Manifold of mappings and regularity properties of half-Lie groups

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In memory of Dr. Hernán Henríquez

Abstract

In the first part, for $p \in [1,\infty]$, we define a smooth manifold structure on the set $AC_{L^p}([a,b],N)$ of absolutely continuous functions $\gamma\colon [a,b] \to N$ with L^p-derivatives for all real numbers $a < b$ and each smooth manifold N modeled on a sequentially complete locally convex topological vector space, such that N admits a local addition. Smoothness of natural mappings between spaces of absolutely continuous functions is discussed, like superposition operators $AC_{L^p}([a,b],N_1) \to AC_{L^p}([a,b],N_2)$, $\eta \mapsto f \circ \eta$, for a smooth map $f\colon N_1 \to N_2$. For $1 \leq p < \infty$ and $r \in \mathbb{N}$ we show that the right half-Lie groups $\mathrm{Diff}^r_K(\mathbb{R}^n)$ and $\mathrm{Diff}^r(M)$ are L^p-semiregular. Here K is a compact subset of $\mathbb{R}^n$ and M is a compact smooth manifold. An L^p-semiregular half-Lie group G admits an evolution map $\mathrm{Evol} : L^p([0,1],T_eG) \to AC_{L^p}([0,1],G)$, where e is the neutral element of G. For the preceding examples, the evolution map Evol is continuous.

In the second part, for a compact manifold with corners M and a finite dimensional smooth manifold without boundary N which admits a local addition, we define a smooth manifold structure on a certain set $\mathcal{F}(M,N)$ of continuous mappings whenever function spaces $\mathcal{F}(U,\mathbb{R})$ on open subsets $U \subseteq [0,\infty)^n$ are given, subject to simple axioms. The construction and properties of spaces of sections and smoothness of natural mappings between the spaces $\mathcal{F}(M,N)$ are discussed, like superposition operators $\mathcal{F}(M,f) : \mathcal{F}(M,N_1) \to \mathcal{F}(M,N_2)$, $\eta \mapsto f \circ \eta$ for smooth maps $f : N_1 \to N_2$.

Zusammenfassung

Im ersten Teil definieren wir für $p \in [1, \infty]$ eine glatte Mannigfaltigkeitsstruktur auf der Menge $AC_{L^p}([a,b], N)$ der absolut stetigen Funktionen $\gamma\colon [a,b] \to N$ mit L^p-Ableitungen für alle reellen Zahlen $a < b$ und jede glatte Mannigfaltigkeit N, die auf einem folgenvollständigen, lokal konvexen topologischen Vektorraum modelliert ist und eine lokale Addition zulässt. Die Glattheit natürlicher Abbildungen zwischen Räumen absolut stetiger Funktionen wird untersucht, wie etwa Superpositionsoperatoren
$AC_{L^p}([a,b], N_1) \to AC_{L^p}([a,b], N_2)$, $\eta \mapsto f \circ \eta$, für eine glatte Abbildung $f\colon N_1 \to N_2$. Für $1 \leq p < \infty$ und $r \in \mathbb{N}$ zeigen wir, dass die rechten Halb-Liegruppen $\mathrm{Diff}_K^r(\mathbb{R}^n)$ und $\mathrm{Diff}^{\,r}(M)$ L^p-semiregulär sind. Hierbei ist K eine kompakte Teilmenge von $\mathbb{R}^n$ und M eine kompakte glatte Mannigfaltigkeit. Eine L^p-semireguläre rechte Halb-Liegruppe G besitzt eine Evolutionsabbildung $\mathrm{Evol} : L^p([0,1], T_eG) \to AC_{L^p}([0,1], G)$, wobei e das Neutralelement von G ist. Für die zuvor genannten Beispiele ist die Evolutionsabbildung Evol stetig.

Im zweiten Teil definieren wir für eine kompakte Mannigfaltigkeit mit Ecken M und eine endlichdimensionale glatte Mannigfaltigkeit ohne Rand N, die eine lokale Addition zulässt, eine glatte Mannigfaltigkeitsstruktur auf gewissen Mengen stetiger Abbildungen $\mathcal{F}(M,N)$, sofern Funktionenräume $\mathcal{F}(U,\mathbb{R})$ auf offenen Teilmengen $U \subseteq [0,\infty)^n$ gegeben sind und einfache Axiome erfüllt werden. Die Konstruktion und Eigenschaften von Räumen von Schnitten sowie die Glattheit natürlicher Abbildungen zwischen den Räumen $\mathcal{F}(M,N)$ werden diskutiert, wie etwa Superpositionsoperatoren
$\mathcal{F}(M,f) : \mathcal{F}(M,N_1) \to \mathcal{F}(M,N_2)$, $\eta \mapsto f \circ \eta$ für glatte Abbildungen $f : N_1 \to N_2$.

Contents

1 Introduction

Manifolds of mappings play an important role in various branches of global analysis, infinite-dimensional geometry and Lie theory. For a compact smooth manifold M, a finite-dimensional manifold N and a non-negative integer r, a smooth manifold structure on the set $C^r(M,N)$ of C^r-functions $\phi\colon M \to N$ was first obtained by Eells (see [9] and the references therein). Later, manifold structures were also obtained on $C^\infty(M,N)$, and for infinite-dimensional manifolds N as long as they admit a local addition, a concept recalled in Definition 1.0.1 (see [30, 25] and the references therein, also [3]).

Manifolds of absolutely continuous functions with values in an infinite-dimensional manifold and regularity properties of half-Lie groups

For a Hilbert manifold M, a smooth manifold structure on the set $AC_{L^2}([0,1],N)$ of absolutely continuous paths with L^2-derivatives in local charts was used by Flaschel and Klingenberg to study closed geodesics in Riemannian manifolds (see [11] and [24], cf. also [41]). For Banach manifolds N admitting a smooth local addition and $p \in [1,\infty]$, a smooth manifold structure on $AC_{L^p}([0,1],N)$ may also be obtained using a method of Krikorian [26] which is similar to Palais' use of Banach section functors [38]. A Lie group structure (and hence a smooth manifold structure) on $AC_{L^p}([0,1],G)$ was obtained in [35] for each Lie group G modeled on a sequentially complete locally convex space (as in [32]), generalizing the case of Fréchet–Lie groups treated in [15].

We construct manifolds of absolutely continuous functions in higher generality. To formulate the main result, let us fix notation.

Definition 1.0.1 Let N be a smooth manifold modeled on real a locally convex space, with tangent bundle TN and its bundle projection $\pi_{TN}\colon TN \to N$. A *local addition* on N is a map

$$\Sigma\colon \Omega \to N,$$

defined on an open set $\Omega \subseteq TN$ which contains the zero-vector $0_p \in T_pN$ for each $p \in N$, such that $\Sigma(0_p) = p$ for all $p \in N$ and

$$\theta\colon \Omega \to N \times N, \quad v \mapsto (\pi_{TN}(v), \Sigma(v))$$

has open image and is a C^∞-diffeomorphism onto its image Ω'.

Throughout the following, $a < b$ are real numbers and $p \in [1,\infty]$. For a definition of absolutely continuous functions with values in a sequentially complete locally convex space E or a smooth manifold N defined thereon, the reader is referred to Definitions 2.1.12 and

2.1.23, respectively (see also [35]). For $\eta \in AC_{L^p}([a,b],N)$, the pointwise operations make

$$\Gamma_{AC}(\eta) := \{\tau \in AC_{L^p}([a,b],TN) \colon \pi_{TN} \circ \tau = \eta\}$$

a vector space; we endow it with a natural topology making it a locally convex topological vector space (cf. Definition 2.2.1). We shall see that

$$\mathcal{V}_\eta \colon := \{\tau \in \Gamma_{AC}(\eta) \colon \tau([a,b]) \subseteq \Omega\}$$

is an open 0-neighborhood in $\Gamma_{AC}(\eta)$. Setting

$$\mathcal{U}_\eta := \{\gamma \in AC_{L^p}([a,b],N) \colon (\eta(t),\gamma(t)) \in \Omega' \text{ for all } t \in [a,b]\},$$

the map

$$\Psi_\eta \colon \mathcal{V}_\theta \to \mathcal{U}_\theta, \quad \tau \mapsto \Sigma \circ \tau$$

is a bijection (see Remark 2.3.3). We show (see Theorem 2.3.5):

Theorem 1.0.2 *For each smooth manifold N modeled on a sequentially complete locally convex space and $p \in [1,\infty]$, the set $AC_{L^p}([a,b],N)$ of all AC_{L^p}-maps $\gamma \colon [a,b] \to N$ admits a smooth manifold structure such that for each local addition $\Sigma \colon \Omega \to N$, the sets $\mathcal{U}_\eta$ are open in $AC_{L^p}([a,b],N)$ for all $\eta \in AC_{L^p}([a,b],N)$ and $\Psi_\eta \colon \mathcal{V}_\eta \to \mathcal{U}_\eta$ is a C^∞-diffeomorphism.*

Using the smooth manifold structures just described, we find:

Theorem 1.0.3 *Let $f \colon N_1 \to N_2$ be a smooth map between smooth manifolds N_1 and N_2 modeled on sequentially complete locally convex spaces such that N_1 and N_2 admit a local addition and $p \in [1,\infty]$. We then have $f \circ \gamma \in AC_{L^p}([a,b],N_2)$ for all $\gamma \in AC_{L^p}([a,b],N_1)$; the map*

$$AC_{L^p}([a,b],f) \colon AC_{L^p}([a,b],N_1) \to AC_{L^p}([a,b],N_2), \quad \gamma \mapsto f \circ \gamma$$

is smooth.

More generally, $AC_{L^p}([a,b],f)$ is C^r for $r \in \mathbb{N} \cup \{0,\infty\}$ whenever f is C^{r+2} (see Proposition 2.3.8).

If, in the situation of Definition 1.0.1, N is a $\mathbb{K}$-analytic manifold modeled on a locally convex topological $\mathbb{K}$-vector space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and $\theta \colon \Omega \to \Omega'$ is a diffeomorphism of $\mathbb{K}$-analytic manifolds, then Σ is called a $\mathbb{K}$-*analytic local addition*. In this case, $AC_{L^p}([a,b],N)$ can be given a $\mathbb{K}$-analytic manifold structure modeled on the locally convex topological $\mathbb{K}$-vector spaces Γ_η with properties as in Theorem 1.0.2, replacing C^∞-diffeomorphisms with diffeomorphisms of $\mathbb{K}$-analytic manifolds there (see Corollary 2.3.6). If N_j is a $\mathbb{K}$-analytic manifold modeled on a sequentially complete locally convex topological $\mathbb{K}$-vector space such that N_j admits a $\mathbb{K}$-analytic local addition for $j \in \{1,2\}$, then the map $AC_{L^p}([a,b],f)$ described in Theorem 1.0.3 is $\mathbb{K}$-analytic for all $p \in [1,\infty]$ (see Corollary 2.3.9).

Manifolds of absolutely continuous paths occur in the regularity theory of infinite-dimensional Lie groups. Consider a Lie group G modeled on a sequentially complete locally convex space, with tangent space $\mathfrak{g} := T_eG$ at the neutral element $e \in G$. For $g \in G$, let $\rho_g\colon G \to G$, $x \mapsto xg$ be the right multiplication with g. Then

$$TG \times G \to TG, \quad (v, g) \mapsto T\rho_g(v) =: v.g$$

is a smooth map and a right action of G on TG. The following concept strengthens "regularity" in the sense of Milnor [32].

Definition 1.0.4 Following [35] (cf. also [15]), G is called *L^p-semiregular* if, for each $\gamma \in \mathcal{L}^p([0,1], \mathfrak{g})$, the initial value problem

$$\dot\eta(t) = \gamma(t).\eta(t), \quad t \in [0,1] \tag{1.0.1}$$

$$\eta(0) = e \tag{1.0.2}$$

has an AC_{L^p}-solution $\eta\colon [0,1] \to G$. Then η is necessarily unique; we call η the *evolution* of γ and write $\mathrm{Evol}([\gamma]) := \eta$. If G is L^p-semiregular and $\mathrm{Evol}\colon L^p([0,1], \mathfrak{g}) \to AC_{L^p}([0,1], G)$ is smooth, then G is called *L^p-regular.*

Remark 1.0.5 If G is modeled on a Fréchet space, (1.0.1) is required to hold for almost all $t \in [0,1]$ with respect to Lebesgue measure. In the general case, η is required to be a Carathéodory solution to (1.0.1), i.e., it solves the corresponding integral equation piecewise in local charts. We mention that a Lie group G is L^p-regular if and only if G is L^p-semiregular and Evol is smooth as map

$$L^p([0,1], \mathfrak{g}) \to C([0,1], G)$$

(see [35]). The latter holds whenever $\mathrm{Evol}\colon L^p([0,1], \mathfrak{g}) \to C([0,1], G)$ is continuous at 0 (see [16, Theorem E]).

Now consider a half-Lie group G modeled on a sequentially complete locally convex space E. Thus G is a group, endowed with a smooth manifold structure modeled on E which makes G a topological group and turns the right translation $\rho_g\colon G \to G$ into a smooth mapping for each $g \in G$ (cf. [6, 29]). Let us use notation as in the case of Lie groups.

Definition 1.0.6 We say that a half-Lie group G is *L^p-semiregular* if the differential equation

$$\dot y(t) = \gamma(t).y(t), \quad t \in [0,1] \tag{1.0.3}$$

satisfies local uniqueness of Carathéodory solutions for each $\gamma \in \mathcal{L}^p([0,1], \mathfrak{g})$ (in the sense of [19]) and the initial value problem (1.0.1) has a Carathéodory solution $\mathrm{Evol}(\gamma) := \eta\colon [0,1] \to G$.

The Lie group $\mathrm{Diff}(M)$ of C^∞-diffeomorphisms of a compact smooth manifold M without boundary is known to be L^1-regular, and also the Lie group $\mathrm{Diff}_K(\mathbb{R}^n)$ of all C^∞-diffeomorphisms $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(x) = x$ for all $x \in \mathbb{R}^n \setminus K$, for each compact subset $K \subseteq \mathbb{R}^n$ (see [15]). For each positive integer r, the following analogues are obtained (see Theorems 2.6.5 and 2.5.3):

Theorem 1.0.7 *Let $1 \leq p < \infty$. For each compact smooth manifold M without boundary and $r \in \mathbb{N}$, the half-Lie group $G := \mathrm{Diff}^r(M)$ of all C^r-diffeomorphisms $\phi\colon M \to M$ is L^p-semiregular. Moreover, its evolution map*

$$\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G)$$

is continuous.

Here $\mathfrak{g}$ is the Banach space of C^r-vector fields on M.

Theorem 1.0.8 *Let $1 \leq p < \infty$ and $r \in \mathbb{N}$. For each positive integer n and compact subset K of $\mathbb{R}^n$, the half-Lie group $G := \mathrm{Diff}^r_K(\mathbb{R}^n)$ of all C^r-diffeomorphisms $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\phi|_{\mathbb{R}^n\setminus K} = \mathrm{id}_{\mathbb{R}^n\setminus K}$ is L^p-semiregular. Moreover, its evolution map*

$$\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G)$$

is continuous.
If we replace L^p with L^∞_{rc} (see Definition 2.1.5) the preceding theorem remains valid.

Here $\mathfrak{g}$ is the Banach space of all C^r-vector fields on $\mathbb{R}^n$ which vanish outside K.

For an L^p-semiregular half-Lie group G admitting a local addition with $1 \leq p < \infty$, the smooth manifold structure on $AC_{L^p}([0,1],G)$ provided by Theorem 1.0.2 makes it possible to discuss continuity properties and differentiability properties of the evolution map as a mapping

$$\mathrm{Evol}\colon L^p([0,1],\mathfrak{g}) \to AC_{L^p}([0,1],G).$$

So far, we have one positive result in this regard:

Theorem 1.0.9 *Let G be a right half-Lie group modeled in a sequentially complete locally convex space space E which admits a local addition and $1 \leq p < \infty$. Let G be L^p-semiregular with continuous evolution map*

$$\mathrm{Evol}_C : L^p([0,1],T_eG), \to C([0,1],G), \quad \gamma \mapsto \mathrm{Evol}_C(\gamma).$$

If the restriction of the right action

$$\tau : T_eG \times G \to TG, \quad (v,g) \mapsto v.g$$

is continuous, then the evolution map

$$\mathrm{Evol} : L^p([0,1],T_eG), \to AC_{L^p}([0,1],G), \quad \gamma \mapsto \mathrm{Evol}(\gamma)$$

is continuous. If G is a right half-Lie group modeled on an integral complete locally convex space E, then if we replace L^p with L^∞_{rc} the result remains valid.

So far, C^0-regularity has been investigated for the half-Lie group $\mathrm{Diff}^r(M)$ in a suitable sense (see the sketch in [31]). Independently, related questions of regularity have been considered by Pierron and Trouvé (see [39]).

Manifolds of mappings associated with real-valued functions spaces

In the second part, we describe a general construction principle for manifold structures on sets of mappings between manifolds when real-valued functions spaces are given, satisfying suitable axioms. The modeling spaces for these manifold structures, which coincide with spaces of sections, are studied at the beginning. Then we study the construction and properties of natural mappings between these manifolds of mappings.
For fixed $m, n \in \mathbb{N}$, we consider an m-dimensional compact smooth manifold with corners M and let N be an n-dimensional smooth manifold without boundary. Following the notation of the work Helge Glöckner and Luis Tárrega [22], we consider a basis of the topology $\mathcal{U}$ of the set $[0, \infty)^m$ satisfying suitable properties (see Definition 3.1.1). Suppose that for each open set $U \in \mathcal{U}$, an integral complete locally convex space $\mathcal{F}(U, \mathbb{R})$ of bounded, continuous real-valued functions is given. Then for each finite-dimensional real vector space E, a set of maps $\mathcal{F}(U, E)$ can be defined in a natural way. If certain axioms are satisfied (see Definition 3.1.5), we say that the family $(\mathcal{F}(U, E))_{U \in \mathcal{U}}$ is suitable for global analysis. Varying the case where M is a smooth manifold without boundary (see [22]), one can define a locally convex space $\mathcal{F}(M, E)$. Moreover, we can also define a set $\mathcal{F}(M, N)$ of N-valued $\mathcal{F}$-functions on the manifold with corners M.

For each function $\gamma : M \to N$ in $\mathcal{F}(M, N)$, we define the real vector space of sections with the pointwise operations

$$\Gamma_{\mathcal{F}}(\gamma) := \{\sigma \in \mathcal{F}(M, TN) : \pi_{TN} \circ \sigma = \gamma\}$$

and we endow it with a natural topology making it an integral complete locally convex topological vector space. We define the set

$$\mathcal{V}_\gamma := \{\sigma \in \Gamma_{\mathcal{F}}(\gamma) : \sigma(M) \subseteq \Omega\},$$

which is open in $\Gamma_{\mathcal{F}}(\gamma)$. Setting

$$\mathcal{U}_\gamma := \{\xi \in \Gamma_{\mathcal{F}}(\gamma) : (\gamma, \xi)(M) \subseteq \Omega'\},$$

the map

$$\Psi_\gamma := \mathcal{F}(M, \Sigma) : \mathcal{V}_\gamma \to \mathcal{U}_\gamma, \quad \sigma \mapsto \Sigma \circ \sigma$$

is a bijection. We show (see Theorem 3.3.3):

Theorem 1.0.10 *Let $\mathcal{U}$ be a good collection of open subsets of $[0, \infty)^m$. If $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ is a family of locally convex spaces suitable for global analysis, then for each m-dimensional compact manifold M with corners and smooth manifold N without boundary which admits a local addition, the set $\mathcal{F}(M, N)$ admits a smooth manifold structure such that the sets $\mathcal{U}_\gamma$ are open in $\mathcal{F}(M, N)$ for all $\gamma \in \mathcal{F}(M, N)$ and for each local addition, the map $\Psi_\gamma \colon \mathcal{V}_\gamma \to \mathcal{U}_\gamma$ is a C^∞-diffeomorphism.*

We show that the point evaluation map $\varepsilon_p : \mathcal{F}(M, N) \to N$ is smooth for each $p \in M$ (see Proposition 3.3.12). For each $v \in T\mathcal{F}(M, N)$ we define the function

$$\Theta_N(v) : M \to TN, \quad \Theta_N(v)(p) := T\varepsilon_p(v).$$

Then with respect to the tangent bundle of $\mathcal{F}(M, N)$ we have:

Proposition 1.0.11 *Let M be an m-dimensional compact smooth manifold with corners, N be an n-dimensional smooth manifold which admits a local addition and π_{TN} : $TN \to N$ its tangent bundle. Then the map*

$$\mathcal{F}(M, \pi_{TN}) : \mathcal{F}(M, TN) \to \mathcal{F}(M, N), \quad \tau \mapsto \pi_{TN} \circ \tau$$

is a smooth vector bundle with fiber $\Gamma_{\mathcal{F}}(\gamma)$ over $\gamma \in \mathcal{F}(M, N)$. Moreover, the map

$$\Theta_N : T\mathcal{F}(M, N) \to \mathcal{F}(M, TN), \quad v \mapsto \Theta_N(v)$$

is an isomorphism of vector bundles.

Using the smooth manifold structures just described, we find:

Proposition 1.0.12 *Let M be an m-dimensional compact smooth manifold with corners, N_1 and N_2 be n-dimensional smooth manifolds which admit local additions (Ω_1, Σ_1) and (Ω_2, Σ_2) respectively. If $f : N_1 \to N_2$ is a smooth map, then the map*

$$\mathcal{F}(M, f) : \mathcal{F}(M, N_1) \to \mathcal{F}(M, N_2), \quad \gamma \mapsto f \circ \gamma,$$

is smooth.

And its tangent map can be characterized in the following way:

Proposition 1.0.13 *Let M be an m-dimensional compact smooth manifold with corners, N_1 and N_2 be finite-dimensional smooth manifolds which admit a local addition. If $f : N_1 \to N_2$ is a smooth map, then the tangent map of*

$$\mathcal{F}(M, f) : \mathcal{F}(M, N_1) \to \mathcal{F}(M, N_2), \quad \gamma \mapsto f \circ \gamma$$

is given by

$$T\mathcal{F}(M, f) = \Theta_{N_2}^{-1} \circ \mathcal{F}(M, Tf) \circ \Theta_{N_1}.$$

2 Manifolds of absolutely continuous functions with values in an infinite-dimensional manifold and regularity properties of half-Lie groups

2.1 Preliminaries

Definition 2.1.1 Let E and F be real locally convex spaces, $U \subseteq E$ be open and $f : U \to F$ be a map. We say that f is C^0 if it is continuous. We say that f is C^1 if f is continuous, the directional derivative

$$df(x,y) := (D_y f)(x) := \lim_{t \to 0} \frac{1}{t}(f(x+ty) - f(x))$$

(with $t \neq 0$) exists in F for all $(x,y) \in U \times E$, and $df : U \times E \to F$ is continuous. Recursively, for $k \in \mathbb{N}$ we say that f is C^k if f is C^1 and $df : U \times E \to F$ is C^{k-1}. We say that f is C^∞ (or smooth) if f is C^k for each $k \in \mathbb{N}$.

Definition 2.1.2 Let E_1, E_2 and F be real locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be open subsets, $r, s \in \mathbb{N} \cup \{0, \infty\}$ and $f : U \times V \to F$ be a map. If the iterated directional derivatives

$$d^{(i,j)} f((x,a), y_1, ..., y_i, b_1, ..., b_j) := (D_{(y_1,0)} ... D_{(y_i,0)} D_{(0,b_1)} ... D_{(0,b_j)} f)(x,a)$$

exist for all $i, j \in \mathbb{N} \cup \{0\}$ such that $i \leq r$ and $j \leq s$, and all $y_1, ..., y_i \in E_1$ and $b_1, ..., b_j \in E_2$, and, we assume that the mappings

$$d^{(i,j)} f : U \times V \times E_1^i \times E_2^j \to F$$

are continuous, then f is called a $C^{r,s}$-map.

Definition 2.1.3 Let X be a locally compact topological space, endowed with a measure $\mu : \mathcal{B}(X) \to [0, \infty]$ on its σ-algebra of Borel sets and let Y be a topological space. A function $\gamma : X \to Y$ is called Lusin μ-measurable (or μ-measurable) if for each compact subset $K \subseteq X$, there exists a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets $K_n \subseteq K$ such that each restriction $\gamma|_{K_n}$ is continuous and $\mu\left(K \setminus \cup_{n \in \mathbb{N}} K_n\right) = 0$.

For details of the construction of Lebesgue spaces, we refer the reader to see [35].

Definition 2.1.4 Let E be a locally convex space, $a < b$ be real numbers, $1 \leq p < \infty$ and $\lambda : \mathcal{B}([a,b]) \to [0, \infty)$ be the restriction of the Lebesgue-Borel measure on $\mathbb{R}$. We define the set $\mathcal{L}^p([a,b],E)$ as the set of all Lusin λ-measurable functions $\gamma : [a,b] \to E$ such that for each continuous seminorm q on E we have

$$q \circ \gamma \in \mathcal{L}^p([a,b],\mathbb{R}).$$

And we endow it with the locally convex topology defined by the family of seminorms

$$\|\cdot\|_{\mathcal{L}^p,q} \colon \mathcal{L}^p([a,b],E) \to [0,\infty[, \quad \|\gamma\|_{\mathcal{L}^p,q} := \|q \circ \gamma\|_{\mathcal{L}^p}.$$

Let $\gamma \sim \eta$ if and only if $\gamma(t) = \eta(t)$ for almost all $t \in [a,b]$ and write $[\gamma]$ for the equivalence class of γ. We define the Hausdorff locally convex space

$$L^p([a,b],E) := \mathcal{L}^p([a,b],E)/[0]$$

with seminorms

$$\|\cdot\|_{L^p,q} \colon L^p([a,b],E) \to [0,\infty[, \quad \|[\gamma]\|_{L^p,q} := \|\gamma\|_{\mathcal{L}^p,q}.$$

Definition 2.1.5 Let E be a locally convex space, $a < b$ be real numbers and $\lambda : \mathcal{B}([a,b]) \to [0,\infty)$ be the restriction of the Lebesgue measure on $\mathbb{R}$. We define the set $\mathcal{L}^\infty([a,b],E)$ of all Lusin λ-measurable, essentially bounded functions $\gamma : [a,b] \to E$. For each continuous seminorm $q : E \to \mathbb{R}^n$ we define the seminorm

$$\|\gamma\|_{\mathcal{L}^\infty,q} := \operatorname*{ess\,sup}_{t\in[a,b]} q \circ \gamma(t).$$

We endow $\mathcal{L}^\infty([a,b],E)$ with the Hausdorff locally convex topology given by these seminorms.
Let $\gamma \sim \eta$ if and only if $\gamma = \eta$ a.e. We define the Hausdorff locally convex space

$$L^\infty([a,b],E) := \mathcal{L}^\infty([a,b],E)/[0]$$

with seminorms

$$\|\cdot\|_{L^\infty,q} \colon L^\infty([a,b],E) \to [0,\infty[, \quad \|[\gamma]\|_{L^\infty,q} := \|\gamma\|_{\mathcal{L}^\infty,q}.$$

We define the vector space $\mathcal{L}^\infty_{rc}([a,b],E)$ of all Borel measurable functions $\gamma : [a,b] \to E$ such that the image $\mathrm{Im}(\gamma)$ has compact and metrizable closure, endowed with the topology induced by $\mathcal{L}^\infty([a,b],E)$. Thus

$$L^\infty_{rc}([a,b],E) := \mathcal{L}^\infty_{rc}([a,b],E)/[0]$$

is a Hausdorff locally convex space.

Remark 2.1.6 Let E be a Frechet space and $p \in [0,1]$. If $L^p_{\mathcal{B}}([0,1],E)$ denotes the Lebesgue space constructed with the set of Borel measurable functions (see [15]), then $L^p_{\mathcal{B}}([0,1],E)$ coincides with $L^p([0,1],E)$ ([35, Proposition 2.10]).

Remark 2.1.7 Let $1 \leq q \leq p < \infty$, then

$$\mathcal{L}^\infty([a,b],E) \subseteq \mathcal{L}^p([a,b],E) \subseteq \mathcal{L}^q([a,b],E) \subseteq \mathcal{L}^1([a,b],E)$$

as a consequence of Hölder's inequality.

Definition 2.1.8 Let E be a locally convex space and $\gamma : [a,b] \to E$ be such that $\alpha \circ \gamma \in \mathcal{L}^1([a,b],\mathbb{R})$ for each $\alpha \in E'$. An element $z \in E$ is called the *weak integral of* γ if

$$\alpha(z) = \int_a^b (\alpha \circ \gamma)(s)ds,$$

for each $\forall \alpha \in E'$. Then z is called the weak integral of γ from a to b, and we write $z =: \int_a^b \gamma(s)ds$.

Definition 2.1.9 Let E be a locally convex space. We say that a sequence $(x_n)_n \subset E$ is a Cauchy sequence if for each $\varepsilon > 0$ and each continuous seminorm q of E, there exists an integer $N_{\varepsilon,q} \in \mathbb{N}$, such that for all $m, n \geq N_{\varepsilon,q}$ we have

$$q(x_m - x_n) < \varepsilon.$$

We say that E is sequentially complete if every Cauchy sequence converge in E.

The following lemma [35, Proposition 2.26] allows us to define absolute continuous functions with vector values.

Lemma 2.1.10 *If E is sequentially complete locally convex space, then for each $\gamma \in \mathcal{L}^1([a,b],E)$, the weak integral $\int_a^b \gamma(s)ds$ exists and the map*

$$\eta : [a,b] \to E, \quad \eta(t) = \int_a^t \gamma(s)ds$$

is continuous.

A related important result of weak integrals is the Mean Value Theorem (see e.g. [14]).

Theorem 2.1.11 *Let E and F be locally convex spaces, $U \subseteq E$ be an open subset, $f : U \to F$ a C^1-map and $x, y \in U$ such that the line segment $\{tx + (1-t)y \in E : t \in [0,1]\}$ is contained in U. Then*

$$f(y) - f(x) = \int_0^1 df(x + t(y-x), y-x)dt.$$

Definition 2.1.12 Let E be a sequentially complete locally convex space and $p \in [1,\infty]$. For $t_0 \in [a,b]$, we say that a function $\eta : [a,b] \to E$ is L^p-absolutely continuous (or just absolutely continuous if there is no confusion) if there exists a $[\gamma] \in L^p([a,b],E)$ such that

$$\eta(t) = \eta(t_0) + \int_{t_0}^t \gamma(s)ds, \quad t \in [a,b]. \tag{2.1.1}$$

We denote the space of all L^p-absolutely continuous functions by $AC_{L^p}([a,b],E)$. Let $t_0 \in [a,b]$ be fixed, since $\eta' := [\gamma]$ is necessarily unique (see [34, Lemma 2.28]), the map

$$\Phi : AC_{L^p}([a,b],E) \to E \times L^p([a,b],E), \quad \eta \mapsto (\eta(t_0), \eta') \tag{2.1.2}$$

is an isomorphism of vector spaces. We endow $AC_{L^p}([a,b],E)$ with the Hausdorff locally convex vector topology which makes Φ an isomorphism of topological vector space (see [34, Definition 3.1]).

The following result will allow us to study differentiability of functions with values in $AC_{L^p}([a,b],E)$.

Lemma 2.1.13 *Let E be a sequentially complete locally convex space and $p \in [1,\infty]$. Then the map*

$$\Psi : AC_{L^p}([a,b],E) \to C([a,b],E) \times L^p([a,b],E), \quad \eta \mapsto (\eta, \eta') \tag{2.1.3}$$

is a linear topological embedding with closed image.

Proof. We let $I : E \times L^p([a,b],E) \to C([a,b],E)$ be the continuous map given by

$$I((x,[\gamma])(t) := x + \int_a^t \gamma(s)ds, \quad t \in [a,b]$$

for each $x \in E$ and $[\gamma] \in L^p([a,b],E)$. Let $\Phi : AC_{L^p}([a,b],E) \to E \times L^p([a,b],E)$, with $t_0 := a$, be the isomorphism of topological vector spaces as above. We consider the map

$$\Theta : E \times L^p([a,b],E) \to C([a,b],E) \times L^p([a,b],E), \quad (x,[\gamma]) \mapsto \big(I(x,[\gamma]),[\gamma]\big)$$

which is continuous.
Moreover, since the evaluation map $\varepsilon_a : C([a,b],E) \to E$, $\eta \mapsto \eta(a)$ is continuous, the map $\big(\Theta|^{\mathrm{Im}(\Theta)}\big)^{-1} = (\varepsilon_a, \mathrm{id}_{L^p})$ is also continuous. Hence Ψ is a topological embedding.
Let $(\eta_\alpha, \eta'_\alpha)_\alpha$ be a net in $\mathrm{Im}(\Theta)$ that converges to $(\eta,[\gamma]) \in C([a,b],E) \times L^p([a,b],E)$. By continuity of ε_a, the net $(\eta_\alpha(a))_\alpha$ converges to $\eta(a) \in E$, and by continuity of Θ, the net $(\Theta(\eta_\alpha(a), \eta'_\alpha))_\alpha$ converges to $(I(\eta(a),[\gamma]),[\gamma]) \in \mathrm{Im}(\Theta)$. Since the net $(\Theta(\eta_\alpha(a), \eta'_\alpha))_\alpha$ also converges to $(\eta,[\gamma])$, we have

$$I(\eta(a),[\gamma]) = \eta.$$

Therefore $\eta' = [\gamma]$ and $(\eta,[\gamma]) \in \mathrm{Im}(\Theta)$. □

Remark 2.1.14 Let $p \in [1,\infty]$. Since the inclusion map $AC_{L^p}([a,b],E) \to C([a,b],E)$ is continuous [35, Lemma 3.2], the topology on $AC_{L^p}([a,b],E)$ is independent of the choice of t_0 and finer than the compact-open topology. Hence the sets

$$AC_{L^p}([a,b],V) := \{\eta \in AC_{L^p}([a,b],E) : \eta([a,b]) \subseteq V\}$$

are open on $AC_{L^p}([a,b],E)$, for each open subset $V \subseteq E$.

For maps between absolute continuous function spaces, we have the following results (see [35]).

Lemma 2.1.15 *Let E be a sequentially complete locally convex space and $p \in [1, \infty]$. For $c, d \in \mathbb{R}$ with $c < d$ we define a map $g : [c, d] \to [a, b]$ via*

$$g(t) = a + \frac{t-c}{d-c}(b-a), \quad t \in [c, d].$$

Then $\eta \circ g \in AC_{L^p}([c, d], E)$ for each $\eta \in AC_{L^p}([a, b], E)$ and the map

$$AC_{L^p}(g, E) : AC_{L^p}([a, b], E) \to AC_{L^p}([c, d], E), \quad \eta \mapsto \eta \circ g$$

is continuous linear.

Lemma 2.1.16 *Let E and F be sequentially complete locally convex spaces, $p \in [1, \infty]$, $V \subseteq E$ be open subset and $f : V \to F$ be a C^1-map. Then $f \circ \eta \in AC_{L^p}([a, b], F)$ for each $\eta \in AC_{L^p}([a, b], V)$.*

Lemma 2.1.17 *Let E and F be sequentially complete locally convex spaces, $p \in [1, \infty]$ and $k \in \mathbb{N} \cup \{0, \infty\}$. Let $V \subseteq E$ be open subset and $f : V \to F$ be a C^{k+2}-map, then the map*

$$f_* := AC_{L^p}([a, b], f) : AC_{L^p}([a, b], V) \to AC_{L^p}([a, b], F), \quad \eta \mapsto f \circ \eta$$

is C^k. Moreover, we have

$$d(f_*)(\eta, \eta_1) = df \circ (\eta, \eta_1)$$

for all $(\eta, \eta_1) \in AC_{L^p}([a, b], V) \times AC_{L^p}([a, b], E)$.

Remark 2.1.18 For sequentially complete locally convex spaces E and F we have

$$AC_{L^p}([a, b], E \times F) \cong AC_{L^p}([a, b], E) \times AC_{L^p}([a, b], F).$$

Definition 2.1.19 Let E and F be complex topological vector spaces, where is F is locally convex, $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping. We say that f is complex analytic if it is continuous and, for each $x \in U$, there exists a 0-neighbouhood $V \subseteq E$ such that $x + V \subseteq U$ and certain continuous homogeneous polynomials $\beta_n : E \to F$ of degree n, such that f admits the expansion: $f(x + y) = \sum_{n=0}^{\infty} \beta_n(y)$, for all $y \in V$.

For our context, we present an application of [4, Proposition 7.7] to our particular case.

Lemma 2.1.20 *Let E and F be complex locally convex spaces, and $f : U \to F$ be a mapping defined on an open subset of E. Then f is complex analytic if and only if f is smooth and the mapping $df(x, \cdot) : E \to F$ is complex linear for each $x \in U$.*

Definition 2.1.21 Let E and F be real locally convex spaces, $U \subseteq E$ be an open subset and $f : U \to F$ be a map. We say that f is real analytic if it extends to an analytic map $f : V \to F_{\mathbb{C}}$ on some open neighborhood V of U in $E_{\mathbb{C}}$, where $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ denotes the complexification of E and F, respectively.

Lemma 2.1.22 *Let E and F be sequentially complete locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in [1, \infty]$, $V \subseteq E$ be an open set and $f : V \to F$ be a $\mathbb{K}$-analytic map. Then the map*

$$f_* := AC_{L^p}([a, b], f) : AC_{L^p}([a, b], V) \to AC_{L^p}([a, b], F), \quad \eta \mapsto f \circ \eta$$

is $\mathbb{K}$-analytic.

Proof. First we consider the case $\mathbb{K} = \mathbb{C}$. By Lemma 2.1.17 the map f_* is smooth and $d(f_*)$ is complex linear in the second variable, hence by Lemma 2.1.20 the map f_* is complex analytic.
If $\mathbb{K} = \mathbb{R}$, then by definition the map f has a complex analytic extension $\tilde{f}$, hence $(\tilde{f})_*$ is the complex analytic extension of f_*, whence f_* is real analytic. $\square$

Definition 2.1.23 Let N be a smooth manifold modeled with on a sequentially complete locally convex space E and $p \in [1, \infty]$. We say that a function $\eta : [a, b] \to N$ is L^p-absolutely continuous if it is continuous and there exists a partition $\{t_0, ..., t_n\}$ of $[a, b]$ such that for each $j \in \{1, ..., n\}$, there exists a chart $\varphi_j : U_j \to V_j$ that verifies

i) $\eta([t_{j-1}, t_j]) \subseteq U_j$.

ii) $\varphi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E)$.

In this case, we say that these charts verify the definition of L^p-absolute continuity for η. If there is no confusion, we simple call η absolutely continuous.

For the case of absolutely continuous functions with values in a manifold, the following facts are available (see [35]).

Lemma 2.1.24 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E and $p \in [1, \infty]$. If $\eta \in AC_{L^p}([a, b], N)$, then*

$$\varphi \circ \eta|_{[\alpha, \beta]} \in AC_{L^p}([\alpha, \beta], E),$$

for each chart $\varphi : U \to V$ of N and each subinterval $[\alpha, \beta] \subseteq [a, b]$ such that $\eta([\alpha, \beta]) \subseteq U$.

Lemma 2.1.25 *Let M and N be smooth manifolds modeled on sequentially complete locally convex spaces and $p \in [1, \infty]$. If $f : M \to N$ is a C^1-map, then $f \circ \eta \in AC_{L^p}([a, b], N)$ for each $\eta \in AC_{L^p}([a, b], M)$.*

2.2 The space of absolutely continuous sections

Definition 2.2.1 Let N be a smooth manifold modeled on a sequentially complete locally convex space E, $\pi_{TN} : TN \to N$ its tangent bundle and $p \in [1, \infty]$. For $\eta \in AC_{L^p}([a, b], N)$ we define the set

$$\Gamma_{AC}(\eta) := \{\sigma \in AC_{L^p}([a, b], TN) : \pi_{TN} \circ \sigma = \eta\} \tag{2.2.1}$$

and we endow it with the pointwise operations, making it a vector space.
Consider a partition $P_n = \{t_0, ..., t_n\}$ of $[a, b]$ and charts $\{(\varphi_i, U_i) : i \in \{1, ..., n\}\}$ of N that verify the definition of absolute continuity for η. Since $\eta([t_{i-1}, t_i]) \subseteq U_i$, we have

$$\sigma([t_{i-1}, t_i]) \subseteq TU_i, \quad \text{for all } \sigma \in \Gamma_{AC}(\eta)$$

for each $i \in \{1, ..., n\}$. We endow $\Gamma_{AC}(\eta)$ with the Hausdorff locally convex vector topology which is the initial topology with respect to the linear mappings

$$h_i : \Gamma_{AC}(\eta) \to AC_{L^p}([t_{i-1}, t_i], E), \quad \sigma \mapsto h_i(\sigma) = d\varphi_i \circ \sigma|_{[t_{i-1}, t_i]} \tag{2.2.2}$$

with $i \in \{1, ..., n\}$.

Proposition 2.2.2 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E, $p \in [1,\infty]$ and $\eta \in AC_{L^p}([a,b],N)$. Let $P = \{t_0,...,t_n\}$ be a partition of $[a,b]$ and $\{(\varphi_i,U_i) : i \in \{1,...,n\}\}$ be charts of N that verify the definition of absolute continuity for η, then the map*

$$\Phi_{\eta,P} : \Gamma_{AC}(\eta) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],E), \quad \sigma \mapsto \left(d\varphi_i \circ \sigma|_{[t_{i-1},t_i]}\right)_{i=1}^{n} \tag{2.2.3}$$

is a linear topological embedding with closed image given by the set of all elements $(\tau_i)_{i=1}^n$ such that

$$\tau_i(t_i) = d\varphi_i \circ (T\varphi_{i+1})^{-1}\Big(\varphi_{i+1}\circ\eta(t_i), \tau_{i+1}(t_i)\Big), \quad \textit{for all } i \in \{1,...,n-1\}.$$

Proof. The linear map $\Phi_{\eta,P}$ is continuous by the previous definition and it is injective. Let $i \in \{1,...,n\}$. If $W_i = \varphi_{i+1}(U_i \cap U_{i+1})$, then the map

$$g_i : W_i \times E \to E, \quad (x,y) \mapsto d\varphi_i \circ (T\varphi_{i+1})^{-1}(x,y),$$

is continuous and linear in the second component. This enables us to define the closed vector subspace K given by the elements

$$(\tau_i)_{i=1}^n \in \prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],E)$$

which verify

$$\tau_i(t_i) = d\varphi_i \circ (T\varphi_{i+1})^{-1}\Big(\varphi_{i+1}\circ\eta(t_i), \tau_{i+1}(t_i)\Big), \quad \text{for all } i \in \{1,...,n-1\}$$

We will show that the image of $\Phi_{\eta,P}$, denoted by $\mathrm{Im}(\Phi_{\eta,P})$, coincides with the closed subspace K. Indeed, the space $\mathrm{Im}(\Phi_{\eta,P})$ is contained in K by definition of $\Phi_{\eta,P}$. Let us consider now $\tau = (\tau_i)_{i=1}^n \in K$. We define the maps

$$\sigma_i : [t_{i-1},t_i] \to TN, \quad s \mapsto (T\varphi_i)^{-1}\Big(\varphi_i\circ\eta(s), \tau_i(s)\Big)$$

and

$$\sigma_\tau : [a,b] \to TN, \quad t \mapsto \sigma_i(t), \text{ for } t \in [t_{i-1},t_i].$$

By the Glueing Lemma, the map σ_τ is continuous. Moreover, by Lemma 2.1.25 each function σ_i is absolutely continuous, hence σ it is too. Since $\pi_{TN}\circ\sigma_\tau = \eta$ we have that $\sigma_\tau \in \Gamma_{AC}(\eta)$ and $\Phi_{\eta,P}(\sigma_\tau) = (\tau_i)_{i=1}^n$, and $\mathrm{Im}(\Phi_{\eta,P}) = K$.
It remains to show that the inverse map

$$\Phi_{\eta,P}^{-1} : \mathrm{Im}(\Phi_{\eta,P}) \to \Gamma_{AC}(\eta), \quad (\tau_i)_{i=1}^n \mapsto \sigma_\tau$$

is continuous. If h_i are the functions that define the topology (Definition 2.2.1), then for each $i \in \{1,...,n\}$ we have $h_i \circ \Phi_{\eta,P}^{-1} = q_i$, where q_i is the continuous linear map

$$q_i : \prod_{j=1}^{n} AC_{L^p}([t_{j-1},t_j],E) \to AC_{L^p}([t_{i-1},t_i],E), \quad (\eta_j)_{j=1}^n \mapsto \eta_i.$$

Hence $\Phi_{\eta,P}^{-1}$ is continuous. □

Remark 2.2.3 From now we consider the map $\Phi_{\eta,P}$ as the homeomorphism

$$\Phi_{\eta,P} : \Gamma_{AC}(\eta) \to \mathrm{Im}(\Phi_{\eta,P}).$$

Corollary 2.2.4 *Let N be a smooth manifold modeled on a E Banach space (resp. Frechet space), $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], N)$. Then the vector space $\Gamma_{AC}(\eta)$ is a Banach space (resp. Frechet space).*

Proof. This follows from the fact that each vector space $AC_{L^p}([t_{i-1}, i], E)$ is a Banach space (resp. Frechet space) □

Proposition 2.2.5 *Let M and N be smooth manifolds modeled on sequentially complete locally convex spaces, $k \in \mathbb{N} \cup \{0, \infty\}$, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], M)$. If $f : M \to N$ is a C^{k+3}-map, then $Tf \circ \sigma \in \Gamma_{AC}(f \circ \eta)$ for each $\sigma \in \Gamma_{AC}(\eta)$. Moreover, the map*

$$\widetilde{f} : \Gamma_{AC}(\eta) \to \Gamma_{AC}(f \circ \eta), \quad \sigma \mapsto Tf \circ \sigma$$

is continuous linear.

Proof. Let E_M and E_N be the modeling spaces of M and N respectively. By Lemma 2.1.16 we have $f \circ \eta \in AC_{L^p}([a, b], N)$ and $Tf \circ \sigma \in AC_{L^p}([a, b], TN)$ for each $\sigma \in \Gamma_{AC}(\eta)$. Since $T_{\eta(t)} f\,(\sigma(t)) \in T_{f\circ\eta(t)} N$ for each $t \in [a, b]$, we have

$$\pi_{TN} \circ (Tf \circ \sigma) = f \circ \eta.$$

Thus $\widetilde{f}(\sigma) \in \Gamma_{AC}(f \circ \eta)$ for each $\sigma \in \Gamma_{AC}(\eta)$. The linearity of $\widetilde{f}$ is trivial.
Without loss of generality, we can choose a partition $P = \{t_0, ..., t_n\}$ of $[a, b]$ such that there exist families of charts $\{(\varphi_i, U_i) : i \in \{1, ..., n\}\}$ and $\{(\phi_i, V_i) : i \in \{1, .., n\}\}$ that verify the definition of absolute continuity for η and $f \circ \eta$ respectively, such that

$$f(U_i) \subseteq V_i, \quad \text{for each } i \in \{1, ..., n\}.$$

For $\sigma \in \Gamma_{AC}(\eta)$ and for each $i \in \{1, ..., n\}$ we denote $\eta_i = \eta|_{[t_{i-1},t_i]}$ and $\sigma_i = \sigma|_{[t_{i-1},t_i]}$. We define the maps

$$F_i : AC_{L^p}([t_{i-1}, t_i], E_M) \to AC_{L^p}([t_{i-1}, t_i], E_N), \quad \tau \mapsto \left(d\phi_i \circ Tf \circ T\varphi_i^{-1}\right) \circ (\varphi_i \circ \eta_i, \tau)$$

and

$$F : \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_M) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_N), \quad (\tau_i)_{i=1}^n \mapsto (F_i(\tau_i))_{i=1}^n$$

which are continuous by Lemma 2.1.17. We will show that

$$F\left(\mathrm{Im}\left(\Phi_{\eta,P}\right)\right) \subseteq \mathrm{Im}\left(\Phi_{f\circ\eta,P}\right).$$

Let $i \in \{1,..,n-1\}$. If

$$\begin{aligned}\tau_i :&= F_i(d\varphi_i \circ \sigma_i)\\ &= (d\phi_i \circ Tf \circ d\varphi_i^{-1})(\varphi_i \circ \eta_i, d\varphi_i \circ \sigma_i)\\ &= d\phi_i \circ Tf \circ \sigma_i\end{aligned}$$

then

$$\begin{aligned}d\phi_i \circ (T\phi_{i+1})^{-1}\Big(\phi_{i+1} \circ (f \circ \eta)(t_i), \tau_{i+1}(t_i)\Big) &= d\phi_i \circ (T\phi_{i+1})^{-1}\Big(\phi_{i+1} \circ (f \circ \eta)(t_i), d\phi_{i+1} \circ Tf \circ \sigma_{i+1}(t_i)\Big)\\ &= d\phi_i \circ Tf \circ \sigma_{i+1}(t_i)\\ &= d\phi_i \circ Tf \circ \sigma_i(t_i)\\ &= \tau_i(t_i)\end{aligned}$$

Hence $F \circ \Phi_{\eta,P}(\sigma) \in \operatorname{Im}(\Phi_{f\circ\eta,P})$ and in consequence

$$\widetilde{f} = \Phi_{f\circ\eta,P}^{-1} \circ F \circ \Phi_{\eta,P}.$$

Thus $\widetilde{f}$ is continuous. □

Remark 2.2.6 The topology of $\Gamma_{AC}(\eta)$ does not depend on the partition or charts chosen. Indeed, since the identity map $\mathrm{id}_M : M \to M$ is smooth, by the previous proposition the map

$$\widetilde{\mathrm{id}_M} : \Gamma_{AC}(\eta) \to \Gamma_{AC}(\mathrm{id}_M \circ \eta), \quad \sigma \mapsto T\mathrm{id}_M \circ \sigma$$

is smooth regardless of the partition or charts chosen. Moreover, this map coincides with the identity map $\mathrm{id}_\Gamma : \Gamma_{AC}(\eta) \to \Gamma_{AC}(\eta)$, $\sigma \mapsto \sigma$.

Remark 2.2.7 For $\eta \in C([a,b],N)$ we endow the vector space

$$\Gamma_C(\eta) = \{\sigma \in C([a,b],TN) : \pi_{TN} \circ \sigma = \gamma\}$$

with the compact-open topology. Since each inclusion

$$AC_{L^p}([t_{i-1},t_i],E) \to C([t_{i-1},t_i],E)$$

is continuous [35, Lemma 3.2], the inclusion map $J_\Gamma : \Gamma_{AC}(\gamma) \to \Gamma_C(\gamma)$ is also continuous. This implies that set

$$\mathcal{V} := \{\sigma \in \Gamma_{AC}(\eta) : \sigma([a,b]) \subseteq V\}$$

is open in $\Gamma_{AC}(\eta)$ for each open subset $V \subseteq TN$.

Proposition 2.2.8 *Let N_1 and N_2 be smooth manifolds modeled on sequentially complete locally convex spaces, $p \in [1,\infty]$ and $\mathrm{pr}_i : N_1 \times N_2 \to N_i$ be the i-projection for $i \in \{1,2\}$. If $\eta_1 \in AC_{L^p}([a,b],N_1)$ and $\eta_2 \in AC_{L^p}([a,b],N_2)$, then the map*

$$\mathcal{P} : \Gamma_{AC}(\eta_1,\eta_2) \to \Gamma_{AC}(\eta_1) \times \Gamma_{AC}(\eta_1), \quad \sigma \mapsto (T\,\mathrm{pr}_1, T\,\mathrm{pr}_2)(\sigma)$$

is a linear homeomorphism.

Proof. By Proposition 2.2.5 the map $\mathcal{P}$ is continuous and clearly linear. Let $P = \{t_0, ..., t_n\}$ be a partition of $[a,b]$; let $\{(\phi_{1,i}, U_{1,i}) \colon i \in \{1,\ldots,n\}\}$ and $\{(\phi_{2,i}, U_{2,i}) \colon i \in \{1,\ldots,n\}\}$ be families of charts of N_1 and N_2, respectively, that verify the definition of absolute continuity for η_1 and η_2, respectively. Then $\eta := (\eta_1, \eta_2) \colon [a,b] \to N_1 \times N_2$ is L^p-absolutely continuous and it is clear that the charts $\{(\phi_{1,i} \times \phi_{2,i}, U_{1,i} \times U_{2,i}) \colon i \in \{1,\ldots,n\}\}$ satisfy the condition of absolute continuity for η. For $j \in \{1,2\}$, consider the linear topological embedding

$$\Phi_{\eta_j,P} \colon \Gamma_{\eta_j} \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_j), \quad \tau \mapsto (d\phi_{j,i} \circ \tau|_{[t_{i-1},t_i]})_{i=1}^{n}$$

where E_j is the modeling space of N_j. Also

$$\Phi_{\eta,P} \colon \Gamma_{\eta} \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_1 \times E_2), \quad \tau \mapsto ((d\phi_{1,i} \times d\phi_{2,i}) \circ \tau|_{[t_{i-1},t_i]})_{i=1}^{n}$$

is a linear topological embedding; here $T_{(x_1,x_2)}(N_1 \times N_2)$ is identified with $T_{x_1}N_1 \times T_{x_2}N_2$. For $i \in \{1,\ldots,n\}$, let

$$\alpha_i \colon AC_{L^p}([t_{i-1}, t_i], E_1) \times AC_{L^p}([t_{i-1}, t_i], E_2) \to AC_{L^p}([t_{i-1}, t_i], E_1 \times E_2)$$

be the map taking a pair (f_1, f_2) of functions to the function $t \mapsto (f_1(t), f_2(t))$; we know that α_i is an isomorphism of topological vector spaces. Then also

$$\alpha \colon \left(\prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_1)\right) \times \left(\prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_2)\right) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], E_1 \times E_2),$$
$$((f_i)_{i=1}^{n}, (g_i)_{i=1}^{n}) \mapsto (\alpha(f_i, g_i))_{i=1}^{n}$$

is an isomorphism of topological vector spaces. If $(f_i)_{i=1}^{n}$ is in the image of $\Phi_{\eta_1,P}$ and $(g_i)_{i=1}^{n}$ is in the image of $\Phi_{\eta_2,P}$, then $\alpha((f_i)_{i=1}^{n}, (g_i)_{i=1}^{n})$ is in the image of $\Phi_{\eta,P}$, as the compatibility at the endpoints can be checked by considering the components in E_1 and E_2. We can therefore define a function

$$\Theta := \Phi_{\eta,P}^{-1} \circ \alpha \circ (\Phi_{\eta_1,P} \times \Phi_{\eta_2,P}) \colon \Gamma_{\eta_1} \times \Gamma_{\eta_2} \to \Gamma_{\eta},$$

which is continuous and linear. We readily check that $\mathcal{P}(\Theta(\sigma,\tau)) = (\sigma,\tau)$ for all $\sigma \in \Gamma_{\eta_1}$ and $\tau \in \Gamma_{\eta_2}$. Hence $\mathcal{P}$ is surjective and thus bijective, with $\mathcal{P}^{-1} = \Theta$ a continuous map. □

Proposition 2.2.9 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E, $p \in [1,\infty]$ and $\eta \in AC_{L^p}([a,b], N)$. For a partition $P = \{t_0, ..., t_n\}$ of $[a,b]$, the map*

$$\rho : \Gamma_{AC}(\eta) \to \prod_{i=1}^{n} \Gamma_{AC}\left(\eta|_{[t_{i-1},t_i]}\right), \quad \sigma \mapsto (\sigma|_{[t_{i-1},t_i]})_{i=1}^{n}$$

is a linear topological embedding with closed image.

Proof. Let $j \in \{1, ..., n\}$, If $P_j = \{t_{j-1}, t_j\}$, since $\eta_j := \eta|_{[t_{j-1},t_j]} \in AC_{L^p}([t_{j-1}, t_j], N)$, we have

$$\rho = \left(\prod_{i=1}^{n} \Phi_{\eta_i, P_i}^{-1} \right) \circ \Phi_{\eta,P}.$$

The image is given by the closed subspace

$$\mathrm{Im}(\rho) = \left\{ (\tau_i)_{i=1}^n \in \prod_{i=1}^{n} \Gamma_{AC}(\eta|_{[t_{i-1},t_i]}) : \tau_i(t_i) = \tau_{i+1}(t_i) \text{ for all } i \in \{1, ..., n-1\} \right\}.$$

Thus $(\rho|^{\mathrm{Im}(\rho)})^{-1} : \mathrm{Im}(\rho) \mapsto \Gamma_{AC}(\eta)$ is well defined and

$$\left(\rho|^{\mathrm{Im}(\rho)}\right)^{-1} = \Phi_{\eta,P}^{-1} \circ \left(\prod_{i=1}^{n} \Phi_{\eta_i,P_i} \right).$$

□

Proposition 2.2.10 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E and $p \in [1, \infty]$. If $g : [c, d] \to [a, b]$ is the map as in Lemma 2.1.15, then $\eta \circ g \in AC_{L^p}([c, d], N)$ for each $\eta \in AC_{L^p}([a, b], N)$. Moreover, if $\eta \in AC_{L^p}([a, b], N)$, then the map*

$$L_g : \Gamma_{AC}(\eta) \to \Gamma_{AC}(\eta \circ g), \quad \sigma \mapsto \sigma \circ g$$

is continuous linear.

Proof. Let $\eta \in AC_{L^p}([a, b], N)$. We will show first that $\eta \circ g \in AC_{L^p}([c, d], N)$. Let $P = \{t_0, ..., t_n\}$ be a partition of $[a, b]$ and $\{(\varphi_i, U_i) : i \in \{1, ..., n\}\}$ charts of N that verify the definition of absolute continuity for η. Since g is a strictly increasing function, we can define a partition $Q = \{s_1, ..., s_n\}$ of $[c, d]$ such that $g(s_i) = t_i$, for each $i \in \{1, ..., n\}$. Moreover, given that

$$\varphi_i \circ (\eta \circ g)|_{[s_{i-1},s_i]} = \left(\varphi_i \circ \eta|_{[t_{i-1},t_i]}\right) \circ g|_{[s_{i-1},s_i]}$$

we have $\eta \circ g \in AC_{L^p}([a, b], E)$ by Lemma 2.1.15. Analogously, we have that $\sigma \circ g \in AC_{L^p}([a, b], TN)$ for each $\sigma \in \Gamma_{AC}(\eta)$ and

$$\pi_{TN} \circ (\sigma \circ g) = \eta \circ g.$$

Hence $L_g(\sigma) \in \Gamma_{AC}(\eta \circ g)$. To see the continuity of L_g, for each $i \in \{1, ..., n\}$ we define the maps

$$G_i : AC_{L^p}([t_{i+1}, t_i], E) \to AC_{L^p}([s_{i+1}, s_i], E), \quad \tau \mapsto \tau \circ g|_{[s_{i+1},s_i]}$$

which are continuous by Lemma 2.1.15. Considering the topological embeddings $\Phi_{\eta,P}$ and $\Phi_{\eta\circ g,Q}$ (as in Proposition 2.2.2) with the same family of charts, if $(\tau_i)_{i=1}^n \in \mathrm{Im}(\Phi_{\eta,P})$

we have

$$\begin{aligned}\tau_i \circ g(s_i) &= \tau_i(t_i)\\ &= d\varphi_i \circ (T\varphi_{i+1})^{-1}\Big(\varphi_{i+1} \circ \eta(t_i), \tau_{i+1}(t_i)\Big)\\ &= d\varphi_i \circ (T\varphi_{i+1})^{-1}\Big(\varphi_{i+1} \circ \eta \circ g(s_i), \tau_{i+1} \circ g(s_i)\Big)\end{aligned}$$

for each $i \in \{1, ..., n-1\}$. If $G = (G_1 \times ... \times G_n)$, then $(G \circ \Phi_{\eta,P})(\sigma) \in \mathrm{Im}(\Phi_{\eta\circ g,Q})$ and

$$L_g = \Phi^{-1}_{\eta\circ g,Q} \circ G \circ \Phi_{\eta,Q}.$$

Hence L_g is continuous and clearly linear. □

Proposition 2.2.11 *Let N be a smooth manifold modeled on sequentially complete locally convex space E, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], N)$. Then the evaluation map*

$$\epsilon : \Gamma_{AC}(\eta) \times [a, b] \to TN, \quad (\sigma, t) \mapsto \sigma(t)$$

is continuous, and linear in the first argument. Moreover, for each $t \in [a, b]$, the point evaluation map

$$\epsilon_t : \Gamma_{AC}(\eta) \to TN, \quad \sigma \mapsto \sigma(t)$$

is smooth.

Proof. The evaluation map

$$\tilde{\epsilon} : \Gamma_C(\eta) \times [a, b] \to TN, \quad (\sigma, t) \mapsto \sigma(t)$$

is continuous and the evaluation map $\tilde{\epsilon}_t : \Gamma_C(\eta) \to TN$, $\sigma \mapsto \sigma(t)$ is smooth for each $t \in [a, b]$ (see [3]). Then $\epsilon = \tilde{\epsilon} \circ (J_\Gamma \times \mathrm{Id}_\mathbb{R})$ and $\epsilon_t = \tilde{\epsilon}_t \circ J_\Gamma$, where $J_\Gamma : \Gamma_{AC}(\eta) \to \Gamma_C(\eta)$ is the inclusion map, which is continuous linear by Remark 2.2.7. □

2.3 Manifolds of absolutely continuous functions

Definition 2.3.1 Let N be a smooth manifold and $\pi_{TN} : TN \to N$ its tangent bundle. A *local addition* is a smooth map $\Sigma : \Omega \to N$ defined on a open neighborhood $\Omega \subseteq TN$ of the zero-section $0_N := \{0_p \in T_pN : p \in N\}$ such that

a) $\Sigma(0_p) = p$ for all $p \in N$.

b) The image $\Omega' := \big(\pi_{TN}, \Sigma\big)(\Omega)$ is open in $N \times N$ and the map

$$\theta_N : \Omega \to \Omega', \quad v \mapsto \big(\pi_{TN}(v), \Sigma(v)\big) \tag{2.3.1}$$

is a C^∞-diffeomorphism.

Moreover, if $T_{0_p}(\Sigma|_{T_pN}) = id_{T_pN}$ for all $p \in N$, we say that the local addition Σ is *normalized*. We denote the local addition as the pair (Ω, Σ).
If $\theta_N : \Omega \to \Omega'$ is a diffeomorphism of $\mathbb{K}$-analytic manifolds, we call $\Sigma : \Omega \to N$ a $\mathbb{K}$-analytic local addition.

Remark 2.3.2 If a smooth manifold N admits a local addition, then also its tangent manifold TN admits a local addition [3, Lemma A.11]. Moreover, each manifold which admits a local addition also admits a normalized local addition [3, Lemma A.14]. From now we will assume that each local addition is normalized.

Remark 2.3.3 Let N be a smooth manifold modeled on a sequentially complete locally convex space E, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], N)$. We define the sets

$$\mathcal{V}_\eta := \{\sigma \in \Gamma_{AC}(\eta) : \sigma([a, b]) \subseteq \Omega\}. \tag{2.3.2}$$

which is open in $\Gamma_{AC}(\eta)$ by Remark 2.2.7 and

$$\mathcal{U}_\eta := \{\gamma \in AC_{L^p}([a, b], N) : (\eta, \gamma)([a, b]) \subseteq \Omega'\}. \tag{2.3.3}$$

Lemma 2.1.25 enable us to define the map

$$\Psi_\eta := AC_{L^p}([a, b], \Sigma) : \mathcal{V}_\eta \to \mathcal{U}_\eta, \quad \sigma \mapsto \Sigma \circ \sigma. \tag{2.3.4}$$

with inverse given by

$$\Psi_\eta^{-1} : \mathcal{U}_\eta \to \mathcal{V}_\eta, \quad \gamma \mapsto \theta_N^{-1} \circ (\eta, \gamma). \tag{2.3.5}$$

The following lemma is just an application of [4, Lemma 10.1] to our particular case.

Lemma 2.3.4 *Let E and F sequentially complete locally convex spaces, $U \subseteq E$ open and $f : U \to F$ a map. If $F_0 \subseteq F$ is a closed vector subspace and $f(U) \subseteq F_0$, then $f : U \to F$ is smooth if and only if $f|^{F_0} : U \to F_0$ is smooth.*

Theorem 2.3.5 *For each smooth manifold N modeled on a sequentially complete locally convex space E which admits a local addition and $p \in [1, \infty]$, the set $AC_{L^p}([a, b], N)$ admits a smooth manifold structure such that the sets $\mathcal{U}_\eta$ are open in $AC_{L^p}([a, b], N)$ for all $\eta \in AC_{L^p}([a, b], N)$ and $\Psi_\eta \colon \mathcal{V}_\eta \to \mathcal{U}_\eta$ is a C^∞-diffeomorphism.*

Proof. We endow $AC_{L^p}([a, b], N)$ with the final topology with respect to the family of maps $\Psi_\eta \colon \mathcal{V}_\eta \to \mathcal{U}_\eta$, for each $\eta \in AC_{L^p}([a, b], N)$. If we define the maps $\Psi_\eta^C \colon \mathcal{V}_\eta^C \to \mathcal{U}_\eta^C$ on the space of continuous functions $C([a, b], N)$ for each $\eta \in C([a, b], N)$ as in Remark 2.3.3, with $\mathcal{V}_\eta \subseteq \Gamma_C(\eta)$, then the final topology on $C([a, b], N)$ coincides with its compact open topology. Hence the inclusion map

$$J : AC_{L^p}([a, b], N) \to C([a, b], N), \quad \gamma \mapsto \gamma$$

is continuous. Moreover, for each $\eta \in AC_{L^p}([a, b], N)$ the set

$$\mathcal{U}_\eta^C := \{\gamma \in C([a, b], N) : (\eta, \gamma)([a, b]) \subseteq \Omega'\}$$

is open in $C([a, b], N)$, whence

$$\mathcal{U}_\eta = \mathcal{U}_\eta^C \cap AC_{L^p}([a, b], N)$$

is open in $AC_{L^p}([a,b],N)$.
The goal is to make the family $\{(\mathcal{U}_\eta, \Psi_\eta^{-1}) : \eta \in AC_{L^p}([a,b],N)\}$ an atlas for $AC_{L^p}([a,b],N)$ for a smooth manifold structure on $AC_{Lp}([a,b],N)$. We need to show that the charts are compatible, i.e., the smoothness of the map

$$\Lambda_{\xi,\eta} := \Psi_\xi^{-1} \circ \Psi_\eta : \Psi_\eta^{-1}(\mathcal{U}_\eta \cap \mathcal{U}_\xi) \subseteq \Gamma_{AC}(\eta) \to \Gamma_{AC}(\xi), \quad \sigma \mapsto \theta_N^{-1} \circ (\xi, \Sigma \circ \sigma) \tag{2.3.6}$$

for each $\eta, \xi \in AC_{L^p}([a,b],N)$ such that the open set

$$\Psi_\eta^{-1}(\mathcal{U}_\eta \cap \mathcal{U}_\xi) = \left(\Psi_\eta^C\right)^{-1}(\mathcal{U}_\eta^C \cap \mathcal{U}_\xi^C) \cap \Gamma_{AC}(\eta)$$

is not empty.
Let $R = \{t_0, ..., t_n\}$ be a partition of $[a,b]$, let $\{(U_{\varphi_i}, \varphi_i) : i \in \{1, ...n\}\}$ and $\{(U_{\phi_i}, \phi_i) : i \in \{1, ...n\}\}$ be charts that verify the definition of absolute continuity for η and ξ, respectively. Denoting $\sigma_i := \sigma|_{[t_{i-1},t_i]}$ for each $\sigma \in AC_{L^p}([a,b],TN)$. We will study the smoothness of the composition

$$\Phi_{\xi,R} \circ \Lambda_{\xi,\eta}|_{[t_{i-1},t_i]} : \Psi_\eta^{-1}(\mathcal{U}_\eta \cap \mathcal{U}_\xi) \to \mathrm{Im}(\Phi_{\xi,R}) \subset \prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],E)$$
$$\sigma \mapsto \left(d\phi_i \circ \Lambda_{\xi,\eta}|_{[t_{i-1},t_i]}(\sigma_i)\right)_{i=1}^{n}$$

that by Lemma 2.3.4, is equivalent to the smoothness of $\Lambda_{\xi,\eta}$. For each $i \in \{1, ..., n\}$, we denote $\eta_i := \eta|_{[t_{i-1},t_i]}$ and $\xi_i := \xi|_{[t_{i-1},t_i]}$ and we have

$$\begin{aligned} d\tilde{\phi_i} \circ \Lambda_{\xi,\eta}|_{[t_{i-1},t_i]}(\sigma_i) &= d\phi_i \circ \theta_N^{-1} \circ (\xi_i, \Sigma \circ \sigma_i) \\ &= d\phi_i \circ \theta_N^{-1} \circ \left(\phi_i^{-1}(\phi_i \circ \xi_i), \Sigma \circ T\varphi_i^{-1} \circ T\varphi_i \circ \sigma_i\right) \\ &= d\phi_i \circ \theta_N^{-1} \circ \left(\phi_i^{-1}(\phi_i \circ \xi_i), \Sigma \circ T\varphi_i^{-1}(\varphi_i \circ \eta_i, d\varphi_i \circ \sigma_i)\right). \end{aligned}$$

Because all of the functions involved are continuous and have an open domain, also the composition

$$H_i(x,y,z) := d\phi_i \circ \theta_N^{-1} \circ \left(\phi_i^{-1}(x), \Sigma \circ T\varphi_i^{-1}(y,z)\right) \tag{2.3.7}$$

has an open domain $\mathcal{O}_i$:. Hence the map $H_i : \mathcal{O}_i \to E$ is smooth. By Lemma 2.1.17, the map

$$AC_{L^p}([t_{i-1},t_i],H_i) : AC_{L^p}([t_{i-1},t_i],\mathcal{O}_i) \to AC_{L^p}([t_{i-1},t_i],E) \quad \alpha \mapsto H_i \circ \alpha$$

is smooth. Doing the identification of products of AC_{L^p} spaces (Remark 2.1.18), if we fix the functions $\phi_i \circ \xi_i$ and $\varphi_i \circ \eta_i$, we have the continuous affine-linear map

$$AC_{L^p}([t_{i-1},t_i],E) \to AC_{L^p}([t_{i-1},t_i],E \times E \times E), \quad \tau \mapsto (\phi_i \circ \xi_i, \varphi_i \circ \eta_i, \tau).$$

We write W_i for the preimage of $AC_{L^p}([t_{i-1},t_i],\mathcal{O}_i)$ under this map. Then the map

$$\Theta_i : W_i \to AC_{L^p}([t_{i-1},t_i],E), \quad \tau \mapsto H_i \circ (\phi_i \circ \xi, \varphi_i \circ \eta, d\varphi_i \circ \tau)$$

is also smooth. Since the maps $h_i : \Gamma_{AC}(\eta) \to AC_{L^p}([t_{i-1}, t_i], E), \sigma \mapsto d\phi_i \circ \sigma_i$ are continuous by definition of the topology, rewriting we have

$$\Phi_{\xi,R} \circ \Lambda_{\xi,\eta}(\sigma) = (\Theta_i \circ h_i(\sigma))_{i=1}^n$$

for each $\sigma \in \Psi_\eta^{-1}(\mathcal{U}_\eta \cap \mathcal{U}_\xi)$, hence $\Lambda_{\xi,\eta}$ is smooth. □

Proceding in the same way, using the fact that compositions of $\mathbb{K}$-analytic maps are $\mathbb{K}$-analytic and using the analytic version of Lemma 2.3.4 (see [14]), we obtain the analogous case.

Corollary 2.3.6 *For each $\mathbb{K}$-analytic manifold N modeled on a sequentially complete locally convex space E which admits a $\mathbb{K}$-analytic local addition and $p \in [1, \infty]$, the set $AC_{L^p}([a, b], N)$ admits a $\mathbb{K}$-analytic manifold structure such that the sets $\mathcal{U}_\eta$ are open in $AC_{L^p}([a, b], N)$ for all $\eta \in AC_{L^p}([a, b], N)$ and $\Psi_\eta \colon \mathcal{V}_\eta \to \mathcal{U}_\eta$ is a $\mathbb{K}$-analytic diffeomorphism.*

Proposition 2.3.7 *Let N be a smooth manifold modeled on a sequentially complete locally convex space which admits a local addition and $1 \leq q \leq p \leq \infty$. Then*

$$AC_{L^\infty}([a, b], N) \subseteq AC_{L^p}([a, b], N) \subseteq AC_{L^q}([a, b], N) \subseteq AC_{L^1}([a, b], N)$$

with smooth inclusion maps.

Proof. Let $\eta \in AC_{L^p}([a, b], N)$. Let $\{t_0, ..., t_n\}$ be a partition of $[a, b]$ and $\{(\varphi_i, U_i) : i \in \{1, ..., n\}\}$ be charts of N that verify the definition of absolute continuity for η.
By [35, Remark 3.2], we know that each inclusion map $AC_{L^p}([t_{i-1}, t_i], E) \to AC_{L^q}([t_{i-1}, t_i], E)$ is continuous linear, hence $AC_{L^p}([a, b], N) \subseteq AC_{L^q}([a, b], N)$. By Proposition 2.2.2, the inclusion is smooth since each map

$$\prod_{i=1}^n AC_{L^p}([t_{i-1}, t_i], E) \to \prod_{i=1}^n AC_{L^q}([t_{i-1}, t_i], E), \quad (\eta_i)_{i=1}^n \mapsto (\eta_i)_{i=1}^n.$$

is continuous linear. □

Proposition 2.3.8 *Let M and N be smooth manifolds modeled on sequentially complete locally convex spaces which admits a local addition, $p \in [1, \infty]$ and $k \in \mathbb{N} \cup \{0, \infty\}$. If $f : M \to N$ is a C^{k+2}-map, then the map*

$$AC_{L^p}([a, b], f) : AC_{L^p}([a, b], M) \to AC_{L^p}([a, b], N), \quad \eta \mapsto f \circ \eta$$

is C^k.

Proof. Let E_M and E_N be the modeleding space of M and N, respectively. Let (Ω_M, Σ_M) and (Ω_N, Σ_N) be local additions on M and N respectively. The map makes sense by Lemma 2.1.25. Let $\eta \in AC_{L^p}([a, b], M)$, $(\mathcal{U}_\eta, \Psi_\eta^{-1})$ and $(\mathcal{U}_{f\circ\eta}, \Psi_{f\circ\eta}^{-1})$ be charts around $\eta \in AC_{L^p}([a, b], M)$ and $f \circ \eta \in AC_{L^p}([a, b], N)$, respectively. We see that the set

$$\Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap AC_{L^p}([a, b], f)^{-1}(\mathcal{U}_{f\circ\eta})\right) = \Gamma_{AC}(\eta) \cap \left(\Psi_\eta^C\right)^{-1}\left(\mathcal{U}_\eta^C \cap AC_{L^p}([a, b], f)^{-1}(\mathcal{U}_{f\circ\eta}^C)\right)$$

is open in $\Gamma_{AC}(\eta)$. If $\theta_N = (\pi_{TN}, \Sigma_N)$, then we define

$$F(\sigma) := \Psi_{f\circ\eta}^{-1} \circ AC_{L^p}([a,b], f) \circ \Psi_\eta(\sigma) = \theta_N^{-1} \circ \Big((f\circ\eta), (f\circ\Sigma_M)\circ\sigma\Big)$$

for all $\sigma \in \Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap AC_{L^p}([a,b], f)^{-1}(\mathcal{U}_{f\circ\eta})\right)$.
Proceeding as the proof of Theorem 2.3.5, choosing the corresponding partition $P = \{t_0, ..., t_n\}$ of $[a,b]$ and the families of charts $\{(U_{\varphi_i}, \varphi_i) : i \in \{1,...,n\}\}$ and $\{(U_{\phi_i}, \phi_i) : i \in \{1,...,n\}\}$ that verify the definition of absolute continuity for η and $f\circ\eta$ respectively, we denote $\sigma_i = \sigma|_{[t_{i-1},t_i]}$. We will study the continuity of the map

$$\Phi_{f\circ\eta} \circ F : \Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap AC_{L^p}([a,b], f)^{-1}(\mathcal{U}_{f\circ\eta})\right) \to \mathrm{Im}(\Phi_{f\circ\eta}), \quad \sigma \mapsto (d\phi_i \circ F(\sigma_i))_{i=1}^n$$

where $\Phi_{f\circ\eta}$ is the topological embedding as in Proposition 2.2.2. For each $i \in \{1,...,n\}$ and $\sigma \in \Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap AC_{L^p}([a,b], f)^{-1}(\mathcal{U}_{f\circ\eta})\right)$ we have

$$\begin{aligned} d\phi_i \circ F(\sigma)|_{[t_{i-1},t_i]} &= d\phi_i \circ \theta_N^{-1} \circ \Big(f\circ\eta, f\circ\Sigma_M\circ\sigma_i\Big) \\ &= d\phi_i \circ \theta_N^{-1} \circ \Big(\phi_i^{-1}\circ\phi_i\circ f\circ\eta, f\circ\Sigma_M\circ T\varphi_i^{-1}(\varphi_i\circ\eta, d\varphi_i\circ\sigma_i)\Big). \end{aligned}$$

Because all of the functions involved are continuous and have an open domain, also the composition

$$H_i(x,y,z) := d\phi_i \circ \theta_N^{-1} \circ \Big(\phi_i^{-1}(x), f\circ\Sigma_M\circ T\varphi_i^{-1}(y,z)\Big) \tag{2.3.8}$$

has an open domain $\mathcal{O}_i$ in $E_N \times E_M \times E_M$. Hence the map $H_i : \mathcal{O}_i \to E_N$ is smooth. Fixing the absolutely continuous functions $\phi_i\circ f\circ\eta$ and $\varphi_i\circ\eta$, we consider the continuous affine-linear map

$$AC_{L^p}([t_{i-1}, i], E_M) \to AC_{L^p}([t_{i-1}, i], E_N \times E_M \times E_M), \quad \tau \mapsto (\phi_i\circ f\circ\eta, \varphi_i\circ\eta, \tau)$$

and we write W_i for the preimage of $AC_{L^p}([a,b], \mathcal{O}_i)$ under this map. Then the map

$$\Theta_i : W_i \to AC_{L^p}([t_{i-1}, t_i], E_N) \quad \tau \mapsto H_i \circ (\phi_i\circ f\circ\eta, \varphi_i\circ\eta, \tau)$$

is C^k. Since the maps $h_i : \Gamma_{AC}(\eta) \to AC_{L^p}([t_{i-1}, t_i], E_M)$, $\sigma \mapsto d\varphi_i \circ \sigma_i$ are continuous linear by definition, rewriting we have

$$\Phi_{f\circ\eta} \circ F(\sigma) = (\Theta_i \circ h_i(\sigma))_{i=1}^n$$

for each $\sigma \in \Psi_\eta^{-1}(\mathcal{U}_\eta \cap \mathcal{U}_\xi)$, hence the map $AC_{L^p}([a,b], f)$ is C^k. □

Proceeding in the same way we have the analogous case.

Corollary 2.3.9 *Let $f : M \to N$ be a $\mathbb{K}$-analytic map between $\mathbb{K}$-analytic manifolds modeled on sequentially complete locally convex spaces which admit $\mathbb{K}$-analytic local additions and $p \in [1, \infty]$. Then the map*

$$AC_{L^p}([a,b], f) : AC_{L^p}([a,b], M) \to AC_{L^p}([a,b], N), \quad \eta \mapsto f\circ\eta$$

is $\mathbb{K}$-analytic.

Remark 2.3.10 The manifold structures for $AC_{L^p}([a,b],N)$ given by different local additions coincide. Indeed, since the identity map $\mathrm{id}_N : N \to N$ is smooth regardless of the chosen local addition, the map

$$AC([a,b],\mathrm{id}_M) : AC_{L^p}([a,b],N) \to AC_{L^p}([a,b],N), \quad \eta \to \mathrm{id}_N \circ \eta$$

is, again, smooth regardless of the chosen local addition in each space.

Remark 2.3.11 The inclusion map $J : AC_{L^p}([a,b],N) \to C([a,b],N)$ is smooth. Indeed, let $(\mathcal{U}_\eta, \Psi_\eta^{-1})$ and $(\mathcal{U}_\eta^C, \left(\Psi_\eta^C\right)^{-1})$ be charts around $\eta \in AC_{L^p}([a,b],N)$ and $\eta \in C([a,b],N)$ respectively, then

$$\Psi_\eta^{-1} \circ J \circ \Psi_\eta^C(\sigma) : \Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap J^{-1}(\mathcal{U}_\eta^C)\right) \subseteq \Gamma_{AC}(\eta) \to \Gamma_C(\eta)$$

is a restriction of the inclusion map $\Gamma_{AC}(\eta) \to \Gamma_C(\eta)$.
Moreover, if $U \subseteq N$ be a open subset, then the manifold structure induced by $AC_{L^p}([a,b],N)$ on the open subset

$$AC_{L^p}([a,b],U) := \{\eta \in AC_{L^p}([a,b],N) : \eta([a,b]) \subseteq U\}.$$

coincides with the manifold structure on $AC_{L^p}([a,b],U)$.

Proposition 2.3.12 *Let N_1 and N_2 be smooth manifolds with local addition modeled on a sequentially complete locally convex spaces which admit a local addition, $p \in [1,\infty]$ and let $\mathrm{pr}_i : N_1 \times N_2 \to N_i$ be the i-th projection for $i \in \{1,2\}$. Then the map*

$$\mathcal{P} : AC_{L^p}([a,b],N_1 \times N_2) \to AC_{L^p}([a,b],N_1) \times AC_{L^p}([a,b],N_2), \quad \eta \mapsto (\mathrm{pr}_1, \mathrm{pr}_2) \circ \eta$$

is a diffeomorphism.

Proof. By the previous remark, if (Ω_1, Σ_1) and (Ω_1, Σ_1) are the local addition on N_1 and N_2 respectively, then we can assume that the local addition on $N_1 \times N_2$ is

$$\Sigma := \Sigma_1 \times \Sigma_2 : \Omega_1 \times \Omega_2 \to N_1 \times N_2$$

where $\Omega_1 \times \Omega_2 \subseteq TN_1 \times TN_2 \cong T(N_1 \times N_2)$. The map $\mathcal{P}$ is smooth as consequence of the smoothness of the maps

$$AC_{L^p}([a,b],\mathrm{pr}_i) : AC_{L^p}([a,b],N_1 \times N_2) \to AC_{L^p}([a,b],N_i),$$

for each $i \in \{1,2\}$.
Let $(\mathcal{U}_{\eta_1} \times \mathcal{U}_{\eta_2}, \Psi_{\eta_1}^{-1} \times \Psi_{\eta_2}^{-1})$ and $(\mathcal{U}_\eta, \Psi_\eta^{-1})$ be charts in $(\eta_1,\eta_2) \in AC_{L^p}([a,b],N_1) \times AC_{L^p}([a,b],N_2)$ and $\mathcal{P}^{-1}(\eta_1,\eta_2) = \eta \in AC_{L^p}([a,b],N_1 \times N_2)$ respectively. Since the map

$$\mathcal{Q} : \Gamma_{AC}(\eta) \to \Gamma_{AC}(\eta_1) \times \Gamma_{AC}(\eta_2), \quad \tau \mapsto (\mathrm{q}_1, \mathrm{q}_2) \circ \tau$$

is an isomorphism of topological vector spaces, where q_1 and q_2 are the corresponding projection of the space, we have

$$\begin{aligned}\Psi_\eta^{-1} \circ \mathcal{P}^{-1} \circ (\Psi_{\eta_1} \times \Psi_{\eta_2})(\sigma_1,\sigma_2) &= (\pi_{N_1 \times N_2}, \Sigma)^{-1} \circ \left(\eta, \mathcal{P}^{-1} \circ (\Sigma_1 \times \Sigma_2)(\sigma_1,\sigma_2)\right) \\ &= (\pi_{N_1 \times N_2}, \Sigma)^{-1} \circ \left(\eta, \Sigma \circ \mathcal{Q}^{-1}(\sigma_1,\sigma_2)\right) \\ &= \mathcal{Q}^{-1}(\sigma_1,\sigma_2)\end{aligned}$$

for all $(\sigma_1,\sigma_2) \in (\Psi_{\eta_1}^{-1} \times \Psi_{\eta_2}^{-1})(\mathcal{U}_{\eta_1} \times \mathcal{U}_{\eta_2} \cap \mathcal{P}(\mathcal{U}_\eta))$. Hence $\mathcal{P}^{-1}$ is smooth. □

Proposition 2.3.13 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition and $p \in [1,\infty]$. For a partition $P = \{t_0, ..., t_n\}$ of $[a,b]$ the map*

$$T : AC_{L^p}([a,b],N) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],N), \quad \eta \mapsto \left(\eta|_{[t_{i-1},t_i]}\right)_{i=1}^{n}$$

is smooth and a smooth diffeomorphism onto a submanifold of $\prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],N)$.

Proof. It is clear that the map T is well defined and injective. Let $\mathrm{Im}(T)$ be the image of the map T. Then

$$\mathrm{Im}(T) = \{(\gamma_i)_{i=1}^n \in \prod_{i=1}^{n} AC_{L^p}([t_{i-1},t_i],N) : \gamma_i(t_i) = \gamma_{i+1}(t_i) \text{ for all } i \in \{1,...,n-1\}\}.$$

Let $\gamma := (\gamma_i)_{i=1}^n \in \prod_{i=1}^n AC_{L^p}([t_{i-1},t_i],N)$. For each $i \in \{1,...,n\}$ let $\Psi_{\gamma_i}^{-1} : \mathcal{U}_i \to \mathcal{V}_i$ be charts around γ_i. Then the map

$$\Psi_\gamma^{-1} := \prod_{i=1}^{n} \Psi_{\gamma_i}^{-1} : \prod_{i=1}^{n} \mathcal{U}_i \to \prod_{i=1}^{n} \mathcal{V}_i$$

is a chart around γ. Let $\eta \in AC_{L^p}([a,b],N)$ and $\tilde{\eta} = T(\eta)$, then $\Psi_{\tilde{\eta}}^{-1} \circ T \circ \Psi_\eta$ is just restriction of the product of the restrictions of the smooth maps

$$\Gamma_{AC}(\eta) \to \prod_{i=1}^{n} \Gamma_{AC}(\eta_i), \quad \sigma \mapsto \left(\sigma|_{[t_{i-1},t_i]}\right)_{i=1}^{n}$$

thus T is a smooth. Now we will show that the image $\mathrm{Im}(T)$ is a submanifold. Let $\gamma = (\gamma_i)_{i=1}^n \in \mathrm{Im}(T)$ with charts as before, then for each $i \in \{1,...,n-1\}$ and $\xi = (\xi_i)_{j=1}^n \in \mathrm{Im}(T) \cap \prod_{j=1}^n \mathcal{U}_j$ we have

$$\begin{aligned}
\Psi_{\gamma_i}^{-1}(\xi_i)(t_i) &= \theta_N^{-1} \circ (\gamma_i, \xi_i)(t_i) \\
&= \theta_N^{-1} \circ (\gamma_i(t_i), \xi_i(t_i)) \\
&= \theta_N^{-1} \circ (\gamma_{i+1}(t_i), \xi_{i+1}(t_i)) \\
&= \Psi_{\gamma_{i+1}}^{-1}(\xi_{i+1})(t_i).
\end{aligned}$$

This implies that if K denotes the vector space

$$K := \{(\sigma_i)_{i=1}^n \in \prod_{i=1}^{n} \Gamma_{AC}(\gamma_i) : \sigma_i(t_i) = \sigma_{i+1}(t_i) \text{ for all } i \in \{1,...,n-1\}\}.$$

Then $\Psi_\gamma^{-1}|_{\mathcal{U}_\gamma \cap \mathrm{Im}(T)} : \mathrm{Im}(\Psi) \cap \prod_{i=1}^n \mathcal{U}_i \to K \cap \prod_{i=1}^n \mathcal{V}_i$ is a chart of $\mathrm{Im}(T)$, making $\mathrm{Im}(T)$ a smooth submanifold and the map $\tilde{T} : AC_{L^p}([a,b],N) \to \mathrm{Im}(T)$, $\eta \mapsto T(\eta)$ a diffeomorphism. □

Remark 2.3.14 Let N be a $\mathbb{K}$-analytic manifold modeled on a sequentially complete locally convex space which admits a $\mathbb{K}$-analytic local addition and $p \in [1, \infty]$. Since every continuous linear operator is analytic, the isomorphism

$$\Gamma_{AC}(\eta) \to K, \quad \sigma \mapsto \left(\sigma|_{[t_{i-1}, t_i]}\right)_{i=1}^{n}$$

is $\mathbb{K}$-analytic, which implies that T in Proposition 2.3.13 is a $\mathbb{K}$-analytic diffeomorphism onto the submanifold $\mathrm{Im}(T)$.

Proposition 2.3.15 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], N)$. If $g : [c, d] \to [a, b]$ is a map as in Lemma 2.1.15, then the map*

$$AC_{L^p}(g, N) : AC_{L^p}([a, b], N) \to AC_{L^p}([c, d], N), \quad \eta \mapsto \eta \circ g$$

is smooth.

Proof. By Proposition 2.2.10 we know that the map is well defined. Let $(\mathcal{U}_\eta, \Psi_\eta^{-1})$ and $(\mathcal{U}_{\eta \circ g}, \Psi_{\eta \circ g}^{-1})$ be charts around $\eta \in AC_{L^p}([a, b], N)$ and $\eta \circ g \in AC_{L^p}([c, d], N)$ respectively, then we have

$$\Psi_{\eta \circ g}^{-1} \circ AC_{L^p}(g, M) \circ \Psi_\eta(\sigma) = \theta_N^{-1} \circ (\eta \circ g, \Sigma \circ (\sigma \circ g))$$

for all $\sigma \in \Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap AC_{L^p}(g, N)^{-1}(\mathcal{U}_{\eta \circ g})\right)$. This set coincides with

$$\Psi_\eta^{-1}\left(\mathcal{U}_\eta \cap C(g, N)^{-1}(\mathcal{U}_{\eta \circ g}^{C})\right)$$

which is open given by the continuity of the map $C(g, N)$. Let $\alpha = \eta \circ g : [c, d] \to N$ and $\tau = \sigma \circ g : [c, d] \to TN$. Then both are absolutely continuous, with $\pi_{TN} \circ \tau = \alpha$, now $\tau \in \Gamma_{AC}(\alpha)$ and

$$\begin{aligned}
\Psi_{\eta \circ g}^{-1} \circ AC_{L^p}(g, M) \circ \Psi_\eta(\sigma) &= \theta_N^{-1} \circ (\alpha, \Sigma \circ \tau) \\
&= \Psi_\alpha^{-1} \circ \Psi_\alpha(\tau) \\
&= \tau \\
&= \sigma \circ g.
\end{aligned}$$

Hence, $\Psi_{\eta \circ g}^{-1} \circ AC_{L^p}(g, M) \circ \Psi_\eta$ is a restriction of the map

$$\Gamma_{AC}(\eta) \to \Gamma_{AC}(\eta \circ g), \quad \sigma \mapsto \sigma \circ g$$

which is continuous linear by Proposition 2.2.10. □

Proposition 2.3.16 *Let M, N and L be smooth manifolds modeled on sequentially complete locally convex spaces which admit a local addition and $p \in [1, \infty]$. If $f : L \times M \to N$ is a C^{k+2}-map and $\gamma \in AC_{L^p}([a, b], L)$ is fixed, then*

$$f_* : AC_{L^p}([a, b], M) \to AC_{L^p}([a, b], N), \quad \eta \mapsto f \circ (\gamma, \eta)$$

is a C^k-map.

Proof. Define the smooth map

$$C_\gamma : AC_{L^p}([a,b],N) \to AC_{L^p}([a,b],L) \times AC_{L^p}([a,b],N), \quad \eta \mapsto (\gamma,\eta)$$

Identifying $AC_{L^p}([a,b],L) \times AC_{L^p}([a,b],N)$ with $AC_{L^p}([a,b],L \times N)$, we have

$$f_* = AC_{L^p}([a,b],f) \circ C_\gamma.$$

Hence f_* is C^k. □

Proposition 2.3.17 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition, $p \in [1,\infty]$ and $t \in [a,b]$. Then the evaluation map*

$$\varepsilon : AC_{L^p}([a,b],N) \times [a,b] \to N, \quad (\eta,t) \mapsto \eta(t)$$

is continuous and the point evaluation map

$$\varepsilon_t : AC_{L^p}([a,b],N) \to N, \quad \eta \mapsto \eta(t)$$

is smooth.

Proof. The evaluation map

$$\varepsilon_c : C([a,b],N) \times [a,b] \to N, \quad (\eta,t) \mapsto \eta(t)$$

is continuous and the point evaluation $(\varepsilon_c)_t : C([a,b],N) \to N$, $\eta \mapsto \eta(t)$ is smooth for each $t \in [a,b]$ (see [1]). Since the inclusion map $J : AC_{L^p}([a,b],N) \to C([a,b],N)$ is smooth, the assumptions follow from the observation that $\varepsilon = \varepsilon_c \circ (J \times \mathrm{Id}_{[a,b]})$ and $\varepsilon_t = (\varepsilon_c)_t \circ J$ for each $t \in [a,b]$. □

Proposition 2.3.18 *Let M, N and L smooth manifolds modeled on a sequentially complete locally convex space such that L admits a local addition and $p \in [1,\infty]$. If $f : L \times M \to N$ is a C^2-map and $\gamma \in AC_{L^p}([a,b],L)$ is fixed, then the map*

$$F : [a,b] \times M \to N, \quad (t,p) \mapsto f(\gamma(t),p)$$

is continuous.

Proof. It follows from the fact that $F = f \circ (\gamma \times \mathrm{id}_M)$. □

Proposition 2.3.19 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition and $p \in [1,\infty]$. For each $q \in N$ the function $\zeta_q : [a,b] \to N$, $t \mapsto q$ is absolutely continuous and the map*

$$\zeta : N \to AC_{L^p}([a,b],N), \quad q \mapsto \zeta_q$$

is smooth and a topological embedding.

Proof. Consider the local addition $\Sigma : \Omega \to N$ and $\theta_N : \Omega \to \Omega'$ as in Definition 2.3.1. Let (U, φ) be a chart around $q \in N$ such that $\{q\} \times U \subseteq \Omega'$ and $(\mathcal{U}_{\zeta_q}, \Psi_{\zeta_q}^{-1})$ be a chart in $\zeta_q \in AC_{L^p}([a,b], N)$.
If $x \in \varphi\left(U \cap \zeta^{-1}(\mathcal{U}_{\zeta_q})\right)$, then for each $t \in [a,b]$ we have

$$\begin{aligned}\Psi_{\zeta_q}^{-1} \circ \zeta \circ \varphi^{-1}(x)(t) &= \theta_N^{-1}\left(\zeta_q(t), \zeta_{\varphi^{-1}(x)}(t)\right)\\ &= \theta_N^{-1}\left(q, \varphi^{-1}(x)\right)\\ &= \theta_N^{-1} \circ (q, \varphi^{-1} \circ \tilde{\zeta}_x(t))\end{aligned}$$

where $\tilde{\zeta}_x : [a,b] \to E$, $t \mapsto x$ is the constant function. Since the map

$$\tilde{\zeta} : E \to AC_{L^p}([a,b], E), \quad x \mapsto \tilde{\zeta}_x$$

is a continuous linear, setting the smooth map

$$h : \varphi(U) \to TN, \quad z \to \theta_N^{-1} \circ (q, \varphi^{-1}(z))$$

we have

$$\Psi_{\zeta_p}^{-1} \circ \zeta \circ \varphi^{-1} = AC_{L^p}([a,b], h) \circ \tilde{\zeta}|_{\phi(U)}.$$

Hence ζ is smooth. Moreover, if $t \in [a,b]$, then $\varepsilon_t \circ \zeta = \mathrm{id}_N : N \to N$. □

Remark 2.3.20 Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition, $p \in [1, \infty]$ and let $TAC_{L^p}([a,b], N)$ be the tangent bundle of $AC_{L^p}([a,b], N)$. Since the point evaluation map $\varepsilon_t : AC_{L^p}([a,b], N) \to N$ is smooth for each $t \in [a,b]$, we have

$$T\varepsilon_t : TAC_{L^p}([a,b], N) \to TN.$$

For each $v \in TAC_{L^p}([a,b], N)$ we define the function

$$\Theta_N(v) : [a,b] \to TN, \quad \Theta_N(v)(t) = T\varepsilon_t(v).$$

Proposition 2.3.21 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits local addition, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a,b], N)$. Then $\Theta_N(v) \in \Gamma_{AC}(\eta)$ for each $v \in T_\eta AC_{L^p}([a,b], N)$ and the map*

$$\Theta_\eta : T_\eta AC_{L^p}([a,b], N) \to \Gamma_{AC}(\eta), \quad v \mapsto \Theta_\eta(v) := \Theta_N|_{T_\eta AC_{L^p}([a,b],N)}(v)$$

is an isomorphism of topological vector spaces.

Proof. Let $\Sigma : \Omega \to N$ be a normalized local addition of N. Since $\Gamma_{AC}(\eta)$ is a vector space, we identify its tangent bundle with $\Gamma_{AC}(\eta) \times \Gamma_{AC}(\eta)$. Let $\Psi_\eta : \mathcal{V}_\eta \to \mathcal{U}_\eta$ be a chart around η such that $\Psi_\eta(0) = \eta$, then

$$T\Psi_\eta : T\mathcal{V}_\eta \simeq \mathcal{V}_\eta \times \Gamma_{AC}(\eta) \to TAC_{L^p}([a,b], N)$$

is a diffeomorphism onto its image. Moreover,

$$T\Psi_\eta : \{0\} \times \Gamma_{AC}(\eta) \to T_\eta AC_{L^p}([a,b],N)$$

is an isomorphism of topological vector spaces. We will show that

$$\Theta_\eta \circ T\Psi_\eta(0,\sigma) = \sigma$$

for each $\sigma \in \Gamma_{AC}(\eta)$. Which is equivalent to show that

$$T\varepsilon_t \circ T\Psi_\eta(0,\sigma) = \sigma(t) \quad \text{for all } t \in [a,b].$$

Working with the geometric point of view of tangent vectors, we see that $(0,\sigma)$ is equivalent to the curve $[s \mapsto s\sigma]$. Hence, for each $t \in [a,b]$ we have

$$\begin{aligned} T\varepsilon_t \circ T\Psi_\eta(0,\sigma) &= T\varepsilon_t \circ T\Psi_\eta([s \mapsto s\sigma]) \\ &= T\varepsilon_t([s \mapsto \Psi_\eta(s\sigma)]) \\ &= T\varepsilon_t([s \mapsto \Sigma(s\sigma)]) \\ &= [s \mapsto \Sigma|_{T_{\eta(t)}N}(s\sigma(t))] \\ &= T_0\Sigma|_{T_{\eta(t)}N}([s \mapsto s\sigma(t)]). \end{aligned}$$

Since Σ is normalized we have $T_0\Sigma|_{T_{\eta(t)}N} = \mathrm{id}_{T_{\eta(t)}N}$ and

$$T\varepsilon_t \circ T\Psi_\eta(0,\sigma) = \sigma(t).$$

In consequence, for each $\sigma \in \Gamma_{AC}(\eta)$, there exists a $v \in T_\eta AC_{L^p}([a,b],N)$ with $v = T\Psi_\eta(0,\sigma)$ such that

$$\Theta_\eta(v) = \sigma.$$

Moreover, the function

$$\Theta_\eta(v) : [a,b] \to TN, \quad t \mapsto \Theta_N(v)(t) = \sigma(t) \in T_{\eta(t)}N$$

is absolutely continuous with $\pi_{TN} \circ \Theta_\eta(v) = \eta$, making the map Θ_η an isomorphism of topological vector spaces. □

Remark 2.3.22 Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits local addition and $p \in [1,\infty]$. Since TN admits local addition and the vector bundle $\pi_{TN} : TN \to N$ is smooth, the map

$$AC([a,b],\pi_{TN}) : AC_{L^p}([a,b],TN) \to AC_{L^p}([a,b],N), \quad \tau \mapsto \pi_{TN} \circ \tau$$

is smooth. Moreover, if $\eta \in AC_{L^p}([a,b],N)$, then

$$AC([a,b],\pi_{TN})^{-1}(\{\eta\}) = \Gamma_{AC}(\eta).$$

The following result follows the same steps as for the case of C^ℓ-maps (with $\ell \geq 0$) from a compact manifold (possibly with rough boundary) to a smooth manifold which admits local addition [3, Theorem A.12].

Proposition 2.3.23 *Let N be a smooth manifold modeled on a sequentially complete locally convex space E which admits a local addition, $p \in [1, \infty]$ and $\pi_{TN} : TN \to N$ its tangent bundle. Then the map*

$$AC([a,b], \pi_{TN}) : AC_{L^p}([a,b], TN) \to AC_{L^p}([a,b], N), \quad \tau \mapsto \pi_{TN} \circ \tau$$

is a smooth vector bundle with fiber $\Gamma_{AC}(\eta)$ over $\eta \in AC_{L^p}([a,b], N)$. Moreover, the map

$$\Theta_N : TAC_{L^p}([a,b], N) \to AC_{L^p}([a,b], TN), \quad v \mapsto \Theta_N(v)$$

is an isomorphism of vector bundles.

Proposition 2.3.24 *Let M and N be a smooth manifolds modeled on sequentially complete locally convex spaces which admits a local addition and $p \in [1, \infty]$. If $f : M \to N$ is a C^3-map, then the tangent map of*

$$AC_{L^p}([a,b], f) : AC_{L^p}([a,b], M) \to AC_{L^p}([a,b], N), \quad \eta \mapsto f \circ \eta$$

is given by

$$TAC_{L^p}([a,b], f) = \Theta_N^{-1} \circ AC_{L^p}([a,b], Tf) \circ \Theta_M.$$

Proof. By Proposition 2.3.8, since f is C^3 we know that $AC_{L^p}([a,b], f)$ is C^1, thus $TAC_{L^p}([a,b], f)$ exists. Let consider the local addition $\Sigma_M : \Omega_M \to M$ and $\eta \in AC_{L^p}([a,b], M)$.
If $\Psi_\eta : \mathcal{V}_\eta \to \mathcal{U}_\eta$ is a chart around η such that $\Psi_\eta(0) = \eta$. We consider the isomorphism of vector space

$$T\Psi_\eta : \{0\} \times \Gamma_{AC}(\eta) \to T_\eta AC_{L^p}([a,b], M).$$

For $t \in [a,b]$ we denote the point evaluation in M and N as ε_t^M and ε_t^N respectively, then for each $\sigma \in \Gamma_{AC}(\eta)$ we have

$$\begin{aligned}
T\varepsilon_t^N \circ TAC_{L^p}([a,b], f) \circ T\Psi_\eta(0, \sigma) &= T\varepsilon_t^N \circ TAC_{L^p}([a,b], f) \circ T\Psi_\eta([s \mapsto s\sigma]) \\
&= T\varepsilon_t^N \circ TAC_{L^p}([a,b], f)([s \mapsto \Sigma_M(s\sigma)]) \\
&= T\varepsilon_t^N([s \mapsto f \circ \Sigma_M(s\sigma)]) \\
&= [s \mapsto \varepsilon_t (f \circ \Sigma_M(s\sigma))] \\
&= [s \mapsto f \circ \Sigma_M(s\sigma(t))] \\
&= Tf \circ T_0\Sigma_M|_{T_{\eta(t)}M}([s\sigma(t)]) \\
&= Tf([s \mapsto s\sigma(t)]) \\
&= AC_{L^p}([a,b], Tf)(\sigma(t)) \\
&= AC_{L^p}([a,b], Tf) \circ T\varepsilon_t^M \circ T\Psi_\eta(0, \sigma).
\end{aligned}$$

Hence

$$\Theta_N \circ TAC_{L^p}([a,b], f) = AC_{L^p}([a,b], Tf) \circ \Theta_M.$$

□

Example 2.3.25 Let $p \in [1, \infty]$. If G is a Lie group modeled in a sequentially locally convex space E, then we already know that the space $AC_{L^p}([a,b], G)$ is a Lie group with Lie algebra given by $AC_{L^p}([a,b], T_eG)$ (see [15, 35]). We will give an alternative proof of this.
Let $e \in G$ be the neutral element, let $L_g : G \to G$, $h \mapsto gh$ be the left translation by $g \in G$ and the action

$$G \times TG \to TG, \quad (g, v_h) \mapsto g.v_h := TL_g(v_h) \in T_{gh}G.$$

If $\varphi : U \subseteq G \to V \subseteq T_eG$ is a chart in e such that $\varphi(e) = 0$, then the set

$$\Omega_\varphi := \bigcup_{g \in G} g.V \subseteq TG$$

and the map

$$\Sigma_\varphi : \Omega_\varphi \to G, \quad v \mapsto \pi_{TG}(v) \left(\varphi^{-1}(\pi_{TG}(v)^{-1}.v)\right)$$

defines a local addition for G (see e.g. [25]); hence $AC_{L^p}([a,b], G)$ is a smooth manifold with charts constructed with $(\Omega_\varphi, \Sigma_\varphi)$.
Let $\mu_G : G \times G \to G$ and $\lambda_G : G \to G$ be the multiplication map and inversion maps on G respectively, we define the multiplication map μ_{AC} and the inversion map λ_{AC} on $AC_{L^p}([a,b], G)$ as

$$\mu_{AC} := AC_{L^p}([a,b], \mu_G) : AC_{L^p}([a,b], G) \times AC_{L^p}([a,b], G) \to AC_{L^p}([a,b], G)$$

and

$$\lambda_{AC} := AC_{L^p}([a,b], \lambda_G) : AC_{L^p}([a,b], G) \to AC_{L^p}([a,b], G)$$

that by Lemma 2.1.17 and Proposition 2.3.8 are smooth.
We observe that for the neutral element $\zeta_e : [a,b] \to G$, $t \mapsto e$ of $AC_{L^p}([a,b], G)$ we have

$$\Gamma_{AC}(\zeta_e) = AC_{L^p}([a,b], T_eG).$$

If $\Psi_{\zeta_e}^{-1} : \mathcal{U}_{\zeta_e} \to \mathcal{V}_{\zeta_e}$ is a chart in $\zeta_e \in AC_{L^p}([a,b], G)$, then we have $\mathcal{U}_{\zeta_e} = AC([a,b], U)$ and $\mathcal{V}_{\zeta_e} = AC([a,b], V)$. Moreover, we see that

$$\begin{aligned}
\Psi_{\zeta_e} \circ AC_{L^p}([a,b], \varphi)(\eta) &= \Sigma_\varphi \circ (\varphi \circ \eta) \\
&= \pi_{TG}(\varphi \circ \eta) \left(\varphi^{-1}(\pi_{TG}(\varphi \circ \eta)^{-1}.\varphi \circ \eta)\right) \\
&= e\varphi^{-1}(e.\varphi \circ \eta) \\
&= \eta.
\end{aligned}$$

This enables us to say that for the neutral element $\zeta_e \in AC_{L^p}([a,b], G)$ the chart is given by

$$AC_{L^p}([a,b], \varphi) : AC_{L^p}([a,b], U) \to AC([a,b], V), \quad \eta \mapsto \varphi \circ \eta.$$

2.4 Semiregularity of right half-Lie groups.

Definition 2.4.1 A group G, endowed with a smooth manifold structure modeled on a locally convex space, is called a right half-Lie group if it is a topological group and if for all $g \in G$, the right translations $\rho_g : G \to G, x \mapsto xg$ are smooth.

Remark 2.4.2 Let G be a right half-Lie group modeled on a sequentially complete locally convex space. We define the right action

$$TG \times G \to TG, \quad (v, g) \mapsto v.g := T\rho_g(v) \tag{2.4.1}$$

and consider its restriction

$$T_eG \times G \to TG, \quad (v, g) \mapsto v.g := T\rho_g(v). \tag{2.4.2}$$

Unlike on Lie groups, on half-Lie groups the latter action may not be smooth. Hence we can not construct a local addition using a convenient chart around the identity (as in Example 2.3.25).

The following proposition is direct application of Proposition 2.3.8.

Proposition 2.4.3 *Let G be a right half-Lie group modeled on a sequentially complete locally convex space which admits local addition and $p \in [1, \infty]$. For $\eta \in AC_{L^p}([a, b], G)$ and $g \in G$ we define the function*

$$\eta.g(t) := \eta(t)g, \quad \textit{for all } t \in [a, b].$$

Then $\eta.g \in AC_{L^p}([a, b], G)$ for each $g \in G$ and the map

$$AC_{L^p}([a, b], \rho_g) : AC_{L^p}([a, b], G) \to AC_{L^p}([a, b], G), \quad \eta \mapsto \eta.g$$

is smooth.

Remark 2.4.4 The smoothness of the map

$$\mathcal{R} : AC_{L^p}([a, b], G) \times G \to AC_{L^p}([a, b], G), \quad (\eta, g) \mapsto \eta.g$$

would imply the smoothness of the multiplication map on G. In fact, since the point evaluation map $\varepsilon_a : AC_{L^p}([a, b], G) \to G$, $\eta \mapsto \eta(a)$ and the map $\zeta : G \to AC_{L^p}([a, b], G)$, $g \mapsto [t \mapsto g]$, are smooth, the multiplication map on G would be smooth as it coincides with the composition

$$\varepsilon_a \circ \mathcal{R} \circ (\zeta, \mathrm{id}_G) : G \times G \to G, \quad (h, g) \mapsto hg.$$

Definition 2.4.5 Let G be a right half-Lie group modeled on a sequentially complete locally convex space E which admits local addition, $p \in [1, \infty]$ and $\eta \in AC_{L^p}([a, b], N)$. Let $P = \{t_0, .., t_n\}$ be a partition of $[a, b]$ and $\{(\varphi_i, U_i) : i \in \{1, ..., n\}\}$ charts of G such that verify the definition of absolute continuity for η. For each $i \in \{1, ..., n\}$ we denote

$$\eta_i := \varphi_i \circ \eta_{[t_{i-1}, t_i]} \in AC_{L^p}([t_{i-1}, t_i], E).$$

Then $\eta_i' \in L^p([t_{i-1}, t_i], E)$. Let $\eta_i' = [\gamma_i]$ with $\gamma_i \in \mathcal{L}^p([t_{i-1}, t_i], E)$. We define $\gamma : [a, b] \to TG$ via

$$\gamma(t) := T(\varphi_i)^{-1}(\eta_i(t), \gamma_i(t)), \text{ if } t \in [t_{i-1}, t_i[\text{ with } i \in \{1, ..., n\},$$

and $\gamma(b) = T(\varphi_n)^{-1}(\eta_n(b), \gamma_n(b))$. Then γ is measurable (see [15]) and we define

$$\dot{\eta} := [\gamma].$$

Definition 2.4.6 Let G be a right half-Lie group modeled in a sequentially complete locally convex space and $p \in [1, \infty]$. We say that G is L^p-semiregular if for each $[\gamma] \in L^p([0, 1], T_e G)$, there exists an AC_{L^p}-Carathéodory solution $\eta_\gamma \in AC_{L^p}([0, 1], G)$ of the equation

$$\dot{y}(t) = \gamma(t).y(t), \quad t \in [0, 1] \tag{2.4.3}$$

$$y(0) = e \tag{2.4.4}$$

such that the differential equation satisfies local uniqueness of Carathéodory solutions in the sense of [19]. In this case, we define the evolution map

$$\text{Evol} : L^p([0, 1], T_e G) \to AC_{L^p}([a, b], G), \quad [\gamma] \mapsto \text{Evol}(\gamma) := \eta_\gamma. \tag{2.4.5}$$

Additionally, if G admits a local addition, we say that G is L^p-regular if G is L^p-semiregular and if the evolution map is smooth. The definition for the case of L^∞_{rc}-semiregularity and L^∞_{rc}-regularity is analogous.

Definition 2.4.7 Let G be a right half-Lie group modeled on a sequentially complete locally convex space and $p \in [1, \infty]$. We say that G is locally L^p-semiregular if there is exists an open 0-neighborhood B of $L^p([0, 1], T_e G)$ such that for each $[\gamma] \in B$ there exists a AC_{L^p}-Carathéodory solution $\eta_\gamma \in AC_{L^p}([0, 1], G)$ of the equation

$$\dot{y}(t) = \gamma(t).y(t), \quad t \in [0, 1]$$

$$y(0) = e$$

and the latter differential equation satisfies local uniqueness of solutions.

For our purpose, we will use the subdivision property [35, Lemma 2.17].

Lemma 2.4.8 *Let E be a locally convex space, $p \in [1, \infty]$ and $[\gamma] \in L^p([0, 1], E)$. For each $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n-1\}$ we define*

$$\gamma_{n,k} : [0, 1] \to E, \quad \gamma_{n,k}(t) := \frac{1}{n}\gamma\left(\frac{k+t}{n}\right).$$

Then $[\gamma_{n,k}] \in L^p([0, 1], E)$. Moreover, for each $[\gamma] \in L^p([0, 1], E)$ and continuous seminorm q on $L^p([0, 1], E)$, we have

$$\sup_{k \in \{0, ..., n-1\}} q(\gamma_{n,k}) \to 0, \quad \text{as } n \to \infty.$$

Lemma 2.4.9 *Let G be a right half-Lie group modeled on a sequentially complete locally convex space which admits local addition and $p \in [1,\infty]$. Then G is locally L^p-semiregular if and only if G is L^p-semiregular.*

Proof. Let G be locally L^p-semiregular. Then there exists an 0-neighborhood B of $L^p([0,1],T_eG)$ such that $\mathrm{Evol}(\beta) \in AC_{L^p}([a,b],G)$ exists for each $[\beta] \in B$. Without loss of generality we assume that

$$B = \{[\beta] \in L^p([0,1],T_eG) : \|[\beta]\|_{L^p,q} < 1\}$$

for some continuous seminorm q on $L^p([0,1],T_eG)$.
Let $[\gamma] \in L^p([0,1],T_eG)$. By Lemma 2.4.8, we find $n \in \mathbb{N}$ such that

$$[\gamma_{n,k}] \in B, \quad \text{for all } k \in \{0,1,...,n-1\}.$$

Since each map

$$\alpha_k : L^p([0,1],T_eG) \to L^p([0,1],T_eG), \quad [\tau] \mapsto [\tau_{n,k}]$$

is continuous linear, there is exists an open γ-neighborhood W_γ such that

$$\alpha_k(W_\gamma) \subseteq B, \quad \text{for all } k \in \{0,1,...,n-1\}.$$

For $[\beta] \in W_\gamma$ and $j \in \{0,...,n-1\}$ we write $\eta_j = \mathrm{Evol}(\beta_{n,j})$ and we define the function $\eta_\beta : [0,1] \to G$ via

$$\eta_\beta(t) := \eta_0(nt), \quad \text{if } t \in [0,1/n]$$

and

$$\eta_\beta(t) := \eta_k(nt-k).\Big(\eta_{k-1}(1)....\eta_0(1)\Big), \quad \text{if } t \in [k/n,(k+1)/n] \text{ with } k \in \{1,..,n-1\}.$$

Then the function η_β is continuous and by Proposition 2.4.3 we have

$$\eta_\beta|_{[k/n,(k+1)/n]} \in AC_{L^p}\left([k/n,(k+1)/n],G\right).$$

Thus $\eta_\beta \in AC_{L^p}([a,b],G)$. If $t \in [0,\frac{1}{n}]$ we have $\eta_\beta(0) = e$ and

$$\begin{aligned}\dot{\eta_\beta}(t) &= \dot{\eta_0}(nt)\\ &= n\beta_{n,0}(nt).\eta_0(nt)\\ &= n\frac{1}{n}\beta\left(\frac{0+nt}{n}\right).\eta_0(nt)\\ &= \beta(t).\eta_\beta(t).\end{aligned}$$

For $k \in \{1,..,n-1\}$ we have

$$\eta_\beta\left(\frac{k}{n}\right) = \eta_k\left(n\frac{k}{n}-k\right).\Big(\eta_{k-1}(1)....\eta_0(1)\Big) = \eta_{k-1}(1)....\eta_0(1)$$

and for $t \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\begin{aligned}\dot{\eta_\beta}(t) &= n\beta_{n,k}(nt-k)\Big(\eta_{k-1}(1)....\eta_0(1)\Big).\eta_k(nt-k)\\ &= \beta(t).\eta_k(nt-k)\Big(\eta_{k-1}(1)....\eta_0(1)\Big)\\ &= \beta(t).\eta_\beta(t).\end{aligned}$$

Thus $\eta_\beta = \mathrm{Evol}(\beta)$ and in particular, $\eta_\gamma = \mathrm{Evol}(\gamma)$. Hence G is L^p-semiregular.
The reciprocal is trivial. □

Definition 2.4.10 Let G be a right half-Lie group modeled on a sequentially complete locally convex space which admits local addition and $p \in [1,\infty]$. We say that G is locally L^p-regular if G is L^p-semiregular and there is exists a 0-neighborhood B of $L^p([0,1], T_eG)$ such that its restricted evolution map $\mathrm{Evol}\,|_B$ is smooth.

The following lemma is just an application of [34, Proposition 4.11] to our case.

Lemma 2.4.11 *Let G be a right half-Lie group modeled in a sequentially complete locally convex space which admit a local addition and $p \in [1,\infty]$. We consider the evolution map with continuous values*

$$\mathrm{Evol}_C : L^p([0,1], T_eG) \to C([0,1], G), \quad [\gamma] \mapsto \mathrm{Evol}_C(\gamma) := \eta_\gamma.$$

Then, Evol_C *is continuous if and only if there exists a* 0*-neighborhood* B *of* $L^p([0,1], T_eG)$ *such that the restricted evolution map* $\mathrm{Evol}_C\,|_B$ *is continuous.*

Proof. Since G is a topological group, the map

$$C([0,1], G) \times C([0,1], G) \to C([0,1], G), \quad (\eta, \xi) \mapsto \eta.(\xi(1))$$

is continuous. Following Lemma 2.4.9, if $[\gamma] \in L^p([0,1], T_eG)$, for each $[\beta] \in W_\gamma$, we have that the construction $\mathrm{Evol}_C(\beta)$ implies that $(\mathrm{Evol}_C)\,|_{W_\gamma}$ is just the product of composition of continuous maps. □

Lemma 2.4.12 *Let E_1, E_2 and F be locally convex spaces, $1 \le p < \infty$ and $U \subseteq E_2$ an open subset. If $r \in \mathbb{N} \cup \{0,\infty\}$ and $f : E_1 \times U \to F$ is a C^r-map such that for each $y \in U$ the map $f(\cdot, y) : E_1 \to F$ is linear, then the map*

$$\tilde{f} : L^p([0,1], E_1) \times C([0,1], U) \to L^p([0,1], F), \quad ([\gamma], \eta) \mapsto f \circ ([\gamma], \eta)$$

is C^r.

Proof. Let $(\gamma_0, \eta_0) \in L^p([0,1], E_1) \times C([0,1], U)$ and $\varepsilon > 0$. Let $\beta : F \to [0,\infty)$ be a continuous semi norm and $K := \eta_0([0,1]) \subseteq U$. Since f is continuous and linear in the first argument, for $y \in K$ there exists a y-neighborhood $V_y \subseteq U$ and a continuous seminorm $\kappa_y : E_1 \to [0,\infty)$ such that

$$f\left(B_1^{\kappa_y}(0) \times V_y\right) \subseteq B_1^\beta(0).$$

By compactness of K, there exists a finite numbers of $y_1, ..., y_n \in K$ such that

$$K \subseteq V := \cup_{i=1}^n V_{y_i}.$$

We define the continuous seminorm $\kappa : E_1 \to [0, \infty)$ by $\kappa := \kappa_{y_1} + ... + \kappa_{y_n}$ and we have

$$\beta\big(f(x, b)\big) \leq \kappa(x), \quad \text{for all } x \in E_1, b \in V$$

and $C([0, 1], V)$ is open in $C([0, 1], U)$. We estimate

$$\begin{aligned}\|\tilde{f}(\gamma_0, \eta_0)\|_{L^p,\beta} &:= \left(\int_0^1 \beta\big(\tilde{f} \circ (\gamma_0(t), \eta_0(t))\big) dt\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \kappa(\gamma_0(t)) dt\right)^{\frac{1}{p}} \\ &\leq \|\gamma_0\|_{L^p,\kappa}.\end{aligned}$$

Hence $\tilde{f}(\gamma_0, \eta_0) \in L^p([0, 1], F)$. Since the space $C([0, 1], E_1)$ is dense in $L^p([0, 1], E_1)$, there exists a $\gamma_c \in C([0, 1], E_1)$ such that

$$\|\gamma_c - \gamma_0\|_{L^p,p} \leq 2\varepsilon/5.$$

Additionally, since the maps

$$C([0, 1], U) \to C([0, 1], F), \quad \eta \mapsto f \circ (\gamma_c, \eta)$$

and the inclusion map $C([0, 1], F) \to L^p([0, 1], F)$ are continuous, we have that

$$C([0, 1], V) \to L^p([0, 1], F), \quad \eta \mapsto f \circ (\gamma_c, \eta)$$

is continuous. Hence there exists an open neighborhood W of $\eta_0 \in C([0, 1], V)$ such that

$$\|\tilde{f}(\gamma_c, \eta) - \tilde{f}(\gamma_c, \eta_0)\|_{L^p,\beta} \leq 2\varepsilon/5, \quad \text{for all } \eta \in W.$$

With this, for each $\eta \in W$ and $\gamma \in L^p([0, 1], E_1)$ with $\|\gamma - \gamma_0\|_{L^p,\kappa} \leq 2\varepsilon/5$ we estimate

$$\begin{aligned}\|\tilde{f}(\gamma, \eta) - \tilde{f}(\gamma_0, \eta_0)\|_{L^p,\beta} &\leq \|\tilde{f}(\gamma - \gamma_c, \eta)\|_{L^p,\beta} + \|\tilde{f}(\gamma_c, \eta) - \tilde{f}(\gamma_0, \eta_0)\|_{L^p,\beta} \\ &\leq \|\tilde{f}(\gamma - \gamma_c, \eta)\|_{L^p,\beta} + \|\tilde{f}(\gamma_c, \eta) - \tilde{f}(\gamma_c, \eta_0)\|_{L^p,\beta} + \|\tilde{f}(\gamma_c - \gamma_0, \eta_0)\|_{L^p,\beta} \\ &\leq \|\gamma - \gamma_c\|_{L^p,\kappa} + \|\tilde{f}(\gamma_c, \eta) - \tilde{f}(\gamma_c, \eta_0)\|_{L^p,\beta} + \|\gamma_c - \gamma_0\|_{L^p,\kappa}.\end{aligned}$$

Since $\|\gamma_c - \gamma_0\|_{L^p,\kappa} \leq 2\varepsilon/10$ and $\|\gamma - \gamma_0\|_{L^p,\kappa} \leq 2\varepsilon/5$ we have

$$\|\gamma - \gamma_c\|_{L^p,\kappa} \leq 2\varepsilon/5$$

and

$$\|\tilde{f}(\gamma, \eta) - \tilde{f}(\gamma_0, \eta_0)\|_{L^p,\beta} \leq 2\varepsilon/5 + 2\varepsilon/5 + 2\varepsilon/10 = \varepsilon$$

Thus $\tilde{f}$ is continuous. By linearity in the first variable the function $\overline{f}$ has a continuous differential $d_1\tilde{f} : L^p([0,1], E_1) \times L^p([0,1], E_1) \times C([0,1], E_2) \to L^p([0,1], F)$ in the first variable. Let consider the map f as a C^1-map. If $x \in E_1$, $y, y_1 \in E_2$ and $t \in \mathbb{R}^\times$, then we have

$$\frac{1}{t}\left(f(x, y + ty_1) - f(x, y)\right) = \int_0^1 d_2 f(x, y + tsy_1, y_1) ds$$

whenever $y + [0,1]ty_1 \subseteq U$. Given that the map $d_2 f : E_1 \times U \times E_2 \to F$ is continuous, identifying $C([0,1], E_1 \times E_2)$ with $C([0,1], E_1) \times C([0,1], E_2)$, we have that the map

$$\widetilde{d_2 f} : L^p([0,1], E_1) \times C([0,1], U) \times C([0,1], E_2) \to L^p([0,1], F), \quad (\gamma, \eta, \eta_1) \mapsto d_2 f(\gamma, \eta, \eta_1)$$

is continuous. For $t \in [0,1]$ we denote

$$g_t : C([0,1], F) \to F, \quad \gamma \mapsto \gamma(t).$$

Since the family of maps $\{g_t : t \in [0,1]\}$ separate points on $C([0,1], F)$, we have that the equality

$$\frac{1}{t}\left(f \circ (\gamma_c, \eta + t\eta_1) - f \circ (\gamma_c, \eta)\right) = \int_0^1 d_2 f(\gamma_c, \eta + ts\eta_1, \eta_1) ds$$

is valid for each $\gamma_c \in C([0,1], E_1)$, $\eta \in C([0,1], U)$, $\eta_1 \in C([0,1], E_2)$ and $t \in \mathbb{R}^\times$ such that

$$\eta + [0,1]t\eta_1 \in C([0,1], U).$$

By density of $C([0,1], E_1)$ on $L^p([0,1], E_1)$ and continuity of $\tilde{f}$, for $\gamma \in L^p([0,1], E)$ the equation verifies

$$\frac{1}{t}\left(\tilde{f}(\gamma, \eta + t\eta_1) - \tilde{f}(\gamma, \eta)\right) = \int_0^1 \widetilde{d_2 f}(\gamma, \eta + ts\eta_1, \eta_1) ds.$$

Let γ, η and η_0 be fixed, then the map $(t, s) \mapsto \widetilde{d_2 f}(\gamma, \eta + ts\eta_1, \eta_1)$ is continuous, including in $t = 0$. Then, taken the limit $t \to 0$ in the equality we obtain

$$d_2\tilde{f}(\gamma, \eta, \eta_1) = \widetilde{d_2 f}(\gamma, \eta, \eta_1).$$

Hence the continuity of $\widetilde{d_2 f}$ implies that the map $\tilde{f}$ is C^1. Proceeding by induction, if f is a C^k-map, since $\tilde{f}$ is linear in the first variable we have

$$\begin{aligned} d\tilde{f}(\gamma, \eta, \gamma_1, \eta_1) &= d_1\tilde{f}(\gamma, \gamma_1, \eta) + d_2\tilde{f}(\gamma, \eta, \eta_1) \\ &= \tilde{f}(\gamma_1, \eta) + \widetilde{d_2 f}(\gamma, \eta, \eta_1). \end{aligned}$$

By the induction hypothesis $\tilde{f}$ and $d_2 f$ are C^{r-1}, hence $\widetilde{d_2 f}$ is C^{r-1} with $\widetilde{d_2 f} = d_2\tilde{f}$, thus $\tilde{f}$ is C^r. □

For the case L^∞_{rc} we recall [15, Proposition 2.3].

Lemma 2.4.13 *Let E_1, E_2 and F be integral complete locally convex spaces and $U \subseteq E_2$ an open subset. If $r \in \mathbb{N} \cup \{0, \infty\}$ and $f : E_1 \times U \to F$ is a C^r-map such that for each $y \in U$ the map $f(\cdot, y) : E_1 \to F$ is linear, then the map*

$$\tilde{f} : L^\infty_{rc}([0,1], E_1) \times C([0,1], U) \to L^\infty_{rc}([0,1], F), \quad ([\gamma], \eta) \mapsto f \circ ([\gamma], \eta)$$

is C^r.

Theorem 2.4.14 *Let G be a right half-Lie group modeled in a sequentially complete locally convex space space E which admits a local addition and $1 \leq p < \infty$. Let G be L^p-semiregular with continuous evolution map*

$$\mathrm{Evol}_C : L^p([0,1], T_eG), \to C([0,1], G) \quad \gamma \mapsto \mathrm{Evol}_C(\gamma).$$

If the restriction of the right action

$$\tau : T_eG \times G \to TG, \quad (v, g) \mapsto v.g$$

is continuous, then the evolution map

$$\mathrm{Evol} : L^p([0,1], T_eG), \to AC_{L^p}([0,1], G), \quad \gamma \mapsto \mathrm{Evol}(\gamma)$$

is continuous. If G is a right half-Lie group modeled in an integral complete locally convex space E, then if we replace L^p with L^∞_{rc} the result remains valid.

Proof. Let $[\gamma] \in L^p([0,1], T_eG)$. Let $\mathcal{P}_{[\gamma]}$ be a open neighborhood of $\mathrm{Evol}_C(\gamma)$ in $C([0,1], G)$, then there exists a partition $P = \{t_0, ..., t_n\}$ of $[0,1]$ and a family of charts $\varphi_i : U_i \to V_i$ such that

$$\mathcal{P}_{[\gamma]} = \bigcap_{i=1}^{n} \{\eta \in C([t_{i-1}, t_i], U_i) : \eta([t_{i-1}, t_i]) \subseteq U_i\}.$$

Let $\mathcal{Q}_{[\gamma]} = \mathrm{Evol}_C^{-1}(\mathcal{P}_{[\gamma]})$, then $\mathcal{Q}_{[\gamma]}$ is an open neighborhood of $[\gamma]$ in $L^p([0,1], T_eG)$. We will show that the map

$$\mathrm{Evol}|_{\mathcal{Q}_{[\gamma]}} : \mathcal{Q}_{[\gamma]} \to \mathcal{P}_{[\gamma]} \cap AC_{L^p}([0,1], G), \quad \gamma \mapsto \mathrm{Evol}(\gamma)$$

is continuous. The map

$$\Phi_1 : \mathcal{Q}_{[\gamma]} \to \prod_{i=1}^{n} L^p([t_{i-1}, t_i], T_eG), \quad [\beta] \mapsto \left([\beta|_{[t_{i-1}, t_i]}]\right)_{i=1}^{n}$$

is a topological embedding. For each $i \in \{1, ..., n\}$ we have

$$\mathrm{Evol}_C(\beta|_{[t_{i-1}, t_i]}) = \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]}, \qquad \text{for all } [\beta] \in \mathcal{Q}_{[\gamma]}$$

and each map $C([t_{i-1}, t_i], U_i) \to C([t_{i-1}, t_i], V_i)$, $\eta \mapsto \varphi_i \circ \eta$ is a homeomorphism. This allow us to define the continuous map

$$\Phi_2 : \mathcal{Q}_{[\gamma]} \to \prod_{i=1}^{n} C([t_{i-1}, t_i], V_i), \quad [\beta] \mapsto \left(\varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]}\right)_{i=1}^{n}.$$

We consider the map

$$f_i : T_eG \times V_i \to E, \quad (v,x) \mapsto d\varphi_i \circ \tau(v, \varphi_i^{-1}(x)).$$

Then each f_i is continuous and for each $x \in V_i$ the function

$$f_i(\cdot, x) : T_eG \to E, \quad v \mapsto f_i(v,x)$$

is linear. Hence by Lemma 2.4.12 the map

$$\tilde{f}_i : L^p([t_{i-1}, t_i], T_eG) \times C([t_{i-1}, t_i], V_i) \to L^p([t_{i-1}, t_i], E), \quad ([\beta], \eta) \mapsto f_i \circ ([\beta], \eta)$$

is continuous. We denote

$$F : \prod_{i=1}^{n} L^p([t_{i-1}, t_i], T_eG) \times C([t_{i-1}, t_i], V_i) \to \prod_{i=1}^{n} L^p([t_{i-1}, t_i], E), \quad ([\beta_i], \eta_i)_{i=1}^n \mapsto (f_i \circ ([\beta_i], \eta_i))_{i=1}^n .$$

Then F is continuous, where for each $i \in \{1, ..., n\}$ we have

$$f_i \circ ([\beta_i], \eta_i) = d\varphi_i \circ \tau([\beta_i], \varphi_i^{-1} \circ \eta_i).$$

This allow us to define the continuous map

$$\begin{aligned} \Phi_3 : \mathcal{Q}_{[\gamma]} &\to \prod_{i=1}^{n} L^p([t_{i-1}, t_i], E) \times C([t_{i-1}, t_i], V_i), \\ \beta &\mapsto \Big(f_i \circ \left(\left[\beta|_{[t_{i-1}, t_i]} \right], \varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right), \varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \Big)_{i=1}^{n} \end{aligned}$$

where for each $i \in \{1, ..., n\}$ we have

$$\begin{aligned} f_i \circ \left(\left[\beta|_{[t_{i-1}, t_i]} \right], \varphi_i \circ \mathrm{Evol}_C |_{[t_{i-1}, t_i]} \right) &= d\varphi_i \circ \tau \left(\left[\beta|_{[t_{i-1}, t_i]} \right], \varphi_i^{-1} \circ \varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right) \\ &= d\varphi_i \circ \tau \left(\left[\beta|_{[t_{i-1}, t_i]} \right], \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right) \\ &= d\varphi_i \circ \left(\mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right)' \\ &= \left(\varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right)' . \end{aligned}$$

Hence

$$\Phi_3([\beta]) = \Big(\left(\varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \right)', \varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1}, t_i]} \Big)_{i=1}^{n}, \quad \text{for all } [\beta] \in \mathcal{Q}_{[\gamma]}.$$

We set the topological embedding (see Lemma 2.1.13)

$$\Psi_i : AC_{L^p}([t_{i-1}, t_i], V_i) \to L^p([t_{i-1}, t_i], V_i) \times C([t_{i-1}, t_i], V_i), \quad \alpha \mapsto (\alpha', \alpha)$$

and

$$\Psi : \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], V_i) \to \prod_{i=1}^{n} L^p([t_{i-1}, t_i], T_eG) \times C([t_{i-1}, t_i], V_i), \quad (\alpha_i)_{i=1}^n \mapsto (\alpha_i', \alpha_i)_{i=1}^n.$$

Thus $\mathrm{Im}(\Phi_3) \subseteq \mathrm{Im}(\Psi)$. For each $i \in \{1, ..., n\}$ we note that

$$\left(\Psi_i|^{\mathrm{Im}(\Psi_i)}\right)^{-1} \left((\varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1},t_i]})', \varphi_i \circ \mathrm{Evol}_C(\beta)|_{[t_{i-1},t_i]}\right) = \varphi_i \circ \mathrm{Evol}(\beta)|_{[t_{i-1},t_i]}$$

for all $[\beta] \in \mathcal{Q}_{[\gamma]}$. We set the continuous map

$$\Phi_4 := \left(\Psi|^{\mathrm{Im}(\Psi)}\right)^{-1} \circ \Phi_3 : \mathcal{Q}_{[\gamma]} \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], V_i), \quad [\beta] \mapsto \left(\varphi_i \circ \mathrm{Evol}(\beta)|_{[t_{i-1},t_i]}\right)_{i=1}^{n}.$$

Let denote the homeomorphism onto its image (see Proposition 2.2.9)

$$\Phi_5 : AC_{L^p}([0,1], G) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], G), \quad \eta \mapsto (\eta|_{[t_{i-1},t_i]})_{i=1}^{n}.$$

and the homeomorphism

$$\Phi_6 : \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], V_i) \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], U_i), \quad \tau_i \mapsto (\varphi_i^{-1} \circ \tau_i)_{i=1}^{n}.$$

We see that

$$\Phi_6 \circ \Phi_4 : \mathcal{Q}_{[\gamma]} \to \prod_{i=1}^{n} AC_{L^p}([t_{i-1}, t_i], U_i), \quad [\beta] \mapsto \left(\mathrm{Evol}(\beta)|_{[t_{i-1},t_i]}\right)_{i=1}^{n}.$$

Since each function $\mathrm{Evol}(\beta)$ is continuous, we have that $(\Phi_6 \circ \Phi_4)\left(\mathcal{Q}_{[\gamma]}\right) \subseteq \mathrm{Im}(\Phi_5)$, hence

$$\mathrm{Evol}\,|_{\mathcal{Q}_{[\gamma]}} = \left(\Phi_5|^{\mathrm{Im}(\Phi_5)}\right)^{-1} \circ \Phi_6 \circ \Phi_4.$$

Thus the evolution map Evol is continuous.
For the case L^∞_{rc}, by Lemma 2.4.13, the proof is analogous since the map

$$\tilde{f}_i : L^\infty_{rc}([t_{i-1}, t_i], T_eG) \times C([t_{i-1}, t_i], V_i) \to L^\infty_{rc}([t_{i-1}, t_i], E), \quad ([\beta], \eta) \mapsto f_i \circ ([\beta], \eta)$$

is continous for each $i \in \{1, ..., n\}$. □

2.5 Semiregularity of $\mathrm{Diff}^r_K(\mathbb{R}^n)$

Let $n, m, r \in \mathbb{N}$. We consider the Fréchet space $C(\mathbb{R}^n, \mathbb{R}^m)$ of continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$, whose topology is generated by the familiy of seminorms

$$\|\cdot\|_L : C(\mathbb{R}^n, \mathbb{R}^m) \to [0, \infty), \quad \phi \mapsto \sup_{x \in L} |\phi(x)|$$

for each non empty compact subset $L \subset \mathbb{R}^n$. Let $C^r(\mathbb{R}^n, \mathbb{R}^m)$ be the Fréchet space of all C^r-maps $\phi : \mathbb{R}^n \to \mathbb{R}^n$, whose topology is the compact-open C^r-topology, i.e., the initial topology with respect to the maps

$$C^r(\mathbb{R}^n, \mathbb{R}^m) \to C(\mathbb{R}^n, \mathbb{R}^m), \quad \phi \mapsto \frac{\partial^\alpha \phi}{\partial x^\alpha}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$. Let $K \subseteq \mathbb{R}^n$ be a non empty compact subset, we define the Banach space of C^r-maps supported in K as

$$C^r_K(\mathbb{R}^n, \mathbb{R}^m) := \{\phi \in C^r(\mathbb{R}^n, \mathbb{R}^m) : \phi|_{\mathbb{R}^n \setminus K} = 0\} \tag{2.5.1}$$

endowed with the induced topology.
We denote derivative map

$$D : C^r_K(\mathbb{R}^n, \mathbb{R}^n) \to C^{r-1}_K(\mathbb{R}^n, \mathbb{R}^{n\times n}), \quad f \mapsto Df := f' \tag{2.5.2}$$

where $f'(x)$ is the Jacobian matrix of f.

Definition 2.5.1 Let $\mathrm{Diff}^r(\mathbb{R}^n)$ be the set of C^r-diffeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$. The set $\mathrm{Diff}^r(\mathbb{R}^n)$ is a group under the composition and we define the subgroup $\mathrm{Diff}^r_K(\mathbb{R}^n)$ of C^r-diffeomorphisms which are supported in a compact set $K \subset \mathbb{R}^n$ as

$$\mathrm{Diff}^r_K(\mathbb{R}^n) := \{\phi \in \mathrm{Diff}^r(\mathbb{R}^n) : \phi - \mathrm{id}_{\mathbb{R}^n} \in C^r_K(\mathbb{R}^n, \mathbb{R}^n)\}. \tag{2.5.3}$$

Let $\mathcal{V}_K := \{\phi - \mathrm{id}_{\mathbb{R}^n} \in C^r_K(\mathbb{R}^n, \mathbb{R}^n) : \phi \in \mathrm{Diff}^r_K(\mathbb{R}^n)\}$, then $\mathcal{V}_K$ is open in $C^r_K(\mathbb{R}^n, \mathbb{R}^n)$ and the map

$$\Phi : \mathrm{Diff}^r_K(\mathbb{R}^n) \to \mathcal{V}_K, \quad \phi \mapsto \phi - \mathrm{id}_{\mathbb{R}^n} \tag{2.5.4}$$

is a global chart for $\mathrm{Diff}^r_K(\mathbb{R}^n)$, turning it into a right half-Lie group modeled on the Banach space $C^r_K(\mathbb{R}^n, \mathbb{R}^n)$ (See [17, Proposition 14.6]). On the set $\mathcal{V}_K$ we define a group multiplication by

$$\phi * \psi := \Phi\left(\Phi^{-1}(\phi) \circ \Phi^{-1}(\psi)\right) = \psi + \phi \circ (\mathrm{id}_{\mathbb{R}^n} + \psi), \quad \text{for each } \phi, \psi \in \mathcal{V}_K$$

with the constant function 0 as neutral element.

Remark 2.5.2 We will study the L^1-semiregularity of $\mathcal{V}_K$ instead of $\mathrm{Diff}^r_K(\mathbb{R}^n)$. Since $\mathcal{V}_K$ is an open set of a locally convex space, we have

$$T\mathcal{V}_K = \mathcal{V}_K \times C^r_K(\mathbb{R}^n, \mathbb{R}^n).$$

For $\psi \in \mathcal{V}_K$ fixed, we have the right translation

$$\rho_\psi : \mathcal{V}_K \to \mathcal{V}_K, \quad \phi \mapsto \psi + \phi \circ (\mathrm{id}_{\mathbb{R}^n} + \psi)$$

and its derivative

$$d\rho_\psi : \mathcal{V}_K \times C^r_K(\mathbb{R}^n, \mathbb{R}^n) \to C^r_K(\mathbb{R}^n, \mathbb{R}^n), \quad (\phi, \varphi) \mapsto \varphi \circ (\mathrm{id}_{\mathbb{R}^n} + \psi).$$

Identifying $T_0\mathcal{V}_K$ with $\{0\} \times C^r_K(\mathbb{R}^n, \mathbb{R}^n)$, the restricted right action is given by

$$(\{0\} \times C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \times \mathcal{V}_K \to T\mathcal{V}_K, \quad ((0, \varphi), \psi) \mapsto d\rho_\psi(0, \varphi) = \varphi \circ (id_{\mathbb{R}^n} + \psi) \tag{2.5.5}$$

Hence the right action is continuous (see e.g. [16]).
If $(\phi, \varphi) \in \mathcal{V}_K \times C^r_K(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$(\phi, \varphi).\psi := T\rho_\psi(\phi, \varphi) = (\phi * \psi, \varphi \circ (\mathrm{id}_{\mathbb{R}^n} + \psi)).$$

If $(0,\gamma) \in \mathcal{L}^1([0,1], T_0\mathcal{V}_K)$, we want to find a function $\eta \in AC_{L^p}([0,1], \mathcal{V}_K)$ such that $\eta' = \gamma.\eta$, i.e.

$$(\eta(t), \eta'(t)) = \Big(\eta(t), \gamma(t) \circ (\mathrm{id}_{\mathbb{R}^n} + \eta(t))\Big), \text{ for almost all } t \in [0,1].$$

In other words, for $\gamma \in \mathcal{L}^1([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n))$, we need a function $\eta \in AC_{L^p}([0,1], \mathcal{V}_K)$ such that $\eta(0) = 0$ and

$$\eta(t) = \int_0^t \gamma(s) \circ (\mathrm{id}_{\mathbb{R}^n} + \eta(s))ds, \text{ for all } t \in [0,1].$$

Setting $\zeta = \mathrm{id}_{\mathbb{R}^n} + \eta(s)$, this is equivalent to

$$\zeta(t) = \mathrm{id}_{\mathbb{R}^n} + \int_0^t \gamma(s) \circ \zeta(s)ds, \text{ for all } t \in [0,1].$$

Let $\varepsilon_x : C^r_K(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$ be the point evaluation map for $x \in \mathbb{R}^n$, then each ε_x is continuous linear and the family of maps $(\varepsilon_x)_{x\in\mathbb{R}^n}$ separate points on $C^r_K(\mathbb{R}^n, \mathbb{R}^n)$, hence the equation holds if and only if the functions $\zeta_x := \varepsilon_x \circ \zeta \in AC_{L^1}([0,1], \mathbb{R}^n)$ satisfy

$$\zeta_x(t) = x + \int_0^t \gamma(s) \circ \zeta_x(s)ds, \text{ for all } t \in [0,1].$$

Theorem 2.5.3 *Let $1 \leq p < \infty$. If $r \in \mathbb{N}$, then the right half-Lie group $\mathit{Diff}^r_K(\mathbb{R}^n)$ is L^p-semiregular. Moreover, the evolution map*

$$\mathrm{Evol} : L^p([0,1], T_e\mathit{Diff}^r_K(\mathbb{R}^n)) \to AC_{L^p}([0,1], \mathit{Diff}^r_K(\mathbb{R}^n)), \quad \gamma \mapsto \eta_\gamma.$$

is continuous.

Proof. Following the discussion in Remark 2.5.2, we will show that $\mathcal{V}_K$ is locally L^p-semiregular.
We define the continuous seminorm

$$\alpha : C^r_K(\mathbb{R}^n, \mathbb{R}^n) \to [0,\infty), \quad \alpha(f) := \|f\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} = \sup_{x\in\mathbb{R}^n} \|Df(x)\|_{op}.$$

For $0 < L < 1$, we denote the open ball centered in $0 \in L^p([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n))$ by

$$B_L := \left\{ [\gamma] \in L^p([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) : \|\gamma\|_{L^p,\alpha} := \left(\int_0^1 (\alpha \circ \gamma)^p(t)dt \right)^{1/p} < L \right\}.$$

For $x \in \mathbb{R}^n$, we define the smooth map

$$c : \mathbb{R}^n \to C([0,1], \mathbb{R}^n), \quad x \mapsto [t \mapsto x].$$

Let $J : L^1([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$ be the continuous linear operator

$$J([\xi])(t) := \int_0^t \xi(s)ds, \text{ for all } t \in [0,1].$$

Since the evaluation map $\varepsilon : C^r_K(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^{\infty,r}$ in the sense of [1] and hence C^r. By Lemma 2.4.12 the map

$$\Phi_n : L^p([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \times C([0,1], \mathbb{R}^n) \to L^p([0,1], \mathbb{R}^n), \quad \Phi_n([\gamma], \zeta) := \varepsilon \circ ([\gamma], \zeta)$$

is well defined and is C^r. We define the operator $T : B_L \times \mathbb{R}^n \times C([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$ via

$$\begin{aligned} T([\gamma], x, \zeta)(t) &:= c_x(t) + J(\Phi_n([\gamma], \zeta))(t) \\ &= x + \int_0^t \gamma(s)\,(\zeta(s))\,ds \end{aligned}$$

for $t \in [0,1]$. Then T is $C^{\infty,\infty,r}$ in the sense of [2].
Let $[\gamma] \in B_L$ and $x \in \mathbb{R}^n$ be fixed, for $\zeta_1, \zeta_2 \in C([0,1], \mathbb{R}^n)$, by the Mean Value Theorem we have

$$\gamma(t) \circ \zeta_2(t) - \gamma(t) \circ \zeta_1(t) = \int_0^1 \Big(D\gamma(t)\Big)(\zeta_1(t) + s(\zeta_2(t) - \zeta_1(t))).(\zeta_2(t) - \zeta_1(t))ds.$$

Thus

$$\begin{aligned} \|T([\gamma], x, \zeta_2) - T([\gamma], x, \zeta_1)\|_\infty &= \sup_{t\in[0,1]} \left\| \int_0^t (\gamma(s) \circ \zeta_2(s) - \gamma(s) \circ \zeta_1(s))ds \right\|_\infty \\ &\leq \sup_{t\in[0,1]} \int_0^1 \left\| \Big(D\gamma(t)\Big)(\zeta_1(t) + s(\zeta_2(t) - \zeta_1(t))).(\zeta_2(t) - \zeta_1(t)) \right\|_\infty ds \\ &\leq \sup_{t\in[0,1]} \int_0^1 \|D\gamma(t)\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} \|\zeta_2 - \zeta_1\|_\infty dt \\ &= \sup_{t\in[0,1]} \int_0^1 \alpha(\gamma(t))dt \|\zeta_2 - \zeta_1\|_\infty \\ &\leq \|\gamma\|_{L^1,\alpha} \|\zeta_2 - \zeta_1\|_\infty \\ &\leq \|\gamma\|_{L^p,\alpha} \|\zeta_2 - \zeta_1\|_\infty \\ &\leq L\|\zeta_2 - \zeta_1\|_\infty. \end{aligned}$$

Hence $T([\gamma], x, \cdot)$ is a C^r-map and an L-contraction. By Banach's Fixed Point Theorem, there exists $\zeta_{[\gamma],x} \in C([0,1], \mathbb{R}^n)$ such that

$$T([\gamma], x, \zeta_{[\gamma],x}) = \zeta_{[\gamma],x}.$$

By [15, Lemma 6.2] the map

$$F_1 : B_L \times \mathbb{R}^n \to C([0,1], \mathbb{R}^n), \quad ([\gamma], x) \mapsto \zeta_{[\gamma],x}$$

is C^r. By the exponential laws, we have:

a) $F_2 : (B_L \times \mathbb{R}^n) \times [0,1] \to \mathbb{R}^n, \quad ([\gamma], x, t) \mapsto \zeta_{[\gamma],x}(t)$ is $C^{r,0}$.

b) $F_3 : B_L \times [0,1] \to C^r(\mathbb{R}^n, \mathbb{R}^n), \quad ([\gamma], t) \mapsto \zeta_{[\gamma],\cdot}(t)$ is $C^{r,0}$.

c) $F_4 : B_L \to C([0,1], C^r(\mathbb{R}^n, \mathbb{R}^n)), \quad [\gamma] \mapsto \zeta_{[\gamma]}$ is continuous.

Let $x \in \mathbb{R}^n \setminus K$ and $\zeta_{[\gamma,x}$ be the fixed point of

$$\zeta_{[\gamma],x}(t) = x + \int_0^t \gamma(s) \left(\zeta_{[\gamma],x}\right)(s) ds.$$

Since $\gamma(t)(x) = 0$ for each $t \in [0,1]$, the constant map c_x is also a fixed point and by uniqueness of solutions we have $\zeta_{[\gamma],x} = c_x$. Let $\mathrm{I} : [0,1] \to C^r(\mathbb{R}^n, \mathbb{R}^n)$, $t \mapsto \mathrm{id}_{\mathbb{R}^n}$ the constant map mapping to the identity map. We define

$$S : B_L \to C([0,1], C^r(\mathbb{R}^n, \mathbb{R}^n)), \quad [\gamma] \mapsto F_4([\gamma]) - \mathrm{I}.$$

Then for each $t \in [0,1]$ we have $S([\gamma])(t) \in C^r_K(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, since S is continuous and $\mathcal{V}_K$ is open, there is exists an 0-neighborhood $B \subseteq B_L$ such that $S(B) \subseteq \mathcal{V}_K$. Hence, we can consider the map S as the continuous map

$$\tilde{S} : B \to C([0,1], \mathcal{V}_K), \quad [\gamma] \mapsto F_4([\gamma]) - \mathrm{I},$$

where, by the discussion in Remark 2.5.2, if $[\gamma] \in B$ we have that $\eta := \tilde{S}([\gamma])$ is solution of

$$\eta(t) = \int_0^t \gamma(s) \circ (\mathrm{id}_{\mathbb{R}^n} + \eta(s)) ds, \text{ for all } t \in [0,1].$$

Hence $\mathrm{Evol}(\gamma) = \eta$. In consequence, $\mathcal{V}_K$ is locally L^p-semiregular and thus, by Lemma 2.4.9, the half-Lie group $\mathrm{Diff}^r_K(\mathbb{R}^n)$ is L^p-semiregular.
Moreover, the evolution map restricted to the 0-neighborhood B is given by

$$\mathrm{Evol}|_B : B \subseteq L^p([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \to AC_{L^p}([0,1], \mathcal{V}_K), \quad \gamma \mapsto \tilde{S}(\gamma).$$

Since $\tilde{S}$ is continuous, by Lemma 2.4.11 the evolution map with continuous values Evol_C is continuous. Moreover, since the restricted right action of $\mathrm{Diff}^r_K(\mathbb{R}^n)$ is continuous, by Theorem 2.4.14, the evolution map

$$\mathrm{Evol} : L^p([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \to AC_{L^p}([0,1], \mathcal{V}_K), \quad \gamma \mapsto \mathrm{Evol}(\gamma)$$

is continuous. □

Proceeding exactly as the case L^p, for the case L^∞_{rc} we have the same result.

Proposition 2.5.4 *If $r \in \mathbb{N}$, then the right half-Lie group $\mathit{Diff}^r_K(\mathbb{R}^n)$ is L^∞_{rc}-semiregular. Moreover, the evolution map*

$$\mathrm{Evol} : L^\infty_{rc}([0,1], T_e\mathit{Diff}^r_K(\mathbb{R}^n)) \to AC_{L^\infty_{rc}}([0,1], \mathit{Diff}^r_K(\mathbb{R}^n)), \quad \gamma \mapsto \eta_\gamma$$

is continuous.

Proof. For $0 < L < 1$, we define the open ball centered in $0 \in L^\infty_{rc}([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n))$ via

$$B_L := \{[\gamma] \in L^\infty_{rc}([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) : \|\gamma\|_{L^\infty_{rc},\alpha} := \operatorname*{ess\,sup}_{t\in[0,1]} \alpha(\gamma(t)) < L\}$$

Following the same notation as Proposition 1.0.7, since the evaluation map ε is $C^{r,\infty}$, by Lemma 2.4.13 the map

$$\Phi_n : L^\infty_{rc}([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \times C([0,1], \mathbb{R}^n) \to L^\infty_{rc}([0,1], \mathbb{R}^n), \quad \Phi_n([\gamma], \zeta) := \varepsilon \circ ([\gamma, \zeta])$$

is C^r. Hence the operator $T : B_L \times \mathbb{R}^n \times C([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$ given by

$$T([\gamma], x, \zeta)(t) := x + \int_0^t \gamma(s)\,(\zeta(s))\,ds, \quad \text{for } t \in [0,1].$$

is a $C^{\infty,\infty,r}$-map. Let $[\gamma] \in L^\infty_{rc}([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n))$ and $x \in \mathbb{R}^n$ be fixed, for $\zeta_1, \zeta_2 \in C([0,1], \mathbb{R}^n)$, by the Mean Value Theorem we have

$$\gamma(t) \circ \zeta_2(t) - \gamma(t) \circ \zeta_1(t) = \int_0^1 \Big(D\gamma(t)\Big)(\zeta_1(t) + s(\zeta_2(t) - \zeta_1(t))).(\zeta_2(t) - \zeta_1(t))ds$$

and

$$\begin{aligned}
\|\Phi_n([\gamma], \zeta_2)(t) - \Phi_n([\gamma], \zeta_1)(t)\|_\infty &= \|\gamma(t) \circ \zeta_2(t) - \gamma(t) \circ \zeta_1(t)\|_\infty \\
&\leq \int_0^1 \|\Big(D\gamma(t)\Big)(\zeta_1(t) + s(\zeta_2(t) - \zeta_1(t))).(\zeta_2(t) - \zeta_1(t))\|_\infty ds \\
&\leq \|D\gamma(t)\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} \|\zeta_2 - \zeta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} \\
&= \alpha(\gamma(t)) \|\zeta_2 - \zeta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}.
\end{aligned}$$

Hence

$$\|\Phi_n([\gamma], \zeta_2) - \Phi_n([\gamma], \zeta_1)\|_{L^\infty_{rc},\infty} \leq \|\gamma\|_{L^\infty_{rc},\alpha} \|\zeta_2 - \zeta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}.$$

Thus $T([\gamma], x, \cdot)$ is a C^r-map and a L-contraction. With this, following the same steps as Proposition 1.0.7 we can show that $\mathrm{Diff}^r_K(\mathbb{R}^n)$ is L^∞_{rc}-semiregular and there exists a 0-neighborhood B in $L^\infty_{rc}([0,1], T_e\mathrm{Diff}^r_K(\mathbb{R}^n))$ such that the restricted map evolution map

$$\mathrm{Evol}_C|_B : B \to AC_{L^\infty_{rc}}([0,1], \mathrm{Diff}^r_K(\mathbb{R}^n)), \quad [\gamma] \mapsto \mathrm{Evol}_C(\gamma)$$

is continuous. By Lemma 2.4.11 the evolution map with continuous values Evol_C is continuous. Moreover, since the restricted right action of $\mathrm{Diff}^r_K(\mathbb{R}^n)$ is continuous, by Theorem 2.4.14, the evolution map

$$\mathrm{Evol} : L^\infty_{rc}([0,1], C^r_K(\mathbb{R}^n, \mathbb{R}^n)) \to AC_{L^\infty_{rc}}([0,1], \mathcal{V}_K), \quad \gamma \mapsto \mathrm{Evol}(\gamma)$$

is continuous. □

2.6 L^p-Semiregularity of Diff $^r(M)$

Let $r \in \mathbb{N}$. Let M be a compact smooth manifold without boundary, g be a smooth Riemannian metric on M and $\exp : \mathcal{D} \subseteq TM \to M$ be the Riemannian exponential map, with $\exp_p := \exp|_{T_pM}$ for each $p \in M$. Let $\pi_{TM} : TM \to M$ be the bundle projection and $\mathcal{W} \subseteq \mathcal{D}$ be an open neighborhood of the zero-section such that the map

$$(\pi_{TM}, \exp) : \mathcal{W} \subseteq TM \to M \times M, \quad v \mapsto (\pi_{TM}(v), \exp(v))$$

is a diffeomorphism into its image. In particular, if $\mathcal{W}_p := \mathcal{W} \cap T_pM$, the map

$$\exp_p|_{\mathcal{W}_p} : \mathcal{W}_p \subseteq T_pM \to \exp_p(\mathcal{W}_p) \subseteq M$$

is a C^∞-diffeomorphism for each $p \in M$.
Let Diff $^r(M)$ be the set of all C^r-diffeomorphism $\phi : M \to M$. For each $\phi \in$ Diff $^r(M)$ we define the Banach space of C^r-sections

$$\Gamma_{C^r}(\phi) := \{X \in C^r(M, TM) : \pi_{TM} \circ X = \phi\}$$

and the open 0-neighborhood

$$\mathcal{V}_\phi := \{X \in \Gamma_{C^r}(\phi) : X(M) \subseteq \mathcal{W}\}.$$

If

$$\mathcal{U}_\phi := \{\exp \circ X \in \text{Diff}^{\,r}(M) : X \in \mathcal{V}_\phi\}$$

then the map

$$\Psi_\phi : \mathcal{U}_\phi \to \mathcal{V}_\phi, \quad \Psi(\phi)(p) := \left(\exp_p|_{\mathcal{W}_p}\right)^{-1}(\phi(p))$$

define a chart for Diff $^r(M)$ on ϕ, with inverse given by

$$\Psi_\phi^{-1} : \mathcal{V}_\phi \to \mathcal{U}_\phi, \quad X \mapsto \exp \circ X,$$

turning Diff $^r(M)$ into a Banach manifold. Moreover, under the composition the manifold Diff $^r(M)$ becomes a right half-Lie group (see e.g. [17]). We see that for the neutral element $e = \text{id}_M$, the tangent space T_eDiff $^r(M)$ coincide with space of C^r-vector fields $X : M \to TM$, denoted by $\mathcal{X}^r(M)$.
The restricted right action

$$\mathcal{X}^r(M) \times \text{Diff}^r_K(\mathbb{R}^n) \to T\text{Diff}^r_K(\mathbb{R}^n), \quad (X, \phi) \mapsto X.\phi = X \circ \phi \tag{2.6.1}$$

is continuous (see e.g. [16]).
Let $[\gamma] \in L^p([0,1], \mathcal{X}^r(M))$, we will study the equation

$$\dot{\eta}(t) = \gamma(t).\eta(t), \quad t \in [0,1] \tag{2.6.2}$$

$$\eta(0) = \text{id}_M. \tag{2.6.3}$$

Definition 2.6.1 Let $(E, \|\cdot\|_E)$ be a normed space and $U \subseteq E$ be a subset. We say that a function $f : [a,b] \times U \to E$ satisfies an L^1-Lipschitz condition if there exists a measurable function $g \in L^1([a,b], \mathbb{R})$ such that

$$\mathrm{Lip}(f(t,\cdot)) \leq g(t), \quad t \in [a,b],$$

where $\mathrm{Lip}(f(t,\cdot)) \in [0,\infty]$ denote the infimum of all Lipschitz constants for $f(t,\cdot) : U \to E$.

Definition 2.6.2 Let M be a C^1-manifold modeled on a normed space E, $J \subseteq \mathbb{R}$ be a non-degenerate interval and $f : J \times M \to TM$ be a function with $f(t,p) \in T_pM$ for all $(t,p) \in J \times M$. We say that f satisfies a local L^1-Lipschitz condition if for all $t_0 \in J$ and $P \in M$, there exists a chart $\kappa : U_\kappa \to V_\kappa$ of M on p and a relatively open subinterval $[a,b] \subseteq J$ which is a neighborhood of t_0 in J such that the map

$$f_\kappa : [a,b] \times V_\kappa \to E, \quad (t,x) \mapsto d\kappa\left(f(t,\kappa^{-1}(y))\right)$$

satisfies an L^1-Lipschitz condition.

Remark 2.6.3 Let $(E, \|\cdot\|)$ be a normed space and $f : [a,b] \times M \to TM$ be a map with $f(t,q) \in T_qM$ for all $(t,q) \in [a,b] \times M$, which satisfies a local L^1-condition. If $\tau, \eta : [a,b] \to M$ are two AC_{L^1}-Carathéodory solutions to

$$y' = f(t,y)$$

satisfying $\tau(t_0) = \eta(t_0)$ for some $t_0 \in [a,b]$, then $\tau = \eta$ [15, Proposition 10.5]. If the initial value problem

$$\begin{aligned} y'(t) &= f(t,y(t)), \quad t \in [a,b] \\ y(0) &= q \end{aligned}$$

has a (necessarily unique) solution $\gamma_q : [a,b] \to M$ for each $q \in M$, then we say that f admits a global flow for initial time t_0 and write

$$F^f_{t,a}(q) := \gamma_q(t),$$

for all $t \in [a,b]$ and $q \in M$.

The following result can be found in the appendix of [15] for the context of AC_{L^1}-functions with values in the Lie group of C^∞-diffeomorphism with compact support $\mathrm{Diff}^\infty_c(M)$. However, this result is also valid in our context.

Theorem 2.6.4 *Let $1 \leq p < \infty$ and $[\gamma] \in L^p([0,1], \mathcal{X}^r(M))$, then map*

$$\overline{\gamma} : [0,1] \times M \to TM, \quad \overline{\gamma}(t,q) := \gamma(t)(q)$$

satisfies a local L^1-Lipschitz condition. Let $\eta \in AC_{L^p}([0,1], \mathit{Diff}^{\,r}(M))$ with $\eta(0) = id_M$. Then $\eta = \mathrm{Evol}([\gamma])$ if and only if f admits global flow F for initial time $t_0 = 0$ and

$$\eta(t)(p) = F^{\overline{\gamma}}_{t,0}(q),$$

for all $t \in [0,1]$ and $q \in M$.

The following result follows the same steps as the proof of the L^1-regularity of $\mathrm{Diff}_c^\infty(M)$ proved in [15, Section 11]. We recommend to the reader to read the reference to see some steps in detail.

Theorem 2.6.5 *Let M be a compact smooth manifold and $1 \leq p < \infty$. If $r \in \mathbb{N}$, then the right half-Lie group $\mathit{Diff}^{\,r}(M)$ is L^p-semiregular. Moreover, the evolution map*

$$\mathrm{Evol} : L^p([0,1], T_e\mathit{Diff}^{\,r}(M)) \to AC_{L^p}([0,1], \mathit{Diff}^{\,r}(M)), \quad [\gamma] \mapsto \eta_{[\gamma]}$$

is continuous.

Proof. For $\phi \in \mathrm{Diff}^{\,r}(M)$, we can choose a finite locally finite cover $(U_i)_{i=1}^n$ of M by relatively compact, open subsets sets and charts $\kappa_i : U_i \to B_5(0) \subseteq \mathbb{R}^n$ such that the family of open sets $\kappa_i^{-1}(B_1(0))$ cover M and $U_i \subseteq \phi^{-1}(U_{\psi_i})$ for some chart $\psi_i : U_{\psi_i} \to V_{\psi_i}$ of M (see e.g. [27]). For $X \in \Gamma_{C^r}(\phi)$ we write

$$X_i := d\psi_i \circ X \circ \kappa_i^{-1} : B_5(0) \to \mathbb{R}^m.$$

Then, for each $\ell \in [1,5]$ the map

$$\rho_\ell : \Gamma_{C^r}(\phi) \to \prod_{i=1}^n C^r(B_\ell(0), \mathbb{R}^m), \quad X \mapsto \rho_\ell(X) := (X_i|_{B_\ell(0)})_{i=1}^n$$

is a linear topological embedding with closed image (see [3]).
Let $\phi = \mathrm{id}_M$, then $\Gamma_{C^r}(\mathrm{id}_M) = \mathcal{X}^r(M)$. Doing the corresponding identifications of product spaces, we have the linear topological embeddings with closed image

$$R_5 := L^p([0,1], \rho_5) : L^p([0,1], \mathcal{X}^r(M)) \to \prod_{i=1}^n L^p([0,1], C^r(B_5(0), \mathbb{R}^m)), \quad \gamma \mapsto \rho_5 \circ \gamma$$

and

$$R_1 := C([0,1], \rho_1) : C([0,1], \mathcal{X}^r(M)) \to \prod_{i=1}^n C([0,1], C^r(B_1(0), \mathbb{R}^m)), \quad \gamma \mapsto \rho_1 \circ \gamma.$$

Before we study the equation for $[\gamma] \in L^p([0,1], \mathcal{X}^r(M))$, the topological embedding R_1 allows us to study it first for

$$[\gamma] \in L^p([0,1], C^r(B_5(0), \mathbb{R}^m)).$$

We consider the continuous seminorm $q : C^r(B_5(0), \mathbb{R}^m) \to [0,\infty)$ given by

$$q(\phi) := \sup_{x \in \overline{B}_4(0)} \left(\|\phi'(x)\|_{op} + \|\phi(x)\|_\infty \right), \quad \text{for all } \phi \in C^r(B_5(0), \mathbb{R}^n).$$

For $0 < L < 1$ we set the open ball

$$B_L := \{\gamma \in L^p([0,1], C^r(B_5(0), \mathbb{R}^m)) : \|\gamma\|_{L^p,q} < L\}.$$

Just like the case for $\mathrm{Diff}_K^r(\mathbb{R}^n)$, we need an AC_{L^p}-Carathéodory solution $\zeta : [0,1] \to B_4(0)$ that verifies the integral equation

$$\zeta'(s) = x + \int_0^t \gamma(s)\,(\zeta(s))\,ds, \quad t \in [0,1],$$

for $[\gamma] \in B_L$ and $x \in B_3(0)$.
We define the operator $T : B_L \times B_3(0) \times C([0,1], B_4(0)) \to C([0,1], B_4(0))$ by

$$T([\gamma], x, \zeta)(t) := x + \int_0^t \gamma(s)\,(\zeta(s))\,ds, \quad t \in [0,1].$$

By Lemma 2.4.12, the map T is $C^{\infty,\infty,r}$ and for each $[\gamma] \in B_L$ and $x \in B_3(0)$ we have

$$\mathrm{Lip}(\Psi([\gamma], x, \cdot)) \leq L.$$

Thus, by Banach's Fixed Point Theorem, for each $([\gamma], x) \in B_L \times B_3(0)$ the map $\Psi([\gamma], x, \cdot)$ has a unique fixed point $\zeta_{[\gamma],x} \in C([0,1], B_4(0))$ and

$$F : B_L \times B_3(0) \to C([0,1], B_4(0)), \quad ([\gamma], x) \mapsto \zeta_{[\gamma],x}$$

is C^r (see [15, Lemma 6.2]). By the exponential law, we have

a) $F_2 : (B_L \times B_3(0)) \times [0,1] \to B_4(0), \quad ([\gamma], t, x) \mapsto \zeta_{[\gamma],x}(t)$ is $C^{r,0}$.

b) $F_3 : B_L \times [0,1] \to C^r(B_3(0), B_4(0)), \quad ([\gamma], t) \mapsto \zeta_{[\gamma]}(t)$ is $C^{r,0}$.

c) $F_4 : B_L \to C([0,1], C^r(B_3(0), B_4(0))), \quad [\gamma] \mapsto \zeta_{[\gamma]}$ is continuous.

Let $\rho_2 : C^r(B_3(0), B_4(0)) \to C^r(B_2(0), B_4(0))$, $\varphi \mapsto \varphi|_{B_2(0)}$, we denote

$$H := C([0,1], \rho_2) \circ F_4 : B_L \to C([0,1], C^r(B_2(0), B_4(0)), \quad [\gamma] \mapsto \rho_2 \circ \zeta_{[\gamma]}.$$

Then H is continuous. We will show that H has absolutely continuous values.
Let consider the open set

$$\lfloor \overline{B_2(0)}, B_4(0) \rfloor_r := \{\varphi \in C^r(B_3(0), \mathbb{R}^m) : \varphi(\overline{B_2(0)}) \subseteq B_4(0)\}.$$

Then the map

$$S : C^r(B_5(0), \mathbb{R}^n) \times \lfloor \overline{B_2(0)}, B_4(0) \rfloor_r \to C^r(B_2(0), \mathbb{R}^m), \quad (\psi, \varphi) \mapsto \psi \circ \varphi$$

is continuous. Moreover, since S is linear in the first variable, by Lemma 2.4.12, the map

$$\tilde{S} : L^p([0,1], C^r(B_5(0), \mathbb{R}^n)) \times C([0,1], \lfloor \overline{B_2(0)}, B_4(0) \rfloor_r) \to L^p([0,1], C^r(B_2(0), \mathbb{R}^m))$$
$$(\alpha, \beta) \mapsto S \circ (\alpha, \beta)$$

is continuous. Hence, for $[\gamma] \in B_L$ we have

$$S \circ (\gamma, H([\gamma])) = \gamma\left(H(\gamma)\right) = \gamma.H([\gamma]) \in L^p([0,1], C^r(B_2(0), \mathbb{R}^m)).$$

This allow us to define the L^p-absolutely continuous function $\tau : [0,1] \to C^r(B_2(0), \mathbb{R}^m)$ given by

$$\tau(t) = \mathrm{id}_{B_2(0)} + \int_0^t \gamma(s)\left(H([\gamma])(s)\right) ds, \quad \text{for each } t \in [0,1].$$

If we consider the point evaluation map $\varepsilon_x : C^r(B_2(0), \mathbb{R}^m) \to \mathbb{R}^m$, $\varphi \mapsto \varphi(x)$, then for each $x \in B_2(0)$ we have

$$\varepsilon_x\left(\tau(t)\right) = \varepsilon_x\left(H([\gamma])(t)\right), \quad \text{for all } t \in [0,1].$$

Since the family of maps $(\varepsilon_x)_{x \in B_2(0)}$ separate points in $C^r(B_2(0), \mathbb{R}^m)$, we have $\tau = H([\gamma])$.
Hence $H[\gamma]$ is absolutely continuous and we can consider the map

$$H_{AC} : B_L \to AC_{L^p}([0,1], C^r(B_2(0), B_4(0)), \quad [\gamma] \mapsto H([\gamma])$$

which is the evolution map for $C^r(B_2(0), B_4(0))$.
Now we will see the case for $L^p([0,1], \mathcal{X}^r(M))$. We define the 0-neighborhood of $L^p([0,1], \mathcal{X}^r(M))$

$$\begin{aligned} \mathcal{B}_L &:= R_5^{-1}\left(\prod_{i=1}^{n} B_L\right) \\ &= \{[\gamma] \in L^p([0,1], \mathcal{X}^r(M)) : (\forall i \in \{1, ..., n\})\ [\gamma_i] \in B_L\} \end{aligned}$$

which is open by continuity of R_5.
Let $[\gamma] \in \mathcal{B}_L$, with $[\gamma_i] \in B_L$ for each $i \in \{1, ..., n\}$. If $m \in M$, there is a chart $\kappa_j : U_j \to B_5(0)$ such that $m \in \kappa_j^{-1}(B_3(0))$ for some $j \in \{1, ..., n\}$. Let $\zeta_{[\gamma],\kappa_j(m)} \in AC_{L^P}([0,1], C^r(B_2(0), B_4(0))$ be the solution of the integral equation

$$\zeta_{[\gamma],\kappa_j(m)}(t) = \kappa_j(m) + \int_0^t \gamma_j(s)\left(\zeta_{[\gamma],\kappa_j(m)}(s)\right) ds, \quad t \in [0,1].$$

Then $\zeta_{[\gamma],\kappa_j(m)}$ is a AC_{L^p}-Carathéodory solution to

$$\begin{aligned} x'(t) &= \gamma_j(t)\left(x(t)\right), \quad t \in [0,1] \\ x(0) &= \kappa_j(m). \end{aligned}$$

Hence the function

$$\eta_{[\gamma],m} : [0,1] \to M, \quad \eta_{[\gamma],m}(t) := \kappa_j^{-1} \circ \zeta_{[\gamma],\kappa_j(m)}(t)$$

is a AC_{L^p}-Carathéodory solution of the equation

$$\begin{aligned} y'(t) &= \gamma(t)\left(y(t)\right), \quad t \in [0,1] \\ y(0) &= m. \end{aligned}$$

Since $\tilde{\gamma}$ satisfies a local L^1-Lipschitz condition, by Remark 2.6.3 the solution of this equation is unique and $\eta_{[\gamma],m}$ is well defined. Moreover, $\tilde{\gamma}$ admits a global flow for initial time $t_0 = 0$, given by

$$F^{\overline{\gamma}}_{t,0}(m) = \eta_{[\gamma],m}(t)$$

for all $t \in [0,1]$, $m \in M$.
Following point 11.16 of [15], for each $i \in \{1,...,n\}$ we can construct an exponential map

$$\exp_i : \mathcal{D}_i \subseteq TB_5(0) \to B_5(0)$$

such that

$$\kappa_i^{-1} \circ \exp_i = \exp \circ T\kappa_i^{-1}|_{\mathcal{D}_i}.$$

There is an open set $\mathcal{O}_i \subseteq \mathcal{D}_i$ containing $\overline{B_4(0)} \times \{0\}$ such that $(\mathrm{pr}_1, \exp_i)(Q_i)$ is open in $B_5(0) \times B_5(0)$ and the map

$$\psi_i := (\mathrm{pr}_1, \exp_i)|_{\mathcal{O}_i} : \mathcal{O}_i \subseteq TB_5(0) \to \psi_i(\mathcal{O}_i) \subseteq B_5(0) \times B_4(0)$$

is a C^∞-diffeomorphism onto its image. Assuming that

$$T\kappa_i^{-1}(\mathcal{O}_i) \subseteq \mathcal{W},$$

since

$$\{(x,x) : x \in \overline{B_5(0)}\} \subseteq (\mathrm{pr}_1, \exp_i)(\mathcal{O}_i),$$

there exists $s_i \in]0,1]$ such that

$$\bigcup_{x \in B_4(0)} \{x\} \times B_{s_i}(x) \subseteq (\mathrm{pr}_1, \exp_i)(\mathcal{O}_i)$$

is a smooth diffeomorphism and there exists an $s_i \in]0,1[$, such that ψ_i^{-1} restrict to a smooth diffeomorphism of the form

$$(\mathrm{Id}_{B_4(0)}, \theta_i) : \bigcup_{x \in B_4(0)} \{x\} \times B_{s_i}(x) \subseteq B_5(0) \times B_5(0) \to \psi_i^{-1}\left(\bigcup_{x \in B_4(0)} \{x\} \times B_{s_i}(x)\right) \subseteq \mathcal{O}_i$$

with open image in $\mathcal{O}_i$. For each $x \in B_4(0)$ we define the set

$$\mathcal{O}_{i,x} := \{y \in \mathbb{R}^m : (x,y) \in \mathcal{O}_i\}.$$

Hence, we have

$$\theta_i(x,\cdot) = \left(\exp(x,\cdot)|_{\mathcal{O}_{i,x}}\right)^{-1}|_{B_{s_i}(0)}.$$

The set

$$Z_{i,x} := \left\{\varphi \in C^r(B_2(0), \mathbb{R}^n) : (\forall x \in \overline{B_1(0)})\ \varphi(x) \in B_{s_i}(x)\right\}$$

is open in $C^r(B_2(0), \mathbb{R}^m)$ and the map

$$(\theta_i)_* : Z_{i,x} \subseteq C^r(B_2(0), \mathbb{R}^m) \to C^r(B_1(0), \mathbb{R}^m), \quad \varphi \mapsto (\theta_i)_*(\varphi) := \theta_i \circ (\mathrm{id}_{B_1(0)}, \varphi)$$

is smooth since θ is smooth [17, Proposition 4.23].
Let $\Psi : \mathcal{U} \to \mathcal{V}$ be a chart on $\mathrm{id}_M \in \mathrm{Diff}^r(M)$, then since ρ_1 is a topological embedding we can assume that

$$\mathcal{V} := \rho_1^{-1}\left(\prod_{i=1}^n \mathcal{V}_i\right)$$

for suitable open 0-neighborhoods $\mathcal{V}_i \subseteq C^r(B_1(0), \mathbb{R}^m)$.
Since $(\theta_i)_*(\mathrm{id}_{B_2(0)}) = 0$, by continuity of $(\theta_i)_*$ there exists open $\mathrm{id}_{B_2(0)}$-neighborhoods $\mathcal{Y}_i \subseteq Z_{i,x}$ such that

$$(\theta_i)_*(\mathcal{Y}_i) \subseteq \mathcal{V}_i.$$

Since $H(0)(t) = \mathrm{id}_{B_2(0)}$ for all $t \in [0,1]$, by continuity of H there exists open 0-neighborhoods $\mathcal{P}_i \subseteq \mathcal{B}_L$ such that

$$H(\mathcal{P}_i) \subseteq C([0,1], \mathcal{Y}_i)$$

If

$$\mathcal{P} := R_5^{-1}\left(\prod_{i=1}^n \mathcal{P}_i\right)$$

then $\mathcal{P}$ is an open 0-neighborhood in $L^p([0,1], \mathcal{X}^r(M))$ with $\mathcal{P} \subseteq \mathcal{B}_L$.
This allow us to do the composition

$$(\theta_i)_* \circ H : B_L \subseteq L^p([0,1], C^r(B_5(0), \mathbb{R}^m) \to C([0,1], C^r(B_1(0), \mathbb{R}^m)$$

Following the same steps of point 1.18 and 1.19 of [15], we have that for each $[\gamma] \in \mathcal{P}$, there exists an unique $\theta_{[\gamma]} \in C([0,1], \mathcal{X}^r(M))$ such that

$$R_1(\theta_{[\gamma]}) = \Big((\theta_i)_*(H([\gamma_i]))\Big)_{i=1}^n.$$

Then $\rho_1\left(\theta_{[\gamma]}(t)\right) \in \prod_{i=1}^n \mathcal{V}_i$, whence $\theta_{[\gamma]}(t) \in \mathcal{V}$. In consequence

$$\Psi^{-1}(\theta_{[\gamma]}(t)) = \exp \circ \theta_{[\gamma]}(t) \in \mathrm{Diff}^{\,r}(M), \quad \text{for all } t \in [0,1].$$

Let $m \in M$ with $x := \kappa_i(m)$, then we have

$$\begin{aligned}
\exp \circ \theta_{[\gamma]}(t)(m) &= \left(\kappa_i^{-1} \circ \exp_i \circ T\kappa_i\right) \circ \theta_{[\gamma]}(t)(\kappa_i^{-1}(x)) \\
&= \kappa_i^{-1} \circ \exp_i \circ (\theta_i)_*(H([\gamma_i]))(t)(x) \\
&= \kappa_i^{-1} \circ \exp_i \circ (\exp_i|_{\mathcal{W}_i})^{-1} \circ H([\gamma_i])(t)(x) \\
&= \kappa_i^{-1} \circ H([\gamma_i])(t)(x) \\
&= \kappa_i^{-1} \circ \zeta_{[\gamma_i],x}(t) \\
&= F_{t,0}^{\tilde{\gamma}}(m).
\end{aligned}$$

Hence, we define

$$\eta_{[\gamma]} := \exp \circ \theta_{[\gamma]} : [0,1] \to \mathrm{Diff}^{\,r}(M).$$

We will show that $\eta_{[\gamma]}$ is absolutely continuous. Since the exponential map exp is smooth, we need to show that $\theta_{[\gamma]}$ is absolutely continuous, but this is equivalent to show that

$$[0,1] \to \prod_{i=1}^{n} C^r(B_1(0), \mathbb{R}^m), \quad t \mapsto \rho_1 \circ \theta_{[\gamma]}(t).$$

But this map coincides with the map

$$[0,1] \to \prod_{i=1}^{n} C^r(B_1(0), \mathbb{R}^m), \quad t \mapsto \left((\theta_i)_* \left(H([\gamma_i])\right)\right)_{i=1}^{n}(t)$$

which is absolutely continuous. Since $(\theta_i)_*$ is smooth and $H([\gamma_i]) \in AC_{L^p}([0,1], C^r(B_2(0), B_4(0)))$, by Lemma 2.1.17, the composition and hence the product are absolutely continuous. Then we have that

$$\theta_{[\gamma]} \in AC_{L^p}([0,1], \mathcal{X}^r(M))$$

whence

$$\eta_{[\gamma]} \in AC_{L^p}([0,1], \operatorname{Diff}^r(M)).$$

In consequence, by Lemma 2.4.9 and Remark 2.6.3, the right half-Lie group $\operatorname{Diff}^r(M)$ is L^p-semiregular. Since the map

$$B_L \to \prod_{i=1}^{n} C([0,1], C^r(B_1(0), \mathbb{R}^n)), \quad [\gamma_i] \mapsto \left((\theta_i)_* \left(H([\gamma_i])\right)\right)_{i=1}^{n}$$

is continuous, the restricted evolution map with continuous values is given by

$$\operatorname{Evol}_C|_{\mathcal{P}} : \mathcal{P} \to AC_{L^p}([0,1], \operatorname{Diff}^r(M)), \quad [\gamma] \mapsto \eta_{[\gamma]} = \exp \circ \theta_{[\gamma]}.$$

Hence, by Lemma 2.4.11 the evolution map Evol_C is continuous and since the restricted right action of $\operatorname{Diff}^r(M)$ is continuous, by Theorem 2.4.14, the evolution map

$$\operatorname{Evol} : L^p([0,1], \mathcal{X}^r(M)) \to AC_{L^p}([0,1], \operatorname{Diff}^r(M)), \quad [\gamma] \mapsto \operatorname{Evol}(\gamma)$$

is continuous. □

Now we will focus in the L^p-semiregularity of the case $\operatorname{Diff}^r(M)$ with M a compact smooth manifold with boundary.

Definition 2.6.6 Let M and N be smooth manifolds with boundaries and suppose $f : \partial M \to \partial N$ is a diffeomorphism, We define adjunction space $M \cup_f N$ as the set formed identifying each $p \in \partial M$ with $f(p) \in \partial N$.

We recall [28, Theorem 9.29].

Theorem 2.6.7 *Let M and N be a smooth manifolds with boundaries and $f : \partial M \to \partial N$ be a diffeomorphism. Then the adjunction space $M \cup_f N$ is a topological manifold without boundary which has a smooth manifold structure such that there are regular domains $M', N' \subseteq M \cup_f N$ diffeomorphic to M and N, respectively, such that $M' \cup N' = M \cup_f N$ and $M' \cap N' = \partial M' = \partial N'$. Moreover, M and N are both compact if and only if $M \cup_f N$ is compact.*

Definition 2.6.8 Let M be a smooth manifold with boundary. If M' denotes a copy of M, we define the double of M as the smooth manifold without boundary

$$DM = M \sqcup_{\mathrm{id}_\partial} M' \tag{2.6.4}$$

where $\mathrm{id}_\partial : \partial M \to \partial M'$ is the identity map.
For $p \in M$, we denote by $(p, 0)$ and $(p, 1)$ the corresponding element in M and M', respectively, and if $p \in \partial M$, then $(p, 0) \sim (p, 1)$. By Theorem 2.6.7, the map

$$\rho : M \to DM, \quad p \mapsto [(p, 0)]$$

is an embedding onto a regular domain of DM which we identify with M.

Definition 2.6.9 Let $r \in \mathbb{N} \cup \{0, \infty\}$ and M be a smooth manifold with boundary. We define the vector space of C^r-stratified vector field on M as

$$\mathcal{X}^r_{str}(M) := \{X \in \mathcal{X}^r(M) : X(\partial M) \subseteq T\partial M\}.$$

We recall [18, Colorally 1.8].

Proposition 2.6.10 *For each $k \in \mathbb{N} \cup \{0, \infty\}$, $n \in \mathbb{N}$, $m \in \{0, ..., d\}$ and locally convex space F, the restriction map*

$$\mathcal{E} : C^k(\mathbb{R}^d, F) \to C^k([0, \infty)^m \times \mathbb{R}^{d-m}, F)$$

has a continuous linear right inverse. Moreover, the restriction map

$$C^k(\mathbb{R}^d, F) \to C^k([0, 1]^d, F)$$

has continuous linear right inverse.

Remark 2.6.11 Let $r \in \mathbb{N} \cup \{0, \infty\}$, $m \in \mathbb{N}$ and M be an m-dimensional compact smooth manifold with boundary. By compactness of M, we find charts $\varphi_i : U_i \to V_i$ of M around points $p_i \in \partial M$ such that $(U_i)_{i=1}^k$ is a finite open cover of ∂M which extends to charts $\widetilde{\varphi}_i : \widetilde{U}_i \to \widetilde{V}_i$ around p_i. For $X \in \mathcal{X}^r_{str}(M)$, we write

$$Y_i := d\varphi_i \circ X \circ \varphi_i^{-1} : V_i \subseteq [0, \infty) \times \mathbb{R}^{m-1} \to \mathbb{R}^m.$$

Without loss of generality, we assume that $V_i = [0, \infty) \times \mathbb{R}^{m-1}$ and we consider the extension $\widetilde{Y}_i : \mathbb{R}^m \to \mathbb{R}^m$ of Y_i given by Proposition 2.6.10. We define

$$\widetilde{X}_i := T\widetilde{\varphi}_i^{-1} \circ \widetilde{Y}_i \circ \widetilde{\varphi}_i : \widetilde{U}_i \to T\widetilde{U}_i.$$

Then $\widetilde{X}_i \in \mathcal{X}^r(\tilde{U}_i)$ and the map

$$\Phi_1 : \mathcal{X}^r_{str}(M) \to \prod_{i=1}^k \mathcal{X}^r(\widetilde{U}_i), \quad X \mapsto (\widetilde{X}_i)_{i=1}^k$$

is continuous linear. Let us consider $\widetilde{U}_{k+1} := M^\circ$, $\widetilde{U}_{k+2} := (M')^\circ$ and the open cover of DM

$$\mathcal{A} := \{\widetilde{U_1}, ..., \widetilde{U_k}, \widetilde{U}_{k+1}, \widetilde{U}_{k+2}\}.$$

Then there exists a partition of the unity subordinate to $\mathcal{A}$, denoted by $\{h_1, ..., h_k, h_{k+1}, h_{k+2}\}$, such that $\text{supp}(h_i) \subseteq \widetilde{U}_i$ for each $i \in \{1, ..., k, k+1, k+2\}$.
Denoting the space of all C^k-vector fields of $\widetilde{U}_i$ with support on $\text{supp}(h_i)$ as $\mathcal{X}_i^r(\widetilde{U}_i)$, we see that the map

$$\mathcal{X}^r(\widetilde{U}_i) \to \mathcal{X}_i^r(\widetilde{U}_i), \quad Y \mapsto h_i Y$$

is continuous linear.
Moreover, if $Z \in \mathcal{X}_i^r(\widetilde{U}_i)$, we can obtain an extension $\mathcal{E}_i(Z) \in \mathcal{X}^r(DM)$ of Z by extending it by 0, and a continuous linear map

$$\mathcal{E}_i : \mathcal{X}_i^r(\widetilde{U}_i) \to \mathcal{X}^r(DM), \quad Z \mapsto \mathcal{E}_i(Z).$$

For $X \in \mathcal{X}_{str}^r(M)$, we write $\widetilde{X}_{k+1} := X|_{M^\circ}$. This enables us to define the extension map

$$\alpha : \mathcal{X}_{str}^r(M) \to \mathcal{X}^r(DM), \quad X \mapsto \alpha(X) := \sum_{i=1}^{k+1} \mathcal{E}_i(h_i \widetilde{X}_i)$$

which is continuous linear.

Remark 2.6.12 Let M be a compact smooth manifiold with boundary and $r \in \mathbb{N} \cup \{\infty\}$. By [21, Proposition 1.3], the set $\text{Diff}^r(M)$ is a embedded submanifold of $C^r(M, DM)$. This allows us to consider the inclusion map restricted onto its image

$$J : \text{Diff}^r(M) \to J(\text{Diff}^r(M)) \subset C^r(M, DM), \quad \phi \mapsto \phi \tag{2.6.5}$$

then J is a diffeomorphism. Since for each $g \in C^r(M, M)$ fixed, the right translation map

$$\rho_{C^r}(g) : C^r(M, DM) \to C^r(M, DM), \quad \phi \mapsto \phi \circ g$$

is smooth. For each $g \in \text{Diff}^r(M)$ fixed, the right translation map

$$\rho(g) : \text{Diff}^r(M) \to \text{Diff}^r(M), \quad \phi \mapsto \phi \circ g$$

can be written as

$$\rho(g) = J^{-1} \circ \rho_{C^r}(J(g)) \circ J.$$

Hence $\text{Diff}^r(M)$ is a right half-Lie group.

Theorem 2.6.13 *Let M be a compact smooth manifold with boundary, $r \in \mathbb{N} \cup \{\infty\}$ and $1 \leq p < \infty$. Then the right half-Lie group $\mathit{Diff}^r(M)$ is L^p-semiregular and the evolution map with continuous values*

$$\text{Evol}_C : L^p([0,1], T_e\mathit{Diff}^r(M)) \to C([0,1], \mathit{Diff}^r(M)), \quad [\gamma] \mapsto \text{Evol}_C(\gamma)$$

is continuous.

Proof. Let consider the map $\alpha : \mathcal{X}_{str}^r(M) \to \mathcal{X}^r(DM)$ as Remark 2.6.11, we define the map

$$\tilde{\alpha} := L^p([0,1], \alpha) : L^p([0,1], \mathcal{X}_{str}^r(M)) \to L^p([0,1], \mathcal{X}^r(DM)), \quad [\gamma] \mapsto [\alpha \circ \gamma];$$

which is linear and continuous. Since DM is a compact smooth manifold without boundary, the right half-Lie group $\mathrm{Diff}^r(DM)$ is L^p-semiregular with continuous evolution map denoted by

$$\mathrm{Evol}_{DM} : L^p([0,1], \mathcal{X}^r(DM)) \to AC_{L^p}([0,1], \mathrm{Diff}^r(DM)), \quad [\gamma] \mapsto \mathrm{Evol}_{DM}(\gamma).$$

Let $[\gamma] \in L^p([0,1], \mathcal{X}^r_{str}(M))$, we define the absolutely continuous function $\xi_\gamma : [0,1] \to \mathrm{Diff}^r(DM)$ by $\xi_\gamma := \mathrm{Evol}_{DM}(\tilde{\alpha}([\gamma]))$. Then ξ_γ is the solution of the equation

$$\begin{aligned} \dot{\xi}_\gamma(t) &= \tilde{\alpha}([\gamma])(t).\xi_\gamma(t), \quad t \in [0,1] \\ \xi_\gamma(0) &= e. \end{aligned}$$

For $[\gamma]$ close to 0, the proof of Theorem 2.6.5 shows that, for each $p \in M$, the function $x_p : [0,1] \to DM$, given by $x_p(t) := \xi_\gamma(t)(p)$ is a solution of the equation

$$\begin{aligned} \dot{x_p}(t) &= \tilde{\alpha}([\gamma])(t) \circ x_p(t), \quad t \in [0,1] \\ x_p(0) &= p, \end{aligned}$$

and this ODE satisfies local uniqueness of Caratheodory solutions. Looking at the compact manifold ∂M without boundary and the vector fields $\gamma(t)|_{\partial M} \in \mathcal{X}^r(\partial M)$ we likewise get a solution $y_p : [0,1] \to \partial M$ for each $p \in \partial M$, for the differential equation

$$\begin{aligned} \dot{y_p}(t) &= \tilde{\alpha}([\gamma])(t) \circ y_p(t), \quad t \in [0,1] \\ y_p(0) &= p. \end{aligned}$$

Then y_p also solves the initial value problem for x_p, whence $x_p = y_p$ by local uniqueness. In consequence $x_p([0,1]) \subseteq \partial M$. This implies that for each $t \in [0,1]$ fixed, we have $\xi_\gamma(t)(M) \subseteq M$, $\xi_\gamma(t)(M') \subseteq M'$ and $\xi_\gamma(t)(\partial M) \subseteq \partial M$. Therefore, we obtain $\xi_\gamma(t)|_M \in \mathrm{Diff}^r(M)$.
Consider the smooth embedding $\iota : \mathrm{Diff}^r(DM) \to C^r(DM, DM)$, $\phi \mapsto \phi$. Then the map

$$\tilde{\iota} := AC_{L^p}([0,1], \iota) : AC_{L^p}([0,1], \mathrm{Diff}^r(DM)) \to AC_{L^p}([0,1], C^r(DM, DM)), \quad \eta \mapsto \iota \circ \eta$$

is smooth. Let $\rho : M \to DM$ be the inclusion map, which is smooth and a diffeomorphism onto its image, then the maps

$$\rho^* := C^r(\rho, DM) : C^r(DM, DM) \to C^r(M, DM), \quad \phi \mapsto \phi \circ \rho = \phi|_M$$

and

$$\tilde{\rho} : AC_{L^p}([0,1], C^r(DM, DM)) \to AC_{L^p}([0,1], C^r(M, DM)), \quad \eta \mapsto \rho^* \circ \eta$$

are smooth.
Consider the restricted inclusion map J as in equation (2.6.5), by Lemma 2.1.25, we define the map

$$\tilde{j} : AC_{L^p}([0,1], J(\mathrm{Diff}^r(M))) \to AC_{L^p}([0,1], \mathrm{Diff}^r(M)), \quad \eta \mapsto J^{-1} \circ \eta.$$

The fact that $\xi_\gamma(t)|_M \in \mathrm{Diff}^r(M)$ for each $t \in [0,1]$, enables us to define the function

$$\zeta_\gamma : [0,1] \to C^r(M, DM), \quad \zeta_\gamma(t) := (\tilde{j} \circ \tilde{\rho} \circ \tilde{\imath})(\xi_\gamma)(t)$$

which is in $AC_{L^p}([0,1], \mathrm{Diff}^r(M))$. Moreover, by definition of α, we have

$$\tilde{\alpha}([\gamma])(t) \circ \zeta_\gamma(t) = \gamma(t) \circ \zeta_\gamma(t).$$

Hence ζ_γ verifies the equation

$$\begin{aligned} &\dot{\zeta}_\gamma(t) = \gamma(t).\zeta_\gamma(t), \quad t \in [0,1] \\ &\zeta_\gamma(0) = e. \end{aligned}$$

Therefore $\mathrm{Diff}^r(M)$ is L^p-semiregular and the evolution map is given by

$$\mathrm{Evol} : L^p([0,1], \mathcal{X}^r_{str}(M)) \to AC_{L^p}([0,1], \mathrm{Diff}^r(M)), \quad [\gamma] \mapsto \left(\tilde{j} \circ \tilde{\rho} \circ \tilde{\imath} \circ \mathrm{Evol}_{DM} \circ \tilde{\alpha}\right)([\gamma]).$$

We consider the inclusion map

$$\omega : AC_{L^p}([0,1], C^r(M, DM)) \to C([0,1], C^r(M, DM)), \quad \eta \mapsto \eta$$

which is smooth, and we define the smooth map

$$\widetilde{\rho}_C : AC_{L^p}([0,1], C^r(DM, DM)) \to C([0,1], C^r(M, DM)), \quad \eta \mapsto \omega(\rho^* \circ \eta).$$

We write

$$\widetilde{j}_C : C([0,1], J(\mathrm{Diff}^r(M))) \to C([0,1], \mathrm{Diff}^r(M)), \quad \eta \to J^{-1} \circ \eta$$

then $\widetilde{j}_C$ is continuous. Therefore, the evolution map with continuous values Evol_C is given by

$$\mathrm{Evol}_C : L^p([0,1], \mathcal{X}^r_{str}(M)) \to C([0,1], \mathrm{Diff}^r(M)), \quad [\gamma] \mapsto \left(\widetilde{j}_C \circ \widetilde{\rho}_C \circ \widetilde{\imath} \circ \mathrm{Evol}_{DM} \circ \widetilde{\alpha}\right)([\gamma])$$

and is continuous since is composition of continuous maps. □

3 Manifolds of mappings associated with real-valued functions spaces

3.1 Preliminaries

Definition 3.1.1 Let $m \in \mathbb{N}$ fixed, a set $\mathcal{U}$ of open subsets of product set $[0,\infty)^m$ will be called a *good collection of open subsets* if the following condition are satisfied:

a) $\mathcal{U}$ is a basis for the topology of $[0,\infty)^m$.

b) If $U \in \mathcal{U}$ and $K \subseteq U$ is a compact non-empty subset, then there exists $V \in \mathcal{U}$ with compact closure $\overline{V}$ in $[0,\infty)^m$ such that $K \subseteq V$ and $\overline{V} \subseteq U$.

c) If $U \subseteq [0,\infty)^m$ is an open set and $W \in \mathcal{U}$ is a relatively compact subset of U, then there exists $V \in \mathcal{U}$ such that V is a relatively compact subset of U and $\overline{W} \subseteq V$.

d) If $\phi : U \to V$ is a C^∞-diffeomorphism between open subsets U and V of $[0,\infty)^m$ and $W \in \mathcal{U}$ is a relatively compact subset of U, then $\phi(W) \in \mathcal{U}$.

Remark 3.1.2 If we consider $\mathcal{U} = \{U \cap [0,\infty)^m : U \text{ is open in } \mathbb{R}^m\}$ then $\mathcal{U}$ defines a good collection of open subsets. This is also true for the case of open and bounded subsets of $\mathbb{R}^m$.

Let U be a open subset of $[0,\infty)^m$, we write $BC(U,\mathbb{R})$ for the vector space of all bounded continuous functions $f : U \to \mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$.

Definition 3.1.3 Let M be a paracompact Hausdorff topological space. A chart $\phi : U \to V$ is a homeomorphism from an open subset $U \subseteq M$ onto an open subset $V \subseteq [0,\infty)^m$. We say that two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are compatibles if $\phi_1(U_1) \cap \phi_2(U_2) = \emptyset$ or the transition map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is smooth.
We say that M is an m-dimensional smooth manifold with corners if M is equipped with a maximal family of charts $\{\phi_i : U_i \to V_i\}_{i\in I}$ such that each pair of chart, are compatible and $M = \cup_{i\in I} U_i$.
We say that N is a smooth manifold if it is a smooth manifold without boundary.

For our context, one important property of smooth manifolds with corners is the existence of cut-off functions.

Lemma 3.1.4 *Let M be a m-dimensional smooth manifold with corners, K be a closed subset of M and U be a open subset of M containing K. Then there exists a smooth function $\xi : M \to [0,1]$ such that $\xi|_K = 1$ and supp$(\xi) \subseteq U$.*

Definition 3.1.5 Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$. For $U \in \mathcal{U}$, the vector subspace $\mathcal{F}(U,\mathbb{R})$ of $BC(U,\mathbb{R})$ will denote a integral complete locally convex space such that the inclusion map $\mathcal{F}(U,\mathbb{R}) \to BC(U,\mathbb{R})$ is continuous.
Let $\{b_1,...,b_n\}$ be a basis for a finite dimensional real vector space E, we define the space

$$\mathcal{F}(U,E) := \sum_{i=1}^{n} \mathcal{F}(U,\mathbb{R})b_i$$

and we endow it with the the locally convex topology making the map

$$\mathcal{F}(U,\mathbb{R})^n \to \mathcal{F}(U,E), \quad (f_1,...,f_n) \mapsto \sum_{i=1}^{n} f_i b_i \tag{3.1.1}$$

an isomorphism of topological vector spaces.
We say that $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ is a family of locally convex spaces suitable for global analysis if the following axioms are satisfied for all finite-dimensional real vector spaces E and F:

(PF) **Pushforward Axiom** For all $U, V \in \mathcal{U}$ such that V is relatively compact in U and each smooth map $f : U \times E \to F$, we have $f_*(\gamma) := f \circ (\mathrm{id}_V, \gamma|_V) \in \mathcal{F}(V,F)$ for all $\gamma \in \mathcal{F}(U,E)$ and the map

$$f_* : \mathcal{F}(U,E) \to F(V,F), \quad \gamma \mapsto f \circ (\mathrm{id}_V, \gamma|_V)$$

is continuous.

(PB) **Pullback Axiom :** Let U be an open subset of $[0,\infty)^m$ and $V, W \in \mathcal{U}$ such that W has compact closure contained in U. Let $\Theta : U \to V$ be a smooth diffeomorphism. Then $\gamma \circ \Theta|_W \in \mathcal{F}(W,E)$ for all $\gamma \in \mathcal{F}(V,E)$ and

$$\mathcal{F}(\Theta|_W, E) : \mathcal{F}(V,E) \to \mathcal{F}(W,E), \quad \gamma \mapsto \gamma \circ \Theta|_W$$

is continuous.

(GL) **Globalization Axiom :** If $U, V \in \mathcal{U}$ with $V \subseteq U$ and $\gamma \in \mathcal{F}(V,E)$ has compact support, then the map $\tilde{\gamma} : U \to E$ defined by

$$\tilde{\gamma}(x) = \begin{cases} \gamma(x), & x \in V \\ 0, & x \in U \setminus \mathrm{supp}(\gamma) \end{cases}$$

is in $\mathcal{F}(U,E)$ and for each compact subset K of V the map

$$e^E_{U,V,K} : \mathcal{F}_K(V,E) \to \mathcal{F}(U,E), \quad \gamma \mapsto \tilde{\gamma}$$

is continuous, where $\mathcal{F}_K(V,E) := \{\gamma \in \mathcal{F}(V,E) : \mathrm{supp}(\gamma) \subseteq K\}$ is endowed with the topology induced by $\mathcal{F}(V,E)$.

(MU) **Multiplication Axiom :** If $U \in \mathcal{U}$ and $h \in C_c^\infty(U, \mathbb{R})$, then $h\gamma \in \mathcal{F}(U, E)$ for all $\gamma \in \mathcal{F}(U, E)$ and the map

$$m_h^E : \mathcal{F}(U, E) \to \mathcal{F}(U, E), \quad \gamma \mapsto h\gamma$$

is continuous.

Remark 3.1.6 Since the map in (3.1.1) is an isomorphism of topological vector spaces, the Axioms (PB), (GL) and (MU) hold in general whenever they hold for $E = \mathbb{R}$. Likewise, Axiom (PF) holds in general whenever it holds for $F = \mathbb{R}$.

Following [22, Remark 3.5], if $\mathcal{U}$ is a good collection of open subsets of $[0, \infty)^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ is a family of locally convex space suitable for global analysis, then we have the following results.

Lemma 3.1.7 *Let $U \subseteq [0, \infty)^\infty$ be an open subset and $V, W \in \mathcal{U}$ such that W has compact closure contained in U and $\Theta : U \to V$ be a smooth diffeomorphism. If $\mathcal{F}(V, \mathbb{R})$ and $\mathcal{F}(W, \mathbb{R})$ are Fréchet spaces such that $\gamma \circ \Theta|_W \in \mathcal{F}(W, \mathbb{R})$ for all $\gamma \in \mathcal{F}(V, \mathbb{R})$, then the map*

$$\mathcal{F}(\Theta|_W, \mathbb{R}) : \mathcal{F}(V, \mathbb{R}) \to \mathcal{F}(W, \mathbb{R}), \quad \gamma \mapsto \gamma \circ \Theta|_W$$

is continuous.

Proof. Let $\gamma \in BC(V, \mathbb{R})$ and $p : \mathbb{R} \to \mathbb{R}$ be a continuous seminorm, then

$$\|\gamma \circ \Theta|_W\|_{\infty,p} := \sup_{x \in W} p(\gamma \circ \Theta|_W(x)) \leq \sup_{z \in V} p(\gamma(z)).$$

Therefore $\gamma \circ \Theta|_W \in BC(W, \mathbb{R})$. We define the continuous linear operator

$$T : BC(\Theta|_W, \mathbb{R}) : BC(V, \mathbb{R}) \to BC(W, \mathbb{R}), \quad \gamma \mapsto \gamma \circ \Theta|_W$$

with $\|T\|_{op} \leq 1$. Hence, its graph graph(T) is closed in $BC(V, \mathbb{R}) \times BC(W, \mathbb{R})$. Since the inclusion map $J : \mathcal{F}(U, \mathbb{R}) \to BC(U, \mathbb{R})$ is continuous, we have

$$\text{graph}(\mathcal{F}(\Theta|_W, \mathbb{R})) = (J \times J)^{-1}(\text{graph}(T)).$$

Then $\mathcal{F}(\Theta|_W, \mathbb{R})$ is continuous by the Closed Graph Theorem. □

Lemma 3.1.8 *If $U \in \mathcal{U}$, $h \in C_c^\infty(U, \mathbb{R})$ and $\mathcal{F}(U, \mathbb{R})$ is a Fréchet space such that $h\gamma \in \mathcal{F}(U, \mathbb{R})$ for all $\gamma \in \mathcal{F}(U, \mathbb{R})$, then the map*

$$m_h : \mathcal{F}(U, \mathbb{R}) \to \mathcal{F}(U, \mathbb{R}), \quad \gamma \mapsto h\gamma$$

is continuous.

Proof. As in the previous lemma, m_h is continuous since the operator

$$M_h : BC(U, \mathbb{R}) \to BC(U, \mathbb{R}), \quad \gamma \mapsto h\gamma$$

is continuous linear, the graph of m_h is closed and therefore, m_h is continuous. □

Lemma 3.1.9 *Let $U, V \in \mathcal{U}$ with $V \subseteq U$ and K be a compact subset of V. Assume that, for each $\gamma \in \mathcal{F}(V, \mathbb{R})$ with support in K, the map $\tilde{\gamma} : U \to \mathbb{R}$ defined by*

$$\tilde{\gamma}(x) = \begin{cases} \gamma(x), & x \in V \\ 0, & x \in U \setminus supp(\gamma) \end{cases}$$

is in $\mathcal{F}(U, \mathbb{R})$. If, moreover, if $\mathcal{F}_K(V, \mathbb{R})$ is a Fréchet space then the map

$$e_{U,V,K} : \mathcal{F}_K(V, \mathbb{R}) \to \mathcal{F}(U, \mathbb{R}), \quad \gamma \mapsto \tilde{\gamma}$$

is continuous

Proof. Likewise to the previous lemmas, if $BC_K(V, \mathbb{R}) := \{\gamma \in BC(V, \mathbb{R}) : supp(\gamma) \subseteq K\}$ then the map

$$BC(V, \mathbb{R}) \to BC_K(U, \mathbb{R}), \quad \gamma \mapsto \tilde{\gamma}$$

which extends functions by 0 is a linear isometry. □

Remark 3.1.10 Since a manifold with corners admits cut-off functions, we can extend the basic consequence of these axioms for the case $\mathbb{R}^m$ (see [22, Section 4]) to our context with corners. Moreover, the proofs are exactly the same. However, the statement of Lemma 3.1.12 is new and we provide a full proof.

Lemma 3.1.11 *Let E and F be finite-dimensional real vector spaces and $U, W \in \mathcal{U}$ such that W is relatively compact in U. If $\Phi : E \to F$ is a smooth map, then $\Phi \circ \gamma|_W \in \mathcal{F}(W, F)$ holds for each $\gamma \in \mathcal{F}(U, E)$ and the map*

$$\mathcal{F}(U, E) \to \mathcal{F}(W, F), \quad \gamma \mapsto \Phi \circ \gamma|_W$$

is continuous. In particular, if $E = F$ and $\Phi = Id_E$, then the restriction map

$$\mathcal{F}(U, E) \to \mathcal{F}(W, E), \quad \gamma \mapsto \gamma|_W$$

is continuous.

Lemma 3.1.12 *Let E and F be finite-dimensional real vector spaces and $U, W \in \mathcal{U}$ such that W is relatively compact in U. If V is an open subset of E and*

$$f \colon V \to F$$

is a smooth map, then the map

$$\mathcal{F}(U/W, f) \colon \{\gamma \in \mathcal{F}(U, E) \colon \gamma(\overline{W}) \subset V\} \to \mathcal{F}(W, F), \quad \gamma \mapsto f \circ \gamma|_W$$

is smooth.

Proof. Given γ_0 in the domain D of $\mathcal{F}(U/W, f)$, we have that $\gamma_0(\overline{W})$ is a compact subset of V. There exists a smooth function $\chi \colon V \to \mathbb{R}$ with compact support $K \subseteq V$ such that $\chi(y) = 1$ for all y in an open subset $Y \subseteq V$ with $\gamma_0(\overline{W}) \subseteq Y$. Then

$$g \colon E \to F, \quad g(y) := \begin{cases} \chi(y) f(y) & \text{if } y \in V; \\ 0 & \text{if } y \in E \setminus K \end{cases}$$

is a smooth function. Since $f|_Y = g|_Y$, we have that

$$f \circ \gamma|_W = g \circ \gamma|_W$$

for all $\gamma \in D$ such that $\gamma(\overline{W}) \subseteq Y$, which is an open neighborhood of γ_0 in D. To see smoothness of $\mathcal{F}(U/W, f)$ on some open neighborhood of γ_0 (which suffices for the proof), we may therefore replace f with g and assume henceforth that $V = E$, whence D is all of $\mathcal{F}(U, E)$. It suffices to show that $\mathcal{F}(U/W, f)$ is C^k for each $k \in \mathbb{N}_0$, and we show this by induction. For the case $k = 0$, see Lemma 3.1.11. Let $k \in \mathbb{N}_0$ now and assume that, for all E, F, U, W and $f\colon V \to F$ as in the lemma, with $V = E$, the map $\mathcal{F}(U/W, f)$ is C^k. We claim that, for all $\gamma, \eta \in \mathcal{F}(U, E)$, the directional derivative

$$d\mathcal{F}(U/W, f)(\gamma, \eta)$$

exists and equals $\mathcal{F}(U/W, df)(\gamma, \eta)$, if we identify the locally convex spaces $\mathcal{F}(U, E) \times \mathcal{F}(U, E)$ and $\mathcal{F}(U, E \times E)$; thus

$$d\mathcal{F}(U/W, f)(\gamma, \eta) = \mathcal{F}(U/W, df)(\gamma, \eta). \tag{3.1.2}$$

If this is true, then

$$d\mathcal{F}(U/W, f) = \mathcal{F}(U/W, df)$$

is C^k by induction and thus continuous, showing that $\mathcal{F}(U/W, f)$ is C^1. Moreover, since $\mathcal{F}(U/W, f)$ is C^1 and $d\mathcal{F}(U/W, f) = \mathcal{F}(U/W, df)$ is C^k, the map $\mathcal{F}(U/W, f)$ is C^{k+1}, which completes the inductive proof. It only remains to prove the claim. To this end, let $\gamma, \eta \in \mathcal{F}(U, E)$. Since $\mathcal{F}(U/W, df)$ is continuous by the case $k = 0$, the map

$$h\colon [0,1] \times [0,1] \to \mathcal{F}(W, F), \quad (t, s) \mapsto df \circ (\gamma + st\eta, \eta)|_W = \mathcal{F}(U/W, df)(\gamma + st\eta, \eta)$$

is continuous. As $\mathcal{F}(W, F)$ is assumed integral complete, for each $t \in [0, 1]$ the continuous path $h(t, \cdot)\colon [0, 1] \to \mathcal{F}(W, F)$ has a weak integral

$$I(t) := \int_0^1 df \circ (\gamma + st\eta, \eta)|_W \, ds$$

in $\mathcal{F}(W, F)$. The function $I\colon [0, 1] \to \mathcal{F}(W, F)$ is continuous by the theorem on parameter-dependent integrals. For $0 \neq t \in [0, 1]$, we consider the difference quotient

$$\Delta(t) = \frac{\mathcal{F}(U/W, f)(\gamma + t\eta) - \mathcal{F}(U/W, f)(\gamma)}{t} = \frac{f \circ (\gamma + t\eta)|_W + f \circ \gamma|_W}{t}.$$

Then

$$\Delta(t) = I(t). \tag{3.1.3}$$

In fact, for each $x \in W$ the point evaluation

$$\varepsilon_x\colon \mathcal{F}(W, F) \to F, \quad \theta \mapsto \theta(x)$$

is a continuous linear map. It therefore commutes with the weak integral and we obtain

$$\begin{aligned} I(t)(x) &= \varepsilon_x(I(t)) = \int_0^1 \varepsilon_x(df \circ (\gamma + st\eta, \eta)|_W)\, ds \\ &= \int_0^1 df(\gamma(x) + st\eta(x), \eta(x))\, ds = \frac{f(\gamma(x) + t\eta(x)) - f(\gamma(x))}{t} \\ &= \Delta(t)(x), \end{aligned}$$

applying the mean value theorem to the smooth function f. Thus (3.1.3) holds. Exploiting the continuity of I, letting $t \to 0$ we obtain

$$\lim_{t\to 0} \Delta(t) = \lim_{t\to 0} I(t) = I(0) = \int_0^1 df \circ (\gamma, \eta)|_W\, ds = df \circ (\gamma, \eta)|_W,$$

establishing (3.1.2). □

Definition 3.1.13 Let U be a open subset of $[0, \infty)^m$ and E be a finite-dimensional real vector space. We let $\mathcal{F}_{\mathrm{loc}}(U, E)$ be the set of all function $\gamma : U \to E$ such that for each $V \in \mathcal{U}$ which is relatively compact in U we have $\gamma|_V \in \mathcal{F}(V, E)$.
We see that each $\gamma \in \mathcal{F}_{\mathrm{loc}}(U, E)$ is continuous and by the previous lemma $\mathcal{F}(U, E) \subseteq \mathcal{F}_{\mathrm{loc}}(U, E)$. We endow $\mathcal{F}_{\mathrm{loc}}(U, E)$ with the initial topology with respect to the family of restriction maps

$$\mathcal{F}_{\mathrm{loc}}(U, E) \to \mathcal{F}(V, E), \quad \gamma \mapsto \gamma|_V$$

where $V \in \mathcal{U}$ which is relatively compact in U. This topology makes $\mathcal{F}_{\mathrm{loc}}(U, E)$ a Hausdorff locally convex space.

Lemma 3.1.14 *Let E be a finite-dimensional vector space. If U and V are open subsets of $[0, \infty)^m$ such that $V \subseteq U$, then $\gamma|_V \in \mathcal{F}_{loc}(V, E)$ for each $\gamma \in \mathcal{F}_{loc}(U, E)$ and the restriction map*

$$\mathcal{F}_{loc}(U, E) \to \mathcal{F}_{loc}(V, E), \quad \gamma \mapsto \gamma|_V$$

is linear and continuous.

Lemma 3.1.15 *Let E and F be finite-dimensional real vector spaces and $U \subseteq [0, \infty)^m$ be open. If $\Phi : E \to F$ is a smooth map, then $\Phi \circ \gamma \in \mathcal{F}_{loc}(U, F)$ holds for each $\gamma \in \mathcal{F}_{loc}(U, E)$ and the map*

$$\mathcal{F}_{loc}(U, E) \to \mathcal{F}_{loc}(U, F), \quad \gamma \mapsto \Phi \circ \gamma$$

is continuous. Moreover, if Q is an open subset of E and $\Psi : Q \to F$ is a smooth map, then $\Psi \circ \gamma \in \mathcal{F}_{loc}(U, F)$ holds for each $\gamma \in \mathcal{F}_{loc}(U, E)$ such that $\gamma(U) \subseteq Q$.

Lemma 3.1.16 *Let E be a finite-dimensional vector space, U and V be open subsets of $[0, \infty)^m$ and $\Theta : U \to V$ be a smooth diffeomorphism. Then $\gamma \circ \Theta \in \mathcal{F}_{loc}(U, E)$ for each $\gamma \in \mathcal{F}_{loc}(V, E)$ and the map*

$$\mathcal{F}(U, E) \to \mathcal{F}(V, E), \quad \gamma \mapsto \gamma \circ \Theta$$

is continuous.

Lemma 3.1.17 *Let E be a finite-dimensional vector space, $U_1, ..., U_n$ be open subsets of $[0,\infty)^m$ and $\gamma_j \in \mathcal{F}_{loc}(U_j, E)$ for $j \in \{1, ..., n\}$ such that*

$$\gamma_j|_{U_i \cap U_j} = \gamma_i|_{U_i \cap U_j}, \quad \text{for all } i, j \in \{1, ..., n\}.$$

If $V \in \mathcal{U}$ is relatively compact in $U_1 \cup ... \cup U_n$, then $\tilde{\gamma} \in \mathcal{F}(V, E)$ holds for the map $\tilde{\gamma} : V \to E$ defined piecewise via $\tilde{\gamma}(x) = \gamma_j(x)$ for $x \in V \cap U_j$.
Moreover, if $\mathcal{E}$ is the vector subspace of $\prod_{j=1}^{n} \mathcal{F}_{loc}(U_j, E)$ given by the n-tuples $(\gamma_1, ..., \gamma_n)$ such that $\gamma_j|_{U_i \cap U_j} = \gamma_i|_{U_i \cap U_j}$, for all $i, j \in \{1, ..., n\}$, endowed with the subspace topology, then the gluing map

$$glue : \mathcal{E} \to \mathcal{F}(V, E), \quad (\gamma_1, ..., \gamma_n) \mapsto \tilde{\gamma}$$

is continuous linear.

Definition 3.1.18 Let M be an m-dimensional compact smooth manifold with corners and N an n dimensional smooth manifold. Let $\mathcal{F}(M, N)$ be the set of all functions $\gamma : M \to N$ such that for each $p \in M$, exist charts $\phi_M : U_M \to V_M$ of M with $V_M \in \mathcal{U}$ and $\phi_N : U_N \to V_N$ a chart of N, such that $p \in U_M$, $\gamma(U_M) \subseteq U_N$ and $\phi_N \circ \gamma \circ \phi_M^{-1} \in \mathcal{F}(V_M, \mathbb{R}^n)$.

Remark 3.1.19 For a compact smooth manifold without boundary M, the properties of maps between $\mathcal{F}$-spaces are studied in Section 5 of [22]. These properties can be extended to the case with corners. We recall the more important results relevant for our context.

Lemma 3.1.20 *Let M be an m-dimensional compact smooth manifold with corners, N be a n-dimensional smooth manifold and $\gamma : M \to N$ be a continuous map. Then $\gamma \in \mathcal{F}(M, N)$ if and only if $\phi_N \circ \gamma \circ \phi_M^{-1} \in \mathcal{F}_{loc}(V_M, \mathbb{R}^n)$ for all charts $\phi_M : U_M \to V_M$ and $\phi_N : U_N \to V_N$ of M and N, respectively, such that $\gamma(U_M) \subseteq U_N$.*

Lemma 3.1.21 *Let $\Phi : N_1 \to N_2$ be a smooth map between finite-dimensional smooth manifolds, and M be a compact smooth manifold. Then $\Phi \circ \eta \in \mathcal{F}(M, N_2)$ for each $\eta \in \mathcal{F}(M, N_1)$.*

Remark 3.1.22 Let M be an n-dimensional compact smooth manifold with corners and E be a finite-dimensional vector space. We give $\mathcal{F}(M, E)$ the initial topology with respect to the maps

$$\mathcal{F}(M, E) \to \mathcal{F}(V_\phi, E), \quad \gamma \mapsto \gamma \circ \phi^{-1}$$

for $\phi : U_\phi \to V_\phi$ in the maximal C^∞ atlas of M.

Lemma 3.1.23 *Let M be a compact smooth manifold with corners and E be a finite-dimensional vector space. For $i \in \{1, ..., k\}$, let $\phi_i : U_i \to V_i$ be charts of M, $W_i \in \mathcal{U}$ be a relatively compact subset of V_i with $M = \cup_{i=1}^{k} \varphi_i^{-1}(W_i)$. Then the linear map*

$$\Theta : \mathcal{F}(M, E) \to \prod_{i=1}^{k} \mathcal{F}(W_i, E), \quad \gamma \mapsto \left(\gamma \circ \phi_i^{-1}|_{W_i}\right)_{i=1}^{k}$$

is a topological embedding with closed image.
The image $Im(\Theta)$ is the set S of all $(\gamma_i)_{i=1}^{k} \in \prod_{i=1}^{k} \mathcal{F}(W_i, E)$ such that $\gamma_i \circ \phi_i(p) = \gamma_j \circ \phi_j(p)$ for all $i, j \in \{1, ..., k\}$ and $p \in \phi_i^{-1}(W_i) \cap \phi_i^{-1}(W_j)$.

Lemma 3.1.24 *Let M be an m-dimensional compact manifold with corners. If E_1 and E_2 are finite-dimensional vector spaces, we consider the projections $pr_j : E_1 \times E_2 \to E_j$, $(x_1, x_2) \mapsto x_j$ for $j \in \{1, 2\}$. Then*

$$(\mathcal{F}(M, pr_1), \mathcal{F}(M, pr_2)) : \mathcal{F}(M, E_1 \times E_2) \to \mathcal{F}(M, E_1) \times \mathcal{F}(M, E_2), \ \gamma \mapsto (pr_1, pr_2) \circ \gamma$$

is an isomorphism of topological vector spaces.

Lemma 3.1.25 *If M is an m-dimensional compact smooth manifold with corners, E and F are finite-dimensional $\mathbb{K}$-vector spaces for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, U is an open subset of E and $g : U \to F$ is $\mathbb{K}$-analytic, the also the map*

$$\mathcal{F}(M, g) : \mathcal{F}(M, U) \to \mathcal{F}(M, F), \quad \gamma \mapsto g \circ \gamma$$

is $\mathbb{K}$-analytic.

3.2 Space of $\mathcal{F}$-sections

Let $m, n \in \mathbb{N}$. We assume that $\mathcal{U}$ is a good collection of open subsets of $[0, \infty)^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ is a family of locally convex spaces suitable for global analysis.
Let M be an m-dimensional compact smooth manifold with corners and N be an n-dimensional smooth manifold. For $\gamma \in \mathcal{F}(M, N)$ we define the set

$$\Gamma_{\mathcal{F}}(\gamma) := \{\sigma \in \mathcal{F}(M, TN) : \pi_{TN} \circ \sigma = \gamma\}$$

and we endow it with the pointwise operations, making it a vector space. We make $\Gamma_{\mathcal{F}}(\gamma)$ a Hausdorff locally convex space, using the initial topology with respect to the family of maps

$$h_{\varphi,\phi} : \Gamma_{\mathcal{F}}(\gamma) \to \mathcal{F}(V_\varphi, \mathbb{R}^n), \quad \sigma \mapsto d\phi \circ \sigma \circ \varphi^{-1}|_W$$

where $\varphi : U_\varphi \to V_\varphi$ is a chart in the maximal C^∞-atlas of M, with $W \in \mathcal{U}$ relatively compact in V_φ and there exists a chart $\phi : U_\phi \to V_\phi$ of N with $\gamma(U_\varphi) \subseteq U_\phi$. These maps make sense because $\gamma(U_\varphi) \subseteq U_\phi$ implies $\sigma(U_\varphi) \subseteq TU_\phi$ for each $\sigma \in \Gamma_{\mathcal{F}}(\eta)$.

Proposition 3.2.1 *Let M be an m-dimensional compact smooth manifold with corners, N be an n-dimensional smooth manifold and $\gamma \in \mathcal{F}(M, N)$. For $i \in \{1, ..., k\}$, let $\varphi_i : U_i \to V_i$ be charts of M such that there exists $W_i \in \mathcal{U}$ relatively compact in V_i with $M = \cup_{i=1}^k \varphi_i^{-1}(W_i)$ and there exists a chart $\phi_i : U_{\phi_i} \to V_{\phi_i}$ such that $\gamma(U_i) \subseteq U_{\phi_i}$. Then the map*

$$\Phi_\gamma : \Gamma_{\mathcal{F}}(\gamma) \to \prod_{i=1}^k \mathcal{F}(W_i, \mathbb{R}^n), \quad \sigma \mapsto \left(d\phi_i \circ \sigma \circ \varphi_i^{-1}|_{W_i}\right)_{i=1}^k$$

is a linear topological embedding with closed image given by the vector subspace of elements $(\tau_i)_{i=1}^n$ such that

$$\tau_i \circ \varphi_i(p) = d\phi_i \circ (T\phi_j)^{-1}\Big(\phi_j \circ \gamma(p), \tau_j \circ \varphi_j(p)\Big)$$

for all $i, j \in \{1, ..., k\}$ and $p \in \varphi_i^{-1}(W_i) \cap \varphi_j^{-1}(W_j)$.

Proof. The map Φ_γ is continuous by definition of the topology on $\Gamma_{\mathcal{F}}(\gamma)$. We denote by S the vector space of elements $(\tau_i)_{i=1}^n$ such that

$$\tau_i \circ \varphi_i(p) = d\phi_i \circ (T\phi_j)^{-1}\Big(\phi_j \circ \gamma(p), \tau_j \circ \varphi_j(p)\Big)$$

for all $i, j \in \{1, .., k\}$ and $p \in \varphi_i^{-1}(W_i) \cap \varphi_j^{-1}(W_j)$.
Clearly $\mathrm{Im}(\Phi_\gamma) \subseteq S$. Let $(\tau_i)_{i=1}^k \in S$, identifying the tangent bundle TV with $V \times \mathbb{R}^n$ for any open subset $V \subseteq \mathbb{R}^n$, we define the function $\sigma : M \to TN$ by

$$\sigma(p) = T\phi_i^{-1}\left(\phi_i \circ \gamma(p), \tau_i \circ \varphi_i(p)\right), \quad \text{if } p \in \varphi_i^{-1}(W_i) \text{ with } i \in \{1, ..., k\}.$$

We will show that the function σ is well defined. Let $p \in \varphi_i^{-1}(W_i) \cap \varphi_j^{-1}(W_j)$, then

$$\begin{aligned} T\phi_i^{-1}\left(\phi_i \circ \gamma(p), \tau_i \circ \varphi_i(p)\right) &= T\phi_j^{-1} \circ T\phi_j \circ T\phi_i^{-1}\left(\phi_i \circ \gamma(p), \tau_i \circ \varphi_i(p)\right) \\ &= T\phi_j^{-1}\left(\phi_j \circ \gamma(p), \tau_j \circ \varphi_j(p)\right) \end{aligned}$$

Hence σ make sense. Moreover, we see that $\pi_{TN} \circ \sigma = \gamma$.
For $\varphi_i|_{\varphi_i^{-1}(W_i)} : \varphi_i^{-1}(W_i) \to W_i$ and $T\phi_i : TU_{\phi_i} \to V_{\phi_i} \times \mathbb{R}^n$ we have

$$\begin{aligned} T\phi_i \circ \sigma \circ \varphi_i|_{\varphi_i^{-1}(W_i)} &= T\phi_i \circ \left(T\phi_i^{-1}\left(\phi_i \circ \gamma, \tau_i \circ \varphi_i\right)\right) \circ \varphi_i|_{\varphi_i^{-1}(W_i)} \\ &= \Big(\phi_i \circ \gamma \circ \varphi_i|_{\varphi_i^{-1}(W_i)}, \tau_i\Big). \end{aligned}$$

Since $W_i \in \mathcal{U}$ we have $\phi_i \circ \gamma \circ \varphi_i|_{\varphi_i^{-1}(W_i)} \in \mathcal{F}(W_i, V_{\phi_i})$ and

$$T\phi_i \circ \sigma \circ \varphi_i|_{\varphi_i^{-1}(W_i)} \in \mathcal{F}(W_i, V_{\phi_i}) \times \mathcal{F}(W_i, \mathbb{R}^n) \cong \mathcal{F}(W_i, V_{\phi_i} \times \mathbb{R}^n).$$

Thus $\sigma \in \Gamma_{\mathcal{F}}(\gamma)$. Evaluating we have

$$\begin{aligned} \Phi_\gamma(\sigma) &= \left(d\phi_i \circ \sigma \circ \varphi_i^{-1}|_{W_i}\right)_{i=1}^k \\ &= \left(d\phi_i \circ T\phi_i^{-1}\left(\phi_i \circ \gamma, \tau_i \circ \varphi_i\right) \circ \varphi_i^{-1}|_{W_i}\right)_{i=1}^k \\ &= \left(d\phi_i \circ T\phi_i^{-1}\left(\phi_i \circ \gamma \circ \varphi_i^{-1}|_{W_i}, \tau_i \circ \varphi_i \circ \varphi_i^{-1}|_{W_i}\right)\right)_{i=1}^k \\ &= \left(d\phi_i\left(\gamma \circ \varphi_i^{-1}|_{W_i}, d\phi_i^{-1} \circ \tau_i\right)\right)_{i=1}^k \\ &= (\tau_i)_{i=1}^k \end{aligned}$$

Hence $S \subseteq \mathrm{Im}(\Phi_\gamma)$. The inverse of Φ_γ is given by $\Phi_\gamma^{-1} : \mathrm{Im}(\Phi_\gamma) \to \Gamma_{\mathcal{F}}(\gamma)$ where

$$\Phi_\gamma^{-1}\Big((\tau_i)_{i=1}^k\Big)(p) = T\phi_i^{-1}\left(\phi_i \circ \gamma(p), \tau_i \circ \varphi_i(p)\right), \quad \text{for } p \in \varphi_i^{-1}(W_i).$$

Consider the arbitrary chart $\alpha : U_\alpha \to V_\alpha$ of M with $W_\alpha \in \mathcal{U}$ relatively compact in V_α and the chart $\beta : U_\beta \to V_\beta$ of N such that $\alpha(U_\alpha) \subseteq U_\beta$. We will show that

$$h_{\alpha,\beta} \circ \Phi_\gamma^{-1} : \mathrm{Im}(\Phi_\gamma) \to \mathcal{F}(W, \mathbb{R}^n), \quad \tau = (\tau_i)_{i=1}^k \mapsto d\beta \circ \sigma_\tau \circ \alpha^{-1}|_W$$

is continuous. Since $M = \cup_{i=1}^k \varphi_i^{-1}(W_i)$, we define the open set

$$Q_j := \alpha\left(U_\alpha \cap \varphi_j^{-1}(W_j)\right).$$

Without loss of generality, we assume that there exists $r \in \{1, .., k\}$ such that $Q_j \neq \emptyset$ for each $j \in \{1, ..., r\}$ and $Q_j = \emptyset$ for each $j \in \{r+1, .., k\}$.
Then, for $\tau \in \mathrm{Im}(\Phi_\gamma)$ and $j \in \{1, ..., r\}$ we have

$$\begin{aligned} d\beta \circ \sigma_\tau \circ \alpha^{-1}|_{Q_j} &= d\beta \circ T\phi_j^{-1}\left(\phi_j \circ \gamma, \tau_j \circ \varphi_j\right) \circ \alpha^{-1}|_{Q_j} \\ &= \mathcal{F}(W_j, d\beta \circ T\phi_j^{-1}) \circ \left(\phi_i \circ \gamma \circ \alpha^{-1}|_{Q_i}, \mathcal{F}(\varphi_j \circ \alpha^{-1}|_{Q_j}, \mathbb{R}^n) \circ \tau_j\right). \end{aligned}$$

Hence $d\beta \circ \sigma_\tau \circ \alpha^{-1}|_{Q_j} \in \mathcal{F}_{\mathrm{loc}}(Q_i, \mathbb{R}^n)$. This enables us to define the continuous map

$$\Lambda : \mathrm{Im}(\Phi_\gamma) \to \prod_{i=1}^r \mathcal{F}_{\mathrm{loc}}(Q_i, \mathbb{R}^n). \quad \tau \mapsto \left(d\beta \circ \sigma_\tau \circ \alpha^{-1}|_{V_i}\right)_{i=1}^r$$

where the image set $\mathrm{Im}(\Lambda)$ coincides with the subspace

$$\left\{(\beta_1, ..., \beta_r) \in \prod_{i=1}^r \mathcal{F}_{\mathrm{loc}}(Q_i, \mathbb{R}^n) : (\forall i, j \in \{1, ..., r\})\, \beta_i|_{Q_i \cap Q_j} = \beta_j|_{Q_i \cap Q_j}\right\}.$$

For $(\beta_1, ..., \beta_r) \in \prod_{i=1}^r \mathcal{F}_{\mathrm{loc}}(Q_i, \mathbb{R}^n)$, we denote the gluing function

$$\beta(x) := \beta_j(x), \quad \text{if } x \in Q_i.$$

For each $W \in \mathcal{U}$ relatively compact in $Q_1 \cup ... \cup Q_r$ we have $\beta|_W \in \mathcal{F}(W, \mathbb{R}^n)$ and the map

$$\mathrm{glue}_W : \mathrm{Im}(\Lambda) \to \mathcal{F}(W, \mathbb{R}^n), \quad (\beta_1, ..., \beta_r) \mapsto \beta|_W$$

is continuous [22, Lemma 4.1]. Therefore $h_{\varphi,\phi} \circ \Phi_\gamma^{-1}$ is continuous since

$$h_{\varphi,\phi} \circ \Phi_\gamma^{-1} = \mathrm{glue}_W \circ \Lambda.$$

Hence Φ_γ^{-1} is continuous. □

Remark 3.2.2 From now we consider the map $\Phi_{\gamma,P}$ as the homeomorphism

$$\Phi_\gamma : \Gamma_{\mathcal{F}}(\gamma) \to \mathrm{Im}(\Phi_\gamma).$$

Corollary 3.2.3 *Let $\gamma \in \mathcal{F}(M, N)$. For $i \in \{1, ..., k\}$, let $\varphi_i : U_i \to V_i$ be charts of M such that there exists $W_i \in \mathcal{U}$ relatively compact in V_i with $M = \cup_{i=1}^k \varphi_i^{-1}(W_i)$ and there exists a chart $\phi_i : U_{\phi_i} \to V_{\phi_i}$ such that $\gamma(U_i) \subseteq U_{\phi_i}$. Then the space $\Gamma_{\mathcal{F}}(\gamma)$ is integral complete. Moreover:*

a) If $\mathcal{F}(W_i, \mathbb{R}^n)$ is a Banach space for all $i \in \{1, .., k\}$, then $\Gamma_{\mathcal{F}}(\gamma)$ i a Banach space with norm $\|\cdot\|_\Gamma$ given by

$$\|\sigma\|_\Gamma := \sum_{i=1}^k \|\left(d\phi_i \circ \sigma \circ \varphi_i^{-1}|_{W_i}\right)\|_{\mathcal{F}(W_i, \mathbb{R}^n)}, \quad \forall \sigma \in \Gamma_{\mathcal{F}}(\gamma).$$

b) If $\mathcal{F}(W_i,\mathbb{R}^n)$ is a Hilbert space for all $i \in \{1,..,k\}$, then $\Gamma_{\mathcal{F}}(\gamma)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_\Gamma$ given by

$$\langle\sigma,\tau\rangle_\Gamma := \sum_{i=1}^{k}\left\langle\left(d\phi_i\circ\sigma\circ\varphi_i^{-1}|_{W_i}\right),\left(d\phi_i\circ\tau\circ\varphi_i^{-1}|_{W_i}\right)\right\rangle_{\mathcal{F}(W_i,\mathbb{R}^n)},\quad \forall\sigma,\tau\in\Gamma_{\mathcal{F}}(\gamma).$$

Proposition 3.2.4 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners, N_1 and N_2 be finite-dimensional smooth manifolds and $\gamma\in\mathcal{F}(M,N_1)$. If $f:N_1\to N_2$ is a smooth function, then $Tf\circ\sigma\in\Gamma_{\mathcal{F}}(f\circ\gamma)$ for each $\sigma\in\Gamma_{\mathcal{F}}(\gamma)$. Moreover, the map*

$$\widetilde{f}:\Gamma_{\mathcal{F}}(\gamma)\to\Gamma_{\mathcal{F}}(f\circ\gamma),\quad \sigma\mapsto Tf\circ\sigma$$

is continuous linear.

Proof. Since f and Tf are smooth, by Lemma 3.1.11 we have $f\circ\gamma\in\mathcal{F}(M,N_2)$ and

$$Tf\circ\sigma\in\mathcal{F}(M,TN_2),\qquad \text{for all } \sigma\in\Gamma_{\mathcal{F}}(\gamma).$$

Since $\sigma(t)\in T_{\gamma(t)}N_1$ for each $t\in[a,b]$, we have $T_{\gamma(t)}f\circ\sigma(t)\in T_{f\circ\gamma(t)}N_2$, thus

$$\pi_{TN_2}\circ(Tf\circ\sigma)=f\circ\gamma$$

and $\widetilde{f}$ is well defined.
Let $\gamma\in\mathcal{F}(M,N_1)$. For $i\in\{1,...,k\}$, let $\varphi_{M,i}:U_{M,i}\to V_{M,i}$ be charts of M such that there exists $W_{M,i}\in\mathcal{U}$ relatively compact in $V_{M,i}$ with $M=\cup_{i=1}^{k}\varphi_{M,i}^{-1}(W_{M,i})$ and there exist chart $\phi_{1,i}:U_{\phi_{1,i}}\to V_{\phi_{1,i}}$ and $\phi_{2,i}:U_{\phi_{2,i}}\to V_{\phi_{2,i}}$ of N_1 and N_2 respectively, such that $\gamma\left(U_{M,i}\right)\subseteq U_{\phi_{1,i}}$ and $f\left(U_{\phi_{1,i}}\right)\subseteq U_{\phi_{2,i}}$. We may assume that $V_{\phi_{1,i}}=\mathbb{R}^{n_1}$ and $V_{\phi_{2,i}}=\mathbb{R}^{n_2}$.
Let $\gamma_i:=\phi_{1,i}\circ\gamma|_{W_{M,i}}$ and $W'_{M,i}\in\mathcal{U}$ be relatively compact in $U_{M,i}$, containing the closure of $W_{M,i}$. For $i\in\{1,...,k\}$ we define the smooth map

$$f_i:=d\phi_{2,i}\circ Tf\circ T\phi_{1,i}^{-1}:TV_{\psi_{1,i}}\subseteq\mathbb{R}^{2n_1}\to\mathbb{R}^{n_2}$$

and

$$G:\prod_{i=1}^{k}\mathcal{F}(W'_{M,i},\mathbb{R}^{2n_1})\to\prod_{i=1}^{k}\mathcal{F}(W_{M,i},\mathbb{R}^{n_2}),\quad (\xi_i)_{i=1}^{k}\mapsto\left(f_i\circ(\gamma_i,\xi_i|_{W_{M,i}})\right)_{i=1}^{k}$$

which is continuous by Lemma 3.1.11. We consider the linear topological embeddings

$$\Phi_\gamma:\Gamma_{\mathcal{F}}(\gamma)\to\prod_{i=1}^{k}\mathcal{F}(W_{M,i},\mathbb{R}^{n_1}),\quad \sigma\mapsto\left(d\phi_{1,i}\circ\sigma\circ\varphi_{M,i}^{-1}|_{W_{M,i}}\right)_{i=1}^{k}$$

and

$$\Phi_{f\circ\gamma} : \Gamma_{\mathcal{F}}(f \circ \gamma) \to \prod_{i=1}^{k} \mathcal{F}(W_{M,i}, \mathbb{R}^{n_2}), \quad \tau \mapsto \Big(d\phi_{i,2} \circ \tau \circ \varphi_{M,i}^{-1}|_{W_{M,i}} \Big)_{i=1}^{k}.$$

Then

$$G(\mathrm{Im}(\Phi_\gamma)) \subseteq \mathrm{Im}(\Phi_{f\circ\gamma}).$$

Indeed, consider $\sigma \in \Gamma_{\mathcal{F}}(\eta)$. Then for each $i \in \{1, ..., k\}$

$$\tau_i := f_i(T\phi_{1,i} \circ \sigma \circ \varphi_{M,i}^{-1}|_{W_{M,i}}) = d\phi_{2,i} \circ Tf \circ \sigma \circ \varphi_{M,i}^{-1}|_{W_{M,i}}.$$

And for all $i, j \in \{1, .., k\}$ and $p \in \varphi_{M,i}^{-1}(W_{M,i}) \cap \varphi_{M,j}^{-1}(W_{M,j})$, we have

$$\begin{aligned}
\tau_i \circ \varphi_{M,i}(p) &= d\phi_{2,i} \circ Tf \circ \sigma \circ \varphi_{M,i}^{-1}|_{W_{M,i}} \circ \varphi_{M,i}(p) \\
&= d\phi_{2,i} \circ Tf \circ \sigma(p) \\
&= d\phi_{2,i}\Big(f \circ \gamma(p), df \circ \sigma(p) \Big) \\
&= d\phi_{2,i} \circ (T\phi_{2,j})^{-1}\Big(\phi_{2,j} \circ f \circ \gamma(p), d\phi_{2,j} \circ df \circ \sigma \circ \varphi_{M,j}^{-1}|_{W_{M,j}} \circ \varphi_{M,j}(p) \Big) \\
&= d\phi_{2,i} \circ (T\phi_{2,j})^{-1}\Big(\phi_{2,j} \circ f \circ \gamma(p), \tau_j \circ \varphi_{M,j}(p) \Big)
\end{aligned}$$

Hence

$$\Big(f_i \circ \Big(d\phi_{1,i} \circ \sigma \circ \varphi_{M,i}^{-1}|_{W_{M,i}} \Big) \Big)_{i=1}^{k} \in \mathrm{Im}(\Phi_{f\circ\gamma}).$$

In consequence

$$\widetilde{f} = \Phi_{f\circ\gamma}^{-1} \circ G \circ \Phi_\gamma.$$

Thus $\widetilde{f}$ is continuous and the linearity is clear. □

Remark 3.2.5 The topology of $\Gamma_{\mathcal{F}}(\gamma)$ does not depend on the chosen family of charts. Indeed, since the identity map $\mathrm{id}_N : N \to N$ is smooth, by previous proposition the map

$$\widetilde{\mathrm{id}_N} : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_{\mathcal{F}}(\mathrm{id}_N \circ \gamma), \quad \sigma \mapsto T\mathrm{id}_M \circ \sigma$$

is smooth regardless of chosen family of charts. Moreover, this map coincides with the identity map $\mathrm{id}_\Gamma : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_{\mathcal{F}}(\gamma)$, $\sigma \mapsto \sigma$.

Remark 3.2.6 Let $\gamma \in BC(M, N)$, we define the space of continuous sections

$$\Gamma_C(\gamma) = \{\sigma \in BC(M, TN) : \pi_{TN} \circ \sigma = \gamma\}$$

endowed with the compact-open topology. For each $i \in \{1, ..., k\}$ the inclusion map

$$J_i : \mathcal{F}(W_i, \mathbb{R}^n) \to BC(W_i, \mathbb{R}^n), \quad \tau_i \mapsto \tau_i$$

is continuous, whence the inclusion map $J : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_C(\gamma)$, $\sigma \mapsto \sigma$ is continuous. This implies that the set

$$\mathcal{V} = \{\sigma \in \Gamma_{\mathcal{F}}(\gamma) : \sigma(M) \subseteq V\}$$

is open in $\Gamma_{\mathcal{F}}(\gamma)$ for each open set $V \subseteq TN$.

Proposition 3.2.7 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be a m-dimensional compact smooth manifold with corners, N_1, N_2 be finite-dimensional smooth manifolds and $\pi_i : N_1 \times N_2 \to N_i$ be the i-projection for $i \in \{1,2\}$. If $\gamma_1 \in \mathcal{F}(M,N_1)$ and $\gamma_2 \in \mathcal{F}(M,N_2)$ then the map*

$$\mathcal{P} : \Gamma_{\mathcal{F}}(\gamma_1 \times \gamma_2) \to \Gamma_{\mathcal{F}}(\gamma_1) \times \Gamma_{\mathcal{F}}(\gamma_1), \quad \sigma \mapsto (T\pi_1, T\pi_2)(\sigma)$$

is a linear homeomorphism.

Proof. Let $f := (\pi_1,\pi_2)$, by Proposition 3.2.4 the map $\mathcal{P}$ is continuous and clearly linear. For $i \in \{1,2\}$ we denote the i-projections by

$$\mathrm{pr}_i : \Gamma_{\mathcal{F}}(\gamma_1) \times \Gamma_{\mathcal{F}}(\gamma_2) \to \Gamma_{\mathcal{F}}(\gamma_i), \quad (\sigma_1,\sigma_2) \mapsto \sigma_i.$$

Let us define the smooth map $\lambda_1 : N_1 \to N_1 \times N_2$ such that

$$T\lambda_1 : TN_1 \to T(N_1 \times N_2), \quad v \mapsto (v,0)$$

and analogously we define $\lambda_2 : N_2 \to N_1 \times N_2$. Then we have

$$\mathcal{P}^{-1} = \mathcal{F}(M,T\lambda_1) \circ \mathrm{pr}_1 + \mathcal{F}(M,T\lambda_2) \circ \mathrm{pr}_2 .$$

Hence $\mathcal{P}^{-1}$ is continuous. □

Proposition 3.2.8 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M_1 and M_2 be compact smooth manifolds with corners and N be a smooth manifold. If $\Theta : M_1 \to M_2$ is a smooth diffeomorphism, then $\gamma \circ \Theta \in \mathcal{F}(M_1,N)$ for each $\gamma \in \mathcal{F}(M_2,N)$. Moreover, the map*

$$L_\Theta : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_{\mathcal{F}}(\gamma \circ \Theta), \quad \sigma \mapsto \sigma \circ \Theta$$

is linear and continuous.

Proof. Let $\gamma \in \mathcal{F}(M_2,N)$. Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ charts of M_1 and M_2 respectively such that $\Theta(U_1) \subseteq U_2$. If $\phi_N : U_N \to V_N$ is a chart of N such that $(\gamma \circ \Theta)(U_1) \subseteq U_N$ then

$$\phi_N \circ (\gamma \circ \Theta) \circ \phi_1^{-1} = \phi_N \circ \gamma \circ \phi_2^{-1} \circ \phi_2 \circ \Theta \circ \phi_1^{-1}.$$

Since $\zeta := \phi_N \circ \gamma \circ \phi_2^{-1} \in \mathcal{F}_{\mathrm{loc}}(V_\psi,\mathbb{R}^n)$ and the map $g := \phi_2 \circ \Theta \circ \phi_1^{-1} : V_\varphi \to V_\psi$ is a smooth diffeomorphism, by Lemma 3.1.16 we have that $\zeta \circ g \in \mathcal{F}_{\mathrm{loc}}(V_\varphi,\mathbb{R}^n)$. Thus

$$\gamma \circ \Theta \in \mathcal{F}(M_1,N).$$

Analogously, we can show that $\sigma \circ \Theta \in \Gamma_{\mathcal{F}}(\gamma \circ \Theta)$ for each $\sigma \in \Gamma_{\mathcal{F}}(\eta)$.
By compactness of M_1 and M_2, for $i \in \{1,...,k\}$ we consider charts $\phi_{1,i} : U_{1,i} \to V_{1,i}$ of M_1 such that there exists $W_{1,i} \in \mathcal{U}$ relatively compact in $V_{1,i}$ with $M_1 = \cup_{i=1}^k \phi_{1,i}^{-1}(W_{1,i})$

and charts $\phi_{2,i} : U_{2,i} \to V_{2,i}$ of M_2 such that there exists $W_{2,i} \in \mathcal{U}$ relatively compact in $V_{2,i}$ with $M_2 = \cup_{i=1}^k \phi_{2,i}^{-1}(W_{2,i})$ such that there exists a chart $\phi_{N,i} : U_{N.i} \to V_{N,i}$ of N such that $\Theta(W_{1,i}) \subseteq W_{2,i}$ and $\gamma(U_{2,i}) \subseteq U_{N,i}$. We define the topological embeddings

$$\Phi_\gamma : \Gamma_{\mathcal{F}}(\gamma) \to \prod_{i=1}^k \mathcal{F}(W_{2,i}, \mathbb{R}^n), \quad \sigma \mapsto \left(d\phi_{N,i} \circ \sigma \circ \phi_{2.i}^{-1}|_{W_{2,i}}\right)_{i=1}^k$$

and

$$\Phi_{\gamma\circ\Theta} : \Gamma_{\mathcal{F}}(\gamma \circ \Theta) \to \prod_{i=1}^k \mathcal{F}(W_{1,i}, \mathbb{R}^n), \quad \sigma \mapsto \Big(d\phi_{N,i} \circ \sigma \circ \Theta \circ \phi_{1,i}^{-1}|_{W_{1,i}}\Big)_{i=1}^k$$

Since the map

$$\Theta_i := \phi_{2.i} \circ \Theta \circ \phi_{1,i}^{-1}|_{W_{1,i}} : W_{1,i} \to \Theta_i(W_{2,i})$$

is a smooth diffeomorphism, the map

$$\mathcal{F}(\Theta_i, \mathbb{R}^n) : \mathcal{F}_{\text{loc}}(\Theta_i(W_{2,i}), \mathbb{R}^n) \to \mathcal{F}_{\text{loc}}(W_{1,i}, \mathbb{R}^n), \quad \tau \mapsto \tau \circ \Theta_i$$

and thus

$$\overline{\Theta} : \prod_{i=1}^k \mathcal{F}_{\text{loc}}(\Theta_i(W_{2,i}), \mathbb{R}^n) \to \prod_{i=1}^k \mathcal{F}_{\text{loc}}(W_{2,i}, \mathbb{R}^n), \quad (\tau_i)_{i=1}^n \mapsto (\tau_i \circ \Theta_i)_{i=1}^k$$

are continuous. We will show that $\overline{\Theta}(\text{Im}(\Phi_\gamma)) \subseteq \text{Im}(\Phi_{\gamma\circ\Theta})$.
For each $i, j \in \{1, ..., k\}$ and $\sigma \in \Gamma_{\mathcal{F}}(\eta)$, if

$$\begin{aligned} \tau_i &:= d\phi_{N,i} \circ \sigma \circ \phi_{2,i}^{-1}|_{W_{2,i}} \circ \Theta_i \\ &= d\phi_{N,i} \circ \sigma \circ \Theta \circ \phi_{1,i}^{-1}|_{W_{1,i}} \end{aligned}$$

then

$$\begin{aligned} \tau_i \circ \phi_{1,i}(p) &= d\phi_{N,i} \circ \sigma \circ \Theta \circ \phi_{1,i}^{-1} \circ \phi_{1,i}(p) \\ &= d\phi_{N,i} \circ \sigma \circ \Theta(p) \\ &= d\phi_{N,i} \circ (T\phi_{N,j})^{-1}\Big(\phi_{N,j} \circ \gamma(p), d\phi_{N,j} \circ \sigma \circ \Theta(p)\Big) \\ &= d\phi_{N,i} \circ (T\phi_{N,j})^{-1}\Big(\phi_{N,j} \circ \gamma(p), \tau_j \circ \phi_{1,j}(p)\Big) \end{aligned}$$

Hence $\overline{\Theta}(\text{Im}(\Phi_\gamma)) \subseteq \text{Im}(\Phi_{\gamma\circ\Theta})$. In consequence, since

$$L_\Theta = \Phi_{\gamma\circ\Theta}^{-1} \circ (\mathcal{F}(\Theta_i, \mathbb{R}^n))_{i=1}^k \circ \Phi_\gamma$$

the map L_Θ is continuous. □

Proposition 3.2.9 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be a compact manifold with corners and N be a smooth manifold. Then the evaluation map*

$$\epsilon : \Gamma_{\mathcal{F}}(\gamma) \times M \to TN, \quad (\sigma, p) \mapsto \sigma(p)$$

is continuous. Moreover, for each $p \in M$ the point evaluation map

$$\epsilon_p : \Gamma_{\mathcal{F}}(\gamma) \to TN, \quad \sigma \mapsto \sigma(p)$$

is smooth, and its co-restriction as a map to $T_{\gamma(p)}N$ is linear.

Proof. Since the evaluation map

$$\tilde{\epsilon} : \Gamma_C(\gamma) \times M \to TN, \quad (\sigma, p) \mapsto \sigma(p)$$

is continuous and the evaluation map $\tilde{\epsilon}_p : \Gamma_C(\gamma) \to TN$, $\sigma \mapsto \sigma(p)$ is smooth for each $p \in M$ (see [3]). Then $\epsilon = \tilde{\epsilon} \circ (J_\Gamma, \mathrm{Id}_{\mathbb{R}})$ and $\epsilon_p = \tilde{\epsilon}_p \circ J_\Gamma$, where $J_\Gamma : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_C(\gamma)$ is the inclusion map, which is smooth by Remark 3.2.6. □

3.3 Manifolds of $\mathcal{F}$-maps on compact manifolds

Remark 3.3.1 Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ is a family of locally convex spaces suitable for global analysis. Let M be m-dimensional compact manifold with corners and N be a smooth manifold which admits a local addition $\Sigma : \Omega \to N$. Let $\gamma \in \mathcal{F}(M,N)$. We define the set

$$\mathcal{V}_\gamma := \{\sigma \in \Gamma_{\mathcal{F}}(\gamma) : \sigma(M) \subseteq \Omega\}.$$

which is open in $\Gamma_{\mathcal{F}}(\gamma)$ (see Remark 3.2.6) and

$$\mathcal{U}_\gamma := \{\xi \in \Gamma_{\mathcal{F}}(\gamma) : (\gamma, \xi)(M) \subseteq \Omega'\}.$$

Lemma 3.1.21 enables us to define the map

$$\Psi_\gamma := \mathcal{F}(M, \Sigma) : \mathcal{V}_\gamma \to \mathcal{U}_\gamma, \quad \sigma \mapsto \Sigma \circ \sigma$$

with inverse given by

$$\Psi_\gamma^{-1} : \mathcal{U}_\gamma \to \mathcal{V}_\gamma, \quad \xi \mapsto \theta_N^{-1} \circ (\gamma, \xi).$$

Moreover, since M is compact, we note that $BC(M,N) = C(M,N)$.

The following lemma is just an application of [4, Lemma 10.1] to our particular case.

Lemma 3.3.2 *Let E and F be finite-dimensional vector spaces, $U \subseteq E$ open and $f : U \to F$ a map. If $F_0 \subseteq F$ is a vector subspace and $f(U) \subseteq F_0$, then $f : U \to F$ is smooth if and only if $f|^{F_0} : U \to F_0$ is smooth.*

Theorem 3.3.3 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis, then for each compact manifold M with corners and smooth manifold N without boundary which admits a local addition, the set $\mathcal{F}(M,N)$ admits a smooth manifold structure such that the sets $\mathcal{U}_\gamma$ are open in $\mathcal{F}(M,N)$ for all $\gamma\in\mathcal{F}(M,N)$ and $\Psi_\gamma\colon\mathcal{V}_\gamma\to\mathcal{U}_\gamma$ is a C^∞-diffeomorphism.*

Proof. We endow $\mathcal{F}(M,N)$ with the final topology with respect to the family $\Psi_\gamma\colon\mathcal{V}_\gamma\to\mathcal{U}_\gamma$ for each $\gamma\in\mathcal{F}(M,N)$. If we define the maps $\Psi_\gamma^C\colon\mathcal{V}_\gamma^C\to\mathcal{U}_\gamma^C$ on the space of continuous functions $C(M,N)$ for each $\gamma\in C(M,N)$ then the final topology on $C(M,N)$ coincides with its topology (the compact-open topology), whence the inclusion map

$$J:\mathcal{F}(M,N)\to C(M,N),\quad \gamma\mapsto\gamma$$

is continuous. Moreover, since

$$\mathcal{U}^C_{J(\gamma)}:=\{\xi\in C(M,N):(J(\gamma),\xi)(M)\subseteq\Omega'\}$$

is open in $C(M,N)$, the set

$$\mathcal{U}_\gamma=\mathcal{U}_\gamma^C\cap\mathcal{F}(M,N)$$

is open in $\mathcal{F}(M,N)$.
The goal is to make to the family $\{(\mathcal{U}_\gamma,\Psi_\gamma^{-1}):\gamma\in\mathcal{F}(M,N)\}$ an atlas for the manifold structure.
Let $\gamma,\xi\in\mathcal{F}(M,N)$, it remains to show that the charts are compatible, i.e. the smoothness of the map

$$\Lambda_{\xi,\gamma}:=\Psi_\xi^{-1}\circ\Psi_\gamma:\Psi_\gamma^{-1}(\mathcal{U}_\gamma\cap\mathcal{U}_\xi)\subseteq\Gamma_{\mathcal{F}}(\gamma)\to\Gamma_{\mathcal{F}}(\xi),\quad \sigma\mapsto\theta_N^{-1}\circ(\xi,\Sigma\circ\sigma).\tag{3.3.1}$$

For $i\in\{1,...,k\}$, let $\varphi_i:U_{M,i}\to V_{M,i}$ be charts of M and $W_{M,i}\in\mathcal{U}$ such that $W_{M,i}$ is relatively compact in V_i with $M=\cup_{i=1}^k\varphi_i^{-1}(W_{M,i})$ and charts $\phi_i^\gamma:U_{N,i}^\gamma\to V_{N,i}^\gamma$ and $\phi_i^\xi:U_{N,i}^\xi\to V_{N,i}^\xi$ of N such that $\gamma\left(U_{M,i}\right))\subseteq U_{N,i}^\gamma$ and $\xi\left(U_{M,i}\right)\subseteq U_{N,i}^\xi$.
We will study the smoothness of the composition

$$\Phi_\xi\circ\Lambda_{\xi,\gamma}:\Psi_\gamma^{-1}(\mathcal{U}_\gamma\cap\mathcal{U}_\xi)\to\mathrm{Im}(\Phi_\xi)\subseteq\prod_{i=1}^k\mathcal{F}(W_{M,i},\mathbb{R}^n),\quad\sigma\mapsto\left(d\phi_i^\xi\circ\Lambda_{\xi,\gamma}(\sigma)\circ\varphi_i^{-1}|_{W_{M,i}}\right)_{i=1}^k$$

which is equivalent to the smoothness of $\Lambda_{\xi,\gamma}$, where Φ_ξ is the linear topological embedding as in Proposition 3.2.1. By Definition 3.1.1 c), we find W'_M in $\mathcal{U}$ such that $\overline{W'_M}$ is relatively compact in V_i and $W'_{M,i}$ contains the closure of $W_{M,i}$.
For each $i\in\{1,...,k\}$ and $\sigma\in\Psi_\gamma^{-1}(\mathcal{U}_\gamma\cap\mathcal{U}_\xi)$ we have

$$d\phi_i^\xi\circ\left(\Psi_\xi^{-1}(\Psi_\gamma(\sigma))\right)\circ\varphi_i^{-1}|_{W'_{M,i}}=d\phi_i^\xi\circ\theta_N^{-1}\circ\left(\xi\circ\varphi_i^{-1}|_{W'_{M,i}},\Sigma\circ\sigma\circ\varphi_i^{-1}|_{W'_{M,i}}\right).$$

Since $\sigma\left(\varphi_i^{-1}(W'_{M,i})\right)\subseteq TU_{\phi_i^\gamma}$ we can do

$$\begin{aligned}\Sigma\circ\sigma\circ\varphi_i^{-1}|_{W'_{M,i}}&=\Sigma\circ(T\phi_i^\gamma)^{-1}\circ T\phi_i^\gamma\circ\sigma\circ\varphi_i^{-1}|_{W'_{M,i}}\\&=\Sigma\circ(T\phi_i^\gamma)^{-1}\left(\phi_i^\gamma\circ\gamma\circ\varphi_i^{-1}|_{W'_{M,i}},d\phi_i^\gamma\circ\sigma\circ\varphi_i^{-1}|_{W'_{M,i}}\right)\end{aligned}$$

and

$$\xi \circ \varphi_i^{-1}|_{W'_{M,i}} = \left(\phi_i^\xi\right)^{-1} \circ \left(\phi_i^\xi \circ \xi \circ \varphi_i^{-1}|_{W'_{M,i}}\right).$$

Because all of the functions involved are continuous and have an open domain, also the composition

$$H_i(x,y,z) := d\phi_i^\xi \circ \theta_N^{-1} \circ \left(\left(\phi_i^\xi\right)^{-1}(x), \Sigma \circ (T\phi_i^\gamma)^{-1}(y,z)\right), \qquad (3.3.2)$$

has an open domain $\mathcal{O}_i$. Hence the map $H_i : \mathcal{O}_i \to E$ is smooth.
By Lemma 3.1.12, the map

$$h_i := \mathcal{F}(W'_{M,i}/W_{M,i}, H_i)$$

is smooth. By the preceding

$$\Phi_\xi \circ \Lambda_{\xi,\gamma} = h_i(\phi_i^\xi \circ \xi \circ \varphi_i^{-1}|_{W'_{M,i}}, \phi_i^\gamma \circ \gamma \circ \varphi_i^{-1}|_{W'_{M,i}}, d\phi_i^\gamma \circ \sigma \circ \varphi_i^{-1}|_{W'_{M,i}}),$$

which is a smooth function of σ, using that the maps

$$\Gamma_{\mathcal{F}}(\gamma) \to \mathcal{F}(W'_{M,i}, \mathbb{R}^n), \quad \sigma \mapsto d\phi_i^\gamma \circ \sigma \circ \varphi_i^{-1}|_{W'_{M,i}}$$

are continuous linear by definition of the topology of $\Gamma_{\mathcal{F}}(\gamma)$. Therefore $\Psi_\xi^{-1} \circ \Psi_\gamma$ is smooth. □

Proceding in the same way, using the fact that composition of $\mathbb{K}$-analytic maps is $\mathbb{K}$-analytic and using the analytic version of Lemma 3.3.2 (see [14]), we can obtain the analogous case.

Corollary 3.3.4 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. For each compact manifold M with corners and $\mathbb{K}$-analytic manifold N without boundary which admits a $\mathbb{K}$-analytic local addition, the set $\mathcal{F}(M,N)$ admits a $\mathbb{K}$-analytic manifold structure such that the sets $\mathcal{U}_\gamma$ are open in $\mathcal{F}(M,N)$ for all $\gamma \in \mathcal{F}(M,N)$ and $\Psi_\gamma \colon \mathcal{V}_\gamma \to \mathcal{U}_\gamma$ is a C^∞-diffeomorphism.*

Proposition 3.3.5 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners, N_1 and N_2 be finite dimensional smooth manifold which admits local addition. If $f : N_1 \to N_2$ is a smooth map, then the map*

$$\mathcal{F}(M,f) : \mathcal{F}(M,N_1) \to \mathcal{F}(M,N_2), \quad \gamma \mapsto f \circ \gamma,$$

is smooth.

Proof. The map is well defined by Lemma 3.1.21. Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be the local addition for N_1 and N_2 respectively. We consider the charts $(\mathcal{U}_\gamma, \Psi_\gamma^{-1})$ and $(\mathcal{U}_{f\circ\gamma}, \Psi_{f\circ\gamma}^{-1})$ in $\gamma \in \mathcal{F}(M,N)$ and $f \circ \gamma \in \mathcal{F}(M,N)$ respectively. We define

$$F(\sigma) := \Psi_{f\circ\gamma}^{-1} \circ \mathcal{F}(M,f) \circ \Psi_\gamma(\sigma) = (\pi_{TN}, \Sigma_2)^{-1} \circ \left(f \circ \gamma, f \circ \Sigma_1 \circ \sigma\right)$$

for all $\sigma \in \Psi_\gamma^{-1}\left(\mathcal{U}_\gamma \cap \mathcal{F}(M,f)^{-1}(\mathcal{U}_{f\circ\gamma})\right)$.
We will proceed as in the proof of the Theorem 3.3.3. For $i \in \{1,...,k\}$, let $\varphi_i : U_{M,i} \to V_{M,i}$ be charts of M, $W_{M,i} \in \mathcal{U}$ such that $W_{M,i}$ is relatively compact in V_i with $M = \cup_{i=1}^k \varphi_i^{-1}(W_{M,i})$ and $\phi_{1,i} : U_{1,i} \to V_{1,i}$ and $\phi_{2,i} : U_{2,i} \to V_{2,i}$ charts of N_1 and N_2 respectively such that $\gamma(U_{M,i}) \subseteq U_{1,i}$ and $(f\circ\gamma)(U_{M,i}) \subseteq U_{2,i}$. We will study the smoothness of the composition

$$\Phi_{f\circ\xi}\circ F : \Psi_\xi^{-1}\left(\mathcal{U}_\gamma \cap \mathcal{F}(M,f)^{-1}(\mathcal{U}_{f\circ\gamma})\right) \to \mathrm{Im}(\Phi_{f\circ\xi}), \quad \sigma \mapsto \left(d\phi_{2,i}\circ F(\sigma)\circ \varphi_{M,i}^{-1}|_{W_{M,i}}\right)_{i=1}^k$$

where $\Phi_{f\circ\xi}$ is the linear topological embedding as in Proposition 3.2.1. Using Definition 3.1.1 c), we find sets $W'_{M,i}$ in $\mathcal{U}$ which are relatively compact in $U_{M,i}$ and contain $W_{M,i}$. For each $i \in \{1,...,k\}$ and $\sigma \in \Psi_\gamma^{-1}\left(\mathcal{U}_\gamma \cap \mathcal{F}(M,f)^{-1}(\mathcal{U}_{f\circ\gamma})\right)$ we have

$$\begin{aligned} d\phi_{2,i}\circ F(\sigma)\circ\varphi_{M,i}^{-1}|_{W'_{M,i}} &= d\phi_{2,i}\circ(\pi_{TN},\Sigma_2)^{-1}\circ\Big(f\circ\gamma, f\circ\Sigma_1\circ\sigma\Big)\circ\varphi_{M,i}^{-1}|_{W'_{M,i}} \\ &= d\phi_{2,i}\circ(\pi_{TN},\Sigma_2)^{-1}\circ\Big(f\circ\gamma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}, f\circ\Sigma_1\circ\sigma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}\Big). \end{aligned}$$

Since $(f\circ\gamma)(U_{M,i}) \subseteq U_{2,i}$ we have

$$f\circ\gamma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}} = \phi_{2,i}^{-1}\circ\left(\phi_{2,i}\circ f\circ\gamma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}\right).$$

And since $\gamma\left(\varphi_{M,i}^{-1}(V_{M,i})\right) \subseteq U_{1,i}$ we have $\sigma\left(\varphi_{M,i}^{-1}(W_{M,i})\right) \subseteq TU_{1,i}$ whence

$$\begin{aligned} f\circ\Sigma_1\circ\sigma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}} &= f\circ\Sigma_1\circ T\phi_{1,i}^{-1}\circ T\phi_{1,i}\circ\sigma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}} \\ &= f\circ\Sigma_1\circ T\phi_{1,i}^{-1}\left(\phi_{1,i}\circ\gamma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}, d\phi_{1,i}\circ\sigma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}\right). \end{aligned}$$

Let

$$H_i(x,y,z) := d\phi_{2,i}\circ(\pi_{TN},\Sigma_2)^{-1}\circ(\phi_{2,i}^{-1}(x), f\circ\Sigma_1\circ T\phi_{1,i}^{-1}(y,z)).$$

Then H_i is defined on an open subset of $\mathbb{R}^{n_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}$ and the $\mathbb{R}^{n_2}$-valued function H_i so obtained is smooth (because it is a composition of smooth functions).
By Lemma 3.1.12, also the corresponding mappings

$$h_i := \mathcal{F}(W'_{M,i}/W_{M,i}, H_i)$$

between functions spaces are smooth. By the above, we have

$$(\Phi_{f\circ\xi}\circ F)(\sigma)h_i\left(\phi_{2,i}\circ f\circ\gamma\circ\varphi_{M,i}^{-1}|_{W'_{M,i}}, \phi_{1,i}\circ\gamma\circ\varphi_{M,i}^{-1}|_{W_{M,i}^{-1}}, d\phi_{1,i}\circ\sigma\circ\varphi_{M,i}^{-1}|_{W_{M,i}^{-1}}\right)$$

which is a smooth function of σ, using that

$$\Gamma_{\mathcal{F}}(\gamma) \to \mathcal{F}(W_{M,i},\mathbb{R}^n), \quad \sigma \mapsto d\phi_{1,i}\circ\sigma\circ\varphi_{M,i}^{-1}|_{W_{M,i}}$$

is a continuous linear map by definition. Therefore $\mathcal{F}(M,f)$ is smooth. □

Applying Lemma 3.1.25 we can obtain the analogous result.

Corollary 3.3.6 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, M be an m-dimensional compact smooth manifold with corners, N_1 and N_2 be n-dimensional $\mathbb{K}$-analytic manifolds with $\mathbb{K}$-analytic local additions (Ω_1,Σ_1) and (Ω_2,Σ_2) respectively. If $f: N_1\to N_2$ is a $\mathbb{K}$-analytic map, then the map*

$$\mathcal{F}(M,f):\mathcal{F}(M,N_1)\to\mathcal{F}(M,N_2),\quad \gamma\mapsto f\circ\gamma,$$

is $\mathbb{K}$-analytic.

Remark 3.3.7 The manifold structures for $\mathcal{F}(M,N)$ given by different local additions are coincide. Indeed, since the identity map $\mathrm{id}_N: N\to N$ is smooth, the map

$$\mathcal{F}(M,\mathrm{id}_N):\mathcal{F}(M,N)\to\mathcal{F}(M,N),\quad \gamma\to \mathrm{id}_M\circ\gamma$$

is smooth regardless of the chosen local addition in each space.

Remark 3.3.8 The inclusion map $J:\mathcal{F}(M,N)\to C([a,b],N)$ is smooth. Indeed, let $(\mathcal{U}_\gamma,\Psi_\gamma^{-1})$ and $(\mathcal{U}^C_{J(\gamma)},\Psi^{-1}_{J(\gamma)})$ be charts in $\gamma\in\mathcal{F}(M,N)$ and $J(\gamma)\in C([a,b],N)$ respectively, then

$$\Psi^{-1}_{J(\gamma)}\circ J\circ\Psi_\gamma^{-1}(\sigma):\Psi_\gamma^{-1}\left(\mathcal{U}_\gamma\cap J^{-1}(\mathcal{U}_{j(\gamma)})\right)\subseteq\Gamma_{\mathcal{F}}(\gamma)\to\Gamma_C(\gamma)$$

is a restriction of the inclusion map $\Gamma_{\mathcal{F}}(\gamma)\to\Gamma_C(\gamma)$.
Moreover, if $U\subseteq N$ is an open subset, then the manifold structure induced by $\mathcal{F}(M,N)$ on the open subset

$$\mathcal{F}(M,U):=\{\gamma\in\mathcal{F}(M,N):\gamma(M)\subseteq U\}.$$

coincides with the manifold structure on $\mathcal{F}(M,U)$.

Proposition 3.3.9 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be a m-dimensional compact smooth manifold with corners, N_1 and N_2 be smooth manifolds which admit local additions, and let $pr_i: N_1\times N_2\to N_i$ be the i-th projection where $i\in\{1,2\}$, then the map*

$$\mathcal{P}:\mathcal{F}(M,N_1\times N_2)\to\mathcal{F}(M,N_1)\times\mathcal{F}(M,N_2),\quad \gamma\mapsto(pr_1,pr_2)\circ\gamma$$

is a diffeomorphism.

Proof. If (Ω_1,Σ_1) and (Ω_1,Σ_1) are the local additions on N_1 and N_2 respectively, then we can assume that the local addition on $N_1\times N_2$ is

$$\Sigma:=\Sigma_1\times\Sigma_2:\Omega_1\times\Omega_2\to N_1\times N_2$$

where $\Omega_1\times\Omega_2\subseteq TN_1\times TN_2\cong T(N_1\times N_2)$. The map $\mathcal{P}$ is smooth as consequence of the smoothness of the maps

$$\mathcal{F}(M,\mathrm{pr}_j):\mathcal{F}(M,N_1\times N_2)\to\mathcal{F}(M,N_i),$$

for each $i \in \{1,2\}$ by the previous results.
Let $(\mathcal{U}_\gamma \times \mathcal{U}_\gamma, \Psi_{\gamma_1}^{-1} \times \Psi_{\gamma_2}^{-1})$ and $(\mathcal{U}_\gamma, \Psi_\gamma^{-1})$ be charts in $(\gamma_1,\gamma_2) \in \mathcal{F}(M,N_1) \times \mathcal{F}(M,N_2)$ and $\mathcal{P}^{-1}(\gamma_1,\gamma_2) = \gamma \in \mathcal{F}(M, N_1 \times N_2)$ respectively. Since the map

$$\mathcal{Q} : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_{\mathcal{F}}(\gamma) \times \Gamma_{\mathcal{F}}(\gamma_2), \quad \tau \mapsto (\mathrm{q}_1, \mathrm{q}_2) \circ \tau$$

where q_1 and q_2 are the corresponding projection of the space, is an diffeomorphism of vector spaces (by Lemma 3.3 and Proposition 3.2.1), we have

$$\begin{aligned}\Psi_\gamma^{-1} \circ \mathcal{P}^{-1} \circ (\Psi_{\gamma_1} \times \Psi_{\gamma_2})(\sigma_1,\sigma_2) &= (\pi_{N_1 \times N_2}, \Sigma)^{-1} \circ \big(\gamma, \mathcal{P}^{-1} \circ (\Sigma_1 \times \Sigma_2)(\sigma_1,\sigma_2)\big) \\ &= (\pi_{N_1 \times N_2}, \Sigma)^{-1} \circ \big(\gamma, \Sigma \circ \mathcal{Q}^{-1}(\sigma_1,\sigma_2)\big) \\ &= \mathcal{Q}^{-1}(\sigma_1,\sigma_2)\end{aligned}$$

for all $(\sigma_1,\sigma_2) \in (\Psi_{\gamma_1}^{-1} \times \Psi_{\gamma_2}^{-1})\,(\mathcal{U}_{\gamma_1} \times \mathcal{U}_{\gamma_2} \cap \mathcal{P}(\mathcal{U}_\gamma))$. Hence $\mathcal{P}^{-1}$ is smooth. □

Proposition 3.3.10 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U \in \mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M_1 and M_2 be m-dimensional compact smooth manifolds with corners and N be an n-dimensional smooth manifold which admits a local addition. If $\Theta : M_1 \to M_2$ is a smooth diffeomorphism, then the map*

$$\mathcal{F}(\Theta, N) : \mathcal{F}(M_2, N) \to F(M_1, N), \quad \gamma \mapsto \gamma \circ \Theta$$

is smooth.

Proof. By Proposition 3.2.8 we know that the map is well defined. Let $(\mathcal{U}_\gamma, \Psi_\gamma^{-1})$ and $(\mathcal{U}_{\gamma\circ\Theta}, \Psi_{\gamma\circ\Theta}^{-1})$ be charts in $\gamma \in \mathcal{F}(M_2,N)$ and $\gamma \circ \Theta \in \mathcal{F}(M_1,N)$ respectively, then we have

$$\Psi_{\gamma\circ\Theta}^{-1} \circ \mathcal{F}(\Theta,N) \circ \Psi_\gamma(\sigma) = \theta_N^{-1} \circ (\gamma \circ \Theta, \Sigma \circ (\sigma \circ \Theta))$$

for all $\sigma \in \Psi_\gamma^{-1}\big(\mathcal{U}_\gamma \cap \mathcal{F}(\Theta,N)^{-1}(\mathcal{U}_{\gamma\circ\Theta})\big)$.
Let $\alpha = \gamma \circ \Theta : M_1 \to N$ and $\tau = \sigma \circ \Theta : M_2 \to TN$, then $\tau \in \Gamma_{\mathcal{F}}(\alpha)$ and

$$\begin{aligned}\Psi_{\gamma\circ\Theta}^{-1} \circ \mathcal{F}(\Theta,N) \circ \Psi_\gamma(\sigma) &= \theta_N^{-1} \circ (\alpha, \Sigma \circ \tau) \\ &= \Psi_\alpha^{-1} \circ \Psi_\alpha(\tau) \\ &= \tau \\ &= \sigma \circ \Theta.\end{aligned}$$

Hence, $\Psi_{\gamma\circ\Theta}^{-1} \circ \mathcal{F}(\Theta,N) \circ \Psi_\gamma$ is a restriction of the map

$$L_\Theta : \Gamma_{\mathcal{F}}(\gamma) \to \Gamma_{\mathcal{F}}(\gamma \circ \Theta), \quad \sigma \to \sigma \circ \Theta$$

which is linear and continuous by Proposition 3.2.8. □

Proposition 3.3.11 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U \in \mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional* *compact smooth manifold with corners. If N, L and K are smooth manifolds which admits local additions and $f : L \times K \to N$ is a smooth map and $\gamma \in \mathcal{F}(M, L)$ is fixed, then*

$$f_* : \mathcal{F}(M, K) \to \mathcal{F}(M, N), \quad \xi \mapsto f \circ (\gamma, \xi)$$

is a smooth map.

Proof. Define the smooth map

$$C_\gamma : \mathcal{F}(M, K) \to \mathcal{F}(M, L) \times \mathcal{F}(M, K), \quad \xi \mapsto (\gamma, \xi).$$

Identifying $\mathcal{F}(M, L) \times \mathcal{F}(M, K)$ with $\mathcal{F}(M, L \times K)$, we have

$$f_* = \mathcal{F}(M, f) \circ C_\gamma.$$

Hence f_* is smooth. □

Proposition 3.3.12 *Let $\mathcal{U}$ be a good collection of open subsets of $[0, \infty)^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners and N be a n-dimensional smooth manifold. Then the evaluation map*

$$\varepsilon : \mathcal{F}(M, N) \times M \to N, \quad (\gamma, p) \mapsto \gamma(p)$$

is continuous. Moreover, for each $p \in M$, the point evaluation map

$$\varepsilon_p : \mathcal{F}(M, N) \to N, \quad \gamma \mapsto \gamma(p)$$

is smooth.

Proof. The evaluation map

$$\varepsilon_c : C(M, N) \times M \to N, \quad (\gamma, p) \mapsto \gamma(p))$$

is $C^{\infty,0}$ with point evaluation $(\varepsilon_c)_p : C(M, N) \to N$, $\gamma \mapsto \gamma(p)$ smooth for each $p \in M$. Since the inclusion map $J : \mathcal{F}(M, N) \to C(M, N)$ is smooth, we have $\varepsilon = \varepsilon_c \circ (J, \mathrm{Id}_M)$ and $\varepsilon_p = (\varepsilon_c)_p \circ J$ for each $p \in M$. □

Proposition 3.3.13 *Let $\mathcal{U}$ be a good collection of open subsets of $[0, \infty)^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be a m-dimensional compact smooth manifold with corners and N be a n-dimensional smooth manifold with local addition. Then, for each $q \in N$, the function $\zeta_q : M \to N$, $p \mapsto q$ is in $\mathcal{F}(M, N)$ and the map*

$$\zeta : N \to \mathcal{F}(M, N), \quad q \mapsto \zeta_q$$

is a smooth topological embedding.

Proof. If $W \in \mathcal{U}$ is relatively compact and $z \in \mathbb{R}^n$, consider the constant function

$$c_z \colon W \to \mathbb{R}^n, \quad x \mapsto z.$$

Then $c_z \in \mathcal{F}(W, \mathbb{R}^n)$. In fact, Definition 3.1.1 c) provides $V \in \mathcal{U}$ such that $\overline{W} \subseteq V$. Then $\eta \colon V \to \mathbb{R}^n$, $x \mapsto 0$ is in $\mathcal{F}(V, \mathbb{R}^n)$. The map $f \colon V \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, y) \mapsto z$ is smooth, whence $c_z = f \circ (\mathrm{id}_W, \eta|_W) \in \mathcal{F}(W, \mathbb{R}^n)$ by the pushforward axiom.
For each $z \in N$, the constant function

$$\zeta_z \colon M \to N, \quad p \mapsto z$$

is in $\mathcal{F}(M, N)$. In fact, if $p \in M$, $\phi_M \colon U_M \to V_M$ is chart for M around p and $\phi_N \colon U_N \to V_N$ a chart for N around $\zeta_z(p) = z$, then Definition 3.1.1 c) provides a relatively compact $\phi_M(p)$-neighborhood $W \subseteq V_M$ such that $W \in \mathcal{U}$. After replacing ϕ_M with its restriction to a map $\phi_M^{-1}(W) \to W$, we may assume that $V_M \in \mathcal{U}$ and V_M is relatively compact. Now $\phi_N \circ \zeta_z \circ \phi_M^{-1}$ is the constant function $W \to \mathbb{R}^n$, $x \mapsto \phi_N(z)$, which is in $\mathcal{F}(W, \mathbb{R}^n)$ as observed above. Thus $\zeta_z \in \mathcal{F}(M, N)$.
In particular, for each $y \in N$, the constant function

$$C_z \colon M \to T_y N, \quad v \mapsto z$$

is an element of $\mathcal{F}(M, T_y N)$, for each $z \in T_y N$. Since $T_y N$ is a finite-dimensional vector space, the linear map

$$C \colon T_y N \to \mathcal{F}(M, T_y N), \quad z \mapsto C_z \tag{3.3.3}$$

is continuous.
Given $y \in N$, consider the constant function $\zeta_y \colon M \to N$, $p \mapsto y$, we define the vector space

$$\Gamma_{\mathcal{F}}(C_y) := \{\tau \in \mathcal{F}(M, TN) : (\forall p \in M)\ \tau(p) \in T_{C_y(p)} N = T_y N\}.$$

We show that

$$\mathcal{F}(M, T_y N) \subseteq \Gamma_{\mathcal{F}}(\zeta_y)$$

with continuous linear inclusion map. The inclusion map $\iota \colon T_y N \to TN$ being smooth, for each $\tau \in \mathcal{F}(M, T_y N)$ we get

$$\tau = \iota \circ \tau = \mathcal{F}(M, \iota)(\tau) \in \mathcal{F}(M, TN)$$

by Lemma 3.1.21. Moreover, $\mathcal{F}(M, \iota)$ (and hence also its co-restriction j to $\Gamma_{\mathcal{F}}(\zeta_y)$) is continuous, by Proposition 3.3.5.
Let $\Sigma \colon \Omega \to N$ be a local addition for N and notation as in Definition 2.3.1 and Remark 3.3.1. We have $V \subseteq \Omega$ for an open 0-neighborhood $V \subseteq T_y N$. Then $U_N := \Sigma(V)$ is an open y-neighborhood in N and $\psi := \Sigma|_V^{U_N} \colon V \to U_N$ is a C^∞-diffeomorphism with

$$\psi^{-1}(u) = \theta_N^{-1}(y, u)$$

for $u \in U_N$. If $\alpha \colon T_y N \to \mathbb{R}^n$ is an isomorphism of vector spaces, then $V_N := \alpha(V)$ is open in $\mathbb{R}^n$ and $\phi_N(u) := \alpha(\psi^{-1}(u))$ defines a chart $\phi_N \colon U_N \to V_N$ of N. For each $v \in V_N$, we have for each $q \in M$

$$(\zeta_y(q), \zeta_{\phi_N^{-1}(v)}(q)) = (y, \phi_N^{-1}(v)) = (y, \psi(\alpha^{-1}(v))) \in \{y\} \times U_N \subseteq \Omega'$$

with
$$\theta_N^{-1}(y,\psi(\alpha^{-1}(v))) = \psi^{-1}(\psi(\alpha^{-1}(v))) = \alpha^{-1}(v).$$
Thus $\zeta_{\phi_N^{-1}(v)} \in \mathcal{U}_{\zeta_y}$ and
$$\Psi_{\zeta_y}^{-1}(\zeta_{\phi_N^{-1}(v)}) = \theta_N^{-1} \circ \Big(\zeta_y, \zeta_{\phi_N^{-1}(v)}\Big)$$
is the constant function $C_{\alpha^{-1}(v)}$. Hence
$$\Psi_{\zeta_y}^{-1} \circ \zeta \circ \phi_N^{-1} = j \circ C \circ \alpha^{-1}|_{V_N},$$
which is a smooth function. Thus ζ is smooth.
Fix $p \in M$. The point evaluation $\varepsilon_p \colon \mathcal{F}(M,N) \to N$, $\gamma \mapsto \gamma(p)$ is smooth and hence continuous. Since $\varepsilon_p \circ \zeta = \mathrm{id}_N$, we deduce that $(\zeta|^{\zeta(N)})^{-1} = \varepsilon_p|_{\zeta(N)}$ is continuous. Thus ζ is a homeomorphism onto its image. □

Remark 3.3.14 Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners, N be an n-dimensional smooth manifold which admits a local addition and let $T\mathcal{F}(M,N)$ be the tangent bundle of $\mathcal{F}(M,N)$. Since the point evaluation map $\varepsilon_p : \mathcal{F}(M,N) \to N$ is smooth for each $p \in M$, we have
$$T\varepsilon_p : T\mathcal{F}(M,N) \to TN.$$
For each $v \in T\mathcal{F}(M,N)$ we define the function
$$\Theta_N(v) : M \to TN, \quad \Theta_N(v)(p) = T\varepsilon_p(v).$$

Proposition 3.3.15 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners, N be an n-dimensional smooth manifold which admits a local addition and $\gamma \in \mathcal{F}(M,N)$. Then $\Theta_N(v) \in \Gamma_{\mathcal{F}}(\gamma)$ for each $v \in T_\gamma\mathcal{F}(M,N)$ and the map*
$$\Theta_\gamma : T_\gamma\mathcal{F}(M,N) \to \Gamma_{\mathcal{F}}(\gamma), \quad v \mapsto \Theta_\gamma(v) := \Theta_N|_{T_\gamma\mathcal{F}(M,N)}(v)$$
is an isomorphism of topological vector spaces.

Proof. Let $\Sigma : \Omega \to N$ be a normalized local addition of N in sense of [3]. Since $\Gamma_{\mathcal{F}}(\gamma)$ is a vector space, we identify its tangent bundle with $\Gamma_{\mathcal{F}}(\gamma) \times \Gamma_{\mathcal{F}}(\gamma)$. Let $\Psi_\gamma : \mathcal{V}_\gamma \to \mathcal{U}_\gamma$ be a chart around γ such that $\Psi_\gamma(0) = \gamma$, then
$$T\Psi_\gamma : T\mathcal{V}_\gamma \simeq \mathcal{V}_\gamma \times \Gamma_{\mathcal{F}}(\gamma) \to T\mathcal{F}(M,N)$$
is a diffeomorphism onto its image. Moreover,
$$T_0\Psi_\gamma : \{0\} \times \Gamma_{\mathcal{F}}(\gamma) \to T_\gamma\mathcal{F}(M,N)$$

is an isomorphism of topological vector spaces. We will show that

$$\Theta_\gamma \circ T\Psi_\gamma(0,\sigma) = \sigma$$

for each $\sigma \in \Gamma_{\mathcal{F}}(\gamma)$. Which is equivalent to show that

$$T\varepsilon_p \circ T\Psi_\gamma(0,\sigma) = \sigma(p) \quad \text{for all } p \in M.$$

Working with the geometric point of view of tangent vectors, we see that $(0,\sigma)$ is equivalent to the curve $[s \mapsto s\sigma]$. Hence, for each $p \in M$ we have

$$\begin{aligned} T\varepsilon_p \circ T\Psi_\gamma(0,\sigma) &= T\varepsilon_p \circ T\Psi_\gamma([s \mapsto s\sigma]) \\ &= T\varepsilon_p([s \mapsto \Psi_\gamma(s\sigma)]) \\ &= T\varepsilon_p([s \mapsto \Sigma(s\sigma)]) \\ &= [s \mapsto \Sigma|_{T_{\gamma(p)}N}(s\sigma(t))] \\ &= T_0\Sigma|_{T_{\gamma(p)}N}([s \mapsto s\sigma(t)]). \end{aligned}$$

Since Σ is normalized we have $T_0\Sigma|_{T_{\gamma(p)}N} = \mathrm{id}_{T_{\gamma(p)}N}$ and

$$T\varepsilon_p \circ T\Psi_\gamma(0,\sigma) = \sigma(p).$$

In consequence, for each $\sigma \in \Gamma_{\mathcal{F}}(\gamma)$, there exists a $v \in T_\gamma\mathcal{F}(M,N)$ with $v = T\Psi_\gamma(0,\sigma)$ such that

$$\Theta_\gamma(v) = \sigma.$$

Moreover, the function

$$\Theta_\gamma(v) : M \to TN, \quad p \mapsto \Theta_N(v)(p) = \sigma(p) \in T_{\gamma(p)}N$$

is in $\mathcal{F}(M,TN)$ with $\pi_{TN} \circ \Theta_\gamma(v) = \gamma$, making the map Θ_γ an isomorphism of topological vector spaces. □

Remark 3.3.16 Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners and N be an n-dimensional smooth manifold which admits a local addition. Since TN admits local addition and the vector bundle $\pi_{TN} : TN \to N$ is smooth, the map

$$\mathcal{F}(M,\pi_{TN}) : \mathcal{F}(M,TN) \to \mathcal{F}(M,N), \quad \tau \mapsto \pi_{TN} \circ \tau$$

is smooth. Moreover, if $\gamma \in \mathcal{F}(M,N)$, then

$$\mathcal{F}(M,\pi_{TN})^{-1}(\{\gamma\}) = \Gamma_{\mathcal{F}}(\gamma).$$

The following result follows the same steps as for the case of C^ℓ-maps (with $\ell \geq 0$) from a compact manifold (possibly with rough boundary) to a smooth manifold which admits local addition [3, Theorem A.12].

Proposition 3.3.17 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners, N be an n-dimensional smooth manifold which admits a local addition and $\pi_{TN} : TN \to N$ its tangent bundle. Then the map*

$$\mathcal{F}(M,\pi_{TN}) : \mathcal{F}(M,TN) \to \mathcal{F}(M,N), \quad \tau \mapsto \pi_{TN}\circ\tau$$

is a smooth vector bundle with fiber $\Gamma_{\mathcal{F}}(\gamma)$ over $\gamma \in \mathcal{F}(M,N)$. Moreover, the map

$$\Theta_N : T\mathcal{F}(M,N) \to F(M,TN), \quad v \mapsto \Theta_N(v)$$

is an isomorphism of vector bundles.

Proposition 3.3.18 *Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be a m-dimensional compact smooth manifold with corners, N_1 and N_2 be a n-dimensional smooth manifold which admits a local addition. If $f : N_1 \to N_2$ is a smooth map, then the tangent map of*

$$\mathcal{F}(M,f) : \mathcal{F}(M,N_1) \to \mathcal{F}(M,N_2), \quad \gamma \mapsto f\circ\gamma$$

is given by

$$T\mathcal{F}(M,f) = \Theta_{N_2}^{-1}\circ\mathcal{F}(M,Tf)\circ\Theta_{N_1}.$$

Proof. Let $\Sigma_1 : \Omega_1 \to N_1$ be a local addition on N_1 and $\gamma \in \mathcal{F}(M,N_1)$.
If $\Psi_\gamma : \mathcal{V}_\gamma \to \mathcal{U}_\gamma$ is a chart on γ such that $\Psi_\gamma(0) = \gamma$, we consider the isomorphism of vector space

$$T\Psi_\gamma : \{0\}\times\Gamma_{\mathcal{F}}(\gamma) \to T_\gamma\mathcal{F}(M,N_1).$$

For $p \in M$ we denote the point evaluation in $\varepsilon_p^i : \mathcal{F}(M,N_i) \to N_i$ for $i \in \{1,2\}$, then for each $\sigma \in \Gamma_{\mathcal{F}}(\gamma)$ we have

$$\begin{aligned}
T\varepsilon_p^2\circ T\mathcal{F}(M,f)\circ T\Psi_\gamma(0,\sigma) &= T\varepsilon_p^2\circ T\mathcal{F}(M,f)\circ T\Psi_\gamma([s\mapsto s\sigma])\\
&= T\varepsilon_p^2\circ T\mathcal{F}(M,f)([s\mapsto \Sigma_1\circ s\sigma])\\
&= T\varepsilon_p^2([s\mapsto f\circ\Sigma_1\circ s\sigma])\\
&= [s\mapsto \varepsilon_p^2(f\circ\Sigma_1\circ s\sigma)]\\
&= [s\mapsto f\circ\Sigma_1(s\sigma(p))]\\
&= Tf\circ T_0\Sigma_1|_{T_{\gamma(p)}N_1}([s\sigma(p)])\\
&= Tf([s\mapsto s\sigma(p)])\\
&= \mathcal{F}(M,Tf)(\sigma(p))\\
&= \mathcal{F}(M,Tf)\circ T\varepsilon_p^1\circ T\Psi_\gamma(0,\sigma).
\end{aligned}$$

Hence

$$\Theta_{N_2}\circ T\mathcal{F}(M,f) = \mathcal{F}(M,Tf)\circ\Theta_{N_1}.$$

□

Example 3.3.19 Let $\mathcal{U}$ be a good collection of open subsets of $[0,\infty)^m$ and $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ be a family of locally convex spaces suitable for global analysis. Let M be an m-dimensional compact smooth manifold with corners and G be an n-dimensional Lie group, then we already know that the space $\mathcal{F}(M,G)$ is a Lie group (see [22]). We will give an alternative proof of this.
Let $e \in G$ be the neutral element, let $L_g : G \to G$, $h \mapsto gh$ be the left translation by $g \in G$ and the action

$$G \times TG \to TG, \quad (g, v_h) \mapsto g.v_h := TL_g(v_h) \in T_{gh}G.$$

If $\varphi : U \subseteq G \to V \subseteq T_eG$ is a chart in e such that $\varphi(e) = 0$, then the set

$$\Omega_\varphi := \bigcup_{g\in G} g.V \subseteq TG$$

is open and the map

$$\Sigma_\varphi : \Omega_\varphi \to G, \quad v \mapsto \pi_{TG}(v)\left(\varphi^{-1}(\pi_{TG}(v)^{-1}.v)\right)$$

defines a local addition for G, hence $\mathcal{F}(M,G)$ is a smooth manifold with charts constructed with $(\Omega_\varphi, \Sigma_\varphi)$. Let $\mu_G : G \times G \to G$ and $\lambda_G : G \to G$ be the multiplication map and inversion maps on G respectively, we define the multiplication map μ_{AC} and the inversion map λ_{AC} on $\mathcal{F}(M,G)$ as

$$\mu_{\mathcal{F}} := \mathcal{F}(M,\mu_G) : \mathcal{F}(M,G) \times \mathcal{F}(M,G) \to \mathcal{F}(M,G)$$

and

$$\lambda_{\mathcal{F}} := \mathcal{F}(M,\lambda_G) : \mathcal{F}(M,G) \to \mathcal{F}(M,G)$$

that by Lemma 3.1.21 and Proposition 3.3.5 are smooth.
We observe that for the neutral element $\zeta_e : M \to G$, $p \mapsto e$ of $\mathcal{F}(M,G)$ we have

$$\Gamma_{\mathcal{F}}(\zeta_e) = \mathcal{F}(M, T_eG).$$

If $\Psi_{\zeta_e}^{-1} : \mathcal{U}_{\zeta_e} \to \mathcal{V}_{\zeta_e}$ is a chart around $\zeta_e \in \mathcal{F}(M,G)$, then we have $\mathcal{U}_{\zeta_e} = \mathcal{F}(M,U)$ and $\mathcal{V}_{\zeta_e} = \mathcal{F}(M,V)$. Moreover, we see that

$$\begin{aligned}\Psi_{\zeta_e} \circ \mathcal{F}(M,\varphi)(\gamma) &= \Sigma_\varphi \circ (\varphi\circ\gamma)\\ &= \pi_{TG}(\varphi\circ\gamma)\left(\varphi^{-1}(\pi_{TG}(\varphi\circ\gamma)^{-1}.\varphi\circ\gamma)\right)\\ &= e\varphi^{-1}(e.\varphi\circ\gamma)\\ &= \gamma.\end{aligned}$$

This enables us to say that for the neutral element $\zeta_e \in \mathcal{F}(M,G)$ the chart is given by

$$\mathcal{F}(M,\varphi) : \mathcal{F}(M,U) \to \mathcal{F}(M,V), \quad \gamma \mapsto \varphi\circ\gamma.$$

Remark 3.3.20 Instead of using the set $[0,\infty)^m$, it is possible to generalize all results to a good collection of open subsets $\mathcal{U}$ of a locally convex, closed subset of $\mathbb{R}^m$, such as half-spaces, all of $\mathbb{R}^m$, or a disjoint union of countably many m-dimensional polytopes.

3.4 Manifolds of λ-Hölder continuous functions

Let $m, n \in \mathbb{N}$, $0 < \lambda \leq 1$ and $U \subset \mathbb{R}^m$ be an open and bounded subset. We say that a function $\eta : U \to \mathbb{R}^n$ is λ-Hölder continuous if there exists a positive constant C such that

$$\|\eta(x) - \eta(y)\| \leq C\|x - y\|^\lambda, \quad \forall x, y \in U.$$

And for each λ-Hölder continuous function we define

$$\|\eta\|_\lambda := \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{\|\eta(x) - \eta(y)\|}{\|x - y\|^\lambda} \right\}.$$

Let $\mathcal{F}_\lambda(U, \mathbb{R}^n)$ be the space of λ-Hölder continuous functions $\eta : U \to \mathbb{R}^n$. By boundedness of the subset U, each function $\eta \in \mathcal{F}_\lambda(U, \mathbb{R}^n)$ is bounded. This allows us to consider the norm on $\mathcal{F}_\lambda(U, \mathbb{R}^n)$

$$\|\eta\|_{\mathcal{F}_\lambda} := \|\eta\|_\infty + \|\eta\|_\lambda.$$

Then $(\mathcal{F}_\lambda(U, \mathbb{R}^n), \|\cdot\|_{\mathcal{F}_\lambda})$ is a Banach space (see e.g. [10]). In particular, if $\lambda = 1$ then $\mathcal{F}_1(U, \mathbb{R})$ denotes the space of Lipschitz continuous functions.
We will denote the inclusion map by $J : \mathcal{F}_\lambda(U, \mathbb{R}^n) \to BC(U, \mathbb{R}^n)$, which is continuous.
Let $\mathcal{U}$ be the family of open and bounded subsets of $\mathbb{R}^m$. For $0 < \lambda \leq 1$ fixed, we consider the family of function spaces $\{\mathcal{F}_\lambda(U, \mathbb{R})\}_{U \in \mathcal{U}}$. We will show that they define a family of locally convex spaces suitable for global analysis.

Remark 3.4.1 Since each space $\mathcal{F}_\lambda(U, \mathbb{R})$ is a Banach space, the axiom (PB) is verified. Indeed, let U be an open subset of $\mathbb{R}^m$ and $V, W \in \mathcal{U}$ such that W has compact closure contained in U and $\Theta : U \to V$ be a C^∞-diffeomorphism. By relative compactness of W, we can consider a finite open cover of convex subsets $(W_i)_{i=1}^k$ for $\overline{W}$ such that $\Theta|_{W_i}$ is Hölder continuous and $\eta \circ \Theta|_{W_i} \in \mathcal{F}_\lambda(W_i, \mathbb{R})$ for each $i \in \{1, ..., k\}$ and $\eta \in \mathcal{F}_\lambda(V, \mathbb{R})$. Therefore,the map $\mathcal{F}_\lambda(\Theta|_W, \mathbb{R})$ makes sense, and by Lemma 3.1.7, the axiom (PB) is verified.

Remark 3.4.2 The axiom (GL) is also verified. In fact, if $U, V \in \mathcal{U}$ with $V \subseteq U$ and $\eta \in \mathcal{F}_\lambda(V, \mathbb{R})$ has compact support. Let $\tilde{\eta} : U \to \mathbb{R}$ be the map defined by extending η by 0, then $\|\tilde{\eta}\|_\lambda = \|\eta\|_\lambda$.
Therefore, the map $e^E_{U,V,K} : \mathcal{F}_K(V, E) \to \mathcal{F}(U, E)$ makes sense, and by Lemma 3.1.8, the axiom (GL) is verified.

Lemma 3.4.3 *If $h \in C^\infty_c(U, \mathbb{R})$, then $h\eta \in \mathcal{F}_\lambda(U, \mathbb{R})$ for each $\eta \in \mathcal{F}_\lambda(U, \mathbb{R})$.*

Proof. Let $\eta \in \mathcal{F}_\lambda(U, \mathbb{R})$. Since the function h is smooth with compact support, is λ-Hölder continuous and the product $h\eta$ is in $\mathcal{F}_\lambda(U, \mathbb{R})$. □

Remark 3.4.4 By Lemma 3.1.9 and Lemma 3.4.3, the axiom (MU) is verified.

Lemma 3.4.5 *Let $U, V \in \mathcal{U}$ such that $V \subseteq U$. Then $\eta|_V \in \mathcal{F}_\lambda(V, \mathbb{R})$ for each $\eta \in \mathcal{F}_\lambda(U, \mathbb{R})$ and the map*

$$\mathcal{F}_\lambda(U, \mathbb{R}) \to \mathcal{F}_\lambda(V, \mathbb{R}), \quad \eta \mapsto \eta|_V$$

is continuous linear.

Proof. This is direct consequence of the properties of the supremum. □

Lemma 3.4.6 *Let $\ell \in \mathbb{N}$ and $V \in \mathcal{U}$ be relatively compact. If $f : \mathbb{R}^\ell \to \mathbb{R}$ is a smooth map, then $f \circ \eta \in \mathcal{F}_\lambda(V, \mathbb{R})$ for each $\eta \in \mathcal{F}_\lambda(V, \mathbb{R}^\ell)$ and the map*

$$\tilde{f} : \mathcal{F}_\lambda(V, \mathbb{R}^\ell) \to \mathcal{F}_\lambda(V, \mathbb{R}), \quad \eta \mapsto f \circ \eta$$

is continuous.

Proof. Let Δ_V denote the diagonal set of $V \times V$. For each $\tau \in \mathcal{F}_\lambda(V, \mathbb{R})$, we define the function

$$h_\tau : (V \times V) \setminus \Delta_V \to \mathbb{R}, \quad (x, y) \mapsto h_\tau(x, y) := \frac{\tau(x) - \tau(y)}{\|x - y\|^\lambda}.$$

Then $h_\tau \in BC((V \times V) \setminus \Delta_V, \mathbb{R})$ with $\|h_\tau\|_\infty = \|\tau\|_\lambda$, hence the linear map

$$\mathcal{F}_\lambda(V, \mathbb{R}) \to BC((V \times V) \setminus \Delta_V, \mathbb{R}), \quad \tau \mapsto h_\tau$$

is continuous linear. Let us consider the map

$$H : \mathcal{F}_\lambda(V, \mathbb{R}) \to BC((V \times V) \setminus \Delta_V, \mathbb{R}), \quad \tau \mapsto h_\tau$$

then H is continuous. This enable us to define the linear map

$$\Phi : \mathcal{F}_\lambda(V, \mathbb{R}) \to BC(V, \mathbb{R}) \times BC((V \times V) \setminus \Delta_V, \mathbb{R}), \quad \tau \mapsto (\tau, H(\tau))$$

which is a topological embedding with closed image. Therefore, if the map $\tilde{f}$ makes sense, its continuity is equivalent to the continuity of

$$F : \mathcal{F}_\lambda(V, \mathbb{R}^\ell) \to BC(V, \mathbb{R}) \times BC((V \times V) \setminus \Delta_V, \mathbb{R}), \quad \eta \mapsto (f \circ \eta, H(f \circ \eta)).$$

First we will show that makes sense, i.e., $F(\eta) \in BC(V, \mathbb{R}) \times BC((V \times V) \setminus \Delta_V, \mathbb{R})$ for each $\eta \in \mathcal{F}_\lambda(V, \mathbb{R}^\ell)$. Since the inclusion map $J : \mathcal{F}_\lambda(V, \mathbb{R}^\ell) \to BC(V, \mathbb{R}^\ell)$ and the map

$$BC(V, \mathbb{R}^\ell) \to BC(V, \mathbb{R}), \quad \eta \mapsto f \circ \eta$$

are continuous, the first component of F

$$F_1 : \mathcal{F}_\lambda(V, \mathbb{R}^\ell) \to BC(V, \mathbb{R}), \quad \eta \mapsto f \circ \eta$$

is continuous. Let us consider the second component of F

$$F_2 : \mathcal{F}_\lambda(V, \mathbb{R}^\ell) \to BC((V \times V) \setminus \Delta_V, \mathbb{R}), \quad \eta \mapsto H(f \circ \eta).$$

Let $\eta \in \mathcal{F}_\lambda(V, \mathbb{R}^\ell)$, then $F_2(\eta)$ is clearly continuous. We will show that $F_2(\eta)$ is bounded. For $(x, y) \in V \times V \setminus \Delta_V$ we have

$$F_2(\eta)(x, y) = H(f \circ \eta)(x, y) = \frac{f(\eta(x)) - f(\eta(y))}{\|x - y\|^\lambda}.$$

Since V is relatively compact, the set $\eta(V)$ can be contained on an open ball $B_{R_\eta}(0)$ for a constant $R_\eta > 0$ large enough. By smoothness, the map f verifies

$$|f(u) - f(v)| \leq L_{f,\eta}\|u - v\|, \quad u, v \in \overline{B_{R_\eta}(0)},$$

for some constant $L_{f,\eta} > 0$. Therefore

$$\|F_2(\eta)\|_\infty \leq L_{f,\eta}\|\eta\|_\lambda.$$

Then $F_2(\eta) \in BC((V \times V) \setminus \Delta_V, \mathbb{R})$. Now we will show that F_2 is continuous in $\eta \in F_\lambda(V, \mathbb{R}^\ell)$. Let $\delta > 0$ and $\gamma \in F_\lambda(V, \mathbb{R}^\ell)$ such that

$$\|\eta - \gamma\|_{\mathcal{F}_\lambda} := \|\eta - \gamma\|_\infty + \|\eta - \gamma\|_\lambda \leq \delta.$$

Then for each $z \in V$ we have

$$\|\eta(z) - \gamma(z)\| \leq \delta,$$

which mean that $\gamma(z) \in B_\delta(\eta(z))$. Therefore

$$\gamma(V) \subseteq \bigcup_{z \in V} B_\delta(\eta(z)).$$

Let $R_\eta > 0$ the constant which verifies $\eta(V) \subseteq B_{R_\eta}(0)$, then $B_\delta(\eta(z)) \subseteq B_{R_\eta+\delta}(0)$ for each $z \in V$. In consequence, $\gamma(V)$ and $\eta(V)$ are contained in $B_{R_\eta+\delta}(0)$ and by smoothness of f, there exists a constant $G_{f,\eta} > 0$ such that

$$|df(u_1, v_1) - df(u_2, v_2)| \leq G_{f,\eta}\|(u_1, v_1) - (u_2, v_2)\| = G_{f,\eta}(\|u_1 - u_2\| + \|v_1 - v_2\|),$$

for each $(u_1, v_1), (u_2, v_2) \in \overline{B_{R_\eta+\delta}(0)} \times \overline{B_{R_\eta+\delta}(0)}$. By the mean value theorem, we have

$$f(u_1) - f(u_2) = \int_0^1 df(u_2 + t(u_1 - u_2), u_1 - u_2)dt, \quad u_1, u_2 \in \overline{B_{R_\eta+\delta}}.$$

Hence, if $\omega := |F_2(\eta)(x, y) - F_2(\gamma)(x, y)|$ then

$$\begin{aligned}
\omega &= \left| \frac{f(\eta(x)) - f(\eta(y))}{\|x - y\|^\lambda} - \frac{f(\gamma(x)) - f(\gamma(y))}{\|x - y\|^\lambda} \right| \\
&= \left| \int_0^1 \left(df\left(\eta(y) + t(\eta(x) - \eta(y)), \frac{\eta(x) - \eta(y)}{\|x - y\|^\lambda}\right) - df\left(\gamma(y) + t(\gamma(x) - \gamma(y)), \frac{\gamma(x) - \gamma(y)}{\|x - y\|^\lambda}\right)\right) dt \right| \\
&\leq G_{f,\eta} \int_0^1 \left\| \left(\eta(y) + t(\eta(x) - \eta(y)), \frac{\eta(x) - \eta(y)}{\|x - y\|^\lambda}\right) - \left(\gamma(y) + t(\gamma(x) - \gamma(y)), \frac{\gamma(x) - \gamma(y)}{\|x - y\|^\lambda}\right) \right\| dt \\
&\leq G_{f,\eta}\left(\int_0^1 \|t(\eta(x) - \gamma(x)) + (1 - t)(\eta(y) - \gamma(y))\|dt + \frac{\|\big(\eta(x) - \gamma(x)\big) - \big(\eta(y) - \gamma(y)\big)\|}{\|x - y\|^\lambda} \right) \\
&\leq G_{f,\eta}(\|\eta - \gamma\|_\infty + \|\eta - \gamma\|_\lambda) \\
&\leq G_{f,\eta}\delta.
\end{aligned}$$

If $\varepsilon = G_{f,\eta}\delta$, we have

$$\|F_2(\eta) - F_2(\gamma)\|_\infty \leq \varepsilon.$$

Therefore, the map F_2 is continuous and in consequence, the map $\tilde{f}$ is continuous. $\square$

Lemma 3.4.7 *Let $U, V \in \mathcal{U}$ such that V is relatively compact in U. If $f : U \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $f_*(\eta) := f \circ (id, \eta|_V) \in \mathcal{F}_\lambda(V, \mathbb{R})$ for all $\eta \in \mathcal{F}_\lambda(U, \mathbb{R}^n)$ and the map*

$$f_* : \mathcal{F}_\lambda(U, \mathbb{R}^n) \to \mathcal{F}_\lambda(V, \mathbb{R}), \quad \eta \mapsto f_*(\eta) = f \circ (id, \eta|_V)$$

is continuous.

Proof. First let assume that $U = \mathbb{R}^m$. Let $\mathrm{id} : V \to \mathbb{R}^m$ be the identity map, then $\mathrm{id} \in \mathcal{F}_\lambda(V, \mathbb{R}^m)$ and by Lemma 3.4.5, the map

$$F_\lambda(\mathbb{R}^m, \mathbb{R}^n) \to F_\lambda(V, \mathbb{R}^m \times \mathbb{R}^n), \quad \eta \mapsto (\mathrm{id}, \eta|_V)$$

is continuous. If $\ell = m + n$, by Lemma 3.4.6, the map

$$F_\lambda(V, \mathbb{R}^m \times \mathbb{R}^n) \to F_\lambda(V, \mathbb{R}), \quad \beta \mapsto f \circ \beta$$

is continuous. Therefore f_* is just the composition of continuous mappings.
Let assume that $U \neq \mathbb{R}^m$. Let $\chi : \mathbb{R}^m \to \mathbb{R}$ be a cut-off function for $\overline{V}$ supported in U (see e.g. [28, Proposition 2.25]); we define

$$g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} \chi(x) f(x, y), & \text{if } x \in U \\ 0, & \text{if } x \in \mathbb{R}^m \setminus \mathrm{supp}(\chi) \end{cases}$$

Then g is smooth and, as before, the map

$$g_* : \mathcal{F}_\lambda(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n) \to \mathcal{F}_\lambda(V, \mathbb{R}), \quad \eta \mapsto g_*(\eta) = g \circ (\mathrm{id}, \eta|_V)$$

is continuous. Moreover, for each $\eta \in \mathcal{F}_\lambda(U, \mathbb{R}^n)$ and $x \in V$ we have

$$\begin{aligned} g_*(\eta)(x) &= g \circ (id, \eta|_V)(x) \\ &= g(x, \eta|_V(x)) \\ &= \chi(x) f(x, \eta|_V(x)) \\ &= f(x, \eta|_V(x)) \\ &= f_*(\eta)(x), \end{aligned}$$

whence $g_* = f_*$. □

Remark 3.4.8 By Lemma 3.4.7, the axiom (PF) is verified.

Combining all these lemmas, we can conclude with the following Lemma.

Lemma 3.4.9 *Let $m \in \mathbb{N}$, $\mathcal{U}$ be the collection of open subsets of $\mathbb{R}^m$ and $0 < \lambda \leq 1$. Then the family of Banach spaces $\{\mathcal{F}_\lambda(U, \mathbb{R})\}_{U \in \mathcal{U}}$ define a family of locally convex spaces suitable for global analysis,*

Definition 3.4.10 Let M and N be finite-dimensional smooth manifolds without boundary and $0 < \lambda \leq 1$. We denote the set $BC^{0,\lambda}(M, N)$ of all functions $\gamma : M \to N$ such that for each $p \in M$, there exist the charts $\phi_M : U_M \to V_M$ of M and $\phi_N : U_N \to V_N$ of N, such that $p \in U_M$, $\gamma(U_M) \subseteq U_N$ and $\phi_N \circ \gamma \circ \phi_M^{-1} \in \mathcal{F}_\lambda(V_M, \mathbb{R}^n)$.

By Lemma 3.4.9 we conclude.

Proposition 3.4.11 *Let $0 < \lambda \leq 1$. For each compact manifold M without boundary and smooth manifold N without boundary which admits local addition, the set $BC^{0,\lambda}(M,N)$ admits a smooth manifold structure.*

Remark 3.4.12 Let N_1 and N_2 be finite-dimensional smooth manifolds without boundary which admit local additions. If $f : N_1 \to N_2$ is a smooth map, then by Proposition 3.3.5, the map

$$BC^{0,\lambda}(M,N_1) \to BC^{0,\lambda}(M,N_2), \quad \gamma \mapsto f \circ \gamma$$

is smooth.

Proposition 3.4.13 *Let M be a compact smooth manifold without boundary and N a smooth manifold without boundary which admits a local addition. If $0 < \beta \leq \lambda \leq 1$, then $\gamma \in BC^{0,\beta}(M,N)$ for each $\gamma \in BC^{0,\lambda}(M,N)$. Moreover, the map*

$$\iota : BC^{0,\lambda}(M,N) \to BC^{0,\beta}(M,N), \quad \gamma \mapsto \gamma$$

is smooth.

Proof. Let $\gamma \in BC^{0,\lambda}(M,N)$, then for each $p \in M$, there exists the charts $\phi_M : U_M \to V_M$ of M and $\phi_N : U_N \to V_N$ of N, such that $p \in U_M$, $\gamma(U_M) \subseteq U_N$ and $\phi_N \circ \gamma \circ \phi_M^{-1} \in \mathcal{F}_\lambda(V_M, \mathbb{R}^n)$. For each $U \in \mathcal{U}$, it is known that for $\beta \leq \lambda$ the linear operator

$$I_U : \mathcal{F}_\lambda(U, \mathbb{R}^n) \to \mathcal{F}_\beta(U, \mathbb{R}^n), \quad \tau \mapsto \tau$$

is continuous. In particular, we have

$$I_{V_M}(\phi_N \circ \gamma \circ \phi_M^{-1}) = \phi_N \circ \gamma \circ \phi_M^{-1} \in \mathcal{F}_\beta(V_M, \mathbb{R}^n).$$

Therefore $\gamma \in BC^{0,\beta}(M,N)$. Now, we consider the charts $(\mathcal{U}_\gamma, \Psi_\gamma^{-1})$ and $(\mathcal{U}_{\iota(\gamma)}, \Psi_{\iota(\gamma)}^{-1})$ in $\gamma \in BC^{0,\lambda}(M,N)$ and $\iota(\gamma) \in BC^{0,\beta}(M,N)$ respectively, then the map

$$\Psi_{\iota(\gamma)}^{-1} \circ \iota \circ \Psi_\gamma : \Psi_\gamma^{-1}\left(\mathcal{U}_\gamma \cap \iota^{-1}(\mathcal{U}_{\iota(\gamma)})\right) \to \Psi_{\iota(\gamma)}\left(\mathcal{U}_\gamma \cap \iota^{-1}(\mathcal{U}_{\iota(\gamma)})\right)$$

given by

$$\Psi_{\iota(\gamma)}^{-1} \circ \iota \circ \Psi_\gamma(\sigma) = (\pi_{TN}, \Sigma_N)^{-1} \circ \Big(\iota(\gamma), \iota(\Sigma_N \circ \sigma)\Big)$$

is just a restriction of the map

$$\tilde{\iota} : \Gamma_{\mathcal{F}_\lambda}(\eta) \to \Gamma_{\mathcal{F}_\beta}(\iota(\eta)), \quad \sigma \mapsto \sigma,$$

which is continuous by Proposition 3.2.1 and continuity of the maps $\{I_U\}_{U \in \mathcal{U}}$. □

Bibliography

[1] Alzaareer, H. and A. Schmeding, *Differentiable mappings on products with different degrees of differentiability in the two factors*, Expo. Math. **33**, (2015); 184–222.

[2] Alzaareer, H., *Differential calculus on multiple products*, Indag. Math. 30 (2019), 1036-1060

[3] Amiri, H., H. Glöckner, and A. Schmeding, *Lie groupoids of mappings taking values in a Lie groupoid*, Arch. Math. (Brno) **56** (2020), 307-356.

[4] Bertram W., H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), 213-282.

[5] Bastiani, A., *Applications différentiables et variétés différentiables de dimension infinie*, J. Anal. Math. **13** (1964), 1–114.

[6] Bauer, M. P. Harms, and P. W. Michor, *Regularity and completeness of half-Lie groups*, to appear in Journal of the EMS (cf. arXiv:2302.01631).

[7] Bochnak J. and J. Siciak, Analytic functions in topological vector spaces, *Stud. Math.*, Stud. Math. 39 (1971), 77-112.

[8] Bourbaki, N.,*Lie Groups and Lie Algebras, Chapters 1 - 3*, Springer, Berlin, 1989.

[9] Eells, J. Jr., *A setting for global analysis*, Bull. Amer. Math. Soc. **72** (1966), 751–807.

[10] Fiorenza R., "Hölder and locally Hölder Continuous Functions, and Open Sets of Class $C^k, C^{k,\lambda}$", Birkhäuser, 2016.

[11] Flaschel, P. and W. Klingenberg, "Riemannsche Hilbertmannigfaltigkeiten," Springer, Berlin, 1972.

[12] Florencio, M., F. Mayoral, and P. J. Paúl, *Spaces of vector-valued integrable functions and localization of bounded subsets*, Math. Nachr. **174** (1995), 89–111.

[13] Floret, K., *Lokalkonvexe Sequenzen mit kompakten Abbildungen*, J. Reine Angew. Math. 247 (1971), 155-195.

[14] Glöckner, H., *Infinite-dimensional Lie groups without completeness restrictions*, pp. 43–59 in: Strasburger, A. et al. (eds.), "Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups," Banach Center Publ. **55**, Warsaw, 2002.

[15] Glöckner, H., *Measurable regularity properties of infinite-dimensional Lie groups*, preprint, arXiv:1601.02568.

[16] Glöckner, H., *Lie groups of real analytic diffeomorphisms are L^1-regular*, Nonlinear Anal. Volume 252 (2025), 113690.

[17] Glöckner, H., *Lie groups over non-discrete topological fields*, preprint, arXiv:math/0408008

[18] Glöckner, H., *Smoothing operators for vector-valued functions and extension operators*, preprint, arXiv:2006.00254

[19] Glöckner, H. and J. Hilgert, *Aspects of control theory on infinite-dimensional Lie groups and G-manifolds*, J. Differ. Equations **343** (2023), 186–232.

[20] Glöckner, H. and K.-H. Neeb, *Infinite Dimensional Lie Groups*, book in preparation.

[21] Glöckner, H. and M. Niestijl, *Jettomorphism groups of polytopes and manifolds with boundary*, manuscript in preparation.

[22] Glöckner H. and L. Tárrega, *Mapping groups associated with real-valued function spaces and direct limits of Sobolev-Lie groups*, J. Lie Theory 33 (2023), 271-296.

[23] Kelley J. L., *General Topology*, Springer, New York, 1975.

[24] Klingenberg, W. P. A., “Riemannian Geometry,” de Gruyter, Berlin, 1995.

[25] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” Amer. Math. Soc., Providence, 1997.

[26] Krikorian, N., *Differentiable structures on function spaces*, Trans. Amer. Math. Soc. **171** (1972), 67–82.

[27] Lang, S., *“Fundamentals of Differential Geometry”*, Springer, 1999.

[28] Lee, J., *“Introduction to Smooth Manifolds”*, Springer, 2000.

[29] Marquis, T. and K.-H. Neeb, *Half-Lie groups*, Transform. Groups **23** (2018), 801–840.

[30] Michor, P. W., *“Manifolds of Differentiable Mappings”*, Shiva, Orpington, 1980.

[31] Michor, P. W., *Manifolds of mappings for continuum mechanics*, pp. 3–75 in: R. Segev et al. (eds.), “Geometric Continuum Mechanics,” Birkhäuser, Cham, 2020.

[32] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: B. S. DeWitt and R. Stora (eds.), “Relativité, groupes et topologie II,” North-Holland, Amsterdam, 1984.

[33] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468.

[34] Nikitin, N., "Regularity Properties of Infinite-Dimensional Lie Groups and Exponential Laws," doctoral dissertation, Paderborn University, 2021 (see nbn-resolving.de/urn:nbn:de:hbz:466:2-39133)

[35] Nikitin, N., "Measurable regularity of infinite-dimensional Lie groups based on Lusin measurability," preprint, arXiv:1904.10928

[36] Omori H., *Infinite-Dimensional Lie Groups*, Amer. Math. Soc., 1997.

[37] Pinaud, M. F., "Manifolds of absolutely continuous functions with values in an infinite-dimensional manifold and regularity properties of half-Lie groups," preprint, arXiv:2409.06512

[38] Palais, R. S., "Foundations of Global Non-Linear Analysis," W. A. Benjamin, New York, 1968.

[39] Pierron, T. and A. Trouvé, "The graded group action framework for sub-riemannian orbit models in shape spaces," preprint, arXiv:2406.16930

[40] W. Rudin, *Functional Analysis*, McGraw Hill, 1991.

[41] Schmeding, A., *Manifolds of absolutely continuous curves and the square root velocity framework*, preprint, arXiv:1612.02604.

[42] Schmeding, A., "An Introduction to Infinite-Dimensional Differential Geometry," Cambridge University Press, Cambridge, 2022.