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Model Predictive Control for output tracking with prescribed performance

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Zusammenfassung

Modellprädiktive Regelung (MPC) stellt einen Eckpfeiler der modernen Regelungstheorie dar und erlaubt die simultane Berücksichtigung von Nebenbedingungen sowie die multikriterielle Optimierung durch iterative Vorhersage und Receding-Horizon-Optimierung. In der Praxis sehen sich MPC-Verfahren jedoch drei wesentlichen Herausforderungen konfrontiert: der Sicherstellung initialer und rekursiver Zulässigkeit (d. h. der dauerhaften Lösbarkeit des zugrundeliegenden Optimierungsproblems), ihrer Robustheit gegenüber Modellabweichungen und unbekannten Störungen sowie Restriktionen bei der Realisierung als Abtastsystem.

Diese Dissertation entwickelt ein innovatives MPC-Framework für nichtlineare, zeitkontinuierliche Systeme, die mittels funktionaler Differentialgleichungen beschrieben werden. Ziel ist die Ausgangsfolgeregelung glatter Referenzsignale innerhalb vorgegebener Fehlertoleranzen zu gewährleisten und die genannten Herausforderungen systematisch zu adressieren.

Im Mittelpunkt steht *Funnel MPC* – ein neuartiger Regelungsansatz, der auf herkömmliche Endbedingungen und restriktiv lange Prädiktionshorizonte verzichtet. Sein Fundament bilden sogenannte Funnel-Penalty-Funktionen: Kostenfunktionen, die Abweichungen des Trackingfehlers von zeitvarianten Toleranzschranken gezielt bestrafen. Angelehnt an Techniken der adaptiven Funnel-Regelung garantiert dieser Ansatz sowohl initiale als auch rekursive Zulässigkeit und gewährleistet zugleich strikte Einhaltung der Soll-Regelgüte.

Darauf aufbauend wird Funnel MPC mit der modellfreien Funnel-Regelung in eine hybride Zwei-Komponenten-Architektur verschmolzen. Diese vereint modellbasierte Optimierung mit adaptiver Ausgangsrückführung, um die konkurrierenden Ziele Optimalität und Robustheit auszubalancieren. Ergebnis ist ein Regler, der die geforderte Regelgüte selbst bei strukturellen Modellungenauigkeiten, unmodellierten Dynamiken und Störungen zuverlässig einhält.

Zur Steigerung der Prädiktionsgenauigkeit integrieren wir ein datengesteuertes Lernverfahren, welches das Systemmodell fortlaufend basierend auf Online-Messungen adaptiert. Diese Komponente reduziert Modell-System-Diskrepanzen kontinuierlich und verbessert auf diese Weise langfristig die Regelgüte, ohne dabei Robustheitsgarantien zu kompromittieren.

Schließlich überführen wir die zeitkontinuierlichen Regelgesetze in eine Abtastimplementierung. Durch Herleitung expliziter Schranken für Abtastrate und Stellaufwand garantieren wir Stabilität unter treppenförmigen Stellsignalen – ein essenzieller Schritt für die praktische Umsetzung auf digitaler Hardware.

Durch systematische Verknüpfung von Zulässigkeit, Robustheit, Lernfähigkeit und Abtastimplementierung entsteht ein ganzheitliches Framework zur Einhaltung vorgegebener Fehlertoleranzen bei der Ausgangsfolgeregelung für eine breite Klasse dynamischer Systeme. Die vorgestellten Ergebnisse ebnen den Weg für zukünftige Entwicklungen im Bereich des lernunterstützten und samplingbasierten, robusten MPC.

Abstract

Model Predictive Control (MPC) is a cornerstone of modern control theory, offering a versatile framework for constraint handling and multi-objective optimisation through iterative prediction and receding-horizon optimisation. However, its practical application can face critical challenges: ensuring initial and recursive feasibility (guaranteeing solvability of the underlying optimisation problem), robustness against system-model mismatches and unknown disturbances, and sampled-data implementation constraints.

This thesis develops a novel MPC framework for a class of non-linear continuous-time systems governed by functional differential equations, targeting output tracking of smooth reference signals within prescribed error bounds, while systematically addressing the aforementioned challenges.

We first introduce *funnel MPC*, a novel algorithm that eliminates reliance on commonly used terminal conditions or restrictive long prediction horizons. At its core are funnel penalty functions – state costs that penalise deviations of the tracking error from prescribed time-varying boundaries. Inspired by adaptive funnel control principles, this framework ensures initial and recursive feasibility while rigorously enforcing tracking performance guarantees.

Building on this foundation, we unify funnel MPC with model-free funnel feedback into a two-component hybrid architecture. This structure synergises model-based optimisation with adaptive feedback compensation, reconciling the competing objectives of optimality and robustness. The resulting controller achieves prescribed tracking performance despite structural model-plant mismatches, unmodelled dynamics, and disturbances.

To further enhance predictive accuracy, we introduce a data-driven learning framework that iteratively refines the model using system measurements. This component enables the controller to mitigate model-plant discrepancies over time, improving long-term performance without compromising robustness guarantees. Bridging theory and practice, we finally formalise the transition from continuous-time control laws to sampled-data implementations, deriving explicit bounds on sampling rates and control effort to guarantee stability under piecewise constant control signals – a critical step toward deploying the algorithm on digital hardware.

By systematically addressing feasibility, robustness, learning integration, and sampled-data implementation, this thesis establishes a cohesive framework to ensure output tracking within prescribed error bounds for a large system class. The results pave the way for future advances in learning-enhanced and sampled-data robust MPC.

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Contents

1 Introduction

Model Predictive Control (MPC) is a versatile, optimisation-based control technique widely recognised for its effectiveness in managing linear and non-linear multi-input multi-output systems, as discussed in textbooks [78, 159]. We also refer to [122] for an overview of the historical development of MPC. A hallmark of MPC is its ability to explicitly incorporate both control and state constraints into the optimisation framework, a feature that has propelled its adoption in diverse applications, see e.g. [154] and also [163]. At its core, MPC leverages a dynamic model of the system to iteratively forecast its behaviour over a finite-time horizon. These predictions enable the controller to solve a receding-horizon Optimal Control Problem (OCP), optimising control inputs to balance competing objectives – such as setpoint tracking or energy efficiency – against hard constraints like actuator saturation or safety-critical state bounds. After applying the first part of this optimal control to the system, the prediction horizon is shifted forward in time and the model is re-initialised with measurement data from the system. Repeating this process ad infinitum forms a closed-loop control system.

Despite its simplicity and conceptual elegance, practical implementation of MPC demands rigorous attention to mathematical foundations. Foremost among these is ensuring both *initial feasibility* (existence of a valid solution of the OCP at startup) and *recursive feasibility*, which guarantees that solvability of the optimal control problem at one time step automatically implies solvability at the successor time instant. Providing these guarantees becomes precarious if the utilised model deviates from the actual system, as MPC relies heavily on model accuracy in order to predict the behaviour of the actual system. Robustness to these discrepancies is a ubiquitous challenge as all models are inherently approximate and real-world systems face unmeasurable disturbances, parametric drift, or unmodelled dynamics. Consequently, designing robust MPC algorithms capable of handling plant-model mismatches and external disturbances remains an active research area. One branch of research focuses on enhancing the MPC algorithm itself via robustification methods to harden the controller against bounded uncertainties, see e.g. [24] for an overview of available techniques. Another branch of research explores adapting the model to ensure robust constraint satisfaction of the actual system. The latter has gained momentum with recent advances in machine learning and spurred interest in integrating techniques like reinforcement learning (RL). On the implementation front, practical limitations persist: sampled-data architectures restrict controllers to discrete-time measurements, and hardware constraints often necessitate piecewise constant control signals, introducing discretisation errors that further complicate theoretical analyses.

This thesis progressively develops an MPC algorithm for continuous-time systems to address output tracking of smooth reference signals with prescribed error bounds. We systematically resolve the aforementioned challenges by first outlining alternative methods and then proposing a novel approach. By integrating principles from the adaptive control technique *funnel control*, we establish initial and recursive feasibility without relying on terminal conditions or restrictive assumptions, such as demanding excessively long prediction horizons. Building on this foundation, we unify the two control strategies into a single, hybrid framework – combining MPC's predictive optimisation with funnel control's adaptability – to ensure robustness against unknown disturbances and structural plant-model mismatches. Next, we investigate the incorporation of a learning mechanism into the framework, enabling data-driven adaptation of the underlying model to refine predictions using system measurements. Finally, we derive sufficient conditions for the sampling rate to guarantee stability when operating the controller in a sampled-data setting with piecewise constant control signals, bridging the gap between theoretical continuity and practical digital implementation.

1.1 Problem formulation

We consider non-linear multi-input multi-output control systems of order $r \in \mathbb{N}$ of the form

$$\begin{aligned} y^{(r)}(t) &= F(\mathbf{T}(y, \dot{y}, \dots, y^{(r-1)})(t), u(t)), \\ \left.\begin{aligned} y|_{[0,t_0]} &= y^0 \in \mathcal{C}^{r-1}([0,t_0], \mathbb{R}^m), \quad &&\text{if } t_0 > 0, \\ \big(y(t_0), \dots, y^{(r-1)}(t_0)\big) &= y^0 \in \mathbb{R}^{rm}, &&\text{if } t_0 = 0, \end{aligned}\right\} \end{aligned} \tag{1.1}$$

with $t_0 \geq 0$, initial trajectory y^0, input $u \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$, and output $y(t) \in \mathbb{R}^m$ at time $t \geq t_0$. Note that u and y have the same dimension $m \in \mathbb{N}$. The system consists of an *unknown* continuous function $F \in \mathcal{C}(\mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m)$ satisfying the so-called *perturbation high-gain property* introduced in Definition 3.1.1 (b), and an *unknown* operator $\mathbf{T}$. The operator $\mathbf{T}$ is causal, locally Lipschitz, and satisfies a bounded-input bounded-output property. These properties will be introduced in detail in Definition 2.2.1 and the system under consideration is characterised in Definition 3.1.1. Note that the system may also incorporate bounded disturbances $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$. They can be modelled as part of the unknown operator $\mathbf{T}$, as we will discuss in Remark 3.1.3 (a). For reasons of simplicity, we however refrain from explicitly including them in equation (1.1).

1.1.1 Control objective

Our objective is to design a control strategy which allows tracking of a given reference trajectory $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ within pre-specified error bounds. To be more precise, the tracking error $t \mapsto e(t) := y(t) - y_{\text{ref}}(t)$ shall evolve within the prescribed performance funnel

$$\mathcal{F}_\psi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \|e\| < \psi(t)\}. \tag{1.2}$$

This funnel is determined by the choice of the function ψ belonging to the set

$$\mathcal{G} := \left\{ \psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \,\middle|\, \inf_{t \geq 0} \psi(t) > 0 \right\}, \tag{1.3}$$

see also Figure 1.1. Note that, for a function $\psi \in \mathcal{G}$, there exists $\lambda > 0$ such that $\psi(t) \geq \lambda$ for all $t \geq 0$. Therefore, signals evolving in $\mathcal{F}_\psi$ are not forced to converge to 0 asymptotically.

Remark 1.1.1. In many practical applications perfect tracking is neither possible nor desired. Usually, the objective rather is to ensure the tracking error to be less than an (arbitrary small) a priori specified constant after a pre-specified period of time and to guarantee that the error does not exceed this bound at a later time. Tracking within a funnel, or in other words practical tracking, is advantageous since it allows tracking for system classes where asymptotic tracking is not possible or requires – when compared to asymptotic tracking – much less control effort. Note that the function ψ is a design parameter, thus its choice is completely up to the designer. Moreover, arbitrary funnel functions – and not restricted to constant or monotonous decreasing funnels – give the user more flexibility in finding a suitable trade-off between tracking performance and control effort. Typically, the specific application dictates the constraints on the tracking error and thus indicates suitable choices for ψ. During safety critical system phases, the funnel will be small, while during non-critical phases the funnel can be widened again to reduce the control effort. •

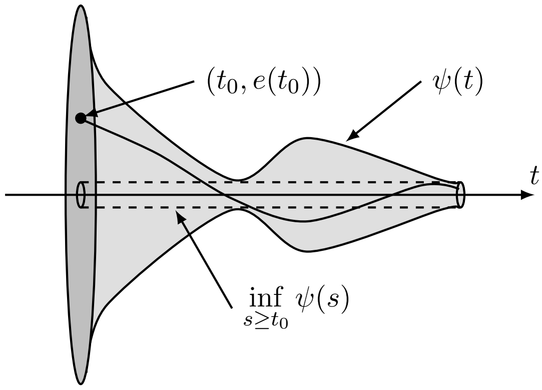


Figure 1.1: Error evolution in a funnel $\mathcal{F}_\psi$ with boundary $\psi(t)$. The figure is based on [33, Fig. 1], adapted to the current setting.

1.1.2 Funnel control

In the context of output reference tracking within prescribed, possibly time-varying, performance boundaries, *funnel control* is an established adaptive high-gain control methodology. The concept has been introduced in the seminal work [95] with the design goal of achieving the control objective laid out in Section 1.1.1 for a broad class of non-linear multi-input, multi-output systems. It has since received a lot of research attention, see e.g. [30, 33, 83]. For a comprehensive literature overview, we also recommend the recent survey paper [31]. The funnel control approach solely invokes certain structural assumptions about the system, namely stable internal dynamics and a known globally defined relative degree with a globally pointwise sign-definite high-frequency matrix. Under these conditions, the adaptive controller offers robustness against disturbances and guarantees specified transient behaviour without relying on explicit knowledge about the system to be controlled. Since it ensures output tracking of reference signals within prescribed performance bounds without having to resort to a model of the system, funnel control proved useful for tracking problems in various applications such as DC-link power flow control, see [173], control of industrial servo-systems, see [82], and temperature control of chemical reactor models, see [97].

Prescribed performance control (PPC) is a methodology closely related to funnel control. It was first introduced in [21]. The core idea of PPC involves transforming the original controlled system into a new state-space representation using predefined performance functions that encode desired transient and, potentially, steady-state behaviours. By ensuring the uniform boundedness of the transformed system's states via appropriate control laws, the tracking problem for the original system is solved – a result that is both necessary and sufficient under this framework. While early PPC designs relied on neural networks to approximate unknown non-linearities, later work in [23] developed an approximation-free scheme tailored for systems in so-called pure feedback form. For a detailed and comprehensive overview of prescribed performance control, we also recommend the survey paper [50]. Although PPC and funnel control share a similar objective – enforcing error trajectories within predefined bounds – they differ in their system classes and structural assumptions. Funnel control applies to systems of the form (1.1), whereas PPC addresses systems structured as:

$$
\begin{aligned}
\dot{x}_k(t) &= f_k(x_1(t), \dots, x_{k+1}(t)), \qquad k = 1, \dots, r-1,\\
\dot{x}_r(t) &= f_r(d(t), x_1(t), \dots, x_r(t), z(t), u(t)),\\
\dot{z}(t) &= g(d(t), x_1(t), \dots, x_r(t), z(t)),\\
y(t) &= x_1(t),
\end{aligned}
$$

with $f_k : \mathbb{R}^{km} \to \mathbb{R}^m$ for $k = 1, \dots, r-1$, $f_r : \mathbb{R}^{n+rm+m} \to \mathbb{R}^m$, $g : \mathbb{R}^{n+rm+q} \to \mathbb{R}^q$, and $d \in L^\infty(\mathbb{R}_{\geq 0}, R^n)$ represents a bounded disturbance. Crucially, both methods operate under minimal system knowledge, requiring only generic structural assumptions rather than explicit functional details. While prescribed performance control presumes the partial derivatives $\frac{\partial f_i}{\partial x_i}$ and $\frac{\partial f_r}{\partial u}$ to be uniformly positive definite, see [23], funnel control assumes the system (1.1) to have bounded-input bounded-state stable internal dynamics and the function F to satisfy the so-called *high-gain property*, see [30]. A key distinction between the two control techniques lies in their information requirements: funnel control relies solely on the output y and its derivatives while PPC necessitates full state feedback. For the latter, this requirement was softened in [63] via the incorporation of a high-gain observer and [22] allows internal dynamics of a certain hierarchical structure, so-called *dynamical uncertainties*. Despite their conceptual overlap, a rigorous comparative analysis of these approaches remains an open research question.

Both funnel control and prescribed performance control face inherent limitations due to their model-free nature. Since neither approach utilises a system model, the controllers lack predictive capabilities, leaving the error evolution within time-varying boundaries uncertain. For instance, the error trajectory may approach the funnel boundary arbitrarily closely, triggering excessively large feedback gains. This can lead to high-magnitude control inputs, peaking signals, and – from an implementation perspective – significant sensitivity to measurement noise. While theoretical guarantees ensure bounded control signals, their precise upper bounds require knowledge of the system and remain a priori unknown. For practical implementation on digital devices, both schemes also demand high sampling rates to maintain feasibility, imposing stringent hardware requirements. The recent work [29] demonstrates that incorporating an internal model into funnel control can markedly enhance performance. This integration reduces noise sensitivity, mitigates extreme gain behaviour, and can even achieve asymptotic tracking without a performance funnel whose width converges to zero. Furthermore, numerical simulations in [32] reveal that an MPC strategy, blending funnel control principles with predictive optimisation, outperforms pure funnel control. This approach achieves smaller control actions and relaxed sampling rate demands for zero-order-hold implementations, highlighting the value of a model integration in the controller design.

Building upon the ideas from [32], this thesis develops an MPC scheme that integrates ideas from funnel control in order to achieve the control objective of output tracking with prescribed performance for systems of the form (1.1). By combining the control methodologies, we circumvent the shortcomings of both individual approaches. This enables us to benefit from the best of both worlds: guaranteed feasibility and robustness (funnel control), and a superior control performance (MPC).

Before we focus on developing this MPC scheme, we would like to give an intuition of the funnel controller's functioning.

Proposition 1.1.2. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a locally Lipschitz continuous function, $\psi \in \mathcal{G}$, $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, and $y^0 \in \mathbb{R}^m$ with $\|y^0 - y_{\mathrm{ref}}(0)\| < \psi(0)$. Then, the application of the output feedback $u(t) := \mu_{\mathrm{FC}}(t, y(t))$ with*

$$\mu_{\mathrm{FC}}(t,y) = -k(t,y)e(t,y), \qquad k(t,y) = \frac{1}{\psi^2(t) - \|e(t,y)\|^2}, \qquad e(t,y) = y - y_{\mathrm{ref}}(t) \quad (1.4)$$

to the system

$$\dot{y}(t) = f(y(t)) + u(t), \quad y(0) = y^0,$$

leads to the closed-loop initial value problem

$$\dot{y}(t) = f(y(t)) - \frac{y(t) - y_{\mathrm{ref}}(t)}{\psi^2(t) - \|y(t) - y_{\mathrm{ref}}(t)\|^2}, \quad y(0) = y^0,$$

which has a solution. Moreover, every solution can be extended to a unique global solution $y : [0,\infty) \to \mathbb{R}^m$ *and both* y *and* u *are bounded with essentially bounded weak derivatives. The tracking error evolves uniformly within the performance funnel, i.e.*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \quad \|e(t)\| \le \psi(t)^{-1} - \varepsilon.$$

In Proposition 3.2.3, we will prove that a more advanced funnel controller design achieves the specified control objective for systems of the form (1.1). The simpler controller (1.4), however, is a special case of the controller methodology proposed in [95] and provides an intuitive demonstration of the underlying principles. When the tracking error $e = y - y_{\text{ref}}$ approaches zero, the control gain $k(t, y(t))$ diminishes, effectively deactivating the controller. Conversely, as the tracking error norm nears the funnel boundary $\psi(t)$, the gain $k(t, y(t))$ grows rapidly, producing a control input $u(t) = -k(t, y(t))e(t, y(t))$ that aggressively steers the error away from the funnel boundary and towards the reference trajectory. Consequently, if the initial tracking error $e(0)$ lies within the funnel boundaries, its evolution remains strictly confined within the funnel $\mathcal{F}_\psi$ for all time.

1.1.3 Model predictive control

The idea of model predictive control (MPC) is, after measuring/obtaining the output $\hat{y} \in \mathcal{C}^{r-1}([\hat{t} - \tau, \hat{t}], \mathbb{R}^m)$ of the system (1.1) over a short time window of length $\tau \ge 0$ at the current time $\hat{t}$ with $\hat{t} - \tau \ge t_0$, to repeatedly calculate a control function $u^\star = u^\star(\cdot; \hat{t}, \hat{y})$ minimising the integral of a *state cost* ℓ on the future time interval $[\hat{t}, \hat{t}+T]$ for $T > 0$, called the *prediction horizon*, and to implement the computed optimal solution $u^\star$ to system (1.1) over an interval of length $\delta < T$, called the *time shift*. The prediction horizon T determines how far ahead the controller plans, while the time shift δ specifies the implementation period before re-optimisation. To make predictions about the future system behaviour and its output and, based on them, to compute optimal control signals, MPC uses a model of the form

$$\begin{aligned} y_{\mathrm{M}}^{(r)}(t) &= F_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(y_{\mathrm{M}}, \dots, y_{\mathrm{M}}^{(r-1)})(t), u(t)), \\ y_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}]} &= \hat{y}_{\mathrm{M}}, \end{aligned} \tag{1.5}$$

where $F_{\mathrm{M}} \in \mathcal{C}(\mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m)$ is a *known* continuous function and $\mathbf{T}_{\mathrm{M}}$ a *known* operator, as a surrogate for the unknown system (1.1). The initial history function $\hat{y}_{\mathrm{M}}$ of the model (1.5) at time $\hat{t} \ge t_0$ is an element of $\mathcal{C}^{r-1}([\hat{t} - \tau, \hat{t}], \mathbb{R}^m)$ and selected based on the system measurement $\hat{y}$, though not necessarily identical to them. Note that we assume that both the output and input dimension $m \in \mathbb{N}$ as well as the order of the differential equation $r \in \mathbb{N}$ match between the system (1.1) and the model (1.5).

Remark 1.1.3. The model (1.5) is intentionally formulated in a quite general form using an abstract operator $\mathbf{T}_{\mathrm{M}}$ and initial values $\hat{y}_{\mathrm{M}}$ given on a time interval of length $\tau \ge 0$ in order to explain the general idea of model predictive control while avoiding the accidental exclusion of particular edge cases. However, it is probably most common to use a control affine multi-input multi-output model of the form

$$\begin{aligned} \dot{x}_{\mathrm{M}}(t) &= f_{\mathrm{M}}(x_{\mathrm{M}}(t)) + g_{\mathrm{M}}(x_{\mathrm{M}}(t))u(t), \qquad x_{\mathrm{M}}(\hat{t}) = \hat{x}_{\mathrm{M}}, \\ y_{\mathrm{M}}(t) &= h_{\mathrm{M}}(x_{\mathrm{M}}(t)), \end{aligned} \tag{1.6}$$

with initial data $\hat{x}_{\mathrm{M}} \in \mathbb{R}^n$ at the current time $\hat{t} \ge t_0$, and functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}^m$. Here, $x_{\mathrm{M}}(t) \in \mathbb{R}^n$ is the state of the model, $y_{\mathrm{M}}(t) \in \mathbb{R}^m$ the model's output, and $u(t) \in \mathbb{R}^m$ the control input. In (1.6), the states x_{M} of the model are laid out in an explicit way contrary to the formulation (1.5) where they are, in a certain sense, hidden within the operator $\mathbf{T}_{\mathrm{M}}$. Many works on MPC use even simpler models, namely linear time invariant (LTI) models, i.e. they assume a model of the form

$$\begin{aligned} \dot{x}_{\mathrm{M}}(t) &= A_{\mathrm{M}} x_{\mathrm{M}}(t) + B_{\mathrm{M}} u(t), \qquad x_{\mathrm{M}}(\hat{t}) = \hat{x}_{\mathrm{M}} \\ y_{\mathrm{M}}(t) &= C_{\mathrm{M}} x_{\mathrm{M}}(t), \end{aligned}$$

with $A_{\mathrm{M}} \in \mathbb{R}^{n \times n}$ and $C_{\mathrm{M}}^\top, B_{\mathrm{M}} \in \mathbb{R}^{n \times m}$. In the later part of this thesis, we will restrict ourselves also to a, in comparison to (1.5), simpler model class, which we will formally define in Definition 2.2.2. It will, however, contain the class of control affine multi-input multi-output models of the form (1.6) (under certain assumptions on f_{M}, g_{M}, and h_{M}) and LTI models as we will see in Examples 2.2.3 and 2.2.4. •

In addition to the model (1.5), another key component of the MPC algorithm is the *stage cost function* ℓ. It formulates rewards for desired model behaviour and penalties for undesired behaviour, which are balanced out during the optimisation process. When solving the problem of tracking a given reference signal y_{ref}, a commonly used stage cost function is

$$\ell : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, \qquad (t, y_{\mathrm{M}}, u) \mapsto \|y_{\mathrm{M}} - y_{\mathrm{ref}}(t)\|^2 + \lambda_u \|u\|^2 \tag{1.7}$$

with $\lambda_u > 0$. While the term $\|y_{\mathrm{M}} - y_{\mathrm{ref}}(t)\|^2$ penalises the distance of the model's output y_{M} to the reference signal y_{ref}, the term $\|u\|^2$ penalises the control effort. The parameter λ_u allows to adjust a suitable trade-off between tracking performance and required control effort. Of course, if a reference input signal u_{ref} is known, the second summand may be replaced by $\|u - u_{\mathrm{ref}}(t)\|^2$.

With the concepts introduced so far at hand, a general MPC algorithm can be formulated as follows.

Algorithm 1.1.4 (MPC).
Given: System (1.1), model (1.5), reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, initial time $t^0 \in \mathbb{R}_{\geq 0}$, boundary $u_{\max} \geq 0$ on the control input, and a stage cost function ℓ as in (1.7).
Set the time shift $\delta > 0$, the prediction horizon $T \geq \delta$, and index $k := 0$.
Define the time sequence $(t_k)_{k \in \mathbb{N}_0}$ by $t_k := t_0 + k\delta$.
Steps:

(a) Obtain a measurement of the state y at current time t_k over the last time period $[t_k - \tau, t_k]$ and set $\hat{y}^k := y|_{[t_k - \tau, t_k]}$. Select an initial value $\hat{y}_{\mathrm{M}}^k$ for the model (1.5) based on the measurement data $\hat{y}^k$.

(b) Compute a solution $u^\star \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$ of the *Optimal Control Problem (OCP)*

$$\begin{aligned} \underset{u \in L^\infty([t_k, t_k+T], \mathbb{R}^m)}{\text{minimise}} \quad & \int_{t_k}^{t_k+T} \ell(t, y_{\mathrm{M}}(t), u(t)) \mathrm{d}t \\ \text{subject to} \quad & y_{\mathrm{M}}^{(r)}(t) = F_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(y_{\mathrm{M}}, \dots, y_{\mathrm{M}}^{(r-1)})(t), u(t)), \\ & y_{\mathrm{M}}|_{[\hat{t} - \tau, \hat{t}]} = \hat{y}_{\mathrm{M}}^k, \\ & \|u(t)\| \leq u_{\max}. \end{aligned} \tag{1.8}$$

(c) Apply the feedback law

$$\mu : [t_k, t_{k+1}) \times \mathcal{C}^r([t_k - \tau, t_k], \mathbb{R}^m) \to \mathbb{R}^m, \quad \mu(t, \hat{y}) = u^\star(t)$$

to system (1.1). Increment k by 1 and go to Step (a).

▲

To ensure a bounded control signal with a maximal predefined control value $u_{\max} \geq 0$, the constraint $\|u\|_\infty \leq u_{\max}$ has been added as an additional constraint to the OCP (1.8). A key reason for the success and popularity of the MPC Algorithm 1.1.4 is its ability to directly take additional constraints into account by either adding them as hard constraints to the OCP (1.8), as done with $\|u\|_\infty \leq u_{\max}$, or by including soft constraints via penalty terms in the cost function ℓ (1.7). For example, one could add the additional constraint

$$\forall\, t \in [\hat{t}, \hat{t} + T]: \quad \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \psi(t) \tag{1.9}$$

to the optimisation problem in order to guarantee that the model's output y_{M} tracks the reference y_{ref} with prescribed performance, cf. [32].

We like to emphasise that both the system (1.1) to be controlled and the model (1.5) used in the MPC Algorithm 1.1.4 are continuous-time (functional) differential equations. While some other MPC approaches focusing on tracking problems also consider continuous-time systems, see e.g. [66], most address discrete-time systems [19, 106, 108, 129].

Difficulties and Drawbacks

Although utilising the stage cost ℓ in (1.7) and constraints (1.9) in Algorithm 1.1.4 might seem like a canonical choice when solving the reference tracking problem with MPC, this approach has several drawbacks. In particular, one has to guarantee initial and recursive feasibility of the MPC Algorithm 1.1.4. This means it is necessary to prove that the optimisation problem (1.8) has initially (i.e. at $t_k = t_0$) and recursively (i.e. at $t_k = t_0 + \delta k$ after k steps of Algorithm 1.1.4) a solution. First of all, one has to show existence of an L^∞-control u bounded by $u_{\max} \geq 0$ which, if applied to a model of type (1.5), guarantees that the model's tracking error

$$e_{\mathrm{M}}(t) := y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t) \tag{1.10}$$

evolves within the performance funnel $\mathcal{F}_\psi$ given by $\psi \in \mathcal{G}$, i.e. fulfilling (1.9). Furthermore, one has to prove that there exists a solution $u^\star$ of the optimisation problem (1.8) and that this solution fulfils (1.9). To show recursive feasibility, it is further necessary to ensure that after applying a solution $u^\star$ of the optimal control problem (1.8) at time $t_k = t^0 + \delta k$ to the system (1.1) the optimisation problem is still well defined at the next time instant $t_{k+1} = t^0 + \delta(k+1)$ when re-initialised with $\hat{y}_{\mathrm{M}}^k$ based on new measurements from the system. We will discuss potential methods to guarantee initial and recursive feasibility and their advantages and disadvantages in more detail in Chapter 2.

Both ensuring feasibility and achieving the control objective are already challenging problems when one assumes the model to coincide with the system. However, since every model, no matter how good, deviates from the actual system and disturbances are omnipresent, we assume the model (1.5) to differ from the system (1.1) rendering the problem even more demanding. One has to account for discrepancies between the model predictions $y_{\mathrm{M}}(t)$ and the actual system output $y(t)$, i.e. the model-plant output mismatch

$$e_{\mathrm{S}}(t) := y(t) - y_{\mathrm{M}}(t). \tag{1.11}$$

In the presented form, the MPC Algorithm 1.1.4 with constraints (1.9) can only achieve that the model's tracking error e_{M} evolves within the funnel $\mathcal{F}_\psi$. However, additional robustification methods need to be utilised in order to compensate for the model-plant output mismatch e_{S} and achieve the control objective for the actual tracking error e as laid out in Section 1.1.1. One aspect that merits particular attention is the selection of the initial value $\hat{y}_{\mathrm{M}}^k$ for the model based on the system measurement $\hat{y}^k$ in Step (a) of Algorithm 1.1.4. To reduce any occurring model-plant mismatch, one ideally would like to initialise the model (1.5) directly with the system measurement $\hat{y}^k$. This, however, might result in a constraint violation of applied restrictions like (1.9) and thus render the optimal control problem (1.8) unsolvable. Although undesirable, it may be necessary to allow for a higher tolerance of the model-plant mismatch in order to mathematically guarantee feasibility and functioning of the control scheme. Although it may seem evident, we would like to point out that initialising the model (1.5) with $\hat{y}_{\mathrm{M}}^k$ based on measurement data $\hat{y}^k$ will result in the model's solution on the interval $[t_k, t_{k+1}]$ not being a continuous extension of the solution on the previous interval $[t_{k-1}, t_k]$. Although these individual solutions fulfil the differential equation (1.5), the entire trajectory is, in general, not a solution. To refer to this trajectory of the concatenated solutions of the model's differential equation (1.5), we will use the term *concatenated solution* of the MPC algorithm. This will be made mathematically precise in Definition 2.4.2. It is clear that the concatenated solution can jump at the time instants t_k because of the initialisation with $\hat{y}_{\mathrm{M}}^k$. Thus, it is not even a continuous function but merely a regulated function. In order to avoid any potential ambiguities, we want to briefly recall the definition.

Definition 1.1.5 (Regulated function)**.** *On an interval I, we call a function $f : I \to \mathbb{R}^n$ a* regulated function*, if the left and right limits $f(t-)$ and $f(t+)$ exist for all interior points $t \in I$ and $f(a-)$ and $f(b+)$ exist whenever $a = \inf I \in I$ or $b = \sup I \in I$. We denote the space of all regulated functions on I by $\mathcal{R}(I, \mathbb{R}^n)$.*

1.2 Structure of this thesis and previously published results

This thesis is subdivided into four parts. Every chapter focuses on one of the laid-out difficulties related to solving the control objective with model predictive control.

Chapter 2 presents the theoretical foundations for solving the output tracking problem as described in Section 1.1.1 using a dedicated MPC algorithm. To this end, it is initially assumed that no disturbances are present and that the model matches the system perfectly. Section 2.1 introduces the general concept of funnel stage cost functions. Incorporating these into the MPC's optimal control problem guarantees that the output tracking error evolves within prescribed performance boundaries. This guarantee is rigorously proven in Section 2.3, following the introduction of the model class in Section 2.2. The resulting funnel MPC Algorithm 2.4.1 is defined in Section 2.4, with the chapter's main result – the proof of its initial and recursive feasibility – being established in Theorem 2.4.3. This proof leverages auxiliary error variables (introduced in Section 2.3.1) to ensure satisfaction of the control objective independently of the model order. It further relies on two key prior results: Theorem 2.3.21, proving the existence of control functions confining the error within funnel boundaries, and Theorem 2.3.26, showing the solvability of the optimal control problem with funnel stage costs. While building upon prior works [1, 2, 3], this chapter significantly extends them by formalising the general concept of funnel stage cost functions and providing detailed proofs for all mathematical aspects, addressing omissions due to page limitations in earlier publications. Furthermore, the auxiliary error framework developed here enabled the proposal of a low-complexity funnel controller for higher-order non-linear systems in [5].

Chapter 3 lifts the standing assumptions made in the previous chapter, namely the absence of disturbances and the perfect model-system alignment. Building upon the ideas from [4], the funnel MPC algorithm is robustified by incorporating the funnel controller as a second controller component. After introducing the system class under consideration in Section 3.1, the structure of the two component control scheme is presented in Section 3.2. The main result is Theorem 3.2.11 showing that the robust funnel MPC Algorithm 3.2.9 achieves the control objective in presence of disturbances and even a structural system model mismatch. Chapter 3 extends the results from [4] to encompass systems and models of higher order with non-linear time delays and potentially infinite-dimensional internal dynamics.

The control scheme is further extended by a (machine) learning component in Chapter 4. The structure of this three-component controller is laid out in Section 4.1. While the funnel controller component mitigates model-plant mismatches, bounded disturbances, and uncertainties, the machine learning component adapts the underlying model to the system data and, thus, improves the contribution of the MPC component over time. Definition 4.1.4 summarises the structural assumptions on a learning algorithm that ensure the successful interplay of the three components. The main result Theorem 4.1.8 shows the functioning of the three-component controller and that this control Algorithm 4.1.6 achieves the control objective as laid out in Section 1.1.1. To illustrate the abstract requirements and assumptions, a possible learning approach is discussed in Section 4.2. The chapter builds upon the work [7] and extends its results to the model and system class of functional differential equations of arbitrary order which was considered in the previous chapters.

In Chapters 2 to 4, it is assumed that the system output can be continuously measured and that an arbitrary measurable control signal can be applied to the system. Chapter 5 lifts this assumption and shows that the robust funnel MPC Algorithm 3.2.9 from Chapter 3 can be modified to achieve the control objective with sampled-data control. In line with the two-component structure of the controller, the chapter is divided into two parts, each of which shows that the respective component can be designed in a sampled data manner. Section 5.1 is based on the work [8] and proposes a funnel controller that achieves the control objective while only measuring the system output at discrete time instants and only applying piecewise constant control signals. Uniform bounds on the required sampling rate and the maximal applied control signal are derived. These results are summed up in Theorem 5.1.3. As a small extension, Section 5.1.1 shows how this controller can also be used as a safety filter for other data-driven control approaches. In Section 5.2, the results are carried over to the funnel MPC scheme. Theorem 5.2.3 shows that the sampled funnel MPC Algorithm 5.2.1, which only applies piecewise constant control signals to the model, achieves the control objective. This section builds upon the work [6] and extends previously published results to higher order models.

Within the course of this dissertation, the scientific article [9] was also published in addition to the mentioned works. In that paper, a mathematical model for a magnetic levitation train is developed. The funnel MPC algorithm from [3] is then applied to the system with the objective of guaranteeing the safe and dependable operation by ensuring that the distance between the magnet and the reaction rail is kept within a given range. The control scheme is then compared to two different control approaches, one linear state feedback controller and a model predictive control scheme with a quadratic cost function, with respect to performance criteria such as robustness, travel comfort, control effort, and computation time in an extensive numerical simulation study. As this dissertation is mainly concerned with the underlying mathematical theory of the funnel MPC algorithm and the work [9] focuses on the modelling aspect of the magnetic levitation systems, we will not present a separate evaluation of the results from [9] in this thesis.

2 Funnel Model Predictive Control

Guaranteeing *initial and recursive feasibility* is essential for the successful application of MPC and it is one of the major challenges. This requires guaranteeing that the optimal control problem is solvable at the initial time step and that solvability at any subsequent step follows recursively. While initial feasibility is often simply assumed, a common strategy to achieve recursive feasibility involves augmenting the OCP with carefully designed terminal conditions – such as terminal costs and constraints – as discussed in [55] and [159]. However, these artificially imposed terminal conditions introduce two key challenges: they raise the computational complexity of solving the OCP and complicate the identification of an initially feasible solution. Consequently, the domain of attraction of the MPC controller may be significantly restricted, as noted in [58, 74]. Furthermore, designing such conditions becomes markedly more intricate under time-varying state constraints [134]. An alternative approach circumvents terminal conditions by leveraging cost controllability principles [60, 185, 198] and employing a sufficiently long prediction horizon, see [40] and reference therein or [65] for an extension to continuous-time systems. Notably, both terminal-condition-based and horizon-based techniques face heightened complexity when applied to systems with time-varying state or output constraints, underscoring the need for tailored solutions in such cases. Additionally, feasibility assurances depend critically on the considered system class: non-linear or uncertain systems often necessitate more conservative designs, such as tube-based methods [140] or adaptive mechanisms for model refinement [88]. For systems with periodic constraints or references (e.g. tracking periodic trajectories), feasibility analysis often requires periodicity-aware terminal sets or horizon lengths [126]. Practical implementations must also contend with computational limits, where inexact solvers or early termination can undermine theoretical guarantees [207]. For systems with persistent infeasibility, slack variables or softened constraints may be introduced, albeit at the expense of performance [102]. Lastly, in economic MPC, where stability is not the primary objective, feasibility frameworks must reconcile transient constraints with long-term economic goals [15].

To overcome the restrictions of mentioned methods to ensure initial and recursive feasibility, *funnel MPC (FMPC)* was originally proposed in [32] and then further developed in [1, 2, 3]. It is an MPC scheme that allows for output tracking of an a priori given reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ such that the tracking error evolves in a pre-specified, potentially time-varying, performance funnel given by a function $\psi \in \mathcal{G}$. The core idea involves replacing ℓ from (1.7) in the MPC Algorithm 1.1.4 with a novel funnel stage cost function $\ell_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \cup \infty$. This function, parametrised by $\lambda_u \in \mathbb{R}_{\geq 0}$, is defined as:

$$\ell_\psi(t, y, u) = \begin{cases} \dfrac{\|y - y_{\mathrm{ref}}(t)\|^2}{\psi(t)^2 - \|y - y_{\mathrm{ref}}(t)\|^2} + \lambda_u \|u\|^2, & \|y - y_{\mathrm{ref}}(t)\| \neq \psi(t), \\ \infty, & \text{else}. \end{cases} \tag{2.1}$$

The term $\frac{\|y - y_{\mathrm{ref}}(t)\|^2}{\psi(t)^2 - \|y - y_{\mathrm{ref}}(t)\|^2}$ penalises the proximity of the tracking error to the funnel boundary, whereas the term $\|u\|$ serves as a penalisation of the control input as in (1.7). The parameter λ_u can be used to adjust the balance between these two control objectives. The cost function ℓ_ψ is motivated by the results on funnel control which we briefly introduced in Section 1.1.2. Contrary to the cost function ℓ as in (1.7), this “funnel-like” stage cost does not directly penalise the norm of the tracking error but its proximity to the funnel boundary ψ and grows unbounded when the error norm approaches ψ.

While many reference tracking MPC approaches focus on ensuring the asymptotic stability of the tracking error – often through terminal constraints or costs – they generally do not enforce strict boundaries on the output signal. For instance, [19, 107] guarantee stability of the tracking error by designing terminal sets and costs around specific reference trajectories, whereas [106] achieves this by relying on a sufficiently long prediction horizon instead of terminal constraints. Similarly, [108] ensures constraint satisfaction for discrete-time systems through stabilisability and detectability assumptions and long horizons. At first glance, tube-based MPC schemes, see e.g. [127, 128], share similarities with funnel MPC, as they confine the tracking error to a controllable and potentially time-varying range. However, their primary goal is to compensate for model uncertainties and disturbances acting on the system by constructing tubes around the reference trajectory. These tubes are often offline-computed and are not user-definable as they must inherently account for system uncertainties; for example [130] dynamically optimises both tubes and reference trajectories based on proximity to tube boundaries.

Control barrier functions (CBFs) determine the control input and enforce safety-critical constraints by solving a quadratic program (QP) that directly regulates the derivative of the barrier function. This ensures that the system remains within a safe set by design, guaranteeing positive invariance and asymptotic stability without requiring predictive optimisation. Due to their simplicity and modularity, CBFs are widely adopted in robotics, see [11, 12] for an overview. Building on this concept, barrier function-based MPC integrates barrier functions into the MPC framework, e.g. [136, 152, 197]. Unlike stand-alone CBFs, this approach incorporates a barrier term directly into the MPC cost function, penalising proximity to constraint boundaries over a prediction horizon. Alternatively, control Lyapunov-barrier functions are incorporated in the MPC scheme to ensure recursively feasibility and stabilisation of the closed-loop system, see e.g. [201]. However, similar to classical MPC, terminal costs or constraints are often still required to ensure recursive feasibility and constraint satisfaction. In contrast, funnel MPC eliminates the need for terminal constraints or costs by employing a unique cost function that diverges as the system output approaches the funnel boundary. It circumvents the reliance on terminal conditions and avoids the need for a sufficiently long prediction horizon entirely.

In this chapter, we will analyse how the utilisation of stage cost functions such as (2.1) in the MPC Algorithm 1.1.4 (which we will then call *funnel MPC*) ensures fulfilment of the control objective: the tracking of a given reference signal y_{ref} with the model's output y_{M} within predefined funnel boundaries ψ. For a large class of models, including but not limited to models with non-linear time delays and potentially infinite-dimensional internal dynamics, we will rigorously prove initial and recursive feasibility of this MPC scheme. This will be achieved without incorporating additional constraints in the optimal control problem (1.8), without imposing additional terminal conditions, and independent of the length of the prediction horizon $T > 0$. For our analysis in this chapter, we will assume that the system (1.1) and the surrogate model (1.5) coincide. In particular, we assume that the model-plant mismatch e_{S} defined in (1.11) is identically zero. Both assumptions will be relaxed in the later parts of this thesis.

2.1 Funnel stage cost functions

The key distinction between funnel MPC and the classical MPC Algorithm 1.1.4 lies in how their respective stage costs penalise the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$. While the classical stage cost function ℓ in (1.7) directly penalises the squared norm of the tracking error e, the funnel MPC stage cost ℓ_ψ in (2.1) imposes a particular penalty tied to the proximity of e to the funnel boundary ψ. This raises a natural question: how does this modified cost function ensure that the tracking error e evolves within the funnel $\mathcal{F}_\psi$ defined by a function $\psi \in \mathcal{G}$ when a solution of the optimisation problem (1.8) is applied to the model (1.5)? If the initial error lies within the funnel, without explicit constraints in the optimal control problem (1.8), then the error e could theoretically still touch or even exceed the boundary and evolve outside of the funnel boundary after some time. The previous work [32] addressed this issue by enforcing explicit hard state constraints of the form (1.9). In contrast, we will show that such constraints are unnecessary. Instead, the specific structure of the cost function ℓ_ψ in (2.1) inherently enforces compliance with the time-varying funnel boundaries. To analyse this mechanism of implicit constraints, we will, in this section, examine the function $\tilde{\ell}_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ defined as

$$\tilde{\ell}_\psi(t,e) = \begin{cases} \dfrac{\|e\|^2}{\psi(t)^2 - \|e\|^2} & \|e\| \neq \psi(t) \\ \infty, & \text{else}, \end{cases} \tag{2.2}$$

which quantifies the "cost" of being close to the funnel boundary. To isolate its essential features from any specific system dynamics or control problem, we examine, for a given $\hat{t} \in \mathbb{R}_{\geq 0}$ and $T > 0$, the integral

$$\int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma(s)) \mathrm{d}s \tag{2.3}$$

evaluated along an arbitrary *Lipschitz path* $\gamma \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)$, i.e. a Lipschitz continuous function γ defined on the interval $[\hat{t}, \hat{t}+T]$ with values in $\mathbb{R}^n$. Identifying the essential aspects of the function $\tilde{\ell}_\psi$ will lead to the Definition 2.1.11 of *funnel penalty functions*. These penalty functions, when used as a stage cost in the optimal control problem (1.8) within the MPC Algorithm 1.1.4, ensure the tracking error e evolves within the funnel $\mathcal{F}_\psi$ given by a function $\psi \in \mathcal{G}$. Restricting our analysis to Lipschitz paths for now allows us to avoid technical complications and to focus on the essential characteristics of the penalty function $\tilde{\ell}_\psi$ and the associated cost function. This simplification is justified because both the model output y_{M} in (1.5) and the reference trajectory $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ are differentiable functions; hence their restrictions to any compact interval $[\hat{t}, \hat{t}+T]$ are Lipschitz continuous (provided $\dot{y}_{\mathrm{M}}$ is bounded). This assumption will later be validated in Propositions 2.2.11 and 2.3.23.

Before proceeding, note a subtle point regarding the interpretation of the integral (2.3). There is a distinction between a function that is Lebesgue integrable (i.e. it belongs to L^1) and a function for which the Lebesgue integral merely exists but which does not necessarily have to be an element of L^1. To make this difference clearer, we call a measurable function $\zeta : B \to \mathbb{R}$ on a Borel set $B \subseteq \mathbb{R}$ *quasi-integrable* if at least one of the Lebesgue integrals

$$\int_B \zeta^+(t)\mathrm{d}t \qquad \text{or} \qquad \int_B \zeta^-(t)\mathrm{d}t$$

(with $\zeta^+ := \max\{\zeta, 0\}$ and $\zeta^- := \max\{-\zeta, 0\}$) is finite. If both integrals diverge, the overall integral is defined to be infinity. In particular, if $\tilde{\ell}_\psi(\cdot, \gamma(\cdot))$ is not quasi-integrable for a given Lipschitz path $\gamma \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)$, then the Lebesgue integral in (2.3) is treated as infinity. Moreover, it may happen that $\psi(t) = \|\gamma(t)\|$ for some $t \in [\hat{t}, \hat{t}+T]$ and with that $\tilde{\ell}_\psi(t, \gamma(t)) = \infty$. If the set of such points does not have Lebesgue measure zero, then the integral (2.3) is infinity as well. In the following, we will prove that if the integral (2.3) is finite, i.e. $\tilde{\ell}_\psi(\cdot, \gamma(\cdot))$ is quasi-integrable over $[\hat{t}, \hat{t}+T]$, and the integral does not diverge, then the Lipschitz path γ must evolve entirely within the funnel $\mathcal{F}_\psi$. To show this, an elementary lemma is proved first.

Lemma 2.1.1. *Let $T > 0$, $\hat{t} \in \mathbb{R}_{\geq 0}$ and $\gamma \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}_{\geq 0})$ be a Lipschitz path. If $\int_{\hat{t}}^{\hat{t}+T} \frac{1}{\gamma(s)} \mathrm{d}s < \infty$, then $\gamma(s) > 0$ for all $s \in [\hat{t}, \hat{t}+T]$.*

Proof. Assume that there exists $t \in (\hat{t}, \hat{t}+T)$ such that $\gamma(t) = 0$. Choose $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon) \subset [\hat{t}, \hat{t}+T]$. Since γ is Lipschitz continuous, we have that

$$\exists\, C > 0 \;\forall\, s \in (t-\varepsilon, t+\varepsilon): \;\; \gamma(s) = |\gamma(s) - \gamma(t)| \leq C\,|s-t|\,.$$

Therefore,

$$\infty > \int_0^T \frac{1}{\gamma(s)} \mathrm{d}s \geq \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\gamma(s)} \mathrm{d}s \geq \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{C\,|s-t|} \mathrm{d}s = \int_{-\varepsilon}^{\varepsilon} \frac{1}{C\,|s|} \mathrm{d}s = \infty,$$

a contradiction. A similar proof applies in the cases $t = \hat{t}$ and $t = \hat{t}+T$. □

Remark 2.1.2. Lemma 2.1.1 is not true for all uniformly continuous functions in general. Consider the example:

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x = 2\sqrt{x}\Big|_0^1 = 2.$$

•

Proposition 2.1.3. *For $\psi \in \mathcal{G}$, $\hat{t} \in \mathbb{R}_{\geq 0}$ and $T > 0$, let $\gamma \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)$ be a Lipschitz path with $(\hat{t}, \gamma(\hat{t})) \in \mathcal{F}_\psi$. Then,*

$$\int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma(s)) \mathrm{d}s < \infty \iff \mathrm{graph}(\gamma) \subset \mathcal{F}_\psi.$$

Proof. We show the two implications separately.
"$\Rightarrow$": We show that $\|\gamma(s)\| < \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$. Since $(\hat{t}, \gamma(\hat{t})) \in \mathcal{F}_\psi$, we have $\left\|\gamma(\hat{t})\right\| < \psi(\hat{t})$. Assume there exists $s \in [\hat{t}, \hat{t}+T]$ with $\|\gamma(s)\| \geq \psi(s)$. By continuity of γ and ψ, there exists

$$\hat{s} := \min\left\{ t \in [\hat{t}, \hat{t}+T] \,\middle|\, \|\gamma(t)\| = \psi(t) \right\}.$$

Note that $\|\gamma(s)\| < \psi(s)$ for all $s \in [\hat{t}, \hat{s})$. Recalling the definition of the Lebesgue integral, see e.g. [160, Def 11.22], $\int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma(s))\,\mathrm{d}s < \infty$ implies $\int_{\hat{t}}^{\hat{t}+T} \left(\tilde{\ell}_\psi(s, \gamma(s))\right)^+ \mathrm{d}s < \infty$. Thus,

$$\begin{aligned}
\int_{\hat{t}}^{\hat{s}} \frac{1}{1 - \frac{\|\gamma(s)\|^2}{\psi(s)^2}} \mathrm{d}s &= \int_{\hat{t}}^{\hat{s}} \frac{\|\gamma(s)\|^2}{\psi(s)^2 - \|\gamma(s)\|^2} + 1 \mathrm{d}s \\
&\leq \int_{\hat{t}}^{\hat{t}+T} \left(\frac{\|\gamma(s)\|^2}{\psi(t)^2 - \|\gamma(s)\|^2} \right)^+ \mathrm{d}s + T \\
&\leq \int_{\hat{t}}^{\hat{t}+T} \left(\tilde{\ell}_\psi(s, \gamma(s))\right)^+ \mathrm{d}s + T < \infty.
\end{aligned}$$

Both the path γ and the funnel function ψ, being an element of $W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$, are Lipschitz continuous functions. Since products and sums of Lipschitz continuous functions on a compact interval are again Lipschitz continuous, we may infer that $1 - \frac{\|\gamma(\cdot)\|^2}{\psi(\cdot)^2}$ is Lipschitz continuous on $[\hat{t}, \hat{s}]$. By definition of $\hat{s}$, it moreover is non-negative. Now, Lemma 2.1.1 yields that it is strictly positive, i.e. $\psi(s)^2 > \|\gamma(s)\|^2$ for all $s \in [\hat{t}, \hat{s}]$, which contradicts the definition of $\hat{s}$.

"$\Leftarrow$": Since $\operatorname{graph}(\gamma) \subset \mathcal{F}_\psi$, we have $\|\gamma(s)\| < \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$. Due to continuity of the involved functions and the compactness of the interval $[\hat{t}, \hat{t}+T]$, there exists $\varepsilon \in (0,1)$ with $\|\gamma(s)\| \leq \psi(s) - \varepsilon$ for all $s \in [\hat{t}, \hat{t}+T]$. Then, $\tilde{\ell}_\psi(s, \gamma(s)) \geq 0$ for all $s \in [\hat{t}, \hat{t}+T]$ and

$$\int_{\hat{t}}^{\hat{t}+T} \left|\tilde{\ell}_\psi(s, \gamma(s))\right| \mathrm{d}s = \int_{\hat{t}}^{\hat{t}+T} \left|\frac{\|\gamma(s)\|^2}{\psi(s)^2 - \|\gamma(s)\|^2}\right| \mathrm{d}s \leq \int_{\hat{t}}^{\hat{t}+T} \frac{\|\psi\|_\infty}{\varepsilon} \mathrm{d}s = T\frac{\|\psi\|_\infty}{\varepsilon} < \infty.$$

This completes the proof. □

Remark 2.1.4. In contrast to funnel MPC, *barrier function based MPC*, see e.g. [136, 197], employs (relaxed) logarithmic barrier functions to penalise states near constraint boundaries. This might initially seem to be merely a subtle difference since both methods involve stage cost functions that grow unbounded as the state approaches the constraint boundaries. However, the distinction has significant theoretical implications. The results in Lemma 2.1.1 and, consequently, Proposition 2.1.3, arise from the non-integrability of $x \mapsto \frac{1}{x}$ over the interval $[0,1]$. Specifically, Proposition 2.1.3 asserts that a finite value of the integral in (2.3) guarantees that any Lipschitz path starting within $\mathcal{F}_\psi$ remains confined to the prescribed funnel boundaries. Consequently, when the stage cost function (2.1) is used in the optimal control problem (1.8) (within the MPC Algorithm 1.1.4), the tracking error $e := y - y_{\text{ref}}$ is ensured to evolve within the funnel boundaries defined by $\psi \in \mathcal{G}$. In contrast, a logarithmic barrier function is integrable over the interval $[0,1]$:

$$\int_0^1 \ln(x^n)\mathrm{d}x = \ln(x^n)x\Big|_0^1 - \int_0^1 x\frac{n}{x}\mathrm{d}x = 0 - n = -n.$$

This integrability implies that logarithmic penalities alone can, in general, not guarantee that a Lipschitz path or, in the context of MPC, the model state always remain within the desired region. As a result, the usage of terminal conditions (costs and constraints) remains essential in the optimal control problem (1.8). •

Proposition 2.1.3 establishes that the integral (2.3) is finite if and only if the Lipschitz path $\gamma \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)$ evolves entirely within the funnel $\mathcal{F}_\psi$, provided γ starts within the funnel. Translating this result to the stage cost ℓ_ψ in (2.1) (to be used in the optimal control problem (1.8)), a finite stage cost guarantees that the model's tracking error $e = y_{\text{M}} - y_{\text{ref}}$ remains within the funnel boundaries defined by ψ. However, the non-linearity and discontinuity of the function ℓ_ψ in (2.1) raise concerns about the solvability of the optimal control problem (1.8) when employing this stage cost function. Moreover, even if a solution exists, additional analysis is required to ensure that its solution also guarantees the evolution of the tracking error within the funnel boundaries. To address this in Theorem 2.3.26, we will construct a sequence of control functions converging to the infimum of the minimisation problem (1.8) and analyse the corresponding sequence of error trajectories. We will then invoke the following Lemma 2.1.5 to prove that if all trajectories in this sequence remain within $\mathcal{F}_\psi$, then their limit will also remain within the funnel boundaries.

Lemma 2.1.5. *Let $\psi \in \mathcal{G}$, $\hat{t} \in \mathbb{R}_{\geq 0}$, $T > 0$ and $\gamma^\star \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)$ be a Lipschitz path with $(\hat{t}, \gamma^\star(\hat{t})) \in \mathcal{F}_\psi$. Further, let $(\gamma_n) \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^n)^{\mathbb{N}}$ be a sequence of Lipschitz paths with $(\hat{t}, \gamma_n(\hat{t})) \in \mathcal{F}_\psi$ for all $n \in \mathbb{N}$ that converges uniformly to $\gamma^\star$. If $\int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma_n(s))\mathrm{d}s$ is uniformly bounded by some constant $M \geq 0$, then*

$$\int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma^\star(s))\mathrm{d}s < \infty.$$

Proof. It suffices to show that $\|\gamma^\star(s)\| < \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$ according to Proposition 2.1.3. Since $(\hat{t}, \gamma^\star(\hat{t})) \in \mathcal{F}_\psi$, we have $\|\gamma^\star(\hat{t})\| < \psi(\hat{t})$. Assume there exists $s \in [\hat{t}, \hat{t}+T]$ with $\|\gamma^\star(s)\| \geq \psi(s)$. By continuity of γ and ψ, there exists

$$\hat{s} := \min\left\{ t \in [\hat{t}, \hat{t}+T] \,\middle|\, \|\gamma^\star(t)\| = \psi(t) \right\}.$$

We have $\operatorname{graph}(\gamma_n) \subset \mathcal{F}_\psi$ for all $n \in \mathbb{N}$ by assumption, cf. Proposition 2.1.3. Since, in addition, $\gamma^\star$ is a bounded function, there exists a compact set $\mathcal{K}$ such that $\operatorname{im}(\gamma^\star) \subset \mathcal{K}$ and $\operatorname{im}(\gamma_n) \subset \mathcal{K}$ for all $n \in \mathbb{N}$. Define the continuously differentiable function

$$\omega : [\hat{t}, \hat{t}+T] \times \mathcal{K} \to \mathbb{R}, \quad (s, x) \mapsto 1 - \frac{\|x\|^2}{\psi(s)^2}.$$

Due to the compactness of $[\hat{t}, \hat{t}+T]$ and $\mathcal{K}$, the function ω is Lipschitz continuous with Lipschitz constant $L_\omega > 0$. We have $\omega(s, \gamma_n(s)) > 0$ for all $n \in \mathbb{N}$ and all $s \in [\hat{t}, \hat{t}+T]$ because $\operatorname{graph}(\gamma_n) \subset \mathcal{F}_\psi$ for all $n \in \mathbb{N}$. Let $L^\star > 0$ be the Lipschitz constant of $\gamma^\star$. Since $\omega(\hat{s}, \gamma^\star(\hat{s})) = 0$, we estimate the following for all $s \in [\hat{t}, \hat{s}]$ and all $n \in \mathbb{N}$.

$$\begin{aligned} \omega(s, \gamma_n(s)) &= |\omega(s, \gamma_n(s))| = |\omega(s, \gamma_n(s)) - \omega(\hat{s}, \gamma^\star(\hat{s}))| \\ &\leq L_\omega \left\| \begin{pmatrix} s - \hat{s} \\ \gamma_n(s) - \gamma^\star(\hat{s}) \end{pmatrix} \right\| = L_\omega \left\| \begin{pmatrix} s - \hat{s} \\ \gamma_n(s) - \gamma^\star(s) + \gamma^\star(s) - \gamma^\star(\hat{s}) \end{pmatrix} \right\| \\ &\leq L_\omega |s - \hat{s}| + L_\omega \|\gamma_n(s) - \gamma^\star(s)\| + L_\omega L^\star |s - \hat{s}|. \end{aligned}$$

Since $\int_{\hat{t}}^{\hat{s}} ((L_\omega + L_\omega L^\star)|s - \hat{s}|)^{-1} \mathrm{d}s = \infty$, there exists $\varepsilon > 0$ with

$$\int_{\hat{t}}^{\hat{s}} ((L_\omega + L_\omega L^\star)|s - \hat{s}| + L_\omega \varepsilon)^{-1} \mathrm{d}s - \hat{s} > M.$$

As a consequence of the uniform convergence of γ_n to $\gamma^\star$, there exists $N \in \mathbb{N}$ such that $\|\gamma_n(s) - \gamma^\star(s)\| < \varepsilon$ for all $n \geq N$ and all $s \in [\hat{t}, \hat{s}]$. Thus, we arrive at the following contradiction for $n \geq N$.

$$\begin{aligned} M &\geq \int_{\hat{t}}^{\hat{t}+T} \tilde{\ell}_\psi(s, \gamma_n(s)) \mathrm{d}s = \int_{\hat{t}}^{\hat{t}+T} \frac{\|\gamma_n(s)\|^2}{\psi(s)^2 - \|\gamma_n(s)\|^2} \mathrm{d}s = \int_{\hat{t}}^{\hat{t}+T} \frac{1}{\omega(s, \gamma_n(s))} - 1 \mathrm{d}s \\ &\geq \int_{\hat{t}}^{\hat{s}} \frac{1}{\omega(s, \gamma_n(s))} - 1 \mathrm{d}s \geq \int_{\hat{t}}^{\hat{s}} \frac{1}{(L_\omega + L_\omega L^\star)|s - \hat{s}| + L_\omega \|\gamma_n(s) - \gamma^\star(s)\|} \mathrm{d}s - \hat{s} \\ &> \int_{\hat{t}}^{\hat{s}} \frac{1}{(L_\omega + L_\omega L^\star)|s - \hat{s}| + L_\omega \varepsilon} \mathrm{d}s - \hat{s} > M. \end{aligned}$$

This completes the proof. □

Proposition 2.1.3 and Lemma 2.1.5 will be essential in proving that the optimal control problem (1.8) using the $\tilde{\ell}_\psi$ from (2.2) has a solution and that this solution ensures that the tracking error e evolves within the funnel $\mathcal{F}_\psi$. To generalise these properties beyond the specific function $\tilde{\ell}_\psi$, the following Definition 2.1.6 introduces the concept of *funnel penalty functions*. These are the functions complying with Proposition 2.1.3 and Lemma 2.1.5.

Definition 2.1.6 (Strict funnel penalty function)**.** *Given $\psi \in \mathcal{G}$, consider a measurable function $\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ whose restriction $\nu_\psi|_{\mathcal{F}_\psi}$ is non-negative and continuous. We call ν_ψ a* strict funnel penalty function *for ψ, if, for all $\hat{t} \in \mathbb{R}_{\geq 0}$, $T > 0$, and every Lipschitz path $\gamma \in \operatorname{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ with $(\hat{t}, \gamma(\hat{t})) \in \mathcal{F}_\psi$ the following holds:*

(F.1) $$\int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma(s)) \mathrm{d}s < \infty \iff \operatorname{graph}(\gamma) \subset \mathcal{F}_\psi.$$

(F.2) If a sequence $(\gamma_n) \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)^{\mathbb{N}}$ with $(\hat{t}, \gamma_n(\hat{t})) \in \mathcal{F}_\psi$ for all $n \in \mathbb{N}$ converges uniformly to γ and there exists $M \geq 0$ such that $\int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma_n(s)) \mathrm{d}s \leq M$ for all $n \in \mathbb{N}$, then

$$\int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma(s)) \mathrm{d}s < \infty.$$

Example 2.1.7. Let $\tilde{\ell}_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ be given as in (2.2) for $\psi \in \mathcal{G}$. For every non-negative function $\nu \in \mathcal{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^m, \mathbb{R})$, the function $\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined by

$$\nu_\psi(t, y) := \tilde{\ell}_\psi(t, y) + \nu(t, y)$$

is a strict funnel penalty function. Since ν is bounded on $\mathcal{F}_\psi$, this is a direct result of Proposition 2.1.3 and Lemma 2.1.5. Consequently, one can model additional soft constraints via the function ν to be penalised in the MPC Algorithm 1.1.4, without losing the property of having a strict funnel penalty function. ◇

In Definition 2.1.6, ν_ψ is called *strict*, because condition (F.1) ensures that the Lipschitz path γ evolves within the interior of the funnel $\mathcal{F}_\psi$, i.e. $\|\gamma(s)\| < \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$. If, in addition, one also wants to allow for equality, i.e. only requires $\|\gamma(s)\| \leq \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$, and replaces $\mathcal{F}_\psi$ in Definition 2.1.6 by

$$\bar{\mathcal{F}}_\psi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \|e\| \leq \psi(t)\},$$

then property (F.2) can be omitted. In this case, property (F.1) and continuity of the function ν_ψ on the larger set $\bar{\mathcal{F}}_\psi$ already imply condition (F.2), as the following Lemma 2.1.8 shows. We therefore call a measurable function $\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ that fulfils property (F.1) for the set $\bar{\mathcal{F}}_\psi$ and whose restriction $\nu_\psi|_{\bar{\mathcal{F}}_\psi}$ is non-negative and continuous a *non-strict funnel penalty function* for ψ.

Lemma 2.1.8. *Let $\psi \in \mathcal{G}$ and $\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ be a non-strict funnel penalty function. Then, ν_ψ fulfils condition (F.2) of Definition 2.1.6 for the set $\bar{\mathcal{F}}_\psi$.*

Proof. Let $\hat{t} \in \mathbb{R}_{\geq 0}$, $T > 0$, and $\gamma \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ be a Lipschitz path with $(\hat{t}, \gamma(\hat{t})) \in \bar{\mathcal{F}}_\psi$. Further, let $M \geq 0$ and $(\gamma_n) \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)^{\mathbb{N}}$ be a to γ uniformly converging sequence with $(\hat{t}, \gamma_n(\hat{t})) \in \mathcal{F}_\psi$ and $\int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma_n(s)) \mathrm{d}s \leq M$ for all $n \in \mathbb{N}$. Due to property (F.1), we have $\mathrm{graph}(\gamma_n) \subset \bar{\mathcal{F}}_\psi$. Recall that, for non-strict funnel penalty functions, Definition 2.1.6 is formulated in terms of $\bar{\mathcal{F}}_\psi$. We show that $\|\gamma(s)\| \leq \psi(s)$ for all $s \in [\hat{t}, \hat{t}+T]$. Assume there exists $\hat{s} \in (\hat{t}, \hat{t}+T]$ with $\|\gamma(\hat{s})\| > \psi(\hat{s})$. Then, there exists $\varepsilon > 0$ with $\|\gamma(\hat{s})\| > \psi(\hat{s}) + \varepsilon$. Since the uniform convergence of (γ_n) towards γ implies pointwise convergence of (γ_n), there exists $K > 0$ such that $\|\gamma(\hat{s}) - \gamma_k(\hat{s})\| < \varepsilon$ for all $k \geq K$. Furthermore, $\|\gamma_k(\hat{s})\| \leq \psi(\hat{s})$ since $\mathrm{graph}(\gamma_k) \subset \bar{\mathcal{F}}_\psi$ for all $k \in \mathbb{N}$. This raises the following contradiction for $k \geq K$

$$\psi(\hat{s}) + \varepsilon < \|\gamma(\hat{s})\| \leq \|\gamma(\hat{s}) - \gamma_k(\hat{s})\| + \|\gamma_k(\hat{s})\| \leq \varepsilon + \psi(\hat{s}).$$

This completes the proof. □

Remark 2.1.9. A strict funnel penalty function ν_ψ cannot be continuous on the whole set $\bar{\mathcal{F}}_\psi$. Otherwise, for a Lipschitz path $\gamma \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ with $\mathrm{graph}(\gamma) \subset \bar{\mathcal{F}}_\psi$ and $\|\gamma(s)\| = \psi(s)$ for some $s \in [\hat{t}, \hat{t}+T]$, the function $t \mapsto \nu_\psi(t, \gamma(t))$ would be bounded and, thus, integrable. Hence, it is clear that the two concepts of a strict and non-strict funnel penalty function are mutually exclusive, meaning a single function cannot be both. •

We have already seen that $\tilde{\ell}_\psi$ as in (2.2) is a strict funnel penalty function. Now, we want to give an example for a non-strict funnel penalty function.

Example 2.1.10. Let $\psi \in \mathcal{G}$ and $\tilde{\nu}_\psi \in \mathcal{C}(\bar{\mathcal{F}}_\psi, \mathbb{R})$ be a non-negative function. Then,

$$\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}, \quad \nu_\psi(t,e) = \begin{cases} \tilde{\nu}_\psi(t,e), & (t,e) \in \bar{\mathcal{F}}_\psi \\ \infty, & \text{else} \end{cases}$$

is a non-strict funnel penalty function. To see this, let $\gamma \in \mathrm{Lip}([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ be a Lipschitz path with $(\hat{t}, \gamma(\hat{t})) \in \bar{\mathcal{F}}_\psi$. We have to show that (F.1) from Definition 2.1.6 holds for $\bar{\mathcal{F}}_\psi$. First, let $\operatorname{graph}(\gamma) \subset \bar{\mathcal{F}}_\psi$. Then, $\nu_\psi(s, \gamma(s)) = \tilde{\nu}_\psi(s, \gamma(s))$ for all $s \in [\hat{t}, \hat{t}+T]$. Due to the continuity of the involved functions, $\tilde{\nu}_\psi(s, \gamma(s))$ is bounded on the compact interval $[\hat{t}, \hat{t}+T]$. Thus, the integral $\int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma(s)) \mathrm{d}s$ is finite. To show the reverse implication, assume now that the integral is finite but that there exists $\hat{s} \in (\hat{t}, \hat{t}+T]$ with $\|\gamma(\hat{s})\| > \psi(\hat{s})$. Then, there exists $\varepsilon > 0$ with $\|\gamma(s)\| > \psi(s)$ for all $s \in [\hat{s} - \varepsilon, \hat{s}]$ because γ and ψ are continuous functions. Hence, the following contradiction arises.

$$\infty > \int_{\hat{t}}^{\hat{t}+T} \nu_\psi(s, \gamma(s)) \mathrm{d}s \geq \int_{\hat{s}-\varepsilon}^{\hat{s}} \nu_\psi(s, \gamma(s)) \mathrm{d}s = \int_{\hat{s}-\varepsilon}^{\hat{s}} \infty \ \mathrm{d}s = \infty.$$

◇

We discussed the essential properties of the function $\tilde{\ell}_\psi$ and summed them up in Definition 2.1.6 of funnel penalty functions. To use this concept in optimal control problems of the form (1.8), we additionally want to be able to penalise the necessary control effort in the cost function. To this end, the following Definition 2.1.11 introduces *funnel stage cost* functions. An example is the funnel MPC stage cost function ℓ_ψ as in (2.1).

Definition 2.1.11 (Funnel stage cost function)**.** *Let $\psi \in \mathcal{G}$ and $\nu_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ be a (non)-strict funnel penalty function. For $\lambda_u \in \mathbb{R}_{\geq 0}$, we call a function*

$$\ell_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}, \quad (t, z, u) \mapsto \nu_\psi(t, z) + \lambda_u \|u\|^2$$

a (non)-strict funnel stage cost*.*

In Definition 2.1.11, the penalisation term for the control input u consists of the squared norm of u multiplied by the parameter λ_u. This parameter allows to adjust a suitable trade-off between tracking performance and required control effort. Note that $\lambda_u = 0$ is explicitly allowed contrary to (1.7). Of course, utilising more sophisticated penalty terms is also possible. One option, for example, is the usage of norms induced by a positive definite matrix. If a reference input signal u_{ref} is known, the second summand may also be replaced by $\|u - u_{\mathrm{ref}}(t)\|^2$. In this work however, we will restrict ourselves to the presented case.

If not explicitly mentioned otherwise, we will use strict funnel stage cost functions in the rest of the present thesis and we will refer to them merely by the term *funnel stage cost*. The presented results generally also remain valid for non-strict funnel stage cost functions, but then only with respect to the set $\bar{\mathcal{F}}_\psi$, i.e. inequalities of the form $\|e\| < \psi(t)$ have to be replaced by $\|e\| \leq \psi(t)$.

Remark 2.1.12. Note that every (non)-strict funnel stage cost ℓ_ψ is non-negative for every element $(t, z, u) \in \mathcal{F}_\psi \times \mathbb{R}^m$. •

2.2 Model class

In the previous Section 2.1, the essential aspects of the cost function ℓ_ψ from (2.1) were identified to introduce the more general concept of funnel stage cost functions. This was done by considering Lipschitz paths in order to conduct this analysis in isolation from any differential equation. In this section, however, we will introduce the class of surrogate models for the system (1.1) to be utilised in the MPC Algorithm 1.1.4. We consider non-linear control affine multi-input multi-output models of order $r \in \mathbb{N}$ of the form

$$\begin{aligned} y_{\mathrm{M}}^{(r)}(t) &= f_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(y_{\mathrm{M}},\ldots,y_{\mathrm{M}}^{(r-1)})(t)\big) + g_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(y_{\mathrm{M}},\ldots,y_{\mathrm{M}}^{(r-1)})(t)\big)u(t), \\ \left.\begin{aligned} y_{\mathrm{M}}|_{[0,t_0]} &= y_{\mathrm{M}}^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m), && \text{if } t_0>0, \\ (y_{\mathrm{M}}(t_0),\ldots,y_{\mathrm{M}}^{(r-1)}(t_0)) &= y_{\mathrm{M}}^0 \in \mathbb{R}^{rm}, && \text{if } t_0 = 0, \end{aligned}\right\} \end{aligned} \tag{2.4}$$

with $t_0 \geq 0$, initial trajectory y_{M}^0, control input $u \in L^\infty_{\mathrm{loc}}([t_0,\infty),\mathbb{R}^m)$, and output $y_{\mathrm{M}}(t) \in \mathbb{R}^m$ at time $t \geq t_0$. Note that, like the system (1.1), u and y_{M} have the same dimension $m \in \mathbb{N}$. The model consists of two locally Lipschitz continuous function $f_{\mathrm{M}} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q,\mathbb{R}^m)$, $g_{\mathrm{M}} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q,\mathbb{R}^{m\times m})$, and an operator $\mathbf{T}_{\mathrm{M}}$. To ensure that the control u can always influence the dynamics, we assume that g_{M} is everywhere point-wise invertible, i.e. g_{M} satisfies $g_{\mathrm{M}}(z) \in \mathrm{GL}_m(\mathbb{R})$ for all $z \in \mathbb{R}^q$. The operator $\mathbf{T}_{\mathrm{M}}$ is causal, locally Lipschitz, satisfies a bounded-input bounded-output and a limited memory property. It is characterised in detail in the following Definition 2.2.1.

Definition 2.2.1 (Operator class $\mathcal{T}^{n,q}_{t_0}$). *For $n, q \in \mathbb{N}$ and $t_0 \in \mathbb{R}_{\geq 0}$, the set $\mathcal{T}^{n,q}_{t_0}$ denotes the class of operators $\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^n) \to L^\infty_{\mathrm{loc}}([t_0,\infty),\mathbb{R}^q)$ for which the following properties hold:*

(T.1) Causality*: $\forall\, y_1, y_2 \in \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^n)\ \forall\, t \geq t_0$:*

$$y_1|_{[0,t]} = y_2|_{[0,t]} \implies \mathbf{T}(y_1)|_{[t_0,t]} = \mathbf{T}(y_2)|_{[t_0,t]}.$$

(T.2) Local Lipschitz*: $\forall\, t \geq t_0\ \forall\, y \in \mathcal{R}([0,t];\mathbb{R}^n)\ \exists\, \Delta, \delta, c > 0\ \forall\, y_1, y_2 \in \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ with $y_1|_{[0,t]} = y_2|_{[0,t]} = y$ and $\|y_1(s) - y(t)\| < \delta$, $\|y_2(s) - y(t)\| < \delta$ for all $s \in [t, t+\Delta]$:*

$$\operatorname*{ess\,sup}_{s\in[t,t+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \leq c \sup_{s\in[t,t+\Delta]} \|y_1(s) - y_2(s)\|.$$

(T.3) Bounded-input bounded-output (BIBO)*: $\forall\, c_0 > 0\ \exists\, c_1 > 0\ \forall\, y \in \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$:*

$$\sup_{t\in\mathbb{R}_{\geq 0}} \|y(t)\| \leq c_0 \implies \sup_{t\in[t_0,\infty)} \|\mathbf{T}(y)(t)\| \leq c_1.$$

(T.4) Limited memory*: $\exists\, \tau \geq 0\ \forall\, t \geq t_0\ \forall\, y_1, y_2 \in \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ with $y_1|_I = y_2|_I$ on the interval $I := [t-\tau,\infty) \cap \mathbb{R}_{\geq 0}$ and $\mathbf{T}(y_1)|_J = \mathbf{T}(y_2)|_J$ on the interval $J := [t-\tau, t] \cap [t_0, t]$:*

$$\mathbf{T}(y_1)|_{[t,\infty)} = \mathbf{T}(y_2)|_{[t,\infty)}.$$

The value τ in is called memory limit *of the operator $\mathbf{T}$.*

Note that an operator $\mathbf{T}_{\mathrm{M}} \in \mathcal{T}^{n,q}_{t_0}$ can model *non-linear time delays*, where t_0 corresponds to the initial delay, and that it can even be the solution operator of an infinite-dimensional dynamical system, e.g. a partial differential equation.

We summarise our assumptions and define the general model class under consideration.

Definition 2.2.2 (Model class $\mathcal{M}^{m,r}_{t_0}$). *We say the model (2.4) belongs to the* model class *$\mathcal{M}^{m,r}_{t_0}$ for $m, r \in \mathbb{N}$, and $t_0 \in \mathbb{R}_{\geq 0}$, written $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0}$, if, for some $q \in \mathbb{N}$, the following holds: $f_{\mathrm{M}} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q,\mathbb{R}^m)$, $g_{\mathrm{M}} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q,\mathbb{R}^{m\times m})$ satisfies $g_{\mathrm{M}}(z) \in GL_m(\mathbb{R})$ for all $z \in \mathbb{R}^q$, and $\mathbf{T}_{\mathrm{M}} \in \mathcal{T}^{rm,q}_{t_0}$.*

Due to their quite technical nature, the properties of the $\mathbf{T}_{\mathrm{M}} \in \mathcal{T}^{n,q}_{t_0}$ deserve some additional explanation. However, before commenting on them in Remark 2.2.5, we briefly discuss a simple example of a model belonging to the considered model class in order to have a familiar picture in mind. In particular, this provides a simple candidate for an operator $\mathbf{T}$.

Example 2.2.3 (Linear time-invariant model)**.** Consider a linear multi-input, multi-output differential equation of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x^0 \\ y(t) &= Cx(t), \end{aligned} \tag{2.5}$$

with $A \in \mathbb{R}^{n\times n}$ and $C^\top, B \in \mathbb{R}^{n\times m}$. Further, assume that this model has a *strict relative degree* $r \in \mathbb{N}$, i.e. $CA^kB = 0$ for all $k < r-1$ and $CA^{r-1}B \in \mathrm{GL}_m(\mathbb{R})$. By [96, Lemma 3.5], there exists an invertible $U \in \mathbb{R}^{n\times n}$ such that with $[z_1^\top, \dots, z_r^\top, \eta^\top]^\top = Ux$ the above system can be transformed into the *(linear) Byrnes-Isidori form*

$$\begin{aligned} \dot{z}_i(t) &= z_{i+1}, & z_i(t_0) &= z_i^0, \\ \dot{z}_r(t) &= \sum_{j=1}^{r} R_j z_j(t) + S\eta + \Gamma u(t), & z_r(t_0) &= z_r^0, \\ \dot{\eta}(t) &= Q\eta(t) + Pz_1(t), & \eta(t_0) &= \eta^0, \end{aligned} \tag{2.6}$$

with output

$$y(t) = z_1(t),$$

where $R_j \in \mathbb{R}^{m\times m}$ for all $j = 1, \dots, r$, $S, P^\top \in \mathbb{R}^{m\times(n-rm)}$, $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$, and $\Gamma = CA^{r-1}B$. Define the linear integral operator $\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)^r \to L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^q)$ by

$$\mathbf{T}(z_1, \dots, z_r)(t) := \sum_{j=1}^{r} R_j z_j(t) + S\left(\mathrm{e}^{Q(t-t_0)}\eta_0 + \int_{t_0}^{t} \mathrm{e}^{Q(t-s)} Pz_1(s)\mathrm{d}s\right).$$

Utilising this operator $\mathbf{T}$ and the Byrnes-Isidori form (2.6), the differential equation (2.5) can be put in the equivalent form (2.4) with initial value

$$(y(t_0), \dot{y}(t_0), \dots, y^{(r-1)}(t_0)) = (Cx^0, CAx^0, \dots, CA^{r-1}x^0).$$

In the following, we examine the properties (T.1)–(T.4) of the operator $\mathbf{T}$. It is easy to see that it satisfies properties (T.1) and (T.2). To also verify the limited memory property (T.4) for $\mathbf{T}$ with $\tau = 0$, let $\hat{t} \geq t_0$ and $(y_1, \dots, y_r), (\tilde{y}_1, \dots, \tilde{y}_r) \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)^r$ with $\mathbf{T}(y_1, \dots, y_r)(\hat{t}) = \mathbf{T}(\tilde{y}_1, \dots, \tilde{y}_r)(\hat{t})$ and $(y_1, \dots, y_r)(t) = (\tilde{y}_1, \dots, \tilde{y}_r)(t)$ for all $t \geq \hat{t}$. Using the shorthand notation

$$V(z_1, \dots, z_r)(t) := \sum_{j=1}^{r} R_j z_j(t) + S\mathrm{e}^{Q(t-t_0)}\eta_0 \quad \text{and} \quad L(z_1)(t) := S\int_{t_0}^{t} \mathrm{e}^{Q(t-s)} Pz_1(s)\mathrm{d}s,$$

we have $\mathbf{T}(z_1, \dots, z_r)(t) = V(z_1, \dots, z_r)(t) + L(z_1)(t)$. It is clear that $V(y_1, \dots, y_r)(t)$ and $V(\tilde{y}_1, \dots, \tilde{y}_r)(t)$ are identical for all $t \geq \hat{t}$. To show the limited memory property (T.4) for the operator $\mathbf{T}$, it therefore is sufficient to show $L(y_1)(t) = L(\tilde{y}_1)(t)$ for all $t \geq \hat{t}$. For $t \geq \hat{t}$, we have

$$\begin{aligned} L(y_1)(t) &= S\int_{t_0}^{t} \mathrm{e}^{Q(t-s)} Py_1(s)\mathrm{d}s \\ &= L(y_1)(\hat{t}) + S\int_{\hat{t}}^{t} \mathrm{e}^{Q(t-s)} Py_1(s)\mathrm{d}s \\ &= \mathbf{T}(y_1, \dots, y_1)(\hat{t}) - V(y_1, \dots, y_r)(\hat{t}) + S\int_{\hat{t}}^{t} \mathrm{e}^{Q(t-s)} Py_1(s)\mathrm{d}s \\ &= \mathbf{T}(\tilde{y}_1, \dots, \tilde{y}_r)(\hat{t}) - V(\tilde{y}_1, \dots, \tilde{y}_r)(\hat{t}) + S\int_{\hat{t}}^{t} \mathrm{e}^{Q(t-s)} P\tilde{y}_1(s)\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= L(\tilde{y}_1)(\hat{t}) + S\int_{\hat{t}}^{t} \mathrm{e}^{Q(t-s)} P\tilde{y}_1(s)\mathrm{d}s \\
&= S\int_{t_0}^{t} \mathrm{e}^{Q(t-s)} P\tilde{y}_1(s)\mathrm{d}s \\
&= L(\tilde{y}_1)(t).
\end{aligned}
$$

Thus, $\mathbf{T}$ has the property (T.4) with $\tau = 0$. Additionally assume that (2.5) has *asymptotically stable zero dynamics*, i.e.

$$
\forall\, \lambda \in \mathbb{C}_{\geq 0}: \quad \det \begin{bmatrix} \lambda I_n - A & B \\ C & 0 \end{bmatrix} \neq 0.
$$

This, also called *minimum phase* property in literature, see e.g. [98, 99], implies that all eigenvalues of the matrix Q in (2.6) are in the open left plane, i.e. $\operatorname{spec}(Q) \subset \mathbb{C}_{<0}$, see [96, Lemma 3.5]. Consequently, the operator $\mathbf{T}$ fulfils the BIBO property (T.3), see also the diagram in [30, Section 2.1.2] nicely illustrating the relationship between the minimum phase property of model (2.5) and the BIBO property of $\mathbf{T}$. ⋄

Example 2.2.4 (Non-linear model with state space representation)**.** Consider a non-linear differential equation of the form

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(t_0) = x^0, \\
y(t) &= h(x(t)),
\end{aligned}
\tag{2.7}
$$

with $t_0 \in \mathbb{R}_{\geq 0}$, $x^0 \in \mathbb{R}^n$, and non-linear functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. We show in the following that (2.7) can, under certain conditions, be put in the form (2.4) and thus is an admissible candidate for a model. We recall the notion of relative degree for the differential equation (2.7), see e.g. [99, Sec. 5.1]. Assuming that f, g, h are sufficiently smooth, the Lie derivative of h along f is defined by $(L_f h)(x) = h'(x)f(x)$ and, successively, we define $L_f^k h = L_f(L_f^{k-1}h)$ with $L_f^0 h = h$. Furthermore, for the matrix-valued function g, we have

$$
(L_g h)(x) = \left[(L_{g_1}h)(x), \ldots, (L_{g_m}h)(x)\right],
$$

where g_i denotes the i-th column of g for $i = 1, \ldots, m$. Then, the differential equation (2.7) is said to have *strict (global) relative degree* $r \in \mathbb{N}$, if

$$
\begin{aligned}
\forall\, k \in \{1, \ldots, r-1\}\ \forall\, x \in \mathbb{R}^n: \ &(L_g L_f^{k-1}h)(x) = 0 \\
\text{and} \quad &(L_g L_f^{r-1}h)(x) \in \mathrm{GL}_m(\mathbb{R}).
\end{aligned}
$$

If (2.7) has relative degree r, then, under the additional assumptions provided in [52, Cor. 5.6], differential equation (2.7) can be transformed into *(non-linear) Byrnes-Isidori form* – a generalisation of (2.6). This means there exists a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that the coordinate transformation $(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t)) = \Phi(x(t))$ puts the differential equation (2.7) into the form

$$
y^{(r)}(t) = p\big(y(t), \ldots, y^{(r-1)}(t), \eta(t)\big) + \Gamma\big(y(t), \ldots, y^{(r-1)}(t), \eta(t)\big)\, u(t), \tag{2.8a}
$$

$$
\dot{\eta}(t) = q\big(y(t), \ldots, y^{(r-1)}(t), \eta(t)\big), \tag{2.8b}
$$

where $p : \mathbb{R}^n \to \mathbb{R}^m$, $q : \mathbb{R}^n \to \mathbb{R}^{n-rm}$, $\Gamma = L_g L_f^{r-1}h : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are continuously differentiable and $(y(t_0), \dot{y}(t_0), \ldots, y^{(r-1)}(t_0), \eta(t_0)) = \Phi(x_0)$. Note that, under these assumptions, the derivatives of the output y of (2.7) are given by $y^{(i)}(t) = (L_f^i h)(x(t))$ for $i = 0, \ldots, r-1$. In the following, we assume the existence of such diffeomorphism

$\Phi : \mathbb{R}^n \to \mathbb{R}^n$ but not necessarily the conditions stated in [52, Cor. 5.6] as these are sufficient but not necessary for the existence of Φ. We further assume that internal dynamics (2.8b) satisfy the following *bounded-input, bounded-state* (BIBS) condition:

$$\forall\, c_0 > 0\ \exists\, c_1 > 0\ \forall\, t_0 \geq 0\ \forall\, \eta^0 \in \mathbb{R}^{n-rm} \forall\, \zeta \in L^\infty_{\text{loc}}([t_0,\infty), \mathbb{R}^{rm}) : \\ \|\eta^0\| + \|\zeta\|_\infty \leq c_0 \implies \|\eta(\cdot; t_0, \eta^0, \zeta)\|_\infty \leq c_1, \quad (2.9)$$

where $\eta(\cdot; t_0, \eta^0, \zeta) : [t_0, \infty) \to \mathbb{R}^{n-rm}$ denotes the unique global solution of (2.8b) when $(y(t), \dots, y^{(r-1)}(t))$ is substituted by ζ. Note that, in view of condition (2.9), the maximal solution $\eta(\cdot; t_0, \eta^0, \zeta)$ can indeed be extended to a global solution, cf. [192, § 10, Thm. XX]. Utilising the unique global solution η of (2.8b), define operator

$$\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm}) \to L^\infty_{\text{loc}}([t_0,\infty), \mathbb{R}^n), \quad \zeta \mapsto \mathbf{T}(\zeta) := (\zeta(\cdot), \eta(\cdot; t_0, \eta^0, \zeta)).$$

It is easy to see that $\mathbf{T}$ satisfies the causality property (T.1). The bounded-input bounded-output property (T.3) is a direct consequence of BIBS condition (2.9) on internal dynamics (2.8b). Utilising the fact that the continuously differentiable function q is local Lipschitz continuous in combination with the BIBS condition (2.9), the local Lipschitz property (T.2) can be verified via straightforward calculations. To verify that it also satisfies the limited memory property (T.4) for $\tau = 0$, let $\hat{t} \geq t_0$ and $\zeta_1, \zeta_2 \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $\mathbf{T}(\zeta_1)(\hat{t}) = \mathbf{T}(\zeta_2)(\hat{t})$ and $\zeta_1(t) = \zeta_2$ for all $t \geq \hat{t}$. As η is the maximal solution of (2.8b), it can be represented for $\zeta \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ as

$$\eta(t; t_0, \eta^0, \zeta) = \eta^0 + \int_{t_0}^{t} q(\zeta(s), \eta(s; t_0, \eta^0, \zeta)) \mathrm{d}s$$

for all $t \geq t_0$. $\mathbf{T}(\zeta_1)(\hat{t}) = \mathbf{T}(\zeta_2)(\hat{t})$ implies

$$\int_{t_0}^{\hat{t}} q(\zeta_1(s), \eta(s; t_0, \eta^0, \zeta_1)) \mathrm{d}s = \int_{t_0}^{\hat{t}} q(\zeta_2(s), \eta(s; t_0, \eta^0, \zeta_2)) \mathrm{d}s.$$

Utilising $\zeta_1|_{[\hat{t},\infty)} = \zeta_2|_{[\hat{t},\infty)}$, we have

$$\begin{aligned}
\mathbf{T}(\zeta_1)(t) &= (\zeta_1(t), \eta^0 + \int_{t_0}^{t} q(\zeta_1(s), \eta(s; t_0, \eta^0, \zeta_1)) \mathrm{d}s) \\
&= (0, \eta^0 + \int_{t_0}^{\hat{t}} q(\zeta_1(s), \eta(s; t_0, \eta^0, \zeta_1)) \mathrm{d}s) + (\zeta_1(t), \int_{\hat{t}}^{t} q(\zeta_1(s), \eta(s; t_0, \eta^0, \zeta_1)) \mathrm{d}s) \\
&= (0, \eta^0 + \int_{t_0}^{\hat{t}} q(\zeta_2(s), \eta(s; t_0, \eta^0, \zeta_2)) \mathrm{d}s) + (\zeta_2(t), \int_{\hat{t}}^{t} q(\zeta_2(s), \eta(s; t_0, \eta^0, \zeta_2)) \mathrm{d}s) \\
&= (\zeta_2(t), \eta^0 + \int_{t_0}^{t} q(\zeta_2(s), \eta(s; t_0, \eta^0, \zeta_2)) \mathrm{d}s) = \mathbf{T}(\zeta_2)(t)
\end{aligned}$$

for $t \geq \hat{t}$. Thus, $\mathbf{T}$ has the property (T.4) with $\tau = 0$. The differential equation (2.7) therefore can be put in the form (2.4) and it is an admissible model. ⋄

Remark 2.2.5. We comment on several aspects of the operator class $\mathcal{T}^{n,q}_{t_0}$ and its properties.

(a) Let $\mathbf{T} \in \mathcal{T}^{n,q}_{t_0}$ and I be the interval $I = [0, \hat{t}]$ or $I = [0, \hat{t})$ for $\hat{t} \in \mathbb{R}_{\geq 0}$. For a given function $\zeta \in \mathcal{R}(I, \mathbb{R}^n)$, let ζ^e denote an arbitrary right extension of ζ on the entire interval of non-negative real numbers, i.e. $\zeta^e \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ with $\zeta^e|_I = \zeta$. By virtue of the causality property (T.1), the restriction of $\mathbf{T}(\zeta^e)$ to the interval I is uniquely determined by the function ζ in the sense that $\mathbf{T}(\zeta^e)|_I$ is independent of the chosen extension ζ^e. This observation made in [95, Remark 2 (iii)] allows us to apply the operator $\mathbf{T}$ in a certain sense to functions $\zeta \in \mathcal{R}(I, \mathbb{R}^n)$ by utilising an arbitrary right extension $\zeta^e \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ instead. We will therefore use throughout this work the following notation. For $s \in I$, we write $\mathbf{T}(\zeta)(s)$ in place of $\mathbf{T}(\zeta^e)(s)$.

(b) The Lipschitz property (T.2) is a rather technical assumption to ensure the existence of a solution of the closed-loop initial value problems (1.1) and (2.4) when a control law is applied, see Theorem 7.0.4. We will summarise the corresponding results, tailored to the context, in Proposition 2.2.8. For the technical details, however, we would also like to refer the reader to the works [94, 95, 161] and also to the Appendix where we recall the solution theory for the class of functional differential equations considered in this thesis. The Lipschitz property will moreover be used in Proposition 2.2.9 in order to prove uniqueness of the solution of the initial value problem (2.4).

(c) To motivate the BIBO property (T.3) of operator $\mathbf{T}$, we consider the example of a differential equation of the form

$$\dot{y}(t) = Ay(t) + Bu(t), \tag{2.10a}$$
$$\dot{\eta}(t) = f(y(t), \eta(t)), \tag{2.10b}$$

with matrices $A \in \mathbb{R}^{n\times n}$ and $B \times \mathbb{R}^{n\times m}$ and a continuously differentiable function $f : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^\ell$. As similarly shown in Example 2.2.3, the differential equation (2.10) can be put in the from (2.4), where $\mathbf{T}$ is the solution operator of the non-linear equation (2.10b). If the matrices (A, B) are controllable, then a stabilising state feedback $u = Ky$ with $K \in \mathbb{R}^{m\times n}$ can be applied to (2.10) and the linear part (2.10a) can be estimated, for $t \geq 0$ and $a, b > 0$, by $\|x(t)\| \leq be^{-at}\|x(0)\|$. Any prespecified a can be realised by the choice of K. However, as stated by Sussmann and Kokotovic in [181], *one cannot, in general, choose K so as to make the number a large without making b large as well.* As first pointed out by Sussmann in [180], the so called *peaking-phenomenon* can cause the non-linear part (2.10b) of the system to have finite escape time even if the system

$$\dot{\eta}(t) = f(0, \eta(t))$$

has 0 as a global asymptotically stable equilibrium. The presumed BIBO property (T.3) of operator $\mathbf{T}$ not only avoids this problem but is even more essential since our control objective is to guarantee that the output y of the system (1.1) respectively the output y_{M} of the model (2.4) evolves within the funnel around the reference signal y_{ref}. Without this assumption and even with perfect tracking, the non-linear dynamics (2.10b) might be unbounded and thus cause an unbounded control effort, or worse, its solution might even have finite escape time.

(d) Compared to previous works on funnel control, see e.g. [30, 31, 95], the limited memory property (T.4) was newly introduced in [2] and is essential in the context of MPC in order to ensure that in Step (a) of each iteration of the MPC Algorithm 1.1.4 only the history of the state of length up to the memory limit $\tau \geq 0$ is utilised, instead of requiring the full signal history, which would be infeasible in practice. Let $\mathbf{T} \in \mathcal{T}_{t_0}^{n,q}$ with memory limit $\tau \geq 0$ and I be the interval $I = [\hat{t}-\tau, \hat{t}+T]$ for $\hat{t} \geq \tau$ and $T \in \mathbb{R}_{\geq 0}$. For given functions $\zeta \in \mathcal{R}(I, \mathbb{R}^n)$ and $\hat{\mathbf{T}} \in L^\infty_{\mathrm{loc}}([\hat{t} - \tau, \hat{t}] \cap [t_0, \hat{t}], \mathbb{R}^q)$, let ${}^e\zeta$ be a left extension of ζ on the interval $[0, \hat{t}]$ with $\mathbf{T}({}^e\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}$. Similar to the observations in (a), the restriction of $\mathbf{T}({}^e\zeta)$ to the interval I is uniquely determined by the functions ζ and $\hat{\mathbf{T}}$ in the sense that $\mathbf{T}({}^e\zeta)|_I$ is independent of the chosen left extension ${}^e\zeta$ due to the limited memory property (T.4). If the value $\hat{\mathbf{T}}$ is fixed, this allows us to write $\mathbf{T}(\zeta)(s)$ in place of $\mathbf{T}({}^e\zeta)(s)$ for $s \in I$ (assuming there exists a left extension ${}^e\zeta$ with $\mathbf{T}({}^e\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}$).

(e) In the literature on funnel control, the used operator $\mathbf{T}$ belonging to $\mathcal{T}_{t_0}^{n,q}$ is usually defined on the space of continuous functions $\mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, see e.g. [30, 95]. In the context of MPC, this is too restrictive. Since the model is re-initialised with data from system measurements in Step (a) at the beginning of every iteration of the MPC Algorithm 1.1.4, one cannot assume continuity of the global solution trajectory of model (2.4). Measurement errors, disturbances, and a potential model-system mismatch will inevitably result in discontinuities at the points of model re-initialisation. To account for this, Definition 2.2.1 generalises the operator's domain to the space of regulated functions $\mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

•

We now define a concept of a solution for the initial value problem (2.4). As we will use this differential equation as a model for the MPC Algorithm 1.1.4, a certain degree of care is required for this definition. Since the initial value problem is solved at every iteration of the algorithm at different time instants $\hat{t} \geq t_0$ with varying initial values based on system measurements obtained in Step (a) of the algorithm, certain theoretical problems arise mainly caused by the domain of operator $\mathbf{T}_{\mathrm{M}}$.

Definition 2.2.6 (Model solution). *Given a control function $u \in L^\infty_{\mathrm{loc}}([\hat{t}, \infty), \mathbb{R}^m)$, a regulated function $x_{\mathrm{M}} = (x_{\mathrm{M},1}, \ldots, x_{\mathrm{M},r})$ with $x_{\mathrm{M},i} : [0, \omega) \to \mathbb{R}^m$, $\omega \in (\hat{t}, \infty]$, $i = 1, \ldots, r$, is called a solution of the initial value problem (2.4) at initial time $\hat{t} \geq t_0 \geq 0$ and with initial data $\hat{x}_{\mathrm{M}} \in \mathcal{R}([\hat{t} - \tau, \hat{t}] \cap [0, \hat{t}], \mathbb{R}^{rm})$ and $\hat{\mathbf{T}}_{\mathrm{M}} \in L^\infty_{\mathrm{loc}}([\hat{t} - \tau, \hat{t}] \cap [t_0, \hat{t}], \mathbb{R}^q)$ with $\tau \geq 0$, if*

$$\begin{aligned} x_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} &= \hat{x}_{\mathrm{M}}, \\ \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} &= \hat{\mathbf{T}}_{\mathrm{M}}, \end{aligned} \tag{2.11}$$

and $x_{\mathrm{M}}|_{[\hat{t},\omega)}$ is absolutely continuous such that, for almost all $t \in [\hat{t}, \omega)$, it fulfils

$$\begin{bmatrix} \dot{x}_{\mathrm{M},1}(t) \\ \vdots \\ \dot{x}_{\mathrm{M},r-1}(t) \\ \dot{x}_{\mathrm{M},r}(t) \end{bmatrix} = \begin{bmatrix} x_{\mathrm{M},2}(t) \\ \vdots \\ x_{\mathrm{M},r}(t) \\ f_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t)\big) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t)\big) \end{bmatrix} u(t). \tag{2.12}$$

A solution x_{M} is said to be maximal *if it has no proper right extension that is also a solution. A maximal solution is called a* response *of the model associated with u and we denote it by $x_{\mathrm{M}}(\cdot; \hat{t}, \hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}, u)$. Its first component $x_{\mathrm{M},1}$ is denoted by $y_{\mathrm{M}}(\cdot; \hat{t}, \hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}, u)$.*

Note that, in (2.11) of Definition 2.2.6, we did not distinguish between the cases $t_0 > 0$ and $t_0 = 0$ as in (2.4), since a larger variety of cases is possible here. Essentially, one needs to distinguished whether the interval $[\hat{t} - \tau, \hat{t}] \cap [0, \hat{t}]$ is a perfect interval or merely a single point. In the latter case, we will implicitly assume a representation of the initial condition as in (2.4) and that $\hat{x}_{\mathrm{M}}$ is an element of $\mathbb{R}^{rm}$ and analogously that $\hat{\mathbf{T}}_{\mathrm{M}}$ is an element of $\mathbb{R}^q$. Or in other words, we interpret the restriction of the considered functions to an interval of the form $I = [\hat{t}]$ as a function evaluation at $\hat{t}$ and identify the function spaces $\mathcal{R}(I, \mathbb{R}^{rm})$ and $L^\infty_{\mathrm{loc}}(I, \mathbb{R}^q)$ with the vector spaces $\mathbb{R}^{rm}$ and $\mathbb{R}^q$, respectively. The parameter $\tau \geq 0$ can be thought of as a memory length of past signal information used to initialise the differential equation.

Remark 2.2.7. The above Definition 2.2.6 contains certain peculiarities and differs from the conventional definition for the solution of an initial value problem. We comment on that.

(a) Definition 2.2.6 uses a solution concept in the sense of *Carathéodory*, see e.g. [192, § 10, Supplement II]. The solution x_{M} does not automatically possess a continuous first derivative and it fulfils differential equation (2.12) on $[\hat{t}, \omega)$ with the exception of a set of Lebesgue measure zero. Equivalently, the solution concept can be formulated using an integral representation instead of equation (2.12). Then, the function x_{M} is considered to be a solution to differential equation (2.4) when satisfying

$$x_{\mathrm{M}}(t) = \hat{x}_{\mathrm{M}}(\hat{t}) + \int_{\hat{t}}^{t} F_{\mathrm{M}}(x_{\mathrm{M}}(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(s)) + G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(s))u(s)\mathrm{d}s, \tag{2.13}$$

for all $t \in [\hat{t}, \omega)$, where $F_{\mathrm{M}}(x_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})) := \left[x_{\mathrm{M},2}, \quad \dots, \quad x_{\mathrm{M},r}, \quad f_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})\big)\right]^{\top}$ and $G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})) := \left[0, \quad \dots, \quad 0, \quad g_{\mathrm{M}}\big(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})\big)\right]^{\top}$. At some instances, we will also use this representation.

(b) Note that in Definition 2.2.6 the solution x_{M} is defined on the whole interval $[0, \omega)$ while the initial values are given at $\hat{t} \geq t_0 \geq 0$ and while x_{M} satisfies the differential equation merely on the interval $[\hat{t}, \omega)$. The rational behind this is for the domain of the solution x_{M} to be in accordance with the domain of the operator $\mathbf{T}_{\mathrm{M}}$ which is $\mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$. By virtue of the causality property (T.1), one can, when solving the initial value problem (2.12) with (2.11), evaluate $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t)$ for $t \in [\hat{t}, \omega)$, in a certain sense, implicitly utilising right extensions x_{M}^{e} as discussed in Remark 2.2.5 (a). However, it is necessary for x_{M} to be also defined in the past, meaning on the interval $[0, \hat{t}]$. The initial values $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ determine, not necessarily uniquely, x_{M} on the interval $[0, \hat{t}]$.

(c) It might seem, at first glance, like $\hat{\mathbf{T}}_{\mathrm{M}}$ is already completely determined by $\hat{x}_{\mathrm{M}}$. The latter however is, in general, not an element of the domain of $\mathbf{T}_{\mathrm{M}}$. In order for $\hat{x}_{\mathrm{M}}$ to be evaluable by the operator $\mathbf{T}_{\mathrm{M}}$, one has to choose a left extension ${}^{e}\hat{x}_{\mathrm{M}}$ on the interval $[0, \hat{t}]$. As this left extension can initially be chosen arbitrarily, the second condition on the initial value in (2.11) entwines the left extension ${}^{e}\hat{x}_{\mathrm{M}}$ with the initial datum $\hat{\mathbf{T}}_{\mathrm{M}}$. As discussed in Remark 2.2.5 (d), if $\tau \geq 0$ is greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, then the left extension ${}^{e}\hat{x}_{\mathrm{M}}$, assuming its existence, is uniquely determined by $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ in the sense that $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t)$ for $t \geq \hat{t}$ does not depend on the chosen extension.

(d) To take the initial value y_{M}^{0} from (2.4) into account at $\hat{t} = t_0$, one can replace in (2.11) the initial value $\hat{x}_{\mathrm{M}}$ with $(y_{\mathrm{M}}^{0}, \dots, {y_{\mathrm{M}}^{0}}^{(r-1)})$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ with $\mathbf{T}_{\mathrm{M}}\big((y_{\mathrm{M}}^{0}, \dots, {y_{\mathrm{M}}^{0}}^{(r-1)})\big)$. However, Definition 2.2.6 is intentionally formulated merely in terms of $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ independent of y_{M}^{0} in order to avoid treating the initial time $\hat{t} = t_0$ as a special case. •

One of the difficulties we will have to address in the following is that the initial values $\hat{x}_{\mathrm{M}}$ and $\mathbf{T}_{\mathrm{M}}$ in (2.11) have to allow for the existence of a solution of initial value problem (2.12). For that $\hat{x}_{\mathrm{M}}$ is required to have an admissible left extension on the interval $[0, \hat{t}]$, i.e. a left extension ${}^{e}\hat{x}_{\mathrm{M}}$ with $\mathbf{T}_{\mathrm{M}}({}^{e}\hat{x}_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$. Moreover, in the context of the MPC Algorithm 1.1.4, the initial values have to be chosen in a way to entwine the initial value problem (2.12) to be solved at the current time instant $\hat{t}$ with the history of x_{M} from the previous iterations and the initial trajectory y_{M}^{0} given at time $t_0 \geq 0$. Before addressing these two questions, we want to consider initial value problem (2.12) individually, independent of the MPC Algorithm 1.1.4, and assume the existence of an admissible left extension for $\hat{x}_{\mathrm{M}}$. In this case, there exists a solution to the initial value problem as the following Proposition 2.2.8 shows.

Proposition 2.2.8. *Consider model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$ and $t_0 \geq 0$. For $\hat{t} \geq t_0$ and a control function $u \in L^{\infty}_{\mathrm{loc}}([\hat{t}, \infty), \mathbb{R}^{m})$, let $\hat{x}_{\mathrm{M}} \in \mathcal{R}([\hat{t} - \tau, \hat{t}] \cap [0, \hat{t}], \mathbb{R}^{rm})$*

and $\hat{\mathbf{T}}_{\mathrm{M}} \in L^{\infty}_{\mathrm{loc}}([\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}],\mathbb{R}^q)$ *with* $\tau \geq 0$. *If* $\hat{x}_{\mathrm{M}}$ *has an admissible left extension* ${}^e\hat{x}_{\mathrm{M}}$ *on the interval* $[0,\hat{t}]$, *i.e.* ${}^e\hat{x}_{\mathrm{M}} \in \mathcal{R}([0,\hat{t}],\mathbb{R}^{rm})$ *with* ${}^e\hat{x}_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\mathrm{M}}$ *and* $\mathbf{T}_{\mathrm{M}}({}^e\hat{x}_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$, *then*

(i) the initial value problem (2.12) *with* (2.11) *has a solution* $x_{\mathrm{M}} : [0,\omega) \to \mathbb{R}^{rm}$ *in the sense of Definition 2.2.6,*

(ii) every solution can be extended to a maximal solution,

(iii) if $x_{\mathrm{M}} : [0,\omega) \to \mathbb{R}^{rm}$ *is a bounded maximal solution, then* $\omega = \infty$.

Proof. For given $\hat{t} \geq t_0$ and $u \in L^{\infty}_{\mathrm{loc}}([\hat{t},\infty),\mathbb{R}^m)$, let $\hat{x}_{\mathrm{M}} \in \mathcal{C}([\hat{t}-\tau,\hat{t}] \cap [0,\hat{t}],\mathbb{R}^{rm})$ and $\hat{\mathbf{T}}_{\mathrm{M}} \in L^{\infty}_{\mathrm{loc}}([\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}],\mathbb{R}^q)$ with $\tau \geq 0$ be arbitrary but fixed. Further, let ${}^e\hat{x}_{\mathrm{M}}$ be an admissible left extension of $\hat{x}_{\mathrm{M}}$ on the interval $[0,\hat{t}]$ and let F_{M} and G_{M} be defined as in (2.13). As a consequence of Corollary 7.0.6 to be found in the Appendix, there exists a solution $x : [0,\omega) \to \mathbb{R}^{rm}$ with $\omega > \hat{t}$ of the initial value problem

$$\begin{aligned} \dot{x}(t) &= F_{\mathrm{M}}(x(t),\mathbf{T}_{\mathrm{M}}(x)(t)) + G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x)(t))u(t), \\ x|_{[0,\hat{t}]} &= {}^e\hat{x}_{\mathrm{M}}, \end{aligned} \tag{2.14}$$

where x fulfils the differential equation for almost all $t \in [\hat{t},\omega)$. Since ${}^e\hat{x}_{\mathrm{M}}$ is an admissible left extension of $\hat{x}_{\mathrm{M}}$, the function x is also a solution of initial value problem (2.12) with (2.11) in the sense of Definition 2.2.6. This shows (i).

Let $\tilde{x} : [0,\omega) \to \mathbb{R}^{rm}$ be an arbitrary solution of the initial value problem (2.12) in the sense of Definition 2.2.6 with initial values $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ as in (2.11). Due to the used solution concept, the function $\tilde{x}$ is also a solution of the initial value problem (2.14) with initial value $x|_{[0,\hat{t}]} = \tilde{x}|_{[0,\hat{t}]}$. According to Corollary 7.0.6, this function can be extended to a maximal solution. Moreover, if any maximal solution of this initial value problem is bounded, then $\omega = \infty$. Since $\tilde{x}|_{[0,\hat{t}]}$ is an admissible left extension of $\hat{x}_{\mathrm{M}}$, both findings carry over to the initial value problem (2.12) with initial values as in (2.11) and the solution definition in the sense of Definition 2.2.6. □

While Proposition 2.2.8 ensures the existence of a solution x_{M} of initial value problem (2.12) with (2.11), it is in general not unique, in particular since x_{M} does not need to comply with the differential equation (2.12) for $t < \hat{t}$. On the interval $[0,\hat{t}]$, it is merely a feasible left extension of $\hat{x}_{\mathrm{M}}$ and, depending on $\mathbf{T}_{\mathrm{M}}$, there might exist several feasible left extensions. However, if τ is greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, then x_{M} is, in a certain sense, unique as the following Proposition 2.2.9 shows. Namely, it is uniquely determined for $t \geq \hat{t}$.

Proposition 2.2.9. *Consider model* (2.4) *with* $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0}$ *and* $t_0 \geq 0$. *For* $\hat{t} \geq t_0$ *and a control function* $u \in L^{\infty}_{\mathrm{loc}}([\hat{t},\infty),\mathbb{R}^m)$, *let* $\hat{x}_{\mathrm{M}} \in \mathcal{R}([\hat{t}-\tau,\hat{t}] \cap [0,\hat{t}],\mathbb{R}^{rm})$ *and* $\hat{\mathbf{T}}_{\mathrm{M}} \in L^{\infty}_{\mathrm{loc}}([\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}],\mathbb{R}^q)$ *with* $\tau \geq 0$. *Let* $x^1_{\mathrm{M}} : [0,\omega_1) \to \mathbb{R}^{rm}$ *and* $x^2_{\mathrm{M}} : [0,\omega_2) \to \mathbb{R}^{rm}$ *be two solutions of the initial value problem* (2.12) *with* (2.11) *in the sense of Definition 2.2.6. If the value* τ *is greater than or equal to the memory limit of* $\mathbf{T}_{\mathrm{M}}$, *then* $x^1_{\mathrm{M}}(t) = x^2_{\mathrm{M}}(t)$ *for all* $t \in [\hat{t},\min\{\omega_1,\omega_2\})$.

Proof. Step 1: We show that x^1_{M} and x^2_{M} coincide on the interval $[\hat{t},\hat{t}+\varepsilon]$ for some $\varepsilon > 0$. As x^1_{M} and x^2_{M} are solutions of the initial value problem (2.12) with (2.11) in the sense of Definition 2.2.6, there exists a feasible left extension of $\hat{x}_{\mathrm{M}}$, i.e. ${}^e\hat{x}_{\mathrm{M}} \in \mathcal{R}([0,\hat{t}],\mathbb{R}^{rm})$ with ${}^e\hat{x}_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\mathrm{M}}$ and $\mathbf{T}_{\mathrm{M}}({}^e\hat{x}_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$. According to the local Lipschitz property (T.2) of operator $\mathbf{T}_{\mathrm{M}}$, there exists constants $\Delta, \delta, c > 0$ such that for all functions $y_1, y_2 \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $y_1|_{[0,\hat{t}]} = y_2|_{[0,\hat{t}]} = {}^e\hat{x}_{\mathrm{M}}$ and $\|y_1(s) - \hat{x}_{\mathrm{M}}(\hat{t})\| < \delta$, $\|y_2(s) - \hat{x}_{\mathrm{M}}(\hat{t})\| < \delta$ for all $s \in [t, t+\Delta]$:

$$\operatorname*{ess\,sup}_{s\in[t,t+\Delta]} \|\mathbf{T}_{\mathrm{M}}(y_1)(s) - \mathbf{T}_{\mathrm{M}}(y_2)(s)\| \leq c \sup_{s\in[t,t+\Delta]} \|y_1(s) - y_2(s)\|.$$

The functions x_{M}^1 and x_{M}^2 are continuous on the interval $[\hat{t}, \min\{\omega_1, \omega_2\})$ and satisfy $x_{\mathrm{M}}^1(\hat{t}) = x_{\mathrm{M}}^2(\hat{t}) = \hat{x}_{\mathrm{M}}(\hat{t})$. Therefore, there exists some $\varepsilon \in (0, \min\{\Delta, \omega_1, \omega_2\})$ such that $\|x_{\mathrm{M}}^1(t) - \hat{x}_{\mathrm{M}}(\hat{t})\| < \delta$ and $\|x_{\mathrm{M}}^2(t) - \hat{x}_{\mathrm{M}}(\hat{t})\| < \delta$ for all $t \in [\hat{t}, \hat{t}+\varepsilon]$.

We will estimate $\|\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(t) - \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(t)\|$ for $t \in [\hat{t}, \hat{t}+\varepsilon]$ in the following. To that end, let $\mathbb{1}_I$ denote the indicator function of an interval $I \subset \mathbb{R}$. Note that, as τ is greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, see property (T.4), we have, for $t \geq \hat{t}$ and $i = 1, 2$,

$$\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^i)(t) = \mathbf{T}_{\mathrm{M}}\left(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^i\right)(t) = \mathbf{T}_{\mathrm{M}}\left(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^i + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\right)(t),$$

where the causality property (T.1) was used in the second equation. Hence, for $t \in [\hat{t}, \hat{t}+\varepsilon]$, it follows

$$\begin{aligned}
&\|\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(t) - \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(t)\| \leq \operatorname*{ess\,sup}_{\nu\in[\hat{t},t]} \|\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(\nu) - \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(\nu)\| \\
&= \operatorname*{ess\,sup}_{\nu\in[\hat{t},t]} \left\|\mathbf{T}_{\mathrm{M}}\Big(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^1 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\Big)(\nu) - \mathbf{T}_{\mathrm{M}}\Big(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^2 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\Big)(\nu)\right\| \\
&\leq \operatorname*{ess\,sup}_{\nu\in[\hat{t},\hat{t}+\Delta]} \left\|\mathbf{T}_{\mathrm{M}}\Big(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^1 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\Big)(\nu) - \mathbf{T}_{\mathrm{M}}\Big(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^2 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\Big)(\nu)\right\| \\
&\leq c \sup_{\nu\in[\hat{t},\hat{t}+\Delta]} \left\|\left(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^1 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\right)(\nu) - \left(\mathbb{1}_{[0,\hat{t})}\,{}^e\hat{x} + \mathbb{1}_{[\hat{t},t]}x_{\mathrm{M}}^2 + \mathbb{1}_{(t,\hat{t}+\Delta]}\hat{x}_{\mathrm{M}}(\hat{t})\right)(\nu)\right\| \\
&= c \sup_{\nu\in[\hat{t},t]} \left\|\mathbb{1}_{[\hat{t},t]}(\nu)x_{\mathrm{M}}^1(\nu) - \mathbb{1}_{[\hat{t},t]}(\nu)x_{\mathrm{M}}^2(\nu)\right\| = c \sup_{\nu\in[\hat{t},t]} \|x_{\mathrm{M}}^1(\nu) - x_{\mathrm{M}}^2(\nu)\|.
\end{aligned}$$

We now estimate $\sup_{\nu\in[\hat{t},\hat{t}+\varepsilon]} \|x_{\mathrm{M}}^1(\nu) - x_{\mathrm{M}}^2(\nu)\|$. To that end, define the continuous function $\zeta : [\hat{t}, \hat{t}+\varepsilon] \to \mathbb{R}$, $t \mapsto \sup_{\nu\in[\hat{t},t]} \|x_{\mathrm{M}}^1(\nu) - x_{\mathrm{M}}^2(\nu)\|$. There exists a compact set K with $x_{\mathrm{M}}^i(t) \in K$ for all $t \in [\hat{t}, \hat{t}+\varepsilon]$ and $i = 1, 2$. Moreover, due to the BIBO property (T.3) of operator $\mathbf{T}_{\mathrm{M}}$, there exists a compact set $\tilde{K}$ with $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^i)(t) \in K$ for all t in $[\hat{t}, \hat{t}+\varepsilon]$ and $i = 1, 2$. As the function f_{M} is an element of $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q, \mathbb{R}^m)$, F_{M} is Lipschitz continuous with constant $L_{F_{\mathrm{M}}} \geq 0$ on the set $K \times \tilde{K}$. Similarly, there exists a Lipschitz constant $L_{G_{\mathrm{M}}} \geq 0$ for the function G_{M} on the set $\tilde{K}$. Using the above considerations and the solution representation (2.13), ζ satisfies the following estimate for all t in $[\hat{t}, \hat{t}+\varepsilon]$.

$$\begin{aligned}
\zeta(t) &= \sup_{\nu\in[\hat{t},t]} \|x_{\mathrm{M}}^1(\nu) - x_{\mathrm{M}}^2(\nu)\| \\
&= \sup_{\nu\in[\hat{t},t]} \Bigg\|\hat{x}_{\mathrm{M}}(\hat{t}) + \int_{\hat{t}}^{\nu} F_{\mathrm{M}}(x_{\mathrm{M}}^1(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s)) + G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s))u(s)\mathrm{d}s \\
&\qquad - \hat{x}_{\mathrm{M}}(\hat{t}) - \int_{\hat{t}}^{\nu} F_{\mathrm{M}}(x_{\mathrm{M}}^2(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s)) + G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s))u(s)\mathrm{d}s\Bigg\| \\
&\leq \sup_{\nu\in[\hat{t},t]} \Bigg(\int_{\hat{t}}^{\nu} \|F_{\mathrm{M}}(x_{\mathrm{M}}^1(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s)) - F_{\mathrm{M}}(x_{\mathrm{M}}^2(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s))\| \,\mathrm{d}s \\
&\qquad + \int_{\hat{t}}^{\nu} \|G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s))u(s) - G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s))u(s)\| \,\mathrm{d}s\Bigg) \\
&\leq \int_{\hat{t}}^{t} \|F_{\mathrm{M}}(x_{\mathrm{M}}^1(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s)) - F_{\mathrm{M}}(x_{\mathrm{M}}^2(s), \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s))\| \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\hat{t}}^{t} \left\| G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s))u(s) - G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s))u(s) \right\| \mathrm{d}s \\
&\leq \int_{\hat{t}}^{t} L_{F_{\mathrm{M}}} \left(\left\| x_{\mathrm{M}}^1(s) - x_{\mathrm{M}}^2(s) \right\| + \left\| \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s)) - \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s)) \right\| \right) \mathrm{d}s \\
&+ \int_{\hat{t}}^{t} L_{G_{\mathrm{M}}} \left\| \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(s)) - \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(s)) \right\| \left\| u|_{[\hat{t},t]} \right\|_{\infty} \mathrm{d}s \\
&\leq \left(L_{F_{\mathrm{M}}}(1+c) + L_{G_{\mathrm{M}}} c \left\| u|_{[\hat{t},\hat{t}+\varepsilon]} \right\|_{\infty} \right) \int_{\hat{t}}^{t} \sup_{s \in [\hat{t},s]} \left\| x_{\mathrm{M}}^1(s) - x_{\mathrm{M}}^2(s) \right\| \mathrm{d}s \\
&\leq \left(L_{F_{\mathrm{M}}}(1+c) + L_{G_{\mathrm{M}}} c \left\| u|_{[\hat{t},\hat{t}+\varepsilon]} \right\|_{\infty} \right) \int_{\hat{t}}^{t} \zeta(s) \mathrm{d}s.
\end{aligned}
$$

Now, Grönwall's inequality yields $\zeta(t) = 0$ for all $t \in [\hat{t}, \hat{t}+\varepsilon]$. This shows $x_{\mathrm{M}}^1(t) = x_{\mathrm{M}}^2(t)$ on the interval $t \in [\hat{t}, \hat{t}+\varepsilon]$.

Step 2: We show that x_{M}^1 and x_{M}^2 coincide on the interval $[\hat{t}, \min\{\omega_1,\omega_2\})$. Suppose $x_{\mathrm{M}}^1|_{[\hat{t},\min\{\omega_1,\omega_2\})} \neq x_{\mathrm{M}}^2|_{[\hat{t},\min\{\omega_1,\omega_2\})}$. Then, there exists a time instant $t \in [\hat{t}, \min\{\omega_1,\omega_2\})$ with $x_{\mathrm{M}}^1(t) \neq x_{\mathrm{M}}^2(t)$. Let

$$
\tilde{t} := \inf \left\{ t \in [\hat{t}, \min\{\omega_1,\omega_2\}) \, \middle| x_{\mathrm{M}}^1(t) \neq x_{\mathrm{M}}^2(t) \right\}.
$$

In view of Step 1, we have $\tilde{t} > \hat{t}$. Since x_{M}^1 and x_{M}^2 are solutions of the initial value problem (2.4) with initial value $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ at initial time $\hat{t}$, we have

$$
x_{\mathrm{M}}^1|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\mathrm{M}} = x_{\mathrm{M}}^2|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} \text{ and } \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}} = \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]}.
$$

Thus, $x_{\mathrm{M}}^1(t) = x_{\mathrm{M}}^2(t)$ for all $t \in [\tilde{t}-\tau, \tilde{t}] \cap [0, \tilde{t}]$. This implies $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)(t) = \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^2)(t)$ for all $t \in [\tilde{t}-\tau, \tilde{t}] \cap [t_0, \tilde{t}]$ because τ is greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, see property (T.4). Define $\tilde{x}_{\mathrm{M}} := x_{\mathrm{M}}^1|_{[\tilde{t}-\tau,\tilde{t}]\cap[0,\tilde{t}]}$ and $\tilde{\mathbf{T}}_{\mathrm{M}} := \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^1)|_{[\tilde{t}-\tau,\tilde{t}]\cap[t_0,\tilde{t}]}$. These functions are elements of $\mathcal{R}([\tilde{t}-\tau,\tilde{t}]\cap[0,\tilde{t}], \mathbb{R}^{rm})$ and $L^\infty_{\mathrm{loc}}([\tilde{t}-\tau,\tilde{t}]\cap[t_0,\tilde{t}], \mathbb{R}^q)$, respectively. Both x_{M}^1 and x_{M}^2 are solutions to the initial value problem (2.4) with initial value $\tilde{x}_{\mathrm{M}}$ and $\tilde{\mathbf{T}}_{\mathrm{M}}$ at initial time $\tilde{t}$ in the sense of Definition 2.2.6. According to Step 1, there exists $\varepsilon > 0$ such that $x_{\mathrm{M}}^1(t) = x_{\mathrm{M}}^2(t)$ for all $t \in [\tilde{t}, \tilde{t}+\varepsilon]$ – a contradiction to the definition of $\tilde{t}$. This completes the proof. □

Remark 2.2.10. Proposition 2.2.9 shows that if τ is greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, then the maximal solution $x_{\mathrm{M}} : [0,\omega) \to \mathbb{R}^{rm}$ of the initial value problem (2.4) with initial value $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ at time $\hat{t}$ is uniquely determined on the interval $[\hat{t}, \omega)$. Since we will consider solutions of the initial value problem (2.12) mostly for $t \geq \hat{t}$, we will speak in this case also of *the* maximal solution and *the* response associated with u when referring to $x_{\mathrm{M}}(\cdot; \hat{t}, \hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}, u)$. •

In the previous Section 2.1, we introduced the concept of funnel penalty function to be used in MPC. Basis for our considerations was the assumption of a Lipschitz continuous solution trajectory of the model (1.5). The following Proposition 2.2.11 shows that this assumption is justified and fulfilled for our model class $\mathcal{M}_{t_0}^{m,r}$.

Proposition 2.2.11. *Consider model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. Let $\hat{t} \geq t_0$, $\tau \in \mathbb{R}_{\geq 0}$, and initial data $\hat{x}_{\mathrm{M}} \in \mathcal{R}([\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}], \mathbb{R}^{rm})$ and $\hat{\mathbf{T}}_{\mathrm{M}} \in L^\infty_{\mathrm{loc}}([\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}], \mathbb{R}^q)$ such that for a control $u \in L^\infty_{\mathrm{loc}}([\hat{t},\infty), \mathbb{R}^m)$ the initial value problem (2.4) has a solution $x_{\mathrm{M}} : [0,\omega) \to \mathbb{R}^{rm}$ with $\omega > \hat{t}$ in the sense of Definition 2.2.6. Then, for every interval length $T \in (0, \omega-\hat{t})$, the restriction $x_{\mathrm{M}}|_{[\hat{t},\hat{t}+T]} : [\hat{t}, \hat{t}+T] \to \mathbb{R}^{rm}$ is a Lipschitz path.*

Proof. We prove the assertion by showing that every component $x_{\mathrm{M},i} : [0,\omega) \to \mathbb{R}^m$ of $x_\mathrm{M} = (x_{\mathrm{M},1},\dots,x_{\mathrm{M},r})$ for $i = 1,\dots,r$ is a Lipschitz continuous function on the interval $[\hat{t},\hat{t}+T]$. By Definition 2.2.6, the function $x_{\mathrm{M},i}$ is continuous on the interval $[\hat{t},\hat{t}+T]$. It therefore is sufficient to show that $\dot{x}_{\mathrm{M},i}$ is an essentially bounded function on the interval $[\hat{t},\hat{t}+T]$. For $i = 1,\dots,r-1$, we have $\dot{x}_{\mathrm{M},i} = x_{\mathrm{M},i+1}$ since x_M fulfils the ordinary differential equation (2.12) on the interval $[\hat{t},\hat{t}+T]$. Due to the compactness of $[\hat{t},\hat{t}+T]$, the continuous function $x_{\mathrm{M},i+1}$ is bounded. For $i = r$, we have

$$\dot{x}_{\mathrm{M},r}(t) = f_\mathrm{M}\big(\mathbf{T}_\mathrm{M}(x_\mathrm{M})(t)\big) + g_\mathrm{M}\big(\mathbf{T}_\mathrm{M}(x_\mathrm{M})(t)\big)u(t)$$

for almost all $t \in [\hat{t},\hat{t}+T]$. The control u as an element of $L^\infty_{\mathrm{loc}}([\hat{t},\infty),\mathbb{R}^m)$ is bounded. Moreover, due to the compactness of the considered interval and the continuity of the involved functions, $f_\mathrm{M}\big(\mathbf{T}_\mathrm{M}(x_\mathrm{M})(t)\big)$ and $g_\mathrm{M}\big(\mathbf{T}_\mathrm{M}(x_\mathrm{M})(t)\big)$ are bounded. Thus, $\dot{x}_{\mathrm{M},r}$ is essentially bounded and the proof is complete. □

2.3 MPC with funnel stage costs

In this section, we will analyse how funnel stage cost functions (introduced in Section 2.1) can be integrated into model predictive control (MPC) to solve the reference tracking problem formulated in Section 1.1.1. We employ a (functional) differential equation of the form (2.4), belonging to the model class $\mathcal{M}^{m,r}_{t_0}$ introduced in Section 2.2, as the predictive model within the MPC framework. To establish *initial and recursive feasibility* of the resulting MPC scheme and its compliance with the control objective, we develop the theoretical groundwork in three key steps:

1. **Relative degree analysis** (Section 2.3.1): We investigate the role of the model's relative degree $r \in \mathbb{N}$ in (2.4) and introduce auxiliary error signals to reduce the complexity of the considered control objective.

2. **Feasibility guarantees** (Section 2.3.2): We address the existence of control signals that solve the tracking problem at every iteration of the MPC Algorithm 1.1.4.

3. **Optimal control problem solvability** (Section 2.3.3): We prove that the optimal control problem using funnel stage costs admits a solution, which inherently satisfies the tracking objective.

While the existence of a solution may appear purely technical, it is non-trivial due to the inherent challenges of funnel stage costs: these functions are highly non-linear and generally discontinuous. Finally, in Section 2.4, we synthesise these results into the funnel MPC Algorithm 2.4.1, ensuring adherence to funnel constraints.

2.3.1 The higher relative degree

We now develop a control framework to address the reference tracking problem outlined in Section 1.1.1, accounting for the relative degree $r \in \mathbb{N}$ of the model (2.4). While the relative degree r might initially appear to be a minor technicality – and extending control strategies from $r = 1$ to $r > 1$ seemingly straightforward – the structural complexity introduced by higher relative degrees poses significant analytical and design challenges. This difficulty is well-documented: in adaptive control, these challenges were highlighted in [146]. Concerning funnel control, the progress was incremental. First proposed for relative degree $r = 1$ systems in 2002 in [95], it took eleven years to extend the framework to $r = 2$ in [83] and further five years to achieve generalisation for arbitrary $r \in \mathbb{N}$ in [33].

To meet the control objective in Section 1.1.1, we introduce auxiliary error variables, circumventing the structural limitations imposed by higher relative degrees. This approach simplifies the design while ensuring compatibility with the funnel stage cost functions discussed in Section 2.1. Define, for $(z_1, \dots, z_r) \in \mathbb{R}^{rm}$ with $z_i \in \mathbb{R}^m$ and for parameters $k_1, \dots, k_{r-1} \in \mathbb{R}_{\geq 0}$, the functions $\xi_i : \mathbb{R}^{rm} \to \mathbb{R}^m$ recursively by

$$\begin{aligned} \xi_1(z_1, \dots, z_r) &:= z_1, \\ \xi_{i+1}(z_1, \dots, z_r) &:= \xi_i(S_m(z_1, \dots, z_r)) + k_i \xi_i(z_1, \dots, z_r), \end{aligned} \tag{2.15}$$

for $i = 1, \dots, r-1$, where

$$S_m : \mathbb{R}^{rm} \to \mathbb{R}^{rm}, \; S_m(z_1, \dots, z_r) := (z_2, \dots, z_r, 0) \tag{2.16}$$

is the left shift operator.

Remark 2.3.1. Using the shorthand notation

$$\chi_r(\zeta)(t) := (\zeta(t), \dot{\zeta}(t), \dots, \zeta^{(r-1)}(t)) \in \mathbb{R}^{rm} \tag{2.17}$$

for a function $\zeta \in W^{r,\infty}(I, \mathbb{R}^m)$ on an interval $I \subset \mathbb{R}_{\geq 0}$ and $t \in I$, we get

$$\begin{aligned} \xi_1(\chi_r(\zeta)(t)) &= \zeta(t), \\ \xi_{i+1}(\chi_r(\zeta)(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}t} \xi_i(\chi_r(\zeta)(t)) + k_i \xi_i(\chi_r(\zeta)(t)) \end{aligned} \tag{2.18}$$

for $i = 1, \dots, r-1$. Furthermore, using the polynomials $p_i(s) = \prod_{j=1}^{i} (s + k_j) \in \mathbb{R}[s]$, the function $\xi_{i+1}(\chi_r(\zeta)(t))$ can be represented as

$$\xi_{i+1}(\chi_r(\zeta)(t)) = p_i(\tfrac{\mathrm{d}}{\mathrm{d}t})\zeta(t)$$

for $i = 1, \dots, r-1$. •

Observe that for a solution x_{M} of the model differential equation (2.4), the auxiliary error variable $\xi_1(x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t))$ coincides with the tracking error $e_{\mathrm{M}}(t) = y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$. Leveraging this equivalence, we solve the tracking problem outlined in Section 1.1.1 by ensuring $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t)$ remains within the set

$$\mathcal{D}_t^{\Psi} := \{ z \in \mathbb{R}^{rm} \mid \|\xi_i(z)\| < \psi_i(t), \; i = 1, \dots, r \} \tag{2.19}$$

for all $t \geq t_0$, where $\Psi := (\psi_1, \dots, \psi_r) \in \mathcal{G}^r$ is a vector of suitable funnel functions. While this approach initially appears to compound the original problem – replacing a single constraint with r time-variant inequalities – it simplifies the task when the funnel functions are strategically designed. Crucially, if a control $u \in L^\infty_{\mathrm{loc}}([\hat{t}, \infty), \mathbb{R}^m)$ applied to the model (2.4) ensures that $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t)$ is an element of the set $\mathcal{D}_t^{\Psi}$ for all $t \geq t_0$, then $x_{\mathrm{M}}(t)$ remains bounded. By Proposition 2.2.8, this guarantees that a maximal solution is indeed a global solution over $[t_0, \infty)$, i.e. it has no finite escape time. In contrast, control strategies that merely confine the tracking error $y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$ to $\mathcal{F}_\psi$ lack this inherent boundedness guarantee, as illustrated by the following example.

Example 2.3.2. The scalar differential equation

$$\ddot{y}(t) = 2\dot{y}^3 + u(t), \qquad y(0) = 0, \quad \dot{y}(0) = \tfrac{1}{2}$$

of order two belongs to the model class $\mathcal{M}_0^{1,2}$. If the constant control $u \equiv 0$ is applied to the differential equation, then the initial value problem has the unique maximal solution $y : [0, 1) \to \mathbb{R}$, $t \mapsto 1 - \sqrt{1-t}$. This constant control allows tracking of the constant reference $y_{\mathrm{ref}} \equiv 1$ within a funnel $\mathcal{F}_\psi$ given a constant funnel function $\psi \equiv 2$ because we have, for all $t \in [0, 1)$,

$$\|y(t) - y_{\mathrm{ref}}(t)\| = \left\|1 - \sqrt{1-t} - 1\right\| = \sqrt{1-t} < 2 = \psi(t).$$

However, the solution has finite escape time and cannot be extended to a global solution as the derivative $\dot{y}(t) = \frac{1}{2\sqrt{1-t}}$ is unbounded and has a pole at $t = 1$. ◇

To ensure that the tracking error $e_{\mathrm{M}}(t) = y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$ evolves within the funnel $\mathcal{F}_\psi$ (defined by a function $\psi \in \mathcal{G}$ as outlined in Section 1.1.1), we construct auxiliary funnel functions $\Psi := (\psi_1, \ldots, \psi_r) \in \mathcal{G}^r$ to simplify the tracking problem. A fundamental prerequisite is that the initial error – the mismatch between the model's initial trajectory y_{M}^0 from (2.4) and the reference trajectory y_{ref} – lies within the funnel, i.e. satisfies $\left\|y_{\mathrm{M}}^0(t_0) - y_{\mathrm{ref}}(t_0)\right\| < \psi(t_0)$. Under this condition, there exists $\gamma \in (0,1)$ such that

$$\left\|y_{\mathrm{M}}^0(t_0) - y_{\mathrm{ref}}(t_0)\right\| \le \gamma^r \psi(t_0). \tag{2.20}$$

Furthermore, the funnel function $\psi \in \mathcal{G}$ satisfies $\dot{\psi}(t) \ge -\alpha\psi(t) + \beta$ for all $t \ge 0$, where $\alpha, \beta > 0$ are constants with $\psi(t_0) \ge \frac{\beta}{\alpha}$, see also (1.3) for the definition of the set $\mathcal{G}$. For completeness, we briefly prove this existence result.

Lemma 2.3.3. *Let $\psi \in \mathcal{G}$, then there exists $\alpha, \beta > 0$ such that*

$$\psi(t_0) \ge \frac{\beta}{\alpha} \quad \text{and} \quad \dot{\psi}(t) \ge -\alpha\psi(t) + \beta \quad \forall\, t \ge 0. \tag{2.21}$$

Proof. We have $\inf_{s\ge 0} \dot{\psi}(s) \le 0$ due to the boundedness of ψ. In the case $\inf_{s\ge 0} \dot{\psi}(s) = 0$, set $\alpha := 1$ and $\beta := \inf_{s\ge 0} \psi(s) > 0$. Then, $\psi(t_0) \ge \frac{\beta}{\alpha}$ and, for all $t \ge 0$,

$$-\alpha\psi(t) + \beta = -\psi(t) + \inf_{s\ge 0} \psi(s) \le 0 \le \inf_{s\ge 0} \dot{\psi}(s) \le \dot{\psi}(t).$$

If $\inf_{s\ge 0} \dot{\psi}(s) < 0$, then set $\alpha := \frac{-\inf_{s\ge 0} \dot{\psi}(s)}{\frac{1}{2}\inf_{s\ge 0} \psi(s)} > 0$ and $\beta := -\inf_{s\ge 0} \dot{\psi}(s) > 0$. Then,

$$-\alpha\psi(t) + \beta = -\alpha\psi(t) + \tfrac{\alpha}{2} \inf_{s\ge 0} \psi(s) \le -\tfrac{\alpha}{2} \inf_{s\ge 0} \psi(s) \le \inf_{s\ge 0} \dot{\psi}(s) \le \dot{\psi}(t)$$

for all $t \ge 0$. Moreover, $\psi(t_0) \ge \frac{\beta}{\alpha}$. This completes the proof. □

Given $\gamma \in (0,1)$ as in (2.20), constants $\alpha, \beta > 0$ satisfying (2.21), and the initial time $t_0 \ge 0$, recursively select parameters $k_1, \ldots, k_{r-1}$ such that

$$\begin{aligned} k_1 &\ge \frac{2\left\|(\dot{y}_{\mathrm{M}}^0 - \dot{y}_{\mathrm{ref}})(t_0)\right\|}{\gamma^{r-1}(1-\gamma)\psi(t_0)} + \frac{2\left(\alpha + \frac{1}{\gamma^{r-1}}\right)}{1-\gamma}, \\ k_i &\ge \frac{2\gamma\left\|\frac{\mathrm{d}}{\mathrm{d}t}\xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))\right\|}{(1-\gamma)\left(\left\|\xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))\right\| + \frac{\beta}{\alpha\gamma^{i-2}}\right)} + \frac{2(1+\alpha)}{1-\gamma} \end{aligned} \tag{2.22}$$

for all $i = 2, \ldots, r-1$, where ξ_i are defined as in (2.15). Using the shorthand notation $\xi_i^0 := \xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))$, define the vector of auxiliary funnel functions Ψ as $\psi_1 := \psi$, and $\psi_2, \ldots, \psi_r$ as follows:

$$\psi_{i+1}(t) := \frac{1}{\gamma^{r-i}}\left(\left\|\dot{\xi}_i^0\right\| + k_i\left\|\xi_i^0\right\|\right) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma^{r-1}} \tag{2.23}$$

for $t \ge 0$ and $i = 1, \ldots, r-1$. Critically, the parameters k_i and functions ψ_i do *only* depend on $\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0)$, i.e. the value of $y_{\mathrm{M}}^0 - y_{\mathrm{ref}}$ and its derivatives at the initial time t_0, rather than the entire trajectories y_{M}^0 and y_{ref}. By construction of ψ_i and by observation (2.18), we have

$$\left\|\xi_i^0\right\| \le \left\|\dot{\xi}_{i-1}^0\right\| + k_{i-1}\left\|\xi_{i-1}^0\right\| < \psi_i(t_0)$$

for all $i = 2, \ldots, r$, and, by assumption (2.20),

$$\left\|\xi_1^0\right\| \le \gamma^r \psi(t_0) < \psi_1(t_0).$$

Therefore, $\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0)$ is an element of $\mathcal{D}_{t_0}^{\Psi}$ as defined in (2.19). Utilising the so constructed parameters k_i from (2.22) and auxiliary funnel functions ψ_i from (2.23), Proposition 2.3.4 establishes the following: If, at time $\hat{t} \geq t_0$, all auxiliary error variables lie within their respective funnels for a function $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$, i.e. $\chi_r(\zeta)(\hat{t}) \in \mathcal{D}_{\hat{t}}^{\Psi}$, and thereafter the last error variable $\xi_r(\chi_r(\zeta)(t))$ evolves within its funnel given by ψ_r, then *all* auxiliary error variables $\xi_i(\chi_r(\zeta)(t))$ remain within their respective funnels given by ψ_i for all $t \geq \hat{t}$. This has remarkable implications for the reference tracking problem from Section 1.1.1. If a control function $u \in L^{\infty}_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ is applied to the model (2.4) and achieves that $\|\xi_r(x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t))\| < \psi_r(t)$ for $t \geq \hat{t}$, then $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^{\Psi}$ for $t \geq \hat{t}$, assuming initially $x_{\mathrm{M}}(\hat{t}) - \chi_r(y_{\mathrm{ref}})(\hat{t}) \in \mathcal{D}_{\hat{t}}^{\Psi}$. This implies, in particular, $\|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$ for $t \geq \hat{t}$ because of the definitions of the set $\mathcal{D}_t^{\Psi}$ in (2.19), the error variables ξ_i in (2.15), and the function $\psi_1 = \psi$. In summary, a control u ensuring that the last auxiliary error variable $\xi_r(x_{\mathrm{M}} - \chi_r(y_{\mathrm{ref}}))$ evolves within the funnel $\mathcal{F}_{\psi_r}$ defined by ψ_r solves the reference tracking problem in Section 1.1.1.

Proposition 2.3.4. *For $\psi \in \mathcal{G}$ and $t_0 \geq 0$, let the parameters $k_i \geq 0$ be given for $i = 1 \ldots, r-1$ as in (2.22) and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathcal{G}^r$ be given as in (2.23). Further, let $\hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be such that $\chi_r(\zeta)(\hat{t}) \in \mathcal{D}_{\hat{t}}^{\Psi}$. If $\|\xi_r(\chi_r(\zeta)(t))\| < \psi_r(t)$ for all $t \in [\hat{t}, s)$ for some $s > \hat{t}$, then $\chi_r(\zeta)(t) \in \mathcal{D}_t^{\Psi}$ for all $t \in [\hat{t}, s)$.*

Proof. Seeking a contradiction, we assume that, for at least one $i \in \{1, \ldots, r-1\}$, there exists $t \in (\hat{t}, s)$ such that $\|\xi_i(\chi_r(\zeta)(t))\| \geq \psi_i(t)$. W.l.o.g. let i be the largest index with this property. In the following, we use the shorthand notation $\xi_i(t) := \xi_i(\chi_r(\zeta)(t))$ and $\xi_i^0 := \xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))$, as before in (2.23). However, we like to emphasise that $\xi_i(t_0) \neq \xi_i^0$ (if $\xi_i(\cdot)$ is defined at t_0) in general, since $\chi_r(\zeta)(t_0) \neq \chi_r(y^0)(t_0)$ is possible. Invoking $\|\xi_i(\hat{t})\| < \psi_i(\hat{t})$ and the continuity of the involved functions, define $t^\star := \min\left\{ t \in [\hat{t}, s) \,\middle|\, \|\xi_i(t)\| = \psi_i(t) \right\}$. Set $\varepsilon := \max\left\{ \sqrt{\tfrac{1}{2}(1+\gamma)}, \left\| \frac{\xi_i(\hat{t})}{\psi_i(\hat{t})} \right\| \right\} \in (0,1)$. Due to continuity of the involved functions, there exists $t_\star := \max\left\{ t \in [\hat{t}, t^\star) \,\middle|\, \left\| \frac{\xi_i(t)}{\psi_i(t)} \right\| = \varepsilon \right\}$. We have $\varepsilon \leq \left\| \frac{\xi_i(t)}{\psi_i(t)} \right\| \leq 1$ for all $t \in [t_\star, t^\star]$. Utilising (2.18) and omitting the dependency on t, we calculate for $t \in [t_\star, t^\star]$:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\xi_i}{\psi_i} \right\|^2 &= \left\langle \frac{\xi_i}{\psi_i}, \frac{\dot{\xi}_i \psi_i - \xi_i \dot{\psi}_i}{\psi_i^2} \right\rangle = \left\langle \frac{\xi_i}{\psi_i}, -\left(k_i + \frac{\dot{\psi}_i}{\psi_i} \right) \frac{\xi_i}{\psi_i} + \frac{\xi_{i+1}}{\psi_i} \right\rangle \\
&\leq -\left(k_i + \frac{\dot{\psi}_i}{\psi_i} \right) \left\| \frac{\xi_i}{\psi_i} \right\|^2 + \left\| \frac{\xi_i}{\psi_i} \right\| \frac{\|\xi_{i+1}\|}{\psi_i} \leq -\left(k_i + \frac{\dot{\psi}_i}{\psi_i} \right) \varepsilon^2 + \frac{\psi_{i+1}}{\psi_i},
\end{aligned}$$

where we used $\|\xi_{i+1}(t)\| \leq \psi_{i+1}(t)$ due to the maximality of i. Now, we distinguish the two cases $i = 1$ and $i > 1$. For $i = 1$, note that $\psi_1 = \psi$ and by properties of $\mathcal{G}$ it follows

$$-\frac{\dot{\psi}(t)}{\psi(t)} \leq \frac{\alpha \psi(t) - \beta}{\psi(t)} \leq \alpha.$$

Furthermore, we have that $\psi(t) \geq \left(\psi(t_0) - \frac{\beta}{\alpha} \right) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}$ for all $t \geq t_0$. Therefore,

$$\begin{aligned}
\frac{\psi_2(t)}{\psi(t)} &\leq \frac{1}{\gamma^{r-1}} \frac{\left(\left\| \dot{\xi}_1^0 \right\| + k_1 \left\| \xi_1^0 \right\| \right) \mathrm{e}^{-\alpha(t-t_0)}}{\left(\psi(t_0) - \frac{\beta}{\alpha} \right) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}} + \frac{\beta}{\alpha \gamma^{r-1} \left(\left(\psi(t_0) - \frac{\beta}{\alpha} \right) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha} \right)} \\
&\leq \frac{1}{\gamma^{r-1}} \frac{\left\| \dot{\xi}_1^0 \right\| + k_1 \left\| \xi_1^0 \right\|}{\psi(t_0)} + \frac{1}{\gamma^{r-1}} \leq \gamma k_1 + \frac{\left\| \dot{\xi}_1^0 \right\|}{\gamma^{r-1} \psi(t_0)} + \frac{1}{\gamma^{r-1}}
\end{aligned}$$

for all $t \geq t_0$, where the estimate $\|\xi_1^0\| \leq \gamma^r \psi(t_0)$ was used, see (2.20). Hence, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\xi_1}{\psi}\right\|^2 \leq -\tfrac{1}{2}(k_1-\alpha)(1+\gamma)+\gamma k_1+\frac{\|\dot{\xi}_1^0\|}{\gamma^{r-1}\psi(t_0)}+\frac{1}{\gamma^{r-1}}$$
$$\leq -\tfrac{1}{2}(1-\gamma)k_1+\alpha+\frac{\|\dot{\xi}_1^0\|}{\gamma^{r-1}\psi(t_0)}+\frac{1}{\gamma^{r-1}} \leq 0$$

for all $t \in [t_\star, t^\star]$, where the last inequality follows from (2.22). Now, consider the case $i > 1$. Then, we have $-\frac{\dot{\psi}_i(t)}{\psi_i(t)} \leq \alpha$ for all $t \geq t_0$ and, invoking that by (2.18)

$$\|\xi_i^0\| \leq \|\dot{\xi}_{i-1}^0\| + k_{i-1}\|\xi_{i-1}^0\|,$$

we find that

$$\frac{\psi_{i+1}(t)}{\psi_i(t)} = \frac{\frac{1}{\gamma^{r-i}}\left(\|\dot{\xi}_i^0\|+k_i\|\xi_i^0\|\right)\mathrm{e}^{-\alpha(t-t_0)}+\frac{\beta}{\alpha\gamma^{r-1}}}{\frac{1}{\gamma^{r-i+1}}\left(\|\dot{\xi}_{i-1}^0\|+k_{i-1}\|\xi_{i-1}^0\|\right)\mathrm{e}^{-\alpha(t-t_0)}+\frac{\beta}{\alpha\gamma^{r-1}}}$$
$$\leq \gamma\frac{\|\dot{\xi}_i^0\|+k_i\|\xi_i^0\|}{\|\dot{\xi}_{i-1}^0\|+k_{i-1}\|\xi_{i-1}^0\|+\frac{\beta}{\alpha\gamma^{i-2}}}+1 \leq \gamma k_i+\gamma\frac{\|\dot{\xi}_i^0\|}{\|\xi_i^0\|+\frac{\beta}{\alpha\gamma^{i-2}}}+1$$

for all $t \geq t_0$. Hence, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\xi_i}{\psi_i}\right\|^2 \leq -\tfrac{1}{2}(k_i-\alpha)(1+\gamma)+\gamma k_i+\gamma\frac{\|\dot{\xi}_i^0\|}{\|\xi_i^0\|+\frac{\beta}{\alpha\gamma^{i-2}}}+1$$
$$\leq -\tfrac{1}{2}(1-\gamma)k_i+\alpha+\gamma\frac{\|\dot{\xi}_i^0\|}{\|\xi_i^0\|+\frac{\beta}{\alpha\gamma^{i-2}}}+1 \leq 0$$

for all $t \in [t_\star, t^\star]$, where the last inequality follows from (2.22). Summarising, in each case the contradiction

$$1 \leq \left\|\frac{\xi_i(t^*)}{\psi_i(t^*)}\right\|^2 \leq \left\|\frac{\xi_i(t_*)}{\psi_i(t_*)}\right\|^2 = \varepsilon^2 < 1$$

arises, which completes the proof. □

The proof of Proposition 2.3.4 not only shows that all auxiliary error variables $\xi_i(\chi_r(\zeta)(t))$ stay within their respective funnels given by ψ_i for $i = 1, \ldots, r-1$, if the last error variable $\xi_r(\chi_r(\zeta)(t))$ evolves within its funnel given by ψ_r, but it moreover shows that the auxiliary error variables $\xi_i(\chi_r(\zeta)(t))$ always uphold an ε-distance to the funnel boundaries ψ_i. For systems with order $r > 1$, this means that the tracking error $y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$ fulfils

$$\|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \varepsilon\psi(t)$$

on every interval $[\hat{t}, s]$ with $s > \hat{t}$, assuming $\|y_{\mathrm{M}}(\hat{t}) - y_{\mathrm{ref}}(\hat{t})\| < \varepsilon\psi(\hat{t})$. We sum up this observation in the following.

Corollary 2.3.5. *For $\psi \in \mathcal{G}$ with $r > 1$, let the parameters $k_i \geq 0$ be given as in (2.22) for $i = 1\ldots, r-1$ and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathcal{G}^r$ be given as in (2.23). Further, let $s > \hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be given such that $\chi_r(\zeta)(t) \in \mathcal{D}_t^\Psi$ for all $t \in [\hat{t}, s]$. There exists $\varepsilon \in (0,1)$, independent of t, s, and ζ, such that if $\|\xi_1(\chi_r(\zeta)(\hat{t}))\| < \varepsilon\psi_1(\hat{t})$, then*

$$\|\xi_1(\chi_r(\zeta)(t))\| < \varepsilon\psi_1(t)$$

for all $t \in [\hat{t}, s]$.

Proof. Setting $\varepsilon := \sqrt{\frac{1}{2}(1+\gamma)}$, this is can be directly seen following the argument for ξ_1 of the proof of Proposition 2.3.4. □

The construction of the parameters k_i in (2.22) and the auxiliary funnel functions ψ_i in (2.23) assumed given and fixed initial trajectory y_{M}^0 for the model (2.4). These parameters and functions were tailored to enable the analysis in Proposition 2.3.4, requiring the funnel functions to be large enough to accommodate the initial errors values $\xi_i^0 := \xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))$ for $i = 1, \ldots, r$. However, this approach imposes intricate constraints on the parameters k_i and time-varying functions ψ_i, complicating their selection. Crucially, the initial trajectory y_{M}^0 – a modelling parameter for the model (2.4) of the original system (1.1) – often admits flexibility. By strategically choosing the function y_{M}^0, we simplify the design of k_i and ψ_i. We therefore present a simplified parameter design in the following. As before, we assume $\psi \in \mathcal{G}$ to be given with associated constants $\alpha, \beta > 0$ fulfilling (2.21). Define $\psi_1 := \psi$ and

$$\begin{array}{rcccl} k_1 = & \ldots & = & k_{r-1} & \geq \alpha + 2, \\ \psi_2(t) = & \ldots & = & \psi_r(t) & := \frac{\beta}{\alpha}. \end{array} \tag{2.24}$$

This yields the simplified constraints:

$$\left\| \xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0)) \right\| < \psi_i(t_0), \quad i = 1, \ldots, r$$

on the initial parameter y_{M}^0. The following Proposition 2.3.6 adapts Proposition 2.3.4 to this streamlined framework.

Proposition 2.3.6. *For $\psi \in \mathcal{G}$, let the parameters $k_i \geq \alpha + 2$ be given for $i = 1, \ldots, r-1$ and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathcal{G}^r$ be given as in (2.24). Let $\hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ such that $\chi_r(\zeta)(\hat{t}) \in \mathcal{D}_{\hat{t}}^{\Psi}$. If $\|\xi_r(\chi_r(\zeta)(t))\| < \frac{\beta}{\alpha}$ for all $t \in [\hat{t}, s)$, $s > \hat{t}$, then $\chi_r(\zeta)(t) \in \mathcal{D}_t^{\Psi}$ for all $t \in [\hat{t}, s)$.*

Proof. We modify the proof of Proposition 2.3.4 to the changed setting. Seeking a contradiction, we assume that there exists $t \in (\hat{t}, s)$ such that $\|\xi_i(\chi_r(\zeta)(t))\| \geq \psi_i(t)$ for at least one $i \in \{1, \ldots, r-1\}$. W.l.o.g. let i be the largest index with this property. We use the shorthand notation $\xi_i(t) := \xi_i(\chi_r(\zeta)(t))$. Define $\varepsilon := \max\left\{ \sqrt{\frac{1}{2}}, \left\| \frac{\xi_i(\hat{t})}{\psi_i(\hat{t})} \right\| \right\} \in (0,1)$. Invoking continuity of ξ_i, there exist time instants $t^\star := \min\left\{ t \in [\hat{t}, s] \,\middle|\, \|\xi_i(t)\| = \psi_i(t) \right\}$ and $t_\star := \max\left\{ t \in [\hat{t}, t^\star) \,\middle|\, \forall\, s \in [t, t^\star] : \|\xi_i(s)\| = \varepsilon \psi_i(s) \right\}$. We separately consider the two cases $i = 1$ and $i > 1$. First, we suppose $i = 1$. Then, note that $\psi(t) \geq \|\xi_1(t)\| \geq \varepsilon \psi(t)$ for all $t \in [t_\star, t^\star]$. By properties of $\psi \in \mathcal{G}$, we have

$$-\frac{\dot{\psi}(t)}{\psi(t)} \leq \frac{\alpha \psi(t) - \beta}{\psi(t)} \leq \alpha,$$

and $\psi(t) \geq \left(\psi(t_0) - \frac{\beta}{\alpha}\right) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha} \geq \frac{\beta}{\alpha}$ for all $t \geq t_0$. Omitting the dependency on t, we calculate for $t \in [t_\star, t^\star]$

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\xi_1}{\psi} \right\|^2 &= \left\langle \frac{\xi_1}{\psi}, \frac{\dot{\xi}_1 \psi - \xi_1 \dot{\psi}}{\psi^2} \right\rangle = \left\langle \frac{\xi_1}{\psi}, -\left(k_1 + \frac{\dot{\psi}}{\psi}\right) \frac{\xi_1}{\psi} + \frac{\xi_2}{\psi} \right\rangle \\ &\leq -\left(k_1 + \frac{\dot{\psi}}{\psi}\right) \left\| \frac{\xi_1}{\psi} \right\|^2 + \frac{\|\xi_1\| \|\xi_2\|}{\psi^2} \leq -\left(k_1 + \frac{\dot{\psi}}{\psi}\right) \frac{1}{2} + \frac{\alpha}{\beta} \|\xi_2\| \\ &\leq -(k_1 - \alpha) \frac{1}{2} + 1 \leq 0, \end{aligned}$$

where we used $k_1 \geq \alpha + 2$ and $\|\xi_2(t)\| \leq \frac{\beta}{\alpha}$ for all $t \in [t_\star, t^\star]$. Thus, upon integration, the contradiction

$$1 = \left\| \frac{\xi_1(t^\star)}{\psi(t^\star)} \right\|^2 \leq \left\| \frac{\xi_1(t_\star)}{\psi(t_\star)} \right\|^2 = \varepsilon^2 < 1$$

arises. Now, we consider the case $\|\xi_i(t)\| \geq \psi_i(t)$ for $i > 1$. By the choice of ε, we have $\beta/\alpha = \psi_i(t) \geq \|\xi_i(t)\| \geq \beta/(\alpha\sqrt{2})$ for all $t \in [t_\star, t^\star]$. Thus, we calculate

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\xi_i\|^2 = \langle \xi_i, \dot{\xi}_i\rangle = \langle \xi_i, -k_i\xi_i + \xi_{i+1}\rangle \leq -k_i\|\xi_i\|^2 + \|\xi_i\|\|\xi_{i+1}\| \leq \tfrac{\beta^2}{\alpha^2}\left(-\tfrac{k_i}{2} + 1\right) \leq 0$$

for $t \in [t_\star, t^\star]$, where we used $k_i \geq 2$ and that $\|\xi_{i+1}(t)\| \leq \frac{\beta}{\alpha}$ for all $t \in [t_\star, t^\star]$ by maximality of i. Hence, the contradiction

$$\tfrac{\beta^2}{\alpha^2} \leq \|\xi_i(t^*)\|^2 \leq \|\xi_i(t_*)\|^2 < \tfrac{\beta^2}{\alpha^2}$$

arises, which completes the proof. □

As in Proposition 2.3.4, the proof of Proposition 2.3.6 shows that the auxiliary error variables $\xi_i(\chi_r(\zeta)(t))$ always uphold an ε-distance to the funnel boundaries ψ_i. Similarly to Corollary 2.3.5, we get the following result for systems with order $r > 1$.

Corollary 2.3.7. *For $\psi \in \mathcal{G}$ with $r > 1$, let the parameters $k_i \geq \alpha + 2$ be given for $i = 1 \dots, r-1$ and $\Psi = (\psi_1, \dots, \psi_r) \in \mathcal{G}^r$ be given as in (2.24). Further, let $s > \hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be given such that $\chi_r(\zeta)(t) \in \mathcal{D}_t^\Psi$ for all $t \in [\hat{t}, s]$. There exists $\varepsilon \in (0,1)$, independent of t, s, and ζ, such that if $\left\|\xi_1(\chi_r(\zeta)(\hat{t}))\right\| < \varepsilon\psi_1(\hat{t})$, then*

$$\|\xi_1(\chi_r(\zeta)(t))\| < \varepsilon\psi_1(t)$$

for all $t \in [\hat{t}, s]$.

Proof. Setting $\varepsilon := \sqrt{\frac{1}{2}}$, this is can be directly seen following the argument for ξ_1 in the proof of Proposition 2.3.6. □

Remark 2.3.8. Building on Proposition 2.3.6, a low-complexity funnel controller for systems of higher-order was proposed in [5]. Contrary to prior works, this control approach eliminates the use of time-varying reciprocal penalty terms, replacing them with constant gains. The simpler controller design has the potential to mitigate numerical issues and enhance its practicality for real-world applications. •

The two presented parameters designs show that there exists a delicate interplay between the choice of initial trajectory y_{M}^0, the parameters $k_i \geq 0$, the associated error variables ξ_i from (2.15), and the corresponding auxiliary funnel functions ψ_i. Even though other parameter designs are conceivable, in the remaining part of this presented thesis, the error variables ξ_i are always defined as in (2.15), the vector $\Psi := (\psi_1, \dots, \psi_r) \in \mathcal{G}^r$ of funnel functions and the corresponding parameters k_i for $i = 1, \dots, r$ are always chosen either according to (2.22) and (2.23) or according to (2.24). We will use the abbreviated notation

$$\Psi \in \mathscr{G} \tag{2.25}$$

to refer to one of the presented cases for the parameter design.

Assumption 2.3.9. We will implicitly always assume that the initial auxiliary errors are within their respective funnels, i.e. $\left\|\xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t_0))\right\| < \psi_i(t_0)$ for $i = 1, \dots, r$.

Remark 2.3.10. Since the control problem is formulated merely for $t \geq t_0$, it is possible that $\left\|\xi_i(\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t))\right\| \geq \psi_i(t)$ for some $t \in [0, t_0)$ and some $i = 1, \dots, r$. To avoid treating this interval as a special case, we assume without loss of generality that $\chi_r(y_{\mathrm{M}}^0 - y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi$ for all $t \in [0, t_0]$ in the remaining part of this presented thesis. It is clear that this is no restriction on the construction of the parameters k_i and auxiliary funnel functions ψ_i in (2.24) since the initial trajectory y_{M}^0 is chosen in order to fit to these parameters. In the parameter design setting (2.22) and (2.23), the initial error values ξ_i^0 and $\dot{\xi}_i^0$ can be replaced by their respective suprema on the compact interval $[0, t_0]$ due to the continuity of the involved functions. •

The following Proposition 2.3.11 summarises the main observation made about the parameter construction, namely that the reference tracking problem from Section 1.1.1 can be solved by a control u that achieves that last auxiliary error variable $\xi_r(x_{\mathrm{M}} - \chi_r(y_{\mathrm{ref}}))$ evolves within the funnel $\mathcal{F}_{\psi_r}$ defined by ψ_r.

Proposition 2.3.11. *Let $\Psi \in \mathscr{G}$, $\hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be given such that $\chi_r(\zeta)(\hat{t}) \in \mathcal{D}^{\Psi}_{\hat{t}}$. If $\|\xi_r(\chi_r(\zeta)(t))\| < \psi_r(t)$ for all $t \in [\hat{t}, s)$ and for some $s > \hat{t}$, then $\chi_r(\zeta)(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, s)$.*

Proof. This is an immediate consequence of Proposition 2.3.4 and Proposition 2.3.6. □

Remark 2.3.12. Note that the results of Proposition 2.3.4, Proposition 2.3.6, and Proposition 2.3.11 also hold true if one allows for $\|\xi_r(\chi_r(\zeta)(t))\| \leq \psi_r(t)$. To be more precise: Let $\Psi \in \mathscr{G}$, $\hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be such that $\|\xi_i(\chi_r(\zeta)(t))\| < \psi_i(t)$ for all $i = 1, \ldots, r-1$. If $\|\xi_r(\chi_r(\zeta)(t))\| \leq \psi_r(t)$ for all $t \in [\hat{t}, s)$ for some $s > \hat{t}$, then $\|\xi_i(\chi_r(\zeta)(t))\| < \psi_i(t)$ for $t \in [\hat{t}, s)$ and all $i = 1, \ldots, r-1$. •

We will sum up the observations made in Corollary 2.3.5 and Corollary 2.3.7 in the following.

Corollary 2.3.13. *Let $r > 1$ and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$. There exists $\varepsilon \in (0, 1)$ with the following property. If, for $s > \hat{t} \geq t_0$, a function $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ fulfils $\chi_r(\zeta)(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, s]$ and $\left\|\xi_1(\chi_r(\zeta)(\hat{t}))\right\| < \varepsilon\psi_1(\hat{t})$, then*

$$\|\xi_1(\chi_r(\zeta)(t))\| < \varepsilon\psi_1(t)$$

for all $t \in [\hat{t}, s]$.

2.3.2 Feasible control signals

Prior to formulating the optimal control problem (OCP) with funnel stage cost functions (to be solved in the MPC Algorithm 1.1.4), we first address the issue of ensuring *initial and recursive feasibility*. In the previous Section 2.3.1, we derived sufficient conditions for a control $u \in L^{\infty}_{\mathrm{loc}}([\hat{t}, \infty), \mathbb{R}^m)$ to solve the reference tracking problem outlined in Section 1.1.1 when applied to the model (2.4) at time $\hat{t} \geq t_0$. However, the existence of such a control function – an essential prerequisite for the successful application of the MPC Algorithm 1.1.4 – remains unverified. We now establish sufficient conditions to guarantee this existence.

Suppose the MPC Algorithm 1.1.4 solves the reference tracking problem up to time $\hat{t} \in [t_0, \infty]$ ensuring $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t)$ is an element of the set $\mathcal{D}^{\Psi}_t$ (from (2.19)) for all $t \leq \hat{t}$, where x_{M} solves the model differential equation (2.4). The concatenated solution trajectory then belongs to the set

$$\mathcal{Y}^{\Psi}_{\hat{t}} := \left\{ \zeta \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm}) \,\middle|\, \forall\, t \in [0, \hat{t}] : \zeta(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}^{\Psi}_t \right\}. \tag{2.26}$$

The concatenated solution trajectory is a solution of the differential equation (2.4) in the sense of Definition 2.2.6 on the intervals of the form $[t_k, t_{k+1}]$ for $k \in \mathbb{N}_0$ and differentiable on these interval. However, it is, in general, in its entirety not a continuous function but merely a regulated function because of the potential non-continuous re-initialisation of the model in Step (a) of Algorithm 1.1.4. Note that, if the concatenated solution trajectory is only defined on a finite interval, then we implicitly assume a right extension of the solution as in Remark 2.2.5 (a) in the definition of $\mathcal{Y}^{\Psi}_{\hat{t}}$ in (2.26). During the execution of the MPC Algorithm 1.1.4, the initial values $\hat{x}_{\mathrm{M}}$ and $\hat{\mathbf{T}}_{\mathrm{M}}$ for the model (2.4) at time $\hat{t}$ as in Definition 2.2.6 are determined by the hitherto existing concatenated solution trajectory. Building on these considerations, we define in the following the set of *feasible initial values* for the model.

Definition 2.3.14 (Feasible initial values $\mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$)**.** *Let $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\Psi \in \mathscr{G}$, and $\tau \geq 0$. Using the notation $I^{\hat{t},\tau}_{t_0} := [\hat{t}-\tau, \hat{t}] \cap [t_0, \hat{t}]$, we define the set of* feasible initial values *for the model* (2.4) *at time $\hat{t} \geq t_0$ as*

$$\mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t}) := \left\{ (\hat{x}_{\rm M}, \hat{\mathbf{T}}_{\rm M}) \in \mathcal{R}(I^{\hat{t},\tau}_0, \mathbb{R}^{rm}) \times L^\infty_{\rm loc}(I^{\hat{t},\tau}_{t_0}, \mathbb{R}^q) \;\middle|\; \exists\, \zeta \in \mathcal{Y}^{\Psi}_{\hat{t}} : \begin{array}{r} \zeta|_{I^{\hat{t},\tau}_0} = \hat{x}_{\rm M}, \\ \mathbf{T}_{\rm M}(\zeta)|_{I^{\hat{t},\tau}_{t_0}} = \hat{\mathbf{T}}_{\rm M} \end{array} \right\}. \tag{2.27}$$

Remark 2.3.15. Note that $\mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ is, for all $\hat{t} \geq t_0$ and $\tau \geq 0$, never empty since $y_{\rm ref} \in \mathcal{Y}^{\Psi}_{\hat{t}}$ and, therefore, the pair $(\chi_r(y_{\rm ref})|_{I^{\hat{t},\tau}_0}, \mathbf{T}_{\rm M}(\chi_r(y_{\rm ref}))|_{I^{\hat{t},\tau}_{t_0}})$ is an element of $\mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$. Moreover, $(\chi_r(y^0_{\rm M})|_{I^{\hat{t},\tau}_0}, \mathbf{T}_{\rm M}(\chi_r(y^0_{\rm M}))|_{I^{\hat{t},\tau}_{t_0}}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ for $\hat{t} = t_0$ according to our assumption in Remark 2.3.10. In the remainder of this thesis, we use the notation $\hat{\mathfrak{X}} := (\hat{x}_{\rm M}, \hat{\mathbf{T}}_{\rm M}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ to refer to the initial values $\hat{x}_{\rm M}$ and $\hat{\mathbf{T}}_{\rm M}$ as a pair, since we will only rarely consider them independently of each other. •

Remark 2.3.16. We want to highlight that choosing a feasible initial value $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ at time $\hat{t}$ for the model (2.4) implies

$$x_{\rm M}(\hat{t}) - \chi_r(y_{\rm ref})(\hat{t}) \in \mathcal{D}^{\Psi}_{\hat{t}}.$$

This means that the tracking error $e_{\rm M}$ and all auxiliary error variables ξ_i as in (2.15) are within their respective funnels at time $\hat{t}$. •

Although the general assumption within this chapter is that the model-plant mismatch e as in (1.11) is always identical to zero, we already want to lay the fundamentals to allow for the initialisation of the model based on measurement data from the system (1.1), as in Step (a) of Algorithm 1.1.4. Therefore, we define an *initialisation strategy* as a function selecting a feasible initial value based on measurements $\hat{x}$ at time $\hat{t} \geq t_0$. In application, one will replace $\hat{x}$ with $\chi_r(y)(\hat{t})$ where y is the output of the system (1.1).

Definition 2.3.17 (Initialisation strategy)**.** *Let $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\Psi \in \mathscr{G}$, and $\tau \geq 0$. Using the notation $I^{\hat{t},\tau}_{t_0} := [\hat{t}-\tau, \hat{t}] \cap [t_0, \hat{t}]$, we call a function*

$$\kappa : \bigcup_{\hat{t} \geq t_0} \mathcal{R}(I^{\hat{t},\tau}_0, \mathbb{R}^{rm}) \to \bigcup_{\hat{t} \geq t_0} \mathcal{R}(I^{\hat{t},\tau}_0, \mathbb{R}^{rm}) \times L^\infty_{\rm loc}(I^{\hat{t},\tau}_{t_0}, \mathbb{R}^q)$$

*with $\kappa(\hat{x}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ for $\hat{x} \in \mathcal{R}(I^{\hat{t},\tau}_0, \mathbb{R}^{rm})$ and $\hat{t} \geq t_0$ a τ-*initialisation strategy *for the model* (2.4)*.*

In view of the limited memory property (T.4) of operator $\mathbf{T}_{\rm M}$, we always utilise a τ-initialisation strategy with τ being greater than or equal to the memory limit of $\mathbf{T}_{\rm M}$. In Definition 2.3.17, the domain of the initialisation strategy κ was chosen to be consistent with its codomain. However, this choice is entirely arbitrary. If deemed beneficial in a given setting, the domain can be adapted such that κ acts on signals defined on different time intervals of a different length, i.e. κ can be defined on $\bigcup_{\hat{t} \geq t_0} \mathcal{R}(I^{\hat{t},\tilde{\tau}}_0, \mathbb{R}^{rm})$ with $\tilde{\tau} \neq \tau$. Changing the domain of κ in such a way does not change the validity of the results presented.

Let a feasible initial value $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ be given for the model (2.4) at time $\hat{t}$. If a control $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$, bounded by some constant $u_{\max} \geq 0$, ensures that $x_{\rm M}(t) - \chi_r(y_{\rm ref})(t)$

evolves within $\mathcal{D}_t^\Psi$ for all t over the next time interval of length $T > 0$, then it is an element of the set

$$\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) := \left\{ u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \;\middle|\; \begin{array}{l} x_{\mathrm{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi \\ \text{for all } t \in [\hat{t}, \hat{t}+T], \|u\|_\infty \leq u_{\max} \end{array} \right\}. \tag{2.28}$$

We want to point out that, in the definition of the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ in (2.28), we implicitly assume that the solution $x_{\mathrm{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u)$ exists on the whole interval $[\hat{t}, \hat{t}+T]$. Due to the construction of funnel functions $\Psi \in \mathscr{G}$ in the previous Section 2.3.1, all functions $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ solve the outlined tracking problem from Section 1.1.1. To guarantee the initial and recursive feasibility of the MPC Algorithm 1.1.4, we therefore want to ensure that the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is always non-empty and that, if a control $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is applied to the model (2.4), then the model's state after applying this control can again be used as a feasible initial value for the model. We first address the latter question.

Theorem 2.3.18. *Consider model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. Let $\tau \geq 0$ be greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$, $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\Psi \in \mathscr{G}$, and $\hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t})$ for $\hat{t} \geq t_0$. Further, let $u_{\max} \geq 0$ and $T > 0$ such that $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$. If a control $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is applied to the model (2.4), then there exists a solution of the initial value problem (2.4) in the sense of Definition 2.2.6 on the interval $[0, \hat{t}+T]$ and every solution $x_{\mathrm{M}} : [0, \hat{t}+T] \to \mathbb{R}^m$ fulfils*

$$\forall\, \delta \in [0,T]\colon (x_{\mathrm{M}}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u)|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[0,\hat{t}+\delta]}, \mathbf{T}(x_{\mathrm{M}}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u))|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[t_0,\hat{t}+\delta]}) \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t}+\delta).$$

Proof. Let $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) = \hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t})$ be arbitrary but fixed. By Definition 2.3.14 there exists a function $\zeta \in \mathcal{Y}_{\hat{t}}^\Psi$ such that $\zeta|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\mathrm{M}}$ and $\mathbf{T}_{\mathrm{M}}(\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$. If a control $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is applied to the model (2.4), then there exists a solution of the initial value problem in the sense of Definition 2.2.6 on the interval $[\hat{t}, \hat{t}+T]$ due to the definition of the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ as in (2.28). Let $x_{\mathrm{M}} : [0, \hat{t}+T] \to \mathbb{R}^m$ be a solution of the initial value problem (2.12). Since x_{M} fulfils the initial conditions (2.11) with $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}})$, we have $\zeta|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\mathrm{M}} = x_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]}$ and $\mathbf{T}_{\mathrm{M}}(\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}} = \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]}$. Define the function $\tilde{\zeta} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ by

$$\tilde{\zeta}(t) = \begin{cases} x_{\mathrm{M}}(t), & t \in [\hat{t}, \hat{t}+T] \\ \zeta(t), & t \in \mathbb{R}_{\geq 0} \setminus [\hat{t}, \hat{t}+T], \end{cases} \tag{2.29}$$

which fulfils $\tilde{\zeta}|_{[\hat{t}-\tau,\hat{t}+\delta]\cap[0,\hat{t}+\delta]} = x_{\mathrm{M}}|_{[\hat{t}-\tau,\hat{t}+\delta]\cap[0,\hat{t}+\delta]}$ for all $\delta \in [0,T]$ and additionally $\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]}$. Since $\tau \geq 0$ is greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$, see property (T.4) in Definition 2.2.1, it follows that

$$\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[t_0,\hat{t}+\delta]} = \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[t_0,\hat{t}+\delta]}$$

for all $\delta \in [0,T]$. By choice of u, we have $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi$ for all $t \in [\hat{t}, \hat{t}+T]$. Hence, $\tilde{\zeta} \in \mathcal{Y}_{\hat{t}+T}^\Psi$. We therefore have

$$(x_{\mathrm{M}}|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[0,\hat{t}+\delta]}, \mathbf{T}(x_{\mathrm{M}})|_{[\hat{t}+\delta-\tau,\hat{t}+\delta]\cap[t_0,\hat{t}+\delta]}) \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t}+\delta)$$

for all $\delta \in [0,T]$. This completes the proof. ∎

Remark 2.3.19. Although Definition 2.3.14 was formulated for arbitrary $\tau \geq 0$, we utilised that τ was greater than or equal to the memory limit of $\mathbf{T}_{\mathrm{M}}$ in Theorem 2.3.18 in order to ensure that the image of the operator $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})$ is independent of the chosen left extension of x_{M}. The function $\tilde{\zeta}$ in (2.29) is, in general, only one of many possible extensions of x_{M}. •

Theorem 2.3.18 shows that the model's state, i.e. the state x_{M} in combination with value of the operator $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})$, is at any point during application of a control function $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ a feasible initial value (assuming $\hat{\mathfrak{X}}$ is a feasible initial value to begin with). This will be essential for the re-initialising of the model during the application of the MPC Algorithm 1.1.4. We now show that if $u_{\max} \geq 0$ is chosen large enough, then $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is non-empty. We first prove Lemma 2.3.20 showing that the functions $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0}$ are bounded for functions ζ evolving within $\mathcal{D}^{\Psi}_{t}$. As a consequence, the dynamics of the model (2.4) are bounded if a control is applied that ensures $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t)$ evolves within $\mathcal{D}^{\Psi}_{t}$.

Lemma 2.3.20. *Consider the model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0}$. Further, let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\Psi \in \mathscr{G}$. Then, there exist constants $f^{\max}_{\mathrm{M}}$, $g^{\max}_{\mathrm{M}}, g^{-1\max}_{\mathrm{M}} \geq 0$ such that for all $\hat{t} \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$:*

$$f^{\max}_{\mathrm{M}} \geq \left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})}) \right\|_{\infty}, \qquad g^{\max}_{\mathrm{M}} \geq \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})}) \right\|_{\infty},$$

and

$$g^{-1\max}_{\mathrm{M}} \geq \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})})^{-1} \right\|_{\infty}.$$

If the function g_{M} is, in addition, positive definite, i.e. $\langle z, g_{\mathrm{M}}(x) z\rangle > 0$ for all $x \in \mathbb{R}^q$ and all $z \in \mathbb{R}^m \setminus \{0\}$, then there exists $g^{\min}_{\mathrm{M}} > 0$ such that for all $z \in \mathbb{R}^m \setminus \{0\}$, all $\hat{t} \in [t_0, \infty]$, and $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$:

$$g^{\min}_{\mathrm{M}} \leq \frac{\left\langle z, g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\chi_r(\zeta))|_{[0,\hat{t})}(t)) z \right\rangle}{\|z\|^2}.$$

Proof. To prove the assertion, we invoke the continuity of the functions $f_{\mathrm{M}}, g_{\mathrm{M}}$ and the resulting boundedness on compact sets. By definition of $\mathcal{Y}^{\Psi}_{\infty}$ and $\mathcal{D}^{\Psi}_{t}$ in (2.26) and (2.19), we have for all $i = 1, \ldots, r$

$$\forall \zeta \in \mathcal{Y}^{\Psi}_{\infty} \forall t \geq 0 : \quad \|\xi_i(\zeta(t) - \chi_r(y_{\mathrm{ref}})(t))\| < \psi_i(t).$$

Due to the definition of the error variables ξ_i in (2.15) there exists an invertible matrix $S \in \mathbb{R}^{rm \times rm}$ such that

$$\begin{pmatrix} \xi_1(\zeta - \chi_r(y_{\mathrm{ref}})) \\ \vdots \\ \xi_r(\zeta - \chi_r(y_{\mathrm{ref}})) \end{pmatrix} = S(\zeta - \chi_r(y_{\mathrm{ref}})). \tag{2.30}$$

Hence, by boundedness of ψ_i and $y^{(i)}_{\mathrm{ref}}$ for all $i = 1, \ldots, r$, there exists a compact set $K \subset \mathbb{R}^{rm}$ with

$$\forall \zeta \in \mathcal{Y}^{\Psi}_{\infty} \forall t \geq 0 : \quad \zeta(t) \in K. \tag{2.31}$$

Invoking the BIBO property of the operator $\mathbf{T}_{\mathrm{M}}$, there exists a compact set $K_q \subset \mathbb{R}^q$ with $\mathbf{T}_{\mathrm{M}}(z)(\mathbb{R}_{\geq 0}) \subset K_q$ for all $z \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $z(\mathbb{R}_{\geq 0}) \subset K$. For arbitrary $\hat{t} \in (0, \infty)$ and $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$, we have $\zeta(t) \in K$ for all $t \in [0, \hat{t})$. For every element $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$, the restriction $\zeta|_{[0,\hat{t})}$ can be extended to a function $\tilde{\zeta} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $\tilde{\zeta}(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$.

We have $\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})(t) \in K_q$ for all $t \in \mathbb{R}_{\geq 0}$ because of the BIBO property of the operator $\mathbf{T}_{\mathrm{M}}$. This implies $\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})}(t) \in K_q$ for all $t \in [0,\hat{t})$ and $\zeta \in \mathcal{Y}_{\hat{t}}^{\Psi}$ since $\mathbf{T}_{\mathrm{M}}$ is causal. Since $f_{\mathrm{M}}(\cdot)$, $g_{\mathrm{M}}(\cdot)$, and $g_{\mathrm{M}}^{-1}(\cdot)$ are continuous, the constants $f_{\mathrm{M}}^{\max} = \max_{z \in K_q} \|f_{\mathrm{M}}(z)\|$, $g_{\mathrm{M}}^{\max} = \max_{z \in K_q} \|g_{\mathrm{M}}(z)\|$ and $g_{\mathrm{M}}^{-1\max} = \max_{z \in K_q} \|g_{\mathrm{M}}(z)^{-1}\|$ are well-defined. For all $\hat{t} \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_{\hat{t}}^{\Psi}$, we have

$$\forall\, t \in [0,\hat{t}): \ \mathbf{T}_{\mathrm{M}}(\zeta)(t) \in K_q.$$

Furthermore, if $g_{\mathrm{M}}(x)$ is positive definite for every $x \in K_q$, then there exists $g_{\mathrm{M}}^{\min} > 0$ such that $g_{\mathrm{M}}^{\min} \leq \frac{\langle z, g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\chi_r(\zeta)))|_{[0,\delta)}(t))z\rangle}{\|z\|^2}$ for all $z \in \mathbb{R}^m \setminus \{0\}$, which proves the assertion. □

To prove the existence of $g_{\mathrm{M}}^{\min} > 0$ in Lemma 2.3.20, it is assumed that g_{M} is positive definite. In general, this is not the case when considering a model $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. However, we will restrict the model class $\mathcal{M}_{t_0}^{m,r}$ and utilise this result in Chapter 5.

We are now in the position to prove the existence of a sufficiently large $u_{\max} \geq 0$ such that the set of controls $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is non-empty.

Theorem 2.3.21. *Consider model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. Let $\tau \geq 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$. Further, let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\Psi \in \mathscr{G}$. Then, there exists $u_{\max} \geq 0$ such that, for $\hat{t} \geq t_0$, $\hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$, and all $T > 0$, we have*

$$\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset. \tag{2.32}$$

Proof. Step 1: We define a candidate value of $u_{\max} \geq 0$. To that end, define, for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$

$$\mu_i^0 := \|\psi_i\|_\infty, \quad \mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j,$$

where $k_i \geq 0$, for $i = 1, \ldots r-1$, are the to Ψ associated constants, which are also used to define the error variables ξ_i as in (2.15). Using the constants $f_{\mathrm{M}}^{\max}$ and $g_{\mathrm{M}}^{-1\max}$ from Lemma 2.3.20, define

$$u_{\max} := g_{\mathrm{M}}^{-1\max}\left(f_{\mathrm{M}}^{\max} + \left\|y_{\mathrm{ref}}^{(r)}\right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\|\dot{\psi}_r\right\|_\infty\right).$$

Step 2: Let $T > 0$, $\hat{t} \geq t_0$, and $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) = \hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$ be arbitrary but fixed. We construct a control function u and show that $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. To this end, for some $u \in L^\infty([\hat{t},\hat{t}+T], \mathbb{R}^m)$, we use the shorthand notation $x_{\mathrm{M}}(t) := x_{\mathrm{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u)$ and $\xi_i(t) := \xi_i(x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t))$ for $i = 1, \ldots, r$. The application of the feedback control

$$u(t) := g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t))^{-1}\left(-f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t)) + y_{\mathrm{ref}}^{(r)}(t) - \sum_{j=1}^{r-1} k_j \xi_j^{(r-j)}(t) + \xi_r(t)\frac{\dot{\psi}_r(t)}{\psi_r(t)}\right)$$

to the system (2.4) leads to a closed-loop system. If this initial value problem is considered on the interval $[\hat{t},\hat{t}+T]$ with initial conditions $(\hat{t},\hat{\mathfrak{X}})$ as in (2.11), then an application of Proposition 2.2.8 yields the existence of a maximal solution $x_{\mathrm{M}} : [0,\omega) \to \mathbb{R}^{rm}$ with $\omega > \hat{t}$ in the sense of Definition 2.2.6. If x_{M} is bounded, then $\omega = \infty$, see Proposition 2.2.8 (iii). In this case, the solution exists on $[0,\hat{t}+T]$. Utilising (2.18), one can show by induction that

$$\xi_r(t) = \xi_1^{(r-1)}(t) + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j-1)}(t).$$

Omitting the dependency on t, we calculate for $t \in [\hat{t}, \omega)$:

$$\begin{aligned}\frac{\dot{\xi}_r\psi_r - \xi_r\dot{\psi}_r}{\psi_r} &= \xi_1^{(r)} + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j)} - \xi_r \frac{\dot{\psi}_r}{\psi_r} \\ &= f_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})) + g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M}))u - y_{\rm ref}^{(r)} + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j)} - \xi_r \frac{\dot{\psi}_r}{\psi_r} = 0.\end{aligned}$$

Therefore,

$$\frac{\rm d}{{\rm d}t}\frac{1}{2}\left\|\frac{\xi_r}{\psi_r}\right\|^2 = \left\langle \frac{\xi_r}{\psi_r}, \frac{\dot{\xi}_r\psi_r - \xi_r\dot{\psi}_r}{\psi_r^2}\right\rangle = 0.$$

We have $\left\|\frac{\xi_r(\hat{t})}{\psi_r(\hat{t})}\right\| < 1$ by the assumption $x_{\rm M}(\hat{t}) - \chi_r(y_{\rm ref})(\hat{t}) \in \mathcal{D}_{\hat{t}}^{\Psi}$, see also Remark 2.3.16. This yields $\left\|\frac{\xi_r(t)}{\psi_r(t)}\right\| < 1$ for all $t \in [\hat{t}, \omega)$. This implies, according to Proposition 2.3.11, $x_{\rm M}(t) - \chi_r(y_{\rm ref})(t) \in \mathcal{D}_t^{\Psi}$ for all $t \in [\hat{t}, \omega)$, i.e. $\|\xi_i(t)\| < \psi_i(t)$ for all $i = 1, \ldots, r$. Thus, $\|\xi_i(t)\| \leq \mu_i^0$ for all $i = 1, \ldots, r$. Invoking boundedness of $y_{\rm ref}^{(i)}$, $i = 0, \ldots, r$, and the relation in (2.30), we may infer that $x_{\rm M}$ is bounded on $[\hat{t}, \omega)$. Hence, $\omega = \infty$. Since $(\hat{x}_{\rm M}, \hat{\mathbf{T}}_{\rm M}) = \hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$, there exists a function $\zeta \in \mathcal{Y}_{\hat{t}}^{\Psi}$ such that $\zeta|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\rm M}$ and $\mathbf{T}_{\rm M}(\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\rm M}$. Moreover, as $x_{\rm M}$ fulfils the initial conditions (2.11), we have $x_{\rm M}(t)|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\rm M}$ and $\mathbf{T}_{\rm M}(x_{\rm M})|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\rm M}$. Define the regulated function $\tilde{\zeta} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ by

$$\tilde{\zeta}(t) = \begin{cases} x_{\rm M}(t), & t \in [\hat{t}, \hat{t} + T] \\ \zeta(t), & t \in \mathbb{R}_{\geq 0} \backslash [\hat{t}, \hat{t} + T]. \end{cases}$$

The function $\tilde{\zeta}$ is an element of $\mathcal{Y}_{\hat{t}+T}^{\Psi}$ because $\zeta \in \mathcal{Y}_{\hat{t}}^{\Psi}$ and $x_{\rm M}(t) - \chi_r(y_{\rm ref})(t) \in \mathcal{D}_t^{\Psi}$ for all $t \in [\hat{t}, \hat{t}+T]$. Therefore, $\left\|f_{\rm M}(\mathbf{T}_{\rm M}(\tilde{\zeta})(t))\right\| \leq f_{\rm M}^{\max}$ and $\left\|g_{\rm M}(\mathbf{T}_{\rm M}(\tilde{\zeta})(t))^{-1}\right\| \leq g_{\rm M}^{-1\max}$ for all $t \in [\hat{t}, \hat{t}+T]$ according to Lemma 2.3.20. Since $\tau \geq 0$ is greater than or equal to the memory limit of operator $\mathbf{T}_{\rm M}$, we have

$$\mathbf{T}_{\rm M}(x_{\rm M})(t) = \mathbf{T}_{\rm M}(\tilde{\zeta})(t)$$

for all $t \in [\hat{t}, \hat{t}+T]$. Thus, $\|f_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))\| \leq f_{\rm M}^{\max}$ and $\left\|g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))^{-1}\right\| \leq g_{\rm M}^{-1\max}$ for all $t \in [\hat{t}, \hat{t}+T]$. Finally, using (2.18) and the definition of μ_i^j, it follows that

$$\left\|\xi_i^{(j+1)}(t)\right\| = \left\|\xi_{i+1}^{(j)}(t) - k_i\xi_i^{(j)}(t)\right\| \leq \mu_{i+1}^j + k_i\mu_i^j = \mu_i^{j+1}$$

inductively for all $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$. Thus, by definition of u and $u_{\max}$, we have $\|u\|_\infty \leq u_{\max}$ and hence $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. □

Remark 2.3.22. Note that $u_{\max} \geq 0$ in Theorem 2.3.21 is independent of the time $\hat{t} \geq t_0$, the initial value $\hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$, and the considered time horizon $T > 0$. It solely depends on the system dynamics, i.e. the functions $(f_{\rm M}, g_{\rm M}, \mathbf{T}_{\rm M}) \in \mathcal{M}_{t_0}^{m,r}$, the reference $y_{\rm ref}$, the funnel functions $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$, and the associated parameters k_i, $i = 1, \ldots, r-1$. •

We have seen in this section that at there exists a control function $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ at every time $\hat{t} \geq t_0$, assuming that the initial value $\hat{\mathfrak{X}}$ is feasible for the model (2.4) and that $u_{\max} \geq 0$ is large enough. Such a control function solves the tracking problem formulated in Section 1.1.1 for the model (2.4). Moreover, it was shown that the state of the model at every time during application of the control u is a feasible initial state for the model. Since the initial value given by $y_{\rm M}^0$ is feasible for the model at time $\hat{t} = t_0$, this means that an iterative application of controls $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ to the model (2.4) solves the tracking problem for the concatenated solution.

2.3.3 Optimal control problem

We are now in the position to address the problem of solving the tracking problem from Section 1.1.1 by means of an optimal control problem utilising the concept of funnel stage cost functions from Section 2.1. Given auxiliary funnel functions $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ around the reference trajectory $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and the corresponding error variables ξ_i for $i = 1, \ldots, r$ as in (2.15), we saw in Section 2.1 that it is sufficient to ensure that the last auxiliary error ξ_r evolves within its funnel given by ψ_r in order to guarantee that all ξ_i evolve within their respective funnels defined by ψ_i for $i = 1, \ldots, r-1$. Therefore, choose a funnel stage cost ℓ_{ψ_r} for the last auxiliary funnel function ψ_r. Let $\tau \geq 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_{\text{M}}$. Then, define for $T > 0$, $\hat{t} \geq t_0$, and $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, the *cost functional* $J^{\Psi}_T(\cdot; \hat{t}, \hat{\mathfrak{X}}) : L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$ by

$$J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) := \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(s, \xi_r(x_{\text{M}}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\text{ref}})(s)), u(s)) \mathrm{d}s. \tag{2.33}$$

Although it is known that, for every $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$, there exists a maximal solution $x_{\text{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u)$, according to Proposition 2.2.8, this solution might have finite escape time, i.e. $\omega < \hat{t} + T$. In this case, and whenever the Lebesgue integral in (2.33) does not exist, (i.e. both the Lebesgue integrals of the positive and negative part of $\ell_{\psi_r}(s, \xi_r(x_{\text{M}}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\text{ref}})(s)), u(s))$ are infinite), then its value is treated as infinity. Further, note that the solution $x_{\text{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u)$ is unique on the interval $[\hat{t}, \hat{t}+T]$, according to Proposition 2.2.9, rendering $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ well-defined. In the following, we will study properties of $J^{\Psi}_T(\cdot; \hat{t}, \hat{\mathfrak{X}})$ a bit more closely and analyse the associated *Optimal Control Problem (OCP)*

$$\underset{\substack{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m), \\ \|u\|_\infty \leq u_{\max}}}{\text{minimise}} \quad J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}), \tag{2.34}$$

where $u_{\max} \geq 0$ is a bound on the maximal control input. We will prove that the OCP (2.34) has a solution and that this solution is an element of $\mathcal{U}_{[\hat{t}, \hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. Thus, it solves the tracking problem from Section 1.1.1, according to our considerations in Section 2.3.2.

The cost function $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ is defined in (2.33) as the integral of funnel stage cost ℓ_{ψ_r} evaluated over the auxiliary error ξ_r. The concept of funnel stage cost functions from Section 2.1 was based on the usage of Lipschitz paths. It has already been proven in Proposition 2.2.11 that the solution trajectories of the model are Lipschitz continuous. It is evident that this is also the case for the auxiliary errors ξ_i. Nevertheless, we will briefly formalise this.

Proposition 2.3.23. *Consider model* (2.4) *with* $(f_{\text{M}}, g_{\text{M}}, \mathbf{T}_{\text{M}}) \in \mathcal{M}^{m,r}_{t_0}$ *with reference trajectory* $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$. *Let* $\Psi \in \mathscr{G}$, $\hat{t} \geq t_0$, $\tau \in \mathbb{R}_{\geq 0}$, *and* $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$. *Moreover, let* $u \in L^\infty_{\text{loc}}([\hat{t}, \infty), \mathbb{R}^m)$ *be a control such that initial value problem* (2.4) *has a solution* $x_{\text{M}} : [0, \omega) \to \mathbb{R}^{rm}$ *with* $\omega > \hat{t}$ *in the sense of Definition 2.2.6. Then, for every* $T \in (0, \omega - \hat{t})$, *the restriction* $\xi_i(x_{\text{M}} - \chi_r(y_{\text{ref}}))|_{[\hat{t}, \hat{t}+T]} : [\hat{t}, \hat{t}+T] \to \mathbb{R}^m$ *is a Lipschitz path for all* $i = 1, \ldots, r$.

Proof. For $y_{\text{ref}} \in W^{1,\infty}$, the function $\chi_r(y_{\text{ref}})(\cdot)$ is a Lipschitz continuous on every compact interval. Therefore, due to the definition of the error variables ξ_i for $i = 1, \ldots, r$ in (2.15), the statement of Proposition 2.3.23 is an immediate consequence of Proposition 2.2.11. □

The following Theorem 2.3.24 not only shows that $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ has a finite value for all $u \in \mathcal{U}_{[\hat{t}, \hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, which is to be expected since this is the set of controls ensuring the evolution of the auxiliary errors ξ_i within their respective funnels, see (2.28), but that $\mathcal{U}_{[\hat{t}, \hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is in fact the set of controls for which $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ is finite.

Theorem 2.3.24. *Consider model (2.4) with* $(f_{\rm M}, g_{\rm M}, \mathbf{T}_{\rm M}) \in \mathcal{M}^{m,r}_{t_0}$ *with reference trajectory* $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$*. Let* $\Psi \in \mathscr{G}$ *and* $\tau \geq 0$ *be greater than or equal to the memory limit of operator* $\mathbf{T}_{\rm M}$*. Further, let* $\hat{t} \geq t_0$*,* $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$*,* $T > 0$*, and* $u_{\max} \geq 0$ *such that* $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$*. Then, the following identity holds:*

$$\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) = \left\{ u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \,\middle|\, J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) < \infty,\ \|u\|_\infty \leq u_{\max} \right\}.$$

Proof. Since ℓ_{ψ_r} is a funnel stage cost, it has the form

$$\ell_{\psi_r} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}, \quad (t, z, u) \mapsto \nu_{\psi_r}(t, z) + \lambda_u \|u\|^2,$$

where ν_{ψ_r} is a funnel penalty function and $\lambda_u \geq 0$, see Definition 2.1.11. Note that, for $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ such that $x_{\rm M}(s; \hat{t}, \hat{\mathfrak{X}}, u)$ satisfies (2.12) for all $s \in [\hat{t}, \hat{t}+T]$, the function $\xi_r(s) := \xi_r(x_{\rm M}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\rm ref})(s)$ is, according to Proposition 2.3.23, a Lipschitz path with $(\hat{t}, \xi_r(\hat{t})) \in \mathcal{F}_{\psi_r}$ since $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, see Remark 2.3.16.

For $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, we have $\|u\|_\infty \leq u_{\max}$ and $x_{\rm M}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\rm ref})(s) \in \mathcal{D}^{\Psi}_t$ for all $s \in [\hat{t}, \hat{t}+T]$. In particular, this implies $(s, \xi_r(s)) \in \mathcal{F}_{\psi_r}$ for all $s \in [\hat{t}, \hat{t}+T]$. Thus,

$$J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) = \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(s, \xi_r(s), u(s)) \mathrm{d}s = \int_{\hat{t}}^{\hat{t}+T} \nu_{\psi_r}(s, \xi_r(s)) + \lambda_u \|u(s)\|^2 \mathrm{d}s < \infty$$

due to the boundedness of u and the property (F.1) of the funnel penalty function ν_{ψ_r}, see Definition 2.1.6.

Conversely, let $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ with $\|u\|_\infty \leq u_{\max}$ such that the cost functional $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) = \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(s, \xi_r(s), u(s)) \mathrm{d}s$ is finite. Since u is bounded and

$$\ell_{\psi_r}(s, \xi_r(s), u(s)) = \nu_{\psi_r}(s, \xi_r(s)) + \lambda_u \|u(s)\|$$

for all $s \in [\hat{t}, \hat{t}+T]$, we have $\int_{\hat{t}}^{\hat{t}+T} \nu_{\psi_r}(s, \xi_r(s)) \mathrm{d}s < \infty$. Thus, $\|\xi_r(s)\| < \psi_r(s)$ for all $s \in [\hat{t}, \hat{t}+T]$ because the property (F.1) of the funnel penalty function ν_{ψ_r}, see Definition 2.1.6. This implies $x_{\rm M}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\rm ref})(s) \in \mathcal{D}^{\Psi}_t$ for all $s \in [\hat{t}, \hat{t}+T]$ according to Proposition 2.3.11. Hence, $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. □

Remark 2.3.25. The following statements hold under the assumptions of Theorem 2.3.24:

(a) $0 \leq J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) < \infty$ for all $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ because the funnel stage cost ℓ_{ψ_r} is non-negative while the error ξ_r evolves within its funnel given by ψ_r, see Remark 2.1.12.

(b) The optimal control problem (2.34) can be reformulated as

$$\operatorname*{minimise}_{u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})} \quad J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}).$$

•

If the initial value $\hat{\mathfrak{X}}$ is feasible for the model (2.4), then any control function u with $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) < \infty$ guarantees that, if applied to the model (2.4), all errors ξ_i remain (strictly) within their respective funnels ψ_i, $i = 1, \ldots, r$. Since $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ is non-negative for all control functions $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, this raises the question as to whether there exists an optimal $u^\star$ which minimises $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ and is a solution to the optimal control problem (2.34). The answer is affirmative and shown in the next theorem.

Theorem 2.3.26. *Consider model (2.4) with* $(f_{\rm M}, g_{\rm M}, \mathbf{T}_{\rm M}) \in \mathcal{M}^{m,r}_{t_0}$ *with reference trajectory* $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$*. Let* $\Psi \in \mathscr{G}$ *and* $\tau \geq 0$ *be greater than or equal to the memory limit of operator* $\mathbf{T}_{\rm M}$*. Further, let* $\hat{t} \geq t_0$*,* $(\hat{x}_{\rm M}, \hat{\mathbf{T}}_{\rm M}) = \hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$*,* $T > 0$*, and* $u_{\max} \geq 0$ *such that* $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$*. Then, there exists a function* $u^\star \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ *such that*

$$J^{\Psi}_T(u^\star; \hat{t}, \hat{\mathfrak{X}}) = \min_{u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}})} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) = \min_{\substack{u \in L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m), \\ \|u\|_\infty \leq u_{\max}}} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}).$$

Proof. The proof essentially follows the lines of [162, Prop. 2.2].
It follows from Remark 2.3.25 that $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) \geq 0$ for all $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. Hence, the infimum $J^\star := \inf_{u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}})} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ exists. Let $(u_k) \in \left(\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})\right)^{\mathbb{N}}$ be a minimising sequence, meaning $J^{\Psi}_T(u_k; \hat{t}, \hat{\mathfrak{X}}) \to J^\star$. By definition of $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, we have $\|u_k\|_\infty \leq u_{\max}$ for all $k \in \mathbb{N}$. Since $L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \subseteq L^2([\hat{t}, \hat{t}+T], \mathbb{R}^m)$, we conclude that (u_k) is a bounded sequence in the Hilbert space L^2. Thus, there exists a function $u^\star \in L^2([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ and a weakly convergent subsequence $u_k \rightharpoonup u^\star$ (which we do not relabel). More precisely, $u_k|_{[\hat{t},t]} \rightharpoonup u^\star|_{[\hat{t},t]}$ weakly in $L^2([\hat{t}, t], \mathbb{R}^m)$ for all $t \in [\hat{t}, \hat{t}+T]$ as a straightforward argument shows. We define $(x_k) := \left(x_{\rm M}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u_k)\right) \in \mathcal{R}([0, \hat{t}+T], \mathbb{R}^n)^{\mathbb{N}}$ as the sequence of associated responses. Note that, although we are only considering the optimal control problem (2.33) on the interval $[\hat{t}, \hat{t}+T]$, the functions $x_{\rm M}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u_k)$ are defined on the entire interval $[0, \hat{t}+T]$ for all $k \in \mathbb{N}$ as they are solutions of the differential equation (2.4) in the sense of Definition 2.2.6. This allows us to formally evaluate $\mathbf{T}_{\rm M}(x_k)$. We show the assertion of Theorem 2.3.26 in the following seven steps.
Step 1: We construct a uniformly bounded sequence of solutions of the differential equation (2.4) in the sense of Definition 2.2.6. Since $(\hat{x}_{\rm M}, \hat{\mathbf{T}}_{\rm M}) = \hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, there exists a function $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$ such that $\zeta|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\rm M}$ and $\mathbf{T}_{\rm M}(\zeta)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\rm M}$. Moreover, we have $x_k|_{[\hat{t}-\tau,\hat{t}]\cap[0,\hat{t}]} = \hat{x}_{\rm M}$ and $\mathbf{T}_{\rm M}(x_k)|_{[\hat{t}-\tau,\hat{t}]\cap[t_0,\hat{t}]} = \hat{\mathbf{T}}_{\rm M}$ because x_k fulfils the initial conditions (2.11). Define the function $\tilde{x}_k \in \mathcal{R}([0, \hat{t}+T], \mathbb{R}^{rm})$ by

$$\tilde{x}_k(t) = \begin{cases} \zeta(t), & t \in [0, \hat{t}) \\ x_k(t), & t \in [\hat{t}, \hat{t}+T]. \end{cases} \tag{2.35}$$

The function $\tilde{x}_k$ is, by construction, a solution of the differential equation (2.12) with initial values $\hat{\mathfrak{X}}$ in the sense of Definition 2.2.6. We have $\tilde{x}_k|_{[\hat{t},\hat{t}+T]} = x_k$ and additionally $\mathbf{T}_{\rm M}(x_k)|_{[\hat{t},\hat{t}+T]} = \mathbf{T}_{\rm M}(\tilde{x}_k)|_{[\hat{t},\hat{t}+T]}$ because $\tau \geq 0$ is greater than or equal to the memory limit of operator $\mathbf{T}_{\rm M}$. Without loss of generality, we therefore assume in the following that x_k has the form (2.35), i.e. we relabel $\tilde{x}_k$ as x_k. By $u_k \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, we have $x_k(t) - \chi_r(y_{\rm ref})(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, \hat{t}+T]$. Thus, $x_k \in \mathcal{Y}^{\Psi}_{\hat{t}+T}$ because $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$. This implies $x_k(t) - \chi_r(y_{\rm ref})(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [0, \hat{t}+T]$. Invoking boundedness of $y^{(i)}_{\rm ref}$, for $i = 0, \ldots, r$, and the relation in (2.30), we may infer that x_k is uniformly bounded on the entire interval $[0, \hat{t}+T]$.
Step 2: We show that the sequence of restrictions $(x_k|_{[\hat{t},\hat{t}+T]})$ is uniformly equicontinuous on the interval $[\hat{t}, \hat{t}+T]$. As x_k is a solution of (2.4) in the sense of Definition 2.2.6, we have

$$x_k(t) = \hat{x}_{\rm M}(\hat{t}) + \int_{\hat{t}}^{t} F_{\rm M}(x_k(s), \mathbf{T}_{\rm M}(x_k)(s)) + G_{\rm M}(\mathbf{T}_{\rm M}(x_k)(s))u_k(s)\mathrm{d}s, \tag{2.36}$$

for all $k \in \mathbb{N}$ and $t \in [\hat{t}, \hat{t}+T]$, where $F_{\rm M}$ and $G_{\rm M}$ are defined as in Definition 2.2.6. Since the sequence (u_k) is bounded, $\bar{u} := \sup_{k\in\mathbb{N}} \|u_k\|_{L^2}$ exists. Furthermore, using the considerations in from Lemma 2.3.20, there exists constants $F^{\max}_{\rm M}, G^{\max}_{\rm M} \geq 0$ such that

$F_{\mathrm{M}}^{\max} \geq \left\| F_{\mathrm{M}}(\tilde{\zeta}, \mathbf{T}_{\mathrm{M}}(\tilde{\zeta})) \right\|_{\infty}$ and $G_{\mathrm{M}}^{\max} \geq \left\| G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})) \right\|_{\infty}$ for all $\tilde{\zeta} \in \mathcal{Y}_{\hat{t}+T}^{\Psi}$. Now, let $\varepsilon > 0$ and define $\varsigma := \min\left\{1, \frac{1}{\varepsilon}(F_{\mathrm{M}}^{\max} + G_{\mathrm{M}}^{\max}\bar{u})\right\}$. Let $k \in \mathbb{N}$ and $t_1, t_2 \in [\hat{t}, \hat{t}+T]$ such that $|t_2 - t_1| < \varsigma^2$. Then, using $x_k \in \mathcal{Y}_{\hat{t}+T}^{\Psi}$ and Hölder's inequality in the third estimate,

$$\begin{aligned}
\|x_k(t_2) - x_k(t_1)\| &\leq \int_{t_1}^{t_2} \|F_{\mathrm{M}}(x_k(s), \mathbf{T}_{\mathrm{M}}(x_k)(s))\| + \|G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x_k)(s))\| \, \|u_k(s)\| \, \mathrm{d}s \\
&\leq F_{\mathrm{M}}^{\max} |t_2 - t_1| + G_{\mathrm{M}}^{\max} \int_{t_1}^{t_2} \|u_k(s)\| \, \mathrm{d}s \\
&\leq F_{\mathrm{M}}^{\max} \sqrt{|t_2 - t_1|} + G_{\mathrm{M}}^{\max} \sqrt{|t_2 - t_1|} \, \|u_k\|_{L^2} \\
&\leq \sqrt{|t_2 - t_1|}(F_{\mathrm{M}}^{\max} + G_{\mathrm{M}}^{\max}\bar{u}) < \varepsilon,
\end{aligned}$$

which shows that $(x_k|_{[\hat{t},\hat{t}+T]})$ is uniformly equicontinuous.

Step 3: By the Arzelà-Ascoli theorem, there exists a function $x^\star \in \mathcal{C}([\hat{t}, \hat{t}+T], \mathbb{R}^{rm})$ and a subsequence (which we do not relabel) such that the restriction $x_k|_{[\hat{t},\hat{t}+T]}$ to the interval $[\hat{t}, \hat{t}+T]$ is uniformly convergent, i.e. $x_k|_{[\hat{t},\hat{t}+T]} \to x^\star$. As in (2.35), we extend $x^\star$ by ζ on the interval $[0, \hat{t}]$ (we do not relabel $x^\star$). By construction (2.35), we have $x_k(t) = \zeta(t) = x^\star(t)$ for all $t \in [t_0, \hat{t})$. Thus, x_k converges uniformly to $x^\star$ on the whole interval $[t_0, \hat{t}+T]$. Now we prove that $x^\star = x_{\mathrm{M}}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u^\star)$, which means to show that $x^\star(t) = \hat{x}_{\mathrm{M}} + \int_{\hat{t}}^{t} F_{\mathrm{M}}(x^\star(s), \mathbf{T}_{\mathrm{M}}(x^\star)(s)) + G_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(x^\star(s)))u^\star(s)\mathrm{d}s$ for all $t \in [\hat{t}, \hat{t}+T]$. On the interval $[\hat{t}, \hat{t}+T]$, the values of $\mathbf{T}_{\mathrm{M}}(x^\star)(s)$ are completely determined by $\hat{\mathfrak{X}}$ and $x^\star|_{[\hat{t},\hat{t}+T]}$ since $\tau \geq 0$ is greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$. The same is true for $\mathbf{T}_{\mathrm{M}}(x_k)(s)$ on the $[\hat{t}, \hat{t}+T]$ for all $k \in \mathbb{N}$. We, therefore, will in the following abuse the notation slightly by only writing $F_{\mathrm{M}}(x^\star(s))$ and $F_{\mathrm{M}}(x_k(s))$ instead of $F_{\mathrm{M}}(x^\star(s), \mathbf{T}_{\mathrm{M}}(x^\star)(s))$ and $F_{\mathrm{M}}(x_k(s), \mathbf{T}_{\mathrm{M}}(x_k(s)))$, respectively. We will use the same shorthand notation for G_{M}. Due to the representation x_k as in (2.36) and since x_k in particular converges pointwise to $x^\star$ and the sequence $(F_{\mathrm{M}}(x_k))$ is uniformly bounded as (x_k) is uniformly bounded and F_{M} is continuous, the bounded convergence theorem gives that

$$\forall\, t \in [\hat{t}, \hat{t}+T] : \int_{\hat{t}}^{t} F_{\mathrm{M}}(x_k(s))\mathrm{d}s \longrightarrow \int_{\hat{t}}^{t} F_{\mathrm{M}}(x^\star(s))\mathrm{d}s.$$

Therefore, it remains to show

$$\forall\, t \in [\hat{t}, \hat{t}+T] : \int_{\hat{t}}^{t} G_{\mathrm{M}}(x_k(s))u_k(s)\mathrm{d}s \longrightarrow \int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star(s))u^\star(s)\mathrm{d}s.$$

The argument s is omitted in the following. Since $G_{\mathrm{M}}(x^\star)$ is bounded on $[\hat{t}, \hat{t}+T]$, it is an element of $L^2([\hat{t}, \hat{t}+T], \mathbb{R}^{n\times m})$, thus the weak convergence of (u_k) implies $\int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star)u_k\mathrm{d}s \to \int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s$ for all $t \in [\hat{t}, \hat{t}+T]$. Therefore, using Hölder's inequality in the second estimate, we obtain, for all $t \in [\hat{t}, \hat{t}+T]$,

$$\begin{aligned}
&\left\| \int_{\hat{t}}^{t} G_{\mathrm{M}}(x_k)u_k - G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s \right\| = \left\| \int_{\hat{t}}^{t} G_{\mathrm{M}}(x_k)u_k + G_{\mathrm{M}}(x^\star)u_k - G_{\mathrm{M}}(x^\star)u_k - G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s \right\| \\
&\quad\leq \int_{\hat{t}}^{t} \|G_{\mathrm{M}}(x_k) - G_{\mathrm{M}}(x^\star)\| \, \|u_k\| \, \mathrm{d}s + \left\| \int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star)u_k - G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s \right\| \\
&\quad\leq \left(\int_{\hat{t}}^{t} \|G_{\mathrm{M}}(x_k) - G_{\mathrm{M}}(x^\star)\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_{\hat{t}}^{t} \|u_k\|^2 \mathrm{d}s \right)^{\frac{1}{2}} + \left\| \int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star)u_k - G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s \right\| \\
&\quad\leq \sup_{m\in\mathbb{N}} \|u_m\|_{L^2} \underbrace{\left(\int_{\hat{t}}^{t} \|G_{\mathrm{M}}(x_k) - G_{\mathrm{M}}(x^\star)\|^2 \mathrm{d}s \right)^{\frac{1}{2}}}_{\to 0} + \underbrace{\left\| \int_{\hat{t}}^{t} G_{\mathrm{M}}(x^\star)u_k - G_{\mathrm{M}}(x^\star)u^\star\mathrm{d}s \right\|}_{\to 0} \to 0.
\end{aligned}$$

Step 4: We show $\|u^\star\|_\infty \le u_{\max}$. To this end, define the sets

$$A_m := \left\{ t \in [\hat{t}, \hat{t}+T] \,\middle|\, \|u^\star(t)\|^2 \ge u_{\max}^2 + \tfrac{1}{m} \right\}, \quad m \in \mathbb{N}.$$

Let $\mathbb{1}_{A_m}$ denote the indicator function of the set A_m, then, since $u_k \rightharpoonup u^\star$, we have that $\langle u_k, \mathbb{1}_{A_m} u^\star \rangle_{L^2} \to \langle u^\star, \mathbb{1}_{A_m} u^\star \rangle_{L^2} = \|\mathbb{1}_{A_m} u^\star\|_{L^2}^2$. On the other hand, by the Cauchy-Schwarz inequality we have that $\langle u_k, \mathbb{1}_{A_m} u^\star \rangle_{L^2} \le \|\mathbb{1}_{A_m} u_k\|_{L^2} \|\mathbb{1}_{A_m} u^\star\|_{L^2}$, thus

$$\|\mathbb{1}_{A_m} u^\star\|_{L^2} = \|\mathbb{1}_{A_m} u^\star\|_{L^2}^{-1} \liminf_{k\to\infty} \langle u_k, \mathbb{1}_{A_m} u^\star \rangle_{L^2} \le \liminf_{k\to\infty} \|\mathbb{1}_{A_m} u_k\|_{L^2}$$

and hence $\int_{A_m} \|u^\star(s)\|^2 \mathrm{d}s \le \liminf_{k\to\infty} \int_{A_m} \|u_k(s)\|^2 \mathrm{d}s$. Since $\|u_k\|_\infty \le u_{\max}$, we then find the following for all $m \in \mathbb{N}$ and $k \in \mathbb{N}$:

$$\lambda(A_m) = \int_{A_m} 1 \mathrm{d}s \le m \int_{A_m} \|u^\star(s)\|^2 - u_{\max}^2 \mathrm{d}s \le m \int_{A_m} \|u^\star(s)\|^2 - \|u_k(s)\|^2 \mathrm{d}s,$$

where λ denotes the Lebesgue measure, thus

$$0 \le \lambda(A_m) \le \liminf_{k\to\infty} m \int_{A_m} \|u^\star(s)\|^2 - \|u_k(s)\|^2 \mathrm{d}s \le 0.$$

Due to the σ-continuity of λ we get

$$\lambda\left(\left\{ t \in [\hat{t}, \hat{t}+T] \,\middle|\, \|u^\star(t)\| > u_{\max} \right\}\right) = \lambda\left(\bigcup_{m\in\mathbb{N}} A_m \right) = \lim_{m\to\infty} \lambda(A_m) = 0.$$

This implies $\|u^\star\|_\infty \le u_{\max}$.
Step 5: We prove $u^\star \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(\hat{t}, \hat{\mathfrak{X}})$, which means to show $x^\star(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi$ for all $t \in [\hat{t}, \hat{t}+T]$. Since ℓ_{ψ_r} is a funnel stage cost, it has the form

$$\ell_{\psi_r} : \mathbb{R}_{\ge 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}, \quad (t, z, u) \mapsto \nu_{\psi_r}(t, z) + \lambda_u \|u\|^2,$$

where ν_{ψ_r} is a funnel penalty function and $\lambda_u \ge 0$, see Definition 2.1.11. The supremum $\sup_{k\in\mathbb{N}} J_T^\Psi(u_k; \hat{t}, \hat{\mathfrak{X}}) < \infty$ exists because $J_T^\Psi(u_k; \hat{t}, \hat{\mathfrak{X}}) \to J^\star$. Thus, due to the uniform boundedness of $\|u_k\|$ and the definition of the function J_T^Ψ, see (2.33), there exists $M \ge 0$ such that

$$\int_{\hat{t}}^{\hat{t}+T} \nu_{\psi_r}(s, \xi_r(x_k(s) - \chi_r(y_{\mathrm{ref}})(s))) \mathrm{d}s \le M$$

for all $k \in \mathbb{N}$. In the following, we use the shorthand notation $\xi_r^k(\cdot) := \xi_r(x_k(\cdot) - \chi_r(y_{\mathrm{ref}})(\cdot))$ and $\xi_r^\star(\cdot) := \xi_r(x^\star(\cdot) - \chi_r(y_{\mathrm{ref}})(\cdot))$. The functions $\xi_r^k(\cdot)$ are, according to Proposition 2.3.23, Lipschitz paths with $(\hat{t}, \xi_r^k(\hat{t})) \in \mathcal{F}_{\psi_r}$ since $\hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t})$, see Remark 2.3.16. The uniform convergence of x_k to $x^\star$ implies the uniform convergence of ξ_r^k to $\xi_r^\star$. Therefore, $(\hat{t}, \xi_r^\star(\hat{t})) \in \mathcal{F}_{\psi_r}$ and $\int_{\hat{t}}^{\hat{t}+T} \nu_{\psi_r}(s, \xi_r^\star(s)) \mathrm{d}s < \infty$ because of the property (F.2) of the funnel penalty function ν_{ψ_r}, see Definition 2.1.6. In particular, this implies $J_T^\Psi(u^\star; \hat{t}, \hat{\mathfrak{X}}) < \infty$. Furthermore, we have $\|\xi_r^\star(s)\| < \psi_r(s)$ for all $s \in [\hat{t}, \hat{t}+T]$ due to the property (F.1) of the function ν_{ψ_r}, see Definition 2.1.6. This implies $x^\star(s) - \chi_r(y_{\mathrm{ref}})(s) \in \mathcal{D}_t^\Psi$ for all $s \in [\hat{t}, \hat{t}+T]$ according to Proposition 2.3.11. Hence, $u^\star \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$.
Step 6: We show $J_T^\Psi(u^\star; \hat{t}, \hat{\mathfrak{X}}) = J^\star$. According to Step 5, we have $\|\xi_r^\star(s)\| < \psi_r(s)$ for all $s \in [\hat{t}, \hat{t}+T]$. By the continuity of the involved functions and the compactness of the interval, there exists $\varepsilon > 0$ such that $\|\xi_r^\star(s)\| \le \psi_r(s) - \varepsilon$ for all $s \in [\hat{t}, \hat{t}+T]$. Moreover, due to the uniform convergence of ξ_r^k to $\xi_r^\star$, there exists $N \in \mathbb{N}$ such that $\left\|\xi_r^k - \xi_r^\star\right\|_\infty < \frac{\varepsilon}{2}$ for $k \ge N$. Thus,

$$\forall\, k \ge N \ \forall\, s \in [\hat{t}, \hat{t}+T] : \quad \left\|\xi_r^k(s)\right\| \le \left\|\xi_r^k(s) - \xi_r^\star(s)\right\| + \left\|\xi_r^\star(s)\right\| < \psi_r(s) - \frac{\varepsilon}{2}.$$

Since the function ν_{ψ_r} restricted to $\mathcal{F}_{\psi_r}$ is continuous, see Definition 2.1.6, the sequence $\left(\nu_{\psi_r}(\cdot,\xi_r^k(\cdot))^i\right)$ therefore is uniformly bounded with $i=1,2$. Hence, the bounded convergence theorem gives that

$$\nu_{\psi_r}(\cdot,\xi_r^k(\cdot))^i \to \nu_{\psi_r}(\cdot,\xi_r^\star(\cdot))^i$$

strongly and, thus, also weakly in $L^2([\hat{t},\hat{t}+T],\mathbb{R})$ for $i=1,2$. Since the L^2-norm is weakly lower semi-continuous and since $J_T^\Psi(u_k;\hat{t},\hat{\mathfrak{X}}) \to J^\star = \inf_{u\in\mathcal{U}_{[\hat{t},\hat{t}+T]}(\hat{t},\hat{\mathfrak{X}})} J_T^\Psi(u;\hat{t},\hat{\mathfrak{X}})$, the following holds.

$$\begin{aligned} J_T^\Psi(u^\star;\hat{t},\hat{\mathfrak{X}}) &= \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(s,\xi_r^\star(s))\mathrm{d}s = \left\|\nu_{\psi_r}(\cdot,\xi_r^\star(\cdot))^{\frac{1}{2}}\right\|_{L^2}^2 + \lambda_u \left\|u^\star\right\|_{L^2}^2 \\ &\le \liminf_{k\to\infty} \left\|\nu_{\psi_r}(\cdot,\xi_r^k(\cdot))^{\frac{1}{2}}\right\|_{L^2}^2 + \liminf_{k\to\infty} \lambda_u \left\|u_k\right\|_{L^2}^2 \le \liminf_{k\to\infty} J_T^\Psi(u_k;\hat{t},\hat{\mathfrak{X}}) = J^\star. \end{aligned}$$

Therefore $J_T^\Psi(u^\star;\hat{t},\hat{\mathfrak{X}}) = \min_{u\in\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}})} J_T^\Psi(u;\hat{t},\hat{\mathfrak{X}})$.

Step 7: We show that $J_T^\Psi(u^\star;\hat{t},\hat{\mathfrak{X}}) = \min_{\substack{u\in L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m),\\ \|u\|_\infty\le u_{\max}}} J_T^\Psi(u;\hat{t},\hat{\mathfrak{X}})$.

Since $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}}) \neq \emptyset$ by assumption this follows from Remark 2.3.25 (ii) and completes the proof. □

Remark 2.3.27. Theorem 2.3.26 shows that there exists a solution $u^\star$ to the optimal control problem (2.34) and that the application of this solution to the model (2.4) solves the tracking problem from Section 1.1.1, i.e. it ensures that the output tracking error $e_{\mathrm{M}}(t) = y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$ evolves within the funnel $\mathcal{F}_\psi$ given by ψ. In application, however, it is often not possible to compute the solution of an OCP. One has to utilise numerical approximations instead. As a consequence of Proposition 2.3.23, every approximation $\tilde{u}$ of $u^\star$, for which the cost function J_T^Ψ is finite, still guarantees that the tracking error evolves within the prescribed performance funnel. •

2.4 The funnel MPC algorithm

We now want to summarise our findings in the funnel MPC Algorithm 2.4.1. With the definitions, concepts, and results so far at hand, we will prove that it is initial and recursive feasible and that the application of this control scheme to the model (2.4) solves the tracking problem laid out in Section 1.1.1, i.e. it ensures that the distance between the model's output y_{M} and a given reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\ge0},\mathbb{R}^m)$ evolves within the funnel $\mathcal{F}_\psi$ given by a function $\psi \in \mathcal{G}$.

Algorithm 2.4.1 (Funnel MPC)**.**
Given:

- Model (2.4) with initial time $t_0 \in \mathbb{R}_{\ge0}$ and initial value $y_{\mathrm{M}}^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m)$, reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\ge0},\mathbb{R}^m)$, signal memory length $\tau \ge 0$,

- a set of funnel boundary function $\Psi = (\psi_1,\dots,\psi_r) \in \mathscr{G}$ with corresponding parameters k_i for $i=1,\dots,r$, input saturation level $u_{\max} \ge 0$, funnel stage cost function ℓ_{ψ_r}, and a τ-initialisation strategy κ as in Definition 2.3.17.

Set the time shift $\delta > 0$, the prediction horizon $T \ge \delta$, index $k := 0$, and $\hat{x}_{\mathrm{M}}^0 := \chi_r(y_{\mathrm{M}}^0)$.
Define the time sequence $(t_k)_{k\in\mathbb{N}_0}$ by $t_k := t_0 + k\delta$.
Steps:

(a) Select initial model state $\mathfrak{X}_k := \kappa(\hat{x}_{\mathrm{M}}^k) \in \mathfrak{I}_{t_0,\tau}^\Psi(t_k)$ at current time t_k based on $\hat{x}_{\mathrm{M}}^k$.

(b) Compute a solution $u_{\mathrm{FMPC},k} \in L^\infty([t_k, t_k+T], \mathbb{R}^m)$ of the Optimal Control Problem (OCP)

$$\underset{\substack{u \in L^\infty([t_k,t_k+T],\mathbb{R}^m),\\ \|u\|_\infty \le u_{\max}}}{\text{minimise}} \quad \int_{t_k}^{t_k+T} \ell_{\psi_r}(s, \xi_r(x_\mathrm{M}(s; t_k, \mathfrak{X}_k, u) - \chi_r(y_\mathrm{ref})(s)), u(s)) \mathrm{d}s. \tag{2.37}$$

(c) Apply the control law

$$\mu : [t_k, t_{k+1}) \times \mathfrak{I}^\Psi_{t_0,\tau}(t_k) \to \mathbb{R}^m, \quad \mu(t, \mathfrak{X}_k) = u_{\mathrm{FMPC},k}(t) \tag{2.38}$$

to model (2.4) with initial time and data $(t_k, \mathfrak{X}_k)$ and obtain, on the interval $I_0^{t_{k+1},\tau} := [t_{k+1} - \tau, t_{k+1}] \cap [0, t_{k+1}]$ a measurement of the model's output and its derivatives $\hat{x}_\mathrm{M}^{k+1} := x_\mathrm{M}(\cdot; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k})|_{I_{t_0}^{t_{k+1},\tau}}$. Increment k by 1 and go to Step (a).

▲

Before proving the correct functioning of the funnel MPC Algorithm 2.4.1 a comment on the solution concept seems in order. At every iteration of Algorithm 2.4.1 the model (2.4) is re-initialised with new initial values $(t_k, \mathfrak{X}_k)$ and a solution x_M on the interval $[t_k, t_k + T]$ while solving the OCP (2.37). Note that this solution is in fact defined on the whole interval $[0, t_k + T]$ according to our understanding of a solution of the initial value problem, see Definition 2.2.6. However, since the in Step (a) selected initial values do not necessarily coincide with the solution of the initial value problem from the previous iteration, applying Algorithm 2.4.1 to the model (2.4) does not result in a closed-loop system with a global solution in the classical sense. In the following Definition 2.4.2, we therefore define a notion of a concatenated solution which takes the re-initialisation into account and is, in a certain sense, a solution of all the considered initial value problems on the subintervals $[t_k, t_{k+1})$.

Definition 2.4.2 (Concatenated model solution). *Let $(f_\mathrm{M}, g_\mathrm{M}, \mathbf{T}_\mathrm{M}) \in \mathcal{M}^{m,r}_{t_0}$ and consider the model (2.4). Let $y_\mathrm{ref} \in W^{r,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$, $\Psi = (\psi_1, \dots, \psi_r) \in \mathscr{G}$, $t_0 \in \mathbb{R}_{\ge 0}$, and $\delta > 0$ be given. Define the sequences $(t_k)_{k \in N_0}$ and $(\mathfrak{X}_k)_{k\in N_0}$ by $t_k := t_0 + k\delta$ and $\mathfrak{X}_k \in \mathfrak{I}^\Psi_{t_0,\tau}(t_k)$. Further, suppose $u \in L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ is a control such that initial value problem (2.4) with initial data $\mathfrak{X}_k$ at time t_k has a solution $x^k_\mathrm{M} : [0, t_{k+1}] \to \mathbb{R}^{rm}$ in the sense of Definition 2.2.6 for every $k \in \mathbb{N}$. We call the function $x_\mathrm{M} : \mathbb{R}_{\ge 0} \to \mathbb{R}^{rm}$ that is piecewise defined by*

$$x_\mathrm{M}(t) = \begin{cases} x^0_\mathrm{M}(t), & t < t_1, \\ x^k_\mathrm{M}(t), & t \in [t_k, t_{k+1}) \end{cases}$$

a concatenated solution *of the initial value problem (2.4) with sequence of initial values $(t_k, \mathfrak{X}_k)_{k\in\mathbb{N}_0}$. Its first $m-$dimensional component $x_{\mathrm{M},1} : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ is denoted by y_M.*

We will now prove the initial and recursive feasibility of the funnel MPC Algorithm 2.4.1 and that applying this algorithm to model (2.4) results in a system that has a concatenated solution in the sense of Definition 2.4.2.

Theorem 2.4.3. *Consider model (2.4) with $(f_\mathrm{M}, g_\mathrm{M}, \mathbf{T}_\mathrm{M}) \in \mathcal{M}^{m,r}_{t_0}$ with initial value $y^0_\mathrm{M} \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$. Let $y_\mathrm{ref} \in W^{r,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$ and $\Psi = (\psi_1, \dots, \psi_r) \in \mathscr{G}$ be given. Further, let $\tau \ge 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_\mathrm{M}$ and choose a τ-initialisation strategy $\kappa : \bigcup_{\hat{t} \ge t_0} \mathcal{R}(I_0^{\hat{t},\tau}, \mathbb{R}^{rm}) \to \bigcup_{\hat{t}\ge t_0} \mathcal{R}(I_0^{\hat{t},\tau}, \mathbb{R}^{rm}) \times L^\infty_{\mathrm{loc}}(I_{t_0}^{\hat{t},\tau}, \mathbb{R}^q)$ as in Definition 2.3.17. Then, there exists $u_{\max} \ge 0$ such that the funnel MPC Algorithm 2.4.1 with $\delta > 0$ and $T \ge \delta$ is initially and recursively feasible, i.e.*

- *at every time instant $t_k := t_0 + k\delta$ for $k \in \mathbb{N}_0$ the OCP (2.37) has a solution $u_{\mathrm{FMPC},k} \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$, and*

- *the model* (2.4) *with applied funnel MPC feedback* (2.38) *has a concatenated solution* $x_{\mathrm{M}} : [0, \infty) \to \mathbb{R}^{rm}$ *in the sense of Definition* 2.4.2.

The corresponding input is given by

$$u_{\mathrm{FMPC}}(t) = u_{\mathrm{FMPC},k}(t),$$

for $t \in [t_k, t_{k+1})$ *and* $k \in \mathbb{N}_0$*. Each global solution* x_{M} *with corresponding output* y_{M} *and input* u_{FMPC} *satisfies:*

(i) the control input is bounded by $u_{\max}$*, i.e.*

$$\forall\, t \geq t_0 : \quad \|u_{\mathrm{FMPC}}(t)\| \leq u_{\max},$$

(ii) the tracking error between the model output and the reference evolves within prescribed boundaries, i.e.

$$\forall\, t \geq t_0 : \quad \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \psi_1(t).$$

Proof. Step 1: According to Remark 2.3.15, the set $\mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ is non-empty for all $\hat{t} \geq t_0$. In particular, $\mathfrak{I}^{\Psi}_{t_0,\tau}(t_0) \neq \emptyset$. Using the notation $I_0^{t_0,\tau} := [t_k - \tau, t_k] \cap [0, t_0]$, let $\hat{x}^0_{\mathrm{M}} := \chi_r(y^0_{\mathrm{M}})|_{I_0^{t_0,\tau}}$ be the measurement of the initial model output y^0_{M} and its derivatives. Note that we identify $\mathcal{C}^r([0, t_0], \mathbb{R}^m)$ with the vector space $\mathbb{R}^{rm}$ if $t_0 = 0$, see two cases $t_0 > 0$ and $t_0 = 0$ in (2.4). Since $\mathfrak{I}^{\Psi}_{t_0,\tau}(t_0) \neq \emptyset$, it is possible to select the initial model state $\mathfrak{X}_0 = \kappa(\hat{x}^0_{\mathrm{M}})$.
Step 2: There exists $u_{\max} \geq 0$ such that, $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$ for all $\hat{t} \geq t_0$, $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, and $T > 0$, according to Theorem 2.3.21. Assume that we have given $\hat{x}^k_{\mathrm{M}} \in \mathcal{R}(I_0^{t_k,\tau}, \mathbb{R}^{rm})$ for $k \in \mathbb{N}_0$, where $I_0^{t_k,\tau} := [t_k - \tau, t_k] \cap [0, t_k]$. $\hat{x}^k_{\mathrm{M}}$ is the model's output and its derivatives on the interval $I_0^{t_k,\tau}$ from the previous iteration of the funnel MPC Algorithm 2.4.1. Since the set $\mathfrak{I}^{\Psi}_{t_0,\tau}(t_k)$ is non-empty according to Remark 2.3.15, there exists a model state $\mathfrak{X}_k := \kappa(\hat{x}^k_{\mathrm{M}}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(t_k)$. Theorem 2.3.26 yields the existence of some function $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ such that the functional J^{Ψ}_T as in (2.33) has a minimum, that is

$$J^{\Psi}_T(u_{\mathrm{FMPC},k}; t_k, \mathfrak{X}_k) = \min_{\substack{u \in L^\infty([t_k,t_k+T],\mathbb{R}^m),\\ \|u\|_\infty \leq u_{\max}}} J^{\Psi}_T(u; t_k, \mathfrak{X}_k).$$

Thus, $u_{\mathrm{FMPC},k}$ is a solution of the OCP (2.37). If the control $u_{\mathrm{FMPC},k}$ is applied to the model (2.4) at initial time t_k initial value $\mathfrak{X}_k$ in Step 2.38, then the initial value problem (2.11) has a solution $x^k_{\mathrm{M}} : [0, t_{k+1}] \to \mathbb{R}^{rm}$ in the sense of Definition 2.2.6 as a consequence of the definition of $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$, see (2.28). As x^k_{M} is defined on the whole interval $[0, t_{k+1}]$, the function $\hat{x}^{k+1}_{\mathrm{M}} := x_{\mathrm{M}}(\cdot; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k})|_{I_{t_0}^{t_{k+1},\tau}} \in \mathcal{R}(I_0^{t_{k+1},\tau}, \mathbb{R}^{rm})$ is well-defined in Step (c) of Algorithm 2.4.1.
Step 3: The recursive application of Step 2 in combination with Step 1 of this proof yield the existence of a sequence $(\mathfrak{X}_k)_{k\in\mathbb{N}_0}$ of initial values $\mathfrak{X}_k \in \mathfrak{I}^{\Psi}_{t_0,\tau}(t_k)$, control signals $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ solving the OCP (2.37), and corresponding solutions $x^k_{\mathrm{M}} : [0, t_{k+1}] \to \mathbb{R}^{rm}$ of the initial value problem (2.11) in the sense of Definition 2.2.6. Hence, the funnel MPC Algorithm 2.4.1 is initially and recursively feasible. Define $u_{\mathrm{FMPC}} \in L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ by $u_{\mathrm{FMPC}}(t) = u_{\mathrm{FMPC},k}(t)$ for $k \in \mathbb{N}_0$ and $x_{\mathrm{M}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{rm}$ by

$$x_{\mathrm{M}}(t) = \begin{cases} x^0_{\mathrm{M}}(t), & t < t_1, \\ x^k_{\mathrm{M}}(t), & t \in [t_k, t_{k+1}). \end{cases}$$

Then, x_{M} is a concatenated solution of the initial value problem (2.4) with sequence of initial values $(t_k, \mathfrak{X}_k)_{k\in\mathbb{N}_0}$ in the sense of Definition 2.4.2.

Step 4: As $u_{\text{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$, we have $\|u_{\text{FMPC},k}\|_\infty \le u_{\max}$ for all $k \in \mathbb{N}_0$. Thus, $\|u_{\text{FMPC}}(t)\| \le u_{\max}$ for all $t \ge t_0$. This shows (i). By definition of the set $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$, all error variables $\xi_i(x_{\text{M}}(t;t_k,\mathfrak{X}_k,u_k^\star) - \chi_r(y_{\text{ref}}))$ from (2.15) evolve within their respective funnels $\mathcal{F}_{\psi_i}$ given by ψ_i for $i = 1,\dots,r$, i.e.

$$\|\xi_i(x_{\text{M}}(t;t_k,\mathfrak{X}_k,u_k^\star) - \chi_r(y_{\text{ref}}))\| < \psi_i(t)$$

for all $t \in [t_k, t_{k+1})$. The output y_{M} is the first $m-$dimensional component of x_{M}, see Definition 2.4.2. According to the definition of ξ_1 as in (2.15), we thus have

$$\|y_{\text{M}}(t) - y_{\text{ref}}(t)\| = \|\xi_1(x_{\text{M}}(t;t_k,\mathfrak{X}_k,u_k^\star) - \chi_r(y_{\text{ref}}))\| < \psi_1(t)$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_0$. This shows (ii) and completes the proof. □

Remark 2.4.4. As the inclined reader has probably already noticed, using the term *recursive feasibility* with respect to the funnel MPC Algorithm 2.4.1 is a bit of a stretch since applying the solution $u_{\text{FMPC},k} \in L^\infty([t_k, t_k+T], \mathbb{R}^m)$ of the optimal control problems (2.37) to model (2.4) with initial data $(t_k, \mathfrak{X}_k)$ is an open-loop control problem for every $k \in \mathbb{N}_0$. These individual problems are only loosely coupled via the initialisation strategy κ. It would be more precise to say that the funnel MPC Algorithm 2.4.1 solves an infinite sequence of open-loop problems. However, Theorem 2.3.18 in combination with Theorem 2.3.26 and Theorem 2.3.24 shows that the funnel MPC Algorithm 2.4.1 ensures that the state of the model at the end of each iteration is again a feasible initial value for the next iteration, i.e.

$$(x_{\text{M}}(\cdot;t_k,\mathfrak{X}_k,u)|_{[t_k-\tau,t_{k+1}]\cap[0,t_{k+1}]}, \mathbf{T}(x_{\text{M}}(\cdot;t_k,\mathfrak{X}_k,u))|_{[t_k-\tau,t_{k+1}]\cap[t_0,\hat{t}+\delta]}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(t_{k+1}).$$

If this model state is always selected in Step (a) of Algorithm 2.4.1, then applying Algorithm 2.4.1 to the model (2.4) results in a closed-loop system with a global solution $x_{\text{M}} : [0,\infty) \to \mathbb{R}^{rm}$ of the initial value problem (2.4) in a more classical sense, meaning the differential equation is fulfilled on the whole interval $[t_0,\infty)$. Using the notation $I_{t_0}^{t_k,\tau} := [t_k - \tau, t_k] \cap [t_0, t_k]$ and setting $\mathfrak{X}_0 := (\chi_r(y_{\text{M}}^0)|_{I_0^{t_0,\tau}}, \mathbf{T}_{\text{M}}(\chi_r(y_{\text{M}}^0))|_{I_{t_0}^{t_0,\tau}})$, it is therefore possible to replace Step (a) of Algorithm 2.4.1, for $k \ge 1$, by

(a') Obtain a measurement of the model state x_{M} and $\mathbf{T}_{\text{M}}(x_{\text{M}})$ of (2.4) on the interval $I_{t_0}^{t_k,\tau}$ and set $\mathfrak{X}_k := (x_{\text{M}}|_{I_0^{t_k,\tau}}, \mathbf{T}(x_{\text{M}})|_{I_{t_0}^{t_k,\tau}})$.

However, utilising an initialisation strategy κ in Step (a) of Algorithm 2.4.1 opens up the possibility of directly applying the funnel MPC algorithm to a system which does not coincide with the model (2.4). In this case, the initial model state is selected based on measurement data from the system's output y and its derivatives. Algorithm 2.4.1 still remains feasible, meaning the optimal control problem (2.37) has a solution and the model output y_{M} is ensured to evolve within the funnel $\mathcal{F}_\psi$. While such guarantees can in such cases not be given for the actual system to be controlled, the controller's performance might still be adequate if system and model only slightly diverge due to measurement errors and small disturbances. In Chapter 3, we will examine in more detail how the funnel MPC Algorithm 2.4.1 can be adapted in order to give guarantees on the tracking error for the actual system in the presence of a model-plant mismatch e_{S} as in (1.11). •

Remark 2.4.5. (a) The OCP (2.37) has neither state nor terminal constraints. Nevertheless, application of the funnel MPC Algorithm 2.4.1 to the model (2.4) ensures the existence of a global solution of the initial value problem in the sense of Definition 2.4.2 or the solution of the closed-loop system if Step (a) is replaced by Step (a') from Remark 2.4.4. However, note that in neither case this solution is unique in general. One of the reasons is that the solution of the OCP (2.37) found in each step may not be unique. The MPC algorithm has to select a particular optimal control. In particular, Theorem 2.4.3 shows that the properties (i) and (ii) are independent of the particular choice made within the MPC algorithm, since they hold for every such solution. However, $x_{\mathrm{M}}|_{[t_k,t_{k-1})}$ is uniquely determined by the choice of $\mathfrak{X}_k$ and $u_k^\star$ for every $k \in \mathbb{N}_0$ as Proposition 2.2.9 shows.

(b) Funnel MPC is initially and recursively feasible for every choice of $T > 0$. Usually, recursive feasibility for model predictive control can only be guaranteed when the prediction horizon is sufficiently long, see e.g. [40], or when additional terminal constraints are added to the OCP, see e.g. [159]. For funnel MPC merely the boundary on the control input $u_{\max} \geq 0$ must be sufficiently large.

•

Remark 2.4.6. While the primary funnel function ψ is user-defined based on application-specific tracking error constraints, the funnel MPC Algorithm 2.4.1 introduces additional parameters – notably the auxiliary funnel functions $\Psi = (\psi_1, \dots, \psi_r) \in \mathscr{G}$ and their associated error gains k_i for $i = 1, \dots, r$ – whose impact on the controller performance warrants discussion.

The functions ψ_i for $i = 1, \dots, r-1$ do not directly influence the controller, as the optimal control problem (2.37) only optimises over a funnel stage cost function ℓ_{ψ_r} linked to the last auxiliary funnel function ψ_r. These auxiliary functions ψ_i are determined by the gains k_i and the constants α, β and γ (determined by the function ψ), as defined in (2.22), (2.23) or (2.24). Consequently, we focus on the effects of k_i and ψ_r:

- Tighter boundary ψ_r improves track precision but restricts the optimiser's flexibility to accommodate secondary objectives, e.g. minimising the control effort.
- Larger k_i values intensify penalisation of error variables ξ_i (see (2.15)), enhancing accuracy at the cost of higher control inputs.
- The minimum bound $u_{\max} \geq 0$ for admissible control inputs depends on k_i and $\sup \psi_i$, as shown in Theorem 2.3.21. However, these derived bounds are typically rather conservative; refining them requires a problem-specific analysis.

Moreover, as noted in Section 2.3.1, the construction of the parameters k_i and ψ_r (via (2.22) and (2.23)) can be simplified using the design (2.24). This simplification replaces the time-varying funnel penalties with a constant cost funnel penalty function in the MPC stage cost ℓ_{ψ_r}, reducing computational complexity. However, the initial model trajectory y_{M}^0 for the model (2.4) must be freely selectable to satisfy the constraints imposed by this simplified design. •

Remark 2.4.7. The proof of the funnel MPC's recursive feasibility hinges primarily on Theorem 2.3.21. Crucially, this result does not depend on the use of *funnel penalty functions*, implying that recursive feasibility can also be guaranteed for the MPC scheme in Algorithm 1.1.4 with alternative stage cost functions $\ell : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m$ – such as the classical quadratic cost function in (1.7) – provided the optimal control problem (1.8) incorporates the additional constraint

$$\forall\, s \in [t_k, t_k + T] : \quad \|\xi_r(x_{\mathrm{M}}(s; t_k, \mathfrak{X}_k, u) - \chi_r(y_{\mathrm{ref}})(s))\| < \psi_r(s).$$

However, without funnel penalty functions, alternative methods are required to ensure the optimal control problem always admits a solution, as the proof of Theorem 2.3.26 relies explicitly on their use. Assuming this is achievable, Algorithm 1.1.4 can also fulfil the control objective outlined in Section 1.1.1. While the funnel MPC (Algorithm Algorithm 2.4.1) is hypothesised to offer superior performance – due to its cost function dynamically penalising proximity to the funnel boundary – a numerical case study [9] found comparable results between the two approaches. A comprehensive comparative analysis, however, remains an open research question. •

2.5 Simulation

This section illustrates the application of the funnel MPC algorithm (Algorithm 2.4.1) using two numerical examples. In this section, we do not distinguish between the actual system and its model. This distinction will be revisited in subsequent chapters. Consequently, we omit the subscript M used to denote model equations in this section.

The MATLAB source code for the simulations performed in this thesis can be found on GITHUB under the link `https://github.com/ddennstaedt/FMPC_Simulation`.

2.5.1 Exothermic chemical reaction

To demonstrate the funnel MPC Algorithm 2.4.1, we consider a model of a chemical reactor where an exothermic reaction Substance-1 → Substance-2 takes place. This example was also used in [97] to study funnel control with input saturation and in [124] to demonstrate the feasibility of the bang-bang funnel controller. According to [187], this type of reactor can be modelled by the following system of equations of order one:

$$\begin{aligned} \dot{x}_1(t) &= c_1\, p(x_1(t), x_2(t), y(t)) + d(x_1^{\text{in}} - x_1(t)),\\ \dot{x}_2(t) &= c_2\, p(x_1(t), x_2(t), y(t)) + d(x_2^{\text{in}} - x_2(t)),\\ \dot{y}(t) &= b\, p(x_1(t), x_2(t), y(t)) - q\, y(t) + u(t), \end{aligned} \tag{2.39}$$

where x_1 is the concentration of the reactant Substance-1, x_2 the concentration of the product Substance-2 and y describes the reactor temperature; u is the feed temperature/coolant control input. Further, the constant $b > 0$ describes the exothermicity of the reaction, $d > 0$ is associated with the dilution rate and $q > 0$ is a constant consisting of the combination of the dilution rate and the heat transfer rate. Further, $c_1 < 0$ and $c_2 \in \mathbb{R}$ are the stoichiometric coefficients and $p : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a locally Lipschitz continuous function with $p(0, 0, t) = 0$ for all $t > 0$ that models the reaction heat. As in [97], we consider for the function p the Arrhenius law

$$p(x_1, x_2, y) = k_0 e^{-\frac{k_1}{y}} x_1, \tag{2.40}$$

where k_0, k_1 are positive parameters. Since $c_1 < 0$, it is easy to see that the subsystem

$$\begin{aligned} \dot{x}_1(t) &= c_1 p(x_1(t), x_2(t), y(t)) + d(x_1^{\text{in}} - x_1(t)),\\ \dot{x}_2(t) &= c_2 p(x_1(t), x_2(t), y(t)) + d(x_2^{\text{in}} - x_2(t)), \end{aligned}$$

satisfies the BIBS condition (2.9) from Example 2.2.4, when y is restricted to the set $\left\{ y \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \,\middle|\, \forall\, t \geq 0 : \; y(t) > 0 \right\}$. We like to emphasise that the control must thus guarantee that y is always positive, which is also from a practical point of view a reasonable objective. The control objective is to steer the reactor's temperature to a certain given reference value $y_{\text{ref}}(t)$ within boundaries given by a function $\psi(t)$. The reactor's temperature should follow a given heating profile specified as

$$y_{\text{ref}}(t) = \begin{cases} y_{\text{ref,start}} + \frac{y_{\text{ref,final}} - y_{\text{ref,start}}}{t_{\text{final}}} t, & t \in [0, t_{\text{final}}),\\ y_{\text{ref,final}}, & t \geq t_{\text{final}}. \end{cases} \tag{2.41}$$

Note that this heating profile has a kink at $t = t_{\text{final}}$. Starting at $y_{\text{ref,start}} = 270\,\text{K}$, the reactor is heated up to $y_{\text{ref,final}} = 337.1\,\text{K}$ within the prescribed time $[0, t_{\text{final}}]$, here we choose $t_{\text{final}} = 2$. The maximal control value is limited to $u_{\max} = 600$. During the heating phase, the tolerated temperature deviation from the heating profile decreases from $\pm 24\,\text{K}$ to $\pm 4.4\,\text{K}$ (time-varying output constraints). After reaching the desired level, the temperature in the reactor is kept constant with deviation of no more than $\pm 4.4\,\text{K}$ after four units of time after beginning of the heating process. We therefore choose the funnel function $\psi \in \mathcal{G}$ given by

$$\psi(t) := 20\mathrm{e}^{-2t} + 4.$$

To achieve the control objective with funnel MPC Algorithm 2.4.1, we use the strict funnel stage cost function $\ell_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$\ell_\psi(t, y, u) = \begin{cases} \dfrac{\|y - y_{\text{ref}}(t)\|}{\psi(t)^2 - \|y - y_{\text{ref}}(t)\|^2} + \lambda_u \|u - 360\|^2, & \|y - y_{\text{ref}}(t)\| \neq \psi(t) \\ \infty, & \text{else}, \end{cases} \tag{2.42}$$

with design parameter $\lambda_u \in \mathbb{R}_{\geq 0}$. This is a slightly modified variant of stage cost function from (2.1). As in [97, 187], the initial data is chosen as $[x_1^0, x_2^0, y^0] = [0.02, 0.9, 270]$ describing the initial concentration of the two substances and the initial reactor temperature. The parameters of the system are

$$\begin{aligned} c_1 &= -1, \quad k_0 = \mathrm{e}^{25}, \quad x_1^{\text{in}} = 1, \quad d = 1.1, \\ c_2 &= 1, \quad k_1 = 8700, \quad x_2^{\text{in}} = 0, \quad q = 1.25, \quad b = 209.2. \end{aligned} \tag{2.43}$$

To demonstrate that the funnel MPC Algorithm 2.4.1 is initially and recursively feasible even for a short prediction horizon, we choose the time shift $\delta = 5 \cdot 10^{-4}$ and $T = 20 \cdot \delta = 10^{-2}$. Due to discretisation, only step functions with constant step length $\mathfrak{r} := \delta$ are considered[1] for the optimal control problem (2.37) of the funnel MPC Algorithm 2.4.1. As perfect system knowledge is assumed here, the model is initialised, at every iteration of the algorithm, with the model's state from the previous iteration. We compare this control approach with the MPC Algorithm 1.1.4 using a standard quadratic cost function

$$\ell(t, y, u) = \|y - y_{\text{ref}}(t)\|^2 + \lambda_u \|u - 360\|^2 \tag{2.44}$$

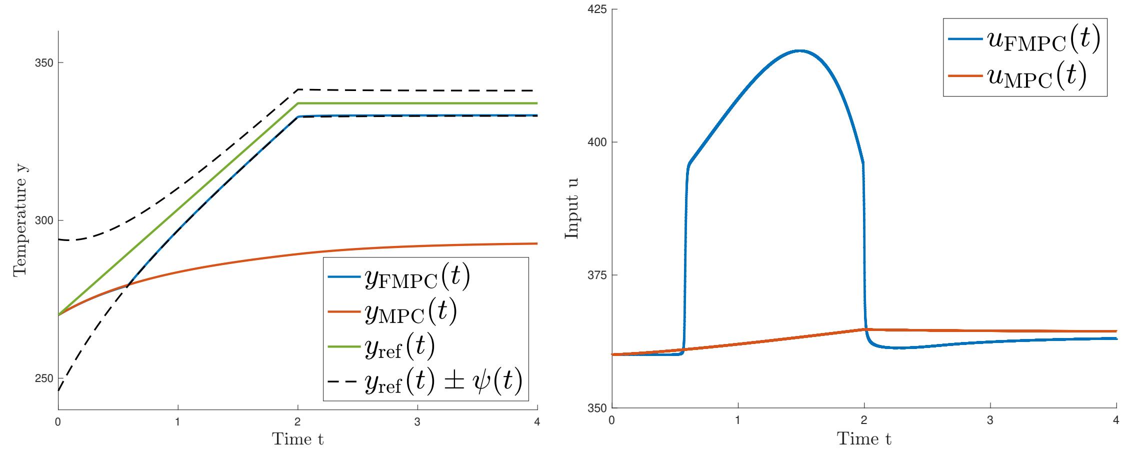
as in (1.7). For both control schemes, the parameter λ_u is chosen as $\lambda_u = 0.1$. The simulations are performed with MATLAB and the toolkit CASADI[2] [16] over the time interval $[0, 4]$ and depicted in Figure 2.1. Figure 2.1a shows that the output of the system evolves within the funnel boundaries when the control signal is generated by the funnel MPC Algorithm 2.4.1 (labelled with y_{FMPC}). The standard MPC Algorithm 1.1.4 with the quadratic stage cost function (2.44) does however not achieve the control objective. The corresponding system output (labelled with y_{MPC}) evolves outside of the prescribed boundaries. This observation is not surprising since no information about the funnel $\mathcal{F}_\psi$ is included in stage cost function (2.44) of the Algorithm 1.1.4. The incorporation of output constraints of the form

$$\forall\, t \in [\hat{t}, \hat{t} + T]: \quad \|y_{\text{M}}(t) - y_{\text{ref}}(t)\| < \psi(t) \tag{2.45}$$

as in (1.9) in the corresponding OCP (1.8) is necessary in order to ensure that MPC with stage cost (2.44) is feasible with the selected prediction horizon T, time shift δ, and design

[1] By a step function on an interval $[a, b]$ with constant step length $\mathfrak{r} > 0$, we mean a mapping $f : [a, b] \to \mathbb{R}$ which is constant on every interval $[a + k\mathfrak{r}, a + (k+1)\mathfrak{r}) \cap [a, b]$ for $k = 0, \ldots, \lceil \frac{b-a}{\mathfrak{r}} \rceil - 1$, see also Definition 5.0.1.

[2] http://casadi.org



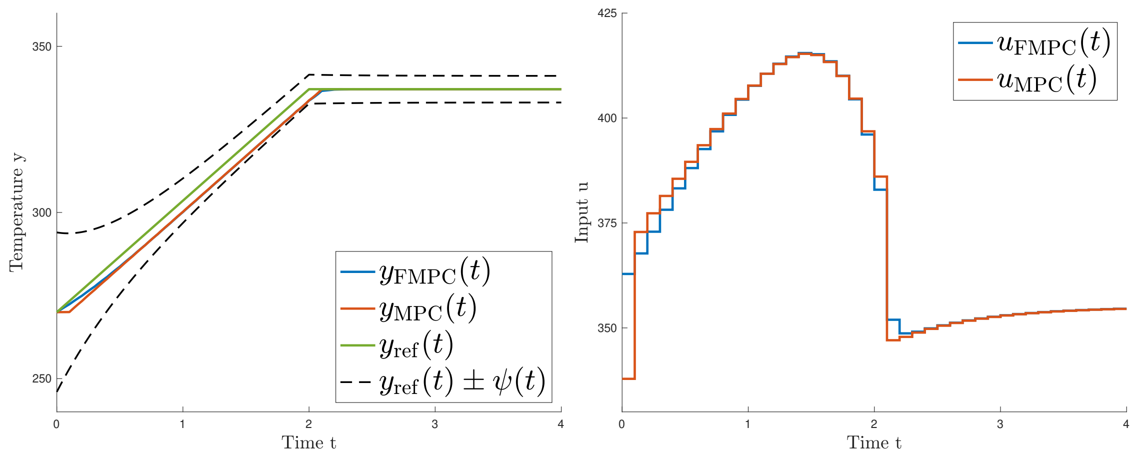
(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 2.1: Simulation of system (2.39) under the control generated by the funnel MPC Algorithm 2.4.1 and the MPC Algorithm 1.1.4 with stage cost (2.44) with parameters $T = 10^{-2}$, $\delta = 5 \cdot 10^{-4}$, and $\lambda_u = 0.1$.

parameter λ_u. In this case, the standard MPC scheme also achieves the control objective in accordance to Remark 2.4.7.

Alternatively, if an appropriately long prediction horizon $T = 1$ is chosen and the penalisation of the control signal is reduced by choosing $\lambda_u = 10^{-4}$, then standard MPC Algorithm 1.1.4 is also able to achieve the control objective by chance. Additionally, the time shift is increased to $\delta = 0.1$. The performance of both control schemes with the longer horizon and time shift and the adapted design parameter $\lambda_u = 10$ is depicted in Figure 2.2. While Figure 2.2a shows the output of the system evolving within the funnel boundaries under the both control schemes, Figure 2.2b shows the corresponding input signals. It is evident that both control techniques generate very similar control signals and



(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 2.2: Simulation of system (2.39) under the control generated by the funnel MPC Algorithm 2.4.1 and the MPC Algorithm 1.1.4 with stage cost (2.44) with parameters $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-4}$.

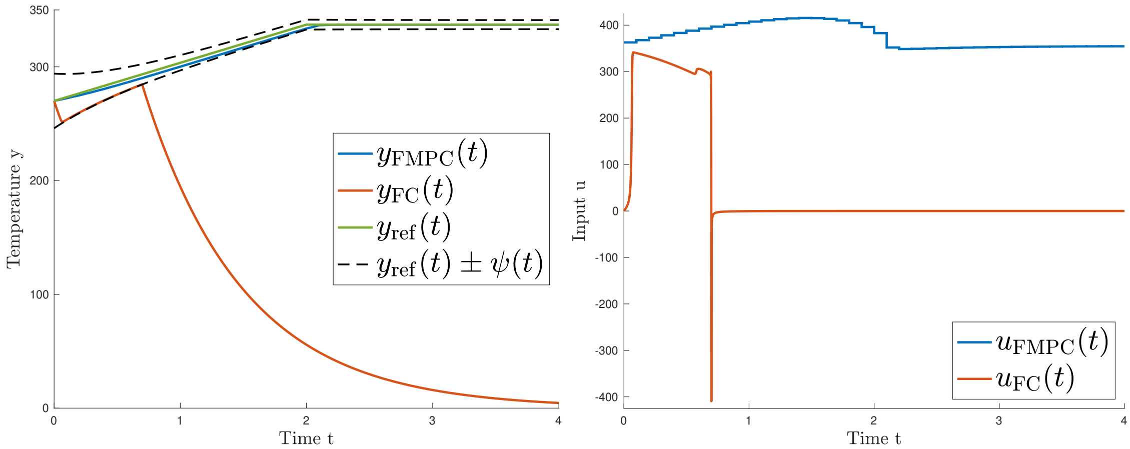
achieve the control objective. However, while it is, to a certain extend, incidental that the tracking error of the system under the control generated by the MPC Algorithm 1.1.4 evolves within $\mathcal{F}_\psi$, the funnel stage cost function (2.42) provably ensures the adherence of the system to the funnel boundaries for the funnel MPC Algorithm 2.4.1, see Theorem 2.3.24.

However, note that the discontinuous funnel penalties function can lead to compatibility issues with standard optimisation frameworks causing the utilised numerical solvers to fail. In this case, the incorporation of additional output constraints like (2.45) in the funnel MPC Algorithm 2.4.1 can mitigate these issues although they are, from a theoretical point of view, redundant.

In the following, we compare the funnel MPC Algorithm 2.4.1 to the funnel controller which was the inspiration for the development and usage of funnel penalty functions of the form (2.1). The original funnel controller proposed in [95] takes the form

$$u_{\mathrm{FC}}(t) = -\frac{1}{\psi(t)^2 - \|e(t)\|^2} e(t). \tag{2.46}$$

For the comparison of the two controllers , we choose, as before, the strict funnel stage cost ℓ_ψ as in (2.42) and the parameters $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-4}$ for the MPC scheme and restrict the set of control functions considered in the OCP (2.37) to step functions with constant step length $\mathfrak{r} := \delta = 0.1$. To numerically compute the solution of the closed-loop system under the both control laws, the explicit four stage Runge-Kutta method (RK4) with a constant step size $h > 0$ is used.



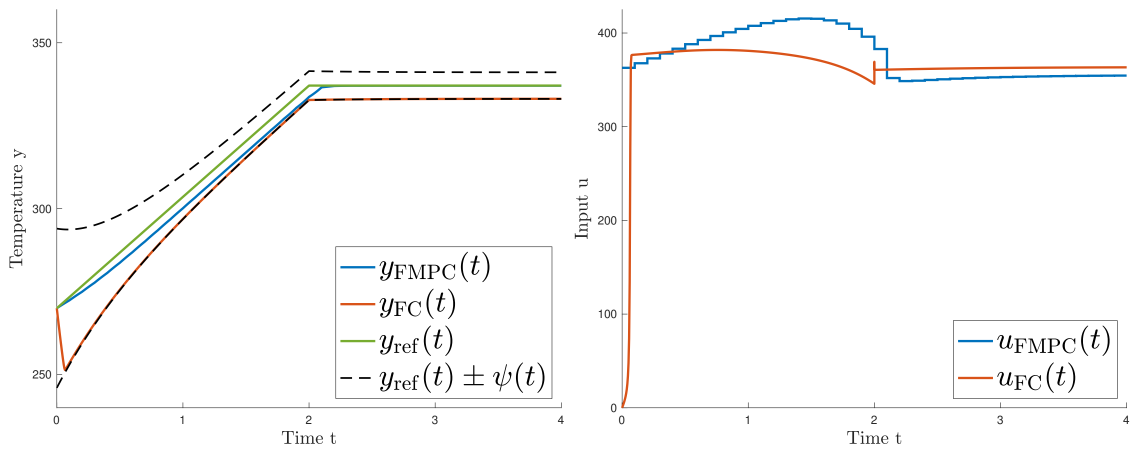
(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 2.3: Simulation of system (2.39) under the control generated by the funnel MPC Algorithm 2.4.1 with parameters $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-4}$ and the funnel control law (2.46) with a constant step size $h = 10^{-3}$.

Figure 2.3 depicts the performance both of the funnel controller (2.46) and the funnel MPC Algorithm 2.4.1 when a constant step size $h = 10^{-3}$ is used in the RK4 method. Figure 2.3a shows that, while initially both control schemes are feasible, the system's output when controlled by the funnel controller (labelled with y_{FC}) breaches the funnel boundary at $t \approx 0.7$ and evolves from then onward outside the prescribed boundaries. The controller reacts at this time instant with a large peak in its control signal, see Figure 2.3b. It is however not able to achieve the control objective on the entire considered time interval. Although the funnel MPC Algorithm 2.4.1 is restricted to step functions as its control signals a relatively wide step length of $\mathfrak{r} = 0.1$ and therefore adapts its control signal significantly less often than the funnel controller, the MPC algorithm is feasible and the system output y_{FMPC} evolves within the performance funnel. Funnel MPC actually still achieves the control objective if an even larger step size of $h = 10^{-2}$ is used to solve the ordinary differential equation.

To ensure that the simulation of the funnel controller (2.46) also achieves the control objective the usage of a smaller step size in the RK4 method is required. The system



(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 2.4: Simulation of system (2.39) under the control generated by the funnel MPC Algorithm 2.4.1 with parameters $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-4}$ and the funnel control law (2.46) with a constant step size $h = 10^{-4}$.

output and the corresponding control signals are depicted in Figure 2.4 when a constant step size $h = 10^{-4}$ is used for the simulation. Figure 2.4a shows the system output under the control of the two approaches evolving within the funnel boundaries and Figure 2.4b depicts the corresponding input signals. It is evident that both control techniques are feasible and achieve the control objective in this case. As the initial tracking error is zero, the funnel controller (2.46) does not act in the beginning. It's input signal u_{FC} is zero and then increases when the system output y_{FC} approaches the funnel boundary. Afterwards, y_{FC} evolves close to the funnel boundary over the entire time interval. The system when controlled by the funnel MPC Algorithm 2.4.1 exhibits a more accurate tracking. The system's output y_{FMPC} evolves closer to the reference signal y_{ref} than y_{FC}. Thanks to its predictive capabilities, the funnel MPC Algorithm 2.4.1 applies already at the beginning a larger control signal u_{FMPC} and does not wait until the system's output is close to the boundary until it reacts. After the system reached the desired temperature $y_{\text{ref,final}}$ at $t_{\text{final}} = 2$, the output y_{FC} tracks the reference signal y_{ref} almost perfectly. The system output y_{FC} under the control of u_{FC} has a constant offset to the reference. It is worth noting that funnel MPC Algorithm 2.4.1 achieves this better performance while applying less control input than the funnel controller (2.46).

2.5.2 Mass-on-car system

For purposes of illustration that funnel MPC Algorithm 2.4.1 can also successfully applied to systems with fixed higher relative degree, i.e. $r > 1$, we consider the example of a mass-on-car system from [172]. This example was also examined in [30] and [32] to compare different versions of funnel control. On a car with mass m_1, to which a force $F = u$ can be applied, a ramp is mounted on which a second mass m_2 moves passively, see Figure 2.5. The second mass is coupled to the car by a spring-damper combination, and the ramp is inclined by a fixed angle $\vartheta \in (0, \pi/2)$. The equations of motion are given by

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos(\vartheta) \\ m_2 \cos(\vartheta) & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ 0 \end{bmatrix}, \tag{2.47}$$

where $z(t)$ is the horizontal position of the car and $s(t)$ is the relative position of the mass on the ramp at time t. The physical constants $k > 0$ and $d > 0$ are the coefficients of the spring and damper, respectively. The horizontal position of the mass on the ramp is the

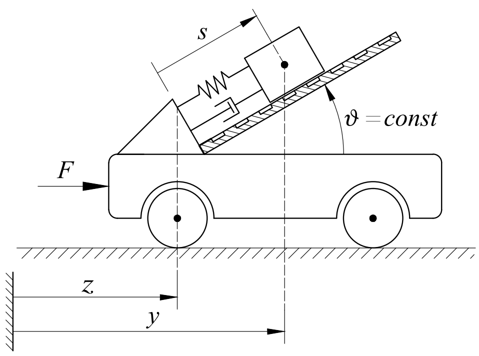


Figure 2.5: Mass-on-car system. The figure is based on the respective figures in [30], and [172].

output y of the system, i.e.

$$y(t) = z(t) + s(t)\cos(\vartheta).$$

The objective is tracking the reference signal $y_{\text{ref}} : t \mapsto \cos(t)$, such that for $\psi \in \mathcal{G}$ the error function $t \mapsto e(t) := y(t) - y_{\text{ref}}(t)$ evolves within the prescribed performance funnel $\mathcal{F}_\psi$, i.e. $\|e(t)\| < \psi(t)$ for all $t \geq 0$. For this example, we choose the funnel boundary function

$$\psi(t) := 5\mathrm{e}^{-2t} + 0.1,$$

which fulfils (2.21) for $\alpha = 2$ and $\beta = 0.2$. By setting $\mu := m_2(m_1 + m_2\sin^2(\vartheta))$, $\mu_1 := \frac{m_1}{\mu}$, and $\mu_2 := \frac{m_2}{\mu}$, the system takes the form (2.5), with

$$x(t) := \begin{bmatrix} z(t) \\ \dot{z}(t) \\ s(t) \\ \dot{s}(t) \end{bmatrix}, \; A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mu_2 k\cos(\vartheta) & \mu_2 d\cos(\vartheta) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(\mu_1+\mu_2)k & -(\mu_1+\mu_2)d \end{bmatrix}, \; B := \begin{bmatrix} 0 \\ \mu_2 \\ 0 \\ -\mu_2\cos(\vartheta) \end{bmatrix}, \; C := \begin{bmatrix} 1 \\ 0 \\ \cos(\vartheta) \\ 0 \end{bmatrix}^\top.$$

As outlined in [30, Sec. 3], the system has global relative degree

$$r = \begin{cases} 2, & \vartheta \in \left(0, \frac{\pi}{2}\right), \\ 3, & \vartheta = 0, \end{cases}$$

bounded-input bounded-output internal dynamics, and the positive scalar high-frequency gain $\Gamma = CA^{r-1}B$, see also Example 2.2.3. For the simulation, we choose the same system parameters

$$m_1 = 4, \quad m_2 = 1, \quad k = 2, \quad d = 1, \quad \vartheta = \frac{\pi}{4} \tag{2.48}$$

and initial values $z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$ as in [30]. Given $\vartheta = \frac{\pi}{4}$, the system has relative degree $r = 2$. Following [98], the system can equivalently be written in the form

$$\begin{aligned} \ddot{y}(t) &= R_1 y(t) + R_2\dot{y}(t) + S\eta(t) + \Gamma u(t) \\ \dot{\eta}(t) &= Q\eta(t) + Py(t), \end{aligned} \tag{2.49}$$

with initial conditions $[y(0), \dot{y}(0)] = [y_0^0, y_1^0] \in \mathbb{R}^2$ and $\eta(0) = \eta^0 \in \mathbb{R}^2$. For the given parameters (2.48), the matrices are

$$R_1 = 0, \;\; R_2 = \frac{8}{9}, \;\; S = \frac{-4\sqrt{2}}{9}\begin{bmatrix} 2 & 1 \end{bmatrix}, \;\; \Gamma = \frac{1}{9}, \;\; Q = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \;\; P = 2\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.50}$$

To apply the funnel MPC Algorithm 2.4.1 to the system (2.49) with relative degree $r = 2$, we choose the construction (2.23) and (2.22) for the auxiliary funnel function ψ_2 and the associated parameter k_1. Straightforward calculations show that ψ_2 takes the form

$$\psi_2(t) := \frac{1}{\gamma} k_1 \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma} \tag{2.51}$$

with $k_1 = 14$ and $\gamma = 0.2$ satisfying (2.20). To achieve the control objective, we use the strict funnel stage cost function $\ell_{\psi_2} : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$\ell_{\psi_2}(t, \zeta, u) = \begin{cases} \dfrac{\|\zeta\|}{\psi_2(t)^2 - \|\zeta\|^2} + \lambda_u \|u\|^2, & \|\zeta\| \neq \psi_2(t) \\ \infty, & \text{else}, \end{cases} \tag{2.52}$$

with design parameter $\lambda_u \in \mathbb{R}_{\geq 0}$. Utilising the error variables $\xi_1(z_1, z_2) := z_1$, and $\xi_2(z_1, z_2) := \xi_1(z_2, 0) + k_1\xi_1(z_1, z_2)$ as in (2.15), the variable ζ in (2.52) is replaced, in the optimal control problem (2.34), by $\xi_2(\chi(y - y_{\text{ref}})(t)) = \dot{e}(t) + k_1 e(t)$, where $e(t) := y(t) - y_{\text{ref}}(t)$.

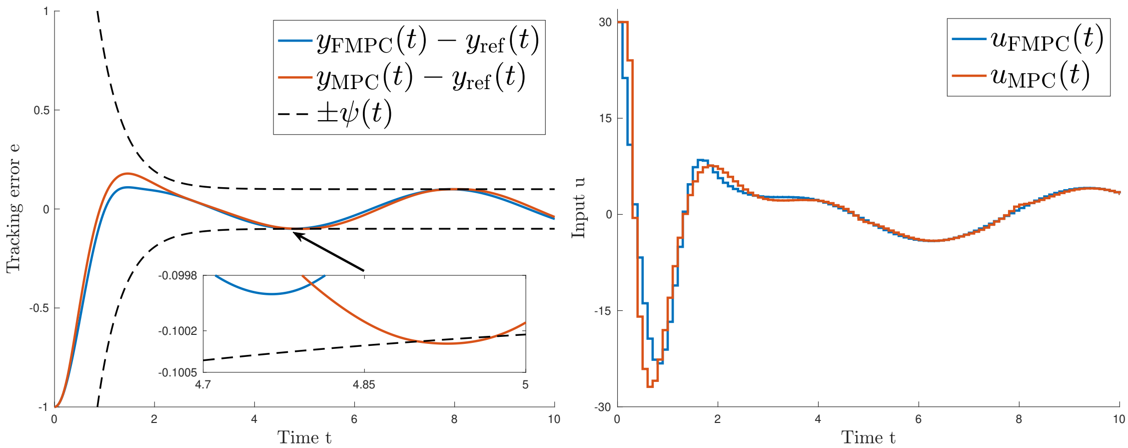
We compare the funnel MPC Algorithm 2.4.1 with the standard MPC Algorithm 1.1.4 using the quadratic cost function

$$\ell(t, y, u) = \|y - y_{\text{ref}}(t)\|^2 + \lambda_u \|u\|^2 \tag{2.53}$$

as in (1.7) and additional output constraints

$$\forall\, t \in [t_k, t_k + T] : \quad \|y(t) - y_{\text{ref}}(t)\| < \psi(t)$$

for $t_k \in \delta\mathbb{N}_0$ as in (1.9) in the OCP (1.8). For both MPC schemes, we choose the prediction horizon $T = 1$, the time shift $\delta = 0.1$, the parameter $\lambda_u = 10^{-4}$, and allow for a maximal control value of $u_{\max} = 30$. Due to discretisation, only step functions with constant step length $\mathfrak{r} := \delta$ are considered when solving the respective optimal control problems. The simulations are performed on the time interval $[0, 10]$ with MATLAB and the toolkit CASADI and displayed in Figure 2.6. While Figure 2.6b shows that the



(a) Tracking error $e = y - y_{\text{ref}}$ with boundary ψ.

(b) Control inputs.

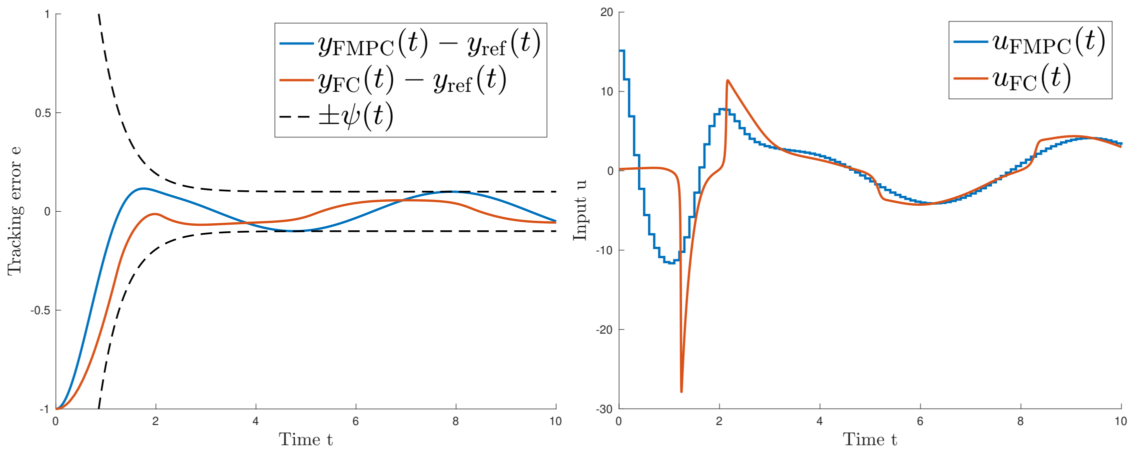
Figure 2.6: Simulation of system (2.49) under the control generated by funnel MPC Algorithm 2.4.1 and the MPC Algorithm 1.1.4 with cost function (1.7) and eq. (1.9). The parameters are $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-4}$.

control signals generated by the two MPC schemes are relatively similar, Figure 2.6a displays that the MPC Algorithm 1.1.4 with cost function (2.53) does, contrary to the funnel MPC Algorithm 2.4.1, not achieve the control objective. Despite the output constraints (1.9), the tracking error $y_{\mathrm{MPC}}(t) - y_{\mathrm{ref}}(t)$ leaves the performance funnel. The reason for this behaviour could be that the solver implements these barriers internally with a certain tolerance level using barrier functions. To ensure an adherence to the funnel boundaries, an adaptation of the parameter λ_u, a smaller step length δ, or a longer prediction horizon T are sufficient as demonstrated in the previous example. Even though the tracking error $y_{\mathrm{FMPC}} - y_{\mathrm{ref}}$ evolves at times close to the funnel boundary, the cost function (2.52) ensures that the control objective is achieved when the system is controlled by the funnel MPC Algorithm 2.4.1.

Now, we compare the funnel MPC Algorithm 2.4.1 to the funnel controller from [30]. For the system (2.49), the funnel control law takes the form

$$\begin{aligned} w(t) &= \frac{\dot{e}(t)}{\psi(t)} + \gamma\left(\frac{e(t)^2}{\psi(t)^2}\right)\frac{e(t)}{\psi(t)}, \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ u_{\mathrm{FC}}(t) &= -\gamma\left(w(t)^2\right) w(t), \end{aligned} \tag{2.54}$$

with $\gamma(s) = \frac{1}{1-s}$ for $s \in [0,1)$. For the funnel MPC scheme, we choose the prediction horizon $T = 1$, the time shift $\delta = 0.1$, the parameter $\lambda_u = 10^{-3}$, and allow for a maximal control value of $u_{\max} = 30$.



(a) Tracking error $e = y - y_{\mathrm{ref}}$ within boundary ψ. (b) Control inputs.

Figure 2.7: Simulation of system (2.49) under the control generated by funnel MPC Algorithm 2.4.1 with parameters $T = 1$, $\delta = 0.1$, and $\lambda_u = 10^{-3}$ and the funnel control law (2.54).

The performance of the funnel controller (2.54) and the funnel MPC Algorithm 2.4.1 is depicted in Figure 2.7. While Figure 2.7a shows the tracking error of the two controllers evolving within the funnel boundaries, Figure 2.7b displays the respective input signals. It is evident that both control techniques are feasible and achieve the control objective. The funnel controller generates a smooth input signal, while the OCP (2.34) of the funnel MPC Algorithm 2.4.1 is solved over step functions with constant step length $\mathfrak{r} := 0.1$. The funnel MPC seemingly takes more advantage of the available error tolerance boundaries resulting in a smaller range of employed control values. Funnel control tends to change the control values very quickly and the control signal shows peaks. The MPC scheme avoids this undesirable behaviour thanks to its predictive capabilities.

3 Robust funnel MPC

Optimisation-based control techniques, such as model predictive control, achieve high-performance control while rigorously adhering to state and input constraints. These schemes – including the funnel MPC Algorithm 2.4.1 developed in Chapter 2 – fundamentally depend on accurate system models. Without such models, essential closed-loop properties – stability, performance, and constraint satisfaction – are generally not preserved. Significant challenges arise from model uncertainties and external disturbances, as even high-fidelity models deviate from real-world systems, while disturbances are omnipresent. Moreover, to mitigate computational complexity, practitioners often opt for simplified, lower-dimensional approximations – such as discretised representations of partial differential equations – over intricate models. For a comprehensive treatment of model order reduction techniques, see for example the textbook [167].

The development of robust MPC methods to address structural model-plant mismatches and external disturbances therefore remains an active research area, see e.g. [51, 109, 156, 179] and the references therein. Key approaches include:

- *Scenario-based optimisation*: Handles uncertainties via sampling a suitable number of randomly selected disturbance realisations in a receding horizon fashion [53].
- *Barrier-augmented MPC*: Ensure states/outputs to remain within safe regions as (relaxed) barrier functions penalise proximity to constraint boundaries. Safety and constraint satisfaction is enforced through dynamic penalty adjustment and inherently accounting for deviations [70, 151, 203].
- *Feedback MPC*: Solves for an optimal and stabilising feedback policy rather than an open-loop input signal [77, 171]. The applied (robust) feedback controller counteracts occurring disturbances between two iterations of the MPC algorithm.
- *Adaptive MPC*: Dynamically updates model parameters online using techniques like moving horizon estimation (MHE) [84], (non)-linear state observers [39, 101, 114], or system identification methods [158] bridging model-system gaps, see e.g. [10, 165]. For a comprehensive overview on adaptive MPC, see also the survey paper [103].
- *Stochastic MPC*: Employs chance constraints or risk-aware formulations for quantifiable probabilistic uncertainties. It offers probabilistic guarantees for systems with measurable noise distributions [115, 141, 176].
- *Learning-augmented MPC*: Integrates data-driven models, such as Gaussian processes or neural networks, to refine predications and quantify uncertainties [18]. We explore this integration in more detail in Chapter 4; see also [88] for a survey.

Central to robust MPC are constraint tightening techniques [59], particularly *tube-based* MPC [117]. To robustly achieve output tracking, these methods construct tubes around reference trajectories to guarantee the actual system output remains within prescribed bounds. For linear systems, foundational work in [140] demonstrates this approach, while non-linear extensions in [67, 109, 156] address geometric and dynamic complexities. Notably, [130] introduces co-optimisation of tubes and reference trajectories, adapting tube geometry based on proximity to boundaries.

To enforce tube invariance, terminal conditions are embedded within the optimisation problem, ensuring recursive feasibility. For linear systems, [62] achieves reference tracking within constant bounds via *robust control invariant (RCI)* sets, which satisfy state, input, and performance constraints. [205] extends this framework to external disturbances, though RCI computation remains non-trivial, with algorithms potentially failing to terminate finitely [62]. For non-linear systems, [177, 204] employ incremental Lyapunov functions and precomputed stabilising feedback laws to ensure control objectives. While effective, these methods face challenges in balancing conservatism and computational tractability, as tube design must inherently account for system uncertainty magnitude.

Despite advancements in robustification methods for MPC, critical challenges persist:

- Computational complexity: Scaling methods for high-dimensional systems [73, 109].
- Conservatism vs performance: Balancing conservatism and performance, in particular in tube-based approaches.
- Safety certification: Ensuring reliability in learning-augmented components [184].

Robust funnel MPC: Bridging prediction and adaptation

To address the challenge of output tracking within prescribed performance boundaries while retaining the predictive power of MPC and the disturbance rejection capabilities of adaptive control, this chapter proposes *robust funnel MPC*. This method relaxes the assumption from Chapter 2 that the system (1.1) and surrogate model (1.5) coincide, explicitly accounting for external disturbances and (structural) model-plant mismatches. The controller synergises two complementary strategies:

1. **Funnel MPC**: Leverages model-based predictions to compute feed-forward control signals.
2. **Funnel control**: A model-free, high-gain adaptive feedback loop (introduced in Section 1.1.2) that refines the control signal using real-time measurements to reject disturbances and compensate mismatches.

The synergy of these techniques ensures arbitrary output constraint satisfaction: the predictive component (funnel MPC) plans trajectories using the surrogate model, while the model-free adaptive component (funnel control) instantaneously compensates for unmodelled dynamics or disturbances. This two component approach marries the predictive power of MPC with the robustness of adaptive feedback, addressing key limitations of stand-alone methods in uncertain environments.

3.1 System class

In this section, we concretise the structural properties of the system (1.1) and formally introduce the system class under consideration. To briefly recapitulate, we consider non-linear multi-input multi-output control systems of order $r \in \mathbb{N}$ of the form

$$\begin{aligned} y^{(r)}(t) &= F(\mathbf{T}(y, \dot{y}, \dots, y^{(r-1)})(t), u(t)), \\ y|_{[0,t_0]} &= y^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m), \quad \text{if } t_0 > 0, \\ \big(y(t_0), \dots, y^{(r-1)}(t_0)\big) &= y^0 \in \mathbb{R}^{rm}, \qquad\qquad\quad \text{if } t_0 = 0, \end{aligned} \tag{1.1 revisited}$$

with $t_0 \geq 0$, initial trajectory y^0, input $u \in L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^m)$, and output $y(t) \in \mathbb{R}^m$ at time $t \geq t_0$. The following definition formalises the properties of the function F and the operator $\mathbf{T}$.

Definition 3.1.1 (System class $\mathcal{N}_{t_0}^{m,r}$). *We say that the system* (1.1) *belongs to the* system class $\mathcal{N}_{t_0}^{m,r}$ *for* $m, r \in \mathbb{N}$, *and* $t_0 \in \mathbb{R}_{\geq 0}$, *written* $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$, *if, for some* $q \in \mathbb{N}$, *the following holds:*

(a) $\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^q)$ *has the* causality *(T.1)*, local Lipschitz *(T.2)*, *and the* bounded-input bounded-output (BIBO) *(T.3) property as defined in Definition 2.2.1.*

(b) $F \in \mathcal{C}(\mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m)$ *has the* perturbation high-gain property, *i.e. for every compact set* $K_m \subset \mathbb{R}^m$ *there exists* $\nu \in (0, 1)$ *such that for every compact set* $K_q \subset \mathbb{R}^q$ *the function*

$$\mathfrak{h} \colon \mathbb{R} \to \mathbb{R}, \;\; s \mapsto \min \left\{ \langle v, F(z, d - sv) \rangle \,|\, d \in K_m, z \in K_q, v \in \mathbb{R}^m, \;\; \nu \leq \|v\| \leq 1 \right\} \tag{3.1}$$

satisfies $\sup_{s \in \mathbb{R}} \mathfrak{h}(s) = \infty$.

We already discussed examples for operators $\mathbf{T}$ satisfying the properties (a) from Definition 3.1.1 in Examples 2.2.3 and 2.2.4. To also gain a better understanding for the high-gain property of the function F in (1.1), we briefly discuss a simple example of a differential equation belonging to the considered system class.

Example 3.1.2. Let $p : \mathbb{R}^n \to \mathbb{R}^n$ and $\Gamma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be continuous non-linear functions. Assume $\Gamma(x) \in \mathrm{GL}_m(\mathbb{R})$ for all $x \in \mathbb{R}^n$. We show that the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$F(x, u) \coloneqq p(x) + \Gamma(x) u \tag{3.2}$$

has the perturbation high-gain property (3.1) if, and only if, $\Gamma(x)$ is sign-definite for all $x \in \mathbb{R}^n$, i.e. the scalar product $\langle v, \Gamma(x) v \rangle$ is positive (negative) for all $v \in \mathbb{R}^m \backslash \{0\}$. We show this equivalence by adapting [30, Sec. 2.1.3] to the given context.

Assume that the function F has the perturbation high-gain property (3.1) and suppose that Γ is not sign-definite. Then, there exists $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m \backslash \{0\}$ with $\langle v, \Gamma(z) v \rangle = 0$. Define $K_m \coloneqq \{0\}$ and $K_n = \{z\}$. For $\nu \in (0, 1)$, set $K_\nu \coloneqq \{ v \in \mathbb{R}^m \,|\, \nu \leq \|v\| \leq 1 \}$. As $\langle v, \Gamma(z) v \rangle = 0$, there exists $\hat{v} \in K_\nu$ with $\langle \hat{v}, \Gamma(z) \hat{v} \rangle = 0$. For $s \in \mathbb{R}$, we have

$$\begin{aligned} \mathfrak{h}(s) &= \min_{v \in K_\nu} \langle v, F(z, -sv) \rangle = \min_{v \in K_\nu} \langle v, p(z) - \Gamma(z) sv \rangle \\ &\leq \nu \|p(z)\| + \min_{v \in K_\nu} -s \langle v, \Gamma(z) v \rangle \leq \nu \|p(z)\| - s \langle \hat{v}, \Gamma(z) \hat{v} \rangle = \nu \|p(z)\| . \end{aligned}$$

This is a contradiction to the perturbation high-gain property (3.1).

Assume Γ is sign-definite. Due to the continuity of $\Gamma(\cdot)$, there exists $\sigma \in \{-1, 1\}$ such that $\sigma \Gamma(z)$ is positive definite for all $z \in \mathbb{R}^n$. We show that the function F has the perturbation high-gain property (3.1). Let $K_m \in \mathbb{R}^m$, $K_n \in \mathbb{R}^n$ be compact sets and set $\nu = \frac{1}{2}$. Define $K_\nu \coloneqq \{ v \in \mathbb{R}^m \,|\, \nu \leq \|v\| \leq 1 \}$. Set $G(z) \coloneqq \frac{\sigma}{2}(\Gamma(z) + \Gamma(z)^\top)$ and let $\lambda_{\min}$ be the smallest eigenvalue of $G(z)$ for all $z \in K_n$ which exists because of the compactness of K_n. Moreover, due to the continuity of the involved functions and the compactness of the considered sets, there exists

$$c \coloneqq \min \left\{ \langle v, p(z) + \Gamma(z) d \rangle \,|\, d \in K_m, z \in K_n, v \in K_\nu \right\} \in \mathbb{R}.$$

Let $(s_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a sequence with $s_j \sigma < 0$ for all $j \in \mathbb{N}$ and $s_j \sigma \to -\infty$ for $j \to \infty$. It follows that

$$\begin{aligned} \mathfrak{h}(s_j) &= \min \left\{ \langle v, F(z, d - s_j v) \rangle \,|\, d \in K_m, z \in K_n, v \in K_\nu \right\} \\ &\geq \min \left\{ \langle v, p(z) + \Gamma(z) d \rangle \,|\, d \in K_m, z \in K_n, v \in K_\nu \right\} \\ &\quad + \min \left\{ -\langle v, \Gamma(z) s_j v \rangle \,|\, z \in K_n, v \in K_\nu \right\} \end{aligned}$$

$$\begin{aligned}&= c + \min\left\{-s_n\sigma\langle v, G(z)v\rangle \mid z \in K_n, v \in K_\nu\right\}\\&\geq c + \min\left\{-s_j\sigma\lambda_{\min}\|v\|^2 \,\middle|\, v \in K_\nu\right\}\\&\geq c - \frac{s_j\sigma\lambda_{\min}}{4}.\end{aligned}$$

Thus, $\mathfrak{h}(s_j) \to \infty$ for $j \to \infty$ proving that the function F has the perturbation high-gain property (3.1).

We saw in Example 2.2.4 that non-linear differential equations of the form

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(t_0) = x^0,\\ y(t) &= h(x(t)),\end{aligned} \tag{2.7 revisited}$$

with $t_0 \in \mathbb{R}_{\geq 0}$, $x^0 \in \mathbb{R}^n$, and non-linear functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$, are admissible candidates for a model by transforming it into the Byrnes-Isidori form (2.8). This was achieved by, among other things, assuming that (2.7) has a strict (global) relative degree $r \in \mathbb{N}$, i.e.

$$\begin{aligned}\forall\, k \in \{1,\dots,r-1\}\ \forall\, x \in \mathbb{R}^n :\ &(L_g L_f^{k-1} h)(x) = 0\\ \text{and}\quad &(L_g L_f^{r-1} h)(x) \in \mathrm{GL}_m(\mathbb{R}).\end{aligned}$$

A consequence of our considerations regarding function F in (3.2) is that (2.7) is also an admissible system if, in addition to the strict relative degree, one assumes $(L_g L_f^{r-1} h)(x)$ to be sign-definite. Similarly, linear time invariant systems given by matrices $A \in \mathbb{R}^{n\times n}$ and $C^\top, B \in \mathbb{R}^{n\times m}$, as discussed in Example 2.2.3, are admissible systems in the sense of Definition 3.1.1 if, in addition to the assumptions from Example 2.2.3, the matrix $CA^{r-1}B$ is sign-definite, where $r > 0$ is the relative degree of the linear system (2.5). ⋄

Remark 3.1.3. We want to comment on a few aspects of the system class $\mathcal{N}_{t_0}^{m,r}$.

(a) For $t_0 \geq 0$ and $m, r \in \mathbb{N}$, let $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$. For $d \in L^\infty([t_0,\infty), \mathbb{R}^p)$, define the operator $\tilde{\mathbf{T}}$ by

$$\tilde{\mathbf{T}}(\zeta)(t) := (d(t), \mathbf{T}(\zeta)(t))$$

for $\zeta \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Straightforward calculations show that $\tilde{\mathbf{T}}$ also fulfils the properties (T.1), (T.2), and (T.3) of Definition 2.2.1. The system class $\mathcal{N}_{t_0}^{m,r}$ therefore implicitly contains differential equations of the form

$$y^{(r)}(t) = F(d(t), \mathbf{T}(y, \dot{y}, \dots, y^{(r-1)})(t), u(t)),$$

with unknown disturbance $d \in L^\infty([t_0,\infty), \mathbb{R}^p)$.

(b) The system class $\mathcal{N}_{t_0}^{m,r}$ allows for the usage of more general operators $\mathbf{T}$ than the model class $\mathcal{M}_{t_0}^{m,r}$ because $\mathbf{T}$ is not required to have the *limited memory* property (T.4) of Definition 2.2.1. Many physical phenomena such as *backlash* and *relay hysteresis*, and *non-linear time delays* can be modelled by means of a general operator $\mathbf{T}$, cf. [30, Sec. 1.2]. Moreover, the operator $\mathbf{T}$ can even be the solution operator of an infinite-dimensional dynamical system, e.g. a partial differential equation. Thus, systems with such internal dynamics can be represented by (1.1), see [36]. For a practically relevant example of infinite-dimensional internal dynamics (modelled by an operator $\mathbf{T}$), we refer to [37], where a moving water tank was subject to funnel control, and the water in the tank was modelled by the linearised Saint-Venant equations. While we deem the limited memory property (T.4) not to be a major restriction posed on the operator $\mathbf{T}_{\mathrm{M}}$ used in the model, it still remains to be verified whether the mentioned examples can also be modelled by an operator with property (T.4).

(c) The perturbation high-gain property of the function F in (b) of Definition 3.1.1 is a modification of the so-called *high-gain* property, see e.g. [30, Def. 1.2], and, at first glance, a stronger assumption. The high-gain property is essential in high-gain adaptive control and, roughly speaking, guarantees that, if a large enough input is applied, the system reacts sufficiently fast. For linear systems, as in Example 2.2.3, having the high-gain property implies that the system can be stabilised via high-gain output feedback, cf. [30, Rem. 1.3]. In order to account for possible bounded perturbations of the input, we require the modified property from (b). It is an open question whether the perturbation high-gain property and the high-gain property are equivalent.

(d) Although there are many systems belonging to both the model class $\mathcal{M}_{t_0}^{m,r}$ from Definition 2.2.2 and the system class $\mathcal{N}_{t_0}^{m,r}$ from Definition 3.1.1, neither the set of admissible models $\mathcal{M}_{t_0}^{m,r}$ is a subset of all considered systems $\mathcal{N}_{t_0}^{m,r}$ nor the opposite is true. Every system $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$ which does not have a control affine representation of the form (2.4) cannot belong to $\mathcal{M}_{t_0}^{m,r}$. On the other hand, Example 3.1.2 shows that differential equations of the form (2.7) are admissible models if $(L_g L_f^{r-1} h)(x)$ is invertible but only admissible systems if $(L_g L_f^{r-1} h)(x)$ is in addition sign definite.

(e) Throughout this thesis, we always assume that the parameters m and r for the system class $\mathcal{N}_{t_0}^{m,r}$ and the model class $\mathcal{M}_{t_0}^{m,r}$ coincide. This means that the system (1.1) and the model (2.4) are of the same order $r \geq 1$ and have the identical output/input dimension $m \geq 1$.

•

In Definition 2.2.6, we introduced a solution concept for the initial value problem (2.4), which is used as model for the funnel MPC Algorithm 2.4.1. Mainly due to the inherent conflict between the domain of the operator and the re-initialisation of the model, it had certain peculiarities distinguishing it from more traditional solution concept. As the system's differential equation (1.1) is not re-initialised during operation of any controller, we utilise conventional solutions in sense of *Carathéodory*. For the sake of completeness, we recall this solution concept.

Definition 3.1.4 (System solution). *For initial trajectory $y^0 \in \mathcal{C}^{(r-1)}([0, t_0], \mathbb{R}^m)$ for $t_0 > 0$ or $y_0 \in \mathbb{R}^{rm}$ in the case $t_0 = 0$ and a control function $u \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$, an absolutely continuous function $x = (x_1, \ldots, x_r) : [0, \omega) \to \mathbb{R}^{rm}$ with $\omega \in (t_0, \infty]$ is called a solution of (1.1) (in the sense of Carathéodory) if*

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), && i = 1, \ldots, r-1, \\ \dot{x}_r(t) &= F(\mathbf{T}(x)(t), u(t)), \end{aligned}$$

for almost all $t \in [t_0, \omega)$ and $x|_{[0,t_0]} = \chi_r(y^0)$ if $t_0 > 0$ or $x(t_0) = y^0$ in the case $t_0 = 0$. A solution is maximal if it has no proper right extension that is also a solution. A maximal solution is also called a response *of the system associated with u and denoted by $x(\cdot; t_0, y^0, u)$. We denote its first component x_1 by $y(\cdot, t_0, y^0, u)$.*

In the Appendix, we show that (1.1) has a solution $x : [0, \omega) \to \mathbb{R}^{rm}$ in the sense of Definition 3.1.4 for every $u \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$ and that every solution can be extended to a maximal solution, see Corollary 7.0.5.

3.2 Controller structure

We propose *robust funnel MPC*, a two component control architecture that synergises the model-based funnel MPC Algorithm 2.4.1 with the *model-free* funnel controller to achieve reference tracking within prescribed boundaries despite a mismatch between the true system (1.1) and the nominal model (2.4). The overall structure is depicted in Figure 3.1. This

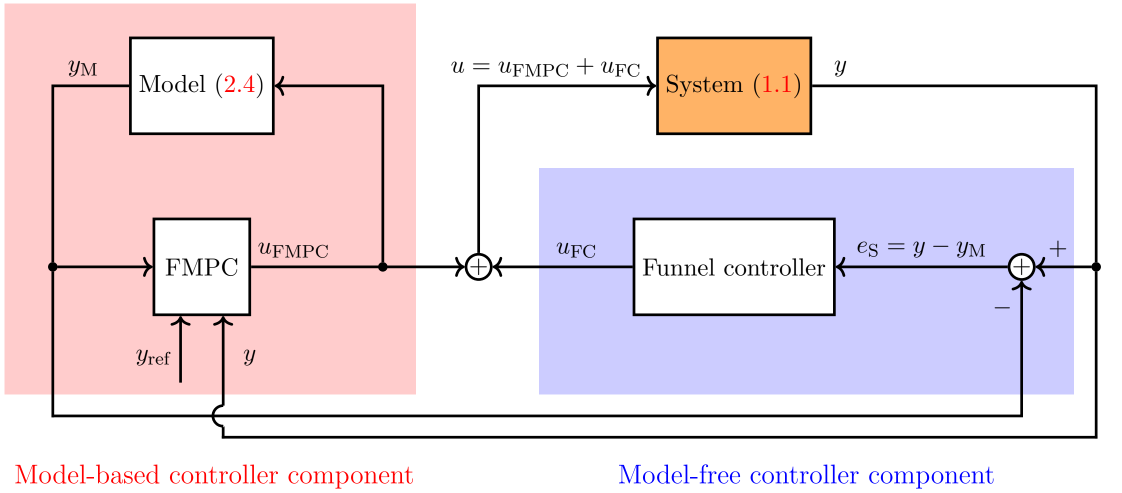


Figure 3.1: Structure of the robust funnel MPC scheme

framework addresses the inherent tension between optimality and robustness by combining the predictive capabilities of MPC with the disturbance rejection of adaptive feedback.

The left (red) block of Figure 3.1 comprises the surrogate model (2.4), the funnel MPC Algorithm 2.4.1, and a given reference trajectory y_{ref}. By Theorem 2.4.3, for any given funnel function $\psi \in \mathcal{G}$, the funnel-MPC controller produces an input u_{FMPC} that minimises the stage cost (2.1) while guaranteeing the model's output y_M tracks y_{ref} within the prescribed funnel ψ, i.e.

$$\|e_M(t)\| = \|y_M(t) - y_{ref}(t)\| < \psi(t) \text{ for all } t \geq t_0.$$

This controller component relies on the model's accuracy but delivers optimality by design. Since the model is chosen by the designer, it is however known exactly.

In contrast, the right block contains the actual system (1.1) and a model-free funnel control loop (blue box in Figure 3.1). Given an arbitrary reference signal $\rho \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and a funnel function $\varphi \in \mathcal{G}$, the funnel control (cf. Section 1.1.2) ensures the system's output y satisfies $\varphi(t)\,\|y(t) - \rho(t)\| < 1$ for all $t \geq t_0$, as shown in [30, 33, 95]. This component requires no model knowledge to compute the control signal u_{FC} and track the reference with predefined accuracy, operating purely on real-time measurements, and is inherently robust to disturbances and system uncertainties, provided the initial error lies within the funnel boundary.

While funnel MPC prioritises optimality through minimisation of a designer-specified cost, funnel control ensures robustness by adaptively rejecting disturbances. Merging these two approaches *robustifies* the funnel MPC scheme against model uncertainties and disturbances. The combined control signal $u = u_{FMPC} + u_{FC}$ sacrifices strict optimality (due to the corrective u_{FC}) and model independence (due to reliance on (2.4)) but achieves a critical balance: the funnel controller remains dormant unless the model-predicted error e_M approaches the funnel boundary ψ. In such critical states – where the model inaccuracies or disturbances threaten constraint violation – u_{FC} activates to realign the system with the model's prediction. By keeping the funnel controller's activation intentionally sparse, it intervenes only as much as necessary to reject disturbances. This minimises deviations from the optimal control signal u_{FMPC}. For instance, if the model inaccurately predicts a disturbance's impact, the funnel controller adjusts u_{FC} instantaneously using high-gain feedback. This ensures the system's output $y(t)$ adheres to constraints even when the model's predictions y_{M} diverge from reality. This minimal intervention strategy preserves near-optimal performance whenever the model is accurate, while enforcing robustness in the presence of mismatches.

Before detailing the precise interconnection and proving that the overall scheme meets the control objective stated in Section 1.1.1, we first outline in more detail the operating principles of the model-free funnel controller. To this end, we will utilise the funnel controller from [30]. This controller uses error variables structurally similar to ξ_i that we have defined in (2.15) to be used by the funnel MPC Algorithm 2.4.1. For $\varphi > 0$, a bijection $\gamma \in \mathcal{C}^1([0,1),[1,\infty))$, $\varepsilon \in (0,1]$, and $z = (z_1,\ldots,z_r) \in \mathbb{R}^{rm}$ with $z_i \in \mathbb{R}^m$, we formally introduce auxiliary error variables e_i for $i = 1,\ldots,r$ in the following. Define

$$e_1(\varphi,z) := \varphi z_1, \quad \mathcal{E}_1^{\varepsilon}(\varphi) := \{ z \in \mathbb{R}^{rm} \,|\, \|e_1(\varphi,z)\| < \varepsilon \},$$

and recursively for $z \in \mathcal{E}_i^{\varepsilon}(\varphi)$ define

$$\begin{aligned} e_{i+1}(\varphi,z) &:= \varphi z_{i+1} + \gamma\left(\|e_i(\varphi,z)\|^2\right) e_i(\varphi,z), \\ \mathcal{E}_{i+1}^{\varepsilon}(\varphi) &:= \{ z \in \mathbb{R}^{rm} \,|\, \|e_i(\varphi,z)\| < \varepsilon,\ j = 1,\ldots,i+1 \}, \end{aligned} \tag{3.3}$$

for $i = 1,\ldots,r-1$. A suitable choice for the bijection is for example $\gamma(s) := 1/(1-s)$. Note that in the definition of e_i and $\mathcal{E}_i^{\varepsilon}(\varphi)$ the value φ can be replaced with a time-varying function $\varphi(\cdot)$ with $\varphi(t) > 0$ for all t. We will make use of this observation.

In Section 2.3.1, we saw that the auxiliary error variables ξ_i introduced in (2.15) have the property that, for $\hat{t} \geq t_0$ and a function $\zeta \in \mathcal{C}^{r-1}([\hat{t},\infty),\mathbb{R}^m)$, all error signals $\xi_i(\chi_r(\zeta)(t))$ for $i = 1,\ldots,r-1$ evolve within their respective funnels given by ψ_i if the last error variable $\xi_r(\chi_r(\zeta)(t))$ evolves within its funnel given by ψ_r, see Proposition 2.3.11. In the following, we show that the error variables e_i in (3.3) exhibit a similar property. To that end, we define, for a function $\varphi \in \mathcal{G}$, the set

$$\mathfrak{Y}_{\hat{t}}^{\varphi} := \left\{ \zeta \in \mathcal{C}^{r-1}(\mathbb{R}_{\geq 0},\mathbb{R}^m) \,\middle|\, \zeta|_{[0,t_0]} = y^0, \forall\, t \in [t_0,\hat{t}) : \chi_r(\zeta)(t) \in \mathcal{E}_r^1(\varphi(t)) \right\}. \tag{3.4}$$

This is the set of all functions $\zeta \in \mathcal{C}^{r-1}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$ coinciding with y^0 and for which $\chi_r(\zeta)$ evolves within $\mathcal{E}_r^1$ on the interval $[t_0,\hat{t})$ for $\hat{t} > t_0$. We show that all error signals $e_i(\varphi(t),\chi_r(\zeta))$ for $i = 1,\ldots,r-1$ evolve within $\mathcal{E}_r^{\varepsilon}(\varphi(t))$ if the norm of the last auxiliary error $e_r(\varphi(t),\chi_r(\zeta))$ remains lower than one for all $t \in [t_0,\hat{t})$ and if all error values e_i at initial time t_0 are an element of $\mathcal{E}_r^{\varepsilon}(\varphi(t_0))$.

Lemma 3.2.1. *Let $\varphi \in \mathcal{G}$, $\gamma \in \mathcal{C}^1([0,1),[1,\infty))$ be a bijection, and $y^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m)$ with $\chi_r(y^0)(t_0) \in \mathcal{E}_r^1(\varphi(t_0))$ be given. Then, there exist constants $\varepsilon_i, \mu_i > 0$ such that for all $\hat{t} \in (t_0,\infty]$ and all $\zeta \in \mathfrak{Y}_{\hat{t}}^{\varphi}$ the functions e_i defined in (3.3) satisfy*

- *i)* $\|e_i(\varphi(t),\chi_r(\zeta)(t))\| \leq \varepsilon_i < 1$,
- *ii)* $\left\|\frac{\mathrm{d}}{\mathrm{d}t} e_i(\varphi(t),\chi_r(\zeta)(t))\right\| \leq \mu_i$,

for all $t \in [t_0,\hat{t}]$ and for all $i = 1,\ldots,r-1$.

Proof. We introduce the constants ε_i, μ_i. Let $\varepsilon_0 = 0$ and $\bar{\eta}_0 := 0$. Utilising the bijectivity of γ, define successively

$$\begin{aligned} &\hat{\varepsilon}_i \in (0,1) \text{ s.t. } \gamma(\hat{\varepsilon}_i^2)\hat{\varepsilon}_i \geq \left\|\frac{\dot{\varphi}}{\varphi}\right\|_{\infty} (1 + \gamma(\varepsilon_{i-1}^2)\varepsilon_{i-1}) + 1 + \bar{\eta}_{i-1}, \\ &\varepsilon_i := \max\{\|e_i(\varphi(t_0),\chi_r(y_0)(t_0))\|, \hat{\varepsilon}_i\} < 1, \\ &\mu_i := \left\|\frac{\dot{\varphi}}{\varphi}\right\|_{\infty} (1 + \gamma(\varepsilon_{i-1}^2)\varepsilon_{i-1}) + 1 + \gamma(\varepsilon_i^2)\varepsilon_i + \bar{\eta}_{i-1}, \\ &\bar{\eta}_i := 2\dot{\gamma}(\varepsilon_i^2)\varepsilon_i^2\mu_i + \gamma(\varepsilon_i^2)\mu_i, \end{aligned} \tag{3.5}$$

for $i = 1, \dots, r-1$. To improve legibility, we use the notation $e_i(t) := e_i(\varphi(t), \chi_r(\zeta)(t))$ for $\zeta \in \mathfrak{Y}_{\hat{t}}^{\varphi}$. Let $\hat{t} \in (t_0, \infty]$ and $\zeta \in \mathfrak{Y}_{\hat{t}}^{\varphi}$ be arbitrary but fixed. We define the auxiliary functions $\eta_i(t) := \gamma(\|e_i(t)\|^2)e_i(t)$, and set $\eta_0(\cdot) = \dot{\eta}_0(\cdot) = 0$. To further increase readability, we omit the dependency of these functions on t in the following. Note that, for $i = 1, \dots, r-1$, each of the error signals defined in (3.3) satisfies

$$\dot{e}_i = \dot{\varphi}\zeta^{(i)} + \varphi\zeta^{(i+1)} + \dot{\eta}_{i-1} = \frac{\dot{\varphi}}{\varphi}(e_i - \eta_{i-1}) + e_{i+1} - \gamma(\|e_i\|^2)e_i + \dot{\eta}_{i-1}$$

for $t \in [t_0, \hat{t})$. We observe

$$\dot{\eta}_i = 2\dot{\gamma}(\|e_i\|^2)\langle e_i, \dot{e}_i\rangle e_i + \gamma(\|e_i\|^2)\dot{e}_i.$$

Seeking a contradiction, we assume that, for at least one $j \in \{1, \dots, r-1\}$, there exists $t^\star \in (t_0, \hat{t})$ such that $\|e_j(t^\star)\|^2 > \varepsilon_j$. W.l.o.g. we assume that this is the smallest possible j. Invoking the assumption $\chi_r(y^0) \in \mathcal{E}_r^1(\varphi(t_0))$ and the continuity of the involved functions, we may define $t_\star := \max\left\{t \in [t_0, t^\star) \,\middle|\, \|e_j(t)\|^2 = \varepsilon_j\right\}$. Then, we calculate

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|e_j\|^2 &= \left\langle e_j, \tfrac{\dot{\varphi}}{\varphi}(e_j - \eta_{j-1}) + e_{j+1} + \dot{\eta}_{j-1} - \gamma(\|e_j\|^2)e_j\right\rangle \\ &\le \|e_j\|\left(\left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \gamma(\varepsilon_{j-1}^2)\varepsilon_{j-1}) + 1 + \bar{\eta}_{j-1} - \gamma(\varepsilon_j^2)\varepsilon_j\right) \le 0\end{aligned}$$

for $t \in [t_\star, t^\star]$. In this estimation, we used the monotonicity of $\gamma(\cdot)$, the definition of ε_j, and the fact that $\dot{\eta}_{j-1}$ is bounded due to the minimality of j. Hence, the contradiction $\varepsilon_j < \|e_j(t^\star)\|^2 \le \|e_j(t_\star)\|^2 = \varepsilon_j$ arises after integration. This yields boundedness of e_j, η_j. Using the derived bounds, we estimate

$$\|\dot{e}_j\| \le \left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \gamma(\varepsilon_{j-1}^2)\varepsilon_{j-1}) + 1 + \gamma(\varepsilon_j^2)\varepsilon_j + \bar{\eta}_{j-1} = \mu_j.$$

We conclude $\|e_i(t)\| \le \varepsilon_i < 1$ and $\|\dot{e}_i(t)\| \le \mu_i$ for all $i = 1, \dots, r-2$ and all $t \in [t_0, \hat{t})$. For $i = r-1$, the same arguments are valid invoking $e_r : [t_0, \hat{t}) \to \mathcal{B}_1$. □

Comparable to result in Corollary 2.3.13 about the error signal ξ_1, Lemma 3.2.1 shows that the auxiliary error signals e_i for $i = 1, \dots, r-1$ maintain a uniform ε distance to the boundary of $\mathcal{E}_r^\varepsilon(\varphi(t))$. If the initial errors are small enough, then the ε_i in (3.5) can be chosen independent of the concrete values of $\|e_i(\varphi(t_0), \chi_r(y_0)(t_0))\|$ for $i = 1, \dots, r-1$. We summarise this in the following.

Corollary 3.2.2. *Let $\varphi \in \mathcal{G}$, $\gamma \in \mathcal{C}^1([0,1), [1,\infty))$ be a bijection, and $r > 1$. Then, there exists $\varepsilon \in (0,1)$ such that for all $y^0 \in \mathcal{C}^{r-1}([0,t_0], \mathbb{R}^m)$ with $\chi_r(y^0)(t_0) \in \mathcal{E}_{r-1}^\varepsilon(\varphi(t_0))$ and for all $\hat{t} \in (t_0, \infty]$ every $\zeta \in \mathfrak{Y}_{\hat{t}}^{\varphi}$ satisfies*

$$\forall\, t \in [t_0, \hat{t}): \quad \chi_r(\zeta)(t) \in \mathcal{E}_{r-1}^\varepsilon(\varphi(t)).$$

Proof. The claim immediately follows from the proof of Lemma 3.2.1 by choosing ε as the minimum of all $\hat{\varepsilon}_i$ in (3.5). □

Building on Lemma 3.2.1, we demonstrate that the funnel control law from [30] guarantees the system (1.1) tracks a given reference signal $\rho \in W^{r,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$ within predefined boundaries governed by a function $\varphi \in \mathcal{G}$. This result is generalised to accommodate bounded disturbances $d \in L^\infty([t_0,\infty), \mathbb{R}^m)$ in the input channel. To achieve this, we leverage the *perturbation high-gain property* defined in Definition 3.1.1 (b), ensuring robustness to such disturbances while maintaining tracking performance.

Proposition 3.2.3. *Consider a system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}^{m,r}_{t_0}$ as in Definition 3.1.1. Let $\mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ be a surjection, $\gamma \in \mathcal{C}^1([0,1),[1,\infty))$ be a bijection. Further, let the functions $y^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m)$, $\rho \in W^{r,\infty}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$, and $\varphi \in \mathcal{G}$ be given such that $\chi_r(y^0-\rho)(t_0) \in \mathcal{E}^1_r(\varphi(t_0))$ and let $d \in L^\infty([t_0,\infty),\mathbb{R}^m)$ be an arbitrary disturbance. Then, the application of*

$$u(t) = (\mathcal{N}\circ\gamma)\left(\|e_r(\varphi(t),\chi_r(e)(t))\|^2\right) e_r(\varphi(t),\chi_r(e)(t)), \quad e(t) := y(t)-\rho(t), \tag{3.6}$$

to the system

$$y^{(r)}(t) = F\big(\mathbf{T}(\chi_r(y)(t), d(t)+u(t)\big), \quad y|_{[0,t_0]} = y^0, \tag{3.7}$$

yields a closed-loop initial value problem, which has a solution, every solution can be maximally extended, and every maximal solution $y : [0,\omega) \to \mathbb{R}^m$ has the following properties

(i) the solution is global, i.e. $\omega = \infty$,

(ii) all signals are bounded, in particular, $u \in L^\infty([t_0,\infty),\mathbb{R}^m)$ and $y \in W^{r,\infty}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$,

(iii) there exists $\varepsilon \in (0,1)$ such that the error signals given by e_i for $i = 1,\dots,r$ as in (3.3) are uniformly bounded by ε, i.e.

$$\forall\, t \geq t_0 : \chi_r(y-\rho)(t) \in \mathcal{E}^\varepsilon_r(\varphi(t)).$$

This implies, in particular, that the tracking error evolves within prescribed error bounds, i.e.

$$\forall\, t \geq t_0 : \|\varphi(t)(y(t)-\rho(t))\| < 1.$$

Proof. We modify the proof of [30, Thm. 1.9] to the current setting.
Step 1: We show the existence of a solution of the feedback-controlled initial value problem (3.7) with funnel control (3.6). To this end, define the set

$$\mathcal{E} := \left\{ (t,z) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{rm} \,\middle|\, z-\chi_r(\rho)(t) \in \mathcal{E}^1_r(\varphi(t)) \right\}$$

where $\mathcal{E}^1_r$ is defined as in (3.3). Moreover, formally define the function $\tilde{F} : \mathcal{E}\times\mathbb{R}^q \to \mathbb{R}^{rm}$ mapping $(t,z,\eta) = (t,z_1,\dots,z_r,\eta)$ to

$$\tilde{F}(t,z,\eta) := \begin{bmatrix} z_2 \\ \vdots \\ z_r \\ F(\eta, d(t) + (\mathcal{N}\circ\gamma)\left(\|e_r(\varphi(t), z-\chi_r(\rho)(t)\|^2\right) e_r(\varphi(t), z-\chi_r(\rho)(t)) \end{bmatrix}.$$

Using the notation $x(t) = \chi_r(y)(t)$, the initial value problem (3.7) with feedback control (3.6) takes the form

$$\dot{x} = \tilde{F}(t,x(t),\mathbf{T}(x)(t)), \quad x|_{[0,t_0]} = \chi_r(y^0) \in \mathcal{C}([0,t_0],\mathbb{R}^{rm}). \tag{3.8}$$

By assumption, we have $(t_0, x(t_0)) \in \mathcal{E}$. Application of Theorem 7.0.4 yields the existence of a maximal solution $x : [0,\omega) \to \mathbb{R}^{rm}$, $\omega \in (t_0,\infty]$ of (3.8) with

$$\operatorname{graph}\left(x|_{[t_0,\omega)}\right) \subset \mathcal{E}.$$

Moreover, the closure of $\operatorname{graph}\left(x|_{[t_0,\omega)}\right)$ is not a compact subset of $\mathcal{E}$.
Step 2: We define several constants for later use. To improve legibility, we use the notation $e_k(t) := e_k(\varphi(t), x(t)-\chi_r(\rho)(t))$ for $k = 1,\dots,r$ and $t \in [t_0,\omega)$ where $e(\cdot,\cdot)$ is defined as

in (3.3). Further, denote with $e(t) := y(t) - \rho(t)$ the tracking error between y (the first m-dimensional component of x) and ρ. For the auxiliary function $\eta_k(t) := \gamma(\|e_k(t)\|^2)e_k(t)$ with $k = 1, \ldots, r-1$, we observe

$$\dot{\eta}_k = 2\dot{\gamma}(\|e_k\|^2)\langle e_k, \dot{e}_k\rangle e_k + \gamma(\|e_k\|^2)\dot{e}_k,$$

omitting the dependency on t. Lemma 3.2.1 yields the existence of $\varepsilon_k, \mu_k > 0$ such that $\|e_k(t)\| \leq \varepsilon_k < 1$ and $\left\|\frac{\mathrm{d}}{\mathrm{d}t}e_k(t)\right\| \leq \mu_k$ for all $t \in [t_0, \omega)$ and all $k = 1, \ldots, r-1$. Thus, there exists $\bar{\eta}_{r-1} \geq 0$ such that $\|\dot{\eta}_{r-1}(t)\| \leq \bar{\eta}_{r-1}$ for all $t \in [t_0, \omega)$ (in the case $r = 1$ set $\eta_0(\cdot) = \dot{\eta}_0(\cdot) = 0$). Moreover, $\|e_k(t)\| \leq \varepsilon_k$ for $k = 1, \ldots r-1$ and $\|e_r(t)\| < 1$ for all $t \in [t_0, \omega)$ implies the boundedness of $x(\cdot)$ in $\mathbb{R}^{rm}$ on the interval $[t_0, \omega)$ because $\chi_r(\rho)(\cdot)$ is bounded by assumption and $\inf_{t \geq 0} \varphi(t) > 0$, see definition of e_k in (3.3). Thus, there exists a compact set $K_q \subset \mathbb{R}^q$ with $\mathbf{T}(x)(t) \in K_q$ for all $t \geq t_0$ according to the bounded-input bounded-output property (T.3) of operator $\mathbf{T}$. Choose a compact set $K_m \subset \mathbb{R}^m$ with $d(t) \in K_m$ for all $t \geq t_0$. As F has the perturbation high-gain property, let $\nu \in (0,1)$ such that the function

$$\mathfrak{h}(s) := \min\left\{\langle v, F(z, d - sv)\rangle \,|\, d \in K_m, z \in K_q, v \in \mathbb{R}^m,\ \nu \leq \|v\| \leq 1\right\}$$

is unbounded from above, see Definition 3.1.1 (b). Due to the unboundedness of the function $\mathfrak{h}$ and the surjectivity of $\mathcal{N} \circ \gamma$, it is possible to choose $\varepsilon_r \in (0,1)$ such that $\varepsilon_r > \max\{\nu, \|e_r(t_0)\|\}$ and

$$\tfrac{1}{2}\mathfrak{h}(\mathcal{N} \circ \gamma(\varepsilon_r^2)) \geq \theta := \left\|\frac{\dot{\varphi}}{\varphi^2}\right\|_\infty (1 + \gamma(\varepsilon_{r-1}^2)\varepsilon_{r-1}) + \left\|\frac{\bar{\eta}_{r-1}}{\varphi}\right\|_\infty + \left\|\rho^{(r)}\right\|_\infty, \tag{3.9}$$

with $\varepsilon_0 = 0$.

Step 3: We show $\|e_r(t)\| \leq \varepsilon_r$ for all $t \in [t_0, \omega)$. Seeking a contradiction, assume there exists $t^\star \in [t_0, \omega)$ with $\|e_r(t^\star)\| > \varepsilon_r$. Due to the continuity of e_r on $[t_0, t^\star]$, there exists

$$t_\star := \sup\left\{t \in [t_0, t^\star) \,|\, \|e_r(t)\| = \varepsilon_r\right\} < t^\star.$$

Then, we have $\|e_r(t)\| \geq \varepsilon_r \geq \nu$ for all $t \in [t_\star, t^\star]$ and $\mathfrak{h}(\mathcal{N} \circ \gamma(\|e_r(t_\star)\|^2)) \geq 2\theta$. Thus, there exists $\tilde{t} \in [t_\star, t^\star]$ such that $\mathfrak{h}(\mathcal{N} \circ \gamma(e_r(t))) \geq \theta$ for all $t \in [t_\star, \tilde{t}]$. Utilising the definition of e_r in (3.3), we have

$$\left\|e^{(r-1)}(t)\right\| = \left\|\frac{1}{\varphi(t)}(e_r(t) - \gamma(e_{r-1}^2)e_{r-1})\right\| < \frac{1}{|\varphi(t)|}\left(1 + \gamma(\varepsilon_{r-1}^2)\varepsilon_{r-1}\right)$$

for all $t \in [t_\star, \tilde{t}]$ and with $e_0(t) = \dot{e}_0(t) = 0$ in the case of $r = 1$. Omitting the dependency on t, we calculate that, for almost all $t \in [t_\star, \tilde{t}]$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|e_r\|^2 &= \langle e_r, \dot{e}_r\rangle \\
&= \left\langle e_r, \dot{\varphi}e^{(r-1)} + \varphi e^{(r)} + \dot{\eta}_{r-1}\right\rangle \\
&= \dot{\varphi}\left\langle e_r, e^{(r-1)}\right\rangle + \varphi\left\langle e_r, F(\mathbf{T}(x), d + u) - \rho^{(r)}\right\rangle + \left\langle e_r, \dot{\eta}_{r-1}\right\rangle \\
&\leq |\dot{\varphi}|\left\|e_r\right\|\left\|e^{(r-1)}\right\| + \left\|e_r\right\|\|\dot{\eta}_{r-1}\| + \varphi\left\|e_r\right\|\left\|\rho^{(r)}\right\| + \varphi\langle e_r, F(\mathbf{T}(x), d + u)\rangle \\
&\leq |\dot{\varphi}|\,|\varphi|(1 + \gamma(\varepsilon_{r-1}^2)\varepsilon_{r-1}) + \bar{\eta}_{r-1} + \varphi\left\|\rho^{(r)}\right\| + \varphi\langle e_r, F(\mathbf{T}(x), d + u)\rangle \\
&\leq \varphi \cdot \left(\left\|\frac{\dot{\varphi}}{\varphi^2}\right\|_\infty (1 + \gamma(\varepsilon_{r-1}^2)\varepsilon_{r-1}) + \left\|\frac{\bar{\eta}_{r-1}}{\varphi}\right\|_\infty + \left\|\rho^{(r)}\right\|_\infty + \langle e_r, F(\mathbf{T}(x), d + u)\rangle\right) \\
&= \varphi \cdot (\theta + \langle e_r, F(\mathbf{T}(x), d + u)\rangle) \\
&= \varphi \cdot \left(\theta + \left\langle e_r, F(\mathbf{T}(x), d + (\mathcal{N} \circ \gamma)(\|e_r\|^2)e_r)\right\rangle\right)
\end{aligned}$$

$$\le \varphi \cdot \left(\theta - \min \left\{ \left\langle v, F\left(z, d - (\mathcal{N} \circ \gamma)\left(\|e_r\|^2\right) v\right)\right\rangle \,\middle|\, \begin{array}{l} d \in K_m, \\ z \in K_q, \\ \nu \le \|v\| \le 1 \end{array} \right\} \right)$$
$$\le \varphi \cdot \left(\theta - \mathfrak{h}\left((\mathcal{N} \circ \gamma)\left(\|e_r\|^2\right)\right)\right) \le 0.$$

Integration yields $\varepsilon < \left\|e_r(\tilde{t})\right\| \le \|e_r(t_\star)\| = \varepsilon$, a contradiction. Therefore, we have $\|e_r(t)\| \le \varepsilon_r$ for all $t \in [t_0, \omega)$.
Step 4: As a consequence of Lemma 3.2.1 and Step 3 $\|e_k(t)\| \le \varepsilon_k$ for all $t \in [t_0, \omega)$ and all $k = 1, \ldots, r$. Choosing $\varepsilon \in (0,1)$ with $\varepsilon > \varepsilon_i$ for all $i = 1, \ldots, r$ shows (iii). By the definition of e_k in (3.3) and the boundedness of the function $\chi_r(\rho)$, the solution x is a bounded function, too. Since the closure of graph $\left(x|_{[t_0,\omega)}\right)$ is not a compact subset of $\mathcal{E}$, this implies $\omega = \infty$ and thereby shows (i). Further, $\|e_r(t)\| \le \varepsilon_r < 1$ implies the boundedness of u in (3.6). Together with the definition of y as the first m-dimensional component of x, see Definition 3.1.4, shows (ii) and completes the proof. □

Remark 3.2.4. The perturbation high-gain property (b) holds for $F \in \mathcal{C}(\mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m)$ if, and only if, for every compact set $K_m \subset \mathbb{R}^m$ there exists $\nu \in (0,1)$ such that, for every compact set $K_q \subset \mathbb{R}^q$, the function $\mathfrak{h}$ defined in (3.1) fulfils

$$\sup_{s>0} \mathfrak{h}(s) = \infty \quad \text{or} \quad \sup_{s<0} \mathfrak{h}(s) = \infty.$$

If $\sup_{s>0} \mathfrak{h}(s) = \infty$ for such K_m, ν and K_q, then we say that F has the *negative-definite perturbation high-gain property* (respectively, *positive-definite perturbation high-gain property* if $\sup_{s<0} \mathfrak{h}(s) = \infty$). If it is a priori known that the negative-definite perturbation high-gain property holds for F, then the surjection $\mathcal{N}$ in (3.6) can be replaced by any surjection $\mathbb{R}_{\ge 0} \to [0, \infty)$. The simplest example is the identity map $s \mapsto s$. The feedback law (3.6) then takes the form $u(t) = \gamma(\|e_r(t)\|^2) e_r(t)$, where $e_r(t) = e_r(\varphi(t), \chi_r(e)(t))$. Similarly, if F has the positive-definite perturbation high-gain property, then the surjection $\mathcal{N}$ in (3.6) can be replaced by an arbitrary surjection $\mathbb{R}_{\ge 0} \to (-\infty, 0]$. •

Proposition 3.2.3 demonstrates that applying the funnel controller u_{FC} (as defined in (3.6)) to the system (1.1) forces the system's output y to track any given reference signal $\rho \in W^{r,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$ within given accuracy bounds governed by a function $\varphi \in \mathcal{G}$. The funnel controller generates its control signal u_{FC} solely from instantaneous measurements of the error signal $e_r(t) = e_r(\varphi(t), \chi_r(y - \rho)(t))$ and requires no model information or look-ahead. However, since the controller lacks predictive capacities, it may yield suboptimal tracking performance or excessive control effort over extended horizons. Crucially, naively deploying the *same* reference signal y_{ref} and funnel function ψ for both the model-based (MPC) and the model-free component (funnel control) risks rendering the MPC signal u_{FMPC} a disruptive disturbance to the funnel controller.

To leverage model-based prediction while retaining the funnel's robustness, we propose a refined integration of the funnel MPC Algorithm 2.4.1 and the funnel controller (3.6). Instead of sharing y_{ref} and ψ, the MPC's predicted model output y_{M} serves as a reference signal for the funnel controller. As depicted in Figure 3.1, the combined controller structure operates as follows:

- **Funnel MPC (red box)**: Computes the control signal $u_{\text{FMPC}}(t)$ and the corresponding model output $y_{\text{M}}(t)$ over the intervals $[t_k, t_{k+1})$ with $t_k \in t_0 + \delta\mathbb{N}_0$ and $\delta > 0$.
- **Funnel controller (blue box)**: Receives y_{M} as its reference, ensuring the system output y tracks y_{M} with prescribed accuracy:

$$\|e_{\text{S}}(t)\| = \varphi(t)\|y(t) - y_{\text{M}}(t)\| < 1.$$

The control signal applied to the system then is $u = u_{\text{FMPC}} + u_{\text{FC}}$. The combined controller leverages the strengths of both components in a complementary framework:

1. **Model accuracy**: When the model output y_{M} aligns perfectly with the system output y, the funnel controller remains inactive ($u_{\text{FC}} = 0$), as the MPC-generated control signal u_{FMPC} alone achieves tracking within prescribed boundaries:
$$\|y(t) - y_{\text{ref}}(t)\| = \|y_{\text{M}}(t) - y_{\text{ref}}(t)\| < \psi(t).$$
Here, the MPC's predictive planning dominates, optimising performance over the horizon without requiring corrective intervention.

2. **Model uncertainty**: Under discrepancies between the model and system, the funnel controller dynamically compensates. The tracking error $e_{\text{S}} \geq 0$ activates u_{FC}, ensuring robustness by enforcing $\varphi(t)\|y(t) - y_{\text{M}}(t)\| < 1$. The magnitude of u_{FC} scales intuitively with the model mismatch – greater deviations demand stronger corrective action, while closer alignment shifts dominance to u_{FMPC}.

This dynamic interaction between the components creates a synergetic self-regulating control hierarchy: The MPC component provides optimal foresight, minimising control effort and improving long-term tracking and the funnel controller acts as a safety layer, guaranteeing transient performance and stability despite uncertainties. Utilising different reference signals (y_{M} for the funnel controller vs. y_{ref} for the funnel MPC component), the design avoids conflict, ensuring u_{FMPC} enhances – rather than disrupts – the funnel controller's corrective role.

3.2.1 Funnel boundary and proper initialisation

The funnel controller (3.6) permits the utilisation of quite general boundary functions $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. We design φ to ensure that the feedback controller not only compensates for model-plant mismatch $e_{\text{S}} = y - y_{\text{M}}$ but also guarantees that the system output y tracks the given reference signal y_{M} within the predefined error bound ψ imposed on the MPC component. To achieve this, we propose

$$\varphi(t) := \frac{1}{\psi(t) - \|y_{\text{M}}(t) - y_{\text{ref}}(t)\|} \tag{3.10}$$

motivated by the following rationale:

- If the MPC component ensures accurate reference tracking (i.e. $y_{\text{M}} \approx y_{\text{ref}}$), then the boundary function for the funnel controller is $\varphi \approx 1/\psi$. This corresponds to a "safe" scenario where larger deviations between the system y and model y_{M} are permissible.

- In safety-critical situations (y_{M} deviates significantly from y_{ref}), φ adaptively tightens the funnel for the model-free controller component, forcing the system to mimic the model and y to closely follow y_{M}. This ensures the MPC's optimal control input affects both dynamics comparably, preventing u_{FMPC} from acting as a disturbance to the funnel controller.

Crucially, deviations between y_{M} and y_{ref} are evaluated relative to the current funnel width ψ: Smaller ψ tolerates less absolute deviation between the system and the model and heightens sensitivity to mismatches, while larger ψ permits greater flexibility. The function φ in (3.10) inherently scales the allowable deviation in relationship to ψ. This proposed design ensures that the total tracking error $e = y - y_{\text{ref}}$ satisfies

$$\|e\| = \|y - y_{\text{M}} + y_{\text{M}} - y_{\text{ref}}\| \leq \underbrace{\|e_{\text{S}}\|}_{<1/\varphi} + \|e_{\text{M}}\| < \psi - \|e_{\text{M}}\| + \|e_{\text{M}}\| = \psi,$$

where time arguments are omitted for clarity.

In the following, we discuss mathematical difficulties arising from this particular choice of funnel φ and reference y_{M}. A notable initial concern is the potential discontinuity of y_{M} (and consequently of the function φ) due to the model's re-initialisation in Step (a) of Algorithm 2.4.1. At each time $t_k = t_0 + \delta\mathbb{N}_0$, the model's initial state in (2.4) is set to $\mathfrak{X}_k$, which may introduce jumps in the concatenated trajectory y_{M}. While Proposition 3.2.3 assumes continuity of φ and ρ, this discontinuity is largely a technical nuance. However, careful initialisation of the combined controller is critical to ensure compatibility between the funnel MPC and funnel controller component.

To preserve the feasibility of Algorithm 2.4.1 (as established in Section 2.3.2), the initial model state $(x_{\mathrm{M}}^k, \mathbf{T}_{\mathrm{M}}^k) = \mathfrak{X}_k$ at time t_k must be an element of $\mathfrak{I}^{\Psi}_{t_0,\tau}(t_k)$. In particular, this implies

$$x_{\mathrm{M}}^k(t_k) - \chi_r(y_{\mathrm{ref}})(t_k) \in \mathcal{D}^{\Psi}_{t_k},$$

as per Remark 2.3.16. Beyond this constraint, the MPC component permits considerable freedom in selecting the initial state $\mathfrak{X}_k$.

To maximise the effectiveness of the funnel MPC component, we want to achieve the control objective of tracking the reference signal y_{ref} primarily through the (piecewise) optimal MPC control signal u_{FMPC}, with ideally minimal funnel controller interventions to correct deviations between the system output y and the model output y_{M}. The model's re-initialisation by $\mathfrak{X}_k$ at each time $t_k = t_0 + \delta\mathbb{N}_0$ is pivotal for maintaining a small model-system mismatch. A sophisticated initialisation strategy, leveraging system output measurements is therefore advisable. Let $y(t_k)$ denote the system output (from (1.1)) and y_{M}^k the model output at time $t_k = t_0 + \delta\mathbb{N}_0$ after initialisation with $\mathfrak{X}_k$, i.e. the first m-dimensional component of $x_{\mathrm{M}}^k(t_k)$. For the funnel controller (3.6) to function correctly when applied to system (1.1) and tracking a given reference ρ within boundaries φ, Proposition 3.2.3 requires

$$\chi_r(y - \rho)(t_k) \in \mathcal{E}^1_r(\varphi(t_k)).$$

This restricts potential choices for the initialisation of the model. The primary mathematical difficulty however lies in ensuring that the funnel controller component remains uniformly bounded on the entire interval $[t_0, \infty)$. Crucially, the maximal control input of (3.6) depends on the maximal value of the error variables e_i as defined in (3.3) for $i = 1, \dots, r$. As we choose ρ to be the model's output y_{M} and φ according to (3.10), these error variables are in a sense "re-initialised" with every initialisation of the model (2.4). While the funnel controller guarantees the boundedness of these error signals between every iteration of the MPC loop, the initialisation of the model with value $\mathfrak{X}_k$ has to ensure that the values $e_i(t_k)$ remain uniformly bounded over all time instants $t_k = t_0 + \delta\mathbb{N}_0$. For the combined controller, this poses the condition

$$\chi_r(y - y_{\mathrm{M}}^k)(t_k) \in \mathcal{E}^{\varepsilon}_r\left(\frac{1}{\psi(t_k) - \|y_{\mathrm{M}}^k - y_{\mathrm{ref}}(t_k)\|}\right)$$

for some $\varepsilon \in (0,1)$. The maximal control input moreover depends on $\left\|y_{\mathrm{M}}^{(r)}\right\|_\infty$, $\left\|\frac{1}{\varphi}\right\|_\infty$, and $\left\|\dot\varphi\right\|_\infty$, see proof of Proposition 3.2.3. For systems of order $r = 1$, boundedness of $\frac{\dot\varphi}{\varphi^2}$ instead of $\dot\varphi$ suffices, see definition of ε_r in the aforementioned proof. The boundedness of $y_{\mathrm{M}}^{(r)}$ directly follows from $\|u_{\mathrm{FMPC}}\|_\infty \le u_{\max}$ and Lemma 2.3.20. Moreover,

$$\|1/\varphi\|_\infty = \|\psi - \|y_{\mathrm{M}} - y_{\mathrm{ref}}\|\|_\infty \le \|\psi\|_\infty + \|y_{\mathrm{M}} - y_{\mathrm{ref}}\|_\infty \le 2\|\psi\|_\infty < \infty$$

since y_{M}, ψ and y_{ref} are bounded. For systems of order $r > 1$, we additionally have to ensure the existence of some $\lambda \in (0,1)$ with

$$\forall\, t \in [t_0, \infty): \quad \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \lambda\psi(t) < \psi(t) \tag{3.11}$$

in order to guarantee the uniform boundedness of $\dot{\varphi}$. While Theorem 2.4.3 only mandates $\|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$ for all $t \in [t_0, \infty)$, Corollary 2.3.13 confirms that (3.11) holds provided the model (2.4) is initialised with a sufficient distance from the funnel boundary, i.e. y_{M}^k fulfils 3.11 at each time instant $t_k \in t_0 + \delta\mathbb{N}_0$.

The following definition formalises the requirements for initialising the model (2.4) in the combined controller (see Figure 3.1).

Definition 3.2.5 (Proper initial values $\mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(\hat{t},\hat{x})$)**.** *Let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\tau \geq 0$, $\varepsilon, \lambda \in (0,1)$, and $\Psi = (\psi_1, \dots, \psi_r) \in \mathscr{G}$. Given the system data $\hat{x} \in \mathbb{R}^{rm}$, we define the set of* proper $(\varepsilon,\ \lambda)$-initial values *for the model* (2.4) *at time $\hat{t} \geq t_0$ as*

$$\mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(\hat{t},\hat{x}) := \left\{ (\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t}) \;\middle|\; \begin{array}{l} \left\|\hat{x}_{\mathrm{M},1}(\hat{t}) - y_{\mathrm{ref}}(\hat{t})\right\| < \lambda \cdot \psi_1(\hat{t}), \\ \hat{x} - \hat{x}_{\mathrm{M}}(\hat{t}) \in \mathcal{E}_r^{\varepsilon}\left(1/\left(\psi_1(\hat{t}) - \left\|\hat{x}_{\mathrm{M},1}(\hat{t}) - y_{\mathrm{ref}}(\hat{t})\right\|\right)\right) \end{array} \right\}.$$

We call $\hat{\mathfrak{X}} \in \mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(\hat{t},\hat{x})$ a proper $(\varepsilon,\ \lambda)$-initialisation *at time $\hat{t}$ given system data $\hat{x} \in \mathbb{R}^{rm}$.*

By system data $\hat{x} \in \mathbb{R}^{rm}$ in Definition 3.2.5, we mean the measurement of the system output and its derivatives at time $\hat{t}$, i.e. we will replace $\hat{x}$ later with $\chi_r(y)(\hat{t})$ where y is the output of the system (1.1). Further note that we implicitly allow $\lambda = 1$ for systems with order $r = 1$ according to our considerations regarding the boundedness of $\frac{\dot{\varphi}}{\varphi^2}$.

Remark 3.2.6. For $\hat{x} \in \mathbb{R}^{rm}$ with $\hat{x} - \chi_r(y_{\mathrm{ref}})(\hat{t}) \in \mathcal{E}_r^{\varepsilon}(1/\psi_1(\hat{t}))$, the set $\mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(\hat{t},\hat{x})$ is non-empty since the pair $(\chi_r(y_{\mathrm{ref}})|_{I_0^{\hat{t},\tau}}, \mathbf{T}_{\mathrm{M}}(\chi_r(y_{\mathrm{ref}}))|_{I_{t_0}^{\hat{t},\tau}})$ is an element of $\mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$, see Remark 2.3.15. •

According to Theorem 2.3.18, the state of the model (2.4) from the previous iteration of the funnel MPC loop can be used to re-initialise the model at every time instant $t_k \in t_0 + \delta\mathbb{N}$, see also Remark 2.4.4. We will see in the proof of Theorem 3.2.11 that it is possible to operate the MPC component of the combined controller as depicted in Figure 3.1 also in such an "open-loop fashion", meaning that no data from the system is handed over to the MPC. To be a bit more precise, we will recursively prove that, during the operation of the combined controller, the state of the model, when initialised with $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k,\hat{x})$ at time t_k, is an element of the set $\mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_{k+1},\hat{x}_{k+1})$ at the next time instant t_{k+1}, where $\hat{x}_k := \chi_r(y)(t_k)$ and $\hat{x}_{k+1} := \chi_r(y)(t_{k+1})$ are the measurements of the output y of the system (1.1) at the respective time instants. In short:

$$(x_{\mathrm{M}}^k|_{[t_{k+1}-\tau, t_{k+1}]\cap[0,t_k]}, \mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}}^k))|_{[t_{k+1}-\tau, t_{k+1}]\cap[t_0,t_{k+1}]}) \in \mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_{k+1},\hat{x}_{k+1}),$$

where $x_{\mathrm{M}}^k := x_{\mathrm{M}}(\cdot; t_k, \mathfrak{X}, u_k)$ is the solution of the model differential equation (2.4) with initial data $\mathfrak{X}$ on the time interval $[t_k, t_k + T]$ when control $u_k \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ is applied to it. When the computing capacity are limited, applying the combined controller with the model predictive control component operating in an open-loop fashion is a simple way of potentially improving the performance of the funnel controller (3.6) without sacrificing speed and ease of implementation as it is possible to pre-compute the MPC's control signal u_{FMPC} in this case.

However, initialising the model predictive controller component with system measurement data sets the control algorithm on a foundation that reflects the current state of the real system (1.1). Such initialisation is therefore crucial to reduce prediction errors made by the model predictive controller component, to minimise the impact of the model-plant mismatch, and to improve the performance of the combined controller. If the system and the model are of order $r = 1$, then it is always possible to find an initialisation $\mathfrak{X}$ at time t_k such that the model output coincides with the system output. To see this, we assume for now that the combined controller as depicted in Figure 3.1 achieves the control objective as laid out in Section 1.1.1 (we will prove in Theorem 3.2.11 that this is actually the case). Let y be the output of the system (1.1). Then,

$$\|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$$

for all $t \in [t_0, t_k]$. Thus, y can be extended to a function $\tilde{y} \in \mathcal{Y}^{\Psi}_{t_k}$ with $\tilde{y}|_{[t_0,t_k]} = y$. This implies $(\tilde{y}|_{I_0^{\hat{t},\tau}}, \mathbf{T}_{\mathrm{M}}(\tilde{y}))|_{I_{t_0}^{\hat{t},\tau}}) \in \mathfrak{I}^{\Psi}_{t_0,\tau}(t_k)$ where $I_{t_0}^{t_k,\tau} := [\hat{t} - \tau, t_k] \cap [t_0, t_k]$. The function $\tilde{y}$ fulfils both

$$\|\tilde{y}(t_k) - y_{\mathrm{ref}}(t_k)\| = \|y(t_k) - y_{\mathrm{ref}}(t_k)\| < \psi(t_k)$$

and

$$\|y(t_k) - \tilde{y}(t_k)\| = 0 < \frac{\varepsilon}{\psi(t_k) - \|y(t_k) - y_{\mathrm{ref}}(t_k)\|}$$

for all $\varepsilon \in (0, 1)$. It is therefore an element of the set $\mathfrak{P}\mathfrak{I}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(\hat{t}, y)$ for $\lambda = 1$ and all $\varepsilon \in (0, 1)$ (note that we allow $\lambda = 1$ in the case $r = 1$), see Definition 3.2.5. It is therefore possible to initialise the model with $(\tilde{y}|_{I_0^{\hat{t},\tau}}, \mathbf{T}_{\mathrm{M}}(\tilde{y}))|_{I_{t_0}^{\hat{t},\tau}})$ at time t_k and the model output then coincides with the system output.

For systems of higher order, it is in general not possible to initialise the model such that $\chi_r(x_{\mathrm{M}})(t_k)$ coincides with the system's measurement data $\chi_r(y)(t_k)$ at time of initialisation $t_k \in t_0 + \delta\mathbb{N}_0$. We illustrate this in the following example.

Example 3.2.7. Consider a scalar system of order $r = 2$. The control objective is to track the constant reference trajectory $y_{\mathrm{ref}}(t) \equiv 0$ within constant boundaries given by the funnel function $\psi \equiv 1$. With the bijection $\gamma(s) := 1/(1 - s)$ for the funnel controller component, the combined controller utilises the error variables given in (2.15), (3.3)

$$\begin{aligned} \xi_1(\chi_r(y_{\mathrm{M}} - y_{\mathrm{ref}})) &= y_{\mathrm{M}}, & \xi_2(\chi_r(y_{\mathrm{M}} - y_{\mathrm{ref}})) &= \dot{y}_{\mathrm{M}} + k y_{\mathrm{M}}, \\ e_1(\varphi, \chi_r(y - y_{\mathrm{M}})) &= \varphi \cdot (y - y_{\mathrm{M}}), & e_2(\varphi, \chi_r(y - y_{\mathrm{M}})) &= \varphi \cdot \left((\dot{y} - \dot{y}_{\mathrm{M}}) + \tfrac{y - y_{\mathrm{M}}}{1 - \|e_1\|^2}\right) \end{aligned}$$

with $\varphi(t) = \frac{1}{\psi(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\|}$. As parameters for the funnel MPC algorithm, we choose the constants $\alpha = 1$, $\beta = 1/6$, and $k_1 = 2 + \alpha = 3$, and the auxiliary funnel $\psi_2 = \frac{\beta}{\alpha}$. Further, assume the system measurement $\chi_r(y) = (y(\hat{t}), \dot{y}(\hat{t})) = (2/3, 0)$ at time $\hat{t} \geq t_0$. When initialising the model with this measurement, i.e. $\chi_r(y_{\mathrm{M}})(\hat{t}) = (y_{\mathrm{M}}(\hat{t}), \dot{y}_{\mathrm{M}}(\hat{t})) := (2/3, 0)$, we have $\varphi(\hat{t}) = 1/(\psi(\hat{t}) - \left|y_{\mathrm{M}}(\hat{t}) - y_{\mathrm{ref}}(\hat{t})\right|) = 3$. Moreover, $e_1(\varphi(\hat{t}), \chi_r(y - y_{\mathrm{M}})(\hat{t})) = 0$ and $e_2(\varphi(\hat{t}), \chi_r(y - y_{\mathrm{M}})(\hat{t})) = 0$. Thus, $\chi_r(y - y_{\mathrm{M}})(\hat{t}) \in \mathcal{E}^{\varepsilon}_r(\varphi(\hat{t}))$ for all $\varepsilon \in (0, 1)$. For the auxiliary variable ξ_1, we have $|\xi_1(\chi_r(y_{\mathrm{M}} - y_{\mathrm{ref}}))| = \left|y(\hat{t})\right| = 2/3 < 1 = \psi(\hat{t})$. However,

$$\xi_2(\chi_r(y_{\mathrm{M}} - y_{\mathrm{ref}})) = \dot{y}_{\mathrm{M}}(\hat{t}) + k_1 y_{\mathrm{M}}(\hat{t}) = k_1 y(\hat{t}) = 2 > 1/6 = \psi_2(\hat{t}).$$

This means $(2/3, 0) \notin \mathcal{D}^{\Psi}_t$. Therefore, there exists no element of $\hat{x}_{\mathrm{M}} \in \mathfrak{P}\mathfrak{I}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(\hat{t}, \chi_r(y))$ coinciding with $\chi_r(y)$ at time $\hat{t}$, i.e. $\hat{x}_{\mathrm{M}}(\hat{t}) = \chi_r(y)$. ◇

Just as there exist a multitude of possibilities to initialise the funnel MPC Algorithm 2.4.1 via an initialisation strategy as defined in Definition 2.3.17, there are also many conceivable methods to select a proper (ε, λ)-initialisation $\hat{\mathfrak{X}} \in \mathfrak{P}\mathfrak{I}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(\hat{t}, \hat{x})$ given measurements $\hat{x} := \chi_r(y)(\hat{t})$ at time $\hat{t}$. A versatile strategy is solving an optimisation problem of the form

$$\underset{(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) \in \mathfrak{P}\mathfrak{I}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(\hat{t},\hat{x})}{\text{minimise}} \quad J(\hat{t}, \hat{x}, \hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}), \tag{3.12}$$

where J is a cost function that takes the desired aspects into account. For example, as it is in general not possible to find initialisation $\mathfrak{X}$ at time t_k such that the model output

$\chi_r(x_\mathrm{M})(t_k)$ coincides with the system output $\chi_r(y)(t_k)$, one could instead minimise the euclidean distance between the two vectors. Another possibility would be to give more weight to the lower derivatives, as these are presumably less affected by disturbances. A large number of potential approaches are conceivable, which can be described by such an optimisation problem.

While many MPC schemes assume access to the full system state, we consider scenarios where only output measurements $\chi_r(y)$ are available. To address the challenge of state estimation in uncertain or disturbed linear discrete-time systems, a Luenberger observer was employed to reconstruct the system state in the works [105, 139]. By integrating this observer with a tube-based MPC framework, the control scheme ensures robust constraint satisfaction and preserves recursive feasibility. This approach demonstrates how observer-based strategies can effectively compensate for state unavailability whilst maintaining closed-loop performance. Clearly, the employment of methods beyond the Luenberger observer like moving horizon estimation (MHE) [84] or non-linear state observers [39, 114] is also conceivable. Similarly, observers can be leveraged to estimate the internal state of the system and thus find more suitable initial states $\hat{\mathbf{T}}_\mathrm{M}$ of the model, i.e. initial values for the operator $\mathbf{T}_\mathrm{M}$. While our analysis is indifferent with regard to the selected initial value, it is clear that the performance of the model predictive component may significantly be improved by accurate estimates of $\mathbf{T}_\mathrm{M}$. The deployment of state observers is particularly well-suited to our problem setting when the structure of the model in (2.4) aligns with the dynamics of the physical system described in (1.1).

Activation function

Minor deviations between the system output $y(t)$ and the predicted model output $y_\mathrm{M}(t)$ are often negligible in practice, posing no risk of violating the funnel boundaries ψ. This is inherently addressed by the design of the function φ in (3.10), as $\varphi \approx 1/\psi$ when $y_\mathrm{M} \approx y_\mathrm{ref}$. From an application standpoint, it may seem advantageous to fully “deactivate” the funnel feedback controller during nominal operation and only engage it in safety-critical scenarios. To this end, we highlight the option of incorporating an *activation function*, i.e. a continuous function $\mathfrak{a} : [0,1] \to [0, \mathfrak{a}^+]$, $\mathfrak{a}^+ > 0$, with $\mathfrak{a}(1) = \mathfrak{a}^+$ into the funnel controller. This continuous function modulates the control signal u_FC based on the magnitude of the error e_r, effectively scaling the gain term $(\mathcal{N} \circ \gamma)$ in the control law (3.6). Crucially, while $\mathfrak{a}(\cdot)$ adjusts the gain magnitude, the adaptive gain mechanism remains unaffected – ensuring it retains the necessary magnitude to enforce error bounds. The use of such an activation function is rigorously justified by the following theoretical result.

Lemma 3.2.8. *Let $\mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ be a surjection, $\gamma \in \mathcal{C}([0,1), [1,\infty))$ be a bijection, and $\mathfrak{a} \in \mathcal{C}([0,1],[0,\mathfrak{a}^+])$ be an activation function with $\mathfrak{a}^+ > 0$ and $\mathfrak{a}(1) = \mathfrak{a}^+$. Then, the function $\tilde{\mathcal{N}} := (\mathfrak{a} \circ \sqrt{\gamma^{-1}}) \cdot \mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ is surjective.*

Proof. $\mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ being a surjection is equivalent to $\limsup_{s\to\infty} \mathcal{N}(s) = \infty$ and $\liminf_{s\to\infty} \mathcal{N}(s) = -\infty$. Since $\lim_{s\to\infty} (\mathfrak{a} \circ \sqrt{\gamma^{-1}})(s) = \mathfrak{a}^+ > 0$, we have

$$\limsup_{s\to\infty} \tilde{\mathcal{N}}(s) = \infty \quad \text{and} \quad \liminf_{s\to\infty} \tilde{\mathcal{N}}(s) = -\infty.$$

This implies that $\tilde{\mathcal{N}} = (\mathfrak{a} \circ \sqrt{\gamma^{-1}}) \cdot \mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ is surjective as well. □

A reasonable and simple choice for an activation function can be

$$\mathfrak{a}(s) = \begin{cases} 0, & 0 \leq s \leq S_\mathrm{crit}, \\ s - S_\mathrm{crit}, & S_\mathrm{crit} \leq s \leq 1, \end{cases}$$

for $S_{\text{crit}} \in (0,1)$. In this particular case we may set $\mathfrak{a}^+ = 1 - S_{\text{crit}}$. In the context of machine learning, in particular, artificial neural networks, this type of functions is known as *rectified linear unit* (ReLU), see e.g. [157] and references therein. Note that $\mathfrak{a}$ defined above satisfies $\mathfrak{a}(S_{\text{crit}}) = 0$, whereby it is a continuous function and thus the funnel controller contributes continuously to the overall control signal.

Lemma 3.2.8 shows that, instead of control law (3.6), it is possible to use the funnel controller u_{FC} with an activation function $\mathfrak{a}$ in Proposition 3.2.3, i.e. the control law

$$u(t) = \mathfrak{a}(\|e_r(t)\|) \cdot (\mathcal{N} \circ \gamma) \left(\|e_r(t)\|^2\right) e_r(t),$$

where $e_r(t) := e_r(\varphi(t), \chi_r(y - \rho)(t))$. In fact, a such scaled funnel controller has already been a potential controller candidate since its development in [95]. However, most examples in the literature utilise the functions $\gamma(s) = 1/(1-s)$ and $\mathcal{N}(s) = s\sin(s)$ for the control law ($\mathcal{N}(s) = \pm s$ in case of a known control direction, see Remark 3.2.4). To our knowledge, [4] was the first work to explicitly mention the possibility to "deactivate" the funnel controller for small error signals.

3.2.2 The robust funnel MPC algorithm

We now consolidate our findings into the robust funnel MPC Algorithm 3.2.9, formally defining the controller structure illustrated in Figure 3.1. Building on the definitions, concepts, and results established thus far, we prove that this scheme is initially and recursively feasible and that its application to the model (2.4) solves the tracking problem formulated in Section 1.1.1. In particular, the scheme guarantees that the deviation between the system output y and a given reference signal $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ evolves within the funnel $\mathcal{F}_\psi$ defined by a function $\psi \in \mathcal{G}$.

Algorithm 3.2.9 (Robust funnel MPC)**.**
Given:

- instantaneous measurements of the output y and its derivatives of system (1.1), initial time $t^0 \in \mathbb{R}_{\geq 0}$, initial trajectory $y^0 \in \mathcal{C}^{(r-1)}([0, t_0], \mathbb{R}^m)$, reference signal $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, funnel function $\psi \in \mathcal{G}$.
- model (2.4), signal memory length $\tau \geq 0$, auxiliary funnel boundary function $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ with corresponding parameters k_i for $i = 1, \ldots, r$, input saturation level $u_{\max} \geq 0$, and funnel stage cost function ℓ_{ψ_r},
- initialisation parameters $\varepsilon, \lambda \in (0,1)$,
- a surjection $\mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and a bijection $\gamma \in \mathcal{C}([0,1), [1,\infty))$.

Set the time shift $\delta > 0$, the prediction horizon $T \geq \delta$, and index $k := 0$.
Define the time sequence $(t_k)_{k \in \mathbb{N}_0}$ by $t_k := t_0 + k\delta$.
Steps:

(a) Obtain a measurement $\hat{x}_k := \chi_r(y)(t_k)$ of the system output y and its derivatives at the current time t_k and choose a *proper (ε,λ)-initialisation* $\mathfrak{X}_k \in \mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_k, \hat{x}_k)$ for the model.

(b) **Funnel MPC**
Compute a solution $u_{\text{FMPC},k} \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$ of the optimal control problem

$$\underset{\substack{u \in L^\infty([t_k, t_k+T], \mathbb{R}^m),\\ \|u\|_\infty \leq u_{\max}}}{\text{minimise}} \int_{t_k}^{t_k+T} \ell_{\psi_r}(s, \xi_r(x_{\text{M}}(s; t_k, \mathfrak{X}_k, u) - \chi_r(y_{\text{ref}})(s)), u(s))\text{d}s. \tag{3.13}$$

Predict the output $y_{\mathrm{M}}^k(t; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k})$ of the model on the interval $[t_k, t_{k+1}]$, and define the adaptive funnel $\varphi_k : [t_k, t_{k+1}] \to \mathbb{R}_{>0}$ by

$$\varphi_k(t) := \frac{1}{\psi_1(t) - \|e_{\mathrm{M}}^k(t)\|}, \tag{3.14}$$

where $e_{\mathrm{M}}^k(t) = y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)$.

(c) **Funnel control**
Using the error variables e_i for $i = 1, \ldots, r$ as in (3.3), define the funnel control law u_{FC} with reference y_{M}^k and funnel function φ_k as in (3.14) by

$$u_{\mathrm{FC},k}(t) := (\mathcal{N} \circ \gamma)(\|e_r(\varphi_k(t), e_{\mathrm{S}}(t))\|^2) e_r(\varphi_k(t), e_{\mathrm{S}}(t)), \tag{3.15}$$

with $e_{\mathrm{S}}(t) = y(t) - y_{\mathrm{M}}^k(t)$. Apply the control law

$$u_k : [t_k, t_{k+1}) \to \mathbb{R}^m, \; u_k(t) = u_{\mathrm{FMPC},k}(t) + u_{\mathrm{FC},k}(t) \tag{3.16}$$

to system (1.1). Increment k by 1 and go to Step (a).

▲

Remark 3.2.10. Algorithm 3.2.9 integrates the funnel MPC Algorithm 2.4.1 (from Chapter 2) with the model-free funnel controller of [30] via Step (c). By employing the model output y_{M} as the reference signal for the funnel controller, the combined scheme leverages the MPC's predictive capabilities even in safety-critical scenarios, while ensuring the MPC's optimal control input u_{FMPC} enhances – rather than disrupts – the funnel controller's operation. Coupled with the funnel function φ (computed using MPC predictions), this guarantees the tracking error remains within the prescribed performance funnel ψ, as formalised in Theorem 3.2.11. The principal mathematical challenges involve ensuring that the funnel MPC algorithm remains feasible under (ε,λ)-initialisation of the model based on system output measurements. To this end, we adapt the results from [30] (resp. Proposition 3.2.3) to the current setting. However, the findings in [30] cannot be directly applied since the reference signal for the funnel controller is assumed to be a priori given and to be continuous – conditions violated in Algorithm 3.2.9 due to the MPC-generated reference y_{M}. •

Theorem 3.2.11. *Consider a system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$ as in Definition 3.1.1 and choose a model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$ as in Definition 2.2.2. Let $t_0 \geq 0$ be the initial time and let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ be given and let $\tau \geq 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$. Further, let $y^0 \in \mathcal{C}^{(r-1)}([0, t_0], \mathbb{R}^m)$ with $\chi_r(y_0 - y_{\mathrm{ref}})(t_0) \in \mathcal{E}_r^1(1/\psi(t_0))$ be the initial trajectory for the system (1.1). Then, there exist $\varepsilon, \lambda \in (0, 1)$ ($\lambda = 1$ in the case $r = 1$), and $u_{\max} \geq 0$ such that the robust funnel MPC Algorithm 3.2.9 with $\delta > 0$ and $T \geq \delta$ is initially and recursively feasible, i.e. at every time instant $t_k := t_0 + k\delta$ for $k \in \mathbb{N}_0$*

- *there exists a proper initialisation $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ and*
- *the OCP (3.13) has a solution $u_{\mathrm{FMPC},k} \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$.*

Moreover, the closed-loop system consisting of the system (1.1) and the feedback law (3.16) has a global solution $y : [0, \infty) \to \mathbb{R}^m$. Each global solution y satisfies that

(i) all signals are bounded, in particular, $u \in L^\infty([t_0, \infty), \mathbb{R}^m)$ and $y \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$,

(ii) the tracking error between the system's output and the reference evolves within prescribed boundaries, i.e.

$$\forall\, t \geq t_0 : \|y(t) - y_{\mathrm{ref}}(t)\| < \psi_1(t).$$

Proof. Step 1: We define the constants λ and $u_{\max}$. To that end, set $\lambda = 1$ if the order of the model (2.4) is $r = 1$. Otherwise, choose $\lambda \in (0,1)$ such that for all $s > \hat{t} \geq t_0$ every function $\zeta \in \mathcal{C}^{r-1}([\hat{t},\infty),\mathbb{R}^m)$ with $\chi_r(\zeta)(t) \in \mathcal{D}_t^\Psi$ and $\left\|\xi_1(\chi_r(\zeta)(\hat{t}))\right\| < \lambda \cdot \psi_1(\hat{t})$ fulfils

$$\|\xi_1(\chi_r(\zeta)(t))\| < \lambda \cdot \psi_1(t)$$

for all $t \in [\hat{t}, s]$. Here, ξ_1 is the first auxiliary error variable used in the funnel MPC Algorithm 2.4.1 as introduced in Section 2.3.1. A constant λ with this properties exists according to Corollary 2.3.13. Further, choose $u_{\max} \geq 0$ such that $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$ for all $\hat{t} \geq t_0$, $\hat{\mathfrak{X}} \in \mathfrak{I}^\Psi_{t_0,\tau}(\hat{t})$, and $T > 0$. Such bound $u_{\max} \geq 0$ exists according to Theorem 2.3.21.
Step 2: Similarly to Lemma 3.2.1, we define several constants for later use. By assumption, we have $\chi_r(y_0 - y_{\rm ref})(t_0) \in \mathcal{E}_r^1(1/\psi_1(t_0))$. Thus, there exists $\bar{\varepsilon} \in (0,1)$ with $\chi_r(y_0 - y_{\rm ref})(t_0) \in \mathcal{E}_r^{\bar{\varepsilon}}(1/\psi_1(t_0))$. In the case of $r > 1$, define

$$\bar{\varphi} := 2\,\|\psi_1\|_\infty \ \text{ and } \ \hat{\varphi} := \frac{\|\dot{\psi}_1\|_\infty + \|\psi_2\|_\infty + k_1\|\psi_1\|_\infty}{\left((1-\lambda)\inf_{s\geq 0}\psi_1(s)\right)^2},$$

where $k_1 \geq 0$ is the first parameter corresponding to the auxiliary funnel function $\Psi \in \mathscr{G}$. Let $\varepsilon_0 = 0$ and $\bar{\eta}_0 := 0$. Utilising the bijectivity of γ, define successively

$$\begin{aligned}
&\hat{\varepsilon}_i \in (0,1) \text{ s.t. } \gamma(\hat{\varepsilon}_i^2)\hat{\varepsilon}_i = \frac{\hat{\varphi}}{\bar{\varphi}}(1 + \gamma(\varepsilon_{i-1}^2)\varepsilon_{i-1}) + 1 + \bar{\eta}_{i-1},\\
&\varepsilon_i := \max\{\bar{\varepsilon}, \hat{\varepsilon}_i\} < 1,\\
&\mu_i := \frac{\hat{\varphi}}{\bar{\varphi}}(1 + \gamma(\varepsilon_{i-1}^2)\varepsilon_{i-1}) + 1 + \gamma(\varepsilon_i^2)\varepsilon_i + \bar{\eta}_{i-1},\\
&\bar{\eta}_i := 2\dot{\gamma}(\varepsilon_i^2)\varepsilon_i^2\mu_i + \gamma(\varepsilon_i^2)\mu_i,
\end{aligned} \tag{3.17}$$

for $i = 1, \ldots, r-1$.
Step 3: We define $\varepsilon \in (0,1)$. To that end, define the set

$$\mathcal{E} := \left\{(t,z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \,\middle|\, z - \zeta(t) \in \mathcal{E}_r^1(1/\psi_1(t)), \zeta \in \mathcal{Y}_\infty^\Psi \right\},$$

with $\mathcal{Y}_\infty^\Psi$ as in (2.26). According to the proof of Lemma 2.3.20, there exists a compact set $\hat{K} \subset \mathbb{R}^{rm}$ with

$$\forall\, \zeta \in \mathcal{Y}_\infty^\Psi \forall\, t \geq 0: \quad \zeta(t) \in \hat{K},$$

see (2.31). Thus, the set $\mathcal{E}$ is bounded. Due to the bounded-input bounded-output property (T.3) in Definition 2.2.1, the operator $\mathbf{T}$ is bounded for all functions $\zeta \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ evolving within $\mathcal{E}$, see also the definition of the set $\mathcal{E}_r^1(1/\psi_1(t))$ in (3.3). Hence, there exists a compact set K with $\mathbf{T}(\zeta) \subset K$ for all $\zeta \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ evolving within $\mathcal{E}$. As F has the perturbation high-gain property, let $\nu \in (0,1)$ such that the function

$$\mathfrak{h}(s) := \min\left\{\langle v, F(z, d - sv)\rangle \,\middle|\, d \in \mathcal{B}_{u_{\max}}, z \in K_q, v \in \mathbb{R}^m,\ \nu \leq \|v\| \leq 1\right\}$$

is unbounded from above, see Definition 3.1.1 (b). Due to the unboundedness of the function $\mathfrak{h}$ and the surjectivity of $\mathcal{N} \circ \gamma$ it is possible to choose $\varepsilon_r \in (0,1)$ such that $\varepsilon_r > \max\{\nu, \|e_r(t_0)\|\}$ and

$$\tfrac{1}{2}\mathfrak{h}(\mathcal{N} \circ \gamma(\varepsilon_r^2)) \geq \theta := \frac{\hat{\varphi}}{\bar{\varphi}^2}(1 + \gamma(\varepsilon_{r-1}^2)\varepsilon_{r-1}) + \frac{\bar{\eta}_{r-1}}{\bar{\varphi}} + f_{\rm M}^{\max} + g_{\rm M}^{\max} u_{\max}, \tag{3.18}$$

where $f_{\rm M}^{\max}$ and $g_{\rm M}^{\max}$ are the constants from Lemma 2.3.20. In the case $r = 1$, replace $\frac{\hat{\varphi}}{\bar{\varphi}^2}$ with $\|\dot{\psi}_1\| + \|\dot{y}_{\rm ref}\|_\infty + f_{\rm M}^{\max} + g_{\rm M}^{\max} u_{\max}$ in (3.18). Choose $\varepsilon \in (0,1)$ with $\varepsilon > \varepsilon_i$ for all $i = 1, \ldots, r$.

Step 4: Let $\delta > 0$ and $T \geq \delta$ be arbitrary but fixed. When applying the robust funnel MPC Algorithm 3.2.9 to the system (1.1), the system's dynamics on each interval $[t_k, t_{k+1}]$ for $t_k = t_0 + k\delta$ and $k \in \mathbb{N}_0$ are given by

$$y_k^{(r)}(t) = F(\mathbf{T}(\chi_r(y_k))(t), u_k(t)), \quad y_k|_{[0,t_k]} = y_{k-1}|_{[0,t_k]} \tag{3.19}$$

where $y_{-1} := y^0$ and u_k is the control given by (3.16). Note that $\chi_r(y_k)(t_k) = \chi_r(y_{k-1})(t_k)$. In the following, we show via induction that the robust funnel MPC Algorithm 3.2.9 is initially and recursively feasible. This means, in particular, that there exists a proper initialisation $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \chi_r(y_{k-1})(t_k))$ at every time instant t_k, that u_k as in (3.16) is well defined on every interval $[t_k, t_{k+1}]$, and that (3.19) has a maximal solution y_k defined on the entire interval $[t_k, t_{k+1}]$.
Step 4.1: When obtaining the measurement of the system's output and its derivatives at the initial time t_0 in Step (a) of the robust funnel MPC Algorithm 3.2.9, we have $\hat{x}_0 = \chi_r(y)(t_0) = \chi_r(y^0)(t_0)$. The construction of ε, which is larger or equal to $\bar{\varepsilon}$, yields $\hat{x}_0 - \chi_r(y_{\mathrm{ref}})(t_0) \in \mathcal{E}_r^{\varepsilon}(1/\psi_1(t_0))$. Thus, the set $\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_0, \chi_r(y^0)(t_k))$ of proper (ε, λ) initial values is non-empty according to Remark 3.2.6.
Step 4.2: Let y_{k-1} be a solution of (3.19) defined on the interval $[0, t_{k-1}]$ for some $k \in \mathbb{N}_0$. Note that $y_{-1} = y^0$ for $k = 0$. Let $\hat{x}_k := \chi_r(y_{k-1})(t_k)$ be the system's output y_{k-1} and its derivatives at time instant t_k. Further, assume that there exists a proper initialisation $(\hat{x}_{\mathrm{M}}^k, \hat{\mathbf{T}}_{\mathrm{M}}^k) := \mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$. We show that the control signal u_k as in (3.16) is well-defined and that when applying u_k to the system (1.1) the initial value problem (3.19) has a solution $y_k : [t_k, t_{k+1}] \to \mathbb{R}^m$. As $\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k) \subset \mathfrak{I}_{t_0,\tau}^{\Psi}(t_k)$, the choice of $u_{\max} \geq 0$ ensures the non-emptiness of the set $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$. Therefore, Theorem 2.3.26 yields the existence of a solution $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ of the OCP (3.13). Let $y_{\mathrm{M}}^k(\cdot; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k}) : [t_k, t_{k+1}] \to \mathbb{R}^m$ be the corresponding output of the model (2.4) when applying the control $u_{\mathrm{FMPC},k}$ with initial time t_k and initial value $\mathfrak{X}_k$ over the time interval $[t_k, t_{k+1}]$. Note that y_{M}^k is, in fact, defined on the whole interval $[0, t_k + T]$ according to the solution concept for the model differential equation (2.4), see Definition 2.2.6. Moreover, y_{M}^k restricted to the interval $[t_k, t_{k+1}]$ is an element of $W^{r,\infty}([t_k, t_{k+1}], \mathbb{R}^m)$. By $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$, we have $\left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\| < \psi_1(t)$ for all $t \in [t_k, t_{k+1}]$. Thus, the function $\varphi_k : [t_k, t_{k+1}] \to \mathbb{R}_{>0}$, $\varphi_k(t) = 1/(\psi_1(t) - \left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\|)$ in (3.14) is well defined. φ_k is bounded with a bounded derivative due to the compactness of the interval $[t_k, t_{k+1}]$. Note that $\hat{x}_k - \hat{x}_{\mathrm{M}}^k \in \mathcal{E}_r^{\varepsilon}(\varphi_0(t_k))$ because $\mathfrak{X}_k$ is a proper initial value, see Definition 3.2.5. Applying the control signal u_k as in (3.16) consisting of sum of $u_{\mathrm{FMPC},k}$ and the funnel control signal $u_{\mathrm{FC},k}$ as in (3.15) with reference y_{M}^k and funnel function φ_k to the system (1.1) with initial value $y_k|_{[0,t_k]} = y_{k-1}|_{[0,t_k]}$ to the loop system (3.19). This initial value problem has a solution $y_k : [t_k, t_{k+1}] \to \mathbb{R}^m$, see Proposition 3.2.3.
Step 4.3: Assuming the existence of a proper initialisation $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$, we show certain bounds for y_{M}^k and φ_k on the interval $[t_k, t_{k+1}]$. $\chi_r(y_{\mathrm{M}}^k - y_{\mathrm{ref}})(t) \in \mathcal{D}_t^{\Psi}$ for all $t \in [t_k, t_{k+1}]$ because $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$, see definition of $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ in (2.28). Since

$$y_{\mathrm{M}}^{(r)}(t) = f_{\mathrm{M}}(\chi_r(y_{\mathrm{M}})(t)) + g_{\mathrm{M}}(\chi_r(y_{\mathrm{M}})(t))u_{\mathrm{FMPC},k}(t)$$

for $t \in [t_k, t_{k+1}]$, the function $y_{\mathrm{M}}^{(r)}$ is bounded on the interval $[t_k, t_{k+1}]$ by $f_{\mathrm{M}}^{\max} + g_{\mathrm{M}}^{\max} u_{\max}$, see Lemma 2.3.20. We observe

$$\|1/\varphi_k(t)\| = \left\|\psi_1(t) - \left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\|\right\| \leq \|\psi_1\|_\infty + \left\|y_{\mathrm{M}}^k - y_{\mathrm{ref}}\right\|_\infty \leq 2\,\|\psi_1\|_\infty = \bar{\varphi}$$

on the interval $[t_k, t_{k+1}]$. Moreover, if the order of the system is $r = 1$, then

$$\left\|\frac{\dot{\varphi}_k(t)}{\varphi_k(t)^2}\right\| \leq \left\|\dot{\psi}(t)\right\| + \left\|\dot{y}_{\mathrm{M}}^k(t)\right\| + \left\|\dot{y}_{\mathrm{ref}}(t)\right\| \leq \left\|\dot{\psi}\right\|_\infty + f_{\mathrm{M}}^{\max} + g_{\mathrm{M}}^{\max} u_{\max} + \left\|\dot{y}_{\mathrm{ref}}\right\|_\infty$$

for almost all $t \in [t_k, t_{k+1}]$. As $(\hat{x}_{\mathrm{M}}^k, \hat{\mathbf{T}}_{\mathrm{M}}^k) := \mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$, we have

$$\left\|\xi_1(\hat{x}_{\mathrm{M}}^k(t_k) - \chi_r(y_{\mathrm{ref}})(t_k))\right\| = \left\|\hat{x}_{\mathrm{M},1}^k(t_k) - y_{\mathrm{ref}}(t_k)\right\| = \left\|y_{\mathrm{M}}^k(t_k) - y_{\mathrm{ref}}(t_k)\right\| < \lambda \cdot \psi_1(t_k).$$

If the order of the system is $r > 1$, this yields $\left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\| < \lambda \cdot \psi_1(t)$ for all $t \in [t_k, t_{k+1}]$ due to the choice of λ, see Corollary 2.3.13. Since $\chi_r(y_{\mathrm{M}}^k - y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi$ for all $t \in [t_k, t_{k+1}]$,

$$\left\|\dot{y}_{\mathrm{M}}^k(t) - \dot{y}_{\mathrm{ref}}(t)\right\| = \left\|\xi_2(\chi_r(y_{\mathrm{M}}^k - y_{\mathrm{ref}})(t)) - k_1\xi_1(\chi_r(y_{\mathrm{M}}^k - y_{\mathrm{ref}})(t))\right\| \le \|\psi_2\|_\infty + k_1 \|\psi_1\|_\infty ,$$

where $k_1 \ge 0$ is the parameter corresponding to the auxiliary error variable ξ_2, see definition of ξ_i in (2.15). Therefore,

$$\|\dot{\varphi}_k(t)\| \le \frac{\left\|\dot{\psi}_1(t)\right\| + \left\|\dot{y}_{\mathrm{M}}^k(t) - \dot{y}_{\mathrm{ref}}(t)\right\|}{\left(\psi_1(t) - \left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\|\right)^2} \le \frac{\left\|\dot{\psi}_1\right\|_\infty + \left\|\psi_2\right\|_\infty + k_1\left\|\psi_1\right\|_\infty}{\left((1-\lambda)\inf_{s\ge 0}\psi(s)\right)^2},$$

for almost all $t \in [t_k, t_{k+1}]$. Note that the derived boundaries for $y_{\mathrm{M}}^{(r)}$, $1/\varphi_k$, $\dot{\varphi}_k$, and $\frac{\dot{\varphi}_k}{\varphi_k^2}$ are independent of the time instant t_k and the particular choice of $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$.

Step 4.4: We show that if $\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ is non-empty, then $\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_{k+1}, \hat{x}_{k+1})$ is non-empty after applying a control u_k as in (3.16) to the system (1.1), where $\hat{x}_k := \chi_r(y_{k-1})(t_k)$ and $\hat{x}_{k+1} := \chi_r(y_k)(t_k)$. Let $(\hat{x}_{\mathrm{M}}^k, \hat{\mathbf{T}}_{\mathrm{M}}^k) := \mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ be an arbitrary but fixed proper initialisation. We have $\hat{x}_k - \hat{x}_{\mathrm{M}}^k(t_k) \in \mathcal{E}_r^\varepsilon(\varphi_k(t_k))$, see Definition 3.2.5. According to Proposition 3.2.3, there exists $\tilde{\varepsilon} \in (0,1)$ with

$$\chi_r(y_k - y_{\mathrm{M}}^k)(t) \in \mathcal{E}_r^{\tilde{\varepsilon}}(\varphi_k(t))$$

for all $t \in [t_k, t_{k+1}]$. In the proof of Proposition 3.2.3, $\tilde{\varepsilon}$ is constructed as the maximum of ε_i, $i = 1, \ldots, r-1$ as defined in (3.5) and ε_r in (3.9). Due to the boundaries derived in Step 4.3, ε as defined in Step 3 fulfils the estimates for $\tilde{\varepsilon}$ in (3.5). Regarding ε_r, note the following. As y_{M}^k can be extended to an element of $\mathcal{Y}_\infty^\Psi$, the function y_k can be extended to a function evolving within the set $\mathcal{E}$. Thus, the bound (3.9) for ε_r can be proven with the same calculations as in the proof of Proposition 3.2.3. Therefore, ε as defined in Step 3 fulfils the estimates for $\tilde{\varepsilon}$ in (3.5) and in (3.9). Or in other words, $\tilde{\varepsilon}$ in Proposition 3.2.3 can be chosen smaller or equal ε from Step 3 of the current proof. Thus, $\chi_r(y_k - y_{\mathrm{M}}^k)(t) \in \mathcal{E}_r^\varepsilon(\varphi_k(t))$ for all $t \in [t_k, t_{k+1}]$. In particular, $\chi_r(y_k - y_{\mathrm{M}}^k)(t_{k+1}) \in \mathcal{E}_r^\varepsilon\left(1/\left(\psi_1(t_{k+1}) - \left\|y_{\mathrm{M}}^k(t_{k+1}) - y_{\mathrm{ref}}(t_{k+1})\right\|\right)\right)$. Further note that $\left\|y_{\mathrm{M}}^k(t) - y_{\mathrm{ref}}(t)\right\| < \lambda \cdot \psi_1(t)$ for all $t \in [t_k, t_{k+1}]$ due to the choice of λ, see Corollary 2.3.13. According to Theorem 2.3.18, we have

$$(\chi_r(y_{\mathrm{M}}^k)|_{[t_{k+1}-\tau, t_{k+1}]\cap[0,t_k]}, \mathbf{T}_{\mathrm{M}}(\chi_r(y_{\mathrm{M}}^k))|_{[t_{k+1}-\tau, t_{k+1}]\cap[t_0,t_{k+1}]}) \in \mathfrak{I}_{t_0,\tau}^\Psi(t_{k+1}).$$

Thus,

$$(\chi_r(y_{\mathrm{M}}^k)|_{[t_{k+1}-\tau, t_{k+1}]\cap[0,t_k]}, \mathbf{T}_{\mathrm{M}}(\chi_r(y_{\mathrm{M}}^k))|_{[t_{k+1}-\tau, t_{k+1}]\cap[t_0,t_{k+1}]}) \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_{k+1}, \hat{x}_{k+1}).$$

Step 4.5: We sum up Step 4. Under the assumption that the set of proper initial values $\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ at time instant t_k is non-empty, we showed in Step 4.2 that one iteration of the robust funnel MPC (3.2.9) can be executed. This means, in particular, that the optimisation problem (3.13) has a solution $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k, t_k+T]}(u_{\max}, \mathfrak{X}_k)$, that the output $y_{\mathrm{M}}^k(t; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k})$ of the model (2.4) exists on the entire interval $[t_k, t_{k+1}]$, and that the adaptive funnel $\varphi_k : [t_k, t_{k+1}] \to \mathbb{R}_{>0}$ given by (3.14) is well-defined. Furthermore, applying the control u_k as defined in (3.16) to the system (1.1) with initial value $y_k|_{[0,t_k]} = y_{k-1}|_{[0,t_k]}$ leads to the loop system which has a maximal solution

$y_k : [t_k, t_{k+1}] \to \mathbb{R}^m$. Utilising the bounds derived in Step 4.3, it was shown in Step 4.4 that $\mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_{k+1}, \hat{x}_{k+1})$ is non-empty after applying a control u_k as in (3.16) to the system (1.1) if $\mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_k, \hat{x}_k)$ is non-empty. Step 4.1 shows that initially the set $\mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_0, \hat{x}_0)$ is non-empty. Therefore, it follows inductively that the robust funnel MPC (3.2.9) can recursively be applied to the system (1.1) and that the closed-loop system consisting of the system (1.1) and the control law (3.16) has a global solution $y : [0, \infty) \to \mathbb{R}^m$.
Step 5: Let $y : [0, \infty) \to \mathbb{R}^m$ be a global solution of the closed-loop system consisting of the system (1.1) and the control law (3.16). We show (i) and (ii). Let $y_{\mathrm{M}} : [0, \infty) \to \mathbb{R}^m$ be the associated concatenated solution of the model differential equation (2.4) with the sequence of initial values $(t_k, \mathfrak{X}_k)_{k\in\mathbb{N}_0}$ and control signals $u_{\mathrm{FMPC},k}$. Further let $\varphi : [t_0, \infty) \to \mathbb{R}_{\geq 0}$ with $\varphi(t) := 1/(\psi_1(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\|)$. Then, $y_{\mathrm{M}}|_{[t_k,t_{k+1})} = y^k_{\mathrm{M}}(\cdot; t_k, \mathfrak{X}_k, u_{\mathrm{FMPC},k})|_{[t_k,t_{k+1})}$ and $\varphi|_{[t_k,t_{k+1})} = \varphi_k|_{[t_k,t_{k+1})}$ for all $k \in \mathbb{N}_0$. We have

$$\chi_r(y - y_{\mathrm{M}})(t) \in \mathcal{E}^{\tilde{\varepsilon}}_r(\varphi(t))$$

for all $t \in [t_0, \infty)$. Since y_{M} and φ are bounded functions, $y \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, see definition of $\mathcal{E}^{\tilde{\varepsilon}}_r$ in (3.3). The funnel MPC signal $u_{\mathrm{FMPC},k}$ is bounded by $u_{\max}$ for all $k \in \mathbb{N}_0$. The funnel control signal $u_{\mathrm{FC},k}$ is bounded by $\mathfrak{h}(\mathcal{N} \circ \gamma(\varepsilon^2))$ for all $k \in \mathbb{N}_0$, see definition of ε_r in (3.18) and the calculations in Step 3 of the proof of Proposition 3.2.3. This shows (i). Moreover, we have

$$\|y(t) - y_{\mathrm{ref}}(t)\| \leq \|y(t) - y_{\mathrm{M}}(t)\| + \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \varphi(t) + \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| = \psi_1(t)$$

for all $t \geq t_0$. This shows (ii) and completes the proof. □

Remark 3.2.12. We comment on the difference between the proposed control scheme and a straightforward combination of a MPC scheme with a feedback control law.

(a) The integration of feed-forward and feedback control is a widely adopted strategy. Prior work in [28, 35] explores combining funnel control with feed-forward methods. Similarly, model predictive control – specifically funnel MPC – can be augmented with a feedback controller. This approach can be implemented in the robust funnel MPC Algorithm 3.2.9 by omitting the feedback loop between the funnel MPC and the system. Instead, at each MPC cycle, the model is re-initialised using only the prior prediction of the model state:

$$\mathfrak{X}_{k+1} := (x^k_{\mathrm{M}}|_{[t_{k+1}-\tau,t_{k+1}]\cap[0,t_k]}, \mathbf{T}_{\mathrm{M}}(x^k_{\mathrm{M}}))|_{[t_{k+1}-\tau,t_{k+1}]\cap[t_0,t_{k+1}]}).$$

This is an element of $\mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_{k+1}, \hat{x}_{k+1})$ independently of ε and λ, making it a special case of a *proper initialisation*. Here, the funnel MPC signal u_{FMPC} can be computed offline via the model and applied as an open-loop control to the system. Concurrently, the feedback controller compensates for errors arising from discrepancies between the model and the physical system.

(b) An alternative to the open-loop operation of Algorithm 3.2.9 involves feedback based on system output measurements $\hat{x}_k := \chi_r(y)(t_k)$. By properly initialising the model with $\mathfrak{X}_k \in \mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,\tau}(t_k, \hat{x}_k)$, two objectives are achieved: recursive feasibility of the MPC scheme is preserved and the model state $\mathfrak{X}_k$ mirrors the system's actual state $\hat{x}_k$. This re-initialisation at each MPC cycle incorporates the impact of the control signal u_{FMPC} on the system. Furthermore, it may enhance the efficacy of the optimal control signal in improving the system's tracking performance.

•

Remark 3.2.13. Theorem 3.2.11 demonstrates that the robust funnel MPC Algorithm 3.2.9 is model-agnostic. For any system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$, the algorithm remains functional regardless of the chosen model (2.4), provided $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. Crucially, the system and model need not share structural similarity. For instance:

- The model may be a lower-dimensional approximation of a higher-dimensional system.
- The model could represent a linearised version of a non-linear system.
- The model might omit time delay effects.

This flexibility ensures applicability across diverse modelling paradigms. •

3.3 Simulation

In this section, we revisit the numerical examples from Section 2.5 to illustrate the robust funnel MPC Algorithm 3.2.9. The MATLAB source code for the performed simulations can be found on GITHUB under the link https://github.com/ddennstaedt/FMPC_Simulation.

Exothermic chemical reaction

To demonstrate the application of the robust funnel MPC Algorithm 3.2.9 by a numerical simulation, we consider again a continuous-time chemical reactor and concentrate on the control goal of steering the reactor's temperature to a predefined reference value $y_{\mathrm{ref}}(t)$ within boundaries given by a function $\psi(t)$. As in Section 2.5.1, we consider a reactor described by the following non-linear system of order one:

$$\begin{aligned} \dot{x}_1(t) &= c_1\, p(x_1(t), x_2(t), y(t)) + d(x_1^{\mathrm{in}} - x_1(t)),\\ \dot{x}_2(t) &= c_2\, p(x_1(t), x_2(t), y(t)) + d(x_2^{\mathrm{in}} - x_2(t)),\\ \dot{y}(t) &= b\, p(x_1(t), x_2(t), y(t)) - q\, y(t) + u(t). \end{aligned} \tag{2.39 revisited}$$

The reactor's temperature should follow a given heating profile specified in (2.41) within tolerance limits defined by the funnel function $\psi(t) := 20\mathrm{e}^{-2t} + 4$. To achieve the control objective with robust funnel MPC Algorithm 3.2.9, we again use the strict funnel stage cost function $\ell_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$\ell_\psi(t, y, u) = \begin{cases} \dfrac{\|y - y_{\mathrm{ref}}(t)\|}{\psi(t)^2 - \|y - y_{\mathrm{ref}}(t)\|^2} + \lambda_u \|u - 360\|^2, & \|y - y_{\mathrm{ref}}(t)\| \neq \psi(t)\\ \infty, & \text{else,} \end{cases} \tag{2.42 revisited}$$

with design parameter $\lambda_u \in \mathbb{R}_{\geq 0}$. We restrict the MPC control signal to $\|u_{\mathrm{FMPC}}\|_\infty \leq 600$. Further, we choose the design parameters $\lambda_u = 10^{-4}$, prediction horizon $T = 1$, and time shift $\delta = 0.1$. However, unlike before, we do not utilise the actual differential equations describing the system (2.39) as a model for the MPC algorithm. Instead, we consider a linearisation of this non-linear reaction process obtained by linearising the Arrhenius function $p(x_1, x_2, y) = k_0 \mathrm{e}^{-\frac{k_1}{y}} x_1$ around the desired final temperature $\bar{y} = 337.1K$ and $x_1 = \frac{1}{2} x_1^{\mathrm{in}}$. This results in

$$p_{\mathrm{lin}}(x_1, x_2, y) = k_0 \mathrm{e}^{-\frac{k_1}{\bar{y}}} x_1 + \frac{k_0 k_1 e^{-\frac{k_1}{\bar{y}}}}{\bar{y}^2} \frac{x_1^{\mathrm{in}}}{2} (y - \bar{y}).$$

Set $a_1 := \frac{k_0 k_1 e^{-\frac{k_1}{\bar y}}}{\bar y^2}\frac{x_1^{\text{in}}}{2}$, $a_2 := k_0 e^{-\frac{k_1}{\bar y}}$ and define the expressions

$$A = \begin{bmatrix} c_1 a_2 - d & 0 & c_1 a_1 \\ c_2 a_2 & -d & c_2 a_1 \\ b a_2 & 0 & b a_1 - q \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad D = \begin{bmatrix} -c_1 a_1 \bar y + d x_{\text{M},1}^{\text{in}} \\ -c_2 a_1 \bar y + d x_{\text{M},2}^{\text{in}} \\ -b a_1 \bar y \end{bmatrix} \in \mathbb{R}^3.$$

Then, with $x_{\text{M}} := [x_{\text{M},1}, x_{\text{M},2}, y_{\text{M}}]^\top \in \mathbb{R}^3$, the model to be used in the funnel MPC controller component is given by

$$\begin{aligned} x_{\text{M}}(t) &= A x_{\text{M}}(t) + B u_{\text{FMPC}}(t) + D, \\ y_{\text{M}}(t) &= C x_{\text{M}}(t), \end{aligned} \tag{3.20}$$

where $C = B^\top = [0, 0, 1] \in \mathbb{R}^{1\times 3}$. We choose the same parameters as in (2.43) and assume initial values of the system and the model to coincide, i.e.

$$[x_1(0), x_2(0), y(0)]^\top = [x_{\text{M},1}(0), x_{\text{M},2}(0), y_{\text{M}}(0)]^\top := [0.02, 0.9, 270]^\top.$$

Due to discretisation, we consider only step functions with a constant step length of $\mathfrak{r} := \delta = 0.1$ to solve the OCP (3.13).

For the control law of funnel control component, we choose the bijection $\gamma(s) = 1/(1-s)$ and the function $\mathcal{N}(s) = -s$. This choice for $\mathcal{N}$ is justified since we assume the control direction to be known, see Remark 3.2.4. This assumption is also realistic from a practical point of view. To additionally demonstrate that the funnel controller can be combined with an activation function $\mathfrak{a}$, as discussed in Section 3.2.1, we interconnect the controller with a ReLU-like map

$$\mathfrak{a}(s) = \begin{cases} 0, & s \le S_{\text{crit}}, \\ s - S_{\text{crit}}, & s \ge S_{\text{crit}}, \end{cases}$$

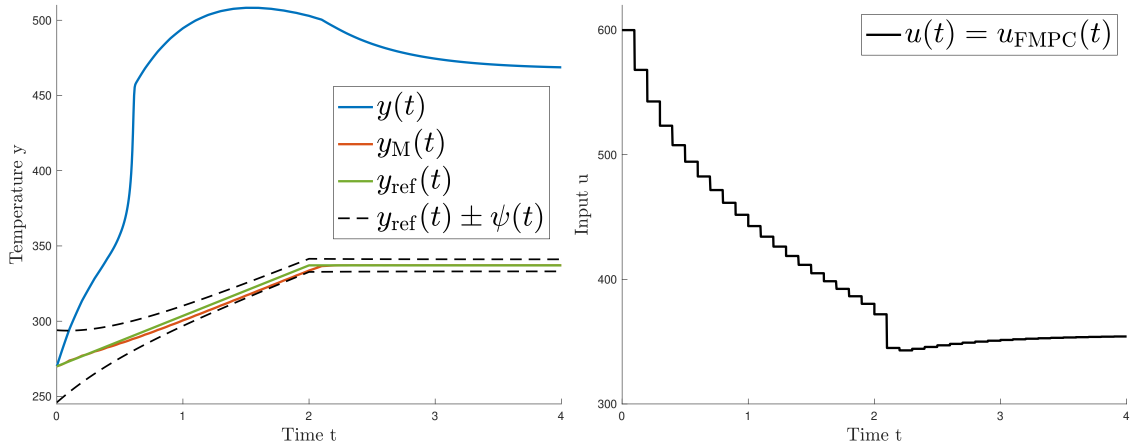
where we choose $S_{\text{crit}} = 0.4$. The funnel controller therefore is only active, if the error $e = y - y_{\text{M}}$ exceeds 40% of the maximal distance to its funnel boundary. We run the simulation on an interval of $[0, 4]$ and consider the following scenarios:

- *Case 1:* Funnel MPC without robustification, i.e. u_{FMPC} is computed via the funnel MPC Algorithm 2.4.1 and applied to the system without an additional funnel control loop. The model is initialised, at every iteration of the algorithm, with the model's state from the previous iteration. The results are shown in Figure 3.2.

- *Case 2:* Robust funnel MPC with a trivial proper re-initialisation, i.e. model is initialised, at every iteration of the algorithm, with the model's state from the previous iteration ($\mathfrak{X}_k = x_{\text{M}}^{k-1}(t_k)$ in Step (a) of Algorithm 3.2.9). The results are depicted in Figure 3.3.

- *Case 3:* Robust funnel MPC with a proper initialisation according to the system's output, i.e. $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, y(t_k))$ is selected such that $y(t_k) = y_{\text{M}}(t_k)$ in Step (a) of Algorithm 3.2.9. To this end, we initialise the model (3.20) with $x_{\text{M}}(t_k) := [x_{\text{M},1}^{k-1}(t_k), x_{\text{M},2}^{k-1}(t_k), y(t_k)]^\top$ at every time instant $t_k \in \delta\mathbb{N}$, i.e. the states $x_{\text{M},1}$ and $x_{\text{M},2}$ remain unchanged during initialisation and $y_{\text{M}}(t_k)$ is set to $y(t_k)$. The results are displayed in Figure 3.4.

In the following figures, the control signal generated via the funnel MPC component is labelled with the subscript FMCP (u_{FMPC}); the signal generated by the additional funnel controller component is labelled with the subscript FC (u_{FC}). Figure 3.2 shows the application of the control signal computed with funnel MPC Algorithm 2.4.1 in Case 1 to the system without an additional funnel control feedback loop. The error

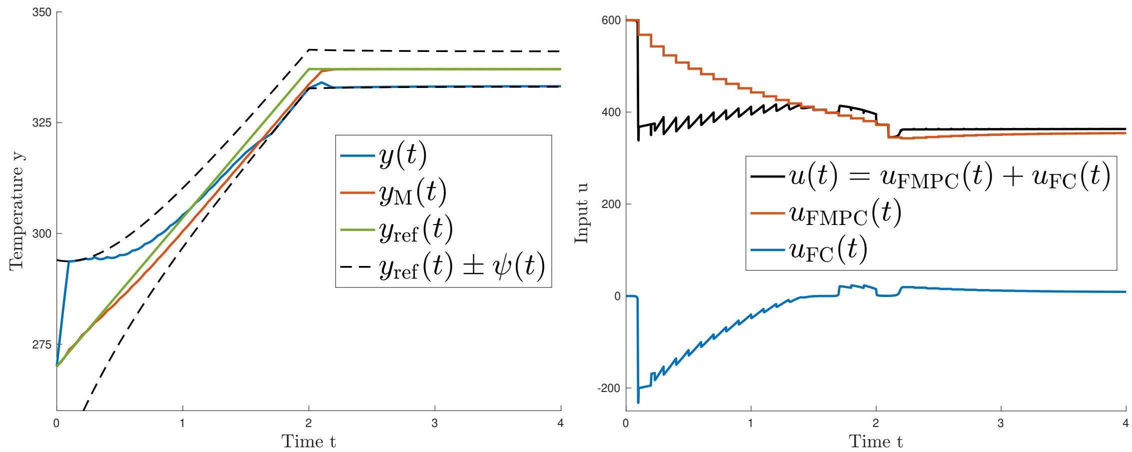


(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 3.2: Simulation of system (2.39) under the control generated by the funnel MPC Algorithm 2.4.1 without additional funnel control feedback loop.

$e_M(t) = y_M(t) - y_{ref}(t)$ between the model's output $y_M(t)$ and the reference $y_{ref}(t)$ evolves within the funnel boundaries $\psi(t)$. However, the control signal computed with funnel MPC using the linear model is not sufficient to achieve that the tracking error $e(t) = y(t) - y_{ref}(t)$ of the non-linear system evolves within the funnel boundaries $\psi(t)$. Obviously, the deviation is induced during the initial phase. After about $t = 2$, the linearised model is a good approximation of the system. In this region, the control u_{FMPC} has a comparable effect on both dynamics; however, the error $y(t) - y_{ref}(t)$ already evolves outside the funnel boundaries $\psi(t)$. Figure 3.3 shows the application of the control signal computed with



(a) Outputs and reference, with boundary ψ.

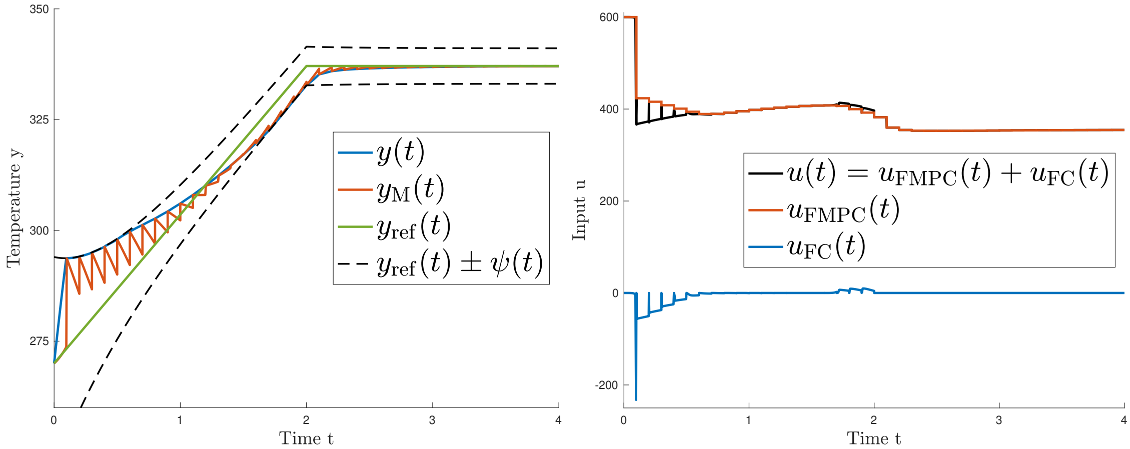
(b) Control inputs.

Figure 3.3: Simulation of system (2.39) under the control generated by the robust funnel MPC Algorithm 3.2.9 with funnel control feedback, without model re-initialisation.

robust funnel MPC Algorithm 3.2.9 in Case 2, i.e. besides the funnel MPC control signal the additional funnel controller is applied in order to guarantee that the error $y(t) - y_{ref}(t)$ evolves within the boundaries $\psi(t)$. Since the model and the system do not coincide, the system evolves differently from the model and hence the funnel controller has to compensate the model-plant mismatch. However, after the system has reached the desired temperature $y_{ref,final}$ at $t_{final} = 2$, the system's states evolve close to the linearisation point of the model (3.20). Hence, the linear model closely approximates the non-linear system (2.39).

Consequently, the control signal u_{FMPC} generated by the MPC controller component is nearly sufficient to maintain the system output y at $y_{\text{ref,final}}$ within the desired temperature range and the funnel controller intervenes only slightly with a small control signal u_{FC}.

Note that, in both Cases 1 and 2, it is possible to pre-compute the control signal u_{FMPC} as no system measurement data is fed back to the funnel MPC component, see Remark 3.2.12 (a).



(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 3.4: Simulation of system (2.39) under the control generated by the robust funnel MPC Algorithm 3.2.9 with funnel control feedback and model re-initialisation.

Figure 3.4 shows the application of Algorithm 3.2.9 in Case 3. Besides the additional application of the funnel controller, the model's state is updated with

$$x_{\text{M}}(t_k) := [x_{\text{M},1}^{k-1}(t_k), x_{\text{M},2}^{k-1}(t_k), y(t_k)]^\top$$

at the beginning of every MPC cycle. This results in $y_{\text{M}}(t_k) = y(t_k)$ at every time instant $t_k \in \delta\mathbb{N}_0$. The internal states $x_{\text{M},1}$ and $x_{\text{M},2}$ of the model remain, however, unchanged during initialisation, i.e. they are initialised with their values from the end of previous iteration. Note that, the proportion of the control signal generated by the MPC component is larger than in the previous case (3.3) and the funnel controller component does overall intervene less. Moreover, after $t \approx 0.5$, the funnel controller is inactive most of the time in Figure 3.4, i.e. the applied control signal can be viewed to be close to *optimal* with respect to the cost function (2.1), since it is computed via the OCP (3.13). In the beginning, the funnel controller however has to compensate for the model inaccuracies in order to ensure that the system's output evolves within the funnel boundaries ψ.

Mass-on-car system

We revisit the example of the mass-on-car system from Section 2.5.2. The relative degree two system is described by the differential equation

$$\begin{aligned} \ddot{y}(t) &= R_1 y(t) + R_2 \dot{y}(t) + S\eta(t) + \Gamma u(t) \\ \dot{\eta}(t) &= Q\eta(t) + Py(t). \end{aligned} \qquad \text{(2.49 revisited)}$$

Assuming the mass $m_2 = 2$, on the ramp inclined by the angle $\vartheta = \frac{\pi}{4}$, is connected to the car with mass $m_1 = 4$ via spring and damper system with spring constant $k = 2$ and damper constant $d = 1$, the matrices R_1, R_2, S, Γ Q, and R have the values as in (2.50). The objective is tracking of the reference signal $y_{\text{ref}}(t) = \cos(t)$ such that the tracking error $y(t) - y_{\text{ref}}(t)$ evolves within the prescribed performance funnel given by the function $\psi \in \mathcal{G}$

with $\psi(t) = 5e^{-2t} + 0.1$. To achieve the control objective with robust funnel MPC Algorithm 3.2.9, we use the strict funnel stage cost function $\ell_{\psi_2} : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$\ell_{\psi_2}(t, \zeta, u) = \begin{cases} \dfrac{\|\zeta\|}{\psi_2(t)^2 - \|\zeta\|^2} + \lambda_u \|u\|^2, & \|\zeta\| \neq \psi_2(t) \\ \infty, & \text{else}, \end{cases} \quad \text{(2.52 revisited)}$$

with design parameter $\lambda_u \in \mathbb{R}_{\geq 0}$ and auxiliary funnel function $\psi_2(t) := \frac{1}{\gamma} k_1 e^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma}$ with $k_1 = 14$ and $\gamma = 0.2$ as in (2.51). For the simulation, the MPC control signal is restricted to $\|u_{\text{FMPC}}\|_\infty \leq u_{\max} = 30$ and we choose the design parameters $\lambda_u = 10^{-4}$, prediction horizon $T = 1$, and time shift $\delta = \frac{T}{12} \approx 0.083$. We assume that the MPC component uses a model with incorrect parameters

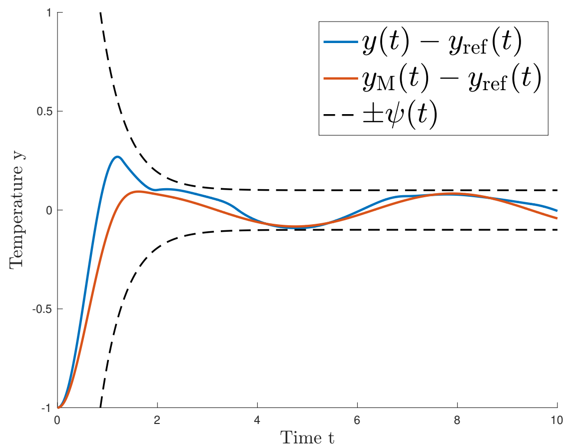
$$m_1 = 6, \quad m_2 = 2, \quad k = 3, \quad d = 0.75,$$

for the mass of the car, the mass, the spring constant, and the damper constant. This results in a differential equation comparable to (2.49). When referring to this model equation, we use the subscript M in the following. Moreover, the OCP (3.13) is restricted to step functions with a constant step length of $\mathfrak{r} := \delta \approx 0.083$ due to discretisation. For the control law of funnel control component, we choose the bijection $\gamma(s) = 1/(1-s)$ and the function $\mathcal{N}(s) = -s$. The funnel feedback law takes the form

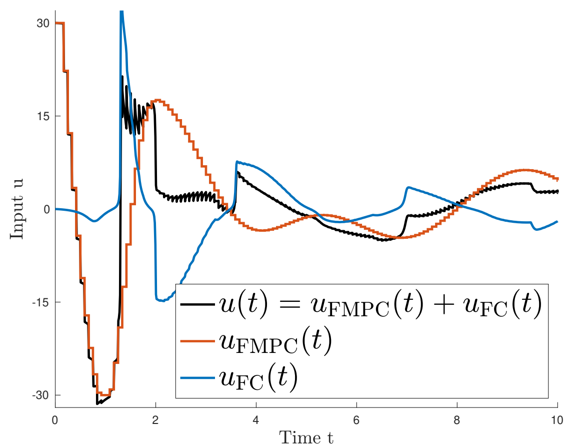
$$\begin{aligned} w(t) &= \varphi(t)\dot{e}_{\text{S}}(t) + \gamma\left(\varphi(t)^2 e_{\text{S}}(t)^2\right)\varphi(t)e_{\text{S}}(t), & e_{\text{S}}(t) &= y(t) - y_{\text{M}}(t), \\ u_{\text{FC}}(t) &= -\gamma\left(w(t)^2\right)w(t), & \varphi(t) &= \frac{1}{\psi(t) - \|y_{\text{M}}(t) - y_{\text{ref}}(t)\|}, \end{aligned} \tag{3.21}$$

where y_{M} is the prediction for the system output computed by the MPC component.

We run the simulation on the interval $[0, 10]$ and the system and the model both use the origin as initial value, i.e. $[y(0), \dot{y}(0), \eta^1(0), \eta^2(0)] = [y_{\text{M}}(0), \dot{y}_{\text{M}}(0), \eta^1_{\text{M}}(0), \eta^2_{\text{M}}(0)] = [0, 0, 0, 0]$. In the following figures, the control signal generated via funnel MPC component is labelled with the subscript FMCP (u_{FMPC}); the signal generated by the additional funnel controller component is labelled with the subscript FC (u_{FC}). Figure 3.5 shows the application of the



(a) Tracking error $e = y - y_{\text{ref}}$ within boundary ψ.



(b) Control inputs.

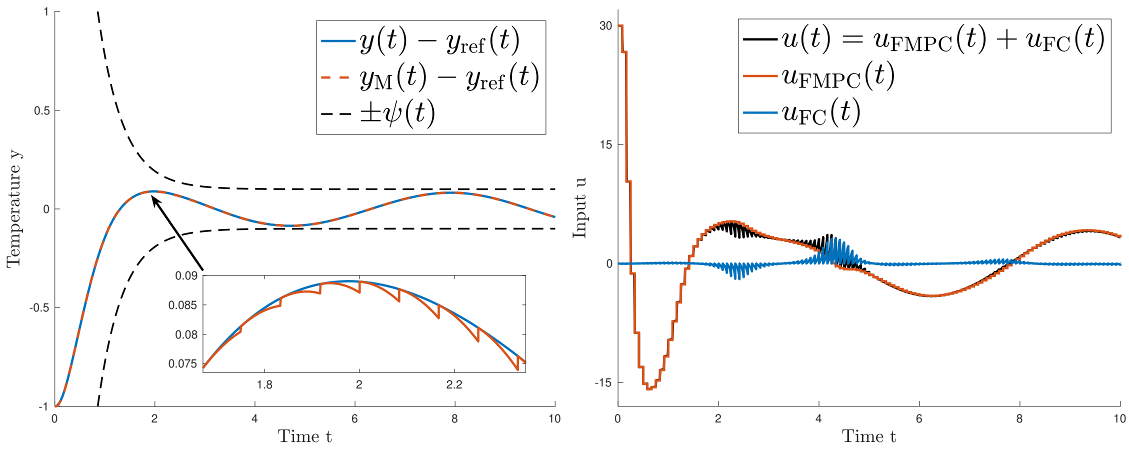
Figure 3.5: Simulation of system (2.49) under the control generated by the robust funnel MPC Algorithm 3.2.9 without model re-initialisation.

control signal computed with the robust funnel MPC Algorithm 3.2.9 to the system (2.49) when the model is not re-initialised with data from the system. The model's state from the previous iteration ($\mathfrak{X}_k = x_{\mathrm{M}}^{k-1}(t_k)$ in Step (a) of Algorithm 3.2.9). The funnel MPC control signal is applied to the system in an open-loop fashion and it is hence possible to pre-compute the control signal u_{FMPC}. As the control signal u_{FC} mitigates the discrepancies between the model's predictions and the system's output, both the model's tracking error $e_{\mathrm{M}}(t) = y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$ and the system's tracking error $e_{\mathrm{S}}(t) = y(t) - y_{\mathrm{ref}}(t)$ evolve within the boundaries given by ψ, see Figure 3.5a. Thus, the controller achieves the control objective. However, the funnel controller is active over the whole considered time interval to compensate for the deviation between the model and system. The resulting control signal $u(t) = u_{\mathrm{FMPC}}(t) + u_{\mathrm{FC}}(t)$ shows large fluctuations with peaks, see Figure 3.5b.

In a second simulation, the model is re-initialised with data from the system but we leave the rest of the setup unchanged. To properly initialise the model in accordance with Definition 3.2.5, we solve, given measurements $\dot{y}(t_k)$ and $\dot{y}(t_k)$ at time $t_k \in \delta\mathbb{N}$, the following optimisation problem at every iteration of Algorithm 3.2.9 following the ideas from (3.12).

$$\begin{aligned}
\underset{y_{\mathrm{M}}^0, \dot{y}_{\mathrm{M}}^0 \in \mathbb{R}}{\text{minimise}} \quad & \left\| y_{\mathrm{M}}^0 - y(t_k) \right\|^2 + \left\| \dot{y}_{\mathrm{M}}^0 - \dot{y}(t_k) \right\|^2 \\
\text{s.t.} \quad & \lambda\psi(t_k) > \left\| y_{\mathrm{M}}^0 - y_{\mathrm{ref}}(t_k) \right\|, \\
& \psi_2(t_k) > \left\| \dot{y}_{\mathrm{M}}^0 - \dot{y}_{\mathrm{ref}}(t_k) + k_1(y_{\mathrm{M}}^0 - y_{\mathrm{ref}}(t_k)) \right\|, \\
& \varepsilon > \hat{\varphi} \left\| y_{\mathrm{M}}^0 - y(t_k) \right\|, \\
& \varepsilon > \left\| \hat{\varphi}(\dot{y}_{\mathrm{M}}^0 - \dot{y}(t_k)) + \gamma \left(\hat{\varphi}^2 (y_{\mathrm{M}}^0 - y(t_k))^2 \right) \hat{\varphi}(y_{\mathrm{M}}^0 - y(t_k)) \right\|, \\
& \hat{\varphi} = \frac{1}{\psi(t_k) - \left\| y_{\mathrm{M}}^0 - y_{\mathrm{ref}}(t_k) \right\|}.
\end{aligned} \tag{3.22}$$

Afterwards, the solution y_{M}^0 and $\dot{y}_{\mathrm{M}}^0$ serves as an initial value for the model's differential equation (2.49) at time $t_k \in \delta\mathbb{N}$. The state η_{M} remains unchanged, i.e. the second differential equation in (2.49) is initialised with the value of η_{M} from the previous iteration. The results are displayed in Figure 3.6. It is evident that the control scheme is feasible and



(a) Tracking error $e = y - y_{\mathrm{ref}}$ within boundary ψ.

(b) Control inputs.

Figure 3.6: Simulation of system (2.49) under the control generated by the robust funnel MPC Algorithm 3.2.9 with proper model re-initialisation via the optimisation problem (3.22).

achieves the control objective. Both errors $y_{\mathrm{M}} - y_{\mathrm{ref}}$ and $y - y_{\mathrm{ref}}$ evolve within the funnel boundaries given by ψ. The model's output y_{M} diverges from the system's output y due to the modelling error. However, it is set back to the system's trajectory at the beginning of every iteration of the robust funnel MPC Algorithm 2.4.1 as Figure 3.6a shows. This results in a control signal in which the predominant portion is contributed by the MPC component. The funnel controller remains mainly inactive only compensating for the modelling errors when the system is in a critical state, i.e. close the to funnel boundary (i.e. for $t \in [4,5]$), see Figure 3.6b. Its contribution is relatively small but suffices to ensure the adherence of the system's output to the prescribed boundaries. The overall control signal is less fluctuating and demonstrates a smaller range of applied control values compared to the previous case.

4 Learning-based robust funnel MPC

MPC fundamentally depends on the availability and accuracy of the model for the underlying dynamical system. However, model-plant mismatches and external disturbances pose significant challenges, driving research into robustification and adaptive strategies. Building on the previous Chapter 3, which introduced *robust funnel MPC* by synergising the funnel MPC Algorithm 2.4.1 with model-free funnel control, this chapter extends the architecture through integrated online learning. The original hybrid approach dynamically compensates for model discrepancies through combined predictive optimisation and reactive feedback, enabling robust tracking even under severe model-plant mismatches.

Complementing direct robustification efforts, an alternative research direction focuses on adapting the underlying model to achieve robust constraint satisfaction. Examples include:

- *Data-Driven model refinement*: Techniques like those in [25] leverage persistently exciting data (cf. [68, 195]) for iterative model updates, while ensuring initial and recursive feasibility. *Set-membership identification* [143] extend this paradigm by bounding model uncertainties using online data, enabling adaptive MPC with guaranteed robust feasibility under bounded disturbances [131].

- *Iterative Learning Control* (ILC): Leveraging historical trial data, ILC refines control inputs cycle-to-cycle for repetitive tasks [45]. Combined with MPC, modern variants improve controller performance in presence of model mismatch and periodic disturbances [89, 132].

- *Gaussian process (GP) integration*: Frameworks, such as [87, 133], combine MPC with Gaussian process regression for probabilistic safety guarantees. The latter incorporates a non-linear autoregressive exogenous model (NARX) model, while the former validates its approach via an autonomous racing case study with chance constraints. Similar methods enable safe learning-based control in robotics [137]. Hybrid *physics-informed machine learning* architectures [155] enhance these approaches by embedding domain knowledge into learned models, reducing data requirements while preserving interpretability [164].

- *Constrained neural networks*: Utilising tubes containing all possible state trajectories [209] restricts neural networks to remain near predefined nominal models. This ensures safe operation despite potential learning failures.

In addition, due to the recent advancements in the field of machine learning, there have been also attempts to utilise such techniques, especially Reinforcement Learning (RL), to learn an optimal control policy and mimic the behaviour of (robust) MPC algorithms [14, 54, 183]. Practical applications include chemical reactor control via industrial MPC implementations [85]. *Transfer learning* can further extend this concept by transferring (safety-critical) control policies across different but related domains, reducing dependence on large number of system-specific data needs [208]. *Predictive safety filters* [189, 190, 191] bridge learning-based control and robust MPC. These filters validate inputs proposed by learning algorithms (e.g. reinforcement learning) against a safety-critical model. If unsafe, inputs are modified as little as necessary to ensure constraint compliance, enabling safe operation while leveraging the benefits of learning-based control. Similar in idea, hybrid frameworks pair data-driven controllers with reactive feedback to safeguard the transient behaviour:

- Policy iteration [76] and Q-learning [8] paired with safeguards,
- Koopman operator-based MPC for non-linear systems [42],
- Data-enabled predictive control (DeePC [61]) leveraging the fundamental lemma by Willems and co-authors [195] for LTI systems [168].

Surveys [46, 88, 188] document the progress of application of various safe learning methods in MPC, yet ensuring (runtime) safety in complex non-linear systems remains challenging – particularly when balancing performance and robustness in uncertain environments.

Building upon these foundations, this chapter extends the robust funnel MPC approach presented in Chapter 3 with a general online learning architecture. This framework continuously improves the surrogate model using historical data – system outputs, model predictions, and applied control signals – from both the model-based funnel MPC and the model-free feedback component. Robust tracking within predefined boundaries is achieved while allowing for:

- **Varying model complexity**: The framework accommodates both fine-tuning of detailed models and learning of entirely unknown dynamics. It handles low-order linear approximations to high-dimensional non-linear models and allows for changes in model dimensionality.
- **Continual improvement**: By refining the model's predictive capability the controller progressively enhances its performance.
- **Methodological agnosticism**: Rather than prescribing a specific learning architecture diverse paradigms and methodologies are supported.

By combining learning techniques with both model-based prediction and adaptive control, this architecture bridges the gap between robustness and adaptability in uncertain environments.

4.1 Controller structure

To achieve the overall control task of output reference tracking within prescribed bounds on the tracking error, we developed a model predictive controller in Chapter 2, which ensures superior controller performance while rigorously maintaining input and output constraints. However, given the inevitability of model-plant mismatches in practice, Chapter 3 augmented this framework with the model-free funnel controller. This addition safeguards the funnel MPC scheme by guaranteeing satisfaction of the output-tracking criterion even under disturbances and model uncertainties. We now introduce a third component – a data-based learning module – integrated alongside funnel MPC and funnel control, see Figure 4.1. This learning component iteratively updates the system model to reduce model-plant mismatch, thereby progressively enhancing overall control performance. A critical challenge lies in ensuring proper functioning of the interplay of these three components, which necessitates the introduction of additional consistency conditions (see Definition 4.1.1 and Definition 4.1.4) for the model updates – the key novelty of this approach compared to the robust funnel MPC Algorithm 3.2.9 (which combines the first two components) proposed in Chapter 3.

For the sake of readability and completeness, we recall the robust funnel MPC Algorithm 3.2.9 and explain the general ideas. In the following, we simplify the explanation

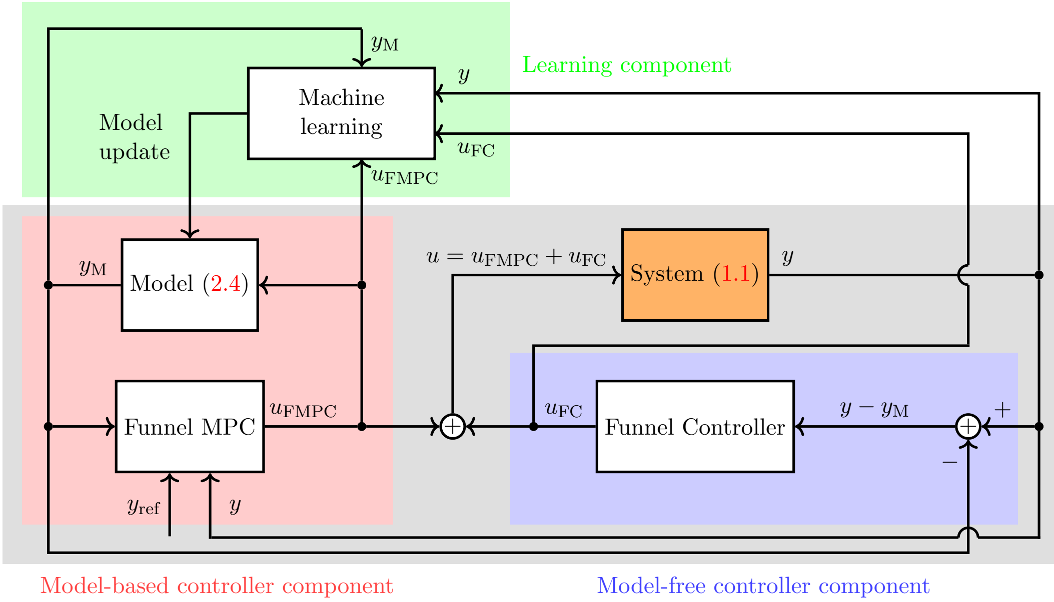


Figure 4.1: Structure of the learning-based robust FMPC scheme. The grey box (containing both the red (funnel MPC) and the blue (funnel control) structures) represents the two-component controller *robust funnel MPC* as discussed in Chapter 3. The green box represents the learning component, which receives the four signals: system output y, model output y_M, funnel MPC control signal u_{FMPC}, and funnel control signal u_{FC}.

and leave out some details in order to improve comprehensibility. We refer the reader to Chapters 2 and 3 for the technical details.

Robust funnel MPC (grey box in Figure 4.1) is a two-component controller that achieves the control objective of tracking a given reference signal $y_{ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ within a prescribed performance funnel $\mathcal{F}_\psi$ given by $\psi \in \mathcal{G}$, as laid out in Section 1.1.1. The controller combines the continuous-time funnel MPC scheme with the adaptive funnel controller. The model-based funnel MPC component (red box in Figure 4.1) uses a model of the form

$$y_M^{(r)}(t) = f_M\big(\mathbf{T}_M(\chi_r(y_M))(t)\big) + g_M\big(\mathbf{T}_M(\chi_r(y_M))(t)\big)u(t)$$

as an approximation of the system (1.1), where $(f_M, g_M, \mathbf{T}_M)$ is an element of the model class $\mathcal{M}_{t_0}^{m,r}$. At time instants $\hat{t} \in t_0 + \delta\mathbb{N}_0$ with $\delta > 0$, the current output $y(\hat{t})$ of the system (1.1) is measured and used to initialise the model, i.e. to select an initial value $\hat{\mathfrak{X}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$. The model is used to predict the future behaviour of the system over the next time interval of length $T > 0$. A control signal $u_{FMPC} \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ satisfying a given bound $u_{max} \geq 0$ on the maximal control value is computed as a solution of a finite horizon optimal control problem. The computed model output y_M when applying u_{FMPC} serves as a prediction for the system behaviour. Utilising a time-varying *funnel penalty function* ℓ_{ψ_r} ensures that the control signal u_{FMPC} achieves the control objective for the model, i.e. the model tracking error $e_M(t) := y_M(t) - y_{ref}(t)$ evolves within the performance funnel $\mathcal{F}_\psi$. Formally, this means that u_{FMPC} is an element of the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{max}, \hat{\mathfrak{X}})$.

The model-free funnel control component (blue box in Figure 3.1 and Figure 4.1) computes an instantaneous control signal u_{FC} based on the deviation between the output y of system (1.1) and the funnel MPC-based predicted y_M. The combined control $u(t) = u_{FMPC}(t) + u_{FC}(t)$ is then applied to the actual system (1.1) at time t. The signal u_{FC} from the funnel controller compensates for occurring disturbances, uncertainties in the model (2.4) and unmodelled dynamics. In other words, the funnel controller ensures that the model-plant mismatch $e_S(t) := y(t) - y_M(t)$ remains small. By doing so, not only the model output y_M tracks the reference signal y_{ref} within prescribed boundaries but also the system output y, i.e. the combined controller achieves the control objective as laid out in Section 1.1.1. Note that the control signal u_{FC} is solely determined by the instantaneous values of the system output y, the funnel function ψ, and the prediction y_M made by the model. Therefore, the model-free component cannot *plan ahead*. This may result in large control values and a rapidly changing control signal if the actual output significantly deviates from its predicted counterpart, where the term *significant* is to be understood in comparison to the current funnel size.

Learning and improving the model is the objective of the third component that we now incorporate in the overall control scheme (green box in Figure 4.1). Since funnel MPC exhibits better controller performance but the robust funnel MPC is able to compensate for model-plant mismatches, it is desirable to improve the model so that, preferably, the control u_{FMPC} is sufficient to achieve the tracking task with prescribed performance for the unknown system while satisfying the input constraints – in other words, it is desirable that the funnel controller component is inactive most of the time. In the following, we identify and establish properties of the learning component such that learning and updating the model preserves the structure necessary for robust funnel MPC Algorithm 3.2.9. We emphasise that, in the present work, we do not focus on a particular learning scheme but develop an abstract learning framework suitable to be combined with robust funnel MPC. In Section 4.2, we discuss a variant of parameter identification as one possible instance of a learning scheme; however, we emphasise that the presented methodology is not restricted to this scheme. As a result, the particular robustness with respect to model-plant mismatches of robust funnel MPC even allows to start with “no model”, e.g. only an integrator chain, and then learn the remaining drift-dynamics.

The idea of the learning component is to use measurement data from the system output y, the model output y_M and its derivatives, i.e. the model state $x_M = \chi_r(y_M)$, the funnel MPC signal u_{FMPC} and the funnel control signal u_{FC} to improve the model used for computation of u_{FMPC} in the next iteration of the MPC algorithm (cf. Figure 4.1). The data $(y, x_M, u_{FMPC}, u_{FC})$ collected up to the time $\hat{t} \geq t_0$ over the interval $[t_0, \hat{t}]$ in order to be used to update the model $(f_M, g_M, \mathbf{T}_M) \in \mathcal{M}_{t_0}^{m,r}$ is an element of the set

$$\mathfrak{S}_{\hat{t}} := \mathcal{C}^{r-1}([t_0, \hat{t}], \mathbb{R}^m) \times \mathcal{R}([t_0, \hat{t}], \mathbb{R}^m)^r \times L^\infty([t_0, \hat{t}], \mathbb{R}^m) \times L^\infty([t_0, \hat{t}], \mathbb{R}^m). \tag{4.1}$$

Note that the image spaces of all signals have the same dimension $m \in \mathbb{N}$ (here we consider x_M to be an element of $\mathcal{R}([t_0, \hat{t}], \mathbb{R}^m)^r$ instead of $\mathcal{R}([t_0, \hat{t}], \mathbb{R}^{rm})$). In order to incorporate an abstract learning scheme $\mathcal{L}$ into the funnel MPC algorithm, it is imperative that, after updating the model, both other controller components – the model-based funnel MPC and the model-free funnel control – maintain functionality. For the functioning of the funnel MPC component, it is necessary to ensure that at every iteration of the MPC scheme that there exists a control signal feasible for the model. Meaning: given $\hat{\mathfrak{x}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$ at time $\hat{t} \in t_0 + \delta\mathbb{N}_0$, there exists a control $u \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ bounded by the constant $u_{max} \geq 0$ that, if applied to the model (2.4), ensures that $x_M(t) - \chi_r(y_{ref})(t)$ evolves within $\mathcal{D}_t^{\Psi}$ for all t over the next time interval of length $T > 0$. In short, the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{max}, \hat{\mathfrak{x}})$ as in (2.28) has to be non-empty given $u_{max} \geq 0$. Theorem 2.3.21 shows that for every model $(f_M, g_M, \mathbf{T}_M) \in \mathcal{M}_{t_0}^{m,r}$ there exists $u_{max} \geq 0$ such that $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{max}, \hat{\mathfrak{x}}) \neq \emptyset$ for all $\hat{t} \geq t_0$, $T > 0$, and $\hat{\mathfrak{x}} \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t})$. However, the difficulty now lies in ensuring that the input saturation level $u_{max} \geq 0$ does not increase over time. We want to a priori choose a uniform $u_{max} \geq 0$ for all models generated by the learning component during the operation of the control algorithm. Moreover, incorporating a learning scheme $\mathcal{L}$ must not lead to a globally unbounded control signal u_{FC} of the model-free funnel controller component.

Establishing the existence of such bound already has been the main challenge in proving the functioning of the robust funnel MPC Algorithm 3.2.9 in Theorem 3.2.11. The bound on the maximal control effort of the funnel controller component u_{FC} derived in the proof of Theorem 3.2.11 depends among other terms on

$$f_{\mathrm{M}}^{\max} + g_{\mathrm{M}}^{\max} u_{\max},$$

where $f_{\mathrm{M}}^{\max} \geq \left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})}) \right\|_\infty$ and $g_{\mathrm{M}}^{\max} \geq \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,\hat{t})}) \right\|_\infty$ for all $\hat{t} \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_{\hat{t}}^{\Psi}$, see (3.18) in the proof of Theorem 3.2.11 and also Lemma 2.3.20. To impose uniform maximal control values on both the model-based and the model-free controller component, we restrict the considered model class $\mathcal{M}_{t_0}^{m,r}$ in the following definition.

Definition 4.1.1 (Restricted model class $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$)**.** *Let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\bar{\rho} \geq 0$, $u_{\max} \geq 0$, and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$. We say that the model (2.4) belongs to the restricted model class $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$ for $m, r \in \mathbb{N}$, and $t_0 \in \mathbb{R}_{\geq 0}$, written $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$, if*

(L.1) $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$,

(L.2) $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}}) \neq \emptyset$ *for all* $\hat{t} \geq t_0$, $T > 0$, *and* $\hat{\mathfrak{x}} \in \mathfrak{I}_{t_0,\hat{t}}^{\Psi}(\hat{t})$,

(L.3) $\bar{\rho} \geq \left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_\infty + \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_\infty u_{\max}$ *for all* $s \in [t_0, \infty]$ *and* $\zeta \in \mathcal{Y}_s^{\Psi}$.

As Definition 4.1.1 restricts the model class $\mathcal{M}_{t_0}^{m,r}$ by the properties (L.2) and (L.3), the question arises for which parameters $u_{\max}, \bar{\rho} \geq 0$ the restricted model class $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$ is non empty. The following lemma gives an answer to this question.

Lemma 4.1.2. *Let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$. For every $u_{\max} > 0$, there exists $\bar{\rho} > 0$ such that $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r} \neq \emptyset$.*

Proof. Given $u_{\max} > 0$, let $\varepsilon > 0$ such that

$$u_{\max} \geq \varepsilon \left(\left\| y_{\mathrm{ref}}^{(r)} \right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\| \dot{\psi}_r \right\|_\infty \right),$$

where k_j with $j = 1, \ldots, r-1$ are the parameters associated to the auxiliary funnel functions $(\psi_1, \ldots, \psi_r)$ and the constants μ_i^j are recursively defined via $\mu_i^0 := \|\psi_i\|_\infty$ and $\mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j$ for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$. Set $f_{\mathrm{M}} \equiv 0$, $\mathbf{T}_{\mathrm{M}} \equiv 0$, and $g_{\mathrm{M}} \equiv \frac{1}{\varepsilon} I_m$, where I_m denotes the identity matrix in $\mathbb{R}^{m \times m}$. Utilising these functions, it is easy to see that $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0}^{m,r}$. According to Theorem 2.3.21, we have $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}}) \neq \emptyset$ for all $\hat{t} \geq t_0$, $T > 0$, and $\hat{\mathfrak{x}} \in \mathfrak{I}_{t_0,\hat{t}}^{\Psi}(\hat{t})$. We choose $\bar{\rho} \geq \frac{1}{\varepsilon} u_{\max}$, then

$$\left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_\infty + \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_\infty u_{\max} = \left\| \tfrac{1}{\varepsilon} I_m \right\|_\infty u_{\max} = \tfrac{1}{\varepsilon} u_{\max} \leq \bar{\rho},$$

for $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_s^{\Psi}$. Therefore, $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$. □

Remark 4.1.3. For order $r = 1$, the set $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$ is non-empty for $u_{\max} = \bar{\rho} = 0$ if $\dot{y}_{\mathrm{ref}} \equiv 0$ and $\dot{\psi} \equiv 0$. Utilising Theorem 2.3.21, this can be easily proven by showing $(0, I_m, 0) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$, where I_m denotes the identity matrix in $\mathbb{R}^{m \times m}$. •

With Definition 4.1.1 at hand, we define a learning scheme $\mathcal{L}$ mapping the signals $(y, x_{\mathrm{M}}, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})$ collected up to the time $\hat{t} \geq t_0$ to a model $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$.

Definition 4.1.4. *(Feasible learning scheme $\mathcal{L}$) Let $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $u_{\max}, \bar{\rho} \geq 0$, and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ such that $\mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}} \neq \emptyset$. We call a function*

$$\mathcal{L} : \bigcup_{t \geq t_0} \mathfrak{S}_t \to \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$$

a $(u_{\max}, \bar{\rho})$-feasible learning scheme for robust funnel MPC.

Remark 4.1.5. The function $\mathcal{L}$ maps the hitherto available data at time $\hat{t}$, i.e. the signals $(\hat{y}, \hat{x}_{\rm M}, \hat{u}_{\rm FMPC}, \hat{u}_{\rm FC}) \in \mathfrak{S}_{\hat{t}}$, to a suitable model $(f_{\rm M}, g_{\rm M}, \mathbf{T}_{\rm M}) \in \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$. Due to the quite abstract nature of Definitions 4.1.1 and 4.1.4, a few comments are in order.

(a) Condition (L.2) in Definition 4.1.1 can be ensured by prescribing two constants $f_{\rm M}^{\max}, g_{\rm M}^{-1\max} \geq 0$ fulfilling

$$f_{\rm M}^{\max} \geq \left\| f_{\rm M}(\mathbf{T}_{\rm M}(\zeta)|_{[0,s)}) \right\|_\infty \quad \text{and} \quad g_{\rm M}^{-1\max} \geq \left\| g_{\rm M}(\mathbf{T}_{\rm M}(\zeta)|_{[0,s)})^{-1} \right\|_\infty$$

for all $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_s^\Psi$. Then, the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is non-empty for all $T > 0$, $\hat{\mathfrak{X}} \in \mathfrak{I}^\Psi_{t_0,\tau}(\hat{t})$, and

$$u_{\max} \geq g_{\rm M}^{-1\max} \left(f_{\rm M}^{\max} + \left\| y_{\rm ref}^{(r)} \right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\| \dot{\psi}_r \right\|_\infty \right),$$

where k_j with $j = 1, \ldots, r-1$ are the parameters associated to the auxiliary funnel functions $(\psi_1, \ldots, \psi_r)$ and the constants μ_i^j are recursively defined via $\mu_i^0 := \|\psi_i\|_\infty$, $\mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j$ for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$, see Theorem 2.3.21. If one additionally prescribes a constant $g_{\rm M}^{\max} \geq 0$ with $g_{\rm M}^{\max} \geq \left\| g_{\rm M}(\mathbf{T}_{\rm M}(\zeta)|_{[0,s)}) \right\|_\infty$ for all $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_s^\Psi$, then condition (L.3) is fulfilled for $\bar{\rho} \geq f_{\rm M}^{\max} + g_{\rm M}^{\max} u_{\max}$.

(b) Condition (L.3) in Definition 4.1.1 guarantees that $y_{\rm M}^{(r)}$ is uniformly bounded by

$$\left\| y_{\rm M}^{(r)}(t) \right\| \leq f_{\rm M}^{\max} + g_{\rm M}^{\max} u_{\max}$$

independent of the chosen model. In Theorem 4.1.8, we use this estimate to prove the uniform boundedness of the control signal $u_{\rm FC}$ generated by the funnel control component.

(c) The function $\mathcal{L}$ in Definition 4.1.1 is defined on the set $\bigcup_{t \geq t_0} \mathfrak{S}_t$. However, it is clear that the domain of $\mathcal{L}$ can be modified to take additional aspects relevant to the control problem into account. We want to comment on certain possibilities.

  (i) The learning scheme $\mathcal{L}$ utilises the entire measured data up to the current time instant $\hat{t}$, meaning the signals $(\hat{y}, \hat{x}_{\rm M}, \hat{u}_{\rm FMPC}, \hat{u}_{\rm FC})$ are collected over the whole interval $[t_0, \hat{t}]$. With increasing time, this results in ever growing memory requirements for the measurements. Obviously, this is not suitable in practice. Thus, it is beneficial to use a sliding window approach and use measurements over a time window of length $\hat{\tau} \geq 0$, i.e. the measurements $(\hat{y}, \hat{x}_{\rm M}, \hat{u}_{\rm FMPC}, \hat{u}_{\rm FC})$ are only defined on the interval $[\hat{t} - \hat{\tau}, \hat{t}] \cap [t_0, \hat{t}]$. However, to avoid introducing another parameter $\hat{\tau}$ and further complicating Definition 4.1.4, we assume in this work that signals are indeed available for the whole time interval $[t_0, \hat{t}]$.

  (ii) In many applications, sufficiently accurate models are often already available. Typically, only specific parameters remain unknown, inaccurately estimated, or require refinement. Furthermore, as most optimisation algorithms inherently require an initialisation point, the current model $(f_M, g_M, \mathbf{T}_M)$ can serve as a natural additional input to the learning module $\mathcal{L}$. This approach achieves dual benefits: reducing computational effort by leveraging prior knowledge, while simultaneously mitigating the risk of algorithmic instability – avoiding abrupt, destabilising changes to the model structure during successive executions of the function $\mathcal{L}$.

(iii) The function $\mathcal{L}$ need not operate solely as a learning algorithm – it can also be utilised to dynamically switch between distinct models within the model-based funnel MPC component of the control framework. For instance, in systems that operate at different setpoints for extended periods, it may be advantageous to employ separate models tailored to each operating regime. Here, $\mathcal{L}$ triggers model switching after setpoint transitions, enabling the use of simpler, locally accurate models rather than relying on a single complex global model. This approach can result in overall improved accuracy while reducing computational overhead.

(d) Since Definition 4.1.4 is rather general, the set of potential learning functions $\mathcal{L}$ can be fairly large and difficult to grasp, including with the restrictions (L.2) and (L.3) in Definition 4.1.1 on the set $\mathcal{M}_{t_0}^{m,r}$. Depending on the specific application, it can therefore be advisable to restrict oneself to a subset of potential models in order to simplify the selection of a suitable function $\mathcal{L}$ and to be able to compare different learning algorithms more easily. In Section 4.2, we will derive conditions for a learning scheme restricted to linear models to be $(u_{\max},\bar{\rho})$-feasible.

(e) Given a system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$, then $\mathbf{T}$ is an operator mapping from $\mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ to $L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^\kappa)$ for some $\kappa \geq 0$, see Definition 3.1.1. For systems with state representation, see Examples 2.2.3 and 2.2.4, this dimension can be interpreted as the dimension of the internal dynamics of the system. The dimension κ is unknown but fixed. In contrast, for the operator $\mathbf{T}_M \in \mathcal{T}_{t_0}^{rm,\nu}$ of the model, the dimension $\nu \in \mathbb{N}_0$ of the model's internal dynamics can be considered as a parameter in the learning step. This means, in order to improve the model such that it "explains" the system measurements, the dimension of the internal state can be varied. Note that $\nu = 0$ (no internal dynamics) is explicitly allowed for the model.

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Now, we summarise the reasoning so far in the following algorithm, which achieves the tracking control objective formulated in Section 1.1.1. It is a modification of the robust funnel MPC Algorithm 3.2.9. Here, the proper re-initialisation of the model at every iteration done in Algorithm 3.2.9 is substituted by the learning component $\mathcal{L}$.

Algorithm 4.1.6 (Learning-based robust funnel MPC)**.**
Given:

- instantaneous measurements of the output y and its derivatives of system (1.1), initial time $t^0 \in \mathbb{R}_{\geq 0}$, initial trajectory $y^0 \in \mathcal{C}^{(r-1)}([0, t_0], \mathbb{R}^m)$, reference signal $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, funnel function $\psi \in \mathcal{G}$,

- auxiliary funnel boundary function $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ with corresponding parameters k_i for $i = 1, \ldots, r$, input saturation level $u_{\max} \geq 0$, parameter $\bar{\rho}$ such that $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r} \neq \emptyset$, initial model $(f_M^0, g_M^0, \mathbf{T}_M^0) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$, and funnel stage cost function ℓ_{ψ_r},

- initialisation parameters $\varepsilon, \lambda \in (0, 1)$ and a $(u_{\max}, \bar{\rho})$-feasible learning scheme $\mathcal{L}$ as in Definition 4.1.4,

- a surjection $\mathcal{N} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and a bijection $\gamma \in \mathcal{C}([0,1),[1,\infty))$.

Set the time shift $\delta > 0$, the prediction horizon $T \geq \delta$, and index $k := 0$.
Define the time sequence $(t_k)_{k\in\mathbb{N}_0}$ by $t_k := t_0 + k\delta$.
Steps:

(a) Obtain a measurement $\hat{x}_k := \chi_r(y)(t_k)$ of the system output y and its derivatives at the current time t_k and choose a *proper (ε, λ)-initialisation* $\mathfrak{X}_k \in \mathfrak{PI}^{\Psi,\varepsilon,\lambda}_{t_0,t_k}(t_k, \hat{x}_k)$.

(b) **Funnel MPC**
Compute a solution $u_{\mathrm{FMPC},k} \in L^\infty([t_k, t_k+T], \mathbb{R}^m)$ of the optimal control problem

$$\underset{\substack{u\in L^\infty([t_k,t_k+T],\mathbb{R}^m),\\ \|u\|_\infty \leq u_{\max}}}{\text{minimise}} \quad \int_{t_k}^{t_k+T} \ell_{\psi_r}(s, \xi_r(x^k_{\mathrm{M}}(s;t_k,\mathfrak{X}_k,u) - \chi_r(y_{\mathrm{ref}})(s)), u(s))\mathrm{d}s \tag{4.2}$$

utilising the model $(f^k_{\mathrm{M}}, g^k_{\mathrm{M}}, \mathbf{T}^k_{\mathrm{M}})$. Predict the state $x^k_{\mathrm{M}}(t;t_k,\mathfrak{X}_k,u_{\mathrm{FMPC},k})$ and output $y^k_{\mathrm{M}}(t;t_k,\mathfrak{X}_k,u_{\mathrm{FMPC},k})$ of the model on the interval $[t_k, t_{k+1}]$, and define the adaptive funnel $\varphi_k : [t_k, t_{k+1}] \to \mathbb{R}_{>0}$ by

$$\varphi_k(t) := \frac{1}{\psi_1(t) - \left\|e^k_{\mathrm{M}}(t)\right\|}, \tag{4.3}$$

where $e^k_{\mathrm{M}}(t) = y^k_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)$.

(c) **Funnel control**
Using the error variables e_i for $i = 1, \ldots, r$ as in (3.3), define the funnel control law u_{FC} with reference y^k_{M} and funnel function φ_k as in (4.3) by

$$u_{\mathrm{FC},k}(t) := (\mathcal{N} \circ \gamma)(\|e_r(\varphi_k(t), e_{\mathrm{S}}(t))\|^2) e_r(\varphi_k(t), e_{\mathrm{S}}(t)), \tag{4.4}$$

with $e_{\mathrm{S}}(t) = y(t) - y^k_{\mathrm{M}}(t)$. Apply the control law

$$u_k : [t_k, t_{k+1}) \to \mathbb{R}^m, \ u_k(t) = u_{\mathrm{FMPC},k}(t) + u_{\mathrm{FC},k}(t) \tag{4.5}$$

to system (1.1).

(d) **Continual learning**
Increment k by 1, find a feasible model

$$\mathcal{L}\big((y, x_{\mathrm{M}}, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{[t_0,t_k]}\big) = (f^k_{\mathrm{M}}, g^k_{\mathrm{M}}, \mathbf{T}^k_{\mathrm{M}})$$

based on the measurement of the signals on the interval $[t_0, t_k]$. Then, go to Step (a).

▲

Remark 4.1.7. We comment on some aspects of the learning-based robust funnel MPC Algorithm 4.1.6.

(a) The signals $(y, x_{\mathrm{M}}, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})$ used for the learning scheme $\mathcal{L}$ during Step (d) are the whole trajectories of the individual functions up to the current time t_{k+1}. This means that y is the solution of the system differential equation (1.1) up to the current time, the control signals u_{FMPC} and u_{FC} are the concatenation of the signals $u_{\mathrm{FC},k}$ and $u_{\mathrm{FMPC},k}$ applied at every interval $[t_i, t_{i+1}]$ for $i = 0, \ldots, k$, and x_{M} is the concatenation of the solutions x^i_{M} of the model differential equation (2.4) with model $(f^i_{\mathrm{M}}, g^i_{\mathrm{M}}, \mathbf{T}^i_{\mathrm{M}})$ and initial value $\mathfrak{X}_i$ on the interval $[t_i, t_{i+1}]$ for $i = 0, \ldots, k-1$. To be more precise:

$$u_{\mathrm{FMPC}}(t) = u_{\mathrm{FMPC},i}(t), \quad u_{\mathrm{FC}}(t) = u_{\mathrm{FC},i}(t), \quad x_{\mathrm{M}}(t) = x^i_{\mathrm{M}}(t;t_i,\mathfrak{X}_i,u_{\mathrm{FMPC},i})$$

for $t \in [t_i, t_{i+1})$ and $i = 0, \ldots, k$. Note that x_{M} is not a concatenated solution in the sense of Definition 2.4.2 as the model $(f^k_{\mathrm{M}}, g^k_{\mathrm{M}}, \mathbf{T}^k_{\mathrm{M}})$ changes at every iteration of the Algorithm 4.1.6.

(b) Let $u_{\max} \geq \left(\left\|y_{\text{ref}}^{(r)}\right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\|\dot{\psi}_r\right\|_\infty\right)$ and $\bar{\rho} = u_{\max}$ where k_j are the parameters associated to the funnel functions $(\psi_1, \ldots, \psi_r)$ for $j = 1, \ldots, r-1$. The constants μ_i^j are recursively defined via $\mu_i^0 := \|\psi_i\|_\infty$, $\mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j$ for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$. Then, the integrator chain

$$y_{\text{M}}^{(r)}(t) = u(t)$$

is a model in $\mathcal{M}_{t_0, u_{\max}, \bar{\rho}}^{m,r}$, see proof of Lemma 4.1.2. The model-based MPC component of the control scheme can operate in this sense "without" a model. It therefore is possible to apply the learning-based robust funnel MPC Algorithm 4.1.6 without an initial model or an offline learning phase.

(c) In practice, it may often not be desirable to update the model $(f_{\text{M}}^k, g_{\text{M}}^k, \mathbf{T}_{\text{M}}^k)$ at every iteration of the Algorithm 4.1.6. Especially, if the execution of the learning procedure is very time-consuming, it may be advantageous to evaluate $\mathcal{L}$ only every i-th iteration for $i > 1$.

(d) Note that, the initialisation in Step (a) of Algorithm 4.1.6 at time $t_k \in t_0 + \delta\mathbb{N}_0$ is independent of the current model $(f_{\text{M}}^k, g_{\text{M}}^k, \mathbf{T}_{\text{M}}^k)$. It only depends on y_{ref}, Ψ, ε, λ, and $\hat{x}_k = \chi_r(y)(t_k)$, see Definition 3.2.5. Instead of $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ for a fixed $\tau \geq 0$ as in Step (a) of the robust funnel MPC Algorithm 3.2.9, we require $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,t_k}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ in (4.2) of Algorithm 4.1.6, i.e. both components of $\mathfrak{X}_k$ are defined on their entire maximal time intervals up to t_k. By doing so, we avoid having to deal with changing memory limits for the operators $\mathbf{T}_{\text{M}}^k$. This set is non empty for $\hat{x}_k$ with $\hat{x}_k - \chi_r(y_{\text{ref}})(t_k) \in \mathcal{E}_r^\varepsilon(1/\psi_1(t_k))$, see Remark 3.2.6. In case all operators generated by the learning scheme $\mathcal{L}$ during Step (d) have a memory limit lower or equal than a pre-specified bound $\bar{\tau} \geq 0$, the initialisation can alternatively be chosen as $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,\bar{\tau}}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ in Step (a) of Algorithm 4.1.6.

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We are now in the position to formulate the main result of this chapter, which extends Algorithm 3.2.9 and the corresponding Theorem 3.2.11 by the learning component.

Theorem 4.1.8. *Consider a system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}_{t_0}^{m,r}$ as in Definition 3.1.1. Let $t_0 \geq 0$ be the initial time, $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, and $\Psi = (\psi_1, \ldots, \psi_r) \in \mathscr{G}$ be given, and let $y^0 \in \mathcal{C}^{(r-1)}([0, t_0], \mathbb{R}^m)$ be the initial trajectory for the system (1.1) with $\chi_r(y_0 - y_{\text{ref}})(t_0) \in \mathcal{E}_r^1(1/\psi(t_0))$. Further, let $u_{\max}, \bar{\rho} > 0$ such that the set of models $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$ is non-empty. There exist $\varepsilon, \lambda \in (0,1)$ such that, for every initial model $(f_{\text{M}}^0, g_{\text{M}}^0, \mathbf{T}_{\text{M}}^0) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$ and for every $(u_{\max}, \bar{\rho})$-feasible learning scheme $\mathcal{L} : \bigcup_{t \geq t_0} \mathfrak{S}_t \to \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$, the robust funnel MPC Algorithm 3.2.9 with $\delta > 0$ and $T \geq \delta$ is initially and recursively feasible, i.e. at every time instant $t_k := t_0 + k\delta$ for $k \in \mathbb{N}_0$*

- *there exists a proper initialisation $\mathfrak{X}_k \in \mathfrak{PI}_{t_0,t_k}^{\Psi,\varepsilon,\lambda}(t_k, \hat{x}_k)$ and*
- *the OCP (3.13) has a solution $u_{\text{FMPC},k} \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$.*

Moreover, the closed-loop system consisting of the system (1.1) and the feedback law (4.5) has a global solution $y : [0, \infty) \to \mathbb{R}^m$. Each global solution y satisfies that

(i) all signals are bounded, in particular, $u \in L^\infty([t_0, \infty), \mathbb{R}^m)$ and $y \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$,

(ii) the tracking error between the system's output and the reference evolves within prescribed boundaries, i.e.

$$\forall\, t \geq t_0 : \|y(t) - y_{\text{ref}}(t)\| < \psi_1(t).$$

Proof. The learning-based robust funnel MPC Algorithm 4.1.6 differs from the robust funnel MPC Algorithm 3.2.9 only in two aspects, the utilisation of the learning scheme $\mathcal{L}$ in Step (d) of the algorithm and the usage of changing models $(f_{\mathrm{M}}^k, g_{\mathrm{M}}^k, \mathbf{T}_{\mathrm{M}}^k)$ in Step (a). We will show how the proof of Theorem 3.2.11 can be adapted to the current setting. However, as the proof of Theorem 3.2.11 does, in large parts, not depend on the used model, we will not repeat all technical details.
Step 1: Let $\delta > 0$ and $T \geq \delta$ be arbitrary but fixed. Note that, the set of controls $\mathcal{U}_{[\hat{t},\hat{t}+T]}(\hat{t},\hat{\mathfrak{X}})$ is non-empty for all $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$, all $\hat{t} \geq t_0$, and all $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\hat{t}}(\hat{t})$ due to property (L.2) of $\mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$, see Definition 4.1.1. Define λ and ε as in Step 1–3 in the proof of Theorem 3.2.11. Then, $\varepsilon \in (0,1)$ is constructed in a way such that we have $\hat{x}_0 - \chi_r(y_{\mathrm{ref}})(t_0) \in \mathcal{E}^{\varepsilon}_r(1/\psi_1(t_0))$ for $\hat{x}_0 := \chi_r(y)(t_0) = \chi_r(y^0)(t_0)$, see definition of ε in the proof of Theorem 3.2.11. Thus, $\mathfrak{P}\mathfrak{I}^{\Psi}_{t_0,\hat{t}}(\hat{t})(t_0, \chi_r(y)(t_0)) \neq \emptyset$, see Remark 3.2.6.
Step 2: Let $\mathcal{L} : \bigcup_{t\geq t_0} \mathfrak{S}_t \to \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$ be a $(u_{\max}, \bar{\rho})$-feasible learning scheme. When applying the learning-based robust funnel MPC Algorithm 4.1.6 to the system (1.1), the system's dynamics on each interval $[t_k, t_{k+1}]$ are given by

$$y_k^{(r)}(t) = F(\mathbf{T}(\chi_r(y_k))(t), u_k(t)), \quad y_k|_{[0,t_k]} = y_{k-1}|_{[0,t_k]} \tag{4.6}$$

where $y_{-1} := y^0$ and u_k is the control law given by (4.5). In Step 4 of the proof of Theorem 3.2.11, it was inductively shown that the robust funnel MPC Algorithm 3.2.9 is initially and recursively feasible. This means, in particular, that there exists a proper initialisation $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}^{\Psi}_{t_0,t_k}(t_k, \chi_r(y_{k-1})(t_k))$ at every time instant t_k, that u_k as in (3.16) is well defined on every interval $[t_k, t_{k+1}]$, and that (4.6) has a maximal solution y_k defined on the entire interval $[t_k, t_{k+1}]$. Step 4 of the proof of Theorem 3.2.11 does not depend on the concrete choice $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$ of the model used on the time interval $[t_k, t_{k+1}]$. Only two of the model's aspects are used within the proof: the non-emptiness of the set $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ and the uniform boundedness of

$$\left\| y_{\mathrm{M}}^{(r)}(t) \right\| = \| f_{\mathrm{M}}(\chi_r(y_{\mathrm{M}})(t)) + g_{\mathrm{M}}(\chi_r(y_{\mathrm{M}})(t)) u_{\mathrm{FMPC},k}(t) \| .$$

The former one is directly fulfilled by property (L.2) of $\mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$, see Definition 4.1.1. The latter one is also satisfied since property (L.3) ensures

$$\left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_{\infty} + \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[0,s)}) \right\|_{\infty} u_{\max} \leq \bar{\rho}$$

for all $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}^{\Psi}_s$. One therefore can adapt Step 4 of the proof of Theorem 3.2.11 to the current setting in order to show that there exists a proper initialisation $\mathfrak{X}_k \in \mathfrak{P}\mathfrak{I}^{\Psi}_{t_0,t_k}(t_k, \chi_r(y_{k-1})(t_k))$ at every time instant t_k, that u_k as in (4.5) is well defined on every interval $[t_k, t_{k+1}]$, and that (4.6) has a maximal solution y_k defined on the entire interval $[t_k, t_{k+1}]$ if the learning-based robust funnel MPC Algorithm 4.1.6 is applied to the system (1.1). The existence of a solution $u_{\mathrm{FMPC},k} \in \mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ of the OCP (4.2) at time instant t_k for $k \in \mathbb{N}_0$ is a direct consequence of Theorem 2.3.26 and the non-emptiness of $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$.
Step 3: The signal $u_{\mathrm{FMPC},k}$ as an element of $\mathcal{U}_{[t_k,t_k+T]}(u_{\max}, \mathfrak{X}_k)$ is bounded by $u_{\max} \geq 0$ for all $k \in \mathbb{N}_0$. The funnel control signal $u_{\mathrm{FC},k}$ is bounded by $\mathfrak{h}(\mathcal{N} \circ \gamma(\varepsilon^2))$ for all $k \in \mathbb{N}_0$ because of the construction of $\varepsilon \in (0,1)$, c.f. Step 5 of the proof of Theorem 3.2.11. Moreover, we have

$$\chi_r(y - y_{\mathrm{M}})(t) \in \mathcal{E}^{\tilde{\varepsilon}}_r(\varphi(t))$$

for all $t \in [t_0, \infty)$ for some $\tilde{\varepsilon} \in (0,1)$, c.f. Step 5 of the proof of Theorem 3.2.11. Since y_{M} and φ are bounded functions, $y \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, see definition of $\mathcal{E}^{\tilde{\varepsilon}}_r$ in (3.3). Finally,

$$\|y(t) - y_{\mathrm{ref}}(t)\| \leq \|y(t) - y_{\mathrm{M}}(t)\| + \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| < \varphi(t) + \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| = \psi_1(t)$$

for all $t \geq t_0$. This shows (ii) and completes the proof. □

Remark 4.1.9. Conditions (L.2) and (L.3) in Definition 4.1.1 ensure $\mathcal{U}_{[t,t+T]}(u_{\max}, \mathfrak{X}_k) \neq \emptyset$ for all $t \geq t_0$ and $T > 0$ and that the funnel control signal $u_{\mathrm{FC},k}$ is uniformly bounded for all $k \in \mathbb{N}$. As the attentive reader might have noticed, it is possible to relax these conditions during operation of the learning-based robust funnel MPC Algorithm 4.1.6. Firstly, it is possible to fix the prediction horizon $T > 0$. Moreover, it is sufficient that the model $(f_{\mathrm{M}}^k, g_{\mathrm{M}}^k, \mathbf{T}_{\mathrm{M}}^k)$ chosen at time instant $t_k \in t_0 + \delta\mathbb{N}_0$ can ensure these properties for all future time $t \geq t_k$. It is not required to ensure them for the past, i.e. $t \leq t_k$. To be precise, one can replace (L.2) at time t_k and with given $T > 0$ by

(L.2') $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$ for all $\hat{t} \geq t_k$ and $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\hat{t}}(\hat{t})$,

and the condition (L.3) can be relaxed by

(L.3') $\bar{\rho} \geq \left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[t_k,s)})\right\|_{\infty} + \left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\zeta)|_{[t_k,s)})\right\|_{\infty} u_{\max}$ for all $s \in [t_k, \infty]$ and $\zeta \in \mathcal{Y}^{\Psi}_s$.

In order to avoid introducing a time dependency and thus an additional parameter which introduces even more technicalities, we refrained from formulating Definition 4.1.1 in this more general way. •

4.2 On learning schemes

In recent years, data-driven control has attracted significant attention, with a proliferation of research contributions in the field. These results can broadly be categorised into control schemes for linear systems and techniques developed for non-linear systems. Bolstered by successful applications [142], powerful numerical methods such as extended dynamic mode decomposition [196], and theoretical advances – including convergence guarantees in the infinite-data limit [113], finite-data error bounds [110], and extensions to stochastic control systems [148] – the Koopman formalism [48], originally proposed in [111], has emerged as a cornerstone for data-driven controller design [75, 149, 178]. Recent work has further extended this framework to model predictive control, establishing rigorous closed-loop guarantees [41, 43, 112]. For linear time-invariant systems, Subspace Predictive Control [69] has gained prominence, while the so-called fundamental lemma by Willems and co-authors [195] enables direct data-driven methods such as DeePC [61]. Complementary approaches include Reinforcement Learning (RL) [182], Gaussian processes for uncertainty-aware designs [87, 104, 133], and SINDY for sparse identification of non-linear dynamics [49]. Deep neural networks (DNNs) have further expanded the scope of data-driven control, enabling approximation of complex dynamics and control policies for high-dimensional systems [54, 153, 170]. Recent advances also address safety-critical scenarios through Hamilton-Jacobi reachability analysis [20].

The structural conditions provided in Definition 4.1.4 can be used to define suitable learning algorithms based on the previously discussed techniques – for linear as well as for non-linear systems.

In this section, we derive sufficient conditions on the parameters of models to be learned in order to make them eligible for a learning scheme $\mathcal{L}$ as defined in Definition 4.1.4. Since in many applications a linear model may serve as a good prediction model, we derive sufficient conditions on the parameters of linear systems of the form

$$\begin{aligned} y_{\mathrm{M}}^{(r)}(t) &= \sum_{j=1}^{r} R_j y_{\mathrm{M}}^{(j-1)}(t) + S\eta + D_1 + \Gamma u(t), \quad && \chi_r(y_{\mathrm{M}})(t_0) = y_{\mathrm{M}}^0, \\ \dot{\eta}(t) &= Q\eta(t) + P y_{\mathrm{M}}(t) + D_2, && \eta(t_0) = \eta^0 \end{aligned} \tag{4.7}$$

where $R_j \in \mathbb{R}^{m\times m}$ for all $j = 1, \ldots, r$, $S, P^{\top} \in \mathbb{R}^{m\times\nu}$, $D_1 \in \mathbb{R}^m$, $D_2 \in \mathbb{R}^{\nu}$, $Q \in \mathrm{S}_{\nu}^{--}$, and $\Gamma \in \mathrm{GL}_m(\mathbb{R})$. We use in the following the notation $R := (R_1, \ldots, R_r)$ and denote the largest eigenvalue of the symmetric negative definite matrix Q by $\lambda_{\max}(Q) < 0$. Define the functions

$$\begin{aligned} f_{D_1} &: \mathbb{R}^\nu \to \mathbb{R}^m, \qquad \eta \mapsto \eta + D_1, \\ g_\Gamma &: \mathbb{R}^\nu \to \mathbb{R}^{m\times m}, \quad \eta \mapsto \Gamma, \end{aligned} \tag{4.8}$$

and the linear integral operator $\mathbf{T}_{R,S,Q,P,D_2,\eta_0} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)^r \to L^\infty_{\mathrm{loc}}([t_0,\infty), \mathbb{R}^\nu)$ by

$$\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(z_1,\ldots,z_r)(t) := \sum_{j=1}^{r} R_j z_j(t) + S\left(\mathrm{e}^{Q(t-t_0)}\eta_0 + \int_{t_0}^{t} \mathrm{e}^{Q(t-s)}\left(P z_1(s) + D_2\right) \mathrm{d}s\right). \tag{4.9}$$

Using these functions, the model (4.7) can be written in the form (2.4), i.e.

$$y_{\mathrm{M}}^{(r)}(t) = f_{D_1}(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(y_{\mathrm{M}}))(t)) + g_\Gamma(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(y_{\mathrm{M}}))(t))u(t)$$

with initial value $\chi_r(y_{\mathrm{M}})(t_0) = y_{\mathrm{M}}^0$. Let $y_{\mathrm{ref}} \in W^{k,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\Psi = (\psi_1,\ldots,\psi_r) \in \mathscr{G}$, let

$$\mathfrak{F} := \left\{ x \in \mathbb{R}^{rm} \,\middle|\, t \geq t_0, x - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi \right\}. \tag{4.10}$$

Due to the boundedness of the involved functions, the set $\mathfrak{F}$ is bounded as well and $\sup_{x\in\mathfrak{F}} \|x\|$ is finite. Thus, for $\bar{y} \geq \sup_{x\in\mathfrak{F}} \|x\|$ and given numbers $\bar{\eta}, \bar{r}, \bar{s}, \bar{\gamma}, \bar{p}, \bar{d} \geq 0$, we define the following set of matrices, where we do not indicate the dependence on the parameters. Let

$$\bar{\mathcal{K}} := \left(\mathbb{R}^{m\times m}\right)^r \times \mathbb{R}^{m\times\nu} \times \mathrm{GL}_m(\mathbb{R}) \times \mathbb{R}^m \times \mathrm{S}_\nu^{--} \times \mathbb{R}^{\nu\times m} \times \mathbb{R}^\nu \times \mathbb{R}^\nu,$$

and define

$$\mathcal{K} := \left\{ (R_1,\ldots,R_r,S,\Gamma,D_1,Q,P,D_2,\eta^0) \in \bar{\mathcal{K}} \,\middle|\, (4.12) \right\}, \tag{4.11}$$

where

$$\begin{aligned} &\|S\| \leq \bar{s}, \quad \|\Gamma\|, \|\Gamma^{-1}\| \leq \bar{\gamma}, \qquad \|D_1\|, \|D_2\| \leq \bar{d}, \qquad \|P\| \leq \bar{p}, \\ &\|\eta^0\| \leq \bar{\eta}, \qquad \lambda_{\max}(Q) \leq -\frac{\bar{p}\bar{y} + \bar{d}}{\bar{\eta}}, \qquad \|R_i\| \leq \bar{r} \text{ for all } i = 1,\ldots,r. \end{aligned} \tag{4.12}$$

Then, we may derive the following statement.

Proposition 4.2.1. *Let $y_{\mathrm{ref}} \in W^{k,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\Psi = (\psi_1,\ldots,\psi_r) \in \mathscr{G}$ with associated parameters $k_i \geq 0$ for $i = 1,\ldots,r-1$, and $\bar{y} \geq \sup_{x\in\mathfrak{F}} \|x\|$. Further, let $\bar{\eta}, \bar{r}, \bar{s}, \bar{\gamma}, \bar{p}, \bar{d} \geq 0$ be given and define recursively $\mu_i^0 := \|\psi_i\|_\infty$, $\mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j$ for $i = 1,\ldots,r-1$ and $j = 0,\ldots,r-i-1$. Choose*

$$u_{\max} \geq \bar{\gamma}\left(r\bar{r}\bar{y} + \bar{s}\bar{\eta} + \bar{d} + \left\|y_{\mathrm{ref}}^{(r)}\right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\|\dot{\psi}_r\right\|_\infty\right),$$

and

$$\bar{\rho} \geq r\bar{r}\bar{y} + \bar{s}\bar{\eta} + \bar{d} + \bar{\gamma} u_{\max}.$$

Then, the set $\mathcal{K}$ defined in (4.11) satisfies the implication

$$(R,S,\Gamma,D_1,Q,P,D_2,\eta^0) \in \mathcal{K} \quad \Longrightarrow \quad (f_{R,S,D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0}) \in \mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}},$$

where f_{R,S,D_1}, g_Γ, and $\mathbf{T}_{R,S,Q,P,D_2,\eta_0}$ are defined as in (4.8) and (4.9).

Proof. Let $(R, S, \Gamma, D_1, Q, P, D_2, \eta^0) \in \mathcal{K}$ be arbitrary but fixed.
Step 1: Repeating the arguments from Example 2.2.3, one can easily see that

$$(f_{R,S,D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0}) \in \mathcal{M}_{t_0}^{m,r}.$$

Step 2: We show properties (L.2) and (L.3) from Definition 4.1.1. Following the reasoning from Remark 4.1.5 (a), it is sufficient to show that

$$r\bar{r}\bar{y} + \bar{s}\bar{\eta} + \bar{d} \geq \left\| f_{D_1}(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta)|_{[0,s)}) \right\|_\infty,$$

and

$$\bar{\gamma} \geq \left\| g_\Gamma(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta)|_{[0,s)}) \right\|_\infty, \quad \bar{\gamma} \geq \left\| g_\Gamma(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta)|_{[0,s)})^{-1} \right\|_\infty$$

for all $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_s^\Psi$. The last two inequalities are trivially fulfilled due to the definition of g_Γ and $\bar{\gamma}$. We show that the first inequality is also satisfied. To this end, let $s \in [t_0, \infty]$ and $\zeta = (\zeta_1, \ldots, \zeta_r) \in \mathcal{Y}_s^\Psi$ with $\zeta_i \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ for $i = 1, \ldots, r$ be arbitrary but fixed. By construction of $\bar{y}$, we have $\|\zeta(t)\| \leq \bar{y}$ for all $t \in [t_0, s]$. Let $\eta(\cdot; t_0, \eta^0, \zeta_1)$ be the maximal solution of the initial value problem

$$\dot{\eta}(t) = Q\eta(t) + D_2 + P\zeta_1(t), \quad \eta(t_0) = \eta^0.$$

For $t \in [t_0, s]$, we calculate

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\left\|\eta(t; t_0, \eta^0, \zeta_1)\right\|^2 &= \eta(t; t_0, \eta^0, \zeta_1)\left(Q\eta(t; t_0, \eta^0, \zeta_1) + P\zeta_1(t) + D_2\right) \\
&\leq \left\|\eta(t; t_0, \eta^0, \zeta_1)\right\| \left(\lambda^+(Q)\left\|\eta(t; t_0, \eta^0, \zeta_1)\right\| + \bar{p}\bar{y} + \|D_2\|\right),
\end{aligned}$$

which is non-positive for $\left\|\eta(t; t_0, \eta^0, \zeta_1)\right\| \geq (\bar{p}\bar{y} + \|D_2\|)/\left|\lambda^+(Q)\right|$ as $\lambda^+(Q) < 0$. Therefore, [118, Thm. 4.3] yields

$$\left\|\eta(t; t_1, \eta^0, \zeta_1)\right\| \leq \max\left\{(\bar{p}\bar{y} + \bar{d})/\left|\lambda^+(Q)\right|, \left\|\eta^0\right\|\right\}$$

for all $t \in [t_0, s]$. By assumption (4.12), we have $\|\eta^0\| \leq \bar{\eta}$ and $|\lambda^+(Q)| \geq (\bar{p}\bar{y} + \bar{d})/\bar{\eta}$. Hence, $\left\|\eta(t; t_1, \eta^0, \zeta_1)\right\| \leq \bar{\eta}$ for all $t \in [t_0, s]$. As $\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta) = \sum_{j=1}^r R_j\zeta_j + S\eta(\cdot; t_1, \eta^0, \zeta_1)$, we estimate

$$\|\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta)(t)\| \leq r\bar{r}\bar{y} + \bar{s}\bar{\eta}.$$

for all $t \in [t_0, s]$. Thus, $\|f_{R,S,D_1}(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\zeta)(t))\| \leq r\bar{r}\bar{y} + \bar{s}\bar{\eta} + \bar{d}$ for all $t \in [t_0, s]$. As $s \in [t_0, \infty]$ and $\zeta \in \mathcal{Y}_s^\Psi$ are arbitrarily chosen, this shows

$$(f_{R,S,D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0}) \in \mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$$

and completes the proof. ☐

With the set of parameters $\mathcal{K}$, the functions $f_{D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0}$, and defined in (4.8) and (4.9), and Proposition 4.2.1, we may define a learning scheme $\mathcal{L}$ mapping from $\bigcup_{\hat{t} \geq t_0} \mathfrak{S}_{\hat{t}}$ to the subset

$$\left\{(f_{D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0}) \,\middle|\, (R, S, \Gamma, D_1, Q, P, D_2, \eta^0) \in \mathcal{K}\right\}$$

of $\mathcal{M}_{t_0,u_{\max},\bar{\rho}}^{m,r}$, defined by

$$\mathcal{L} : ((y, x_{\mathrm{M}}, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{[t_0,\hat{t}]}) \mapsto (f_{D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0})$$

for some $\hat{t} \geq t_0$, where $(f_{D_1}, g_\Gamma, \mathbf{T}_{R,S,Q,P,D_2,\eta_0})$ is determined by the solution of an optimisation problem involving measurements of the system data y and the applied control signals u_{FMPC} and u_{FC} over the time interval $[t_0, \hat{t}]$ of the form

$$\begin{aligned} \underset{(R,S,\Gamma,D_1,Q,P,D_2,\eta^0)\in\mathcal{K}}{\text{minimise}} \quad & J((y,z)|_{[t_0,\hat{t}]}) \\ \text{s.t. } \chi_r(z)(t_0) = z^0, & \\ z^{(r)}(t) \quad & = f_{D_1}(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(z))(t)) \\ & \quad + g_\Gamma(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(z))(t))(u_{\text{FMPC}} + u_{\text{FC}})(t), \end{aligned} \tag{4.13}$$

where $J(\cdot)$ is a suitable cost function. Here, $\hat{t}$ refers to time of the execution of the learning algorithm, i.e. the current time instant t_k during operation of the learning-based robust funnel MPC Algorithm 4.1.6. Note that solving the differential equation $z^{(r)}(t) = f_{D_1}(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(z))(t)) + g_\Gamma(\mathbf{T}_{R,S,Q,P,D_2,\eta_0}(\chi_r(z))(t))(u_{\text{FMPC}} + u_{\text{FC}})(t)$ is equivalent to solving the linear differential equation with the state (4.7).

Remark 4.2.2. In application, measurement of the system data y (and its derivatives) and the applied control signals u_{FMPC} and u_{FC} is only available at discrete time instants $t_0 + i\mathfrak{r}$ with $\mathfrak{r} > 0$ and $i \in \mathbb{N}$. In this case, it is reasonable to replace the control used in constraints (4.13) by the piecewise constant $\tilde{u}$ defined as

$$\tilde{u}(t) = (u_{\text{FMPC}} + u_{\text{FC}})(t_0 + (i-1)\mathfrak{r}),$$

for $t \in [t_0 + (i-1)\mathfrak{r}, t_0 + i\mathfrak{r})$ and all $i \leq (\hat{t} - t_0)/\mathfrak{r}$, and use a cost function $J(\cdot)$ which evaluates y and z only at time instants $t_0 + i\mathfrak{r}$. In the following Chapter 5, we will discuss the matter of using piecewise constant control signals for the overall control problem in more detail. However, we want to discuss, in the following, some possible choices for the cost function $J(\cdot)$ when only discrete measurements are available.

(a) $J((y,z)|_{[t_0,\hat{t}]}) \coloneqq \sum_{i=0}^{\lfloor(\hat{t}-t_0)/\mathfrak{r}\rfloor} a_i \|\chi_r(z)(t_0+i\mathfrak{r}) - \chi_r(y)(t_0+i\mathfrak{r})\|^2$ with weights $a_i \geq 0$. The idea is to find a model in the set $\mathcal{K}$ which minimises the weighted squared measured output errors. The weights a_i reflect the relative importance of the measurements $\chi_r(y)(t_0 + i\mathfrak{r})$. In certain cases, it might be beneficial to weight data points that are far in the past lower than current data points. By choosing $a_i > 0$ for all $i > 0$, all measured past data is taken into account. With increasing runtime of the algorithm, this results in a growing complexity of the optimisation problem, computation time, and required memory space for the measurements. Therefore, this is not suitable in practice. Thus, it is beneficial to use a moving horizon estimation approach and only take the last N measurements into account and set $a_i = 0$ for $i < \lfloor(\hat{t} - t_0)/\mathfrak{r}\rfloor - N$. In application, one has to find a good balance between considering many data points (large N), thus having a probably more accurate model, and low computation time and memory requirements (small N). This is comparable to the sliding window approach discussed in Remark 4.1.5(c)(i).

(b) If the computation of the solution of the optimisation problem has to be done very quickly, it is also possible to only consider the last measurement $y(t_0 + \lfloor(\hat{t} - t_0)/\mathfrak{r}\rfloor\mathfrak{r})$. Thus, one might choose the cost function

$$J((y,z)|_{[t_0,\hat{t}]}) \coloneqq \|\chi_r(z)(t_0 + \lfloor(\hat{t} - t_0)/\mathfrak{r}\rfloor\mathfrak{r}) - \chi_r(y)(t_0 + \lfloor(\hat{t} - t_0)/\mathfrak{r}\rfloor\mathfrak{r})\|^2.$$

The idea is to find a model, which best explains the last MPC period in terms of output error, i.e., a model on the prediction interval $[t_k, t_{k+1}]$ so that, with $\mathfrak{r} = \delta$, the error $\|\chi_r(z)(t_{k+1}) - \chi_r(y)(t_{k+1})\|$ at the end of the interval is minimal.

(c) In addition, it is worth considering to include the used model in the cost function as discussed in Remark 4.1.5(c)(ii). This can be done by adding regularisation terms for the model parameters in the cost function. For the parameter vector $\mathcal{K}_i = (R_i, S_i, \Gamma_i, D_{1,i}, Q_i, P_i, D_{2,i}, \eta_i^0) \in \mathcal{K}$, one could either penalise the weighted distance of $\mathcal{K}_i$ to a priori known parameters $\mathcal{K}^\star = (R^\star, S^\star, \Gamma^\star, D^\star, Q^\star, P^\star, D_2^\star, \eta^{0\star})$ and thus allow only small adaptions of the a priori known model or penalise the change of parameters $\mathcal{K}_i$ such that the model does only change slightly between two learning steps. This results in a cost function of the form

$$J((y,z)|_{[t_0,t]}) := \sum_{i=0}^{\lfloor (\hat{t}-t_0)/\mathfrak{r} \rfloor} \left(a_i \| \chi_r(z)(t_0 + i\mathfrak{r}) - \chi_r(y)(t_0 + i\mathfrak{r}) \|^2 + \sum_{j=1}^{8} b_i^j \| (\mathcal{K}_i^j - \tilde{\mathcal{K}}^j) \| \right),$$

where $\tilde{\mathcal{K}} = \mathcal{K}^\star$ or $\tilde{\mathcal{K}} = \mathcal{K}_{i-1}$ and with weights $a_i, b_i^j \geq 0$. Here the expressions $\mathcal{K}_i^j, \tilde{\mathcal{K}}_i^j$ with $j = 1, \ldots, 8$ refer to the j^{th} entry of the tuple $\mathcal{K}_i$, $\tilde{\mathcal{K}}_i$, respectively; for instance, $\mathcal{K}_i^2 = S_i$.

•

Remark 4.2.3. The bounds for $u_{\max}$ and $\bar{\rho}$ derived Proposition 4.2.1 are rather conservative and can clearly be improved. However, Proposition 4.2.1 exemplifies how to construct a subset of models belonging to $\mathcal{M}^{m,r}_{t_0,u_{\max},\rho}$ by prescribing bounds $f_{\rm M}^{\max}, g_{\rm M}^{\max}, g_{\rm M}^{-1\max} \geq 0$ on the dynamics. Proposition 4.2.1 relies on the following abstract idea. For a compact set $K \subset \mathbb{R}^\nu$, choose a set of operators $\mathcal{T} \subseteq \mathcal{T}_{t_0}^{rm,\nu}$ with

$$\forall\, \mathbf{T}_{\rm M} \in \mathcal{T}\ \forall\, \zeta \in \mathcal{Y}_\infty^\Psi : \quad \mathbf{T}_{\rm M}(\zeta)(\mathbb{R}_{\geq 0}) \subset K. \tag{4.14}$$

Moreover, consider only functions $f_{\rm M} \in (\mathbb{R}^\nu, \mathbb{R}^m)$ and $g_{\rm M} \in (\mathbb{R}^\nu, \mathbb{R}^{m\times m})$ satisfying $g_{\rm M}(z) \in \mathrm{GL}_m(\mathbb{R})$ for all $z \in \mathbb{R}^\nu$ and

$$\|f_{\rm M}(x)\| \leq f_{\rm M}^{\max}, \quad \|g_{\rm M}(x)\| \leq g_{\rm M}^{\max}, \quad \|g_{\rm M}(x)^{-1}\| \leq g_{\rm M}^{-1\max}$$

for all $x \in K$. Using this approach, one can construct a set of models of the form

$$y_{\rm M}^{(r)}(t) = p\big(\chi_r(y_{\rm M})(t), \eta(t)\big) + \Gamma\big(\chi_r(y_{\rm M})(t), \eta(t)\big)\, u(t), \tag{4.15a}$$
$$\dot{\eta}(t) = q\big(\chi_r(y_{\rm M})(t), \eta(t)\big), \tag{4.15b}$$

with $p : \mathbb{R}^{rm\times\nu} \to \mathbb{R}^m$, $q : \mathbb{R}^{rm\times\nu} \to \mathbb{R}^\nu$, $\Gamma : \mathbb{R}^{rm\times\nu} \to \mathbb{R}^{m\times m}$, belonging to $\mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$ where $u_{\max}$ and $\bar{\rho}$ are given as in Remark 4.1.5 (a). We already saw in Example 2.2.4 that these models belong to $\mathcal{M}^{m,r}_{t_0}$. The main difficulty lies in constructing a compact set $K \subset \mathbb{R}^\nu$ and ensuring (4.14). The matter ultimately comes down to finding a uniform bound $\bar{\eta} \geq 0$ of

$$\|\eta(t; t_0, \eta^0, \zeta)\| \leq \bar{\eta}$$

for all $t \geq t_0$ and all $\zeta \in \mathcal{Y}_\infty^\Psi$ where $\eta(t; t_0, \eta^0, \zeta)$ is the global solution of the equation (4.15b) where $\chi_r(y_{\rm M})(t)$ is replaced by ζ. One way to verify the satisfaction of such a uniform bound is to apply [118, Thm. 4.3], which states the following. Assume there exists $V \in \mathcal{C}^1(\mathbb{R}^\nu, \mathbb{R}_{\geq 0})$ with $V(\eta) \to \infty$ as $\|\eta\| \to \infty$ and, for $q \in \mathcal{C}(\mathbb{R}^{rm} \times \mathbb{R}^\nu, \mathbb{R}^\nu)$, $V'(\eta) \cdot q(z, \eta) \leq 0$ for all $z \in \mathfrak{F}$ as in (4.10) and $\eta \in \mathbb{R}^\nu$ with $\|\eta\| > \tilde{\eta}$ for a predefined value $\tilde{\eta} \geq 0$. Then, $\|\eta(t; t_0, \eta^0, \zeta)\| \leq \max\{\eta^0, \tilde{\eta}\}$ for all $t \geq t_0$, all $\eta^0 \in \mathbb{R}^\nu$, and all $\zeta \in \mathcal{Y}_\infty^\Psi$. Hence, fixing $V(\cdot)$ and $\tilde{\eta} > 0$ in advance can be used to restrict choices of $q(\cdot)$ satisfying $\|\eta(\cdot; t_0, \eta^0, \zeta)\|_\infty \leq \bar{\eta}$. We made use of this fact in the proof of Proposition 4.2.1. •

4.3 Simulation

In this section, we illustrate the application of the learning-based robust funnel MPC Algorithm 4.1.6 to the numerical examples from Section 2.5. The MATLAB source code for the performed simulations can be found on GITHUB under the link https://github.com/ddennstaedt/FMPC_Simulation.

Exothermic chemical reaction

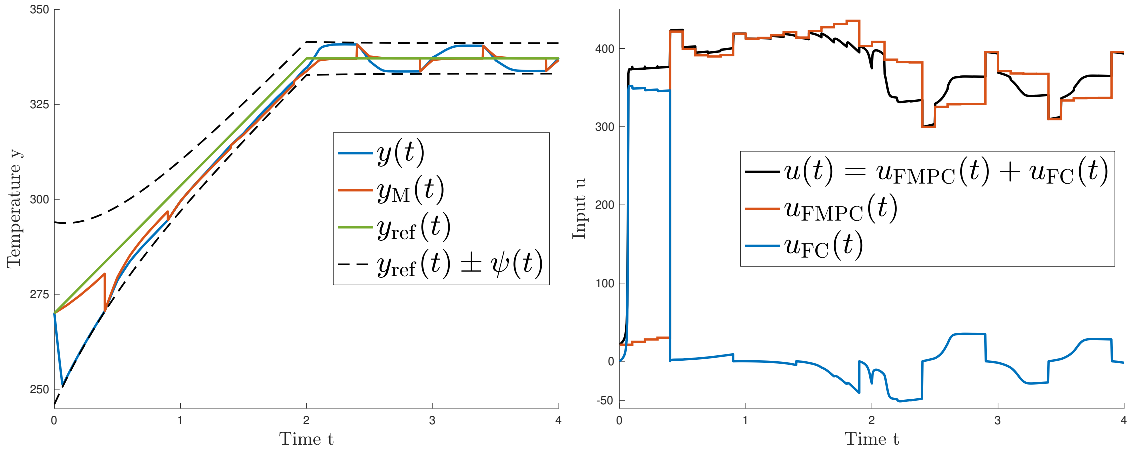
To demonstrate the functioning of the robust funnel MPC Algorithm 2.4.1, we revisit the example of a continuous chemical reactor from Section 2.5.1. The system is described by the following non-linear differential equation:

$$\begin{aligned}\dot{x}_1(t) &= c_1\, p(x_1(t), x_2(t), y(t)) + d(x_1^{\mathrm{in}} - x_1(t)),\\ \dot{x}_2(t) &= c_2\, p(x_1(t), x_2(t), y(t)) + d(x_2^{\mathrm{in}} - x_2(t)),\\ \dot{y}(t) &= b\, p(x_1(t), x_2(t), y(t)) - q\, y(t) + u(t),\end{aligned} \qquad \text{(2.39 revisited)}$$

where the function p is the Arrhenius law (2.40), the parameters are given in (2.43), and the initial data is $[x_1^0, x_2^0, y^0] = [0.02, 0.9, 270]$. Following the given heating profile $y_{\mathrm{ref}}(t)$ given in (2.41) within boundaries defined by the funnel function $\psi(t) := 20\mathrm{e}^{-2t} + 4$, the control objective is to steer the reactor's temperature y to a certain desired constant value $y_{\mathrm{ref,final}}$. To achieve the control objective with the learning-based robust funnel MPC Algorithm 4.1.6, we consider linear models of order $r = 1$ of the form (4.7) with $R, D_1 \in \mathbb{R}$, $S, D_2^\top, P^\top \in \mathbb{R}^{1\times 2}$, and $Q \in \mathbb{R}^{2\times 2}$. To learn the model from the measured data, we use linear regression subject to the constraints introduced in Definitions 4.1.1 and 4.1.4. Hence, feasibility of the data-based models is guaranteed by Proposition 4.2.1. We assume $\Gamma = 1$ and, as initial model, we choose $R = D_1 = 0 \in \mathbb{R}$, $S = D_2^\top = P^\top = 0 \in \mathbb{R}^{1\times 2}$, $Q = 0 \in \mathbb{R}^{2\times 2}$, and $\eta^0 = [x_1^0, x_1^0] = [0.02, 0.9]$, which represents an integrator chain with decoupled internal dynamics. To improve this (deliberately poorly chosen) model over time, we adapt the matrices over a compact set $\mathcal{K}$ as in (4.11) at every fifth time step t_k by minimising the model-plant mismatch based on the data of the last system output $y(t_{k-1})$, i.e. we solve the optimisation problem

$$\begin{aligned}&\underset{(R,S,1,D_1,Q,P,D_2,\eta(0))\in\mathcal{K}}{\text{minimize}} \quad \|y_{\mathrm{M}}(t_k) - y(t_k)\|^2\\ &\text{s.t.} \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\eta(t)\\ y_{\mathrm{M}}(t)\end{bmatrix} = \begin{bmatrix}S & R\\ Q & P\end{bmatrix}\begin{bmatrix}\eta(t)\\ y_{\mathrm{M}}(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} u(t) + \begin{bmatrix}D_1\\ D_2\end{bmatrix},\\ &\qquad \begin{bmatrix}\eta(t_{k-1})\\ y_{\mathrm{M}}(t_{k-1})\end{bmatrix} = \begin{bmatrix}\eta(0)\\ y(t_{k-1})\end{bmatrix},\end{aligned}$$

where $u(t) = u_{\mathrm{FMPC}}(t_{k-1}) + u_{\mathrm{FC}}(t_{k-1})$ which was applied to the model at the last time step t_{k-1} and $\eta(0) := [x_1^0, x_1^0]$ is the vector of initial concentrations of the substances x_1 and x_2. As before, we choose the strict funnel stage cost $\ell_\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined in (2.42) with $\lambda_u = 10^{-4}$, the prediction horizon $T = 1$, and time shift $\delta = 0.1$ for the funnel MPC component of the control algorithm and restrict the OCP (4.2) to step functions with a constant step length of $\delta = 0.1$. We choose for the set $\mathcal{K}$ as in (4.11) the parameters in (4.12) as $\bar{r} = 1.3$, $\bar{s} = 1.4$, $\bar{\eta} = 0.91$, $\bar{\gamma} = 1$, $\bar{p} = 1/400$, $\bar{d} = 2.5$, and $\bar{y} = 341.4$. We have $\|\dot{y}_{\mathrm{ref}}\|_\infty = 33.55$ given by the heating profile and $\|\dot{\psi}\|_\infty = 40$ by choice of the funnel function. Thus, we restrict the funnel MPC control signal to $\|u_{\mathrm{FMPC}}\|_\infty \leq u_{\max} := 600$ to satisfy the requirements of Proposition 4.2.1 for $\bar{\rho} = 1125$. The learning scheme is therefore $(u_{\max},\bar{\rho})$-feasible. For the control law of funnel control component, we choose the bijection $\gamma(s) = 1/(1-s)$ and the function $\mathcal{N}(s) = -10s$.



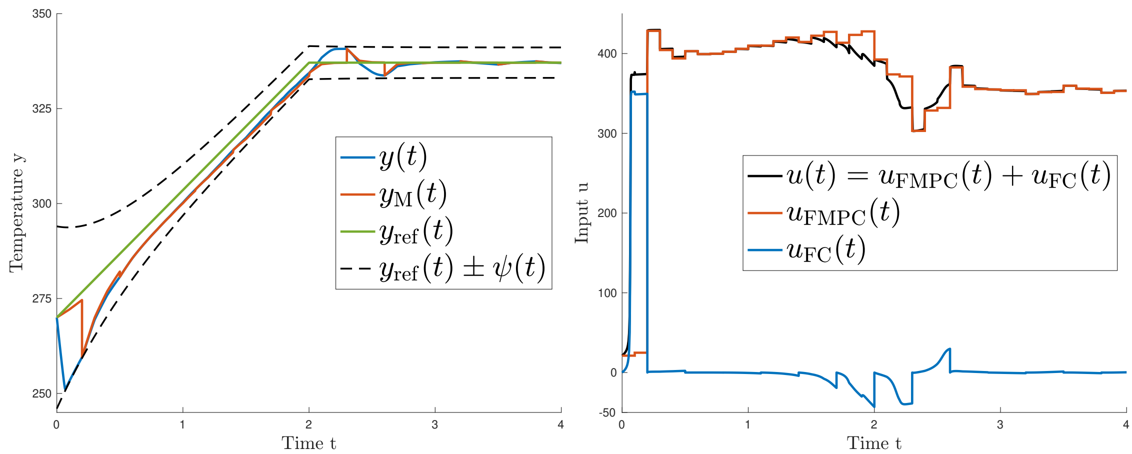
(a) Outputs and reference, with boundary ψ.

(b) Control inputs.

Figure 4.2: Simulation of system (2.39) under the control generated by the learning-based robust funnel MPC Algorithm 4.1.6 with model update every five iterations.

Figure 4.2 shows the control signals and the system and model output errors, respectively. It is evident that both $y_M - y_{ref}$ and $y - y_{ref}$ remain within the predefined funnel boundaries ψ. Before the first learning step for $t \in [0, 0.5)$, the tracking error $y - y_{ref}$ and the predicted error $y_M - y_{ref}$ diverge due to the poor quality of the initial model. However, since the tracking error is not close to the funnel boundary, the funnel controller remains inactive in the beginning and only reacts when the tracking error is close to the boundary. After the first learning step, the general direction of the predicted tracking error is consistent with the actual tracking error. The funnel controller still has to slightly compensate for the model inaccuracies in order to guarantee that the tracking error remains within the boundaries, but with a significantly smaller contribution to the control signal. After each learning step, the model output jumps y_M to the system output y due to the newly updated model. The control signal u_{FC} is zero after each learning step since the system and model output coincide, and it becomes larger afterwards to compensate for the model inaccuracy. After the heating phase, the model, only being updated every five iterations of the MPC algorithm, does not adequately describe the system dynamics. The funnel controller therefore has to compensate these inaccuracies during the whole operation of the algorithm, but with a significantly smaller control signal than before the first learning step. In a second simulation, we update the model every third time step t_k instead of every fifth but leave rest of the controller configuration unchanged. The results are depicted in Figure 4.3. As one can see, the combined controller is able to achieve the control objective. Before the initial learning step, the funnel controller has to compensate for the inaccuracies of the model with a large control signal comparable to the setting before. Already after the first update of the model, the principal portion of the control signal is generated by the MPC component. The funnel controller only has to intervene during the transition heating phase (before $t_{final} = 2$) to the constant temperature phase of the system (after $t_{final} = 2$). Thenceforth, the linear model is adequate to predict the system behaviour and the control signal computed by funnel MPC is sufficient to achieve the tracking objective. In contrast to the case before, the funnel controller remains mainly inactive after $t \approx 2.6$. This shows that the “quality” of the learning scheme and the update frequency of the model can have a significant impact on the controller behaviour and its performance. The more accurate the model is, the less control is required by the funnel controller to mitigate the model-system mismatch. However, updating the model more frequently can lead to increased computation costs.

We note that this example merely serves as an illustration that the learning-based robust



(a) Outputs and reference, with boundary ψ. (b) Control inputs.

Figure 4.3: Simulation of system (2.39) under the control generated by the learning-based robust funnel MPC Algorithm 4.1.6 with model update every three iterations.

funnel MPC Algorithm 4.1.6 can be combined with any ($u_{\max}$,$\bar{\rho}$)-feasible learning scheme $\mathcal{L}$. We do not claim that the learning algorithm used is superior to other methods.

Mass-on-car system

To illustrate that the learning-based funnel MPC Algorithm 4.1.6 can be successfully applied to systems with relative degree $r > 1$, we revisit the example of the mass-on-car system from Section 2.5.2. Assuming the mass $m_2 = 2$, on the ramp inclined by the angle $\vartheta = \frac{\pi}{4}$, is connected to the car with mass $m_1 = 4$ via a spring and damper system with spring constant $k = 2$ and damper constant $d = 1$, the system can be described by the differential equation

$$\begin{aligned} \ddot{y}(t) &= R_1 y(t) + R_2 \dot{y}(t) + S\eta(t) + \Gamma u(t) \\ \dot{\eta}(t) &= Q\eta(t) + Py(t). \end{aligned} \qquad \text{(2.49 revisited)}$$

with matrices given in (2.50). The objective is to track the reference signal $y_{\text{ref}}(t) = \cos(t)$ such that the tracking error $y(t) - y_{\text{ref}}(t)$ evolves within the prescribed performance funnel given by the function $\psi \in \mathcal{G}$ with $\psi(t) = 5\mathrm{e}^{-2t} + 0.1$. To achieve this control objective with the learning-based robust funnel MPC Algorithm 4.1.6, we use the strict funnel stage cost function $\ell_{\psi_2} : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as defined in (2.52). For the simulation, the MPC control signal is further restricted to $\|u_{\text{FMPC}}\|_\infty \leq u_{\max} = 30$ and we choose the design parameters $\lambda_u = 10^{-4}$, prediction horizon $T = 0.5$, and time shift $\delta = \frac{T}{20} = 0.025$. For the model-free component of the controller, we use a slightly modified form of the control law (3.21):

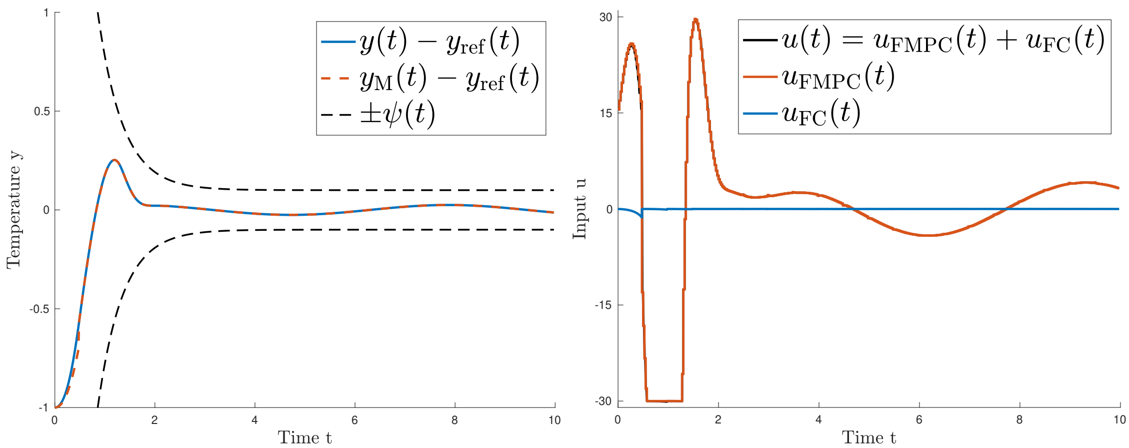
$$\begin{aligned} w(t) &= \varphi(t)\dot{e}_{\mathrm{S}}(t) + \gamma\left(\varphi(t)^2 e_{\mathrm{S}}(t)^2\right)\varphi(t)e_{\mathrm{S}}(t), & e_{\mathrm{S}}(t) &= y(t) - y_{\mathrm{M}}(t), \\ u_{\mathrm{FC}}(t) &= -2\gamma\left(w(t)^2\right)w(t), & \varphi(t) &= \frac{1}{\psi(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\|}, \end{aligned}$$

where y_{M} is the prediction for the system output computed by the MPC component.

Similar to [32], where this problem was studied in the context of model identification for the learning component during runtime, we assume knowledge about the structure of the system, but only limited information about its parameters. We assume to know $m_1, m_2 \in [0.5, 10]$ and $k, d \in [0.5, 5]$. As an initial model, we choose the parameters $m_1 = 6$, $m_2 = 2$, $k = 3$,and $d = 0.75$. To learn or update the model parameters, we take measurements of the system's input-output data $((u_{\text{FMPC}}+u_{\text{FC}})(ih), y(ih))$ for $h = 2.5\cdot 10^{-4}$ and $i \in \mathbb{N}_0$ and update the model every twentieth iteration of the MPC algorithm, i.e. at $t \in T\mathbb{N}$, by solving the optimisation problem

$$\begin{aligned}
\underset{\substack{m_1,m_2\in[0.5,10],\\ k,d\ \in[0.5,5]}}{\text{minimise}} \quad & \sum_{i=0}^{2000j} \|\tilde{y}_{\text{M}}(ih) - y(ih)\|^2 \\
\text{s.t.} \quad & z(0) = 0 \text{ and for all } i = 1,\ldots,2000j: \\
& z(ih) = z(h; z((i-1)h), (u_{\text{FMPC}} + u_{\text{FC}})((i-1)h)), \\
& \tilde{y}_{\text{M}}(ih) = [1,0,0,0]z(ih),
\end{aligned}$$

at every time $t = 2000jh$ for $j \in \mathbb{N}$, where $z = [y, \dot{y}, \eta_1, \eta_2]^\top$ denotes the state of the mass-on-car system (2.49) and $z(\cdot; z((i-1)h), (u_{\text{FMPC}}+u_{\text{FC}})((i-1)h))$ denotes its solution under the initial condition $z(0) = z((i-1)h)$ and with constant control $u(\cdot) \equiv (u_{\text{FMPC}}+u_{\text{FC}})((i-1)h)$. Since only the interval $[0, 10]$ is considered for the simulation, the entire history of input-output data is considered in the optimisation problem instead of a moving horizon approach. After every execution of this learning scheme, the model is properly initialised by solving the optimisation problem (3.22). Between two updates of the model, the MPC component's control signal u_{FMPC} is applied to the system in open-loop fashion, i.e. the model is initialised with its state from the previous iteration as initial value. All simulations are depicted in Figure 4.4. It is evident that the control scheme is feasible and achieves the



(a) Tracking error $e = y - y_{\text{ref}}$ within boundary ψ.

(b) Control inputs.

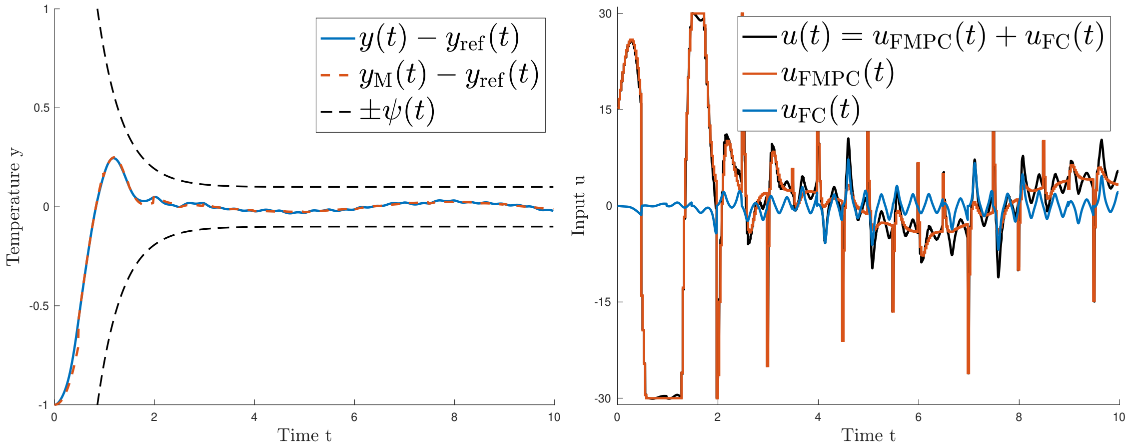
Figure 4.4: Simulation of system (2.49) under the control generated by the learning-based robust funnel MPC Algorithm 4.1.6

control objective. Both errors $y_{\text{M}} - y_{\text{ref}}$ and $y - y_{\text{ref}}$ evolve within the funnel boundaries given by ψ, see Figure 4.4a. While the model output y_{M} and the system y initially diverge, both trajectories evolve almost identically already following the first model update at $t = 0.5$. Note that already after the first learning step, the quality of the model is apparently good enough such that the funnel controller remains henceforth inactive and does not have to compensate for model errors. The control signal primarily consists of the control u_{FMPC} generated by the model-based controller component, see Figure 4.4b.

In a second simulation, we add an artificial additive disturbance d to the differential equation, i.e. the system takes the form

$$\begin{aligned}
\ddot{y}(t) &= R_1 y(t) + R_2 \dot{y}(t) + S\eta(t) + \Gamma u(t) + d(t) \\
\dot{\eta}(t) &= Q\eta(t) + Py(t).
\end{aligned} \tag{4.16}$$

The disturbance is unknown to the controller and, for the simulation, we choose the periodic disturbance $d(t) = \cos(20 \cdot t)$ and leave the controller as it is. The results are depicted in Figure 4.5. The controller evidently still achieves the control objective.



(a) Tracking error $e = y - y_{\mathrm{ref}}$ within boundary ψ.

(b) Control inputs.

Figure 4.5: Simulation of disturbed system (4.16) under the control generated by the learning-based robust funnel MPC Algorithm 4.1.6

The two tracking errors $y_{\mathrm{M}} - y_{\mathrm{ref}}$ and $y - y_{\mathrm{ref}}$ evolve within the funnel boundaries given by ψ, see Figure 4.5a. The system output y closely tracks the model y_{M}, which in turn tracks the given reference y_{ref} within the prescribed funnel boundaries, despite the added disturbance. Contrary to the prior case, the funnel controller remains active during the whole operation of the controller. It has to compensate the high-frequency additive disturbance. However, its contribution remains relatively modest. The predominant portion of the control signal consists of the MPC component's control action suggesting that the learning component still successfully identifies the underlying system dynamics.

5 Sampled-data robust funnel MPC

When applying control strategies to real-world systems, both model predictive and adaptive control algorithms are nowadays commonly implemented on digital devices. Unlike the idealised, continuously measured signals assumed in classical control theory, practical digital controllers only measure system outputs at discrete sampling intervals. Consequently, the controller observes the plant at discrete time points, computes a new input, and then holds that input constant until the next sample – introducing two fundamental challenges. First, dynamics or disturbances occurring between samples may go undetected, and high-frequency components can alias as lower-frequency behaviour if the sampling rate violates the Nyquist–Shannon criterion [175]. Second, because most digital hardware can usually only generate piecewise-constant inputs, the controller cannot apply an arbitrarily varying (dis-)continuous actuation signal, potentially degrading performance relative to a continuous design. As a result, the controller must be implemented as a sampled-data controller, specifically designed to operate under these discrete-time conditions. In its simplest form, a sampled-data controller samples the system output at regular intervals and uses this information to compute a control action. This control action is then held constant over the entire sampling period, only updating at the next sampling time. Although conceptually straightforward, this arrangement requires careful attention to preserve stability and performance. Potential challenges include:

- *Stability criteria shift*: Stability of (linear) discrete systems require poles inside the unit circle (vs. left half-plane in continuous-time). Discretisation can alter pole locations, destabilising an otherwise stable design, see Example 5.1.1.
- *Model discretisation errors*: Converting a continuous system to a discrete-time model – for example via Zero-order-Hold (ZoH) approximations – introduces approximation errors that can degrade accuracy [147, 206].
- *Performance loss*: Sampled-data controllers can reduce performance of the closed-loop systems [123] and exhibit slower responsiveness, increased overshoot [144], and steady-state errors [56].
- *Inter-sample constraint violation*: When safety or performance constraints must hold continuously, a controller updated only at discrete time instants can inadvertently violate them due to insufficiently fast sampling [44, 202].

These issues have motivated a rich body of research in digital control, see [17, 116]. To mitigate discretisation effects and balance trade-offs between sampling frequency, computational load, and performance in digital implementations, several mitigation techniques have been developed:

- *Sampled-data redesign*: Explicitly account for discrete-time dynamics during controller synthesis, rather than simply discretising a continuous design [79, 80].
- *Multi-rate sampling*: Use varying sampling frequencies for subsystems with different time scales [138, 145].
- *Event-triggered and self-triggered Control*: Update control actions only when certain conditions are met (e.g. when errors exceed thresholds) rather than at fixed intervals, reducing computational load [86].

By accounting for digital implementation from the outset, these approaches help bridge the gap between continuous-time theory and real-world sampled-data systems.

In this chapter, we show that it is possible to modify the robust funnel MPC Algorithm 3.2.9 from Chapter 3 such that it achieves the output tracking problem with prescribed performance as outlined in Section 1.1.1 with sampled-data control. In contrast to the robust funnel MPC from Chapter 3, the space of admissible controls is restricted to step functions, i.e. the control signal can only change finitely often between two sampling instants. Thus, the control signal applied to the system has the form

$$u(t) \equiv u_i \qquad \forall\, t \in [t_i, t_{i+1}),\ i \in \mathbb{N}_0,$$

where the data to compute the control signal u_i is collected at sample times $(t_i)_{i\in\mathbb{N}_0}$. To introduce the control scheme properly, we formally define step functions in the following definition.

Definition 5.0.1 (Step function)**.** *Let $I \subset \mathbb{R}$ be an interval of the form $I = [a, b]$ with $b > a$ or $I = [a, \infty)$. We call a strictly increasing sequence $\mathcal{P} = (t_i)_{i\in\mathbb{N}_0}$ with $\lim_{i\to\infty} t_i = \infty$ and $t_0 = a$ a partition of I. The norm of $\mathcal{P}$ is defined as $|\mathcal{P}| := \sup\{t_{i+1} - t_i \,|\, i \in \mathbb{N}_0\}$. A function $f : I \to \mathbb{R}^m$ is called step function with partition $\mathcal{P}$ if f is constant on every interval $[t_i, t_{i+1}) \cap I$ for all $i \in \mathbb{N}_0$. We denote the space of all step functions on I with partition $\mathcal{P}$ by $\mathcal{T}_{\mathcal{P}}(I, \mathbb{R}^m)$.*

Note that in the case of finite intervals $I = [a, b]$ with $b > a$, Definition 5.0.1 can also be formulated using finite sequences $\mathcal{P} = (t_i)_{i=0}^N$ with $N \in \mathbb{N}$ and $t_N = b$. However, using infinite sequences every partition $\mathcal{P}$ of $[a, \infty)$ is also a partition of $[a, b]$ for all $b > a$. Using this observation simplifies formulating our results. Further, note that Definition 5.0.1 allows for the usage of a non-uniform step length, i.e. for $\mathfrak{r}_i := |t_{i-1} - t_i|$ we allow $\mathfrak{r}_i \neq \mathfrak{r}_j$ for $i \neq j$, where $i, j \in \mathbb{N}$. However, in practice, a uniform step length will be used often.

The robust funnel MPC Algorithm 3.2.9 from Chapter 3 consists of two components, the model-free funnel controller (3.6) and the model based funnel MPC Algorithm 2.4.1, see also Figure 3.1. In the following Sections 5.1 and 5.2, we restrict ourselves to showing that both components individually can be designed to work with the restricted space of step functions as control signals. However, we refrain from integrating both controllers in one single control scheme like done for the robust funnel MPC Algorithm 3.2.9 and proven in Theorem 3.2.11. The arguments and considerations for such an integration are the same as in Chapter 3 and do not provide any new insights into the underlying issue. The restriction to step functions merely adds another level of technicalities.

For the controller design in this chapter, we restrict both the class of potential systems $\mathcal{N}^{m,r}_{t_0}$ and associated models $\mathcal{M}^{m,r}_{t_0}$. For both the system and the model, we consider non-linear multi-input multi-output differential equations of order $r \in \mathbb{N}$ of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(\mathbf{T}(\chi_r(y))(t)\big) + g\big(\mathbf{T}(\chi_r(y))(t)\big)u(t),\\ y|_{[0,t_0]} &= y^0 \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m), \end{aligned} \tag{5.1}$$

where $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q, \mathbb{R}^m)$, $g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^q, \mathbb{R}^{m\times m})$, and $\mathbf{T} \in \mathcal{T}^{rm,q}_{t_0}$. In addition, we assume that the matrix valued function g is strictly positive definite, that is

$$\forall x \in \mathbb{R}^q\ \forall z \in \mathbb{R}^m \setminus \{0\} : \quad \langle z, g(x)z\rangle > 0.$$

Note that by replacing u in (5.1) by $-u$ all results presented in this chapter remain valid if g is strictly negative definite. Note further that, while some authors only use the term *strictly positive definite* for symmetric matrices, we do not assume $g(x)$ to be symmetric.

We use the notation $(f, g, \mathbf{T}) \in \mathfrak{N}^{m,r}_{t_0}$ to refer to a system, respectively a model, of the form (5.1) with the aforementioned properties. When it is necessary to distinguish between the system and the model, we will use an index M, i.e. f_{M}, g_{M}, $\mathbf{T}_{\mathrm{M}}$, to refer to the model's functions as done in Chapter 2. However, we want to emphasise that the system and the model are not assumed to be identical even though we use the same class $\mathcal{N}_{t_0}^{m,r}$ of functions for the system and the model. In order to avoid having to differentiate between the model and system class in this chapter, we assume for the sake of simplicity that $\mathbf{T} \in \mathcal{T}_{t_0}^{rm,q}$ However, all presented results hold true if the operator $\mathbf{T}$ of the system merely fulfils the *causality* (T.1), *local Lipschitz* (T.2), and the *bounded-input bounded-output (BIBO)* (T.3) property as defined in Definition 2.2.1. It is not required to fulfil the *limited memory* property (T.4).

Remark 5.0.2. As previously pointed out in Remark 3.1.3 (a), an unknown disturbance $d \in L^\infty([t_0,\infty),\mathbb{R}^p)$ in the system (5.1) can be modelled in terms of the operator $\mathbf{T} \in \mathcal{T}_{t_0}^{rm,q}$. Systems of the form

$$y^{(r)}(t) = f\big(d(t), \mathbf{T}(\chi_r(y))(t)\big) + g\big(d(t), \mathbf{T}(\chi_r(y))(t)\big)u(t)$$

are therefore implicitly contained in the system class $\mathcal{N}_{t_0}^{m,r}$. •

5.1 Funnel control with zero-order-hold

Funnel control is an adaptive high-gain control methodology guaranteeing satisfaction of a priori fixed, possibly time-varying output constraints while only imposing structural assumptions but not requiring knowledge about the system dynamics, see e.g. [30] and the survey paper [31]. However, the availability of the system's output as a continuous-time signal and the ability to continuously adapt the input signal is pivotal for its functioning, cf. Propositions 1.1.2 and 3.2.3.

Although funnel control has been successfully implemented in a sampled-data system with Zero-order-Hold (ZoH) for a sufficiently small sampling time in [32], we are not aware of any results prior to [8] rigorously showing that the output signal stays within the prescribed boundaries for ZoH funnel control. In this section, we present the in [8] proposed sampled-data feedback controller with ZoH. We show that the controller ensures output tracking of a given reference signal within prescribed, possibly time-varying performance bounds – at every time instant meaning that also the intersampling behaviour is fully taken into account. To balance the need for a sufficiently large feedback gain for output tracking and avoidance of overshooting (which could violate error bounds within one sampling period), we use results from the previous chapters to infer uniform bounds on sampling rates and control inputs. This allows us to ensure that the imposed output constraints are satisfied along the closed loop leveraging coarse bounds on the system dynamics. To the best of our knowledge, in funnel control uniform bounds on the input signal are only known if the region of feasible initial values is further restricted *and* the dynamics are known [30]. While there have been several attempts to deal with the closely related issue of input saturation [27, 90, 97] and bang-bang controller designs [124, 125] exhibiting similarities to our approach, an analysis of combining a ZoH with funnel control has not been conducted prior to the work [8].

Before presenting the results from [8], we want to motivate why applying a controller in a sampled-data fashion to a system poses additional challenges. When applied to system (5.1), a high-gain feedback controller, e.g. the funnel controller, achieves the control objective as laid out in Section 1.1.1 if the gain is large enough. When applied in a sample-and-hold form, however, such approaches can fail if the gain or the sampling time is too large, respectively. To see this consider the following example.

Example 5.1.1. Consider the scalar linear system

$$\dot{x}(t) = ax(t) + u(t),$$

with $a \in \mathbb{R}$. As is well known, every linear feedback $u(t) = -kx(t)$ with $k > |a|$ stabilises the system. If u is applied in a sample-and-hold form with sampling rate $\mathfrak{r} > 0$, then the solution at the time instants $i\mathfrak{r}$ with $i \in \mathbb{N}$ has the form

$$x((i+1)\mathfrak{r}) = \mathrm{e}^{a\mathfrak{r}} x(i\mathfrak{r}) - \tfrac{1}{a}\big(\mathrm{e}^{a\mathfrak{r}} - 1\big) k x(i\mathfrak{r}) = \big(e^{a\mathfrak{r}} - \tfrac{1}{a}\big(\mathrm{e}^{a\mathfrak{r}} - 1\big)k\big) x(i\mathfrak{r}).$$

Thus, for $k > \left| a \frac{\mathrm{e}^{a\mathfrak{r}}+1}{\mathrm{e}^{a\mathfrak{r}}-1} \right|$, we have $|x((i+1)\mathfrak{r})| > |x(i\mathfrak{r})|$. Therefore, the system is unstable, even if the initial uncontrolled system is stable, i.e. $a < 0$. ◇

To design a zero-order-hold control strategy able to achieve the control objective using data only collected at discrete time instants given a partition $\mathcal{P} = (t_i)_{i \in \mathbb{N}_0}$ of the interval $[t_0, \infty)$ we utilise the auxiliary error variables e_k for $k = 1, \ldots, r$ as in (3.3). As in Chapter 3, they are recursively given for $\varphi > 0$, a bijection $\gamma \in \mathcal{C}^1([0,1),[1,\infty))$, and $z = (z_1, \ldots, z_r) \in \mathbb{R}^{rm}$ with $z_k \in \mathbb{R}^m$ by

$$e_1(\varphi, z) := \varphi z_1, \quad e_{k+1}(\varphi, z) := \varphi z_{k+1} + \gamma\left(\|e_k(\varphi, z)\|^2\right) e_k(\varphi, z), \qquad (3.3 \text{ revisited})$$

for $k = 1, \ldots, r-1$. For details we refer to Chapter 3. Given a funnel function $\varphi \in \mathcal{G}$ and a reference trajectory $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, we use in the following the short notation $e_r(t) := e_r(\varphi(t), \chi_r(y - y_{\mathrm{ref}})(t))$, where y is the output of the system (5.1). We propose the following controller structure for $i \in \mathbb{N}_0$

$$\forall t \in [t_i, t_i + \mathfrak{r}) : \; u_{\mathrm{ZoH}}(t) = \begin{cases} 0, & \|e_r(t_i)\| < \iota, \\ -\nu \frac{e_r(t_i)}{\|e_r(t_i)\|^2}, & \|e_r(t_i)\| \geq \iota, \end{cases} \tag{5.2}$$

where $\iota \in (0,1)$ is an *activation threshold*, and $\nu > 0$ is the *input gain*. In Theorem 5.1.3, we derive lower bounds on the input gain ν and upper bounds on the maximal sampling time, i.e. $\mathfrak{r} = |\mathcal{P}|$, which ensure that the control objective is achieved when applying the controller (5.2) to the system (5.1). We do this by showing $e_r(t) \in \mathcal{B}_1$ for all $t \geq t_0$. Thus, the control signal u_{ZoH} is then uniformly bounded since

$$\forall t \geq t_0 : \quad \|u_{\mathrm{ZoH}}(t)\| \leq \frac{\nu}{\iota}.$$

The controller design can be considered to be similar to funnel control, see [30, 33, 95], in terms of its ability to achieve output reference tracking within predefined error boundaries, as well as concerning the used intermediate error variables (3.3). On the other hand, contrary to the standard funnel controller, the feedback law (5.2) is a normalised linear sample-and-hold output feedback with uniformly bounded sampling rate. A further essential difference to continuous funnel control is that in the present approach the control objective is achieved by using estimates about the system dynamics, while in continuous-time funnel control no such information is used to the price that the maximal control effort cannot be estimated a priori.

In order to formulate and prove the main result of [8] about feasibility of the proposed ZoH controller (5.2), we recall some results from the previous Chapters 2 and 3. To ensure that the controller achieves the control objective, namely that the system output y tracks a given reference signal y_{ref} with prescribed performance in terms of a function $\varphi \in \mathcal{G}$, we show that the norm of the axillary error variables e_k for $k = 1, \ldots, r$ as in (3.3) evaluated along $\chi_r(y - y_{\mathrm{ref}})$ is always below one, i.e.

$$\forall\, k = 1, \ldots, r \;\forall\, t \geq t_0 : \quad \|e_k(\varphi(t), \chi_r(y - y_{\mathrm{ref}})(t))\| < 1. \tag{5.3}$$

Assuming that (5.3) is fulfilled at the initial time $t = t_0$, Lemma 3.2.1 states that all error signals $e_k(\varphi(t), \chi_r(y - y_{\mathrm{ref}})(t))$ for $k = 1, \ldots, r$ satisfy (5.3) for all $t \geq t_0$ given that the norm of the last auxiliary error $e_r(\varphi(t), \chi_r(y - y_{\rm ref})(t))$ remains below one for all $t \geq t_0$. In the proof of Proposition 3.2.3, we used this result to show that the funnel controller (3.6) achieves the control objective. In a similar fashion, we derive bounds on the input gain ν and upper bounds on the sampling rate, i.e. $\mathfrak{r} = |\mathcal{P}|$, that ensure the feasibility of the proposed ZoH controller (5.2) by guaranteeing that the norm of the last auxiliary error $e_r(\varphi(t), \chi_r(y - y_{\rm ref})(t))$ remains bounded by one. In addition to the mentioned statement, Lemma 3.2.1 states that $\|e_k(\varphi, \chi_r(\zeta))\|$ for $k = 1, \ldots, r-1$ remain bounded away from one by some $\varepsilon \in (0,1)$ for all signals $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ and all $\hat{t} \geq t_0$, where $\mathfrak{Y}^{\varphi}_{\hat{t}}$ is the set of all functions coinciding with y_0 on the interval $[0, t_0]$ and fulfilling (5.3) where $y - y_{\rm ref}$ is replaced by ζ, see (3.4) This yields the existence of a compact set in which all functions $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ evolve until $\hat{t}$. The existence of such a compact set allows us to adapt Lemma 2.3.20 to the current setting stating that the system (5.1) with $(f, g, \mathbf{T}) \in \mathfrak{N}^{m,r}_{t_0}$ is uniformly bounded for every $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ and all $\hat{t} \geq t_0$.

Lemma 5.1.2. *Consider the system (5.1) with $(f, g, \mathbf{T}) \in \mathfrak{N}^{m,r}_{t_0}$ and reference trajectory $y_{\rm ref} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$. Let $\varphi \in \mathcal{G}$. Then, there exist constants $f^{\max}$, $g^{\max} \geq 0$ such that for all $\hat{t} \in (t_0, \infty]$ and $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$:*

$$f^{\max} \geq \left\| f(\mathbf{T}_{\rm M}(\chi_r(\zeta))|_{[0,\hat{t})}) \right\|_\infty, \qquad g^{\max} \geq \left\| g(\mathbf{T}_{\rm M}(\chi_r(\zeta))|_{[0,\hat{t})}) \right\|_\infty.$$

Moreover, there exists $g^{\min} > 0$ such that for all $z \in \mathbb{R}^m \setminus \{0\}$ and all $\hat{t} \in (t_0, \infty]$:

$$g^{\min} \leq \frac{\left\langle z, g(\mathbf{T}(\chi_r(\zeta))|_{[0,\hat{t})}(t)) z \right\rangle}{\|z\|^2}.$$

Proof. To prove the assertion, we adapt the proof of Lemma 2.3.20 to the current setting. According to Lemma 3.2.1, there exist constants $\varepsilon_k > 0$ such that all functions $\zeta \in \mathfrak{Y}^{\varphi}_{\infty}$ fulfil $\|e_k(\varphi(t), \chi_r(\zeta)(t))\| \leq \varepsilon_k < 1$ for all $t \in [t_0, \infty)$ and all $k = 1, \ldots, r-1$. Hence, by boundedness of φ and $y^{(i)}_{\rm ref}$ for all $i = 1, \ldots, r$, there exists a compact set $K \subset \mathbb{R}^{rm}$ with

$$\forall\, \zeta \in \mathfrak{Y}^{\varphi}_{\infty} \forall\, t \geq 0: \quad \chi_r(\zeta)(t) \in K.$$

Invoking the BIBO property of the operator $\mathbf{T}$, there exists a compact set $K_q \subset \mathbb{R}^q$ with $\mathbf{T}(z)(\mathbb{R}_{\geq 0}) \subset K_q$ for all $z \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $z(\mathbb{R}_{\geq 0}) \subset K$. For arbitrary $\hat{t} \in (t_0, \infty)$ and $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$, we have $\chi_r(\zeta)(t) \in K$ for all $t \in [0, \hat{t}]$. For every element $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ the restriction $\chi_r(\zeta)|_{[0,\hat{t})}$ can be extended to a function $\tilde{\zeta} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $\tilde{\zeta}(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$. We have $\mathbf{T}(\tilde{\zeta})(t) \in K_q$ for all $t \in \mathbb{R}_{\geq 0}$ because of the BIBO property of the operator $\mathbf{T}$. This implies $\mathbf{T}(\chi_r(\zeta))|_{[0,\hat{t})}(t) \in K_q$ for all $t \in [0, \hat{t})$ and $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ since $\mathbf{T}$ is causal. Since $f(\cdot)$ and $g(\cdot)$ are continuous, the constants $f^{\max}_{\rm M} = \max_{z \in K_q} \|f_{\rm M}(z)\|$ and $g^{\max}_{\rm M} = \max_{z \in K_q} \|g_{\rm M}(z)\|$ are well-defined. For all $\hat{t} \in (0, \infty]$ and $\zeta \in \mathfrak{Y}^{\varphi}_{\hat{t}}$ we have

$$\forall\, t \in [0, \hat{t}): \ \mathbf{T}(\chi_r(\zeta))(t) \in K_q.$$

Furthermore, since $g(x)$ is positive definite for every $x \in K_q$, there exists $g^{\min} > 0$ such that $g^{\min} \leq \frac{\langle z, g(\mathbf{T}(\chi_r(\zeta))|_{[0,\hat{t})}(t)) z \rangle}{\|z\|^2}$ for all $z \in \mathbb{R}^m \setminus \{0\}$. This completes the proof. $\square$

A consequence of Lemma 5.1.2 is that the dynamics of system (5.1) are bounded if a control is applied that ensures that all error signals $e_k(\varphi(t), \chi_r(y - y_{\rm ref})(t))$ for $k = 1, \ldots, r$ satisfy (5.3). In the following Theorem 5.1.3, we use these bounds to derive an input gain $\nu > 0$ large enough to counteract the system dynamics. When applying the ZoH controller (5.2) to the system the large enough gain guarantee that the norm of the auxiliary error signal $e_r(\varphi(t), \chi_r(y - y_{\text{ref}})(t))$ decreases at the sampling instants t_i, if the error signal is greater than or equal to the activation threshold $\iota \in (0, 1)$, see Step 2.b in the proof of Theorem 5.1.3. Based on bound on the system dynamics and the maximal control value applied to the system, we compute a uniform bound on the sampling time $\mathfrak{r} = |\mathcal{P}|$ required to avoid overshooting of the error signal between two sampling instants.

Theorem 5.1.3. *Given a reference $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and a funnel function $\varphi \in \mathcal{G}$, consider the system (5.1) with $(f, g, \mathbf{T}) \in \mathfrak{N}^{m,r}_{t_0}$. Assume that the initial trajectory $y^0 \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$ satisfies $\chi_r(y^0 - y_{\text{ref}})(t_0) \in \mathcal{E}^1_r(\varphi(t_0))$, i.e. the error variables in (3.3) satisfy $\|e_k(\varphi(t_0), \chi_r(y^0 - y_{\text{ref}})(t_0))\| < 1$ for all $k = 1, \ldots, r$. With the constants given in (3.5) and in Lemma 5.1.2, set*

$$\kappa_0 := \left\| \frac{\dot{\varphi}}{\varphi} \right\|_\infty (1 + \gamma(\varepsilon^2_{r-1})\varepsilon_{r-1}) + \|\varphi\|_\infty (f^{\max} + \|y^{(r)}_{\text{ref}}\|_\infty) + \bar{\eta}_{r-1},$$

and choose the input gain

$$\nu > \frac{2\kappa_0}{g^{\min} \inf_{s \geq 0} \varphi(s)}.$$

Further, for an activation threshold $\iota \in (0, 1)$, define the constant $\kappa_1 := \kappa_0 + \|\varphi\|_\infty \frac{\nu}{\iota} g^{\max}$ and let $\mathcal{P}$ be a partition of the interval $[t_0, \infty)$ for which the maximal sampling time $\mathfrak{r} := |\mathcal{P}|$ fulfils

$$0 < \mathfrak{r} \leq \min \left\{ \frac{\kappa_0}{\kappa_1^2}, \frac{1 - \iota}{\kappa_0} \right\}. \tag{5.4}$$

Then, the ZoH controller (5.2) applied to a system (5.1) yields

$$\|e_k(\varphi(t), \chi_r(y - y_{\text{ref}})(t))\| < 1$$

for all $k = 1, \ldots, r - 1$ and $\|e_r(t)\| \leq 1$ for all $t \geq t_0$. This is initial and recursive feasibility of the ZoH control law (5.2). In particular, the tracking error $e := y - y_{\text{ref}}$ satisfies $\|e(t)\| < 1/\varphi(t)$ for all $t \geq t_0$.

Proof. The proof consists of two main steps. In the first step, we establish the existence of a solution of the initial value problem (5.1), (5.2). In the second step, we show feasibility of the proposed control law, i.e. all error variables $e_k(\varphi(t), \chi_r(y - y_{\text{ref}})(t))$, $k = 1, \ldots, r$ are bounded by one. Thus, the tracking error evolves within the funnel boundaries given by φ. In the following, we use the shorthand notation $e_k(t) := e_k(\varphi(t), \chi_r(y - y_{\text{ref}})(t))$.
Step 1: The application of the control signal (5.2) to system (5.1) leads to an initial value problem. If this problem is considered on the interval $[t_0, \mathfrak{r}]$, then there exists a unique maximal solution on $[t_0, \omega)$ with $\omega \in (t_0, \mathfrak{r}]$. If all error variables $e_k(t)$ evolve within the set $\mathcal{B}_1$ for all $t \in [t_0, \omega)$, then $\|\chi_r(y)(\cdot)\|$ is bounded on the interval $[t_0, \omega)$ and, as a consequence of the BIBO condition of the operator, $\mathbf{T}(\cdot)$ is bounded as well. Then $\omega = \mathfrak{r}$, cf. [192, § 10, Thm. XX] and there is nothing else to show. Seeking a contradiction, assume the existence of $t \in [t_0, \omega)$ such that $\|e_k(t)\| \geq 1$ for at least one $k = 1, \ldots, r$. Invoking Lemma 3.2.1, it remains only to show that the last error variable e_r satisfies $\|e_r(t)\| \leq 1$ for all $t \in [t_0, \omega)$. Before doing so, record the following observation. For $\eta_{r-1}(t) := \gamma(\|e_{r-1}(t)\|^2)e_{r-1}(t)$ and $z(\cdot) := \mathbf{T}(\chi(y))(\cdot)$, we calculate

$$\begin{aligned} \dot{e}_r(t) - \varphi(t)g(z(t))u(t) &= \dot{\varphi}(t)e^{(r-1)}(t) + \varphi(t)e^{(r)}(t) + \dot{\eta}_{r-1}(t) - \varphi(t)g(z(t))u \\ &= \frac{\dot{\varphi}(t)}{\varphi(t)}(e_r(t) - \eta_{r-1}(t)) + \dot{\eta}_{r-1}(t) + \varphi(t)(f(z(t)) - y^{(r)}_{\text{ref}}(t)) =: J(t). \end{aligned} \tag{5.5}$$

Step 2: We show $\|e_r(t)\| \leq 1$ for all $t \in [t_0, \omega)$. We separately investigate the two cases $\|e_r(t_0)\| < \iota$ and $\|e_r(t_0)\| \geq \iota$.

Step 2.a: Consider $\|e_r(t_0)\| < \iota$. In this case, the constant control signal $u(t) = u_{\text{ZoH}}(t) = 0$ is applied to the system. Seeking a contradiction, we suppose that there exists a time instant $t^* := \inf\{t \in (t_0, \omega) \,|\, \|e_r(t)\| > 1\}$. For the function $J(\cdot)$ introduced in (5.5), we observe $\|J|_{[t_0,t^*)}\|_\infty \leq \kappa_0$ according to Lemmata 3.2.1 and 5.1.2. Then, we calculate

$$\begin{aligned} 1 = \|e_r(t^*)\| &\leq \|e_r(t_0)\| + \int_{t_0}^{t^*} \|\dot{e}_r(s)\| \mathrm{d}s \\ &= \|e_r(t_0)\| + \int_{t_0}^{t^*} \|J(s)\| \mathrm{d}s \\ &\leq \|e_r(t_0)\| + \int_{t_0}^{t^*} \kappa_0 \mathrm{d}s < \iota + \kappa_0 \omega < 1, \end{aligned}$$

where $t^* < \omega \leq \mathfrak{r} < (1-\iota)/\kappa_0$ was used. This contradicts the definition of t^*.
Step 2.b: Consider $\|e_r(t_0)\| \geq \iota$. In this case, $u(t) = u_{\text{ZoH}}(t) = -\nu e_r(t_0)/\|e_r(t_0)\|^2$ is applied to the system. We show again $\|e_r(t)\| \leq 1$ for all $t \in [t_0, \omega)$. To this end, seeking a contradiction, we suppose the existence of $t^* = \inf\{(t_0, \omega) \,|\, \|e_r(t)\| > 1\}$. For the function $J(\cdot)$, we observe $\|J|_{[t_0,t^*)}\|_\infty \leq \kappa_0$ according to Lemmata 3.2.1 and 5.1.2. Moreover, $\|\dot{e}_r|_{[t_0,t^*]}\| \leq \kappa_1$ due to equation (5.5) and the bound $\|u_{\text{ZoH}}\|_\infty \leq \frac{\nu}{\iota}$. Invoking the initial conditions and continuity of the involved functions, and (5.5), we calculate for $t \in [t_0, t^*]$:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2} \|e_r(t)\|^2 = \langle e_r(t), \dot{e}_r(t) \rangle &= \left\langle e_r(t_0) + \int_{t_0}^{t} \dot{e}_r(s) \mathrm{d}s, \dot{e}_r(t) \right\rangle \\ &\leq \|e_r(t_0)\| \|J(t)\| + \omega \|\dot{e}_r|_{[t_0,t^*]}\|_\infty^2 + \varphi(t) \langle e_r(t_0), g(z(t)) u_{\text{ZoH}}(t) \rangle \\ &= \|e_r(t_0)\| \|J(t)\| + \omega \|\dot{e}_r|_{[t_0,t^*]}\|_\infty^2 - \varphi(t) \nu \tfrac{\langle e_r(t_0), g(z(t)) e_r(t_0) \rangle}{\|e_r(t_0)\|^2} \\ &\leq \|e_r(t_0)\| \kappa_0 + \omega \|\dot{e}_r|_{[t_0,t^*]}\|_\infty^2 - \inf_{s \geq t_0} \varphi(s) g_{\min} \nu \\ &\leq \kappa_0 + \omega \kappa_1^2 - \inf_{s \geq t_0} \varphi(s) g_{\min} \nu \\ &\leq 2\kappa_0 - \inf_{s \geq t_0} \varphi(s) g_{\min} \nu < 0. \end{aligned}$$

Here, the second line holds true due to $t^* < \omega \leq \mathfrak{r}$, the penultimate line via the definition of $\mathfrak{r}$, and the last line by definition of ν. In particular, this yields $\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_r(t)|_{t=t_0}\|^2 < 0$, by which $t^* > t_0$. Therefore, we find the contradiction $1 = \|e_r(t^*)\|^2 < \|e_r(t_0)\|^2 \leq 1$. Repeated application of the arguments in *Steps 1* and *2* on the interval $[t_i, t_i + \mathfrak{r}]$, $i \in \mathbb{N}$, yields recursive feasibility. □

The maximal sampling time $\mathfrak{r}$ in (5.4) strongly depends on the evolution of the funnel function and on the reference y_{ref}. This gives the possibility of dynamically adapting the sampling time, e.g. in the case of setpoint transition, where the reference is constant y_{ref}^0 in the first period and constant $y_{\text{ref}}^1 \neq y_{\text{ref}}^0$ in the last period. At the setpoints the sampling time can be larger than during the transition.

The parameter $\iota \in (0,1)$ in (5.2) is an "activation threshold" to set the control input to zero for small tracking errors, akin to the idea of using funnel control with an activation function as discussed in Section 3.2.1, the λ-tracker [92], or more broadly event- and self-triggered controller designs, see e.g. [86] and references therein. The activation threshold ι is chosen by the designer and divides the funnel for the tracking error in a safe and a safety critical region. A large value of ι implies that the controller will be inactive for a wide range of values of the last error variable, which, in case of relative degree one, means inactivity for a wide range of the tracking error, while still guaranteeing transient accuracy.

Applying a zero-input to the system (5.1) while the tracking error is within the safe region is mainly done for mathematical reasons as it simplifies the proof of Theorem 5.1.3.

In many situations it might be beneficial to apply different bounded control signal instead. One potential strategy is to simply *hold the input*, i.e. to apply the control value $u(t_{i-1})$ of the last sampling period. As pointed out in [166], neither of these two strategies is consistently superior to the other. However, more sophisticated strategies may choose the control value according to some data informativity framework [186] and can outperform the controller (5.2). In the following Section 5.1.1, we give a short outlook on how such data-driven approaches can be safeguarded by the proposed controller (5.2).

An explicit bound on the control input can be computed in advance, since $\|u\|_\infty \le \nu/\iota$. This bound depends on the system parameters derived in Lemma 5.1.2. However, precise knowledge about the functions f, g and the operator $\mathbf{T}$ is not necessary. Mere (conservative) estimates on the bounds $f^{\max}$, $g^{\max}$, and $g^{\min}$ in Lemma 5.1.2 are sufficient to guarantee the functioning of the ZoH controller (5.2).

The controller (5.2) only requires for its functioning measurement data of the system's output and its derivatives at discrete time instants t_i. It therefore overcomes the funnel controller's requirement of the availability of continuous output signal. However, the reliance of the controller (5.2) on the derivatives of the system's output can still be problematic in application as those signals are very sensitive to noise and might require the usage of numerical differentiation algorithms. For systems of order $r = 2$ the control approach (5.2) was adapted in [120] to overcome this issue and to only rely on the output signal at discrete time instants but not on its derivatives. However, a generalisation to higher-order systems is still outstanding.

5.1.1 Safeguarded data-based control

Dividing the funnel for the tracking error in a safe and a safety critical region opens up the possibility for the controller (5.2) to act as a safety filter for data-driven approaches and (online) learning techniques, which have gained a lot of popularity recently. These techniques, despite their superior performance, often lack rigorous constraint satisfaction, which is especially important in safety-critical applications like medical devices and human-robot interaction, see e.g. [46]. We also refer to [13] and [184] for an overview of the challenges employing learning-based approaches to safety-critical systems; and for challenges and recent results in the field of continual learning, we refer to the two comprehensive surveys [174, 193].

To address the challenge of ensuring constraint satisfaction while leveraging the benefits of learning-based control, the field of safe learning has gained prominence and several safety frameworks have been proposed [71, 88], employing various approaches like control barrier functions [11], Hamilton-Jacobi reachability analysis [20, 57], Model Predictive Control (MPC) [18], and Lyapunov stability [150]. Predictive safety filters, as exemplified in [189, 190], verify control input signals against a model to ensure compliance with prescribed constraints. Similar ideas are also used in the learning-based robust funnel MPC Algorithm 4.1.6 from Chapter 4 as the funnel controller compensates for the model inaccuracies of the model based controller component. The model-free controller component serves as a safety filter for the learning component which updates (or even replaces) the model at runtime while being employed in the funnel MPC algorithm. In [76] the funnel controller from [30] in combination with an activation function as presented in Section 3.2.1 was used in a comparable manner as a safety filter for a model-free Reinforcement Learning (RL) control algorithm, namely the Proximal Policy Optimisation (PPO) algorithm from [169]. In a similar manner, the funnel controller was utilised to ensure safety guarantees for Koopman operator-based MPC scheme in [42].

To utilise the controller (5.2) as a safety filter, the idea is to apply a data-driven control algorithm to the system (5.1) and temporarily interrupt its learning and control process when the activation threshold is surpassed, resorting to the pure feedback control with

ZoH, see Figure 5.1. The combination of a data-driven control algorithm with the ZoH

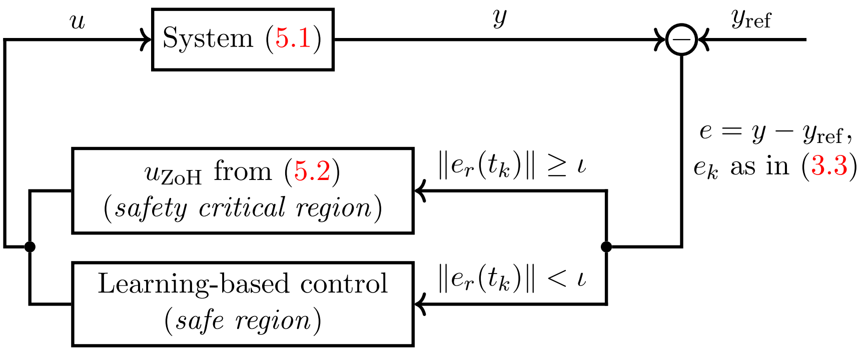


Figure 5.1: Schematic structure of the combined controller (5.6).

feedback control (5.2) can be formulated in the following switched control strategy.

$$\forall t \in [t_i, t_{i+1}) : u(t) = \begin{cases} u_{\text{data}}(t), & \|e_r(t_i)\| < \iota, \\ -\nu \frac{e_r(t_i)}{\|e_r(t_i)\|^2}, & \|e_r(t_i)\| \geq \iota. \end{cases} \tag{5.6}$$

Since the calculations in the proof of Theorem 5.1.3 involve worst case estimates, the application of $u(t) \neq 0$ for $t \in [t_i, t_i + \mathfrak{r})$, if $\|e_r(t_i)\| < \iota$ requires adaption of the sampling time $\mathfrak{r}$. The following Theorem 5.1.4 formalises this observation.

Theorem 5.1.4. *Given a reference $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and a function $\varphi \in \mathcal{G}$, consider a system (5.1) with $(f, g, \mathbf{T}) \in \mathfrak{N}^{m,r}_{t_0}$. Assume the initial trajectory $y^0 \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$ satisfies $\chi_r(y^0 - y_{\text{ref}})(t_0) \in \mathcal{E}^1_r(\varphi(t_0))$. Let the constants on the system dynamics be given as in Lemma 5.1.2, and, for an activation threshold $\iota \in (0,1)$, κ_0, κ_1 and ν be given as in Theorem 5.1.3. Further, for $u_{\max} \geq 0$, let $\mathcal{P}$ be a partition of the interval $[t_0, \infty)$ for which the maximal sampling time $\mathfrak{r} := |\mathcal{P}|$ satisfies*

$$0 < \mathfrak{r} \leq \min \left\{ \frac{\kappa_0}{\kappa_1^2}, \frac{1 - \iota}{\kappa_0 + \|\varphi\|_\infty g_{\max} u_{\max}} \right\}.$$

If $\|u_{\text{data}}\|_\infty \leq u_{\max}$, then the combined controller (5.6) applied to a system (5.1) yields

$$\|e_k(\varphi(t), \chi_r(y - y_{\text{ref}})(t))\| < 1$$

for all $k = 1, \ldots, r-1$ and $\|e_r(t)\| \leq 1$ for all $t \geq t_0$. This is initial and recursive feasibility of the ZoH control law (5.6). In particular, the tracking error $e := y - y_{\text{ref}}$ satisfies $\|e(t)\| < 1/\varphi(t)$ for all $t \geq t_0$.

Proof. By adapting the sampling time $\mathfrak{r}$ the statement follows with the same proof as for Theorem 5.1.3. □

Remark 5.1.5. The control schemes applied when $\|e_r(t_i)\| < \iota$ is not required to achieve any tracking guarantees. The only requirement is that the control signal u_{data} satisfies $\|u_{\text{data}}\|_\infty \leq u_{\max}$ for given $u_{\max} \geq 0$. In particular, this means that *any* controller (predictive, or learning-based, or model inversion-based, or locally stabilising) applied in the safe region given it satisfies the input constraints defined by $u_{\max}$. Moreover, a control scheme applied in the safe region is not even supposed to be suitable for the system to be controlled. This means that it is possible to apply, for example, controllers designed for discrete-time systems to the continuous-time system to be controlled. Maintenance of the tracking behaviour is still ensured by Theorem 5.1.4. •

The versatility of the proposed framework (5.6) has been demonstrated in [8, 168] through its application to prominent data-driven predictive control schemes, specifically data-driven model predictive control and Reinforcement Learning (RL). The data-driven MPC scheme presented therein builds on Willems et al.'s so-called fundamental lemma [195], which enables a non-parametric description of the system's input-output behaviour from measurement data, see also [68, 135] and the references therein. This combined control approach elevates standard MPC to a data-enabled predictive control scheme, cf. [26, 61]. In [8], Q-learning – first developed [194] and now a cornerstone of RL supporting many derivative algorithms [100] – illustrates how the controller (5.2) combines with model-free RL techniques. This integration both safeguards the learning process and enhances the control signal via the strategy (5.6). Although Theorem 5.1.4 requires a shorter sampling period to ensure compliance with the control objective, the two-component data-driven controller (5.6) outperformed the pure feedback controller (5.2) in both cases.

5.1.2 Simulation

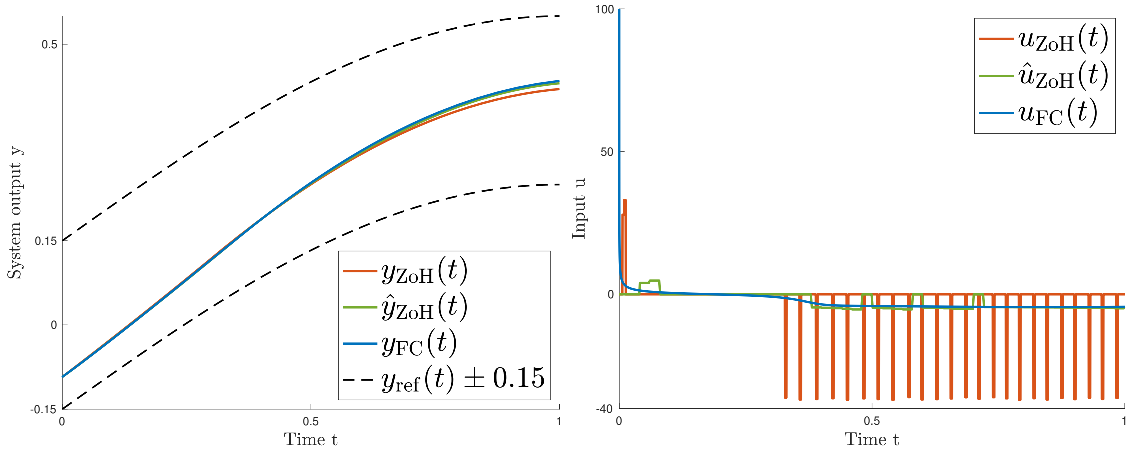
For the purpose of illustration, we revisit the mass-on-car system [172] from Section 2.5.2, and compare the ZoH controller (5.2) with the funnel controller presented in [30]. Given the parameters $m_1 = 1$, $m_2 = 2$, spring constant $k = 1$, damping $d = 1$, and angle $\vartheta = \pi/4$, the system takes the form

$$\begin{aligned} \ddot{y}(t) &= R_1 y(t) + R_2 \dot{y}(t) + S\eta(t) + \Gamma u(t) \\ \dot{\eta}(t) &= Q\eta(t) + Py(t), \end{aligned} \tag{2.49 revisited}$$

with initial conditions $[y(0), \dot{y}(0)] = [y_0^0, y_1^0] \in \mathbb{R}^2$ and $\eta(0) = \eta^0 \in \mathbb{R}^2$ for

$$R_1 = 0, \quad R_2 = \frac{1}{4}, \quad S = \frac{-\sqrt{2}}{8}\begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \Gamma = \frac{1}{4}, \quad Q = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad P = \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We simulate output reference tracking of the signal $y_{\text{ref}}(t) = 0.4\sin(\frac{\pi}{2}t)$ for $t \in [0, 1]$, transporting the mass m_2 on the car from position 0 to 0.4 within chosen error boundaries of ± 0.15. We choose the activation threshold $\iota = 0.75$. With these parameters a brief calculation (using the variation of constants formula for the internal dynamics) yields $f^{\max} \le 1.4$, $g^{\max} = g^{\min} = 0.25$, and hence, the sampling time $\mathfrak{r} \le 3.2 \cdot 10^{-3}$, and the input gain $\nu \ge 27.78$, which guarantee success of the tracking task according to Theorem 5.1.3. Choosing the smallest ν, this already results in $\|u_{\text{ZoH}}\|_\infty \le \nu/\iota \le 37.04$. We start with a small initial tracking error of $y(0) = -0.0925$, and $\dot{y}(0) = \dot{y}_{\text{ref}}(0)$. The simulation of the controller (5.2) in comparison to the continuous-time funnel controller [30] is displayed in Figure 5.2. The corresponding signals of the continuous-time funnel controller have the subscript FC, i.e. y_{FC} and u_{FC}. Since simulating the ZoH controller (5.2) is by chance also successful for $\mathfrak{r} = 2.0 \cdot 10^{-2}$ and $\nu = 4$ – beyond the theoretical bounds derived in Theorem 5.1.3 – the corresponding signals are also displayed and have a circumflex, i.e. $\hat{y}_{\text{ZoH}}$ and $\hat{u}_{\text{ZoH}}$. Figure 5.2b shows the system's output alongside the reference trajectory within the error tolerance bounds. Note that although the control input is discontinuous for the control law (5.2), the output signal remains continuous due to integration. The corresponding input signals are shown in Figure 5.2a. The three considered controllers achieve the tracking task. The ZoH input consists of separated pulses for two primary reasons. First, the control law (5.2) uses (undirected) worst-case estimates $g^{\min}, g^{\max}$ and $f^{\max}$ to compute the input signal. Hence, the control signal is at many time instants unnecessary large; however, it is ensured that the control signal always sufficiently large. Second, (5.2) includes the activation threshold ι, rendering the controller is inactive when the tracking error is small. If the tracking error exceeds this threshold at a sampling instant, the applied input is sufficiently large (due to the worst



(a) Outputs and reference, with error boundary. (b) Control inputs.

Figure 5.2: Simulation of system (2.49) under the control of the zero-order-hold control law (5.2) and the funnel controller [30].

case estimations) to force the error back below the threshold by the next sampling instant. Thus, at this time instant the input is determined to be zero. Consequently, the worst-case estimations combined with the ZoH implementation inevitably produce a peaky control signal. The control signal $\hat{u}_{\mathrm{ZoH}}$ (green) is also peaky, but exhibits smaller magnitude (due to smaller ν) and larger pulse width (due to larger $\mathfrak{r}$). Overall, $\hat{u}_{\mathrm{ZoH}}$ is comparable to u_{FC}. The successful simulation with these parameters suggests potential for finding better estimates of sufficient control parameters $\nu, \mathfrak{r}$ in future work. The control performance could also be enhanced using the extension discussed in Section 5.1.1. Note that the control signal u_{FC} also has a large initial peak, with $\|u_{\mathrm{FC}}\|_\infty \approx 100$. For the simulation, we used MATLAB. The corresponding source code can be found on GITHUB under the link `https://github.com/ddennstaedt/FMPC_Simulation`. For the integration of the dynamics, the routine `ode15s` with $\mathrm{AbsTol} = \mathrm{RelTol} = 10^{-6}$ and adaptive step size was utilised. To simulate the system behaviour under control of the funnel controller [30], `ode15s` produces a maximal step size of $\approx 3.99 \cdot 10^{-2}$ and a minimal step size of $\approx 1.21 \cdot 10^{-6}$. Thus, the largest step is about twelve times larger than $\mathfrak{r}$, and the smallest time step is about 4000 times smaller than $\mathfrak{r}$. Due to the worst case estimates used in the proof of Theorem 5.1.3 to derive the bounds for $\mathfrak{r}$, the proposed framework (5.6) requires a higher sampling rate than the funnel controller u_{FC} during most of the time. However, there are currently no results regarding an upper limit for the sampling rate of u_{FC}. Especially for unfavourable initial values, it can become arbitrarily large.

5.2 Sampled-data funnel MPC

In this section, we adapt the funnel MPC Algorithm 2.4.1 – designed to achieve the control objective outlined in Section 1.1.1 – to operate under sampled-data constraints. Unlike the prior learning-based and robust funnel MPC formulations explored in Chapters 2 to 4, the space of admissible controls is now restricted to step functions, where the control signal may only change finitely often between two sampling instants.

Sampling can have a profound impact on both the stability and performance of both linear and non-linear model predictive control schemes, as analysed in [199]. Consequently, a variety of sampled-data MPC schemes for continuous-time systems have been developed [72, 200]. Notably, [206] derives discrete-time model approximations for continuous-time systems, whose solution error scales with the sampling time. Complementary approaches like event-triggered MPC [47] further optimise digital implementations by updating control actions only when necessary, reducing computational overhead without sacrificing stability.

In contrast to these existing frameworks, we reformulate the funnel MPC Algorithm 2.4.1 as a sampled-data scheme building on the ZoH-funnel controller framework developed in Section 5.1. By constraining controls to step functions, we propose the following modification of the funnel MPC Algorithm 2.4.1.

Algorithm 5.2.1 (Sampled-data funnel MPC).
Given: Model (2.4) with initial time $t_0 \in \mathbb{R}_{\geq 0}$ and initial value $y_{\mathrm{M}}^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m)$, reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$, signal memory length $\tau \geq 0$, a set of funnel boundary function $\Psi = (\psi_1,\dots,\psi_r) \in \mathscr{G}$ with corresponding parameters k_i for $i = 1,\dots,r$, input saturation level $u_{\max} \geq 0$, a maximal step length $\mathfrak{r} > 0$, funnel stage cost function ℓ_{ψ_r}, and a τ-initialisation strategy κ as in Definition 2.3.17.
Set the time shift $\delta > 0$, the prediction horizon $T \geq \delta$, index $k := 0$, and $\hat{x}_{\mathrm{M}}^0 := \chi_r(y_{\mathrm{M}}^0)$. Choose a partition $\mathcal{P} = (t_i)_{i\in\mathbb{N}_0}$ of the interval $[t_0,\infty)$ with $|\mathcal{P}| \leq \mathfrak{r}$ and which contains $(t_0 + i\delta)_{i\in\mathbb{N}_0}$ as a subsequence.
Define the time sequence $(\hat{t}_k)_{k\in\mathbb{N}_0}$ by $\hat{t}_k := t_0 + k\delta$.
Steps:

(a) Select initial model state $\mathfrak{X}_k := \kappa(\hat{x}_{\mathrm{M}}^k) \in \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t}_k)$ at current time $\hat{t}_k$ based on $\hat{x}_{\mathrm{M}}$.

(b) Compute a solution $u_{\mathrm{FMPC},k} \in \mathcal{T}_{\mathcal{P}}([\hat{t}_k,\hat{t}_k+T],\mathbb{R}^m)$ of

$$\underset{\substack{u\in\mathcal{T}_{\mathcal{P}}([\hat{t}_k,\hat{t}_k+T],\mathbb{R}^m),\\ \|u\|_\infty \leq u_{\max}}}{\text{minimise}} \int_{\hat{t}_k}^{\hat{t}_k+T} \ell_{\psi_r}(s,\xi_r(x_{\mathrm{M}}(s;\hat{t}_k,\mathfrak{X}_k,u) - \chi_r(y_{\mathrm{ref}})(s)),u(s))\mathrm{d}s. \tag{5.7}$$

(c) Apply the control law

$$\mu : [\hat{t}_k,\hat{t}_{k+1}) \times \mathfrak{I}_{t_0,\tau}^{\Psi}(\hat{t}_k) \to \mathbb{R}^m, \quad \mu(t,\hat{x}_{\mathrm{M}}^k) = u_{\mathrm{FMPC},k}(t) \tag{5.8}$$

to model (5.1) with initial time and data $(t_k,\mathfrak{X}_k)$ and obtain, on the interval $I_0^{t_{k+1},\tau} := [t_{k+1}-\tau,t_{k+1}] \cap [0,t_{k+1}]$ a measurement of the model's output and its derivatives $\hat{x}_{\mathrm{M}}^{k+1} := x_{\mathrm{M}}(\cdot;t_k,\mathfrak{X}_k,u_{\mathrm{FMPC},k})|_{I_{t_0}^{t_{k+1},\tau}}$. Increment k by 1 and go to Step (a).

▲

Remark 5.2.2. Note that while the time shift $\delta > 0$ is an upper bound for the step length $\mathfrak{r} > 0$ of the control signals, δ is allowed to be larger than $\mathfrak{r}$ under the condition that the partition $\mathcal{P}$ contains $(t_0 + i\delta)_{i\in\mathbb{N}_0}$ as a subsequence. In this case, several control signals are applied to the system between two steps of the MPC Algorithm 5.2.1. This can also be interpreted as a multi-step MPC scheme, cf. [199]. •

Theorem 5.2.3. *Consider model (5.1) with $(f_{\mathrm{M}},g_{\mathrm{M}},\mathbf{T}_{\mathrm{M}}) \in \mathfrak{N}_{t_0}^{m,r}$ with initial trajectory $y_{\mathrm{M}}^0 \in \mathcal{C}^{r-1}([0,t_0],\mathbb{R}^m)$. Let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$ and $\Psi = (\psi_1,\dots,\psi_r) \in \mathscr{G}$ be given. Further, let $\tau \geq 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$ and $\kappa : \bigcup_{\hat{t}\geq t_0} \mathcal{R}(I_0^{\hat{t},\tau},\mathbb{R}^{rm}) \to \bigcup_{\hat{t}\geq t_0} \mathcal{R}(I_0^{\hat{t},\tau},\mathbb{R}^{rm}) \times L^\infty_{\mathrm{loc}}(I_{t_0}^{\hat{t},\tau},\mathbb{R}^q)$ be an τ-initialisation strategy as in Definition 2.3.17. Then, there exists $u_{\max} \geq 0$ and a maximal step length $\mathfrak{r} > 0$ such that the sampled-data funnel MPC Algorithm 5.2.1 with $\delta > 0$, $T \geq \delta$, and a partition $\mathcal{P}$ of the interval $[t_0,\infty)$ with $|\mathcal{P}| \leq \mathfrak{r}$ is initially and recursively feasible, i.e.*

- *the OCP (5.7) has a solution $u_{\mathrm{FMPC},k} \in \mathcal{T}_{\mathcal{P}}([\hat{t}_k,\hat{t}_k+T],\mathbb{R}^m)$ at every time instant $\hat{t}_k := t_0 + \delta k$ for $k \in \mathbb{N}_0$, and*

- *the model (5.1) with applied funnel MPC feedback (5.8) has a concatenated solution $x_{\mathrm{M}} : [0,\infty) \to \mathbb{R}^{rm}$ in the sense of Definition 2.4.2.*

The corresponding input is given by

$$u_{\text{FMPC}}(t) = u_{\text{FMPC},k}(t),$$

for $t \in [\hat{t}_k, \hat{t}_{k+1})$ *and* $k \in \mathbb{N}_0$. *Each global solution* x_{M} *with corresponding output* y_{M} *and input* u_{FMPC} *satisfies:*

(i) the control input is bounded by $u_{\max}$, *i.e.*

$$\forall\, t \geq t_0 : \quad \|u_{\text{FMPC}}(t)\| \leq u_{\max},$$

(ii) the tracking error between the model output and the reference evolves within prescribed boundaries, i.e.

$$\forall\, t \geq t_0 : \quad \|y_{\text{M}}(t) - y_{\text{ref}}(t)\| < \psi_1(t).$$

To prove Theorem 5.2.3, we reformulate certain results from Chapter 2 adapted to the changed setting. Most importantly, one has to show that there exists a step function u that, if applied to the model (2.4) at time $\hat{t}$, ensures that $x_{\text{M}}(t) - \chi_r(y_{\text{ref}})(t)$ evolves within $\mathcal{D}_t^{\Psi}$ for all t over the next time interval of length $T > 0$. For a step function with partition $\mathcal{P}$ to achieve this objective it has to be an element of

$$\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}}) := \mathcal{T}_{\mathcal{P}}([\hat{t}, \hat{t}+T], \mathbb{R}^m) \cap \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}}). \tag{5.9}$$

Theorem 2.3.21 shows that there exists a bound $u_{\max} \geq 0$ on the control input such that the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}})$ is non-empty. To prove that there exists a step function $u \in \mathcal{T}_{\mathcal{P}}([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ with a uniform minimal step length $\mathfrak{r} > 0$ that is an element of $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}})$, we utilise ideas from Theorem 5.1.3. The difficulty lies in the usage of different auxiliary error variables. Theorem 5.1.3 shows that there exists a piece-wise constant control ensuring the evolution of $\chi_r(y - y_{\text{ref}})(t)$ within the set $\mathcal{E}_r^1(\varphi(t))$ for all $t \geq t_0$. To be used in the discrete funnel MPC Algorithm 5.2.1, this result has to be also verified utilising the error signals ξ_i as in (2.15) (the set $\mathcal{E}_r^1(\varphi(t))$ is defined in terms of the error variables e_i in (3.3)).

Lemma 5.2.4. *Consider model (5.1) with* $(f_{\text{M}}, g_{\text{M}}, \mathbf{T}_{\text{M}}) \in \mathfrak{N}^{m,r}_{t_0}$. *Let* $\tau \geq 0$ *be greater than or equal to the memory limit of operator* $\mathbf{T}_{\text{M}}$. *Further, let* $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *and* $\Psi \in \mathscr{G}$. *Then, there exists* $u_{\max} \geq 0$ *and* $\mathfrak{r} > 0$ *such that, for* $\hat{t} \geq t_0$, $\hat{\mathfrak{x}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, $T > 0$, *and every partition* $\mathcal{P}$ *of the interval* $[\hat{t}, \hat{t}+T]$ *with* $|\mathcal{P}| \leq \mathfrak{r}$, *we have*

$$\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{x}}) \neq \emptyset.$$

Proof. To prove the existence of a step function achieving the control objective, we combine the ideas from Theorem 5.1.3 and Lemma 2.3.20 in the following.
Step 1: We define $u_{\max} \geq 0$ and $\mathfrak{r}$. As in the proof of Theorem 2.3.21, define, for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$,

$$\mu_i^0 := \|\psi_i\|_\infty, \quad \mu_i^{j+1} := \mu_{i+1}^j + k_i \mu_i^j,$$

where $k_i \geq 0$ are the to Ψ associated constants, which are also used to define the error variables ξ_i as in (2.15). Utilising the constants $f_{\text{M}}^{\max}$, $g_{\text{M}}^{\max}$, and $g_{\text{M}}^{\min}$ from Lemma 2.3.20, define

$$\kappa_0 := \left\| \tfrac{1}{\psi_r} \right\|_\infty \left(f_{\text{M}}^{\max} + \left\| y_{\text{ref}}^{(r)} \right\|_\infty + \sum_{j=1}^{r-1} k_j \mu_j^{r-j} + \left\| \dot{\psi} \right\|_\infty \right)$$

and choose an input gain

$$\nu > \frac{2\kappa_0 \inf_{s \geq t_0} \psi_r(s)}{g_{\text{M}}^{\min}}.$$

With $\kappa_1 := \kappa_0 + 2 \left\| \frac{1}{\psi_r} \right\|_\infty g_{\mathrm{M}}^{\max} \nu$, we define the constants

$$\mathfrak{r} := \min \left\{ \frac{\kappa_0}{\kappa_1^2}, \frac{1}{2\kappa_0} \right\} \quad \text{and} \quad u_{\max} := 2\nu.$$

All parameters are chosen in a similar fashion as in Theorem 5.1.3.
Step 2: Let $T > 0$, $\hat{t} \geq t_0$, and $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) = \hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ be arbitrary but fixed. Further, let $\mathcal{P} = (t_i)_{i \in \mathbb{N}_0}$ be a partition of the interval $[\hat{t}, \infty)$ with $|\mathcal{P}| \leq \mathfrak{r}$. Note that by $\mathcal{P}$ is also a partition of the interval $[\hat{t}, \hat{t}+T]$ by being a partition of the interval $[\hat{t}, \infty)$, see Definition 5.0.1. We construct a control step function u and show that $u \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. To this end, for some $u \in L^\infty([\hat{t}, \infty), \mathbb{R}^m)$, we use the shorthand notation $x_{\mathrm{M}}(t) := x_{\mathrm{M}}(t; \hat{t}, \hat{\mathfrak{X}}, u)$ and $\xi_i(t) := \xi_i(x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t))$ for $i = 1, \ldots, r$. The application of the ZoH feedback control

$$u_{\mathrm{ZoH}}(t) = \begin{cases} 0, & \left\| \frac{\xi_r(t_i)}{\psi_r(t_i)} \right\| < \frac{1}{2} \\ -\nu \frac{\psi_r(t_i)\xi_r(t_i)}{\|\xi_r(t_i)\|^2}, & \left\| \frac{\xi_r(t_i)}{\psi_r(t_i)} \right\| \geq \frac{1}{2}, \end{cases} \tag{5.10}$$

to the system (5.1) leads to a closed-loop system. If this initial value problem is considered on the interval $[\hat{t}, \hat{t} + T]$ with initial conditions $(\hat{t}, \hat{\mathfrak{X}})$ as in (2.11), then an application of Proposition 2.2.8 yields the existence of a maximal solution $x_{\mathrm{M}} : [0, \omega) \to \mathbb{R}^{rm}$ in the sense of Definition 2.2.6. If x_{M} is bounded, then $\omega = \infty$, see Proposition 2.2.8 (iii). In this case, the solution exists on $[0, \hat{t} + T]$.
Step 3: We show that $\left\| \frac{\xi_r(t_i)}{\psi_r(t_i)} \right\| < 1$ for $i \in \mathbb{N}_0$ implies $\left\| \frac{\xi_r(t)}{\psi_r(t)} \right\| < 1$ for all $t \in [t_i, t_{i+1}]$. Seeking a contradiction, suppose that there exists a maximal $i \in \mathbb{N}_0$ such that we have $\left\| \frac{\xi_r(t)}{\psi_r(t)} \right\| < 1$ for all $t \in [\hat{t}, t_i]$ and $\left\| \frac{\xi_r(t)}{\psi_r(t)} \right\| \geq 1$ for some $t \in (t_i, t_{i+1})$. Then, there exists

$$t^* := \inf \left\{ t \in (t_i, t_{i+1}) \;\middle|\; \left\| \frac{\xi_r(t)}{\psi_r(t)} \right\| \geq 1 \right\}.$$

We have $\left\| \frac{\xi_r(\hat{t})}{\psi_r(\hat{t})} \right\| < 1$ by the assumption $x_{\mathrm{M}}(\hat{t}) - \chi_r(y_{\mathrm{ref}})(\hat{t}) \in \mathcal{D}^{\Psi}_{\hat{t}}$, see also Remark 2.3.16. This yields $\left\| \frac{\xi_r(t)}{\psi_r(t)} \right\| < 1$ for all $t \in [\hat{t}, t^\star)$. This implies, according to Proposition 2.3.11, $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, t^\star)$, i.e. $\|\xi_i(t)\| < \psi_i(t)$ for all $i = 1, \ldots, r$. Thus, $\|\xi_i(t)\| \leq \mu_i^0$ for all $i = 1, \ldots, r$. Invoking boundedness of $y_{\mathrm{ref}}^{(i)}$, $i = 0, \ldots, r$, and the relation in (2.30), we may infer that x_{M} is bounded on $[\hat{t}, t^\star]$. Hence, $\omega > t^\star$. Since $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) = \hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, there exists a function $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$ such that $\zeta|_{[\hat{t}-\tau,\hat{t}] \cap [0,\hat{t}]} = \hat{x}_{\mathrm{M}}$ and $\mathbf{T}_{\mathrm{M}}(\zeta)|_{[\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$. Moreover, the function x_{M} fulfils $x_{\mathrm{M}}(t)|_{[\hat{t}-\tau,\hat{t}] \cap [0,\hat{t}]} = \hat{x}_{\mathrm{M}}$ and $\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})|_{[\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}]} = \hat{\mathbf{T}}_{\mathrm{M}}$ because x_{M} satisfies the initial conditions (2.11). Define the function $\tilde{\zeta} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ by

$$\tilde{\zeta}(t) = \begin{cases} x_{\mathrm{M}}(t), & t \in [\hat{t}, t^\star) \\ \zeta(t), & t \in \mathbb{R}_{\geq 0} \setminus [\hat{t}, t^\star). \end{cases}$$

Then, $\tilde{\zeta}$ is an element of $\mathcal{Y}^{\Psi}_s$ for all $s \in [\hat{t}, t^\star)$ because $\zeta \in \mathcal{Y}^{\Psi}_{\hat{t}}$ and $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, t^\star)$. Hence, we have $\left\| f_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})(t)) \right\| \leq f_{\mathrm{M}}^{\max}$ and $\left\| g_{\mathrm{M}}(\mathbf{T}_{\mathrm{M}}(\tilde{\zeta})(t))^{-1} \right\| \leq g_{\mathrm{M}}^{-1\,\max}$ for all $t \in [\hat{t}, t^\star)$ according to Lemma 2.3.20. Since $\tau \geq 0$ is greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$, we have

$$\mathbf{T}_{\mathrm{M}}(x_{\mathrm{M}})(t) = \mathbf{T}_{\mathrm{M}}(\tilde{\zeta})(t)$$

for all $t \in [\hat{t}, t^\star)$. Thus, $\|f_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))\| \le f_{\rm M}^{\max}$ and $\|g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))^{-1}\| \le g_{\rm M}^{-1\max}$ for all $t \in [\hat{t}, t^\star)$. Using (2.18) and the definition of μ_i^j, it follows that

$$\forall\, t \in [\hat{t}, t^\star) : \left\|\xi_i^{(j+1)}(t)\right\| = \left\|\xi_{i+1}^{(j)}(t) - k_i \xi_i^{(j)}(t)\right\| \le \mu_{i+1}^j + k_i \mu_i^j = \mu_i^{j+1}$$

inductively for all $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-i-1$. Utilising again (2.18), it follows by induction that

$$\xi_r(t) = \xi_1^{(r-1)}(t) + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j-1)}(t).$$

Omitting the dependency on t, we calculate for $t \in [\hat{t}, t^\star)$:

$$\begin{aligned}
\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r} = \frac{\dot{\xi}_r \psi_r - \xi_r \dot{\psi}_r}{\psi_r^2} &= \frac{1}{\psi_r}\left(\xi_1^{(r)} + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j)} - \xi_r \frac{\dot{\psi}_r}{\psi_r}\right) \\
&= \frac{1}{\psi_r}\left(f_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})) + g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M}))u - y_{\rm ref}^{(r)} + \sum_{j=1}^{r-1} k_j \xi_j^{(r-j)} - \xi_r \frac{\dot{\psi}_r}{\psi_r}\right).
\end{aligned} \tag{5.11}$$

We now consider the two cases $\left\|\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\| < \frac{1}{2}$ and $\left\|\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\| \ge \frac{1}{2}$ separately.

Step 3.a: We consider $\left\|\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\| < \frac{1}{2}$. By definition (5.10), we have $u_{\rm ZoH}(t) = 0$ for all $t \in [t_i, t^\star)$. With (5.11), we have $\left\|\left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\| \le \kappa_0$ for $t \in [\hat{t}, t^\star)$. Thus, we calculate

$$1 = \left\|\tfrac{\xi_r(t^\star)}{\psi_r(t^\star)}\right\| \le \left\|\tfrac{\xi_r(t_i)}{\psi_r(t_i)}\right\| + \int_{t_i}^{t^\star} \left\|\left(\tfrac{\rm d}{{\rm d}t}\tfrac{\xi}{\psi_r}\right)(s)\right\| {\rm d}s \le \left\|\tfrac{\xi_r(t_i)}{\psi_r(t_i)}\right\| + \int_{t_i}^{t^\star} \kappa_0 {\rm d}s < \tfrac{1}{2} + \kappa_0 \mathfrak{r} \le 1,$$

where $t^\star < \mathfrak{r} \le \frac{1}{2\kappa_0}$ was used. This contradicts the definition of $t^\star$.

Step 3.b: We consider $\left\|\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\| \ge \frac{1}{2}$. Therefore, we have $u_{\rm ZoH}(t) = -\nu \frac{\psi_r(t_i)\xi_r(t_i)}{\|\xi_r(t_i)\|^2}$ for all $t \in [t_i, t^\star)$. With (5.11), we have $\left\|\left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\| \le \kappa_1$ for all $t \in [\hat{t}, t^\star)$. Moreover, for the expression $J(t) := \left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t) - \frac{1}{\psi_r(t)} g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))u(t)$, we have $\|J(t)\| \le \kappa_0$ for all $t \in [\hat{t}, t^\star)$. Thus, we calculate

$$\begin{aligned}
\tfrac{\rm d}{{\rm d}t}\tfrac{1}{2}\left\|\frac{\xi_r(t)}{\psi_r(t)}\right\|^2 &= \left\langle \frac{\xi_r(t)}{\psi_r(t)}, \left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\rangle \\
&= \left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)} + \int_{t_i}^{t}\left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(s){\rm d}s, \left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\rangle \\
&= \left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)}, \left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\rangle + \left\langle \int_{t_i}^{t}\left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(s){\rm d}s, \left(\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right)(t)\right\rangle \\
&\le \left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)}, J(t) + \frac{1}{\psi_r(t)} g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))u(t)\right\rangle + (t - t_i)\left\|\frac{\rm d}{{\rm d}t}\frac{\xi_r}{\psi_r}\right\|_\infty^2 \\
&\le \|J(t)\| + \left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)}, \frac{1}{\psi_r(t)} g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))u(t)\right\rangle + \mathfrak{r}\kappa_1^2 \\
&= \kappa_0 - \left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)}, \frac{1}{\psi_r(t)} g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))\nu\frac{\psi_r(t_i)\xi_r(t_i)}{\|\xi_r(t_i)\|^2}\right\rangle + \mathfrak{r}\kappa_1^2 \\
&\le 2\kappa_0 - \frac{\nu}{\psi_r(t)}\frac{\|\xi_r(t_i)\|^2}{\psi_r(t_i)^2}\left\langle \frac{\xi_r(t_i)}{\psi_r(t_i)}, g_{\rm M}(\mathbf{T}_{\rm M}(x_{\rm M})(t))\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\rangle \\
&\le 2\kappa_0 - \frac{\nu}{\inf_{s \ge t_0}\psi_r(s)} g_{\rm M}^{\min} < 0.
\end{aligned}$$

In particular, this yields $\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left\|\frac{\xi_r}{\psi_r}\right\|^2\right)(t_i) < 0$, by which $t^* > t_i$. Therefore, we find the contradiction $1 = \left\|\frac{\xi_r(t^*)}{\psi_r(t^*)}\right\|^2 < \left\|\frac{\xi_r(t_i)}{\psi_r(t_i)}\right\|^2 < 1$.

Step 4: As $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, we have $\left\|\frac{\xi_r(\hat{t})}{\psi_r(\hat{t})}\right\| < 1$, see Remark 2.3.16. By induction, Step 3 yields $\left\|\frac{\xi_r(t)}{\psi_r(t)}\right\| < 1$ for all $t \in [\hat{t}, \omega)$. This implies, according to Proposition 2.3.11, $x_{\mathrm{M}}(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}^{\Psi}_t$ for all $t \in [\hat{t}, \omega)$, i.e. $\|\xi_i(t)\| < \psi_i(t)$ for all $i = 1, \ldots, r$. Invoking boundedness of $y^{(i)}_{\mathrm{ref}}$, $i = 0, \ldots, r$, and the relation in (2.30), we may infer that x_{M} is bounded on $[\hat{t}, \omega)$. Thus, $\omega = \infty$. Also note that the ZoH feedback control u_{ZoH} in (5.10) fulfils $\|u_{\mathrm{ZoH}}\|_\infty \le u_{\max}$. Since the partition $\mathcal{P}$ is also a partition of the interval $[\hat{t}, \hat{t} + T]$, we have $u_{\mathrm{ZoH}} \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$. This completes the proof. □

To prove the functioning of the discrete funnel MPC Algorithm 5.2.1, we further have to show that the optimisation problem (5.7) has a solution. To this end, we recall, for $T > 0$, $\hat{t} \ge t_0$, and $\hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$ with $\tau \ge 0$ being greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$, the definition of cost functional $J^{\Psi}_T(\cdot; \hat{t}, \hat{\mathfrak{X}}) : L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$ given by

$$J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) := \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(s, \xi_r(x_{\mathrm{M}}(s; \hat{t}, \hat{\mathfrak{X}}, u) - \chi_r(y_{\mathrm{ref}})(s)), u(s))\mathrm{d}s. \qquad \text{(2.33 revisited)}$$

We prove that $J^{\Psi}_T(\cdot; \hat{t}, \hat{\mathfrak{X}})$, when restricted to the set $\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ as in (5.9) has a minimum by adapting Theorem 2.3.26 to the changed setting.

Lemma 5.2.5. *Consider model (5.1) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathfrak{N}^{m,r}_{t_0}$ with reference trajectory $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$. Let $\Psi \in \mathscr{G}$ and $\tau \ge 0$ be greater than or equal to the memory limit of operator $\mathbf{T}_{\mathrm{M}}$. Further, let $\hat{t} \ge t_0$, $(\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) = \hat{\mathfrak{X}} \in \mathfrak{I}^{\Psi}_{t_0,\tau}(\hat{t})$, $T > 0$, $u_{\max} \ge 0$, and $\mathcal{P}$ be a partition of the interval $[\hat{t}, \hat{t} + T]$ such that $\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \neq \emptyset$. Then, there exists a function $u^\star \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ such that*

$$J^{\Psi}_T(u^\star; \hat{t}, \hat{\mathfrak{X}}) = \min_{u \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) = \min_{\substack{u \in \mathcal{T}_{\mathcal{P}}([\hat{t},\hat{t}+T], \mathbb{R}^m),\\ \|u\|_\infty \le u_{\max}}} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}).$$

Proof. We adapt the proof of Theorem 2.3.26. Since $\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}}) \subset \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, the set $\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ is non-empty by assumption. By Theorem 2.3.26, there exists a control $u \in \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$ minimising the functional $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$. Thus, the infimum $J^\star := \inf_{u \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})} J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}})$ exists as well. Let $(u_k)_{k\in\mathbb{N}_0} \in \left(\mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})\right)^{\mathbb{N}_0}$ be a minimising sequence, meaning $J^{\Psi}_T(u; \hat{t}, \hat{\mathfrak{X}}) \to J^\star$. As $\mathcal{P} = (t_k)_{k\in\mathbb{N}_0}$ is a partition of the interval $[\hat{t}, \hat{t} + T]$, we have $t_0 = \hat{t}$ and there exists a minimal $N \in \mathbb{N}_0$ with $t_n > \hat{t} + T$ for all $n > N$. Define $u_{i,k} := u_k(t_i)$ for $i = 0, \ldots, N$. For every $i = 0, \ldots, N$, $(u_{i,k})_{k\in\mathbb{N}_0}$ is a sequence in $\mathbb{R}^m$ with $\|u_{i,k}\| \le u_{\max}$ for all $k \in \mathbb{N}$. Thus, it has a limit point $u^\star_i \in \mathbb{R}^m$. The function $u^\star$ defined by $u^\star|_{[t_i,t_{i+1})\cap[\hat{t},\hat{t}+T]} := u^\star_i$ is an element of $\mathcal{T}_{\mathcal{P}}([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ with $\|u^\star\| \le u_{\max}$. Up to subsequence, u_k converges uniformly to $u^\star$. We define $(x_k) := \left(x_{\mathrm{M}}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u_k)\right) \in \mathcal{R}([0, \hat{t} + T], \mathbb{R}^n)^{\mathbb{N}}$ as the sequence of associated responses. Repeating Steps 2 and 3 of the proof of Theorem 2.3.26, the sequence (x_k) has a subsequence (which we do not relabel) that uniformly converges to $x^\star = x_{\mathrm{M}}(\cdot; \hat{t}, \hat{\mathfrak{X}}, u^\star)$. It remains to show that $u^\star \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})$, $J^\star = J^{\Psi}_T(u^\star; \hat{t}, \hat{\mathfrak{X}})$, and that $J^{\Psi}_T(u^\star; \hat{t}, \hat{\mathfrak{X}}) = \min_{u \in \mathcal{U}^{\mathcal{P}}_{[\hat{t},\hat{t}+T]}(u_{\max}, \hat{\mathfrak{X}})}$. These statements follow along the lines of Steps 5–7 of the proof of Theorem 2.3.26. □

We are now in the position to summarise our results in the proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. Using the results of Lemmata 5.2.4 and 5.2.5 proving Theorem 5.2.3 is a straightforward adaptation of the proof of Theorem 2.4.3 to the changed context. □

5.2.1 Simulation

To illustrate the theoretical results by a numerical example, we consider a torsional oscillator with two flywheels, which are connected by a rod, see Figure 5.3. Such a system can be interpreted as a simple model of a driving train, cf. [64]. The equations of motion for the

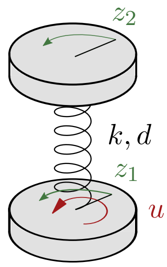


Figure 5.3: Torsional oscillator. The figure is based on [64, Fig. 2.7], edited to the case of two flywheels for the present purpose.

torsional oscillator are given by

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1(t) \\ \ddot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -d & d \\ d & -d \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} + \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

where for $i = 1, 2$ (the index 1 refers to the lower flywheel) z_i is the rotational position of the flywheel, $I_i > 0$ is the inertia, $d, k > 0$ are damping and torsional-spring constant, respectively. We aim to control the oscillator such that the lower flywheel follows a given velocity profile. Hence, we choose $y(t) = \dot{z}_1(t)$ as the output. To remove the rigid-body motion from the dynamics, we introduce $\hat{z} := z_1 - z_2$. With this new variable, setting $x := [\hat{z}, \dot{z}_1, \dot{z}_2]$ the dynamics can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) = \dot{z}_1(t), \end{aligned}$$

where

$$\tilde{A} := \begin{bmatrix} 0 & 1 & -1 \\ -k & -d & d \\ k & d & -d \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_2 \end{bmatrix},$$
$$A := M^{-1}\tilde{A}, \qquad B := M^{-1}\tilde{B}, \quad C := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Using standard techniques, see e.g. [91] and also Example 2.2.3, the reduced dynamics of the torsional oscillator can then be written in Byrnes-Isidori form (2.6)

$$\begin{aligned} \dot{y}(t) &= Ry(t) + S\eta(t) + \Gamma u(t), \\ \dot{\eta}(t) &= Q\eta(t) + Py(t), \end{aligned} \tag{5.12}$$

where η is the internal state, and

$$R = \frac{-d}{I_1}, \quad S = \frac{1}{I_1}\begin{bmatrix} k & d \end{bmatrix}, \quad Q = \frac{1}{I_2}\begin{bmatrix} 0 & I_2 \\ -k & -d \end{bmatrix}, \quad P = \frac{1}{I_2}\begin{bmatrix} -I_2 \\ d \end{bmatrix}.$$

Note that Q is a stable matrix, i.e. its eigenvalues are on the left half plane. Thus, the internal dynamics are bounded-input bounded-state stable. The high-gain matrix is given by $\Gamma := CB = 1/I_1 > 0$. For the purpose of simulation, we choose the reference

$$y_{\text{ref}}(t) = \frac{250}{2}\left(1 + \frac{1}{\sqrt{2\pi}}\int_0^t \mathrm{e}^{-\frac{1}{2}(s-3)^2}\mathrm{d}s\right),$$

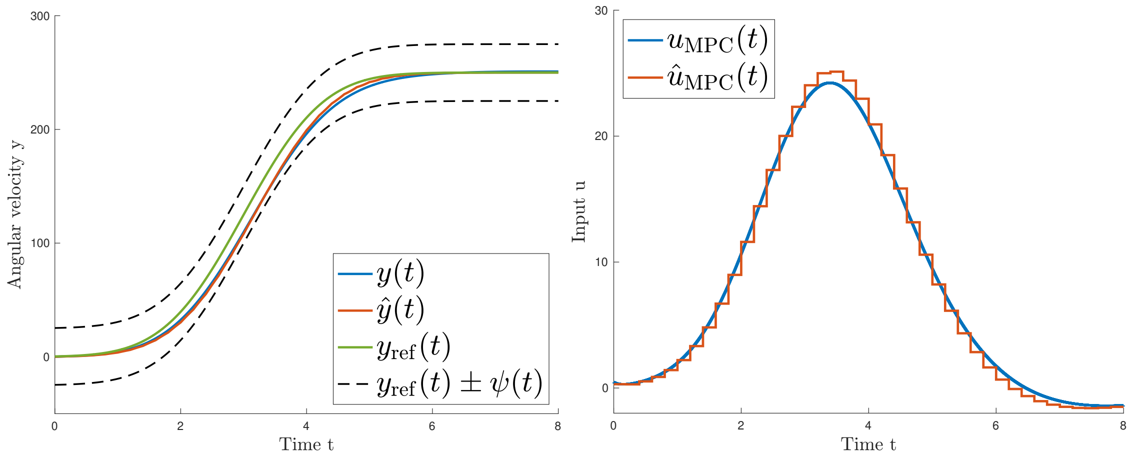
which is a modified version of the error function (ERF) and represents a smooth transition from zero rotation to an (approximately) constant angular velocity of 250 rotations per unit time. Thus, $\|y_{\text{ref}}\|_\infty \leq 250$, $\|\dot{y}_{\text{ref}}\|_\infty = 250/\sqrt{2\pi}$. Inserting the dimensionless parameters $I_1 = 0.136$, $I_2 = 0.12$, $k = 0.1$, and $d = 0.16$, and invoking the reference y_{ref} and the constant error tolerance $\psi \equiv 25$ (we allow a deviation of 10%), we may derive worst case bounds on the system dynamics by estimating the explicit solution of the linear equations (5.12). We compute these bounds in order to estimate a sufficiently large $u_{\max} \geq 0$ as in the proof of Lemma 5.2.4. For the sake of simplicity, we will assume $\eta(0) = 0$, which does not cause loss of generality. For $\|y\|_\infty \leq \|y_{\text{ref}}\|_\infty + \psi$, we estimate

$$\forall\, t \geq 0 : \quad \|\eta(t)\| \leq \frac{M}{\mu}\|P\|(\|y_{\text{ref}}\|_\infty + \psi),$$

where $M := \sqrt{\|K^{-1}\|\|K\|}$ and $\mu := 1/(2\|K\|)$, and $K \in \mathbb{R}^{2\times 2}$ solves the Lyapunov equation $KQ + Q^\top K + I_2 = 0$. Inserting the values, we find that the estimates for step length of the control signal $\mathfrak{r}$ and maximal control provided in the proof of Lemma 5.2.4 are satisfied with $\mathfrak{r} = 0.002$, and $u_{\max} = 267$. We choose the time shift $\delta = \mathfrak{r}$, i.e. a constant control is applied to the system between two iterations of the sampled-data funnel MPC Algorithm 5.2.1. Further, the prediction horizon is set as $T = 10\delta$. For the purpose of simulation, we use the non-strict funnel penalty function

$$\ell(t, y, u) = \begin{cases} \|y - y_{\text{ref}}\|^2 + \lambda_u\|u\|^2, & \|y - y_{\text{ref}}\| \leq \psi(t) \\ \infty, & \text{else}, \end{cases}$$

with $\lambda_u = 10^{-1}$. The results are depicted in Figure 5.4. While Figure 5.4 displays the system's output evolving within the funnel boundary, Figure 5.4b shows the corresponding control signals.



(a) Outputs and reference, with error boundary. (b) Control inputs.

Figure 5.4: Simulation of system (5.12) under the control generated by Algorithm 5.2.1 with $\delta = \mathfrak{r} = 0.002$ and $\delta = \mathfrak{r} = 0.2$.

We stress that the estimates for $\mathfrak{r}$ and $u_{\max}$ in the proof of Lemma 5.2.4 are very conservative. To demonstrate this aspect, we run a second simulation, where we chose

$\delta = \mathfrak{r} = 0.2$, $T = 1$, and $u_{\max} = 30$. The results of this simulation are labelled as $\hat{y}, \hat{u}$, respectively. With this much larger uniform step length, the tracking objective can be satisfied as well, cf. Figure 5.4. Note that the maximal applied control value is in both cases much smaller than the (conservative) estimate $u_{\max}$ satisfying Lemma 5.2.4. These simulations suggest that the bounds derived Lemma 5.2.4 leave room for improvement. As before, all simulations have been performed with MATLAB using the CASADI framework. The corresponding source code can be found on GITHUB under the link https://github.com/ddennstaedt/FMPC_Simulation.

6 Outlook

In this thesis, the concept of funnel model predictive control is presented, which integrates ideas from the adaptive high-gain control technique funnel control in a model predictive control scheme. Building upon the framework outlined in Chapter 2, three extensions are subsequently introduced in Chapter 3 through 5. The following section summarises the main results and provides a brief outlook on future research directions.

Funnel model predictive control represents a novel MPC approach to output tracking for a class of non-linear multi-input multi-output systems governed by functional differential equations. By combining the predictive capabilities of MPC with concepts of the adaptive funnel control technique, this framework guarantees prescribed transient performance – ensuring the tracking error remains within user-defined, time-varying boundaries for smooth reference signals. Central to its efficacy are *funnel penalty functions*, which dynamically penalise the error trajectory's distance to the funnel boundaries eliminating the need for conventional mechanisms such as terminal conditions, artificially extended prediction horizons, or restrictive output constraints to ensure initial and recursive feasibility.

A critical assumption underpinning funnel MPC is the availability of sufficiently large control values, quantified by $u_{\max} \geq 0$. While Theorem 2.3.21 establishes existence of such a bound, its current formulation is inherently conservative and computationally intractable – limiting practical applicability. Addressing this, future research should prioritise:

1. **Refinement of estimates**: Existing bounds on $u_{\max} \geq 0$, derived as worst-case guarantees independent of the prediction horizon $T > 0$, likely obscure potential synergies between T and the required control effort. A rigorous exploration of T's role – particularly in balancing transient performance against input magnitude – could yield tighter, horizon-dependent bounds.

2. **Parametric sensitivity analysis**: A systematic characterisation of how auxiliary parameters (e.g. funnel shape, error variables ξ_i, weighting parameters k_i) influence feasibility and performance would enhance design flexibility.

3. **Fixed-input feasibility**: Developing mechanisms to ensure recursive feasibility under a priori fixed control limits $u_{\max} \geq 0$ remains a pivotal challenge for implementation.

4. **Cost function simplification**: Investigating whether the weighted sum of the tracking error $e_{\mathrm{M}} = y_{\mathrm{M}} - y_{\mathrm{ref}}$ and its derivatives in the funnel penalty function for higher order systems can be reduced to the sole error signal e_{M} – while ensuring initial and recursive feasibility provided $T > 0$ is chosen large enough – would simplify the algorithm's complexity.

5. **Generalisation of model class**: The presented results hold for models with a strict global relative degree. Since funnel control has been successfully generalised to systems with vector relative degree [34, 121], it is worth investigating a corresponding generalisation of the funnel MPC framework.

6. **Numerical implementation**: The incompatibility of discontinuous funnel penalties functions with standard optimisation frameworks (e.g. CasADi) needs to be

addressed. Future work should explore the development of smooth approximations or custom solvers tailored to funnel penalty functions in order to ensure fast numerical convergence while adhering to funnel boundaries and maintaining feasibility guarantees.

Beyond these technical refinements, broader questions remain unanswered. A comprehensive benchmarking study comparing funnel MPC against classical MPC variants remains an open research question. Furthermore, extending the developed principles to alternative control objectives, such as safety-critical set invariance (e.g. confining states to prescribed safe regions), presents fertile ground for further theoretical and applied investigations.

Robust funnel MPC synergises funnel MPC and model-free adaptive funnel control into a two-component architecture. This hybrid scheme bridges the often-competing priorities of optimality and robustness, achieving prescribed tracking performance even under structural model-plant mismatches and unknown disturbances.

- **Funnel MPC** prioritises optimality by minimising a designer-specified cost functional over receding horizons.
- **Funnel control** ensures robustness through adaptive disturbance rejection, activated only when necessary.

Key to their compatibility is the strategic design of the funnel controller's reference signal and boundary, derived from the MPC's predictions. This ensures the components complement rather than conflict. Further refinement can be achieved via an activation function, which sparsely engages the funnel controller to minimally perturb the optimal MPC signal while rejecting disturbances.

The framework periodically updates the model with system measurements via proper initialisation. While theoretically generalisable, this process remains cumbersome in practice, prompting the question: Can initialisation be streamlined without compromising robustness? Future research will focus on extracting criteria to find explicit and beneficial proper initialisation strategies.

Further open challenges and future directions include:

1. **Unified model-system classes**: The model and system currently require distinct classes of differential equation. While the model is assumed to have a control affine representation, the function F describing the system dynamics has the perturbation high-gain property. A unification of these two classes would broaden applicability.
2. **Explicit combined input bounds**: While the MPC component's control input is bounded by $u_{\max} \geq 0$, the model-free funnel controller lacks explicit a-priori bounds. Deriving a composite bound for the combined scheme is critical for safety-critical applications.
3. **Derivative-free operation**: The funnel controller's reliance on output derivatives poses practical challenges with noisy measurements. Integrating a *funnel pre-compensator* [38, 119] – to estimate derivatives or bypass their need – warrants exploration.
4. **Order flexibility**: The proposed framework mandates matching relative degrees for model and system. Relaxing this constraint could enable simplified models (e.g. lower-order approximations) for complex systems.

Learning-based robust funnel MPC extends the robust funnel MPC framework by integrating a versatile online learning architecture. This approach continuously refines the surrogate model using historical data – system outputs, model predictions, and applied control signals – drawn from both the model-based funnel MPC and the model-free feedback component. It ensures robust tracking within predefined (time-varying) performance boundaries while accommodating:

- **Varying model complexity**, from simplified approximations to high-fidelity representations.
- **Continual improvement** via iterative data assimilation.
- **Methodological agnosticism**, allowing integration of diverse learning paradigms.

By combining learning techniques with both model-based prediction and adaptive control, this framework bridges the gap between robustness and adaptability in uncertain environments.

While the current formulation is abstract and theoretical, future research will address critical open questions:

1. **Learning scheme efficacy**: What defines an effective learning scheme? How can controller performance improvement be rigorously verified?
2. **Technique compatibility**: Which established methods – Willems' fundamental lemma, Koopman operator theory, or neural networks – can effectively be used to leverage the collected data?
3. **Feasibility guarantees**: How can feasibility be rigorously proven for advanced learning algorithms?
4. **Prior knowledge integration**: How should existing system knowledge inform the learning architecture?

Sampled-data robust funnel MPC demonstrates how output tracking with prescribed performance can be achieved while restricting admissible controls to piecewise constant step functions. The key contribution is *explicit uniform bounds* on sampling rates and maximal control effort for both the funnel MPC and model-free funnel controller. This is an important step to bridge the gap between continuous-time theory and real-world sampled-data implementations. For the funnel controller, we further showed that its Zero-order-Hold implementation can serve as a safety filter for learning-based control architectures. However, effective deployment requires addressing the reliance on noise-sensitive output derivative measurements – a critical challenge for future work. While foundational, the derived bounds remain highly conservative. Relaxing these estimates is essential for practical applicability. Additionally, the current system and model classes (tailored for sampled-data control) represent subsets of those in prior chapters. Generalising these results to broader classes of systems/models remains an open problem.

To advance digital implementation, three key questions arise:

- Can the continuous-time cost function (currently integral-based) used in the funnel MPC algorithm be efficiently discretised with uniform error bounds?
- Can the algorithms be redeveloped entirely for discrete-time systems, bypassing continuous-time computations?
- How might a discrete-time theory for funnel control and funnel MPC be formulated?

Presently, all theoretical guarantees assume continuous-time dynamics. A discrete-time counterpart – for both components – remains unexplored. Furthermore, learning techniques specifically tailored for sampled-data systems – such as those leveraging intermittent measurements or quantised data – could prove particularly advantageous in enhancing adaptability while preserving robustness. The integration of such methods also promises to be an interesting direction for future research.

Appendix

The existence of solutions of the differential equations is essential for both the system (1.1) and model (2.4). From an application point of view, the question of the solution's existence is often not of interest or merely seen as a technical detail. However, it is of utmost importance mathematically as the foundation of all further investigations and results. Although several works, see e.g. [93, 94, 95, 161], have already provided answers to this question for systems similar to the ones considered in this thesis, we would like to provide a rigorous proof in this work as well for the sake of completeness.

To this end, we consider the initial value problem

$$\begin{aligned} \dot{y}(t) &= F(t, y(t), \mathbf{T}(y)(t)), \\ y|_{[0,t_0]} &= y^0 \in \mathcal{C}([0, t_0], \mathbb{R}^n), \end{aligned} \tag{7.1}$$

and will prove the existence of solutions for this initial value problem in the following. For the sake of generality, we want to analyse the problem with the function F being only defined on a domain, i.e. a non-empty connected relatively open set, but not necessarily on the whole space. Thus, let $\mathcal{D}_1 \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ be non-empty, connected, relatively open sets with $(t_0, y^0(t_0)) \in \mathcal{D}_1$. Assume that $F : \mathcal{D}_1 \times \mathbb{R}^q \to \mathbb{R}^n$ is a *Carathéodory function*, i.e. it has the following properties for every compact interval I, $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$ with $I \times \bar{\mathcal{B}}_\varepsilon(x^0) \subset \mathcal{D}_1$ and every compact set $K \subset \mathbb{R}^q$:

(C.1) $F(t, \cdot, \cdot) : \bar{\mathcal{B}}_\varepsilon(x^0) \times K \to \mathbb{R}^n$ is continuous for almost all $t \in I$,

(C.2) $F(\cdot, x, z) : I \to \mathbb{R}^n$ is measurable for all fixed $(x, z) \in \bar{\mathcal{B}}(x^0) \times K$,

(C.3) there exists an integrable function $\kappa : I \to \mathbb{R}_{\geq 0}$ such that $\|F(t, x, z)\| \leq \kappa(t)$ for almost all $t \in I$ and all $(x, z) \in \bar{\mathcal{B}}_\varepsilon(x^0) \times K$.

This notion of a Carathéodory function is based on the definition in [93, Appendix B].

We assume that the operator $\mathbf{T}$ only acts on functions evolving within a domain $\mathcal{D}_2 \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. To formally define the properties of $\mathbf{T}$, we impose on $\mathcal{D}_2$ that, for all $t \in \mathbb{R}_{\geq 0}$, there exists $x \in \mathbb{R}^n$ with $(t, x) \in \mathcal{D}_2$ and define the set of regulated functions evolving in $\mathcal{D}_2$ as $\mathcal{RD}_2 := \{y \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \,|\, \mathrm{graph}(y) \subset \mathcal{D}_2\}$. Then, we assume the operator $\mathbf{T} : \mathcal{RD}_2 \to L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^q)$ to fulfil the following properties:

(T.1') $\forall\, y_1, y_2 \in \mathcal{RD}_2 \ \forall\, t \geq t_0$:

$$y_1|_{[0,t]} = y_2|_{[0,t]} \implies \mathbf{T}(y_1)|_{[t_0,t]} = \mathbf{T}(y_2)|_{[t_0,t]}.$$

(T.2') $\forall\, t \geq t_0 \ \forall\, y \in \mathcal{R}([0, t]; \mathbb{R}^n)$ with $\mathrm{graph}(y) \subset \mathcal{D}_2 \ \exists\, \Delta, \delta, c > 0 \ \forall\, y_1, y_2 \in \mathcal{RD}_2$ with $y_1|_{[0,t]} = y_2|_{[0,t]} = y$ and $\|y_1(s) - y(t)\| < \delta$, $\|y_2(s) - y(t)\| < \delta$ for all $s \in [t, t + \Delta]$:

$$\operatorname*{ess\,sup}_{s \in [t, t+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \leq c \sup_{s \in [t, t+\Delta]} \|y_1(s) - y_2(s)\|\,.$$

(T.3') For every $t \geq t_0$ and every family $(K_s)_{s \in [0,t]}$ of compact sets $K_s \subset \mathbb{R}^n$ such that $\bigcup_{s \in [0,t]} K_s$ is a bounded set and $\bigcup_{s \in [0,t]} \{s\} \times K_s \subset \mathcal{D}_2$, there exists $c_1 > 0$ such that for all $y \in \mathcal{RD}_2$:

$$\mathrm{graph}(y) \subset \bigcup_{s \in [0,t]} \{s\} \times K_s \implies \sup_{s \in [t_0, t)} \|\mathbf{T}(y)(s)\| \leq c_1,$$

The properties (T.1'), (T.2'), and (T.3') adapt (T.1), (T.2), and (T.3) from Definition 2.2.1 to accommodate the restriction of $\mathbf{T}$ to the set of functions with codomain $\mathcal{D}_2$. While defining the operator $\mathbf{T}$ only on a set of functions restricted to a domain for a system of the form (7.1) was already considered in [81], a proof for the existence of solutions was omitted. Since the usage of a such modified operator could be of interest in application and future research work, we want to provide a proof in the following. It is clear that modified properties (T.1') and (T.2') are equivalent to their original counterparts in the case $\mathcal{D}_2 = \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. This is not the case for (T.3'). It is a weaker assumption on $\mathbf{T}$ as the following lemma shows.

Lemma 7.0.1. *Let $\mathcal{D}_2 = \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $\mathbf{T} : \mathcal{R}\mathcal{D}_2 \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^q)$ be an operator with property (T.1'). If $\mathbf{T}$ has property (T.3), then it also satisfies property (T.3'). The opposite is in general not true.*

Proof. Let $\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^q)$ satisfying (T.1') and (T.3). We show $\mathbf{T}$ has property (T.3'). Let $t \geq t_0$ and $(K_s)_{s \in [0,t]}$ be a family of compact sets $K_s \subset \mathbb{R}^n$ with $\bigcup_{s \in [0,t]} K_s$ being bounded. There exists $c_0 > 0$ with $\bigcup_{s \in [0,t]} K_s \subset \bar{\mathcal{B}}_{c_0}$. Due to property (T.3), there exists $c_1 > 0$ with such that

$$\sup_{t \in [t_0, \infty)} \|\mathbf{T}(y)(t)\| \leq c_1$$

for all $y \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ with $\sup_{t \in \mathbb{R}_{\geq 0}} \|y(t)\| \leq c_0$. Let $y \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ be a function with $\operatorname{graph}(y) \subset \bigcup_{s \in [0,t]} \{s\} \times K_s$. Define $\tilde{y} \in \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ by $\tilde{y}(s) = y(s)$ for $s \in [0, t]$ and $\tilde{y}(s) = y(t)$ for $s > t$. Due to the causality property (T.1'), we have

$$\sup_{s \in [t_0, t]} \|\mathbf{T}(y)(s)\| = \sup_{s \in [t_0, t]} \|\mathbf{T}(\tilde{y})(s)\| \leq \sup_{s \in [t_0, \infty)} \|\mathbf{T}(\tilde{y})(s)\| \leq c_1.$$

This shows that $\mathbf{T}$ fulfils (T.3').

To show that the opposite is in general not true, we consider a counter example. Define $\mathbf{T} : \mathcal{R}(\mathbb{R}_{\geq 0}, \mathbb{R}) \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R})$ by $\mathbf{T}(y)(t) := \int_0^t y(\tau) \mathrm{d}\tau$ for $t \in [t_0, \infty)$. It is clear that $\mathbf{T}$ is a causal, i.e. it fulfils property (T.1'). We show that $\mathbf{T}$ has also property (T.3'). To this end, let $t \geq t_0$ and $(K_s)_{s \in [0,t]}$ be a family of compact sets $K_s \subset \mathbb{R}^n$ with $\bigcup_{s \in [0,t]} K_s$ being bounded. There exists $c_0 > 0$ with $\bigcup_{s \in [0,t]} K_s \subset \bar{\mathcal{B}}_{c_0}$. Let $y \in \mathcal{R}\mathcal{D}_2$ with $\operatorname{graph}(y) \subset \bigcup_{s \in [0,t]} \{s\} \times K_s$ and set $c_1 := t c_0$. Then, we have

$$\sup_{s \in [t_0, t]} \|\mathbf{T}(y)(s)\| = \sup_{s \in [t_0, t]} \left\| \int_0^s y(\tau) \mathrm{d}\tau \right\| \leq \int_0^t \|y\|_\infty \mathrm{d}\tau \leq t c_0 = c_1.$$

Thus, $\mathbf{T}$ has property (T.3'). However, for the constant function $\tilde{y} \equiv 1$, we have $\mathbf{T}(y)(t) \to \infty$ for $t \to \infty$. Therefore, $\mathbf{T}$ does not have the bounded-input bounded-output property (T.3). □

With the assumed properties of F and $\mathbf{T}$ at hand, we define a solution of the initial value problem (7.1) in the virtue of [95, Section 5] as follows.

Definition 7.0.2. *For $y^0 \in \mathcal{C}([0, t_0], \mathbb{R}^n)$ with $(t_0, y^0(t_0)) \in \mathcal{D}_1$ and $\operatorname{graph}(y^0) \subset \mathcal{D}_2$, a function $y : [0, \omega) \to \mathbb{R}^n$ with $\omega \in (t_0, \infty]$ and $[t_0, \omega) \subset I$ is called a* solution *of the initial value problem* (7.1)*, if $y|_{[0,t_0]} = y^0$ and*

$$\forall\, t \in [t_0, \omega): \quad y(t) = y^0(t_0) + \int_{t_0}^t F(s, y(s), \mathbf{T}(y)(s)) \mathrm{d}s.$$

A solution $y : [0, \omega) \to \mathbb{R}^n$ is said to be maximal *if it has no proper right extension that is also a solution.*

Note that we identify $\mathcal{C}([0,t_0],\mathbb{R}^n)$ with $\mathbb{R}^n$ if $t_0 = 0$ in Definition 7.0.2. Moreover, given a function $y \in \mathcal{R}([0,\omega),\mathbb{R}^n)$ with $\omega < \infty$ and $\operatorname{graph}(y) \subset \mathcal{D}_2$, $\mathbf{T}(y)(t)$ is interpreted for $t \in [0,\omega)$ as the evaluation of $\mathbf{T}(y^e)(t)$ for an arbitrary right extension $y^e \in \mathcal{R}\mathcal{D}_2$ of y, as elaborated in Remark 2.2.5 (a).

Remark 7.0.3. Although it was not mentioned explicitly in Definition 7.0.2, a solution $y : [0,\omega) \to \mathbb{R}^n$ of the initial value problem (7.1) has the following properties:

(i) $y_{[t_0,\omega)}$ is absolutely continuous,

(ii) $(t, y(t)) \in \mathcal{D}_1 \cap \mathcal{D}_2$ for all $t \in [t_0,\omega)$.

•

With the definition of solutions of the initial value problem (7.1) established, we now present a key existence theorem.

Theorem 7.0.4. *Consider the initial value problem* (7.1) *where $F : \mathcal{D}_1 \times \mathbb{R}^q \to \mathbb{R}^n$ is a Carathéodory function and the operator $\mathbf{T} : \mathcal{R}\mathcal{D}_2 \to L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^q)$ has the properties (T.1'), (T.2'), and (T.3'). Let $y^0 \in \mathcal{C}([0,t_0],\mathbb{R}^n)$ with $y^0(t_0) \in \mathcal{D}_1$ and* $\operatorname{graph}(y^0) \subset \mathcal{D}_2$. *Then,*

(i) the initial value problem (7.1) *has a solution $y : [0,\omega) \to \mathbb{R}^n$ with $\omega > t_0$ in the sense of Definition 7.0.2,*

(ii) every solution can be extended to a maximal solution,

(iii) if F is locally essentially bounded and $y \in \mathcal{C}([0,\omega),\mathbb{R}^n)$ is a maximal solution, then the closure of $\operatorname{graph}(y|_{[t_0,\omega)})$ *is not a compact subset of $\mathcal{D}_1 \cap \mathcal{D}_2$.*

Proof. We adapt the proof of [93, Theorem 7.1] to the current setting.
Step 1: We have $y^0 \in \mathcal{C}([0,t_0],\mathbb{R}^n)$ with $\operatorname{graph}(y^0) \subset \mathcal{D}_2$. Thus, using property (T.2') of operator $\mathbf{T}$, there exist $\Delta, \delta, c > 0$ such that, for all $y_1, y_2 \in \mathcal{R}\mathcal{D}_2$ with $y_1|_{[0,t_0]} = y^0 = y_2|_{[0,t_0]}$ and $\left\|y_1(s) - y^0(t_0)\right\| < \delta$, $\left\|y_2(s) - y^0(t_0)\right\| < \delta$, we have for all $s \in [t_0, t_0+\Delta]$:

$$\operatorname*{ess\,sup}_{s\in[t_0,t_0+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \le c \sup_{s\in[t_0,t_0+\Delta]} \|y_1(s) - y_2(s)\| \,.$$

Both $\Delta > 0$ and $\delta > 0$ can be chosen sufficiently small such that

$$[t_0, t_0+\Delta] \times \bar{\mathcal{B}}_\delta(y^0(t_0)) \subset \mathcal{D}_1 \cap \mathcal{D}_2.$$

For $t \in [0, t_0+\Delta]$ define the compact set

$$K_t := \begin{cases} \{y^0(t)\}\,, & t \in [0,t_0), \\ \bar{\mathcal{B}}_\delta(y^0(t_0)), & t \in [t_0, t_0+\Delta]. \end{cases}$$

Then, $\bigcup_{t\in[0,t_0+\Delta]} K_t$ is a bounded set and $\bigcup_{t\in[0,t_0+\Delta]} \{t\} \times K_t \subset \mathcal{D}_2$. By property (T.3') of operator $\mathbf{T}$, there exists $c_1 > 0$ such that, for $y \in \mathcal{R}\mathcal{D}_2$ with $\operatorname{graph}(y) \subset \bigcup_{t\in[0,t_0+\Delta]} \{t\} \times K_t$, we have $\|\mathbf{T}(y)(t)\| < c_1$ for all $t \in [t_0, t_0+\Delta]$. Note that, for every right extension $y^e \in \mathcal{R}([0,t_0+s],\mathbb{R}^n)$ of y^0, $s \in [0,\Delta]$, with $y^e([t_0,t_0+s]) \subset \bar{\mathcal{B}}_\delta(y^0(t_0))$, there exists a function $\hat{y}^e \in \mathcal{R}\mathcal{D}_2$ with $\hat{y}^e|_{[0,t_0+s]} = y^e$ and $\operatorname{graph}(\hat{y}^e|_{[0,t_0+\Delta]}) \subset \bigcup_{t\in[0,t_0+\Delta]} \{t\} \times K_t$. Thus,

$$\forall\, t \in [t_0, t_0+s] : \quad \|\mathbf{T}(y^e)(t)\| = \|\mathbf{T}(\hat{y}^e)(t)\| \le \sup_{t\in[t_0,t_0+\Delta]} \|\mathbf{T}(\hat{y}^e)(t)\| < c_1 \tag{7.2}$$

because of the causality property (T.1') of operator $\mathbf{T}$. We will use this observation later. As F is a Carathéodory function, property (C.3) yields the existence of an integrable function $\kappa : [t_0, t_0+\Delta] \to \mathbb{R}_{\ge 0}$ with

$$\forall\, (t,x,z) \in [t_0, t_0+\Delta] \times \bar{\mathcal{B}}_\delta(y^0(t_0)) \times \bar{\mathcal{B}}_{c_1} : \quad \|F(t,x,z)\| \le \kappa(t).$$

Define $\gamma : [0, t_0 + \Delta] \to \mathbb{R}_{\geq 0}$ by

$$\gamma(t) := \begin{cases} 0, & t \in [0, t_0), \\ \int_{t_0}^{t} \kappa(s)\mathrm{d}s, & t \in [t_0, t_0 + \Delta]. \end{cases}$$

There exists $\tau > 0$ such that $\gamma(t_0 + \tau) < \delta$. We define a sequence $(y_n) \in \mathcal{C}([0, t_0 + \Delta], \mathbb{R}^n)^{\mathbb{N}}$ as follows

$$y_n(t) = \begin{cases} y^0(t), & t \in [0, t_0], \\ y^0(t_0), & t \in (t_0, t_0 + \tau/n], \\ y^0(t_0) + \int_{t_0}^{t-\tau/n} F(s, y_n(s), \mathbf{T}(y_n)(s))\mathrm{d}s, & t \in (t_0 + \tau/n, t_0 + \tau]. \end{cases}$$

By construction, y_n is a right extension of y^0 with $y_n(t) \in \bar{\mathcal{B}}_\delta(y^0(t_0))$ for all $t \in [t_0, t_0 + \tau/n]$ and $n \in \mathbb{N}$. Thus, $\|\mathbf{T}(y_n)(t)\| < c_1$ for all $t \in [t_0, t_0 + \tau/n]$ and $n \in \mathbb{N}$ because of the observation made in (7.2). Therefore,

$$\left\|y_n(t) - y^0(t_0)\right\| \leq \int_{t_0}^{t-\tau/n} \|F(s, y_n(s), \mathbf{T}(y_n)(s))\| \,\mathrm{d}s \leq \int_{t_0}^{t-\tau/n} \kappa(s)\mathrm{d}s = \gamma(t - \tau/n) \quad (7.3)$$

for $t \in [t_0 + \tau/n, t_0 + \tau]$ and $n \in \mathbb{N}$. Since $\gamma(t - \tau/n) < \delta$, we infer $y_n(t) \in \bar{\mathcal{B}}_\delta(y^0(t_0))$ and $\|\mathbf{T}(y_n)(t)\| < c_1$ for all $t \in [t_0, t_0 + \tau]$. Thus, $\|F(t, y_n(t), \mathbf{T}(y_n)(t)\| \leq \kappa(t)$ for all $t \in [t_0, t_0 + \tau]$ and all $n \in \mathbb{N}$.

We will prove that the sequence $(y_n)_{n\in\mathbb{N}}$ is equicontinuous. To this end, let $\varepsilon > 0$ be arbitrary but fixed. The function γ is uniformly continuous on the compact interval $[t_0, t_0 + \tau]$. Thus, there exists $\bar{\delta} > 0$ such that

$$|\gamma(t) - \gamma(s)| < \varepsilon$$

for all $t, s \in [t_0, t_0 + \tau]$ with $|t - s| < \bar{\delta}$. Let $n \in \mathbb{N}$ and $t, s \in [t_0, t_0 + \tau]$ with $|t - s| < \bar{\delta}$. We assume $s \leq t$ without loss of generality and consider three cases. First, if $t_0 \leq s \leq t \leq t_0 + \tau/n$, then $y_n(s) = y_n(t) = y^0(t_0)$. Thus, $\|y_n(s) - y_n(t)\| = 0$. Second, if $t_0 \leq s \leq t_0 + \tau/n \leq t$, then

$$\|y_n(t) - y_n(s)\| = \left\|y_n(t) - y^0(t_0)\right\| = \gamma(t - \tau/n) < \varepsilon,$$

where estimate (7.3) was used. Third, if $t_0 + \tau/n \leq s \leq t$, then

$$\|y_n(t) - y_n(s)\| \leq |\gamma(t - \tau/n) - \gamma(s - \tau/n)| < \varepsilon.$$

As $y_n|_{[0,t_0]} = y^0$ for all n, the sequence $(y_n)_{n\in\mathbb{N}}$ is therefore equicontinuous. By the Arzelà-Ascoli theorem, there exists a function $y \in \mathcal{C}([0, t_0 + \tau], \mathbb{R}^n)$ and a subsequence (which we do not relabel) such that y_n is uniformly convergent, i.e. $y_n \to y$. Clearly, $y|_{[0,t_0]} = y^0$ and $y([t_0, t_0 + \tau]) \subset \bar{\mathcal{B}}_\delta(y^0(t_0)) \subset \mathcal{D}_1 \cap \mathcal{D}_2$ since $\bar{\mathcal{B}}_\delta(y^0(t_0))$ is compact and $y_n([t_0, t_0 + \tau]) \subset \bar{\mathcal{B}}_\delta(y^0(t_0))$ for all $n \in \mathbb{N}$.

Since the $\mathbf{T}$ is local Lipschitz continuous, see property (T.2'), $\lim_{n\to\infty} \mathbf{T}(y_n)(t) = \mathbf{T}(y)(t)$ for almost all $t \in [0, t_0 + \tau]$. Thus,

$$\lim_{n\to\infty} F(t, y_n(t), \mathbf{T}(y_n)(t)) = F(t, y(t), \mathbf{T}(y)(t))$$

for almost all $t \in [0, t_0 + \tau]$ since $F(t, \cdot, \cdot)$ is continuous, according to property (T.1'). As $\|F(t, y_n(t), \mathbf{T}(y_n)(t)\| < \kappa(t)$ for all $t \in [t_0, t_0 + \tau]$, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{t_0}^{t} F(s, y_n(s), \mathbf{T}(y_n)(s))\mathrm{d}s = \int_{t_0}^{t} F(s, y(s), \mathbf{T}(y)(s))\mathrm{d}s$$

for all $t \in [t_0, t_0 + \tau]$. Note that

$$\begin{aligned} y_n(t) &= y^0(t_0) + \int_{t_0}^{t-\tau/n} F(s, y_n(s), \mathbf{T}(y_n)(s))\mathrm{d}s \\ &= y^0(t_0) + \int_{t_0}^{t} F(s, y_n(s), \mathbf{T}(y_n)(s))\mathrm{d}s - \int_{t-\tau/n}^{t} F(s, y_n(s), \mathbf{T}(y_n)(s))\mathrm{d}s \end{aligned}$$

for all $t \in (t_0 + \tau/n, t_0 + \tau]$ and $n \in \mathbb{N}$. For the limit $n \to \infty$, we conclude

$$y(t) = \begin{cases} y^0(t), & t \in [0, t_0], \\ y^0(t_0) + \int_{t_0}^{t} F(s, y(s), \mathbf{T}(y)(s))\mathrm{d}s, & t \in [t_0, t_0 + \tau]. \end{cases}$$

Therefore, y is a solution of the initial value problem (7.1) in the sense of Definition 7.0.2 proving assertion (i).

Step 2: We prove assertion (ii). Let $y : [0, \omega) \to \mathbb{R}^n$ be a solution of the initial value problem (7.1). Define the set

$$\mathcal{E} := \left\{ (\hat{\omega}, \zeta) \,\middle|\, \hat{\omega} \geq \omega, \zeta \in \mathcal{C}([0, \hat{\omega}), \mathbb{R}^n) \text{ is a solution of (7.1) with } \zeta|_{[0,\omega)} = y \right\}.$$

This is basically the set of all right extensions of y that are also a solution of (7.1). As $(\omega, y) \in \mathcal{E}$, this set is non-empty. The relation $\preceq$ given by

$$(\omega_1, y_1) \preceq (\omega_2, y_2) \quad \iff \quad \omega_1 \leq \omega_2 \wedge y_1 = y_2|_{[0,\omega_1)}$$

defines a partial order on $\mathcal{E}$. Let Ω be a chain in $\mathcal{E}$, i.e. a totally ordered subset of $\mathcal{E}$. Define $\omega^\star := \sup \{ \hat{\omega} \,|\, (\hat{\omega}, \zeta) \in \Omega \}$. Further, define $y^\star \in \mathcal{C}([0, \omega^\star), \mathbb{R}^n)$ by $y^\star|_{[0,\hat{\omega})} = \zeta$ for $(\hat{\omega}, \zeta) \in \mathcal{P}$. Then, $(\omega^\star, y^\star) \in \Omega$ and $(\hat{\omega}, \zeta) \preceq (\omega^\star, y^\star)$ for all $(\hat{\omega}, \zeta) \in \mathcal{P}$, i.e. $(\omega^\star, y^\star)$ is an upper bound of Ω. Zorn's lemma yields the existence of an maximal element of $\mathcal{E}$. By the construction of $\mathcal{E}$ this is a maximal extension of y that is also a solution. This proves (ii).

Step 3: We prove assertion (iii). Assume that F is locally essentially bounded and let $y \in \mathcal{C}([0, \omega), \mathbb{R}^n)$ be a maximal solution. Seeking a contradiction, suppose that the closure of $\operatorname{graph}(y|_{[t_0,\omega)})$ is a compact subset of $\mathcal{D}_1 \cap \mathcal{D}_2$. This implies, in particular, that $[t_0, \omega)$ is a bounded interval. As y is bounded, property (T.3') implies the boundedness of $\mathbf{T}(y)|_{[t_0,\omega)}$. The local essential boundedness of F yields the existence of $c_2 > 0$ with $\|\dot{y}(t)\| = \|F(t, y(t), \mathbf{T}(y)(t)\| \leq c_2$ for all $t \in [t_0, \omega)$. Hence, y is uniformly continuous on the interval $[t_0, \omega)$. There thus exists a right extension $y^e \in \mathcal{C}([0, \omega], \mathbb{R}^n)$ of y with $\operatorname{graph}(y^e|_{[t_0,\omega]}) \subset \mathcal{D}_1 \cap \mathcal{D}_2$. In particular, $(\omega, y^e(\omega)) \in \mathcal{D}_1 \cap \mathcal{D}_2$. Assertion (i) yields the existence of a solution $\hat{y} : [0, \hat{\omega}) \to \mathbb{R}^n$ with $\hat{\omega} > \omega$ of the initial value problem

$$\begin{aligned} \dot{y}(t) &= F(t, y(t), \mathbf{T}(y)(t)), \\ y|_{[0,\omega]} &= y^e. \end{aligned}$$

As $\hat{y}|_{[0,\omega)} = y$, the function $\hat{y}$ is a proper extension of y and also a solution of the initial value problem (7.1). This contradicts the maximality of y and completes the proof. □

By reducing a higher order system of the form

$$\begin{aligned} y^{(r)}(t) &= F(t, \chi_r(y)(t), \mathbf{T}(\chi_r(y))(t)), \\ y|_{[0,t_0]} &= y^0 \in \mathcal{C}([0, t_0], \mathbb{R}^n), \end{aligned} \tag{7.4}$$

with $r > 1$ to a system of order one, Theorem 7.0.4 can clearly be also applied to such systems. Note that we now identify $\mathcal{C}([0, t_0], \mathbb{R}^n)$ with $\mathbb{R}^{rn}$ in the case $t_0 = 0$. In the spirit of Definition 3.1.4, a solution of the initial value problem (7.4) is an absolutely continuous function $x = (x_1, \ldots, x_r) : [0, \omega) \to \mathbb{R}^{rm}$ with $\omega \in (t_0, \infty]$ fulfilling

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), & i &= 1, \ldots, r-1, \\ \dot{x}_r(t) &= F(t, \chi_r(x)(t), \mathbf{T}(x)(t)), \end{aligned} \tag{7.5}$$

for almost all $t \in [t_0, \omega)$ and $x|_{[0,t_0]} = \chi_r(y^0)$ (resp. $x(t_0) = y^0$ in the case $t_0 = 0$).

Theorem 7.0.4 yields as a straightforward corollary the existence of solutions of the initial value problem (1.1) for the considered system class if a control $u \in L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ is applied.

Corollary 7.0.5. *Consider system (1.1) with $(F, \mathbf{T}) \in \mathcal{N}^{m,r}_{t_0}$ at initial time $t_0 \geq 0$. Let $y^0 \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$ be an initial trajectory and $u \in L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ be a control function. Then,*

(i) the initial value problem (1.1) has a solution $x : [0, \omega) \to \mathbb{R}^{rm}$ in the sense of Definition 3.1.4,

(ii) every solution can be extended to a maximal solution,

(iii) if $x : [0, \omega) \to \mathbb{R}^{rm}$ is a bounded maximal solution, then $\omega = \infty$.

Proof. The assertions follow directly from Theorem 7.0.4 since $\mathbf{T}$ has the properties (T.1'), (T.2'), and (T.3'), and since it is easy to see that the function

$$\tilde{F} : \mathbb{R}_{\geq 0} \times \mathbb{R}^q \to \mathbb{R}^m, \quad (t, z) \mapsto \tilde{F}(t, z) := F(u(t), z),$$

is Carathéodory function. □

The same holds true for class of models $\mathcal{M}^{m,r}_{t_0}$ we considered in this thesis.

Corollary 7.0.6. *Consider model (2.4) with $(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}}) \in \mathcal{M}^{m,r}_{t_0}$ at initial time $t_0 \geq 0$. Let $y^0_{\mathrm{M}} \in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$ be an initial trajectory and $u \in L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^m)$ be a control function. Then,*

(i) the initial value problem (2.12) has a solution $x_{\mathrm{M}} : [0, \omega) \to \mathbb{R}^{rm}$ in the sense of (7.5),

(ii) every solution can be extended to a maximal solution,

(iii) if $x_{\mathrm{M}} : [0, \omega) \to \mathbb{R}^{rm}$ is a bounded maximal solution, then $\omega = \infty$.

Nomenclature

The following notation is used in this thesis to refer to commonly known mathematical concepts

$\mathbb{N}$	set of positive integers
$\mathbb{N}_0$	$:= \mathbb{N} \cup \{0\}$, set of non-negative integers
$\mathbb{Z}$	ring of integers
$X^{\mathbb{N}}$	set of sequences with elements in a set X
$\mathbb{R}$	field of real numbers
$\mathbb{R}_{\geq 0}$	$:= [0, \infty)$, set of non-negative real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{C}_{\geq 0 (>0)}$	$:= \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \ (>0)\}$, the complex (open) right half-plane
$\mathbb{C}_{\leq 0 (<0)}$	$:= \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0 \ (<0)\}$, the complex (open) left half-plane
$\mathbb{R}[s]$	ring of polynomials with coefficients in $\mathbb{R}$ and indeterminate s
$\mathbb{R}^n$	vector space of real-valued ordered n-tuples with $n \in \mathbb{N}$
$\langle \cdot, \cdot \rangle$	Euclidean scalar product in $\mathbb{R}^n$
$\lVert \cdot \rVert$	Euclidean norm in $\mathbb{R}^n$
$\mathcal{B}_\varepsilon(x^0)$	$:= \{x \in \mathbb{R}^n \mid \lVert x - x^0 \rVert < \varepsilon\}$, open ball around $x^0 \in \mathbb{R}^n$ with radius $\varepsilon > 0$
$\bar{\mathcal{B}}_\varepsilon(x^0)$	$:= \{x \in \mathbb{R}^n \mid \lVert x - x^0 \rVert \leq \varepsilon\}$, closed ball around $x^0 \in \mathbb{R}^n$ with radius $\varepsilon > 0$
$\mathcal{B}_\varepsilon$	$:= \mathcal{B}_\varepsilon(0)$, open ball around the origin with radius $\varepsilon > 0$
$\bar{\mathcal{B}}_\varepsilon$	$:= \bar{\mathcal{B}}_\varepsilon(0)$, closed ball around the origin with radius $\varepsilon > 0$
$\mathbb{R}^{n \times m}$	vector space of real-valued $n \times m$ matrices with $n, m \in \mathbb{N}$
I_n	the identity matrix in $\mathbb{R}^{n \times n}$
$\lVert A \rVert$	$:= \sup_{\lVert x \rVert = 1} \lVert Ax \rVert$, induced operator norm for $A \in \mathbb{R}^{n \times m}$
$\mathrm{S}_n(\mathbb{R})$	set of symmetric $\mathbb{R}^{n \times n}$-matrices
$\mathrm{S}_n^{--}(\mathbb{R})$	set of symmetric negative definite $\mathbb{R}^{n \times n}$-matrices
$\mathrm{GL}_n(\mathbb{R})$	group of invertible $\mathbb{R}^{n \times n}$ matrices
$\det(A)$	determinant of a square matrix $A \in \mathbb{R}^{n \times n}$

$\mathrm{spec}(A)$	spectrum of a square matrix $A \in \mathbb{R}^{n\times n}$ (set of a eigenvalues of A)
$\lambda_{\max}(A)$	largest eigenvalue of a symmetric matrix $A \in \mathrm{S}_n$
$\lambda_{\min}(A)$	smallest eigenvalue of a symmetric matrix $A \in \mathrm{S}_n$
$\lfloor x \rfloor$	$:= \max\{n \in \mathbb{Z} \mid n \leq x\}$, the floor function for real numbers $x \in \mathbb{R}$
$\circ$	composition of functions
$\mathrm{graph}(f)$	$:= \{(x, f(x)) \in X \times Y \mid x \in X\}$, the graph of a function $f : X \to Y$
$f\vert_A$	restriction of the function $f : X \to Y$ to the subset A of the set X
$\mathbb{1}_A$	indicator function $\mathbb{1}_A : X \to \{0, 1\}$ of a subset A of a set X
$\lambda(\cdot)$	Lebesgue measure
f^+	$:= \max\{f, 0\}$, positive part of a function $f : X \to \mathbb{R}$
f^-	$:= \max\{-f, 0\}$, negative part of a function $f : X \to \mathbb{R}$
$\mathcal{T}_{\mathcal{P}}(I, \mathbb{R}^n)$	space of step functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$ with partition $\mathcal{P}$, see Definition 5.0.1
$\mathcal{R}(I, \mathbb{R}^n)$	space of regulated functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$, see Definition 1.1.5
$\mathrm{Lip}(V, \mathbb{R}^n)$	space of Lipschitz continuous functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$
$\mathrm{Lip}_{\mathrm{loc}}(V, \mathbb{R}^n)$	space of locally Lipschitz continuous functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$
$\mathcal{C}^p(V, \mathbb{R}^n)$	linear space of p-times continuously differentiable functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$ and $p \in \mathbb{N}_0 \cup \{\infty\}$
$\mathcal{C}(V, \mathbb{R}^n)$	$:= \mathcal{C}^0(V, \mathbb{R}^n)$, linear space of continuous functions $f : V \to \mathbb{R}^n$, $V \subset \mathbb{R}^m$
$L^p(I, \mathbb{R}^n)$	space of measurable p-integrable functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$ with norm $\lVert\cdot\rVert_{L^p}$ and $p \in \mathbb{N}$
$\langle \cdot, \cdot \rangle_{L^2}$	scalar product in $L^2(I, \mathbb{R}^n)$
$x_k \rightharpoonup x^\star$	weak convergence of $(x_k) \in L^2(I, \mathbb{R}^n)^{\mathbb{N}}$ to $x^\star \in L^2(I, \mathbb{R}^n)$
$L^\infty(I, \mathbb{R}^n)$	space of measurable essentially bounded functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$
$\operatorname*{ess\,sup}_{t\in I} \lVert f(t) \rVert$	essential supremum of a measurable function $f : I \to \mathbb{R}^n$
$\lVert f \rVert_\infty$	$:= \operatorname{ess\,sup}_{t\in I} \lVert f(t) \rVert$, norm of $f \in L^\infty(I, \mathbb{R}^n)$
$L^\infty_{\mathrm{loc}}(I, \mathbb{R}^n)$	space of measurable locally bounded functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$
$W^{k,\infty}(I, \mathbb{R}^n)$	Sobolev space of all k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ over an interval $I \subset \mathbb{R}$ such that $f, \dots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)$

Notation

Unless explicitly stated otherwise, the following symbols introduced and used in this work always have the meaning given below.

t_0	≥ 0, initial time
$\hat{t}$	$\geq t_0$, arbitrary time instant
$\mathcal{N}^{m,r}_{t_0}$	system class with $m, r \in \mathbb{N}$, see Definition 3.1.1
$(F, \mathbf{T})$	$\in \mathcal{N}^{m,r}_{t_0}$
y	output of system (1.1)
y^0	$\in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$, initial system trajectory
$\mathcal{T}^{n,q}_{t_0}$	operator class as defined in Definition 2.2.1, with $n, q \in \mathbb{N}$
τ	≥ 0, memory limit of operator $\mathbf{T}_{\mathrm{M}} \in \mathcal{T}^{n,q}_{t_0}$
$\mathcal{M}^{m,r}_{t_0}$	model class with $m, r \in \mathbb{N}$, see Definition 2.2.2
$\mathcal{M}^{m,r}_{t_0,u_{\max},\bar{\rho}}$	restricted model class with $\bar{\rho}, u_{\max} \geq 0$, see Definition 4.1.1
$(f_{\mathrm{M}}, g_{\mathrm{M}}, \mathbf{T}_{\mathrm{M}})$	$\in \mathcal{M}^{m,r}_{t_0}$, model used in the MPC algorithm
x_{M}	solution of the model differential equation (2.4), see Definition 2.2.6
y_{M}	output of model (2.4)
y^0_{M}	$\in \mathcal{C}^{r-1}([0, t_0], \mathbb{R}^m)$, initial model trajectory
y_{ref}	$\in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, reference trajectory
e	$:= y - y_{\mathrm{ref}}$, tracking error
e_{M}	$:= y_{\mathrm{M}} - y_{\mathrm{ref}}$, model's tracking error (1.10)
e_{S}	$:= y - y_{\mathrm{M}}$, model-system output mismatch (1.11)
$\mathcal{G}$	$:= \left\{ \psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \,\middle|\, \inf_{t \geq 0} \psi(t) > 0 \right\}$, set of admissible funnel functions as defined in (1.3)
ψ	$\in \mathcal{G}$, funnel boundary function
α, β	> 0, constants associated to $\psi \in \mathcal{G}$ satisfying (2.21)
$\mathcal{F}_\psi$	$:= \{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \lVert e \rVert < \psi(t) \}$, funnel for $\psi \in \mathcal{G}$, defined in (1.2)
$\bar{\mathcal{F}}_\psi$	$:= \{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \lVert e \rVert \leq \psi(t) \}$, for $\psi \in \mathcal{G}$

δ	> 0, time-shift of the MPC algorithm
T	$\geq \delta > 0$, prediction horizon of the MPC algorithm
$(t_k)_{k\in\mathbb{N}_0}$	time sequence defined by $t_k := t_0 + k\delta$ for $k \in \mathbb{N}_0$
$u_{\max}$	≥ 0, input saturation level of the MPC algorithm
$u_{\mathrm{FMPC},k}$	$\in L^\infty([t_k, t_k+T], \mathbb{R}^m)$, solution of the optimal control problem (2.37)
x_{M}^k	solution of the model differential equation on the interval $[t_k, t_{k+1}]$
y_{M}^k	predicted output of model (2.4) on the interval $[t_k, t_{k+1}]$
$u_{\mathrm{FMPC}}(t)$	$= u_{\mathrm{FMPC},k}(t)$ for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_0$
ν_ψ	funnel penalty function for $\psi \in \mathcal{G}$, see Definition 2.1.6
ℓ_ψ	funnel stage cost function for $\psi \in \mathcal{G}$, see Definition 2.1.11
$J_T^\Psi(\cdot;\hat{t},\hat{\mathfrak{X}})$	$L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m) \to \mathbb{R}\cup\{\infty\}$, cost functional as defined in (2.33)
$I_{t_0}^{\hat{t},\tau}$	$:= [\hat{t}-\tau,\hat{t}] \cap [t_0,\hat{t}]$ for $\hat{t} \geq t_0 \geq 0$ and $\tau \geq 0$
χ_r	shorthand notation for $\chi_r(\zeta)(t) := (\zeta(t), \dot{\zeta}(t), \ldots, \zeta^{(r-1)}(t)) \in \mathbb{R}^{rm}$ for a function $\zeta \in W^{r,\infty}(I,\mathbb{R}^m)$ as defined in (2.17)
S_m	left shift operator as defined in (2.16)
ξ_i	auxiliary error variable as defined in (2.15) for $i = 1,\ldots,r$
Ψ	$:= (\psi_1,\ldots,\psi_r) \in \mathscr{G}$, auxiliary funnel functions, see (2.25)
k_i	≥ 0, parameter associated to ψ_i and ξ_i for $i = 1,\ldots,r$
$\mathcal{D}_t^\Psi$	$:= \{ z \in \mathbb{R}^{rm} \mid \|\xi_i(z)\| < \psi_i(t),\ i = 1,\ldots,r \}$ for $t \in \mathbb{R}_{\geq 0}$ and $\Psi \in \mathcal{G}^r$ as defined in (2.19)
$\mathcal{Y}_{\hat{t}}^\Psi$	$:= \left\{ \zeta \in \mathcal{R}(\mathbb{R}_{\geq 0},\mathbb{R}^{rm}) \,\middle|\, \forall\, t \in [0,\hat{t}] : \zeta(t) - \chi_r(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^\Psi \right\}$ for $\hat{t} \geq t_0$ and $\Psi \in \mathscr{G}$, defined in (2.26)
$\mathfrak{I}_{t_0,\tau}^\Psi(\hat{t})$	set of feasible initial values at time $\hat{t} \geq t_0$ for $\Psi \in \mathscr{G}$ and $\tau \geq 0$, see Definition 2.3.14
$\mathfrak{P}\mathfrak{I}_{t_0,\tau}^{\Psi,\varepsilon,\lambda}(\hat{t},\hat{x})$	proper initial model state for $\varepsilon,\lambda \in (0,1)$ at time $\hat{t} \geq t_0$ given system data $\hat{x} \in \mathbb{R}^{rm}$, see Definition 3.2.5
$\hat{\mathfrak{X}}$	$:= (\hat{x}_{\mathrm{M}}, \hat{\mathbf{T}}_{\mathrm{M}}) \in \mathfrak{I}_{t_0,\tau}^\Psi(\hat{t})$, model initial value at time $\hat{t} \geq t_0$
$\mathfrak{X}_k$	$:= (x_{\mathrm{M}}^k, \mathbf{T}_{\mathrm{M}}^k) \in \mathfrak{I}_{t_0,\tau}^\Psi(t_k)$, model initial state at time t_k
κ	initialisation strategy for the model, see Definition 2.3.17
$\mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}})$	set of feasible controls on the interval $[\hat{t},\hat{t}+T]$ as defined in (2.28)
$\mathcal{U}_{[\hat{t},\hat{t}+T]}^{\mathcal{P}}(u_{\max},\hat{\mathfrak{X}})$	$:= \mathcal{T}_{\mathcal{P}}([\hat{t},\hat{t}+T],\mathbb{R}^m) \cap \mathcal{U}_{[\hat{t},\hat{t}+T]}(u_{\max},\hat{\mathfrak{X}})$, see (5.9)

$f_{\mathrm{M}}^{\max}, g_{\mathrm{M}}^{\max}, g_{\mathrm{M}}^{-1\max}, g^{\min}$	≥ 0, bounds on the dynamics of the model, see Lemma 2.3.20
$f^{\max}, g^{\max}, g^{\min}$	≥ 0, bounds on the dynamics of the system, see Lemma 5.1.2
$\mathcal{N}$	$\in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$, surjection used in the funnel controller
γ	$\in \mathcal{C}^1([0,1),[1,\infty))$, bijection used in the funnel controller
$e_i(\varphi, z)$	auxiliary error variable for $\varphi > 0$, $z \in \mathbb{R}^{rm}$, $i = 1, \dots, r$, see (3.3)
$\mathcal{E}_i^{\varepsilon}(\varphi)$	$:= \left\{ z \in \mathbb{R}^{rm} \,\middle|\, \|e_j(\varphi, z)\| < \varepsilon,\ j = 1, \dots, i \right\}$ for $i \in \{1, \dots, r\}$ and $\varepsilon \in (0,1]$, see (3.3)
$\mathfrak{Y}_{\hat{t}}^{\varphi}$	$:= \left\{ \zeta \in \mathcal{C}^{r-1}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \,\middle|\, \zeta\|_{[0,t_0]} = y^0, \forall\, t \in [t_0, \hat{t}) : \chi_r(\zeta)(t) \in \mathcal{E}_r^1(\varphi(t)) \right\}$, see (3.4)
$\varphi(t)$	$:= \frac{1}{\psi(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\|}$, boundary for funnel controller component based on model prediction, see (3.10)
φ_k	$[t_k, t_{k+1}] \to \mathbb{R}_{\geq 0}$, adaptive funnel based on model prediction, see (3.14)
$u_{\mathrm{FC},k}(t)$	$:= (\mathcal{N} \circ \gamma)(\|e_r(\varphi_k(t), e_{\mathrm{S}}(t))\|^2) e_r(\varphi_k(t), e_{\mathrm{S}}(t))$, control law of the funnel controller component on the interval $[t_k, t_{k+1})$, see (3.15)
$u_{\mathrm{FC}}(t)$	$= u_{\mathrm{FC},k}(t)$ for $t \in [t_k, t_{k+1})$ for $k \in \mathbb{N}_0$
$\mathfrak{a}$	$: [0,1] \to [0, \mathfrak{a}^+]$ with $\mathfrak{a}^+ > 0$, activation function, see Section 3.2.1
$\mathfrak{S}_{\hat{t}}$	$:= \mathcal{C}^r([t_0, \hat{t}], \mathbb{R}^m) \times \mathcal{R}([t_0, \hat{t}], \mathbb{R}^m)^r \times L^\infty([t_0, \hat{t}], \mathbb{R}^m) \times L^\infty([t_0, \hat{t}], \mathbb{R}^m)$ for $\hat{t} \geq t_0$, see (4.1)
$\mathcal{L}$	$\bigcup_{t \geq t_0} \mathfrak{S}_t \to \mathcal{M}_{t_0, u_{\max}, \bar{\rho}}^{m,r}$, feasible learning scheme, see Definition 4.1.4
$\mathfrak{N}_{t_0}^{m,r}$	model/system class considered in Chapter 5 for the sampled-data robust funnel MPC
u_{ZoH}	funnel controller with zero-order-hold, defined in (5.2)
ν	≥ 0, input gain of u_{ZoH}
ι	≥ 0, activation threshold of u_{ZoH}
$\mathfrak{r}$	> 0, sampling time of u_{ZoH} and the sampled-data funnel MPC Algorithm 5.2.1

Bibliography

All scientific articles written in the context of this dissertation are listed below in the section Publications and Preprints. All cited publications can be found thereafter in References.

Publications

In connection with the work on this dissertation, the following articles were published in international scientific journals.

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