Analysis of taxis processes in markedly irregular frameworks

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Researched and written at the Institute of Mathematics of Paderborn University under the supervision of

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and submitted to the PhD board for mathematics of the Faculty of Computer Science, Electrical Engineering and Mathematics of Paderborn University in September 2025 to attain the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.).

Submitted:	8 September 2025
Defended:	12 December 2025

Abstract

This thesis is concerned with systems of partial differential equations from mathematical biology that model the directed movement of microscopic cells along the gradient of a chemical substance in numerous and varied settings. All modeling frameworks considered in this thesis trace their lineage back to the seminal work by Keller and Segel studying the chemotactic behavior of *dictyostelium discoideum* slime mold but present additional mathematical challenges due to irregularities introduced by some of the modified dynamics in addition to the intricacies already inherent to the original Keller–Segel system. In this context, we are then interested in exploring whether said systems admit global (classical or potentially only weaker) solutions and, if so, what qualitative properties said solutions exhibit.

The first framework we study in this regard is a chemotaxis-Navier-Stokes model with potentially rotational flux in two-dimensional domains with no-flux boundary conditions, which was first introduced by Tuval et al. in response to experimental observations about the interaction between organisms and their surrounding liquid. In this setting, we show that it is possible to construct certain generalized solutions to said system, which eventually become classical and stabilize to their expected steady states. As a crucial ingredient to the derivation of these long-time behavior properties without needing undue additional assumptions, we derive two optimal functional inequalities based on the Trudinger–Moser inequality via variational methods.

The second framework considered in this thesis is a haptotaxis model in bounded domains of two or three dimensions that represents the invasive movement of cancer cells into healthy tissue while also undergoing a non-Fickian myopic diffusion process. Under a global positivity assumption on the diffusion and taxis tensor, we first show that classical solution exist for the system. Under much more relaxed assumption on the diffusion and taxis tensor encompassing cases of spatial degeneracy on small sets, we then show that it is still possible to construct weak solutions to the same model.

Finally for our third framework, we look at two well-known variations on the original Keller–Segel system in bounded domains of two or higher dimensions but with highly irregular initial data. More specifically, we consider the attractive-repulsive Keller–Segel system as well as the chemotaxis-consumption system. In these settings, we show that even if these systems start in such a highly irregular state, e.g. a Dirac measure, classical solutions to the systems can still be constructed and in fact become immediately smooth as opposed to the persistence of singularities observed in some simplified versions of the original Keller–Segel system.

Zusammenfassung

Diese Arbeit beschäftigt sich mit System partieller Differentialgleichungen aus der mathematischen Biologie, welche eingesetzt werden, um das gerichtete Bewegungsverhalten mikroskopischer Zellen entlang von Stoffkonzentrationsgradienten in vielzähligen und unterschiedlichen Situationen zu modellieren. Dabei können zwar alle von uns betrachteten Modelle ihre Abstammung auf das viel untersuchte, klassische Modell von Keller und Segel für die Analyse des chemotaktischen Aggregationsverhalten von *Dictyostelium discoideum* Schleimpilzen zurückführen, aber stellen dennoch gänzlich neue mathematische Herausforderungen aufgrund neuer oder veränderter Dynamiken und der zugehörigen potentiellen Irregularitäten dar. In dieser Arbeit liegt unseren Fokus darauf zu bestimmen, ob es trotzdem möglich ist für besagte Modelle globale (klassische oder potentiell nur schwächere) Lösungen zu konstruieren und, wenn ja, zu untersuchen, welche qualitativen Eigenschaften diese Lösungen haben.

Das erste Modell, das wir in dieser Hinsicht betrachten, ist ein Chemotaxis-Navier-Stokes-System mit potentiell nichtdiagonalen Flussanteilen in zweidimensionalen, beschränkten Gebieten. Systeme dieser Art wurde von Tuval et al. unter anderem als Antwort auf neue experimentelle Beobachtungen zu der Interaktion von Organismen mit der umgebenden Flüssigkeit vorgeschlagen. In diesem Kontext zeigen wir, dass es möglich ist verallgemeinerte Lösungen für das Modell zu konstruieren, die nach einer potentiellen Wartezeit klassisch werden und sich auf langen Zeitskalen ihrem erwarteten Equilibriumszustand annähern. Eine kritische Zutat zu unserer Herleitung dieser qualitativen Eigenschaften ohne Notwendigkeit für weiterer Annahmen an die Systemparameter sind zwei optimale Funktionalungleichungen auf Basis der Trudinger–Moser Ungleichung, die wir mit Methoden der Variationsrechnung nachweisen.

Als das zweite Modell dieser Arbeit betrachten wir ein Haptotaxis-System in zwei- und dreidimensionalen, beschränkten Gebieten, welches aus der Modellierung des invasiven Verhalten von Krebszellen in gesundes Gewebe bei gleichzeitiger nicht-Fickscher, kurzsichtiger Diffusion hervorgegangen ist. Unter einer globalen Positivätsannahme an den Diffusions- und Taxistensor zeigen wir zunächst, dass globale, klassische Lösungen für das System existieren. Wir schwächen dann unsere Annahmen an besagten Tensor deutlich ab, so dass auch Fälle von lokaler, räumlicher Degeneriertheit auf kleinen Mengen behandelt werden können, und zeigen dann, dass es trotzdem noch möglich ist schwache Lösungen für das gleiche System zu konstruieren.

Zuletzt betrachten wir zwei klassische Varianten des originalen Keller–Segel Systems in zwei und höherdimensionalen, beschränkten Gebieten: Das attraktiv-repulsive Keller–Segel System und das Chemotaxis-Verbrauch-System. Im Gegensatz zum Löwenanteil bisheriger Untersuchungen dieser Modelle versehen wir beide Systeme allerdings mit potentiell stark irregulären Anfangsdaten wie z.B. einem Dirac-Maß. Wir zeigen dann, dass es trotzdem möglich ist klassische Lösungen zu konstruieren, welche sogar unmittelbar glatt werden. Dies steht in direktem Kontrast zur früheren Resultaten, dass chemotaktischer Kollaps dieser Art in bestimmten Vereinfachungen des originalen Keller–Segel Systems persistent sein kann.

Original publications

The results and corresponding arguments outlined in this thesis have previously been published as [43], [44], [45], [46], [47] and [48] in various peer-reviewed journals. While we have rearranged, extended and modified the aforementioned manuscripts for a more integrated presentation and reading experience, this thesis naturally still shares large passages of text with said publications and any direct quotations from this set of articles will not be explicitly pointed out during the course of this thesis.

Our introductory first chapter naturally compiles relevant information from all of these papers but the remaining chapters can be fairly directly mapped to their corresponding publications in the following way:

Chapter 2 and 3:

[43] Heihoff, F.: *Global mass-preserving solutions for a two-dimensional chemotaxis system with rotational flux components coupled with a full Navier–Stokes equation.* Discrete & Continuous Dynamical Systems - B, 25(12):4703–4719, 2020. doi:10.3934/dcdsb.2020120.

[47] Heihoff, F.: *Two New Functional Inequalities and Their Application to the Eventual Smoothness of Solutions to a Chemotaxis-Navier–Stokes System with Rotational Flux.* SIAM Journal on Mathematical Analysis, 55(6):7113–7154, 2023. doi:10.1137/22m1531178.

Chapter 4:

[46] Heihoff, F.: *Global solutions to a haptotaxis system with a potentially degenerate diffusion tensor in two and three dimensions.* Nonlinearity, 36(2):1245–1278, 2023. doi:10.1088/1361-6544/acadcb.

Chapter 5:

[44] Heihoff, F.: *On the Existence of Global Smooth Solutions to the Parabolic–Elliptic Keller–Segel System with Irregular Initial Data.* Journal of Dynamics and Differential Equations, 35(2):1693–1717, 2021. doi:10.1007/s10884-021-09950-y.

[45] Heihoff, F.: *Does strong repulsion lead to smooth solutions in a repulsion-attraction chemotaxis system even when starting with highly irregular initial data?* Discrete and Continuous Dynamical Systems - B, 28(10):5216–5243, 2023. doi:10.3934/dcdsb.2022245.

Chapter 6:

[48] Heihoff, F.: *Can a chemotaxis-consumption system recover from a measure-type aggregation state in arbitrary dimension?* Proceedings of the American Mathematical Society, 152(12):5229–5247, 2024. doi:10.1090/proc/16988.

Acknowledgments

First and foremost, I would like to express my earnest gratitude to my supervisor, Michael Winkler, for giving me the opportunity to spend the last few years working on not only the research presented in this thesis but many related topics as well. His steady support, patience and valuable advice made the sometimes frustrating process of pushing at the boundary between the known and the unknown genuinely enjoyable. Fostering an atmosphere promoting the free sharing of ideas and open collaboration by example, he has shaped a working group that it is truly a joy to be a part of. In this vein, I would be remiss to not acknowledge the many great colleagues and friends I have worked with and enjoyed the company of over the years. I would like to especially show my appreciation here for Tobias Black, Mario Fuest and Johannes Lankeit. Their many insightful comments and our many in-depth discussions have been a crucial boon in elevating my research in general and this thesis in particular.

I also want to express my sincere gratitude for being regularly invited as a speaker to the many recent incarnations of the *International Workshop on Mathematical Analysis of Chemotaxis* by the organizers Silvia Frassu, Mario Fuest, Sachiko Ichida, Johannes Lankeit, Monica Marras, Masaaki Mizukami, Guiseppe Viglialoro and especially Michael Winkler and Tomomi Yokota as the original creators of the workshop series. Attending these events not only afforded me the opportunity to present my findings to a broad audience but also allowed me to meet many new friends and colleagues as well as participate in many fascinating discussions in often interesting locales. Through our shared enthusiasm for cross-diffusion problems, the workshops further served as an invaluable source of inspiration and motivation. For all of this, I am truly grateful.

As it has faithfully accompanied me since I started my bachelor's studies in mathematics all those years ago and as it still is very much my academic home, I want to express my sincere appreciation for Paderborn University. Over the years, the university has provided me not only with the opportunity to study many facets of mathematics under some great educators but also facilities and funding that supported me in exploring the topics in this thesis and beyond.

I further want to thank the Deutsche Forschungsgemeinschaft for supporting me through their funding of the research projects *Emergence of structures and advantages in cross-diffusion systems* (411007140) and *Fine structures in interpolation inequalities and application to parabolic problems* (462888149) without which this research could not have been conducted.

Moreover, I would like to thank the anonymous reviewers as well as journal editors for their valuable feedback and support during the publishing process of many of the results making up this thesis.

Last but by no means least, I want to thank my family and friends for always providing much appreciated support while kindly indulging my various eccentricities not only during the research period that ultimately culminated in this document but uncountable

times beforehand as well. Especially thinking back on the trials endured by most of us during the recent pandemic but also just the general ups and downs of research and life, I am extremely grateful for the necessary energy, optimism and confidence this support has provided me with, enabling me to push forward in research as well as life in general at my own pace.

Contents

1 Introduction

1.1 On the modeling and dynamics of chemotaxis

Both in the realm of the microscopic as well as the macroscopic, the behavior of many biological entities is shaped in key ways by the intricacies of their movement mechanisms. It thus stands to reason that better understanding said mechanisms should in turn yield deeper understanding of the organisms themselves. Especially in the domain of individual cells, migration along gradients of a diffusive or nondiffusive substance, commonly called chemotactic or haptotactic movement, respectively, is of some prominence in many biological processes. This can range from bacteria being simply attracted to a food or fuel source to complex self-organization patterns facilitated by the production of a signal chemical. Spurred on by the seminal research papers due to Keller and Segel (cf. [57], [58]) investigating exactly this kind of self-organized aggregation behavior of the *dictyostelium discoideum* slime mold by way of modeling the organism's dynamics as a coupled system of partial differential equations, recent decades have seen ever increasing interest in this type of modeling technique for various systems involving similar migration patterns.

The popularity of this kind of approach to understanding certain biological processes as well as its mathematical allure can naturally already be illustrated well using the original Keller–Segel system, which (slightly simplified) reads as

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n \nabla c), \\ c_t = \Delta c - c + n. \end{cases} \tag{KS}$$

It is generally posed in either the whole space or a bounded domain with additional no-flux boundary conditions, which is the setting we will focus on in this thesis. In the above system, the function n represents the population of the slime mold and the function c the concentration of the chemoattractant at a specific location x and time t. Both the slime mold population as well as the chemical are modeled to be diffusive substances using the Laplacian terms in either equation while the attractive chemotaxis interaction is represented by the cross diffusion term $-\nabla \cdot (n \nabla c)$. The second equation further features the production term $+n$ and decay term $-c$ modeling that the attractant is volatile and decays over time but is produced by the slime mold to facilitate self-organization.

While the number of explicitly modeled interactions present in the system is fairly small as we have just seen, it has nonetheless been shown that model solutions exhibit rather subtle dynamics due to the interplay between the stabilizing effects of diffusion and the aggregation promoting effects of the cross-diffusion term. More importantly, the behavior of solutions to the classic Keller–Segel system seem to match experimental observations, that is to say some solutions to the above system grow arbitrarily large at certain points in the considered domain in finite time while mass is always preserved, which can be readily interpreted as aggregation. This kind of solution behavior is typically called finite-time blowup. In the case of the Keller–Segel system whether blowup occurs can depend on various factors. In fact in one-dimensional domains, classical solutions to the Keller–Segel system are always global and bounded (cf. [92]) while in three or higher dimensional domains there e.g. exists a dense set of radially symmetric initial states that lead to finite-time blowup (cf. [126]). In two-dimensional domains, the blowup behavior becomes much more subtle. If the initial state of n has a mass less than 4π (or 8π for radially symmetric initial data), globally bounded classical solutions to (KS) always exist (cf. [89]). Conversely for almost any initial mass greater than the aforementioned critical value 4π, unbounded solutions to (KS) have been successfully constructed (cf. [51]), which in the radially symmetric setting have even been shown to blow up in finite time (cf. [50], [81], [82]). In the parabolic-elliptic version of the Keller–Segel system, this critical mass phenomenon in two-dimensional domains has been even more conclusively proven (cf. [85], [86]). Strikingly, it has further been shown that certain blowup solutions in fact approach Dirac-type singularities (cf. [88]), which in some simplified settings can even be shown to persist past blowup time (cf. [10]). This naturally gives even more credence to the aggregation interpretation.

As this struggle of stabilization versus aggregation is fundamental to the rich dynamics of the Keller–Segel system, it should come as no surprise that understanding how this battle plays out in similar scenarios featuring cross-diffusion has been an exciting area of recent study. In this thesis, we will explore how aspects of this contest between the existence of bounded global classical solutions and blowup play out in various closely related but markedly irregular frameworks featuring added or modified dynamics compared to the classic Keller–Segel system.

1.2 Framework 1: A chemotaxis-fluid model with non-rotational flux

The first such cross-diffusion framework we will consider here is the chemotaxis-fluid model

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\nabla c), \\ c_t + u\cdot\nabla c = \Delta c - f(c)n, \\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0. \end{cases} \tag{CF}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^2$ with no-flux boundary conditions and appropriate parameter functions $S : \Omega\times[0,\infty)^2 \to \mathbb{R}^{2\times 2}$, $f : [0,\infty) \to [0,\infty)$ and $\phi : \Omega \to \mathbb{R}$.

The introduction of models of this type by Tuval et al. in [115] was motivated by the experimental observation due to Dombrowski et al. that a population of *Bacillus subtilis* generate speeds of fluid movement after aggregation that seemed to be insufficiently explained by considering each cell in isolation (cf. [28]). This challenges the up to that point standing assumption that fluid-cell interaction can be disregarded because each cell has only negligible influence on the fluid. Thus one of the key changes to the original Keller–Segel system here is the addition of a full Navier–Stokes fluid model, which is coupled to the chemotaxis system via the buoyancy term $n\nabla\phi$ and the convection terms $u\cdot\nabla n$ as well as $u\cdot\nabla c$. Apart from this, the second equation features a consumption term $-f(n)c$ instead of the original production and decay terms $-c+n$ as the chemical in this model is supposed to be a food or fuel source as opposed to a method of self-organization for the organisms. Lastly, the model also features a general sensitivity function S as part of the taxis term, which allows it to represent a much broader spectrum of taxis scenarios by choosing specific sensitivity functions. This can range from the classic Keller–Segel taxis interaction, over a taxis interaction incorporating certain volume-filling effects to even fully matrix-valued sensitivity functions with potentially rotational flux. Due to analytical findings of long-time homogenization in settings with scalar-valued sensitivities (cf. e.g. [127]), which conflict with experimental observations of more intricate structure formation (cf. [28]), the matrix-valued case has seen increased interest from a modeling standpoint in recent years. More specifically as experimental settings suggest that new structural patterns originate near the boundary of the observed domain (cf. [28]), modern modeling approaches often introduce rotational flux components near said boundary, leading to a sensitivity function S of roughly the form

$$S = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } a > 0,\ b \in \mathbb{R}$$

with significant non-diagonal entries near the boundary (cf. [142], [143]).

Challenges of the model. The added dynamics described above pose some significant new challenges when establishing solution theory for (CF) as well as when exploring long-time behavior of any thus obtained solutions compared to the original Keller–Segel model. These complications centrally stem from allowing our sensitivity function S to be matrix-valued and fairly general as well as considering a full Navier–Stokes fluid model as opposed to the easier to handle Stokes model or no added fluid model at all. To elaborate on these unique difficulties, let us now remind ourselves of some prior discoveries in related settings to establish proper context.

For scalar sensitivities S, there exist many results about similar systems to (CF), with or without fluid interaction, concerning global existence (cf. [20], [29], [125], [147]) and long-time behavior (cf. [127], [131]) in two-dimensional settings due to some very convenient energy inequalities. In three-dimensional domains, there are generally only less ambitious existence results available, likely due to the problematic Navier–Stokes equation (cf. [125], [130], [131]). In the matrix-valued case, the aforementioned energy inequalities are no longer available. This makes analysis of especially the first equation in (CF) highly difficult. Therefore to our knowledge, prior work concerning non-scalar sensitivities has either hinged on some strong assumptions about S or the initial data (cf. [16], [18], [120], [121], [122]), on adding sufficiently strong nonlinear diffusion to the first equation (cf. [128]) or only constructing generalized solutions (cf. [132]). Even in the fluid-free version of (CF) without imposing any strong assumptions on S, global smooth solutions in the two-dimensional case seem to have thus far only been constructed under significant smallness conditions for c_0 (cf. [67]) and, if we allow for general initial data and space dimension, only global generalized solutions (similar to those in Definition 3.3.1) seem to be available (cf. [129]).

If we remove the nonlinear convection term and simplify the fluid model to a Stokes equation in (CF), existence and eventual smoothness of generalized solutions can be shown (cf. [134]). Sadly the methods used to establish this seem to not translate to the full Navier–Stokes case.

Our results. Overcoming these challenges, we show that there exist generalized solutions (cf. Definition 3.3.1) and that such generalized solutions in fact eventually become smooth and stabilize as the food source gets depleted. This is done under fairly mild assumptions on both the initial data as well as the system parameters f, S and ϕ. The precise theorem as well as the full proofs are laid out in Chapter 3.

New functional equalities. As a key ingredient to the proof of this result, we will spend Chapter 2 on the derivation of optimal versions (along one axis of optimization) of two functional inequalities stemming from the well-known Trudinger–Moser inequality (cf. [21], [84], [114]). More precisely we will show that there exists $\beta_0 > 0$ such that

$$\int_G \varphi(\psi - \overline{\psi}) \le \frac{1}{a} \int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) + \frac{a}{4\beta_0} \left\{ \int_G \psi \right\} \int_G |\nabla\varphi|^2 \tag{F1}$$

and

$$\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) \le \frac{1}{\beta_0} \left\{ \int_G \psi \right\} \int_G |\nabla \ln(\psi)|^2 \tag{F2}$$

hold for all $a > 0$ and sufficiently regular functions φ, ψ in two-dimensional domains of appropriate regularity. While similar inequalities have found previous application in chemotaxis settings (cf. [89], [135]), prior versions still featured an additive integral term. In our arguments, this would have lead to an additional small-mass condition on n_0 for our eventual smoothness and stabilization result, which in light of our new versions of these functional inequalities would have been purely technical as opposed to an essential property of the model. Interest in inequalities of this type is naturally not restricted to only chemotaxis applications in planar domains but similar functional inequalities have also been considered on spherical domains and in related settings (cf. [27], [91], [140]).

Even though we developed the above functional inequalities to solve an immediate problem in our efforts to explore the long-time behavior properties of solutions to (CF), the result seems of some independent interest given that it explores the boundaries of the crucial gap between the Sobolev space $W^{1,2}$ embedding into every L^p space with finite p but not embedding into L^∞ in two-dimensional domains. In fact, this particular gap often seems to be the locus of particular intricate dynamics. This is e.g. evident in the fact that the critical mass phenomenon of the Keller–Segel system in two-dimensional domains, which we mentioned before, is inextricably connected to another critical constant of the Trudinger–Moser inequality (cf. [51], [89]). Widening our field of view slightly for another example, there also exist blowing-up solutions to the mean field equation (cf. [96] or [3] for a similar discussion in a slightly different setting), where the mentioned blowup occurs as again a critical system parameter approaches a value connected to the Trudinger–Moser inequality. Given that these functional inequalities have already proven their usefulness in our concrete setting and are of a fairly general form, they might prove to be a useful tool for future explorations of various problems in two-dimensional settings.

1.3 Framework 2: Haptotaxis with a potentially degenerate diffusion and taxis tensor

The second taxis process we study in this thesis is the invasive movement of tumor cells into healthy tissue along gradients of tissue density during the progression of certain types of cancer (cf. [100]), which is naturally of extensive interest from a medical perspective. Differing from the oxygen-consuming bacteria discussed in the previous section, cancer cells are generally modeled as following a haptotactic as opposed to chemotactic movement mechanism (cf. [19]). Notably not only in the narrower setting of cancer cell modeling but more general mathematical taxis models, by far the most attention at this point has been paid to approaches employing a Fickian diffusive movement model for the entities in question, which assumes some homogeneity of the underlying medium. But bolstered by experiments observing cell aggregation near interfaces between grey and white matter in mouse brains (cf. [14]), it has recently been suggested that especially in more heterogeneous environments, such as brain tissue, cell movement might be better described by non-Fickian diffusion (cf. [9]), which is far less mathematically studied in these taxis settings.

This being the case, we choose exactly such a haptotaxis model of cancer invasion featuring non-Fickian *myopic diffusion*, which was introduced in [30], as our second focal point in this thesis. More specifically, we consider the system

$$\begin{cases} n_t = \nabla \cdot (\mathbb{D}\nabla n + n\nabla \cdot \mathbb{D}) - \chi\nabla \cdot (n\mathbb{D}\nabla c) + \mu n(1 - n^{r-1}), \\ c_t = -nc \end{cases} \tag{DH}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \{2, 3\}$, with a no-flux boundary condition and appropriate parameters $\chi > 0$, $\mu > 0$, $r \geq 2$ and $\mathbb{D} : \overline{\Omega} \to \mathbb{R}^{N\times N}$, $\mathbb{D}$ positive semidefinite on $\overline{\Omega}$. The first equation models the invading cancer cells moving according to the aforementioned myopic diffusion, which is represented by the term $\nabla \cdot (\mathbb{D}\nabla u + u\nabla \cdot \mathbb{D})$, as well as according to haptotaxis, which is represented by the term $-\chi\nabla \cdot (u\mathbb{D}\nabla w)$. Apart from this, the equation further incorporates a logistic source to represent the proliferation of cancer cells. The second equation models the remaining healthy tissue cells and only features a consumption term.

The key feature of interest in the above system from both an application as well as a mathematical perspective is of course the parameter matrix $\mathbb{D}$, which represents a space dependent coupled diffusion and taxis tensor. In practice, this tensor can be derived from the underlying tissue structure by employing biomedical imaging methods (cf. [30]) and represents the influence of said underlying structure on the movement of cells through it. To account for situations of both locally very dense as well as locally very sparse tissue, which both occur in concrete applications and hinder cell movement significantly, we allow $\mathbb{D}$ to be potentially degenerate. Notably in one dimension, solutions to a closely related system with degenerate diffusion have already been shown (cf. [133]) to reflect the aggregation behavior in interface regions seen in experiments (cf. [14]) while to our knowledge in systems with non-degenerate diffusion long-time behavior results seem to generally be restricted to convergence of solutions to spatially constant states (cf. e.g. [119]). This seems to indicate that models of this kind featuring degenerate diffusion could potentially be a better representation of real world behavior, but to our knowledge have thus far only been mathematically studied in one dimension. Therefore, establishing existence theory for such models in higher dimensions (but under weak assumptions on the parameters) seems a natural next step to explore their applicability and usefulness to the analysis of real world scenarios.

Our results. Our main results are twofold: First, we establish the existence of global classical solutions given a uniform positivity condition for $\mathbb{D}$, which allows us to basically treat it as we would any other elliptic diffusion operator, as well as a condition ensuring sufficient regularizing influence of the logistic source term (cf. Theorem 4.1.1). Second, we establish that it is still possible to construct fairly standard weak solutions under much more relaxed conditions for $\mathbb{D}$. More specifically, we drop the assumption that $\mathbb{D}$ must be globally positive in $\overline{\Omega}$ and replace it with a set of assumptions much more tailored to our methods for constructing said weak solutions, which are strictly weaker than the prior positivity assumption in allowing for matrices that are in some (small) parts of Ω only positive semidefinite (cf. Theorem 4.1.13). The proofs of both of these results are laid out in full detail in Chapter 4.

Prior work. To give some context for the above results, we will now give a brief summary of some relevant literature regarding haptotaxis in general and our system in particular. For a broader overview, we refer the reader to the general haptotaxis survey found in [118].

Let us first note that for the one-dimensional case, where $\mathbb{D}$ simplifies to a real-valued function, there are already some results available for a variant of our scenario without a logistic source term (including potential spatial degeneracy) dealing with existence theory as well as long-time behavior (cf. [133], [136], [137]). Weak solutions have also been constructed in very similar haptotaxis systems featuring porous-medium type and signal-dependent degeneracies as opposed to spatial ones (cf. [149]).

Regarding haptotaxis systems with non-degenerate diffusion operators, e.g. $\mathbb{D} \equiv 1$ in our system, global existence and sometimes boundedness theory has been studied in various closely related settings (cf. [15], [70], [78], [108], [116], [117], [139]). Notably, these systems often feature an additional equation modeling a diffusive (potentially attractive) chemical and the fixed parameter choice $r = 2$ for the logistic term in addition to the more regular diffusion. In many of these scenarios, it has further been established that solutions converge to their constant steady states (cf. [71], [78], [95], [108], [119], [148]) under varied but sometimes restrictive assumptions. There has also been some analysis of haptotaxis with tissue remodeling, which is represented in the model by some additional source terms in the equation for c (cf. [94], [104], [109]).

Apart from haptotaxis models, there has also been significant analysis of chemotaxis models featuring degenerate diffusion (cf. [52], [66], [141] including degeneracies depending on the cell density itself).

Lastly, let us just briefly mention that the regularizing effects of logistic source terms we rely on for our result have already been very well-documented in various chemotaxis systems (cf. [62], [124] among many others) as well as haptotaxis systems (cf. [110]).

1.4 Framework 3: Measure-valued initial data in attractive-repulsive chemotaxis as well as chemotaxis-consumption models

For our final topic of this thesis, we will investigate an attractive-repulsive chemotaxis system as well as a chemotaxis-consumption system regarding their ability to recover from blowup. More precisely, we are interested in whether in these popular systems blowup states always persist in the form of Dirac-type singularities as observed in the original Keller–Segel system under some circumstances (cf. [10], [65]) or whether instead solutions to the systems can immediately become smooth again. While most prior investigations into blowup behavior ask whether a classical solution to a chemotaxis system can ever reach a measure-valued blowup state after some finite time, we approach study of blowup in a sense in reverse here as we presuppose a measure-valued initial blowup state and then ask whether classical solutions can still be constructed that connect to said initial state in a reasonable fashion.

Attraction-repulsion chemotaxis system. The first model we consider in this regard is the attraction-repulsion chemotaxis system

$$\begin{cases} n_t = \Delta n - \chi \nabla \cdot (n \nabla c_a) + \xi \nabla \cdot (n \nabla c_r), \\ \tau c_{at} = \Delta c_a + \alpha n - \beta c_a, \\ \tau c_{rt} = \Delta c_r + \gamma n - \delta c_r \end{cases} \tag{AR}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, with parameters $\chi, \xi \geq 0$, $\alpha, \beta, \gamma, \delta > 0$ and $\tau \in \{0, 1\}$ as well as no-flux boundary conditions. Such variations of the original Keller–Segel model, which not only feature an attractive but also a repulsive chemical, have recently been introduced in an effort to e.g. understand Alzheimer's disease (cf. [79]) or describe quorum-sensing effects observed in certain kinds of bacteria (cf. [93]). The first equation models the movement of the organisms in question toward the attractive chemical represented by c_a at rate χ and away from the repulsive chemical represented by c_r at rate ξ. As seen in the second and third equation, both chemicals are produced by the organisms at rates depending on the choice of α and γ, respectively, and decay over time at rates β and δ, respectively. Notably, the second and third equation can either be of parabolic or elliptic type depending on the choice of parameter τ. This choice is generally interpreted as the chemicals either conforming to evolution at similar time scales as the organisms in the parabolic case or the chemicals reacting almost immediately to changes in organism concentration in the elliptic case.

Prior work. From an intuitive standpoint and in many cases very much by design, one would expect a sufficiently strong repulsive influence to counteract the aggregation behavior often underlying finite-time blowup and thus leading more readily to the global existence of classical solutions. In fact given regular initial data, this intuition seems to be supported by prior mathematical analysis as it has been shown that, if the repulsive taxis in the system (AR) is strictly stronger than its attractive counterpart in the sense that $\xi\gamma - \chi\alpha > 0$, then global classical solutions exist in two dimensions if $\tau = 1$ and in arbitrary dimension if $\tau = 0$ (cf. [76], [105]). It has further been shown that under potentially additional parameter restrictions said solutions are even globally bounded or exhibit certain large-time stabilization properties (cf. [54], [56], [74], [77]). Conversely if attraction dominates over repulsion, the blowup results already established for attraction-only systems (cf. e.g. [85], [86], [123], [126]) seem to largely translate to the competition case (cf. [55], [63], [105]). Naturally apart from the system in (AR), which stays fairly close to the original Keller–Segel system, many of its canonical variations have also been explored as well. To mention a few, there has been some consideration of models, in which the attractant is consumed instead of produced (cf. [32], [33]), in which the taxis mechanisms further interact with a logistic source term (cf. [69], [146]), or in which the movement mechanisms feature some form of nonlinearity (cf. [22], [32], [75]). Apart from this, there has also been some analysis of the interaction between attraction and consumption in the whole space case (cf. [87], [144]).

Let us further briefly mention that there is another prominent setting which at times deals with interaction of attraction and repulsion, namely predator-prey models. Here, the predators, which are generally modeled by the first of two diffusive equations, are attracted by the prey. The prey in turn, which is modeled by the second of the aforementioned equations, is repelled by the predators. If both taxis mechanisms are present in such a setting however, the situation seems to be much less clear cut than in the attraction-repulsion model seen in (AR) as even the construction of (potentially only generalized) solutions seems to be rather challenging given that cross-diffusion is present in both equations (cf. [36], [37], [111]). Due to this, most efforts in this area focus only on one of the two mechanisms and remove the other.

Our results. Given any measure-valued initial data for the first solution component and sufficiently regular $W^{1,r}(\Omega)$ data for the second and third solution components, we show that the system in (AR) admits global classical solutions in the two-dimensional parabolic-parabolic and parabolic-elliptic case as long as the repulsive forces are sufficiently stronger than their attractive counterparts. For the three-dimensional parabolic-elliptic case, we are still able to construct similar solutions but additionally need to assume that the initial data for the first solution component is slightly more regular than $L^1(\Omega)$ in addition to the strong repulsion assumption. The precise conditions on the system parameters and initial data as well as the full construction can be found in Chapter 5.

Chemotaxis-consumption system. To neatly bookend our discussions in this thesis, we return to a chemotaxis-consumption system (albeit this time without a coupled fluid model and in arbitrarily high dimensions). More precisely, as our last system of consideration we investigate

$$\begin{cases} n_t = \Delta n - \chi \nabla \cdot (n \nabla c), \\ c_t = \Delta c - cn \end{cases} \tag{C}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, with $\chi > 0$ and no-flux boundary conditions. Even when not coupled to a fluid equation, models of this type are still often used to e.g. model certain bacteria that are attracted by oxygen (cf. [115]) as already elaborated upon previously.

Prior work. Under a smallness assumption for the initial state of the second solution component, it has been shown that bounded global classical solutions to (C) with smooth initial data exist in domains of arbitrary dimension (cf. [6], [103]). Removing the initial data condition, a classical existence result of this type still holds in two-dimensional domains (cf. [53]) and it has further been shown that weak solutions exist in three-dimensional domains, which eventually become smooth (cf. [107]). There has also been extensive exploration of chemotaxis-consumption models with modified taxis or diffusion mechanisms (cf. e.g. [17], [67], [68], [145]), added production terms (cf. e.g. [64], [68]), as well as models coupled with a fluid equation (cf. e.g. [4], [130], [135] for some relevant existence theory and [72], [135] for discussions of long-time behavior in this setting).

Our results. Building on this base of knowledge, we show that given any measure-valued initial data for the first solution component as well as $L^\infty(\Omega)$ initial data satisfying the smallness condition

$$0 < \|c_0\|_{L^\infty(\Omega)} < \frac{\pi}{\chi}\sqrt{\frac{2}{N}}$$

for the second solution component, global classical solutions to (C) can still be constructed in domains of arbitrary dimension N in a way that connects them to the initial data in a reasonable way. To our knowledge, the above condition on c_0 seems weaker than those required in previous works on (C) even when assuming smooth initial data. We present the full theorem and proof in Chapter 6.

1.5 Notation

Concluding this introduction, we will give a brief overview over some relevant notation used in this thesis. To this end, let $U \subseteq \mathbb{R}^k, V \subseteq \mathbb{R}^j$, $k, j \in \mathbb{N}$ as well as let $m, n \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\alpha \in (0, 1)$ for the remainder of this section.

We will call a matrix $M \in \mathbb{R}^{k\times k}$, positive definite or positive semidefinite if and only if it is symmetric and $x \cdot Mx > 0$ or $x \cdot Mx \geq 0$ for all $x \in \mathbb{R}^k \setminus \{0\}$, respectively. We will further write $M_1 \geq M_2$ if and only if $x \cdot M_1 x \geq x \cdot M_2 x$ for all $x \in \mathbb{R}^k$, where $M_1, M_2 \in \mathbb{R}^{k\times k}$. For a positive semidefinite matrix $M \in \mathbb{R}^{k\times k}$, $\sqrt{M}$ is the standard matrix square root. Further, $M + s := M + sI$ and $M \geq s$ if and only if $M \geq sI$ for all $M \in \mathbb{R}^{k\times k}$ and $s \in \mathbb{R}$, where $I \in \mathbb{R}^{k\times k}$ is the identity matrix. The relations $<$, $\leq$ and $>$ between two matrices as well as a matrix and a scalar are defined in the same way. Any matrix properties are always to be understood in a pointwise fashion when applied to a matrix-valued function.

For the purposes of this thesis, we will represent strong and weak derivatives in the same way. Whenever possible, we mean strong derivatives. We will use standard notation for partial derivatives, e.g. $\partial_x f, \frac{\partial}{\partial x} f$ or f_x for the derivative of f with respect to x. ∇f is the gradient of f, $\nabla \cdot f$ is the divergence of f and Δf is the Laplacian of f. We use standard notation, i.e. $\int$, for integration and always mean the Lebesgue integral with the Lebesgue measure or surface measure for surface integrals, respectively, if no explicit measure is specified. In surface integrals as well as boundary conditions, the symbol $\nu = \nu(x)$ always denotes the unit outward normal vector of the currently considered domain at that point x on the boundary. The absolute value $|\cdot|$ applied to a vector $x \in \mathbb{R}^k$ always denotes the Euclidean norm of x, applied to a matrix $M \in R^{k\times k}$ always denotes the Frobenius norm of M and applied to a measurable set U always denotes the Lebesgue measure of U.

In the following paragraph, U and V are always assumed to be sets with nonempty interior that are a (potentially proper) subset of the closure of their interior, i.e. nonempty open sets, the closure of nonempty open sets or essentially anything in between. We then say that f is an m-times continuously differentiable real-valued function over such a set U if f is m-times continuously differentiable on the interior of U and f as well as its first m derivatives can be continuously extended to the entirety of U.

$C^0(U)$ is the space of continuous real-valued functions over U with

$$\|f\|_{C^0(U)} \coloneqq \sup_{x\in U} |f(x)| \qquad \text{if } U \text{ is compact.}$$

$C^m(U)$ is the space of m-times continuously differentiable real-valued functions over U with

$$\|f\|_{C^m(U)} \coloneqq \max_{|\beta|\leq m} \sup_{x\in U} \left|\frac{\partial^{|\beta|}}{\partial x^\beta} f(x)\right| \qquad \text{if } U \text{ is compact.}$$

$C^\infty(U)$ is the space of arbitrarily often continuously differentiable real-valued functions over U and $C_c^\infty(U)$ is the space of such functions f with compact support

$$\operatorname{supp} f \coloneqq \overline{\{x \in U \mid f(x) \neq 0\}}.$$

$C^\alpha(U)$ is the space of locally α-Hölder continuous real-valued functions with

$$\|f\|_{C^\alpha(U)} \coloneqq \|f\|_{C^0(U)} + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} \;\middle|\; x, y \in U, x \neq y\right\} \qquad \text{if } U \text{ is compact.}$$

$C^{m+\alpha}(U)$ is the space of functions $f \in C^m(U)$ such that the m-th derivatives of f are locally α-Hölder continuous with

$$\|f\|_{C^{m+\alpha}(U)} \coloneqq \|f\|_{C^m(U)} + \max_{|\beta|=m} \left\|\frac{\partial^{|\beta|}}{\partial x^\beta} f\right\|_{C^\alpha(U)} \qquad \text{if } U \text{ is compact.}$$

$C^{m,n}(U \times V)$ is the space of real-valued functions defined on $U \times V$ that are m-times continuously differentiable with respect to U and n-times continuously differentiable with respect to V. For compact sets U and V, the parabolic Hölder space $C^{\alpha,\frac{\alpha}{2}}(U \times V)$ consists of functions $f \in C^0(U \times V)$ such that the following parabolic Hölder norm is finite:

$$\|f\|_{C^{\alpha,\frac{\alpha}{2}}(U\times V)} \coloneqq \|f\|_{C^0(U\times V)} + \sup\left\{\frac{|f(x,t)-f(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}} \;\middle|\; \begin{matrix} x, y \in U, \\ t, s \in V, \end{matrix}\; x \neq y \text{ or } t \neq s\right\}$$

For definitions of higher order versions of this type of parabolic Hölder norm and associated space (e.g. $C^{2+\alpha,1+\frac{\alpha}{2}}(U \times V)$), we refer the reader to [61, pp. 7-8] as spaces and norms of this type will mostly find use in this thesis in the context of Schauder theory cited from the aforementioned reference.

The Lebesgue spaces of real-valued functions $L^p(U)$ are defined in the standard way with

$$\|f\|_{L^p(U)} \coloneqq \left(\int_U |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \qquad \text{if } p < \infty$$

or

$$\|f\|_{L^\infty(U)} \coloneqq \operatorname{ess\,sup}_{x\in U} |f(x)| \qquad \text{if } p = \infty.$$

$L^{|z|\ln(|z|+1)}(U)$ is the corresponding Orlicz space (with the Luxemburg norm) associated with the N-function (satisfying a global Δ_2-condition) given by $z \mapsto |z|\ln(|z|+1)$ (cf. [2, Chapter 8], [59]). If U has finite measure, it is straightforward to show that $L^{|z|\ln(|z|+1)}(U)$ embeds continuously into $L^1(U)$ (by e.g. using the property of the Luxemburg norm seen in (9.21) from [59]). $L^p_{\mathrm{loc}}(U)$ is the space of measurable real-valued functions f on U such that $\|f\|_{L^p(K)}$ is finite for all compact sets $K \subseteq U$.

$\mathcal{M}_+(U)$ is the set of all positive Radon measures on U equipped with the vague topology (cf. e.g. [7, Chapter IV]). When necessary, we will interpret nonnegative functions $f \in L^1(U)$ as elements of $\mathcal{M}_+(U)$ by treating them as a density function relative to the standard Lebesgue measure. For example, we say a sequence of nonnegative functions $(f_i)_{i\in\mathbb{N}} \subseteq L^1(U)$ converges to a measure $\mu \in \mathcal{M}_+(U)$ in the vague topology as $i \to \infty$ if and only if $\int_U \varphi(x) f_i(x)\,\mathrm{d}x \to \int_U \varphi(x)\,\mathrm{d}\mu(x)$ as $i \to \infty$ for all $\varphi \in C^0(U)$ with compact support.

In this paragraph, U is always assumed to be a domain, i.e. a nonempty, open and connected set. We then denote the standard Sobolev spaces as $W^{m,p}(U)$ (cf. [2]) with

$$\|f\|_{W^{m,p}(U)} \coloneqq \begin{cases} \left(\sum_{|\beta|\le m} \|\frac{\partial^{|\beta|}}{\partial x^\beta} f\|^p_{L^p(U)}\right)^{\frac{1}{p}} & \text{for } p < \infty \\ \max_{|\beta|\le m} \|\frac{\partial^{|\beta|}}{\partial x^\beta} f\|_{L^\infty(U)} & \text{for } p = \infty \end{cases}.$$

These are the spaces of m-times weakly differentiable $L^p(U)$ functions with weak derivatives in $L^p(U)$. If $p < \infty$, these spaces can be equivalently characterized as the completion of $C^\infty(U)$ regarding the norm $\|\cdot\|_{W^{m,p}(U)}$ (cf. [2, Theorem 3.17]). We then define the spaces $W^{m,p}_0(U)$ as the closure of $C^\infty_c(U)$ in $W^{m,p}(U)$ or equivalently as the subsets of $W^{m,p}(U)$ where an appropriate boundary trace is 0 if U is of sufficient boundary regularity (cf. [2, Theorem 5.36 and Theorem 5.37]).

While we have thus far only talked about spaces of real-valued functions, let us briefly address the case of functions with values in some more general Banach space X, which will come up fairly regularly during this thesis. In this case, we write variants of the above spaces with values in X instead of $\mathbb{R}$ following the convention seen in e.g. $C^0(U;X)$ or $C^\alpha(U;X)$. When referring to the spaces $L^p(U;X)$ and $L^p_{\mathrm{loc}}(U;X)$, we always mean the standard Bochner–Lebesgue spaces. Any norms introduced above (not featuring derivatives) are adapted to this case by simply replacing absolute values in them by the corresponding norm associated with X as necessary. If $X = \mathbb{R}^k$ or $X = \mathbb{R}^{k\times k}$, then derivatives will always be understood componentwise. When applying a norm defined above for real-valued functions to such a vector-valued or matrix-valued function, we always take the absolute value of the function at every point $x \in U$ first and then apply said norm.

For variants of the above spaces used in the study of fluid equations, i.e. $L^p_\sigma(U)$ and $W^{m,p}_{0,\sigma}(U)$, as well as associated projection operators $\mathcal{P} = \mathcal{P}_p$, see Remark 3.6.6 for the relevant definitions. For variants of the above spaces used in our study of spatially degenerate parabolic equations, i.e. $L^p_{\mathbb{D}}(U)$, $W^{1,p}_{\mathbb{D}}(U)$, $W^{1,p}_{\text{div}}(U;\mathbb{R}^{k\times k})$, as well as associated projections operators P_i, $i \in \{1,2\}$, see Definition 4.1.7.

We always write the dual space of any Banach space X as X^*. Moreover, we denote convergence of a sequence $(f_i)_{i\in\mathbb{N}} \subseteq X$ to f as $i \to \infty$ by writing $f_i \to f$ and weak convergence to f as $i \to \infty$ by writing $f_i \rightharpoonup f$.

For any sectorial operator A (e.g. the negative Neumann Laplacian $-\Delta$ or the Stokes operator), we will write the semigroup generated by $-A$ as $(e^{-tA})_{t\geq 0}$ or $e^{-tA}\varphi$ when applied to a function φ. We further denote the spectrum of A as $\sigma(A)$. If all elements of $\sigma(A)$ have positive real part (e.g. in the case of the Stokes operator), we write the fractional powers of A as A^β and their domains as $D(A^\beta)$ with $\beta > 0$ (cf. [49], [80]).

For space and clarity reasons, we will often abbreviate terms. We will e.g. sometimes write n, c instead of $n(x,t)$, $c(x,t)$. Differential operators like the Laplace operator, divergence operator or gradient operator always only apply to the space variable x. Inside a time integral with e.g. integration variable s, the functions n, c will depend on said integration variable s instead of a constant time t if nothing else is explicitly stated.

2 Two new functional inequalities based on the Trudinger–Moser inequality

2.1 Main result

As a crucial prerequisite for the next chapter as well as a result of individual merit, we will devote this chapter to deriving the following functional inequalities:

Theorem 2.1.1. *For any piecewise C^2, bounded, finitely connected domain $G \subseteq \mathbb{R}^2$ with a finite number of vertices and a minimum interior angle $\theta_G \in (0, \pi]$ at its vertices, there exists a constant $\beta_0 \in (0, 2\theta_G)$ such that*

$$\int_G \varphi(\psi - \overline{\psi}) \leq \frac{1}{a} \int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) + \frac{a}{4\beta_0} \left\{\int_G \psi\right\} \int_G |\nabla\varphi|^2 \tag{F1}$$

for all $\varphi \in W^{1,2}(G)$, positive $\psi \in L^p(G)$ with $p > 1$ and $a > 0$ and

$$\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) \leq \frac{1}{\beta_0} \left\{\int_G \psi\right\} \int_G |\nabla \ln(\psi)|^2 \tag{F2}$$

for all positive $\psi \in L^p(G)$ with $p > 1$ and $\ln(\psi) \in W^{1,2}(G)$. Here, $\overline{\psi} \coloneqq \frac{1}{|G|} \int_G \psi$.

Remark 2.1.2. The restrictions on the regularity of the domain G in Theorem 2.1.1 originate directly from the version of the Trudinger–Moser inequality from [21] laid out in Theorem 2.3.1, which serves as the basis for our arguments in this chapter. In accordance with [21], Theorem 2.1.1 also encompasses domains that are entirely C^2 and thus have no vertices. In this case, θ_G is equal to π. While the characterization of a piecewise C^2, bounded, finitely connected domain in $\mathbb{R}^2$ with a finite number of vertices and a minimum interior angle $\theta_G \in (0, \pi]$ at its vertices is left somewhat vague in [21] by not giving a formal definition or reference for some of the more ambiguous properties, any reasonable characterization of such a domain G should certainly imply the segment and cone conditions for G as defined in [2, Paragraph 3.21 and Paragraph 4.6], which are the only other boundary regularity conditions needed for the remaining arguments in this chapter. The segment condition is used here to ensure that the set $C^1(\overline{G})$ is dense in $W^{1,2}(G)$ (cf. [2, Theorem 3.22]) and the cone condition is used here to ensure that the set $W^{1,2}(G)$ embeds continuously and compactly into $L^p(G)$ for all $p \in [1, \infty)$ (cf. [2, Theorem 4.12 and Theorem 6.3]). The latter directly implies the Poincaré inequality

$$\|\varphi\|_{L^2(G)} \le C\|\nabla\varphi\|_{L^2(G)} \qquad \text{for all } \varphi \in W^{1,2}(G) \text{ with } \int_G \varphi = 0$$

for some constant $C > 0$ under the same cone condition (see the proof of [138, Theorem 7.7] for the relevant argument), which is the final result used in this chapter that requires additional boundary regularity. Therefore, no further restrictions on the domain G than those already present in Theorem 2.3.1 are necessary for our proof of Theorem 2.1.1.

Remark 2.1.3. For functions ψ of higher regularity (e.g. $C^1(\overline{G})$), the inequality in (F2) can be rewritten as

$$\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) \le \frac{1}{\beta_0}\left\{\int_G \psi\right\}\int_G \frac{|\nabla\psi|^2}{\psi^2},$$

which is how we will use it in Chapter 3.

Remark 2.1.4. Let us further note that many types of domains commonly encountered in the study of partial differential equations are always finitely connected as a consequence of some stronger regularity property typically assumed of such domains. Indeed, star-shaped domains (and thus convex domains) are straightforward examples of finitely connected domains as they are easily shown to be simply connected. While perhaps less intuitively obvious but even more widely applicable, many standard boundary regularity properties central to the study of Sobolev spaces and partial differential equations also ensure that the domain in question must already be finitely connected as long as the boundary of the domain is also compact. This is due to the fact that common characterizations of e.g. Lipschitz or smooth boundary regularity (cf. e.g. [2, Paragraphs 4.4 to 4.11], [26, Section 2.3.4] or [138, Section I.2] for a more in depth discussion on how various boundary regularity properties relate to each other) entail that the boundary can be covered by a family of connected open sets as a consequence of the boundary being e.g. locally representable as a graph of a sufficiently regular function or the boundary being a embedded submanifold of sufficient regularity. Due to compactness, the boundary must thus be a subset of a finite union of connected sets and can thus only have finitely many connected components. As each interior hole of the domain corresponds to a connected component of the boundary, this naturally directly implies that the domain can only have finitely many interior holes and is thus finitely connected.

As a matter of convenience, G will always be a piecewise C^2, bounded, finitely connected domain in $\mathbb{R}^2$ with a finite number of vertices and a minimum interior angle $\theta_G \in (0, \pi]$ at its vertices for the remainder of this chapter. As already mentioned in Remark 2.1.2 if G has no vertices, then θ_G is equal to π in accordance with [21].

2.2 Approach

Both (F1) and (F2) are ultimately a consequence of a standard corollary of the Trudinger–Moser inequality (cf. [84], [114]), namely the inequality

$$\int_G e^{\varphi} \le C_G \exp\left(\frac{1}{4\beta} \int_G |\nabla\varphi|^2 + \frac{1}{|G|} \int_G \varphi \right) \qquad \text{for all } \varphi \in W^{1,2}(G) \tag{2.2.1}$$

which holds for all $\beta \in (0, 2\theta_G]$ as well as some fixed $C_G \ge |G|$ (This lower bound for C_G directly follows from setting $\varphi := 0$). Although the above inequality is fairly easy to derive if optimality of the constants is not necessarily an objective, our aim here will be minimizing the constant C_G as this is central to the derivation of our new inequalities.

To achieve such a minimization of C_G in planar domains, we begin by employing techniques from the calculus of variations to first find a minimizer φ_β of the functional

$$J_\beta(\varphi) := \frac{1}{4\beta} \int_G |\nabla\varphi|^2 + \frac{1}{|G|} \int_G \varphi - \ln\left(\frac{1}{|G|} \int_G e^{\varphi} \right) \qquad \text{for all } \varphi \in W^{1,2}(G),$$

which arises in a natural way from (2.2.1) after some rearrangement, for each $\beta \in (0, 2\theta_G]$.

Having found such minimizers, we then show that they solve a Neumann problem corresponding to the equation

$$-\frac{1}{2\beta}\Delta\varphi_\beta = -\frac{1}{|G|} + \frac{e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}}$$

in a weak sense. As a property inherent to their construction, we can immediately see that all minimizers are bounded in $W^{1,2}(G)$ independent of β. We further note that, when β becomes small, the Laplacian on the left-hand side of the above equation becomes arbitrarily strong, which has the following consequence: If we restrict ourselves to solutions, which are normalized to $\int_G \varphi_\beta = 0$ and are bounded in $W^{1,2}(G)$ by a fixed constant independent of β, then as $\beta \searrow 0$ the only solution that fulfills these constraints is $\varphi_\beta \equiv 0$. Combining these insights, we can then conclude that our minimizers must be equal to zero almost everywhere as well if β is sufficiently small. But this directly gives us $J_\beta \ge J_\beta(\varphi_\beta) = J_\beta(0) = 0$ for sufficiently small β, which implies (2.2.1) with $C_G = |G|$.

2.3 The Trudinger–Moser inequality

We will start the derivation of our new functional inequalities by reminding ourselves of an already well-known inequality first pioneered by Trudinger in [114] and then later refined by Moser in [84, Theorem 1], which will serve as the starting point for all further considerations. As it is somewhat more convenient for our purposes, we use the more recent formulation of the same inequality by Chang and Yang in [21, Proposition 2.3], which can be extended from $C^1(\overline{G})$ to $W^{1,2}(G)$ by a straightforward density argument (cf. [2, Theorem 3.22]):

Theorem 2.3.1. *Let $G \subseteq \mathbb{R}^2$ be a piecewise C^2, bounded, finitely connected domain with a finite number of vertices and a minimum interior angle $\theta_G \in (0, \pi]$ at its vertices. Then there exists a constant $C_G \geq |G|$ such that, for all $\varphi \in W^{1,2}(G)$ with*

$$\int_G |\nabla \varphi|^2 \leq 1 \text{ and } \int_G \varphi = 0$$

and $0 < \beta \leq 2\theta_G$, we have

$$\int_G e^{\beta \varphi^2} \leq C_G.$$

As the above restrictions on φ can be somewhat inconvenient, we will now prove a standard corollary to the Trudinger–Moser inequality eliminating said restrictions at the cost of some corresponding terms on the right-hand side and a slightly different term on the left-hand side of the inequality:

Corollary 2.3.2. *For each $0 < \beta \leq 2\theta_G$ and $\varphi \in W^{1,2}(G)$, we have*

$$\int_G e^{\varphi} \leq C_G \exp\left(\frac{1}{4\beta} \int_G |\nabla \varphi|^2 + \frac{1}{|G|} \int_G \varphi \right) \tag{2.3.1}$$

with C_G from Theorem 2.3.1.

Proof. As (2.3.1) is trivially true if $\|\nabla \varphi\|_{L^2(G)} = 0$ with $C_G = |G|$ due to functions with this property being constant almost everywhere, we can assume that $\|\nabla \varphi\|_{L^2(G)} > 0$ for the remainder of this proof without loss of generality. Then by using Young's inequality to see that

$$\varphi - \overline{\varphi} \leq |\varphi - \overline{\varphi}| \leq \beta \left(\frac{\varphi - \overline{\varphi}}{\|\nabla \varphi\|_{L^2(G)}} \right)^2 + \frac{1}{4\beta} \|\nabla \varphi\|^2_{L^2(G)}$$

with $\overline{\varphi} := \frac{1}{|G|} \int_G \varphi$, we directly gain from Theorem 2.3.1 that

$$\int_G e^{\varphi - \overline{\varphi}} \leq C_G \exp\left(\frac{1}{4\beta} \int_G |\nabla \varphi|^2 \right)$$

or further that

$$\int_G e^{\varphi} \leq C_G \exp\left(\frac{1}{4\beta} \int_G |\nabla \varphi|^2 + \frac{1}{|G|} \int_G \varphi \right)$$

for all $0 < \beta \leq 2\theta_G$ and $\varphi \in W^{1,2}(G)$. □

As integrals of the form $\int_G e^{\varphi}$ will naturally play a significant role in the following arguments, let us briefly note that the above corollary ensures that said integrals are always positive and finite if $\varphi \in W^{1,2}(G)$, which makes them reasonably straightforward to handle.

2.4 A variational approach to minimizing C_G

Understanding the relationship of the constants C_G and β in Corollary 2.3.2 will be the linchpin to our proof of Theorem 2.1.1. While there have been considerable efforts to achieve the above inequality for optimal, meaning large, values of β in many different contexts (cf. [1], [23], [84], for instance), we will be more interested in how small we can make $C_G \geq |G|$ at the cost of only allowing for smaller values of β. To our knowledge, minimizing C_G has thus far only been considered on the sphere $\mathbb{S}^n$ and in related settings (cf. [27], [91], [140], for instance).

Therefore, what we are now essentially looking at is a minimization problem, which we will handle using variational methods. Concerning which functional to minimize, we let ourselves be guided by a similar approach in [39, Theorem 18.2.1] to minimizing the constant C_G on the sphere $\mathbb{S}^2$ (cf. [27] for an overview about proof techniques on $\mathbb{S}^2$). Thus for each $\beta \in (0, 2\theta_G]$, we will analyze the following functional:

$$J_\beta(\varphi) := \frac{1}{4\beta} \int_G |\nabla\varphi|^2 + \frac{1}{|G|} \int_G \varphi - \ln\left(\frac{1}{|G|} \int_G e^\varphi\right) \qquad \text{for all } \varphi \in W^{1,2}(G). \tag{2.4.1}$$

As $J_\beta \geq 0$ immediately implies that (2.3.1) holds with $C_G = |G|$, which is the smallest possible value for C_G in said inequality, it will be our aim for the remainder of this section to show that minimizers φ_β for J_β exist and that, for sufficiently small β, they have the property $J_\beta(\varphi_\beta) = 0$.

To do this, let us now first consider a basic lower boundedness and coerciveness property of J_β directly following from Corollary 2.3.2:

Lemma 2.4.1. *There exists a constant $C \geq 0$ such that*

$$J_\beta(\varphi) \geq \frac{1}{8\beta} \int_G |\nabla\varphi|^2 - C \geq -C$$

for all $\varphi \in W^{1,2}(G)$ and $\beta \in (0, \theta_G]$.

Proof. Let $\beta \in (0, \theta_G]$ and $\gamma := 2\beta \in (\beta, 2\theta_G]$. Then we know from Corollary 2.3.2 that

$$-\ln\left(\frac{1}{|G|} \int_G e^\varphi\right) \geq -\frac{1}{4\gamma} \int_G |\nabla\varphi|^2 - \frac{1}{|G|} \int_G \varphi - \ln\left(\frac{C_G}{|G|}\right).$$

for all $\varphi \in W^{1,2}(G)$ after some minor rearrangement. If we now apply this to J_β, we see that

$$\begin{aligned} J_\beta(\varphi) &\geq \left(\frac{1}{4\beta} - \frac{1}{4\gamma}\right) \int_G |\nabla\varphi|^2 - \ln\left(\frac{C_G}{|G|}\right) \\ &= \frac{1}{8\beta} \int_G |\nabla\varphi|^2 - \ln\left(\frac{C_G}{|G|}\right) \end{aligned}$$

for all $\varphi \in W^{1,2}(G)$, which completes the proof. □

This property now enables us to find a minimizer φ_β for each J_β by first allowing us to construct a minimizing sequence for each functional and then arguing that said sequences converge in certain topologies to some limit function in $W^{1,2}(G)$. We then only need to further show that said convergence properties lead to sufficient estimates to ensure that the limit object is in fact an actual minimizer.

Moreover by utilizing the then established minimizer property, we can additionally show that each φ_β solves a certain weak elliptic Neumann boundary value problem as a first step in our efforts to show that $J_\beta(\varphi_\beta) = 0$.

Lemma 2.4.2. *For each $\beta \in (0, \theta_G]$, there exists a function $\varphi_\beta \in W^{1,2}(G)$ with*

$$\int_G \varphi_\beta = 0$$

and

$$\frac{1}{2\beta} \int_G \nabla\varphi_\beta \cdot \nabla\psi = -\frac{1}{|G|} \int_G \psi + \frac{\int_G \psi e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}} \tag{2.4.2}$$

for all $\psi \in W^{1,2}(G)$, which is a minimizer of J_β, meaning that

$$\inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) = J_\beta(\varphi_\beta).$$

Proof. We fix $\beta \in (0, \theta_G]$. We know from Lemma 2.4.1 that $\inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) \geq -C$ for some $C > 0$ and we can therefore choose a (minimizing) sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq W^{1,2}(G)$ such that

$$J_\beta(\varphi_k) \to \inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi)$$

as $k \to \infty$. Without loss of generality, we can further assume that

$$\int_G \varphi_k = 0 \qquad \text{for all } k \in \mathbb{N}$$

because it is easily seen that J_β is invariant under the addition of constants to its argument. Because the sequence $(J_\beta(\varphi_k))_{k\in\mathbb{N}}$ converges, it is bounded and thus Lemma 2.4.1 implies that the sequence

$$\left(\int_G |\nabla\varphi_k|^2 \right)_{k\in\mathbb{N}}$$

is bounded as well. As we know that $\int_G \varphi_k = 0$ for all $k \in \mathbb{N}$, the Poincaré inequality (cf. [138, Theorem 7.7]) implies that therefore the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(G)$ as well. Without loss of generality (by choosing fitting subsequences), this allows us to assume that there exists a function $\varphi_\beta \in W^{1,2}(G)$ with

$$\begin{cases} \varphi_k \rightharpoonup \varphi_\beta & \text{in } W^{1,2}(G) \\ \varphi_k \to \varphi_\beta & \text{in } L^1(G) \text{ and } L^2(G) \end{cases} \tag{2.4.3}$$

as $k \to \infty$ by standard compactness arguments (cf. [2, Theorem 6.3]). The above $L^1(G)$ convergence then ensures that $\int_G \varphi_\beta = 0$. Further due to the mean value theorem, we can now observe that

$$\begin{aligned}\left|\int_G e^{\varphi_k} - \int_G e^{\varphi_\beta}\right| &\le \int_G |e^{\varphi_k} - e^{\varphi_\beta}| \\ &\le \int_G |\varphi_k - \varphi_\beta| e^{|\varphi_k| + |\varphi_\beta|} \\ &\le \|\varphi_k - \varphi_\beta\|_{L^2(G)} \left(\int_G e^{2|\varphi_k| + 2|\varphi_\beta|}\right)^{\frac{1}{2}}\end{aligned}$$

for all $k \in \mathbb{N}$. Using the $L^2(G)$ convergence from (2.4.3) as well as the fact that $\int_G e^{2|\varphi_k|+2|\varphi_\beta|}$ is uniformly bounded due to Corollary 2.3.2 and the $W^{1,2}(G)$ bound for the sequence already established prior, this directly implies that

$$\int_G e^{\varphi_k} \to \int_G e^{\varphi_\beta}$$

as $k \to \infty$.

Using this convergence property combined with (2.4.3), we then see that

$$\begin{aligned}\inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) &= \lim_{k\to\infty} J_\beta(\varphi_k) \\ &= \frac{1}{4\beta} \liminf_{k\to\infty} \int_G |\nabla \varphi_k|^2 + \frac{1}{|G|} \lim_{k\to\infty} \int_G \varphi_k - \ln\left(\frac{1}{|G|} \lim_{k\to\infty} \int_G e^{\varphi_k}\right) \\ &\ge J_\beta(\varphi_\beta)\end{aligned}$$

and therefore that

$$J_\beta(\varphi_\beta) = \inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi).$$

Thus, φ_β is a minimizer of J_β.

It now only remains to show that φ_β is also a weak solution of the Neumann problem corresponding to (2.4.2). To this end, let now $\psi \in W^{1,2}(G)$ be fixed but arbitrary. We then consider the function

$$f\colon (-1,1) \to \mathbb{R}, \quad t \mapsto J_\beta(\varphi_\beta + t\psi),$$

which has a global minimum in 0 by our observations about J_β. Further

$$\begin{aligned}f(t) = &\frac{1}{4\beta}\int_G |\nabla\varphi_\beta|^2 + t\frac{1}{2\beta}\int_G \nabla\varphi_\beta \cdot \nabla\psi + t^2 \frac{1}{4\beta}\int_G |\nabla\psi|^2 \\ &+ \frac{1}{|G|}\int_G \varphi_\beta + t\frac{1}{|G|}\int_G \psi - \ln\left(\frac{1}{|G|}\int_G e^{\varphi_\beta + t\psi}\right).\end{aligned}$$

One easily sees that f is differentiable as it is mostly a polynomial in t and the remaining terms are amenable to results about the differentiation of parameter integrals (note that

Corollary 2.3.2 can be used to establish the necessary integrability properties). The minimality property of f in 0 therefore implies that

$$0 = f'(0) = \frac{1}{2\beta}\int_G \nabla\varphi_\beta \cdot \nabla\psi + \frac{1}{|G|}\int_G \psi - \frac{\int_G \psi e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}},$$

which gives us (2.4.2) and thus completes the proof. □

Having now constructed the minimizers φ_β, the key to showing that for sufficiently small β we have $J_\beta \geq J_\beta(\varphi_\beta) = 0$ is understanding the weak elliptic Neumann problem

$$\begin{cases} -\frac{1}{2\beta}\Delta\varphi_\beta = -\frac{1}{|G|} + \frac{e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}} & \text{on } G, \\ \nabla\varphi_\beta \cdot \nu = 0 & \text{on } \partial G. \end{cases}$$

In this regard, the two most crucial insights about the above system as well as the minimizers are the following: First, the minimizers are bounded in $W^{1,2}(G)$ independent of β as a consequence of their minimization property. Second by reducing the value of β, we can make the Laplacian in the above system arbitrarily strong when compared to the source terms on the right, which manifests as the following property: When only considering solutions that are normalized to $\int_G \varphi_\beta = 0$ and are bounded in $W^{1,2}(G)$ by some constant $C > 0$ independent of β, we can increase the strength of the Laplacian to such a degree that at some point the only member of the aforementioned solution class is $\varphi_\beta \equiv 0$.

Combined, this means that, for sufficiently small β, the minimizers φ_β must be almost everywhere equal to zero as well. But this directly implies $J_\beta(\varphi_\beta) = J_\beta(0) = 0$.

We will now make these ideas precise to prove the following lemma.

Lemma 2.4.3. *There exists $\beta_0 \in (0, \theta_G]$ such that*

$$\inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) = 0$$

for all $\beta \in (0, \beta_0]$.

Proof. For each $\beta \in (0, \theta_G]$, let φ_β be the minimizer of J_β constructed in Lemma 2.4.2. First note that there exists a constant $K_1 > 0$ such that

$$0 = J_\beta(0) \geq \inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) = J_\beta(\varphi_\beta) \geq \frac{1}{8\beta}\int_G |\nabla\varphi_\beta|^2 - K_1$$

and therefore that

$$\int_G |\nabla\varphi_\beta|^2 \leq 8\beta K_1 \leq 8\theta_G K_1$$

for all $\beta \in (0, \theta_G]$ by Lemma 2.4.1. As a direct consequence of this as well as Corollary 2.3.2, the Hölder inequality and the Poincaré inequality with constant $K_2 > 0$ (cf. [138, Theorem 7.7]), we gain that

$$\begin{aligned}\left\| e^{|\varphi_\beta|} \right\|_{L^3(G)}^3 = \int_G e^{3|\varphi_\beta|} &\leq C_G \exp\left(\frac{9}{8\theta_G} \int_G |\nabla \varphi_\beta|^2 + \frac{3}{|G|} \int_G |\varphi_\beta| \right) \\ &\leq C_G \exp\left(9K_1 + \frac{3}{\sqrt{|G|}} \|\varphi_\beta\|_{L^2(G)} \right) \\ &\leq C_G \exp\left(9K_1 + \frac{3K_2}{\sqrt{|G|}} \|\nabla\varphi_\beta\|_{L^2(G)} \right) \\ &\leq C_G \exp\left(9K_1 + 3K_2 \left(\frac{8\theta_G K_1}{|G|} \right)^{\frac{1}{2}} \right) =: K_3 \end{aligned} \tag{2.4.4}$$

for all $\beta \in (0, \theta_G]$. We now further observe that

$$\int_G e^{\varphi_\beta} = \frac{|G|}{|G|} \int_G e^{\varphi_\beta} \geq |G| \exp\left(\frac{1}{|G|} \int_G \varphi_\beta \right) = |G| \tag{2.4.5}$$

for all $\beta \in (0, \theta_G]$ because of Jensen's inequality and the fact that $\int_G \varphi_\beta = 0$. We then set $\psi = \varphi_\beta$ in (2.4.2) to see that

$$\frac{1}{2\beta} \int_G |\nabla \varphi_\beta|^2 = -\frac{1}{|G|} \int_G \varphi_\beta + \frac{\int_G \varphi_\beta e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}} \quad \text{for all } \beta \in (0, \theta_G]. \tag{2.4.6}$$

As a first consequence of (2.4.6) and the fact that $\int_G \varphi_\beta = 0$, we gain that

$$\int_G \varphi_\beta e^{\varphi_\beta} \geq 0$$

and therefore that

$$\frac{\int_G \varphi_\beta e^{\varphi_\beta}}{\int_G e^{\varphi_\beta}} \leq \frac{\int_G \varphi_\beta e^{\varphi_\beta}}{|G|}$$

for all $\beta \in (0, \theta_G]$ because of (2.4.5). If we then apply this to (2.4.6), we see that

$$\begin{aligned}\frac{1}{2\beta} \int_G |\nabla \varphi_\beta|^2 &\leq \frac{1}{|G|} \left(\int_G \varphi_\beta (e^{\varphi_\beta} - 1) \right) \leq \frac{1}{|G|} \left(\int_G |\varphi_\beta| |e^{\varphi_\beta} - e^0| \right) \leq \frac{1}{|G|} \int_G |\varphi_\beta|^2 e^{|\varphi_\beta|} \\ &\leq \frac{1}{|G|} \|\varphi_\beta^2\|_{L^{\frac{3}{2}}(G)} \left\| e^{|\varphi_\beta|} \right\|_{L^3(G)} = \frac{1}{|G|} \|\varphi_\beta\|_{L^3(G)}^2 \left\| e^{|\varphi_\beta|} \right\|_{L^3(G)} \\ &\leq \frac{K_3^{\frac{1}{3}}}{|G|} \|\varphi_\beta\|_{L^3(G)}^2 \leq \frac{K_3^{\frac{1}{3}} K_4^2}{|G|} \|\nabla \varphi_\beta\|_{L^2(G)}^2 = \frac{K_3^{\frac{1}{3}} K_4^2}{|G|} \int_G |\nabla \varphi_\beta|^2 \end{aligned}$$

for all $\beta \in (0, \theta_G]$ by (2.4.4), the mean value theorem, the Hölder inequality and a Poincaré-Sobolev inequality with constant $K_4 > 0$ (cf. [138, Theorem 7.7], [2, Theorem 4.12]). If β is now smaller than or equal to

$$\beta_0 := \min\left(\frac{|G|}{4K_3^{\frac{1}{3}} K_4^2}, \theta_G \right),$$

we gain that

$$\int_G |\nabla \varphi_\beta|^2 = 0$$

from the previous inequality, which implies that $\varphi_\beta = 0$ almost everywhere as $\int_G \varphi_\beta = 0$. Therefore

$$\inf_{\varphi \in W^{1,2}(G)} J_\beta(\varphi) = J_\beta(\varphi_\beta) = J_\beta(0) = 0$$

for all $\beta \in (0, \beta_0]$, which completes the proof. □

This new insight now allows us to significantly improve upon Corollary 2.3.2 (along one specific axis) by manipulating some of the terms in the functional J_β defined in (2.4.1) to gain the following:

Corollary 2.4.4. *For each* $0 < \beta \le \beta_0$ *and* $\varphi \in W^{1,2}(G)$*, we have*

$$\int_G e^\varphi \le |G| \exp\left(\frac{1}{4\beta}\int_G |\nabla\varphi|^2 + \frac{1}{|G|}\int_G \varphi\right) \tag{2.4.7}$$

with β_0 *from Lemma 2.4.3.*

2.5 Proving our new functional inequalities

After this brief excursion into the calculus of variations and the theory of elliptic problems, we will now refocus our efforts on proving Theorem 2.1.1 using our optimal Corollary 2.4.4 and similar methods as those seen in [135, Section 2.1]:

Proof of Theorem 2.1.1. Let $\varphi \in W^{1,2}(G)$, $\psi \in L^p(G)$ with $p > 1$, let ψ be positive and let $m := \int_G \psi > 0$. Observe now for any $a > 0$ that

$$\begin{aligned}
\ln\left(\int_G e^{a\varphi}\right) &= \ln\left(\int_G e^{a\varphi}\frac{m}{\psi}\frac{\psi}{m}\right) \\
&\ge \int_G \left(\ln(e^{a\varphi}) + \ln\left(\frac{m}{\psi}\right)\right)\frac{\psi}{m} \\
&= \frac{a}{m}\int_G \varphi\psi - \frac{1}{m}\int_G \psi \ln\left(\frac{\psi}{m}\right)
\end{aligned}$$

by Jensen's inequality. Note that our choices of function spaces for φ and ψ ensure that all the integrals are well defined. If we now combine this with Corollary 2.4.4 (applied to $a\varphi$) and multiply by $\frac{m}{a}$, we get that

$$\int_G \varphi\psi \le \frac{1}{a}\int_G \psi \ln\left(\frac{\psi}{m}\right) + \frac{m}{a}\ln\left(|G|\exp\left(\frac{a^2}{4\beta_0}\int_G |\nabla\varphi|^2 + \frac{a}{|G|}\int_G \varphi\right)\right)$$

$$= \frac{1}{a}\int_G \psi \ln\left(\frac{\psi}{m}\right) + \frac{am}{4\beta_0}\int_G |\nabla\varphi|^2 + \frac{m}{|G|}\int_G \varphi + \frac{m}{a}\ln(|G|)$$
$$= \frac{1}{a}\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) + \frac{a}{4\beta_0}\left\{\int_G \psi\right\}\int_G |\nabla\varphi|^2 + \frac{m}{|G|}\int_G \varphi$$

or further that

$$\int_G \varphi(\psi - \overline{\psi}) \le \frac{1}{a}\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) + \frac{a}{4\beta_0}\left\{\int_G \psi\right\}\int_G |\nabla\varphi|^2$$

after some rearranging with $\overline{\psi} \coloneqq \frac{1}{|G|}\int_G \psi$, which is (F1) exactly.

For $\psi \in L^p(G)$ with $p > 1$, ψ positive with $\ln(\psi) \in W^{1,2}(G)$, we now set

$$\varphi \coloneqq \ln\left(\frac{\psi}{\overline{\psi}}\right) \qquad \text{and} \qquad a \coloneqq 2$$

in the previous inequality to get that

$$\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) - \overline{\psi}\int_G \ln\left(\frac{\psi}{\overline{\psi}}\right) \le \frac{1}{2}\int_G \psi \ln\left(\frac{\psi}{\overline{\psi}}\right) + \frac{1}{2\beta_0}\left\{\int_G \psi\right\}\int_G |\nabla \ln(\psi)|^2.$$

Because by Jensen's inequality we have

$$\int_G \ln\left(\frac{\psi}{\overline{\psi}}\right) \le |G| \ln\left(\frac{\frac{1}{|G|}\int_G \psi}{\overline{\psi}}\right) = |G|\ln(1) = 0,$$

this directly implies the inequality (F2). □

3 Eventually smooth generalized solutions to a chemotaxis-fluid system with rotational flux

3.1 Main result

In this chapter, we turn our attention to the chemotaxis-fluid system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - f(c)n, \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla \cdot u = 0. \end{cases} \tag{CF}$$

in a bounded domain $\Omega \subseteq \mathbb{R}^2$ with a smooth boundary. We further add the boundary conditions

$$\nabla n \cdot \nu = n(S(x,n,c)\nabla c) \cdot \nu, \;\; \nabla c \cdot \nu = 0, \;\; u = 0 \quad \text{for all } x \in \partial\Omega, t > 0 \tag{CFB}$$

and initial conditions

$$n(x,0) = n_0(x), \;\; c(x,0) = c_0(x), \;\; u(x,0) = u_0(x) \quad \text{for all } x \in \Omega \tag{CFI}$$

for initial data with the properties

$$\begin{cases} n_0 \in C^\iota(\overline{\Omega}) & \text{for some } \iota > 0 \text{ and with } n_0 > 0 \text{ in } \overline{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega) & \text{with } c_0 > 0 \text{ in } \Omega, \\ u_0 \in D(A_2^\vartheta) & \text{for some } \vartheta \in (\frac{1}{2}, 1) \end{cases} \tag{3.1.1}$$

to (CF). Here, A_2 denotes the Stokes operator on the Hilbert space $L^2_\sigma(\Omega)$ as defined in Remark 3.6.6.

Assuming further that the functions f, S and ϕ, which parameterize (CF), have the properties that

$$f \in C^1([0,\infty)) \quad \text{with} \quad f(0) = 0 \quad \text{and} \quad f(c) > 0 \quad \text{for all } c \in (0,\infty), \tag{3.1.2}$$

that, for $S = (S_{ij})_{i,j\in\{1,2\}}$,

$$S_{ij} \in C^2(\overline{\Omega} \times [0,\infty) \times [0,\infty)) \quad \text{for } i,j \in \{1,2\}, \tag{3.1.3}$$

that

$$|S(x,n,c)| \le S_0(c) \quad \text{for all } (x,n,c) \in \overline{\Omega} \times [0,\infty)^2 \tag{3.1.4}$$

and some nondecreasing $S_0 : [0,\infty) \to [0,\infty)$ as well as that

$$\phi \in W^{2,\infty}(\Omega), \tag{3.1.5}$$

we show that the above system admits global generalized solutions. We will further show that the solutions we construct become eventually smooth and stabilize. That is, we prove the following theorem:

Theorem 3.1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with a smooth boundary. Assume further that f, S, ϕ satisfy (3.1.2)–(3.1.5) and the initial data (n_0, c_0, u_0) have the properties outlined in (3.1.1). Then the system (CF) with initial data and boundary conditions (CFI) and (CFB) has a global mass-preserving generalized solution (n,c,u) in the sense of Definition 3.3.1 below and there exists a time $t_0 > 0$ such that*

$$(n,c,u) \in C^{2,1}(\overline{\Omega} \times [t_0,\infty)) \times C^{2,1}(\overline{\Omega} \times [t_0,\infty)) \times C^{2,1}(\overline{\Omega} \times [t_0,\infty); \mathbb{R}^2). \tag{3.1.6}$$

Further, there exists $P \in C^{1,0}(\overline{\Omega} \times [t_0,\infty))$ such that (n,c,u,P) is a classical solution of (CF) on $\Omega \times (t_0,\infty)$ with boundary conditions (CFB).

Additionally,

$$n(\cdot,t) \to \frac{1}{|\Omega|}\int_\Omega n_0, \quad c(\cdot,t) \to 0, \quad u(\cdot,t) \to 0 \tag{3.1.7}$$

in $C^2(\overline{\Omega})$ or $C^2(\overline{\Omega}; \mathbb{R}^2)$, respectively, as $t \to \infty$.

3.2 Approach

As will be not be the last time in this thesis, our arguments in this chapter will be based on first defining a family of approximated versions of the system in question indexed by some parameter $\varepsilon \in (0,1)$ as seen in (CF$_\varepsilon$), which easily admit some approximate solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$. We then derive sufficient uniform bounds to construct our desired generalized solution in the sense of Definition 3.3.1 as the limit of their approximate counterparts along a suitable sequence of parameters ε by standard compact embedding results for appropriate function spaces. Working on this approximate level will further afford us the luxury of being able to largely stay in the realm of classical solutions to derive even our additional long-time smoothing and stabilization results for these generalized solutions at the cost of our arguments only applying to exactly the solutions we construct here, which we do not prove to be necessarily unique.

At a fundamental level, these arguments will rest on some important pieces of a priori information, which we derive in Section 3.4.2 by utilizing some key testing procedures and making careful use of the new functional inequalities from Theorem 2.1.1. We begin this section by making use of the favorable consumption term in the second equation in (CF_ε) to gain the following fairly strong bounds for the second solution components c_ε:

$$\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \|c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \quad \text{and} \quad \int_t^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,t) \tag{3.2.1}$$

for all $p \in [1,\infty]$, $t \ge 0$, $s \in [0,t]$ and $\varepsilon \in (0,1)$. By testing the first equation in (CF_ε) with $\frac{1}{n_\varepsilon}$, we then gain the following only fairly weak, yet crucial, a priori information for the first solution components n_ε:

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le \int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} \le C \qquad \text{for all } \varepsilon \in (0,1) \tag{3.2.2}$$

Employing the new functional inequality in (F2), we can then use this information to gain further uniform bounds of the form

$$\int_0^\infty \int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{3.2.3}$$

Using the above bound for n_ε combined with the new functional inequality in (F1) to control the buoyancy term in the third equation in (CF_ε), we further gain that

$$\sup_{t\in(0,\infty)} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C, \qquad \int_0^\infty \int_\Omega |u_\varepsilon|^2 \le C \quad \text{and} \quad \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le C \tag{3.2.4}$$

holds for all $\varepsilon \in (0,1)$.

Notably, time-local version of many of these bounds could also be achieved by using non-optimal versions of our functional inequalities from (F1) and (F2) as e.g. found in [135, Section 2.1]. While this would pose no additional challenge for our existence theory, it would naturally make the above estimates much less useful for our long-time behavior analysis. Our corresponding arguments could essentially be adapted to still work with such weaker a priori information if we introduced an additional small mass assumption but our improved functional inequalities and the thus resulting better a priori information make such an additional (and in this light purely technical) small mass assumption unnecessary.

Construction of generalized solutions. Following in the footsteps of a similar construction carried out for the Stokes case in [132], we use the above a priori bounds to derive some additional space-time integrability properties for the time derivatives of $\ln(n_\varepsilon+1)$, c_ε, u_ε in Lemma 3.5.1 to facilitate an argument based on the Aubin–Lions lemma in Lemma 3.5.2 to construct solution candidate functions n, c and u as limits of the approximate solutions along a suitable sequence of parameters ε. As this approach inherently only yields fairly weak convergence properties for the first solution components and only for terms of the form $\ln(n_\varepsilon+1)$, we then make use of the bound in (3.2.3) to improve the convergence properties for the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in Lemma 3.5.3 by employing the de la Vallée Poussin criterion for uniform integrability as well as the

Vitali convergence theorem. Finally by revisiting an argument based on Steklov averages from [132] to further improve the convergence properties of the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in Lemma 3.5.4, we can prove that all necessary solution properties in fact survive the limit process in Lemma 3.5.5 and thus the limit functions are indeed a mass-preserving generalized solution in the sense of Definition 3.3.1.

Eventual smoothness and stabilization. As in our arguments eventual stabilization and smoothing will be inextricably linked, our first step in proving both of these properties will be deriving a fairly strong uniform convergence property for the second solution components c_ε. In fact using our functional inequality from (F1) combined with (3.2.1) and (3.2.3), we can show in Lemma 3.6.1 that the mass of c_ε and thus every $L^p(\Omega)$-norm of c_ε with finite p must become arbitrarily small as $t \to \infty$ in a uniform fashion. As a direct consequence of this and again (3.2.1), we further gain that $\int_t^\infty \int_\Omega |\nabla c_\varepsilon|^2$ becomes uniformly small as $t \to \infty$ as well. While the former is already a great indication of our expected stabilization behavior, it is the latter that will prove crucial for our further arguments here.

We can now introduce what is arguably the very core of our remaining arguments in this chapter, namely the (eventual) energy functional

$$\mathcal{F}_\varepsilon(t) \coloneqq \int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2C}\int_\Omega |u_\varepsilon|^2. \tag{3.2.5}$$

By analyzing the time evolution of this functional, we can show that, if $\mathcal{F}_\varepsilon$ ever becomes small at some point in time, it in fact stays similarly small from that time on. More precisely, we show in Lemma 3.6.3 that, if $\mathcal{F}_\varepsilon(t_0) \le \frac{\delta}{8C}$ and $\int_{t_0}^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{\delta}{8C^2}$ holds for sufficiently small $\delta > 0$ at some uniform point in time $t_0 > 0$, then not only will the functional itself stay smaller than δ for all $t > t_0$ but also space-time integrals of some key dissipative terms connected to the functional will become small as well. It is this conditionality of our central energy functional where the link between stabilization and smoothing in our arguments reveals itself most clearly as the above means that we can only gain a uniform energy bound once we are already close to stabilizing in our approach.

That it is always possible to find a uniform $t_0 > 0$ for any $\delta > 0$ such that prerequisites for Lemma 3.6.3 are fulfilled is now an easy consequence of (3.2.1)–(3.2.4) as well as the long-time behavior properties for c_ε we already discussed two paragraphs ago. Therefore, $\mathcal{F}_\varepsilon$ must become uniformly small as $t \to \infty$, which not only essentially provides us with some already pretty reasonable convergence properties for our solution but also crucially with some improved uniform a priori information after a potential waiting time.

This new a priori information then proves to be an important foothold to derive much better $L^p(\Omega)$-type bounds for our solution components as well as some of their derivatives by way of a bootstrap procedure in Section 3.6.3. Based on the methods presented in [135], we begin this procedure by deriving better bounds for n_ε and u_ε via testing-based methods immediately followed by the derivation of even stronger bounds for all solution components based on semigroup methods. Expanding on this further, we then devote Section 3.6.4 to the derivation of $C^{\alpha,\frac{\alpha}{2}}$-type parabolic Hölder bounds for all solution components via a mixture of semigroup methods inspired by [34] and [120, Lemma 3.4] as well as standard parabolic regularity theory. Shifting from the approximate level to our concrete generalized solution now, we then use these fairly strong bounds and the compact embedding properties of such Hölder spaces to show in Section 3.6.5 that our generalized solution must have been of a correspondingly high level of regularity after some appropriate waiting time as well and in fact satisfy a more standard notion of weak solution than Definition 3.3.1. This then allows us to apply parabolic regularity theory to our generalized solutions directly to show that they must have already been classical after said waiting time. We employ this argument on the level of our limit function as opposed to still working with the approximate system to avoid the regularizations in (CF_ε) complicating our application of this higher order regularity theory.

Finally using the strong eventual boundedness properties derived for our generalized solutions during the arguments laid out in the previous paragraph combined with the already mentioned uniform eventual smallness of $\mathcal{F}_\varepsilon$, we can fairly quickly derive our desired stabilization properties in Section 3.6.6 as well. As this was the last piece of the puzzle still missing, this essentially completes our proof of Theorem 3.1.1.

3.3 Generalized solution concept

We begin our arguments proper by introducing the central solution concept we will be concerned with in this chapter. Because of the similarities of the problem discussed in this chapter to the pure Stokes case seen in [132] and our desire to not unnecessarily duplicate effort, we let ourselves be guided by the generalized solution concept presented in said reference, which we of course slightly adapt to the full Navier–Stokes case. This reads as follows:

Definition 3.3.1. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, be a bounded domain with a smooth boundary and $f \in C^0([0,\infty))$, $S \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^{N\times N})$ as well as $\phi \in W^{1,\infty}(\Omega)$ some parameter functions. We then call a triple of functions

$$\begin{aligned} &n \in L^\infty((0,\infty); L^1(\Omega)), \\ &c \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \cap L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)) \quad \text{and} \\ &u \in L^\infty_{\mathrm{loc}}([0,\infty); (L^2(\Omega))^2) \cap L^2_{\mathrm{loc}}([0,\infty); (W^{1,2}_0(\Omega))^2) \end{aligned} \tag{3.3.1}$$

with $n \geq 0$, $c \geq 0, \nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$,

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \quad \text{for a.e. } t > 0 \tag{3.3.2}$$

and

$$\ln(n+1) \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)) \tag{3.3.3}$$

a global mass-preserving generalized solution of (CF), (CFB), (CFI) and (3.1.1) if the inequality

$$\begin{aligned} &-\int_0^\infty\int_\Omega \ln(n+1)\varphi_t - \int_\Omega \ln(n_0+1)\varphi(\cdot,0) \\ &\geq \int_0^\infty\int_\Omega \ln(n+1)\Delta\varphi + \int_0^\infty\int_\Omega |\nabla \ln(n+1)|^2\varphi \\ &\quad - \int_0^\infty\int_\Omega \frac{n}{n+1}\nabla \ln(n+1)\cdot(S(x,n,c)\nabla c)\varphi \\ &\quad + \int_0^\infty\int_\Omega \frac{n}{n+1}(S(x,n,c)\nabla c)\cdot\nabla\varphi \\ &\quad + \int_0^\infty\int_\Omega \ln(n+1)(u\cdot\nabla\varphi) \end{aligned} \tag{3.3.4}$$

holds for all nonnegative $\varphi \in C_c^\infty(\overline{\Omega}\times[0,\infty))$ with $\nabla\varphi\cdot\nu = 0$ on $\partial\Omega\times[0,\infty)$, if further

$$\int_0^\infty\int_\Omega c\varphi_t + \int_\Omega c_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi + \int_0^\infty\int_\Omega nf(c)\varphi - \int_0^\infty\int_\Omega c(u\cdot\nabla\varphi) \tag{3.3.5}$$

holds for all $\varphi \in L^\infty(\Omega\times(0,\infty))\cap L^2((0,\infty);W^{1,2}(\Omega))$ having compact support in $\overline{\Omega}\times[0,\infty)$ with $\varphi_t \in L^2(\Omega\times(0,\infty))$, and if finally

$$-\int_0^\infty\int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\varphi + \int_0^\infty\int_\Omega n\nabla\phi\cdot\varphi \tag{3.3.6}$$

holds for all $\varphi \in C_c^\infty(\Omega\times[0,\infty);\mathbb{R}^2)$ with $\nabla\cdot\varphi = 0$ on $\Omega\times[0,\infty)$.

Remark 3.3.2. It can be shown that generalized solutions of this type become classical if the functions n, c and u as well as the relevant parameter functions f, S and ϕ are sufficiently regular. The argument for this can be sketched as follows: For the solution components u and c, this is standard as both satisfy the common variational formulation of their respective equations while the solution component n presents us with somewhat more of a challenge as it only satisfies a very specific integral inequality and mass conservation property. That this is already sufficient has for instance been argued in [129, Lemma 2.1] for the case $u \equiv 0$ and the argument transfers easily.

3.4 Approximate solutions

Similar to e.g. the approach seen in [132], the key to our construction of generalized solutions with the desired eventual smoothness and stabilization properties will be a family of regularized versions of the problem in (CF) with (CFB) and (CFI) as this will allow us to essentially work with global classical solutions for large parts of this chapter.

3.4.1 Regularized versions of (CF)

We define the aforementioned regularized problems as follows: We first fix families of functions $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ and $(\chi_\varepsilon)_{\varepsilon\in(0,1)}$ with

$$\rho_\varepsilon \in C_c^\infty(\Omega) \quad \text{such that} \quad \begin{aligned}&0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \quad \text{and}\\ &\rho_\varepsilon \nearrow 1 \text{ pointwise in } \Omega \text{ as } \varepsilon \searrow 0\end{aligned}$$

and

$$\chi_\varepsilon \in C_c^\infty([0,\infty)) \quad \text{such that} \quad \begin{aligned}&0 \le \chi_\varepsilon \le 1 \text{ in } [0,\infty) \quad \text{and}\\ &\chi_\varepsilon \nearrow 1 \text{ pointwise in } [0,\infty) \text{ as } \varepsilon \searrow 0\end{aligned}$$

constructed by standard methods. For $\varepsilon \in (0,1)$, we then define

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)\chi_\varepsilon(n)S(x,n,c) \qquad \text{for all } (x,n,c) \in \Omega \times [0,\infty)^2$$

and consider the following initial boundary value problems:

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega,\ t > 0,\\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon), & x \in \Omega,\ t > 0,\\ u_{\varepsilon t} + (u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t > 0,\\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0,\\ \nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu = 0, \ \ u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0,\\ n_\varepsilon(x,0) = n_0(x),\ c_\varepsilon(x,0) = c_0(x),\ u_\varepsilon(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{CF$_\varepsilon$}$$

This regularized version of (CF) with (CFB) and (CFI) then easily admits a global classical solution because it replaces the more involved no-flux boundary condition seen in (CFB) with standard Neumann boundary conditions and makes the first equation accessible to comparison arguments with a non-zero constant to gain a global upper bound for n_ε, which would be much harder to achieve otherwise. This of course only works under similar assumptions on the parameter functions f, S, ϕ as stated at the beginning of this chapter. As the techniques to achieve such an existence result for the approximated system above are fairly well-documented and do not appreciably differ for our case in comparison to e.g. the case studied in [125], we will only give a short sketch to justify the following lemma:

Lemma 3.4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with a smooth boundary. Then for $\varepsilon \in (0,1)$, initial data with regularity properties as seen in (3.1.1) and f, S, ϕ satisfying (3.1.2)–(3.1.5), there exist functions*

$$\begin{aligned} n_\varepsilon, c_\varepsilon &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),\\ u_\varepsilon &\in C^0(\overline{\Omega} \times [0,\infty); \mathbb{R}^2) \cap C^{2,1}(\overline{\Omega} \times (0,\infty); \mathbb{R}^2),\\ P_\varepsilon &\in C^{1,0}(\Omega \times (0,\infty)) \end{aligned}$$

such that $n_\varepsilon > 0$, $c_\varepsilon \ge 0$ on $\overline{\Omega} \times (0,\infty)$ and $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (CF$_\varepsilon$).

Proof. Standard contraction mapping methods in an appropriate setting (as e.g. seen in [125, Lemma 2.1] for a similar system) provide us with a classical solution for (CF_ε) on a space-time cylinder $\Omega \times [0, T_{\max,\varepsilon})$ with some maximal $T_{\max,\varepsilon} \in (0, \infty]$ and a blowup criterion of the following type:

$$\text{If } T_{\max,\varepsilon} < \infty,$$
$$\text{then } \limsup_{t \nearrow T_{\max,\varepsilon}} \Big(\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A_2^\vartheta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \Big) = \infty.$$

Here, q is some number greater than 2 and ϑ is as in (3.1.1). Nonnegativity and positivity on $\overline{\Omega} \times (0, T_{\max,\varepsilon})$ of c_ε and n_ε respectively are immediately ensured by the maximum principle. Note now further that standard comparison arguments can be used to immediately gain a global upper bound for n_ε on the whole space-time cylinder because we defined S_ε to be zero for sufficiently large values of n. This already rules out blowup regarding n_ε. As the second equation in (CF_ε) is generally fairly unproblematic in this regard as well, similar boundedness results can be achieved for c_ε (cf. Lemma 3.4.2). Regarding the possible blowup of c_ε or u_ε, we can look at the prior work done in Section 4.2 of [125], where it has been proven that the two norms in the blowup criterion regarding c_ε and u_ε respectively are bounded for much weaker prerequisites as already established here. Note that this is mostly done using the second and third equation of the system studied in said reference, which are apart from some slight generalizations the same as the second and third equation in (CF_ε). Only one step in the reference uses an energy inequality not available to us to establish a bound for $\int_\Omega |\nabla c_\varepsilon|^2$, which in our case can be easily gained by a straightforward testing procedure for the second equation in (CF_ε) without using said energy inequality. Thus we can fully rule out blowup and $T_{\max,\varepsilon} = \infty$ must always hold. □

For the remainder of this chapter, we now fix a smoothly bounded domain $\Omega \subseteq \mathbb{R}^2$, some initial values (n_0, c_0, u_0) with regularity properties as seen in (3.1.1) and parameter functions f, S and ϕ satisfying (3.1.2)–(3.1.5). We then further fix a corresponding family of solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon \in (0,1)}$ to (CF_ε) as constructed in Lemma 3.4.1.

3.4.2 Basic a priori estimates

As the starting point for both the construction of our generalized solutions as well as the arguments yielding their more intricate qualitative properties, we now derive some fairly standard uniform a priori estimates for our fixed family of approximate solutions by testing methods.

Lemma 3.4.2. *The mass conservation property*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \tag{3.4.1}$$

holds for all $t > 0$, $\varepsilon \in (0,1)$ *and, for each* $p \in [1,\infty]$, *the inequality*

$$\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \|c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \tag{3.4.2}$$

holds for all $t \ge s \ge 0$ *and* $\varepsilon \in (0,1)$. *We further have that*

$$\int_t^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,t) \tag{3.4.3}$$

for all $t > 0$, $\varepsilon \in (0,1)$ *and there exists* $C > 0$ *such that*

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le \int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} \le C \tag{3.4.4}$$

for all $\varepsilon \in (0,1)$.

Proof. The mass conservation property in (3.4.1) is immediately evident after integration of the first equation in (CF_ε). Further, testing the second equation in (CF_ε) with c_ε^{p-1} for $p \in [1,\infty)$ gives us

$$\frac{1}{p}\int_\Omega c_\varepsilon^p(\cdot,t) + (p-1)\int_s^t \int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + \int_s^t \int_\Omega n_\varepsilon c_\varepsilon^{p-1} f(c_\varepsilon) = \frac{1}{p}\int_\Omega c_\varepsilon^p(\cdot,s)$$

for all $t \ge s \ge 0$, which immediately yields (3.4.2) for finite p as well as (3.4.3). The case $p = \infty$ in (3.4.2) then follows by taking the limit $p \to \infty$. To now derive (3.4.4), we test the first equation in (CF_ε) with $\frac{1}{n_\varepsilon}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln(n_\varepsilon) = \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} - \int_\Omega \frac{\nabla n_\varepsilon}{n_\varepsilon}\cdot S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \qquad \text{for all } t > 0,$$

which we then further improve to

$$\frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} \le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln(n_\varepsilon) + \frac{S_0^2(\|c_0\|_{L^\infty(\Omega)})}{2}\int_\Omega |\nabla c_\varepsilon|^2$$

for all $t > 0$ by application of Young's inequality, (3.1.4) and (3.4.2). Due to (3.4.1), (3.4.3) and the fact that $\ln(x) \le x$ for all $x > 0$, time integration of the above then results in (3.4.4). □

As reflected by the above lemma, strong uniform a priori information for the first solution components seems much harder to achieve than e.g. for the second solution components. It thus seems prudent to make use of what little information we nonetheless have to the best of our ability. This is where our improved functional inequalities from the previous chapter will make a crucial contribution to allow us to derive a global space-time integral bound slightly stronger than L^1 for the first solution components. By another application of the functional inequalities from Theorem 2.1.1, this in turn allows us to control the buoyancy term in the third equation in (CF_ε) to gain a similar space-time integral bound of L^2-type for the third solution components and their gradients. Notably, previous versions of said same inequalities (cf. e.g. [135, Section 2.1]), which contain an additional mass term, would still yield time-local variants of the results below. While such bounds would be sufficient to construct our generalized solutions, it crucially would not allow us to gain our desired eventual smoothness result without introducing an additional small mass condition on the first solution component, which is avoided by using our improved variants of said functional inequalities.

Corollary 3.4.3. *There exists $C > 0$ such that*

$$\int_0^\infty \int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) \le C$$

for all $\varepsilon \in (0,1)$ with $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$.

Proof. Combining the inequalities in (3.4.1) and (3.4.4) from Lemma 3.4.2 with the functional inequality in (F2) from Theorem 2.1.1 directly yields this. □

Lemma 3.4.4. *There exists $C > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \tag{3.4.5}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ as well as

$$\int_0^\infty \int_\Omega |u_\varepsilon|^2 \le C, \quad \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le C \tag{3.4.6}$$

for all $\varepsilon \in (0,1)$.

Proof. We first test the third equation in (CF_ε) with u_ε to gain that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 &= -\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega n_\varepsilon \nabla\phi \cdot u_\varepsilon \\ &= -\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega (n_\varepsilon - \overline{n_0})(\nabla\phi \cdot u_\varepsilon) \end{aligned}$$

with $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$ for all $t > 0$ and $\varepsilon \in (0,1)$ using that $\int_\Omega \nabla\phi \cdot u_\varepsilon = \int_{\partial\Omega} \phi(u_\varepsilon \cdot \nu) - \int_\Omega \phi \nabla \cdot u_\varepsilon = 0$ due to the fourth and fifth lines in (CF_ε). We can then improve this via our functional inequality in (F1) from Theorem 2.1.1 to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 \le -\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{a}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{a}{4\beta_0}\left\{\int_\Omega n_0\right\}\int_\Omega |\nabla(\nabla\phi \cdot u_\varepsilon)|^2 \tag{3.4.7}$$

for any $a > 0$ and all $t > 0$ as well as $\varepsilon \in (0,1)$. Using that there exists $C_\mathrm{p} > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \le C_\mathrm{p}\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega}) \text{ with } \varphi = 0 \text{ on } \partial\Omega$$

due to the well-known Poincaré inequality (cf. [12, p. 290]), we now further note that

$$\begin{aligned}\int_\Omega |\nabla(\nabla\phi\cdot u_\varepsilon)|^2 &\le 2\int_\Omega |\nabla\phi|^2|\nabla u_\varepsilon|^2 + 2\int_\Omega |D^2\phi|^2|u_\varepsilon|^2 \\ &\le 2\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + 2\|D^2\phi\|^2_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2 \le K_1\int_\Omega |\nabla u_\varepsilon|^2\end{aligned}$$

with $K_1 := 2\|\nabla\phi\|^2_{L^\infty(\Omega)} + 2\|D^2\phi\|^2_{L^\infty(\Omega)}C_{\mathrm{p}}^2$ for all $t > 0$ and $\varepsilon \in (0,1)$, where $D^2\phi$ is the Hessian of ϕ. If we now choose

$$a := \frac{2\beta_0}{K_1}\left\{\int_\Omega n_0\right\}^{-1}$$

in the inequality in (3.4.7), we gain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 \le -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{K_1}{2\beta_0}\left\{\int_\Omega n_0\right\}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right)$$

for all $t > 0$ and $\varepsilon \in (0,1)$. After time integration and some rearranging, we then further see that

$$\int_\Omega |u(\cdot,t)|^2 + \int_0^t\int_\Omega |\nabla u_\varepsilon|^2 \le \int_\Omega |u_0|^2 + \frac{K_1}{\beta_0}\left\{\int_\Omega n_0\right\}\int_0^t\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right)$$

for all $t > 0$ and $\varepsilon \in (0,1)$. By the integrability property laid out in Corollary 3.4.3 and the fact that Jensen's inequality ensures that $\int_\Omega n_\varepsilon \ln(\frac{n_\varepsilon}{\overline{n_0}}) \ge 0$, there then moreover exists a constant $K_2 > 0$ such that

$$\int_\Omega |u(\cdot,t)|^2 + \int_0^t\int_\Omega |\nabla u_\varepsilon|^2 \le \int_\Omega |u_0|^2 + \frac{K_1K_2}{\beta_0}\left\{\int_\Omega n_0\right\}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. This together with one last application of the Poincaré inequality completes the proof. □

3.5 Construction of generalized solutions

3.5.1 Constructing the limit functions (n, c, u)

Having now established suitable baseline bounds for all three of our solution components in the previous section, we can begin the final preparations for the construction of limit functions for our family of approximate solutions as $\varepsilon \searrow 0$. We do this by proving some additional, albeit still fairly weak, boundedness results for the time derivatives of the three families $(\ln(n_\varepsilon + 1))_{\varepsilon\in(0,1)}$, $(c_\varepsilon)_{\varepsilon\in(0,1)}$ and $(u_\varepsilon)_{\varepsilon\in(0,1)}$ as this will provide the last prerequisite for some central applications of the Aubin–Lions lemma (cf. [112, p. 215]):

Lemma 3.5.1. *For all $T > 0$, there exists a constant $C(T) > 0$ such that*

$$\int_0^T \|\partial_t \ln(n_\varepsilon(\cdot,t)+1)\|_{(W_0^{2,2}(\Omega))^*}\,\mathrm{d}t \le C(T), \tag{3.5.1}$$

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*}\,\mathrm{d}t \le C(T) \text{ and} \tag{3.5.2}$$

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W_{0,\sigma}^{2,2}(\Omega))^*}\,\mathrm{d}t \le C(T) \tag{3.5.3}$$

for all $\varepsilon \in (0,1)$.

Proof. We will only give this proof in full detail for (3.5.3) and then just provide a sketch for (3.5.1) and (3.5.2) as all three inequalities can be proven in quite similar fashion and more detailed proofs for (3.5.1) and (3.5.2) can also be found in [132].

We first fix $\psi \in W_{0,\sigma}^{2,2}(\Omega)$ and then test the third equation in (CF_ε) with ψ to gain that

$$\int_\Omega u_{\varepsilon t}\cdot\psi = -\int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\psi - \int_\Omega \nabla u_\varepsilon \nabla\psi - \int_\Omega P_\varepsilon(\nabla\cdot\psi) + \int_\Omega n_\varepsilon\nabla\phi\cdot\psi$$

for all $t > 0$ and $\varepsilon \in (0,1)$ after some partial integration steps. This then leads to

$$\begin{aligned}
&\left|\int_\Omega u_{\varepsilon t}\cdot\psi\right| \\
&\le \int_\Omega |u_\varepsilon||\nabla u_\varepsilon||\psi| + \int_\Omega |\nabla u_\varepsilon||\nabla\psi| + \int_\Omega n_\varepsilon|\nabla\phi||\psi| \\
&\le \|u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} \\
&\quad + \|\nabla\phi\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\int_\Omega n_\varepsilon \\
&\le \left\{\left(\|u_\varepsilon\|_{L^2(\Omega)}+1\right)\|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|\nabla\phi\|_{L^\infty(\Omega)}\int_\Omega n_0\right\}\left(\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\right)
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ by using the Cauchy–Schwarz inequality and the fact that $\nabla\cdot\psi = 0$. By employing Young's inequality, the fact that $W^{2,2}(\Omega)$ embeds continuously into $L^\infty(\Omega)$ and (3.4.5) from Lemma 3.4.4, we see that there exist constants $K_1, K_2 > 0$ such that

$$\int_0^T \|u_{\varepsilon t}\|_{(W_{0,\sigma}^{2,2}(\Omega))^*} \le \int_0^T \left(K_1\int_\Omega |\nabla u_\varepsilon|^2 + K_2\right)$$

for all $\varepsilon \in (0,1)$. Another application of Lemma 3.4.4 and specifically (3.4.6) therein then directly gives us the desired bound for the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$.

By testing the first equation in (CF_ε) with $\frac{\psi}{n_\varepsilon+1}$ and the second equation in (CF_ε) with ψ for any $\psi \in W^{2,2}(\Omega)$, we gain that

$$\left|\int_\Omega \partial_t\ln(n_\varepsilon+1)\cdot\psi\right| \le K_3\left\{1 + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2\right\}\times$$

$$\Big(\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big)$$

and

$$\Big|\int_\Omega c_{\varepsilon t}\cdot\psi\Big| \le K_3\Big\{1+\int_\Omega|\nabla c_\varepsilon|^2+\int_\Omega|u_\varepsilon|^2\Big\}\Big(\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big)$$

for all $t > 0$, $\varepsilon \in (0,1)$ and some $K_3 > 0$ by similar techniques as seen above or in the proof of Lemma 3.4 in [132]. Combining these two inequalities with Lemma 3.4.2 and Lemma 3.4.4 then yields the remaining two bounds for the families $(n_\varepsilon)_{\varepsilon\in(0,1)}$, $(c_\varepsilon)_{\varepsilon\in(0,1)}$ and therefore completes the proof. □

This then allows us to use essentially three applications of the Aubin–Lions lemma (cf. [112, p. 215]) to prove the following sequence selection and convergence result:

Lemma 3.5.2. *There exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq (0,1)$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ such that*

$$\begin{cases} n_\varepsilon \to n & \textit{a.e. in } \Omega\times(0,\infty),\\ \ln(n_\varepsilon+1) \rightharpoonup \ln(n+1) & \textit{in } L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega)),\\ c_\varepsilon\to c & \textit{in } L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)) \textit{ and a.e. in } \Omega\times(0,\infty),\\ c_\varepsilon(\cdot,t)\to c(\cdot,t) & \textit{in } L^2(\Omega) \textit{ for a.e. } t>0,\\ c_\varepsilon \rightharpoonup c & \textit{in } L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega)),\\ u_\varepsilon\to u & \textit{in } (L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)))^2 \textit{ and a.e. in } \Omega\times(0,\infty) \textit{ and}\\ u_\varepsilon \rightharpoonup u & \textit{in } L^2_{\mathrm{loc}}([0,\infty);W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{3.5.4}$$

as $\varepsilon=\varepsilon_j\searrow 0$ and a triple of limit functions (n,c,u) defined on $\Omega\times(0,\infty)$ and satisfying $n,c\ge 0$ almost everywhere and $\nabla\cdot u = 0$.

Proof. By applying Lemma 3.4.2 and Lemma 3.5.1 as well as the Aubin–Lions lemma (cf. [112, p. 215]), we immediately gain relative compactness of the families $(\ln(n_\varepsilon+1))_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega))$ with respect to the weak topology and in $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and therefore $L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty))$ with respect to the default topology. Moreover, by the boundedness properties presented in Lemma 3.4.4 and Lemma 3.5.1 and another application of the Aubin–Lions lemma to the triple of function spaces

$$W^{1,2}_{0,\sigma}(\Omega)\subseteq L^2_\sigma(\Omega)\subseteq (W^{2,2}_{0,\sigma}(\Omega))^*,$$

we gain relative compactness of the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $(L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)))^2$ with respect to the default topology and in $L^2_{\mathrm{loc}}([0,\infty);W^{1,2}_{0,\sigma}(\Omega))$ with respect to the weak topology because of the inequalities in (3.4.5), (3.4.6) and (3.5.3). By multiple standard subsequence extraction arguments, we can therefore construct a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ with the convergence properties seen in (3.5.4).

Note that the nonnegativity properties directly transfer from the approximate functions because of pointwise convergence while $\nabla \cdot u = 0$ is directly ensured by u being an element of $L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{0,\sigma}(\Omega))$. □

For the remainder of this chapter, we now fix (n, c, u) as constructed in Lemma 3.5.2 as well as the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ used in said construction.

3.5.2 Proving the generalized solution properties of (n, c, u)

While the convergence properties outlined in Lemma 3.5.2 are already almost sufficient for proving that the limit functions found in said lemma are in fact solutions in the sense of Definition 3.3.1, we still need to prove that the sequences converge in slightly stronger ways to fully complete this argument. In this vein, we first focus on the sequence $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ as it thus far exhibits the weakest convergence properties. To improve this, we will now show that $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ converges towards n in an L^1 fashion as well. While we can find a proof for this in [132], which should still work in our case without any modification, we present a somewhat shorter argument here based on Corollary 3.4.3 and standard results about uniform integrability.

Lemma 3.5.3. *The limit function n has the further convergence property*

$$n_\varepsilon \to n \qquad \text{in } L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \text{ as } \varepsilon = \varepsilon_j \searrow 0.$$

Proof. Fix $T > 0$. For

$$G(t) := t \ln\left(\frac{t}{\overline{n_0}}\right) + \overline{n_0} > 0 \qquad \text{for all } t \geq 0,$$

observe that

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty$$

and there exists a constant $K > 0$ such that

$$\int_0^T \int_\Omega G(|n_\varepsilon|) = \int_0^T \int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \int_0^T \int_\Omega \overline{n_0} \leq K + \overline{n_0}|\Omega| T$$

for all $\varepsilon \in (0,1)$ by Corollary 3.4.3. Therefore, the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ fulfills the De La Vallée Poussin criterion for uniform integrability on $\Omega \times (0,T)$ (cf. [25, p. 24]).

Because Lemma 3.5.2 furthermore ensures pointwise convergence of the sequence $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ to n almost everywhere as $j \to \infty$, all prerequisites for the Vitali convergence theorem (cf. [25, p. 23]) are therefore met and it directly provides us with the desired $L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ convergence as $T > 0$ was arbitrary. □

Though we have already ensured certain weak convergence properties for the first derivatives of the sequence $(c_{\varepsilon_j})_{j\in\mathbb{N}}$ in Lemma 3.5.2, they appear to be insufficient to ensure convergence of the type of terms seen in (3.3.4) stemming from the modeling of the chemotactic interaction. Thus as the final step of our construction, we will show that the gradients of the sequence $(c_{\varepsilon_j})_{j\in\mathbb{N}}$ not only converge in a weak sense but also already in a strong sense using a technique based on an approximation by Steklov averages. As our application of this technique here is very similar to arguments presented for the proof of Lemma 4.4 in [132] and Lemma 8.2 in [129], we will only give a sketch of the critical steps to prove the following lemma.

Lemma 3.5.4. *The limit function c satisfies (3.3.5) for all $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ having compact support in $\overline{\Omega} \times [0,\infty)$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$ and the convergence property*

$$\nabla c_\varepsilon \to \nabla c \quad \text{in } L^2_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \text{ as } \varepsilon = \varepsilon_j \searrow 0$$

holds.

Proof. Note first that, due to the convergence properties seen in Lemma 3.5.2 and the fact that all c_ε satisfy (3.3.5) for appropriate φ as a consequence of them being a classical solution of the corresponding partial differential equation, it is easy to see that c satisfies (3.3.5) for appropriate φ as well by passing to the limit $\varepsilon \searrow 0$.

We further observe that the convergence properties for $(c_\varepsilon)_{\varepsilon\in(0,1)}$ shown in Lemma 3.5.2 already give us that

$$\int_0^T \int_\Omega |\nabla c|^2 \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2$$

for all $T > 0$ and we therefore only need to show that a similar estimate from below also holds.

Our argument to achieve this will be twofold: First, we construct a family of test functions φ that are essentially time averaged versions of c with approximated initial data, which sufficiently approximate c itself. This ensures the necessary regularity for the family of test functions to be used in (3.3.5) and after some limit processes we gain that

$$\frac{1}{2}\int_\Omega c^2(\cdot,T) - \frac{1}{2}\int_\Omega c_0^2 + \int_0^T \int_\Omega |\nabla c|^2 \ge -\int_0^T \int_\Omega ncf(c) \tag{3.5.5}$$

for all $T \in (0,\infty)\setminus N$ whereby N is a null set such that $(0,\infty)\setminus N$ contains only Lebesgue points of the map

$$(0,\infty) \to [0,\infty), \quad t \mapsto \int_\Omega c^2(x,t)\,\mathrm{d}x.$$

We only consider the aforementioned Lebesgue points to ensure that certain time averages converge properly. For the full details, see e.g. Lemma 8.1 in [129].

Second by potentially enlarging N, we can further assume that outside of N the integral $\int_\Omega c_\varepsilon^2(\cdot,t)$ converges to $\int_\Omega c^2(\cdot,t)$ as $\varepsilon = \varepsilon_j \searrow 0$ because of Lemma 3.5.2 without loss of generality. Combining this with (3.5.5), uniform L^∞ boundedness of the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ from Lemma 3.4.2 and the L^1 convergence property from Lemma 3.5.3 then yields that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla c|^2 &\geq -\frac{1}{2}\int_\Omega c^2(\cdot,T) + \frac{1}{2}\int_\Omega c_0^2 - \int_0^T\int_\Omega ncf(c)\\ &= \lim_{\varepsilon=\varepsilon_j\searrow 0}\left\{-\frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,T) + \frac{1}{2}\int_\Omega c_0^2 - \int_0^T\int_\Omega n_\varepsilon c_\varepsilon f(c_\varepsilon)\right\}\\ &= \lim_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega |\nabla c_\varepsilon|^2\end{aligned}$$

for all $T\in(0,\infty)\setminus N$. This is exactly our desired lower bound for almost every $T\in(0,\infty)$, which is sufficient to complete the proof. □

Having prepared all the necessary convergence results, we can now prove the first half of Theorem 3.1.1, namely that there exist global mass-preserving generalized solutions to (CF) with initial data and boundary conditions (CFI) and (CFB).

Lemma 3.5.5. *The triple of limit functions (n,c,u) is a global mass-preserving generalized solution to (CF) with initial data and boundary conditions (CFI) and (CFB) in the sense of Definition 3.3.1.*

Proof. We have already established some of the properties needed for Definition 3.3.1 in Lemma 3.5.2 as well as Lemma 3.5.4 and therefore now only need to show that (3.3.1), (3.3.2), (3.3.4) and (3.3.6) also hold. The mass conservation property in (3.3.2) follows directly from our L^1 convergence result in Lemma 3.5.3 and the mass conservation property of the approximate solutions seen in Lemma 3.4.2. This then further ensures that n is of the appropriate regularity for (3.3.1) while the remaining regularity properties for c and u are provided by the convergence results in Lemma 3.5.2, the uniform L^∞ bound for the sequence $(c_{\varepsilon_j})_{j\in\mathbb{N}}$ in Lemma 3.4.2 and the uniform L^2 bound for the sequence $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ in Lemma 3.4.4.

It is further easy to see that the approximate solutions satisfy both (3.3.4) and (3.3.6) by partial integration and use of the boundary conditions in (CF_ε). We therefore only need to further argue that these solution properties survive the necessary limit process. For most terms in the integral equality in (3.3.6) concerning u, this is fairly straightforward to show using the convergence properties established in Lemma 3.5.2 but we will nonetheless give the full argument for at least the newly introduced term (compared to the Stokes case) as an example. This is especially pertinent as we needed to establish stronger convergence properties for u to handle this term compared to [132], namely standard L^2 as opposed to weak L^2 convergence.

We first fix $\varphi\in C_0^\infty(\Omega\times[0,\infty);\mathbb{R}^2)$. Then there exists $T>0$ such that

$$\mathrm{supp}(\varphi)\subseteq\Omega\times[0,T].$$

We now observe that

$$\begin{aligned}
&\left| \int_0^\infty \int_\Omega (u_\varepsilon \otimes u_\varepsilon) \cdot \nabla\varphi - \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi \right| \\
&\leq \|\nabla\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left[\int_0^T \int_\Omega |u_\varepsilon - u||u_\varepsilon| + \int_0^T \int_\Omega |u||u_\varepsilon - u| \right] \\
&\leq \|\nabla\varphi\|_{L^\infty(\Omega\times(0,\infty))} \times \\
&\quad \left[\|u_\varepsilon - u\|_{L^2(\Omega\times(0,T))} \|u_\varepsilon\|_{L^2(\Omega\times(0,T))} + \|u\|_{L^2(\Omega\times(0,T))} \|u_\varepsilon - u\|_{L^2(\Omega\times(0,T))} \right]
\end{aligned}$$

for all $\varepsilon \in (0, 1)$, which ensures that

$$\int_0^\infty \int_\Omega (u_\varepsilon \otimes u_\varepsilon) \cdot \nabla\varphi \to \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0$$

because of the $L^2_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty))$ convergence of the sequence $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ towards u.

As it has been pretty thoroughly discussed in the proof of Theorem 1.1 in [132] that (3.3.4) similarly survives a corresponding limit process given similar convergence properties as proven here, we will not go into further depth regarding this point and refer the reader to the reference. □

3.6 Proving the eventual smoothness and stabilization properties of (n, c, u)

Given that we have now successfully constructed the triple of functions (n, c, u) and proven that said triple is in fact a proper generalized solution in the sense of Definition 3.3.1, we turn our attention to proving that our solutions enjoy the further eventual smoothness and stabilization properties outlined in Theorem 3.1.1.

3.6.1 Eventual uniform smallness of the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^p(\Omega)$ for all $p \in [1, \infty)$

As the second equation in $(\mathrm{CF}_\varepsilon)$ is in many ways the easiest to handle, it is not surprising that the first fairly strong result of this section is in fact concerned with the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$. Namely, we will now prove that the $L^p(\Omega)$-norms of said family are not only monotonically decreasing for $p \in [1, \infty)$ as seen in Lemma 3.4.2, but that they in fact tend to zero for $t \to \infty$ in an ε-independent fashion. Similar to our prior results in Corollary 3.4.3 and Lemma 3.4.4, this is again heavily based on our new functional inequalities from Theorem 2.1.1.

Lemma 3.6.1. *For each $p \in [1, \infty)$ and $\delta > 0$, there exists a time $t_0 = t_0(\delta, p) > 0$ such that*

$$\|c_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq \delta$$

for all $t \geq t_0$ *and* $\varepsilon \in (0,1)$.

Proof. As our first step, we integrate the second equation in (CF_ε) to gain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\varepsilon = -\int_\Omega n_\varepsilon f(c_\varepsilon) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

which implies that

$$\int_0^\infty \int_\Omega n_\varepsilon f(c_\varepsilon) \leq \int_\Omega c_0 \qquad \text{for all } \varepsilon\in(0,1) \tag{3.6.1}$$

by time integration. We now rewrite $\int_\Omega f(c_\varepsilon)$ as follows:

$$\int_\Omega f(c_\varepsilon) = \frac{1}{\overline{n_0}}\left[\int_\Omega (\overline{n_0}-n_\varepsilon)f(c_\varepsilon) + \int_\Omega n_\varepsilon f(c_\varepsilon)\right]$$

for all $t>0$ and $\varepsilon\in(0,1)$ with $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$. This then allows us to the apply functional inequality in (F1) from Theorem 2.1.1 (setting $\varphi := -f(c_\varepsilon)$, $\psi := n_\varepsilon$ and $a := 2$) to further see that

$$\begin{aligned}\int_\Omega f(c_\varepsilon) &\leq \frac{1}{\overline{n_0}}\left[\frac{1}{2}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2\beta_0}\left\{\int_\Omega n_0\right\}\int_\Omega |-\nabla f(c_\varepsilon)|^2 + \int_\Omega n_\varepsilon f(c_\varepsilon)\right] \\ &\leq \frac{1}{\overline{n_0}}\left[\frac{1}{2}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{K_1^2}{2\beta_0}\left\{\int_\Omega n_0\right\}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon f(c_\varepsilon)\right]\end{aligned}$$

for all $\varepsilon\in(0,1)$ with β_0 as in Theorem 2.1.1 and $K_1 := \|f'\|_{L^\infty([0,\|c_0\|_{L^\infty(\Omega)}])}$. Considering the integrability properties in Corollary 3.4.3, Lemma 3.4.2 and (3.6.1), there must therefore exist a constant $K_2>0$ such that

$$\int_0^\infty \int_\Omega f(c_\varepsilon) \leq K_2 \qquad \text{for all } \varepsilon\in(0,1). \tag{3.6.2}$$

We now fix $\delta>0$ and then let $\xi := \frac{\delta}{2|\Omega|}$. Because f is positive outside of zero and continuous, there must exist a constant $K_3>0$ such that

$$f(y) \geq K_3 \qquad \text{for all } y\in[\xi, \|c_0\|_{L^\infty(\Omega)}].$$

Because (3.6.2) implies that

$$\frac{1}{t_0}\int_0^{t_0}\int_\Omega f(c_\varepsilon) \leq \frac{K_3}{\|c_0\|_{L^\infty(\Omega)}}\frac{\delta}{2} \qquad \text{for all } \varepsilon\in(0,1)$$

with

$$t_0 := \frac{K_2\|c_0\|_{L^\infty(\Omega)}}{K_3}\frac{2}{\delta} > 0,$$

we can, for each $\varepsilon\in(0,1)$, find at least one $t_\varepsilon\in(0,t_0)$ such that

$$\int_\Omega f(c_\varepsilon(\cdot,t_\varepsilon)) \leq \frac{K_3}{\|c_0\|_{L^\infty(\Omega)}}\frac{\delta}{2}.$$

Using this, we then gain that

$$\begin{aligned}\int_\Omega c_\varepsilon(\cdot,t_\varepsilon) &= \int_{\{c_\varepsilon(\cdot,t_\varepsilon)\le\xi\}} c_\varepsilon(\cdot,t_\varepsilon) + \int_{\{c_\varepsilon(\cdot,t_\varepsilon)>\xi\}} c_\varepsilon(\cdot,t_\varepsilon) \\ &\le |\Omega|\xi + \frac{\|c_0\|_{L^\infty(\Omega)}}{K_3}\int_\Omega f(c_\varepsilon(\cdot,t_\varepsilon)) \\ &\le \frac{\delta}{2} + \frac{\delta}{2} = \delta \qquad \text{for all } \varepsilon\in(0,1)\end{aligned}$$

and therefore

$$\int_\Omega c_\varepsilon(\cdot,t) \le \delta \qquad \text{for all } \varepsilon\in(0,1) \text{ and } t\ge t_0$$

because of the monotonicity properties for the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ seen in Lemma 3.4.2 and the fact that $t_\varepsilon < t_0$ for all $\varepsilon\in(0,1)$. This is exactly our desired result for $p=1$ and, because Lemma 3.4.2 further gives us a global uniform $L^\infty(\Omega)$ bound for the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$, our desired result also follows for $p>1$ by interpolation. □

By combining the above lemma with (3.4.3) from Lemma 3.4.2, we then immediately gain an important corollary about the gradients of the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$.

Corollary 3.6.2. *For each $\delta>0$, there exists a time $t_0 = t_0(\delta) > 0$ such that*

$$\int_{t_0}^\infty\int_\Omega |\nabla c_\varepsilon|^2 \le \delta$$

for all $\varepsilon\in(0,1)$.

Proof. Because of Lemma 3.6.1, there exists $t_0>0$ such that

$$\int_\Omega c_\varepsilon^2(\cdot,t_0) \le 2\delta \qquad \text{for all } \varepsilon\in(0,1).$$

The inequality (3.4.3) from Lemma 3.4.2 then immediately implies our desired result. □

3.6.2 Eventual uniform smallness of a key functional and its associated norms

As our next step, we will now show that, if the functional $\mathcal{F}_\varepsilon$ seen in (3.6.3) is small at some time $t_0>0$, it in fact stays at least somewhat small from there on out. While the functional itself is already composed of some key integrals, the argument yielding this result also gives us that time-space integrals over some higher-order derivatives of our solution components become small when considered only from t_0 onward. This approach is inspired by similar methods seen in [135].

Lemma 3.6.3. *There exist constants $\delta_0 > 0$, $C \geq 1$ such that the following holds for all $\delta \in (0, \delta_0)$ and $\varepsilon \in (0,1)$:*

If there exists $t_0 > 0$ such that the functional

$$\mathcal{F}_\varepsilon(t) \coloneqq \int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2C}\int_\Omega |u_\varepsilon|^2 \quad \text{for all } t > 0, \tag{3.6.3}$$

where $\overline{n_0} \coloneqq \frac{1}{|\Omega|}\int_\Omega n_0$, has the property

$$\mathcal{F}_\varepsilon(t_0) \leq \frac{\delta}{8C} \tag{3.6.4}$$

and further the inequality

$$\int_{t_0}^\infty \int_\Omega |\nabla c_\varepsilon|^2 \leq \frac{\delta}{8C^2} \tag{3.6.5}$$

holds, then

$$\mathcal{F}_\varepsilon(t) \leq \delta \tag{3.6.6}$$

for all $t \geq t_0$ as well as

$$\int_{t_0}^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \leq \delta, \quad \int_{t_0}^\infty \int_\Omega |\Delta c_\varepsilon|^2 \leq \delta \quad \text{and} \quad \int_{t_0}^\infty \int_\Omega |\nabla u_\varepsilon|^2 \leq \delta. \tag{3.6.7}$$

Proof. Before we present the actual core of this proof, let us first fix some necessary constants to streamline later arguments and make sure that there are no hidden interdependencies:

Let first $C_\text{p} > 0$ be the constant used in the following well-known Poincaré inequalities on Ω (cf. [12, p. 290]):

$$\|\varphi\|_{L^2(\Omega)} \leq C_\text{p}\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega}) \text{ with } \varphi = 0 \text{ on } \partial\Omega \tag{3.6.8}$$

and

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq C_\text{p}\|\Delta\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^2(\overline{\Omega}) \text{ with } \nabla\varphi\cdot\nu = 0 \text{ on } \partial\Omega \tag{3.6.9}$$

Let now $C_\text{s} > 0$ be the constant in the following combination of a Sobolev inequality with another Poincaré inequality (cf. [12, p. 313]):

$$\int_\Omega |\varphi - \overline{\varphi}|^2 \leq C_\text{s}\left\{\int_\Omega |\nabla\varphi|\right\}^2 \qquad \text{for all } \varphi \in C^1(\overline{\Omega}) \text{ with } \overline{\varphi} \coloneqq \frac{1}{|\Omega|}\int_\Omega \varphi \tag{3.6.10}$$

Finally, let $C_\text{gni} > 0$ be such that the inequality

$$\int_\Omega |\nabla\varphi|^4 \leq C_\text{gni}\left\{\int_\Omega |\nabla\varphi|^2\right\}\int_\Omega |\Delta\varphi|^2 \qquad \text{for all } \varphi \in C^2(\overline{\Omega}) \text{ with } \nabla\varphi\cdot\nu = 0 \text{ on } \partial\Omega, \tag{3.6.11}$$

which can be derived from the Gagliardo–Nirenberg inequality (cf. [90]), elliptic regularity theory (cf. [34, Theorem 19.1]), the Poincaré inequality (cf. [12, p. 312]) and (3.6.9), holds.

Let us further fix the basic constants

$$K_1 \coloneqq S_0(\|c_0\|_{L^\infty(\Omega)}), \qquad K_2 \coloneqq \|f\|_{C^1([0,\|c_0\|_{L^\infty(\Omega)}])},$$
$$K_3 \coloneqq \|\nabla\phi\|_{L^\infty(\Omega)}, \qquad K_4 \coloneqq 2K_1^2 + 2K_2^2 + K_2,$$

which only depend on the parameters of the system (CF) and the initial data c_0. Then let

$$C \coloneqq \max\left(2K_3^2 C_{\mathrm{p}}^2 C_{\mathrm{s}} m_0, K_4 \frac{m_0}{|\Omega|}, 1\right), \quad K_5 \coloneqq C_{\mathrm{s}} m_0 K_4^2 + C \quad \text{and} \quad \delta_0 \coloneqq \frac{1}{4K_5 C_{\mathrm{gni}}}$$

with $m_0 \coloneqq \int_\Omega n_0$. For the actual core of this proof, we now fix $\varepsilon \in (0,1)$, $\delta \in (0, \delta_0)$ and $t_0 > 0$ such that (3.6.4) and (3.6.5) hold with C and δ_0 as fixed above. We then test each of the first three equations in (CF$_\varepsilon$) with certain appropriate test functions:

We test the first equation with $\ln(n_\varepsilon)$ and use Young's inequality to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) &= -\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \nabla n_\varepsilon \cdot S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \\ &\leq -\frac{7}{8}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 2K_1^2 \int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon \end{aligned} \tag{3.6.12}$$

for all $t > 0$. We then test the second equation with $-\Delta c_\varepsilon$ to see that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\varepsilon|^2 &= -\int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon + \int_\Omega n_\varepsilon f(c_\varepsilon)\Delta c_\varepsilon \\ &= -\int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega (\nabla u_\varepsilon \nabla c_\varepsilon)\cdot\nabla c_\varepsilon - \frac{1}{2}\int_\Omega u_\varepsilon \cdot \nabla|\nabla c_\varepsilon|^2 \\ &\quad - \int_\Omega f(c_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega n_\varepsilon f'(c_\varepsilon)|\nabla c_\varepsilon|^2 \\ &\leq -\int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C}\int_\Omega |\nabla u_\varepsilon|^2 + C\int_\Omega |\nabla c_\varepsilon|^4 \\ &\quad + \frac{1}{8}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + (2K_2^2 + K_2)\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon \end{aligned} \tag{3.6.13}$$

for all $t > 0$ by again using Young's inequality and the fact that

$$\int_\Omega u_\varepsilon \cdot \nabla|\nabla c_\varepsilon|^2 = -\int_\Omega (\nabla \cdot u_\varepsilon)|\nabla c_\varepsilon|^2 = 0$$

because $\nabla \cdot u_\varepsilon \equiv 0$. Similar to the proof of Lemma 3.4.4, we test the third equation with u_ε to see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 = -\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega (n_\varepsilon - \overline{n_0})(\nabla\phi \cdot u_\varepsilon)$$

$$
\begin{aligned}
&\leq -\int_\Omega |\nabla u_\varepsilon|^2 + \frac{K_3^2 C_{\mathrm{p}}^2}{2}\int_\Omega (n_\varepsilon - \overline{n_0})^2 + \frac{1}{2C_{\mathrm{p}}^2}\int_\Omega |u_\varepsilon|^2 \\
&\leq -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{K_3^2 C_{\mathrm{p}}^2 C_{\mathrm{s}}}{2}\left\{\int_\Omega |\nabla n_\varepsilon|\right\}^2 \\
&\leq -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{K_3^2 C_{\mathrm{p}}^2 C_{\mathrm{s}}}{2}\left\{\int_\Omega n_\varepsilon\right\}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \\
&\leq -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{C}{4}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}
\end{aligned}
\tag{3.6.14}
$$

for all $t > 0$ by using Young's inequality, the Hölder inequality, (3.6.8) and (3.6.10).

The inequalities in (3.6.12) and (3.6.13) then combine to give us

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\varepsilon|^2 \\
&\leq -\frac{3}{4}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} - \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C}\int_\Omega |\nabla u_\varepsilon|^2 + C\int_\Omega |\nabla c_\varepsilon|^4 + K_4\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon
\end{aligned}
$$

for all $t > 0$. The most critical term here is $\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon$, which we therefore further estimate in a similar fashion to the arguments seen in (3.6.14) as

$$
\begin{aligned}
\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon &= \int_\Omega |\nabla c_\varepsilon|^2 (n_\varepsilon - \overline{n_0}) + \frac{m_0}{|\Omega|}\int_\Omega |\nabla c_\varepsilon|^2 \\
&\leq C_{\mathrm{s}} m_0 K_4 \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{4C_{\mathrm{s}} m_0 K_4}\int_\Omega (n_\varepsilon - \overline{n_0})^2 + \frac{m_0}{|\Omega|}\int_\Omega |\nabla c_\varepsilon|^2 \\
&\leq C_{\mathrm{s}} m_0 K_4 \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{4K_4}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{m_0}{|\Omega|}\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0,
\end{aligned}
$$

to gain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\varepsilon|^2 \\
&\leq -\frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} - \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C}\int_\Omega |\nabla u_\varepsilon|^2 + K_5\int_\Omega |\nabla c_\varepsilon|^4 + \mathcal{G}_\varepsilon(t)
\end{aligned}
$$

with $\mathcal{G}_\varepsilon(t) := K_4 \frac{m_0}{|\Omega|}\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2$ for all $t > 0$. This combined with (3.6.14) then finally gives us

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2C}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\varepsilon|^2 \\
&\leq -\frac{1}{4}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} - \int_\Omega |\Delta c_\varepsilon|^2 - \frac{1}{4C}\int_\Omega |\nabla u_\varepsilon|^2 + K_5\int_\Omega |\nabla c_\varepsilon|^4 + \mathcal{G}_\varepsilon(t),
\end{aligned}
$$

which can be further rewritten as

$$
\mathcal{F}_\varepsilon'(t) + \frac{1}{4}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C}\int_\Omega |\nabla u_\varepsilon|^2 \leq K_5\int_\Omega |\nabla c_\varepsilon|^4 + \mathcal{G}_\varepsilon(t)
$$

for all $t > 0$. Because we further know that

$$\int_\Omega |\nabla c_\varepsilon|^4 \le C_{\text{gni}} \left\{ \int_\Omega |\nabla c_\varepsilon|^2 \right\} \int_\Omega |\Delta c_\varepsilon|^2 \le 2C_{\text{gni}} \mathcal{F}_\varepsilon(t) \int_\Omega |\Delta c_\varepsilon|^2$$

due to (3.6.11) and the fact that Jensen's inequality ensures that $\int_\Omega n_\varepsilon \ln(\frac{n_\varepsilon}{n_0}) \ge 0$, we finally gain that

$$\mathcal{F}_\varepsilon'(t) + \frac{1}{4} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + (1 - 2K_5 C_{\text{gni}} \mathcal{F}_\varepsilon(t)) \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C} \int_\Omega |\nabla u_\varepsilon|^2 \le \mathcal{G}_\varepsilon(t)$$

for all $t > 0$. We then let

$$S_\varepsilon \coloneqq \{ T > t_0 \mid \mathcal{F}_\varepsilon(t) \le \delta \text{ for all } t \in (t_0, T) \},$$

which is non-empty because of the continuity of $\mathcal{F}_\varepsilon$, (3.6.4) and the fact that $C \ge 1$. For all $T \in S_\varepsilon$ and $t \in (t_0, T)$, we know that

$$\mathcal{F}_\varepsilon'(t) + \frac{1}{4} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C} \int_\Omega |\nabla u_\varepsilon|^2 \le \mathcal{G}_\varepsilon(t), \tag{3.6.15}$$

because

$$(1 - 2K_5 C_{\text{gni}} \mathcal{F}_\varepsilon(t)) \ge (1 - 2K_5 C_{\text{gni}} \delta) \ge (1 - 2K_5 C_{\text{gni}} \delta_0) = \frac{1}{2}.$$

If we now integrate from t_0 to t in (3.6.15), we gain that

$$\begin{aligned}
&\mathcal{F}_\varepsilon(t) + \frac{1}{4} \int_{t_0}^t \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{2} \int_{t_0}^t \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C} \int_{t_0}^t \int_\Omega |\nabla u_\varepsilon|^2 \\
&\le \mathcal{F}_\varepsilon(t_0) + \int_{t_0}^\infty \mathcal{G}_\varepsilon(s) \, \mathrm{d}s \le \frac{\delta}{8C} + K_4 \frac{m_0}{|\Omega|} \int_{t_0}^\infty \int_\Omega |\nabla c_\varepsilon|^2 \\
&\le \frac{\delta}{8C} + \frac{\delta}{8C} \frac{m_0 K_4}{C|\Omega|} \le \frac{\delta}{4C} \le \frac{\delta}{4}
\end{aligned} \tag{3.6.16}$$

for all $T \in S_\varepsilon$ and $t \in (t_0, T)$ because of (3.6.4) and (3.6.5) as well as the fact that $C \ge 1$ and $\frac{m_0 K_4}{|\Omega| C} \le 1$ by definition of C. Therefore, $S_\varepsilon = [t_0, \infty)$ due to the continuity of S_ε and thus (3.6.16) holds on the entire interval $[t_0, \infty)$.

This then directly implies (3.6.6) and (3.6.7) and consequently completes the proof. □

We can now use the above insight in combination with the various properties derived in Lemma 3.4.2, Corollary 3.4.3, Lemma 3.4.4 and Corollary 3.6.2 to show that the smallness conditions for the functional $\mathcal{F}_\varepsilon$ and gradient of c_ε are in fact achievable for every $\delta > 0$ at some ε-independent time t_0 and that therefore the functional and thus its individual components become small in a uniform fashion. As an added bonus, we naturally also gain some uniform higher order smallness information for some of the associated dissipative terms, which will prove useful later on.

Lemma 3.6.4. *For each $\delta > 0$, there exists $t_0 = t_0(\delta) > 0$ such that*

$$\int_\Omega n_\varepsilon(\cdot,t)\ln\left(\frac{n_\varepsilon(\cdot,t)}{\overline{n_0}}\right) \le \delta, \quad \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le \delta, \quad \int_\Omega |u_\varepsilon(\cdot,t)|^2 \le \delta$$

for all $t > t_0$, $\varepsilon \in (0,1)$ with $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$ and

$$\int_{t_0}^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le \delta, \quad \int_{t_0}^\infty \int_\Omega |\Delta c_\varepsilon|^2 \le \delta, \quad \int_{t_0}^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le \delta$$

for all $\varepsilon \in (0,1)$.

Proof. Let $\delta_0 > 0$, $C > 0$ and the functional $\mathcal{F}_\varepsilon$ be as in Lemma 3.6.3. Without loss of generality, we assume $\delta < \delta_0$. Because of Corollary 3.6.2, we find $t_1 > 0$ such that

$$\int_{t_1}^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{\delta}{8C^2} \qquad \text{for all } \varepsilon \in (0,1).$$

Because of Corollary 3.4.3, Lemma 3.4.2, and Lemma 3.4.4, we further know that there exists a constant $K_1 > 0$ such that

$$\begin{aligned}&\frac{1}{t-t_1}\int_{t_1}^t \mathcal{F}_\varepsilon(s)\,\mathrm{d}s \\ \le&\frac{1}{t-t_1}\left[\int_0^\infty\int_\Omega n_\varepsilon \ln\left(\frac{n_\varepsilon}{\overline{n_0}}\right) + \frac{1}{2}\int_0^\infty\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2C}\int_0^\infty\int_\Omega |u_\varepsilon|^2\right] \le \frac{K_1}{t-t_1}\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $t > t_1$. This implies that, for $t_0 := \frac{8CK_1}{\delta} + t_1$ and all $\varepsilon \in (0,1)$, we have

$$\frac{1}{t_0 - t_1}\int_{t_1}^{t_0} \mathcal{F}_\varepsilon(s)\,\mathrm{d}s \le \frac{\delta}{8C}.$$

Therefore for each $\varepsilon \in (0,1)$, there must exist $t_\varepsilon \in (t_1, t_0)$ such that

$$\mathcal{F}_\varepsilon(t_\varepsilon) \le \frac{\delta}{8C}.$$

This directly implies our desired result from t_ε and therefore from t_0 onward for all $\varepsilon \in (0,1)$ by application of Lemma 3.6.3. □

3.6.3 Eventual uniform bounds for $\|n_\varepsilon\|_{L^\infty(\Omega)}$, $\|\nabla c_\varepsilon\|_{L^p(\Omega)}$, $\|A^\beta u_\varepsilon\|_{L^p(\Omega)}$ for $p \in [1,\infty)$ and $\beta \in (\frac{1}{2}, 1)$ via bootstrap arguments

The eventual smallness and integrability results of the previous section now give us a critical foothold to establish even better uniform a priori bounds for all three solution components of our approximate solutions from some large time $t_0 > 0$ onward. The methods used for this will be a combination of testing procedures and semigroup methods used in a fairly standard bootstrap process.

Our first step of this section will therefore be to improve our thus far very weak bounds for the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ to $L^p(\Omega)$ bounds for arbitrary but finite p. This is done mostly by testing the first equation in $(\mathrm{CF}_\varepsilon)$ with n_ε^{p-1} and using the results of the previous section to argue that from some time $t_0 > 0$ onward the following is true: In any time interval of length 1, there exists at least one time, at which n_ε is uniformly bounded in $L^p(\Omega)$, and, from that time on, the growth of $\int_\Omega n_\varepsilon^p(\cdot, t)$ is at most exponential, whereby the bound and all the growth parameters are independent of the choice of interval. Taken together, these two facts directly imply our desired result.

This approach as well as the one used in the lemma immediately following it are again inspired by similar methods seen in [135].

Lemma 3.6.5. *There exists $t_0 > 0$ such that, for each $p \in (1, \infty)$, there is $C(p) > 0$ with*

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \le C(p)$$

for all $t > t_0$ and $\varepsilon \in (0, 1)$.

Proof. Due to Lemma 3.6.4, there exists $t_0 > 1$ such that

$$\int_{t_0-1}^{\infty}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le 1, \quad \int_{t_0-1}^{\infty}\int_\Omega |\Delta c_\varepsilon|^2 \le 1 \tag{3.6.17}$$

and

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \le 1 \quad \text{for all } t > t_0 - 1 \tag{3.6.18}$$

for all $\varepsilon \in (0, 1)$.

We fix $p \in (1, \infty)$ and $t_1 > t_0$. Because

$$\int_{t_1-1}^{t_1}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le 1 \quad \text{for all } \varepsilon \in (0, 1),$$

there must, for each $\varepsilon \in (0, 1)$, exist $t_\varepsilon \in (t_1 - 1, t_1)$ such that

$$\int_\Omega \frac{|\nabla n_\varepsilon(\cdot, t_\varepsilon)|^2}{n_\varepsilon(\cdot, t_\varepsilon)} \le 1.$$

Due to the Gagliardo–Nirenberg inequality, this implies

$$\begin{aligned}
\|n_\varepsilon(\cdot, t_\varepsilon)\|_{L^p(\Omega)} &= \|n_\varepsilon^{\frac{1}{2}}(\cdot, t_\varepsilon)\|_{L^{2p}(\Omega)}^2 \\
&\le K_1 \left[\|\nabla n_\varepsilon^{\frac{1}{2}}(\cdot, t_\varepsilon)\|_{L^2(\Omega)}^{\alpha} \|n_\varepsilon^{\frac{1}{2}}(\cdot, t_\varepsilon)\|_{L^2(\Omega)}^{1-\alpha} + \|n_\varepsilon^{\frac{1}{2}}(\cdot, t_\varepsilon)\|_{L^2(\Omega)} \right]^2 \\
&\le 2K_1 \left[\left(\int_\Omega \frac{|\nabla n_\varepsilon(\cdot, t_\varepsilon)|^2}{n_\varepsilon(\cdot, t_\varepsilon)} \right)^{\alpha} \left(\int_\Omega n_0 \right)^{1-\alpha} + \int_\Omega n_0 \right] \\
&\le 2K_1 \left[\left(\int_\Omega n_0 \right)^{1-\alpha} + \int_\Omega n_0 \right] =: K_2
\end{aligned} \tag{3.6.19}$$

with some $K_1 > 0$ and $\alpha := \frac{p-1}{p} \in (0,1)$ for all $\varepsilon \in (0,1)$. As our next step, a standard testing procedure yields that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p &= -(p-1)\int_\Omega |\nabla n_\varepsilon|^2 n_\varepsilon^{p-2} + (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon \cdot S_\varepsilon(\cdot, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \\ &\le -\frac{p-1}{2}\int_\Omega |\nabla n_\varepsilon|^2 n_\varepsilon^{p-2} + K_3\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon^p \\ &\le -\frac{2(p-1)}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_3\|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \|n_\varepsilon^{\frac{p}{2}}\|_{L^4(\Omega)}^2\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ with $K_3 := \frac{p-1}{2}S_0^2(\|c_0\|_{L^\infty(\Omega)})$. According to the Gagliardo–Nirenberg inequality (cf. [90]), well-known elliptic regularity theory (cf. [34, Theorem 19.1]), the Poincaré inequality (cf. [12]) and (3.6.18), there exists a constant $K_4 > 0$ with

$$\|n_\varepsilon^{\frac{p}{2}}\|_{L^4(\Omega)}^2 \le K_4\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}\|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)} + K_4\|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2$$

for all $t > 0$ and $\varepsilon \in (0,1)$ and

$$\|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \le K_4\|\Delta c_\varepsilon\|_{L^2(\Omega)}\|\nabla c_\varepsilon\|_{L^2(\Omega)} \le K_4\|\Delta c_\varepsilon\|_{L^2(\Omega)}$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ implying that

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \\ &\le -\frac{2(p-1)}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_3K_4^2\|\Delta c_\varepsilon\|_{L^2(\Omega)}\left(\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}\|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)} + \|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2\right) \\ &\le -\frac{(p-1)}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_5\int_\Omega |\Delta c_\varepsilon|^2\int_\Omega n_\varepsilon^p + \int_\Omega n_\varepsilon^p \\ &\le K_5\int_\Omega |\Delta c_\varepsilon|^2\int_\Omega n_\varepsilon^p + \int_\Omega n_\varepsilon^p \qquad \text{for all } t > t_0 - 1 \text{ and } \varepsilon \in (0,1)\end{aligned}$$

with $K_5 := (\frac{p^2}{4(p-1)} + \frac{1}{4})K_3^2K_4^4$. This differential inequality combined with (3.6.17) and (3.6.19), then gives us that

$$\int_\Omega n_\varepsilon^p \le K_2^p \exp\left(pK_5\int_{t_\varepsilon}^t\int_\Omega |\Delta c_\varepsilon|^2 + p(t - t_\varepsilon)\right) \le K_2^p \exp\left(pK_5 + p\right) =: K_6$$

for all $t \in (t_\varepsilon, t_\varepsilon + 1)$ because $t_\varepsilon > t_1 - 1 > t_0 - 1$ by a standard comparison argument for all $\varepsilon \in (0,1)$. As further $t_1 \in (t_\varepsilon, t_\varepsilon + 1)$ due to $t_\varepsilon \in (t_1 - 1, t_1)$, this implies

$$\int_\Omega n_\varepsilon^p(\cdot, t_1) \le K_6.$$

Because $t_1 > t_0$ was arbitrary and K_6 is independent of t_1, this completes the proof. □

Given that we have now established quite a strong set of bounds for the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$, which will make the $n_\varepsilon\nabla\phi$ term in the third equation of $(\mathrm{CF}_\varepsilon)$ much more manageable, we will now turn our attention to said equation.

For this, let us briefly introduce some definitions and results used in the theory of fluid equations, which were already alluded to when talking about initial data regularity in the introduction (cf. (3.1.1)) and will now become an important tool.

Remark 3.6.6. We define $L^p_\sigma(\Omega)$ as the space of all solenoidal functions in $(L^p(\Omega))^2$, or more precisely

$$L^p_\sigma(\Omega) := \left\{ f \in (L^p(\Omega))^2 \;\middle|\; \nabla \cdot f = 0 \right\}$$

for all $p \in (1, \infty)$ with $\nabla\cdot$ interpreted as a distributional derivative. We then let $W^{m,p}_{0,\sigma}(\Omega) := (W^{m,p}_0(\Omega))^2 \cap L^p_\sigma(\Omega)$ for all $m \in \mathbb{N}$ and $p \in (1, \infty)$. As proven in e.g. [38, Section III.1], there further exists a unique, continuous projection

$$\mathcal{P}_p : (L^p(\Omega))^2 \to L^p_\sigma(\Omega)$$

called the Helmholtz projection for all $p \in (1, \infty)$. In fact, $\mathcal{P}_2$ is an orthogonal projection (cf. [101, Section II.2.5]). As the construction of this projection in [38] essentially rests on solving a certain elliptic Neumann problem, it can be shown that for sufficiently regular functions (e.g. $C^0(\overline{\Omega})$) all Helmholtz projections introduced above are in fact identical and therefore we will often simply write $\mathcal{P}$ for the projection and leave out the subscript when appropriate.

Using this, we define the Stokes operator on $L^p_\sigma(\Omega)$ as

$$A_p := -\mathcal{P}_p \Delta$$

with $D(A_p) := (W^{2,p}(\Omega))^2 \cap W^{1,p}_{0,\sigma}(\Omega)$ (cf. [40], [101, Section III.2]) for all $p \in (1, \infty)$. As the above definition is based on the Helmholtz projection, all operators A_p are again identical when applied to a sufficiently regular function (e.g. $C^2(\overline{\Omega})$). Thus, we will often omit the subscript here as well and just write A when appropriate. In [40] and [41], it has been shown that, for all $p \in (1, \infty)$, A_p is sectorial (in fact its spectrum is contained in $(0, \infty)$), that $-A_p$ generates a bounded analytic semigroup $(e^{-tA_p})_{t \geq 0}$ of class C_0 on $D(A_p)$ and that the fractional powers A^α_p of A_p exist for all $\alpha \in (0, 1)$. Due to the regularity theory for the stationary Stokes equation (cf. e.g. [38, Lemma IV.6.1]), the Stokes operator further has the following property: For each $p \in (1, \infty)$, there exists $C(p) > 0$ such that

$$\|\varphi\|_{W^{2,p}(\Omega)} \leq C(p) \|A_p \varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in D(A_p). \tag{3.6.20}$$

Notably, this means that the norm $\|A_p \cdot\|_{L^p(\Omega)}$ is equivalent to the standard Sobolev norm $\|\cdot\|_{W^{2,p}(\Omega)}$ on $D(A_p)$ for all $p \in (1, \infty)$. Therefore, we will from hereon out consider $\|A_p \cdot\|_{L^p(\Omega)}$ to be the default norm of the space $D(A_p)$. In a similar vein when talking about the domains of the fractional powers $D(A^\alpha_p)$, $p \in (1, \infty)$, $\alpha \in (0, 1)$, we will from now on always assume these spaces to be equipped with the corresponding norm $\|A^\alpha_p \cdot\|_{L^p(\Omega)}$. Framed in this way, the spaces $D(A^\alpha_p)$ then have rather favorable continuous embedding properties into certain Sobolev and Hölder spaces due to standard semigroup theory (cf. [41] or [49, p. 39]) and the regularity property in (3.6.20), which will be useful on multiple occasions.

After this brief excursion, let us now return to our actual objective, namely the derivation of an $L^2(\Omega)$ bound for the gradients of the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$. Structurally this proof is very similar to the one for Lemma 3.6.5 in that we again establish boundedness for the gradients at one time in every time interval of length 1 (from some time $t_0 > 0$ onward) and then derive an additional growth restriction in said interval, whereby both times all parameters are again independent of the choice of interval.

Lemma 3.6.7. *There exist $t_0 > 0$ and $C > 0$ such that*

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C$$

for all $t > t_0$ and $\varepsilon \in (0,1)$.

Proof. We first fix $t_0 > 1$ and $K_1 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le K_1, \quad \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le 1 \qquad \text{for all } t > t_0 - 1 \tag{3.6.21}$$

and

$$\int_{t_0-1}^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le 1 \tag{3.6.22}$$

for all $\varepsilon \in (0,1)$ according to Lemma 3.6.5 and Lemma 3.6.4.

Let $t_1 > t_0$ be fixed but arbitrary. Then for each $\varepsilon \in (0,1)$, there exists $t_\varepsilon \in (t_1 - 1, t_1)$ such that

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t_\varepsilon)|^2 \le 1 \tag{3.6.23}$$

due to (3.6.22). We may now further fix $K_2 > 0$ such that

$$\|u_\varepsilon\|^2_{L^\infty(\Omega)} \le K_2\|Au_\varepsilon\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)} \le K_2\|Au_\varepsilon\|_{L^2(\Omega)} \tag{3.6.24}$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ due to (3.6.21), the Gagliardo–Nirenberg inequality and the regularity property in (3.6.20). We then apply the Helmholtz projection to the third equation in $(\mathrm{CF}_\varepsilon)$ and test with Au_ε to see that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_\varepsilon|^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |A^{\frac{1}{2}}u_\varepsilon|^2 = -\int_\Omega |Au_\varepsilon|^2 - \int_\Omega Au_\varepsilon \cdot \mathcal{P}\left[(u_\varepsilon\cdot\nabla)u_\varepsilon\right] + \int_\Omega Au_\varepsilon \cdot \mathcal{P}\left[n_\varepsilon\nabla\phi\right] \\
&\le -\frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \int_\Omega |\mathcal{P}\left[(u_\varepsilon\cdot\nabla)u_\varepsilon\right]|^2 + \int_\Omega |\mathcal{P}\left[n_\varepsilon\nabla\phi\right]|^2 \\
&\le -\frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \int_\Omega |(u_\varepsilon\cdot\nabla)u_\varepsilon|^2 + \int_\Omega |n_\varepsilon\nabla\phi|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + K_1^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 \\
&\leq -\frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + K_2 \|Au_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + K_1^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 \\
&\leq \frac{K_2^2}{2}\left(\int_\Omega |\nabla u_\varepsilon|^2\right)^2 + K_1^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2
\end{aligned}
$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ by using (3.6.24), Young's inequality and some fundamental properties of the fractional powers of the Stokes operator (cf. [49], [101, Lemma III.2.2.1]). This further implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_\varepsilon|^2 \leq K_3 \left(\int_\Omega |\nabla u_\varepsilon|^2\right)^2 + K_4 \qquad \text{for all } t > t_0 - 1 \text{ and } \varepsilon \in (0,1)
$$

with $K_3 := K_2^2$, $K_4 := 2K_1^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2$. This differential inequality combined with (3.6.23) and (3.6.22) then gives us that

$$
\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \\
&\leq \left(\int_\Omega |\nabla u_\varepsilon(\cdot,t_\varepsilon)|^2\right) \exp\left(K_3 \int_{t_\varepsilon}^t \int_\Omega |\nabla u_\varepsilon|^2\right) + K_4 \int_{t_\varepsilon}^t \exp\left(K_3 \int_s^t \int_\Omega |\nabla u_\varepsilon|^2\right) \mathrm{d}s \\
&\leq (1+K_4)e^{K_3} =: K_5
\end{aligned}
$$

for all $t \in (t_\varepsilon, t_\varepsilon + 1)$ and $\varepsilon \in (0,1)$ because $t_\varepsilon > t_1 - 1 > t_0 - 1$ by a standard comparison argument. As further $t_1 \in (t_\varepsilon, t_\varepsilon + 1)$ due to the fact that $t_\varepsilon \in (t_1 - 1, t_1)$, this implies

$$
\int_\Omega |\nabla u_\varepsilon(\cdot,t_1)|^2 \leq K_5.
$$

Given that $t_1 > t_0$ was arbitrary and K_5 is independent of t_1, this completes the proof. □

As already seen in the proof above, deriving an $L^2(\Omega)$ bound for the gradients of the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is equivalent to deriving a bound for $\|A^{\frac{1}{2}} u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$, which is fairly easy to check by using fundamental properties of the fractional powers of the Stokes operator (cf. [49], [101, Lemma III.2.2.1]). As our next step then, we now want to expand on this by proving stronger bounds of the form $\|A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C$ because the embedding properties of $D(A_2^{\frac{1}{2}})$ are not quite sufficient for our later arguments, namely the derivation of certain Hölder-type bounds. This is done mostly by using the above results in combination with semigroup methods and well-known smoothing properties of the Stokes semigroup (cf. [42], [49, p. 26]).

Lemma 3.6.8. *There exists $t_0 > 0$ such that, for each $\beta \in (\frac{1}{2}, 1)$ and $p \in (2, \infty)$, there is $C(\beta, p) > 0$ with*

$$
\|A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C(\beta, p)
$$

for all $t > t_0$ and $\varepsilon \in (0,1)$.

Proof. We fix $\beta \in (\frac{1}{2}, 1)$ and $p \in (2, \infty)$. We can then fix a time $t_0 > 1$ independently of p and a constant $K_1 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le 1, \quad \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le K_1, \quad \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le K_1$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ due to Lemma 3.6.4, Lemma 3.6.5 and Lemma 3.6.7. Lastly, we fix $q \in (2,p)$ such that

$$\beta + \frac{1}{q} - \frac{1}{p} < 1. \tag{3.6.25}$$

The Sobolev embedding theorem (cf. [26, Theorem 2.72]) implies that there further exists $K_2 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^{\frac{q(p+q)}{p-q}}(\Omega)} \le K_2, \quad \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le K_2$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ due to us working in two dimensions and due to the previously established bounds.

Let now $t_1 > t_0$ be fixed but arbitrary. Then relying on the smoothing and continuity properties of the Stokes semigroup $(e^{-tA})_{t\ge0}$ and the Helmholtz projection $\mathcal{P}_p$ (cf. [38], [42, p. 201], [49, p. 26]), we estimate each u_ε using the variation-of-constant representation of the third equation in $(\mathrm{CF}_\varepsilon)$ on (t_1-1, t_1) after projecting with $\mathcal{P}$ as follows:

$$\begin{aligned}
&\| A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\
&= \Big\| A^\beta e^{-(t-(t_1-1))A} u_\varepsilon(\cdot, t_1-1) - \int_{t_1-1}^t A^\beta e^{-(t-s)A} \mathcal{P}\left[(u_\varepsilon(\cdot,s)\cdot\nabla) u_\varepsilon(\cdot,s)\right] \mathrm{d}s \\
&\qquad + \int_{t_1-1}^t A^\beta e^{-(t-s)A}\mathcal{P}\left[n_\varepsilon(\cdot,s)\nabla\phi\right] \mathrm{d}s \Big\|_{L^p(\Omega)} \\
&\le K_3(t-(t_1-1))^{-\beta} \|u_\varepsilon(\cdot,t_1-1)\|_{L^p(\Omega)} \\
&\quad + K_3 \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} \|\mathcal{P}\left[(u_\varepsilon(\cdot,s)\cdot\nabla)u_\varepsilon(\cdot,s)\right]\|_{L^q(\Omega)} \,\mathrm{d}s \\
&\quad + K_3 \int_{t_1-1}^t (t-s)^{-\beta} \|\mathcal{P}\left[n_\varepsilon(\cdot,s)\nabla\phi\right]\|_{L^p(\Omega)} \,\mathrm{d}s \\
&\le K_5(t-(t_1-1))^{-\beta} + K_5 \\
&\quad + K_3K_4 \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} \|u_\varepsilon(\cdot,s)\|_{L^{\frac{q(p+q)}{p-q}}(\Omega)} \|\nabla u_\varepsilon(\cdot,s)\|_{L^{\frac{p+q}{2}}(\Omega)} \,\mathrm{d}s \\
&\le K_5(t-(t_1-1))^{-\beta} + K_5 + K_2K_3K_4 \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} \|\nabla u_\varepsilon(\cdot,s)\|_{L^{\frac{p+q}{2}}(\Omega)} \,\mathrm{d}s
\end{aligned} \tag{3.6.26}$$

with some $K_3, K_4 > 0$ and $K_5 := \max(K_2K_3, \frac{K_1K_3K_4\|\nabla\phi\|_{L^\infty(\Omega)}}{1-\beta})$ for all $t \in (t_1-1, t_1]$ and $\varepsilon \in (0,1)$.

Interpolation using the Hölder inequality combined with the fact that $D(A_p^\beta)$ embeds continuously into $W^{1,p}(\Omega)$ (cf. [49, p. 39] or [41]) because $\beta > \frac{1}{2}$ then gives us $K_6 > 0$ such that

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^{\frac{p+q}{2}}(\Omega)} \le \|\nabla u_\varepsilon(\cdot,t)\|^{1-\alpha}_{L^2(\Omega)} \|\nabla u_\varepsilon(\cdot,t)\|^{\alpha}_{L^p(\Omega)} \le K_1^{1-\alpha} K_6 \|A^\beta u_\varepsilon(\cdot,t)\|^\alpha_{L^p(\Omega)}$$

for all $t > t_0 - 1$ and $\varepsilon \in (0,1)$ with $\alpha := (\frac{1}{2} - \frac{2}{p+q})(\frac{1}{2} - \frac{1}{p})^{-1} \in (0,1)$. In (3.6.26), this then results in

$$\begin{aligned} \|A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le{}& K_5(t-(t_1-1))^{-\beta} + K_5 \\ &+ K_7 \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} \|A^\beta u_\varepsilon(\cdot,s)\|^\alpha_{L^p(\Omega)} \,\mathrm{d}s \end{aligned} \tag{3.6.27}$$

for all $t \in (t_1 - 1, t_1]$ and $\varepsilon \in (0,1)$ with $K_7 := K_1^{1-\alpha} K_2 K_3 K_4 K_6$.

Let now

$$M_\varepsilon := \sup_{s \in (t_1-1, t_1]} (s-(t_1-1))^\beta \|A^\beta u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} < \infty.$$

for all $t \in (t_1 - 1, t_1]$ and $\varepsilon \in (0,1)$. With this definition, we can conclude from (3.6.27) that

$$\begin{aligned} &(t-(t_1-1))^\beta \|A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\ &\le 2K_5 + K_7 M_\varepsilon^\alpha (t-(t_1-1))^\beta \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} (s-(t_1-1))^{-\alpha\beta} \,\mathrm{d}s \end{aligned} \tag{3.6.28}$$

for all $t \in (t_1 - 1, t_1]$ and $\varepsilon \in (0,1)$. As both of the exponents in the remaining integral are greater than -1 due to our choice of q, a straightforward estimation yields $K_8 > 0$ such that

$$\int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} (s-(t_1-1))^{-\alpha\beta} \,\mathrm{d}s \le K_8 (t-(t_1-1))^{1-\beta-\frac{1}{q}+\frac{1}{p}-\alpha\beta}$$

and thus

$$(t-(t_1-1))^\beta \int_{t_1-1}^t (t-s)^{-\beta-\frac{1}{q}+\frac{1}{p}} (s-(t_1-1))^{-\alpha\beta} \,\mathrm{d}s \le K_8$$

for all $t \in (t_1 - 1, t_1]$ and $\varepsilon \in (0,1)$ again due to our choice of q. If we apply this to (3.6.28), we find that

$$M_\varepsilon \le 2K_5 + K_7 K_8 M_\varepsilon^\alpha \le 2K_5 + (1-\alpha)(K_7K_8)^{\frac{1}{1-\alpha}} + \alpha M_\varepsilon$$

due to Young's inequality and therefore that

$$M_\varepsilon \le \frac{2K_5}{1-\alpha} + (K_7K_8)^{\frac{1}{1-\alpha}} =: K_9$$

for all $\varepsilon \in (0,1)$. This further gives us that

$$\|A^\beta u_\varepsilon(\cdot,t_1)\|_{L^p(\Omega)} \le M_\varepsilon \le K_9 \quad \text{ for all } \varepsilon \in (0,1)$$

and thus completes the proof as $t_1 > t_0$ was arbitrary. □

Given the regularity properties of the fractional powers of the Stokes operator, we gain the following corollary, which translates the above abstract bounds into more familiar settings.

Corollary 3.6.9. *There exists $t_0 > 0$ such that, for each $\alpha \in (0,1)$, there is $C(\alpha) > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le \|u_\varepsilon(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \le C(\alpha)$$

for all $t > t_0$ and $\varepsilon \in (0,1)$.

Proof. We first choose $p \in (2,\infty)$ and $\beta \in (\frac{1}{2},1)$ such that

$$1 + \alpha < 2\beta - \frac{2}{p},$$

which is always possible.

Lemma 3.6.8 then gives us $t_0 > 0$ and $K_1 > 0$ such that

$$\|A^\beta u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le K_1$$

for all $t > t_0$. Note that t_0 given to us by the lemma is in fact independent of β and p. By well-known continuous embedding property of $D(A_p^\beta)$ into $C^{1+\alpha}(\overline{\Omega})$ seen for example in [41] or [49, p. 39], this already implies our desired result. □

As our next step, we will now use semigroup methods to prove some additional bounds for the families $(n_\varepsilon)_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in a fairly standard and therefore brief fashion:

Lemma 3.6.10. *There exists $t_0 > 0$ such that, for each $p \in (1,\infty]$, there is $C(p) > 0$ with*

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p)$$

for all $t > t_0$ and $\varepsilon \in (0,1)$.

Proof. We use the variation-of-constant representation combined with standard semigroup smoothness estimates from [123, Lemma 1.3] to estimate the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ as follows:

$$\begin{aligned}
&\|\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\
&\le \left\| \nabla e^{\Delta} c_\varepsilon(\cdot,t-1) - \int_{t-1}^t \nabla e^{(t-s)\Delta}(u_\varepsilon\cdot\nabla c_\varepsilon)\,\mathrm{d}s - \int_{t-1}^t \nabla e^{(t-s)\Delta} f(c_\varepsilon) n_\varepsilon \,\mathrm{d}s \right\|_{L^p(\Omega)} \\
&\le K_1\|c_0\|_{L^p(\Omega)} + K_1\int_{t-1}^t (1+(t-s)^{-1+\frac{1}{p}})\times \\
&\qquad \left\{ \|u_\varepsilon\|_{L^\infty(\Omega)}\|\nabla c_\varepsilon\|_{L^2(\Omega)} + \|f\|_{L^\infty([0,\|c_0\|_{L^\infty(\Omega)}])}\|n_\varepsilon\|_{L^2(\Omega)} \right\} \mathrm{d}s
\end{aligned}$$

with some constant $K_1 > 0$ for all $t > 1$. This already implies our desired result for all finite p and sufficiently large $t_0 > 0$ because of Corollary 3.6.9, Lemma 3.6.4 and Lemma 3.6.5. For the case $p = \infty$, we use a similar approach as before to derive

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq K_2 \|c_0\|_{L^\infty(\Omega)} + K_2 \int_{t-1}^{t} (1 + (t-s)^{-\frac{3}{4}}) \times \\ \Big\{ \|u_\varepsilon\|_{L^\infty(\Omega)} \|\nabla c_\varepsilon\|_{L^4(\Omega)} + \|f\|_{L^\infty([0, \|c_0\|_{L^\infty(\Omega)}])} \|n_\varepsilon\|_{L^4(\Omega)} \Big\} \, \mathrm{d}s$$

with some constant $K_2 > 0$ for all $t > 1$. Using this very result for $p = 4$ as proven above and the same lemmas as before then completes the proof. □

Lemma 3.6.11. *There exists $t_0 > 0$ and $C > 0$ such that*

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

for all $t > t_0$ and $\varepsilon \in (0, 1)$.

Proof. Using the variation-of-constants representation applied to the family $(n_\varepsilon)_{\varepsilon \in (0,1)}$ and combining it with semigroup smoothness estimates from [123, Lemma 1.3] yields that

$$\begin{aligned}
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} &= \Big\| \, e^{\Delta} n_\varepsilon(\cdot, t-1) - \int_{t-1}^{t} e^{(t-s)\Delta} (u_\varepsilon \cdot \nabla n_\varepsilon) \, \mathrm{d}s \\
&\quad - \int_{t-1}^{t} e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon S_\varepsilon(\cdot, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon) \, \mathrm{d}s \, \Big\|_{L^\infty(\Omega)} \\
&= \Big\| \, e^{\Delta} n_\varepsilon(\cdot, t-1) - \int_{t-1}^{t} e^{(t-s)\Delta} \nabla \cdot (u_\varepsilon n_\varepsilon) \, \mathrm{d}s \\
&\quad - \int_{t-1}^{t} e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon S_\varepsilon(\cdot, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon) \, \mathrm{d}s \, \Big\|_{L^\infty(\Omega)} \\
&\leq K_1 \|n_\varepsilon(\cdot, t-1)\|_{L^p(\Omega)} + K_1 \int_{t-1}^{t} (1 + (t-s)^{-\frac{3}{4}}) \times \\
&\quad \Big\{ \, \|u_\varepsilon\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^4(\Omega)} + S_0(\|c_0\|_{L^\infty(\Omega)}) \|n_\varepsilon\|_{L^8(\Omega)} \|\nabla c_\varepsilon\|_{L^8(\Omega)} \, \Big\} \, \mathrm{d}s
\end{aligned}$$

with some constant $K_1 > 0$ for all $t > 1$ because $\nabla \cdot u_\varepsilon = 0$. Combining this with Corollary 3.6.9, Lemma 3.6.5 and Lemma 3.6.10 then gives us the desired bound by similar arguments as in the previous lemma. □

One immediate consequence of this lemma is an additional global space-time integrability property for the gradients of n_ε. This property will prove useful when later arguing that n is in fact a weak solution of its associated differential equation as a step in the process of proving its more classical solution properties.

Lemma 3.6.12. *There exists* $t_0 > 0$ *and* $C > 0$ *such that*

$$\int_{t_0}^{\infty} \int_\Omega |\nabla n_\varepsilon|^2 \le C$$

for all $\varepsilon \in (0,1)$.

Proof. Due to Lemma 3.6.11, there exist $t_0 > 0$ and $K_1 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K_1 \quad \text{for all } t \ge t_0 \text{ and } \varepsilon \in (0,1). \tag{3.6.29}$$

Testing the first equation in $(\mathrm{CF}_\varepsilon)$ with n_ε in a similar fashion as in the proof of Lemma 3.6.5 and applying Young's inequality immediately yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^2 &\le -\frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{S_0^2(\|c_0\|_{L^\infty(\Omega)})}{2}\int_\Omega |\nabla c_\varepsilon|^2 n_\varepsilon^2 \\ &\le -\frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + K_2\int_\Omega |\nabla c_\varepsilon|^2\end{aligned}$$

with $K_2 := \frac{1}{2}K_1^2 S_0^2(\|c_0\|_{L^\infty(\Omega)})$ for all $t > t_0$ and $\varepsilon \in (0,1)$. Time integration combined with Lemma 3.4.2 and (3.6.29) then directly gives us our desired result. □

3.6.4 Establishing baseline uniform parabolic Hölder bounds for the families $(n_\varepsilon)_{\varepsilon\in(0,1)}$, $(c_\varepsilon)_{\varepsilon\in(0,1)}$ and $(u_\varepsilon)_{\varepsilon\in(0,1)}$

As our next step in the journey towards a proof of Theorem 3.1.1, we will now transition from only establishing uniform space bounds for the solution components of our approximate solutions to fully parabolic Hölder-type bounds. Establishing such bounds will then allow us to use the well-known compact embedding properties of such spaces to argue that, at least from some large time onward, our generalized solutions were in fact of a similarly high level of regularity.

We will start by establishing a uniform $C^\alpha([t,t+1];C^{1+\alpha}(\overline{\Omega}))$-type bound for the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ as u_ε plays a role in all three equations of $(\mathrm{CF}_\varepsilon)$ due to the convection terms. While the step from the bounds already established in the previous sections to uniform parabolic Hölder bounds is often achieved by employing well-known and ready-made parabolic regularity theory, we are not aware of such a result that fits the third equation in $(\mathrm{CF}_\varepsilon)$ and gives us the type of bound desired here.

Similar to the methods seen in e.g. [34] and [120, Lemma 3.4], we will therefore use a different approach that is based on the regularity properties of the fractional powers of the Stokes operator and the variation-of-constants representation of the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$, not unlike what we did in the proof of Lemma 3.6.8. The key difference to similar previous efforts in this chapter is that we apply these methods to difference terms of the form $u_\varepsilon(\cdot,t_2) - u_\varepsilon(\cdot,t_1)$ instead of u_ε itself.

Lemma 3.6.13. *There exists $t_0 > 0$ such that, for each $\beta \in (0,1)$, $p \in (2,\infty)$, there is a constant $C(\beta,p) > 0$ such that*

$$\|A^\beta\,[\,u_\varepsilon(\cdot,t_2) - u_\varepsilon(\cdot,t_1)\,]\,\|_{L^p(\Omega)} \leq C(\beta,p)(t_2-t_1)^{\frac{1-\beta}{2}}$$

for all $t_2 > t_1 > t_0$ with $t_2 - t_1 \leq 1$ and $\varepsilon \in (0,1)$.

Proof. We fix $\beta \in (0,1)$ and $p \in (2,\infty)$. We then further fix $t_0 > 0$ independently of β and p and $K_1 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq K_1, \qquad \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq K_1,$$
$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq K_1, \qquad \|A^{\frac{\beta+1}{2}} u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq K_1$$

for all $t \geq t_0$ and $\varepsilon \in (0,1)$ according to Lemma 3.6.5, Corollary 3.6.9 and Lemma 3.6.8. This then implies for

$$F_\varepsilon(x,t) \coloneqq \mathcal{P}\,[\,-(u_\varepsilon(x,t)\cdot\nabla)u_\varepsilon(x,t) + \nabla\phi(x)\cdot n_\varepsilon(x,t)\,] \qquad \text{for all } (x,t) \in \Omega \times [0,\infty)$$

that

$$\begin{aligned}
&\|F_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\
&\leq K_2\,\|-(u_\varepsilon(\cdot,t)\cdot\nabla)u_\varepsilon(\cdot,t) + \nabla\phi\cdot n_\varepsilon(\cdot,t)\,\|_{L^p(\Omega)} \\
&\leq K_2|\Omega|^{\frac{1}{p}}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + K_2\|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\
&\leq |\Omega|^{\frac{1}{p}}K_1^2K_2 + K_1K_2\|\nabla\phi\|_{L^\infty(\Omega)} \eqqcolon K_3
\end{aligned}$$

for all $t > t_0$ and $\varepsilon \in (0,1)$ with some constant $K_2 > 0$ due to the continuity of the Helmholtz projection $\mathcal{P}_p$.

Let now $t_2 > t_1 > t_0$ be such that $t_2 - t_1 \leq 1$. Using the variation-of-constants representation of u_ε with respect to the semigroup $(e^{-tA})_{t\geq 0}$, we then observe that

$$\begin{aligned}
&\left\|A^\beta\,[\,u_\varepsilon(\cdot,t_2) - u_\varepsilon(\cdot,t_1)\,]\right\|_{L^p(\Omega)} \\
&\leq \left\|A^\beta e^{-(t_2-t_0)A}u(\cdot,t_0) - A^\beta e^{-(t_1-t_0)A}u(\cdot,t_0)\right\|_{L^p(\Omega)} \\
&+ \left\|\int_{t_0}^{t_2} A^\beta e^{-(t_2-s)A}F_\varepsilon(\cdot,s)\,\mathrm{d}s - \int_{t_0}^{t_1} A^\beta e^{-(t_1-s)A}F_\varepsilon(\cdot,s)\,\mathrm{d}s\right\|_{L^p(\Omega)} \\
&\eqqcolon D_1 + D_2 \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned}$$

Using well-known smoothing properties of the Stokes semigroup (cf. [49, Theorem 1.4.3]) combined with the defining fact that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}e^{-tA}\varphi = -Ae^{-tA}\varphi$$

for all $t > 0$ and $\varphi \in C^2(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$ and $\nabla \cdot \varphi = 0$ on Ω and the fundamental theorem of calculus, we now estimate D_1 as follows:

$$\begin{aligned} D_1 &= \left\| A^\beta \int_{t_1}^{t_2} A e^{-(s-t_0)A} u(\cdot, t_0) \, \mathrm{d}s \right\|_{L^p(\Omega)} \\ &= \left\| \int_{t_1}^{t_2} A^{\frac{\beta+1}{2}} e^{-(s-t_0)A} A^{\frac{\beta+1}{2}} u(\cdot, t_0) \, \mathrm{d}s \right\|_{L^p(\Omega)} \\ &\le K_4 \int_{t_1}^{t_2} (s-t_0)^{-\frac{\beta+1}{2}} \|A^{\frac{\beta+1}{2}} u(\cdot, t_0)\|_{L^p(\Omega)} \, \mathrm{d}s \le K_1 K_4 \int_{t_1}^{t_2} (s-t_0)^{-\frac{\beta+1}{2}} \, \mathrm{d}s \\ &= \frac{2K_1K_4}{1-\beta} ((t_2-t_0)^{\frac{1-\beta}{2}} - (t_1-t_0)^{\frac{1-\beta}{2}}) \le K_5 (t_2-t_1)^{\frac{1-\beta}{2}} \end{aligned} \tag{3.6.30}$$

for all $\varepsilon \in (0,1)$ with $K_4 > 0$ being the smoothing constant from [49, Theorem 1.4.3] and $K_5 := \frac{2K_1K_4}{1-\beta}$.

By a similar argument, we gain for D_2 that

$$\begin{aligned} D_2 &\le K_6 \int_{t_1}^{t_2} (t_2-s)^{-\beta} \|F_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\ &\quad + \left\| \int_{t_0}^{t_1} A^\beta \left[e^{-(t_2-s)A} - e^{-(t_1-s)A} \right] F_\varepsilon(\cdot, s) \, \mathrm{d}s \right\|_{L^p(\Omega)} \\ &\le \frac{K_3K_6}{1-\beta} (t_2-t_1)^{1-\beta} + \left\| \int_{t_0}^{t_1} A^\beta \int_{t_1}^{t_2} A e^{-(\sigma-s)A} F_\varepsilon(\cdot, s) \, \mathrm{d}\sigma \, \mathrm{d}s \right\|_{L^p(\Omega)} \\ &\le \frac{K_3K_6}{1-\beta} (t_2-t_1)^{1-\beta} + K_3K_6 \int_{t_0}^{t_1} \int_{t_1}^{t_2} (\sigma-s)^{-1-\beta} \, \mathrm{d}\sigma \, \mathrm{d}s \\ &= \frac{K_3K_6}{1-\beta} (t_2-t_1)^{1-\beta} - \frac{K_3K_6}{\beta} \int_{t_0}^{t_1} (t_2-s)^{-\beta} - (t_1-s)^{-\beta} \, \mathrm{d}s \\ &= \frac{K_3K_6}{1-\beta} (t_2-t_1)^{1-\beta} \\ &\quad + \frac{K_3K_6}{\beta(1-\beta)} \left[(t_2-t_1)^{1-\beta} - (t_2-t_0)^{1-\beta} - (t_1-t_1)^{1-\beta} + (t_1-t_0)^{1-\beta} \right] \\ &\le K_7 (t_2-t_1)^{1-\beta} \end{aligned} \tag{3.6.31}$$

for all $\varepsilon \in (0,1)$ with $K_6 > 0$ being the smoothing constant from [49, Theorem 1.4.3] and $K_7 := K_3K_6\frac{1+\beta}{\beta(1-\beta)}$. Note here that the last step was made possible by the fact that $(t_2-t_0)^{1-\beta} \ge (t_1-t_0)^{1-\beta}$.

Because $t_2 - t_1 \le 1$, we further know that

$$(t_2-t_1)^{1-\beta} = (t_2-t_1)^{\frac{1-\beta}{2}} (t_2-t_1)^{\frac{1-\beta}{2}} \le (t_2-t_1)^{\frac{1-\beta}{2}}.$$

Therefore the two estimates (3.6.30) and (3.6.31) complete the proof. □

By again using similar methods to the proof of Corollary 3.6.9 as well as using said corollary itself, we can now derive our desired parabolic Hölder bound for the third solution components u_ε.

Corollary 3.6.14. *There exists* $t_0 > 0$ *such that, for each* $\alpha \in (0, \frac{1}{5})$*, there is a constant* $C(\alpha) > 0$ *with*

$$\|u_\varepsilon\|_{C^\alpha([t,t+1];C^{1+\alpha}(\overline{\Omega}))} \leq C(\alpha)$$

for all $t > t_0$ *and* $\varepsilon \in (0, 1)$*.*

Proof. Let $\beta := 1 - 2\alpha \in (\frac{3}{5}, 1)$ and $p \in (2, \infty)$ be such that

$$1 + \alpha < 2\beta - \frac{2}{p},$$

which is always possible because $1 + \alpha < \frac{6}{5} < 2\beta$. Then Corollary 3.6.9 as well as Lemma 3.6.13 give us a parameter independent time $t_0 > 0$ and constant $K_1 > 0$ such that

$$\|u_\varepsilon(\cdot, t_1)\|_{C^{1+\alpha}(\overline{\Omega})} \leq K_1 \tag{3.6.32}$$

and

$$\|A^\beta\, [\, u_\varepsilon(\cdot, t_2) - u_\varepsilon(\cdot, t_1)\,]\,\|_{L^p(\Omega)} \leq K_1(t_2 - t_1)^\alpha$$

for all $\varepsilon \in (0, 1)$ and $t_2 > t_1 > t_0$ with $t_2 - t_1 \leq 1$, which by the continuous embedding of $D(A_p^\beta)$ into $C^{1+\alpha}(\overline{\Omega})$ (cf. [49, p. 39] or [41]) implies that

$$\|u_\varepsilon(\cdot, t_2) - u_\varepsilon(\cdot, t_1)\|_{C^{1+\alpha}(\overline{\Omega})} \leq K_1 K_2 (t_2 - t_1)^\alpha \tag{3.6.33}$$

for all $\varepsilon \in (0, 1)$ with some $K_2 > 0$. Combining (3.6.32) and (3.6.33) then implies the desired result. □

To now prove similar, albeit slightly weaker, parabolic Hölder bounds for the first two solution components of our approximate solutions, we will employ the ready-made parabolic regularity theory of [97] to the first two equations in (CF_ε).

Lemma 3.6.15. *There exist* $t_0 > 0$*,* $\alpha \in (0, 1)$ *and* $C > 0$ *such that*

$$\|n_\varepsilon\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|c_\varepsilon\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} \leq C$$

for all $t > t_0$ *and* $\varepsilon \in (0, 1)$*.*

Proof. As preparations for the proof, let us now fix $t_0 > 0$ and $K_1 > 0$ such that

$$\begin{aligned} &\|n_\varepsilon\|_{L^\infty(\Omega\times[t_0-1,\infty))} \leq K_1, \quad \|\nabla c_\varepsilon\|_{L^\infty(\Omega\times[t_0-1,\infty))} \leq K_1, \\ &\|c_\varepsilon\|_{L^\infty(\Omega\times[t_0-1,\infty))} \leq K_1, \quad \|u_\varepsilon\|_{L^\infty(\Omega\times[t_0-1,\infty))} \leq K_1 \end{aligned} \tag{3.6.34}$$

due to Lemma 3.6.11, Lemma 3.6.10, Lemma 3.4.2 and Corollary 3.6.9.

Let us now check that our approximate solutions conform to the prerequisites used for

Theorem 1.3 in [97]. Framed in the notation of the reference, the first two equations in $(\mathrm{CF}_\varepsilon)$ considered in isolation translate to

$$a(x,t,y,z) := z - n_\varepsilon(x,t)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon(x,t) - u_\varepsilon(x,t)n_\varepsilon(x,t), \qquad b(x,t,y,z) := 0,$$
$$a(x,t,y,z) := z - u_\varepsilon(x,t)c_\varepsilon(x,t), \qquad b(x,t,y,z) := -n_\varepsilon(x,t)f(c_\varepsilon(x,t)) \tag{3.6.35}$$

for $(x,t,y,z) \in \Omega \times [t_0-1,\infty) \times \mathbb{R} \times \mathbb{R}^2$, respectively. We can then choose the parameters in the reference to be mostly constants, which only depend on K_1 due to (3.6.34) and (3.6.35), or to be trivial. We further choose $p := 2$, $\hat{r} := \infty$, $\hat{q} := 2$, $\kappa_1 := \frac{1}{2}$. The remaining conditions to use Theorem 1.3 from [97] are then easy to check and we can therefore apply it to the first and second equation in $(\mathrm{CF}_\varepsilon)$ to gain $\alpha \in (0,1), K_2 > 0$ such that

$$\|n_\varepsilon\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|c_\varepsilon\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} \le K_2$$

for all $t > t_0$ and $\varepsilon \in (0,1)$. As Theorem 1.3 makes it explicit, how K_2 and α depend on the chosen parameters, it is easy to verify that both constants are in fact independent of t. This completes the proof. □

3.6.5 Deriving $C^{2+\alpha,1+\frac{\alpha}{2}}$-type parabolic Hölder regularity properties for (n, c, u)

We now transition from proving properties of the approximate solutions to the generalized solutions (n,c,u) themselves. This is done mostly to mitigate problems stemming from the approximated sensitivity term S_ε when applying higher order parabolic regularity theory to the approximate solutions. More specifically, said problems stem from the fact that the standard regularity theory found in [61] and [73] would require all S_ε to be uniformly bounded in some appropriate Hölder spaces to yield uniform higher order Hölder space bounds for the solutions themselves but our method of approximation for the sensitivity function S does not necessarily yield such uniform bounds.

Our first step then is to translate the uniform parabolic Hölder regularity properties of the approximate solutions derived in the previous section to our generalized solution (n,c,u) by using the well-known compact embedding properties of Hölder spaces. Notably, the argument to achieve this requires us to potentially redefine the functions n, c and u on a null set to make sure they coincide with a new set of locally Hölder-continuous limit functions. For ease of notation, we assume without loss of generality that the triple (n,c,u) was already chosen in Lemma 3.5.2 in such a way as to not need any such redefinition here. This is possible because it can be easily seen that any changes to n, c or u on a null set do not interfere with the properties derived in Section 3.5 and all arguments in this section thus far have been restricted to the approximate level.

Lemma 3.6.16. *There exist $t_0 > 0$, $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|n\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|c\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|u\|_{C^{\alpha}([t,t+1];C^{1+\alpha}(\overline{\Omega}))} \le C \tag{3.6.36}$$

for all $t > t_0$*.*

Proof. As we already know that (n, c, u) are the (almost everywhere) pointwise limits of the approximate solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ along a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, the compact embedding properties of the Hölder spaces $C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times [t, t+1])$, $C^\beta([t, t+1]; C^{1+\beta}(\overline{\Omega}))$ into similar spaces with slightly smaller parameters combined with Corollary 3.6.14 and Lemma 3.6.15 directly yield our desired result (see the paragraph leading into this lemma for the reason why we can assume without loss of generality that no redefinition of the functions n, c and u on a null set is necessary here). □

While the weak solution properties described in Definition 3.3.1 for the second and third equation are fairly standard and therefore very accessible to ready-made regularity theory, the integral inequalities used for the first solution component n in said definition are not compatible with such standard regularity results. Therefore, our second step in this section is arguing that n fulfills a similar weak solution property to the other two solution components from some time $t_0 > 0$ onward due to the strong a priori bounds derived in the previous sections.

Lemma 3.6.17. *There exists* $t_0 > 0$ *such that* n *is a weak solution of the first equation in (CF) on* $\Omega \times [t_0, \infty)$ *with no-flux boundary conditions in the sense that* $n \in C^0(\overline{\Omega} \times [t_0, \infty)) \cap L^2_{\mathrm{loc}}([t_0, \infty); W^{1,2}(\Omega))$ *and*

$$\begin{aligned}&\int_{t_0}^\infty \int_\Omega n\varphi_t + \int_\Omega n(\cdot, t_0)\varphi(\cdot, t_0)\\ =& \int_{t_0}^\infty \int_\Omega \nabla n \cdot \nabla\varphi - \int_{t_0}^\infty \int_\Omega nS(\cdot, n, c)\nabla c \cdot \nabla\varphi - \int_{t_0}^\infty \int_\Omega n(u \cdot \nabla\varphi)\end{aligned} \tag{3.6.37}$$

holds for all $\varphi \in C_c^\infty(\overline{\Omega} \times [t_0, \infty))$*.*

Proof. We begin by fixing $t_0 > 0$ and $K > 0$ such that

$$\begin{aligned}&\|n_\varepsilon\|_{L^\infty(\Omega\times[t_0,\infty))} \le K, &&\|n_\varepsilon\|_{L^2_{\mathrm{loc}}([t_0,\infty);W^{1,2}(\Omega))} \le K,\\ &\|c_\varepsilon\|_{L^2_{\mathrm{loc}}([t_0,\infty);W^{1,2}(\Omega))} \le K, &&\|u_\varepsilon\|_{L^\infty(\Omega\times[t_0,\infty))} \le K\end{aligned} \tag{3.6.38}$$

due to Lemma 3.6.11, Lemma 3.6.12, Lemma 3.4.2 and Corollary 3.6.9. Among other things, this then allows us to assume without loss of generality (by potentially choosing a subsequence) that

$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{and} \quad \nabla c_\varepsilon \rightharpoonup \nabla c \qquad \text{in } L^2_{\mathrm{loc}}(\Omega \times [t_0, \infty)) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.6.39}$$

This combined with Lemma 3.6.16 then immediately gives us that $n \in C^0(\overline{\Omega} \times [t_0, \infty)) \cap L^2_{\mathrm{loc}}([t_0, \infty); W^{1,2}(\Omega))$.

It is further easy to see that the approximate solutions satisfy

$$\begin{aligned}&\int_{t_0}^\infty \int_\Omega n_\varepsilon \varphi_t + \int_\Omega n_\varepsilon(\cdot,t_0)\varphi(\cdot,t_0)\\ =& \int_{t_0}^\infty \int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi - \int_{t_0}^\infty \int_\Omega n_\varepsilon S_\varepsilon(\cdot,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \cdot \nabla\varphi - \int_{t_0}^\infty \int_\Omega n_\varepsilon(u_\varepsilon \cdot \nabla\varphi)\end{aligned} \tag{3.6.40}$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \times [t_0,\infty))$. We therefore now only need to show that all of the above integral terms converge along the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ to the corresponding terms in (3.6.37). For the first two and the last term in (3.6.40), convergence is immediately assured by the dominated convergence theorem in combination with some of the bounds from (3.6.38). The convergence of the third term in (3.6.40) follows directly from (3.6.39). For the remaining fourth term in (3.6.40), consider first that (3.6.38) combined with the (almost everywhere) pointwise convergence of the approximate solutions implies that $n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla\varphi \to nS(x,n,c)\nabla\varphi$ in $L^2(\Omega\times[t_0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$ due to the dominated convergence theorem and the fact that the approximate sensitivities S_ε converge to S in a pointwise fashion as $\varepsilon \searrow 0$. This then combined with the weak convergence properties from (3.6.39) ensures the convergence of the last remaining term and therefore completes the proof. □

Given that now n, c, and u each fulfill a quite standard weak solution property for their corresponding equations and are already of fairly high regularity (cf. Lemma 3.6.16), our final step of this section is to use the well-known uniqueness and regularity theory from [61] and [73] (for the first two equations) as well as [101] and [102] (for the third equation) combined with a standard cutoff function argument to remove the influence of initial data regularity (cf. also Lemma 5.5.4 for the same basic technique) to argue that all three solution components were already bounded in some $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [t,t+1])$ spaces. Our application of said Schauder theory largely follows the approach shown in [18, Lemma 5.7] but will deviate slightly from this precedent to make it more explicit that our constants are uniform in time as needed for our later long-time behavior results.

Lemma 3.6.18. *There exist $t_0 > 0$, $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|c\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} \le C$$

for all $t > t_0$.

Proof. Due to requiring some temporal leeway for the removal of initial data regularity conditions from the Schauder theory at the core of the following arguments by using cutoff functions, we need to choose a time $t_0 > 0$ in this lemma that is slightly larger than the times t_0 found in Lemmas 3.6.16 and 3.6.17. More specifically, we use the two aforementioned lemmas to choose $t_0 > 1$, $\alpha_1 \in (0,1)$ and $K_1 > 0$ such that

$$\|n\|_{C^{\alpha_1,\frac{\alpha_1}{2}}(\overline{\Omega}\times[t,t+1])} + \|c\|_{C^{\alpha_1,\frac{\alpha_1}{2}}(\overline{\Omega}\times[t,t+1])} + \|u\|_{C^{\alpha_1}([t,t+1];C^{1+\alpha_1}(\overline{\Omega}))} \le K_1 \tag{3.6.41}$$

holds for all $t > t_0 - 1$ and such that n is a weak solution on $\Omega \times [t_0 - 1, \infty)$ in the sense laid out in Lemma 3.6.17.

By framing the functions n and c as solutions to partial differential equations of the form $-\varphi_t + \nabla \cdot (A(x, t, \varphi, \nabla\varphi)) + B(x, t, \varphi, \nabla\varphi) = 0$ with boundary condition $A(x, t, \varphi, \nabla\varphi) \cdot \nu = 0$ and A and B defined in a very similar way to a and b in (3.6.35), both solution components become available to the parabolic regularity theory laid out in [73, Theorem 1.1]. When combined with the bounds established in (3.6.41) as well as a cutoff function argument to remove initial data conditions, said parabolic regularity result then yields additional Hölder bounds for c, which in turn allow us to apply the same regularity theory again to gain similar bounds for n. In this way, we can find $\alpha_2 \in (0, 1)$ and $K_2 > 0$ such that

$$\|n\|_{C^{\alpha_2+1, \frac{\alpha_2+1}{2}}(\overline{\Omega} \times [t,t+1])} + \|c\|_{C^{\alpha_2+1, \frac{\alpha_2+1}{2}}(\overline{\Omega} \times [t,t+1])} \leq K_2 \tag{3.6.42}$$

for all $t > t_0 - \frac{1}{2}$.

We now consider the first two equations in (CF) in the standard form $\varphi_t = \Delta\varphi + F(x, t)$ with boundary condition $\nabla\varphi \cdot \nu = G(x, t)$, where F and G are appropriately chosen source and boundary terms, respectively. This makes n and c accessible to the well-known Schauder theory found in [61, p. 170 and p. 320] combined with another cutoff function argument. By leveraging the bounds established in (3.6.42), we can thus apply this regularity theory first to c and then to n to find $\alpha_3 \in (0, 1)$ and $K_3 > 0$ such that

$$\|n\|_{C^{2+\alpha_3, 1+\frac{\alpha_3}{2}}(\overline{\Omega} \times [t,t+1])} + \|c\|_{C^{2+\alpha_3, 1+\frac{\alpha_3}{2}}(\overline{\Omega} \times [t,t+1])} \leq K_3$$

for all $t > t_0$.

Let us now briefly elaborate on two somewhat subtle but important reasons for us first using [73, Theorem 1.1] to establish some intermediate Hölder bounds for n and c before applying the theory from [61] in the paragraph above. First, the boundary term $G(x, t) = n(x, t)(S(x, n(x, t), c(x, t))\nabla c(x, t)) \cdot \nu(x)$ for the first equation in (CF) features a potentially nonlinear dependency on n due to the arbitrary matrix-valued function S and thus the boundary condition $\nabla n \cdot \nu = G(x, t)$ can not generally be rewritten as $\mathcal{B}n = 0$ with a linear boundary operator $\mathcal{B}$ that is independent of n. This means that, for the standard regularity theory from [61] to be usable, we need a $C^{1+\beta, \frac{1+\beta}{2}}$-type bound for n and thus G on the boundary, which is provided by [73] under less stringent requirements on n. Second while it would be possible to streamline the above argument somewhat by reinterpreting some of the source terms (e.g. $u \cdot \nabla n$ or $u \cdot \nabla c$) as parts of the relevant elliptic operators to apply the results from [61] earlier, the standard theory laid out in [61, p. 170 and p. 320] does not explicitly account (in the actual text of the relevant theorems) for how bounds of the coefficients of the considered operator ultimately play into the final constants obtained in the theorems and it would thus be much less obvious that our constants are uniform in time.

Finally, we now turn our attention to the third equation in (CF), which can be framed as the standard evolution Stokes problem $\varphi_t = \Delta\varphi + \nabla P + F(x,t)$ with $\nabla \cdot \varphi = 0$ and boundary condition $\varphi = 0$ as well as the specific source term $F(x,t) = -(u(x,t) \cdot \nabla)u(x,t) + n(x,t)\nabla\phi(x)$. Due to (3.6.41), we already know that there exist $\alpha_4 \in (0,1)$ and $K_4 > 0$ such that $\|F\|_{C^{\alpha_4,\frac{\alpha_4}{2}}(\overline{\Omega}\times[t,t+1])} \leq K_4$ for all $t > t_0 - 1$. This then allows us to combine the uniqueness result for weak solutions laid out in [101, Theorem V.1.5.3] with the regularity theory from [102, Proposition 1.1] and another cutoff function argument to conclude that there exist $\alpha_5 \in (0,1)$ and $K_5 > 0$ such that

$$\|u\|_{C^{2+\alpha_5,1+\frac{\alpha_5}{2}}} \leq K_5,$$

for all $t > t_0$, which completes the proof. □

3.6.6 Stabilization of (n, c, u)

Having now essentially established all our desired regularity properties for the generalized solutions in the previous lemma, we now focus on proving the remaining long-time stabilization properties for n, c and u in $C^2(\overline{\Omega})$.

Lemma 3.6.19. *The generalized solution (n, c, u) has the long-time stabilization property in (3.1.7).*

Proof. Lemma 3.6.18 directly gives us $\alpha \in (0,1)$, $t_0 > 0$ and $K_1 > 0$ such that

$$\|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|c\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} \leq K_1$$

for all $t > t_0$.

Let us further fix $K_2 > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{L^1(\Omega)} \leq K_2 \sqrt{\int_\Omega \varphi} \cdot \sqrt{\int_\Omega \varphi \ln\left(\frac{\varphi}{\overline{\varphi}}\right)}$$

for all nonnegative $\varphi \in C^0(\overline{\Omega})$ with $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ according to a Cziszár–Kullback or Pinsker-type inequality (cf. [24]). Let now $\delta > 0$. By Lemma 3.6.4 and the inequality above, there therefore exists $t_\delta > t_0$ such that

$$\|n_\varepsilon(\cdot,t) - \overline{n_0}\|_{L^1(\Omega)} \leq K_2 \sqrt{\int_\Omega n_0} \cdot \sqrt{\int_\Omega n_\varepsilon(\cdot,t) \ln\left(\frac{n_\varepsilon(\cdot,t)}{\overline{n_0}}\right)} < \frac{\delta}{2} \tag{3.6.43}$$

for all $t > t_\delta$ and $\varepsilon \in (0,1)$. Further for each $t > t_\delta > t_0$, there exists an $\varepsilon(t) \in (0,1)$ such that

$$\|n(\cdot,t) - n_{\varepsilon(t)}(\cdot,t)\|_{L^1(\Omega)} < \frac{\delta}{2}$$

because of, for example, the (almost everywhere) pointwise convergence of the approximate solutions to the generalized solutions combined with the dominated convergence theorem (using a constant majorant as established by Lemma 3.6.11). Combining the above two inequalities then results in

$$\|n(\cdot,t)-\overline{n_0}\|_{L^1(\Omega)} \le \|n(\cdot,t)-n_{\varepsilon(t)}(\cdot,t)\|_{L^1(\Omega)} + \|n_{\varepsilon(t)}(\cdot,t)-\overline{n_0}\|_{L^1(\Omega)} < \delta$$

for all $t > t_\delta$ and therefore $n(\cdot,t) \to \overline{n_0}$ in $L^1(\Omega)$ as $t \to \infty$.

As the start of a proof by contradiction, we assume now that $n(\cdot,t)$ does not converge to $\overline{n_0}$ in $C^2(\overline{\Omega})$ as $t \to \infty$. Then there must exist a constant $K_3 > 0$ and a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\|n(\cdot,t_k)-\overline{n_0}\|_{C^2(\overline{\Omega})} > K_3 \quad \text{for all } k \in \mathbb{N}. \tag{3.6.44}$$

As the family

$$(n(\cdot,t_k))_{k\in\mathbb{N}}$$

is furthermore uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$ by K_1, an application of the Arzelà–Ascoli theorem yields that the sequence $(t_k)_{k\in\mathbb{N}}$ has a subsequence, along which $n(\cdot,t_k)$ converges to some limit value in $C^2(\overline{\Omega})$. As we already know that $n(\cdot,t_k)$ converges to $\overline{n_0}$ in $L^1(\Omega)$ as $k \to \infty$ by prior arguments, the above $C^2(\overline{\Omega})$ limit must be $\overline{n_0}$ as well. This is a contraction to (3.6.44) and therefore we have proven that $n(\cdot,t) \to \overline{n_0}$ as $t \to \infty$ in $C^2(\overline{\Omega})$.

As we have proven similar uniform convergence properties to (3.6.43) for $c_\varepsilon(\cdot,t)$ and $u_\varepsilon(\cdot,t)$ in Lemma 3.6.1 and Lemma 3.6.4, the above argument can basically be reused verbatim to prove the remaining two convergence properties in (3.1.7). This completes the proof. □

3.7 Proof of the main theorem

As we have at this point collected all of the necessary parts, we can now present the proof of our main result in a fairly swift fashion.

Proof of Theorem 3.1.1. Let (n,c,u) be the functions constructed in Lemma 3.5.2. In Lemma 3.5.5, we have then shown that these functions form a global mass-preserving generalized solution in the sense of Definition 3.3.1, which constitutes the first part of Theorem 3.1.1. As a direct consequence of Lemma 3.6.18, we can then find $t_0 > 0$ such that (3.1.6) holds, meaning that our generalized solutions were in fact already smooth all along from some $t_0 > 0$ onwards. As we further know that n, c and u fulfill a standard weak solution property due to Lemma 3.5.5 and Lemma 3.6.17 for their corresponding equations in (CF), standard solution theory then yields that our solutions must have already been classical on $\Omega \times (t_0,\infty)$ and that an associated pressure function P for the fluid equation can be constructed (cf. [61, p. 170 and p. 320], [101, Theorem V.1.8.3]). Finally, Lemma 3.6.19 yields the long-time stabilization property seen in (3.1.7). This completes the proof. □

4 Weak solutions to a haptotaxis system with a potentially degenerate diffusion and taxis tensor

4.1 Main result

In this chapter, we consider the potentially degenerate haptotaxis system

$$\begin{cases} n_t = \nabla \cdot (\mathbb{D}\nabla n + n\nabla \cdot \mathbb{D}) - \chi\nabla \cdot (n\mathbb{D}\nabla c) + \mu n(1 - n^{r-1}), \\ c_t = -nc \end{cases} \tag{DH}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \{2, 3\}$, with the no-flux boundary condition

$$(\mathbb{D}\nabla n) \cdot \nu = \chi(n\mathbb{D}\nabla c) \cdot \nu - n(\nabla \cdot \mathbb{D}) \cdot \nu \qquad \text{for all } x \in \partial\Omega, t > 0 \tag{DHB}$$

and appropriate parameters $\chi > 0$, $\mu > 0$, $r \geq 2$ and $\mathbb{D} : \overline{\Omega} \to \mathbb{R}^{N\times N}$, $\mathbb{D}$ positive semidefinite on $\overline{\Omega}$.

Our results concerning the system in (DH) with boundary condition (DHB) are twofold. We will first derive the following existence result concerning global classical solutions in two and three dimensions under the assumptions that $\mathbb{D}$ and the initial data are sufficiently regular, $\mathbb{D}$ is positive definite on $\overline{\Omega}$ and the logistic source term is sufficiently strong.

Theorem 4.1.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \in \{2, 3\}$, be a bounded domain with a smooth boundary, $\chi \in (0, \infty)$, $\mu \in (0, \infty)$, $r \in [2, \infty)$ and $\mathbb{D} \in C^2(\overline{\Omega}; \mathbb{R}^{N\times N})$. We further assume that $\mathbb{D}$ is (symmetric and) positive definite on $\overline{\Omega}$ and satisfies $(\nabla \cdot \mathbb{D}) \cdot \nu = 0$ on $\partial\Omega$. Let $n_0, c_0 \in C^{2+\vartheta}(\overline{\Omega})$, $\vartheta \in (0, 1)$, be some initial data with $n_0, c_0 > 0$ on $\overline{\Omega}$ and $(\mathbb{D}\nabla n_0) \cdot \nu = (\mathbb{D}\nabla c_0) \cdot \nu = 0$ on $\partial\Omega$.*

If either $r > 2$ or $\mu \geq \chi\|c_0\|_{L^\infty(\Omega)}$, then there exists a unique pair of positive functions $n, c \in C^{2,1}(\overline{\Omega} \times [0, \infty))$ such that (n, c) is a global classical solution to (DH) on $\Omega \times (0, \infty)$ with the no-flux boundary condition (DHB) and initial data (n_0, c_0).

This result, while of course also of independent interest, will then serve as a building block for the construction of weak solutions to the same system under much more relaxed restrictions on $\mathbb{D}$ and the initial data. Chiefly, global positivity of the matrix $\mathbb{D}$ is not necessarily needed anymore and is instead replaced by a set of much weaker regularity assumptions, which are precisely tailored to our specific approach.

The first such regularity property concerns the divergence of $\mathbb{D}$ (applied columnwise) and how it can be estimated by the (potentially degenerate) scalar product induced by $\mathbb{D}$ on $\mathbb{R}^N$ at each point $x \in \Omega$.

Definition 4.1.2. Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary. We then say a positive semidefinite matrix-valued function $\mathbb{D} = (\mathbb{D}_1 \dots \mathbb{D}_N) \in L^1(\Omega; \mathbb{R}^{N\times N})$ with $\nabla \cdot \mathbb{D} := (\nabla \cdot \mathbb{D}_1, \dots, \nabla \cdot \mathbb{D}_N) \in L^1(\Omega; \mathbb{R}^N)$ allows for a *divergence estimate* with exponent $\beta \in [\frac{1}{2}, 1)$ if there exists $A \geq 0$ such that

$$\left|\int_\Omega (\nabla \cdot \mathbb{D}) \cdot \Phi\right| \leq A \left(\int_\Omega (\Phi \cdot \mathbb{D}\Phi)^\beta + 1\right) \tag{4.1.1}$$

for all $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^N)$.

Remark 4.1.3. Note that, if $\mathbb{D} \in C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$, $\mathbb{D}$ allowing for a divergence estimate with exponent $\beta \in (\frac{1}{2}, 1)$ implies that $\nabla \cdot \mathbb{D} \in L^{\frac{2\beta}{2\beta-1}}(\Omega; \mathbb{R}^N) \subseteq L^2(\Omega; \mathbb{R}^N)$. This stems from the fact that the estimate in (4.1.1) essentially implies that the functional $\Phi \mapsto \int_\Omega (\nabla \cdot \mathbb{D}) \cdot \Phi$ is an element of $(L^{2\beta}(\Omega; \mathbb{R}^N))^*$, which is isomorphic to $L^{\frac{2\beta}{2\beta-1}}(\Omega; \mathbb{R}^N)$.

It is fairly easy to verify that any smooth, positive definite $\mathbb{D}$ allows for such an estimate with the optimal exponent $\beta = \frac{1}{2}$. Let us therefore now briefly illustrate that the above property is also achievable for less regular $\mathbb{D}$, which are e.g. at some points in Ω only positive semidefinite, by giving some examples. While we will not necessarily fully explore these examples and leave out some of the more cumbersome corner cases for ease of presentation, they will accompany us throughout this section as a tool to give some intuition for later introduced definitions as well as to give concrete examples for degenerate cases in which weak solutions can still be constructed.

Example 4.1.4. We will first take a look at the prototypical case of a matrix-valued function $\mathbb{D}_1$ on a ball with a single degenerate point in the origin, or more precisely we will consider $\mathbb{D}_1(x) := |x|^s I$ on $\Omega := B_1(0) \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, I being the identity matrix and s being some positive real number.

As $\nabla \cdot \mathbb{D}_1(x) = \nabla(|x|^s) = s|x|^{s-2}x$ almost everywhere, we can estimate

$$\begin{aligned}\left|\int_\Omega (\nabla \cdot \mathbb{D}_1) \cdot \Phi\right| \leq \int_\Omega |(\nabla \cdot \mathbb{D}_1) \cdot \Phi| \leq s \int_\Omega |x|^{s-1} |\Phi| &= s \left\| |x|^{\frac{s}{2}-1} (\Phi \cdot |x|^s I \Phi)^{\frac{1}{2}} \right\|_{L^1(\Omega)} \\ &\leq s \left\| |x|^{\frac{s}{2}-1} \right\|_{L^{\frac{2\beta}{2\beta-1}}(\Omega)} \left\| (\Phi \cdot |x|^s I \Phi)^{\frac{1}{2}} \right\|_{L^{2\beta}(\Omega)} \\ &\leq s \left\| |x|^{\frac{s}{2}-1} \right\|_{L^{\frac{2\beta}{2\beta-1}}(\Omega)} \left(\int_\Omega (\Phi \cdot \mathbb{D}_1 \Phi)^\beta + 1\right)\end{aligned}$$

for all $\beta \in (\frac{1}{2}, 1)$ and $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^N)$ using the Hölder inequality as well as Young's inequality. As $|x|^{\frac{s}{2}-1} \in L^{\frac{2\beta}{2\beta-1}}(\Omega)$ if and only if $\frac{2\beta}{2\beta-1}(\frac{s}{2} - 1) > -N$, the prototypical case discussed above fulfills the divergence estimate for all $\beta \in (\frac{N}{s-2+2N}, \infty) \cap (\frac{1}{2}, 1)$. Note that for $s > 2 - N$, which in two or more dimensions is always ensured, the set $(\frac{N}{s-2+2N}, \infty) \cap (\frac{1}{2}, 1)$ is never empty, and therefore our prototypical example always has the discussed property for all positive s and some appropriate β.

Example 4.1.5. To illustrate that our framework also supports analysis of singularities occurring on higher dimensional manifolds, let us further consider the similar prototypical example $\mathbb{D}_2(x_1, \dots, x_N) := |x_1|^s I$ on the same set Ω with s now being a real number greater than 1. As here $\nabla \cdot \mathbb{D}_2(x_1, \dots, x_N) = (s|x_1|^{s-2}x_1, 0, \dots, 0)$ almost everywhere, we gain that $\mathbb{D}_2$ has the property laid out in Definition 4.1.2 for all $\beta \in (\frac{1}{s}, \infty) \cap (\frac{1}{2}, 1)$ by a similar argument as for the previous example.

As to be expected in both of the above examples, smaller values of s result in the divergence estimate only holding for ever larger exponents β. As we will see in our theorem regarding the existence of weak solutions at the end of this section, these larger values of β will necessitate stronger regularizing influence from the logistic source term to compensate for our arguments to work.

Example 4.1.6. To complement the previous discussion, we will lastly give a nondiagonal example, which is only ever degenerate regarding one of its eigenvalues, namely

$$\mathbb{D}_3(x_1, x_2) := \begin{bmatrix} (x_1 + 2)|x_2| & -\frac{1}{2}|x_2|x_2 \\ -\frac{1}{2}|x_2|x_2 & 1 \end{bmatrix} \qquad \text{on } \Omega = B_1(0) \subseteq \mathbb{R}^2.$$

We first note that $\det(\mathbb{D}_3(x_1, x_2)) = |x_2|(x_1 + 2 - \frac{|x_2|^3}{4}) \geq \frac{3}{4}|x_2|$ on Ω. Therefore by Silvester's criterion, $\mathbb{D}_3$ is positive definite on Ω if $x_2 \neq 0$. If $x_2 = 0$, it is easy to see that $\mathbb{D}_3$ has the eigenvalues 0 and 1 and is thus still positive semidefinite but not positive definite anymore. Because $\nabla \cdot \mathbb{D}_3 = 0$ almost everywhere on Ω as can be seen by a simple computation, $\mathbb{D}_3$ moreover trivially allows for a divergence estimate with exponent $\beta = \frac{1}{2}$.

Before we can now approach the second regularity property of this section as well as properly define what we in fact mean by weak solutions in this chapter, we need to first introduce a set of function spaces. Said spaces are generally fairly straightforward generalizations of standard Sobolev and Lebesgue spaces incorporating $\mathbb{D}$ as well as some spaces derived from them, which are more specific to our setting. We will further take the introduction of said spaces as an opportunity to present some of their most important properties for our purposes immediately after defining them. For a more thorough discussion of e.g. the degenerate Sobolev spaces introduced below, we refer the reader to [99].

Definition 4.1.7. Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary and $p \in [1, \infty)$.

We then define the Sobolev-type space

$$W^{1,p}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \coloneqq \left\{ M \in L^p(\Omega; \mathbb{R}^{N\times N}) \;\middle|\; \nabla \cdot M \in L^p(\Omega; \mathbb{R}^N) \right\}$$

with the norm

$$\|M\|_{W^{1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^{N\times N})} \coloneqq \|M\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\nabla \cdot M\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Herein, the divergence of a square matrix-valued function $M = (M_1 \, \dots \, M_N)$ is defined as $\nabla \cdot M \coloneqq (\nabla \cdot M_1, \dots, \nabla \cdot M_N)$.

Let now $\mathbb{D} \in C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$ be positive semidefinite everywhere. We then define the Lebesgue-type space $L^p_{\mathbb{D}}(\Omega)$ as the set of all measurable $\mathbb{R}^N$-valued functions Φ on Ω with finite seminorm

$$\|\Phi\|_{L^p_{\mathbb{D}}(\Omega)} \coloneqq \left(\int_\Omega (\Phi \cdot \mathbb{D}\Phi)^{\frac{p}{2}} \right)^{\frac{1}{p}}$$

modulo all of those functions with $\|\Phi\|_{L^p_{\mathbb{D}}(\Omega)} = 0$ in the same vein as the standard Lebesgue spaces.

Furthermore, we define the Sobolev-type spaces $W^{1,p}_{\mathbb{D}}(\Omega)$ as the completion of the function space $C^\infty(\overline{\Omega}; \mathbb{R})$ in the norm

$$\|\varphi\|_{W^{1,p}_{\mathbb{D}}(\Omega)} \coloneqq \|\varphi\|_{L^p(\Omega)} + \|\nabla\varphi\|_{L^p_{\mathbb{D}}(\Omega)}$$

in the same vein as the standard Sobolev spaces. It is straightforward to see that each space $W^{1,p}_{\mathbb{D}}(\Omega)$ can be interpreted as a subspace of $L^p(\Omega) \times L^p_{\mathbb{D}}(\Omega)$ in a natural way and thus elements of these spaces can be written as tuples (φ, Φ). Therefore, there exist the natural continuous projections

$$P_1 : W^{1,p}_{\mathbb{D}}(\Omega) \to L^p(\Omega) \quad \text{and} \quad P_2 : W^{1,p}_{\mathbb{D}}(\Omega) \to L^p_{\mathbb{D}}(\Omega)$$

associated with this representation.

Remark 4.1.8. We will now give a brief overview of the properties the above spaces retain from the standard Sobolev and Lebesgue spaces as well as some of the differences. As most of the proofs translate directly from standard Sobolev theory or are laid out in [99], we will only list the properties we are interested in without extensive argument.

First of all by construction, $W^{1,p}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N})$, $L^p_{\mathbb{D}}(\Omega)$ and $W^{1,p}_{\mathbb{D}}(\Omega)$ are Banach spaces, which are reflexive if $p \in (1, \infty)$, by essentially the same arguments as for the standard Sobolev and Lebesgue spaces and, for $p = 2$, they are in fact Hilbert spaces with the natural inner products. It is further easy to see that, if (φ, Φ) is a strong or weak limit of a sequence $((\varphi_k, \Phi_k))_{k\in\mathbb{N}}$ in $W^{1,p}_{\mathbb{D}}(\Omega)$, the function $\varphi \in L^p(\Omega)$ coincides with the pointwise almost everywhere limit of the sequence $(\varphi_k)_{k\in\mathbb{N}}$ if it exists due to P_1 being continuous regarding both topologies and well-known results about strong and weak convergence in $L^p(\Omega)$.

As opposed to the standard Sobolev spaces, the spaces $W^{1,p}_{\mathbb{D}}(\Omega)$ can not in general be understood as subspaces of the spaces $L^p(\Omega)$ as P_1 is not always injective. (For an example of this, see [99, p. 1877]). Given that this can be problematic when deriving analogues to the (compact) embedding properties of Sobolev spaces for our weaker variants, let us now briefly note that, under sufficient regularity assumptions for $\mathbb{D}$, the spaces $W^{1,p}_{\mathbb{D}}(\Omega)$ do in fact embed into the spaces $L^p(\Omega)$. In particular if $p = 2$, which is the parameter choice we are most interested here, this is the case if $\sqrt{\mathbb{D}} \in W^{1,2}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N})$ according to Lemma 8 from [99]. In fact, this is achieved by actually recovering some weak derivative type properties for $P_2(\varphi)$, $\varphi \in W^{1,2}_{\mathbb{D}}(\Omega)$, and then using the uniqueness of weak derivatives to argue that $P_2(\varphi)$ and therefore φ itself is uniquely identified by $P_1(\varphi)$.

While it presents a slight abuse of notation, we will in a similar fashion to [99] use φ to mean $P_1(\varphi) \in L^p(\Omega)$ for elements $\varphi \in W^{1,p}_{\mathbb{D}}(\Omega)$ when unambiguous and generally use the convention $\nabla\varphi = P_2(\varphi)$ even if $\nabla\varphi$ is not necessarily the actual weak derivative. If φ is additionally an element of $C^1(\overline{\Omega})$, we will always assume $\nabla\varphi$ to be equal to the standard derivative, of course.

Having introduced the necessary function spaces, we can now clearly state the second and last regularity property for $\mathbb{D}$ we are interested in. It is a simple compact embedding property, which is mainly used in this chapter to facilitate application of the well-known Aubin–Lions lemma.

Definition 4.1.9. Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary. We say a positive semidefinite matrix-valued function $\mathbb{D} \in C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$ allows for a *compact $L^1(\Omega)$ embedding* if $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1(\Omega)$, i.e. P_1 is injective and each bounded sequence in $W^{1,2}_{\mathbb{D}}(\Omega)$ has a subsequence that converges in $L^1(\Omega)$.

Remark 4.1.10. Let us briefly note that any $\mathbb{D}$, which is equal to zero on any open subset U of Ω, cannot fulfill the property laid out in Definition 4.1.9 as it is well documented that $L^2(U)$, which is equal to $W^{1,2}_{\mathbb{D}}(U)$ in this case, does not embed compactly into $L^1(U)$.

We will now give some additional criteria for the above compact embedding property to not only make our results easier to use in applications but also to help us prove that the examples discussed above in fact fulfill it.

Lemma 4.1.11. *Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary and U be a relatively closed set in Ω of measure zero. Moreover, let $\mathbb{D} \in C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$ be positive semidefinite.*

If $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1_{\mathrm{loc}}(\Omega \setminus U)$, i.e. P_1 is injective and each bounded sequence in $W^{1,2}_{\mathbb{D}}(\Omega)$ has a subsequence that converges in $L^1_{\mathrm{loc}}(\Omega \setminus U)$, then $\mathbb{D}$ allows for *a compact $L^1(\Omega)$ embedding.*

Further if $\mathbb{D}$ is positive definite on $\Omega\setminus U$, then $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1_{\mathrm{loc}}(\Omega\setminus U)$ in the above sense and thus $\mathbb{D}$ allows for a compact $L^1(\Omega)$ embedding.

Proof. To prove the first half of our result, we assume that $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1_{\mathrm{loc}}(\Omega \setminus U)$ in the above sense. As this already ensures that P_1 is injective, we only need to show that any bounded sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq W^{1,2}_{\mathbb{D}}(\Omega)$ has a subsequence that converges in $L^1(\Omega)$. To do this, we use our assumed compact embedding property to choose a subsequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ and measurable function $\varphi : \Omega \to \mathbb{R}$ such that $\varphi_{k_j} \to \varphi$ in $L^1(K_i)$ as $j \to \infty$ for all $i \in \mathbb{N}$, where $(K_i)_{i\in\mathbb{N}}$ is an increasing sequence of compact sets such that $\bigcup_{i\in\mathbb{N}} K_i = \Omega\setminus U$. We can then employ a standard diagonal sequence argument to gain yet another subsequence, which we will again call $(\varphi_{k_j})_{j\in\mathbb{N}}$ for convenience, with the property that $\varphi_{k_j} \to \varphi$ almost everywhere in $\Omega \setminus U$ and thus almost everywhere in all of Ω as $j \to \infty$ because U is a null set. Given that the thus constructed subsequence is further bounded in $L^2(\Omega)$ due to it being bounded in $W^{1,2}_{\mathbb{D}}(\Omega)$, we can use Vitali's theorem and the de la Vallée Poussin criterion for uniform integrability (cf. [25, pp. 23-24]) to conclude that $\varphi_{k_j} \to \varphi$ in $L^1(\Omega)$ as well, yielding the first half of our result.

To prove the second half of our result, we assume that $\mathbb{D}$ is positive definite on $\Omega \setminus U$ and then need to show that $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1_{\mathrm{loc}}(\Omega \setminus U)$ in the above sense. To do this, we first choose a countable family of open balls $(B_i)_{i\in\mathbb{N}}$ such that $\overline{B_i} \subseteq \Omega \setminus U$ and $\bigcup_{i\in\mathbb{N}} B_i = \Omega \setminus U$, which is possible by e.g. considering the open balls centered at rational points with rational radii as $\Omega \setminus U$ is open. As $\mathbb{D}$ is positive definite and continuous on each compact set $\overline{B_i} \subseteq \Omega \setminus U$, there exists a constant $M_i > 0$ such that $\frac{1}{M_i} \leq \mathbb{D} \leq M_i$ on B_i for each $i \in \mathbb{N}$. Therefore, the norms of the spaces $W^{1,2}(B_i)$ and $W^{1,2}_{\mathbb{D}}(B_i)$ are equivalent for all $i \in \mathbb{N}$.

Let us now verify that P_1 is injective. To do this, let $\varphi \in W^{1,2}_{\mathbb{D}}(\Omega)$ be such that $\|\varphi\|_{L^2(\Omega)} = 0$. If we restrict φ to a ball B_i, the above implies that $\|\varphi\|_{L^2(B_i)} = 0$ and thus $\|\varphi\|_{W^{1,2}(B_i)} = 0$ because φ is an element of $W^{1,2}_{\mathbb{D}}(B_i) = W^{1,2}(B_i)$, which in turn implies that $\|\varphi\|_{W^{1,2}_{\mathbb{D}}(B_i)} = 0$ for all $i \in \mathbb{N}$. As our collection of balls was in fact countable, this then directly implies that $\|\varphi\|_{W^{1,2}_{\mathbb{D}}(\Omega)} = 0$ by the σ-subadditivity property of measures. Thus P_1 is injective. To then show that this embedding is further compact, we again begin by fixing a bounded sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq W^{1,2}_{\mathbb{D}}(\Omega)$. We then first note that our sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(B_i)$ for each $i \in \mathbb{N}$ as well and that, due to the Rellich–Kondrachov theorem (cf. [2, Theorem 6.3]), the spaces $W^{1,2}(B_i)$ embed compactly into $L^1(B_i)$. Thus a standard diagonal sequence argument yields a subsequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ and measurable function $\varphi : \Omega \setminus U \to \mathbb{R}$ such that $\varphi_{k_j} \to \varphi$ in $L^1(B_i)$ as $j \to \infty$ for all $i \in \mathbb{N}$ as our family of balls was countable. As any compact set $K \subseteq \Omega \setminus U$ is contained in the union of finitely many balls B_i, it further follows that $\varphi_{k_j} \to \varphi$ in $L^1(K)$ as $j \to \infty$ for all compact sets $K \subseteq \Omega \setminus U$. We have thus shown that $\varphi_{k_j} \to \varphi$ in $L^1_{\mathrm{loc}}(\Omega \setminus U)$ as $j \to \infty$ and therefore $W^{1,2}_{\mathbb{D}}(\Omega)$ embeds compactly into $L^1_{\mathrm{loc}}(\Omega \setminus U)$. This completes our argument. □

While we have now invested some effort into formalizing the restrictions on $\mathbb{D}$ necessary for our later construction of weak solutions, we have yet to clarify what we in fact mean by a weak solution to (DH). Let us now rectify this in the following definition.

Definition 4.1.12. Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary and let $\chi \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $r \in \mathbb{R}$. Let $\mathbb{D} \in W^{1,q}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \cap C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$ be (symmetric and) positive semidefinite everywhere, $q \in (1,\infty)$ and $p := \max(2, r, \frac{q}{q-1})$. Let further $n_0, c_0 \in L^1(\Omega)$ be some initial data.

We then call a tuple of functions

$$
\begin{aligned}
n &\in L^1_{\mathrm{loc}}([0,\infty); W^{1,1}_{\mathbb{D}}(\Omega)) \cap L^p_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)),\\
c &\in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{\mathbb{D}}(\Omega))
\end{aligned}
$$

a weak solution of (DH) with boundary condition (DHB), initial data n_0, c_0 and the above parameters if

$$
\begin{aligned}
\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla n \cdot \mathbb{D}\nabla\varphi + \int_0^\infty \int_\Omega n(\nabla\cdot\mathbb{D})\cdot\nabla\varphi \\
- \chi \int_0^\infty \int_\Omega n\nabla c \cdot \mathbb{D}\nabla\varphi - \mu \int_0^\infty \int_\Omega n(1-|n|^{r-1})\varphi
\end{aligned}
$$

and

$$
\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega nc\varphi
$$

hold for all $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$.

As we have at this point clearly defined our target and some of the necessary preconditions, let us now fully state the second main theorem we endeavor to prove in this chapter.

Theorem 4.1.13. *Let $\Omega \subseteq \mathbb{R}^N$, $N \in \{2,3\}$, be a bounded domain with a smooth boundary, $\chi \in (0,\infty)$, $\mu \in (0,\infty)$, $\beta \in [\frac{1}{2}, 1)$, $r \in [2,\infty)$ with $\frac{\beta}{1-\beta} \leq r$ and $\mathbb{D} \in W^{1,2}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \cap C^0(\overline{\Omega}; \mathbb{R}^{N\times N})$ be (symmetric and) positive semidefinite everywhere. Let further $\mathbb{D}$ allow for a divergence estimate with exponent β (cf. Definition 4.1.2) and let $\mathbb{D}$ allow for a compact $L^1(\Omega)$ embedding (cf. Definition 4.1.9). Finally, let $n_0 \in L^{|z|\ln(|z|+1)}(\Omega)$ and $c_0 \in L^\infty(\Omega)$ be some initial data with $\sqrt{c_0} \in W^{1,2}(\Omega)$ and $n_0 \geq 0$, $c_0 \geq 0$ almost everywhere.*

Then there exist a.e. nonnegative functions

$$
n \in L^{\frac{2r}{r+1}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{2r}{r+1}}_{\mathbb{D}}(\Omega)) \cap L^r_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)), \tag{4.1.2}
$$

$$
c \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{\mathbb{D}}(\Omega)) \cap L^\infty(\Omega \times (0,\infty)) \tag{4.1.3}
$$

such that (n,c) is a weak solution to (DH) with boundary condition (DHB) in the sense of Definition 4.1.12.

Remark 4.1.14. In light of the above theorem, we will now take another look at the three examples introduced previously in this section. Let us first note that the second criterion in Lemma 4.1.11 is fairly easy to verify for all three examples and thus all three allow for a compact $L^1(\Omega)$ embedding. As already discussed in Example 4.1.4, the matrix-valued function $\mathbb{D}_1(x) := |x|^s I$, $s > 0$, on $\Omega := B_1(0) \subseteq \mathbb{R}^N$, $N \in \{2,3\}$, allows for a divergence estimate for all $\beta \in (\frac{N}{s-2+2N}, \infty) \cap (\frac{1}{2}, 1) \neq \emptyset$. Indeed for any

$$r \in [2,\infty) \quad \text{with} \quad r > \frac{N}{s-2+N} = \frac{\frac{N}{s-2+2N}}{1 - \frac{N}{s-2+2N}}, \tag{4.1.4}$$

we can find $\beta \in (\frac{1}{2}, 1)$ such that $r \geq \frac{\beta}{1-\beta}$ and such that $\mathbb{D}_1$ allows for a divergence estimate with exponent β because the function $x \mapsto \frac{x}{1-x}$ is continuous in $\frac{N}{s-2+2N} \in (0,1)$. Note that, if $\frac{N}{s-2+2N} < \frac{2}{3}$, we can always choose $\beta = \frac{2}{3}$ as $r \geq 2 = \frac{\frac{2}{3}}{1-\frac{2}{3}}$. Similarly if we assume that

$$r \in [2,\infty) \quad \text{with} \quad r > \frac{1}{s-1} = \frac{\frac{1}{s}}{1 - \frac{1}{s}}, \tag{4.1.5}$$

then there exists $\beta \in (\frac{1}{2}, 1)$ such that $r \geq \frac{\beta}{1-\beta}$ and such that the matrix-valued function $\mathbb{D}_2(x_1, \dots, x_N) := |x_1|^s I$, $s > 1$, on $\Omega := B_1(0) \subseteq \mathbb{R}^N$, $N \in \{2,3\}$, from Example 4.1.5 allows for a divergence estimate with exponent β by essentially the same argument. Notably, $\mathbb{D}_3$ from Example 4.1.6 always allows for a divergence estimate with exponent $\frac{1}{2}$ and thus no additional condition on $r \geq 2$ is necessary as $2 \geq 1 = \frac{\frac{1}{2}}{1-\frac{1}{2}}$. Therefore, Theorem 4.1.13 means that, for sufficiently regular initial data n_0, c_0 and if either $\mathbb{D} = \mathbb{D}_1$ and r and s satisfy (4.1.4) or $\mathbb{D} = \mathbb{D}_2$ and r and s satisfy (4.1.5) or $\mathbb{D} = \mathbb{D}_3$, weak solutions to (DH) with boundary condition (DHB) in fact exist.

4.2 Approach

For the derivation of Theorem 4.1.1, we begin by using standard contraction mapping methods to gain local solutions with an associated blowup criterion as the operator in the first equation is strictly elliptic for a globally positive matrix-valued function $\mathbb{D}$. We then immediately transition to analyzing the function $a := ne^{-\chi c}$, which together with c solves the closely related problem (DH*). We do this because, in a sense, this transformation eliminates the problematic cross-diffusive term from the first equation by integrating it into the function a and its associated diffusion dynamics. Using a fairly classic Moser-type iteration argument, we then establish an $L^\infty(\Omega)$ bound for a, which translates back to n. Using this bound combined with two testing procedures then yields a further $W^{1,4}(\Omega)$ bound for c, which together with the already established bound for n is sufficient to ensure that finite-time blowup is in fact impossible in two and three dimensions and thus completes the proof of our first result.

Regarding the proof of Theorem 4.1.13, we begin by approximating the initial data, the matrix-valued function $\mathbb{D}$ as well as the logistic source term in such a way as to make the already established global classical existence result applicable to the in this way approximated versions of (DH) with boundary condition (DHB). For the family of solutions $(n_\varepsilon, c_\varepsilon)_{\varepsilon\in(0,1)}$ gained in this fashion, we then establish a bound of the form

$$\int_\Omega n_\varepsilon \ln(n_\varepsilon) + \int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon} + \int_0^t \int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + \int_0^t \int_\Omega n_\varepsilon^{r+\varepsilon} \ln(n_\varepsilon) \le C$$

by way of an energy-type inequality, which already proved useful in the one-dimensional case discussed in [133]. Using this as a baseline, we derive the bounds necessary for applications of the Aubin–Lions compact embedding lemma to gain our desired weak solutions as limits of the approximate ones.

4.3 Existence of classical solutions

As the existence of classical solutions to (DH) with boundary condition (DHB), apart from being an interesting result on its own merits, plays an important role in our construction of their weak counterparts, we will in this section first focus on their derivation. In fact, our ultimate goal for this section will be the proof of our first main result, namely Theorem 4.1.1. The methods presented here will in many ways mirror those for similar systems with a standard Laplacian as diffusion operator. We mainly verify that the differing elements in our systems do not impede said methods.

To this end, we now fix a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \{2,3\}$ and system parameters $\chi \in (0,\infty)$, $\mu \in (0,\infty)$, $r \in [2,\infty)$ and $\mathbb{D} \in C^2(\overline{\Omega}; \mathbb{R}^{N\times N})$. We further assume that $\mathbb{D}$ is in fact positive definite everywhere and has the property $(\nabla \cdot \mathbb{D}) \cdot \nu = 0$ on $\partial\Omega$. Given these assumptions, we can fix $M \ge 1$ such that

$$\frac{1}{M} \le \mathbb{D} \le M, \quad \|\nabla \cdot \mathbb{D}\|_{L^\infty(\Omega)} \le M \quad \text{and} \quad \|\nabla \cdot (\nabla \cdot \mathbb{D})\|_{L^\infty(\Omega)} \le M. \tag{4.3.1}$$

We also fix some initial data $n_0, c_0 \in C^{2+\vartheta}(\overline{\Omega})$, $\vartheta \in (0,1)$, with $(\mathbb{D}\nabla n_0)\cdot\nu = (\mathbb{D}\nabla c_0)\cdot\nu = 0$ on $\partial\Omega$ and $n_0 > 0$, $c_0 > 0$ on $\overline{\Omega}$.

Comparing the very strong regularity assumptions for $\mathbb{D}$ in this section to the much weaker ones in the following section devoted to the construction of weak solutions, the question why the gap in assumed regularity between these sections is as large as it is naturally presents itself. Let us therefore briefly address this issue. It is certainly possible to derive most of the a priori estimates that are used in this section to argue that blowup of local solutions is impossible under similarly specific regularity assumptions as seen in Definition 4.1.2 or Definition 4.1.9 (albeit with some additions). But generalizing the theory employed by us to first gain said local solutions with less regular $\mathbb{D}$ would necessitate something similar to standard parabolic Schauder and semigroup theory but for potentially very degenerate operators, which is not a focus of this thesis as establishing such results would likely necessitate a rather extensive detour.

4.3.1 Existence of local solutions

After this introductory paragraph giving our rationale for the assumptions on $\mathbb{D}$ in this section, we will now focus on the construction of local solutions to the system (DH) with boundary condition (DHB) as a first step in constructing global ones. As for an everywhere positive definite matrix-valued function $\mathbb{D}$, the diffusion operator in the first equation is strictly elliptic and therefore accessible to most of the same existence and regularity theory as the Laplacian, we will not go into detail concerning the construction of local solutions but rather refer the reader to a local existence and uniqueness result for a similar haptotaxis system with our operator replaced by the Laplacian in [109], which can be fairly easily adapted.

Lemma 4.3.1. *There exist $T_{\max} \in (0,\infty]$ and a unique pair of positive functions $n, c \in C^{2,1}(\overline{\Omega}\times[0,T_{\max}))$ such that (n,c) is a classical solution to (DH) with boundary condition (DHB) as well as initial data (n_0, c_0) on $\Omega \times (0, T_{\max})$ and satisfies the following blowup criterion:*

$$\text{If } T_{\max} < \infty, \ \text{ then } \limsup_{t \nearrow T_{\max}} \Big(\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,N+1}(\Omega)} \Big) = \infty. \tag{4.3.2}$$

For ease of further discussion, we now fix such a maximal local solution (n, c) on $\Omega \times (0, T_{\max})$ with initial data (n_0, c_0) and the parameters as stated in the above introductory paragraphs for the remainder of this section.

Before diving into the derivation of more substantial bounds for the above solution, we derive a straightforward mass bound for the first solution component as well as an $L^\infty(\Omega)$ bound for the second solution component. These bounds will not only prove useful when ruling out blowup in this section but also serve as a baseline for bounds derived in our later efforts focused on the construction of weak solutions.

Lemma 4.3.2. *The inequalities*

$$\int_\Omega n(\cdot,t) \le \mu|\Omega|t + \int_\Omega n_0 \quad \text{ and } \quad \|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)}$$

hold for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation in (DH) and applying partial integration yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n = \mu \int_\Omega n(1 - n^{r-1}) \le \mu|\Omega|$$

for all $t \in (0, T)$ and therefore immediately give us the first half of our result by time integration. Given that further $c_t \le 0$ due to the second equation in (DH), the second half of our result follows directly as well. □

4.3.2 A priori estimates

The next natural step after establishing local solutions with an associated blowup criterion is of course arguing that finite-time blowup is impossible and the maximal local solutions were in fact global all along. To do this, we will devote this section to a set of a priori estimates, which increase in strength as the section goes on until they rule out blowup of both n and c.

As is not uncommon in the analysis of these kinds of haptotaxis systems (cf. [110]), we will from now consider the function $a := ne^{-\chi c}$ defined on $\overline{\Omega} \times [0, T_{\max})$ and its associated initial data $a_0 := n_0 e^{-\chi c_0}$ defined on $\overline{\Omega}$ in addition to the actual solution components n and c themselves. A simple computation then shows that (a, c) is a classical solution of the following related system:

$$\begin{cases} a_t = e^{-\chi c}\nabla \cdot (e^{\chi c}\mathbb{D}\nabla a) + e^{-\chi c}\nabla \cdot (ae^{\chi c}(\nabla \cdot \mathbb{D})) & \\ \qquad + \mu a(1 - a^{r-1}e^{\chi(r-1)c}) + \chi a^2 c e^{\chi c} & \text{on } \Omega \times (0, \infty), \\ c_t = -ae^{\chi c}c & \text{on } \Omega \times (0, \infty), \\ 0 = (\mathbb{D}\nabla a) \cdot \nu = -a(\nabla \cdot \mathbb{D}) \cdot \nu & \text{on } \partial\Omega \times (0, \infty), \\ a(\cdot, 0) = a_0 > 0, \ \ c(\cdot, 0) = c_0 > 0 & \text{on } \Omega. \end{cases} \tag{DH*}$$

Notably, the boundary condition for a in the penultimate line of (DH*), which we gain as a consequence of (DHB), simplifies to $(\mathbb{D}\nabla a) \cdot \nu = 0$ on $\partial\Omega$ due to our standing assumption that $(\nabla \cdot \mathbb{D}) \cdot \nu = 0$ on $\partial\Omega$ in this section.

The key property of the above system, which makes it so useful for our purposes, is that it, in a sense, eliminates the taxis term or at least the explicit gradient of c from the first equation (by essentially integrating it into a and its associated diffusion dynamics). This alleviates many of the common problems associated with the taxis term in testing or semigroup-based approaches used to derive a priori estimates. A second useful property of this transformation is that, by definition, bounds that do not involve derivatives are easily translated back from a to n as we will see later. Note however that, as soon as we want to propagate bounds about the gradient of a back to n, the complications introduced by the taxis term come back into play, making this transformation much less useful for endeavors of this kind.

We now begin by translating the baseline estimates given in Lemma 4.3.2 to our newly defined function a as we will henceforth focus on (a, c) as our central object of analysis for quite some time. We will further for the remainder of this section work under the assumption that $T_{\max} < \infty$ as this is exactly the case we want to rule out by leading this assumption to a contradiction using the blowup criterion in (4.3.2).

Corollary 4.3.3. *If $T_{\max} < \infty$, there exists $C > 0$ such that*

$$\int_\Omega a \le C$$

for all $t \in (0, T_{\max})$.

Proof. As $\int_\Omega a = \int_\Omega n e^{\chi c} \leq e^{\chi \|c\|_{L^\infty(\Omega)}} \int_\Omega n$ holds, this is an immediate consequence of Lemma 4.3.2 if $T_{\max} < \infty$. □

In preparation for a later Moser-type iteration argument for the first solution component a (cf. [5] and [83] for some early as well as [35] and [106] for some more contemporary examples of this technique), which will later be used to rule out its finite-time blowup, we will now derive a recursive inequality for terms of the form $\int_\Omega a^p$. This recursion will in fact allow us to estimate each term of the form $\int_\Omega a^p$ by terms of the form $(\int_\Omega a^{\frac{p}{2}})^2$, which will prove sufficient to later gain an $L^\infty(\Omega)$ bound for a. The method employed to gain said recursion is testing the first equation in (DH*) with $e^{\chi c} a^{p-1}$ followed by some estimates based on the Gagliardo–Nirenberg inequality.

To facilitate this derivation of said recursion, we will from now on assume that the regularizing influence of the logistic source term in the first equation of (DH) is sufficiently strong, or more precisely we assume that either $r > 2$ or μ is sufficiently large in comparison to χ and the $L^\infty(\Omega)$ norm of c_0. However at this point and therefore for the whole of the Moser-type iteration argument, we will not use our assumed restriction to two or three dimensions just yet.

Lemma 4.3.4. *If $T_{\max} < \infty$ and either $r > 2$ or $r = 2$ and $\mu \geq \chi \|c_0\|_{L^\infty(\Omega)}$, then there exists a constant $C > 0$ such that*

$$\sup_{t \in (0, T_{\max})} \int_\Omega a^p \leq C \max\left(\int_\Omega a_0^p,\ C^{p+1},\ p^C \left(\sup_{t \in (0, T_{\max})} \int_\Omega a^{\frac{p}{2}} \right)^2 \right)$$

for all $p \geq 2$.

Proof. We test the first equation in (DH*) with $e^{\chi c} a^{p-1}$ and apply partial integration to see that

$$\begin{aligned}
\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega e^{\chi c} a^p &= \int_\Omega e^{\chi c} a^{p-1} a_t + \frac{\chi}{p} \int_\Omega c_t e^{\chi c} a^p = \int_\Omega e^{\chi c} a^{p-1} a_t - \frac{\chi}{p} \int_\Omega c e^{2\chi c} a^{p+1} \\
&= \int_\Omega a^{p-1} \nabla \cdot (e^{\chi c} \mathbb{D} \nabla a) + \int_\Omega a^{p-1} \nabla \cdot (a e^{\chi c} (\nabla \cdot \mathbb{D})) \\
&\quad + \mu \int_\Omega e^{\chi c} a^p - \mu \int_\Omega e^{r\chi c} a^{p-1+r} + \chi \frac{p-1}{p} \int_\Omega c e^{2\chi c} a^{p+1} \\
&= -(p-1) \int_\Omega e^{\chi c} a^{p-2} (\nabla a \cdot \mathbb{D} \nabla a) - (p-1) \int_\Omega e^{\chi c} a^{p-1} ((\nabla \cdot \mathbb{D}) \cdot \nabla a) \\
&\quad + \mu \int_\Omega e^{\chi c} a^p - \mu \int_\Omega e^{r\chi c} a^{p-1+r} + \chi \frac{p-1}{p} \int_\Omega c e^{2\chi c} a^{p+1}
\end{aligned} \tag{4.3.3}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$. Given the properties of $\mathbb{D}$ seen in (4.3.1), we can use Young's inequality to further estimate that

$$- (p-1) \int_\Omega e^{\chi c} a^{p-2} (\nabla a \cdot \mathbb{D} \nabla a) - (p-1) \int_\Omega e^{\chi c} a^{p-1} ((\nabla \cdot \mathbb{D}) \cdot \nabla a)$$

$$\begin{aligned}
&\leq -\frac{p-1}{M}\int_\Omega e^{\chi c}a^{p-2}|\nabla a|^2 + M(p-1)\int_\Omega e^{\chi c}a^{p-1}|\nabla a| \\
&\leq -\frac{p-1}{2M}\int_\Omega e^{\chi c}a^{p-2}|\nabla a|^2 + 2M^3(p-1)\int_\Omega e^{\chi c}a^{p} \\
&\leq -\frac{p-1}{p^2}\frac{2}{M}\int_\Omega e^{\chi c}|\nabla a^{\frac{p}{2}}|^2 + 2M^3\,p\int_\Omega e^{\chi c}a^{p} \\
&\leq -\frac{1}{p}\frac{1}{M}\int_\Omega e^{\chi c}|\nabla a^{\frac{p}{2}}|^2 + 2M^3\,p\int_\Omega e^{\chi c}a^{p}
\end{aligned}$$

as well as more elementary that

$$\chi\frac{p-1}{p}\int_\Omega ce^{2\chi c}a^{p+1} \leq \chi\|c_0\|_{L^\infty(\Omega)}\int_\Omega e^{2\chi c}a^{p+1}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$, which when applied to (4.3.3) results in

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi c}a^p + \frac{1}{p}\frac{1}{M}\int_\Omega e^{\chi c}|\nabla a^{\frac{p}{2}}|^2 \\
\leq&(\mu + 2M^3\,p)\int_\Omega e^{\chi c}a^p - \mu\int_\Omega e^{r\chi c}a^{p-1+r} + \chi\|c_0\|_{L^\infty(\Omega)}\int_\Omega e^{2\chi c}a^{p+1}
\end{aligned} \tag{4.3.4}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$. If $r > 2$, we can now further estimate that

$$\begin{aligned}
&-\mu\int_\Omega e^{r\chi c}a^{p-1+r} + \chi\|c_0\|_{L^\infty(\Omega)}\int_\Omega e^{2\chi c}a^{p+1} \\
&\leq -\mu\int_\Omega e^{r\chi c}a^{p-1+r} + \chi\|c_0\|_{L^\infty(\Omega)}\int_\Omega e^{r\chi c}a^{p+1} \\
&\leq \chi\|c_0\|_{L^\infty(\Omega)}\left(\frac{\chi\|c_0\|_{L^\infty(\Omega)}}{\mu}\right)^{\frac{p+1}{r-2}} e^{r\chi\|c_0\|_{L^\infty(\Omega)}}|\Omega| \leq K_1^{p+1}
\end{aligned}$$

with

$$K_1 \coloneqq \left(\chi\|c_0\|_{L^\infty(\Omega)}e^{r\chi\|c_0\|_{L^\infty(\Omega)}}|\Omega| + 1\right)\left(\frac{\chi\|c_0\|_{L^\infty(\Omega)}}{\mu}\right)^{\frac{1}{r-2}}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$ by Young's inequality. If, however, $r = 2$ and $\mu \geq \chi\|c_0\|_{L^\infty(\Omega)}$, it is immediately obvious that

$$-\mu\int_\Omega e^{r\chi c}a^{p-1+r} + \chi\|c_0\|_{L^\infty(\Omega)}\int_\Omega e^{2\chi w}a^{p+1} \leq 0 \leq K_1^{p+1}$$

with $K_1 \coloneqq 1$ for all $t \in (0, T_{\max})$ and $p \geq 2$. Therefore, we can in both cases conclude from (4.3.4) that

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi c}a^p + \frac{1}{p}\frac{1}{M}\int_\Omega e^{\chi c}|\nabla a^{\frac{p}{2}}|^2 &\leq (\mu + 2M^3\,p)\int_\Omega e^{\chi c}a^p + K_1^{p+1} \\
&\leq p\,K_2\int_\Omega a^p + K_1^{p+1}
\end{aligned} \tag{4.3.5}$$

with $K_2 \coloneqq (\mu + 2M^3)e^{\chi\|c_0\|_{L^\infty(\Omega)}}$ for all $t \in (0, T_{\max})$ and $p \geq 2$.

We can now use the Gagliardo–Nirenberg inequality as well as Young's inequality to fix a constant $K_3 > 0$ such that

$$\begin{aligned}\int_\Omega a^p = \|a^{\frac{p}{2}}\|_{L^2(\Omega)}^2 &\leq K_3 \|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\alpha} \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{2(1-\alpha)} + K_3 \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^2 \\ &\leq \frac{1}{p^2}\frac{1}{M}\frac{1}{K_2}\int_\Omega |\nabla a^{\frac{p}{2}}|^2 + ((p^2 M K_2)^{\frac{\alpha}{1-\alpha}} K_3^{\frac{1}{1-\alpha}} + K_3)\left(\int_\Omega a^{\frac{p}{2}}\right)^2 \\ &\leq \frac{1}{p^2}\frac{1}{M}\frac{1}{K_2}\int_\Omega e^{\chi c}|\nabla a^{\frac{p}{2}}|^2 + K_4 p^{K_4}\left(\int_\Omega a^{\frac{p}{2}}\right)^2\end{aligned}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$ with

$$\alpha \coloneqq \frac{1}{1 + \frac{2}{N}} \in (0,1)$$

and $K_4 \coloneqq \max(\frac{2\alpha}{1-\alpha}, (MK_2)^{\frac{\alpha}{1-\alpha}} K_3^{\frac{1}{1-\alpha}} + K_3)$. Applying this to (4.3.5) then implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi c}a^p \leq K_2 K_4 p^{K_4+2}\left(\int_\Omega a^{\frac{p}{2}}\right)^2 + pK_1^{p+1} \leq K_2 K_4 p^{K_4+2}\left(\int_\Omega a^{\frac{p}{2}}\right)^2 + (2K_1)^{p+1}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$. Time integration then yields

$$\begin{aligned}\int_\Omega a^p(\cdot,t) \leq \int_\Omega e^{\chi c}a^p(\cdot,t) &\leq T_{\max} K_2 K_4 p^{K_4+2}\left(\sup_{s\in(0,T_{\max})}\int_\Omega a^{\frac{p}{2}}(\cdot,s)\right)^2 \\ &\quad + T_{\max}(2K_1)^{p+1} + e^{\chi\|c_0\|_{L^\infty(\Omega)}}\int_\Omega a_0^p\end{aligned}$$

for all $t \in (0, T_{\max})$ and $p \geq 2$ as $T_{\max} < \infty$, which after estimating the sum on the right-hand side by thrice the maximum of its summands completes the proof. □

We will now proceed to give the actual iteration argument yielding an $L^\infty(\Omega)$-type bound for a and therefore n, which is sufficient to rule out finite-time blowup for the first solution component n.

Lemma 4.3.5. *If $T_{\max} < \infty$ and either $r > 2$ or $r = 2$ and $\mu \geq \chi\|c_0\|_{L^\infty(\Omega)}$, then there exists a constant $C > 0$ such that*

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{and therefore} \quad \|n(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{\chi\|c_0\|_{L^\infty(\Omega)}}$$

for all $t \in (0, T_{\max})$.

Proof. Let $p_i := 2^i$, $i \in \mathbb{N}_0$, and $J_i := \sup_{t\in(0,T_{\max})} \left(\int_\Omega a^{p_i}(\cdot,t)\right)^{\frac{1}{p_i}}$. Then J_0 is finite because of Corollary 4.3.3 and the fact that $p_0 = 1$. We further know that

$$\|a_0\|_{L^{p_i}(\Omega)} \le (1+|\Omega|)\|a_0\|_{L^\infty(\Omega)} =: K_1.$$

Due to Lemma 4.3.4, we can conclude that there exists a constant $K_2 \ge 1$ such that the numbers J_i satisfy the following recursion:

$$J_i \le K_2^{\frac{1}{p_i}} \max\left(\|a_0\|_{L^{p_i}(\Omega)},\ K_2^{\frac{p_i+1}{p_i}},\ p_i^{\frac{K_2}{p_i}} J_{i-1} \right) \quad \text{for all } i \in \mathbb{N}.$$

Iterating this recursion finitely many times ensures that all J_i are finite.

If there exists an incrementing sequence of indices $i \in \mathbb{N}$, along which the inequality $J_i \le \max(K_1K_2, K_2^3)$ holds, we immediately gain our desired result by taking the limit of J_i along said sequence. Thus, we can now assume that there exists $i_0 \in \mathbb{N}$ with

$$J_i \ge \max(K_1K_2, K_2^3) > \begin{cases} K_2^{\frac{1}{p_i}} \|a_0\|_{L^{p_i}(\Omega)} \\ K_2^{\frac{1}{p_i}} K_2^{\frac{p_i+1}{p_i}} \end{cases} \quad \text{for all } i \ge i_0$$

to cover the remaining case. Given these assumptions, the above recursion simplifies to

$$J_i \le (p_iK_2)^{\frac{K_2}{p_i}} J_{i-1} \le K_3^{\frac{1}{\sqrt{p_i}}} J_{i-1}$$

for all $i \ge i_0$ with some $K_3 > 0$ (only depending on K_2) as the function $z \mapsto (zK_2)^{\frac{K_2}{\sqrt{z}}}$ is bounded on $[1,\infty)$. By now again iterating this recursion finitely many times, we gain that

$$J_i \le K_3^{\sum_{j=i_0}^{i} \frac{1}{\sqrt{p_j}}} J_{i_0-1} \tag{4.3.6}$$

for all $i \ge i_0$. As

$$\sum_{j=i_0}^{i} \frac{1}{\sqrt{p_j}} = \sum_{j=i_0}^{i} \left(\frac{1}{\sqrt{2}}\right)^j \le \sum_{j=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^j < \infty$$

for all $i \ge i_0$ due to the series on the right side being of geometric type, we can conclude from (4.3.6) that the sequence J_i is uniformly bounded. Therefore, taking the limit $i \to \infty$ gives us our desired bound for a. As $n = ae^{\chi c}$, the corresponding bound for n follows directly from this and Lemma 4.3.2. □

To now establish that finite-time blowup of the $W^{1,4}(\Omega)$-norm of the second solution component c is equally as impossible, we will begin by testing the first equation in (DH*) with $-\nabla\cdot(\mathbb{D}\nabla a)$ and combining the result with a differential inequality enjoyed by $\int_\Omega |\nabla c|^4$. The key to extracting a sufficiently strong bound for c is to then use the strength of the absorptive terms originating from the fully elliptic operator $-\nabla\cdot(\mathbb{D}\nabla\cdot)$ to counteract the influence of potentially destabilizing terms due to the haptotaxis interaction. Note that the ellipticity of the operator is ensured because we assume that $\mathbb{D}$ is positive definite everywhere in $\overline{\Omega}$.

Lemma 4.3.6. *If $T_{\max} < \infty$ and either $r > 2$ or $r = 2$ and $\mu \geq \chi \|c_0\|_{L^\infty(\Omega)}$, then there exists a constant $C > 0$ such that*

$$\|\nabla c(\cdot, t)\|_{L^4(\Omega)} \leq C$$

for all $t \in (0, T_{\max})$.

Proof. Given Lemma 4.3.5, we can fix a constant $K_1 \geq 1$ such that

$$\|a(\cdot, t)\|_{L^\infty(\Omega)} \leq K_1 \quad \text{and} \quad \int_\Omega \left(a^2(\cdot, t) + a^{2r}(\cdot, t) + a^4(\cdot, t)\right) \leq K_1 \tag{4.3.7}$$

for all $t \in (0, T_{\max})$. Using the Gagliardo–Nirenberg inequality and standard regularity estimates (cf. [34, Theorem 19.1] or [80, Theorem 3.1.1]) for the elliptic operator $-\nabla \cdot (\mathbb{D}\nabla\,\cdot\,)$ (with Neumann-type boundary conditions), we can fix a constant $K_2 \geq 1$ such that

$$\int_\Omega |\nabla \varphi|^4 \leq K_2 \left(\int_\Omega |\nabla \cdot (\mathbb{D}\nabla\varphi)|^2 + \int_\Omega |\varphi|^2\right) \|\varphi\|^2_{L^\infty(\Omega)}$$

for all $\varphi \in C^2(\overline{\Omega})$ with $(\mathbb{D}\nabla\varphi) \cdot \nu = 0$ on $\partial\Omega$. This in turn implies that

$$\int_\Omega |\nabla a|^4 \leq K_3 \left(\int_\Omega |\nabla \cdot (\mathbb{D}\nabla a)|^2 + 1\right) \tag{4.3.8}$$

for all $t \in (0, T_{\max})$ with $K_3 \coloneqq K_1^3 K_2$.

After establishing these preliminaries, we now note that the first equation in (DH*) can also be written as

$$\begin{aligned} a_t = {} & \nabla \cdot (\mathbb{D}\nabla a) + \chi \nabla c \cdot \mathbb{D}\nabla a + \nabla \cdot (a(\nabla \cdot \mathbb{D})) + \chi a(\nabla c \cdot (\nabla \cdot \mathbb{D})) \\ & + \mu a(1 - a^{r-1} e^{\chi(r-1)c}) + \chi a^2 c e^{\chi c}. \end{aligned}$$

We then test this variant of said equation with $-\nabla\cdot(\mathbb{D}\nabla a)$ and employ partial integration (using the fact that $(\nabla \cdot \mathbb{D}) \cdot \nu = 0$ on $\partial\Omega$) as well as Young's inequality to conclude that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nabla a \cdot \mathbb{D}\nabla a) &= \int_\Omega (\nabla a_t \cdot \mathbb{D}\nabla a) \\ &= \int_\Omega \nabla(\nabla \cdot (\mathbb{D}\nabla a)) \cdot \mathbb{D}\nabla a + \chi \int_\Omega \nabla(\nabla c \cdot \mathbb{D}\nabla a) \cdot \mathbb{D}\nabla a \\ &\quad + \int_\Omega \nabla(\nabla \cdot (a\nabla \cdot \mathbb{D})) \cdot \mathbb{D}\nabla a + \chi \int_\Omega \nabla(a\nabla c \cdot (\nabla \cdot \mathbb{D})) \cdot \mathbb{D}\nabla a \\ &\quad + \int_\Omega \nabla \left(\mu a(1 - a^{r-1} e^{\chi(r-1)c}) + \chi a^2 c e^{\chi c}\right) \cdot \mathbb{D}\nabla a \\ &\leq -\frac{1}{2}\int_\Omega |\nabla \cdot (\mathbb{D}\nabla a)|^2 + 2\chi^2 \int_\Omega |\nabla c \cdot \mathbb{D}\nabla c||\nabla a \cdot \mathbb{D}\nabla a| \\ &\quad + 2\int_\Omega |\nabla \cdot (a\nabla \cdot \mathbb{D})|^2 + 2\chi^2 \int_\Omega a^2 |\nabla c|^2 |\nabla \cdot \mathbb{D}|^2 \end{aligned}$$

$$+ K_4 \int_\Omega \left(a^2 + a^{2r} + a^4\right) \tag{4.3.9}$$

for all $t \in (0, T_{\max})$ with $K_4 := 8 \max\left(\mu, \mu e^{\chi(r-1)\|c_0\|_{L^\infty(\Omega)}}, \chi\|c_0\|_{L^\infty(\Omega)} e^{\chi\|c_0\|_{L^\infty(\Omega)}}\right)^2$. Using the bounds outlined in (4.3.1) and (4.3.7), we can now further derive that

$$\begin{aligned} 2\chi^2 \int_\Omega |\nabla c \cdot \mathbb{D}\nabla c||\nabla a \cdot \mathbb{D}\nabla a| &\le 2\chi^2 M^2 \int_\Omega |\nabla c|^2 |\nabla a|^2 \\ &\le 8\chi^4 M^4 K_3 \int_\Omega |\nabla c|^4 + \frac{1}{8K_3}\int_\Omega |\nabla a|^4 \end{aligned}$$

and

$$\begin{aligned} 2\int_\Omega |\nabla \cdot (a\nabla \cdot \mathbb{D})|^2 &\le 4\int_\Omega |\nabla a|^2 |\nabla \cdot \mathbb{D}|^2 + 4\int_\Omega a^2 |\nabla \cdot (\nabla \cdot \mathbb{D})|^2 \\ &\le 4M^2 \left(\int_\Omega |\nabla a|^2 + \int_\Omega a^2\right) \\ &\le 4M^2 \left(\int_\Omega |\nabla a|^2 + K_1\right) \\ &\le \frac{1}{8K_3}\int_\Omega |\nabla a|^4 + 32M^4K_3 + 4M^2K_1 \end{aligned}$$

and

$$2\chi^2 \int_\Omega a^2 |\nabla c|^2 |\nabla \cdot \mathbb{D}|^2 \le \chi^2 M^2 \left(\int_\Omega a^4 + \int_\Omega |\nabla c|^4\right) \le \chi^2 M^2 K_1 \left(\int_\Omega |\nabla c|^4 + 1\right)$$

for all $t \in (0, T_{\max})$. Applying these three estimates combined with the second bound in (4.3.7) to (4.3.9) then yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nabla a \cdot \mathbb{D}\nabla a) \le -\frac{1}{2}\int_\Omega |\nabla \cdot (\mathbb{D}\nabla a)|^2 + \frac{1}{4K_3}\int_\Omega |\nabla a|^4 + K_5 \int_\Omega |\nabla c|^4 + K_6 \tag{4.3.10}$$

for all $t \in (0, T_{\max})$ with $K_5 := 8\chi^4 M^4 K_3 + \chi^2 M^2 K_1$ and $K_6 := 32M^4K_3 + 4M^2K_1 + \chi^2 M^2 K_1 + K_1 K_4$.

As our second step, we now obtain the following estimate for the time derivative of certain gradient terms of the second solution component c as follows:

$$\begin{aligned} \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^4 &= \int_\Omega |\nabla c|^2 \nabla c \cdot \nabla c_t = -\int_\Omega |\nabla c|^2 \nabla c \cdot \nabla(a e^{\chi c} c) \\ &= -\int_\Omega |\nabla c|^4 a e^{\chi c}(\chi c + 1) - \int_\Omega |\nabla c|^2 (\nabla c \cdot \nabla a) e^{\chi c} c \\ &\le K_7 \int_\Omega |\nabla c|^3 |\nabla a| \le K_7 \int_\Omega |\nabla c|^4 + K_7 \int_\Omega |\nabla a|^4 \end{aligned}$$

for all $t \in (0, T_{\max})$ with $K_7 := \|c_0\|_{L^\infty(\Omega)} e^{\chi\|c_0\|_{L^\infty(\Omega)}}$.

Now combining this with (4.3.10) (using an appropriate scaling factor) we gain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nabla a \cdot \mathbb{D}\nabla a) + \frac{1}{16K_3K_7}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^4$$

$$\le -\frac{1}{2}\int_\Omega |\nabla\cdot(\mathbb{D}\nabla a)|^2 + \frac{1}{2K_3}\int_\Omega |\nabla a|^4 + K_8\int_\Omega |\nabla c|^4 + K_6$$

for all $t \in (0, T_{\max})$ with $K_8 \coloneqq K_5 + \frac{1}{4K_3}$. The application of (4.3.8) to the inequality above then yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nabla a\cdot\mathbb{D}\nabla a) + \frac{1}{16K_3K_7}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c|^4 \\
&\le K_8\int_\Omega |\nabla c|^4 + K_6 + \frac{1}{2} \\
&\le K_9\left(\frac{1}{2}\int_\Omega (\nabla a\cdot\mathbb{D}\nabla a) + \frac{1}{16K_3K_7}\int_\Omega |\nabla c|^4\right) + K_6 + \frac{1}{2}
\end{aligned}$$

with $K_9 \coloneqq 16K_3K_7K_8$ for all $t \in (0, T_{\max})$, which, by a standard comparison argument and the assumption that $T_{\max}$ is finite, directly gives us our desired result. □

Remark 4.3.7. The result of the above lemma only ensures that finite-time blowup of the second solution component's gradient according to (4.3.2) is impossible in two and three dimensions. In fact, it is at this point and only this point in this section, where our restriction to two or three dimensions becomes necessary. This, of course, in turn means that any extension of the results of this section to a higher dimensional setting would only need to extend the above argument to one providing better bounds for the gradient of c.

Given that Lemma 4.3.5 and Lemma 4.3.6 rule out any kind of finite-time blowup for our local solutions, the proof of the first central result of this chapter can now be stated quite succinctly.

Proof of Theorem 4.1.1. If we assume $T_{\max} < \infty$, Lemma 4.3.5 and Lemma 4.3.6 in combination contradict the consequence of the blowup criterion in (4.3.2) in this case. Therefore, $T_{\max} = \infty$ and thus the local solutions constructed in Lemma 4.3.1 must be in fact global. This is sufficient to prove Theorem 4.1.1 as the fixed assumptions of this section were in fact identical to those of said theorem. □

Remark 4.3.8. It is still possible to construct global classical solutions in the two-dimensional case even when removing the logistic growth term from (DH) by using methods that have previously been used when for example dealing with standard diffusion and some slightly modified versions of our arguments (cf. [8]).

Essentially, the argument boils down to using an estimate of the form

$$\|n\|_{L^3(\Omega)}^3 \le \varepsilon\|n\|_{W^{1,2}(\Omega)}^2\|n\ln(n)\|_{L^1(\Omega)} + C(\varepsilon)\|n\|_{L^1(\Omega)}$$

with $\varepsilon > 0$ being potentially arbitrarily small (cf. [11, p. 1199]) in combination with an additional baseline $\int_\Omega n\ln(n)$ estimate based on a well-known energy-type inequality (cf.

Lemma 4.4.2) to establish an $L^2(\Omega)$ bound. From there, the arguments are very similar to the Moser-type iteration argument presented above, only with some slight complications added, which are easily surmountable. Lemma 4.3.6 translates basically verbatim.

We decided not to present this result here as it will not be needed for our later construction of weak solutions and is not appreciably different from what we have done here or has already been done in the standard diffusion case.

4.4 Existence of weak solutions

We have at this point established all the classical existence theory we want to address in this chapter and therefore will now transition to our construction of weak solutions, which is in part based on said classical theory.

4.4.1 Approximate solutions

Our construction of weak solutions will centrally rely on approximation of said solutions by classical solutions, which solve a suitably regularized version of the original problem. As we already derived global existence of classical solutions for the system (DH) with boundary condition (DHB) under very strong assumptions on $\mathbb{D}$, we of course want to construct our weak solutions under much weaker assumptions on $\mathbb{D}$. In fact, the central regularization employed by us will be concerned with approximating a potentially quite irregular $\mathbb{D}$ by matrix-valued functions $\mathbb{D}_\varepsilon$ that are sufficiently regular to ensure classical existence of solutions. Apart from this, we will use approximated initial data. We will also slightly modify the logistic source term to ensure $r > 2$ in our approximated system because we can then further eliminate the assumption concerning the parameters χ and μ needed for the classical theory when $r = 2$. One central advantage of this approach is that our approximate systems are very close to the system we actually want to construct solutions for and thus our regularizations only minimally interfere with the structures present in the system, which we want to exploit for e.g. a priori information.

To now make all of this more explicit, we begin by fixing a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \{2, 3\}$, and system parameters $\chi \in (0, \infty)$, $\mu \in (0, \infty)$, $r \in [2, \infty)$. We also fix some a.e. nonnegative initial data $n_0 \in L^{|z| \ln(|z|+1)}(\Omega)$ and $c_0 \in C^0(\overline{\Omega})$ with $\sqrt{c_0} \in W^{1,2}(\Omega)$. We further fix $\mathbb{D} \in W^{1,2}_{\mathrm{div}}(\Omega; \mathbb{R}^{N \times N}) \cap C^0(\overline{\Omega}; \mathbb{R}^{N \times N})$ with the following properties:

- $\mathbb{D}$ is positive semidefinite everywhere.
- $\mathbb{D}$ allows for a divergence estimate with exponent $\beta \in [\frac{1}{2}, 1)$ and constant $A > 0$ such that $\frac{\beta}{1-\beta} \leq r$ (cf. Definition 4.1.2).
- $\mathbb{D}$ allows for a compact $L^1(\Omega)$ embedding (cf. Definition 4.1.9).

As for any $\beta \in [\frac{1}{2}, \frac{2}{3}]$ the condition $\frac{\beta}{1-\beta} \leq r$ is always fulfilled independent of our choice of $r \in [2, \infty)$ and as it is easy to see that, if $\mathbb{D}$ allows for a divergence estimate in accordance with Definition 4.1.2, it also allows for a divergence estimate with any larger exponent, we can assume that the parameter β seen in the second of the above properties is in fact an element of $[\frac{2}{3}, 1) \subseteq (\frac{1}{2}, 1)$ without loss of generality. Then according to Remark 4.1.3, the aforementioned divergence estimate directly implies that

$$\mathbb{D} \in W^{1,q}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \subseteq W^{1,2}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \subseteq W^{1,\frac{r}{r-1}}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N})$$

with $q \coloneqq \frac{2\beta}{2\beta-1} < \infty$.

Having fixed $\mathbb{D}$ with the properties laid out above, we now want to also fix an approximate family $(\mathbb{D}_\varepsilon)_{\varepsilon\in(0,1)}$ of more regular versions of $\mathbb{D}$ as already mentioned. But as this approximation should ideally uniformly retain most of the structural properties in some reasonable way while being regular enough to work with our already established classical existence theory, the construction of such a family is rather involved. Thus, we will prove this approximation result as the following lemma:

Lemma 4.4.1. *There exists an approximate family* $(\mathbb{D}_\varepsilon)_{\varepsilon\in(0,1)} \subseteq C^2(\overline{\Omega}; \mathbb{R}^{N\times N})$ *with* $\mathbb{D}_\varepsilon$ *positive definite on* $\overline{\Omega}$, $(\nabla \cdot \mathbb{D}_\varepsilon) \cdot \nu = 0$ *on* $\partial\Omega$ *for all* $\varepsilon \in (0,1)$ *and*

$$\mathbb{D}_\varepsilon \to \mathbb{D} \quad \textit{in} \quad W^{1,q}_{\mathrm{div}}(\Omega; \mathbb{R}^{N\times N}) \cap C^0(\overline{\Omega}; \mathbb{R}^{N\times N}) \qquad \textit{as } \varepsilon \searrow 0. \tag{4.4.1}$$

We can further choose this family in such a way as to ensure that

$$\left| \int_\Omega (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \Phi \right| \leq B \left(\int_\Omega (\Phi \cdot \mathbb{D}_\varepsilon \Phi)^\beta + 1 \right) \tag{4.4.2}$$

with $B \coloneqq A + 1$ *and*

$$\mathbb{D} + \varepsilon \leq \mathbb{D}_\varepsilon \leq \mathbb{D} + 3\varepsilon \tag{4.4.3}$$

for all $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^N)$ *and* $\varepsilon \in (0,1)$. *Here,* $A > 0$, $\beta \in (\frac{1}{2}, 1)$ *and* $q = \frac{2\beta}{2\beta-1} \in (2, \infty)$, *are the constants fixed at the beginning of this section.*

Proof. This proof is a two-step process. We first approximate $\mathbb{D}$ in our desired function space with the appropriate boundary conditions and then, as a second step, we show that, with only slight modification, we can gain the remaining properties from that approximation.

For the initial approximation, we assume without loss of generality that $\mathbb{D}$ is smooth. We can do this as it is well-known that a standard convolution-based argument would give us a smooth approximation of $\mathbb{D}$ in our desired space, which we can then approximate again to gain all additional desired properties. In our case, the key property not covered by such a convolution-based method is that we want all our approximate matrices to fulfill a very specific boundary condition. Therefore, we will now demonstrate how an approximation of a smooth $\mathbb{D}$ by matrices with exactly this property can be achieved using the continuity properties of semigroups associated with carefully chosen sectorial operators (cf. [31]).

To this end, we fix functions $d'_{i,j}$ such that

$$d_{i,j} = \begin{cases} d'_{i,i} + \sum_{l,k=1}^{N} d'_{l,k}, & \text{if } i = j, \\ d'_{i,j}, & \text{if } i \neq j \end{cases} \tag{4.4.4}$$

for all $i, j \in \{1, \dots, N\}$, where $(d_{i,j})_{i,j\in\{1,\dots,N\}} := \mathbb{D}$. As can be easily seen, the functions $d'_{i,j}$ are linear combinations of the components of $\mathbb{D}$ and therefore smooth as well. We then set $d'_{i,j,\varepsilon} = e^{\varepsilon A_{i,j}} d'_{i,j}$, $\varepsilon \in (0,1)$, where $A_{i,j}$ is the Laplacian on $C^0(\overline{\Omega})$ with boundary operator $\mathcal{B}_{i,j}(x, D) := \sum_{l=1}^{N} \nu_l(x) D_l + \frac{1}{2}\nu_j(x) D_i + \frac{1}{2}\nu_i(x) D_j$, which corresponds to the boundary condition $\nabla\varphi \cdot \nu + \frac{1}{2}(\partial_{x_i}\varphi)\nu_j + \frac{1}{2}(\partial_{x_j}\varphi)\nu_i = 0$, $\varphi \in C^1(\overline{\Omega})$, and domain $D(A_{i,j})$ as given in [80, Corollary 3.1.24 (ii)]. Let us briefly note that we introduced the functions $d'_{i,j}$ to ensure that the operators $A_{i,j}$ have non-tangential boundary conditions and are therefore sectorial (cf. [80], [113]) while still being able to choose the boundary conditions in such a way as to allow for the computation in the next paragraph.

Due to the well-known continuity properties of said semigroup (cf. [49], [80]), we know that $d'_{i,j,\varepsilon} \to d'_{i,j}$ and therefore $d_{i,j,\varepsilon} \to d_{i,j}$ in $W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$ as $\varepsilon \searrow 0$ with $d_{i,j,\varepsilon}$ defined in an analogous fashion to (4.4.4). Thus, $\mathbb{D}_\varepsilon := (d_{i,j,\varepsilon})_{i,j\in\{1,\dots,N\}} \to \mathbb{D}$ in our desired way. Further,

$$\begin{aligned} (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \nu &= \sum_{i,j=1}^{N} (\partial_{x_j} d_{i,j,\varepsilon})\nu_i = \sum_{i,j=1, i\neq j}^{N} (\partial_{x_j} d_{i,j,\varepsilon})\nu_i + \sum_{i=1}^{N} (\partial_{x_i} d_{i,i,\varepsilon})\nu_i \\ &= \sum_{i,j=1, i\neq j}^{N} (\partial_{x_j} d'_{i,j,\varepsilon})\nu_i + \sum_{i=1}^{N} \left(\partial_{x_i} d'_{i,i,\varepsilon} + \sum_{l,k=1}^{N} \partial_{x_i} d'_{l,k,\varepsilon} \right) \nu_i \\ &= \sum_{i,j=1}^{N} \left(\tfrac{1}{2}(\partial_{x_j} d'_{i,j,\varepsilon})\nu_i + \tfrac{1}{2}(\partial_{x_i} d'_{i,j,\varepsilon})\nu_j \right) + \sum_{l,k=1}^{N} \nabla d'_{l,k,\varepsilon} \cdot \nu \\ &= \sum_{i,j=1}^{N} \left(\nabla d'_{i,j,\varepsilon} \cdot \nu + \tfrac{1}{2}(\partial_{x_j} d'_{i,j,\varepsilon})\nu_i + \tfrac{1}{2}(\partial_{x_i} d'_{i,j,\varepsilon})\nu_j \right) = 0 \end{aligned} \tag{4.4.5}$$

on $\partial\Omega$ for all $\varepsilon \in (0,1)$ due to the prescribed boundary conditions of the operators $A_{i,j}$. Thus, we have constructed a suitable approximate family for $\mathbb{D}$ with the correct boundary conditions.

As our second step, we will now fix one such family of approximations of $\mathbb{D}$ and call it $(\mathbb{D}'_\varepsilon)_{\varepsilon\in(0,1)}$, as we still want to slightly modify it. We can assume that

$$\|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} \leq \varepsilon \quad \text{and} \quad \|\nabla \cdot \mathbb{D}'_\varepsilon - \nabla \cdot \mathbb{D}\|_{L^{\frac{2\beta}{2\beta-1}}(\Omega)} \leq \varepsilon^{\frac{1}{2}}$$

for all $\varepsilon \in (0,1)$ without loss of generality. If we then set $\mathbb{D}_\varepsilon := \mathbb{D}'_\varepsilon + \|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} + \varepsilon$, we can ensure that

$$\mathbb{D} + \varepsilon = \mathbb{D}_\varepsilon - \mathbb{D}_\varepsilon + \mathbb{D} + \varepsilon = \mathbb{D}_\varepsilon - \mathbb{D}'_\varepsilon + \mathbb{D} - \|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)}$$

$$\leq \mathbb{D}_\varepsilon + \|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} - \|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} = \mathbb{D}_\varepsilon$$

and

$$\begin{aligned}\mathbb{D}_\varepsilon &= \mathbb{D} - \mathbb{D} + \mathbb{D}_\varepsilon = \mathbb{D} + \mathbb{D}'_\varepsilon - \mathbb{D} + \|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} + \varepsilon \\ &\leq \mathbb{D} + 2\|\mathbb{D}'_\varepsilon - \mathbb{D}\|_{L^\infty(\Omega)} + \varepsilon \leq \mathbb{D} + 3\varepsilon\end{aligned}$$

for all $\varepsilon \in (0,1)$ without affecting any of the desired properties that we already derived as we only modify $\mathbb{D}'_\varepsilon$ by adding spatially constant terms that converge to zero as $\varepsilon \searrow 0$. This gives us (4.4.3).

To derive the divergence estimate, we first observe that

$$\begin{aligned}&\left| \int_\Omega (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \Phi - \int_\Omega (\nabla \cdot \mathbb{D}) \cdot \Phi \right| \\ \leq &\int_\Omega |\nabla \cdot \mathbb{D}_\varepsilon - \nabla \cdot \mathbb{D}||\Phi| \leq \|\nabla \cdot \mathbb{D}_\varepsilon - \nabla \cdot \mathbb{D}\|_{L^{\frac{2\beta}{2\beta-1}}(\Omega)} \|\Phi\|_{L^{2\beta}(\Omega)} \\ \leq &\|\varepsilon^{\frac{1}{2}} \Phi\|_{L^{2\beta}(\Omega)} \leq \int_\Omega (\Phi \cdot \varepsilon \Phi)^\beta + 1 \leq \int_\Omega (\Phi \cdot \mathbb{D}_\varepsilon \Phi)^\beta + 1\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^N)$. We can then further estimate

$$\begin{aligned}\left| \int_\Omega (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \Phi \right| \leq &\left| \int_\Omega (\nabla \cdot \mathbb{D}) \cdot \Phi \right| + \left| \int_\Omega (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \Phi - \int_\Omega (\nabla \cdot \mathbb{D}) \cdot \Phi \right| \\ \leq &A \left(\int_\Omega (\Phi \cdot \mathbb{D}\Phi)^\beta + 1 \right) + \left(\int_\Omega (\Phi \cdot \mathbb{D}_\varepsilon \Phi)^\beta + 1 \right) \\ \leq &(A+1) \left(\int_\Omega (\Phi \cdot \mathbb{D}_\varepsilon \Phi)^\beta + 1 \right)\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^N)$ using our assumed divergence estimate for $\mathbb{D}$ and (4.4.3). This gives us (4.4.2) and thus completes the proof. □

We now fix an approximate family $(\mathbb{D}_\varepsilon)_{\varepsilon \in (0,1)}$ for $\mathbb{D}$ as constructed in the above lemma as well as the associated constant B. We then proceed to construct our approximate initial data. To do this, we first fix families $(n_{0,\varepsilon})_{\varepsilon \in (0,1)}$, $(c'_{0,\varepsilon})_{\varepsilon \in (0,1)} \subseteq C^3(\overline{\Omega})$ of positive functions with $(\mathbb{D}_\varepsilon \nabla n_{0,\varepsilon}) \cdot \nu = (\mathbb{D}_\varepsilon \nabla c'_{0,\varepsilon}) \cdot \nu = 0$ on $\partial\Omega$ and $\|c'_{0,\varepsilon}\|_{L^\infty(\Omega)} \leq \|\sqrt{c_0}\|_{L^\infty(\Omega)} + 1$ for all $\varepsilon \in (0,1)$ as well as

$$n_{0,\varepsilon} \to n_0 \qquad \text{in } L^{|z| \ln(|z|+1)}(\Omega) \subseteq L^1(\Omega), \tag{4.4.6}$$

$$c'_{0,\varepsilon} \to \sqrt{c_0} \qquad \text{in } W^{1,2}(\Omega) \tag{4.4.7}$$

as $\varepsilon \searrow 0$. These families can again be constructed by using convolutions (cf. e.g. [2, Theorem 8.21] for the Orlicz space case) or by a similar semigroup-based method as seen before in the much more challenging case of the family $(\mathbb{D}_\varepsilon)_{\varepsilon \in (0,1)}$. Positivity of both families can further be achieved by first approximating the function in a nonnegative way, which is a property of both convolution and semigroup-based methods, and then adding ε to the resulting approximation as a secondary step. Lastly, the $L^\infty(\Omega)$ bound for the family $(c'_{0,\varepsilon})_{\varepsilon\in(0,1)}$ is a straightforward consequence of e.g. the maximum principle in the case of a semigroup-based approximation.

We then let $c_{0,\varepsilon} := (c'_{0,\varepsilon})^2 \in C^3(\overline{\Omega})$ for all $\varepsilon \in (0,1)$ and, because of the properties already established for the family $(c'_{0,\varepsilon})_{\varepsilon\in(0,1)}$, it is straightforward to derive that $c_{0,\varepsilon} > 0$ on $\overline{\Omega}$, $(\mathbb{D}_\varepsilon \nabla c_{0,\varepsilon}) \cdot \nu = 0$ on $\partial\Omega$ and

$$c_{0,\varepsilon} \to c_0 \qquad \text{in } L^p(\Omega) \text{ for all } p \in [1,\infty), \tag{4.4.8}$$

$$\sqrt{c_{0,\varepsilon}} \to \sqrt{c_0} \qquad \text{in } W^{1,2}(\Omega) \tag{4.4.9}$$

as $\varepsilon \searrow 0$.

One important consequence of the above approximations is that we can fix a uniform constant $M > 0$ such that

$$\|\nabla \cdot \mathbb{D}_\varepsilon\|_{L^2(\Omega)} \le M, \quad \|\mathbb{D}_\varepsilon\|_{L^\infty(\Omega)} \le M \tag{4.4.10}$$

and

$$\int_\Omega n_{0,\varepsilon} \le M, \quad \int_\Omega n_{0,\varepsilon} \ln(n_{0,\varepsilon}) \le \int_\Omega n_{0,\varepsilon} \ln(n_{0,\varepsilon} + 1) \le M, \tag{4.4.11}$$

$$\|c_{0,\varepsilon}\|_{L^\infty(\Omega)} \le M, \quad \int_\Omega \frac{\nabla c_{0,\varepsilon} \cdot \mathbb{D}_\varepsilon \nabla c_{0,\varepsilon}}{c_{0,\varepsilon}} \le M \tag{4.4.12}$$

for all $\varepsilon \in (0,1)$.

We then consider the approximate systems

$$\begin{cases} n_{\varepsilon t} = \nabla \cdot (\mathbb{D}_\varepsilon \nabla n_\varepsilon + n_\varepsilon \nabla \cdot \mathbb{D}_\varepsilon) \\ \qquad\quad - \chi \nabla \cdot (n_\varepsilon \mathbb{D}_\varepsilon \nabla c_\varepsilon) + \mu n_\varepsilon (1 - n_\varepsilon^{r+\varepsilon-1}) & \text{on } \Omega \times (0,\infty), \\ c_{\varepsilon t} = -n_\varepsilon c_\varepsilon & \text{on } \Omega \times (0,\infty), \\ (\mathbb{D}_\varepsilon \nabla n_\varepsilon) \cdot \nu = \chi (n_\varepsilon \mathbb{D}_\varepsilon \nabla c_\varepsilon) \cdot \nu - n_\varepsilon (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \nu & \text{on } \partial\Omega \times (0,\infty) \\ n_\varepsilon(\cdot,0) = n_{\varepsilon,0}, \quad c_\varepsilon(\cdot,0) = c_{\varepsilon,0} & \text{on } \Omega \end{cases} \tag{DH$_\varepsilon$}$$

and use our already established classical existence theory from Theorem 4.1.1 to now fix positive global classical solutions $(n_\varepsilon, c_\varepsilon)$ to the above system for each $\varepsilon \in (0,1)$ and the remainder of this section. Note that as $r + \varepsilon > 2$, we do not need to make additional assumptions on the parameters χ and μ to ensure that said existence theory is applicable.

4.4.2 Uniform a priori estimates

We will now derive the bounds necessary to ensure compactness of our families of approximate classical solutions in function spaces conducive to the construction of our desired weak solutions to (DH) with boundary condition (DHB) as limits of said approximate solutions along a suitable sequence of $\varepsilon \in (0,1)$.

Apart from the baseline established in Lemma 4.3.2 for the classical existence theory, which can be easily translated to our approximate solutions in an ε-independent fashion, we will now derive some extended bounds based on an energy-type inequality as an additional baseline for later arguments in this section. This type of energy inequality was already used in the one-dimensional case in [133].

Lemma 4.4.2. *For each $T > 0$, there exists a constant $C = C(T) > 0$ such that*

$$\int_\Omega n_\varepsilon \ln(n_\varepsilon) + \int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon} + \int_0^t \int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + \int_0^t \int_\Omega n_\varepsilon^{r+\varepsilon} \ln(n_\varepsilon) \leq C$$

holds for all $t \in (0,T)$ and all $\varepsilon \in (0,1)$.

Proof. Fix $T > 0$.

By then testing the first equation in $(\mathrm{DH}_\varepsilon)$ with $\ln(n_\varepsilon)$ we gain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \ln(n_\varepsilon) - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon = \int_\Omega n_{\varepsilon t} \ln(n_\varepsilon) \\
&= \int_\Omega \ln(n_\varepsilon)\nabla\cdot(\mathbb{D}_\varepsilon \nabla n_\varepsilon + n_\varepsilon \nabla\cdot\mathbb{D}_\varepsilon) - \chi\int_\Omega \ln(n_\varepsilon)\nabla\cdot(n_\varepsilon \mathbb{D}_\varepsilon \nabla c_\varepsilon) \\
&\quad + \mu\int_\Omega n_\varepsilon(1 - n_\varepsilon^{r+\varepsilon-1})\ln(n_\varepsilon) \\
&= -\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} - \int_\Omega (\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon + \chi\int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \\
&\quad + \mu\int_\Omega n_\varepsilon(1 - n_\varepsilon^{r+\varepsilon-1})\ln(n_\varepsilon) \\
&\leq -\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + B\int_\Omega (\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon)^\beta + B + \chi\int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \\
&\quad + \mu\int_\Omega n_\varepsilon(1 - n_\varepsilon^{r+\varepsilon-1})\ln(n_\varepsilon) \\
&\leq -\frac{1}{2}\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + 2^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} \int_\Omega n_\varepsilon^{\frac{\beta}{1-\beta}} + B + \chi\int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \\
&\quad + \mu\int_\Omega n_\varepsilon(1 - n_\varepsilon^{r+\varepsilon-1})\ln(n_\varepsilon)
\end{aligned} \tag{4.4.13}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ by partial integration, use of the no-flux boundary condition and the divergence estimate in (4.4.2) combined with Young's inequality. We can then further gain from the second equation in $(\mathrm{DH}_\varepsilon)$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon} \\
&= \int_\Omega \frac{\nabla c_{\varepsilon t} \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon} - \frac{1}{2}\int_\Omega \frac{c_{\varepsilon t}(\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon)}{c_\varepsilon^2} \\
&= -\int_\Omega \frac{n_\varepsilon(\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon)}{c_\varepsilon} - \int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon + \frac{1}{2}\int_\Omega \frac{n_\varepsilon(\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon)}{c_\varepsilon}
\end{aligned}$$

$$= -\frac{1}{2}\int_\Omega \frac{n_\varepsilon(\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon)}{c_\varepsilon} - \int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \le -\int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \tag{4.4.14}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Combining (4.4.13) and (4.4.14) now allows us to further estimate as follows due to the critical $\int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon$ terms in both equations neutralizing each other given the correct coefficients:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega n_\varepsilon \ln(n_\varepsilon) - \int_\Omega n_\varepsilon + \frac{\chi}{2}\int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon}\right\} + \frac{1}{2}\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} \\
&\le 2^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} \int_\Omega n_\varepsilon^{\frac{\beta}{1-\beta}} + B + \mu \int_\Omega n_\varepsilon \ln(n_\varepsilon) - \mu \int_\Omega n_\varepsilon^{r+\varepsilon} \ln(n_\varepsilon)
\end{aligned} \tag{4.4.15}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$.

As $\frac{\beta}{1-\beta} \le r$ by assumption, there exists a constant $K > 0$ (independent of ε) such that

$$2^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} z^{\frac{\beta}{1-\beta}} - \frac{\mu}{2} z^{r+\varepsilon} \ln(z) \le 2^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} z^{\frac{\beta}{1-\beta}} - \frac{\mu}{2} z^{r} \ln(z) \le K$$

for all $z \ge 0$ and $\varepsilon \in (0,1)$. Given this, we can then further estimate in (4.4.15) to see that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega n_\varepsilon \ln(n_\varepsilon) + \frac{\chi}{2}\int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon}\right\} + \frac{1}{2}\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+\varepsilon}\ln(n_\varepsilon) \\
&\le \mu \int_\Omega n_\varepsilon \ln(n_\varepsilon) + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon + K|\Omega| + B
\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Time integration in combination with Gronwall's inequality and the uniform $L^1(\Omega)$ bound for n_ε due to Lemma 4.3.2 as well as the uniform initial data bounds from (4.4.11) and (4.4.12) then yields our desired result as the above differential inequality essentially means that the growth of the considered terms can be at most exponential. □

We now further extract some relevant but straightforward additional bounds for our approximate solutions from the previous lemma.

Corollary 4.4.3. *For each $T > 0$, there exists $C = C(T) > 0$ such that*

$$\int_0^T \int_\Omega n_\varepsilon^{r+\varepsilon} \ln(n_\varepsilon^{r+\varepsilon}) \le C, \qquad \int_0^T \int_\Omega n_\varepsilon^{r} \ln(n_\varepsilon) \le C, \tag{4.4.16}$$

$$\int_0^T \|n_\varepsilon^{\frac{1}{2}}(\cdot,s)\|^2_{W^{1,2}_{\mathbb{D}}(\Omega)} \,\mathrm{d}s \le C, \tag{4.4.17}$$

$$\int_0^T \|n_\varepsilon(\cdot,s)\|^{\frac{2r}{r+1}}_{W^{1,\frac{2r}{r+1}}_{\mathbb{D}}(\Omega)} \,\mathrm{d}s \le \int_0^T \|n_\varepsilon(\cdot,s)\|^{\frac{2r}{r+1}}_{W^{1,\frac{2r}{r+1}}_{\mathbb{D}_\varepsilon}(\Omega)} \,\mathrm{d}s \le C \tag{4.4.18}$$

and

$$\int_0^T \|c_\varepsilon(\cdot,s)\|^2_{W^{1,2}_{\mathbb{D}}(\Omega)} \,\mathrm{d}s \le \int_0^T \|c_\varepsilon(\cdot,s)\|^2_{W^{1,2}_{\mathbb{D}_\varepsilon}(\Omega)} \,\mathrm{d}s \le C \tag{4.4.19}$$

for all $\varepsilon \in (0,1)$.

Proof. Fix $T > 0$.

Then given that

$$\int_\Omega \nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \leq \|c_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega \frac{\nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon}{c_\varepsilon}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ as well as knowing that $\mathbb{D} \leq \mathbb{D}_\varepsilon$ according to (4.4.3) for all $\varepsilon \in (0,1)$, Lemma 4.4.2 combined with Lemma 4.3.2 and (4.4.12) yields (4.4.19).

As

$$z^r \ln(z) \leq z^{r+\varepsilon} \ln(z) \quad \text{and} \quad z^{r+\varepsilon} \ln(z^{r+\varepsilon}) = (r+\varepsilon) z^{r+\varepsilon} \ln(z) \leq (r+1) z^{r+\varepsilon} \ln(z) + 1$$

for all $z \geq 0$ and $\varepsilon \in (0,1)$, the result (4.4.16) follows directly from Lemma 4.4.2.

To address the last remaining results in (4.4.17) and (4.4.18), we now note that

$$\int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} = 4 \int_\Omega \nabla n_\varepsilon^{\frac{1}{2}} \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon^{\frac{1}{2}}$$

and

$$\begin{aligned}
\int_\Omega (\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon)^{\frac{\frac{2r}{r+1}}{2}} &= \int_\Omega n_\varepsilon^{\frac{r}{r+1}} \left(\frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} \right)^{\frac{r}{r+1}} \\
&\leq \int_\Omega n_\varepsilon^r + \int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} \\
&\leq r \int_\Omega n_\varepsilon^r \ln(n_\varepsilon) + |\Omega| + \int_\Omega \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon}
\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ due to Young's inequality. Given this, the results in (4.4.17) and (4.4.18) also follow directly from Lemma 4.4.2 and the fact that $\mathbb{D} \leq \mathbb{D}_\varepsilon$ for all $\varepsilon \in (0,1)$ according to (4.4.3). □

By another testing procedure for the first equation in (DH_ε), which is very similar to the one already used in the proof of Lemma 4.4.2, we will now derive another space-time integral bound for n_ε and its gradient.

Lemma 4.4.4. *For each $T > 0$, there exists a constant $C = C(T) > 0$ such that*

$$\int_0^T \int_\Omega n_\varepsilon^{-\frac{3}{2}} (\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon) \leq C$$

for all $\varepsilon \in (0,1)$.

Proof. Fix $T > 0$.

We first note that

$$\left| \int_\Omega n_\varepsilon^{-\frac{1}{2}} (\nabla \cdot \mathbb{D}_\varepsilon) \cdot \nabla n_\varepsilon \right|$$

$$
\begin{aligned}
&= 2\left|\int_\Omega (\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon^{\frac12}\right| \le 2B\int_\Omega\left(\nabla n_\varepsilon^{\frac12}\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon^{\frac12}\right)^\beta + 2B \\
&\le 2B\int_\Omega\left(\frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon}\right)^\beta + 2B \le 2B\int_\Omega \frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon} + 2B(1+|\Omega|) \qquad (4.4.20)
\end{aligned}
$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$ due to (4.4.2).

We then further test the first equation in (DH_ε) with $-n_\varepsilon^{-\frac12}$ to derive that

$$
\begin{aligned}
-2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^{\frac12} &= -\int_\Omega n_\varepsilon^{-\frac12} n_{\varepsilon t} \\
&= -\int_\Omega n_\varepsilon^{-\frac12}\nabla\cdot(\mathbb{D}_\varepsilon\nabla n_\varepsilon + n_\varepsilon\nabla\cdot\mathbb{D}_\varepsilon) + \chi\int_\Omega n_\varepsilon^{-\frac12}\nabla\cdot(n_\varepsilon\mathbb{D}_\varepsilon\nabla c_\varepsilon) \\
&\quad - \mu\int_\Omega n_\varepsilon^{\frac12}(1-n_\varepsilon^{r+\varepsilon-1}) \\
&= -\frac12\int_\Omega n_\varepsilon^{-\frac32}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon) - \frac12\int_\Omega n_\varepsilon^{-\frac12}((\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon) \\
&\quad + \frac{\chi}{2}\int_\Omega n_\varepsilon^{-\frac12}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon) - \mu\int_\Omega n_\varepsilon^{\frac12}(1-n_\varepsilon^{r+\varepsilon-1}) \\
&\le -\frac12\int_\Omega n_\varepsilon^{-\frac32}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon) + \frac12\left|\int_\Omega n_\varepsilon^{-\frac12}((\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon)\right| \\
&\quad + \frac{\chi}{4}\int_\Omega\frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon} + \frac{\chi}{4}\int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon + \mu\int_\Omega n_\varepsilon^{r+\varepsilon-\frac12}
\end{aligned}
$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$ by partial integration, use of the no-flux boundary condition and the Cauchy–Schwarz inequality combined with Young's inequality. This then immediately implies

$$
\begin{aligned}
&\frac12\int_\Omega n_\varepsilon^{-\frac32}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon) \\
&\le 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^{\frac12} + \frac12\left|\int_\Omega n_\varepsilon^{-\frac12}(\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon)\right| \\
&\quad + \frac{\chi}{4}\int_\Omega\frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon} + \frac{\chi}{4}\int_\Omega\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon + \mu\int_\Omega n_\varepsilon^{r+\varepsilon-\frac12} \qquad (4.4.21)
\end{aligned}
$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$. As Young's inequality as well as the fact that $z\le z\ln(z)+1$ for all $z\ge0$ further yields that

$$
\int_\Omega n_\varepsilon^{r+\varepsilon-\frac12} \le \int_\Omega n_\varepsilon^{r+\varepsilon} + |\Omega| \le \int_\Omega n_\varepsilon^{r+\varepsilon}\ln(n_\varepsilon^{r+\varepsilon}) + 2|\Omega|
$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$, the inequality in (4.4.21) combined with the already established bounds from Lemma 4.3.2, Lemma 4.4.2 and (4.4.11) as well as (4.4.20) gives us our desired estimate after an integration in time. □

As our final preparation for a now soon following compactness argument (based on the Aubin–Lions lemma), which is used to construct the candidates for our weak solutions, we will prepare uniform integrability estimates for the time derivatives of $n_\varepsilon^{1/2}$ and c_ε.

Note that the construction of a solution candidate for the second solution component c could likely be achieved by less powerful means. But as we will already need to employ fairly extensive compact embedding arguments to handle the first solution components n_ε anyway and deriving the necessary additional uniform bounds for c_ε is trivial, we will use the same compactness argument for the second solution component as well for the sake of uniformity of presentation.

Lemma 4.4.5. *For each $T > 0$, there exists $C = C(T) > 0$ such that*

$$\int_0^T \|(n_\varepsilon^{\frac{1}{2}})_t(\cdot,t)\|_{(W^{N+1,2}(\Omega))^*}\,\mathrm{d}t \le C \quad \textit{and} \quad \int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W^{N+1,2}(\Omega))^*}\,\mathrm{d}t \le C$$

for all $\varepsilon \in (0,1)$.

Proof. Fix $T > 0$.

We then begin by noting that the bound for $c_{\varepsilon t}$ is an immediate and straightforward consequence of Lemma 4.3.2 combined with (4.4.12) and the second equation in (DH$_\varepsilon$) as well as the fact that $W^{N+1,2}(\Omega)$ embeds continuously into $L^\infty(\Omega)$.

We now focus our attention on deriving the $(n_\varepsilon^{\frac{1}{2}})_t$ bound. To this end, we test the first equation in (DH$_\varepsilon$) with $n_\varepsilon^{-\frac{1}{2}}\varphi$, $\varphi \in C^\infty(\overline{\Omega})$, and apply partial integration to see that

$$\begin{aligned}
2\left|\int_\Omega (n_\varepsilon^{\frac{1}{2}})_t\varphi\right| &= \left|\int_\Omega n_\varepsilon^{-\frac{1}{2}} n_{\varepsilon t}\varphi\right| \\
\le{}& \left|\int_\Omega n_\varepsilon^{-\frac{1}{2}}\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi\right| + \frac{1}{2}\left|\int_\Omega n_\varepsilon^{-\frac{3}{2}}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon)\varphi\right| \\
&+ \left|\int_\Omega n_\varepsilon^{\frac{1}{2}}((\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla\varphi)\right| + \frac{1}{2}\left|\int_\Omega n_\varepsilon^{-\frac{1}{2}}((\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla n_\varepsilon)\varphi\right| \\
&+ \chi\left|\int_\Omega n_\varepsilon^{\frac{1}{2}}\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi\right| + \frac{\chi}{2}\left|\int_\Omega n_\varepsilon^{-\frac{1}{2}}(\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon)\varphi\right| + \mu\left|\int_\Omega n_\varepsilon^{\frac{1}{2}}(1-n_\varepsilon^{r+\varepsilon-1})\varphi\right| \\
\le{}& \left|\int_\Omega n_\varepsilon^{-\frac{1}{2}}\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi\right| + \frac{1}{2}\left|\int_\Omega n_\varepsilon^{-\frac{3}{2}}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon)\varphi\right| \\
&+ 2\left|\int_\Omega n_\varepsilon^{\frac{1}{2}}((\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla\varphi)\right| + \left|\int_\Omega (\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla(n_\varepsilon^{\frac{1}{2}}\varphi)\right| \\
&+ \chi\left|\int_\Omega n_\varepsilon^{\frac{1}{2}}\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi\right| + \frac{\chi}{2}\left|\int_\Omega n_\varepsilon^{-\frac{1}{2}}(\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon)\varphi\right| + \mu\left|\int_\Omega n_\varepsilon^{\frac{1}{2}}(1-n_\varepsilon^{r+\varepsilon-1})\varphi\right|
\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Using that

$$\begin{aligned}
\left|\int_\Omega (\nabla\cdot\mathbb{D}_\varepsilon)\cdot\nabla(n_\varepsilon^{\frac{1}{2}}\varphi)\right| &\le B\int_\Omega \left(\nabla(n_\varepsilon^{\frac{1}{2}}\varphi)\cdot\mathbb{D}_\varepsilon\nabla(n_\varepsilon^{\frac{1}{2}}\varphi)\right)^\beta + B \\
&\le B\int_\Omega \nabla(n_\varepsilon^{\frac{1}{2}}\varphi)\cdot\mathbb{D}_\varepsilon\nabla(n_\varepsilon^{\frac{1}{2}}\varphi) + B(|\Omega|+1)
\end{aligned}$$

$$\leq \frac{B}{2}\int_\Omega |\varphi|^2 \frac{\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon}{n_\varepsilon} + 2B\int_\Omega n_\varepsilon \nabla\varphi\cdot\mathbb{D}_\varepsilon\nabla\varphi + B(|\Omega|+1)$$

due to (4.4.2) and (4.4.10) as well as the Hölder and Young's inequality, we can further estimate to conclude that

$$\begin{aligned}
&2\left|\int_\Omega (n_\varepsilon^{\frac{1}{2}})_t\varphi\right| \\
&\leq \left(\int_\Omega \nabla\varphi\cdot\mathbb{D}_\varepsilon\nabla\varphi\right)^{\frac{1}{2}}\left(\int_\Omega \frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon}\right)^{\frac{1}{2}} + \frac{\|\varphi\|_{L^\infty(\Omega)}}{2}\int_\Omega n_\varepsilon^{-\frac{3}{2}}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon) \\
&\quad + 2\|\nabla\varphi\|_{L^\infty(\Omega)}\left(\int_\Omega n_\varepsilon + \int_\Omega |\nabla\cdot\mathbb{D}_\varepsilon|^2\right) \\
&\quad + \frac{B\|\varphi\|^2_{L^\infty(\Omega)}}{2}\int_\Omega \frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon} + 2B\int_\Omega n_\varepsilon\nabla\varphi\cdot\mathbb{D}_\varepsilon\nabla\varphi + B(|\Omega|+1) \\
&\quad + \chi\left(\int_\Omega n_\varepsilon\nabla\varphi\cdot\mathbb{D}_\varepsilon\nabla\varphi\right)^{\frac{1}{2}}\left(\int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon\right)^{\frac{1}{2}} \\
&\quad + \frac{\chi\|\varphi\|_{L^\infty(\Omega)}}{2}\left(\int_\Omega \frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon}\right)^{\frac{1}{2}}\left(\int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon\right)^{\frac{1}{2}} \\
&\quad + \mu\|\varphi\|_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^{\frac{1}{2}} + \mu\|\varphi\|_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^{r+\varepsilon-\frac{1}{2}} \\
&\leq K\left(\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)} + \|\varphi\|^2_{L^\infty(\Omega)} + \|\nabla\varphi\|^2_{L^\infty(\Omega)}\right)\times \\
&\qquad \left(\int_\Omega \frac{\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon}{n_\varepsilon} + \int_\Omega n_\varepsilon^{-\frac{3}{2}}(\nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla n_\varepsilon)\right. \\
&\qquad \left. + \int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon + \int_\Omega n_\varepsilon^{r+\varepsilon}\ln(n_\varepsilon^{r+\varepsilon}) + 1\right)
\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ with some appropriate constant $K > 0$ only dependent on Ω, μ, r, χ, B, T and M. Given the above inequality, the remainder of our desired result follows from Lemma 4.4.2, Corollary 4.4.3 and Lemma 4.4.4 as well as the continuous embedding of $W^{N+1,2}(\Omega)$ into $W^{1,\infty}(\Omega)$ and density of $C^\infty(\overline{\Omega})$ in $W^{N+1,2}(\Omega)$. □

4.4.3 Construction of weak solutions

Having prepared all the necessary bounds, we will now construct the solution candidates by using various compact embedding arguments to gain them as the limit of our approximate solutions.

Lemma 4.4.6. *There exist a null sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq (0,1)$ and a.e. nonnegative functions*

$$\begin{aligned}
n &\in L^{\frac{2r}{r+1}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{2r}{r+1}}_{\mathbb{D}}(\Omega)) \cap L^r_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)), \\
c &\in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{\mathbb{D}}(\Omega)) \cap L^\infty(\Omega\times(0,\infty)),
\end{aligned}$$

such that

$$n_\varepsilon \to n \qquad \text{in } L^r_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.4.22}$$
$$n_\varepsilon^{r+\varepsilon} \to n^r \qquad \text{in } L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.4.23}$$
$$n_\varepsilon \rightharpoonup n \qquad \text{in } L^{\frac{2r}{r+1}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{2r}{r+1}}_{\mathbb{D}}(\Omega)), \tag{4.4.24}$$
$$c_\varepsilon \to c \qquad \text{in } L^p_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \text{ for all } p \in [1,\infty) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.4.25}$$
$$c_\varepsilon \rightharpoonup c \qquad \text{in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{\mathbb{D}}(\Omega)) \tag{4.4.26}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Proof. Given that both of the families $(n_\varepsilon^{\frac{1}{2}})_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ are bounded in the space $L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{\mathbb{D}}(\Omega))$ according to Corollary 4.4.3 and similarly both of the families $((n_\varepsilon^{\frac{1}{2}})_t)_{\varepsilon\in(0,1)}$ and $(c_{\varepsilon t})_{\varepsilon\in(0,1)}$ are bounded in the space $L^1_{\mathrm{loc}}([0,\infty); (W^{N+1,2}(\Omega))^*)$ according to Lemma 4.4.5, we can apply the Aubin–Lions lemma (cf. [112, p. 215]) to the above families using the triple of embedded spaces $W^{1,2}_{\mathbb{D}}(\Omega) \subseteq L^1(\Omega) \subseteq (W^{N+1,2}(\Omega))^*$. Note that this is only possible as the first embedding is in fact compact by our assumptions (cf. Definition 4.1.9). Therefore, there exists a null sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq (0,1)$ and functions $\tilde{n}, c : \overline{\Omega} \times [0,T) \to \mathbb{R}$ such that

$$n_\varepsilon^{\frac{1}{2}} \to \tilde{n} \quad \text{and} \quad c_\varepsilon \to c \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); L^1(\Omega)) \text{ and therefore in } L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$. This sequence is constructed by applying the Aubin–Lions lemma countably infinitely many times on time intervals of the form $[0,T]$, $T \in \mathbb{N}$, combined with a straightforward extension and diagonal sequence argument. We can further choose the above sequence in such way as to ensure that $n_\varepsilon^{\frac{1}{2}} \to \tilde{n}$ and $c_\varepsilon \to c$ pointwise almost everywhere as $\varepsilon = \varepsilon_j \searrow 0$ by potentially switching to another subsequence. Due to the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ furthermore being uniformly bounded in $L^\infty(\Omega \times (0,\infty))$ (cf. (4.4.12) and Lemma 4.3.2), the above convergence properties directly imply (4.4.25) as well as the fact that c is nonnegative almost everywhere and $c \in L^\infty(\Omega \times (0,\infty))$.

We now set $n := \tilde{n}^2$ and observe that the above almost everywhere pointwise convergence for the already constructed sequences then ensures that

$$n_\varepsilon \to n \quad \text{and} \quad n_\varepsilon^{r+\varepsilon} \to n^r \quad \text{a.e. pointwise}$$

as $\varepsilon = \varepsilon_j \searrow 0$. This immediately gives us non-negativity of n as well. Further as for every $T > 0$ there exists $K = K(T) > 0$ such that

$$\int_0^T \int_\Omega n_\varepsilon^r |\ln(n_\varepsilon)| \le K \quad \text{and} \quad \int_0^T \int_\Omega n_\varepsilon^{r+\varepsilon} |\ln(n_\varepsilon^{r+\varepsilon})| \le K$$

for all $\varepsilon \in (0,1)$ according to Corollary 4.4.3, we can use Vitali's theorem and the de la Vallée Poussin criterion for uniform integrability (cf. [25, pp. 23-24]) to gain the convergence properties in (4.4.22) and (4.4.23).

The remaining weak convergence properties in (4.4.24) and (4.4.26) then follow immediately by another similar but fairly standard subsequence extraction argument as the respective families of functions are bounded in the relevant spaces according to Corollary 4.4.3.

As all not yet explicitly established regularity properties for n and c directly follow from the convergence properties and we have at this point proven all said properties, this completes the proof. $\square$

For the remainder of this section, we will now fix the functions n and c as well as the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ constructed in the preceding lemma. While the convergence properties derived in Lemma 4.4.6 are in fact already sufficient to allow us to translate the weak solution property from our approximate solutions to our now established solution candidates, we will as a last effort prior to the proof of Theorem 4.1.13 derive some more specifically tailored convergence properties to handle some of the more complex terms in the weak solution definition.

Lemma 4.4.7. *The convergence properties*

$$\int_0^\infty \int_\Omega \nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla\varphi \to \int_0^\infty \int_\Omega \nabla n \cdot \mathbb{D}\nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{4.4.27}$$

and

$$\int_0^\infty \int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla\varphi \to \int_0^\infty \int_\Omega n\nabla c \cdot \mathbb{D}\nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{4.4.28}$$

hold for all $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$.

Proof. Fix $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$ and $T > 0$ such that $\operatorname{supp}(\varphi) \subseteq \overline{\Omega} \times [0,T)$. We can then fix a constant $K_1 = K_1(T) \geq 1$ such that

$$\int_0^T \int_\Omega (\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon)^{\frac{r}{r+1}} \leq K_1 \quad \text{and} \quad \int_0^T \int_\Omega \nabla c_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla c_\varepsilon \leq K_1$$

for all $\varepsilon \in (0,1)$ according to Corollary 4.4.3. This implies that

$$\begin{aligned} \|\nabla n_\varepsilon\|_{L^1(\Omega\times(0,T))} &\leq (|\Omega|+1)\left(\int_0^T \int_\Omega (\nabla n_\varepsilon \cdot \nabla n_\varepsilon)^{\frac{r}{r+1}}\right)^{\frac{r+1}{2r}} \\ &\leq (|\Omega|+1)\left(\frac{1}{\varepsilon^{\frac{r}{r+1}}}\int_0^T \int_\Omega (\nabla n_\varepsilon \cdot \mathbb{D}_\varepsilon \nabla n_\varepsilon)^{\frac{r}{r+1}}\right)^{\frac{r+1}{2r}} \leq \frac{K_2}{\varepsilon^{\frac{1}{2}}} \end{aligned} \tag{4.4.29}$$

and

$$\|\nabla c_\varepsilon\|_{L^2(\Omega\times(0,T))} = \left(\int_0^T \int_\Omega \nabla c_\varepsilon \cdot \nabla c_\varepsilon\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le (|\Omega|+1)\left(\frac{1}{\varepsilon}\int_0^T\int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon\right)^{\frac{1}{2}}\\
&\le \frac{K_2}{\varepsilon^{\frac{1}{2}}}
\end{aligned}
\tag{4.4.30}
$$

for all $\varepsilon\in(0,1)$ with $K_2 = K_2(T) \coloneqq K_1(|\Omega|+1)$ due to the Hölder inequality and the estimate in (4.4.3). We then observe that

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi - \int_0^T\int_\Omega \nabla n\cdot\mathbb{D}\nabla\varphi\right|\\
&\le \left|\int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi - \int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}\nabla\varphi\right|\\
&\quad + \left|\int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega \nabla n\cdot\mathbb{D}\nabla\varphi\right|\\
&\le \|\nabla n_\varepsilon\|_{L^1(\Omega\times(0,T))}\|\mathbb{D}_\varepsilon-\mathbb{D}\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega\times(0,T))}\\
&\quad + \left|\int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega \nabla n\cdot\mathbb{D}\nabla\varphi\right|\\
&\le 3K_2\varepsilon^{\frac{1}{2}}\|\nabla\varphi\|_{L^\infty(\Omega\times(0,T))} + \left|\int_0^T\int_\Omega \nabla n_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega \nabla n\cdot\mathbb{D}\nabla\varphi\right|
\end{aligned}
$$

for all $\varepsilon\in(0,1)$ because of (4.4.3) and (4.4.29). This inequality immediately implies (4.4.27) due to the weak convergence property in (4.4.24) presented in Lemma 4.4.6.

We now similarly estimate that

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi - \int_0^T\int_\Omega n\nabla c\cdot\mathbb{D}\nabla\varphi\right|\\
&\le \left|\int_0^T\int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi - \int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi\right|\\
&\quad + \left|\int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla\varphi - \int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}\nabla\varphi\right|\\
&\quad + \left|\int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega n\nabla c\cdot\mathbb{D}\nabla\varphi\right|\\
&\le \left(\int_0^T\int_\Omega \nabla c_\varepsilon\cdot\mathbb{D}_\varepsilon\nabla c_\varepsilon\right)^{\frac{1}{2}}\left(\int_0^T\int_\Omega (n-n_\varepsilon)^2(\nabla\varphi\cdot\mathbb{D}_\varepsilon\nabla\varphi)\right)^{\frac{1}{2}}\\
&\quad + \|n\|_{L^2(\Omega\times(0,T))}\|\nabla c_\varepsilon\|_{L^2(\Omega\times(0,T))}\|\mathbb{D}_\varepsilon-\mathbb{D}\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega\times(0,T))}\\
&\quad + \left|\int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega n\nabla c\cdot\mathbb{D}\nabla\varphi\right|\\
&\le K_1^{\frac{1}{2}}(\|\mathbb{D}\|_{L^\infty(\Omega)}+3)^{\frac{1}{2}}\|\nabla\varphi\|_{L^\infty(\Omega)}\|n-n_\varepsilon\|_{L^2(\Omega\times(0,T))}
\end{aligned}
$$

$$+ 3K_2\varepsilon^{\frac{1}{2}}\|n\|_{L^2(\Omega\times(0,T))}\|\nabla\varphi\|_{L^\infty(\Omega\times(0,T))}$$
$$+ \left|\int_0^T\int_\Omega n\nabla c_\varepsilon\cdot\mathbb{D}\nabla\varphi - \int_0^T\int_\Omega n\nabla c\cdot\mathbb{D}\nabla\varphi\right|$$

for all $\varepsilon \in (0,1)$ because of (4.4.3) and (4.4.30). Due to the convergence properties in (4.4.22) and (4.4.26) as well as the fact that $r \geq 2$ and therefore $n \in L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty))$, the above estimate implies (4.4.28) and thus completes the proof. □

As all convergence properties necessary to argue that n and c are in fact our desired weak solution have been established, we can now present the at this point fairly short proof of our second main existence result, namely Theorem 4.1.13.

Proof for Theorem 4.1.13. We first note that n, c are already sufficiently regular to ensure that (4.1.2) and (4.1.3) hold due to Lemma 4.4.6.

It is further straightforward to verify that $(n_\varepsilon, c_\varepsilon)_{\varepsilon\in(0,1)}$ are weak solutions in the sense of Definition 4.1.12 for all $\varepsilon \in (0,1)$ with only slightly different parameters. Consequently, we only now need to confirm that all the terms in the weak solution definition converge to their counterparts without ε. For all the terms that are structurally identical in both the approximated case as well as in the weak solution definition we want to achieve for n and c, this is covered by Lemma 4.4.6 as well as the convergence properties of the initial data laid out in (4.4.6) and (4.4.8). The terms that differ because $\mathbb{D}$ was replaced by $\mathbb{D}_\varepsilon$ are covered by either Lemma 4.4.7 or (4.4.1) combined with (4.4.22) from Lemma 4.4.6. Finally, the logistic terms $\int_0^\infty\int_\Omega n_\varepsilon(1-n_\varepsilon^{r-1+\varepsilon})\varphi$ occurring in the weak solution definition for our approximate solutions converge to their proper counterpart $\int_0^\infty\int_\Omega n(1-n^{r-1})\varphi$ due to (4.4.22) and (4.4.23) from Lemma 4.4.6 as well. We have now discussed that all the terms occurring in the weak solution definition of the approximate solutions converge to the correct terms for our solution candidates. Therefore, (n,c) is a weak solution of the type described in Definition 4.1.12. □

5 Smooth solutions to an attractive-repulsive chemotaxis system with measure-valued initial data

5.1 Main result

In this chapter, we will look at a recent extension of the seminal Keller–Segel system, which adds a second repulsive chemical to the mix. More precisely, we consider the system

$$\begin{cases} n_t = \Delta n - \chi\nabla\cdot(n\nabla c_a) + \xi\nabla\cdot(n\nabla c_r), \\ \tau c_{at} = \Delta c_a + \alpha n - \beta c_a, \\ \tau c_{rt} = \Delta c_r + \gamma n - \delta c_r \end{cases} \tag{AR}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, with parameters $\chi, \xi \geq 0$, $\alpha, \beta, \gamma, \delta > 0$ and $\tau \in \{0, 1\}$ and the Neumann boundary conditions

$$0 = \nabla n \cdot \nu = \nabla c_a \cdot \nu = \nabla c_r \cdot \nu \qquad \text{for all } x \in \partial\Omega, t > 0. \tag{ARB}$$

While this system has been fairly thoroughly explored given initial data of high regularity as expanded upon in the introduction, we will show that even when the models starts out in what is essentially a blown-up state, that is with measure-valued initial data for the first solution component, it is still possible to construct smooth solutions to the system assuming the regularizing forces are stronger than their aggregation promoting counterparts. More specifically, we will spend this chapter proving the following result:

Theorem 5.1.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with a smooth boundary, $\chi, \xi \geq 0$ and $\alpha, \beta, \gamma, \delta > 0$ as well as $\tau \in \{0, 1\}$. Let further $n_0 \in \mathcal{M}_+(\overline{\Omega})$ be an initial datum with $m := n_0(\overline{\Omega}) > 0$. If $\tau = 1$, let further $c_{a,0}, c_{r,0} \in W^{1,r}(\Omega)$ with $r \in (\frac{6}{5}, 2)$ be some additional nonnegative initial data. If*

$$\tau = 1, \quad N = 2 \quad \text{and} \quad \xi\gamma - \chi\alpha \geq 0 \text{ or} \tag{S1}$$

$$\tau = 0, \quad N = 2 \quad \text{and} \quad \xi\gamma - \chi\alpha \geq -\frac{C_{\mathrm{S2}}}{m} \text{ or} \tag{S2}$$

$$\tau = 0, \quad N = 3 \quad \text{and} \quad \xi\gamma - \chi\alpha \geq 0 \text{ as well as } n_0 \in L^\kappa(\Omega) \text{ with } \kappa \in (1, 2), \tag{S3}$$

where $C_{\mathrm{S2}} > 0$ is a constant only depending on the domain Ω, then there exist nonnegative functions $n \in C^{2,1}(\overline{\Omega} \times (0, \infty))$ and $c_a, c_r \in C^{2,\tau}(\overline{\Omega} \times (0, \infty))$ solving (AR) with boundary *conditions (ARB) classically on* $\Omega \times (0, \infty)$*. They further attain their initial data in the following fashion:*

$$n(\cdot, t) \to n_0 \qquad \text{in } \mathcal{M}_+(\overline{\Omega}), \tag{5.1.1}$$

$$c_a(\cdot, t) \to c_{a,0} \qquad \text{in } W^{1,r}(\Omega) \text{ if } \tau = 1, \tag{5.1.2}$$

$$c_r(\cdot, t) \to c_{r,0} \qquad \text{in } W^{1,r}(\Omega) \text{ if } \tau = 1 \tag{5.1.3}$$

as $t \searrow 0$*, where we interpret the functions* $n(\cdot, t)$*,* $t > 0$*, as the positive Radon measures* $n(x, t)\mathrm{d}x$ *with* $\mathrm{d}x$ *being the standard Lebesgue measure on* $\overline{\Omega}$*.*

5.2 Approach

Similar to the approach seen in previous chapters, the construction of our desired solution will be based on approximating them by a family of solutions $(n_\varepsilon, c_{a,\varepsilon}, c_{r,\varepsilon})_{\varepsilon \in (0,1)}$, for which global existence is much easier to establish. To this end, we will spend the next section approximating our initial data by smooth functions in a fashion convenient for later arguments and then prove that with such smooth initial data classical solutions to (AR) with boundary conditions (ARB) exist globally as an extension of the arguments presented in [76] and [105]. Additionally in this section, we also introduce the functions $c_\varepsilon := \xi c_{r,\varepsilon} - \chi c_{a,\varepsilon}$ for each $\varepsilon \in (0, 1)$, which allow us to transform the system (AR) with boundary conditions (ARB) to the system (AR*), because the second system will prove more convenient for some of the later arguments. The remainder of this chapter will then be devoted to deriving uniform a priori bounds for exactly these approximate solutions to facilitate a central compact embedding argument, which will serve as the source of our actual solutions, as well as to ensure that the thus constructed solutions have our desired continuity properties at $t = 0$.

Naturally any substantial a priori estimates have to necessarily decay toward $t = 0$ if they are to be uniform in the approximation parameter ε as our initial data are very irregular. Thus, the a priori estimates serving as the linchpins of all further considerations will roughly have the form

$$G(n_\varepsilon(\cdot, t), c_\varepsilon(\cdot, t)) \leq Ct^{-\lambda} \tag{5.2.1}$$

or

$$\int_0^t s^\lambda \, G(n_\varepsilon(\cdot, s), c_\varepsilon(\cdot, s)) \, \mathrm{d}s \leq C \tag{5.2.2}$$

for some $\lambda > 0$, where G is some key norm or functional and the parameter λ represents how severely the estimate decays close to $t = 0$. The derivation of these estimates will take place in Section 5.4 and, while most of the arguments afterward will handle both the parabolic-parabolic and parabolic-elliptic cases in a fairly integrated fashion, said derivation will use very different methods depending on the value of τ.

For the parabolic-elliptic case, the argument boils down to testing the first equation in (AR) with n_ε^{p-1} and then after two partial integrations directly replacing the resulting

$\Delta c_{r,\varepsilon}$ and $\Delta c_{a,\varepsilon}$ terms by lower order expressions using the elliptic second and third equations in (AR). Applying carefully chosen interpolation inequalities as well as elliptic regularity theory to this then allows us to derive a differential inequality with super-linear decay for $\int_\Omega n_\varepsilon^p$, which is sufficient to yield an estimate of the type (5.2.1) for $\int_\Omega n_\varepsilon^p$ with any $p \in (1, \infty)$.

In the parabolic-parabolic case, our approach hinges on the use of the well-known energy-type functional $\mathcal{F}_\varepsilon(t) \coloneqq \zeta \int_\Omega n_\varepsilon \ln(n_\varepsilon) + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2$ with $\zeta \coloneqq \xi\gamma - \chi\alpha$. But as this functional cannot necessarily be uniformly bounded at $t = 0$ due to our initial data potentially not having finite energy, we further decouple it from the initial data by multiplying it with t^λ, $\lambda > 0$. Using testing based methods, analysis of this functional then not only yields an estimate of type (5.2.1) for the functional itself but crucially also a bound of type (5.2.2) for the dissipative terms $\int_\Omega |\Delta c_\varepsilon|^2$ and $\zeta \int_\Omega |\nabla \sqrt{n_\varepsilon}|^2$. Notably while the first set of bounds could also be achieved by a similar super-linear decay approach as described in the previous paragraph, the latter bounds seem to be much more conveniently accessible by analyzing the aforementioned time-dampened version of the functional $\mathcal{F}_\varepsilon$. And importantly, it is in fact exactly said latter bounds that will allow us to prove our desired continuity properties at $t = 0$ in this scenario, as well as allow us to derive a set of uniform bounds for $\int_\Omega n_\varepsilon^2$ away from $t = 0$ to give us a similar starting point to the parabolic-elliptic case for the next section.

In Section 5.5, we then use the uniform bounds for $\int_\Omega n_\varepsilon^N$ away from $t = 0$, which we have at this point established in all scenarios, as the basis for a bootstrap argument taking us all the way to uniform bounds for the first solution components in $C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times [t_0, t_1])$ and uniform bounds for the second and third solution components in $C^{2+\theta,\tau+\frac{\theta}{2}}(\overline{\Omega} \times [t_0, t_1])$ with $t_1 > t_0 > 0$. We do this mostly using the variation-of-constants representation of the involved equations or corresponding elliptic regularity theory as well as fairly standard Hölder regularity theory from [61], [73] as well as [97]. Due to the compact embedding properties of Hölder spaces this immediately allows us to construct our desired solutions (n, c_a, c_r) as limits of the thus far discussed approximate ones and argue that they classically solve (AR) with boundary conditions (ARB) as the resulting strong convergence properties safely transfer any solution properties from the approximate solutions to their limits.

It thus only remains to be shown that said solutions are connected to our initial data in the fashion outlined in (5.1.1)–(5.1.3), which will be the main subject of Section 5.6. To do this for the first solution component, we essentially start by proving that the approximate solutions are uniformly continuous at $t = 0$ in the sense of (5.1.1). We accomplish this by showing that the space-time integral $\int_0^t \|n_\varepsilon(\cdot, s)\nabla c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s$ related to the chemotaxis mechanism becomes uniformly small as t goes to zero in all scenarios. That it is possible to prove such a property chiefly depends on the degradation toward zero of our estimates derived in Section 5.4 to be sufficiently benign. Notably, the magnitude of the degradation parameters naturally depends on the regularity of our initial data in the sense that better regularity leads to smaller degradation toward zero and thus the derivation of the aforementioned bound is, however indirect, the source of our initial data regularity needs in Theorem 5.1.1. We then use said property combined with the first equation in (AR) and the fundamental theorem of calculus to gain our desired uniform continuity property, which by virtue of the already established convergence properties translates immediately to our actual solutions. Using a convenient property of our initial data approximation, a similar argument built on semigroup methods grants us the properties in (5.1.2) and (5.1.3) in the parabolic-parabolic case.

5.3 Approximate solutions

From here on out, we fix the system parameters $\chi, \xi \geq 0$, $\alpha, \beta, \gamma, \delta > 0$ and $\tau \in \{0, 1\}$ as well as a domain $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, with a smooth boundary for the remainder of the chapter. We further fix some initial data $n_0 \in \mathcal{M}_+(\overline{\Omega})$ with $m := n_0(\overline{\Omega}) > 0$ as well as nonnegative $c_{a,0}, c_{r,0} \in W^{1,r}(\Omega)$ with $r \in (\frac{6}{5}, 2)$ if $\tau = 1$. Moreover if n_0 is additionally assumed to be an element of $L^\kappa(\Omega)$ for some $\kappa \in (1, 2)$, we also fix this parameter κ. Otherwise, we let κ be equal to 2 so κ is defined in all scenarios for convenience of notation in some later arguments.

To construct the solutions laid out in Theorem 5.1.1, we will use a family of approximate solutions, which will later be argued to converge to our desired solutions. This section will thus be devoted to the construction of said approximate solutions and to facilitate this we will begin by approximating our potentially highly irregular initial data by smooth functions, which will in fact be the only regularization necessary.

Thus, we now fix a family of approximate positive initial data $(n_{0,\varepsilon})_{\varepsilon \in (0,1)} \subseteq C^\infty(\overline{\Omega})$ such that

$$n_{0,\varepsilon} \to n_0 \text{ in } \mathcal{M}_+(\overline{\Omega}) \text{ as } \varepsilon \searrow 0 \quad \text{as well as} \quad \int_\Omega n_{0,\varepsilon} = n_0(\overline{\Omega}) = m \text{ for all } \varepsilon \in (0,1), \tag{5.3.1}$$

where we interpret the functions $n_{0,\varepsilon}$ as the positive Radon measures $n_{0,\varepsilon}(x)\mathrm{d}x$ with $\mathrm{d}x$ being the standard Lebesgue measure on $\overline{\Omega}$.

Remark 5.3.1. Let us give a brief argument as to how such an approximation of Radon measures can be achieved: It is fairly easy to see that approximating Dirac measures δ_x by smooth functions in this way is indeed possible given sufficient boundary regularity (e.g. by using $f_\varepsilon(y) := C(\varepsilon)e^{-\frac{1}{\varepsilon}|x-y|^2}$ with $C(\varepsilon) > 0$ being some normalization constant under the assumption that Ω satisfies e.g. a fairly weak cone property as laid out in [138, Definition 2.2]). This implies that the Dirac measures are contained in the closure (relative to the vague topology) of the set $\mathcal{F} := \{\varphi \in C^\infty(\overline{\Omega}) \mid \int_\Omega \varphi = 1, \varphi \geq 0\}$. Further, one can show that the Dirac measures are the extreme points of the convex set of probability measures $\mathcal{M}_1(\overline{\Omega}) \subseteq \mathcal{M}_+(\overline{\Omega})$, which is compact in the vague topology. This makes it accessible to the Krein–Milman theorem (cf. [98, Theorem 3.23]) implying that

$$\mathcal{M}_1(\overline{\Omega}) = \overline{\operatorname{conv}(\{\delta_x \mid x \in \overline{\Omega}\})} \subseteq \overline{\mathcal{F}} \subseteq \mathcal{M}_1(\overline{\Omega})$$

and therefore that $\mathcal{M}_1(\overline{\Omega}) = \overline{\mathcal{F}}$ (see also [7, Corollary 30.5]). As $\mathcal{M}_+(\overline{\Omega})$ is metrizable (cf. [7, Theorem 31.5]) and thus the elements of the closure of any set $A \subseteq \mathcal{M}_+(\overline{\Omega})$ can be characterized as the limits of sequences entirely contained in A, the above observations are sufficient to gain our desired approximation after a straightforward scaling argument.

If n_0 is additionally an element of $L^\kappa(\Omega)$, we let $n_{0,\varepsilon} := e^{\varepsilon\Delta} n_0 \in C^\infty(\overline{\Omega})$ for all $\varepsilon \in (0,1)$ instead, where $(e^{t\Delta})_{t\geq 0}$ is the Neumann heat semigroup on Ω. By the continuity, positivity and mass conservation properties of said semigroup, this not only directly ensures the previously prescribed properties but also that

$$n_{0,\varepsilon} \to n_0 \quad \text{ in } L^\kappa(\Omega) \text{ as } \varepsilon \searrow 0 \tag{5.3.2}$$

as well. Similarly if $\tau = 1$, we let

$$c_{a,0,\varepsilon} := e^{\varepsilon(\Delta - \beta)} c_{a,0} := e^{-\varepsilon\beta} e^{\varepsilon\Delta} c_{a,0} \in C^\infty(\overline{\Omega}) \tag{5.3.3}$$

and

$$c_{r,0,\varepsilon} := e^{\varepsilon(\Delta - \delta)} c_{r,0} := e^{-\varepsilon\delta} e^{\varepsilon\Delta} c_{r,0} \in C^\infty(\overline{\Omega}) \tag{5.3.4}$$

for all $\varepsilon \in (0,1)$. These approximate functions are nonnegative due to the maximum principle and have the following convergence property due to the continuity of the semigroup at $t = 0$:

$$c_{a,0,\varepsilon} \to c_{a,0} \quad \text{and} \quad c_{r,0,\varepsilon} \to c_{r,0} \quad \text{ in } W^{1,r}(\Omega) \text{ as } \varepsilon \searrow 0. \tag{5.3.5}$$

As a convenient by-product of this construction, we also gain that

$$\int_\Omega c_{a,0,\varepsilon} \leq \int_\Omega c_{a,0} \quad \text{and} \quad \int_\Omega c_{r,0,\varepsilon} \leq \int_\Omega c_{r,0} \tag{5.3.6}$$

again due to the mass conservation property of the Neumann heat semigroup.

As it will not only be a useful tool in arguing that our approximate solutions are in fact global, but will also play a key part in many of our later derivations of a priori estimates, we will now introduce the following transformation for classical solutions to (AR) with boundary conditions (ARB): For any such solution (n, c_a, c_r), we let

$$c(x,t) := \xi c_r(x,t) - \chi c_a(x,t) \quad \text{and} \quad \zeta := \xi\gamma - \chi\alpha \in \mathbb{R} \quad \text{as well as} \quad \sigma := \chi(\beta - \delta) \in \mathbb{R} \tag{5.3.7}$$

for all $x \in \overline{\Omega}$, $t \in [0, \infty)$. Then (n, c, c_a) solves the related system

$$\begin{cases} n_t = \Delta n + \nabla \cdot (n \nabla c) & \text{on } \Omega \times (0,\infty), \\ \tau c_t = \Delta c - \delta c + \zeta n + \sigma c_a & \text{on } \Omega \times (0,\infty), \\ \tau c_{at} = \Delta c_a + \alpha n - \beta c_a & \text{on } \Omega \times (0,\infty), \\ 0 = \nabla n \cdot \nu = \nabla c \cdot \nu = \nabla c_a \cdot \nu & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{AR*}$$

In part using this, we can now formulate the necessary existence result for our approximate solutions.

Lemma 5.3.2. *There exists $\overline{C_{S2}} > 0$ such that, if we assume (S1), (S2) with $C_{S2} \leq \overline{C_{S2}}$ or (S3), then the following holds:*

For each $\varepsilon \in (0, 1)$, there exist a positive function $n_\varepsilon \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$ and nonnegative functions $c_{a,\varepsilon}, c_{r,\varepsilon} \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$ such that $(n_\varepsilon, c_{a,\varepsilon}, c_{r,\varepsilon})$ is a classical solution to (AR) on $\Omega \times (0, \infty)$ with boundary conditions (ARB) and initial data $n_{0,\varepsilon}$ as well as initial data $c_{a,0,\varepsilon}$ and $c_{r,0,\varepsilon}$ if $\tau = 1$. Further,

$$\int_\Omega n_\varepsilon(\cdot, t) = \int_\Omega n_{0,\varepsilon} = n_0(\overline{\Omega}) =: m \tag{5.3.8}$$

as well as

$$\int_\Omega c_{a,\varepsilon}(\cdot, t) \leq \max\left(\frac{\alpha m}{\beta}, \tau \int_\Omega c_{a,0}\right) \text{ and } \int_\Omega c_{r,\varepsilon}(\cdot, t) \leq \max\left(\frac{\gamma m}{\delta}, \tau \int_\Omega c_{r,0}\right) \tag{5.3.9}$$

for all $t \in (0, \infty)$.

Proof. As the system in (AR) with boundary conditions (ARB) is the same as the one discussed in [105], we can in fact use Lemma 3.1 from said reference to ensure that, for each $\varepsilon \in (0, 1)$, nonnegative local solutions exist on some time interval $(0, T_{\max,\varepsilon})$ and, if $T_{\max,\varepsilon} < \infty$, then $\limsup_{t \nearrow T_{\max}} \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. The mass conservation property in (5.3.8) and mass boundedness properties in (5.3.9) then immediately follow by integrating the equations in (AR). Positivity of n_ε is further a direct consequence of the maximum principle.

If $\xi\gamma - \chi\alpha > 0$, finite-time blowup of the above solutions has already been ruled out in [76, Theorem 1.1] or [105, Theorem 2.1] for all of our scenarios.

If $\xi\gamma - \chi\alpha = 0$, then $\zeta = 0$ and thus the second equation in the closely related system (AR*) does not directly depend on n_ε anymore, which makes the system much less challenging. In two and three dimensions, we can thus use either elliptic regularity theory or semigroup methods to first conclude that there exist bounds for $c_{a,\varepsilon}$ in $L^p(\Omega)$ for all $p \in (1, N)$ using the mass conservation property in (5.3.8), which by the same line of reasoning gives us a bound for c_ε in $W^{1,p}(\Omega)$ for any $p \in (1, \infty)$ and c_ε defined as in (5.3.7) (cf. Lemma 5.5.1 for a similar argument). Similar to the reasoning employed in Lemma 5.5.2, another semigroup-based argument then allows us to rule out finite-time blowup in this case altogether.

If $\xi\gamma - \chi\alpha < 0$, we are necessarily in Scenario (S2). As thus $N = 2$, we can use the same elliptic regularity theory for $L^1(\Omega)$ source terms (cf. [13, Lemma 23]) we will later also use in Lemma 5.4.8 to fix a constant $K_1(p) > 0$ for each $p \in (1, \infty)$ such that $\int_\Omega {c_{r,\varepsilon}}^p \leq K_1(p)$ on $(0, T_{\max,\varepsilon})$ due to the mass conservation property in (5.3.8). Further using a straightforward consequence of the Gagliardo–Nirenberg inequality found e.g. later in Lemma 5.4.7, we then fix $K_2(p) > 0$ for each $p \in (1, \infty)$ such that

$$\int_\Omega n_\varepsilon^{p+1} \leq K_2(p) m \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_2(p) m^{p+1} \tag{5.3.10}$$

for all $\varepsilon \in (0,1)$ and $t \in (0, T_{\max,\varepsilon})$, where $K_2(p)$ only depends on p and the domain Ω. Having fixed these constants, we now let $\overline{C_{\mathrm{S2}}} \coloneqq \frac{1}{4K_2(8)}$.

We then test the first equation in (AR) with n_ε^{p-1}, use partial integration and then use the second and third equation in (AR), the estimate in (S2) with $C_{\mathrm{S2}} \le \overline{C_{\mathrm{S2}}}$ as well as Young's inequality to gain

$$\begin{aligned}
\frac{1}{p(p-1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n_\varepsilon^p &= -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\chi}{p} \int_\Omega \nabla n_\varepsilon^p \cdot \nabla c_{a,\varepsilon} - \frac{\xi}{p} \int_\Omega \nabla n_\varepsilon^p \cdot \nabla c_{r,\varepsilon} \\
&= -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 - \frac{\chi}{p} \int_\Omega n_\varepsilon^p \Delta c_{a,\varepsilon} + \frac{\xi}{p} \int_\Omega n_\varepsilon^p \Delta c_{r,\varepsilon} \\
&= -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 - \frac{\zeta}{p} \int_\Omega n_\varepsilon^{p+1} - \frac{\chi\beta}{p} \int_\Omega c_{a,\varepsilon} n_\varepsilon^p + \frac{\xi\delta}{p} \int_\Omega c_{r,\varepsilon} n_\varepsilon^p \\
&\le -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\overline{C_{\mathrm{S2}}}}{mp} \int_\Omega n_\varepsilon^{p+1} + \frac{\xi\delta}{p} \int_\Omega c_{r,\varepsilon} n_\varepsilon^p \\
&\le -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{2\overline{C_{\mathrm{S2}}}}{mp} \int_\Omega n_\varepsilon^{p+1} + K_3(p) m^p \int_\Omega {c_{r,\varepsilon}}^{p+1} \\
&\le -\frac{4}{p^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{2\overline{C_{\mathrm{S2}}}}{mp} \int_\Omega n_\varepsilon^{p+1} + K_1(p+1) K_3(p) m^p
\end{aligned} \tag{5.3.11}$$

for all $\varepsilon \in (0,1)$, $t \in (0, T_{\max,\varepsilon})$ and $p \in (2,\infty)$ with $K_3(p) \coloneqq (\frac{\xi\delta}{p})^{p+1} (\frac{\overline{C_{\mathrm{S2}}}}{p})^{-p}$. If we now apply (5.3.10) to the above and set $p = 8$, we gain

$$\frac{1}{56} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n_\varepsilon^8 \le K_4$$

for all $\varepsilon \in (0,1)$ and $t \in (0, T_{\max,\varepsilon})$ with $K_4 \coloneqq (\frac{1}{16} + K_1(9)K_3(8))m^8$ by our choice of $\overline{C_{\mathrm{S2}}}$. Thus by time integration and standard elliptic regularity theory (cf. [34]) applied to the second and third equation in (AR), it follows that there exists $K_5 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^8(\Omega)} \le K_5, \quad \|c_{a,\varepsilon}(\cdot,t)\|_{W^{2,8}(\Omega)} \le K_5 \quad \text{as well as} \quad \|c_{r,\varepsilon}(\cdot,t)\|_{W^{2,8}(\Omega)} \le K_5$$

for all $\varepsilon \in (0,1)$ and $t \in (0, T_{\max,\varepsilon})$ if $T_{\max,\varepsilon}$ is finite. With c_ε defined as in (5.3.7), this directly gives us

$$\|c_\varepsilon(\cdot,t)\|_{W^{2,8}(\Omega)} \le K_5(|\xi| + |\chi|) \eqqcolon K_6$$

for all $\varepsilon \in (0,1)$ and $t \in (0, T_{\max,\varepsilon})$ if $T_{\max,\varepsilon}$ is finite. Using the variation-of-constants representation for n_ε corresponding to the first equation in (AR*) and the smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]), it further follows that

$$\begin{aligned}
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le \left\| e^{t\Delta} n_{0,\varepsilon} + \int_0^t e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon(\cdot,s) \nabla c_\varepsilon(\cdot,s)) \,\mathrm{d}s \right\|_{L^\infty(\Omega)} \\
&\le K_6 \|n_{0,\varepsilon}\|_{L^\infty(\Omega)} + K_7 \int_0^t (1 + (t-s)^{-\frac{3}{4}}) \|n_\varepsilon(\cdot,t) \nabla c_\varepsilon(\cdot,s)\|_{L^4(\Omega)} \,\mathrm{d}s \\
&\le K_6 \|n_{0,\varepsilon}\|_{L^\infty(\Omega)} + K_5 K_6 K_7 \int_0^t (1 + s^{-\frac{3}{4}}) \,\mathrm{d}s
\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $t \in (0, T_{\max,\varepsilon})$ with some appropriate constant $K_7 > 0$ if $T_{\max,\varepsilon}$ is finite. As the remaining integral is bounded for $t \in (0, T_{\max,\varepsilon})$ if $T_{\max,\varepsilon}$ is finite, the above inequality in fact rules out finite-time blowup in this scenario as well and thus completes the proof. □

As we will henceforth always work in at least one of the three scenarios (S1), (S2) with a constant smaller than the constant $\overline{C_{\mathrm{S2}}}$ introduced in the above lemma or (S3), we will also fix the solutions constructed above as $(n_\varepsilon, c_{a,\varepsilon}, c_{r,\varepsilon})$ for all $\varepsilon \in (0,1)$ as a matter of convenience for the remainder of the chapter. We further always correspondingly define ζ, σ and c_ε as in (5.3.7).

5.4 A priori estimates degrading toward zero

As is typical for a construction of this kind, we will now spend the remainder of this chapter deriving sufficient a priori estimates to gain our desired solutions as limits of the approximate solutions fixed in the previous section. Notably as a consequence of the low initial data regularity, most of these bounds necessarily need to decay as $t \searrow 0$ if they are to be independent of ε. While we will see that the degree to which this decay happens is not important to ensure that our limit functions solve (AR) with boundary conditions (ARB) on $\Omega \times (0,\infty)$, we will in fact need more qualitative information regarding the decay to ensure that the limit functions are continuous at $t = 0$ in the sense laid out in (5.1.1), (5.1.2) and (5.1.3). Therefore, the aim of this section is to derive exactly such uniform a priori information for n_ε, $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ and c_ε.

As the parabolic-parabolic and parabolic-elliptic cases call for very different methods to establish this important baseline information, we will address them here separately.

5.4.1 The parabolic-parabolic case

Before we approach the derivation of the titular a priori estimates decaying close to time zero for the parabolic-parabolic case, we first derive the best to be expected Sobolev bounds for $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ and c_ε that hold up to time zero corresponding with the initial data regularity for $c_{a,0}$ and $c_{r,0}$ as well as the mass conservation property in (5.3.8). These bounds will later serve as a useful baseline for interpolation.

Lemma 5.4.1. *Assume we are in Scenario (S1). Then there exists $C > 0$ such that*

$$\|c_{a,\varepsilon}(\cdot,t)\|_{W^{1,r}(\Omega)} \le C, \quad \|c_{r,\varepsilon}(\cdot,t)\|_{W^{1,r}(\Omega)} \le C, \quad \|c_\varepsilon(\cdot,t)\|_{W^{1,r}(\Omega)} \le C$$

for all $t \in [0,\infty)$ and $\varepsilon \in (0,1)$ with $r \in (\frac{6}{5}, 2)$ as fixed at the beginning of Section 5.3.

Proof. Given the convergence properties in (5.3.5), we can fix $K_1 > 0$ such that

$$\|c_{a,\varepsilon}(\cdot, 0)\|_{W^{1,r}(\Omega)} \le K_1$$

for all $\varepsilon \in (0,1)$. Then using the variation-of-constants representation of the second equation in (AR) and known smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]), we can fix $K_2 > 0$ such that

$$\begin{aligned}
&\|\nabla c_{a,\varepsilon}(\cdot, t)\|_{L^r(\Omega)} \\
&\le \left\| \nabla e^{t(\Delta-\beta)} c_{a,\varepsilon}(\cdot, 0) + \alpha \int_0^t \nabla e^{(t-s)(\Delta-\beta)} n_\varepsilon(\cdot, s)\,\mathrm{d}s \right\|_{L^r(\Omega)} \\
&\le K_2 \|\nabla c_{a,\varepsilon}(\cdot, 0)\|_{L^r(\Omega)} + \alpha K_2 \int_0^t (1 + (t-s)^{\frac{1}{r}-\frac{3}{2}}) e^{-\beta(t-s)} \|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)}\,\mathrm{d}s \\
&\le K_1 K_2 + \alpha m K_2 \int_0^t (1 + s^{\frac{1}{r}-\frac{3}{2}}) e^{-\beta s}\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, \infty)$ and $\varepsilon \in (0,1)$. As due to $r \in (1,2)$ the remaining integral term is bounded independent of t, our desired bound for $c_{a,\varepsilon}$ follows from the above by combining it with the mass bound from (5.3.9) and e.g. the Poincaré inequality. By essentially the same reasoning, we obtain a corresponding bound for $c_{r,\varepsilon}$. The bound for c_ε then follows immediately as c_ε is merely a linear combination of $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$. □

Since in Scenario (S1) we work in a two-dimensional setting, the Sobolev embedding theorem directly yields the following corollary to the above, which we will use to handle the case $\sigma \ne 0$ whenever necessary.

Corollary 5.4.2. *Assume we are in Scenario (S1). Then for each $p \in [1, \frac{2r}{2-r}]$, there exists $C = C(p) > 0$ such that*

$$\|c_{a,\varepsilon}(\cdot, t)\|_{L^p(\Omega)} \le C$$

for all $t \in [0, \infty)$ and $\varepsilon \in (0,1)$.

To now gain the central result of this section, we will employ an energy-type argument based on the familiar functional $\mathcal{F}_\varepsilon(t) := \zeta \int_\Omega n_\varepsilon \ln(n_\varepsilon) + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2$ multiplied by t^λ with a sufficiently strong dampening exponent λ to make it initial-data independent while ensuring that the resulting additional terms can still be absorbed by the dissipative terms. This will not only allow us to derive a bound for said energy-type functional itself but more importantly some dampened space-time integral bounds corresponding to higher order terms of n_ε and c_ε, which will prove crucial to connect our desired solutions to their initial data. Further note that this argument centrally relies on the restriction $\zeta = \xi\gamma - \chi\alpha \ge 0$ from (S1) to ensure that the functional is always bounded from below and in fact this is the main reason this restriction is necessary for Scenario (S1).

Lemma 5.4.3. *Assume we are in Scenario (S1). Then there exists* $\lambda \in (0, \frac{2}{3})$ *such that, for each* $T > 0$*, there is* $C = C(T) > 0$ *with*

$$\zeta \int_\Omega n_\varepsilon(\cdot, t) \ln(n_\varepsilon(\cdot, t)) \le Ct^{-\lambda} \quad \text{and} \quad \int_\Omega |\nabla c_\varepsilon(\cdot, t)|^2 \le Ct^{-\lambda} \tag{5.4.1}$$

as well as

$$\zeta \int_0^t s^\lambda \int_\Omega \frac{|\nabla n_\varepsilon(x, s)|^2}{n_\varepsilon(x, s)} \,\mathrm{d}x \,\mathrm{d}s \le C \quad \text{and} \quad \int_0^t s^\lambda \int_\Omega |\Delta c_\varepsilon(x, s)|^2 \,\mathrm{d}x \,\mathrm{d}s \le C \tag{5.4.2}$$

for all $t \in (0, T)$ *and* $\varepsilon \in (0, 1)$*.*

Proof. We begin by fixing $\lambda \in (0, \frac{2}{3})$ such that

$$\lambda - \frac{2}{r} > -1,$$

which is possible as $r > \frac{6}{5}$.

We now test the first equation in (AR*) with $t^\lambda \ln(n_\varepsilon)$ and use partial integration to gain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\lambda \int_\Omega n_\varepsilon \ln(n_\varepsilon)\right] = -t^\lambda \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} - t^\lambda \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \lambda t^{\lambda-1} \int_\Omega n_\varepsilon \ln(n_\varepsilon) \tag{5.4.3}$$

for all $t \in (0, \infty)$ and $\varepsilon \in (0, 1)$. We then fix $p \in (1, 2)$ such that

$$\frac{1-p}{2-p} + \lambda > 0$$

and employ the Gagliardo–Nirenberg inequality as well as Young's inequality combined with the mass conservation property in (5.3.8) to find $K_1 > 0$ such that

$$\begin{aligned} \int_\Omega n_\varepsilon \ln(n_\varepsilon) &\le \frac{1}{e(p-1)} \int_\Omega n_\varepsilon^p = \frac{1}{e(p-1)} \|n_\varepsilon^{\frac{1}{2}}\|_{L^{2p}(\Omega)}^{2p} \\ &\le K_1 \|\nabla n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}^{2p-2} + K_1 = \frac{K_1}{2^{2p-2}} \left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\right)^{p-1} + K_1 \\ &\le \frac{t}{2\lambda} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + K_2 t^{\frac{1-p}{2-p}} + K_1 \end{aligned}$$

for all $t \in (0, \infty)$ and $\varepsilon \in (0, 1)$ with $K_2 := 2^{\frac{1-p}{2-p}} \lambda^{\frac{p-1}{2-p}} K_1^{\frac{1}{2-p}}$. We now apply this to (5.4.3) to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\lambda \int_\Omega n_\varepsilon \ln(n_\varepsilon)\right] \le -\frac{t^\lambda}{2} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} - t^\lambda \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \lambda K_2 t^{\frac{1-p}{2-p}+\lambda-1} + \lambda K_1 t^{\lambda-1} \tag{5.4.4}$$

for all $t \in (0, \infty)$ and $\varepsilon \in (0, 1)$.

As our next step, we test the second equation in (AR*) with $-t^\lambda \Delta c_\varepsilon$ and use partial integration to gain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^\lambda}{2}\int_\Omega |\nabla c_\varepsilon|^2\right] \\ &= -t^\lambda \int_\Omega |\Delta c_\varepsilon|^2 - \delta t^\lambda \int_\Omega |\nabla c_\varepsilon|^2 + \zeta t^\lambda \int_\Omega \nabla c_\varepsilon \cdot \nabla n_\varepsilon \\ &\quad - \sigma t^\lambda \int_\Omega c_{a,\varepsilon} \Delta c_\varepsilon + \tfrac{\lambda}{2} t^{\lambda-1} \int_\Omega |\nabla c_\varepsilon|^2 \end{aligned} \tag{5.4.5}$$

for all $t \in (0,\infty)$ and $\varepsilon \in (0,1)$. Another application of the Gagliardo–Nirenberg inequality as well as Young's inequality combined with elliptic regularity theory and the baseline bound in $W^{1,r}(\Omega)$ established in Lemma 5.4.1 further yields $K_3 > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla c_\varepsilon|^2 = \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} \le K_3 \|\Delta c_\varepsilon\|^{2-r}_{L^2(\Omega)} + K_3 = K_3\left(\int_\Omega |\Delta c_\varepsilon|^2\right)^{1-\frac{r}{2}} + K_3 \\ \le \frac{t}{\lambda}\int_\Omega |\Delta c_\varepsilon|^2 + K_4 t^{1-\frac{2}{r}} + K_3 \end{aligned}$$

for all $t \in (0,\infty)$ and $\varepsilon \in (0,1)$ with $K_4 := K_3^{\frac{2}{r}} \lambda^{\frac{2}{r}-1}$. If we now apply this as well as Young's inequality and Corollary 5.4.2 to (5.4.5), we see that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^\lambda}{2}\int_\Omega |\nabla c_\varepsilon|^2\right] \\ &\le -\frac{t^\lambda}{2}\int_\Omega |\Delta c_\varepsilon|^2 + \zeta t^\lambda \int_\Omega \nabla c_\varepsilon \cdot \nabla n_\varepsilon - \sigma t^\lambda \int_\Omega c_{a,\varepsilon}\Delta c_\varepsilon + \frac{\lambda K_4}{2} t^{\lambda - \frac{2}{r}} + \frac{\lambda K_3}{2} t^{\lambda-1} \\ &\le -\frac{t^\lambda}{4}\int_\Omega |\Delta c_\varepsilon|^2 + \zeta t^\lambda \int_\Omega \nabla c_\varepsilon \cdot \nabla n_\varepsilon + \sigma^2 K_5^2 t^\lambda + \frac{\lambda K_4}{2} t^{\lambda - \frac{2}{r}} + \frac{\lambda K_3}{2} t^{\lambda-1} \end{aligned}$$

for all $t \in (0,\infty)$ and $\varepsilon \in (0,1)$ with $K_5 > 0$ as provided by Corollary 5.4.2 because $\frac{2r}{2-r} \ge 2$. Combining this with an appropriately scaled (5.4.4) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\zeta t^\lambda \int_\Omega n_\varepsilon \ln(n_\varepsilon) + \frac{t^\lambda}{2}\int_\Omega |\nabla c_\varepsilon|^2\right] + \zeta \frac{t^\lambda}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{t^\lambda}{4}\int_\Omega |\Delta c_\varepsilon|^2 \\ &\le K_6\left[t^{\frac{1-p}{2-p}+\lambda-1} + t^{\lambda-\frac{2}{r}} + t^{\lambda-1} + t^\lambda\right] \end{aligned}$$

for all $t \in (0,\infty)$ and $\varepsilon \in (0,1)$ with $K_6 := \max(\zeta\lambda K_2, \frac{\lambda K_4}{2}, \zeta\lambda K_1 + \frac{\lambda K_3}{2}, \sigma^2 K_5^2)$. Given that our choices of p and λ ensure that the exponents on the right side of the above inequality are all larger than negative one and $\zeta \ge 0$, our desired result follows immediately by time integration. □

The above result now leads directly into a straightforward but important corollary.

Corollary 5.4.4. *Assume we are in Scenario (S1). Then for each $T > 0$, there exists $C = C(T) > 0$ such that*

$$\int_0^t s^{2\lambda} \int_\Omega |\nabla c_\varepsilon(x,s)|^4 \,\mathrm{d}x \,\mathrm{d}s \le C$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ with λ as in Lemma 5.4.3.

Proof. Using the Gagliardo–Nirenberg inequality, we see that

$$\begin{aligned} t^{2\lambda} \int_\Omega |\nabla c_\varepsilon|^4 = t^{2\lambda} \|\nabla c_\varepsilon\|_{L^4(\Omega)}^4 &\le K t^{2\lambda} \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + K t^{2\lambda} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^4 \\ &= K \left(t^\lambda \int_\Omega |\Delta c_\varepsilon|^2 \right) \left(t^\lambda \int_\Omega |\nabla c_\varepsilon|^2 \right) + K \left(t^\lambda \int_\Omega |\nabla c_\varepsilon|^2 \right)^2 \end{aligned}$$

for all $\varepsilon \in (0,1)$ and $t \in (0,\infty)$ with some appropriate $K > 0$. Given the bounds already established in Lemma 5.4.3, this immediately gives us our desired result. □

While we have now already established all the necessary estimates to ensure continuity of our desired solutions at time $t = 0$, we will now derive a further $L^2(\Omega)$ estimate for n_ε away from $t = 0$ to facilitate the uniform handling of both the elliptic and parabolic cases in the next section, which is devoted to the derivation of regularity away from $t = 0$. This is achieved by adapting a standard testing argument for parabolic-parabolic repulsion systems in a similar manner to the above energy-based arguments.

Lemma 5.4.5. *Assume we are in Scenario (S1). Then for each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exists $C = C(t_1, t_0) > 0$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C$$

for all $t \in (t_0, t_1)$ and $\varepsilon \in (0,1)$.

Proof. We begin by fixing $t_1 > t_0 > 0$ and then use Corollary 5.4.4 to further fix $K_1 = K_1(t_0, t_1) > 0$ such that

$$\int_{\frac{t_0}{2}}^{t_1} \int_\Omega |\nabla c_\varepsilon|^4 \le K_1$$

for all $\varepsilon \in (0,1)$.

We now test the first equation in (AR*) with $(t - \frac{t_0}{2})^2 n_\varepsilon$ to gain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{(t-\frac{t_0}{2})^2}{2} \int_\Omega n_\varepsilon^2 \right] \\ &= -(t-\tfrac{t_0}{2})^2 \int_\Omega |\nabla n_\varepsilon|^2 - (t-\tfrac{t_0}{2})^2 \int_\Omega n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon + (t-\tfrac{t_0}{2}) \int_\Omega n_\varepsilon^2 \end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{(t-\frac{t_0}{2})^2}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{(t-\frac{t_0}{2})^2}{2}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^2 + (t-\tfrac{t_0}{2})\int_\Omega n_\varepsilon^2 \\
&\leq -\frac{(t-\frac{t_0}{2})^2}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{(t-\frac{t_0}{2})^2}{2}\left(\int_\Omega n_\varepsilon^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{1}{2}} + (t-\tfrac{t_0}{2})\int_\Omega n_\varepsilon^2 \qquad (5.4.6)
\end{aligned}
$$

for all $t \in (\frac{t_0}{2}, t_1)$ and $\varepsilon \in (0,1)$ by applying the Young and Hölder inequalities.

Using the Gagliardo–Nirenberg inequality combined with the mass conservation property in (5.3.8), we now fix $K_2 > 1$ such that

$$
\begin{aligned}
\int_\Omega n_\varepsilon^2 = \|n_\varepsilon\|_{L^2(\Omega)}^2 &\leq K_2\|\nabla n_\varepsilon\|_{L^2(\Omega)} + K_2 = K_2\sqrt{\int_\Omega |\nabla n_\varepsilon|^2} + K_2 \\
&\leq \frac{(t-\frac{t_0}{2})}{4}\int_\Omega |\nabla n_\varepsilon|^2 + (t-\tfrac{t_0}{2})^{-1}K_2^2 + K_2
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega n_\varepsilon^4 &= \|n_\varepsilon\|_{L^4(\Omega)}^4 \\
&\leq K_2^2\|\nabla n_\varepsilon\|_{L^2(\Omega)}^2\|n_\varepsilon\|_{L^2(\Omega)}^2 + K_2^2 \\
&= K_2^2\left(\int_\Omega |\nabla n_\varepsilon|^2\right)\left(\int_\Omega n_\varepsilon^2\right) + K_2^2
\end{aligned}
$$

for all $t \in (\frac{t_0}{2}, t_1)$ and $\varepsilon \in (0,1)$. Applying these inequalities to (5.4.6) then yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(t-\frac{t_0}{2})^2}{2}\int_\Omega n_\varepsilon^2\right] \\
&\leq -\frac{(t-\frac{t_0}{2})^2}{4}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{K_2(t-\frac{t_0}{2})^2}{2}\left(\int_\Omega |\nabla n_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega n_\varepsilon^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{1}{2}} \\
&\quad + \frac{K_2(t-\frac{t_0}{2})^2}{2}\left(\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{1}{2}} + K_3 \\
&\leq K_4(t-\tfrac{t_0}{2})^2\left(\int_\Omega |\nabla c_\varepsilon|^4\right)\left(\int_\Omega n_\varepsilon^2\right) + K_4\left(\int_\Omega |\nabla c_\varepsilon|^4\right) + K_4 \\
&= K_4\left(\int_\Omega |\nabla c_\varepsilon|^4\right)\left(1 + (t-\tfrac{t_0}{2})^2\int_\Omega n_\varepsilon^2\right) + K_4
\end{aligned}
$$

for all $t \in (\frac{t_0}{2}, t_1)$ and $\varepsilon \in (0,1)$ with $K_3 = K_3(t_0,t_1) := K_2^2 + (t_1 - \frac{t_0}{2})K_2$ and $K_4 = K_4(t_0,t_1) := \max(\frac{K_2^2}{4}, K_3 + (t_1 - \frac{t_0}{2})^4)$. Setting $y_\varepsilon(t) := 1 + (t-\frac{t_0}{2})^2\int_\Omega n_\varepsilon^2(\cdot,t)$ for all $t \in [\frac{t_0}{2}, t_1)$, the above implies that the function y_ε satisfies

$$
y_\varepsilon'(t) \leq 2K_4\left(1 + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^4\right)y_\varepsilon(t)
$$

for all $t \in (\frac{t_0}{2}, t_1)$ and $\varepsilon \in (0,1)$ as well as $y_\varepsilon(\frac{t_0}{2}) = 1$ for all $\varepsilon \in (0,1)$. By a standard comparison with the explicit solution to the differential equality corresponding to the above inequality, this then yields

$$\begin{aligned}(t-\tfrac{t_0}{2})^2\int_\Omega n_\varepsilon^2(\cdot,t) &\le 1+(t-\tfrac{t_0}{2})^2\int_\Omega n_\varepsilon^2(\cdot,t) = y_\varepsilon(t)\\ &\le \exp\left(2K_4(t-\tfrac{t_0}{2})+2K_4\int_{\frac{t_0}{2}}^t\int_\Omega|\nabla c_\varepsilon(\cdot,s)|^4\,\mathrm{d}s\right)\\ &\le \exp\left(2K_4(t_1-\tfrac{t_0}{2})+2K_1K_4\right) =: K_5\end{aligned}$$

for all $t\in(\frac{t_0}{2},t_1)$ and $\varepsilon\in(0,1)$. Thus

$$\int_\Omega n_\varepsilon^2(\cdot,t)\le\frac{4K_5}{t_0^2}$$

for all $t\in(t_0,t_1)$ and $\varepsilon\in(0,1)$, which is sufficient to complete the proof. □

5.4.2 The parabolic-elliptic case

We now turn our attention to the parabolic-elliptic case. As our methods here will not be based on the energy inequality used in the previous section and we also do not need any dissipative information of type seen in e.g. (5.4.2) to facilitate later arguments, our approach to gain initial data independent estimate for higher order norms will differ from the previous section somewhat. In fact instead of multiplying the norms we are interested in by a dampening term of the form t^λ as before, we will instead derive a superlinearly decaying ODI for said norms. More specifically we will show that $L^p(\Omega)$ norms for n_ε are subsolutions of a system of the form

$$\begin{cases} y'(t) = -Ay^\alpha(t)+B, & t>0,\\ y(0)=y_0. \end{cases} \tag{5.4.7}$$

Using a sequence of straightforward comparison arguments, it is then easy to see that such subsolutions satisfy an estimate independent of the initial data but decaying toward $t=0$ similar to those of the previous section.

Lemma 5.4.6. *For each $A>0, B\ge 0$ and $\alpha>1$, there exists $C=C(A,B,\alpha)>0$ such that the following holds: The solution $y\in C^0([0,\infty))\cap C^1((0,\infty))$ of (5.4.7) with initial data $y_0\ge 0$ has the property*

$$y(t)\le Ct^{\frac{1}{1-\alpha}}+C \qquad \textit{for all } t>0.$$

Proof. We fix $A>0, B\ge 0, \alpha>1$ and initial data $y_0\ge 0$.

The Picard–Lindelöf theorem immediately ensures that a solution y to (5.4.7) exists locally and is unique. Global existence and nonnegativity of y then follow by comparing with a sufficiently large constant or zero.

If $y_0 \le (B/A)^{\frac{1}{\alpha}}$, then by comparison with a constant function of value $(B/A)^{\frac{1}{\alpha}}$, we immediately know that

$$y(t) \le \left(\tfrac{B}{A}\right)^{\frac{1}{\alpha}} \qquad \text{for all } t > 0. \tag{5.4.8}$$

As this is already sufficient for our desired result for any such small y_0, we will now focus on the remaining case of $y_0 > (B/A)^{\frac{1}{\alpha}}$. In this case, we immediately gain

$$y(t) > \left(\tfrac{B}{A}\right)^{\frac{1}{\alpha}} \qquad \text{for all } t > 0$$

by essentially the same comparison argument. Given this, we define $z(t) \coloneqq y(t) - (B/A)^{\frac{1}{\alpha}} > 0$ for all $t \ge 0$ and then note that

$$\begin{aligned} z'(t) = y'(t) &= -Ay^\alpha(t) + B \\ &= -A\left(y(t) - \left(\tfrac{B}{A}\right)^{\frac{1}{\alpha}} + \left(\tfrac{B}{A}\right)^{\frac{1}{\alpha}}\right)^\alpha + B \\ &\le -Az^\alpha(t) \qquad \text{for all } t > 0. \end{aligned}$$

By now comparing z with the explicit solution to the initial value problem $w' = -Aw^\alpha$, $w(0) = z(0)$, we can conclude that

$$z(t) \le \left(A(\alpha-1)t + z^{1-\alpha}(0)\right)^{\frac{1}{1-\alpha}} \le (A(\alpha-1))^{\frac{1}{1-\alpha}}\, t^{\frac{1}{1-\alpha}} \qquad \text{for all } t > 0$$

and therefore that

$$y(t) \le (A(\alpha-1))^{\frac{1}{1-\alpha}}\, t^{\frac{1}{1-\alpha}} + \left(\tfrac{B}{A}\right)^{\frac{1}{\alpha}} \qquad \text{for all } t > 0. \tag{5.4.9}$$

As we have now covered all necessary cases, combining (5.4.8) and (5.4.9) completes the proof with

$$C \coloneqq \max\left((A(\alpha-1))^{\frac{1}{1-\alpha}},\, (B/A)^{\frac{1}{\alpha}} \right).$$ □

As a second prerequisites for our later derivations, we will now further prove some straightforward consequences of the Gagliardo–Nirenberg inequality.

Lemma 5.4.7. *For each $p \in (1,\infty)$, there exists $C = C(p) > 0$ such that*

$$\int_\Omega n_\varepsilon^{p+\frac{2}{N}} \le Cm^{\frac{2}{N}} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + Cm^{p+\frac{2}{N}} \tag{5.4.10}$$

and

$$\left(\int_\Omega n_\varepsilon^p\right)^{1+\frac{2}{N(p-1)}} \le Cm^{\frac{2p}{N(p-1)}} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + Cm^{p+\frac{2p}{N(p-1)}} \tag{5.4.11}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ with m as defined in (5.3.8).

Proof. Fix $p \in (1, \infty)$. As then

$$\frac{1}{2+\frac{4}{Np}} + \frac{1}{N} \geq \frac{1}{2+\frac{4}{N}} + \frac{1}{N} = \frac{2+\frac{4}{N}+N}{2N+4} \geq \frac{2+N}{2N+4} = \frac{1}{2} \quad \text{and} \quad \frac{2}{p} \leq 2 \leq 2 + \frac{4}{Np},$$

we can use the Gagliardo–Nirenberg inequality, or rather a variant of it from [70, Lemma 2.3], which allows for some of the parameters to be from the interval $(0,1)$ in a way not covered by the original inequality, to gain $K_1 > 0$ such that

$$\|n_\varepsilon^{\frac{p}{2}}\|_{L^{2+\frac{4}{Np}}(\Omega)} \leq K_1 \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\alpha} \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-\alpha} + K_1 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ with

$$\alpha = \frac{\frac{p}{2} - \frac{1}{2+\frac{4}{Np}}}{\frac{p}{2} + \frac{1}{N} - \frac{1}{2}} = \frac{\frac{p+\frac{2}{N}-1}{2+\frac{4}{Np}}}{\frac{1}{2}(p+\frac{2}{N}-1)} = \frac{2}{2+\frac{4}{Np}}.$$

This then implies that

$$\begin{aligned}\int_\Omega n_\varepsilon^{p+\frac{2}{N}} = \|n_\varepsilon^{\frac{p}{2}}\|_{L^{2+\frac{4}{Np}}(\Omega)}^{2+\frac{4}{Np}} &\leq K_2 \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{Np}} + K_2 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2+\frac{4}{Np}} \\ &= K_2 m^{\frac{2}{N}} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_2 m^{p+\frac{2}{N}} \end{aligned} \tag{5.4.12}$$

with $K_2 := (2K_1)^{2+\frac{4}{Np}}$ due to the mass conservation property seen in Lemma 5.3.2.

Because moreover

$$\frac{1}{N} + \frac{1}{2} \geq \frac{1}{2} \quad \text{and} \quad \frac{2}{p} \leq 2,$$

the same Gagliardo–Nirenberg type inequality from [70] is again applicable and gives us $K_3 > 0$ such that

$$\|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)} \leq K_3 \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\beta} \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-\beta} + K_3 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ with

$$\beta = \frac{\frac{p}{2} - \frac{1}{2}}{\frac{p}{2} + \frac{1}{N} - \frac{1}{2}} = \frac{1}{1 + \frac{2}{N(p-1)}}.$$

This implies that

$$\begin{aligned}\left(\int_\Omega n_\varepsilon^p\right)^{1+\frac{2}{N(p-1)}} &= \|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\left(1+\frac{2}{N(p-1)}\right)} \\ &\leq K_4 \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{N(p-1)}} + K_4 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2\left(1+\frac{2}{N(p-1)}\right)}\end{aligned}$$

$$= K_4 m^{\frac{2p}{N(p-1)}} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_4 m^{p+\frac{2p}{N(p-1)}} \tag{5.4.13}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ with $K_4 \coloneqq (2K_3)^{2\left(1+\frac{2}{N(p-1)}\right)}$ due to the mass conservation property seen in Lemma 5.3.2.

Combining (5.4.12) and (5.4.13) then completes the proof. □

Having now established the above prerequisites, our first step proper toward the goals of this section entails the derivation of bounds for $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ and c_ε up to $t = 0$ as a baseline for future interpolation. This time our argument is naturally based on elliptic regularity theory as opposed to its parabolic counterpart in the previous section.

Lemma 5.4.8. *Assume we are in Scenario (S2) with $C_{\mathrm{S2}} \le \overline{C_{\mathrm{S2}}}$ and $\overline{C_{\mathrm{S2}}}$ as introduced in Lemma 5.3.2 or in Scenario (S3). Then for each $p \in (1, \frac{N}{N-1})$, there exists $C_1 = C_1(p) > 0$ such that*

$$\|c_{a,\varepsilon}(\cdot,t)\|_{W^{1,p}(\Omega)} \le C_1, \quad \|c_{r,\varepsilon}(\cdot,t)\|_{W^{1,p}(\Omega)} \le C_1, \quad \|c_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} \le C_1$$

and, for each $q \in (1, \frac{N}{N-2})$, there exists $C_2 = C_2(q) > 0$ such that

$$\|c_{a,\varepsilon}(\cdot,t)\|_{L^q(\Omega)} \le C_2, \quad \|c_{r,\varepsilon}(\cdot,t)\|_{L^q(\Omega)} \le C_2, \quad \|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C_2$$

for all $t \in [0,\infty)$ and $\varepsilon \in (0,1)$.

Proof. This is a direct consequence of the mass conservation and boundedness properties in (5.3.8) and (5.3.9) combined with the elliptic regularity theory for $L^1(\Omega)$ source terms from [13, Lemma 23] applied to the second and third equation in (AR) and the embedding properties of Sobolev spaces as well as the fact that c_ε is merely a linear combination of $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$. □

Given this baseline, we now turn our attention to the main argument of this section, which is naturally where most of the assumptions in (S2) and (S3) will become important. Notably, it is here where the final value of C_{S2} is being fixed as a refinement of our choice of $\overline{C_{\mathrm{S2}}}$ in Lemma 5.3.2.

Lemma 5.4.9. *There exists a constant $C_{\mathrm{S2}} > 0$ smaller than the constant $\overline{C_{\mathrm{S2}}}$ introduced in Lemma 5.3.2 such that, if we are in Scenario (S2) with said constant C_{S2} or in Scenario (S3), the following holds:*

For each $T > 0$, there exists $C = C(T) > 0$ such that

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C t^{-\frac{N(p-1)}{2}}$$

for $p \in \{N, \frac{5}{2}, \frac{3\kappa}{4\kappa-3}\}$ and, if $n_0 \in L^\kappa(\Omega)$,

$$\|n_\varepsilon(\cdot,t)\|_{L^\kappa(\Omega)} \le C$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$.

Proof. To make sure that our choice of C_{S2} is sound, we will now fix C_{S2} before starting our argument proper. To this end, we fix $K_1 > 0$ such that

$$\int_\Omega n_\varepsilon^{p+\frac{2}{N}} \le K_1 m^{\frac{2}{N}} \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_1 m^{p+\frac{2}{N}} \tag{5.4.14}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$ using Lemma 5.4.7. We then choose $C_{\mathrm{S2}} > 0$ such that $C_{\mathrm{S2}} \le \frac{2}{pK_1}$ for all $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$ and such that it is smaller than the constant $\overline{C_{\mathrm{S2}}}$ introduced in Lemma 5.3.2.

Similar to some of the arguments presented in Lemma 5.3.2 and due to the fact that both of our considered scenarios are of parabolic-elliptic type, we begin by testing the first equation in (AR) with n_ε^{p-1} and then use the second and third equation in (AR) to gain that

$$\begin{aligned}\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p &= -\frac{4}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 - \frac{\chi}{p}\int_\Omega n_\varepsilon^p \Delta c_{a,\varepsilon} + \frac{\xi}{p}\int_\Omega n_\varepsilon^p \Delta c_{r,\varepsilon} \\ &= -\frac{4}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 - \frac{\zeta}{p}\int_\Omega n_\varepsilon^{p+1} - \frac{\chi\beta}{p}\int_\Omega c_{a,\varepsilon} n_\varepsilon^p + \frac{\xi\delta}{p}\int_\Omega c_{r,\varepsilon} n_\varepsilon^p \\ &\le -\frac{4}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 - \frac{\zeta}{p}\int_\Omega n_\varepsilon^{p+1} + \frac{\xi\delta}{p}\int_\Omega c_{r,\varepsilon} n_\varepsilon^p \end{aligned} \tag{5.4.15}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in (1,\infty)$. If we are in Scenario (S3), $\zeta \ge 0$ and thus

$$-\frac{\zeta}{p}\int_\Omega n_\varepsilon^{p+1} \le 0$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in (1,\infty)$. If we are in Scenario (S2), we can apply (5.4.14) to see that

$$-\frac{\zeta}{p}\int_\Omega n_\varepsilon^{p+1} \le \frac{C_{\mathrm{S2}}}{mp}\int_\Omega n_\varepsilon^{p+1} \le \frac{2}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{2m^p}{p^2}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$ by our previous choice of C_{S2} and the fact that in this scenario $N = 2$. Thus in both scenarios, we gain

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \le -\frac{2}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\xi\delta}{p}\int_\Omega c_{r,\varepsilon} n_\varepsilon^p + \frac{2m^p}{p^2} \tag{5.4.16}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$ from (5.4.15).

By employing Young's inequality again combined with (5.4.14), we further see that

$$\begin{aligned}\frac{\xi\delta}{p}\int_\Omega c_{r,\varepsilon} n_\varepsilon^p &\le K_2 \int_\Omega {c_{r,\varepsilon}}^{1+\frac{Np}{2}} + \frac{1}{p^2 m^{\frac{2}{N}} K_1}\int_\Omega n_\varepsilon^{p+\frac{2}{N}} \\ &\le K_2 \int_\Omega {c_{r,\varepsilon}}^{1+\frac{Np}{2}} + \frac{1}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{m^p}{p^2}\end{aligned}$$

with an appropriate $K_2 > 0$, which applied to (5.4.16) yields

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \leq -\frac{1}{p^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_2\int_\Omega {c_{r,\varepsilon}}^{1+\frac{Np}{2}} + \frac{3m^p}{p^2}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$. By applying the straightforward consequence of the Gagliardo–Nirenberg inequality found in Lemma 5.4.7 to the above, we can then find $K_3 > 0$ such that

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \leq -\frac{1}{K_3}\left(\int_\Omega n_\varepsilon^p\right)^{1+\frac{2}{N(p-1)}} + K_2\int_\Omega {c_{r,\varepsilon}}^{1+\frac{Np}{2}} + K_3 \tag{5.4.17}$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$. Yet another application of the Gagliardo–Nirenberg inequality in a fashion very similar to Lemma 5.4.7 then further yields $K_4 > 0$ such that

$$\|c_{r,\varepsilon}\|_{L^{1+\frac{Np}{2}}(\Omega)} \leq K_4\|c_{r,\varepsilon}\|_{W^{2,p}}^{\theta}\|c_{r,\varepsilon}\|_{L^q(\Omega)}^{1-\theta}$$

with $q = q(p) := (N(N-1) + 2\frac{N}{p})(2+\frac{N}{p})^{-1} \in (1, \frac{N}{N-2})$ and $\theta = \theta(p) := \frac{p}{2}(1+\frac{Np}{2})^{-1} \in (0,1)$ for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$. Combining this with elliptic regularity theory and the baseline established in Lemma 5.4.8 further yields $K_5, K_6 > 0$ such that

$$\int_\Omega {c_{r,\varepsilon}}^{1+\frac{Np}{2}} \leq K_5\|n_\varepsilon\|_{L^p(\Omega)}^{\frac{p}{2}} = K_5\left(\int_\Omega n_\varepsilon^p\right)^{\frac{1}{2}} \leq \frac{1}{2K_2K_3}\left(\int_\Omega n_\varepsilon^p\right)^{1+\frac{2}{N(p-1)}} + K_6$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$. Applying this to (5.4.17) gives us that

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \leq -\frac{1}{2K_3}\left(\int_\Omega n_\varepsilon^p\right)^{1+\frac{2}{N(p-1)}} + K_3 + K_2K_6$$

for all $t \in (0,\infty)$, $\varepsilon \in (0,1)$ and $p \in \{N, \frac{5}{2}, \kappa, \frac{3\kappa}{4\kappa-3}\}$. From this, our desired results follow directly from either Lemma 5.4.6, which provides us with an initial data independent bound decaying towards zero for ODIs with superlinear decay, or by a straightforward time integration if $n_0 \in L^\kappa(\Omega)$ as this implies that the family $(n_{0,\varepsilon})_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^\kappa(\Omega)$ due to (5.3.2). □

5.5 Estimates away from $t = 0$ and the construction of our solution candidates

Our aim for this section is to use the uniform bound for n_ε in $L^N(\Omega)$ and the uniform bound for ∇c_ε in $L^1(\Omega)$ away from $t = 0$, which we established in the previous section for both parabolic-parabolic as well as parabolic-elliptic scenarios, as basis for a bootstrap argument to derive sufficient Hölder bounds for n_ε, $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$ and thus construct a candidate for our desired solution via limit process. Notably, we will strive to prove sufficiently strong bounds such that said limit solutions immediately inherit the classical solution properties from the approximate solutions. To do this, we will generally focus on the more challenging parabolic cases and only give references for how similar results can be achieved in the elliptic cases where applicable.

We begin this process by first establishing bounds for $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ and c_ε in all $W^{1,p}(\Omega)$ with finite p by either semigroup methods or elliptic regularity theory. Following this, we then use said bounds for a semigroup-based argument to derive a uniform bound for n_ε in $L^\infty(\Omega)$.

Lemma 5.5.1. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then for each $p \in (1,\infty)$ and $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exists $C = C(p, t_0, t_1) > 0$ such that*

$$\|c_{a,\varepsilon}(\cdot,t)\|_{W^{1,p}(\Omega)} \leq C, \quad \|c_{r,\varepsilon}(\cdot,t)\|_{W^{1,p}(\Omega)} \leq C \quad \textit{and} \quad \|c_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} \leq C$$

for all $t \in (t_0, t_1)$.

Proof. Fix $t_1 > t_0 > 0$ and $p \in (1,\infty)$.

Then due to either Lemma 5.4.1 and Lemma 5.4.5 or Lemma 5.4.8 and Lemma 5.4.9 depending on the scenario, there exists $K_1 = K_1(t_0,t_1) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^N(\Omega)} \leq K_1 \quad \text{and} \quad \|\nabla c_{a,\varepsilon}(\cdot,t)\|_{L^1(\Omega)} \leq K_1$$

for all $t \in [\frac{t_0}{2}, t_1)$.

If $\tau = 1$, we can use the variation-of-constants representation of the second equation in (AR) and well-known smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]) to gain a constant $K_2 > 0$ such that

$$\begin{aligned}
\|\nabla c_{a,\varepsilon}(\cdot,t)\|_{L^p(\Omega)} &= \left\| \nabla e^{\left(t-\frac{t_0}{2}\right)(\Delta-\beta)} c_{a,\varepsilon}(\cdot,\tfrac{t_0}{2}) + \alpha \int_{t_0/2}^t \nabla e^{(t-s)(\Delta-\beta)} n_\varepsilon(\cdot,s)\,\mathrm{d}s \right\|_{L^p(\Omega)} \\
&\leq K_1K_2\left(1 + (t-\tfrac{t_0}{2})^{\frac{N}{2p}-\frac{N}{2}}\right) + \alpha K_1K_2 \int_{t_0/2}^t \left(1 + (t-s)^{\frac{N}{2p}-1}\right)\mathrm{d}s \\
&\leq K_1K_2\left(1 + (\tfrac{t_0}{2})^{\frac{N}{2p}-\frac{N}{2}}\right) + \alpha K_1K_2(t_1 - \tfrac{t_0}{2}) + \alpha K_1K_2\tfrac{2p}{N}(t_1-\tfrac{t_0}{2})^{\frac{N}{2p}}
\end{aligned}$$

for all $t \in (t_0,t_1)$ and $\varepsilon \in (0,1)$, which gives us the first of our desired bounds when combined with the mass boundedness property in (5.3.9) and the Poincaré inequality. Essentially the same argument can be applied to $c_{r,\varepsilon}$ to achieve the second bound. As c_ε is merely a linear combination of $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$, the bound for c_ε then follows immediately and completes the proof.

If $\tau = 0$, the same bounds follow directly from elliptic regularity theory (cf. [34]) combined with the Sobolev embedding theorem. □

Lemma 5.5.2. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then for each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exists $C = C(t_0, t_1) > 0$ such that*

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C$$

for all $t \in (t_0, t_1)$ and $\varepsilon \in (0, 1)$.

Proof. Fix $t_1 > t_0 > 0$.

Then according to Lemma 5.4.5 or Lemma 5.4.9 as well as Lemma 5.5.1, there exists $K_1 = K_1(t_0, t_1) > 0$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^N(\Omega)} \le K_1 \quad \text{and} \quad \|\nabla c_\varepsilon(\cdot, t)\|_{L^{4N}(\Omega)} \le K_1$$

for all $t \in [\frac{t_0}{2}, t_1)$. We then further define

$$M_\varepsilon \coloneqq \sup_{t \in (t_0/2, t_1)} \|(t - \tfrac{t_0}{2})^{\frac{1}{2}} n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} < \infty.$$

Using the variation-of-constants representation for the first equation in (AR*) combined with the smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]) yields a constant $K_2 > 0$ such that

$$\begin{aligned}
&\left\|(t - \tfrac{t_0}{2})^{\frac{1}{2}} n_\varepsilon(\cdot, t)\right\|_{L^\infty(\Omega)} \\
&= \left\| (t - \tfrac{t_0}{2})^{\frac{1}{2}} \left[e^{\left(t - \frac{t_0}{2}\right)\Delta} n_\varepsilon(\cdot, \tfrac{t_0}{2}) + \int_{t_0/2}^{t} e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon(\cdot, s) \nabla c_\varepsilon(\cdot, s)) \,\mathrm{d}s \right] \right\|_{L^\infty(\Omega)} \\
&\le K_2 \|n_\varepsilon(\cdot, \tfrac{t_0}{2})\|_{L^N(\Omega)} + K_2 (t - \tfrac{t_0}{2})^{\frac{1}{2}} \int_{t_0/2}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) \|n_\varepsilon(\cdot, s) \nabla c_\varepsilon(\cdot, s)\|_{L^{2N}(\Omega)} \,\mathrm{d}s \\
&\le K_1 K_2 + K_1 K_2 (t - \tfrac{t_0}{2})^{\frac{1}{2}} \int_{t_0/2}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) \|n_\varepsilon(\cdot, s)\|_{L^N(\Omega)}^{\frac{1}{4}} \|n_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^{\frac{3}{4}} \,\mathrm{d}s \\
&\le K_1 K_2 + K_1^{\frac{5}{4}} K_2 (t - \tfrac{t_0}{2})^{\frac{1}{2}} M_\varepsilon^{\frac{3}{4}} \int_{t_0/2}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) (s - \tfrac{t_0}{2})^{-\frac{3}{8}} \,\mathrm{d}s
\end{aligned}$$

for all $t \in (\frac{t_0}{2}, t_1)$ and $\varepsilon \in (0, 1)$. As further

$$\begin{aligned}
&\int_{t_0/2}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) (s - \tfrac{t_0}{2})^{-\frac{3}{8}} \,\mathrm{d}s \\
&= \int_{t_0/2}^{t_{\mathrm{mid}}} \left(1 + (t-s)^{-\frac{3}{4}}\right) (s - \tfrac{t_0}{2})^{-\frac{3}{8}} \,\mathrm{d}s + \int_{t_{\mathrm{mid}}}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) (s - \tfrac{t_0}{2})^{-\frac{3}{8}} \,\mathrm{d}s \\
&\le \left(1 + (\tfrac{t}{2} - \tfrac{t_0}{4})^{-\frac{3}{4}}\right) \int_{t_0/2}^{t_{\mathrm{mid}}} (s - \tfrac{t_0}{2})^{-\frac{3}{8}} \,\mathrm{d}s + (\tfrac{t}{2} - \tfrac{t_0}{4})^{-\frac{3}{8}} \int_{t_{\mathrm{mid}}}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right) \mathrm{d}s \\
&= \tfrac{13}{5} (\tfrac{t}{2} - \tfrac{t_0}{4})^{\frac{5}{8}} + \tfrac{28}{5} (\tfrac{t}{2} - \tfrac{t_0}{4})^{-\frac{1}{8}}
\end{aligned}$$

with $t_{\text{mid}} := \frac{1}{2}\left(t + \frac{t_0}{2}\right)$ and thus

$$(t - \tfrac{t_0}{2})^{\frac{1}{2}} \int_{t_0/2}^{t} \left(1 + (t-s)^{-\frac{3}{4}}\right)(s - \tfrac{t_0}{2})^{-\frac{3}{8}}\,\mathrm{d}s$$
$$\leq \sqrt{2}\left(\tfrac{13}{5}(\tfrac{t_1}{2} - \tfrac{t_0}{4})^{\frac{9}{8}} + \tfrac{28}{5}(\tfrac{t_1}{2} - \tfrac{t_0}{4})^{\frac{3}{8}}\right) =: K_3 = K_3(t_0, t_1)$$

for all $t \in (\frac{t_0}{2}, t_1)$, our previous considerations directly yield

$$M_\varepsilon \leq K_4 M_\varepsilon^{\frac{3}{4}} + K_4 \leq \frac{3}{4} M_\varepsilon + \frac{K_4^4}{4} + K_4$$

with $K_4 = K_4(t_0, t_1) := \max(K_1 K_2, K_1^{\frac{5}{4}} K_2 K_3)$ and thus

$$M_\varepsilon \leq K_4^4 + 4K_4,$$

which readily implies our desired result. □

We now use the above bounds to apply standard regularity theory by Porzio and Vespri (cf. [97]), which conveniently is already formulated in such a way as to work with minimal initial data regularity, to gain uniform space-time Hölder bounds for all three solution components.

Lemma 5.5.3. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then for each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exist $C = C(t_0, t_1) > 0$ and $\theta = \theta(t_0, t_1) \in (0, 1)$ such that*

$$\|n_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \leq C \tag{5.5.1}$$

and

$$\|c_{a,\varepsilon}\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \leq C \quad \text{as well as} \quad \|c_{r,\varepsilon}\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \leq C \tag{5.5.2}$$

for all $\varepsilon \in (0, 1)$.

Proof. Reframing the first equation in (AR*) as

$$n_{\varepsilon t} - \nabla \cdot (a_\varepsilon(x, t, n_\varepsilon, \nabla n_\varepsilon)) = b(x, t, n_\varepsilon, \nabla n_\varepsilon)$$

with $a_\varepsilon(x, t, \varphi, \Phi) := \Phi + n_\varepsilon(x, t)\nabla c_\varepsilon(x, t)$ and $b(x, t, \varphi, \Phi) := 0$ for all $(x, t, \varphi, \Phi) \in \Omega \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^N$ makes it available to the results about Hölder regularity proven in [97]. In fact, the bounds established in Lemma 5.5.1 and Lemma 5.5.2 are already sufficient to gain the first half of our desired result by [97, Theorem 1.3].

Similarly if $\tau = 1$, the second equation in (AR) can be rewritten as

$$\partial_t c_{a,\varepsilon} - \nabla \cdot (a(x, t, c_{a,\varepsilon}, \nabla c_{a,\varepsilon})) = b_\varepsilon(x, t, c_{a,\varepsilon}, \nabla c_{a,\varepsilon})$$

with $a(x,t,\varphi,\Phi) := \Phi$ and $b_\varepsilon(x,t,\varphi,\Phi) := \alpha n_\varepsilon(x,t) - \beta c_{a,\varepsilon}(x,t)$ for all $(x,t,\varphi,\Phi) \in \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N$, making it available to essentially the same argument as above due to the bounds in Lemma 5.5.1 and Lemma 5.5.2. Given the similar structure of the third equation in (AR) this also holds true for $c_{r,\varepsilon}$ and thus gives us the second half of our result.

If $\tau = 0$, then our desired bounds for $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$ can be derived as follows: Given that we have already proven the first half of this results, we can fix $K_1 > 0$ and $\theta \in (0, \frac{1}{2})$ such that

$$\|n_\varepsilon\|_{C^{2\theta,\theta}(\overline{\Omega}\times[t_0,t_1])} \le K_1,$$

which by a straightforward calculation yields a second constant $K_2 > 0$ such that

$$\|n_\varepsilon\|_{C^{\frac{\theta}{2}}([t_0,t_1];C^\theta(\overline{\Omega}))} \le K_2$$

for all $\varepsilon \in (0,1)$. We can then use Theorem 3.1 from [60, p. 135] to gain $K_3 > 0$ such that

$$\|c_{a,\varepsilon}(\cdot,t)\|_{C^{2+\theta}(\overline{\Omega})} \le K_3 \|n_\varepsilon(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le K_2 K_3$$

for all $t \in [t_0,t_1]$ and $\varepsilon \in (0,1)$. Note further that

$$0 = \Delta(c_{a,\varepsilon}(\cdot,t) - c_{a,\varepsilon}(\cdot,s)) - \beta(c_{a,\varepsilon}(\cdot,t) - c_{a,\varepsilon}(\cdot,s)) + \alpha(n_\varepsilon(\cdot,t) - n_\varepsilon(\cdot,s)) \text{ on } \Omega$$

as well as

$$\nabla(c_{a,\varepsilon}(\cdot,t) - c_{a,\varepsilon}(\cdot,s)) \cdot \nu = 0 \text{ on } \partial\Omega$$

for all $t,s > 0$ and $\varepsilon \in (0,1)$. Essentially the same elliptic regularity argument yields $K_4 > 0$ such that

$$\|c_{a,\varepsilon}(\cdot,t) - c_{a,\varepsilon}(\cdot,s)\|_{C^{2+\theta}(\overline{\Omega})} \le K_4 \|n_\varepsilon(\cdot,t) - n_\varepsilon(\cdot,s)\|_{C^\theta(\overline{\Omega})} \le K_2 K_4 (t-s)^{\frac{\theta}{2}}$$

for all $t,s \in [t_0,t_1]$ and $\varepsilon \in (0,1)$. As $C^{\frac{\theta}{2}}([t_0,t_1];C^{2+\theta}(\overline{\Omega}))$ embeds continuously into $C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])$ and the exact same argument works for $c_{r,\varepsilon}$ as well, this is sufficient to complete the proof. □

Using the parabolic regularity theory from [61] and [73] now allows us to gain the final bounds necessary for our solution construction, namely higher order space-time Hölder bounds. As said regularity theory requires higher initial data regularity than we have uniformly available, we combine it with a straightforward cutoff function argument to decouple it from the initial data.

Lemma 5.5.4. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then for each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exist $C = C(t_0,t_1) > 0$ and $\theta = \theta(t_0,t_1) \in (0,1)$ such that*

$$\|n_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \le C$$

and

$$\|c_{a,\varepsilon}\|_{C^{2+\theta,\tau+\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \le C \quad \textit{as well as} \quad \|c_{r,\varepsilon}\|_{C^{2+\theta,\tau+\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \le C$$

for all $\varepsilon \in (0,1)$.

Proof. Fix $t_1 > t_0 > 0$. We then further fix a cutoff function $\rho \in C^\infty([\frac{t_0}{4}, t_1])$ with

$$\rho \equiv 0 \text{ on } [\tfrac{t_0}{4}, \tfrac{t_0}{2}], \quad \rho(t) \in [0,1] \text{ for all } t \in [t_0/2, t_0] \quad \text{and} \quad \rho \equiv 1 \text{ on } [t_0, t_1].$$

We now begin by treating $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$. If $\tau = 1$, we first let $\tilde{c}_{a,\varepsilon}(x,t) \coloneqq \rho(t)c_{a,\varepsilon}(x,t)$ for all $x \in \overline{\Omega}$, $t \in [t_0/4, t_1]$ and $\varepsilon \in (0,1)$. These functions then are classical solutions of

$$\begin{cases} \partial_t \tilde{c}_{a,\varepsilon} = \Delta \tilde{c}_{a,\varepsilon} + f_\varepsilon(x,t) & \text{on } \Omega \times (\frac{t_0}{4}, t_1), \\ \nabla \tilde{c}_{a,\varepsilon} \cdot \nu = 0 & \text{on } \partial\Omega \times (\frac{t_0}{4}, t_1), \\ \tilde{c}_{a,\varepsilon}(\cdot, \frac{t_0}{4}) = 0 & \text{on } \Omega \end{cases}$$

with $f_\varepsilon(x,t) \coloneqq [\rho'(t) - \beta\rho(t)]c_{a,\varepsilon}(x,t) + \alpha\rho(t)n_\varepsilon(x,t)$ for all $x \in \overline{\Omega}$, $t \in [t_0/4, t_1]$ and $\varepsilon \in (0,1)$. As the bounds established in Lemma 5.5.3 ensure that there exist $K_1 = K_1(t_0,t_1) > 0$ and $\theta_1 = \theta_1(t_0,t_1) \in (0,1)$ such that

$$\|f_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_1$$

for all $\varepsilon \in (0,1)$, we can apply standard parabolic regularity theory from [61, p. 170 and p. 320] to gain constants $K_2 = K_2(t_0,t_1) > 0$ and $\theta_2 = \theta_2(t_0,t_1) \in (0,1)$ such that

$$\|\tilde{c}_{a,\varepsilon}\|_{C^{2+\theta_2,\tau+\frac{\theta_2}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_2$$

for all $\varepsilon \in (0,1)$. As $\rho \equiv 1$ on $[t_0,t_1]$, this directly implies

$$\|c_{a,\varepsilon}\|_{C^{2+\theta_2,\tau+\frac{\theta_2}{2}}(\overline{\Omega}\times[t_0,t_1])} \le K_2$$

for all $\varepsilon \in (0,1)$. By essentially the same argument, we can further find $K_3 = K_3(t_0,t_1) > 0$ and $\theta_3 = \theta_3(t_0,t_1) \in (0,1)$ such that

$$\|c_{r,\varepsilon}\|_{C^{2+\theta_3,\tau+\frac{\theta_3}{2}}(\overline{\Omega}\times[t_0,t_1])} \le K_3$$

for all $\varepsilon \in (0,1)$.

If $\tau = 0$, then the corresponding bounds follow directly from standard elliptic regularity theory (cf. [60]) by the exact same argument as presented in proof of Lemma 5.5.3.

These arguments of course immediately also ensure the existence of constants $K_4 = K_4(t_0,t_1) > 0$ and $\theta_4 = \theta_4(t_0,t_1) \in (0,1)$ such that

$$\|c_\varepsilon\|_{C^{2+\theta_4,\tau+\frac{\theta_4}{2}}(\overline{\Omega}\times[t_0,t_1])} \le K_4 \tag{5.5.3}$$

for all $\varepsilon \in (0,1)$ as c_ε is a linear combination of $c_{a,\varepsilon}$ and $c_{r,\varepsilon}$.

Given that we have now established the second half of our result, we can focus our attention on n_ε. We again begin by setting $\tilde{n}_\varepsilon(x,t) := \rho(t)n_\varepsilon(x,t)$ for all $x \in \overline{\Omega}$, $t \in [t_0/4, t_1]$ and $\varepsilon \in (0,1)$. Then these functions are a classical solution of the system

$$\begin{cases} \tilde{n}_{\varepsilon t} = \nabla \cdot (A_\varepsilon(x,t,\tilde{n}_\varepsilon,\nabla\tilde{n}_\varepsilon)) + B_\varepsilon(x,t,\tilde{n}_\varepsilon,\nabla\tilde{n}_\varepsilon) & \text{on } \Omega \times (\frac{t_0}{4}, t_1), \\ A_\varepsilon(x,t,\tilde{n}_\varepsilon,\nabla\tilde{n}_\varepsilon) \cdot \nu = 0 & \text{on } \partial\Omega \times (\frac{t_0}{4}, t_1), \\ \tilde{n}_\varepsilon(\cdot, \frac{t_0}{4}) = 0 & \text{on } \Omega \end{cases}$$

with $A_\varepsilon(x,t,\varphi,\Phi) := \Phi + \rho(t)n_\varepsilon(x,t)\nabla c_\varepsilon(x,t)$ and $B_\varepsilon(x,t,\varphi,\Phi) := \rho'(t)n_\varepsilon(x,t)$ for all $(x,t,\varphi,\Phi) \in \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N$ and $\varepsilon \in (0,1)$, which is compatible with the notation from [73]. As Lemma 5.5.3 and (5.5.3) ensure that there exist $K_5 = K_5(t_0,t_1) > 0$ and $\theta_5 = \theta_5(t_0,t_1) \in (0,1)$ such that

$$\|n_\varepsilon\|_{L^\infty(\Omega\times(t_0/4,t_1))} \le K_5, \quad \|\rho n_\varepsilon \nabla c_\varepsilon\|_{C^{\theta_5,\frac{\theta_5}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_5$$

and

$$\|B_\varepsilon\|_{L^\infty(\Omega\times(t_0/4,t_1)\times\mathbb{R}\times\mathbb{R}^N)} \le K_5$$

after a standard time shift for all $\varepsilon \in (0,1)$, we can apply [73, Theorem 1.1] to gain $K_6 = K_6(t_0,t_1) > 0$ and $\theta_6 = \theta_6(t_0,t_1) \in (0,1)$ such that

$$\|\nabla\tilde{n}_\varepsilon\|_{C^{\theta_6,\frac{\theta_6}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_6 \quad \text{and thus} \quad \|\nabla n_\varepsilon\|_{C^{\theta_6,\frac{\theta_6}{2}}(\overline{\Omega}\times[t_0,t_1])} \le K_6 \tag{5.5.4}$$

for all $\varepsilon \in (0,1)$.

Slightly reframing the above system to make it available to the results presented in [61], $\tilde{n}_\varepsilon$ is also a classical solution to

$$\begin{cases} \tilde{n}_{\varepsilon t} = \Delta\tilde{n}_\varepsilon + f_\varepsilon(x,t) & \text{on } \Omega \times (\frac{t_0}{4}, t_1) \\ \nabla\tilde{n}_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega \times (\frac{t_0}{4}, t_1) \\ \tilde{n}_\varepsilon(\cdot, \frac{t_0}{4}) = 0 & \text{on } \Omega \end{cases}$$

with $f_\varepsilon(x,t) := \rho(t)\nabla n_\varepsilon(x,t)\cdot\nabla c_\varepsilon(x,t) + \rho(t)n_\varepsilon(x,t)\Delta c_\varepsilon(x,t) + \rho'(t)n_\varepsilon(x,t)$ for all $x \in \overline{\Omega}$, $t \in [t_0/4, t_1]$ and $\varepsilon \in (0,1)$. Due to the bounds established in Lemma 5.5.3, (5.5.3) as well as (5.5.4), there exist $K_7 = K_7(t_0,t_1) > 0$ and $\theta_7 = \theta_7(t_0,t_1) \in (0,1)$ such that

$$\|f_\varepsilon\|_{C^{\theta_7,\frac{\theta_7}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_7$$

after a standard time shift for all $\varepsilon \in (0,1)$. This then allows us to apply the same regularity theory from [61, p. 170 and p. 320] as before to find $K_8 = K_8(t_0,t_1) > 0$ and $\theta_8 = \theta_8(t_0,t_1) \in (0,1)$ such that

$$\|\tilde{n}_\varepsilon\|_{C^{2+\theta_8,1+\frac{\theta_8}{2}}(\overline{\Omega}\times[t_0/4,t_1])} \le K_8 \quad \text{and thus} \quad \|n_\varepsilon\|_{C^{2+\theta_8,1+\frac{\theta_8}{2}}(\overline{\Omega}\times[t_0,t_1])} \le K_8$$

for all $\varepsilon \in (0,1)$, which completes the proof with $\theta = \theta(t_0,t_1) := \min(\theta_2,\theta_3,\theta_8)$ and an appropriate constant C. □

Using these bounds, we now construct our desired solution candidates as follows:

Lemma 5.5.5. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then there exist nonnegative functions $n \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ and $c_a, c_r \in C^{2,\tau}(\overline{\Omega} \times (0,\infty))$ as well as a null sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that*

$$n_\varepsilon \to n \qquad \text{in } C^{2,1}(\overline{\Omega} \times [t_0, t_1]) \tag{5.5.5}$$

$$c_{a,\varepsilon} \to c_a \qquad \text{in } C^{2,\tau}(\overline{\Omega} \times [t_0, t_1]) \tag{5.5.6}$$

$$c_{r,\varepsilon} \to c_r \qquad \text{in } C^{2,\tau}(\overline{\Omega} \times [t_0, t_1]) \tag{5.5.7}$$

for all $t_1 > t_0 > 0$ as $\varepsilon = \varepsilon_j \searrow 0$ and such that (n, c_a, c_r) is a classical solution to (AR) with boundary conditions (ARB).

Proof. Given the bounds established in Lemma 5.5.4, the convergence properties in (5.5.5) to (5.5.7) as well as the existence of nonnegative functions n, c_a, c_r of appropriate regularity are an immediate consequence of the compact embedding properties of Hölder spaces combined with a straightforward diagonal sequence argument. As our approximate solutions n_ε, $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ are already classical solutions to (AR) with boundary conditions (ARB) and given the now established fairly strong convergence properties in (5.5.5) to (5.5.7), our desired solution properties immediately transfer from the approximate solutions to their limits n, c_a and c_r. This completes the proof. □

5.6 Continuity at $t = 0$

Having now constructed solution candidates, which already solve (AR) with boundary conditions (ARB) classically, the only thing that remains to be shown is that said candidates are connected to the initial data in the way outlined in (5.5.5) to (5.5.7). We first focus on (5.5.5) as it is not only relevant in all of our scenarios but also the most challenging of the continuity properties. In fact, the key to deriving this property is to first show that our approximate solutions are continuous at $t = 0$ in an appropriate and ε independent fashion. To do this, we first show that a space-time integral of a quantity connected to the taxis becomes uniformly small as t goes to zero.

Lemma 5.6.1. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). Then there exist $C > 0$ and $\theta > 0$ such that*

$$\int_0^t \|n_\varepsilon(\cdot, s)\nabla c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s \le Ct^\theta$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$.

Proof. We begin by considering Scenario (S1) and more specifically we start by treating the simpler case of ζ being equal to zero. Here, we can use the variation-of-constants representation of the second equation in (AR*), smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3 (ii)]), (5.3.5) combined with the Sobolev inequality and Corollary 5.4.2 to find $K_1 > 0$ such that

$$\begin{aligned}\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| \nabla e^{t(\Delta-\delta)} c_\varepsilon(\cdot,0) + \sigma \int_0^t \nabla e^{(t-s)(\Delta-\delta)} c_{a,\varepsilon}(\cdot,s)\,\mathrm{d}s \right\|_{L^\infty(\Omega)} \\ &\le K_1(1 + t^{-\frac{1}{2}-\frac{2-r}{2r}}) + |\sigma| K_1 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{2-r}{2r}})\,\mathrm{d}s \\ &\le 2K_1 t^{-\frac{1}{r}} + 2|\sigma| K_1 \int_0^t (t-s)^{-\frac{1}{r}}\,\mathrm{d}s \\ &= 2K_1 t^{-\frac{1}{r}} + 2\tfrac{r}{r-1}|\sigma| K_1 t^{\frac{r-1}{r}} \le 2(1+|\sigma|)\tfrac{r}{r-1} K_1 t^{-\frac{1}{r}}\end{aligned}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$ as $\zeta = 0$. Due to the mass conservation property in (5.3.8), this directly implies

$$\begin{aligned}\int_0^t \|n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)}\,\mathrm{d}s &\le m \int_0^t \|\nabla c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\le 2m(1+|\sigma|)\tfrac{r}{r-1} K_1 \int_0^t s^{-\frac{1}{r}}\,\mathrm{d}s \\ &= 2m(1+|\sigma|)(\tfrac{r}{r-1})^2 K_1 t^{\frac{r-1}{r}}\end{aligned}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$, which sufficiently addresses this case as $r > 1$.

Let us now consider the more subtle case of ζ being positive in Scenario (S1). Here, we first fix $\lambda \in (0, \frac{2}{3})$ as in Lemma 5.4.3 and an appropriately small constant $\theta \in (0,1)$ such that

$$-\frac{3}{2}\lambda - \theta > -1 + \theta \quad \text{as well as} \quad -\frac{2}{3}\lambda - \frac{1}{3}\theta > -1 + \theta.$$

Using Young's inequality, we then see that

$$\|n_\varepsilon(\cdot,t)\nabla c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \int_\Omega n_\varepsilon^{\frac{4}{3}} + t^{2\lambda+\theta} \int_\Omega |\nabla c_\varepsilon|^4 \tag{5.6.1}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. Due to Corollary 5.4.4, we already know that there exists $K_2 > 0$ such that

$$\int_0^t s^{2\lambda+\theta} \int_\Omega |\nabla c_\varepsilon(x,s)|^4\,\mathrm{d}x\,\mathrm{d}s \le t^\theta \int_0^t s^{2\lambda} \int_\Omega |\nabla c_\varepsilon(x,s)|^4\,\mathrm{d}x\,\mathrm{d}s \le K_2 t^\theta \tag{5.6.2}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. Therefore we can now focus on the remaining term involving n_ε in (5.6.1). To this end, we use the Gagliardo–Nirenberg inequality as well as the mass conservation property in (5.3.8) to gain $K_3 > 0$ such that

$$t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \int_\Omega n_\varepsilon^{\frac{4}{3}} = t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \|n_\varepsilon^{\frac{1}{2}}\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{8}{3}} \le K_3 t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \right)^{\frac{1}{3}} + K_3 t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta}$$

$$
\begin{aligned}
&= K_3 t^{-\lambda-\frac{2}{3}\theta}\left(t^{\lambda+\theta}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\right)^{\frac{1}{3}} + K_3 t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \\
&\le K_3 t^{\lambda+\theta}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + K_3 t^{-\frac{3}{2}\lambda-\theta} + K_3 t^{-\frac{2}{3}\lambda-\frac{1}{3}\theta} \\
&\le K_3 t^{\lambda+\theta}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 2K_3 t^{-1+\theta}
\end{aligned}
$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$, where the last step is facilitated by our choice of θ. Thus by application of Lemma 5.4.3, we find $K_4 > 0$ such that

$$
\begin{aligned}
\int_0^t s^{-\frac{2}{3}\lambda-\frac{1}{3}\theta}\int_\Omega n_\varepsilon^{\frac{4}{3}}(x,s)\,\mathrm{d}x\,\mathrm{d}s &\le K_3\int_0^t s^{\lambda+\theta}\int_\Omega \frac{|\nabla n_\varepsilon(x,s)|^2}{n_\varepsilon(x,s)}\,\mathrm{d}x\,\mathrm{d}s + 2K_3\int_0^t s^{-1+\theta}\,\mathrm{d}s \\
&\le K_3 t^\theta \int_0^t s^{\lambda}\int_\Omega \frac{|\nabla n_\varepsilon(x,s)|^2}{n_\varepsilon(x,s)}\,\mathrm{d}x\,\mathrm{d}s + 2K_3\int_0^t s^{-1+\theta}\,\mathrm{d}s \\
&\le K_3 K_4 t^\theta + \frac{2K_3}{\theta} t^\theta
\end{aligned}
$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$ as $\zeta > 0$. Combining this with the similar estimate (5.6.2) and the estimate (5.6.1) then directly yields our desired result for Scenario (S1) with $\zeta > 0$.

We next look at Scenario (S2). Using Lemma 5.4.9 and Lemma 5.4.8, we here immediately obtain $K_5 > 0$ such that

$$
\begin{aligned}
\int_0^t \|n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)}\,\mathrm{d}s &\le \int_0^t \|n_\varepsilon(\cdot,s)\|_{L^{\frac{5}{2}}(\Omega)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^{\frac{5}{3}}(\Omega)}\,\mathrm{d}s \\
&\le K_5\int_0^t s^{-\frac{3}{2}\cdot\frac{2}{5}}\,\mathrm{d}s = \frac{5K_5}{2}\,t^{\frac{2}{5}}
\end{aligned}
$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$, which completes the proof for this case.

Lastly, we will handle Scenario (S3) as follows: According to Lemma 5.4.9, there exists a uniform bound for n_ε in $L^\kappa(\Omega)$ on the time interval $(0,1)$ and thus by elliptic regularity theory there further exist uniform bounds for $c_{a,\varepsilon}$, $c_{r,\varepsilon}$ and therefore c_ε in $W^{2,\kappa}(\Omega)$ on the same time interval. Using the Sobolev inequality, this means we can fix $K_6 > 0$ such that

$$
\|\nabla c_\varepsilon(\cdot,t)\|_{L^{\frac{3\kappa}{3-\kappa}}(\Omega)} \le K_6
$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. Having established this, we can then use Lemma 5.4.9 to find $K_7 > 0$ such that

$$
\begin{aligned}
\int_0^t \|n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)}\,\mathrm{d}s &\le \int_0^t \|n_\varepsilon(\cdot,s)\|_{L^{\frac{3\kappa}{4\kappa-3}}(\Omega)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^{\frac{3\kappa}{3-\kappa}}(\Omega)}\,\mathrm{d}s \\
&\le K_6 K_7\int_0^t s^{-\frac{3-\kappa}{2\kappa}}\,\mathrm{d}s = \frac{2\kappa K_5 K_6}{3\kappa-3}\,t^{\frac{3\kappa-3}{2\kappa}}
\end{aligned}
$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. As $\frac{3\kappa-3}{2\kappa} > 0$, this in fact completes the proof. □

Using the fundamental theorem of calculus and the first equation in (AR*), we now derive the following uniform continuity property for our approximate solutions from the above result.

Lemma 5.6.2. *Assume we are in Scenario (S1), Scenario (S2) with* $C_{S2} > 0$ *as in Lemma 5.4.9 or in Scenario (S3). Then for each* $\varphi \in C^0(\overline{\Omega})$ *and* $\eta > 0$*, there exists* $t_0 = t_0(\varphi, \eta) \in (0, 1)$ *such that*

$$\left| \int_\Omega n_\varepsilon(\cdot, t)\varphi - \int_\Omega n_{0,\varepsilon}\varphi \right| \le \eta$$

for all $t \in (0, t_0)$ *and* $\varepsilon \in (0, 1)$*.*

Proof. Let $\eta > 0$ and $\varphi \in C^0(\overline{\Omega})$ be fixed but arbitrary.

As the set of functions $\psi \in C^2(\overline{\Omega})$ with $\nabla\psi \cdot \nu = 0$ on $\partial\Omega$ is dense in $C^0(\overline{\Omega})$ (which can be seen by e.g. approximating any $\rho \in C^0(\overline{\Omega})$ by functions $\rho_\lambda := e^{\lambda\Delta}\rho$ with $\lambda > 0$, where $(e^{t\Delta})_{t\ge 0}$ is the Neumann heat semigroup, in a similar way to our approach in Section 5.3), we can further fix such a function ψ with

$$\|\varphi - \psi\|_{L^\infty(\Omega)} \le \frac{\eta}{3m}. \tag{5.6.3}$$

Using the fundamental theorem of calculus, the first equation in (AR*), partial integration and Lemma 5.6.1, we can then fix $K_1 > 0$ and $\theta > 0$ such that

$$\begin{aligned}
\left| \int_\Omega n_\varepsilon(\cdot, t)\psi - \int_\Omega n_{0,\varepsilon}\psi \right| &= \left| \int_0^t \int_\Omega n_{\varepsilon t}\psi \right| = \left| \int_0^t \int_\Omega \Delta n_\varepsilon \psi + \int_0^t \int_\Omega \nabla \cdot (n_\varepsilon \nabla c_\varepsilon)\psi \right| \\
&= \left| \int_0^t \int_\Omega n_\varepsilon \Delta\psi - \int_0^t \int_\Omega (n_\varepsilon \nabla c_\varepsilon) \cdot \nabla\psi \right| \\
&\le m\|\Delta\psi\|_{L^\infty(\Omega)} t + \|\nabla\psi\|_{L^\infty(\Omega)} \int_0^t \|n_\varepsilon(\cdot, s)\nabla c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&\le m\|\Delta\psi\|_{L^\infty(\Omega)} t + K_1 \|\nabla\psi\|_{L^\infty(\Omega)} t^\theta
\end{aligned}$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$. Thus we can find a sufficiently small $t_0 > 0$ such that

$$\left| \int_\Omega n_\varepsilon(\cdot, t)\psi - \int_\Omega n_{0,\varepsilon}\psi \right| \le \frac{\eta}{3}$$

for all $t \in (0, t_0)$ and $\varepsilon \in (0, 1)$.

From this as well as (5.6.3), we can further conclude that

$$\begin{aligned}
\left| \int_\Omega n_\varepsilon(\cdot, t)\varphi - \int_\Omega n_{0,\varepsilon}\varphi \right| \le &\left| \int_\Omega n_\varepsilon(\cdot, t)\varphi - \int_\Omega n_\varepsilon(\cdot, t)\psi \right| \\
&+ \left| \int_\Omega n_\varepsilon(\cdot, t)\psi - \int_\Omega n_{0,\varepsilon}\psi \right|
\end{aligned}$$

$$+\left|\int_\Omega n_{0,\varepsilon}\psi - \int_\Omega n_{0,\varepsilon}\varphi\right|$$
$$\le m\frac{\eta}{3m} + \frac{\eta}{3} + m\frac{\eta}{3m} = \eta$$

for all $t \in (0, t_0)$ and $\varepsilon \in (0, 1)$, which is exactly our desired result. □

We now only need to argue that the above property survives the limit process undertaken in Lemma 5.5.5 to gain the first of the continuity properties at $t = 0$ for our solution candidates.

Lemma 5.6.3. *Assume we are in Scenario (S1), Scenario (S2) with $C_{\mathrm{S2}} > 0$ as in Lemma 5.4.9 or in Scenario (S3). If n is the function constructed in Lemma 5.5.5, then*

$$n(\cdot, t) \to n_0 \quad \text{in} \quad \mathcal{M}_+(\overline{\Omega})$$

as $t \searrow 0$, where we interpret the functions $n(\cdot, t)$, $t > 0$, as the positive Radon measures $n(x, t)\mathrm{d}x$ with $\mathrm{d}x$ being the standard Lebesgue measure on $\overline{\Omega}$.

Proof. Let $\eta > 0$ and $\varphi \in C^0(\overline{\Omega})$ be fixed but arbitrary.

According to Lemma 5.6.2, we can then fix $t_0 > 0$ such that

$$\left|\int_\Omega n_\varepsilon(\cdot, t)\varphi - \int_\Omega n_{0,\varepsilon}\varphi\right| \le \frac{\eta}{3}$$

for all $t \in (0, t_0)$ and $\varepsilon \in (0, 1)$.

Using (5.3.1) as well as Lemma 5.5.5, we can further, for each $t \in (0, t_0)$, find $\varepsilon(t) \in (0, 1)$, such that

$$\left|\int_\Omega n_{0,\varepsilon(t)}\varphi - \int_{\overline{\Omega}} \varphi \,\mathrm{d}n_0\right| \le \frac{\eta}{3} \quad \text{and} \quad \left|\int_\Omega n(\cdot, t)\varphi - \int_\Omega n_{\varepsilon(t)}(\cdot, t)\varphi\right| \le \frac{\eta}{3}.$$

Combining the above estimates then yields

$$\begin{aligned}\left|\int_\Omega n(\cdot, t)\varphi - \int_{\overline{\Omega}} \varphi \,\mathrm{d}n_0\right| &\le \left|\int_\Omega n(\cdot, t)\varphi - \int_\Omega n_{\varepsilon(t)}(\cdot, t)\varphi\right| \\ &\quad + \left|\int_\Omega n_{\varepsilon(t)}(\cdot, t)\varphi - \int_\Omega n_{0,\varepsilon(t)}\varphi\right| \\ &\quad + \left|\int_\Omega n_{0,\varepsilon(t)}\varphi - \int_{\overline{\Omega}} \varphi \,\mathrm{d}n_0\right| \\ &\le \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta\end{aligned}$$

for all $t \in (0, t_0)$. This completes the proof. □

As our last significant step of this chapter, we will now treat the remaining continuity properties at $t = 0$ for the parabolic-parabolic case. Similar to our prior arguments in this section, we will do this by first showing that the approximate solutions are uniformly continuous at $t = 0$ by way of semigroup methods and then translate this continuity property to our limit solutions.

Lemma 5.6.4. *Assume we are in Scenario (S1). If c_a, c_r are the functions constructed in Lemma 5.5.5, then*

$$c_a(\cdot, t) \to c_{a,0} \quad \text{and} \quad c_r(\cdot, t) \to c_{r,0} \quad \text{in} \quad W^{1,r}(\Omega)$$

as $t \searrow 0$ with $r \in (\frac{6}{5}, 2)$ as fixed at the beginning of Section 5.3.

Proof. Let $\eta > 0$ be fixed but arbitrary.

Using the variation-of-constants representation of the second equation in (AR), we first note that

$$\begin{aligned}
&\|c_{a,\varepsilon}(\cdot, t) - c_{a,0,\varepsilon}\|_{W^{1,r}(\Omega)} \\
&\leq \|e^{t(\Delta-\beta)} c_{a,0,\varepsilon} - c_{a,0,\varepsilon}\|_{W^{1,r}(\Omega)} + \alpha \int_0^t \|e^{(t-s)(\Delta-\beta)} n_\varepsilon(\cdot, s)\|_{W^{1,r}(\Omega)} \,\mathrm{d}s
\end{aligned}$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$.

To treat the first summand, we recall our definition of $c_{a,0,\varepsilon}$ in (5.3.3) and standard properties of the Neumann heat semigroup to gain $K_1 > 0$ such that

$$\begin{aligned}
\|e^{t(\Delta-\beta)} c_{a,0,\varepsilon} - c_{a,0,\varepsilon}\|_{W^{1,r}(\Omega)} &= \|e^{t(\Delta-\beta)} e^{\varepsilon(\Delta-\beta)} c_{a,0} - e^{\varepsilon(\Delta-\beta)} c_{a,0}\|_{W^{1,r}(\Omega)} \\
&= \|e^{\varepsilon(\Delta-\beta)} (e^{t(\Delta-\beta)} c_{a,0} - c_{a,0})\|_{W^{1,r}(\Omega)} \\
&\leq K_1 \|e^{t(\Delta-\beta)} c_{a,0} - c_{a,0}\|_{W^{1,r}(\Omega)}
\end{aligned}$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$. Due to further continuity properties of the Neumann heat semigroup and the fact that $c_{a,0} \in W^{1,r}(\Omega)$, this then allows us to fix $t_1 > 0$ such that

$$\|e^{t(\Delta-\beta)} c_{a,0,\varepsilon} - c_{a,0,\varepsilon}\|_{W^{1,r}(\Omega)} \leq \frac{\eta}{6}$$

for all $t \in (0, t_1)$ and $\varepsilon \in (0, 1)$.

Regarding the second summand, we begin by employing the smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]) to obtain $K_2 > 0$ such that

$$\begin{aligned}
\int_0^t \|e^{(t-s)(\Delta-\beta)} n_\varepsilon(\cdot, s)\|_{W^{1,r}(\Omega)} \,\mathrm{d}s &\leq K_2 \int_0^t (1 + (t-s)^{-\frac{3}{2}+\frac{1}{r}}) \|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&= mK_2 t + \tfrac{mK_2}{\frac{1}{r}-\frac{1}{2}} \, t^{\frac{1}{r}-\frac{1}{2}}
\end{aligned}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. As $r \in (\frac{6}{5}, 2)$, we can therefore fix $t_2 > 0$ such that

$$\alpha \int_0^t \|e^{(t-s)(\Delta-\beta)} n_\varepsilon(\cdot, s)\|_{W^{1,r}(\Omega)} \leq \frac{\eta}{6}$$

for all $t \in (0, t_2)$ and $\varepsilon \in (0,1)$. Thus

$$\|c_{a,\varepsilon}(\cdot, t) - c_{a,0,\varepsilon}\|_{W^{1,r}(\Omega)} \leq \frac{\eta}{3} \tag{5.6.4}$$

for all $t \in (0, t_0)$ and $\varepsilon \in (0,1)$ with $t_0 \coloneqq \min(t_1, t_2)$.

Due to (5.3.5) and Lemma 5.5.5, we can, for each $t \in (0,1)$, further find $\varepsilon(t) \in (0,1)$ such that

$$\|c_{a,0,\varepsilon(t)} - c_{a,0}\|_{W^{1,r}(\Omega)} \leq \frac{\eta}{3} \quad \text{and} \quad \|c_a(\cdot, t) - c_{a,\varepsilon(t)}(\cdot, t)\|_{W^{1,r}(\Omega)} \leq \frac{\eta}{3}.$$

Using these estimates as well as (5.6.4), we then observe that

$$\begin{aligned}
\|c_a(\cdot, t) - c_{a,0}\|_{W^{1,r}(\Omega)} &\leq \|c_a(\cdot, t) - c_{a,\varepsilon(t)}(\cdot, t)\|_{W^{1,r}(\Omega)} \\
&\quad + \|c_{a,\varepsilon(t)}(\cdot, t) - c_{a,0,\varepsilon(t)}\|_{W^{1,r}(\Omega)} \\
&\quad + \|c_{a,0,\varepsilon(t)} - c_{a,0}\|_{W^{1,r}(\Omega)} \\
&\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta
\end{aligned}$$

for all $t \in (0, t_0)$. Therefore, we have proven our desired result for c_a. Essentially the same argument applied to c_r then completes the proof. □

5.7 Proof of the main theorem

As we have already given all the necessary arguments in previous lemmata, the proof of our main theorem can now be given quite succinctly.

Proof of Theorem 5.1.1. Let n, c_a, c_r be the functions constructed in Lemma 5.5.5. By construction, said functions are nonnegative classical solutions to (AR) with boundary conditions (ARB) of appropriate regularity and, by Lemma 5.6.3 and Lemma 5.6.4, they have our desired continuity properties at $t = 0$. □

6 Immediate smoothing in a chemotaxis-consumption system starting from measure-type initial data in arbitrary dimension

6.1 Main result

This chapter is devoted to the study of the chemotaxis-consumption system

$$\begin{cases} n_t = \Delta n - \chi \nabla \cdot (n \nabla c), \\ c_t = \Delta c - cn \end{cases} \tag{C}$$

in a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N > 0$, with $\chi > 0$ and the Neumann boundary conditions

$$0 = \nabla n \cdot \nu = \nabla c \cdot \nu \qquad \text{for all } x \in \partial\Omega, t > 0. \tag{CB}$$

In a similar vein to the previous chapter, we are again interested in whether solutions to (C) with boundary conditions (CB) are able to recover when starting out in a blowup state. In fact, we show that, under a very similar smallness assumption to the classical existence case for the system but with measure-valued initial data, it is still possible to construct global classical solutions. More specifically, we prove the following theorem:

Theorem 6.1.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with a smooth boundary and let $\chi > 0$.*

Then for all initial data $n_0 \in \mathcal{M}_+(\overline{\Omega})$ with $n_0(\overline{\Omega}) > 0$ and nonnegative initial data $c_0 \in L^\infty(\Omega)$ with

$$0 < \|c_0\|_{L^\infty(\Omega)} < \frac{\pi}{\chi}\sqrt{\frac{2}{N}}, \tag{6.1.1}$$

there exist nonnegative functions $n, c \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ such that (n,c) is a classical solution of (C) with Neumann boundary conditions (CB) on $\Omega \times (0,\infty)$ and such that

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t > 0. \tag{6.1.2}$$

They further attain their initial data in the following fashion:

$$n(\cdot,t) \to n_0 \qquad \textit{in } \mathcal{M}_+(\overline{\Omega}) \tag{6.1.3}$$

$$c(\cdot,t) \to c_0 \qquad\qquad \textit{in } L^p(\Omega) \textit{ for all } p \in (1,\infty) \tag{6.1.4}$$

as $t \searrow 0$, *where we interpret the functions* $n(\cdot,t)$, $t > 0$, *as the positive Radon measures* $n(x,t)\mathrm{d}x$ *with* $\mathrm{d}x$ *being the standard Lebesgue measure on* $\overline{\Omega}$.

Remark 6.1.2. In addition to the above, the solution component c also attains its initial data c_0 in the weak-* topology on $L^\infty(\Omega)$ as a consequence of (6.1.2) and (6.1.4). This can be quickly seen by the following proof by contradiction: Assume that $c(\cdot,t)$ does not converge to c_0 in the weak-* topology on $L^\infty(\Omega)$ as $t \searrow 0$. Then there must exist $C > 0$, a function $\varphi \in L^1(\Omega)$ and a sequence of times $(t_k)_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $t_k \searrow 0$ as $k \to \infty$ and

$$\left| \int_\Omega c(\cdot,t_k)\varphi - \int_\Omega c_0\varphi \right| \geq C \qquad \text{for all } k \in \mathbb{N}. \tag{6.1.5}$$

As a consequence of (6.1.4), we can then choose a subsequence $(t_{k_j})_{j\in\mathbb{N}}$ such that $c(\cdot,t_{k_j})$ converges to c_0 pointwise almost everywhere as $j \to \infty$. The bound in (6.1.2) combined with Lebesgue's dominated convergence theorem (using $|\varphi|\|c_0\|_{L^\infty(\Omega)}$ as the majorant) then ensures that $\int_\Omega c(\cdot,t_{k_j})\varphi \to \int_\Omega c_0\varphi$ as $j \to \infty$, which contradicts (6.1.5) and thus completes the proof.

6.2 Approach

Following a similar approach as the previous chapter, we will construct our desired solutions as the limit of a family of approximate solutions $(n_\varepsilon, c_\varepsilon)_{\varepsilon\in(0,1)}$. We will further be guided by both the classical existence results for small data solutions to (C) with boundary conditions (CB) and smooth initial data by Tao (cf. [103]) and Baghaei and Khelghati (cf. [6]).

To construct the aforementioned approximate solutions, we first smoothly approximate our initial data and then apply the ready-made local existence theory from [103] to the otherwise unchanged system (C) with boundary conditions (CB). Ruling out finite-time blowup for these local solutions will be a convenient by-product of our other arguments. Using a slightly adapted version of the key result of [6], we then derive the following uniform differential inequality in Lemma 6.4.2:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) + \frac{4\delta(p-1)}{p^2}\int_\Omega \varphi(c_\varepsilon)|\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\delta}{p}\int_\Omega n_\varepsilon^p \varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \leq 0 \tag{6.2.1}$$

with some small $\delta \in (0,1)$ and $\varphi(s) \coloneqq e^{z(s)}$, $s \in [0, \|c_0\|_{L^\infty(\Omega)}]$ for all $p \in (1, \frac{N}{2}+\delta]$, where z is a solution to the differential equation in (6.4.1) with carefully chosen parameters. It is this argument that necessitates the introduction of the smallness condition in (6.1.1). Notably while in [6] any a priori information of dissipative type thereby obtained was discarded, it will play a crucial role in our later derivations, which is the key reason making the modifications to the original argument necessary.

In view of the potential irregularity of our initial data, we naturally cannot expect the functional $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$ to be uniformly bounded at $t = 0$. Fortunately, the second term in (6.2.1), which is of dissipative type, suggests that our functional enjoys immediate boundedness properties independent of the initial state as long as there is at least a uniform mass bound available similar to e.g. the immediate smoothing properties of solutions to the heat equation. Instead of using a standard comparison argument to show that this is in fact the case, we will use a similar technique as used in the previous chapter to address the parabolic-parabolic case, which will allow us to further extract some additional a priori information for the third term in (6.2.1). More specifically by multiplying our functional by t^λ with a sufficiently large $\lambda > 0$ and analyzing the time evolution of the new functional obtained in this way, we gain the following uniform bounds on any time interval $(0, T)$ in Lemma 6.4.4:

$$t^\lambda \int_\Omega n_\varepsilon^p(x,t)\,\mathrm{d}x \le C(T,p,\lambda) \tag{6.2.2}$$

and

$$\int_0^t s^\lambda \int_\Omega n_\varepsilon(x,s)^p |\nabla c_\varepsilon(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T,p,\lambda) \tag{6.2.3}$$

In Section 6.5, we then use (6.2.2) as the basis of a straightforward bootstrap argument employing semigroup theory as well as standard parabolic regularity theory to gain sufficiently strong parabolic Hölder space bounds for our approximate solutions. Using standard compact embedding properties of Hölder spaces, this then allows us to construct functions $n, c \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ as the limit of our approximate solutions by diagonal sequence argument, which immediately inherit their classical solution properties from said approximate solutions. Notably, the improvement of the initial data condition in our result stems at its core from us starting this bootstrap procedure at a uniform-in-time bound for n_ε in a function space just slightly above $L^{\frac{N}{2}}(\Omega)$ and not in any smaller function space.

In Section 6.6, it now remains to show that the functions n and c also satisfy the continuity properties in (6.1.3) and (6.1.4), respectively. Our approach here is twofold. We first argue that our approximate solutions were already uniformly continuous in $t = 0$ in appropriate topologies and then only need to show that this uniform continuity property survives the limit process. While this is fairly straightforward for the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$, treating the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is a bit more involved due to the problematic taxis term in the first equation of (C). To overcome this problem and control the taxis term, we make decisive use of (6.2.3) to show in Lemma 6.6.1 that $\int_0^t \|n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)}\,\mathrm{d}s$ becomes uniformly small as $t \searrow 0$.

6.3 Approximate solutions

For the remainder of this chapter, we fix a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, with a smooth boundary, the system parameter $\chi > 0$ as well as the initial data $n_0 \in \mathcal{M}_+(\overline{\Omega})$ and $c_0 \in L^\infty(\Omega)$ with $m := n_0(\overline{\Omega}) > 0$, $c_0 \geq 0$ and $c_0 \not\equiv 0$. Constants and parameters in the results and proofs of this chapter will only implicitly depend on the above parameters. Any other dependencies will be made explicit.

As already expanded upon in the previous section, we will construct our desired solutions as the limit of a family of approximate solutions obtained by using the already established local existence theory for (C) with boundary conditions (CB) from [103]. To this end, we begin by fixing a family $(n_{0,\varepsilon})_{\varepsilon\in(0,1)} \in C^\infty(\overline{\Omega})$ such that all $n_{0,\varepsilon}$ are positive, that

$$n_{0,\varepsilon} \to n_0 \text{ in } \mathcal{M}_+(\overline{\Omega}) \text{ as } \varepsilon \searrow 0 \quad \text{and that} \quad \int_\Omega n_{0,\varepsilon} = n_0(\overline{\Omega}) =: m \text{ for all } \varepsilon \in (0,1). \tag{6.3.1}$$

For more details on how such an approximation can be achieved, we refer the reader to Remark 5.3.1 in the previous chapter. We further set $c_{0,\varepsilon} := e^{\varepsilon\Delta}c_0 \in C^\infty(\overline{\Omega})$ for each $\varepsilon \in (0,1)$, where $(e^{t\Delta})_{t\geq 0}$ is the Neumann heat semigroup. Due to the maximum principle, this immediately guarantees positivity of all $c_{0,\varepsilon}$ and further that

$$\|c_{0,\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \tag{6.3.2}$$

for all $\varepsilon \in (0,1)$. Moreover, the continuity properties of the heat semigroup ensure that

$$c_{0,\varepsilon} \to c_0 \quad \text{in } L^p(\Omega) \text{ for all } p \in (1,\infty) \tag{6.3.3}$$

as $\varepsilon \searrow 0$.

As already mentioned, this now allows us to construct local solutions to (C) with boundary conditions (CB) and initial data $(n_{0,\varepsilon}, c_{0,\varepsilon})$ using [103, Lemma 2.1].

Lemma 6.3.1. *For each $\varepsilon \in (0,1)$, there exists a maximal existence time $T_{\max,\varepsilon} \in (0,\infty]$ and a nonnegative classical solution*

$$(n_\varepsilon, c_\varepsilon) \in \Big(C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon}))\Big)^2$$

to (C) on $\Omega \times (0, T_{\max,\varepsilon})$ with Neumann boundary conditions (CB) and initial data $(n_{0,\varepsilon}, c_{0,\varepsilon})$ as well as the following blowup criterion:

$$\textit{If } T_{\max,\varepsilon} < \infty, \textit{ then } \limsup_{t \nearrow T_{\max,\varepsilon}} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{6.3.4}$$

Further,

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_{0,\varepsilon} = n_0(\overline{\Omega}) = m \quad \textit{and} \quad \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_{0,\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \tag{6.3.5}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof. This lemma is a direct consequence of [103, Lemma 2.1]. □

For the remainder of this chapter, we now fix the approximate solutions $(n_\varepsilon, c_\varepsilon)$ established in the previous lemma as well as their maximal existence time $T_{\max,\varepsilon}$ for all $\varepsilon \in (0, 1)$.

6.4 A priori estimates up to $t = 0$

In this section, we will prove some key uniform a priori estimates for the approximate solutions up to $t = 0$. As the arguments to achieve such estimates will in part be built on the techniques first developed in [103] and later refined in [6] to construct classical solutions to (C) with boundary conditions (CB) and smooth initial data, let us briefly reiterate the core ideas of said references to give some context. The key innovation of [103] is to investigate the time evolution of the functional $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$ with $\varphi(s) := e^{z(s)}$ and $z(s) := (\beta s)^2$, where β is carefully chosen to facilitate pointwise estimates for φ and certain multiples of its first and second derivatives to allow any term suggesting growth of the functional to be absorbed by either of two terms of dissipative type using Young's inequality. Notably, said estimates only work for small values of s and are thus the reason arguments in this lineage generally require a smallness condition on c_0 of the type seen in (6.1.1).

In [6], Baghaei and Khelghati then showed that this approach can be extended to work under a much weaker smallness condition on the initial data by making even more effective use of the subtle structures present in the differential equation solved by the functional. In fact, their key observation is that, if z is a solution to the ordinary differential equation

$$z''(s) = \frac{A}{C}(z'(s))^2 + \frac{B}{C}z'(s) + \frac{D}{C} \tag{6.4.1}$$

with carefully chosen parameters A, B, C and D, the terms suggesting growth of the functional together with their dissipative counterparts are equal to an integral of the form $-\int_\Omega (f(\cdot,t) - g(\cdot,t))^2 \leq 0$ due to the binomial theorem. In their original argument, this transformation required full use of the dissipative terms and thus yielded no a priori information in this regard. As this kind of dissipative information will prove crucial to our later arguments, some slight modifications to their original construction of z will thus be necessary.

Lemma 6.4.1. *Let $\delta \in [0, 1)$ and $p \in (1, \infty)$. Then the function*

$$z(s) := z_{p,\delta}(s) := \int_0^s \left[-\frac{B}{2A} + \frac{\sqrt{4AD - B^2}}{2A} \tan\left(\frac{\sqrt{4AD - B^2}}{2C} \tau \right) \right] \mathrm{d}\tau \tag{6.4.2}$$

with

$$\begin{aligned} A = A(p,\delta) &:= 4 - \frac{4(p-1)(1-\delta)^2}{p} > 0, \quad & B = B(p,\delta) &:= -4\delta\chi(p-1) \leq 0, \\ C = C(p,\delta) &:= \frac{4(p-1)(1-\delta)^2}{p} > 0, & D = D(p,\delta) &:= \chi^2(p-1)^2 > 0, \end{aligned} \tag{6.4.3}$$

is well-defined and smooth for all $s \in [0, s_0]$ *with*

$$0 < s_0 < \frac{\pi}{\chi}\frac{1}{\sqrt{p}}\sqrt{\frac{(1-\delta)^3}{2\delta p + (1-\delta)}} \tag{6.4.4}$$

and solves the ordinary differential equation in (6.4.1) on $(0, s_0)$*. Lastly,*

$$z'(s) \geq -\frac{B}{2A} \geq 0 \quad \textit{and} \quad z''(s) \geq \frac{4AD - B^2}{4AC} > 0 \tag{6.4.5}$$

holds for all $s \in (0, s_0)$*.*

Proof. We first note that

$$\begin{aligned} 4AD - B^2 &= 4\left(4 - \frac{4(p-1)(1-\delta)^2}{p}\right)\chi^2(p-1)^2 - 16\delta^2\chi^2(p-1)^2 \\ &= 16\chi^2\frac{(p-1)^2}{p}\left[p - (p-1)(1-\delta)^2 - p\delta^2\right] \\ &= 16\chi^2\frac{(p-1)^2(1-\delta)}{p}\left[p(1+\delta) - (p-1)(1-\delta)\right] \\ &= 16\chi^2\frac{(p-1)^2(1-\delta)}{p}\left[2\delta p + (1-\delta)\right] > 0 \end{aligned}$$

and thus further that

$$0 \leq \frac{\sqrt{4AD - B^2}}{2C}s = \frac{\chi\sqrt{p}}{2}\sqrt{\frac{2\delta p + (1-\delta)}{(1-\delta)^3}}s < \frac{\pi}{2}$$

for all $s \in [0, s_0]$ due to (6.4.4), which ensures that z is well-defined on $[0, s_0]$ and smooth. As $z''(s) = \frac{4AD-B^2}{4AC}\left[1 + \tan^2\left(\frac{\sqrt{4AD-B^2}}{2C}s\right)\right]$, this also directly yields (6.4.5). Finally because the solution property in (6.4.1) is purely a consequence of the structure present in (6.4.2) and largely independent of the specific choice of constants, it can be derived in the exact same fashion as in [6]. □

Even after the above modifications to the approach, we still evidently cannot expect the functional $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$ to be uniformly bounded at $t = 0$. Nonetheless, a differential inequality for said functional of a similar type to the one used in [6] or [103] will still prove invaluable.

Lemma 6.4.2. *If c_0 satisfies (6.1.1), then there exists $\delta \in (0,1)$ such that*

$$\varphi(s) \coloneqq e^{z(s)}$$

with $z \equiv z_{p,\delta}$ as in Lemma 6.4.1 is well-defined for all $p \in (1, \frac{N}{2}+\delta]$ and $s \in [0, \|c_0\|_{L^\infty(\Omega)}]$ and the inequality

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) + \frac{4\delta(p-1)}{p^2}\int_\Omega \varphi(c_\varepsilon)|\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\delta}{p}\int_\Omega n_\varepsilon^p \varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \le 0 \tag{6.4.6}$$

holds for all $p \in (1, \frac{N}{2}+\delta]$, $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. According to our assumption that c_0 satisfies (6.1.1), we can fix a small $\delta \in (0,1)$ such that

$$\|c_0\|_{L^\infty(\Omega)} < \frac{\pi}{\chi}\frac{1}{\sqrt{\frac{N}{2}+\delta}}\sqrt{\frac{(1-\delta)^3}{\delta N + 2\delta^2 + (1-\delta)}} \le \frac{\pi}{\chi}\frac{1}{\sqrt{p}}\sqrt{\frac{(1-\delta)^3}{2\delta p + (1-\delta)}}$$

for all $p \in (1, \frac{N}{2}+\delta]$. By Lemma 6.4.1, this directly ensures that $z_{\delta,p}$ and thus φ are well-defined on $[0, \|c_0\|_{L^\infty(\Omega)}]$ for all $p \in (1, \frac{N}{2}+\delta]$.

Applying the differential equations from (C) as well as partial integration to the terms resulting from time differentiation of the functional $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$, we then gain that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) \\
&= \int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon) n_{\varepsilon t} + \frac{1}{p}\int_\Omega n_\varepsilon^p \varphi'(c_\varepsilon) c_{\varepsilon t} \\
&= \int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)\Delta n_\varepsilon - \chi\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)\nabla\cdot(n_\varepsilon\nabla c_\varepsilon) \\
&\quad + \frac{1}{p}\int_\Omega n_\varepsilon^p\varphi'(c_\varepsilon)\Delta c_\varepsilon - \frac{1}{p}\int_\Omega n_\varepsilon^{p+1}\varphi'(c_\varepsilon)c_\varepsilon \\
&= -(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2 - \int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla n_\varepsilon \\
&\quad + \chi(p-1)\int_\Omega n_\varepsilon^{p-1}\varphi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla n_\varepsilon + \chi\int_\Omega n_\varepsilon^p\varphi'(c_\varepsilon)|\nabla c_\varepsilon|^2 \\
&\quad - \int_\Omega n_\varepsilon^{p-1}\varphi'(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \frac{1}{p}\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 - \frac{1}{p}\int_\Omega n_\varepsilon^{p+1}\varphi'(c_\varepsilon)c_\varepsilon
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Given that the nonnegativity of $\varphi' = z'\varphi$ due to (6.4.5) implies that $-\frac{1}{p}\int_\Omega n_\varepsilon^{p+1}\varphi'(c_\varepsilon)c_\varepsilon \le 0$, it further follows that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon) + \delta(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{\delta}{p}\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \\
&\le -(1-\delta)(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \left|\chi(p-1)\varphi(c_\varepsilon)-2\varphi'(c_\varepsilon)\right| n_\varepsilon^{p-1}|\nabla n_\varepsilon||\nabla c_\varepsilon| \\
&-\int_\Omega \left(\frac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon)\right) n_\varepsilon^p|\nabla c_\varepsilon|^2
\end{aligned} \tag{6.4.7}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$ after some slight rearrangement.

Regarding the rightmost term in the above inequality, the fact that

$$
\varphi'(c_\varepsilon)=z'(c_\varepsilon)\varphi(c_\varepsilon)
$$

as well as

$$
\varphi''(c_\varepsilon)=z''(c_\varepsilon)\varphi(c_\varepsilon)+z'(c_\varepsilon)\varphi'(c_\varepsilon)=\left[z''(c_\varepsilon)+(z'(c_\varepsilon))^2\right]\varphi(c_\varepsilon)
$$

combined with the ordinary differential equation in (6.4.1) then yields

$$
\begin{aligned}
&\frac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon) \\
&=\frac{1-\delta}{p}\left(z''(c_\varepsilon)+(z'(c_\varepsilon))^2-\frac{p\chi}{1-\delta}z'(c_\varepsilon)\right)\varphi(c_\varepsilon) \\
&=\frac{1-\delta}{pC}\left((A+C)(z'(c_\varepsilon))^2+\frac{B(1-\delta)-p\chi C}{(1-\delta)}z'(c_\varepsilon)+D\right)\varphi(c_\varepsilon),
\end{aligned}
$$

where $A=A(p,\delta)$, $B=B(p,\delta)$, $C=C(p,\delta)$ and $D=D(p,\delta)$ are the constants defined in (6.4.3). By directly using the definition of said constants, we now see that $A+C=4$, $D=\chi^2(p-1)^2$ as well as

$$
\frac{B(1-\delta)-p\chi C}{(1-\delta)}=\frac{-4\delta\chi(p-1)(1-\delta)-4\chi(p-1)(1-\delta)^2}{(1-\delta)}=-4\chi(p-1)
$$

and thus that

$$
\frac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon)=\frac{1-\delta}{pC}\left(2z'(c_\varepsilon)-\chi(p-1)\right)^2\varphi(c_\varepsilon)\geq 0. \tag{6.4.8}
$$

This then allows us to complete the square on the right-hand side of (6.4.7) to gain

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon)+\delta(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2+\frac{\delta}{p}\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \\
&\leq-\int_\Omega\left(\sqrt{(1-\delta)(p-1)\varphi(c_\varepsilon)}n_\varepsilon^{\frac{p}{2}-1}|\nabla n_\varepsilon|-\sqrt{\tfrac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon)}n_\varepsilon^{\frac{p}{2}}|\nabla c_\varepsilon|\right)^2 \\
&+\int_\Omega\left[\left|\chi(p-1)\varphi(c_\varepsilon)-2\varphi'(c_\varepsilon)\right|-2\sqrt{(1-\delta)(p-1)\varphi(c_\varepsilon)\left(\tfrac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon)\right)}\right]\times \\
&\qquad n_\varepsilon^{p-1}|\nabla n_\varepsilon||\nabla c_\varepsilon|
\end{aligned}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Now noting that

$$
2\sqrt{(1-\delta)(p-1)\varphi(c_\varepsilon)\left(\tfrac{1-\delta}{p}\varphi''(c_\varepsilon)-\chi\varphi'(c_\varepsilon)\right)}
$$

$$= 2\sqrt{\frac{(1-\delta)^2(p-1)}{pC}(2z'(c_\varepsilon) - \chi(p-1))^2\varphi^2(c_\varepsilon)}$$
$$= |2\varphi'(c_\varepsilon) - \chi(p-1)\varphi(c_\varepsilon)|$$

due to (6.4.8), the definition of C in (6.4.3) and the fact that $\varphi' = z'\varphi$, it follows that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^p\varphi(c_\varepsilon) + \delta(p-1)\int_\Omega n_\varepsilon^{p-2}\varphi(c_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{\delta}{p}\int_\Omega n_\varepsilon^p\varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \le 0$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. This directly implies our desired result because $n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 = \frac{4}{p^2}|\nabla n_\varepsilon^{\frac{p}{2}}|^2$. □

Remark 6.4.3. The original argument by Tao from [103] can be similarly adapted to gain essentially the same result as above but only under the stricter initial data condition $0 < \|c_0\|_{L^\infty(\Omega)} < \frac{2}{3N\chi}$.

Having now derived the above differential inequality, we can see that it not only provides a favorable monotonicity property for the functional under consideration but also yields a dissipative term of the form $\int_\Omega \varphi(c_\varepsilon)|\nabla n_\varepsilon^{\frac{p}{2}}|^2$, which is similar to terms often used to derive the immediate smoothing properties of e.g. the heat equation as $\varphi(c_\varepsilon)$ is uniformly bounded from below and above. In our case, this dissipative character of the above inequality will in fact lead to a similar immediate uniform boundedness property for the functional independent of its initial value (and only relying on a uniform mass bound). While we could again use this to derive our desired boundedness property by showing that the functional is a subsolution to an ordinary differential equation with superlinear decay as in Section 5.4.2, we will instead take the same approach as in Section 5.4.1 to also gain some time-weighted a priori information about the third term in (6.4.6). This information will later be crucial to prove that our construction yields solutions that are still connected to the initial data in a reasonable fashion.

Lemma 6.4.4. *If c_0 satisfies (6.1.1), then there exists $\delta > 0$ such that the following holds:*

For each $T \in (0,\infty)$, $p \in (1, \frac{N}{2} + \delta]$ and $\lambda > \frac{N}{2}(p-1)$, there exists $C = C(T,p,\lambda) > 0$ such that

$$t^\lambda \int_\Omega n_\varepsilon^p(x,t)\,\mathrm{d}x \le C \tag{6.4.9}$$

and

$$\int_0^t s^\lambda \int_\Omega n_\varepsilon^p(x,s)|\nabla c_\varepsilon(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \le C \tag{6.4.10}$$

for all $t \in (0,T) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. We begin by fixing $\delta > 0$ as in Lemma 6.4.2, $p \in (1, \frac{N}{2}+\delta]$ as well as $\lambda > \frac{N}{2}(p-1)$. Multiplying the functional $\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)$ by t^λ, with φ as in Lemma 6.4.2, and differentiating regarding the time variable yields

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\lambda \int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)\right] + \frac{4\delta(p-1)}{p^2} t^\lambda \int_\Omega \varphi(c_\varepsilon)|\nabla n_\varepsilon^{\frac{p}{2}}|^2 + \frac{\delta}{p} t^\lambda \int_\Omega n_\varepsilon^p \varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \\
&\le \frac{\lambda}{p} t^{\lambda-1} \int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)
\end{aligned} \tag{6.4.11}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ due to (6.4.6). Noting that due to the first half of (6.4.5) we know that

$$\varphi(c_\varepsilon(\cdot,t)) = e^{z(c_\varepsilon(\cdot,t))} \le e^{z(\|c_0\|_{L^\infty(\Omega)})} =: K_1 = K_1(p)$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ with z as constructed in Lemma 6.4.1, we can apply a version of the Gagliardo–Nirenberg inequality allowing for a conveniently expanded parameter range (cf. [70, Lemma 2.3]) to the problem term on the right-hand side of the above inequality to gain $K_2 = K_2(p) > 0$ such that

$$\begin{aligned}
\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) &\le K_1 \|n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \le K_1 K_2 \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p-1)}{p-1+\frac{2}{N}}} \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{\frac{4}{N}}{p-1+\frac{2}{N}}} + K_1 K_2 \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\le K_3 \left(\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2\right)^{\frac{p-1}{p-1+\frac{2}{N}}} + K_3
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $K_3 = K_3(p) \coloneqq K_1 K_2 \max(m^{2p/(Np-N+2)}, m^p)$ due to the mass conservation property in (6.3.5). Using Young's inequality and the fact that $\varphi \ge e^{z(0)} = 1$ due to (6.4.5), the above then yields

$$\begin{aligned}
\int_\Omega n_\varepsilon^p \varphi(c_\varepsilon) &\le \tfrac{4\delta(p-1)}{\lambda p} t \int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_4 t^{-\frac{N}{2}(p-1)} + K_3 \\
&\le \tfrac{4\delta(p-1)}{\lambda p} t \int_\Omega \varphi(c_\varepsilon)|\nabla n_\varepsilon^{\frac{p}{2}}|^2 + K_4 t^{-\frac{N}{2}(p-1)} + K_3
\end{aligned}$$

with $K_4 = K_4(p,\lambda) \coloneqq (\frac{\lambda p}{4\delta(p-1)})^{\frac{N}{2}(p-1)} K_3^{\frac{N}{2}(p-1+\frac{2}{N})}$, which applied to (6.4.11) results in

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\lambda \int_\Omega n_\varepsilon^p \varphi(c_\varepsilon)\right] + \frac{\delta}{p} t^\lambda \int_\Omega n_\varepsilon^p \varphi''(c_\varepsilon)|\nabla c_\varepsilon|^2 \le K_5 (t^{\lambda-\frac{N}{2}(p-1)-1} + t^{\lambda-1})$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $K_5 = K_5(p,\lambda) \coloneqq \frac{\lambda}{p}\max(K_3, K_4)$. As our choice of λ ensures that

$$\lambda - 1 \ge \lambda - \tfrac{N}{2}(p-1) - 1 > -1,$$

time integration combined with the fact that $\varphi \ge 1$ and $\varphi'' = (z'' + (z')^2)\varphi \ge z'' \ge \frac{4AD-B^2}{4AC} =: K_6 = K_6(p) > 0$ due to (6.4.5) then completes the proof with $C(T,p,\lambda) \coloneqq \frac{pK_5}{\delta(\lambda-\frac{N}{2}(p-1))}\max(1, \frac{1}{K_6})(T^{\lambda-\frac{N}{2}(p-1)} + T^\lambda)$. □

6.5 Construction of a solution candidate

Using the uniform bounds for the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{N}{2}+\delta}(\Omega)$ from Lemma 6.4.4 as a baseline, we now devote this section to deriving sufficient parabolic Hölder bounds for our approximate solutions to construct a solution candidate for our main theorem by employing suitable compact embedding properties of said Hölder spaces. As we have already presented this type of bootstrap argument fairly extensively in the previous chapter, we will keep most of the proofs in this section fairly brief.

Our first step is to improve the bounds gained from (6.4.9) using semigroup methods in a similar fashion to Lemmas 5.5.1 and 5.5.2.

Lemma 6.5.1. *If c_0 satisfies (6.1.1), then there exists $q > N$ such that the following holds:*

For each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exists $C = C(t_0, t_1) > 0$ such that

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \le C$$

for all $t \in (t_0, t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. We begin by fixing $t_0 > 0$ and $t_1 > t_0$. According to Lemma 6.4.4, we can then fix $q > N$ as well as $K_1 = K_1(t_0, t_1) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^{\frac{q}{2}}(\Omega)} < K_1$$

for all $t \in (t_0/2, t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Using the variation-of-constants representation of c_ε as well as the smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]), we can then find $K_2 = K_2(t_0, t_1) > 0$ such that

$$\begin{aligned}
&\|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \\
&\le \|\nabla e^{(t-\frac{t_0}{2})\Delta} c_\varepsilon(\cdot,\tfrac{t_0}{2})\|_{L^q(\Omega)} + \int_{t_0/2}^t \|\nabla e^{(t-s)\Delta} n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \,\mathrm{d}s \\
&\le K_2(t-\tfrac{t_0}{2})^{-\frac{1}{2}} \|c_\varepsilon(\cdot,\tfrac{t_0}{2})\|_{L^q(\Omega)} + K_2 \int_{t_0/2}^t (t-s)^{-\frac{1}{2}-\frac{N}{2q}} \|n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^{\frac{q}{2}}(\Omega)} \,\mathrm{d}s \\
&\le K_2(\tfrac{t_0}{2})^{-\frac{1}{2}} |\Omega|^{\frac{1}{q}} \|c_0\|_{L^\infty(\Omega)} + K_1K_2\|c_0\|_{L^\infty(\Omega)} (\tfrac{1}{2} - \tfrac{N}{2q})^{-1} (t_1 - \tfrac{t_0}{2})^{\frac{1}{2}-\frac{N}{2q}}
\end{aligned}$$

for all $t \in (t_0, t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ as $q > N$ ensures that $-\frac{1}{2} - \frac{N}{2q} > -1$. Combined with (6.3.5), this completes the proof. □

Lemma 6.5.2. *If c_0 satisfies (6.1.1), then, for all $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exists $C = C(t_0, t_1) > 0$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t \in (t_0, t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. We begin by fixing $t_0 > 0$ and $t_1 > t_0$. Due to Lemma 6.4.4 and Lemma 6.5.1, we can then fix $q > N$ and $K_1 = K_1(t_0, t_1) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^{\frac{q}{2}}(\Omega)} \le K_1 \quad \text{and} \quad \|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le K_1$$

for all $t \in [t_0/2, t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Let then $r \in (N, q)$ be such that $r > \frac{q}{2}$ and further

$$M_\varepsilon(T) \coloneqq \sup_{t\in(t_0/2,t_1)\cap(0,T)} \left\|(t-\tfrac{t_0}{2})^{\frac{N}{q}} n_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} < \infty$$

for all $T \in (t_0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Now using the variation-of-constants representation of n_ε, the smoothing properties of the Neumann heat semigroup (cf. [123, Lemma 1.3]) as well as the Hölder inequality in a similar way to the proof of Lemma 5.5.2, we can find $K_2 = K_2(t_0, t_1) > 0$ such that

$$\begin{aligned}
&\left\|(t-\tfrac{t_0}{2})^{\frac{N}{q}} n_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} \\
&\le \left\|(t-\tfrac{t_0}{2})^{\frac{N}{q}} e^{(t-\frac{t_0}{2})\Delta} n_\varepsilon(\cdot,\tfrac{t_0}{2})\right\|_{L^\infty(\Omega)} \\
&\quad + \left\|\chi(t-\tfrac{t_0}{2})^{\frac{N}{q}} \int_{t_0/2}^t e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s))\,\mathrm{d}s\right\|_{L^\infty(\Omega)} \\
&\le K_2\|n_\varepsilon(\cdot,\tfrac{t_0}{2})\|_{L^{\frac{q}{2}}(\Omega)} + K_2(t-\tfrac{t_0}{2})^{\frac{N}{q}} \int_{t_0/2}^t (t-s)^{-\frac12-\frac{N}{2r}} \|n_\varepsilon(\cdot,s)\nabla c_\varepsilon(\cdot,s)\|_{L^r(\Omega)}\,\mathrm{d}s \\
&\le K_1K_2 + K_1K_2(t-\tfrac{t_0}{2})^{\frac{N}{q}} \int_{t_0/2}^t (t-s)^{-\frac12-\frac{N}{2r}} \|n_\varepsilon(\cdot,s)\|_{L^{\frac{qr}{q-r}}(\Omega)}\,\mathrm{d}s \\
&\le K_1K_2 + K_1^{\frac{q+r}{2r}} K_2(t-\tfrac{t_0}{2})^{\frac{N}{q}} \int_{t_0/2}^t (t-s)^{-\frac12-\frac{N}{2r}} \|n_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{3r-q}{2r}}\,\mathrm{d}s \\
&\le K_1K_2 + K_1^{\frac{q+r}{2r}} K_2(M_\varepsilon(T))^{\frac{3r-q}{2r}}(t-\tfrac{t_0}{2})^{\frac{N}{q}} \int_{t_0/2}^t (t-s)^{-\frac12-\frac{N}{2r}}(s-\tfrac{t_0}{2})^{-\frac{N}{q}\frac{3r-q}{2r}}\,\mathrm{d}s \qquad (6.5.1)
\end{aligned}$$

for all $t \in (t_0/2, t_1) \cap (0,T)$, $T \in (t_0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Note here that $\frac{qr}{q-r} > \frac{q}{2}$ and $\frac{3r-q}{2r} \in (0,1)$ as $q > r > \frac{q}{2}$.

Now applying the linear substitution $s \mapsto (t-\frac{t_0}{2})s + \frac{t_0}{2}$ to the last remaining integral term yields

$$\begin{aligned}
&\int_{t_0/2}^t (t-s)^{-\frac12-\frac{N}{2r}}(s-\tfrac{t_0}{2})^{-\frac{N}{q}\frac{3r-q}{2r}}\,\mathrm{d}s \\
&= (t-\tfrac{t_0}{2})^{1-\frac12-\frac{N}{2r}-\frac{N}{q}\frac{3r-q}{2r}} \int_0^1 (1-s)^{-\frac12-\frac{N}{2r}} s^{-\frac{N}{q}\frac{3r-q}{2r}}\,\mathrm{d}s = K_3(t-\tfrac{t_0}{2})^{\frac12-\frac{3N}{2q}}
\end{aligned}$$

for all $t \in (t_0/2, t_1)$ with $K_3 \coloneqq B(1-\frac{N}{q}\frac{3r-q}{2r}, \frac12-\frac{N}{2r})$, where B is the beta function. Notably this is only possible because $q > r > N$ ensures that $-\frac12-\frac{N}{2r} > -1$ as well as

$-\frac{N}{q}\frac{3r-q}{2r} > -\frac{3r-q}{2r} > -1$ and thus that all of the integrals involved are finite. Applying the above substitution to (6.5.1), we then gain that

$$M_\varepsilon(T) \leq K_4 + K_4(M_\varepsilon(T))^{\frac{3r-q}{2r}} \tag{6.5.2}$$

for all $T \in (t_0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ with

$$K_4 = K_4(t_0,t_1) \coloneqq \max(K_1K_2, K_1^{\frac{q+r}{2r}} K_2K_3(t_1 - \tfrac{t_0}{2})^{\frac{1}{2}-\frac{N}{2q}})$$

as $q > N$ implies that $\frac{1}{2} - \frac{3N}{2q} + \frac{N}{q} = \frac{1}{2} - \frac{N}{2q} > 0$. As $\frac{3r-q}{2r} \in (0,1)$, the inequality in (6.5.2) immediately implies that

$$M_\varepsilon(T) \leq K_5$$

for all $T \in (t_0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $K_5 = K_5(t_0,t_1) \coloneqq 2K_4 + (2K_4)^{\frac{2r}{q-r}}$ by Young's inequality. From this, it directly follows that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq (\tfrac{t_0}{2})^{-\frac{N}{q}} K_5$$

for all $t \in (t_0,t_1) \cap (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. □

The above lemma is now sufficient to rule out finite-time blowup for our approximate solutions according to the blowup criterion in (6.3.4).

Corollary 6.5.3. *If c_0 satisfies (6.1.1), then $T_{\max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$.*

The remainder of the bootstrap argument is now a straightforward combination of standard parabolic regularity theory from the literature and we will thus keep the following proof relatively brief as it follows roughly the same outline as Section 5.5.

Lemma 6.5.4. *If c_0 satisfies (6.1.1), then, for each $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$, there exist $C = C(t_0,t_1) > 0$ and $\theta = \theta(t_0,t_1) \in (0,1)$ such that*

$$\|n_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \leq C \quad \text{and} \quad \|c_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t_0,t_1])} \leq C$$

for all $\varepsilon \in (0,1)$.

Proof. Using the bounds established in Lemma 6.5.1 and Lemma 6.5.2 allows us to apply standard parabolic Hölder regularity theory (cf. [97, Theorem 1.3]) to find $\theta_1 = \theta_1(t_0,t_1) \in (0,1)$ such that both $(n_\varepsilon)_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ are uniformly bounded in $C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega} \times [\frac{t_0}{8}, t_1])$, where the bounds here and similar bounds later in the proof only depend on t_0 and t_1. Using further parabolic Hölder regularity theory from [61, p. 170 and p. 320] combined with a straightforward cutoff function argument, we then gain $\theta_2 = \theta_2(t_0,t_1) \in (0,1)$ such that the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega} \times [\frac{t_0}{4}, t_1])$. We can then use a theorem due to Lieberman (cf. [73, Theorem 1.1]) and another cutoff function argument to find $\theta_3 = \theta_3(t_0, t_1) \in (0,1)$ such that the family $(\nabla n_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{\theta_3, \frac{\theta_3}{2}}(\overline{\Omega} \times [\frac{t_0}{2}, t_1])$. This then yet again allows us to apply regularity results from [61, p. 170 and p. 320] in a similar fashion as before to gain $\theta_4 = \theta_4(t_0, t_1) \in (0,1)$ such that the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{2+\theta_4, 1+\frac{\theta_4}{2}}(\overline{\Omega} \times [t_0, t_1])$.

Setting $\theta = \theta(t_0, t_1) := \min(\theta_2, \theta_4)$ then completes the proof. □

Using the above bounds combined with the Arzelà–Ascoli compact embedding theorem, we can now construct candidates for our desired solutions as limits of their approximate counterparts. Given the strength of the bounds outside of $t = 0$ in the above lemma, it is then easy to see that all solution properties apart from those concerned with the initial data immediately translate from our approximate solutions to their limits.

Lemma 6.5.5. *If c_0 satisfies (6.1.1), then there exist a null sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq (0,1)$ as well as nonnegative functions $n, c \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ such that*

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \tag{6.5.3}$$

for all $t > 0$, such that

$$n_\varepsilon \to n \quad \text{in } C^{2,1}(\overline{\Omega} \times [t_0, t_1]), \tag{6.5.4}$$

$$c_\varepsilon \to c \quad \text{in } C^{2,1}(\overline{\Omega} \times [t_0, t_1]) \tag{6.5.5}$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0 > 0$ as $\varepsilon = \varepsilon_j \searrow 0$, and such that (n, c) is a classical solution to (C) with boundary conditions (CB) on $\Omega \times (0, \infty)$.

Proof. For every $k \in \mathbb{N}$, we can find $\theta_k \in (0,1)$ such that the families $(n_\varepsilon)_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ are uniformly bounded in $C^{2+\theta_k, 1+\frac{\theta_k}{2}}(\overline{\Omega} \times [\frac{1}{k}, k])$ as a consequence of Lemma 6.5.4. As the spaces $C^{2+\theta_k, 1+\frac{\theta_k}{2}}(\overline{\Omega} \times [\frac{1}{k}, k])$ embed compactly into the spaces $C^{2,1}(\overline{\Omega} \times [\frac{1}{k}, k])$ for all $k \in \mathbb{N}$ due to e.g. the Arzelà–Ascoli theorem, a straightforward diagonal sequence argument then allows us to construct functions $n, c \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ with the convergence properties seen in (6.5.4) and (6.5.5) by successively extracting convergent subsequence in $C^{2,1}(\overline{\Omega} \times [\frac{1}{k}, k])$ for increasing values of $k \in \mathbb{N}$. As the resulting convergence properties ensure that all terms in the system (C) converge pointwise on $\Omega \times (0,\infty)$ and the terms in the boundary conditions (CB) converge pointwise on $\partial\Omega \times (0,\infty)$, respectively, it also follows immediately that (n, c) is a classical solution to (C) with boundary conditions (CB) as this was already the case for each $(n_\varepsilon, c_\varepsilon)$. Nonnegativity of n and c as well as the upper bound (6.5.3) are further properties inherited from the approximate solutions due to our strong convergence properties and the basic solution properties laid out in Lemma 6.3.1. □

6.6 Continuity in $t = 0$

Having now constructed our solution candidate (n, c) in Lemma 6.5.5, it remains to show that said solution candidate is connected to the initial data (n_0, c_0) in a sensible fashion, i.e. we want to show that n and c are continuous in $t = 0$ in some appropriate topology and with the correct values at $t = 0$. As we will see, this will prove rather more involved for the first solution component than the second solution component due to some additional effort necessary to handle the taxis term. Nonetheless in both cases, the approach is the same at a fundamental level. We first show that our approximate solutions were already uniformly continuous in $t = 0$ in some appropriate sense and then show that this property survives the limit process due to its uniformity.

We begin by treating the first solution component n. As our first step toward deriving the aforementioned uniform continuity property, we will make crucial use of the bound (6.4.10) from Lemma 6.4.4 to show that a certain space-time integral connected to the taxis becomes uniformly small as the upper bound of the integration time interval approaches zero.

Lemma 6.6.1. *If c_0 satisfies (6.1.1), then there exist $C > 0$ and $\alpha > 0$ such that*

$$\int_0^t \|n_\varepsilon(\cdot, s)\nabla c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s \le Ct^\alpha$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$.

Proof. Let $\delta \in (0, 1)$ be as in Lemma 6.4.4. We then fix $\lambda \in (\frac{\delta}{2}, 1)$ and $\alpha \in (0, 1)$ such that

$$\lambda + \alpha < 1 - \alpha.$$

Using Young's inequality as well as our choice of α, we now note that

$$\begin{aligned}
&\int_0^t \|n_\varepsilon(\cdot, s)\nabla c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&\le \int_0^t s^{\lambda+\alpha} \int_\Omega n_\varepsilon^{\frac{N+\delta}{N}}(x, s)|\nabla c_\varepsilon(x, s)|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t s^{-\lambda-\alpha} \int_\Omega n_\varepsilon^{\frac{N-\delta}{N}}(x, s) \,\mathrm{d}x \,\mathrm{d}s \\
&\le t^\alpha \int_0^t s^\lambda \int_\Omega n_\varepsilon^{\frac{N+\delta}{N}}(x, s)|\nabla c_\varepsilon(x, s)|^2 \,\mathrm{d}x \,\mathrm{d}s + m^{\frac{N-\delta}{N}} |\Omega|^{\frac{\delta}{N}} \int_0^t s^{\alpha-1} \,\mathrm{d}s \\
&= t^\alpha \int_0^t s^\lambda \int_\Omega n_\varepsilon^{\frac{N+\delta}{N}}(x, s)|\nabla c_\varepsilon(x, s)|^2 \,\mathrm{d}x \,\mathrm{d}s + \tfrac{1}{\alpha} m^{\frac{N-\delta}{N}} |\Omega|^{\frac{\delta}{N}} t^\alpha
\end{aligned} \tag{6.6.1}$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$ due to the mass conservation property in (6.3.5). As $\lambda > \frac{\delta}{2} = \frac{N}{2}(\frac{N+\delta}{N} - 1)$ and $\frac{N+\delta}{N} = 1 + \frac{\delta}{N} \le \frac{N}{2} + \delta$, we can then use Lemma 6.4.4 to gain $K > 0$ such that

$$\int_0^t s^\lambda \int_\Omega n_\varepsilon^{\frac{N+\delta}{N}}(x, s)|\nabla c_\varepsilon(x, s)|^2 \,\mathrm{d}x \,\mathrm{d}s \le K$$

for all $t \in (0, 1)$ and $\varepsilon \in (0, 1)$, which combined with (6.6.1) completes the proof. □

Using the fundamental theorem of calculus, the above property now almost immediately translates to uniform continuity of the first solution component of the approximate solutions in $t = 0$ in the vague topology. Due to the convergence properties already proven in Lemma 6.5.5 as well as due to the properties of our initial data approximation, this then readily gives us our desired continuity property for n.

Lemma 6.6.2. *If c_0 satisfies (6.1.1) and n is the function constructed in Lemma 6.5.5, then*

$$n(\cdot,t) \to n_0 \quad \text{in } \mathcal{M}_+(\overline{\Omega})$$

as $t \searrow 0$, where we interpret the functions $n(\cdot,t)$, $t > 0$, as the positive Radon measures $n(x,t)\mathrm{d}x$ with $\mathrm{d}x$ being the standard Lebesgue measure on $\overline{\Omega}$.

Proof. Let $\eta > 0$ and $\varphi \in C^2(\overline{\Omega})$ with $\nabla\varphi \cdot \nu = 0$ on $\partial\Omega$ be fixed but arbitrary.

Using the fundamental theorem of calculus, the first equation in (C), partial integration as well as Lemma 6.6.1, we gain $K > 0$ and $\alpha > 0$ such that

$$\begin{aligned}
\left|\int_\Omega n_\varepsilon(\cdot,t)\varphi - \int_\Omega n_{0,\varepsilon}\varphi\right| &= \left|\int_0^t \int_\Omega n_{\varepsilon t}\varphi\right| = \left|\int_0^t \int_\Omega \left[\Delta n_\varepsilon - \chi\nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\right]\varphi\right| \\
&= \left|\int_0^t \int_\Omega n_\varepsilon \Delta\varphi + \chi\int_0^t\int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla\varphi\right| \\
&\le m\|\Delta\varphi\|_{L^\infty(\Omega)}t + \chi\|\nabla\varphi\|_{L^\infty(\Omega)}\int_0^t\int_\Omega \|n_\varepsilon\nabla c_\varepsilon\|_{L^1(\Omega)} \\
&\le m\|\Delta\varphi\|_{L^\infty(\Omega)}t + \chi K\|\nabla\varphi\|_{L^\infty(\Omega)}t^\alpha
\end{aligned}$$

for all $t \in (0,1)$ and $\varepsilon \in (0,1)$. Thus, we can fix $t_0 \in (0,1)$ such that

$$\left|\int_\Omega n_\varepsilon(\cdot,t)\varphi - \int_\Omega n_{0,\varepsilon}\varphi\right| \le \frac{\eta}{3}$$

for all $t \in (0,t_0)$ and $\varepsilon \in (0,1)$. Using the convergence properties from (6.3.1) as well as Lemma 6.5.5, we can then, for each $t \in (0,t_0)$, fix $\varepsilon(t) \in (0,1)$ such that

$$\left|\int_\Omega n_{0,\varepsilon(t)}\varphi - \int_{\overline{\Omega}} \varphi\,\mathrm{d}n_0\right| \le \frac{\eta}{3} \quad \text{and} \quad \left|\int_\Omega n(\cdot,t)\varphi - \int_\Omega n_{\varepsilon(t)}(\cdot,t)\varphi\right| \le \frac{\eta}{3}.$$

Combining the above estimates, we then conclude that

$$\begin{aligned}
\left|\int_\Omega n(\cdot,t)\varphi - \int_{\overline{\Omega}}\varphi\,\mathrm{d}n_0\right| &\le \left|\int_\Omega n(\cdot,t)\varphi - \int_\Omega n_{\varepsilon(t)}(\cdot,t)\varphi\right| \\
&\quad + \left|\int_\Omega n_{\varepsilon(t)}(\cdot,t)\varphi - \int_\Omega n_{0,\varepsilon(t)}\varphi\right| \\
&\quad + \left|\int_\Omega n_{0,\varepsilon(t)}\varphi - \int_{\overline{\Omega}}\varphi\,\mathrm{d}n_0\right| \le \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta
\end{aligned}$$

for all $t \in (0,t_0)$. Given that the set of all functions $\varphi \in C^2(\overline{\Omega})$ with $\nabla\varphi\cdot\nu = 0$ on $\partial\Omega$ is dense in $C^0(\overline{\Omega})$ (which can be seen by e.g. approximating any $\rho \in C^0(\overline{\Omega})$ by functions $\rho_\lambda := e^{\lambda\Delta}\rho$ with $\lambda > 0$, where $(e^{t\Delta})_{t\ge 0}$ is the Neumann heat semigroup, in a similar way to our approach in Section 6.3), this immediately implies our desired result. □

To now prove a corresponding continuity property in $t = 0$ for the second solution component, we will use semigroup methods combined with the baseline bounds established in Lemma 6.3.1 to facilitate an argument similar to the one laid out above. Notably, the following reasoning is somewhat streamlined by our choice of initial data approximation for c_0, which is inherently compatible with the action of the Neumann heat semigroup.

Lemma 6.6.3. *If c_0 satisfies (6.1.1) and c is the function constructed in Lemma 6.5.5, then*

$$c(\cdot, t) \to c_0 \qquad \text{in } L^p(\Omega) \text{ for all } p \in [1, \infty)$$

as $t \searrow 0$.

Proof. We fix $p \in [1, \infty)$. Let then $\eta > 0$ be fixed but arbitrary.

Using the variation-of-constants representation of the second equation in (C) combined with the baseline bounds established in Lemma 6.3.1 and the smoothing properties of the Neumann heat semigroup as well as the fact that $c_{0,\varepsilon} = e^{\varepsilon\Delta} c_0$ by definition, we can now estimate as follows:

$$\begin{aligned}
\|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^1(\Omega)} &\leq \|c_{0,\varepsilon} - e^{t\Delta} c_{0,\varepsilon}\|_{L^1(\Omega)} + \int_0^t \left\| e^{(t-s)\Delta} n_\varepsilon(\cdot, s) c_\varepsilon(\cdot, s) \right\|_{L^1(\Omega)} \mathrm{d}s \\
&\leq \|e^{\varepsilon\Delta}(c_0 - e^{t\Delta} c_0)\|_{L^1(\Omega)} + K \int_0^t \|n_\varepsilon(\cdot, s) c_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \mathrm{d}s \\
&\leq K \|c_0 - e^{t\Delta} c_0\|_{L^1(\Omega)} + mK \|c_0\|_{L^\infty(\Omega)} t
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$ with some $K > 0$. Due to the further continuity properties of the Neumann heat semigroup in $t = 0$, the above allows us to find $t_0 > 0$ such that

$$\|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq 2^{1-p} \|c_0\|_{L^\infty(\Omega)}^{1-p} \frac{\eta^p}{3^p}$$

for all $\varepsilon \in (0, 1)$ and $t \in (0, t_0)$. By application of the Hölder inequality, we then further gain that

$$\begin{aligned}
\|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^p(\Omega)} &\leq \|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^1(\Omega)}^{\frac{1}{p}} \|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}^{\frac{p-1}{p}} \\
&\leq 2^{\frac{p-1}{p}} \|c_{0,\varepsilon} - c_\varepsilon(\cdot, t)\|_{L^1(\Omega)}^{\frac{1}{p}} \|c_0\|_{L^\infty(\Omega)}^{\frac{p-1}{p}} \\
&\leq \frac{\eta}{3}
\end{aligned} \tag{6.6.2}$$

for all $\varepsilon \in (0, 1)$ and $t \in (0, t_0)$.

Using the convergence properties laid out in (6.3.3) as well as Lemma 6.5.5, we can, for each $t \in (0, t_0)$, find $\varepsilon(t) \in (0, 1)$ such that

$$\|c_0 - c_{0,\varepsilon(t)}\|_{L^p(\Omega)} \leq \frac{\eta}{3} \quad \text{and} \quad \|c(\cdot, t) - c_{\varepsilon(t)}(\cdot, t)\|_{L^p(\Omega)} \leq \frac{\eta}{3}.$$

Combined with (6.6.2), this gives us that

$$\begin{aligned}
&\|c_0 - c(\cdot,t)\|_{L^p(\Omega)} \\
&\le \|c_0 - c_{0,\varepsilon(t)}\|_{L^p(\Omega)} + \|c_{0,\varepsilon(t)} - c_{\varepsilon(t)}(\cdot,t)\|_{L^p(\Omega)} + \|c_{\varepsilon(t)}(\cdot,t) - c(\cdot,t)\|_{L^p(\Omega)} \\
&\le \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta
\end{aligned}$$

for all $t \in (0, t_0)$. As $\eta > 0$ was arbitrary, this completes the proof. □

6.7 Proof of the main theorem

Having at this point proven all parts of Theorem 6.1.1 individually, its proof can now be presented in a rather swift fashion.

Proof of Theorem 6.1.1. Assume that c_0 satisfies (6.1.1). Then let $n, c \in C^{2,1}(\overline{\Omega}\times(0,\infty))$ be the functions constructed in Lemma 6.5.5 under this assumption. According to the same lemma, (n, c) is already a solution to (C) with boundary conditions (CB) on $\Omega \times (0,\infty)$ and fulfills (6.1.2) due to (6.5.3). Lemma 6.6.2 and Lemma 6.6.3 then further ensure that the continuity properties in (6.1.3) and (6.1.4) hold for n and c as well. With this, all necessary properties for n and c are proven. □

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