliques in GL(4, 2) and GL(5, 2).

(σ, τ)	Bound of Thm. 6.4.4	LP bound
$((21^2), \varnothing)$	630	420
$((1^2), (2))$	105	84
$((2), (2))$	210	168
$((32), \varnothing)$	156 240	139 500
$((31^2), \varnothing)$	78 120	53 010
$((221), \varnothing)$	39 060	24 180
$((21^3), \varnothing)$	19 530	11 718
$((3), (2))$	26 040	19 530
$((21), (2))$	6 510	3 550
$((1^3), (2))$	3 255	2 604
$((1), (22))$	1 085	805

6.5 Existence results

In this section we show that, for a partition σ, nonnegative integers $\tau_2 \geq \tau_3 \geq \cdots$, and sufficiently large n, there exist (σ, τ)-transitive sets in $\mathrm{GL}(n, q)$ that are arbitrarily small compared to $\mathrm{GL}(n, q)$, where $\tau_1 = n - |\sigma| - \tau_2 - \tau_2 - \cdots$. In view of Corollary 6.2.8, it suffices to consider $((t), (n-t))$-transitive sets in $\mathrm{GL}(n, q)$. For brevity, we shall call such a set a *t-design* in $\mathrm{GL}(n, q)$. In Section 6.6 we study these objects in more detail.

We give a recursive construction of t-designs in $\mathrm{GL}(n, q)$ using $\{1, 2 \ldots, t\}$-designs, or *t-designs* for short, in the q-Johnson scheme $J_q(n, k)$. Recall from Theorem 2.2.13 that a t-design in $J_q(n, k)$ is a subset D of $J_q(n, k)$ such that the number of elements in D containing a given t-space of $\mathbb{F}_q^n$ is independent of the particular choice of this t-space. Our construction can be understood as a q-analog of the construction given in [MS06, Section 6] for the symmetric group S_n.

Let $V = \mathbb{F}_q^n$ and, for a k-space U of V, let $\mathrm{GL}(U)$ be the general linear group of U, which is of course isomorphic to $\mathrm{GL}(k, q)$. Fix a k-space U of V and an $(n-k)$-space W of V such that

$$V = U \oplus W.$$

For our recursive construction, we need three ingredients: a t-design Y in $\mathrm{GL}(U)$, a t-design Z in $\mathrm{GL}(W)$, and a t-design D in $J_q(n, k)$. For each $B \in D$, there are $q^{k(n-k)}$ complementary spaces, namely $(n-k)$-spaces C with $V = B \oplus C$. We denote the collection of such spaces by C_B. For each $B \in D$, we fix an isomorphism $g_B : U \to B$ and, for each $B \in D$ and each $C \in C_B$, we fix an isomorphism $h_{B,C} : W \to C$.

Note that, given a pair (B, C) with $B \in D$ and $C \in C_B$, then every pair of isomorphisms (y, z), where $y \colon B \to B$ and $z \colon C \to C$, can be uniquely extended to an isomorphism on V by linearity. We denote this extension by (y, z). Hence, if $v \in V$, then there are unique $b \in B$ and $c \in C$ with $v = b + c$ and we have

$$(y, z)(v) = y(b) + z(c).$$

The following lemma contains a recursive construction of t-designs in $\mathrm{GL}(n, q)$.

Lemma 6.5.1. *Let Y be a t-design in $\mathrm{GL}(U)$, let Z be a t-design in $\mathrm{GL}(W)$, and let D be a t-design in $J_q(n, k)$. Then the set*

$$\{(g_B \circ y, h_{B,C} \circ z) : y \in Y, z \in Z, B \in D, C \in C_B\} \tag{6.5}$$

is a t-design in $\mathrm{GL}(V)$.

Note that, taking $Y = \mathrm{GL}(U)$, $Z = \mathrm{GL}(W)$, and $D = J_q(n, k)$, the set constructed in Lemma 6.5.1 equals $\mathrm{GL}(V)$.

Example 6.5.2. By [BKL05] there exists a 2-design in $J_2(6, 3)$ of cardinality 279. Taking Y and Z to be isomorphic to $\mathrm{GL}(3, 2)$ in Lemma 6.5.1, we obtain a 2-design in $\mathrm{GL}(6, 2)$ of cardinality $\frac{1}{5}|\mathrm{GL}(6, 2)|$.

To prove Lemma 6.5.1, we shall need the following well known result about designs in $J_q(n, k)$, in which $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient from (2.1).

Lemma 6.5.3 ([Suz90, Lemma 2.1], [KP15, Fact 1.5]). *Let D be a t-design in $J_q(n, k)$ and let i, j be nonnegative integers satisfying $i + j \le t$. Let I be an i-space of V and let J be a j-space of V such that $I \cap J = \{0\}$. Then the number*

$$m_{i,j} = |\{B \in D : I \le B \wedge B \cap J = \{0\}\}|$$

is independent of the particular choice of I and J and given by

$$m_{i,j} = |D|\, q^{j(k-i)} \frac{\begin{bmatrix} n-i-j \\ k-i \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q \begin{bmatrix} n \\ t \end{bmatrix}_q}.$$

We are now ready to prove Lemma 6.5.1.

Proof (of Lemma 6.5.1): Choose t-tuples $(v_1, v_2, \dots, v_t)$ and $(v'_1, v'_2, \dots, v'_t)$ of linearly independent vectors of V. Suppose that exactly i of the vectors $v_1, v_2, \dots, v_t$ are in U. After reordering we can assume that these are $v_1, v_2, \dots, v_i$. Then the remaining $j = t - i$ vectors $v_{i+1}, v_{i+2}, \dots, v_t$ are outside U, namely they belong to complementary spaces of U.

The number of elements $B \in D$ containing $v'_1, v'_2, \dots, v'_i$, but none of the vectors $v'_{i+1}, v'_{i+2}, \dots, v'_t$, equals the constant $m_{i,j}$ given in Lemma 6.5.3 and, for each such B, there are $q^{k(n-k-j)}$ complementary spaces $C \in C_B$ containing the remaining j vectors. Fix a pair (C, B) with these properties. Write $v_\ell = u_\ell + w_\ell$ with $u_\ell \in U$ and $w_\ell \in W$

for all ℓ and note that our assumption implies that $v_\ell = u_\ell$ for all $\ell \le i$. There is a constant r_i such that there are exactly r_i elements $y \in Y$ taking v_ℓ to $g_B^{-1}(v'_\ell)$ for all $\ell \le i$. For each such $y \in Y$, there is a constant s_j such that there are exactly s_j elements $z \in Z$ taking w_ℓ to

$$h_{B,C}^{-1}(v'_\ell - g_B(y(u_\ell)))$$

for all $\ell > i$.

Hence the total number of automorphisms in (6.5) taking the tuple $(v_1, v_2, \dots, v_t)$ to the tuple $(v'_1, v'_2, \dots, v'_t)$ equals

$$m_{i,j}\, r_i\, s_j\, q^{k(n-k-j)}.$$

We have to show that this number is independent of i. Lemma 6.5.3 implies that

$$(q^k - q^i)\, m_{i,j} = (q^n - q^{k+j-1})\, m_{i+1,j-1}$$

and it is readily verified that

$$r_i = (q^k - q^i)\, r_{i+1}$$

for $i \le t-1$ and

$$s_j = (q^{n-k} - q^j)\, s_{j+1}$$

for $j \le t-1$. By combining these identities we find that

$$m_{i+1,j-1}\, r_{i+1}\, s_{j-1}\, q^{k(n-k-j+1)} = m_{i,j}\, r_i\, s_j\, q^{k(n-k-j)},$$

which completes the proof. □

The following existence result for t-designs in $J_q(n,k)$ was obtained by Fazeli, Lovett, and Vardy [FLV14], using the probabilistic approach of Kuperberg, Lovett, and Peled [KLP17].

Lemma 6.5.4. *If $k > 12(t+1)$ and $n \ge ckt$ for some universal constant c, then there exists a t-design in $J_q(n,k)$ of cardinality at most $q^{12(t+1)n}$.*

We now use the recursive construction in Lemma 6.5.1 together with Lemma 6.5.4 to obtain the following existence result for t-designs in $\mathrm{GL}(n,q)$.

Theorem 6.5.5. *Let t be a positive integer and let $\epsilon > 0$. Then, for all sufficiently large n, there exists a t-design Y in $\mathrm{GL}(n,q)$ satisfying $|Y|/|\mathrm{GL}(n,q)| < \epsilon$.*

Proof: Fix $k > 12(t+1)$. We apply Lemma 6.5.1 with $Y = \mathrm{GL}(U)$ and $Z = \mathrm{GL}(W)$. Then from Lemma 6.5.4 we obtain the existence of a t-design in $\mathrm{GL}(n,q)$ of cardinality at most

$$N = |\mathrm{GL}(k,q)| \cdot |\mathrm{GL}(n-k,q)|\, q^{k(n-k)} q^{12(t+1)n},$$

provided that $n \geq ckt$ for the constant c of Lemma 6.5.4. Note that we have

$$\frac{N}{|\mathrm{GL}(n,q)|} = \frac{q^{12(t+1)n}}{\left[{n \atop k}\right]_q} < \frac{q^{12(t+1)n}}{q^{k(n-k)}}.$$

Since $k > 12(t+1)$, this number tends to zero as n tends to infinity. □

By combining Theorem 6.5.5 and Corollary 6.2.8 we obtain an existence result for general (σ, τ)-transitive sets in $\mathrm{GL}(n,q)$.

Corollary 6.5.6. *Let $(\sigma, \tilde{\tau}) \in \Theta_{t,q}$ and let $\epsilon > 0$. Then for all sufficiently large n, there exists a (σ, τ)-transitive set Y in $\mathrm{GL}(n,q)$ satisfying $|Y|/|\mathrm{GL}(n,q)| < \epsilon$, where $\tau = (n - |\sigma| - |\tilde{\tau}|, \tilde{\tau}_1, \tilde{\tau}_2, \dots)$.*

6.6 Designs, codes, and orthogonal polynomials

So-called P- and Q-polynomial association schemes are closely related to orthogonal polynomials in the sense that their eigenvalues and dual eigenvalues, respectively, arise as evaluations of such polynomials (see [BI84] or [Del73], for example). The conjugacy class association scheme of $\mathrm{GL}(n,q)$ does not have these properties. Nevertheless, there is still a relationship to certain orthogonal polynomials, namely the *Al-Salam-Carlitz polynomials*.

First, we recall and establish some basic properties of these polynomials and then apply these results to subsets of $\mathrm{GL}(n,q)$.

6.6.1 Al-Salam-Carlitz polynomials

Definition 6.6.1. The *Al-Salam-Carlitz polynomials* are given by

$$U_k^{(a)}(x) = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \prod_{i=0}^{j-1} (x - aq^i).$$

They were introduced in [ASC65] and some properties can be found in [Chi78] and [Kim97]. We are only interested in the case $a = 1$ and write $U_k(x)$ for $U_k^{(1)}(x)$. These polynomials satisfy the recurrence relation

$$U_{k+1}(x) = (x - 2q^k)U_k(x) + q^{k-1}(1 - q^k)U_{k-1}(x) \quad \text{for } k \geq 0$$

with the initial condition $U_{-1}(x) = 0$ and $U_0(x) = 1$. The first polynomials are

$$U_1(x) = x - 2$$
$$U_2(x) = x^2 - 2(q+1)x + 3q + 1$$
$$U_3(x) = x^3 - 2(q^2+q+1)x^2 + (3q^3+4q^2+4q+1)x - 2q(2q^2+q+1).$$

An equivalent definition of the Al-Salam-Carlitz polynomials is

$$\sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q U_k(x) = \prod_{i=0}^{j-1}(x - q^i) \quad \text{for } j = 0, 1, \dots. \tag{6.6}$$

This follows from the inversion formula

$$\sum_{k=j}^{\ell} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} \ell \\ k \end{bmatrix}_q = \delta_{j\ell}, \tag{6.7}$$

which in turn can be obtained from the q-binomial theorem.

The Al-Salam-Carlitz polynomials are q-analogs of the *Charlier polynomials* and are orthogonal with respect to a q-analog of a Poisson distribution, whose k-th moment is

$$\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q, \tag{6.8}$$

the number of subspaces of a k-dimensional vector space over $\mathbb{F}_q$. Let θ denote the class function of $\mathrm{GL}(n, q)$ given by

$$\theta(g) = q^{n - \mathrm{rk}(g-I)}$$

for each $g \in \mathrm{GL}(n, q)$, where I is the identity of $\mathrm{GL}(n, q)$. Let w_i be the number of elements $g \in \mathrm{GL}(n, q)$ satisfying $\theta(g) = q^i$. Explicit expressions for w_i were obtained by Rudvalis and Shinoda in an unpublished work [RS88] and by Fulman [Ful99], which shows that

$$w_i = \frac{|\mathrm{GL}(n,q)|}{|\mathrm{GL}(i,q)|} \sum_{k=0}^{n-i} \frac{(-1)^k q^{\binom{k}{2}}}{q^{ki}\,|\mathrm{GL}(k,q)|}. \tag{6.9}$$

We shall later see that this expression also follows from our results (see Remark 6.6.8).

The class function θ defines a discrete random variable on $\mathrm{GL}(n, q)$ and it was shown in [FS16] that its k-th moment equals (6.8), provided that $k \le n$. Hence the Al-Salam-Carlitz polynomials also satisfy the orthogonality relation

$$\sum_{i=0}^{n} w_i\, U_k(q^i) U_\ell(q^i) = 0 \quad \text{for } k \ne \ell \text{ and } k + \ell \le n. \tag{6.10}$$

(It follows from Theorem 6.6.2 that, for $k = \ell$ and $2k \le n$, the evaluation of the left-hand side is $|\mathrm{GL}(k, q)| \cdot |\mathrm{GL}(n, q)|$.)

With every polynomial $f(x) = f_n x^n + \cdots + f_1 x + f_0$ in $\mathbb{R}[x]$ we associate the class function $f(\theta) = f_n \theta^n + \cdots + f_1\theta + f_0$. This induces an algebra homomorphism from $\mathbb{R}[x]$ to the set of class functions of $\mathrm{GL}(n,q)$. Let $\zeta^j = \zeta^{(j,0)}$ be the permutation character on ordered j-tuples of linearly independent elements of $\mathbb{F}_q^n$ from Section 4.3. So that ζ^0 is the trivial character of $\mathrm{GL}(n,q)$. Note that

$$\zeta^j = \prod_{i=0}^{j-1}(\theta - q^i).$$

Hence we have

$$U_k(\theta) = \sum_{j=0}^{k}(-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \zeta^j \quad \text{for } k = 0, 1, \dots, n \tag{6.11}$$

and by (6.6)

$$\zeta^j = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q U_k(\theta) \quad \text{for } j = 0, 1, \dots, n. \tag{6.12}$$

For $0 \le k \le n/2$, we now decompose $U_k(\theta)$ into irreducible characters of $\mathrm{GL}(n,q)$.

Theorem 6.6.2. *For $0 \le k \le n/2$, the decomposition of $U_k(\theta)$ into irreducible characters is*

$$U_k(\theta) = \sum_{\underline{\nu} \in \Lambda_k} \chi^{\underline{\nu}}(1)\, \chi^{r(\underline{\nu})}, \tag{6.13}$$

where $r(\underline{\nu})$ is the element $\underline{\lambda} \in \Lambda_n$ that agrees with $\underline{\nu}$ except on $X-1$, where it is $\underline{\lambda}(X-1) = (n-k, \underline{\nu}(X-1)_1, \underline{\nu}(X-1)_2, \dots)$, namely $\underline{\lambda}(X-1)$ is obtained from $\underline{\nu}(X-1)$ by inserting a row with $n-k$ boxes in the Young diagram of $\underline{\nu}(X-1)$. In particular $U_k(\theta)$ is a character.

Proof: Since $U_0(\theta)$ is just the trivial character, (6.13) holds for $k=0$. Let m be an integer satisfying $1 \le m \le n/2$ and suppose that (6.13) holds for all k satisfying $0 \le k \le m-1$. We show that (6.13) then also holds for $k = m$.

Recall from Section 1.1 that the inner product on class functions ϕ and ψ of $\mathrm{GL}(n,q)$ is given by

$$\langle \phi, \psi \rangle = \frac{1}{|\mathrm{GL}(n,q)|} \sum_{g \in \mathrm{GL}(n,q)} \phi(g)\overline{\psi(g)}.$$

It follows from the orthogonality relation (6.10) that

$$\langle U_k(\theta), U_\ell(\theta) \rangle = 0 \quad \text{for } 0 \le k < \ell \le n/2.$$

From (6.12) we have $\langle \zeta^m, U_k(\theta)\rangle = \langle U_k(\theta), U_k(\theta)\rangle$ for all k satisfying $1 \le k \le n/2$. Since $U_k(\theta)$ is a character for all k satisfying $0 \le k \le m-1$, we find from (6.12) that $U_m(\theta)$ decomposes into those irreducible characters that occur in the decomposition of ζ^m, but not in the decomposition of $U_0(\theta), U_1(\theta), \dots, U_{m-1}(\theta)$, hence not in the decomposition of ζ^{m-1}.

As in the proof of Lemma 4.3.3 we have

$$\zeta^m = \sum_{\underline{\nu}\in\Lambda_m} \chi^{\underline{\nu}}(1)\,(\chi^{\underline{\nu}} \odot 1_{\mathrm{GL}(n-m,q)}),$$

where $1_{\mathrm{GL}(n-m,q)}$ is the trivial character of $\mathrm{GL}(n-m,q)$. Note that from Lemma 4.2.10 it follows that the Littlewood-Richardson coefficient $c^{\mu}_{\nu,(n-m)}$ is either 0 or 1 and it equals 1 precisely when the Young diagram of μ is obtained from that of ν by adding $n-m$ cells no two of which are in the same column. Hence by Lemma 4.2.11 the character $\chi^{\underline{\nu}} \odot 1_{\mathrm{GL}(n-m,q)}$ decomposes into those irreducible characters $\chi^{\underline{\lambda}}$ for which $\underline{\lambda}$ agrees with $\underline{\nu}$ except on $X-1$ and $\underline{\lambda}(X-1)$ is obtained from $\underline{\nu}(X-1)$ by adding $n-m$ boxes to the Young diagram of $\underline{\nu}(X-1)$ no two of which in the same column. Hence the irreducible characters occurring in the decomposition of ζ^m but not in the decomposition of ζ^{m-1} are precisely $\chi^{r(\underline{\nu})}$ with multiplicity $\chi^{\underline{\nu}}(1)$, where $\underline{\nu} \in \Lambda_m$. □

By combining Theorem 6.6.2 and (6.12), we obtain the decomposition into irreducible characters of ζ^j for $0 \le j \le n/2$. This result strengthens Lemma 4.3.9 for $(\sigma,\tau) = ((t),(n-t))$ and $t \le n/2$.

Corollary 6.6.3. *For $0 \le j \le n/2$ the decomposition of ζ^j into irreducible characters is*

$$\zeta^j = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q \sum_{\underline{\nu}\in\Lambda_k} \chi^{\underline{\nu}}(1)\,\chi^{r(\underline{\nu})},$$

where $r(\underline{\nu})$ is as in Theorem 6.6.2.

6.6.2 Designs and codes

Henceforth we call a $((t),(n-t))$-transitive subset of $\mathrm{GL}(n,q)$ a *t-design*. Thus a t-design in $\mathrm{GL}(n,q)$ is transitive on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$. We also call an $((n-d+1),(d-1))$-clique a *d-code*. Hence, for all distinct elements x, y of a d-code, there is no $(n-d+1)$-tuple of linearly independent elements of $\mathbb{F}_q^n$ fixed by $x^{-1}y$. This implies that $\mathrm{rk}(x-y) \ge d$ for all distinct x, y in a d-code.

Theorems 6.2.3 and 6.4.3 specialise in these cases as follows.

Corollary 6.6.4. *Let Y be a subset of $\mathrm{GL}(n,q)$ with inner distribution $(a_{\underline{\mu}})$ and dual distribution $(b_{\underline{\lambda}})$. Then Y is a t-design if and only if*

$$b_{\underline{\lambda}} = 0 \quad \text{for each } \underline{\lambda} \in \Lambda_n \text{ satisfying } n-t \le \underline{\lambda}(X-1)_1 < n$$

and a d-code if and only if

$$a_{\underline{\mu}} = 0 \quad \text{for each } \underline{\mu} \in \Lambda_n \text{ satisfying } n-d+1 \le \underline{\mu}(X-1)'_1 < n.$$

Note that the mapping $(x, y) \mapsto \operatorname{rk}(x - y)$ is a metric on $\mathrm{GL}(n, q)$. Accordingly, for a subset Y of $\mathrm{GL}(n, q)$, we define the *distance distribution* to be the tuple $(A_i)_{0\le i\le n}$, where

$$A_i = \frac{1}{|Y|}\left|\{(x, y) \in Y \times Y : \operatorname{rk}(x - y) = i\}\right|$$

and the *dual distance distribution* to be the tuple $(A'_k)_{0\le k\le n}$, where

$$A'_k = \sum_{i=0}^{n} U_k(q^{n-i})A_i.$$

Note that

$$A'_k = \frac{1}{|Y|}\sum_{x,y\in Y} U_k(q^{n-\operatorname{rk}(x-y)}). \tag{6.14}$$

We now characterise t-designs in terms of zeros in its dual distance distribution.

Proposition 6.6.5. *Let Y be a subset of* $\mathrm{GL}(n, q)$ *with dual distance distribution (A'_k) and let t be an integer satisfying $1 \le t \le n$. If Y is a t-design, then $A'_k = 0$ for all k satisfying $1 \le k \le t$. Moreover the converse also holds if $t \le n/2$. That is, if $t \le n/2$ and $A'_k = 0$ for all k satisfying $1 \le k \le t$, then Y is a t-design.*

Proof: First suppose that Y is a t-design. From (6.14) and (6.11) we have

$$A'_k = \frac{1}{|Y|}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q \sum_{x,y\in Y}\zeta^j(x^{-1}y). \tag{6.15}$$

By Lemma 4.3.3, the permutation character ζ^j decomposes into those irreducible characters $\chi^{\underline{\lambda}}$ for which $\underline{\lambda}(X - 1)_1 \ge n - j$. Moreover, since ζ^j is a permutation character, it contains the trivial character with multiplicity 1. From Corollary 6.6.4 we then find that the inner sum in (6.15) equals $|Y|^2$ for all j satisfying $0 \le j \le t$. Hence we have, for all k satisfying $0 \le k \le t$,

$$A'_k = |Y|\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q = |Y|\,\delta_{k,0},$$

using (6.7) together with elementary manipulations.

Now, for each k satisfying $0 \le k \le n/2$, we find from (6.14), Theorem 6.6.2, and (4.21) that

$$\begin{aligned} A'_k &= \frac{1}{|Y|}\sum_{\underline{\nu}\in\Lambda_k}\chi^{\underline{\nu}}(1)\sum_{x,y\in Y}\chi^{r(\underline{\nu})}(x^{-1}y) \\ &= \sum_{\underline{\nu}\in\Lambda_k}\frac{\chi^{\underline{\nu}}(1)}{\chi^{r(\underline{\nu})}(1)}\,b_{r(\underline{\nu})}, \end{aligned}$$

where $r(\underline{\nu})$ is as in Theorem 6.6.2. Suppose that t satisfies $1 \le t \le n/2$ and that $A'_k = 0$ for all k satisfying $1 \le k \le t$. Since $\chi^{\underline{\nu}}(1)/\chi^{r(\underline{\nu})}(1)$ is positive, we find that

$b_{r(\underline{\nu})} = 0$ for all $\underline{\nu} \in \Lambda_k$ and hence $b_{\underline{\lambda}} = 0$ for all $\underline{\lambda} \in \Lambda_n$ satisfying $n-t \le \underline{\lambda}(X-1)_1 < n$. Corollary 6.6.4 then implies that Y is a t-design. □

Theorem 6.4.4 specialises as follows.

Corollary 6.6.6. *Let Y be a subset of* $\mathrm{GL}(n,q)$ *and let d and t be the largest integers such that Y is a d-code and a t-design. Then*

$$\prod_{i=0}^{t-1}(q^n - q^i) \le |Y| \le \prod_{i=0}^{n-d}(q^n - q^i).$$

Moreover, if equality holds in one of the bounds, then equality also holds in the other and this case happens if and only if $d = n - t + 1$.

The upper bound in Corollary 6.6.6 is a q-analog of a corresponding well known bound $n(n-1)\cdots d$ for permutation codes from Theorem 3.3.3. The bounds in Corollary 6.6.6 can be achieved. Namely from Example 6.2.2 it follows that a Singer cycle in $\mathrm{GL}(n,q)$ gives an n-code in $\mathrm{GL}(n,q)$ of size $q^n - 1$ and from Section 6.3 we have that A_7 inside $\mathrm{GL}(4,2)$ is a 2-code of size 2520.

It turns out that the distance distribution of a subset Y of $\mathrm{GL}(n,q)$ is uniquely determined by its parameters, provided that Y is a t-design and a d-code, where $d \ge n-t$. The following result generalises (6.9).

Theorem 6.6.7. *Suppose that Y is a t-design and an $(n-t)$-code in* $\mathrm{GL}(n,q)$. *Then the distance distribution (A_i) of Y satisfies*

$$A_{n-i} = \sum_{j=i}^{t}(-1)^{j-i} q^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\frac{|Y|}{\prod_{k=0}^{j-1}(q^n - q^k)} - 1 \right)$$

for each $i \in \{0, 1, \dots, n-1\}$.

Proof: We have

$$A'_k = \sum_{i=0}^{n} U_k(q^i) A_{n-i}.$$

Multiply both sides by $\begin{bmatrix} j \\ k \end{bmatrix}_q$, sum over k, and use (6.6) to find that

$$\sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q A'_k = \sum_{i=0}^{n} A_{n-i} \prod_{k=0}^{j-1}(q^i - q^k).$$

Since Y is an $(n-t)$-code, we have $A_1 = \cdots = A_{n-t-1} = 0$ and, since Y is a t-design, we find by Proposition 6.6.5 that $A'_1 = \cdots = A'_t = 0$. Moreover we have $A_0 = 1$ and $A'_0 = |Y|$ and therefore

$$|Y| - \prod_{k=0}^{j-1}(q^n - q^k) = \sum_{i=0}^{t} A_{n-i} \prod_{k=0}^{j-1}(q^i - q^k)$$

for each $j \in \{1, 2, \dots, t\}$. The identity

$$\prod_{k=0}^{j-1}(q^i - q^k) = \begin{bmatrix} i \\ j \end{bmatrix}_q (q^j - 1)\cdots(q^j - q^{j-1})$$

gives

$$\begin{aligned}\sum_{i=0}^{t} A_{n-i}\begin{bmatrix} i \\ j \end{bmatrix}_q &= \frac{|Y| - \prod_{k=0}^{j-1}(q^n - q^k)}{\prod_{k=0}^{j-1}(q^j - q^k)} \\ &= \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\frac{|Y|}{\prod_{k=0}^{j-1}(q^n - q^k)} - 1\right)\end{aligned}$$

for each $j \in \{1, 2, \dots, t\}$. Now the desired result follows from (6.7). □

Remark 6.6.8. Consider $Y = \mathrm{GL}(n,q)$ having inner distribution (A_i), so that $A_{n-i} = w_i$. Since Y is a 1-code and an n-design, Theorem 6.6.7 gives

$$A_{n-i} = \sum_{j=i}^{n-1}(-1)^{j-i}q^{\binom{j-i}{2}}\begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\prod_{k=j}^{n-1}(q^n - q^k) - 1\right).$$

Now a lengthy, but straightforward, calculation reveals that $A_{n-i} = w_i$, given in (6.9). Note that the proof of Theorem 6.6.7 uses only the (easy) forward direction of Proposition 6.6.5 and not the decomposition in Theorem 6.6.2. Hence our proof of Theorem 6.6.7 and therefore of (6.9) is self-contained.

Note that the upper bound in Corollary 6.6.6 is at most

$$q^{n(n-d+1)}.$$

We close this section by showing that there exist d-codes almost as large as this upper bound. Our construction uses so-called *linear maximum rank distance codes* with minimum distance d, which are $\mathbb{F}_q$-subspaces Z of $\mathbb{F}_q^{n\times n}$ of dimension $n(n-d+1)$, such that $\mathrm{rk}(x-y) \ge d$ for all distinct $x, y \in Z$. Such objects exist for all integers d satisfying $1 \le d \le n$ [Del78, Theorem 6.3].

Proposition 6.6.9. *For each d satisfying $1 \le d \le n$, there exists a d-code in* $\mathrm{GL}(n,q)$ *of size at least*

$$\left(1 - \frac{1}{q-1}\right) q^{n(n-d+1)}.$$

For $q = 2$ there exists a d-code in $\mathrm{GL}(n,q)$ *of size at least* $q^{n(n-d)}$.

Proof: Consider a linear maximum rank distance code Z in $\mathbb{F}_q^{n\times n}$ with minimum distance d. We show that $Z \cap \mathrm{GL}(n,q)$ has the required properties. It is well known [Del78, Theorem 5.6] that the number of matrices in Z of rank i depends only on the parameters q, n, and d. In particular the number of invertible matrices in Z equals

$$N = \sum_{j=0}^{n-d} (-1)^j C_j,$$

where

$$C_j = q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (q^{n(n-d+1-j)} - 1).$$

It follows that $C_j / C_{j+1} \geq q^j$ for each $j \in \{0, 1, \ldots, n-d-1\}$ and therefore $C_0, C_1, \ldots, C_{n-d}$ is nonincreasing. Hence we have

$$\begin{aligned} N \geq C_0 - C_1 &= (q^{n(n-d+1)} - 1) - \frac{q^n - 1}{q-1}(q^{n(n-d)} - 1) \\ &\geq \frac{q-2}{q-1} q^{n(n-d+1)} + \frac{q^{n(n-d)} - 1}{q-1}, \end{aligned}$$

as required. □

6.7 Open Problem

In [KLP17], Kuperberg, Lovett, and Peled proved with the help of the probabilistic method the existence of small *orthogonal arrays*, t-(v, k, λ) designs, and small *t-wise permutations*, where the latter are subsets of the symmetric group that are transitive on $(n-t, 1^t)$-tabloids.

In [Ern20] the method of [KLP17] was used to obtain a stronger result on the asymptotic existence of small transitive subsets of permutations, which also improves the existence result from Theorem 3.4.6 for transitive sets in the symmetric group.

Theorem 6.7.1 ([Ern20]). *Let $\sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_k$ be a nonincreasing sequence of positive integers and let $\delta, \varepsilon > 0$. Then for all sufficiently large $\sigma_1 \geq \sigma_2$ there exists a subset Y of the symmetric group S_n, where $n = \sigma_1 + \sigma_2 + \ldots, +\sigma_k$, that is transitive on $(\sigma_1, \sigma_2, \ldots, \sigma_k)$-tabloids satisfying $|Y| / \left(\frac{(Cn)^{C(\sigma_2 + \ldots + \sigma_k) + \delta}}{(\sigma_2! \cdots \sigma_k!)^{14}} \right) < \varepsilon$ for a constant C.*

In Corollary 6.5.6 we obtain the existence of a (σ, τ)-transitive sets Y in the finite general linear group $\mathrm{GL}(n, q)$ with $|Y|$ growing more slowly than $|\mathrm{GL}(n, q)|$. Knowing the result for the symmetric group motivates the following open question for a q-analog setting.

Open Problem. Is there a constant $C_n < |\mathrm{GL}(n, q)|$ such that there exists a transitive set Y in $\mathrm{GL}(n, q)$ with $|Y|$ growing more slowly than C_n ?

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