Paderborn University

Dissertation

Erdős Sieves and Dynamics

by

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for the fulfillment of the degree of

Dr. rer. nat.

submitted to the

Faculty of Computer Science, Electrical Engineering and Mathematics

supervisor:

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January 2026

Abstract

We study collections of congruence classes modulo an infinite set of pairwise coprime ideals in an étale $\mathbb{Q}$-algebra K, called *sieves*, together with the associated sets of R-free numbers, consisting of elements of $\mathcal{O}_K$ not contained in any class of a sieve R. This framework generalizes several classical examples, including $\mathcal{B}$-free integers, squarefree values of polynomials, and carefree couples.

We investigate the associated dynamical systems, in particular the space of admissible sets Ω_R and the orbit closure X_R, with the aim of extending Sarnak's program beyond previously studied $\mathcal{B}$-free systems. For this purpose, we introduce the notion of "weak light tails" and analyze sieves satisfying this and related properties.

We prove a generalized version of Sarnak's program for sieves in étale $\mathbb{Q}$-algebras, including general Erdős $\mathcal{B}$-free systems. Furthermore, we show that the measure-theoretic dynamical system (Ω_R, S, ν_R) is always isomorphic to a rotation on a compact abelian group. We also introduce minimal sieves to describe the relation between X_R and Ω_R and to address uniqueness questions for R-free numbers. Finally, we apply these methods to number-theoretic problems such as sumsets of R-free numbers, squarefree polynomial values, and an ergodic prime number theorem for R-free numbers under weak light tails.

Zusammenfassung

Wir untersuchen Familien von Kongruenzklassen modulo einer unendlichen Menge paarweise teilerfremder Ideale in einer étalen $\mathbb{Q}$-Algebra K, die wir *Siebe* nennen, sowie die zugehörigen Mengen R-freier Zahlen, bestehend aus Elementen von $\mathcal{O}_K$, die in keiner Klasse eines Siebs R liegen. Dieser Rahmen verallgemeinert klassische Beispiele wie $\mathcal{B}$-freie ganze Zahlen, quadratfreie Werte von Polynomen und carefree couples.

Wir analysieren die zugehörigen dynamischen Systeme, insbesondere den Raum zulässiger Mengen Ω_R und den Orbitabschluss X_R, mit dem Ziel, Sarnaks Programm über bisher untersuchte $\mathcal{B}$-freie Systeme hinaus zu erweitern. Dazu führen wir den Begriff "weak light tails" ein und untersuchen Siebe mit dieser und verwandten Eigenschaften.

Wir beweisen eine verallgemeinerte Version von Sarnaks Programm für Siebe in étalen $\mathbb{Q}$-Algebren, einschließlich allgemeiner Erdős $\mathcal{B}$-freier Systeme. Außerdem zeigen wir, dass das maßtheoretische dynamische System (Ω_R, S, ν_R) stets isomorph zu einer Rotation auf einer kompakten abelschen Gruppe ist. Zudem führen wir minimale Siebe ein, um die Beziehung zwischen X_R und Ω_R sowie Eindeutigkeitsfragen für R-freie Zahlen zu beschreiben. Abschließend wenden wir diese Methoden auf zahlentheoretische Probleme wie Summenmengen R-freier Zahlen, quadratfreie Polynomwerte und einen ergodischen Primzahlsatz für R-freie Zahlen an.

"De facto, que aproveita ao homem ganhar o mundo inteiro,
se vier a perder-se e a arruinar-se a si próprio?"
— Lc 9:25

Contents

1. Introduction

The objective of this work is to study sets of algebraic integers obtained by removing congruence classes modulo an infinite family of ideals $\mathcal{B}$ from an ambient space, together with the associated dynamics. Such sets have been studied over $\mathbb{Z}$ for the past century, mainly in the case where all congruence classes are 0. The canonical example of such sets would be the squarefree integers. To obtain these, one takes the set $\mathcal{B} = \{p^2\mathbb{Z} : p \text{ prime}\}$, and then takes the integers that are not congruent to 0 modulo p^2 for any prime p. But many sets can be obtained in such a manner. For example, the set of primes corresponds to positive integers that are not congruent to 0 modulo pq for any primes p and q. One of the most basic questions about such sets is related to their density. In [12], Besicovitch showed that there are sets $\mathcal{B}$ of ideals of $\mathbb{Z}$ such that the set of integers not in $b\mathbb{Z}$ for every $b\mathbb{Z} \in \mathcal{B}$ does not have a well defined density. Yet, Erdős and Davenport showed in [22] that such sets always have a logarithmic density.

Going beyond $\mathbb{Z}$, sets of this form have also been studied over $\mathbb{Z}^n$ and over the ring of integers $\mathcal{O}_K$ of some number field K. For example, the set of $l-$free numbers in a number field, can be obtained by removing from $\mathcal{O}_K$ all integers which are not congruent to 0 modulo $\mathfrak{b}$ for any ideal $\mathfrak{b}$ contained in the set $\mathcal{B} = \{\mathfrak{p}^l : \mathfrak{p} \text{ prime of } \mathcal{O}_K\}$. Also the set of *visible lattice points*, which is the set of pairs $(x, y) \in \mathbb{Z}^2$ where x and y are coprime, can be obtained from $\mathbb{Z}^2$ by removing every element of the ideals in $\mathcal{B} = \{p\mathbb{Z} \times p\mathbb{Z} : p \text{ prime}\}$. In order to be able to deal with all such examples, we will work with sets $\mathcal{B}$ of ideals in the ring of integers $\mathcal{O}_K$, where K is taken to be an étale $\mathbb{Q}-$algebra (i.e. the product of finitely many number fields, see Definition 2.5).

We are interested in what happens when one chooses a set of ideals $\mathcal{B}$, and removes congruence classes modulo $\mathfrak{b} \in \mathcal{B}$ distinct from only 0. We formalize this with the notion of *sieves*. By a sieve over an étale $\mathbb{Q}-$algebra K we mean a pair $(\mathcal{B}_R, (R_\mathfrak{b})_{\mathfrak{b}\in\mathcal{B}_R})$, where $\mathcal{B}_R$ are a set of pairwise coprime ideals in $\mathcal{O}_K$, and each $R_\mathfrak{b}$ is a set of congruence classes mod $\mathfrak{b}$. We have chosen to restrict ourselves to the case where $\mathcal{B}_R$ is a set of pairwise coprime ideals, as this case is already complex enough in itself.

Having defined a sieve R, we want to study the set of $R-$free numbers

$$\mathcal{F}_R := \mathcal{O}_K \setminus \left(\bigcup_{\mathfrak{b}\in\mathcal{B}_R} R_\mathfrak{b} \right).$$

When a sieve is of the form $R_\mathfrak{b} = \mathfrak{b}$ for all $\mathfrak{b} \in \mathcal{B}_R$, we will refer to it as a $\mathcal{B}-$free system. We have just described how in the case of $\mathcal{B}-$free systems (especially over $\mathbb{Q}$), quite a lot has been done in regards to studying the density of $\mathcal{F}_R$. In contrast, other types of sets that can be realized as the $R-$free numbers for some sieve R have been far less studied.

Yet, from a number theoretic point of view, there are many interesting sets that are of this type. An example that we investigate in Section 9 corresponds to the set of squarefree values of polynomials. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial. Studying the set of values x for which $f(x)$ is squarefree is a problem that has been investigated by number theorists for almost one century (see for example [17],[34],[35],[41] or [46]). We provide an overview of the story of this problem in Section 9.2. Notice how, if we let R^f be the sieve with congruence classes modulo p^2 given by the set

$$R_p^f = \{x \in \mathbb{Z}/p^2\mathbb{Z} : f(x) \equiv 0 \mod p^2\} + p^2\mathbb{Z}, \tag{1}$$

we have that $\mathcal{F}_{R^f}$ is exactly the set of values x for which $f(x)$ is squarefree.

If R is a $\mathcal{B}-$free system with $\mathcal{B}_R = \mathcal{B}$, we will write $\mathcal{F}_\mathcal{B}$ for $\mathcal{F}_R$. In the last twenty five years, there has been extensive study of shift spaces related to these sets (see [7] or [65] for some early examples). There are two related shift spaces that one tends to study. In the case of $\mathcal{B}-$free systems they usually agree, but this does not hold for general sieves (we dedicate Section 8 of this thesis to this topic). Assuming that $\mathcal{B}$ is a set of ideals of $\mathcal{O}_K$, both spaces live inside $\{0,1\}^{\mathcal{O}_K}$ (which we identify with the power set of $\mathcal{O}_K$) where there is a shift action $S_a(A) = -a + A$ for any $a \in \mathcal{O}_K$ and set $A \subset \mathcal{O}_K$. The first space, which we denote by $X_\mathcal{B}$, is the closure of the orbit of $\mathcal{F}_\mathcal{B}$ under S, inside of $\{0,1\}^{\mathcal{O}_K}$. The second is the collection of all $\mathcal{B}-$admissible sets, which are sets $A \subset \mathcal{O}_K$ such that $A + \mathfrak{b} \neq \mathcal{O}_K$ for all $\mathfrak{b} \in \mathcal{B}$. We will denote this set by $\Omega_\mathcal{B}$.

In the seminal paper [68], Sarnak conjectured that, denoting by μ the Möbius function, we have that if (X,T) is a topological dynamical system with entropy 0 (see Section 2.2 for the definitions), we have, for all $x \in X$ and $f \in C(X)$, that

$$\sum_{n \leq N} \mu(n) f(T^n x) = o(N).$$

Note that this generalizes many famous results in number theory, for example in the case where X is a single point, this is just the Prime Number Theorem (see Theorem 2.2). In order to tackle this problem, Sarnak started by considering the shift spaces associated to the squarefree numbers (note that μ^2 is the characteristic function of the squarefree numbers). Letting $\mathcal{B} = \{p^2\mathbb{Z} : p \text{ prime}\}$, Sarnak announced the following results.

(1) There is a measure $\nu_\mathcal{B}$ (the Mirsky measure) for which $\mathcal{F}_\mathcal{B}$ is a generic point (see Definition 2.27) of the system $(X_\mathcal{B}, S, \nu_\mathcal{B})$.
(2) The topological entropy of $(X_\mathcal{B}, S)$ is $\frac{6}{\pi^2}$.
(3) $X_\mathcal{B} = \Omega_\mathcal{B}$.
(4) The system $(X_\mathcal{B}, S)$ is proximal (see Equation (8)) and its unique minimal system is $\{\emptyset\}$.
(5) There is a nontrivial joining of the systems $(X_\mathcal{B}, S)$ and (G,T) (see Definition 2.10), where G is the group $\prod_p \mathbb{Z}/p^2\mathbb{Z}$ and T is the rotation on G defined by $T(g)_p := g_p + 1$.

These five points together form what is referred to in the literature as *Sarnak's program.* Sarnak didn't publish detailed proofs of these results, but his notes inspired many authors that further generalized these results to many different contexts. We now give a short overview of some of these developments.

In [19], Cellarosi and Sinai investigated the squarefree flow $(X_\mathcal{B}, S, \nu_\mathcal{B})$, and showed that it is isomorphic to $(G, T, \mathbb{P})$, where $\mathbb{P}$ denotes the Haar measure of G. Peckner showed in [63] that the system $(X_\mathcal{B}, S)$ has a unique measure of maximum entropy. In [1], El Abdalaoui, Lemańczyk and De La Rue generalized points (1), (2) and (3) of Sarnak's program for Erdős $\mathcal{B}-$free systems over $\mathbb{Q}$. These are $\mathcal{B}-$free systems where the elements of $\mathcal{B}$ are pairwise coprime and $\sum_{b\mathbb{Z}\in\mathcal{B}} 1/b$ is finite. They also showed that there is an isomorphism between the system $(X_\mathcal{B}, S, \nu_\mathcal{B})$ and a rotation of the group $\prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$. The paper [26] of

Dymek, Kasjan, Kułaga-Przymus, and Lemańczyk provides an extensive treatment of all $\mathcal{B}-$free systems over $\mathbb{Q}$ (including the case where the elements of $\mathcal{B}$ are not pairwise coprime).

These results were then generalized for $\mathcal{B}-$free systems over different étale $\mathbb{Q}-$algebras K. In the case where $K = \mathbb{Q}^n$, Pleasants and Huck in [65] proved points (1), (2) and (3) of Sarnak's program for $k-$free lattice points. This corresponds to the case where $\mathcal{B} = \{p^k\mathbb{Z} \times \cdots \times p^k\mathbb{Z} : p \text{ prime}\}$. Additionally, they showed that the system $(X_{\mathcal{B}}, S, \nu_{\mathcal{B}})$ is isomorphic to a rotation of the group $G = \prod_p \mathbb{Z}^n/(p^k\mathbb{Z}\times\cdots\times p^k\mathbb{Z})$. In the case of number fields, Cellarosi and Vinogradov proved a generalization of point (1) when K is a number field with ring of integers $\mathcal{O}_K$ and $\mathcal{B} = \{\mathfrak{p}^k : \mathfrak{p} \text{ prime ideal of } \mathcal{O}_K\}$ and showed that the system $(\Omega_{\mathcal{B}}, S, \nu_{\mathcal{B}})$ is isomorphic to a rotation of a compact group. Finally, in [2], the authors generalized all of Sarnak's program to the case where K is a number field, and $\mathcal{B}$ is a set of ideals of $\mathcal{O}_K$ that are pairwise coprime and such that $\sum_{\mathfrak{b}\in\mathcal{B}} N(\mathfrak{b})^{-1}$ is finite (where $N(\mathfrak{b}) = |\mathcal{O}_K/\mathfrak{b}|$ is the norm of $\mathfrak{b}$). An older version of this pre-print was lacking a step in a key result. This was solved by adding a proof of Lemma 5.3 in this work to [2], resulting in an updated version where the author of this Thesis is also co-author of [2].

The notion of a sieve in this context was introduced by Gundlach and Klüners in [42], while generalizing the results of [8]. For a sieve R, they define the space X_R, corresponding to the closure of the orbit of $\mathcal{F}_R$ under S inside of $\{0,1\}^{\mathcal{O}_K}$. They also defined the space Ω_R of admissible sets, which are those $A \subset \mathcal{O}_K$ such that $-A + R_{\mathfrak{b}} \neq \mathcal{O}_K$ for all $\mathfrak{b} \in \mathcal{B}_R$. Their work focused mostly on studying the symmetry group of the system (Ω_R, S), and dealt with no measure theoretical dynamics. Since sieves generalize $\mathcal{B}-$free systems, the first objective of this thesis was to prove a version of Sarnak's program for general sieves. In attempting to do this, we had to define a number of properties for sieves. The most important of these, were the notions of an Erdős sieve, as well as of a sieve with strong and weak light tails.

We say a sieve is *Erdős*, if

$$\sum_{\mathfrak{b}\in\mathcal{B}_R} \frac{|R_{\mathfrak{b}}|}{N(\mathfrak{b})} < \infty.$$

For any sieve, we are free to order $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$, in which case we write $R_i := R_{\mathfrak{b}_i}$. Given a Følner sequence I_N (see Definition 2.21), we say that an Erdős sieve R has *weak light tails* for I_N if

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i>L} R_i \setminus \bigcup_{j\leq L} R_j \right) = 0,$$

where $\overline{d}_I$ is the upper density with respect to I_N (see Definition 3.13). We also say that R has *strong light tails* if

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i>L} R_i \right) = 0.$$

Strong light tails implies weak light tails, but not the other way around. In Example 5.14 we provide a sieve that has weak light tails for $I_N = [0, N]$, but does not have strong light tails. In general, sieves with weak light tails have well behaved dynamical systems, but the property may not be stable (for example, removing just one congruence class from an Erdős sieve R with weak light tails for some I_N can produce a sieve without weak light tails for any Følner sequence).

We now present the results obtained. To our knowledge, no work had been done on the systems (X_R, S) and (Ω_R, S) previous to [42], except for showing that when R is a sieve over $\mathbb{Q}$, the system (Ω_R, S) has a unique measure of maximum entropy for certain R, see Theorem 2.2.25 in [52] (a more general result was also obtained in [26]).

Our first major result (see Theorem 3.21) is the following generalization of point (1) of Sarnak's program, which connects the system (Ω_R, S, ν_R), where ν_R is the Mirsky measure in Ω_R (see Definition 3.9), the weak light tails property and the set $\mathcal{F}_R$.

Theorem 1.1. *Let R be an Erdős sieve. For a given Følner sequence I_N, the following are equivalent.*

(1) *$\mathcal{F}_R$ is a generic point of (Ω_R, S, ν_R) with respect to I_N.*
(2) *R has weak light tails with respect to I_N.*
(3) *The set $\mathcal{F}_R$ has a density with respect to I_N given by*

$$d_I(\mathcal{F}_R) = \prod_{\mathfrak{b} \in \mathcal{B}_R} \left(1 - \frac{|R_\mathfrak{b}|}{N(\mathfrak{b})}\right).$$

Point (2) of Sarnak's program for sieves was already shown as Lemma 5.13 in [42]. In Theorem 4.1, we provide a more general result that implies that the topological entropy of (Ω_R, S) agrees with the one given in Lemma 5.13 of [42]. Point (3) of Sarnak's program does not have a straightforward generalization for general sieves. Sieves with strong light tails are our most 'well behaved' sieves, and even for these, it might be the case that $X_R \neq \Omega_R$, as shown in Example 8.1. Yet, in Section 8 we provide a number of results which clarify the relation between X_R and Ω_R when R is a sieve with weak light tails. We first show the following result, given in Theorem 8.4.

Theorem 1.2. *Let R be an Erdős sieve. Then, there is a Følner sequence I_N with respect to which R has weak light tails if and only if*

$$\nu_R(X_R) = 1.$$

In order to present our other major result from Section 8, we must introduce the concept of a *minimal* sieve. By using different ideals, the same sets can be expressed as the union of congruence classes, for example $\{0, 2\} + 4\mathbb{Z}$ is the same set as $2\mathbb{Z}$, or $\{0, 2, 3, 4\} + 6\mathbb{Z}$ is the same as $2\mathbb{Z} \cup 3\mathbb{Z}$. This means that there can exist two sieves R and R' that are effectively the same, but that are technically distinct because $\mathcal{B}_R \neq \mathcal{B}_{R'}$. In this case, we define the notion of a contraction (see Definition 6.5) which takes a sieve R and returns a sieve R' where the congruence classes being sieved out are exactly the same, but every ideal in $\mathcal{B}_{R'}$ divides some unique ideal in $\mathcal{B}_R$. When a sieve can no longer be contracted, we say it is minimal. We have the following result (see Theorem 8.12).

Theorem 1.3. *Let R and R' be minimal Erdős sieves with weak light tails for some (not necessarily common) Følner sequences. The following are equivalent.*

(1) *$\mathcal{B}_R = \mathcal{B}_{R'}$, and for every $\mathfrak{b} \in \mathcal{B}_R$, there is some $\delta_\mathfrak{b} \in \mathcal{O}_K$ such that $R_\mathfrak{b} = \delta_\mathfrak{b} + R'_\mathfrak{b}$,*
(2) *$X_R = X_{R'}$,*
(3) *$\Omega_R = \Omega_{R'}$.*

We also used the notion of minimal sieve to characterize equivalence of sieves. We say R and R' are equivalent, and write $R \sim R'$, if $\mathcal{F}_R = \mathcal{F}_{R'}$. The following theorem (see Theorem 6.21) describes the relation of uniqueness between a sieve R and $\mathcal{F}_R$.

Theorem 1.4. *Let R be an Erdős sieve.*

- *There exists a minimal Erdős sieve R' such that $R \sim R'$.*
- *If R has weak light tails for some Følner sequence I_N, there exists a minimal Erdős sieve R' with weak light tails for I_N, such that if W is minimal and $W \sim R$, then $W = R'$, or W does not have weak light tails for any Følner sequence.*
- *If R has strong light tails for I_N, then there exists a unique minimal sieve R' (which will have strong light tails for I_N) such that $R \sim R'$.*

In Theorem 4.2 and Theorem 4.6, respectively, we prove versions of points (4) and (5) of Sarnak's program for general sieves. As it was done in [19] or [2] for $\mathcal{B}-$free systems, we also show that for every sieve R there is a group $G_{R,F}$ and a rotation T^F of this group such that we have an isomorphism of dynamical systems, which constitutes the central result of section 7 (see Theorem 7.16).

Theorem 1.5. *Let R be an Erdős sieve. For any $\mathfrak{b} \in \mathcal{B}_R$, let*

$$F(R_{\mathfrak{b}}) = \{x \in \mathcal{O}_K : x + R_{\mathfrak{b}} = R_{\mathfrak{b}}\},$$

and define the group

$$G_{R,F} := \prod_{\mathfrak{b} \in \mathcal{B}_R} \mathcal{O}_K / F(R_{\mathfrak{b}}).$$

Letting T^F be the action of $\mathcal{O}_K$ on $G_{R,F}$ given by $T^F_a(g)_{\mathfrak{b}} = g_{\mathfrak{b}} + a$ and $\mathbb{P}^F$ the Haar measure on $G_{R,F}$, we have that (Ω_R, S, ν_R) is isomorphic to $(G_{R,F}, T^F, \mathbb{P}^F)$.

Given $g \in G_{R,F}$, if we write $R(g)$ to be the sieve defined by

$$R(g)_{\mathfrak{b}} = g_{\mathfrak{b}} + R_{\mathfrak{b}},$$

then the isomorphism is exactly the map that sends g to $\mathcal{F}_{R(g)}$.

Many of our results required that R has weak light tails for some Følner sequence. For any étale $\mathbb{Q}-$algebra K one can define a metric $\|\cdot\| : K \to \mathbb{R}_{\geq 0}$ (see Equation (2)) so that the balls B_N containing all $x \in \mathcal{O}_K$ such that $\|x\| \leq N$ form a Følner sequence. In Theorem 5.7 we show the following result.

Theorem 1.6. *Let R be an Erdős sieve over an étale $\mathbb{Q}-$algebra K such that there is a finite set $A \subset \mathcal{O}_K$ for which $R_{\mathfrak{b}} \subset A + \mathfrak{b}$ for every $\mathfrak{b} \in \mathcal{B}_R$. Then R has strong light tails for B_N.*

In particular, every $\mathcal{B}-$free system has strong light tails for B_N, and so our results generalize the previous literature for $\mathcal{B}-$free systems over any étale $\mathbb{Q}-$algebra.

In Section 9 we provide applications to number theory of our work into Sarnak's program for sieves. In [61] Moreira, Richter and Robertson showed that for any $C \subset \mathbb{N}$ such that $\overline{d}(C) > 0$, there are infinite $A, B \subset \mathbb{N}$ such that $A + B \subset C$. Host has showed in [48] that there are sets C with $\overline{d}(C) > 0$ such that if there are infinite A and B such that $A+B \subset C$, then we must have $\overline{d}(A) = \overline{d}(B) = 0$. For this reason, it is interesting to study sets $C \subset \mathbb{N}$ such that there are $A, B \subset \mathbb{N}$ which satisfy $A + B \subset C$ and $\overline{d}(B) > 0$. We show that $R-$free numbers for sieves with strong light tails provide plenty of examples of such sets.

Theorem 1.7. *Let A be a subset of $\mathbb{Z}$ and R an Erdős sieve such that*

$$\prod_{b\mathbb{Z} \in \mathcal{B}_R} \left(1 - \frac{|-A + R_b|}{b}\right) > 0.$$

Then, there exists a sequence $g \in G_{R,F}$ and some $B \subset \mathbb{Z}$ with $d(B) > 0$ such that

$$A + B \subset \mathcal{F}_{R(g)}.$$

Inspired by the work in [72] we also show, using a result from [11], a Prime Number Theorem for $R-$free numbers. Let v_p be the $p-$adic valuation of the integers, that is, $v_p(m)$ is the largest non-negative integer k such that $p^k \mid m$, and write

$$\Omega(m) = \sum_{p \text{ prime}} v_p(m).$$

We have the following result (see Theorem 9.20).

Theorem 1.8. *Let R be an Erdős sieve with weak light tails for $I_N = [1, N]$. Let (X, T) be a uniquely ergodic dynamical system, and μ its unique invariant measure. Then for every function $f \in C(X)$ and $x \in X$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m \in \mathcal{F}_R \cap I_N} f(T^{\Omega(m)} x) = d_I(\mathcal{F}_R) \int_X f \, d\mu.$$

This thesis is divided as follows. In Section 2 we present the preliminary results necessary for the remainder of the thesis. We start by introducing some basic facts about number fields and étale $\mathbb{Q}-$algebras, and then prove some statements related to counting lattice points in balls. We then introduce the main tools and results from both topological and measure theoretical dynamics, that we use in studying the dynamical systems (Ω_R, S, ν_R). In Section 3 we define sieves and the systems X_R and Ω_R. We also define the Mirsky measure ν_R and show some basic properties of the system (Ω_R, S, ν_R). We provide a proof of Theorem 1.1, which generalizes point (1) of Sarnak's program. Additionally, we show that there are sets of the form $\mathcal{F}_R$ that don't have logarithmic density, and make progress in a related question by Erdős.

Section 4 is dedicated to dealing with the parts of Sarnak's program related to the topological dynamical systems (Ω_R, S). In particular we show generalizations of points (2), (4) and (5) of Sarnak's program, and as an application show that, for any sieve R, there is always some Følner sequence I_N with respect to which R does not have weak light tails. In Section 5, we investigate the weak and strong light tail conditions. We first relate the weak light tails of R and the occurrence of repeated patterns in the set $\mathcal{F}_R$. We then prove Theorem 1.6 and provide a number of different examples of sieves over $\mathbb{Q}$ with strong light tails for $B_N = [-N, N]$. We end this section by investigating what happens (with respect to preserving weak light tails) to a sieve after removing or addition of ideals. We also show that a number of additional properties imply that a sieve with weak light tails must also have strong light tails.

In Section 6 we study the equivalence of sieves. We define contractions and dilations (the operation opposite to contractions), as well as the notion of a minimal sieve, in order to show Theorem 1.4. We then use dilations and contractions to define a notion of "union of sieves", which we use to compute the density of some interesting sets in $\mathbb{Z} \times \mathbb{Z}$. The objective of Section 7 is to determine the isomorphism classes of the systems (Ω_R, S, ν_R). We start by expliciting when, for two Erdős sieves R and R', one has $\Omega_R = \Omega_{R'}$. We then provide two proofs of Theorem 1.5. The first one, which is completely original and uses sieves, has the advantage of being conceptually much clearer. The second one follows closely the proof of similar results in the previous literature. In the end of the Section 7 we compute the spectrum of (Ω_R, S, ν_R), which, by using the Halmos-von Neumman Theorem, allows us to characterize the isomorphism classes of these systems. In Section 8 we study the set X_R, showing both Theorem 1.2 and Theorem 1.3.

The last sections is dedicated to number theoretic questions. In Section 9 we present applications of our results. We show Theorem 1.7 and conjecture that for every sieve R with strong light tails for B_N, there are infinite sets A, B such that $d(B) > 0$ and $A + B \subset \mathcal{F}_R$. We then consider sieves of the form R^f (as defined in Equation (1)) when f is an irreducible polynomial, and show a number of results about such sieves, which imply that the squarefree values of many such polynomials must have repeated patterns (see Corollary 9.11). We end this section with a proof of Theorem 1.8, and show how this result generalizes the results in [72].

Acknowledgments

I am very thankful to Dr. Jürgen Klüners, who supervised this Thesis, for allowing me to study this fascinating area of mathematics. His many comments and questions greatly helped me on this path of discovery, that I am glad to have trekked. I am also extremely thankful to Dr. Joanna Kułaga-Przymus, for her revisions of many past versions of this work, as well as for inviting me to visit Tórun. Both her comments and this visit greatly improved the quality of the final work. I would also like to thank Dr. Fabian Gundlach, Dr. Aurelia Dymek, and Dr. Michael Baake for many productive discussions, which gave me new insights in both Number Theory and Dynamical Systems. Ich möchte mich auch bei Günther und Gisela Mertens bedanken, die mich in Paderborn so herzlich aufgenommen haben. Finalmente, gostava de agradecer aos meus pais, por me terem sempre apoiado.

2. Preliminaries

Throughout, the natural numbers $\mathbb{N}$ will mean the integers greater than or equal to 1. For the union of this set with $\{0\}$, we will write $\mathbb{N}_0$. The following is a well know result in analysis, that due to how often we will use in this work, we write down as a lemma (see Exercise 1.32 in [36]).

Lemma 2.1. *If $0 < a_i < 1$ are real numbers, then*

$$\sum_{i\geq 1} a_i < \infty \Leftrightarrow \prod_{i\geq 1}(1-a_i) > 0.$$

2.1. **Ideals in Number Fields.** By a number field K we mean a finite field extension of the rationals $\mathbb{Q}$, such as the quadratic number fields $\mathbb{Q}[\sqrt{d}]$, or the cyclotomic fields $\mathbb{Q}[e^{2\pi i/n}]$. We say K has degree n if looking at it as a vector space over $\mathbb{Q}$, it has dimension n. For example, if d is not a square, then $\mathbb{Q}[\sqrt{d}]$ has degree 2, and $\mathbb{Q}[e^{2\pi i/n}]$ always has degree $\varphi(n)$, where φ is the Euler totient function. It is well known (see for example Theorem 51 in the Appendix B of [58]) that for every K there is some $\alpha \in K$ such that $K = \mathbb{Q}[\alpha]$.

We denote by $\mathcal{O}_K$ the ring of integers of K, that is, those $x \in K$ that are roots of a monic polynomial with coefficients in $\mathbb{Z}$. As a ring, $\mathcal{O}_K$ is always a Dedekind domain, meaning that any non-zero ideal can be uniquely factorized as a product of non-zero prime ideals, which correspond to the ideals not contained in any other proper ideal. As an additive group, $\mathcal{O}_K$ is always isomorphic to $\mathbb{Z}^n$ (see Theorem 2 in Section 2 of [58]), so we can think of it as a lattice. By a lattice in $\mathcal{O}_K$, we mean a free $\mathbb{Z}-$module of rank n in $\mathcal{O}_K$. Each non-zero ideal $\mathfrak{b}$ of $\mathcal{O}_K$ is a lattice, and we denote by the *norm* of an ideal the quantity

$$N(\mathfrak{b}) := [\mathcal{O}_K : \mathfrak{b}] = |\mathcal{O}_K/\mathfrak{b}|,$$

which is always finite. For any two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have $N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b})$.

The same way we have the Prime Number Theorem over $\mathbb{Q}$, we have the Prime Ideal Theorem over any number field, as proven by Landau in the second part of [54].

Theorem 2.2. *Let K be a number field with ring of integers $\mathcal{O}_K$. Denoting by $\pi_K(X)$ the number of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ such that $N(\mathfrak{p}) \leq X$, we have*

$$\lim_{X\to\infty} \frac{\pi_K(X)}{X/\log(X)} = 1.$$

Whenever A and B are two subsets of an abelian semigroup (like $\mathbb{N}$ or $\mathcal{O}_K$ for some number field K), we will write

$$A + B := \{a + b : a \in A, b \in B\},$$

with $A + \emptyset = \emptyset$. Given two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_K$, we say they are coprime if $\mathfrak{a} + \mathfrak{b} = \mathcal{O}_K$. In this case, we will write $(\mathfrak{a}, \mathfrak{b}) = 1$. For ideals we have that $\mathfrak{a} \mid \mathfrak{b}$ is equivalent to $\mathfrak{b} \subset \mathfrak{a}$, so we have that the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{b}$, which we write as $(\mathfrak{a}, \mathfrak{b})$, equals $\mathfrak{a} + \mathfrak{b}$. Given a set $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$ of ideals, we will denote by $\mathrm{lcm}(\{\mathfrak{b}_1, \ldots, \mathfrak{b}_k\})$ the greatest ideal that is contained in all of $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$, that is

$$\mathrm{lcm}(\{\mathfrak{b}_1, \ldots, \mathfrak{b}_k\}) = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k.$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, this corresponds to $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$.

For a number field K of degree n, there are n distinct embeddings of K into $\mathbb{C}$. Writing $K = \mathbb{Q}[\alpha]$, and letting f be the minimal polynomial of α, these correspond to the field homomorphisms determined by $\phi(\alpha) = \theta$, where θ is one of the n distinct roots of f in $\mathbb{C}$

We then consider the Minkowski embedding $\sigma : \mathcal{O}_K \to \mathbb{C}^n$ given by

$$\sigma(x) := (\phi(x))_{\phi \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})}.$$

We can define a (vector field) norm on K by taking the norm inherited from the supremum norm in the Minkowski embedding, that is

$$\|x\| := \sup_{\phi \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})} |\phi(x)|. \tag{2}$$

We write for any $N \in \mathbb{R}_{\geq 0}$

$$B_N := \{x \in \mathcal{O}_K : \|x\| \leq N\}. \tag{3}$$

Example 2.3. If $K = \mathbb{Q}$ we clearly have $B_N = [-N, N]$. For another example, write $\zeta = e^{2\pi i/3}$ and say that $K = \mathbb{Q}[\zeta]$. This is a cyclotomic field, so $\mathcal{O}_K = \mathbb{Z}[\zeta]$. Taking some $x \in \mathcal{O}_K$, we have $x = a_1 + a_2\zeta$ where $a_1, a_2 \in \mathbb{Z}$ and

$$\sigma(x) = (a_1 + a_2\zeta, a_1 + a_2\zeta^2).$$

From this, one easily computes that

$$B_N = \{a_1 + a_2\zeta \in \mathcal{O}_K : [a_1^2 + a_2^2 - a_1a_2]^{1/2} \leq N\}.$$

Given some $x \in K$, we will also write

$$B_N(x) := \{y \in \mathcal{O}_K : \|x - y\| \leq N\}.$$

For any lattice $\Gamma \subset \mathcal{O}_K$, we write

$$\lambda_i(\Gamma) := \inf\{N \in \mathbb{R}_{\geq 0} : B_N \text{ contains } i \text{ linearly independent elements of } \Gamma\}.$$

Ideals of a number ring don't behave exactly like arbitrary lattices. Given functions f and g, we will write $f(x) \ll_a g(x)$ if there is some constant C_a depending only on the object a (for example, the field K or its degree n), such that $f(x) \leq C_a g(x)$ for all x. We will use the notation $f(x) \asymp_a g(x)$ if $f(x) \ll_a g(x)$ and $g(x) \ll_a f(x)$. One of the properties that distinguishes ideals from general lattices is that we always have that $\lambda_1(\mathfrak{b}) \asymp_K \lambda_n(\mathfrak{b})$, as the next lemma demonstrates. The collection of lattices $\{k\mathbb{Z} \times \mathbb{Z} : k \in \mathbb{N}\}$ inside of $\mathbb{Z}^2$ shows that this is not the case for general lattices, since we have $\lambda_1(k\mathbb{Z} \times \mathbb{Z}) = 1$ and $\lambda_2(k\mathbb{Z} \times \mathbb{Z}) = k$.

Lemma 2.4. *Let K be a number field of degree n. Then for any ideal $\mathfrak{b}$ of $\mathcal{O}_K$, we have*

$$\lambda_1(\mathfrak{b}) \asymp_K \lambda_n(\mathfrak{b}) \asymp_K N(\mathfrak{b})^{\frac{1}{n}}.$$

Proof. This is Corollary 4 in [37]. They show that, writing d_K for the discriminant of K, we always have

$$d_K^{\max(0, \frac{1}{n} - \frac{1}{2m})} N(\mathfrak{b})^{1/n} \ll_n \lambda_m(\mathfrak{b}) \ll_n d_K^{\min(\frac{1}{2n-2m+2}, \frac{1}{n})} N(\mathfrak{b})^{1/n}.$$

Since d_K and n only depend on K, the result follows. □

Notice that while Lemma 2.4 does not hold for lattices in general, Minkowski's second theorem (see Theorem 2E in [69]) shows that for any lattice $\Gamma \subset \mathcal{O}_K$,

$$[\mathcal{O}_K : \Gamma] \asymp_K \lambda_1(\Gamma) \dots \lambda_n(\Gamma). \tag{4}$$

Definition 2.5. *An étale $\mathbb{Q}-$algebra K is a product of finitely many finite extensions of $\mathbb{Q}$, that is, K is an algebra that can be written as the cartesian product $K_1 \times \cdots \times K_m$ of number fields K_i with coordinate-wise sum and product.*

Given an étale $\mathbb{Q}-$algebra $K = K_1 \times \cdots \times K_m$, the ring of integers of K is then

$$\mathcal{O}_K = \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_m}.$$

As in the number field case, every non-zero ideal of $\mathcal{O}_K$ can be uniquely factored into the product of prime ideals, with the prime ideals in $\mathcal{O}_K$ being of the form $P_1 \times \cdots \times P_m$, where P_i is a prime ideal of $\mathcal{O}_{K_i}$ for a unique i, and $P_j = \mathcal{O}_{K_j}$ for all remaining indexes. Hence, all ideals I of $\mathcal{O}_K$ can be written in the form $I = I_1 \times \cdots \times I_m$. We will say that I is an *invertible ideal*, if $I_i \neq (0)$ for every $1 \leq i \leq m$. This means that if I is an invertible ideal, then I is a lattice, and $[\mathcal{O}_K : I] < \infty$.

Given ideals $I = I_1 \times \cdots \times I_m$ and $J = J_1 \times \cdots \times J_m$, we have

$$I + J = (I_1 + J_1) \times \cdots \times (I_m + J_m) \text{ and } IJ = (I_1 J_1) \times \cdots \times (I_m J_m).$$

In particular, we say the ideals are coprime and write $(I, J) = 1$, if $I + J = \mathcal{O}_K$, or equivalently, if $(I_i, J_i) = 1$ for every i. Similarly, we say that I divides J and write $I \mid J$ if $I_i \mid J_i$ for every i. Notice that the Chinese Remainder Theorem also holds in this context, meaning that if $\mathfrak{b}_1, \dots, \mathfrak{b}_l$ are a sequence of pairwise coprime non-zero ideals of $\mathcal{O}_K$, then

$$\mathcal{O}_K / \prod_{i=1}^{l} \mathfrak{b}_i \cong \prod_{i=1}^{l} \mathcal{O}_K / \mathfrak{b}_i,$$

with the isomorphism being given by the map that sends $x + \prod_{i=1}^{l} \mathfrak{b}_i$ into $(x + \mathfrak{b}_i)$. This is because if we write each $\mathfrak{b}_i$ as $\mathfrak{b}_i^{(1)} \times \cdots \times \mathfrak{b}_i^{(j)} \times \cdots \times \mathfrak{b}_i^{(m)}$, then we can break $\mathcal{O}_K / \prod_{i=1}^{l} \mathfrak{b}_i$ into the (Cartesian) product of $\mathcal{O}_{K_j} / \prod_{i=1}^{l} \mathfrak{b}_i^{(j)}$ over all $1 \leq j \leq m$. Applying the Chinese Remainder Theorem for each number field K_j, and then taking the Cartesian product of $\prod_{i=1}^{l} \mathcal{O}_{K_j} / \mathfrak{b}_i^{(j)}$ gives $\prod_{i=1}^{l} \mathcal{O}_K / \mathfrak{b}_i$

We will use the following lemma to count the points in a particular congruence class modulo a lattice.

Lemma 2.6. *Let K be an étale $\mathbb{Q}-$algebra of degree n and Γ a lattice of $\mathcal{O}_K$. For any $a \in \mathcal{O}_K$, define*

$$T_a(N) := |\{x \in B_N : x \equiv a \mod \Gamma\}|.$$

We have

$$T_a(N) = \frac{|B_N|}{[\mathcal{O}_K : \Gamma]} + O_K\left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Gamma) \cdots \lambda_i(\Gamma)}\right).$$

For the proof of the above lemma, we need a result from [73]. Before we formulate it, we need to introduce some notation. Let $\mathrm{Lip}(n, c, M, L)$ be the family of all sets $S \subset \mathbb{R}^n$ such that there exist maps $\phi_1, \dots, \phi_M \colon [0,1]^{n-c} \to \mathbb{R}^n$ such that

- for each $1 \leq j \leq M$, ϕ_j is Lipschitz with constant L with respect to the corresponding Euclidean norms,
- $S \subset \bigcup_{1 \leq j \leq M} \phi_j([0,1]^{d-c})$.

Additionally, given a lattice $\Lambda \subset \mathbb{R}^d$, we denote by $\det(\Lambda)$ the Lebesgue measure of any fundamental domain of Λ. The following result corresponds to Theorem 5.4 in [73].

Theorem 2.7. *If $\Lambda \subset \mathbb{R}^n$ is a lattice and $S \subset \mathbb{R}^d$ is bounded with $\partial S \in Lip(n, 1, M, L)$ then*

$$|S \cap \Lambda| = \frac{\mathrm{vol}(S)}{\det(\Lambda)} + C \cdot M \left(1 + \max_{1 \leq i < n} \frac{L^i}{\lambda_1(\Lambda) \cdots \lambda_i(\Lambda)}\right),$$

where C is a constant depending only on n.

Proof of Proposition 2.6. Consider $S = -t + [-N, N]^n$ (where t will be chosen later). Clearly, ∂S is in $\mathrm{Lip}(n, 1, 2n, 2N)$, since each of the $2n$ faces that contribute to the boundary of ∂S can be parameterized by a map with Lipschitz constant $2N$ (for example, take the map $\phi\colon [0,1]^{n-1} \to [-N, N]^{n-1} \times \{N\}$ given by $\phi(x_1, \ldots, x_{n-1}) = -t + (2Nx_1 - N, \ldots, 2Nx_{n-1} - N, N))$. Moreover, $\mathrm{vol}(S) = (2N)^n$. It follows by Theorem 2.7 that

$$|[-N, N]^n \cap (t + \Lambda)| = |S \cap \Lambda| = \frac{2^n N^n}{\det(\Lambda)} + \mathrm{O}_n \left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Lambda) \cdots \lambda_i(\Lambda)}\right) \tag{5}$$

for any lattice $\Lambda \subset \mathbb{R}^d$.

Using σ for the Minkowski embedding of K in $\mathbb{R}^n$, notice that

$$|B_N \cap (\Gamma + a)| = |\sigma^{-1}([-N, N]^n) \cap (\sigma^{-1}(\sigma(\Gamma)) + \sigma^{-1}(\sigma(a)))| = |[-N, N]^n \cap (\sigma(\Gamma) + \sigma(a))|.$$

Therefore Equation (5) with $t = \sigma(a)$ and $\Lambda = \sigma(\Gamma)$ yields the following:

$$\begin{aligned} |B_N \cap (\Gamma + a)| &= \frac{2^n N^n}{\det(\sigma(\Gamma))} + \mathrm{O}_d \left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\sigma(\Gamma)) \cdot \ldots \cdot \lambda_i(\sigma(\Gamma))}\right) \\ &= \frac{2^n N^n}{\det(\sigma(\Gamma))} + \mathrm{O}_d \left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Gamma) \cdot \ldots \cdot \lambda_i(\Gamma)}\right). \end{aligned} \tag{6}$$

In particular, for $a = 0$ and $\Gamma = \mathcal{O}_K$, we have

$$|B_N| = \frac{2^n N^n}{\det(\sigma(\mathcal{O}_K))} + \mathrm{O}(N^{n-1}). \tag{7}$$

By Proposition 5.2 in Chapter 1 of [62], together with the remaining discussion in Section 5 of that chapter, we have that for any nonzero lattice Γ of $\mathcal{O}_K$, there exists a constant c_K, depending only on K, such that $\det(\sigma(\Gamma)) = c_K[\mathcal{O}_K : \Gamma]$. In particular $\det(\sigma(\mathcal{O}_K)) = c_K$. Using (7) it follows that

$$\frac{2^n N^n}{\det(\sigma(\Gamma))} = \frac{2^n N^n}{c_K[\mathcal{O}_K : \Gamma]} = (|B_N| + \mathrm{O}(N^{n-1})) \cdot \frac{1}{[\mathcal{O}_K : \Gamma]}.$$

Combining this with (6), we conclude that

$$\begin{aligned}|B_N \cap (\Gamma + a)| &= (|B_N| + \mathrm{O}(N^{n-1})) \cdot \frac{1}{[\mathcal{O}_K : \Gamma]} + \mathrm{O}_n\left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Gamma) \cdot \ldots \cdot \lambda_i(\Gamma)}\right) \\ &= \frac{|B_N|}{[\mathcal{O}_K : \Gamma]} + \mathrm{O}_n\left(\frac{N^{n-1}}{[\mathcal{O}_K : \Gamma]}\right) + \mathrm{O}_n\left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Gamma) \cdot \ldots \cdot \lambda_i(\Gamma)}\right) \\ &= \frac{|B_N|}{[\mathcal{O}_K : \Gamma]} + \mathrm{O}_K\left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\Gamma) \cdot \ldots \cdot \lambda_i(\Gamma)}\right)\end{aligned}$$

where the last equality follows by (4) after noticing that $\lambda_n(\Gamma) \geq 1$, since $\|x\| \geq 1$ for any $x \in \mathcal{O}_K$. ☐

Remark 2.8. In particular, if $\mathfrak{b}$ is a non-zero ideal in $\mathcal{O}_K$, then we have

$$T_a(N) = \frac{|B_N|}{N(\mathfrak{b})} + O_K\left(1 + \sum_{i=1}^{n-1} \frac{N^i}{\lambda_1(\mathfrak{b}) \cdots \lambda_i(\mathfrak{b})}\right).$$

Using Lemma 2.4, we know that for every i, $\lambda_i(\mathfrak{b}) \asymp_K N(\mathfrak{b})^{1/n}$ and so we have

$$T_a(N) = \frac{|B_N|}{N(\mathfrak{b})} + O_K\left(1 + \left(\frac{N}{N(\mathfrak{b})^{1/n}}\right)^{n-1}\right).$$

2.2. **Dynamical Systems.** We will now define the notions from the theory of dynamical systems that we will use in our work, and present a number of standard results that we will need. Given a group G and a topological space X, we say that a map $T : G \times X \to X$ is an action of G on X by homemorphisms, if the maps $T_g(x) := T(g, x)$ are always homeomorphisms, and they satisfy

(1) $T_{\mathbf{0}}(x) = x$ for all $x \in X$, where $\mathbf{0}$ is the identity of G,
(2) $T_{g_1}(T_{g_2}(x)) = T_{g_1 g_2}(x)$ for any $g_1, g_2 \in G$.

Definition 2.9. *A topological dynamical system is a pair (X, T), where X is a compact metric topological space, and T is an action of a group G on X by homeomorphisms.*

In general, we will only work with groups G that are isomorphic to $\mathbb{Z}^n$, so in this section G is always assumed to be finitely generated, abelian, and locally compact. In this section, we will denote the metric of X by d.

A morphism ϕ of topological dynamical systems between (X, T) and (Y, S), is a continuous map from X to Y such that $\phi(T_g(x)) = S_g(\phi(x))$ for every $x \in X$ and $g \in G$. If ϕ is surjective, then we say that ϕ is a *factor* morphism, and that (Y, S) is a factor of (X, T). Given a subset $A \subset X$, we say that A is $T-$invariant if $T_g(A) = A$ for every $g \in G$.

Definition 2.10. *Given two topological dynamical systems (X, T) and (Y, S), we say that a set $J \subset X \times Y$ is a joining, if it is closed, invariant under $(T \times S)$, and has full projections on both coordinates. If $J \subsetneq X \times Y$, we say that J is a non trivial joining (see Section 3 in* [40]*).*

We say that a space (X, T) is *minimal*, if X does not have a proper closed $T-$invariant subset. Equivalently, X is minimal if the orbit $\{T_g(x) : g \in G\}$ of every $x \in X$ is dense. Using Zorn's Lemma, one can easily show that every space has a non-empty minimal subsystem (see Theorem 4 in the first chapter of [3]).

We say a system is *distal* if for any $x \neq y$, we have

$$\liminf_{g \in G} d(T_g(x), T_g(y)) > 0.$$

On the other hand, we say that a system is *proximal* if for any $x, y \in X$, we have

$$\liminf_{g \in G} d(T_g(x), T_g(y)) = 0. \tag{8}$$

We will need the following two results about proximality.

Lemma 2.11. *If the system (X, T) is proximal, then there exists a unique x_0 such that $T_g(x_0) = x_0$ for all $g \in G$.*

Proof. We follow the proof presented in page 175 of [50]. Using the definition of proximality, it is clear that if there is a fixed point, then it is unique. We now show that there is a fixed point. We consider a non-empty minimal subset M of X. For any $x \in M$, the set $\overline{\{T_g(x) : g \in G\}}$ is equal to M by minimality. Take any $h \in G$. By proximality, there is a sequence g_i such that $(T_{g_i}(x), T_{g_i}(T_h(x)))$ must converge to some pair (y, y). Since x is in the orbit of y, we can assume that $y = x$. Then, we get that $T_h(T_{g_i}(x))$ converges to both x and $T_h(x)$, so we must have $x = T_h(x)$. It follows that every point of M must be fixed by T, so $M = \{x_0\}$, which is the unique fixed point of X. $\square$

Given a set $S \subset G$, we say that it is *syndetic* if there is some compact set K such that $S + K = G$. We have the following result (see [2]).

Lemma 2.12. *Let (X, T) be a dynamical system for which there is some x_0 such that $T_g(x_0) = x_0$ for all $g \in G$. Then, the following are equivalent.*

(1) *For all $x, y \in X$ and $\epsilon > 0$, the set $\{g \in G : d(T_g(x), T_g(y)) < \epsilon\}$ is syndetic.*
(2) *For all $x \in X$ and $\epsilon > 0$, the set $\{g \in G : d(T_g(x), x_0) < \epsilon\}$ is syndetic.*

If any of these conditions hold, then (X, T) is proximal.

Definition 2.13. *We say (X, T) is ergodic if every $T-$invariant closed proper subset of X is nowhere dense (see Definition II.2 in* [39]*). We say it is topologically weakly mixing if $X \times X$ is ergodic.*

We now shift our focus to measure theoretical dynamical systems.

Definition 2.14. *A measure theoretical dynamical system is a quadruple $(X, \mathcal{F}, \mu, T)$, such that $(X, \mathcal{F}, \mu)$ is a probability space, and T is an action of a group G on X by measurable functions that preserve μ, that is, a map $T : G \times X \to X$ such that for every $A \in \mathcal{F}$ and $g \in G$, we have*

$$\mu(T_g^{-1}(A)) = \mu(A).$$

Alternatively, we also say that μ is $T-$invariant, if T preserves μ. Most times we will write the dynamical system $(X, \mathcal{F}, \mu, T)$ as (X, μ, T) instead, since μ will always be assumed to be defined on the Borel $\sigma-$algebra of X (which is the $\sigma-$algebra generated by the open sets of X).

A morphism between dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ is a map $\phi : X' \to Y$ where X' is a subset of X such that $\mu(X') = 1$, satisfying the properties:

(1) For all $x \in X'$ and $g \in G$, $\phi(T_g(x)) = S_g(\phi(x))$.
(2) The function ϕ is measurable, and $\nu = \phi_*(\mu)$,

where the *pushforward* of a measure μ by a map ϕ is the measure defined by the equation

$$\phi_*(\mu)(U) = \mu(\phi^{-1}(U))$$

for all measurable $U \in \mathcal{G}$. An isomorphism of the dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ is a bijective map $\phi : X' \to Y'$, where X' and Y' are subsets of X, Y satisfying $\mu(X') = 1$ and $\nu(Y') = 1$ respectively, for which (1) and (2) hold.

If there is a morphism ϕ between the systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$, then we say that Y is a factor of X. Notice that (2) implies that $\nu(\phi(X)) = 1$, so whenever ϕ is a morphism, there must be sets X' and Y' of full measure such that ϕ restricted to X' is surjective on Y'.

Definition 2.15. *Given a topological dynamical system (X, T), we say that a $T-$invariant Borel measure μ is ergodic if for any measurable set U such that*

$$\mu(U \Delta T_g(U)) = 0$$

for every $g \in G$, we have $\mu(U) \in \{0, 1\}$.

Here, Δ denotes the symmetric difference $A \Delta B = (A \cup B) \setminus (A \cap B)$ for sets A, B. In particular, if for all $g \in G$ we have $T_g(U) = U$ and μ is ergodic, then $\mu(U) \in \{0, 1\}$. Notice that if $(X, \mathcal{F}, \mu, T)$ is an ergodic system (that is, a measure theoretical dynamical system such that μ is ergodic for (X, T)), and $(Y, \mathcal{G}, \nu, S)$ is one of its factors, then it must also be ergodic.

Definition 2.16. *Given a topological dynamical system (X, T), we write $\mathcal{M}(X)$ for the set of probability measures on X, and $\mathcal{M}^T(X)$ for the set of $T-$invariant probability measures on X.*

The following is a classical result about $\mathcal{M}^T$ (see Theorem 8.4 in [28]).

Theorem 2.17. *The set $\mathcal{M}^T(X)$ is a non-empty convex closed subset of $\mathcal{M}(X)$. Additionally, a measure μ is an extremal point of $\mathcal{M}^T(X)$ if and only if it is ergodic.*

By an extremal point, we mean a point z such that if z can be written as $z = \alpha x + \beta y$ with $\alpha + \beta = 1$, $\alpha, \beta > 0$, then $z = x = y$.

The fact that $\mathcal{M}^T(X)$ is always non-empty is known as the Krylov–Bogolyubov theorem (see Corollary 4.2 in [28]). As a corollary of Theorem 2.17, note that for every dynamical system (X, T), there must exist some ergodic measure.

From Theorem 2.17, one should expect any $\mu \in \mathcal{M}(X)$ to be an "average" of distinct ergodic measures. This idea is made rigorous by Choquet's Theorem which states that if C is a compact metrizable convex set in a locally convex topological vector space, then there is a probability measure μ supported in the extremal points of C such that for every $x \in X$,

$$x = \int z d\mu(z).$$

This implies the following result (see Theorem 8.20 in [28], or Theorem 4.2.6 in first chapter of [43]), known as *ergodic decomposition*.

Theorem 2.18. *Let (X,T) be a topological dynamical system, and μ a $T-$invariant measure. There is a Lebesgue space A with measure λ, and a partition of X into $T-$invariant subsets X_α for $\alpha \in A$, as well as invariant measures μ_α supported in X_α, such that for all $f \in C(X)$,*

$$\int_X f \, d\mu = \int_A \left(\int_{X_\alpha} f \, d\mu_\alpha \right) \, d\lambda(\alpha).$$

A particularly important class of dynamical systems are minimal rotations. Let X be a compact metrizable abelian group with identity element $\mathbf{0}$ and Haar measure m_X. We call a topological dynamical system (X,T) where $T_g(x) = T_g(\mathbf{0}) + x$ a *rotation* of X. We write m_X for the Haar measure on X and $\mathbf{0}$ for the identity in X. Since m_X is invariant under all translations, it is $T-$invariant. We say a system (X,T) is *uniquely ergodic*, if $M(X)$ contains a unique measure (by Theorem 2.17, this measure must be ergodic). The following result shows that a rotation (X,T) is uniquely ergodic if and only if it is a minimal system. The proof we provide is an adaptation of the proof of Theorem 4.14 in [28], where the statement is only made for $G = \mathbb{Z}$.

Theorem 2.19. *Let (X,T) be a rotation of X. The following are equivalent.*

(1) *The system (X,T) is uniquely ergodic.*
(2) *T is ergodic for m_X.*
(3) *The set $\{T_g(\mathbf{0}) : g \in G\}$ is dense in X.*

Proof. It is clear that (1) implies (2). To show that (2) implies (3), consider the set $Y = \overline{\{T_g(\mathbf{0}) : g \in G\}}$. It is a closed subgroup of X that is T invariant. If $Y \neq X$, then there would be $|X/Y|$ cosets all with the same measure (by definition of the Haar measure). If $m_X(Y) = 0$, then we would get a cover of X by countably many sets of measure 0, which is absurd. Therefore, by ergodicity, we must have $m_X(Y) = 1$ and $|X/Y| = 1$, that is $Y = X$, which implies that $\{T_g(\mathbf{0}) : g \in G\}$ is dense.

To show that (3) implies (1), we will show that if μ is a $T-$invariant measure, then μ is invariant by translation of any $y \in X$, which by uniqueness of the Haar measure, implies that $\mu = m_X$. Indeed, take any $y \in X$ and $f \in C(X)$. Fixing some $\epsilon > 0$, the fact that $\{T_g(\mathbf{0}) : g \in G\}$ is dense in X implies that there is some $a \in G$ such that $d(T_a(\mathbf{0}), y)$ is small enough that

$$|f(T_a(\mathbf{0}) + x) - f(y + x)| < \epsilon$$

for every $x \in X$. Take any measurable set A. Since μ is T invariant, we have

$$\int_A f(x)\, d\mu = \int_A f(T_a(x))\, d\mu.$$

Therefore, we get

$$\left|\int_A f(y+x) - f(x)\, d\mu\right| = \left|\int_A f(y+x) - f(T_a(x))\, d\mu\right| \leq \int_A |f(y+x) - f(T_a(\mathbf{0})+x)|\, d\mu < \epsilon.$$

It follows that μ is invariant under translation by any $y \in X$, so $\mu = m_X$. $\square$

Given a measure dynamical system (X, T, μ), we can consider the induced Koopman representation U of G on $L^2(X, \mu)$ given by

$$U_g(f)(x) = f(T_{-g}x).$$

We can associate to any dynamical system the point spectrum σ_p of the corresponding Koopman representation, which is given by

$$\sigma_p(X, T, \mu) = \left\{\chi \in \widehat{G} \;:\; \exists\, f \in L^2(X, \mu) \setminus \{0\} \text{ such that } U_g f = \chi(g)\, f \text{ for all } g \in G\right\}.$$

where $\widehat{G}$ denotes the group of characters of G, that is, the homomorphisms $\chi : G \to S^1$ (where S^1 denotes the unitary circle in $\mathbb{C}$).

A system is said to have *discrete spectrum* if $L^2(X)$ has an orthonormal basis formed by eigenfunctions of U (that is, those $f \in L^2(X)$ such that $U_g(f) = \chi(g)f$ for some character χ). When working on systems with discrete spectrum, the Halmos-von Neumann Theorem is a powerful tool for determining which systems are isomorphic. The theorem states the following (see Theorem 5.5 in [45]).

Theorem 2.20. *Let (X_1, T_1, μ_1) and (X_2, T_2, μ_2) be ergodic systems with discrete spectrum. The systems are isomorphic if and only if $\sigma_p(X_1, T_1, \mu_1) = \sigma_p(X_2, T_2, \mu_2)$.*

The spectrum of the Koopman representation is an invariant of dynamical systems (meaning that it agrees for any two isomorphic measure theoretical dynamical systems), independent of whether or not they have discrete spectrum. Another invariant of topological dynamical systems is their *entropy*, which we now define. To do this, we first define what a Følner sequence is.

Definition 2.21. *We say that a sequence of finite, non-empty sets $I_N \subset G$, $N \geq 1$, is a Følner sequence if for every $x \in G$,*

$$\lim_{N\to\infty} \frac{|(x + I_N)\Delta I_N|}{|I_N|} = 0.$$

As an example, when $G = \mathbb{Z}$, the sequence $I_N = [0, N]$ is a Følner sequence. Additionally, notice that if I_N is any Følner sequence, and F_N is given by $F_N = a_N + I_N$ for some $a_N \in \mathcal{O}_K$, then given $x \in \mathcal{O}_K$ we have

$$\lim_{N\to\infty} \frac{|(x + F_N)\Delta F_N|}{|F_N|} = \lim_{N\to\infty} \frac{|a + ((x + I_N)\Delta I_N)|}{|I_N|} = 0,$$

and so F_N is again a Følner sequence.

For a collection of finite partitions $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of X, their *join* is defined by

$$\bigvee_{i=1}^{n} \mathcal{P}_i := \{A_1 \cap A_2 \cap \cdots \cap A_n : A_i \in \mathcal{P}_i\}.$$

Given a finite partition $\mathcal{P} = \{A_1, \ldots, A_n\}$, let

$$H(\mathcal{P}, \mu) := -\sum_{i=1}^{n} \mu(A_i) \log(\mu(A_i)).$$

By writing for any finite $F \subset G$,

$$\mathcal{P}^F := \bigvee_{g \in F} T_g^{-1}(\mathcal{P}),$$

we can define measure theoretical entropy as follows.

Definition 2.22. *Given a measure theoretical dynamical system (X, T, μ) and any Følner sequence I_N in G, we denote by measure theoretical (or Komogorov-Sinai) entropy the quantity*

$$h(X, T, \mu) = \limsup_{\mathcal{P}} \liminf_{N} \frac{1}{|I_N|} H(\mathcal{P}^{I_N}, \mu)$$

where $\mathcal{P}$ runs across all measurable finite partitions of X.

Lemma 2.23. *Let ϕ be a factor map between (X, T, μ) and (Y, S, ν). Then*

$$h(Y, S, \nu) \leq h(X, T, \mu).$$

Proof. Let $\mathcal{P}$ be any finite partition of Y. We have that $\phi^{-1}(\mathcal{P})$ is a finite partition of X, so, for any I_N,

$$H(\mathcal{P}^{I_N}, \nu) = -\sum_{A \in \mathcal{P}^{I_N}} \nu(A) \log(\nu(A)) = -\sum_{A \in \mathcal{P}^{I_N}} \mu(\phi^{-1}(A)) \log(\mu(\phi^{-1}(A))) = H(\phi^{-1}(\mathcal{P})^{I_N}, \mu),$$

which implies that

$$\liminf_{N} H(\mathcal{P}^{I_N}, \nu) = \liminf_{N} H(\phi^{-1}(\mathcal{P})^{I_N}, \mu) \leq h(X, T, \mu). \qquad \square$$

Consequently, if (X, T, μ) has zero entropy, then every one of its factor systems will also have zero entropy.

Similarly to what we have done for measure theoretical dynamical systems, we can also define (topological) entropy for topological dynamical systems. Let (X, T) be a topological dynamical system. Given a collection of open covers $\mathcal{U}_i$ of X, let

$$\bigvee_{i=1}^{n} \mathcal{U}_i := \{A_1 \cap A_2 \cap \cdots \cap A_n : A_i \in \mathcal{U}_i\}.$$

Again, given any finite set $F \subset G$, and an open cover of X, we will write

$$\mathcal{P}^F := \bigvee_{g \in F} T_g^{-1}(\mathcal{P}).$$

For a cover $\mathcal{U}$, we denote by $N(\mathcal{U})$ the cardinality of the smallest subcover of $\mathcal{U}$ (that is, the cardinality of the smallest subset of $\mathcal{U}$ that also covers X). We can then define topological entropy as follows.

Definition 2.24. *Given a topological dynamical system (X, T) and any Følner sequence I_N in G, we denote by topological entropy the quantity*

$$\mathrm{h}_{top}(X,T) = \limsup_{\mathcal{U}} \liminf_{N} \frac{1}{|I_N|} N(\mathcal{U}^{I_N})$$

where $\mathcal{U}$ runs across all finite open covers of X.

When X is a shift space, that is, $X \subset \mathcal{A}^G$ for some finite set $\mathcal{A}$ and T is defined by

$$(T_a x)(m) = x(m-a),$$

then we can consider its *patch counting entropy.* Let $x|_I$ denote the restriction of the function $x : G \to \mathcal{A}$ to some $I \subset G$. The patch counting entropy is given by

$$\mathrm{h}_{\mathrm{pc}}(X,T) = \limsup_{N\to\infty} \frac{\log(|\{x|_{-I_N} : x \in X\}|)}{|I_N|},$$

independent of the the chosen Følner sequence I_N. We have the following result (see Example 9.41 in [50]).

Lemma 2.25. *If (X, T) is a shift space, then $\mathrm{h}_{pc}(X,T) = \mathrm{h}_{top}(X,T)$.*

Topological and measure theoretical entropies are related by the Variational Principle (see Theorem 9.48 in [50]).

Theorem 2.26. *Let (X, T) be a topological dynamical system. Let $\mathcal{M}^T(X)$ be the set of $T-$invariant measures (as in Definition 2.16). We have*

$$\mathrm{h}_{top}(X,T) = \limsup_{\mu\in\mathcal{M}^T(X)} h(X,T,\mu).$$

We finish this section with the definition of a *generic* point.

Definition 2.27. *Let (X, T, μ) be a dynamical system and I_N a Følner sequence. We say that $x \in X$ is generic with respect to I_N if for every continuous function $f \in C(X)$ we have*

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{a\in I_N} f(T_a(x)) = \int_X f \, d\mu.$$

The Ergodic Theorem gives conditions for almost every point in a system to be generic. The following very general form was shown by Lindenstrauss in [56]. We say that a Følner sequence I_N is *tempered* if there is some constant C such that for all N,

$$\left| \bigcup_{L<N} -I_L + I_N \right| < C|I_N|. \tag{9}$$

Note that every Følner sequence has a tempered subsequence (see Proposition 1.4 in [56]). The Pointwise Ergodic Theorem states the following.

Theorem 2.28. *Let (X, T, μ) be an ergodic dynamical system, and I_N a tempered Følner sequence. Then, for every $f \in L^1(X)$, we have*

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{a\in I_N} f(T_a(x)) = \int_X f d\mu$$

for $\mu-$almost ever $x \in X$.

Remark 2.29. In particular, we have that given an ergodic measure μ and a tempered Følner sequence, the set $\mathrm{Gen}(\mu, I_N)$ of generic points with respect to I_N satisfies $\mu(\mathrm{Gen}(\mu, I_N)) = 1$ (see Corollary 8 in [48]).

In a compact space X, a sequence μ_N of probability measures μ_N is said to *converge weakly* to μ if for all $f \in C(X)$,

$$\lim_{N\to\infty} \int_X f\ d\mu_N = \int_X f\ d\mu.$$

Hence, it is clear that $x \in \mathrm{Gen}(\mu, I_N)$ is equivalent to the sequence of measures

$$\nu_N = \frac{1}{|I_N|} \sum_{a\in I_N} \delta_{T_a(x)}$$

converging weakly to μ (where $\delta_{T_a(x)}$ is the measure that assigns 1 to A if $T_a(x) \in A$, and 0 otherwise). In particular, if we have two ergodic measures μ and ν, we must have $\mathrm{Gen}(\mu, I_N) \cap \mathrm{Gen}(\nu, I_N) = \emptyset$, so two ergodic measures are always mutually singular. This is, there are sets A and B (corresponding to $\mathrm{Gen}(\mu, I_N)^c$ and $\mathrm{Gen}(\nu, I_N)^c$ respectfully), such that $X = A \cup B$, and for any measurable set M, $\mu(A \cap M) = 0$ and $\nu(B \cap M) = 0$.

Lemma 2.30. *Let (X, T) be a uniquely ergodic system, with unique measure μ. Then, for every Følner sequence I_N,*

$$\mathrm{Gen}(\mu, I_N) = X.$$

Proof. We fix a Følner sequence I_N and some $x \in X$. Let $\nu_N := \frac{1}{|I_N|} \sum_{a\in I_N} \delta_{T_a(x)}$ be a sequence in $\mathcal{M}(X)$ (as in Definition 2.16). By Prokhorov's theorem (see Theorems 5.1 and 5.2 in [14]), the space $\mathcal{M}(X)$ with the weak-* topology is sequentially compact, given that X is a compact metric space.

Consequently, there is a subsequence ν_{N_i} that converges weakly to some measure $\nu \in \mathcal{M}(X)$. We claim that ν is $T-$invariant. Indeed, taking any $b \in G$, we have that the sequence $(T_b)_*\nu_{N_i}$ converge weakly to $(T_b)_*\nu$. For any measurable A, we have

$$|\nu_{N_i}(A) - (T_b)_*\nu_{N_i}(A)| = \frac{1}{|I_{N_i}|} \left| \sum_{a\in I_{N_i} \Delta T_{-b}(I_{N_i})} \delta_{T_a(x)}(A) \right| \leq \frac{|I_{N_i} \Delta T_{-b}(I_{N_i})|}{|I_{N_i}|},$$

which will go to 0 as i goes to infinity, since I_N is a Følner sequence. Therefore, $(T_b)_*\nu_{N_i}$ and ν_{N_i} must be weakly converging to the same limit, so $\nu = (T_b)_*\nu$ for every b, that is, ν is $T-$invariant.

We have $x \in \mathrm{Gen}(\nu, I_N)$, but since (X, T) is uniquely ergodic, we must have that $\nu = \mu$. Since this holds independently of x or I_N, the result follows. $\square$

Given a point $x \in X$, let $\mathbb{O}_T(x)$ be the orbit closure of x, that is

$$\mathbb{O}_T(x) := \overline{\{T_a(x) : a \in G\}}.$$

We will use the following property of generic points.

Lemma 2.31. *Let (X, T, μ) be an ergodic dynamical system. If there exists a Følner sequence I_N such that $x \in \mathrm{Gen}(\mu, I_N)$ then*

$$\mu\left(\mathbb{O}_T(x)\right) = 1.$$

Proof. Define the sequence of measures

$$\mu_N := \frac{1}{|I_N|} \sum_{a \in I_N} \delta_{T_a(x)}.$$

Since x is generic with respect to I_N, we know that the sequence μ_N converges weakly to μ. We have that $\mu_N(\{T_a(x) : a \in G\}) = 1$ for every N.

We now use Portmanteau's theorem (see Theorem 2.1 in [14]), which states that if μ_N converges weakly to μ, and C is a closed set, then

$$\limsup_N \mu_N(C) \leq \mu(C).$$

By applying it with $C = \mathbb{O}_T(x)$, we get

$$1 = \limsup_N \mu_N(\mathbb{O}_T(x)) \leq \mu(\mathbb{O}_T(x)),$$

which shows the desired implication. $\square$

3. Sieves and $R-$free numbers

In this section we introduce the basic objects with which we are going to work. Throughout, we assume that K is an étale $\mathbb{Q}-$algebra of degree n. We then define a sieve R over K as a set of congruence classes modulo an infinite number of pairwise coprime ideals of $\mathcal{O}_K$. Our objective is to study $\mathcal{F}_R$, the set of elements of $\mathcal{O}_K$ that do not belong to any of the classes contained in R, that is, the elements that are not sieved by R. In order to do this, we consider two shift spaces inside of $\{0,1\}^{\mathcal{O}_K}$, where the shift action S of $\mathcal{O}_K$ in $\{0,1\}^{\mathcal{O}_K}$ is defined in Equation (10).

The first space we define (in Definition 3.5) is Ω_R, the space of all $R-$admissible sets. Next we define X_R, which corresponds to the orbit closure of $\mathcal{F}_R$ under the shift action S. All this generalizes what has been done in the $\mathcal{B}-$free systems context, which corresponds to the case where all congruence classes in our sieve are of the form $0 + \mathfrak{b}$ for some ideal $\mathfrak{b}$ of $\mathcal{O}_K$.

We show that Ω_R contains X_R, and that we can define an ergodic measure on Ω_R which we denote by ν_R (the Mirsky measure). In order to do this, we first define the odometer G_R of R. This is a compact group where $\mathcal{O}_K$ acts by an action T. We define a map $\varphi_R : G_R \to \Omega_R$, which allows us to set ν_R as the pushforward of $\mathbb{P}$ (the Haar measure on G_R) by φ_R. In Lemma 3.12 we show that (Ω_R, S, ν_R) is a factor system of $(G_R, T, \mathbb{P})$, which shows this system is ergodic with measure theoretic entropy 0.

We then define when a sieve has weak light tails, a property which will be fundamental for the remainder of this work. We conclude the first subsection with Theorem 3.21, where we connect the density of $\mathcal{F}_R$ (see Definition 3.13) with the genericity of the point $\mathcal{F}_R \in \Omega_R$, and the weak light tails of the sieve R. This works as a generalization of point (1) of Sarnak's program.

In the second subsection we investigate some properties related to the density of sets that can be represented as $\mathcal{F}_R$ for some sieve R. We show that there are sieves such that $\mathcal{F}_R$ does not have logarithmic density, and solve a particular case of a problem of Erdős.

3.1. **Shift Spaces.**

We start with the definition of sieve.

Definition 3.1. *A sieve over an étale $\mathbb{Q}-$algebra K is a pair $(\mathcal{B}_R, (R_\mathfrak{b})_{\mathfrak{b}\in\mathcal{B}_R})$ where $\mathcal{B}_R$ is an infinite set of pairwise coprime invertible ideals of $\mathcal{O}_K$, and each $R_\mathfrak{b}$ is a set of the form $R_\mathfrak{b} = S_\mathfrak{b} + \mathfrak{b}$ with $S_\mathfrak{b}$ finite sets such that $R_\mathfrak{b} \neq \mathcal{O}_K$ for every $\mathfrak{b} \in \mathcal{B}_R$.*

We say that a sieve R is *supported* on the set $\mathcal{B}_R$. Note that we allow $R_\mathfrak{b}$ to be empty for any number of ideals $\mathfrak{b}$, as long as there are still infinitely many ideals $\mathfrak{b}$ such that $R_\mathfrak{b} \neq \emptyset$. Yet, when proving results we will always assume that $R_\mathfrak{b} \neq \emptyset$ for all $\mathfrak{b} \in \mathcal{B}_R$. In many cases it will be convenient to assume that $\mathcal{B}_R$ is ordered, so that $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$, in which case we will write $R_i = S_i + \mathfrak{b}_i$. As an example, let $\mathcal{B} = \{p_i^2 : i \in \mathbb{N}\}$, where p_i denotes the i-th prime. We can describe the squarefree sieve as the sieve R supported on $\mathcal{B}$ and defined by $R_i = p_i^2\mathbb{Z}$.

A $\mathcal{B}-$free system[1] denotes a sieve R such that $R_{\mathfrak{b}} = \mathfrak{b}$ for every $\mathfrak{b} \in \mathcal{B}_R$. In this work we will not consider generalizations of sieves where one allows for $\mathcal{B}_R$ to be a set of non-coprime ideals. Since this case has been treated in the literature for $\mathcal{B}-$free systems (see for example [26]), we still sometimes feel the need to refer to this case. Therefore, we will use the term *pseudosieve* for a pair $(\mathcal{B}_R, (R_{\mathfrak{b}})_{\mathfrak{b}\in\mathcal{B}_R})$ where $\mathcal{B}$ is an infinite set of (not necessarily pairwise coprime) invertible ideals and every $R_{\mathfrak{b}}$ can be written as $A_{\mathfrak{b}} + \mathfrak{b}$ for some $A_{\mathfrak{b}}$, with $R_{\mathfrak{b}} \neq \mathcal{O}_K$.

Given a sieve R and some ideal $\mathfrak{b}$, we will write $|R_{\mathfrak{b}}|$ for the cardinality of the image of $R_{\mathfrak{b}}$ in $\mathcal{O}_K/\mathfrak{b}$. This means that its complement $R^c_{\mathfrak{b}} = \mathcal{O}_K \setminus R_{\mathfrak{b}}$ satisfies $|R^c_{\mathfrak{b}}| = N(\mathfrak{b}) - |R_{\mathfrak{b}}|$. More generally, when A is a subset of $\mathcal{O}_K$, we will write $|A + \mathfrak{b}|$ for the cardinality of the image A in $\mathcal{O}_K/\mathfrak{b}$. We then define the *volume* of a set $A \subset \mathcal{O}_K/\mathfrak{b}$ as $\mathrm{vol}(A) := |A + \mathfrak{b}|/N(\mathfrak{b})$.

Definition 3.2. *We say a sieve R is Erdős if*

$$\sum_{\mathfrak{b}\in\mathcal{B}_R} \mathrm{vol}(R_{\mathfrak{b}}) < \infty.$$

These are so named because in the $\mathcal{B}-$free system literature, a system is called *Erdős* if the elements of $\mathcal{B}$ are pairwise coprime and $\sum_{\mathfrak{b}\in\mathcal{B}} 1/N(\mathfrak{b}) < \infty$, after Erdős studied such systems (over $\mathbb{Q}$) in [31].

One of the main purposes of our work is to study sets which can be realized as the $R-$free numbers for some sieve, that is, those that are not sieved by any $R_{\mathfrak{b}}$.

Definition 3.3. *Given a sieve R over an étale $\mathbb{Q}-$algebra K, we denote by the $R-$free numbers the set*

$$\mathcal{F}_R := \mathcal{O}_K \setminus \bigcup_{\mathfrak{b}\in\mathcal{B}_R} R_{\mathfrak{b}}.$$

For example, let R be a sieve such that $\mathcal{B}_R = \{p^k\mathbb{Z} : p \text{ prime}\}$ and defined by $R_p = p^k\mathbb{Z}$ for some integer $k \geq 2$. Then $\mathcal{F}_R$ corresponds to numbers that are not divisible by p^k for any prime, that is, the $k-$free numbers.

In this work, we will identify the space $\{0,1\}^{\mathcal{O}_K}$ with the set of subsets of $\mathcal{O}_K$ (by identifying the indicator function $\mathbb{1}_A$ with the set A). This set is a compact topological space equipped with the product topology on countably many copies of the discrete space $\{0,1\}$. Note that this topology is metrizable, and is induced by the metric d such that for $X, Y \subset \mathcal{O}_K$, we have

$$d(X,Y) = \min(1, \sup\{\epsilon > 0 : X \cap B_\epsilon = Y \cap B_\epsilon\}^{-1}).$$

That is, under this metric, two sets are close if they are identical in a big neighborhood around the origin. We define the shift map S in $\{0,1\}^{\mathcal{O}_K}$ as the action of $\mathcal{O}_K$ on this space given by

$$S_a(A) = A - a. \tag{10}$$

We will define dynamics on two *shift spaces*, which are closed, $S-$invariant subsets of $\{0,1\}^{\mathcal{O}_K}$. The first space we will consider is the set Ω_R of $R-$*admissible sets*.

[1] We will always use the term $\mathcal{B}-$free system to mean sieves that can be obtained by sieving out only ideals. This is different from the usual literature, where the term refers to the associated topological or measure theoretical dynamical systems instead.

Definition 3.4. *Given a sieve R over an étale $\mathbb{Q}-$algebra K, we say a set $A \subset \mathcal{O}_K$ is $R-$admissible if for every $\mathfrak{b} \in \mathcal{B}_R$, $-A + R_{\mathfrak{b}} \neq \mathcal{O}_K$.*

Equivalently, we say that A is an $R-$admissible set if for every $\mathfrak{b} \in \mathcal{B}_R$ there is some $\delta_{\mathfrak{b}} \in \mathcal{O}_K$ such that $(A + \delta_{\mathfrak{b}}) \cap R_{\mathfrak{b}} = \emptyset$. By taking $\delta_{\mathfrak{b}} = 0$ for every $\mathfrak{b}$, we see that $\mathcal{F}_R$ is always $R-$admissible. Additionally, it is clear that if A is $R-$admissible, so is $S_a(A)$ for any $a \in \mathcal{O}_K$.

Definition 3.5. *The set $\Omega_R \subset \{0,1\}^{\mathcal{O}_K}$ denotes the set of all $R-$admissible sets. We write X_R for the orbit closure of $\mathcal{F}_R$ in $\{0,1\}^{\mathcal{O}_K}$, that is,*

$$X_R = \mathbb{O}_S(\mathcal{F}_R) = \overline{\{S_a(\mathcal{F}_R) : a \in \mathcal{O}_K\}}.$$

Since X_R is a closed subset of a compact space, it is also compact. The same holds for Ω_R, as we show in the following lemma.

Lemma 3.6. *Ω_R is closed.*

Proof. Assume that A is not an admissible set. By definition, there must be some $\mathfrak{b} \in \mathcal{B}_R$ such that $-A + R_{\mathfrak{b}} = \mathcal{O}_K$. Let N be big enough that $(-A \cap B_N) + \mathfrak{b} = \mathcal{O}_K$. For any admissible set B, we necessarily have $(-B \cap B_N) \neq (-A \cap B_N)$, so $d(A, \Omega_R) > 1/N$. ☐

Remark 3.7. This implies that $X_R \subset \Omega_R$, given that $\{S_a(\mathcal{F}_R) : a \in \mathcal{O}_K\} \subset \Omega_R$, and so the closure of the orbit must also be contained in Ω_R, as this space is closed.

Since X_R and Ω_R are both compact, (Ω_R, S) and (X_R, S) are topological dynamical systems.

Definition 3.8. *Given a sieve R over an étale $\mathbb{Q}-$algebra K, we define its odometer as the dynamical system $(G_R, T, \mathbb{P})$, where*

$$G_R := \prod_{\mathfrak{b} \in \mathcal{B}_R} \mathcal{O}_K/\mathfrak{b},$$

the action T of $\mathcal{O}_K$ on G_R is given by

$$T_a((g_{\mathfrak{b}})_{\mathfrak{b} \in \mathcal{B}_R}) = (a + g_{\mathfrak{b}})_{\mathfrak{b} \in \mathcal{B}_R},$$

and $\mathbb{P}$ is the Haar measure on G_R.

The measure $\mathbb{P}$ has a simple description. Given a set $A = \{(\mathfrak{b}_1, a_1), \ldots, (\mathfrak{b}_m, a_m)\}$ where the $\mathfrak{b}_i \in \mathcal{B}_R$ are distinct and $a_i \in \mathcal{O}_K$, let C_A be the cylinder set

$$C_A = \{g \in G_R : \forall_{1 \leq i \leq m} \; g_{\mathfrak{b}_i} \equiv a_i \mod \mathfrak{b}_i\}.$$

These sets generate the Borel $\sigma-$algebra of G_R, and for any such A, $\mathbb{P}(C_A)$ is given by

$$\mathbb{P}(C_A) = \prod_{i=1}^{m} \frac{1}{N(\mathfrak{b}_i)}.$$

Fixing some ordering of $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$, and for any $g \in G_R$, writing $g_i := g_{\mathfrak{b}_i}$, we can define a metric on G_R by

$$d(g,h) := \sum_i \frac{\rho_i(g,h)}{2^i},$$

where $\rho_i(g,h) = 0$ if $g_i = h_i$ and 1 otherwise. We will write $\mathbf{0}$ for the identity of G_R. For any integer $L \geq 1$ and tuple $(g_1, \dots, g_L)$, the Chinese Remainder Theorem states that there is some $a \in \mathcal{O}_K$, such that $T_a(\mathbf{0})_i = g_i$ for any integer $1 \leq i \leq L$. Therefore, it is clear that $\overline{\{T_a(\mathbf{0}) : a \in \mathcal{O}_K\}} = G_R$, which means that (G_R, T) is a rotation of a group, and so a uniquely ergodic system by Theorem 2.19.

We now define a measure on Ω_R, known as the Mirsky measure (since this measure will associate to a set A the frequency of the pattern A in $\mathcal{F}_R$, which Mirsky first considered in [59] for the $r-$free numbers). In order to do this, we consider the map $\varphi_R : G_R \to \Omega_R$, which we will define by the relation

$$a \in \varphi_R(g) \Leftrightarrow \forall_{\mathfrak{b} \in \mathcal{B}_R} \; a + g_{\mathfrak{b}} \notin R_{\mathfrak{b}}. \tag{11}$$

Note that the image of φ_R does indeed lie in Ω_R, since by definition $(\varphi_R(g) + g_{\mathfrak{b}}) \cap R_{\mathfrak{b}} = \emptyset$ for every $\mathfrak{b} \in \mathcal{B}_R$. The elements of $\varphi_R(\mathbf{0})$ are those a such that $a \notin R_{\mathfrak{b}}$ for every $\mathfrak{b} \in \mathcal{B}_R$, that is,

$$\varphi_R(\mathbf{0}) = \mathcal{F}_R.$$

Definition 3.9. *Given a sieve R, we define the Mirsky measure ν_R on Ω_R as the pushforward of $\mathbb{P}$ in Ω_R by φ_R, that is $\nu_R(A) := \mathbb{P}(\varphi_R^{-1}(A))$ for any measurable set A.*

This measure is defined on the Borel $\sigma-$algebra of Ω_R, which is generated by *cylinder sets.*

Definition 3.10. *Given a sieve R and disjoint sets A, B, we define the cylinder set $C^R_{A,B}$ as the set*

$$C^R_{A,B} := \{Y \in \Omega_R : A \subset Y, Y \cap B = \emptyset\}.$$

The topology in Ω_R is generated by such sets with A and B finite, so they are open. Simultaneously, if A and B are finite, $C^R_{A,B}$ is closed. Indeed, if Y_m is a sequence of sets in $C^R_{A,B}$ converging to some $Y \in \Omega_R$, and N is so big that $A \cup B \subset B_N$, then $d(Y, Y_m) < 1/N$ implies that Y and Y_m agree in B_N, so $A \subset Y$ and $B \cap Y = \emptyset$, so $Y \in C^R_{A,B}$. That is, these $C^R_{A,B}$ are clopen (sets that are both open and closed), so it follows that $\mathbb{1}_{C_{A,B}}$ is a continuous function whenever A and B are finite.

In particular, $C^R_{A,\emptyset}$ corresponds to the $R-$admissible sets that contain A. In this case, ν_R is given by

$$\nu_R(C^R_{A,\emptyset}) = \mathbb{P}(\varphi_R^{-1}(C_{A,\emptyset})) = \mathbb{P}(\{g \in G_R : \forall_{\mathfrak{b} \in \mathcal{B}_R} \; (g_{\mathfrak{b}} + A) \cap R_{\mathfrak{b}} = \emptyset\}) = \prod_{\mathfrak{b} \in \mathcal{B}_R} \left(1 - \frac{|-A + R_{\mathfrak{b}}|}{N(\mathfrak{b})}\right). \tag{12}$$

Using the following lemma (which is Lemma 2.3 from [1]), it follows that when A and B are finite we have

$$\nu_R(C^R_{A,B}) = \sum_{A \subset D \subset A \cup B} (-1)^{|D \setminus A|} \prod_{\mathfrak{b} \in \mathcal{B}_R} \left(1 - \frac{|-D + R_{\mathfrak{b}}|}{N(\mathfrak{b})}\right). \tag{13}$$

Lemma 3.11. *Let μ be a probability measure is a space Ω, and let E_x be a sequence of measurable sets. Denote $F_x := \Omega \setminus E_x$. Then, given finite disjoint sets A and B, we have*

$$\mu\left(\bigcap_{a\in A} E_a \cap \bigcap_{b\in B} F_b\right) = \sum_{A\subset D\subset A\cup B} (-1)^{|D\setminus A|}\mu\left(\bigcap_{d\in D} E_d\right).$$

We will now study some of the properties of the system (Ω_R, S, ν_R). We start with the following result.

Lemma 3.12. *The dynamical system (Ω_R, S, ν_R) is a factor of $(G_R,T,\mathbb{P})$.*

Proof. We show that φ_R is the factor map between these systems. It is a measure preserving map by the definition of ν_R. So we are left with showing that $\varphi_R \circ T = S \circ \varphi_R$. This corresponds to showing that for any $x, a \in \mathcal{O}_K$, $a \in \varphi_R(g + x)$ if and only if $a \in -x + \varphi_R(g)$. But $a \in \varphi_R(g + x)$ is equivalent to $(a + x) + g_{\mathfrak{b}} \notin R_{\mathfrak{b}}$ for all $\mathfrak{b} \in \mathcal{B}_R$, which is equivalent to $(a + x) \in \varphi_R(g)$, so the result holds. ☐

Since $(G_R,T,\mathbb{P})$ is ergodic, it follows that (Ω_R, S, ν_R) also is. Additionally, since $(G_R,T,\mathbb{P})$ is a minimal rotation of a compact group, we have $h(G_R, T, \mathbb{P}) = 0$ (see Definition 2.16 for the definition of measure theoretical entropy). By Lemma 2.23, it follows that

$$h(\Omega_R, S, \nu_R) = 0. \tag{14}$$

Definition 3.13. *Given a set $A \subset \mathcal{O}_K$ and a Følner sequence I_N, we define the upper and lower densities with respect to I_N to be respectively,*

$$\overline{d}_I(A) := \limsup_{N\to\infty} \frac{|A \cap I_N|}{|I_N|} \text{ and } \underline{d}_I(A) := \liminf_{N\to\infty} \frac{|A \cap I_N|}{|I_N|}.$$

If these agree, we write the limit as $d_I(A)$, which we call the density of A with respect to I_N.

In the case where $I_N = B_N$ (as defined in Equation (3)), we omit the I, and write $\overline{d}(A), \underline{d}(A), d(A)$ for the corresponding densities.

Since we will be interested in computing the densities of $\mathcal{F}_R$ for different sieves R, we will use the following lemma which allows us to compute the density of the elements that miss a set of congruence classes.

Lemma 3.14. *Let $L \geq 1$ be an integer, I_N a Følner sequence, $\mathfrak{b}_1, \dots, \mathfrak{b}_L$ a collection of pairwise coprime ideals. For each i, take $A_i \subset \mathcal{O}_K$ and let $R_i = A_i + \mathfrak{b}_i$. If*

$$C_L := \{x \in \mathcal{O}_K : \forall_i\, x \notin R_i\},$$

then

$$\frac{1}{|I_N|} \sum_{a\in I_N} \mathbb{1}_{C_L}(a) \to d_I(C_L) = \prod_{i=1}^{L} \left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right),$$

as $N \to \infty$.

Proof. Let $\mathfrak{c}_L$ denote the product of all the $\mathfrak{b}_i$, and consider the topological dynamical system (X,T) where $X = \mathcal{O}_K/\mathfrak{c}_L$ and $T_a(x) = x + a$ is an action of $\mathcal{O}_K$ on X. Since (X,T) is a minimal rotation of a compact group, Theorem 2.19 implies that it is uniquely ergodic. The unique $T-$invariant probability measure is given by

$$\mathbb{P}(A) = \frac{|A|}{N(\mathfrak{c}_L)}.$$

Let $\mathfrak{C}_L$ for the image of C_L in X. By the Chinese Remainder Theorem, there is a bijection between $\mathfrak{C}_L$ and tuples $(x_1,\dots,x_L)$ with $x_i \in \mathcal{O}_K/\mathfrak{b}_i$ such that $x_i \notin R_i$ for every i. Therefore $|\mathfrak{C}_L| = \prod_i (N(\mathfrak{b}_i) - |R_i|)$, and so

$$\mathbb{P}(\mathfrak{C}_L) = \frac{|\mathfrak{C}_L|}{N(\mathfrak{c}_L)} = \prod_{i=1}^{L}\left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right).$$

By Lemma 2.30 the point $\mathbf{0}$ belongs to $\mathrm{Gen}(\mathbb{P}, I_N)$ for any Følner sequence I_N, that is,

$$\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L}(a) = \frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{\mathfrak{C}_L}(T_a(\mathbf{0})) \to \mathbb{P}(\mathfrak{C}_L) = \prod_{i=1}^{L}\left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right)$$

as N goes to infinity. □

We want to know when the convergence in Lemma 3.14 holds as we take L to infinity. We now introduce the notion of weak light tails, which we will show is exactly the condition which determines whether this holds or not. This will be the key ingredient to generalizing point (1) of Sarnak's program to Erdős sieves.

Definition 3.15. *We say a sieve R supported on $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$ has* weak light tails *for the Følner sequence I_N (or, with respect to I_N) if it is Erdős, and*

$$\lim_{L\to\infty}\overline{d}_I\left(\bigcup_{i>L} R_i \setminus \bigcup_{j\le L} R_j\right) = 0.$$

Equivalently, we say an Erdős sieves has weak light tails for I_N if

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{|\{x \in I_N : x \in R_i \text{ for some } i \text{ with } i > L, x \notin R_i \text{ if } i \le L\}|}{|I_N|} = 0.$$

In Proposition 3.25, we provide an example of a sieve R such that $\mathcal{F}_R$ does not have a density with respect to B_N. This shows that the limit supremum used on this definition is required as sets of form $\bigcup_{i>L} R_i$ may not have a density.

We say that such sieves have *weak* light tails, because we will also consider another property of sieves, which we denote by *strong light tails.* As the name implies, a sieve with strong light tails will have weak light tails.

Definition 3.16. *We say a sieve R has* strong light tails *for a Følner sequence I_N (or, with respect to I_N), if it is Erdős and*

$$\lim_{L\to\infty}\overline{d}_I\left(\bigcup_{i>L} R_i\right) = 0.$$

Equivalently, we say an Erdős sieve R has strong light tails for I_N if

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{|\{x\in I_N : x\in R_i \text{ for some } i \text{ with } i>L\}|}{|I_N|}=0.$$

Clearly strong light tails imply weak light tails, but as we will show in Example 5.14, the converse does not hold. Although the definition of these properties assumes that $\mathcal{B}_R$ was ordered, both properties are invariant under reordering of $\mathcal{B}_R$, as we will show in Lemma 5.15 and Lemma 5.16.

Our use of the term *light tails* to describe such sieves is inspired by the use of the term in Section 1.1 of [26]. In this paper the authors defined a $\mathcal{B}-$free system with light tails as a set $\{b_1, b_2, \dots\}\subset\mathbb{N}$ such that

$$\lim_{L\to\infty}\overline{d}\left(\bigcup_{i>L} b_i\mathbb{Z}\right)=0.$$

It is clear that all such sets correspond to sieves with strong light tails.

Example 3.17. For an example of a sieve without weak light tails for any I_N, let R be the sieve over $\mathbb{Q}$ and supported on $\mathcal{B}=\{p_i^2 : i\in\mathbb{N}\}$, defined by

$$R_i=\{-i,i\}+p_i^2\mathbb{Z}.$$

We have for any $x\in\mathbb{Z}\setminus\{0\}$ that $x\in R_{|x|}$, so given $L\geq 1$ we have

$$|I_N\cap\bigcup_{i>L}R_i|\geq|I_N|-L.$$

On the other hand, by Lemma 3.14, we have that

$$|I_N\setminus\bigcup_{i\leq L}R_i|=|I_N|\prod_{i=1}^{L}\left(1-\frac{2}{p_i^2}\right)+o(|I_N|).$$

Consequently, we have that

$$\lim_{N\to\infty}\frac{|I_N\cap\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j|}{|I_N|}\geq\lim_{N\to\infty}\frac{\left(|I_N|\left(1-\prod_{i=1}^{L}\left(1-\frac{2}{p_i^2}\right)\right)-L-o(|I_N|)\right)}{|I_N|}=1-\prod_{i=1}^{L}\left(1-\frac{2}{p_i^2}\right)$$

which is bigger than 0 for every L. Therefore the sieve R does not have weak light tails for any Følner sequence.

On the other hand, let $I_N=[0,N]$, and consider the sieve R given by

$$R_i=p_i^2\mathbb{Z},$$

that is, the squarefree sieve. It has strong light tails for I_N as a consequence of Lemma 5.3. Note that for every sieve R there is some Følner sequence I_N such that R does not have weak light tails with respect to I_N, as we will show in Remark 4.3.

We are now in a position to generalize point (1) of Sarnak's program to Erdős sieves, by expliciting when $\mathcal{F}_R$ is generic in (Ω_R, S, ν_R). We start by showing two lemmas. The following very simple lemma shows that point (1) of Sarnak's program is a theorem of interest from a number theoretic point of view.

Lemma 3.18. *Let A, B, X be subsets of $\mathcal{O}_K$. We have*

$$\sum_{x\in X} \mathbb{1}_{C^R_{A,B}}(S_x(\mathcal{F}_R)) = |\{x \in X : x + A \subset \mathcal{F}_R, (x+B) \cap \mathcal{F}_R = \emptyset\}|.$$

Proof. If $\mathbb{1}_{C^R_{A,B}}(S_x(\mathcal{F}_R))$ is 1, then $-x + \mathcal{F}_R$ is an admissible set containing A and disjoint from B, which is equivalent to $x + A \subset \mathcal{F}_R$ and $x + B \cap \mathcal{F}_R = \emptyset$, since $S_x(\mathcal{F}_R)$ is always admissible. □

This allows us to relate the weak light tails of R and $d_I(\mathcal{F}_R)$, which we will use to show our generalization of point (1) of Sarnak's program.

Lemma 3.19. *Let R be a sieve and I_N a Følner sequence. Then*

$$\bar{d}_I(\mathcal{F}_R) \leq \nu_R(C^R_{\{0\},\emptyset}) = \prod_{\mathfrak{b}\in\mathcal{B}_R} \left(1 - \frac{|R_\mathfrak{b}|}{N(\mathfrak{b})}\right).$$

For an Erdős sieve R, $d_I(\mathcal{F}_R)$ exists and equals $\nu_R(C^R_{\{0\},\emptyset})$ if and only if R has weak light tails for I_N.

Proof. Notice that $a \in \mathcal{F}_R$ is equivalent to $a + \{0\} \subset \mathcal{F}_R$. Therefore, by Lemma 3.18, we have

$$\frac{|I_N \cap \mathcal{F}_R|}{|I_N|} = \frac{|\{a \in I_N : a + \{0\} \subset \mathcal{F}_R\}|}{|I_N|} = \frac{1}{|I_N|}\sum_{a\in I_N} \mathbb{1}_{C^R_{\{0\},\emptyset}}(S_a(\mathcal{F}_R)).$$

Since $\varphi_R(\mathbf{0}) = \mathcal{F}_R$ and φ_R is a factor map (by Lemma 3.12), we have that

$$S_a(\mathcal{F}_R) = S_a(\varphi_R(\mathbf{0})) = \varphi_R(T_a(\mathbf{0})),$$

and so $S_a(\mathcal{F}_R) \in C^R_{\{0\},\emptyset}$ is equivalent to $T_a(\mathbf{0}) \in \varphi_R^{-1}(C^R_{\{0\},\emptyset})$, hence we get

$$\frac{|I_N \cap \mathcal{F}_R|}{|I_N|} = \frac{1}{|I_N|}\sum_{a\in I_N} \mathbb{1}_{\varphi_R^{-1}(C^R_{\{0\},\emptyset})}(T_a(\mathbf{0})). \tag{15}$$

We would like to evaluate the sum in the right hand side as N goes to infinity using Lemma 2.30, given that (G_R, T) is a uniquely ergodic system. The problem is that φ_R is not continuous, so $\mathbb{1}_{\varphi_R^{-1}(C^R_{\{0\},\emptyset})}$ might not be continuous. We must therefore approach this set through clopen sets. Let us order the support $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$. We have that

$$\varphi_R^{-1}(C^R_{\{0\},\emptyset}) = \{g \in G_R : g_i \notin R_i \text{ for all } i\}.$$

By defining

$$C_L := \{g \in G_R : g_i \notin R_i \text{ for all } i \leq L\},$$

we have that C_L is a clopen subset of G_R, so $\mathbb{1}_{C_L}$ is a continuous function on G_R. By the same argument used in Lemma 3.14 (simply choosing X to be G_R instead), we have

$$\lim_{N\to\infty} \frac{1}{|I_N|}\sum_{a\in I_N} \mathbb{1}_{C_L}(T_a(\mathbf{0})) = \mathbb{P}(C_L) = \prod_{i=1}^{L}\left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right).$$

Note that $\varphi_R^{-1}(C^R_{\{0\},\emptyset}) \subset C_L$ for every L. We approach the set $\varphi_R^{-1}(C^R_{\{0\},\emptyset})$ through the sets C_L, by noticing that for any integer $L \geq 1$ we have the disjoint union

$$C_L = \varphi_R^{-1}(C^R_{\{0\},\emptyset}) \cup (C_L \setminus \varphi_R^{-1}(C^R_{\{0\},\emptyset})).$$

Using this, along with Equation (15), gives

$$\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L}(T_a(\mathbf{0}))=\frac{|\mathcal{F}_R\cap I_N|}{|I_N|}+\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L\setminus\varphi_R^{-1}(C^R_{\{0\},\emptyset})}(T_a(\mathbf{0})). \tag{16}$$

It follows that

$$\frac{|\mathcal{F}_R\cap I_N|}{|I_N|}\leq\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L}(T_a(\mathbf{0}))$$

holds for every N. Taking the limit supremum on N gives

$$\bar{d}_I(\mathcal{F}_R)\leq\prod_{i=1}^{L}\left(1-\frac{|R_i|}{N(\mathfrak{b}_i)}\right).$$

Since this holds for every L, we can take the limit as L goes to infinity, which gives

$$\bar{d}_I(\mathcal{F}_R)\leq\nu_R(C^R_{\{0\},\emptyset})$$

as we wanted.

Returning to Equation (16), notice that $T_a(\mathbf{0})\in C_L\setminus\varphi_R^{-1}(C^R_{\{0\},\emptyset})$ is equivalent to $a\notin\mathcal{F}_R$ and $a\notin R_j$ for all $j\leq L$. That is, $\mathbb{1}_{C_L\setminus\varphi_R^{-1}(C^R_{\{0\},\emptyset})}(T_a(\mathbf{0}))=1$ if and only if $a\in\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j$, and so

$$\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L\setminus\varphi_R^{-1}(C^R_{\{0\},\emptyset})}(T_a(\mathbf{0}))=\frac{\left|I_N\cap\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right|}{|I_N|}.$$

Hence, if on Equation (16) we take the limit supremum on N, we obtain

$$\prod_{i=1}^{L}\left(1-\frac{|R_i|}{N(\mathfrak{b}_i)}\right)\leq\bar{d}_I(\mathcal{F}_R)+\bar{d}_I\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right).$$

Taking the limit as L goes to infinity, we get that

$$0\leq\nu_R(C^R_{\{0\},\emptyset})-\bar{d}_I(\mathcal{F}_R)\leq\lim_{L\to\infty}\bar{d}_I\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right). \tag{17}$$

On the other hand, rewriting Equation (16) as

$$\frac{|\mathcal{F}_R\cap I_N|}{|I_N|}=\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L}(T_a(\mathbf{0}))-\frac{\left|I_N\cap\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right|}{|I_N|}$$

and taking the limit infimum on N we get

$$\underline{d}_I(\mathcal{F}_R)\geq\prod_{i=1}^{L}\left(1-\frac{|R_i|}{N(\mathfrak{b}_i)}\right)-\bar{d}_I\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right).$$

Taking the limit as L goes to infinity, this gives

$$\nu_R(C^R_{\{0\},\emptyset})\geq\underline{d}_I(\mathcal{F}_R)\geq\nu_R(C^R_{\{0\},\emptyset})-\lim_{L\to\infty}\bar{d}_I\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\leq L}R_j\right). \tag{18}$$

Hence, if R has weak light tails for I_N, Equation (17) gives $\nu_R(C^R_{\{0\},\emptyset}) = \bar{d}_I(\mathcal{F}_R)$ and Equation (18) gives $\nu_R(C^R_{\{0\},\emptyset}) = \underline{d}_I(\mathcal{F}_R)$, so $d_I(\mathcal{F}_R)$ exists and equals $\nu_R(C^R_{\{0\},\emptyset})$. On the other hand, if $d_I(\mathcal{F}_R)$ exists and equals $\nu_R(C^R_{\{0\},\emptyset})$, then rewriting Equation (16) as

$$\frac{\left|I_N \cap \bigcup_{i>L} R_i \setminus \bigcup_{j\leq L} R_j\right|}{|I_N|} = \frac{1}{|I_N|}\sum_{a\in I_N} \mathbb{1}_{C_L}(T_a(\mathbf{0})) - \frac{|\mathcal{F}_R \cap I_N|}{|I_N|},$$

we have that the right hand side is the sum of two sequences in N with a well defined limit, so we can take the limit as N goes to infinity to get

$$d_I\left(\bigcup_{i>L} R_i \setminus \bigcup_{j\leq L} R_j\right) = \prod_{i=1}^{L}\left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right) - \nu_R(C^R_{\{0\},\emptyset}).$$

Taking L to go to infinity, the right hand side goes to 0, so R will have weak light tails with respect to I_N. □

Remark 3.20. It may happen that a sieve is such that $d_I(\mathcal{F}_R)$ is well defined, but is different from $\nu_R(C^R_{\{0\},\emptyset})$, in which case it does not have weak light tails by Lemma 3.19 (for an example look at the sieve W' in Example 5.14). It can also happen that a sieve is such that the upper density $\bar{d}_I(\mathcal{F}_R)$ is equal to $\nu_R(C^R_{\{0\},\emptyset})$, but this still does not imply that R has weak light tails (see Example 3.28).

Notice that in the proof of Lemma 3.19, we never use that R is Erdős. By Lemma 2.1, R being Erdős is equivalent to $\nu_R(C^R_{\{0\},\emptyset}) > 0$. Hence, if R is not Erdős, we have $d_I(\mathcal{F}_R) = 0$ for every Følner sequence I_N. This shows that if we didn't restrict our definition of sieve with weak light tails to Erdős sieves, we would have that every sieve R such that $\sum_{\mathfrak{b}\in\mathcal{B}_R} \mathrm{vol}(R_\mathfrak{b}) = \infty$ would have weak light tails with respect to every Følner sequence I_N. When R is such a sieve, we have that $\nu_R = \delta_\emptyset$, so (Ω_R, S, ν_R) is not an interesting system. For this reason, we restrict our study of sieves with weak light tails to only Erdős sieves.

We are now in a position to show the following theorem, which is our generalization of point (1) of Sarnak's program. For the proof that (2) implies (1), we use ideas similar to those in the proof of Theorem 4.1 in [1] and Theorem A (i) in [2].

Theorem 3.21. *Let R be an Erdős sieve. For a given Følner sequence I_N, the following are equivalent.*

1. *$\mathcal{F}_R$ is a generic point of (Ω_R, S, ν_R) with respect to I_N.*
2. *R has weak light tails with respect to I_N.*
3. *The set $\mathcal{F}_R$ has a density with respect to I_N given by*

$$d_I(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset}) = \prod_{\mathfrak{b}\in\mathcal{B}_R}\left(1 - \frac{|R_\mathfrak{b}|}{N(\mathfrak{b})}\right).$$

Proof. We have shown that (2) and (3) are equivalent in Lemma 3.19. To show that (1) implies (3), notice that $C^R_{\{0\},\emptyset}$ is clopen in Ω_R. Hence, if $\mathcal{F}_R$ is generic for I_N, then by Lemma 3.18 we have

$$d_I(\mathcal{F}_R) = \lim_{N\to\infty} \frac{1}{|I_N|}\sum_{x\in I_N} \mathbb{1}_{C^R_{\{0\},\emptyset}}(S_x(\mathcal{F}_R)) = \nu_R(C^R_{\{0\},\emptyset}),$$

so (3) holds.

It remains to show that (2) implies (1). First, notice that when A, B are finite, $\mathbb{1}_{C^R_{A,B}}$ is a continuous function. Since $\mathbb{1}_{C^R_{A,B}}\mathbb{1}_{C^R_{X,Y}} = \mathbb{1}_{C^R_{A\cap X, B\cup Y}}$, the functions $\mathbb{1}_{C^R_{A,B}}$ generate a subalgebra of the space of continuous functions over Ω_R. Clearly, this algebra separates points, since if $X \neq Y$, taking $x \in X \setminus Y$, we have $\mathbb{1}_{C^R_{\{x\},\emptyset}}(X) = 1$ but $\mathbb{1}_{C^R_{\{x\},\emptyset}}(Y) = 0$. By the Stone-Weierstrass theorem, it follows that the functions of the form $\mathbb{1}_{C^R_{A,B}}$ span a dense subalgebra of the set of continuous functions over Ω_R.

Consequently we only need to show the result for functions of this type. In fact, using Lemma 3.11 we see that we only have to do it for functions of the form $\mathbb{1}_{C^R_{A,\emptyset}}$. Arguing as in the proof of Lemma 3.19 we have $\mathbb{1}_{C^R_{A,\emptyset}}(S_x(\mathcal{F}_R)) = \mathbb{1}_{\varphi_R^{-1}(C^R_{A,\emptyset})}(T_x(\mathbf{0}))$, and so we reduce the problem to showing that

$$\frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{\varphi_R^{-1}(C^R_{A,\emptyset})}(T_x(\mathbf{0})) \to \nu_R(C^R_{A,\emptyset}) = \mathbb{P}(\varphi_R^{-1}(C^R_{A,\emptyset}))$$

as N goes to infinity.

The idea is again to use the fact that every point in G_R is generic for the system $(G_R, T, \mathbb{P})$, as this is a minimal rotation of a compact group. Yet, since φ_R is not continuous, we must approach the set $\varphi^{-1}(C^R_{A,\emptyset})$ by clopen sets to use this fact.

To do this, we look at the set $\varphi^{-1}(C^R_{A,\emptyset})$. If we order $\mathcal{B}_R$, we see that $\varphi^{-1}(C^R_{A,\emptyset})$ corresponds to those $g \in G_R$ such that $g_i + a \notin R_i$ for every $i \in \mathbb{N}$ and $a \in A$, that is,

$$\varphi^{-1}(C^R_{A,\emptyset}) = \{g \in G_R : g_i \notin -A + R_i \text{ for all } i\}.$$

Let

$$C_L := \{g \in G : g_i \notin -A + R_i \text{ for all } i \text{ such that } i \leq L\}.$$

It is clear that $\varphi_R^{-1}(C^R_{A,\emptyset}) \subset C_L$ for every L, and therefore

$$C_L = \varphi_R^{-1}(C^R_{A,\emptyset}) \cup (C_L \setminus \varphi_R^{-1}(C^R_{A,\emptyset})).$$

We now proceed as in Lemma 3.19. We have that the C_L are clopen, so using Lemma 2.30 we know that $\mathbf{0}$ is generic in $(G_R, T, \mathbb{P})$ with respect to I_N, and so

$$\lim_{N\to\infty}\frac{1}{|I_N|}\sum_{a\in I_N}\mathbb{1}_{C_L}(T_a(\mathbf{0})) = \mathbb{P}(C_L) = \prod_{i=1}^{L}\left(1 - \frac{|-A+R_i|}{N(\mathfrak{b}_i)}\right).$$

Therefore, by Equation (12) we have

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{C_L}(T_x(\mathbf{0})) = \nu_R(C^R_{A,\emptyset}).$$

Since

$$\frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{\varphi_R^{-1}(C^R_{A,\emptyset})}(T_x(\mathbf{0})) = \frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{C_L}(T_x(\mathbf{0})) - \frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{C_L\setminus\varphi_R^{-1}(C^R_{A,\emptyset})}(T_x(\mathbf{0})),$$

the result will follow if we can show that

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{1}{|I_N|}\sum_{x\in I_N}\mathbb{1}_{C_L\setminus\varphi_R^{-1}(C^R_{A,\emptyset})}(T_x(\mathbf{0})) = 0,$$

which corresponds to showing that

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i>L} (-A+R_i) \setminus \bigcup_{i\le L} (-A+R_i) \right) = 0.$$

This is because $T_x(0) \notin \varphi_R^{-1}(C^R_{A,\emptyset})$ occurs if and only if there is some i such that $x \in -A+R_i$, and $T_x(0) \in C_L$ is equivalent to $x \notin -A+R_i$ for every $i \le L$.

We now notice that

$$\begin{aligned}\left| I_N \cap \bigcup_{i>L} (-A+R_i) \setminus \bigcup_{i\le L} (-A+R_i) \right| &\le \sum_{a\in A} \left| I_N \cap \bigcup_{i>L} (-a+R_i) \setminus \bigcup_{i\le L} (-A+R_i) \right| \\ &\le \sum_{a\in A} \left| I_N \cap \bigcup_{i>L} (-a+R_i) \setminus \bigcup_{i\le L} (-a+R_i) \right| \\ &= \sum_{a\in A} \left| (I_N + a) \cap \bigcup_{i>L} R_i \setminus \bigcup_{i\le L} R_i \right|.\end{aligned}$$

Using the fact that for any three sets X, Y, Z we have $|X \cap Y| \le |X \cap Z| + |Y \Delta Z|$, we get (taking $X = \bigcup_{i>L} R_i \setminus \bigcup_{i\le L} R_i$, $Y = (I_N + a)$ and $Z = I_N$) that

$$\left| (I_N + a) \cap \bigcup_{i>L} R_i \setminus \bigcup_{i\le L} R_i \right| \le \left| I_N \cap \bigcup_{i>L} R_i \setminus \bigcup_{i\le L} R_i \right| + |(I_N+a)\Delta I_N|,$$

and so all we need to show is that

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{|A|}{|I_N|} \left| I_N \cap \bigcup_{i>L} R_i \setminus \bigcup_{i\le L} R_i \right| + \sum_{a\in A} \frac{|(I_N+a)\Delta I_N|}{|I_N|} = 0.$$

The first term will go to 0 since R has weak light tails, while the second will go to 0 since I_N is a Følner sequence. □

Remark 3.22. We will be computing $d_I(\mathcal{F}_R)$ for many different sieves using Theorem 3.21. From a number theoretic point of view, one would like to give bounds of the form

$$|\mathcal{F}_R \cap I_N| = |I_N| \nu_R(C^R_{\{0\},\emptyset}) + E(N)$$

where E is an explicit function for which we can say more than $E(N) = o(|I_N|)$.

Yet, with our methods, this is not possible to do. The ergodic theory result we used to show Lemma 3.14 does not provide us with any error bounds. We now give a different proof of this lemma for the particular case $I_N = B_N$, this time providing error bounds.

As in Lemma 3.14, we let $\mathfrak{b}_1, \dots, \mathfrak{b}_L$ be a collection of pairwise coprime ideals, and define the ideal $\mathfrak{c}_L = \prod_{1\le i\le L} \mathfrak{b}_i$. We also take a collection of sets R_i of the form $R_i = A_i + \mathfrak{b}_i$, and write

$$C_L = \{a \in \mathcal{O}_K : \forall_{1\le i\le L}\, a \notin R_i\}.$$

Let $\mathfrak{C}_L$ denote the image of C_L in $\mathcal{O}_K/\mathfrak{c}_L$. Every $a \in B_N \cap C_L$ will be congruent to a unique $c \in \mathfrak{C}_L$ modulo $\mathfrak{c}_L$, so

$$\frac{1}{|B_N|}\sum_{a\in B_N} \mathbb{1}_{C_L}(a) = \frac{1}{|B_N|}\sum_{c\in\mathfrak{C}_L} |\{a \in B_N : a \equiv c \mod \mathfrak{c}_l\}|.$$

Applying the result in Remark 2.8 it follows that

$$\frac{1}{|B_N|}\sum_{a\in B_N} \mathbb{1}_{C_L}(a) = \frac{1}{|B_N|}\sum_{c\in\mathfrak{C}_L} \left(\frac{|B_N|}{N(\mathfrak{c}_L)} + O_K\left(1 + \left(\frac{N}{N(\mathfrak{c}_L)^{1/n}}\right)^{n-1}\right).\right),$$

which we can rewrite as

$$\frac{1}{|B_N|}\sum_{a\in B_N} \mathbb{1}_{C_L}(a) = \frac{|\mathfrak{C}_L|}{N(\mathfrak{c}_L)}\left(1 + O_K\left(\frac{1}{|B_N|} + N(\mathfrak{c}_L)^{1/n}\frac{N^{n-1}}{|B_N|}\right)\right).$$

We see that taking N to infinity, the error term goes to 0, providing an alternative proof of Lemma 3.14. But $N(\mathfrak{c}_L)$ goes to infinity as L goes to infinity, so the convergence is not uniform.

3.2. **Density of $R-$free numbers.** The objective of this subsection is to prove a number of facts about the densities of sets of the form $\mathcal{F}_R$. Most of the results in this section don't have a good analogue over number fields, so everything is done over $\mathbb{N}$. In particular, we will use the notation $d(A)$ for the density along the Følner sequence $I_N = [1, N]$.

We start by showing in Proposition 3.25 that there are sieves R such that $\mathcal{F}_R$ does not have logarithmic density, which implies that there are sieves such that $\mathcal{F}_R$ does not have density. We then deal in Theorem 3.26 with a particular case of a problem of Erdős (see [15]), which gives us the following insight (as explained in Remark 3.27). Let R be an Erdős sieve, and S the set of representatives of smallest norm of the congruence classes in R. If $d(S) = 0$, then R has weak light tails for $I_N = [1, N]$. We also use Theorem 3.26 in Example 3.28 to show that there are sieves R such that $\overline{d}(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset})$, but such that R does not have weak light tails.

We now define logarithmic density.

Definition 3.23. *Given a set $A \subset \mathbb{N}$, we define the logarithmic density of $\delta(A)$ as the quantity*

$$\delta(A) = \lim_{N\to\infty} \frac{1}{\log(N)} \sum_{m\in A\cap[1,N]} \frac{1}{m}.$$

The upper and lower densities $\overline{\delta}$ and $\underline{\delta}$ are then defined by taking the limsup and liminf respectively.

It is well known that for any $A \subset \mathbb{N}$, the following inequalities always hold:

$$\underline{d}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{d}(A).$$

In particular, if $d(A)$ exists, then $\delta(A)$ exists and equals $d(A)$.

A powerful tool when working with $\mathcal{B}-$free systems, where $\mathcal{B}$ is a set of possibly non-pairwise coprime integers, is the Davenport–Erdős Theorem (see for example the proof of Theorem 4.1 in [26] for a use of the theorem in this context). Given a set $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$, let

$$\mathcal{F}_{b_1,\dots,b_k} = \left(\bigcup_{i=1}^{k} b_i\mathbb{Z} \right)^c.$$

We will also use the notation $\mathcal{F}_{\mathcal{B}}$ for the set obtained by sieving by every class $b_i\mathbb{Z}$. The Davenport–Erdős theorem states the following (see [22]).

Theorem 3.24. *Let $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$ be a collection of distinct integers. Then, $\delta(\mathcal{F}_{\mathcal{B}})$ exists and we have*

$$\delta(\mathcal{F}_{\mathcal{B}}) = \overline{d}(\mathcal{F}_{\mathcal{B}}) = \lim_{k\to\infty} d(\mathcal{F}_{b_1,\dots,b_k}).$$

We now show that there is no Davenport–Erdős theorem for general sieves, by providing a sieve R for which $\delta(\mathcal{F}_R)$ does not exist[2].

Proposition 3.25. *There is a sieve R for which $\delta(\mathcal{F}_R)$ does not exist.*[3]

Proof. Start by defining $S = \bigcup_{i=0}^{\infty}]2^{2^{2i}}, 2^{2^{(2i+1)}}]$ and $I_N = [1, N]$. We have $\bar{\delta}(S) = 2/3$ and $\underline{\delta}(S) = 1/3$. Indeed, since $f(x) = 1/x$ is a non-increasing function, we can bound the Riemann sum by certain integrals which gives

$$\sum_{u<m\le v} \frac{1}{m} = \log(v) - \log(u) + c$$

for some small constant c. It is now easy to see that by considering the set S intersected with intervals of the form $[1, 2^{2^{(2k+1)}}]$, we have

$$\bar{\delta}(S) = \lim_{k\to\infty} \frac{1}{2^{2k+1}\log(2)} \sum_{i\le k} \sum_{2^{2^{2i}}<m\le 2^{2^{(2i+1)}}} \frac{1}{m} = \lim_{k\to\infty} \frac{1}{2^{2k+1}\log(2)} \sum_{i\le k} (4^i \log(2) + c) = \frac{2}{3}.$$

On the other hand, by considering the intersection of S with sets of the form $[1, 2^{2^{(2k)}}]$ we have

$$\underline{\delta}(S) = \lim_{k\to\infty} \frac{1}{2^{2k}\log(2)} \sum_{i<k} \sum_{2^{2^{2i}}<m\le 2^{2^{(2i+1)}}} \frac{1}{m} = \lim_{k\to\infty} \frac{1}{4^k\log(2)} \sum_{i<k} (4^i \log(2) + c) = \frac{1}{3}.$$

We now consider the sieve R defined by $R_i = i\mathbb{1}_S(i) + p_i^4\mathbb{Z}$, where $\mathbb{1}_S$ denotes the characteristic function of S. We want to show that $\delta(\mathcal{F}_R)$ is not well defined.

To do this, first notice that any number in $\mathcal{F}_R$ must be in S^c, so we get $\mathcal{F}_R \cap I_N \subset S^c \cap I_N$. Next, notice that for a number x to be sieved out from I_N, either $x \in S \cap I_N$, or $x = i\mathbb{1}_S(i) + p_i^4 k$, with some $k \ge 1$. Since $x \le N$, we get $i \le i\mathbb{1}_S(i) + p_i^4 k \le N$. Consequently, we get that

$$\sum_{m\in S^c\cap I_N} \frac{1}{m} - \sum_{i\le N} \sum_{1\le k\le N/p_i^4} \frac{1}{i + kp_i^4} \le \sum_{m\in\mathcal{F}_R\cap I_N} \frac{1}{m}.$$

[2]In particular, this shows that $d(\mathcal{F}_R)$ does not exist, which implies that Proposition 2.2 in [29] is incorrect.

[3]Independent from us, user "Leeham" in the comment section of [16] used a Large Language Model to find another example of such a set. Their investigation was motivated by the fact that prior to the 12/01/2026, it was incorrectly claimed in this source that finding such a sieve was an open problem of Erdős.

By using the fact that $\sum_{1\le k\le N}\frac{1}{k}\le \log(N)+1$, we can bound the double sum by

$$\sum_{i\le N}\sum_{1\le k\le N/p_i^4}\frac{1}{i+kp_i^4}\le \sum_{i\le N}\frac{1}{p_i^4}\sum_{1\le k\le N/p_i^4}\frac{1}{k}\le \sum_{i\le N}\frac{1}{p_i^4}(\log(N)+1).$$

The series $\sum_i p_i^{-4}$ can be bounded by $\zeta(4)-1$, so, using the fact that $\mathcal{F}_R\cap I_N\subset S^c\cap I_N$, we have

$$\sum_{m\in S^c\cap I_N}\frac{1}{m}-(\zeta(4)-1)(\log(N)+1)\le \sum_{m\in\mathcal{F}_R\cap I_N}\frac{1}{m}\le \sum_{m\in S^c\cap I_N}\frac{1}{m}.$$

We have that $\zeta(4)-1<1/10$, so taking $N_i=2^{2^{2i+1}}$ and $M_i=2^{2^{2i}}$, we have that for i big enough,

$$\frac{1}{3}-\frac{1}{10}+\epsilon_i\le \frac{1}{\log(N_i)}\sum_{m\in\mathcal{F}_R\cap I_{N_i}}\frac{1}{m}\le\frac{1}{3}+\epsilon_i,$$

and

$$\frac{2}{3}-\frac{1}{10}+\epsilon_i\le \frac{1}{\log(M_i)}\sum_{m\in\mathcal{F}_R\cap I_{M_i}}\frac{1}{m}\le\frac{2}{3}+\epsilon_i,$$

where $|\epsilon_i|\to 0$ as i increases. For sufficiently big i, these two intervals will not intersect, so $\mathcal{F}_R$ cannot have a well defined logarithmic density. □

The key thing here is that our sieve R was built in such a way that the set $\{\min_{x\in R_i}|x| : i\in\mathbb{N}\}$ does not have logarithmic density. But what would happen if we didn't sieve out these elements, that is, if we only sieved out those $i+kp_i^4$ with $k\ge 1$. Should this set have logarithmic density? Below we quote Erdős, making this exact question. We then answer the question affirmatively, in the case where we are sieving out by a sieve such that $\sum_i\frac{|R_i|}{b_i}<\infty$.

The following is the exact quote of Erdős, taken from [33] (this problem was initially proposed by Erdős in [30] as Point (26), it also appears as both Problems number 25 and 486 in the Erdős Problems website, see [15] and [16]).

> Let $n_1<n_2<\dots$ be an arbitrary sequence of integers. Besicovitch proved more than 60 years ago that the set of the multiples of the n_i does not have to have a density. In those prehistoric days this was a great surprise. Davenport and I proved that the set of multiples of the $\{n_i\}$ have a logarithmic density and the logarithmic density equals the lower density of the set of multiples of the $\{n_i\}$. Now the following question is perhaps of interest: Exclude one or several residues mod n_i (where only the integers $\ge n_i$ are excluded). Is it true that the logarithmic density of the integers which are not excluded always exists? This question seems difficult even if we only exclude one residue mod n_i for every n_i.

We now show that if R_i is the set of congruence classes mod n_i being sieved out and we have $\sum_i|R_i|/n_i<\infty$, then the set of non-excluded integers has a well defined density.

Theorem 3.26. *Let $\mathcal{B} = \{b_1, b_2, \dots\}$ be a sequence of positive integers. For each i, let R_i be sets of the form*

$$R_i = \{r_i^{(k)} : 0 \leq k \leq c(i)\} + b_i\mathbb{N}$$

where $0 \leq r_i^{(k)} < b_i$. Then, if $\sum_i |R_i|/b_i < \infty$, we have

$$d\left((\bigcup_i R_i)^c\right) = \lim_{L\to\infty} d\left((\bigcup_{i=1}^{L} R_i)^c\right).$$

Proof. Let $I_N = [0, N]$ and define

$$\mathcal{F}_R = (\bigcup_i R_i)^c.$$

Since

$$\mathcal{F}_R^c = \bigcup_{l>L} R_l \cup \bigcup_{i=1}^{L} R_i,$$

the union not necessarily being disjoint, we simply have to show that

$$\lim_{L\to\infty} \overline{d}_I(\bigcup_{i>L} R_i) = 0,$$

and the result will follow. For every $x \in R_i$, we have that $x \geq b_i$, consequently, for any fixed N, there are only finitely many i such that $R_i \cap I_N \neq \emptyset$. Additionally, we have that $|R_i \cap I_N| \leq N|R_i|/b_i$, given that $|\{r_i^{(k)} + jb_i : j \in \mathbb{N}_0\} \cap I_N| \leq 1 + N|R_i|/b_i$, and we are certainly removing the element $r_i^{(k)}$ for each k. Consequently, we have that

$$\left|\bigcup_{i>L} R_i \cap I_N\right| \leq \sum_{i>L} |R_i \cap I_N| \leq N \sum_{i>L} \frac{|R_i|}{b_i}.$$

By our hypothesis, the series converges, so after dividing by N taking the limit of N to infinity, and then taking L to infinity, this will go to 0. Notice that

$$\lim_{L\to\infty} d\left((\bigcup_{i=1}^{L} R_i)^c\right)$$

is always well defined, since it is the limit of a monotonic sequence. ∎

Remark 3.27. In the case where the b_i are pairwise coprime, Theorem 3.26 shows that

$$d\left((\bigcup_i R_i)^c\right) = \prod_i \left(1 - \frac{|R_i|}{b_i}\right).$$

Writing $R_i' = \bigcup_{1\leq k\leq c(i)} r_i^{(k)} + b_i\mathbb{Z}$ (using the same notation as in the statement of Theorem 3.26), we have that R' is an Erdős sieve. If S denotes the set of all $r_i^{(k)}$, we see that $((\bigcup_i R_i)^c \setminus S) \cap \mathbb{N} = \mathcal{F}_{R'} \cap \mathbb{N}$. Consequently, if $d(S) = 0$, then $d(\mathcal{F}_{R'}) = \prod_i \left(1 - \frac{|R_i|}{b_i}\right)$, which implies that R' has weak light tails for $I_N = [1, N]$ by Theorem 3.21.

A question that arises from Lemma 3.19 is if $\overline{d}_I(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset})$ for some sieve R, does this imply that $d_I(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset})$? This is not the case as we show in the following example.

Example 3.28. Let A be a set with lower density 0 but positive upper density, and let R be the sieve defined by $R_i = i\mathbb{1}_A(i) + p_i^2\mathbb{Z}$. Let $\mathcal{F}'_R = \mathbb{N} \setminus \bigcup_i \{i\mathbb{1}_A(i) + jp_i^2 : j \in \mathbb{N}\}$. Writing $I_N = [1, N]$, we have shown in the proof of Theorem 3.26 that $d_I(\mathcal{F}'_R) = \nu_R(C^R_{\{0\},\emptyset})$. We have that

$$|\mathcal{F}_R \cap I_N| \geq |\mathcal{F}'_R \cap I_N| - |A \cap I_N|.$$

Taking a sequence N_i so that $|A \cap I_{N_i}|/|I_{N_i}|$ tends to 0, we get that

$$\lim_i \frac{|\mathcal{F}_R \cap I_{N_i}|}{|I_{N_i}|} \geq \lim_i \frac{|\mathcal{F}'_R \cap I_{N_i}|}{|I_{N_i}|} - \frac{|A \cap I_{N_i}|}{|I_{N_i}|} = \prod_i \left(1 - \frac{1}{p_i^2}\right),$$

so $\overline{d}_I(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset})$. On the other hand, we always have that $|\mathcal{F}_R \cap I_N| \leq |A^c \cap I_N|$, so

$$\frac{|\mathcal{F}_R \cap I_N|}{|I_N|} \leq 1 - \frac{|A \cap I_N|}{|I_N|}.$$

By choosing a set A such that $\overline{d}(A) > 1 - \prod_i \left(1 - \frac{1}{p_i^2}\right)$, we get that

$$\underline{d}(\mathcal{F}_R) < \prod_i \left(1 - \frac{1}{p_i^2}\right),$$

and so we cannot have that $d(\mathcal{F}_R)$ is well defined.

4. Topological Dynamical Systems

In this section we investigate some of the properties of the topological dynamical system (Ω_R, S). In particular, we will show generalizations of points $(2), (4)$ and (5) of Sarnak's Program, as described in the introduction. As a consequence, we are able to better characterize the structure of $\mathcal{F}_R$, the set of invariant measures of (Ω_R, S), along with other properties of this dynamical system.

We start with point (2), of computing the entropy of (Ω_R, S). This was already done in Lemma 5.13 of [42], so, inspired by Theorem 3.16 of [2], we prove a more general result which we use in Lemma 7.18. We then show point (4) that (X_R, S) is proximal. As part of this result, we show that for every Erdős sieve R there is some Følner sequence I_N such that $\overline{d}_I(\mathcal{F}_R) = 0$. As pointed out in Remark 4.3, this shows that no Erdős sieve has weak light tails for every Følner sequence. We finish by showing point (5), that there is a non-trivial joining between (Ω_R, S) and (G_R, T). As we explain in Remark 4.7, this shows that (Ω_R, S) cannot be topologically weakly mixing.

We now show point (2). We start by setting notation for the more general context in which we will show this result. Let $\underline{s} = (s_1, s_2, \dots)$ be a sequence such that $s_i \leq |R_i^c|$ for every i. We follow the notation of [2], and define the set

$$Y_{R,\geq \underline{s}} := \{A \in \Omega_R : |A + \mathfrak{b}_i| \leq |R_i^c| - s_i\}, \tag{19}$$

where $|A + \mathfrak{b}_i|$ is the cardinality of this set in $\mathcal{O}_K/\mathfrak{b}_i$. Notice that taking $\underline{0} = (0, 0, 0, \dots)$ to be the sequence that is constantly 0, we have $Y_{R,\geq \underline{0}} = \Omega_R$.

Theorem 4.1. *Let R be any sieve and $\underline{s}$ a sequence such that $s_i \leq |R_i^c|$ for every $i \in \mathbb{N}$. Then,*

$$\mathrm{h}_{top}(Y_{R,\geq \underline{s}}, S) = \log(2) \prod_i \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right).$$

In particular,

$$\mathrm{h}_{top}(\Omega_R, S) = \log(2) \prod_i \left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)} \right).$$

Proof. Our proof follows closely the one used in the proof of Lemma 5.14 in [42]. By Lemma 2.25, we have to show that

$$\lim_{N\to\infty} \frac{\log_2(|\{E \subset B_N : E \in Y_{R,\geq \underline{s}}\}|)}{|B_N|} = \prod_i \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right).$$

We start by showing that this is the correct upper bound. Fix some positive integer M, and let

$$G_M = \prod_{i \leq M} \mathcal{O}_K/\mathfrak{b}_i.$$

Given sets $D_1, \dots, D_M$ such that $D_i \subset R_i^c$ and $|D_i| = s_i$, define for any $\delta = (\delta_1, \dots, \delta_M) \in G_M$ the sets

$$U_{D_1,\dots,D_M}(\delta) = \left(\mathcal{O}_K \setminus \bigcup_{i \leq M} (\delta_i + (R_i \cup D_i)) \right) \cap B_N.$$

Any $E \subset B_N$ such that $E \in Y_{R,\geq \underline{s}}$ must be a subset of some $U_{D_1,\ldots,D_M}(\delta)$, and so we have that

$$|\{E \subset B_N : E \in Y_{R,\geq \underline{s}}\}| \leq \sum_{\delta \in G_M} \sum_{D_1,\ldots,D_M} |\{E \subset U_{D_1,\ldots,D_M}(\delta)\}| = \sum_{\delta \in G_M} \sum_{D_1,\ldots,D_M} 2^{|U_{D_1,\ldots,D_M}(\delta)|},$$

where the second sum runs over all possible choices of sets $D_i \subset R_i^c$ such that $|D_i| = s_i$. Lemma 3.14 tells us that

$$|U_{D_1,\ldots,D_M}(\delta)| = |B_N| \prod_{i=1}^{M} \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right) + o(|B_N|),$$

but the error term depends on the set $U_{D_1,\ldots,D_M}(\delta)$. Since there are only finitely many such sets, there is a constant C_M dependent only on M such that for any such set we have

$$|U_{D_1,\ldots,D_M}(\delta)| \leq |B_N| \prod_{i=1}^{M} \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right) + C_M f(N),$$

where $f = o(|B_N|)$. Therefore,

$$\log_2(|\{E \subset B_N : E \in Y_{R,\geq \underline{s}}\}|) \leq \log_2 \left(|G_M| \prod_{i=1}^{M} \binom{|R_i^c|}{s_i} \right) + |B_N| \prod_{i=1}^{M} \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right) + C_M f(N),$$

which implies that

$$\lim_{N\to\infty} \frac{\log_2(|\{E \subset B_N : E \in Y_{R,\geq \underline{s}}\}|)}{|B_N|} \leq \prod_{i=1}^{M} \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right).$$

For the lower bound, take any choice of $A_i \subset R_i^c$ such that $|A_i| = s_i$ and define $R_i' = R_i \cup A_i$. Note that $\Omega_{R'} \subset Y_{R,\geq \underline{s}}$. We pick according to a uniform distribution an element t_i of $\mathcal{O}_K/\mathfrak{b}_i$ independently for every i. For any $x \in \mathcal{O}_K$, the probability that $x \notin t_i + R_i'$ for some i is $1 - \mathrm{vol}(R_i')$. Hence, for any fixed N, the expected number of $x \in B_N$ such that $x \notin t_i + R_i'$ for any i is $|B_N| \prod_i (1 - \mathrm{vol}(R_i'))$. Consequently, there must be some choice of t_i for which

$$|B_N \setminus \bigcup_i (t_i + R_i')| \geq |B_N| \prod_i (1 - \mathrm{vol}(R_i')).$$

Noticing that any subset of $\mathcal{O}_K \setminus \bigcup_i (t_i + R_i')$ is $R'-$admissible, we have

$$\log_2(|\{E \subset B_N : E \in \Omega_{R'}\}|) \geq \log_2(|\{E : E \subset B_N \setminus \bigcup_i (t_i + R_i')\}|) = |B_N \setminus \bigcup_i (t_i + R_i')|,$$

and so, using the fact that

$$1 - \mathrm{vol}(R_i') = 1 - \frac{|R_i| + s_i}{N(\mathfrak{b}_i)} = \frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)},$$

we get, by combining these equations, that

$$\log_2(|\{E \subset B_N : E \in Y_{R,\geq \underline{s}}\}|) \geq \log_2(|\{E \subset B_N : E \in \Omega_{R'}\}|) \geq |B_N| \prod_i \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)} \right),$$

which concludes the proof. □

We now generalize point (4) of Sarnak's program. In the case of $\mathcal{B}-$free systems, a large generalization of this result was given in Theorem 1.3 of [27]. We show the following for sieves.

Theorem 4.2. *For a sieve R over an étale $\mathbb{Q}-$algebra K the following holds.*

(1) $\emptyset \in X_R$.
(2) (Ω_R, S) *is proximal.*
(3) *For any lattice* $\Gamma \subset \mathcal{O}_K$*, and* $a \in \mathcal{O}_K$*,* $a + \Gamma \not\subset \mathcal{F}_R$.
(4) *We have*

$$\inf\{\overline{d}_I(\mathcal{F}_R) : I_N \text{ is a Følner sequence in } \mathcal{O}_K\} = 0.$$

Proof. We start by showing that (1) holds, and that all remaining statements follow from this. To say that $\emptyset \in X_R$ is equivalent to showing that for every N there is some a_N such that $(a_N + B_N) \cap \mathcal{F}_R$ is empty. We now show this holds since $\mathcal{F}_R$ is obtained from removing infinitely many congruence classes of ideals which are pairwise coprime.

Fix some N, and write $B_N = \{x_1, \ldots, x_l\}$. For every i, take some a_i such that $a_i \in R_i$. By the Chinese Remainder Theorem, there is some x such that $x + x_i \equiv a_i \mod \mathfrak{b}_i$ for each $1 \leq i \leq l$. This means that $x + x_i \notin \mathcal{F}_R$ for every i, that is, $(x + B_N) \cap \mathcal{F}_R = \emptyset$. Consequently, $\emptyset \in X_R$.

We now show that (1) implies (2). Because $S_a(\emptyset) = \emptyset$ for all a, we see that $\emptyset$ is the unique fixed point in (Ω_R, S). By Lemma 2.12, it is enough to show that for any $A \in \Omega_R$ the set $\{a \in \mathcal{O}_K : d(a + A, \emptyset) < \epsilon\}$ is syndetic.

Let R' be the sieve supported on $\mathcal{B}_R$ and defined by $R'_{\mathfrak{b}} = (A + \mathfrak{b})^c$. If A is in Ω_R, then we must have $A + \mathfrak{b} \neq \mathcal{O}_K$ for every $\mathfrak{b} \in \mathcal{B}_R$, so R' is indeed a well defined sieve. Additionally, we have that $a \notin R'_{\mathfrak{b}}$ for every $a \in A$ and $\mathfrak{b} \in \mathcal{B}_R$, so $A \subset \mathcal{F}_{R'}$. It follows that

$$d(a + A, \emptyset) \leq d(a + \mathcal{F}_{R'}, \emptyset),$$

and so we have

$$\{a \in \mathcal{O}_K : d(a + A, \emptyset) < \epsilon\} \supset \{a \in \mathcal{O}_K : d(a + \mathcal{F}_{R'}, \emptyset) < \epsilon\}.$$

Hence, it is enough to show that the set $\{a \in \mathcal{O}_K : d(a + \mathcal{F}_{R'}, \emptyset) < \epsilon\}$ is syndetic. To do this, notice that this follows from showing that $\{a \in \mathcal{O}_K : \mathcal{F}_{R'} \cap (a + B_N) = \emptyset\}$ is syndetic for every N. We have shown that $\emptyset \in X_{R'}$, so we know that for every N there is some a_N such that $\mathcal{F}_{R'} \cap (a_N + B_N) = \emptyset$. By definition of $\mathcal{F}_{R'}$, this means that for every $f \in a_N + B_N$ there is some $\mathfrak{b}_f$ and $b_f \in R'_{\mathfrak{b}_f}$ such that f is congruent to $b_f \mod \mathfrak{b}_f$. Let $\mathfrak{b}$ be the product of all the $\mathfrak{b}_f$. For any $b \in \mathfrak{b}$, we have that

$$f + b \equiv f \mod \mathfrak{b}_f,$$

and so $f + b \in R'_{\mathfrak{b}_f}$. It follows that $(a_N + B_N + \mathfrak{b}) \cap \mathcal{F}_{R'} = \emptyset$.

Consequently, we have $a_N + \mathfrak{b} \subset \{a \in \mathcal{O}_K : (a + B_N) \cap \mathcal{F}_{R'} = \emptyset\}$, so we just have to show that $a_N + \mathfrak{b}$ is syndetic. But taking any finite set such that $C + \mathfrak{b} = \mathcal{O}_K$, we have $a_N + \mathfrak{b} + C = \mathcal{O}_K$, so $\{a \in \mathcal{O}_K : (a + B_N) \cap \mathcal{F}_{R'} = \emptyset\}$ must also be syndetic. We conclude that (Ω_R, S) is proximal.

If (1) holds, then for any N there exists some a_N such that $(a_N + B_N) \cap \mathcal{F}_R = \emptyset$. Given a lattice Γ of $\mathcal{O}_K$ and $a \in \mathcal{O}_K$, we have that $|(a_N + B_N) \cap (a + \Gamma)| = |B_N \cap (a - a_N + \Gamma)|$. By Lemma 2.6, we have

$$|B_N \cap (a - a_N + \Gamma)| \gg_\Gamma N^n,$$

hence, for sufficiently large N we have $(a_N + B_N) \cap (a + \Gamma) \neq \emptyset$, and so $a + \Gamma \not\subset \mathcal{F}_R$. Point (3) follows.

Similarly, if (1) holds, then taking a sequence a_N such that $(a_N + B_N) \cap \mathcal{F}_R = \emptyset$ for all N, we have that $I_N := a_N + B_N$ is a Følner sequence, and we have

$$d_I(\mathcal{F}_R) = \lim_{N\to\infty} \frac{|I_N \cap \mathcal{F}_R|}{|I_N|} = \lim_{N\to\infty} \frac{0}{|I_N|} = 0.$$

We conclude that point (4) holds.

□

Remark 4.3. Since $X_R \subset \Omega_R$, we have by point (2) that (X_R, S) is always a proximal dynamical system.

Additionally, point (4) implies that for every Erdős sieve R there is at least one Følner sequence I_N such that R does not have weak light tails for I_N. This is because if R is Erdős, then $\nu_R(C^R_{\{0\},\emptyset}) > 0$. Because we can find I_N such that $d_I(\mathcal{F}_R) = 0 < \nu_R(C^R_{\{0\},\emptyset})$, Theorem 3.21 implies that R does not have weak light tails with respect to I_N. Equivalently, this shows that there is no sieve R that has weak light tails with respect to every Følner sequence I_N.

As a corollary of Theorem 4.2 we also get the following result about the translation symmetries of $\mathcal{F}_R$.

Corollary 4.4. *Let R be a sieve over an étale $\mathbb{Q}-$algebra of degree n. If $\mathcal{F}_R \neq \emptyset$, there are at most $n-1$ $\mathbb{Q}-$independent x_i such that $x_i + \mathcal{F}_R = \mathcal{F}_R$. In particular, if R is a sieve over $\mathbb{Q}$, there is no $x \neq 0$ such that $x + \mathcal{F}_R = \mathcal{F}_R$.*

Proof. If $x + \mathcal{F}_R \subset \mathcal{F}_R$, then $2x + \mathcal{F}_R = x + (x + \mathcal{F}_R) \subset x + \mathcal{F}_R \subset \mathcal{F}_R$. By induction, we see that $x\mathbb{Z} + \mathcal{F}_R \subset \mathcal{F}_R$. If there were at least n $\mathbb{Q}-$independent x_i such that $x_i + \mathcal{F}_R \subset \mathcal{F}_R$, then we would have that

$$x_1\mathbb{Z} + \cdots + x_n\mathbb{Z} + \mathcal{F}_R \subset \mathcal{F}_R.$$

But due to the linear independence of the x_i, the set $\Gamma = x_1\mathbb{Z} + \cdots + x_n\mathbb{Z}$ is a lattice inside $\mathcal{O}_K$. Taking any $x \in \mathcal{F}_R$, we would have that $x + \Gamma \subset \mathcal{F}_R$, contradicting point (3) of Theorem 4.2.

The claim when R is a sieve over $\mathbb{Q}$ is clear, since if $m \neq 0$, then $m\mathbb{Z}$ will be a lattice in $\mathbb{Z}$. □

Remark 4.5. In general, if K is an étale $\mathbb{Q}-$algebra of degree $n \geq 2$, there will be sieves such that $x_i + \mathcal{F}_R = \mathcal{F}_R$ for $(n-1)$ $\mathbb{Q}-$independent x_i. For example, consider the sieve R over the Gaussian primes $\mathbb{Z}[i]$ (so $i = \sqrt{-1}$) supported on $\mathcal{B}_R = \{p\mathbb{Z}[i] : p \text{ is prime}, p \equiv 3 \mod 4\}$ and given by

$$R_p = \{0, 1, \ldots, p-1\} + p\mathbb{Z}[i] = \mathbb{Z} + (ip)\mathbb{Z}$$

whenever $p\mathbb{Z}[i] \in \mathcal{B}_R$. Since these primes are inert in $\mathbb{Z}[i]$, each of the ideals $p\mathbb{Z}[i]$ is prime, so they are pairwise coprime. The $R-$free numbers for this set are given by

$$\mathcal{F}_R = \{a + mi : a \in \mathbb{Z}, m \text{ is an integer divisible only by primes } p \text{ such that } p \not\equiv 3 \mod 4\},$$

so it is clear that $1 + \mathcal{F}_R = \mathcal{F}_R$.

We conclude this section with the generalization of point (5) of Sarnak's program. The proof presented is similar to the proof of Theorem E in [2].

Theorem 4.6. *The systems* (Ω_R, S) *and* (G_R, T) *have a non-trivial joining (see Definition 2.10).*

Proof. Let $J^* \subset \Omega_R \times G$ denote the set

$$J^* = \left\{ (A, g) : A \in \Omega_R, g \in \prod_i (-A + R_i)^c \right\},$$

and let J be the closure of J^*. We claim that J is the desired joining. First, notice that given $(A, g) \in J^*$, $(S_a \times T_a)(A, g) = (-a + A, a + g)$. But, if $g_i \in (-A + R_i)^c$, then

$$a + g_i \in a + (-A + R_i)^c = (a - A + R_i)^c = (-(-a + A) + R_i)^c.$$

Therefore, $S \times T$ sends J^* into J^*, which by continuity implies that $(S_a \times T_a)(J) \subset J$ for every a. Since the map is clearly invertible, it follows that $(S_a \times T_a)(J) = J$.

Next, note that $\emptyset \in \Omega_R$, so $\{\emptyset\} \times G \subset J^*$, which means that J has full projection into G. By definition, for every admissible A, $-A + R_i \neq \mathcal{O}_K$, so for every i, there is some $g_i \in (-A + R_i)^c$. This defines for every A at least one element of G such that $(A, g) \in J$, so J also has a full projection onto Ω_R.

It remains to show that $J \neq \Omega_R \times G$. If A is an admissible set such that $0 \in A$ and $(A, g) \in J^*$, then notice that

$$g \in \left(G \setminus \prod_i R_i \right).$$

Assume that we have a sequence $(A_i, g_i) \in J^*$ such that the A_i are converging to $\{0\}$ in Ω_R. Then, for i big enough, we must have that $0 \in A_i$, since $d(\{0\}, A) = 1$ for every admissible A such that $0 \notin A$. It follows that for big enough i, all the g_i will be contained in $G \setminus \prod_i R_i$, so

$$\left(\{0\} \times \prod_i R_i \right) \cap J = \emptyset. \qquad \square$$

Remark 4.7. As pointed out in [2], Theorem 4.6 implies that (Ω_R, S) is not a topologically weakly mixing system (see Definition 2.13). This follows directly from Theorem II.3 in [39], where it is shown that if a system is topologically weakly mixing, then it has no non-trivial joining with a minimal distal system (which (G_R, T) is, since $d(T_a(x), T_a(y)) = d(x, y)$).

5. Sieves with Light Tails

The objective of this section is to study sieves with weak and strong light tails. It is divided in three subsections. In the first subsection we provide motivation, from a number theoretic point of view, for why one would like to know if a particular sieve has weak light tails. We show in Theorem 5.1 that if we know that R has weak light tails (for some Følner sequence I_N) and A is a finite $R-$admissible set, then the set of x such that $x + A \subset \mathcal{F}_R$ has positive density (with respect to I_N). In Theorem 5.2 we show that for an infinite admissible set A, the same will happen if and only if a particular sieve R' is Erdős and has weak light tails (for the respective I_N).

The second subsection is dedicated to providing examples of sieves with strong light tails for B_N. The major result of this subsection is Theorem 5.7 where we show that if $\mathcal{B}$ is a collection of pairwise coprime ideals of $\mathcal{O}_K$, for some étale $\mathbb{Q}-$algebra K, such that $\sum_{\mathfrak{b}\in\mathcal{B}} N(\mathfrak{b})^{-1} < \infty$, then the sieve R supported on $\mathcal{B}_R$ and defined by $R_{\mathfrak{b}} = \mathfrak{b}$ has strong light tails for B_N. In Example 5.11 and Example 5.13 we present sieves that have strong light tails for B_N, but where $|R_i|$ goes to infinity as we let i grow.

In the third subsection we start by providing an example of a sieve with weak light tails for some I_N, that does not have strong light tails for B_N (see Example 5.14). The remainder of this subsection is dedicated to characterizing such sieves, and how certain operations in sieves alter their properties. In Lemma 5.15 we show that the weak light tails property is invariant under permutation of the elements of $\mathcal{B}_R$, and in Lemma 5.16 we show that the same holds for strong light tails. We prove in Lemma 5.17 that adding an ideal to $\mathcal{B}_R$ preserves weak light tails. We show that removing ideals does not preserve this property, but in Theorem 5.25 we demonstrate that if R is a sieve with weak light tails which after the removal of any finite number of ideals from $\mathcal{B}_R$ maintains this property, then in fact R must have strong light tails. Theorem 5.29 and Theorem 5.28 provide examples of two additional properties that together with weak light tails imply that a sieve R must have strong light tails.

5.1. Repeated Patterns in $\mathcal{F}_R$.

It is a powerful thing to state that an Erdős sieve R has weak/strong light tails with respect to some Følner sequence I_N. Indeed, assume that A is any finite set, and we want to know whether the pattern A repeats infinitely in $\mathcal{F}_R$. For example, taking $A = \{0, 2\}$, this is the same as asking if there are infinitely many twin $R-$free numbers (as in, $x \in \mathcal{F}_R$ such that $x + 2$ is also in $\mathcal{F}_R$). For finite A, if we know that R has weak light tails with respect for some I_N, this is just a matter of knowing if A is $R-$admissible or not.

If A is not admissible, then there cannot be any $x \in \mathcal{F}_R$ such that $x + A \subset \mathcal{F}_R$, since there will be some $\mathfrak{b} \in \mathcal{B}_R$ such that $-A + R_{\mathfrak{b}} = \mathcal{O}_K$, and so taking $a \in A \cap (R_{\mathfrak{b}} - x)$, we will always have that $a + x \notin \mathcal{F}_R$. On the other hand, for any finite and admissible A, Lemma 3.18 gives

$$d_I(\{x \in I_N : x + A \subset \mathcal{F}_R\}) = \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{x\in I_N} \mathbb{1}_{C^R_{A,\emptyset}}(S_x(\mathcal{F}_R)).$$

If R has weak light tails with respect to I_N, then Theorem 3.21 (together with Equation (12)) implies that this limit equals

$$\nu_R(C^R_{A,\emptyset}) = \prod_{\mathfrak{b}\in\mathcal{B}_R} \left(1 - \frac{|-A+R_{\mathfrak{b}}|}{N(\mathfrak{b})}\right).$$

If R is Erdős this product will always be bigger than 0 since by definition if A is admissible, then $|-A+R_{\mathfrak{b}}| < N(\mathfrak{b})$, and if A is finite then the product is positive by Lemma 2.1 (using the fact that $|-A+R_{\mathfrak{b}}| \leq |A||R_{\mathfrak{b}}|$). We have just proven the following theorem.

Theorem 5.1. *Let R be an Erdős sieve with weak light tails for some Følner sequence I_N, and A a finite admissible set of R. Then,*

$$d_I(\{x : x+A\subset\mathcal{F}_R\}) = \prod_i \left(1 - \frac{|-A+R_i|}{N(\mathfrak{b}_i)}\right) > 0.$$

In particular, there are infinitely many x such that $x+A\subset\mathcal{F}_R$.

In the case that A is infinite, $C^R_{A,\emptyset}$ is no longer clopen, and so we cannot use the weak light tails of R to show that there are infinitely many x such that $x+A\subset\mathcal{F}_R$ if $\nu_R(C^R_{A,\emptyset}) > 0$. Yet, we can still use Lemma 3.19 to say the following.

Theorem 5.2. *Let R be a sieve and A an $R-$admissible set. For any Følner sequence I_N, we have*

$$\overline{d}_I(\{x\in\mathcal{O}_K : x+A\subset\mathcal{F}_R\}) \leq \nu_R(C^R_{A,\emptyset}).$$

Additionally, $d_I(\{x : x+A\subset\mathcal{F}_R\}) = \nu_R(C^R_{A,\emptyset}) > 0$ holds if and only if the sieve R' supported on $\mathcal{B}_R$ and defined by

$$R'_{\mathfrak{b}} = -A + R_{\mathfrak{b}}$$

is Erdős and has weak light tails for I_N.

Proof. We have that $x+A\subset\mathcal{F}_R$ is equivalent to $(x+A)\cap R_{\mathfrak{b}} = \emptyset$ for every $\mathfrak{b}$, which is the same as $\{x\}\cap(-A+R_{\mathfrak{b}}) = \{x\}\cap R'_{\mathfrak{b}} = \emptyset$, so we see that

$$\{x\in\mathcal{O}_K : x+A\subset\mathcal{F}_R\} = \mathcal{F}_{R'}.$$

From Lemma 3.19, it is now clear that the result will follow if we show that $\nu_R(C^R_{A,\emptyset}) = \nu_{R'}(C^{R'}_{\{0\},\emptyset})$. From the definition of ν_R, it is therefore enough to show that $\varphi_R^{-1}(C^R_{A,\emptyset}) = \varphi_{R'}^{-1}(C^{R'}_{\{0\},\emptyset})$ (since $\mathcal{B}_R = \mathcal{B}_{R'}$, we have that $G_R = G_{R'}$, so this equality makes sense).

Assume that $\varphi_R(g)\in C^R_{A,\emptyset}$. This is equivalent to $A\subset\varphi_R(g)$. From the definition of φ_R, we see that this is equivalent to

$$\forall_{a\in A}\forall_i\, a+g_i\notin R_i \Leftrightarrow \forall_i\forall_{a\in A}\, g_i\notin -a+R_i \Leftrightarrow \forall_i\, g_i\notin -A+R_i \Leftrightarrow \forall_i\, g_i\notin R'_i,$$

which is equivalent to $0\in\varphi_{R'}(g)$. It follows that $\varphi_R(g)\in C^R_{A,\emptyset}$ if and only if $\varphi_{R'}(g)\in C^{R'}_{\{0\},\emptyset}$, which concludes the proof. $\square$

5.2. **Sieves with Strong Light Tails.** We now investigate conditions which imply that a sieve R has strong light tails for the Følner sequence B_N. We start by showing that this holds for all Erdős $\mathcal{B}-$free systems over an étale $\mathbb{Q}-$algebra. We need to first demonstrate that this holds for number fields, and then show the result in its full generality. In this section, we always assume that $\mathcal{B}_R$ is ordered, and so we write R_i for $R_{\mathfrak{b}_i}$.

Lemma 5.3. *Let R be a sieve over a number field K of degree n, such that there is a finite set T for which $R_i \subset T + \mathfrak{b}_i$ for every i. If $\sum_i \frac{1}{N(\mathfrak{b}_i)} < \infty$, then R is Erdős and has strong light tails for B_N.*

Proof. It is enough to prove the result for the sieve R' defined by $R'_i = T + \mathfrak{b}_i$. Notice that

$$\bigcup_i (T + \mathfrak{b}_i) = T + \bigcup_i \mathfrak{b}_i.$$

If $x \in \left(T + \bigcup_{i>L} \mathfrak{b}_i\right) \cap B_N$, then there are $t \in T, b_j \in \mathfrak{b}_j$ such that $x = t + b_j$, $\|x\| \leq N$, which implies $\|b_j\| \leq N + \|t\|$. Hence, taking $M := \max_{t \in T} \|t\|$, it is clear that

$$\left| \left(T + \bigcup_{i>L} \mathfrak{b}_i \right) \cap B_N \right| \leq |T| \left| \bigcup_{i>L} \mathfrak{b}_i \cap B_{N+M} \right|,$$

so, without loss of generality, we can assume that R is such that $R_i = \mathfrak{b}_i$.

We are assuming that $\sum_i \frac{1}{N(\mathfrak{b}_i)} < \infty$, so R is Erdős. We now show that it has strong light tails. Let $x \in B_N \setminus \{0\}$. If $x \in \mathfrak{b}_i$, then we must have that $\lambda_1(\mathfrak{b}_i) \leq \|x\| \leq N$. Therefore, we have that

$$\left| B_N \cap \bigcup_{i>L} \mathfrak{b}_i \right| \leq 1 + \sum_{\substack{i:\lambda_1(\mathfrak{b}_i) \leq N \\ i>L}} |\{x \in B_N \setminus \{0\} : x \in R_i\}|,$$

which, after applying Lemma 2.6 gives

$$\left| B_N \cap \bigcup_{i>L} \mathfrak{b}_i \right| \leq 1 + \sum_{\substack{i:\lambda_1(\mathfrak{b}_i) \leq N \\ i>L}} \left(\frac{|B_N|}{N(\mathfrak{b}_i)} + O\left(1 + \sum_{j=1}^{n-1} \frac{N^j}{\lambda_1(\mathfrak{b}_i) \cdots \lambda_j(\mathfrak{b}_i)} \right) \right).$$

We have to deal with three distinct sums separably, and show that once we divide by $|B_N|$, and take the limit of N and then L to infinity, these will go to 0. First, notice that

$$\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i) \leq N \\ i>L}} \frac{|B_N|}{N(\mathfrak{b}_i)} \leq \lim_{L\to\infty} \sum_{i>L} \frac{1}{N(\mathfrak{b}_i)} = 0,$$

as the series converges by hypothesis, so the first sum is dealt with.

We next have to show that

$$\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i) \leq N \\ i>L}} 1 = 0.$$

By Lemma 2.4, there is some C dependent only on K such that if $N(\mathfrak{b}_i) \leq CN^n$, then $\lambda_1(\mathfrak{b}_i) \leq N$. Therefore, the sum is bounded up to a constant multiple by

$$\frac{1}{|B_N|} \sum_{i:N(\mathfrak{b}_i)\leq CN^n} 1$$

for some C depending only on K. Since all the $\mathfrak{b}_i$ are coprime, the number of ideals in $\mathcal{B}$ with norm smaller than CN^n must be bounded by the number of prime ideals with norm smaller than CN^n. By Theorem 2.2, this number is bounded by $cN^n/\log(N)$ for some constant c depending only on K. Consequently, it follows that

$$\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i)\leq N \\ i>L}} 1 \ll \lim_{L\to\infty} \lim_{N\to\infty} \frac{N^n}{|B_N|\log(N)} = 0,$$

as we wanted to show.

We are left with showing that for any $1 \leq j \leq n-1$, we have

$$\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i)\leq N \\ i>L}} \frac{N^j}{\lambda_1(\mathfrak{b}_i)\cdots\lambda_j(\mathfrak{b}_i)} = 0.$$

Fix j. Using Equation (4) we have that

$$\frac{N^j}{\lambda_1(\mathfrak{b}_i)\cdots\lambda_j(\mathfrak{b}_i)} \asymp_K \frac{\lambda_{j+1}(\mathfrak{b}_i)\cdots\lambda_n(\mathfrak{b}_i)N^j}{N(\mathfrak{b}_i)}.$$

By Lemma 2.4, we know that $\lambda_1(\mathfrak{b}_i) \asymp_K \lambda_n(\mathfrak{b}_i)$, so there is some C depending only on K, such that $\lambda_n(\mathfrak{b}_i) \leq C\lambda_1(\mathfrak{b}_i)$. Therefore, for any i such that $\lambda_1(\mathfrak{b}_i) \leq N$, we have that $\lambda_n(\mathfrak{b}_i) \leq CN$. Since $\lambda_j(\mathfrak{b}) \leq \lambda_n(\mathfrak{b})$ for every j, we have

$$\frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i)\leq N \\ i>L}} \frac{\lambda_{j+1}(\mathfrak{b}_i)\cdots\lambda_n(\mathfrak{b}_i)N^j}{N(\mathfrak{b}_i)} \leq \frac{1}{|B_N|} \sum_{\substack{i:\lambda_n(\mathfrak{b}_i)\leq CN \\ i>L}} \frac{\lambda_n(\mathfrak{b}_i)^{n-j}N^j}{N(\mathfrak{b}_i)} \leq \frac{N^n}{|B_N|} \sum_{\substack{i:\lambda_n(\mathfrak{b}_i)\leq CN \\ i>L}} \frac{C^{n-j}}{N(\mathfrak{b}_i)}.$$

The term $N^n/|B_N|$ is bounded by a constant only depending on K, so it follows that

$$\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{|B_N|} \sum_{\substack{i:\lambda_1(\mathfrak{b}_i)\leq N \\ i>L}} \frac{\lambda_{j+1}(\mathfrak{b}_i)\cdots\lambda_n(\mathfrak{b}_i)N^j}{N(\mathfrak{b}_i)} \ll \lim_{L\to\infty} \sum_{i>L} \frac{1}{N(\mathfrak{b}_i)} = 0.$$

Since all these limits go to 0, we conclude that

$$\lim_{L\to\infty} d\left(\bigcup_{i>L} R_i\right) = 0,$$

as we wanted to show. □

Remark 5.4. Lemma 5.3 together with Theorem 3.21 imply that, over any number field K, if a sieve R is an Erdős $\mathcal{B}-$free system, then $\mathcal{F}_R$ has a well defined density. The fact that the elements of $\mathcal{B}_R$ are pairwise coprime was fundamental in our proof. Indeed, in case this does not hold, this result will fail even for pseudosieves over $\mathbb{Q}$. A famous example was presented by Besicovitch in [12], of a $\mathcal{B}-$free system such that for the corresponding pseudosieve R the density of $\mathcal{F}_R$ is not well defined.

The proof of this result when K is an étale $\mathbb{Q}-$algebra uses the same ideas as the proof of Lemma 5.3. But when $K = K_1 \times K_2 \times \cdots \times K_l$ is as étale $\mathbb{Q}-$algebra, we may have an infinite collection of ideals of the form $\mathcal{O}_{K_1} \times \mathfrak{b}_i$ where $\mathfrak{b}_i$ are nonzero proper ideals of $\mathcal{O}_{K_2 \times \ldots K_L}$. Clearly, it is not the case that $\lambda_1(\mathcal{O}_{K_1} \times \mathfrak{b}_i) \asymp \lambda_n(\mathcal{O}_{K_1} \times \mathfrak{b}_i)$ as we had in the number field case. To deal with this, we will use the following lemma.

Lemma 5.5. *Let K be an étale $\mathbb{Q}$-algebra, and L a number field. Let R be a sieve over K, supported on a set $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots\}$. Let F_N be an arbitrary Følner sequence in $\mathcal{O}_L$, and $\mathfrak{c}_1, \mathfrak{c}_2, \ldots$ be any infinite sequence of pairwise coprime nonzero ideals of $\mathcal{O}_L$.*

If R is Erdős and has weak (respectively, strong) light tails for some Følner sequence I_N in $\mathcal{O}_K$, the sieve R' over $K \times L$ supported on $\{\mathfrak{b}_1 \times \mathfrak{c}_1, \mathfrak{b}_2 \times \mathfrak{c}_2, \ldots\}$ and defined by $R'_i = R_i \times \mathcal{O}_K + \mathfrak{b}_i \times \mathfrak{c}_i$ is also Erdős, and has weak (respectively, strong) light tails over $I_N \times F_N$.

Proof. It is easy to show that if I_N and F_N are Følner sequences, then so is $I_N \times F_N$. Because

$$\mathrm{vol}(R'_i) = \frac{|R_i| N(\mathfrak{c}_i)}{N(\mathfrak{b}_i) N(\mathfrak{c}_i)} = \mathrm{vol}(R_i),$$

if R is Erdős, so is R'.

We have that

$$|\{(x, y) \in I_N \times F_N : (x, y) \in R'_i \text{ for some } i > L\}| = |\{x \in I_N : x \in R_i \text{ for some } i > L\}||F_N|$$

with the analogous result applying if we wanted to show that R' has weak light tails. It follows that if R has weak or strong light tails for I_N, so does R' for $I_N \times F_N$. $\square$

Remark 5.6. Notice that if R has strong light tails for some I_N, and R' is another sieve such that $R'_i \subset R_i$ for all i, then

$$\overline{d}_I \left(\bigcup_{i>L} R'_i \right) \leq \overline{d}_I \left(\bigcup_{i>L} R_i \right) = 0,$$

so R' also has strong light tails for I_N. The same does not hold assuming only that R has weak light tails for I_N, as is demonstrated by the sieves W and W' in Example 5.14.

Consequently, as a corollary of Lemma 5.5, we have that if R is a sieve over an étale $\mathbb{Q}-$algebra K with strong light tails for some I_N, and R' is another sieve over some étale $\mathbb{Q}-$algebra L, then the sieve W defined by $W_i = R_i \times R'_i$ will have strong light tails for any Følner sequence in $\mathcal{O}_{K \times L}$ of the form $I_N \times F_N$ with F_N a Følner sequence in $\mathcal{O}_L$.

Additionally, note that for our sequence $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \ldots$ of ideals of $\mathcal{O}_L$, we are free to take $\mathfrak{c}_i = \mathcal{O}_L$ for every i, since any ideal of $\mathcal{O}_L$ is coprime to $\mathcal{O}_L$.

With this, we can now generalize Lemma 5.3 to the case where K is an étale $\mathbb{Q}-$algebra. Up to now, we have always worked over some fixed K, so there was no ambiguity when we wrote B_N (as defined in

Equation (3)). In order to avoid any ambiguities, in the following proof we will use the notation B_N^K for the set of elements of $\mathcal{O}_K$ that have norm smaller than or equal to N.

Theorem 5.7. *Let R be a sieve over an étale $\mathbb{Q}-$algebra K of degree n, such that $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$ and there is a finite set T for which $R_i \subset T + \mathfrak{b}_i$ for every i. If $\sum_i \frac{1}{N(\mathfrak{b}_i)} < \infty$, then R is Erdős and has strong light tails for B_N.*

Proof. By proceeding as in the proof of Lemma 5.3, we can assume without loss of generality that $R_i = \mathfrak{b}_i$ for every $i \in \mathbb{N}$. Writing $K = K_1 \times \cdots \times K_l$, we show the result by induction in l. The case $l = 1$ corresponds to Lemma 5.3. Now, assume that it holds for $l - 1$. We write $K = K' \times K_l$, and $\mathfrak{c}_i = \mathfrak{a}_i \times \mathfrak{b}_i$ where $\mathfrak{a}_i$ are ideals of $\mathcal{O}_{K'}$ and $\mathfrak{b}_i$ are ideals of $\mathcal{O}_{K_l}$, so that $\mathcal{B}_R = \{\mathfrak{a}_i \times \mathfrak{b}_i : i \in \mathbb{N}\}$. We may have $\mathfrak{a}_i = \mathcal{O}_{K'}$ or $\mathfrak{b}_i = \mathcal{O}_{K_l}$, but we can't have both simultaneously. Hence, we have the disjoint union

$$\mathcal{B}_R = \bigcup_{j=1}^{3} \{\mathfrak{a}_i \times \mathfrak{b}_i : i \in I_j\},$$

where $I_1 = \{i : \mathfrak{a}_i = \mathcal{O}_{K'}\}, I_2 = \{i : \mathfrak{b}_i = \mathcal{O}_{K_l}\}$ and $I_3 = \{i : \mathfrak{a}_i \neq \mathcal{O}_{K'}, \mathfrak{b}_i \neq \mathcal{O}_{K_l}\}$.

If any of the sets I_j is finite, we don't have to consider them, since the strong light tails property is clearly invariant under changing (including removal) of any finite number of R_i. Hence, we see that the proof of this theorem reduces to showing that the sieves $R^{(j)}$ supported on $\mathcal{B}_j := \{\mathfrak{c}_i : i \in I_j\}$ and defined by $R_i^{(j)} = \mathfrak{c}_i$ all have strong light tails with respect to B_N^K for those j such that I_j is infinite.

The fact that $\sum_i \frac{1}{N(\mathfrak{c}_i)} < \infty$ implies that $\sum_{i \in I_j} \frac{1}{N(\mathfrak{a}_i)N(\mathfrak{b}_i)} < \infty$ for all j. Assume that I_1 is infinite. This would imply that $\sum_{i \in I_1} \frac{1}{N(\mathfrak{b}_i)} < \infty$, which by Lemma 5.3 means that the sieve $Z^{(1)}$ supported on $\mathcal{B}_{Z^{(1)}} = \{\mathfrak{b}_i : i \in I_1\}$ and defined by $Z_i^{(1)} := \mathfrak{b}_i$ over K_l has strong light tails for $B_N^{K_l}$. Consequently, by Lemma 5.5, the sieve $R^{(1)}$ will have strong light tails with respect to $B_N^K = B_N^{K'} \times B_N^{K_l}$. The same argument can be used to show that $R^{(2)}$ has strong light tails for B_N^K, this time using the induction hypothesis which tells us that because $\sum_{i \in I_2} \frac{1}{N(\mathfrak{a}_i)} < \infty$, the sieve $Z_i^{(2)} := \mathfrak{a}_i$ (with $i \in I_2$) over K' has strong light tails for $B_N^{K'}$.

Therefore, we have reduced the problem to showing that $R^{(3)}$ has strong light tails for B_N^K if I_3 is infinite. We will use the exact same ideas already present in the proof of Lemma 5.3. In order for the notation not to become overwhelming, we will just do the proof for the case $l = 2$, but it should be clear that the exact same can be done for arbitrary l. We also assume simply that $I_3 = \mathbb{N}$, so that $R = R^{(3)}$.

Let $x \in B_N^K \setminus \{0\}$. This means that writing $x = (x_1, x_2)$ with $x_1 \in K_1$ and $x_2 \in K_2$, we have $x_1 \in B_N^{K_1}$ and $x_2 \in B_N^{K_2}$. Therefore, if $x \in \mathfrak{a}_i \times \mathfrak{b}_i$, then $\lambda_1(\mathfrak{a}_i) \leq \|x_1\| \leq N$ and $\lambda_1(\mathfrak{b}_i) \leq \|x_2\| \leq N$. Proceeding as in Lemma 5.3, we get that

$$\left| B_N^K \cap \bigcup_{i > L} \mathfrak{a}_i \times \mathfrak{b}_i \right| \leq 1 + \sum_{\substack{i:\lambda_1(\mathfrak{a}_i) \leq N \\ \lambda_1(\mathfrak{b}_i) \leq N \\ i > L}} \left(\frac{|B_N^K|}{N(\mathfrak{a}_i \times \mathfrak{b}_i)} + O\left(1 + \sum_{j=1}^{n-1} \frac{N^j}{\lambda_1(\mathfrak{a}_i \times \mathfrak{b}_i) \cdots \lambda_j(\mathfrak{a}_i \times \mathfrak{b}_i)}\right)\right).$$

We again have to verify that after dividing by $|B_N^K|$ and taking the limit of N, and then L to infinity, this expression will go to 0. It is clear that

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}\frac{|B_N^K|}{N(\mathfrak{a}_i\times\mathfrak{b}_i)}\le\lim_{L\to\infty}\sum_{i>L}\frac{1}{N(\mathfrak{a}_i\times\mathfrak{b}_i)}=0,$$

so we now deal with the remaining terms.

We have that

$$\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}1\le\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ i>L}}1+\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{b}_i)\le N\\ i>L}}1.$$

Since the $\mathfrak{a}_i$ are pairwise coprime, and so are the $\mathfrak{b}_i$, we can bound these sums using Theorem 2.2, and conclude that

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}1=0.$$

Fix any $1\le j\le n-1$. We are left with showing that

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}\frac{N^j}{\lambda_1(\mathfrak{a}_i\times\mathfrak{b}_i)\cdots\lambda_j(\mathfrak{a}_i\times\mathfrak{b}_i)}=0.$$

Using Equation (4), we have that

$$\frac{N^j}{\lambda_1(\mathfrak{a}_i\times\mathfrak{b}_i)\cdots\lambda_j(\mathfrak{a}_i\times\mathfrak{b}_i)}\asymp_K\frac{\lambda_{j+1}(\mathfrak{a}_i\times\mathfrak{b}_i)\cdots\lambda_n(\mathfrak{a}_i\times\mathfrak{b}_i)N^j}{N(\mathfrak{a}_i)N(\mathfrak{b}_i)}.$$

Let n_1 and n_2 denote the degrees of K_1 and K_2 respectively such that $n=n_1+n_2$. For any $1\le k\le n$ we have that $\lambda_k(\mathfrak{a}_i\times\mathfrak{b}_i)\le\lambda_{n_1}(\mathfrak{a}_i)+\lambda_{n_2}(\mathfrak{b}_i)$. By Lemma 2.4, we know that there are constants C_1 and C_2 such that $\lambda_{n_1}(\mathfrak{a}_i)\le C_1\lambda_1(\mathfrak{a}_i)$ and $\lambda_{n_2}(\mathfrak{b}_i)\le C_2\lambda_1(\mathfrak{b}_i)$. It follows that

$$\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}\frac{\lambda_{j+1}(\mathfrak{a}_i\times\mathfrak{b}_i)\cdots\lambda_n(\mathfrak{a}_i\times\mathfrak{b}_i)N^j}{N(\mathfrak{a}_i)N(\mathfrak{b}_i)}\le\sum_{\substack{i:\lambda_{n_1}(\mathfrak{a}_i)\le C_1N\\ \lambda_{n_2}(\mathfrak{b}_i)\le C_2N\\ i>L}}\frac{O_j(\lambda_{n_1}(\mathfrak{a}_i)^{n-j}+\lambda_{n_2}(\mathfrak{b}_i)^{n-j})N^j}{N(\mathfrak{a}_i)N(\mathfrak{b}_i)}.$$

Since we are summing over i's such that $\lambda_{n_1}(\mathfrak{a}_i)\le C_1N$ and $\lambda_{n_2}(\mathfrak{b}_i)\le\lambda_{n_2}(\mathfrak{b}_i)\le C_2N$, the numerator must be $\ll_j(C_1+C_2)^{n-j}N^n$. Consequently, we get that

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{|B_N^K|}\sum_{\substack{i:\lambda_1(\mathfrak{a}_i)\le N\\ \lambda_1(\mathfrak{b}_i)\le N\\ i>L}}\frac{N^j}{\lambda_1(\mathfrak{a}_i\times\mathfrak{b}_i)\cdots\lambda_j(\mathfrak{a}_i\times\mathfrak{b}_i)}\ll_j\lim_{L\to\infty}\sum_{i>L}\frac{1}{N(\mathfrak{a}_i)N(\mathfrak{b}_i)}=0,$$

which concludes our proof. $\square$

Example 5.8. Let $\mathcal{B}_R = \{p\mathbb{Z} \times p\mathbb{Z} : p \text{ prime}\}$ and define R by $R_p = p\mathbb{Z} \times p\mathbb{Z}$. Since $\sum_p 1/p^2 < \infty$, Theorem 5.7 implies that R has strong light tails for B_N. The set of visible lattice points corresponds to $\mathcal{F}_R$, and we get the well known result that

$$d(\mathcal{F}_R) = \prod_p \left(1 - \frac{1}{p^2}\right).$$

We now provide other examples of sieves with strong light tails for B_N. The idea here is that if, for some given N, we can control the number of i such that $R_i \cap B_N \neq \emptyset$, then we may be able to show that R has strong light tails for B_N. One way of doing this is assuming that all elements of R_i have norm bigger than N for all i bigger than some function f. Then, assuming that f grows slowly enough, we are able to show that R has strong light tails. Indeed, this was what we did in the proof of Theorem 5.7 using $f(i) = \lambda_1(\mathfrak{b}_i)$. More generally, we have the following lemma.

Lemma 5.9. *Let R be an Erdős sieve over an étale $\mathbb{Q}-$algebra of degree n. If there is some function f and constant $C > 0$ such that $f(i) \leq C \min_{a \in R_i} \|a\|$ and*

$$\sum_{i : f(i) \leq N} \lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i) \operatorname{vol}(R_i) = o(N^{n-j+1}),$$

for every $1 \leq j \leq n$, then R has strong light tails for the Følner sequence $I_N = B_N$.

Proof. If $x \in B_N \cap R_i$, then we get that $f(i) \leq C\|x\| \leq CN$, so we have

$$\left| B_N \cap \bigcup_{i > L} R_i \right| = \sum_{\substack{i : f(i) < CN \\ i > L}} |R_i \cap B_N|.$$

The set R_i is the disjoint union of sets of the form $a + \mathfrak{b}_i$. By Lemma 2.6 together with Equation (4), we have

$$|(a + \mathfrak{b}_i) \cap B_N| = \frac{|B_N|}{N(\mathfrak{b}_i)} + O\left(\sum_{j=1}^{n} \frac{\lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i)}{N(\mathfrak{b}_i)} N^{j-1}\right).$$

Hence, we get

$$\left| B_N \cap \bigcup_{i > L} R_i \right| \ll |B_N| \sum_{i > L} \operatorname{vol}(R_i) + \sum_{j=1}^{n} N^{j-1} O\left(\sum_{\substack{i : f(i) < CN \\ i > L}} \frac{\lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i)}{N(\mathfrak{b}_i)} |R_i| \right).$$

Our hypothesis is that R is so that

$$N^{j-1} \sum_{i : f(i) \leq CN} \frac{\lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i)}{N(\mathfrak{b}_i)} |R_i| = N^{j-1} o(N^{n-j+1}) = o(N^n).$$

Therefore, it is clear that

$$\lim_{L \to \infty} \lim_{N \to \infty} \frac{\left|B_N \cap \bigcup_{i > L} R_i\right|}{|B_N|} = \lim_{L \to \infty} \sum_{i > L} \operatorname{vol}(R_i) = 0$$

since R is Erdős. ∎

Using Lemma 5.9, we now provide other examples of sieves over $\mathbb{Q}$ with strong light tails with respect to B_N.

Example 5.10. Let R be a sieve such that $|R_i| = 1$ for every i, and assume that there is some function f such that $f(i) \leq \min_{a \in R_i} |a|$ with $f(i) \gg i^{1+\epsilon}$ for some $\epsilon > 0$. Using Lemma 5.9, we can easily show that R has strong light tails for B_N.

Indeed, since $|R_i| = 1$ for every i, all we have to show is that $\sum_{i: f(i) \leq N} 1 = o(N)$. That is, we have to show that

$$|\{i : f(i) \leq N\}| = o(N).$$

But since $f(i) \gg i^{1+\epsilon}$, it is clear that $|\{i : f(i) \leq N\}| \ll N^{1/(1+\epsilon)} = o(N)$. In particular, the sieve R defined by $R_i = i^2 + p_i^2\mathbb{Z}$ will have strong light tails.

Up to now, we have only given examples of sieves with strong light tails such that $|R_i|$ is bounded by a constant. If we assume that $|R_i|$ does not grow too fast, we can also provide examples of sieves R which have strong light tails for B_N.

Example 5.11. Let v be the arithmetic function defined by $v(p^k) = p$, extended to all of $\mathbb{N}$ by being multiplicative. It is clear from the Prime Number Theorem, that if $\mathcal{B} = \{b_i : i \in \mathbb{N}\}$ is a set of pairwise coprime numbers, we have

$$|\{i : v(b_i) \leq N\}| \ll \frac{N}{\log(N)}.$$

Suppose that R is a sieve such that $v(b_i) \ll \min_{a \in R_i} |a|$ and $\max_{i: v(b_i) \leq N} |R_i| = o(\log(N))$. Then R has strong light tails for the Følner sequence $I_N = B_N$.

Indeed, by Lemma 5.9 it is enough to show that

$$\sum_{i: v(\mathfrak{b}_i) \leq N} |R_i| = o(N).$$

We have

$$\sum_{i: v(\mathfrak{b}_i) \leq N} |R_i| \leq \left(\max_{i: v(b_i) \leq N} |R_i| \right) |\{i : v(b_i) \leq N\}| \ll \left(\max_{i: v(b_i) \leq N} |R_i| \right) \frac{N}{\log(N)},$$

so the result follows from our hypothesis on the size of $|R_i|$.

As an example, it follows that the sieve R defined by $R_i = \{jp_i : 1 \leq j \leq \lceil \sqrt{\log(i)} \rceil\} + p_i^2\mathbb{Z}$ has strong light tails for B_N.

Note that there are sieves for which $|R_i|$ grows linearly or even faster, that still have strong light tails, as we will show in the following theorem. Here, we assume that R satisfies the condition

$$\min_{x \in R_i} \|x\| \geq \left(\frac{N(\mathfrak{b}_i)}{C|R_i|} \right)^{1/n}$$

for some C. Let us explain why this is a natural condition to consider. If R is any sieve, there is always some sequence δ_i such that defining a sieve R' by $R'_i = \delta_i + R_i$, we have $\min_{x \in R'_i} \|x\| \geq \left(\frac{N(\mathfrak{b}_i)}{C|R'_i|} \right)^{1/n}$. From the dynamical point of view, R and the sieve R' obtained in this matter are connected (this is made precise in Lemma 7.3), so from the strong light tails of R' we can extract information about R.

Theorem 5.12. *Let R be a sieve over a number field K of degree n. If R satisfies*

$$\sum_i \frac{|R_i|^{\frac{2}{n}}}{N(\mathfrak{b}_i)} < \infty,$$

and

$$\min_{x \in R_i} \|x\| \geq \left(\frac{N(\mathfrak{b}_i)}{C|R_i|} \right)^{1/n}$$

for some C, then R has strong light tails for B_N.

Proof. By Lemma 5.9, we just have to show that for every j, it holds that

$$\sum_{i: \frac{N(\mathfrak{b}_i)}{C|R_i|} \leq N^n} \lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i) \operatorname{vol}(R_i) = o(N^{n-j+1})$$

By Lemma 2.4, we have $\lambda_n(\mathfrak{b}_i) \leq cN(\mathfrak{b}_i)^{1/n}$ for some constant c, and so, if we have that $\frac{N(\mathfrak{b}_i)}{C|R_i|} \leq N^n$ for some i, then $\lambda_n(\mathfrak{b}_i) \leq cN(\mathfrak{b}_i)^{1/n} \leq C'|R_i|^{1/n}N$, where $C' := cC^{1/n}$. Therefore,

$$\sum_{i: \frac{N(\mathfrak{b}_i)}{C|R_i|} \leq N^n} \lambda_j(\mathfrak{b}_i) \cdots \lambda_n(\mathfrak{b}_i) \operatorname{vol}(R_i) \leq \sum_{i: \lambda_n(\mathfrak{b}_i) \leq C'|R_i|^{1/n}N} \lambda_n(\mathfrak{b}_i)^{n-j} \frac{|R_i|}{N(\mathfrak{b}_i)} \leq N^{n-j} \sum_i \frac{|R_i|^{1+(n-j)/n}}{N(\mathfrak{b}_i)}.$$

Since by hypothesis this series converges for every j, the result follows. □

Example 5.13. Let R be a sieve defined by $R_i = \{p_i^3 + j : 1 \leq j \leq p_i\} + p_i^4\mathbb{Z}$. We have that $|R_i| = p_i$, and $\min_{x \in R_i} |x| = p_i^3$. Since $p_i^4/p_i = p_i^3$ and

$$\sum_i \frac{|R_i|^2}{p_i^4} = \sum_i \frac{1}{p_i^2} < \zeta(2) < \infty,$$

this sieve has strong light tails for B_N by Theorem 5.12.

5.3. **Weak and Strong Light Tails.** We will now study a series of natural operations one can do on sieves, and investigate when they preserve weak or strong light tails. We start with the guiding example of a sieve which has weak light tails, but does not have strong light tails.

Example 5.14. Let $\mathcal{B} = \{p_i^2 : i \in \mathbb{N}\}$, and R the sieve supported on $\mathcal{B}$ and defined by

$$R_i = 1 + 4(i-1) + p_i^2\mathbb{Z}.$$

We consider the Følner sequence $I_N = [0, N]$.

If $i \in 1 + 4\mathbb{N}_0$ then $i \in R_{(i-1)/4+1}$. Therefore, given any positive N, L, we have

$$\left| I_N \cap \bigcup_{i>L} R_i \right| \geq \frac{N}{4} - 1 - L,$$

since we count every element of $(1 + 4\mathbb{N}_0) \cap [0, N]$, except for possibly those in R_j with $j < L$. It follows that

$$\overline{d}_I \left(\bigcup_{i>L} R_i \right) \geq \frac{1}{4}$$

for all L, so the sieve R does not have strong light tails.

On the other hand, if $x \in I_N$ is such that $x \in R_i$ for some $i > L$, but $x \notin R_1$, then $x \geq 1+4(i-1)+p_i^2k$ with $k \geq 1$. Since $x \leq N$, it follows that if R_i intersects I_N, then $1+4(i-1)+p_i^2 \leq N$. In particular, it is clear that if $p_i > N$, then $R_i \cap I_N \subset \{1+4(i-1)\} \subset R_1$.

Since there are at most $1+\frac{N}{b}$ numbers congruent to a class mod b in $[0, N]$, we get that

$$\left| I_N \cap \bigcup_{i>L} R_i \setminus \bigcup_{j\leq L} R_j \right| \leq \sum_{\substack{i:p_i<N \\ i>L}} \left(1+\frac{N}{p_i^2}\right) \leq N \sum_{i>L} \frac{1}{p_i^2} + \pi(N),$$

where $\pi(N)$ denotes the prime counting function.

Dividing by $|I_N| = N+1$ and taking N to infinity, we are left with the series $\sum_{i>L} 1/p_i^2$, and since this is convergent, when we let L grow to infinity this term will go to 0. It follows that R has weak light tails.

Consider now the sieve W, also supported on $\{p_i^2 : i \in \mathbb{N}\}$ with

$$W_1 = 1+4\mathbb{Z} \quad W_{2i} = 1+4(i-1)+p_{2i}^2\mathbb{Z} \quad W_{2i+1} = 1-4(i-1)+p_{2i+1}^2\mathbb{Z}.$$

The sieve W is constructed so that $W_1 \subset \bigcup_{i\geq 2} W_i$ but by the exact same argument as for R, we can show that it has weak light tails for I_N. Consider then the sieve W' supported on $\{p_{i+1}^2 : i \in \mathbb{N}\}$ and defined by $W'_i = W_{i+1}$ (alternatively, we could define $W'_1 = \emptyset$ and $W'_i = W_i$). Since

$$\bigcup_{i\geq 1} W_i = \bigcup_{i\geq 2} W_i,$$

it follows that $\mathcal{F}_W = \mathcal{F}_{W'}$. But by Theorem 3.21, W' does not have weak light tails for I_N, since

$$d_I(\mathcal{F}_{W'}) = d_I(\mathcal{F}_W) = \prod_{i\geq 1}\left(1-\frac{1}{p^2}\right) < \prod_{i\geq 2}\left(1-\frac{1}{p^2}\right) = \nu_{W'}(C^{W'}_{\{0\},\emptyset}).$$

In this example, we took a sieve W and removed an element of $\mathcal{B}_W$ to obtain a new sieve $\mathcal{B}_{W'}$. There are a number of operations we can do on a sieve R by manipulating the set $\mathcal{B}_R$ on which it is supported. First, the definition of weak light tails presupposes that $\mathcal{B}_R$ has been ordered, and we now show that this is not problematic, since the weak light tails property is invariant under any reordering of $\mathcal{B}_R$.

Lemma 5.15. *Let R be an Erdős sieve supported on a set $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$ with weak light tails for some Følner sequence I_N and σ a permutation of $\mathbb{N}$. If R' is the sieve defined by $R'_i = R_{\sigma(i)}$, then R' also has weak light tails for I_N.*

Proof. By Lemma 3.19, it is enough to show that $d_I(\mathcal{F}_{R'}) = \nu_{R'}(C^{R'}_{\{0\},\emptyset})$. It is clear that $\mathcal{F}_{R'} = \mathcal{F}_R$, so since R has weak light tails we get $d_I(\mathcal{F}_{R'}) = d_I(\mathcal{F}_R) = \nu_R(C^R_{\{0\},\emptyset})$. Using Equation (12), we see that

$$\nu_R(C^R_{\{0\},\emptyset}) = \prod_i (1-\mathrm{vol}(R_i)) = \prod_i (1-\mathrm{vol}(R_{\sigma(i)})) = \nu_{R'}(C^{R'}_{\{0\},\emptyset}),$$

given that the product is absolutely convergent, and so invariant under reordering. This concludes the proof. $\square$

We can also show that the same holds for the strong light tails property.

Lemma 5.16. *Let R be an Erdős sieve with strong light tails for some Følner sequence I_N and σ a permutation of $\mathbb{N}$. If R' is the sieve defined by $R'_i = R_{\sigma(i)}$, then R' also has strong light tails with respect to I_N.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(L) = \min_{i \geq L} \sigma(i).$$

This is defined so that if $i \geq L$, then $\sigma(i) \geq f(L)$. Consequently, we have

$$\bigcup_{i \geq L} R'_i \subset \bigcup_{i:\sigma(i) \geq f(L)} R_{\sigma(i)} \subset \bigcup_{j \geq f(L)} R_j.$$

Since σ is a bijection, we have that $\lim_{L\to\infty} f(L) = \infty$, given that for any k, there are only k solutions to $f(x) \leq k$. It follows that

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i \geq L} R'_i \right) \leq \lim_{L\to\infty} \overline{d}_I \left(\bigcup_{j \geq f(L)} R_j \right) = 0$$

using that R has strong light tails for I_N. □

Another operation we can do on a sieve R is to add congruence classes to it. There are two possibilities here. We can take a new ideal $\mathfrak{b}$ coprime to all of the elements of $\mathcal{B}_R$, and add to R some $R_\mathfrak{b}$, producing a new sieve R' supported on $\mathcal{B}_R \cup \{\mathfrak{b}\}$ such that $\mathcal{F}_{R'} = \mathcal{F}_R \cap R_\mathfrak{b}^c$. Alternatively, we could also take some $\mathfrak{b}$ already in $\mathcal{B}_R$, and some $x \notin R_\mathfrak{b}$, and then consider the sieve R' such that $R'_\mathfrak{b} = R_\mathfrak{b} \cup \{x + \mathfrak{b}\}$. In the next lemma we show that the weak light tails property is invariant under both of these operations.

Lemma 5.17. *Let R be an Erdős sieve supported on the set $\mathcal{B}_R = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots\}$. Let R' be a sieve supported on $\mathcal{B}_{R'} = \{\mathfrak{b}_0\} \cup \mathcal{B}$, with $R'_i = R_i$ for $i \geq 1$. Additionally, let W be a sieve over $\mathcal{B}$ such that $R_1 \subset W_1$ and $W_i = R_i$ for $i > 1$. If R has weak light tails for some Følner sequence I_N, then so do R' and W.*

Proof. Since $R'_i = R_i$ for all $i \geq 1$, we have that

$$\overline{d}_I \left(\bigcup_{i > L} R_i \setminus \bigcup_{1 \leq j \leq L} R_j \right) \geq \overline{d}_I \left(\bigcup_{i > L} R'_i \setminus \bigcup_{0 \leq j \leq L} R'_j \right),$$

so it follows that if R has weak light tails for I_N, then so does R'.

Similarly, if $W_1 \supset R_1$, then

$$\overline{d}_I \left(\bigcup_{i > L} R_i \setminus \bigcup_{1 \leq j \leq L} R_j \right) \geq \overline{d}_I \left(\bigcup_{i > L} W_i \setminus \bigcup_{1 \leq j \leq L} W_j \right),$$

so if R has weak light tails for I_N, then so does W. □

Remark 5.18. The strong light tails property is also invariant under these operations. Indeed, it is clear from the definition that any operation (such as addition or removal of congruence classes) that affects only finitely many $R_{\mathfrak{b}}$ will preserve strong light tails. This is distinct from weak light tails, since the removal of congruence classes from a sieve may not preserve weak light tails. This was already shown in Example 5.14, where we defined a sieve W and a sieve W' obtained from W by removing W_1. We have shown that W has weak light tails for $I_N = [0, N]$, but W' does not have weak light tails.

As a consequence of Lemma 5.17, we are able to compute the density of $\mathcal{F}_R$ intersected with some congruence class.

Lemma 5.19. *Let $\mathcal{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \cdots\}$ be an infinite set of pairwise coprime ideals. Let R be an Erdős sieve over $\mathcal{B}$ with weak light tails for I_N. If $\mathfrak{b}$ is an ideal coprime to every ideal in $\mathcal{B}$, then for every $x, b \in \mathcal{O}_K$,*

$$d_I((x + \mathcal{F}_R) \cap (b + \mathfrak{b})) = \frac{d_I(\mathcal{F}_R)}{N(\mathfrak{b})}.$$

Additionally, we have for any i that

$$d_I(\mathcal{F}_R \cap (x + \mathfrak{b}_i)) = \frac{d_I(\mathcal{F}_R)}{|R_i^c|}$$

if $x + \mathfrak{b}_i \notin R_i$

Proof. We consider the sieve R' defined by $R'_0 = (b - x + \mathfrak{b})^c$ and $R'_i = R_i$ for $i > 0$. Since R has weak light tails, so does R' by Lemma 5.17. We have

$$\mathcal{F}_{R'} = \left(\bigcup_{i>0} R_i \cup R'_0 \right)^c = \mathcal{F}_R \cap (b - x + \mathfrak{b}),$$

and so $d_I((x+\mathcal{F}_R) \cap (b+\mathfrak{b})) = d_I(x+\mathcal{F}_{R'}) = d_I(\mathcal{F}_{R'})$, which is equal to $d_I(\mathcal{F}_R)N(\mathfrak{b})^{-1}$ by Theorem 3.21.

To show the second result, consider the sieve W defined by $W_i = (x + \mathfrak{b}_i)^c$, and $W_j = R_j$ if $j \neq i$. We have

$$\mathcal{F}_R \cap (x + \mathfrak{b}_i) = \left(\bigcap_{j \neq i} R_j^c \cap R_i^c \right) \cap (x + \mathfrak{b}_i) = \bigcap_{j \neq i} R_j^c \cap (R_i^c \cap (x + \mathfrak{b}_i)) = \bigcap_{j \neq i} W_j^c \cap (x + \mathfrak{b}_i) = \mathcal{F}_W.$$

We have that $R_i \subset W_i$. Therefore, by Lemma 5.17, if R has weak light tails so does W (we can assume that $i = 1$ by Lemma 5.15), and we get

$$d_I(\mathcal{F}_W) = d_I(\mathcal{F}_R) \left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right)^{-1} \left(1 - \frac{N(\mathfrak{b}_i) - 1}{N(\mathfrak{b}_i)}\right) = d_I(\mathcal{F}_R) \frac{N(\mathfrak{b}_i) - (N(\mathfrak{b}_i) - 1)}{N(\mathfrak{b}_i) - |R_i|} = \frac{d_I(\mathcal{F}_R)}{|R_i^c|}. \qquad \square$$

More generally, suppose that we have a collection of sets $A_i \subset \mathcal{O}_K$ and we want to know the density of $\mathcal{F}_R$ intersected with $A_i + \mathfrak{b}_i$. Using Lemma 5.19, this density can be computed as follows.

Lemma 5.20. *Let R be an Erdős sieve with weak light tails for some Følner sequence I_N. Let $A_{i_j} \subset R_{i_j}^c$ be a collection of finite sets, with $1 \le j \le k$, such that for any distinct $a, b \in A_{i_j}$, $a + \mathfrak{b}_{i_j} \ne b + \mathfrak{b}_{i_j}$. Then*

$$d_I\left(\mathcal{F}_R \cap \bigcap_{j=1}^{k} (A_{i_j} + \mathfrak{b}_{i_j})\right) = d_I(\mathcal{F}_R) \prod_{j=1}^{k} \frac{|A_{i_j}|}{|R_{i_j}^c|}.$$

Proof. For any distinct $a, b \in A_{i_j}$ we have $a + \mathfrak{b}_{i_j} \ne b + \mathfrak{b}_{i_j}$, so

$$d_I(\mathcal{F}_R \cap (A_{i_j} + \mathfrak{b}_{i_j})) = \sum_{a \in A_{i_j}} d_I(\mathcal{F}_R \cap (a + \mathfrak{b}_{i_j})) = d_I(\mathcal{F}_R) \frac{|A_{i_j}|}{|R_{i_j}^c|}$$

by Lemma 5.19. The result follows by induction. Assume that it holds for $k-1$. Take the sieve R' defined by $R'_i = (A_{i_j} + \mathfrak{b}_{i_j})^c$ if $i = i_j$ for some $1 \le j \le k-1$ and $R'_i = R_i$ otherwise. By repeatedly using Lemma 5.17, we have that R' has weak light tails. We get

$$d_I\left(\mathcal{F}_R \cap \bigcap_{j=1}^{k} (A_{i_j} + \mathfrak{b}_{i_j})\right) = d_I(\mathcal{F}_{R'} \cap (A_{i_k} + \mathfrak{b}_{i_k})) = d_I(\mathcal{F}_{R'}) \frac{|A_{i_k}|}{|R_{i_k}^c|} = d_I(\mathcal{F}_R) \prod_{j=1}^{k} \frac{|A_{i_j}|}{|R_{i_j}^c|},$$

the last equality following from the induction hypothesis. □

Example 5.21. Let R be the squarefree sieve $R_i = p_i^2\mathbb{Z}$. Then it is well known that $d(\mathcal{F}_R) = \zeta(2)^{-1} = 6/\pi^2$. By Lemma 5.20 it follows that the density of even squarefree numbers is $(2/3)\zeta(2)^{-1} = 4/\pi^2$ (this result was shown in [49] for example).

Definition 5.22. *We say that a sieve R satisfies the local global principle, if after choosing any finite subset S of $\mathcal{B}$ and $x_\mathfrak{b}$ such that $x_\mathfrak{b} \notin R_\mathfrak{b}$ for any $\mathfrak{b} \in S$, we can find infinitely many $y \in \mathcal{F}_R$ such that $y \equiv x_\mathfrak{b} \mod \mathfrak{b}$.*

In [42] (from which we take the name 'local global principle' for this property of R) it is pointed out that there is a close relation between light tails of R and the local global principle of R. By Lemma 5.20, sieves with weak light tails for some I_N not only satisfy the local global principle, but in fact, for any choice of S and $x_\mathfrak{b}$, the density of $y \in \mathcal{F}_R$ congruent to $x_\mathfrak{b}$ for every $\mathfrak{b} \in S$ is given by

$$d_I(\mathcal{F}_R) \prod_{\mathfrak{b} \in S} \frac{1}{|R_\mathfrak{b}^c|}. \tag{20}$$

Yet, this is not an equivalence, as we show in the next example.

Example 5.23. Consider the sieves W and W' from Example 5.14. We know that W has weak light tails for $I_N = [0, N]$, while W' does not. We have that $\mathcal{B}_W = \{p_i^2 : i \in \mathbb{N}\}$, while $\mathcal{B}_{W'} = \mathcal{B}_W \setminus \{4\}$. Take any finite $S \subset \mathbb{N} \setminus \{1\}$, and choose $x_i \notin W'_i$ for $i \in S$. We want to show that the density of those $y \in \mathcal{F}_{W'}$ not congruent to $x_i \mod p_i^2$ is given by Equation (20), in spite of this sieve not having weak light tails for I_N.

Indeed, since $\mathcal{F}_W = \mathcal{F}_{W'}$, and W has weak light tails, we know that the density of such y is given by

$$d_I(\mathcal{F}_W) \prod_{i \in S} \frac{1}{|W_i^c|} = d_I(\mathcal{F}_{W'}) \prod_{i \in S} \frac{1}{|(W'_i)^c|},$$

as we wanted to show.

This means that the local global property does not imply weak light tails. This was to be expected, since while the local global property does not depend on a Følner sequence I_N, weak light tails does (as shown by Remark 4.3). Yet, this example shows that even a stronger form of the local global principle depending on some I_N, where we assume that the density (with respect to I_N) of $y \in \mathcal{F}_R$ that are congruent to $x_\mathfrak{b} \mod \mathfrak{b}$, for some finitely many $x_\mathfrak{b} \notin R^c_\mathfrak{b}$, is given by Equation (20), does not imply that R has weak light tails for I_N.

In general, the local global property is far less well behaved than the weak light tails property. Yet, contrarily to weak light tails, the local global principle is preserved by removing ideals from $\mathcal{B}_R$. Let R be a sieve satisfying the local global principal, and R^1 the sieve supported on $\mathcal{B}_R \setminus \{\mathfrak{b}_1\}$ and defined by $R^1_i = R_{i+1}$. Then R^1 is supported over the set $\mathcal{B}_R \setminus \{\mathfrak{b}_1\}$, and because we are sieving out less, we have $\mathcal{F}_R \subset \mathcal{F}_{R^1}$. It is then clear that R^1 must also satisfy the local global principle. Any finite $S \subset \mathcal{B}_R \setminus \{\mathfrak{b}_1\}$ is a subset of $\mathcal{B}_R$, so given any number of congruence classes modulo $\mathfrak{b}$ with $\mathfrak{b}$ in S, we can find a solution in $\mathcal{F}_R$ by the local global principle of R, which will also be in $\mathcal{F}_{R^1}$.

Contrary to weak light tails, the local global principle for sieves is not preserved by adding ideals (which we have shown holds for weak light tails in Lemma 5.17) as the following example shows.

Example 5.24. Let R be the sieve defined by $R_1 = \{0, 1\}+4\mathbb{Z}$, $R_{2i} = 1+4i+p_{2i}^2\mathbb{Z}$ and $R_{2i+1} = 1-4i+p_{2i}^2\mathbb{Z}$. Proceeding as we have done in Example 5.14, we can show that R has weak light tails for B_N. Let R' be the sieve supported on $\{p_i : i \geq 2\}$ and defined by $R'_i = R_i$ if $i \geq 2$. The sieve R' satisfies the local global principle because R does. Yet, the sieve R'' defined by $R''_1 = 0+4\mathbb{Z}$ and $R''_i = R_i$ for $i \geq 2$ will not satisfy the local global principle, since the only element of $1 + 4\mathbb{Z}$ in $\mathcal{F}_{R''}$ is 1, even though $(1 + 4\mathbb{Z}) \cap R''_1 = \emptyset$. Since R'' was obtained from R' by adding to it the congruence class $0+4\mathbb{Z}$, it follows that the local global principle is not preserved by adding congruence classes. Additionally, since R'' can be obtained from R by removing the congruence class $1 + 4\mathbb{Z}$ from R_1, it follows that the local global principle is also not preserved by removing classes.

Below we summarize how the different operations preserve (or don't) each property.

	Local global property	Weak light tails for I_N	Strong light tails for I_N
Adding congruence classes	does not preserve	preserves	preserves
Removing congruence classes	does not preserve	does not preserve	preserves
Removing ideals	preserves	does not preserve	preserves

TABLE 1. How different operations change the properties of an Erdős sieve R.

As can be seen, the strong light tails property is much more stable. Now, say that we have a sieve R with weak light tails for some I_N, such that the weak light tails are preserved after repeatedly removing congruence classes. It is natural to ask if this sieve must have strong light tails. We now show that this is the case.

Theorem 5.25. *Let R be an Erdős sieve and I_N any Følner sequence. We have that R has strong light tails for I_N if and only if for every L, the sieve R^L supported on $\mathcal{B}_{R^L} = \{\mathfrak{b}_{L+1}, \mathfrak{b}_{L+2}, \dots\}$ and defined by $R_i^L = R_{L+i}$ has weak light tails for I_N.*

Proof. From the definition of strong light tails, it is clear that this property is invariant, under any change of a finite number of the R_i. Therefore, if R has strong light tails, so does every R^L, which implies that every R^L will have weak light tails.

We now show the opposite implication. Since R is Erdős, so is every R^L. By definition, $\mathcal{F}_{R^L}$ is equal to $\left(\bigcup_{j>L} R_j\right)^c$, so

$$d_I\left(\bigcup_{j>L} R_j\right) = d_I(\mathcal{F}_{R^L}^c) = 1 - d_I(\mathcal{F}_{R^L}) = 1 - \prod_{j>L}\left(1 - \frac{|R_j|}{N(\mathfrak{b}_j)}\right),$$

where in the last step we use the fact that R^L has weak light tails and apply Theorem 3.21. Since R is Erdős, the product goes to 1 as L goes to infinity, which concludes the proof. $\square$

Notice that if we replaced our hypothesis that R^L has weak light tails for every L, by the fact that there is a sequence L_i such that R^{L_i} has weak light tails for every i, the result would still hold. Note also that in order for this result to hold, we must be able to remove any number of ideals from R, with it retaining the weak light tails property. It is not true that if R has weak light tails for B_N, and for every $\mathfrak{b} \in \mathcal{B}$, the sieve $W^{(\mathfrak{b})}$ supported on $\mathcal{B}_R \setminus \{\mathfrak{b}\}$ by $W_{\mathfrak{a}}^{(\mathfrak{b})} = R_{\mathfrak{a}}$ also has weak light tails, then R has strong light tails. We provide an example.

Example 5.26. Let R be the sieve defined by $R_1 = 2\mathbb{Z}$, $R_2 = 3\mathbb{Z}$ with

$$R_{2i} = 6i + p_i^2\mathbb{Z} \quad \text{and} \quad R_{2i+1} = -6i + p_i^2\mathbb{Z}$$

for any $i \geq 2$. It is not hard to show that R has weak light tails by proceeding as in Example 5.14. Additionally, any sieve obtained from R by removing exactly one ideal will also have weak light tails. This is because if R' is a sieve obtained from R by removing exactly one ideal, then there is still some i such that $6\mathbb{Z} \subset R'_i$ given that $6\mathbb{Z}$ is a subset of both R_1 and R_2.

Yet, it is clear that R does not have strong light tails for B_N, as we have that

$$\left|B_N \cap \bigcup_{i>L} R_i\right| > \frac{N}{3} - L - 2.$$

Theorem 5.25 points to us the necessity of studying which sieves with weak light tails remain with weak light tails after removing ideals. We now study such properties. Our guiding example is the sieve R given in Example 5.14, which has weak but not strong light tails. The key feature of this sieve, is that one of the congruence classes in R is sieved out by the other classes. We want to show that when this does not happen for any class then R will have strong light tails. We start with the following lemma.

Lemma 5.27. *Let R and R' be Erdős sieves both supported on the set $\mathcal{B} = \{\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{b}_2, \dots\}$ such that $R'_0 = R_0 \cup (x + \mathfrak{b}_0)$ for some $x \notin R_0$, and $R'_i = R_i$ for every $i > 0$ (R_0 is allowed to be the empty set here). If R' has weak light tails for I_N as well as*

$$\lim_{L\to\infty} \overline{d}_I \left((x + \mathfrak{b}_0) \cap \bigcup_{i>L} R_i \setminus \bigcup_{1\le j\le L} R_j \right) = 0, \tag{21}$$

then R has weak light tails for I_N.

Proof. We start by writing $X_L = R_0^c \cap R_1^c \cap \cdots \cap R_L^c$ with $L \geq 1$. Then, the light tails condition for R can be written as

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{|\mathcal{F}_R^c \cap X_L \cap I_N|}{|I_N|} = 0.$$

We decompose $\mathcal{F}_R^c$ into those elements which are in $x + \mathfrak{b}_0$, and those which are not, to obtain

$$|\mathcal{F}_R^c \cap X_L \cap I_N| = |(\mathcal{F}_R^c \cap (x + \mathfrak{b}_0)) \cap X_L \cap I_N| + |(\mathcal{F}_R^c \cap (x + \mathfrak{b}_0)^c) \cap X_L \cap I_N|,$$

and since $(x + \mathfrak{b}_0)^c \cap R_0^c = (R'_0)^c$, we get that

$$(\mathcal{F}_R^c \cap (x + \mathfrak{b}_0)^c) \cap X_L \cap I_N = I_N \cap \bigcup_{i>L} R'_i \setminus \bigcup_{0\le j\le L} R'_j.$$

It follows that because R' has weak light tails, we have

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{|(\mathcal{F}_R^c \cap (x + \mathfrak{b}_0)^c) \cap X_L \cap I_N|}{|I_N|} = 0.$$

On the other hand, because $(x + \mathfrak{b}_0) \cap (R_0)^c = x + \mathfrak{b}_0$, we have that

$$(x + \mathfrak{b}_0) \cap X_L = (x + \mathfrak{b}_0) \setminus \bigcup_{1\le j\le L} R_j.$$

so we have

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{|(\mathcal{F}_R^c \cap (x + \mathfrak{b}_0)) \cap X_L \cap I_N|}{|I_N|} = \lim_{L\to\infty} \overline{d}_I \left((x + \mathfrak{b}_0) \cap \bigcup_{i>L} R_i \setminus \bigcup_{1\le j\le L} R_j \right) = 0,$$

where the second equality is by hypothesis. The result follows. ☐

With this lemma, it is now easy to show the following theorem, which formalizes the idea that if R is a sieve with weak light tails, and no congruence class is being sieved out by the others, then R has strong light tails.

Theorem 5.28. *If R is an Erdős sieve with weak light tails for I_N, such that for every $j \in \mathbb{N}$ and $x_j \in R_j$, we have*

$$\lim_{L\to\infty} \overline{d}_I \left((x_j + \mathfrak{b}_j) \cap \bigcup_{i>L} R_i \right) = 0, \tag{22}$$

then R has strong light tails for I_N.

Proof. We see that if R is an Erdős sieve such that, for every $x_j \in R_j$, Equation (22) holds, then the same will happen for a sieve R' obtained from R by removing an ideal. If R satisfies Equation (22), then Equation (21) also holds for R, so if R is a sieve with weak light tails, and R' is the sieve obtained by removing $x_j + \mathfrak{b}_j$ from R, Lemma 5.27 shows that R' will have weak light tails. The result then follows directly from Theorem 5.25. $\square$

Theorem 5.28 shows that if R has weak light tails for I_N, but not strong light tails, then there must be some $j \in \mathbb{N}$ and $x_j \in R_j$ such that $(x_j + \mathfrak{b}_j)$ has a subset of positive I_N-density that is sieved out by the R_i with $i \neq j$. If R is such a sieve, then by taking some y such that $x_j \notin y + R_j$, we would expect the sieve R' defined by $R'_j = y + R_j$ and $R'_i = R_i$ for $i \neq j$ not to have weak light tails for I_N.

It is natural to ask for the inverse of this, that is, if we have a sieve R with weak light tails for I_N such that translations of R_i preserve the weak light tails condition, must it have strong light tails? Indeed, this is the case, as we now show.

Theorem 5.29. *Let R be an Erdős sieve with weak light tails with respect to some Følner sequence I_N. Suppose that for any finite set $A \subset \mathbb{N}$ and choice of $x_i \in \mathcal{O}_K$ for $i \in A$, the sieve R' defined by $R'_i = x_i + R_i$ with $i \in A$, and $R'_i = R_i$ otherwise, also has weak light tails. Then R has strong light tails for I_N.*

Proof. We want to show that for any M, the sieve R^M defined by $R^M_i = R_{M+i}$ has weak light tails, and then apply Theorem 5.25. This is equivalent to showing that

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i>L} R_i \setminus \bigcup_{M<j\leq L} R_j \right) = 0.$$

Our hypothesis implies that for any choice of $x_j \in \mathcal{O}_K$ with $1 \leq j \leq M$ we have

$$\lim_{L\to\infty} \overline{d}_I \left(\bigcup_{i>L} R_i \setminus \left(\bigcup_{1\leq j\leq M} (x_j + R_j) \cup \bigcup_{M<j\leq L} R_j \right) \right) = 0.$$

Let G_M be the finite set defined by

$$G_M = \prod_{j=1}^{M} \mathcal{O}_K/\mathfrak{b}_j.$$

Then, from the equality

$$\bigcup_{i>L} R_i \setminus \left(\bigcup_{1\leq j\leq M} (x_j + R_j) \cup \bigcup_{M<j\leq L} R_j \right) = \left(\bigcup_{i>L} R_i \cap \bigcap_{M<j\leq L} R_j^c \right) \cap \bigcap_{1\leq j\leq M} (x_j + R_j)^c,$$

it is clear that each of these sets is contained in $\bigcup_{i>L} R_i \setminus \bigcup_{M<j\leq L} R_j$, and that the result will follow if we show that

$$\bigcup_{(x_1,\ldots,x_m)\in G_M} \left(\bigcup_{1\leq j\leq M} (x_j + R_j)^c \right) = \mathcal{O}_K.$$

To show this, let y_j be a collection of elements of $\mathcal{O}_K$ chosen so that $y_j \in R_j^c$ for every $1 \leq j \leq M$. Let y be any arbitrary element of $\mathcal{O}_K$ and pick the unique $x \in G_M$ such that $x_j \equiv y - y_j \mod \mathfrak{b}_j$ for every j. Then $y \in x_j + y_j + \mathfrak{b}_j \subset x_j + R_j^c$ for every j, which shows that y is in $\bigcap_{1 \leq j \leq M} (x_j + R_j)^c$. This concludes the proof. $\square$

Remark 5.30. Since any change in finitely many R_i does not affect the strong light tails property the result in Theorem 5.29 is an equivalence.

Additionally, this shows that if R has weak light tails for some I_N, but not strong light tails, then there is a finite set $A \subset \mathbb{N}$ and a collection of $x_i \in \mathcal{O}_K$ for $i \in A$ such that the sieve R' defined by $R'_i = x + R_i$ if $i \in A$ (and $R'_i = R_i$ otherwise) does not have weak light tails for I_N.

6. A Theory of Sieves

In this section we study sieves as objects of interest in themselves. Our overarching objective is to study sets of the form $\mathcal{F}_R$ for some sieve R, so one of the first questions we ask is when two sieves produce the same set $\mathcal{F}_R$. To study this, we define two sieves to be equivalent if they sieve out the same elements.

Definition 6.1. *We say two sieves R and R' over the same étale $\mathbb{Q}-$algebra are equivalent and write $R \sim R'$ if $\mathcal{F}_R = \mathcal{F}_{R'}$.*

We want to know when two sieves are equivalent. We start with the following consideration. Notice that $2\mathbb{Z}$ and $\{0,2\} + 4\mathbb{Z}$ are the same sets in $\mathbb{Z}$. Therefore, two sieves R and R', defined by $R_1 = 2\mathbb{Z}$, $R'_1 = \{0,2\} + 4\mathbb{Z}$ and $R_i = R'_i$ for $i > 1$ will be equivalent. Yet, they will not be the same, as $\mathcal{B}_R \neq \mathcal{B}_{R'}$. This leads us to define the notion of a minimal sieve, whose elements of $\mathcal{B}_R$ are as small as they possibly can norm wise (so for example, R' cannot be a minimal sieve, as we could replace R'_1 by $2\mathbb{Z}$ and $2 < 4$).

This section is divided in two subsections. In the first, we define an operation of dilation which changes a sieve into another sieve whose ideals in the support have bigger norm (as in the example above). We then define the notion of a contraction, which is the opposite of a dilation. A minimal sieve is then defined as a sieve that cannot be contracted. In Lemma 6.6 we show that the weak and strong light tails properties are preserved by dilations and contractions. We then show in Lemma 6.10 that every sieve can be repeatedly contracted until we obtain an equivalent minimal sieve. This shows that when we want to make statements about the uniqueness of $\mathcal{F}_R$, we have to restrict ourselves to the class of minimal sieves. Once we do this, we can make multiple claims about the uniqueness of $\mathcal{F}_R$, depending on whether R has weak or strong light tails, that are summarized in Theorem 6.21. What distinguishes the case where R has weak light tails, and the case where R has strong light tails, is Theorem 6.19, which states that when R has strong light tails and $R' \sim R$, then R' must also have strong light tails. In Remark 6.18 we show that the same does not happen if R only has weak light tails.

In the second subsection, we define the union of two sieves R and R', which corresponds to the sieve W such that $\mathcal{F}_W$ is the intersection of $\mathcal{F}_R$ and $\mathcal{F}_{R'}$. We make use of the results in the first subsection to define this object for a number of sieves, without having to restrict ourselves to the case where $\mathcal{B}_R = \mathcal{B}_{R'}$. We show in Lemma 6.26 that this operation preserves weak light tails, which allows us to show that the $R-$free numbers for certain sieves have the expected density, simply by counting the size of $|R_i|$ for each i. A well known example is given in Example 6.29. This is significant, since in general such counting can only provide us with an heuristic, as Theorem 3.21 shows.

6.1. **Minimal Sieves.**

Given a sieve R, there are two main ways of producing an equivalent sieve R'. The composition of these will be what we will call a dilation. The first operation to consider, is one where we take a sieve R, and for each $\mathfrak{b}_i \in \mathcal{B}_R$ choose $\mathfrak{c}_i$ such that $\mathfrak{b}_i \mid \mathfrak{c}_i$ and the $\mathfrak{c}_i$ are pairwise coprime. Then, $\mathfrak{b}_i + \mathfrak{c}_i = \mathfrak{b}_i$, and so a sieve R' supported on $\mathcal{B}_{R'} = \{\mathfrak{c}_i : i \in \mathbb{N}\}$ with $R'_i = R_i + \mathfrak{c}_i = R_i$ is equivalent to R. We give two examples of this operation.

Example 6.2. Let R be the squarefree sieve supported on $\mathcal{B}_R = \{p^2\mathbb{Z} : p \text{ prime}\}$ and such such that $R_p = p^2\mathbb{Z}$. It is equivalent to the sieve R' supported on $\mathcal{B}_{R'} = \{p^3\mathbb{Z} : p \text{ prime}\}$ and defined by

$$R'_p = \{jp^2 : 0 \leq j < p\} + p^3\mathbb{Z} = p^2\mathbb{Z} + p^3\mathbb{Z}.$$

Alternatively, let q_i denote the $i-$th prime congruent to $1 \mod 4$, and W the sieve $W_i = q_i^2\mathbb{Z}$. Taking r_i to be the primes congruent to $3 \mod 4$, it is clear that W is equivalent to the sieve W' supported on $\mathcal{B}_{W'} = \{q_i^2 r_i^2 : i \in \mathbb{N}\}$ and defined by

$$W'_i = \{jq_i^2 : 0 \leq j < r_i^2\} + q_i^2 r_i^2\mathbb{Z} = q_i^2\mathbb{Z} + q_i^2 r_i^2\mathbb{Z}.$$

The second operation we can apply on R is as follows. Take a finite set $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_m\}$ of elements of $\mathcal{B}_R$, and write $\mathfrak{c} = \prod_{i=1}^m \mathfrak{b}_i$. Consider the sieve R' supported on $(\mathcal{B}_R \cup \{\mathfrak{c}\}) \setminus \{\mathfrak{b}_1, \ldots, \mathfrak{b}_m\}$ where we have removed from R all the $R_{\mathfrak{b}_i}$, and instead have $R'_{\mathfrak{c}} = \bigcup R_{\mathfrak{b}_i} + \mathfrak{c}$. Since there was some $x_i \notin R_{\mathfrak{b}_i}$ for every i, the Chinese Remainder Theorem guarantees that $R'_{\mathfrak{c}} \neq \mathcal{O}_K$. Take the following example.

Example 6.3. Let R be an Erdős $\mathcal{B}-$free system. Let $\mathcal{P}$ be a partition of $\mathbb{N}$ into finite sets (so that $\mathcal{P}$ is a collection of disjoint finite sets that together cover $\mathbb{N}$). For any element A of $\mathcal{P}$, let $b_A := \prod_{i\in A} b_i$, and write $\mathcal{B}' := \{b_A\mathbb{Z} : A \in \mathcal{P}\}$. The sieve R' supported on $\mathcal{B}'$, and defined by $R'_A = \bigcup_{i\in A} b_i\mathbb{Z} + b_A\mathbb{Z}$ is equivalent to R, as $b_i\mathbb{Z} + b_A\mathbb{Z} = b_i\mathbb{Z}$ for any $i \in A$, and so

$$\mathcal{F}^c_{R'} = \bigcup_{A\in\mathcal{P}} R'_A = \bigcup_{A\in\mathcal{P}} \bigcup_{i\in A} R_i = \bigcup_{i\in\mathbb{N}} R_i = \mathcal{F}^c_R.$$

We can combine these two operation on sieves, to obtain the following general operation that we call *dilation*.

Definition 6.4. *Let R be a sieve supported on $\mathcal{B}_R$, and consider some partition $\mathcal{P}$ of $\mathbb{N}$ into finite sets. For every $A \in \mathcal{P}$, let $\mathfrak{a}_A$ be some ideal satisfying $(\mathfrak{a}_A, \mathfrak{b}_i) = 1$ if $i \notin A$, and such that for any $B \in \mathcal{P}$, $(\mathfrak{a}_A, \mathfrak{a}_B) = 1$ if $A \neq B$. Finally, let $\mathfrak{c}(A) = lcm(\{\mathfrak{a}_A\} \cup \{\mathfrak{b}_i : i \in A\})$ and $\mathcal{B}' := \{\mathfrak{c}(A) : A \in \mathcal{P}\}$. Given this initial data we define the corresponding dilation of R to be the sieve R' supported on $\mathcal{B}'$ and defined by*

$$R'_A = \bigcup_{i\in A} R_i + \mathfrak{c}(A).$$

We remark that since $\mathfrak{b}_i \mid \mathfrak{c}(A)$ for every $i \in A$, R'_A is just the union of all the R_i with $i \in A$. The choice to write the "$+\mathfrak{c}(A)$" is a stylistic choice, so that we always write $R_{\mathfrak{b}}$ in the form $R_{\mathfrak{b}} = S + \mathfrak{b}$.

In order to show that a dilation does indeed define a sieve, we have to show that the elements of $\mathcal{B}'$ are pairwise coprime. To see this, note that if $(\mathfrak{c}(A), \mathfrak{c}(B)) \neq 1$, then one of the following four things must happen: either $(\mathfrak{a}_A, \mathfrak{a}_B) \neq 1$, $(\mathfrak{a}_A, \mathfrak{b}_j) \neq 1$ for some $j \in B$,$(\mathfrak{a}_B, \mathfrak{b}_i) \neq 1$ for some $i \in A$, or $(\mathfrak{b}_i, \mathfrak{b}_j) \neq 1$ for some $i \in A$ and $j \in B$. The first three options cannot happen by the definition of dilation, and the last one cannot happen since the elements of $\mathcal{B}_R$ are assumed to be pairwise coprime. Furthermore, R' is equivalent to R, since $\mathfrak{c}(A) + \mathfrak{b}_i = \mathfrak{b}_i$ for any $i \in A$, and so

$$\mathcal{F}^c_{R'} = \bigcup_{A\in\mathcal{P}} R'_A = \bigcup_{A\in\mathcal{P}} \bigcup_{i\in A} R_i = \mathcal{F}^c_R.$$

On the other hand, given some sieve R and $\mathfrak{b} \in \mathcal{B}_R$, it may be possible to write

$$R_{\mathfrak{b}} = \bigcup_{i \in A} S_i + \mathfrak{b}_i,$$

where $A \subset \mathbb{N}$ and $S_i \subset \mathcal{O}_K$ are non-empty finite sets, and the $\mathfrak{b}_i$ form a set of pairwise coprime ideals distinct from $\mathfrak{b}$ such that $\mathfrak{b}_i \mid \mathfrak{b}$. In this case, we can define a sieve R' supported on $\mathcal{B} \setminus \{\mathfrak{b}\} \cup \{\mathfrak{b}_1, \ldots, \mathfrak{b}_{|A|}\}$ such that $R'_{\mathfrak{b}_i} = S_i + \mathfrak{b}_i$, and $R \sim R'$. We call this operation a *contraction* of R, since it is always the inverse of a dilation. Indeed, if R is our initial sieve, and we choose a finite set A, and an ideal $\mathfrak{b}$ such that $\operatorname{lcm}(\{\mathfrak{b}_i : i \in A\}) | \mathfrak{b}$, then by dilating we will obtain a sieve R' supported on $\mathcal{B} \cup \{\mathfrak{b}\} \setminus \{\mathfrak{b}_1, \ldots, \mathfrak{b}_{|A|}\}$ such that $R'_{\mathfrak{b}} = \bigcup_{i \in A} R_i + \mathfrak{b}_i$. By contracting, it is clear we get the original sieve.

More generally, because a dilation is defined as an operation on all ideals simultaneously, we also define a (general) contraction as the repetition of this operation for each ideal for which it is possible.

Definition 6.5. *Let R be a sieve. We say a sieve R' is a contraction of R if there is a partition $\mathcal{P}$ of $\mathcal{B}_{R'}$ into finite sets and a bijection from $\mathcal{B}_R$ to $\mathcal{P}$ sending $\mathfrak{b}$ to $A_{\mathfrak{b}} \in \mathcal{P}$, such that either $A_{\mathfrak{b}} = \{\mathfrak{b}\}$ and $R_{\mathfrak{b}} = R'_{\mathfrak{b}}$ or*

$$R_{\mathfrak{b}} = \bigcup_{\mathfrak{b}' \in A_{\mathfrak{b}}} R'_{\mathfrak{b}'},$$

where for every $\mathfrak{b}' \in A_{\mathfrak{b}}$ we have that $\mathfrak{b}' \mid \mathfrak{b}$, but $\mathfrak{b}' \neq \mathfrak{b}$.

We show that dilations and contractions preserve the Erdős and light tail conditions for any Følner sequence I_N.

Lemma 6.6. *Let R be a sieve and R' a dilation of R. Then R' will be an Erdős sieve with weak (respectively, strong light tails) if and only if R is Erdős and has weak (respectively, strong light tails) for a Følner sequence I_N.*

Proof. We will use the same notation as in Definition 6.4. Since R' is a dilation of R, there must exist some partition $\mathcal{P}$ of $\mathbb{N}$ into non-empty finite sets, and some collection $\{\mathfrak{a}_A : A \in \mathcal{P}\}$, such that writing $\mathfrak{c}(A) = \operatorname{lcm}(\{\mathfrak{a}_A\} \cup \{\mathfrak{b}_i : i \in A\})$ and $\mathcal{C} = \{\mathfrak{c}(A) : A \in \mathcal{P}\}$, the sieve R' is supported on $\mathcal{C}$ and for any $\mathfrak{c} \in \mathcal{C}$ we have

$$R'_{\mathfrak{c}} = \bigcup_{\mathfrak{b} | \mathfrak{c}} R_{\mathfrak{b}} + \mathfrak{c}.$$

Take any $\mathfrak{c} \in \mathcal{C}$, and suppose that $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$ are the element of $\mathcal{B}_R$ that divide $\mathfrak{c}$. Then,

$$\operatorname{vol}(R'_{\mathfrak{c}}) = \frac{|\bigcup_{i=1}^k R_{\mathfrak{b}_i} + \mathfrak{c}|}{N(\mathfrak{c})} \leq \sum_{i=1}^k \frac{|R_{\mathfrak{b}_i} + \mathfrak{c}|}{N(\mathfrak{c})} = \sum_{i=1}^k \operatorname{vol}(R_{\mathfrak{b}_i}).$$

Since every $\mathfrak{b}$ divides at least one $\mathfrak{c}$, it follows that $\sum_{\mathfrak{c} \in \mathcal{C}} \operatorname{vol}(R'_{\mathfrak{c}}) \leq \sum_{\mathfrak{b} \in \mathcal{B}} \operatorname{vol}(R_{\mathfrak{b}})$, so R' is Erdős if R is. On the other hand, if $x \notin R'_{\mathfrak{c}}$, then $x \notin R_{\mathfrak{b}}$ for every $\mathfrak{b} \mid \mathfrak{c}$. It follows that

$$\prod_{\mathfrak{b} | \mathfrak{c}} (1 - \operatorname{vol}(R_{\mathfrak{b}})) \leq (1 - \operatorname{vol}(R'_{\mathfrak{c}})),$$

and so $\prod_{\mathfrak{b} \in \mathcal{B}} (1 - \operatorname{vol}(R_{\mathfrak{b}})) \leq \prod_{\mathfrak{c} \in \mathcal{C}} (1 - \operatorname{vol}(R'_{\mathfrak{c}}))$, which implies that R is Erdős if R' is.

Next we show that R' has weak/strong light tails if and only if R does. By Lemmas 5.15 and 5.16, we are free to reorder $\mathcal{B}_R$ and $\mathcal{C}$, so we order them in such a way, that there is a function f defined by the rule that $\mathfrak{b}_i \mid \mathfrak{c}_j$ is equivalent to $f(j) \leq i < f(j+1)$. Then we have

$$R'_L = \bigcup_{f(L) \leq j < f(L+1)} R_j,$$

and so

$$\bigcup_{i>L} R'_i = \bigcup_{j \geq f(L+1)} R_j.$$

Similarly, we have that

$$\bigcup_{i \leq L} R'_i = \bigcup_{j < f(L+1)} R_j,$$

which gives the equality

$$\bigcup_{i>L} R'_i \setminus \bigcup_{j \leq L} R'_j = \bigcup_{i \geq f(L+1)} R_i \setminus \bigcup_{j \leq f(L+1)} R_j,$$

We now show the equivalence in the case of strong light tails. If R is Erdős and has strong light tails, then the right hand side of

$$\overline{d}_I \left(\bigcup_{i>L} R'_i \right) = \overline{d}_I \left(\bigcup_{j \geq f(L+1)} R_j \right), \tag{23}$$

is a subsequence of a sequence that converges to 0, and so the left hand side must converge to 0, that is, R' must have strong light tails. On the other hand, notice that the sequence

$$a_L = \overline{d}_I \left(\bigcup_{i>L} R_i \right)$$

is monotonically non increasing. If R' has strong light tails, then Equation (23) shows that a_L has a subsequence that converges to 0, which implies that the entire sequence must converge to 0, so R must also have strong light tails.

The equivalence in the case of weak light tails follows analogously, only having to point out that the sequence

$$b_L = \overline{d}_I \left(\bigcup_{i>L} R_i \setminus \bigcup_{j \leq L} R_j \right)$$

is also monotonically non increasing, since

$$\bigcup_{i>L+1} R_i \setminus \bigcup_{j \leq L+1} R_j \subset \bigcup_{i>L} R_i \setminus \bigcup_{j \leq L} R_j$$

for every L. □

This motivates us to define the notion of a *minimal* sieve.

Definition 6.7. *We say a sieve R is minimal if for every $\mathfrak{b} \in \mathcal{B}_R$, there is no collection of pairwise coprime ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_r$ distinct from $\mathfrak{b}$ with $\mathfrak{b}_i \mid \mathfrak{b}$ for every $1 \leq i \leq r$, and finite sets $S_i \subset \mathcal{O}_K$ such that*

$$R_{\mathfrak{b}} = \bigcup_{i=1}^{r} (S_i + \mathfrak{b}_i).$$

Equivalently, we say that R is minimal if it has no contractions or, if there is no sieve R' such that R is a dilation of R'.

Remark 6.8. Throughout this section, we will always assume that given a sieve R and $\mathfrak{b} \in \mathcal{B}_R$, we have $R_{\mathfrak{b}} \neq \emptyset$. The reason is as follows: consider a sieve R' such that $\mathcal{B}_{R'} = \mathcal{B}_R \cup A$, with $R'_{\mathfrak{b}} = R_{\mathfrak{b}}$ if $\mathfrak{b} \in \mathcal{B}_R$ and $R'_{\mathfrak{b}} = \emptyset$ if $\mathfrak{b} \in A$, to be a dilation of R. A sieve obtained from R by removing those $\mathfrak{b}$ in $\mathcal{B}_R$ such that $R_{\mathfrak{b}} = \emptyset$ is then a contraction of R. In particular, a minimal sieve R will then necessarily, as a sieve which has no contractions, be a sieve such that $R_{\mathfrak{b}} \neq \emptyset$ for every $\mathfrak{b} \in \mathcal{B}_R$.

Example 6.9. The simplest example of minimal sieves are those coming from $\mathcal{B}-$free systems. Say that $R_{\mathfrak{b}} = \mathfrak{b}$ for every $\mathfrak{b} \in \mathcal{B}_R$. Taking any $\mathfrak{b}' \mid \mathfrak{b}$ distinct from $\mathfrak{b}$, we cannot possibly have $x + \mathfrak{b}' \subset R_{\mathfrak{b}} = \mathfrak{b}$, given that $x + \mathfrak{b}'$ will contain $[\mathfrak{b} : \mathfrak{b}'] > 1$ congruence classes mod $\mathfrak{b}$.

For another example, consider a sieve R given by $R_p = p^2\mathbb{Z}$ for every prime different from 5, and such that $R_5 = \{0, 5, 6, 10, 15, 20\} + 25\mathbb{Z}$. Then, $R_5 = 5\mathbb{Z} \cup (\{6\} + 25\mathbb{Z})$, which cannot be written as the union of congruence classes mod 5 (the only number that divides 25 distinct from 1 and 25). Since $p^2\mathbb{Z}$ cannot be written as the union of congruence classes mod p, it follows that R is minimal.

Finally, let R be the sieve defined by $R_1 = \{1, 2, 7, 13, 17, 19, 25\} + 30\mathbb{Z}$, and $R_i = p_{i+3}^2\mathbb{Z}$ for $i \geq 2$. Then, $R_1 = (1 + 6\mathbb{Z}) \cup (2 + 15\mathbb{Z})$. Since 6 and 15 are not coprime, and no other combination of the form $x + b\mathbb{Z}$ is contained in R_1 with $b \mid 30$, it follow that R is minimal, despite of the fact that for every $x \in R_1$, there is some $\mathfrak{a} \mid 30\mathbb{Z}$ such that $x + \mathfrak{a} \subset R_1$.

For an example of non-minimal sieves, let W be a sieve such that $W_5 = \{0, 5, 10, 15, 20\} + 25\mathbb{Z}$. We can write $W_5 = 5\mathbb{Z}$, so W is not minimal. Alternatively, consider a sieve W' such that $W'_6 = \{0, 2, 3, 4\} + 6\mathbb{Z}$. Since $W'_6 = 2\mathbb{Z} \cup 3\mathbb{Z}$, we also see that W' is not minimal.

Intuitively, for any sieve R, we can keep contracting it, until we end up with a minimal sieve. In the next lemma we show that this is indeed the case.

Lemma 6.10. *Let R be an Erdős sieve. There is a minimal sieve R' equivalent to R. Additionally, if R has weak/strong light tails for some Følner sequence I_N, then so does R'.*

Proof. Fix any $\mathfrak{b} \in \mathcal{B}_R$. The sieve R' can be constructed recursively using the following algorithm. If $R_{\mathfrak{b}}$ cannot be written as the union of some $W_{\mathfrak{b}'}$, with $\mathfrak{b}'$ in some finite set $A_{\mathfrak{b}}$ such that $\mathfrak{b}' \mid \mathfrak{b}$ and $\mathfrak{b}' \neq \mathfrak{b}$, then we add $\mathfrak{b}$ to $\mathcal{B}_{R'}$ and set $R'_{\mathfrak{b}} := R_{\mathfrak{b}}$. Otherwise, we replace $\mathfrak{b}$ in $\mathcal{B}_R$ by all of the $\mathfrak{b}' \in A_{\mathfrak{b}}$, and set $R_{\mathfrak{b}'} := W_{\mathfrak{b}'}$. Repeatedly applying this process now for every $R_{\mathfrak{b}'}$, we will eventually terminate in such a way that no $R'_{\mathfrak{b}'}$ can be further contracted for any $\mathfrak{b}' \in \mathcal{B}_{R'}$ that divides our original $\mathfrak{b}$, and that the union of $R'_{\mathfrak{b}'}$ for all such $\mathfrak{b}'$ equals $R_{\mathfrak{b}}$. The number of steps until termination is finite, as it cannot be greater than the number of divisors of $\mathfrak{b}$. Repeating the process now for each $\mathfrak{b} \in \mathcal{B}_R$ will produce the entire minimal sieve R'.

Letting $A_{\mathfrak{b}}$ be the set of all $\mathfrak{b}' \in \mathcal{B}_{R'}$ that divide $\mathfrak{b}$, we will have that the collection $\{A_{\mathfrak{b}}\}_{\mathfrak{b} \in \mathcal{B}_R}$ is a partition of $\mathcal{B}_{R'}$, and $R_{\mathfrak{b}} = \bigcup_{\mathfrak{b}' \in A_{\mathfrak{b}}} R'_{\mathfrak{b}'}$. It follows that R is a dilation of R', and so by Lemma 6.6, the sieve R is Erdős with weak/strong light tails for I_N if and only if R' also is. $\square$

Example 6.11. Consider the sieve R with $\mathcal{B}_R = \{p_{2i}p_{2i+1}^2 : i \in \mathbb{N}\}$ defined by

$$R_i = \{jp_{2i+1}^2 : 0 \leq j < p_{2_i}\} + p_{2i}p_{2i+1}^2\mathbb{Z}.$$

This sieve is equivalent to the sieve R' defined by $R'_i = p_{2i+1}^2\mathbb{Z}$, which is minimal (as shown in Example 6.9). Also, since R' has strong light tails (by Lemma 5.3), so does R.

We now want to study the extent to which any particular sieve is uniquely equivalent to a minimal sieve. The weak light tails property will play an important role here. We start with two lemmas, which will help us characterize when $R \sim R'$.

Lemma 6.12. *Let R be an Erdős sieve. If there is some ideal $\mathfrak{c}$ coprime to every element of $\mathcal{B}_R$ such that $y + \mathfrak{c} \subset \mathcal{F}_R^c$ for some $y \in \mathcal{O}_K$, then R does not have weak light tails for any Følner sequence.*

Proof. Fix some integer $L \geq 1$. Due to our hypothesis, we have that for every N,

$$\left| \left(\bigcup_{i=L+1}^{\infty} R_i \setminus \bigcup_{i=1}^{L} R_i \right) \cap I_N \right| \geq \left| \left((y + \mathfrak{c}) \setminus \bigcup_{i=1}^{L} R_i \right) \cap I_N \right|.$$

By applying Lemma 3.14 (for the sets $(y + \mathfrak{c})^c, R_1, \ldots, R_L$), we get that

$$\lim_{N \to \infty} \frac{1}{|I_N|} \left| \left((y + \mathfrak{c}) \setminus \bigcup_{i=1}^{L} R_i \right) \cap I_N \right| = \frac{1}{N(\mathfrak{c})} \prod_{i=1}^{L} \left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)} \right).$$

Since R is Erdős, the limit converges to some positive constant bigger than 0 as we take L to infinity, which concludes the proof. $\square$

Remark 6.13. Assume that R is an Erdős sieve for which there is some $x \notin R_1$ such that

$$x + \mathfrak{b}_1 \subset \bigcup_{i \geq 2} R_i.$$

Then, because $(x + \mathfrak{b}_1) \cap R_1 = \emptyset$, the computation done in Lemma 6.12 also implies that R does not have weak light tails for any Følner sequence.

Lemma 6.14. *Let R and R' be Erdős sieves such that R has weak light tails for some Følner sequence I_N. If $R \sim R'$, then for any $\mathfrak{b}' \in \mathcal{B}_{R'}$ we have*

$$R'_{\mathfrak{b}'} \subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}}.$$

In particular, there must be some $\mathfrak{b} \in \mathcal{B}_R$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$.

Proof. We start by fixing some $\mathfrak{b}' \in \mathcal{B}_{R'}$ and show that if there is no $\mathfrak{b} \in \mathcal{B}_R$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$, then $R \not\sim R'$. Indeed, if this is the case, then $(\mathfrak{b}', \mathfrak{b}) = 1$ for every $\mathfrak{b} \in \mathcal{B}_R$. Take any b such that $b + \mathfrak{b}' \subset R_{\mathfrak{b}'}$. Lemma 5.19 implies that $\mathcal{F}_R \cap (b + \mathfrak{b}') \neq \emptyset$, but $\mathcal{F}_{R'} \subset (b + \mathfrak{b}')^c$. It follows that $\mathcal{F}_R \not\subset \mathcal{F}_{R'}$, so $R \not\sim R'$.

In order to prove the result, we show that if there is some $\mathfrak{b}' \in \mathcal{B}_{R'}$ such that

$$R'_{\mathfrak{b}'} \not\subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}},$$

then $R \not\sim R'$. To do this, we assume that $R \sim R'$, and we will reach a contradiction.

If $R \sim R'$, then $R'_{\mathfrak{b}'} \subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R} R_{\mathfrak{b}}$. Writing $\mathfrak{c} = \operatorname{lcm}(\{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1\} \cup \{\mathfrak{b}'\})$, and taking some x such that $x \in R'_{\mathfrak{b}'} \setminus \bigcup_{(\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}}$, we have that $x + \mathfrak{b}' \subset R'_{\mathfrak{b}'}$ and $(x + \mathfrak{b}) \cap R_{\mathfrak{b}} = \emptyset$ for every $\mathfrak{b}$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$. Therefore,

$$(x + \mathfrak{c}) \cap \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}} = \emptyset \tag{24}$$

and it follows that

$$x + \mathfrak{c} \subset R'_{\mathfrak{b}'} \setminus \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}} \subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R} R_{\mathfrak{b}} \setminus \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}} = \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') = 1} R_{\mathfrak{b}}. \tag{25}$$

However, this cannot happen. Let W be the sieve supported on the set

$$\mathcal{B}_W = \{\mathfrak{c}\} \cup \{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') = 1\}$$

and defined by

$$W_{\mathfrak{c}} = \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}} + \mathfrak{c},$$

with $W_{\mathfrak{b}} = R_{\mathfrak{b}}$ if $\mathfrak{b} \in \mathcal{B}_W$ is different from $\mathfrak{c}$. We see that W is a dilation of R. Therefore, if R has weak light tails for I_N, so does W by Lemma 6.6. Yet, we can rewrite Equation (24) as $(x + \mathfrak{c}) \cap W_{\mathfrak{c}} = \emptyset$ and Equation (25) as $(x + \mathfrak{c}) \subset \bigcup_{\mathfrak{b} \neq \mathfrak{c}} W_{\mathfrak{b}}$, which by Remark 6.13 implies that W does not have weak light tails for any Følner sequence. We obtain the desired contradiction. $\square$

We can use Lemma 6.14 to provide a property that allows us to determine whether two sieves with weak light tails are equivalent.

Lemma 6.15. *Let R and R' be Erdős Sieves with weak light tails for some Følner sequence (not necessarily common to both). Then $R \sim R'$ holds if and only if for every $\mathfrak{b}' \in \mathcal{B}_{R'}$, there is some $\mathfrak{b} \in \mathcal{B}_R$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$, with*

$$R'_{\mathfrak{b}'} \subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}},$$

and if for every $\mathfrak{b} \in \mathcal{B}_R$, there is some $\mathfrak{b}' \in \mathcal{B}_{R'}$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$, with

$$R_{\mathfrak{b}} \subset \bigcup_{\mathfrak{b}' \in \mathcal{B}_{R'} : (\mathfrak{b}, \mathfrak{b}') \neq 1} R'_{\mathfrak{b}'}.$$

Proof. If $R \sim R'$, then because both sieves have weak light tails, the fact that R and R' satisfy this property follows directly from Lemma 6.14. Hence, all we have to show is that if this property holds, then $R \sim R'$. But this is clear, since for every $\mathfrak{b} \in \mathcal{B}_R$ and $\mathfrak{b}' \in \mathcal{B}_{R'}$, we are assuming that $R_\mathfrak{b} \subset \mathcal{F}_{R'}^c$ and $R'_{\mathfrak{b}'} \subset \mathcal{F}_R^c$, which shows that $\mathcal{F}_R = \mathcal{F}_{R'}$, and so $R \sim R'$. □

With this, we can show that if R has weak light tails, then the minimal sieve R' obtained by repeated contraction of R is the unique minimal sieve equivalent to R with weak light tails. In order to prove this result, we need the following lemma.

Lemma 6.16. *Let $\mathfrak{b}$ be an ideal of $\mathcal{O}_K$, and $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$ a collection of ideals such that $(\mathfrak{b}, \mathfrak{b}_i) \neq 1$. Let $R_{\mathfrak{b}_i}$ be a collection of congruence classes modulo $\mathfrak{b}_i$ such that $R_{\mathfrak{b}_i} \neq \mathcal{O}_K$. If*

$$x + \mathfrak{b} \subset \bigcup_{i=1}^{r} R_{\mathfrak{b}_i},$$

then there is some j such that $x + \mathfrak{b} \subset R_{\mathfrak{b}_j}$.

Proof. The first step is to show that if $x + \mathfrak{b} \subset R_{\mathfrak{b}_1} \cup R_{\mathfrak{b}_2}$, then $x + \mathfrak{b}$ is contained in either $R_{\mathfrak{b}_1}$ or $R_{\mathfrak{b}_2}$. Once we have this, then for any collection of ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$, we can consider $\mathfrak{c} = \prod_{i=2}^r \mathfrak{b}_i$ and $R_\mathfrak{c} = \bigcup_{i=2}^r R_{\mathfrak{b}_i}$, such that the condition $x + \mathfrak{b} \subset \bigcup_i R_{\mathfrak{b}_i}$ becomes $x + \mathfrak{b} \subset R_{\mathfrak{b}_1} \cup R_\mathfrak{c}$. This will now imply that either $x + \mathfrak{b} \subset R_{\mathfrak{b}_1}$ or $x + \mathfrak{b} \subset R_\mathfrak{c}$. Using induction on $R_\mathfrak{c}$, we conclude that there is some j such that $x + \mathfrak{b} \subset R_{\mathfrak{b}_j}$.

By replacing $R_{\mathfrak{b}_i}$ by $R_{\mathfrak{b}_i} - x$, we reduce the problem to showing that if $\mathfrak{b} \subset R_{\mathfrak{b}_1} \cup R_{\mathfrak{b}_2}$, then $\mathfrak{b}$ is a subset of either $R_{\mathfrak{b}_1}$ or $R_{\mathfrak{b}_2}$. To show this, let us assume to the contrary, that $\mathfrak{b}$ is not a subset of either of these sets, but it is a subset of their union. Let $\phi : \mathcal{O}_K/\mathfrak{b}_1\mathfrak{b}_2 \to \mathcal{O}_K/\mathfrak{b}_1 \oplus \mathcal{O}_K/\mathfrak{b}_2$ be the map that sends x to $(x + \mathfrak{b}_1, x + \mathfrak{b}_2)$. Since $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are coprime, the Chinese Remainder Theorem implies that it is a bijection. Let $\iota : \mathfrak{b} \to \mathcal{O}_K/\mathfrak{b}_1\mathfrak{b}_2$ be the map that sends x to $x + \mathfrak{b}_1\mathfrak{b}_2$. Our hypothesis is that $\iota(\mathfrak{b}) \subset \phi^{-1}(R_{\mathfrak{b}_1} \times \mathcal{O}_K/\mathfrak{b}_2) \cup \phi^{-1}(\mathcal{O}_K/\mathfrak{b}_2 \times R_{\mathfrak{b}_2})$, but there are $x, y \in \mathfrak{b}$ such that $x \notin R_{\mathfrak{b}_1}$ and $y \notin R_{\mathfrak{b}_2}$.

Notice that the map ϕ is a bijection between $\iota(\mathfrak{b})$ and elements of the form $(x + \mathfrak{b}_1, y + \mathfrak{b}_2)$ with $x, y \in \mathfrak{b}$. Indeed, it is injective by the Chinese Remainder Theorem, and it is clear that all elements of $\phi(\iota(\mathfrak{b}))$ are of the form given. So, to see the bijectivity, it is enough to check that both the domain and image have the same size. The image of $\mathfrak{b}$ inside of $\mathcal{O}_K/\mathfrak{c}$ for any ideal $\mathfrak{c}$ has cardinality $N(\mathfrak{c})/N\left[(\mathfrak{b}, \mathfrak{c})\right]$ (here $N\left[(\mathfrak{b}, \mathfrak{c})\right]$ corresponds to the norm of the ideal $\mathfrak{b} + \mathfrak{c}$), so the result follows from

$$\frac{N(\mathfrak{b}_1\mathfrak{b}_2)}{N\left[(\mathfrak{b}, \mathfrak{b}_1\mathfrak{b}_2)\right]} = \frac{N(\mathfrak{b}_1)}{N\left[(\mathfrak{b}, \mathfrak{b}_1)\right]} \frac{N(\mathfrak{b}_2)}{N\left[(\mathfrak{b}, \mathfrak{b}_2)\right]},$$

which is a consequence of $\mathfrak{b}_1, \mathfrak{b}_2$ being coprime, and multiplicativity of the norm.

Therefore, it cannot happen that there are $x, y \in \mathfrak{b}$ such that $x \notin R_{\mathfrak{b}_1}$ and $y \notin R_{\mathfrak{b}_2}$, since under these conditions $\phi^{-1}(x + \mathfrak{b}_1, y + \mathfrak{b}_2)$ is not in $\phi^{-1}(R_{\mathfrak{b}_1} \times \mathcal{O}_K/\mathfrak{b}_2) \cup \phi^{-1}(\mathcal{O}_K/\mathfrak{b}_2 \times R_{\mathfrak{b}_2})$, but we have just shown it must be in $\iota(\mathfrak{b})$. □

We can now show that if R has weak light tails for I_N, we can contract it until we get a minimal sieve with weak light tails for I_N, and that this sieve is the unique minimal sieve equivalent to R that has weak light tails for some Følner sequence.

Theorem 6.17. *Let R be an Erdős sieve. If there exists a minimal sieve R' equivalent to R with weak light tails for any Følner sequence I_N, then it is the unique minimal sieve with weak light tails for I_N that is equivalent to R.*

Proof. Let W and W' be two minimal sieves equivalent to R with weak light tails for some (possibly distinct) Følner sequences, supported on the sets $\mathcal{B}_W$ and $\mathcal{B}_{W'}$ respectively. We want to show that $W = W'$. By transitivity, we have $W \sim W'$, and so Lemma 6.15 tells us that for any $\mathfrak{b} \in \mathcal{B}_W$, we have

$$W_{\mathfrak{b}} \subset \bigcup_{\mathfrak{c} \in \mathcal{B}_{W'} : (\mathfrak{b}, \mathfrak{c}) \neq 1} W'_{\mathfrak{c}}.$$

Take any $x \in W_{\mathfrak{b}}$. By Lemma 6.16, we have that $x + \mathfrak{b}$ is contained in some $W'_{\mathfrak{c}}$. Hence, we have that $x + \mathfrak{b} + \mathfrak{c} \subset W'_{\mathfrak{c}} + \mathfrak{c}$, which means that $x + (\mathfrak{b}, \mathfrak{c}) \subset W'_{\mathfrak{c}}$. Since $W \sim W'$, it follows that $x + (\mathfrak{b}, \mathfrak{c}) \subset \mathcal{F}^c_W$. We now show that this implies that $x + (\mathfrak{b}, \mathfrak{c}) \subset W_{\mathfrak{b}}$. Take any $y \in (\mathfrak{b}, \mathfrak{c})$. If $(x + y + \mathfrak{b}) \cap W_{\mathfrak{b}} = \emptyset$, then because $x + y + \mathfrak{b} \subset \mathcal{F}^c_W$, Remark 6.13 would imply that W does not have weak light tails for any Følner sequence. Since this is not the case, it follows that $(x + y + \mathfrak{b}) \subset W_{\mathfrak{b}}$ for any $y \in (\mathfrak{b}, \mathfrak{c})$, so we must have $x + (\mathfrak{b}, \mathfrak{c}) \subset W_{\mathfrak{b}}$.

Repeating this for every $x \in W_{\mathfrak{b}}$, we get that there are $\mathfrak{c}_1, \dots, \mathfrak{c}_r \in \mathcal{B}_{W'}$ with $(\mathfrak{b}, \mathfrak{c}_i) \neq 1$, such that $x + (\mathfrak{b}, \mathfrak{c}_i) \subset W_{\mathfrak{b}}$. We get $W_{\mathfrak{b}} = \bigcup_{i=1}^r A_i + (\mathfrak{b}, \mathfrak{c}_i)$ for some finite sets A_i (corresponding to those $x \in W_{\mathfrak{b}}$ such that $x + (\mathfrak{b}, \mathfrak{c}_i) \subset W_{\mathfrak{b}}$). Since W is minimal, this implies that there is at least one i such that $(\mathfrak{b}, \mathfrak{c}_i) = \mathfrak{b}$, which implies that $\mathfrak{b} \mid \mathfrak{c}_i$. Since the elements of $\mathcal{B}_{W'}$ are pairwise coprime, $\mathfrak{c}_i$ must be the unique ideal in $\mathcal{B}_{W'}$ that is not coprime to $\mathfrak{b}$, and so $W_{\mathfrak{b}} \subset W'_{\mathfrak{c}_i}$. Let us denote this $\mathfrak{c}_i$ by $\mathfrak{c}$.

By reversing the argument, now using the minimality of W', we will get that there is a unique ideal $\mathfrak{b}' \in \mathcal{B}_W$ such that $\mathfrak{c} \mid \mathfrak{b}'$ and $W'_{\mathfrak{c}} \subset W_{\mathfrak{b}'}$. This together with $\mathfrak{b} \mid \mathfrak{c}$ implies that $\mathfrak{b} = \mathfrak{b}'$, and so, in fact, we must have $\mathfrak{b} = \mathfrak{c}$ and $W_{\mathfrak{b}} = W'_{\mathfrak{c}}$. Repeating this for every ideal, we conclude that $\mathcal{B}_W = \mathcal{B}_{W'}$, and that W and W' are the same sieve. Consequently, a sieve R can only be equivalent to one unique minimal sieve with weak light tails for some I_N. ☐

Remark 6.18. First, notice that an Erdős sieve may be not equivalent to any minimal sieve with weak light tails. Indeed, the sieve $R_i = \{-i, i\} + p_i^2\mathbb{Z}$ from Example 3.17 is such that $d_I(\mathcal{F}_R) = 0$ for every Følner sequence, so if an Erdős sieve R' with weak light tails for I_N were to be equivalent to it, we would satisfy $0 = d_I(\mathcal{F}_R) = d_I(\mathcal{F}_{R'}) > 0$, which is absurd.

On the other hand, notice that this theorem does not imply that if a sieve is minimal and has weak light tails, that then there are no other minimal sieves equivalent to it, just that these sieves will not have weak light tails. Take the sieves W and W' from Example 5.14. We have that W has weak light tails, while W' does not, but both are minimal and they are equivalent. This shows that although W and W' are equivalent, they cannot be obtained from one another by contractions or dilations.

Additionally, notice that if R has weak light tails for some sequence I_N, and R' is this unique minimal sieve with weak light tails for I_N, then R will have weak light tails for some other Følner sequence F_N if and only if R' also does.

Yet, if we assume that a sieve R has strong light tails, then there is a unique minimal sieve R' such that $R \sim R'$. This is a consequence of Theorem 6.17 together with the following result, which is of interest in itself.

Theorem 6.19. *Let R and R' be sieves, such that R is Erdős and has strong light tails for some Følner sequence I_N. If $R \sim R'$, then R' is also Erdős with strong light tails for I_N.*

Proof. We will assume that both $\mathcal{B}_R$ and $\mathcal{B}_{R'}$ are ordered. Lemma 2.1 implies that

$$\nu_R(C^R_{\{0\},\emptyset}) = \prod_i \left(1 - \frac{|R_i|}{N(\mathfrak{b}_i)}\right) > 0,$$

if and only if R is Erdős. Since R has strong light tails, Lemma 3.19 gives us

$$0 < d_I(\mathcal{F}_R) = d_I(\mathcal{F}_{R'}) \leq \nu_{R'}(C_{\{0\},\emptyset}),$$

which means that R' must be Erdős.

We now show that R' has strong light tails for I_N. By Lemma 6.14, we know that since $R \sim R'$ and R has strong light tails, we have for every $\mathfrak{b}' \in \mathcal{B}_{R'}$,

$$R'_{\mathfrak{b}'} \subset \bigcup_{\mathfrak{b} \in \mathcal{B}_R : (\mathfrak{b}, \mathfrak{b}') \neq 1} R_{\mathfrak{b}}. \tag{26}$$

Define a function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(i) := \min\{j \in \mathbb{N} : (\mathfrak{b}_j, \mathfrak{b}'_i) \neq 1\}.$$

Notice that for any j, there are at most a finite number of $i's$ such that $g(i) = j$ (with an upper bound given by the number of prime divisors of $\mathfrak{b}_j$). It follows that

$$\liminf_{l \to \infty} g(l) = \infty.$$

By Equation (26) every R'_i is contained in $\bigcup_{j:(\mathfrak{b}_j, \mathfrak{b}'_i) \neq 1} R_j$. It follows that for any i we have

$$R'_i \subset \bigcup_{j \geq g(i)} R_j,$$

and so, writing $G(L) = \min_{l \geq L} g(l)$, we have

$$\bigcup_{i \geq L} R'_i \subset \bigcup_{j \geq G(L)} R_j,$$

and so

$$\overline{d}_I\left(\bigcup_{i \geq L} R'_i\right) \leq \overline{d}_I\left(\bigcup_{j \geq G(L)} R_j\right).$$

Since $\liminf_{l\to\infty} g(l) = \infty$, we have that $G(L)$ must go to infinity as we increase L. Since R has strong light tails, the right hand side will go to 0 as we take L to infinity, so R' must also have strong light tails. □

Remark 6.20. Note that $R \sim R'$ with R an Erdős sieve, does not necessarily imply that R' is Erdős. For example, $R_i = \{-i, i\} + p_i^2\mathbb{Z}$ is clearly Erdős. Taking any non Erdős sieve R' such that $\mathcal{F}_{R'} = \mathcal{F}_R$, for example $R'_p = (\mathbb{Z} \setminus p\mathbb{Z}) + p\mathbb{Z}$, we see that $R \sim R'$, but R' is not Erdős.

We summarize the results of this subsection in the following theorem.

Theorem 6.21. *Let R be an Erdős sieve.*

- *There exists a minimal Erdős sieve R' such that $R \sim R'$.*
- *If R has weak light tails for some Følner sequence I_N, there exists a minimal Erdős sieve R' with weak light tails for I_N, such that if W is minimal and $W \sim R$, then $W = R'$, or W does not have weak light tails for any Følner sequence.*
- *If R has strong light tails for I_N, then there exists a unique minimal sieve R' (which will have strong light tails for I_N) such that $R \sim R'$.*

Here, we write for two sieves R and R' that $R = R'$, if $\mathcal{B}_R = \mathcal{B}_{R'}$, and $R_{\mathfrak{b}} = R'_{\mathfrak{b}}$ for every $\mathfrak{b} \in \mathcal{B}_R$. Since every Erdős $\mathcal{B}-$free system is a minimal sieve which has strong light tails by Theorem 5.7, we get the following corollary.

Corollary 6.22. *Let R and R' be an Erdős $\mathcal{B}-$free systems over an étale $\mathbb{Q}-$algebra K. Then $R \sim R'$ if and only if $R = R'$.*

6.2. **Union of Sieves.** Suppose that we are given two sieves R and R', which we can assume to be minimal. We want to define the notion of union of sieves. If R and R' are defined over the same set of ideals $\mathcal{B}$, then we can define this simply as $(R \cup R')_i := R_i \cup R'_i + \mathfrak{b}_i$.

More generally, given two sieves R and R', it might be possible to dilate them in such a way that we obtain sieves W and W', such that $\mathcal{B}_W = \mathcal{B}_{W'}$, and then we can define $(R \cup R')_{\mathfrak{c}} = W_{\mathfrak{c}} \cup W'_{\mathfrak{c}}$ for $\mathfrak{c} \in \mathcal{B}_W$.

We therefore need to find a suitable set $\mathcal{C}$ on which both W and W' will be supported. The obvious choice would be $\mathcal{B}_R \cup \mathcal{B}_{R'}$, but we want $\mathcal{C}$ to be made of pairwise coprime ideals, so in general this won't do. Surely, if $\mathfrak{b} \in \mathcal{B}_R$ is such that $(\mathfrak{b}, \mathfrak{b}') = 1$ for every $\mathfrak{b}' \in \mathcal{B}_{R'}$, then it would make sense to add $\mathfrak{b}$ to $\mathcal{C}$. If this is not the case, we might still have that for example $(\mathfrak{b}, \mathfrak{b}') \neq 1$ for some unique $\mathfrak{b}' \in \mathcal{B}'$ (such that $\mathfrak{b}$ is also the unique element of $\mathcal{B}$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$). Then it would still make sense to add $\mathfrak{c} = \mathrm{lcm}(\mathfrak{b}, \mathfrak{b}')$ to $\mathcal{C}$, and define $W_{\mathfrak{c}} = R_{\mathfrak{b}} + \mathfrak{c}$ and $W'_{\mathfrak{c}} = R'_{\mathfrak{b}'} + \mathfrak{c}$.

These considerations lead us to the following definition. We define a graph $\mathcal{G}_{R,R'}$, whose vertices will be the elements of $\mathcal{B}_R \cup \mathcal{B}_{R'}$, and where there is an edge between $\mathfrak{b}$ and $\mathfrak{b}'$ if $(\mathfrak{b}, \mathfrak{b}') \neq 1$. It is clear that there can only be an edge between vertices if the corresponding ideals do not come from the same set (so $\mathcal{G}_{R,R'}$ will always be a bipartite graph). The graph $\mathcal{G}_{R,R'}$ will be the union of distinct connected components, which may or may not be infinite. We will now use this graph to define the common base for our sieves, assuming that $\mathcal{G}_{R,R'}$ does not contain an infinite component.

Definition 6.23. *Let R and R' be two sieves. Let $\mathcal{G}_{R,R'}$ be the graph with set of vertices $\mathcal{B}_R \cup \mathcal{B}_{R'}$, and an edge between $\mathfrak{b}, \mathfrak{b}'$ if $(\mathfrak{b}, \mathfrak{b}') \neq 1$. Let $\mathcal{C}^*(R, R')$ denote the collection of connected components of $\mathcal{G}_{R,R'}$. Given $c \in \mathcal{C}^*(R, R')$ finite, we define*

$$\mathfrak{c}(c) := lcm(\{\mathfrak{b} : \mathfrak{b} \in c\}).$$

If every element of $\mathcal{C}^*(R, R')$ is finite, then we set $\mathcal{C}(R, R') = \mathfrak{c}(\mathcal{C}^*(R, R'))$ and say that R and R' have a common basis.

We must verify that the ideals in $\mathcal{C}(R, R')$ are indeed pairwise coprime. Suppose that we can find some distinct $\mathfrak{a}_1, \mathfrak{a}_2$ such that $(\mathfrak{a}_1, \mathfrak{a}_2) \neq 1$. Then, there would be distinct connected components c_1 and c_2 of $\mathcal{G}_{R,R'}$ such that $\mathfrak{a}_i = \mathfrak{c}(c_i)$, and $(\mathfrak{c}(c_1), \mathfrak{c}(c_2)) \neq 1$. But it is a property of the least common multiple, that if A and B are finite sets, and $(\text{lcm}(A), \text{lcm}(B)) \neq 1$, then $(a, b) \neq 1$ for some $a \in A$ and $b \in B$. This would imply that there is an edge between elements of c_1 and c_2, which is impossible, as these were taken to be distinct connected components.

Remark 6.24. It might happen that two sieves don't have a common basis. Consider the example where we define two sieves by $R_i = p_{2i-1}p_{2i}\mathbb{Z}$ and $R'_i = p_{2i}p_{2i+1}\mathbb{Z}$. We see that for any i, $p_{2i-1}p_{2i}$ is not coprime to $p_{2i}p_{2i+1}$, which is not coprime to $p_{2i+1}p_{2i+2)}$.

Additionally, two sieves might not have have a common base, but be equivalent to sieves that do have a common base. Let q_i denote the $i-$th prime that is congruent to $1 \mod 4$, and r_i the $i-$th prime that is congruent to $3 \mod 4$. Let R be the sieve supported on the set $\mathcal{B}_R = \{q_i^2 r_i^2\mathbb{Z} : i \in \mathbb{N}\}$ and defined by $R_i = q_i^2\mathbb{Z} + q_i^2 r_i^2\mathbb{Z}$. Define also a sieve R', with support on the set $\mathcal{B}_{R'} = \{q_i^2 r_{i+1}^2\mathbb{Z} : i \in \mathbb{N}\}$, by $R'_i = r_{i+1}^2\mathbb{Z} + q_i^2 r_{i+1}^2\mathbb{Z}$. These sieves don't have a common basis, since for every i, $(q_i^2 r_i^2, q_i^2 r_{i+1}) \neq 1$ and $(q_{i+1}^2 r_{i+1}^2, q_i^2 r_{i+1}) \neq 1$, meaning that $\mathcal{G}_{R,R'}$ will be a connected infinite graph. Yet, we see that R is equivalent to the sieve W such that $\mathcal{B}_W = \{q_i^2\mathbb{Z} : i \in \mathbb{N}\}$ and $W_i = q_i^2\mathbb{Z}$, and R' is equivalent to the sieve W' such that $\mathcal{B}_{W'} = \{r_i^2\mathbb{Z} : i \in \mathbb{N}\}$ with $W'_1 = \emptyset$ and $W'_i = r_i^2\mathbb{Z}$ otherwise. Although R and R' don't have a common base, W and W' do have one, since $\mathcal{G}_{W,W'}$ is a graph with no edges.

Note that if R and R' are two sieves, and W, W' are contractions of R and R' respectively, then there is a map from $\mathcal{G}_{W,W'}$ into $\mathcal{G}_{R,R'}$ that sends to $\mathfrak{b} \in \mathcal{B}_R$ all those $\mathfrak{c} \in \mathcal{B}_W$ that divide $\mathfrak{b}$ (and similarly for ideals of $\mathcal{B}_{R'}$ and $\mathcal{B}_{W'}$). Additionally, if there is an edge from $\mathfrak{c} \in \mathcal{B}_W$ to $\mathfrak{c}' \in \mathcal{B}_{W'}$, this will be sent to the edge between the $\mathfrak{b} \in \mathcal{B}_R$ divisible by $\mathfrak{c}$ and the $\mathfrak{b}' \in \mathcal{B}_{R'}$ divisible by $\mathfrak{c}'$. It follows that if there is an infinite connected component in $\mathcal{G}_{W,W'}$, it will be sent by this map into one in $\mathcal{G}_{R,R'}$. Hence, two sieves can only have a common base if the minimal sieves to which they equivalent to (from Lemma 6.10) have a common basis.

If R and R' have a common basis, we define the sieves W and W' by

$$W_{\mathfrak{c}(c)} = \bigcup_{\mathfrak{b} \in c \cap \mathcal{B}_R} (R_{\mathfrak{b}} + \mathfrak{c}(c)) \qquad W'_{\mathfrak{c}(c)} = \bigcup_{\mathfrak{b}' \in c \cap \mathcal{B}_{R'}} (R'_{\mathfrak{b}'} + \mathfrak{c}(c)).$$

Then W is equivalent to R, since $R_{\mathfrak{b}} + \mathfrak{c}(c) = R_{\mathfrak{b}}$, and every $\mathfrak{b} \in \mathcal{B}_R$ will belong to one connected component of $\mathcal{G}_{R,R'}$. Notice that it may happen that for some $c \in \mathcal{C}^*(R, R')$, we have $c \cap \mathcal{B}_R = \emptyset$ or

$c \cap \mathcal{B}_{R'} = \emptyset$. In this case, we will have $W_{\mathfrak{c}(c)} = \emptyset$ or $W'_{\mathfrak{c}(c)} = \emptyset$. Additionally, if R and R' are Erdős with weak/strong light tails for some I_N, then Lemma 6.6 guarantees that so are W and W'.

This allows us to define the union of R and R' as the union of W and W'. That is, as the sieve $R \cup R'$ defined over $\mathcal{C}$ by

$$(R \cup R')_{\mathfrak{c}(c)} = W_{\mathfrak{c}(c)} \cup W'_{\mathfrak{c}(c)} = \bigcup_{\mathfrak{b} \in c \cap \mathcal{B}_R} (R_{\mathfrak{b}} + \mathfrak{c}(c)) \cup \bigcup_{\mathfrak{b}' \in c \cap \mathcal{B}_{R'}} (R'_{\mathfrak{b}'} + \mathfrak{c}(c)).$$

If $(R \cup R')_{\mathfrak{c}(c)} \neq \mathcal{O}_K$ for every $c \in \mathcal{C}^*$, then we say that the union of R and R' is well defined.

Example 6.25. Let K be an imaginary quadratic number field, and for every p that splits in K, let $\mathfrak{p}_p$ be the prime such that $\mathfrak{p}_p \overline{\mathfrak{p}}_p = p\mathcal{O}_K$. Define a sieve R as the $\mathcal{B}-$free system supported on the set $\mathcal{B}_R = \{\mathfrak{p}_p^2 : p \text{ splits in } \mathcal{O}_K\}$ and a sieve R' as the $\mathcal{B}-$free system supported on $\mathcal{B}_{R'} = \{\overline{\mathfrak{p}}_p^2 : p \text{ splits in } \mathcal{O}_K\}$. Then, we get a sieve $R \cup R'$ which is the $\mathcal{B}-$free system supported on $\mathcal{B}_R \cup \mathcal{B}_{R'}$.

For another example, let q_i and r_i be as in Remark 6.24, and define the sieves R and R' by $R_i = q_i + q_i^2\mathbb{Z}$, $R'_i = r_i + r_i^2\mathbb{Z}$. Then, we have that $R \cup R'$ is the sieve supported on $\{q_i^2 r_i^2 \mathbb{Z} : i \in \mathbb{N}\}$ and defined by $(R \cup R')_i = ((q_i + q_i^2\mathbb{Z}) \cup (r_i + r_i^2\mathbb{Z})) + q_i^2 r_i^2 \mathbb{Z}$.

On the other hand, if R and R' are sieves such that $R_1 = 0 + 2\mathbb{Z}$ and $R'_1 = 1 + 2\mathbb{Z}$, then their union is not well defined, since $R_1 \cup R'_1 = \mathbb{Z}$.

Given any $\mathfrak{b} \in \mathcal{B}_R \cup \mathcal{B}_{R'}$, take the $\mathfrak{c} \in \mathcal{C}$ such that $\mathfrak{b} \mid \mathfrak{c}$. Then, it is clear from the definition that $R_{\mathfrak{b}} = R_{\mathfrak{b}} + \mathfrak{c} \subset (R \cup R')_{\mathfrak{c}}$. It follows that

$$\mathcal{F}_{R \cup R'} = \left(\bigcup_{\mathfrak{c} \in \mathcal{C}} (R \cup R')_{\mathfrak{c}} \right)^c = \left(\bigcup_{\mathfrak{b} \in \mathcal{B}} R_{\mathfrak{b}} \cup \bigcup_{\mathfrak{b} \in \mathcal{B}'} R'_{\mathfrak{b}} \right)^c = \mathcal{F}_R \cap \mathcal{F}_{R'}.$$

Our interest in the union of sieves is twofold. First, as we have just shown, the $(R \cup R')-$free elements correspond to the intersection of $\mathcal{F}_R$ and $\mathcal{F}_{R'}$. Second, the union of sieves allows us to take sieves with weak/strong light tails, and obtain new sieves with the same properties. This is shown in the following lemma.

Lemma 6.26. *Let R and R' be two sieves for which their union is well defined and I_N some Følner sequence. The sieve $R \cup R'$ is Erdős if and only if both R and R' are Erdős. If both R and R' have weak light tails for I_N, then so does $R \cup R'$. Additionally, $R \cup R'$ will have strong light tails for some I_N if and only if both R and R' have strong light tails for I_N.*

Proof. By dilating R and R' if necessary, we are free to assume that R and R' are defined over a common base $\mathcal{B}$, and that $(R \cup R')_{\mathfrak{b}} = R_{\mathfrak{b}} \cup R_{\mathfrak{b}'}$. Hence, using the fact that

$$\max(|R_{\mathfrak{b}}|, |R'_{\mathfrak{b}}|) \leq |(R \cup R')_{\mathfrak{b}}| \leq |R_{\mathfrak{b}}| + |R'_{\mathfrak{b}}|,$$

we see that $R \cup R'$ is Erdős if and only if both R and R' also are.

Similarly, we have that, ordering $\mathcal{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots\}$,

$$\max\left(\left| I_N \cap \bigcup_{i>L} R_i \right|, \left| I_N \cap \bigcup_{i>L} R'_i \right| \right) \leq \left| I_N \cap \bigcup_{i>L} (R \cup R')_i \right| \leq \left| I_N \cap \bigcup_{i>L} R_i \right| + \left| I_N \cap \bigcup_{i>L} R'_i \right|,$$

so $(R \cup R')$ will have strong light tails for I_N if and only if both R and R' have strong light tails with respect to I_N.

Note that if $x \notin (R \cup R')_i$, then both $x \notin R_i$ and $x \notin R'_i$, and so

$$\left| I_N \cap \bigcup_{i>L} (R \cup R')_i \setminus \bigcup_{j \le L} (R \cup R')_j \right| \le \left| I_N \cap \bigcup_{i \ge L} R_i \setminus \bigcup_{j<L} R_j \right| + \left| I_N \cap \bigcup_{i>L} R'_i \setminus \bigcup_{j<L} R'_j \right|.$$

Therefore if both R and R' have weak light tails for I_N, so does $R \cup R'$. □

Remark 6.27. Contrarily to the strong light tails property, it might be the case that $R \cup R'$ has weak light tails for I_N, without R and R' having both weak light tails for I_N. Consider the example where R is defined by $R_1 = 0 + 4\mathbb{Z}$ and $R_i = 1 + 4i + p_i^2\mathbb{Z}$ and R' is defined by $R'_1 = 1 + 4\mathbb{Z}$ and $R'_i = p_i^2\mathbb{Z}$ for any $i > 1$. As in Example 5.14, we can show that R does not have weak light tails for B_N (but R' has strong light tails for B_N by Lemma 5.3). The sieve W defined by $W_1 = \{0, 1\} + 4\mathbb{Z}$, $W_i = \{0, 1 + 4i\} + p_i^2\mathbb{Z}$ is the union of R and R'. It has weak light tails for B_N, as it can be written as the union of two sieves with weak light tails, T and T', defined by $T_1 = 1 + 4\mathbb{Z}, T'_1 = 0 + 4\mathbb{Z}$, $T_i = R_i$ and $T'_i = R'_i$ for $i > 1$.

We now provide natural examples where one might consider the union of sieves. We start with an example over $\mathbb{Q}$.

Example 6.28. Let R be the sieve defined by $R_p = 2^{p-1} + p^2\mathbb{Z}$. We will show that it has strong light tails for B_N. By Fermat's little theorem, we know that $2^{p-1} \equiv 1 \mod p$, and so $2^{p-1} \in \{1 + jp : 0 \le j < p\} + p^2\mathbb{Z}$. If there are only finitely many Wieferich primes (those for which $2^{p-1} \equiv 1 \mod p^2$), then by what we showed in Example 5.11, the sieve W defined over the non Wieferich primes by $W_p = 2^{p-1} + p^2\mathbb{Z}$ has strong light tails in B_N, since $\min_{a \in W_p} |a| = p - 1$. We can then obtain the sieve R from W by adding finitely many ideals, so Lemma 5.17 implies that R has strong light tails. If there are infinitely many Wieferich primes, then the sieve W' defined over Wieferich primes by $W'_p = 1 + p^2\mathbb{Z}$ has strong light tails for B_N due to Lemma 5.3, and so, either there are not infinitely many Wieferich primes, in which case R can be obtained from W' by adding finitely many ideals, or R is the union of the sieves W and W'. In either case, using Lemmas 5.17 or 6.26 respectively, R must have strong light tails for B_N.

We also provide an example over $\mathbb{Q} \times \mathbb{Q}$.

Example 6.29. In [60], the notion of a *carefree couple* is defined as a pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, such that x and y are coprime and x is squarefree. The sieve R defined by $R_p = p\mathbb{Z} \times p\mathbb{Z}$ is, as pointed out in Example 5.8 such that $(x, y) \in \mathcal{F}_R$ is equivalent to x and y being coprime. The sieve R' defined by $R'_p = p^2\mathbb{Z} \times \mathbb{Z}$ is such that $(x, y) \in \mathcal{F}_{R'}$ is equivalent to $x \notin p^2\mathbb{Z}$ for every p, that is, to x being squarefree. Consequently, the set of carefree couples corresponds to $\mathcal{F}_R \cap \mathcal{F}_{R'} = \mathcal{F}_{R \cup R'}$.

To see that $R \cup R'$ is well defined, notice that R is equivalent to the sieve W defined by

$$W_p = \{(jp, 0) : 0 \le j \le (p - 1)\} + p^2\mathbb{Z} \times p\mathbb{Z}$$

and R' is equivalent to the sieve W' defined by

$$W'_p = \{(0, j) : 0 \le j \le p - 1\} + p^2\mathbb{Z} \times p\mathbb{Z}.$$

Hence, R and R' have a common basis, and writing $(R \cup R')_p = W_p \cup W'_p + p^2\mathbb{Z} \times p\mathbb{Z}$, we see that $|(R \cup R')_p| = 2p - 1 < p^3$, so $R \cup R'$ is well defined and Erdős.

Note that both R and R' have strong light tails for B_N by Theorem 5.7. Consequently, by Lemma 6.26, $R \cup R'$ has strong light tails for B_N. We can therefore compute the density of carefree pairs to be

$$d(\mathcal{F}_{R \cup R'}) = \prod_p \left(1 - \frac{2p-1}{p^3}\right) = \frac{1}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right)$$

as was also shown in [60].

If instead we want to consider the set of *strongly carefree couples*, where (x, y) is squarefree and y is also squarefree, we could instead consider the sieve

$$Z_p = \{(jp, kp) : 0 \leq j, k \leq (p-1)\} \cup \{(j, 0) : 0 \leq j \leq p^2 - 1\} \cup \{(0, k) : 0 \leq k \leq p^2 - 1\} + p^2\mathbb{Z} \times p^2\mathbb{Z},$$

which is such that $\mathcal{F}_Z$ is the set of strongly carefree couples. Again by Theorem 5.7 this is the union of three sieves with strong light tails with respect to B_N, so we can use Lemma 6.26 to conclude that it has strong light tails for B_N. Since $|Z_p| = 3p^2 - (p + p + 1) + 1 = 3p^2 - 2p$, we conclude that the density of strongly carefree couples is

$$d(\mathcal{F}_Z) = \prod_p \left(1 - \frac{3p^2 - 2p}{p^4}\right) = \frac{1}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right).$$

As these examples make it clear, the theory of the union of sieves allows us to turn heuristic arguments into rigorous proofs when trying to compute the density of sets that can be realized as the $R-$free numbers for some sieve R. The drawback is of course that we cannot give any error bounds, as explained in Remark 3.22. But if all we need is to show that a certain set is infinite, showing it has positive density is obviously enough.

7. Spectrum and Equivalence of Dynamical Systems

In this section, we investigate the dynamical system (Ω_R, S, ν_R). This section is divided in three subsections. In the first we determine when $\Omega_R = \Omega_{R'}$ for two Erdős sieves R and R', and show (see Theorem 7.8) that whenever this is the case, then $\nu_R = \nu_{R'}$. In particular, we first show in Lemma 7.3 that when $\mathcal{B}_R = \mathcal{B}_{R'}$, then $\Omega_R = \Omega_{R'}$ if and only if for every $\mathfrak{b} \in \mathcal{B}_R$ there is some $\delta_\mathfrak{b} \in \mathcal{O}_K$ such that $R_\mathfrak{b} = \delta_\mathfrak{b} + R'_\mathfrak{b}$ (this fact had already been shown in [42]). We then show in Lemma 7.5 that both Ω_R and ν_R are invariant under dilations and contractions. We finish by showing in Lemma 7.6 that if R and R' are minimal, then $\Omega_R = \Omega_{R'}$ implies that $\mathcal{B}_R = \mathcal{B}_{R'}$. These three facts together completely characterize when $\Omega_R = \Omega_{R'}$, and allow us to easily show Theorem 7.8.

The second section is dedicated to showing that for every Erdős sieve R, there exists a compact group $G_{R,F}$ and a system $(G_{R,F}, T^F, \mathbb{P}^F)$ that is isomorphic to (Ω_R, S, ν_R) and such that $(G_{R,F}, T^F)$ is a rotation of a group (as described in Theorem 2.19). We provide two distinct proofs. The first one is completely original and conceptually simpler using the theory of sieves. We restrict ourselves to minimal sieves R, and show in Lemma 7.13 that the set $\mathcal{S}_R$, containing all sets of the form $\mathcal{F}_{R'}$ where R' is an Erdős sieve with weak light tails for B_N and satisfying $\Omega_R = \Omega_{R'}$, is such that $\nu_R(\mathcal{S}_R) = 1$. We then define for each $g \in G_{R,F}$ a sieve $R(g)$, and show in that the map sending g to $\mathcal{F}_{R(g)}$ is the desired isomorphism. In the second proof, inspired by what had been done in previous literature, one shows using Theorem 4.1 that a space Y_R such that $\mathcal{S}_R \subset Y_R$ satisfies $\nu_R(Y_R) = 1$, and then shows that we have a well behaved map defined in Y_R that sends $A \in Y_R$ to the unique g such that $A \subset \mathcal{F}_{R(g)}$.

We conclude by determining when (Ω_R, S, ν_R) is isomorphic to $(\Omega_{R'}, S, \nu_{R'})$. Using the results of the second subsection and the Halmos-von Neumman Theorem (Theorem 2.20), this can be done by computing the spectrum of (Ω_R, S, ν_R), which is exactly what we do in the the third subsection.

7.1. Equality of Dynamical Systems.

We want to characterize when $\Omega_R = \Omega_{R'}$, and show that whenever this is the case, then $\nu_R = \nu_{R'}$. We start by doing this for sieves supported on the same set, which was already done in [42]. We provide a proof since some of the lemmas used to prove this result are needed for when we extend it.

Lemma 7.1. *Let R be an Erdős sieve. For every $\mathfrak{b} \in \mathcal{B}_R$, there is some $R-$admissible set A such that $A + \mathfrak{b} = R^c_\mathfrak{b}$.*

Proof. Fix some $\mathfrak{b} \in \mathcal{B}_R$, and define the set

$$\Delta = \left\{ \mathfrak{b}' \in \mathcal{B}_R : \frac{|R_{\mathfrak{b}'}|}{N(\mathfrak{b}')} \geq \frac{1}{|R^c_\mathfrak{b}|} \right\}.$$

Because R is Erdős, we have that Δ is finite. Therefore, we can use the Chinese Remainder Theorem to find a_j with $1 \leq j \leq |R^c_\mathfrak{b}|$ such that each a_j belongs to a different congruence class not in $R_\mathfrak{b}$, and $a_j \notin R_{\mathfrak{b}'}$ for any $\mathfrak{b}' \in \Delta$.

Let A be the set containing each of these a_j. We will show that it is admissible, which implies the result. For any $\mathfrak{b}' \in \Delta \cup \{\mathfrak{b}\}$, we have by definition of A that $A \cap R_{\mathfrak{b}'} = \emptyset$, so we are left with showing that for $\mathfrak{b}' \notin \Delta \cup \{\mathfrak{b}\}$, we have $-A + R_{\mathfrak{b}'} \neq \mathcal{O}_K$. But for any such $\mathfrak{b}'$, we have $|A|\frac{|R_{\mathfrak{b}'}|}{N(\mathfrak{b}')} < 1$. Therefore, the set $-A + R_{\mathfrak{b}'}$ cannot cover $\mathcal{O}_K$, and so A is admissible. $\square$

Using this we can show the following lemma.

Lemma 7.2. *Let R, R' be two Erdős sieves supported on the same set $\mathcal{B}$. Then, $\Omega_R \subset \Omega_{R'}$ if and only if for every i, there is some δ_i such that $\delta_i + R'_i \subset R_i$.*

Proof. Assume that for every i there is some δ_i such that $R'_i \subset (R_i - \delta_i)$. For any A that is $R-$admissible there is some ϵ_i such that $(\epsilon_i + A) \cap R_i = \emptyset$, so we have

$$(\epsilon_i - \delta_i + A) \cap R'_i \subset (\epsilon_i - \delta_i + A) \cap (R_i - \delta_i) = \emptyset,$$

which means that A is also R'-admissible.

Let us now prove the other implication. By Lemma 7.1 we can, for each i, find some $R-$admissible set A such that $A + \mathfrak{b}_i = R_i^c$. Since $\Omega_R \subset \Omega_{R'}$, A is also $R'-$admissible, so there is some δ_i such that $A + \delta_i \subset (R'_i)^c$, which means that $-\delta_i + R'_i \subset R_i$. $\square$

It is now easy to show the following equivalence.

Lemma 7.3. *Let R and R' be two Erdős sieves supported on the same base $\mathcal{B}$. Then $\Omega_R = \Omega_{R'}$ if and only if for every i, there is some δ_i such that $\delta_i + R_i = R'_i$.*

Proof. If there is some δ_i such that $\delta_i + R_i = R'_i$, then $A \cap (\delta_i + R_i) = A \cap R'_i$, so it is clear A is in Ω_R if and only if it is in $\Omega_{R'}$.

On the other hand, assume that $\Omega_R = \Omega_{R'}$. By Lemma 7.2, for every i, there are some δ_i, δ'_i such that $\delta_i + R_i \subset R'_i$, and $\delta'_i + R'_i \subset R_i$. But this implies that $\delta_i + \delta'_i + R'_i \subset R'_i$, and since $|\delta_i + \delta'_i + R'_i| = |R'_i|$, it follows that they must be the same. Therefore, we get the relations

$$R_i \subset -\delta_i + R'_i = \delta'_i + R'_i \subset R_i,$$

which imply that $R_i = \delta'_i + R'_i$. $\square$

As a corollary, we get the following result.

Lemma 7.4. *Let R and R' be Erdős sieves. If $\Omega_R = \Omega_{R'}$ and $\mathcal{B}_R = \mathcal{B}_{R'}$, then $\nu_R = \nu_{R'}$.*

Proof. By Lemma 7.3, there is a sequence δ_i of elements of $\mathcal{O}_K$ such that $R_i = -\delta_i + R'_i$ for each i. Let $G := G_R = G_{R'}$ be the group defined in Definition 3.8 and consider the map $V : G \to G$, such that $V(g)_i = g_i + \delta_i$. Notice that $\varphi_R = \varphi_{R'} \circ V$, given that

$$y \in \varphi_R(g) \Leftrightarrow \forall_j \ g_j + y \notin R_j \Leftrightarrow \forall_j \ V(g)_j + y \notin \delta_j + R_j \Leftrightarrow \forall_j \ V(g)_j + y \notin R'_j \Leftrightarrow y \in \varphi_{R'}(V(g)).$$

For any measurable U we get

$$\nu_{R'}(U) = \mathbb{P}(\varphi_{R'}^{-1}(U)) = \mathbb{P}(V(\varphi_R^{-1}(U))) = \mathbb{P}(\varphi_R^{-1}(U)) = \nu_R(U),$$

using the fact that V preserves the measure of G. $\square$

When R and R' are not supported on the same set, we would like to have a similar result. If we could show that $\Omega_R = \Omega_{R'}$ already implies that $\mathcal{B}_R = \mathcal{B}_{R'}$, then we could simply remove this hypothesis from Lemma 7.4. However, this is not the case, since, as we now show, dilations don't change Ω_R or ν_R. Afterwards, we will show that if we restrict ourselves to considering minimal sieves, then it is the case that $\Omega_R = \Omega_{R'}$ implies that $\mathcal{B}_R = \mathcal{B}_{R'}$.

Lemma 7.5. *Let R be an Erdős sieve, and R' a dilation of R. Then $\Omega_R = \Omega_{R'}$ and $\nu_R = \nu_{R'}$. In particular, the identity map is an isomorphism of the dynamical systems (Ω_R, S, ν_R) and $(\Omega_{R'}, S, \nu_{R'})$.*

Proof. Let R be an Erdős sieve. To define a dilation, let $\mathcal{P}$ be a partition of $\mathbb{N}$, $\mathcal{A}$ a collection of ideals $\mathfrak{a}_A$ indexed on $\mathcal{P}$ such that $(\mathfrak{a}_A, \mathfrak{a}_B) = 1$ if $A \neq B$, and $\mathcal{C}$ the collection of ideals of the form $\mathfrak{c}(A) = \operatorname{lcm}(\{\mathfrak{a}_A\} \cup \{\mathfrak{b}_i : i \in A\})$ for $A \in \mathcal{P}$. Let R' be the associated dilation, that is, the sieve supported on $\mathcal{C}$ defined by

$$R'_A = \bigcup_{i \in A} R_i + \mathfrak{c}(A).$$

First, we point out that $\Omega_R = \Omega_{R'}$. To see this, take any $B \in \Omega_R$. For every i, there is some δ_i such that $(\delta_i + B + \mathfrak{b}_i) \cap R_i = \emptyset$. Using the Chinese Remainder Theorem, define for every $A \in \mathcal{P}$ some δ_A such that $\delta_A \equiv \delta_i \mod \mathfrak{b}_i$ for every $i \in A$. Then

$$(\delta_A + B) \cap R'_A = \bigcup_{i \in A} (\delta_A + B) \cap R_i \subset \bigcup_{i \in A} (\delta_A + B + \mathfrak{b}_i) \cap R_i = \bigcup_{i \in A} (\delta_i + B + \mathfrak{b}_i) \cap R_i = \emptyset, \tag{27}$$

so B is R'-admissible. Conversely, if B is $R'-$admissible, then, for any A, there is some δ_A such that $(\delta_A + B) \cap R'_A = \emptyset$, and so by the first equality in Equation (27), we have $(\delta_A + B) \cap R_i = \emptyset$ for every $i \in A$.

We now have to show that $\nu_R = \nu_{R'}$. To do this, we consider

$$G_R = \prod_i \mathcal{O}_K/\mathfrak{b}_i \quad \text{and} \quad G_{R'} = \prod_{A \in \mathcal{P}} \mathcal{O}_K/\mathfrak{c}(A),$$

and consider the map $V : G_{R'} \to G_R$ that is the product of the maps $V_A : \mathcal{O}_K/\mathfrak{c}(A) \to \prod_{i \in A} \mathcal{O}_K/\mathfrak{b}_i$ given by $V_A(x + \mathfrak{c}_A) = (x + \mathfrak{b}_i)_{i \in A}$. We have that $\varphi_R \circ V = \varphi_{R'}$, since for any $a \in \mathcal{O}_K$,

$$a \in \varphi_{R'}(g) \Leftrightarrow \forall_{A \in \mathcal{P}}\ a + g_A \notin R'_A \Leftrightarrow \forall_{A \in \mathcal{P}}\ \forall_{i \in A}\ a + g_A \notin R_i \Leftrightarrow \forall_i\ a + V(g)_i \notin R_i \Leftrightarrow a \in \varphi_R(V(g)),$$

where we are using the fact that $g_A + \mathfrak{b}_i = V(g)_i$ when $i \in A$.

Denoting the Haar measure in G_R and $G_{R'}$ by $\mathbb{P}_R$ and $\mathbb{P}_{R'}$ respectively, it remains to show that $\mathbb{P}_R(U) = \mathbb{P}_{R'}(V^{-1}(U))$, since then it follows that for any measurable subset of $\Omega_{R'} = \Omega_R$,

$$\nu_{R'}(U) = \mathbb{P}_{R'}(\varphi_{R'}^{-1}(U)) = \mathbb{P}_{R'}(V^{-1}(\varphi_R^{-1}(U))) = \mathbb{P}_R(\varphi_R^{-1}(U)) = \nu_R(U).$$

In order to show that $\mathbb{P}_{R'}(U) = \mathbb{P}_R(V^{-1}(U))$, it is enough to prove that this holds for all cylinder sets

$$C(x_1, \ldots, x_k) := \{g \in G_R : g_i \equiv x_i \mod \mathfrak{b}_i\}$$

for every $k \geq 1$. Fix any k, and sequence $x_1, \dots, x_k$. We have that $\mathbb{P}_R(C(x_1, \dots, x_k)) = \prod_{i=1}^k N(\mathfrak{b}_i)^{-1}$, so now we have to show that this is also the value of $\mathbb{P}_{R'}(V^{-1}(C(x_1, \dots, x_k)))$. Let $A_1, \dots, A_l$ be elements of $\mathcal{P}$ that cover the set $\{1, 2, \dots, k\}$. By independence, we have that

$$\mathbb{P}_{R'}(V^{-1}(C(x_1, \dots, x_k))) = \prod_{i=1}^{l} \mathbb{P}_{R'}(V_{A_i}^{-1}(C^{A_i}(x_1, \dots, x_k))),$$

where

$$C^{A_i}(x_1, \dots, x_k) = \{g \in \prod_{j \in A_i} \mathcal{O}_K/\mathfrak{b}_j : g_j \equiv x_j \mod \mathfrak{b}_j \text{ for every } j \in A_i \cap [1, \dots, k]\}.$$

Therefore, the result will follow if we can show that for any i

$$\mathbb{P}_{R'}(V_{A_i}^{-1}(C^{A_i}(x_1, \dots, x_k))) = \prod_{j \in A_i \cap [1,k]} N(\mathfrak{b}_j)^{-1}.$$

The ideal $\mathfrak{c}(A_i) = \operatorname{lcm}(\{\mathfrak{a}_{A_i}\} \cup \{\mathfrak{b}_j\}_{j \in A_i})$ can be written as the product of coprime ideals $\prod_{j \in A_i} \mathfrak{b}_i$ and some $\mathfrak{a}'_{A_i}$, which is uniquely defined since $\mathcal{O}_K$ has unique factorization of ideals into prime ideals. Hence we have the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_K/\mathfrak{c}(A_i) \arrow[r, "\phi_1"] \arrow[rrr, bend right, "V_{A_i}"'] & \mathcal{O}_K/\mathfrak{a}'(A_i) \times \mathcal{O}_K/\prod_{j \in A_i} \mathfrak{b}_j \arrow[r, "\pi"] & \mathcal{O}_K/\prod_{j \in A_i} \mathfrak{b}_j \arrow[r, "\phi_2"] & \prod_{j \in A_i} \mathcal{O}_K/\mathfrak{b}_j
\end{tikzcd}$$

where ϕ_1, ϕ_2 are isomorphisms obtained by the Chinese Remainder Theorem, and π is the projection on the second coordinate. From this, and using the fact that ϕ_1, ϕ_2 are bijections, we see that

$$|V_{A_i}^{-1}(C^{A_i}(x_1, \dots, x_k))| = |\pi^{-1}(\phi_2^{-1}(C^{A_i}(x_1, \dots, x_k)))| = N(\mathfrak{a}'(A_i))|C^{A_i}(x_1, \dots, x_k)|.$$

It is clear that $|C^{A_i}(x_1, \dots, x_k)| = \prod_{j \in A_i \cap [k+1, \infty[} N(\mathfrak{b}_i)$, and so

$$\mathbb{P}_{\mathcal{C}}(V_{A_i}^{-1}(C^{A_i}(x_1, \dots, x_k))) = \frac{|V_{A_i}^{-1}(C^{A_i}(x_1, \dots, x_k))|}{N(\mathfrak{c}(A_i))} = \frac{N(\mathfrak{a}'(A_i)) \prod_{j \in A_i \cap [k+1, \infty[} N(\mathfrak{b}_j)}{N(\mathfrak{a}'(A_i)) \prod_{j \in A_i} N(\mathfrak{b}_j)} = \prod_{j \in A_i \cap [1,k]} \frac{1}{N(\mathfrak{b}_j)}$$

as we wanted to show. This implies that $\nu_R = \nu'_{R'}$, which concludes the proof of the lemma. □

We now show that if $\Omega_R = \Omega_{R'}$, for minimal sieves R and R', then they must be supported on the same set. This will allow us to show that if $\Omega_R = \Omega_{R'}$ then $\nu_R = \nu_{R'}$.

Lemma 7.6. *Let R and R' be minimal Erdős sieves. If $\Omega_R = \Omega_{R'}$, then $\mathcal{B}_R = \mathcal{B}_{R'}$.*

Proof. Take any $\mathfrak{b} \in \mathcal{B}_R$. We claim that there must be some $\mathfrak{b}' \in \mathcal{B}_{R'}$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$. To show this we assume that $(\mathfrak{b}, \mathfrak{b}') = 1$ for every $\mathfrak{b}' \in \mathcal{B}_{R'}$, and show that this implies that $\Omega_R \neq \Omega_{R'}$. In order to do so, we proceed similarly to how we did in the proof of Lemma 7.1 to find some $A \in \Omega_{R'}$ that is not in Ω_R.

We consider

$$\Delta := \{\mathfrak{b}' \in \mathcal{B}_{R'} : \frac{|R'_{\mathfrak{b}'}|}{N(\mathfrak{b}')} \geq \frac{1}{N(\mathfrak{b})}\}.$$

Since R' is Erdős, Δ is finite. Using the Chinese Remainder Theorem, we can find a finite set A such that $A + \mathfrak{b} = \mathcal{O}_K$, and $A \subset (R'_{\mathfrak{b}'})^c$ for every $\mathfrak{b}' \in \Delta$. This means that $A \notin \Omega_R$, but we now show that it is in $\Omega_{R'}$. For ideals $\mathfrak{b} \in \Delta$, we know by definition of A that $A \subset (R'_{\mathfrak{b}'})^c$. For ideals $\mathfrak{b} \notin \Delta$ we procced as in in Lemma 7.1. For every such ideal we have that $|A|\frac{|R'_{\mathfrak{b}'}|}{N(\mathfrak{b}')} < 1$, so $A + R'_{\mathfrak{b}'}$ cannot cover $\mathcal{O}_K$, and so $A \in \Omega_{R'}$. We obtain the desired contradiction, so for any $\mathfrak{b} \in \mathcal{B}_R$, there must be some $\mathfrak{b}' \in \mathcal{B}_{R'}$ such that $(\mathfrak{b}, \mathfrak{b}') \neq 1$.

We now write $V_{\mathfrak{b}} := \{\mathfrak{b}' \in \mathcal{B}_{R'} : (\mathfrak{b}, \mathfrak{b}') \neq 1\}$, which must be a non-empty set. We will show that it equals $\{\mathfrak{b}\}$. The first step is to show that there exists some $x \in \mathcal{O}_K$ such that

$$R_{\mathfrak{b}} \subset x + \bigcup_{\mathfrak{b}' \in V_{\mathfrak{b}}} R'_{\mathfrak{b}'}. \tag{28}$$

Writing $\mathfrak{c} = \mathrm{lcm}(\{\mathfrak{b}\} \cup V_{\mathfrak{b}})$, this is equivalent to showing that

$$R_{\mathfrak{b}} + \mathfrak{c} \subset x + \bigcup_{\mathfrak{b}' \in V_{\mathfrak{b}}} R'_{\mathfrak{b}'} + \mathfrak{c}.$$

Consider the sieve W supported on $\mathcal{B}_W = (\mathcal{B}_{R'} \cup \{\mathfrak{c}\}) \setminus V_{\mathfrak{b}}$ and defined by $W_{\mathfrak{c}} = \bigcup_{\mathfrak{b}' \in V_{\mathfrak{b}}} R'_{\mathfrak{b}'} + \mathfrak{c}$ and $W_{\mathfrak{b}'} = R'_{\mathfrak{b}'}$ for any other $\mathfrak{b}' \in \mathcal{B}_W$. This is a dilation of R', therefore Lemma 7.5 implies that $\Omega_W = \Omega_{R'} = \Omega_R$. By Lemma 7.1, we can find some finite $A \in \Omega_W$ such that $A + \mathfrak{c} = (W_{\mathfrak{c}})^c$. Assume that for all $x \in \mathcal{O}_K$, we have

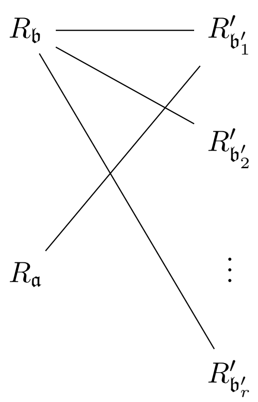
$$R_{\mathfrak{b}} + \mathfrak{c} \not\subset x + \bigcup_{\mathfrak{b}' \in V_{\mathfrak{b}}} R'_{\mathfrak{b}'} + \mathfrak{c} = x + W_{\mathfrak{c}}.$$

Then we get that $(R_{\mathfrak{b}} + \mathfrak{c}) \cap x + (W_{\mathfrak{c}})^c \neq \emptyset$ for all $x \in \mathcal{O}_K$. This implies that

$$\emptyset \neq R_{\mathfrak{b}} \cap (x + A + \mathfrak{c}) \subset R_{\mathfrak{b}} \cap (x + A + \mathfrak{b})$$

for all $x \in \mathcal{O}_K$, which shows that A is not in Ω_R, contradicting the fact that $\Omega_W = \Omega_R$. Consequently, there must be some $x \in \mathcal{O}_K$ such that Equation (28) holds, as we wanted to show.

'Visually', the proof now looks as follows. As in Section 6, we consider a bipartite graph with edges labeled $R_{\mathfrak{b}}$ or $R'_{\mathfrak{b}'}$, with an edge between $R_{\mathfrak{b}}$ and $R'_{\mathfrak{b}'}$ if $(\mathfrak{b}, \mathfrak{b}') \neq 1$. We first claim that, writing $V_{\mathfrak{b}}$ as $\{\mathfrak{b}'_1, \ldots, \mathfrak{b}'_r\}$ and taking some $\mathfrak{a} \in \mathcal{B}_R$ different from $\mathfrak{b}$, this graph cannot have a subgraph that looks as follows.



This means that this graph can be written as the union of disjoint graphs of the form

$$\begin{tikzcd}
R_{\mathfrak{b}} \arrow[r, no head] \arrow[dr, no head] \arrow[dddr, no head] & R'_{\mathfrak{b}'_1} \\
& R'_{\mathfrak{b}'_2} \\
& \vdots \\
& R'_{\mathfrak{b}'_r}
\end{tikzcd}$$

or the equivalent mirrored graphs (meaning that for some $R'_{\mathfrak{b}'}$, it will be connected to some $R_{\mathfrak{b}_1}, \ldots, R_{\mathfrak{b}_l}$), and then we will use the minimality of R and R' to show that we must have $r = 1$.

More formally, take $y \in R_{\mathfrak{b}}$. By Equation (28), there is some x (not depending on y) such that $y - x + \mathfrak{b} \subset \bigcup_{\mathfrak{b}' \in V_{\mathfrak{b}}} R'_{\mathfrak{b}'}$. By Lemma 6.16, it follows that $y - x + \mathfrak{b} \subset R'_{\mathfrak{b}'_i}$ for some i. Consequently, we must have that $y - x + (\mathfrak{b}, \mathfrak{b}'_i) \subset R'_{\mathfrak{b}'_i}$. We now claim that there must be some x_i such that $y - x_i + (\mathfrak{b}, \mathfrak{b}'_i) \subset R_{\mathfrak{b}}$ if $y - x \subset R'_{\mathfrak{b}'_i}$.

Indeed, assume that $y - x - t + (\mathfrak{b}, \mathfrak{b}'_i) \not\subset R_{\mathfrak{b}}$ for all $t \in \mathcal{O}_K$. We know that there is some $t'_i \in \mathcal{O}_K$ such that

$$y - x + (\mathfrak{b}, \mathfrak{b}'_i) \subset R'_{\mathfrak{b}'_i} \subset t'_i + \bigcup_{\mathfrak{a} \in \mathcal{B}_R : (\mathfrak{a}, \mathfrak{b}'_i) \neq 1} R_{\mathfrak{a}},$$

using Equation (28) applied to $R'_{\mathfrak{b}'_i}$. If $y - x - t'_i + (\mathfrak{b}, \mathfrak{b}'_i) \not\subset R_{\mathfrak{b}}$, then there must be some $z \in (\mathfrak{b}, \mathfrak{b}'_i)$ such that $y - x - t'_i + z + \mathfrak{b} \not\subset R_{\mathfrak{b}}$. Consequently, we would have, using Lemma 6.16, that $y - x - t'_i + \mathfrak{b} \subset R_{\mathfrak{a}}$ for some $\mathfrak{a} \in \mathcal{B}_R$, distinct from $\mathfrak{b}$. But this is impossible, since this would imply that $\mathcal{O}_K = \mathfrak{a} + \mathfrak{b} \subset R_{\mathfrak{a}}$. It follows that if $y - x \subset R'_{\mathfrak{b}'_i}$, then taking $x_i = x + t'_i$, we must have $y - x_i + (\mathfrak{b}, \mathfrak{b}'_i) \subset R_{\mathfrak{b}}$.

Choosing representatives $y_1, \ldots, y_{|R_{\mathfrak{b}}|}$ for $R_{\mathfrak{b}}$ in $\mathcal{O}_K$, let A_i be the set of y_j such that $y_j + \mathfrak{b} \subset R'_{\mathfrak{b}'_i}$. Using the Chinese Remainder Theorem, we can find some x such that $x \equiv x_i \mod (\mathfrak{b}, \mathfrak{b}'_i)$ for all i (with the x_i whose existence we showed in the last paragraph). We get that

$$R_{\mathfrak{b}} \subset \bigcup_{i=1}^{r} A_i + (\mathfrak{b}, \mathfrak{b}'_i) = \bigcup_{i=1}^{r} A_i + x - x_i + (\mathfrak{b}, \mathfrak{b}'_i) = x + \bigcup_{i=1}^{r} A_i - x_i + (\mathfrak{b}, \mathfrak{b}'_i) \subset x + R_{\mathfrak{b}}.$$

As finite subsets of $\mathcal{O}_K/\mathfrak{b}$, both $R_{\mathfrak{b}}$ and $x + R_{\mathfrak{b}}$ have the same cardinality, so $R_{\mathfrak{b}} \subset x + R_{\mathfrak{b}}$ implies equality. Consequently, we get that

$$\bigcup_{i=1}^{r} A_i + (\mathfrak{b}, \mathfrak{b}'_i) = R_{\mathfrak{b}}.$$

By minimality of R, we conclude that there must be some i such that $(\mathfrak{b}, \mathfrak{b}'_i) = \mathfrak{b}$, which by coprimality of the $\mathfrak{b}'_i$, shows that there must be a unique $\mathfrak{b}'$ such that $V_{\mathfrak{b}} = \{\mathfrak{b}'\}$ and $\mathfrak{b} \mid \mathfrak{b}'$. Using the minimality of R', we can use the same argument to show that the set of those $\mathfrak{a} \in \mathcal{B}_R$ such that $(\mathfrak{a}, \mathfrak{b}') \neq 1$ must also have only one element which is divisible by $\mathfrak{b}'$. This unique element must be $\mathfrak{b}$, and since $\mathfrak{b} \mid \mathfrak{b}'$ and $\mathfrak{b}' \mid \mathfrak{b}$, we must have an equality $\mathfrak{b} = \mathfrak{b}'$, which shows that $\mathcal{B}_R = \mathcal{B}_{R'}$ as we wanted to show. □

Together, Lemmas 7.3 and 7.6 give the following theorem.

Theorem 7.7. *Let R and R' be minimal Erdős sieves. Then $\Omega_R = \Omega_{R'}$ if and only if $\mathcal{B}_R = \mathcal{B}_{R'}$, and for every $\mathfrak{b} \in \mathcal{B}_R$, there is some $\delta_\mathfrak{b} \in \mathcal{O}_K$ such that $R_\mathfrak{b} = \delta_\mathfrak{b} + R'_\mathfrak{b}$.*

Since every sieve is equivalent to some minimal sieve, this means that for every R there is some minimal sieve R' such that their associated dynamical systems are isomorphic. We get the following result.

Theorem 7.8. *Let R and R' be Erdős sieves. If $\Omega_R = \Omega_{R'}$, then $\nu_R = \nu_{R'}$.*

Proof. We can assume without loss of generality that R and R' are minimal, since by Lemma 6.10 these sieves can be contracted until they are minimal, and by Lemma 7.5 both Ω_R and ν_R are preserved by contractions.

Hence, we can apply Lemma 7.6 to conclude that $\mathcal{B}_R = \mathcal{B}_{R'}$. Since $\Omega_R = \Omega_{R'}$, the result now follows from Lemma 7.4. □

7.2. **Isomorphisms of Dynamical Systems.** We now show that the system (Ω_R, S, ν_R) is isomorphic to a rotation on a compact group. This generalizes what had been previously done (see for example [1], [2], [20] or [26]), but the use of sieves significantly simplifies the proof of this result.

A sketch of the proof when R is a $\mathcal{B}-$free system goes as follows. Given some $g \in G_R$, let $R(g)$ be the sieve defined by

$$R(g)_i = -g_i + R_i. \tag{29}$$

Note that $\varphi_R(g) = \mathcal{F}_{R(g)}$. Let G'_R be the set of those $g \in G_R$ such that $R(g)$ has strong light tails for B_N. By showing that if R is minimal, then $R(g)$ is minimal, Theorem 6.21 shows that φ_R restricted to G'_R will be a bijection, and so the result will follow if we show that

$$\nu_R(\{\mathcal{F}_{R(g)} \in \Omega_R : R(g) \text{ has strong light tails for } B_N\}) = 1. \tag{30}$$

The reason why this sketch does not work for general sieves, is that it does not deal with two technical hurdles, with which we will now address. First, the sieve R may not be minimal. But this is easily dealt with, since by Lemma 6.10 there is a minimal sieve R' that can be obtained from R by successive contractions, and by Lemma 7.5 the systems (Ω_R, S, ν_R) and $(\Omega_{R'}, S, \nu_{R'})$ will be isomorphic. Therefore, we are free to assume that R is minimal.

The second hurdle comes from assuming that the map sending g to $R(g)$ is a bijection. This is clear when R is a $\mathcal{B}-$free system, but for general sieves it may be the case that we have $x + R_i = R_i$ without $x \in \mathfrak{b}_i$.

Example 7.9. Take a sieve such that $R_1 = 2\mathbb{Z} = \{0, 2\} + 4\mathbb{Z}$. Then, $2 + R_1 = R_1$, although $2 \notin 4\mathbb{Z}$. Taking some $A \in Y_R$ such that $A + 4\mathbb{Z} = \{0, 2\} + 4\mathbb{Z}$, we have that $1 + A + 4\mathbb{Z} = 3 + A + 4\mathbb{Z}$, despite both having empty intersection with R_1.

Note that in this example the sieve R is not minimal. Indeed, in the special case of sieves over $\mathbb{Q}$, if R is a minimal, then $x + R_i = R_i$ implies that $x \in \mathfrak{b}_i$. This is because if $x + R_i = R_i$, then $x\mathbb{Z} + R_i = R_i$, which means that $R_i = \bigcup_{r \in R_i} r + (x, b_i)\mathbb{Z}$. Since R is minimal, this requires that $(x, b_i) = b_i$, and so $x \in b_i\mathbb{Z}$.

Yet, if R is not a sieve over $\mathbb{Q}$, this is no longer the case. This is because $x\mathbb{Z}$ stops being an ideal of $\mathcal{O}_K$. We provide an example.

Example 7.10. Let R be a sieve over $\mathbb{Q}[i]$ for which $R_1 = \{i, 1+i\} + 2\mathbb{Z}[i]$. We have that

$$\mathcal{O}_{\mathbb{Q}[i]} = \{0, 1, i, 1+i\} + 2\mathbb{Z}[i],$$

and $2\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i]$ is the square of a prime. The set R_1 contains elements from both congruence classes modulo $(1+i)\mathbb{Z}[i]$ so it is minimal. Yet, we have that $1 + R_1 = R_1$, in spite of $1 \notin 2\mathbb{Z}[i]$.

To solve this, the key insight is to notice that those x such that $x + R_i = R_i$ form a (additive) subgroup of $\mathcal{O}_K$ that contains $\mathfrak{b}_i$. Indeed, if $x + R_i = R_i$, then $-x + R_i = R_i$, and it is clear that if x, y belong to this subgroup, so does $x + y$. Defining for each R_i the set

$$F(R_i) = \{x \in \mathcal{O}_K : x + R_i = R_i\}$$

of those elements of $\mathcal{O}_K$ that fix R_i, we now define the group $G_{R,F}$ by

$$G_{R,F} := \prod_i \mathcal{O}_K / F(R_i). \tag{31}$$

If, given $g, g' \in G_R$, we have $R(g) = R'(g)$, we get that for all $i \in \mathbb{N}$, $-g_i + R_i = -g'_i + R_i$, which implies that $(g'_i - g_i) \in F(R_i)$, and so g and g' must be mapped into the same element of $G_{R,F}$ under the map that sends $(g_i)_{i\in\mathbb{N}} \in G_R$ to $(g_i + F(R_i))_{i\in\mathbb{N}} \in G_{R,F}$.

Writing

$$G'_{R,F} := \{g \in G_{R,F} : R(g) \text{ has strong light tails for } B_N\} \tag{32}$$

we now have that by Theorem 6.21 the map from $G'_{R,F}$ to Ω_R that sends g to $\mathcal{F}_{R(g)}$ is a bijection. We are now missing two things, in order to show that (Ω_R, S, ν_R) is isomorphic to $(G_{R,F}, T^F, \mathbb{P}^F)$, where T^F is the action of $\mathcal{O}_K$ in $G_{R,F}$ defined by $T^F_a(g)_i = g_i + a$, and $\mathbb{P}^F$ is the Haar measure in $G_{R,F}$.

The first is that the map $\varphi_{R,F} : G_{R,F} \to \Omega_R$, that sends g to $\mathcal{F}_{R(g)}$ is a factor map. The second is Equation (30). We start with the first. An equivalent way of defining $\varphi_{R,F}$ is though the equivalence

$$a \in \varphi_{R,F}(g) \Leftrightarrow \forall_i \ (a + g_i) \cap R_i = \emptyset.$$

In Lemma 3.12, we showed that (Ω_R, S, ν_R) is a factor of $(G_R, T, \mathbb{P})$. We now show that the same holds true for the system $(G_{R,F}, T, \mathbb{P}^F)$.

Lemma 7.11. *The map $\varphi_{R,F}$ is a factor map from $(G_{R,F}, T^F, \mathbb{P}^F)$ to (Ω_R, S, ν_R).*

Proof. We have to show that for any measurable $U \subset G_{R,F}$, $\nu_R(U) = \mathbb{P}^F(\varphi_{R,F}^{-1}(U))$ and that

$$S \circ \varphi_{R,F} = \varphi_{R,F} \circ T^F.$$

Then, the result will automatically follow, since the image of $\varphi_{R,F}$ in Ω_R will have measure 1.

We start by defining $\phi_i : \mathcal{O}_K/\mathfrak{b}_i \to \mathcal{O}_K/F(R_i)$ to be the projection $\phi_i(x) = x + F(R_i)$ which is well defined since $\mathfrak{b}_i \subset F(R_i)$. By taking the product of all the ϕ_i, we obtain a map $\phi : G_R \to G_{R,F}$. We have that $\varphi_R = \varphi_{R,F} \circ \phi$, given that $a \in \varphi_R(g)$ is equivalent to $\forall_i\ (a + g_i) \cap R_i = \emptyset$, which is equivalent to $\forall_i\ (a + F(R_i) + g_i) \cap R_i = \emptyset$ by definition of $F(R_i)$. We get the following commutative diagram.

$$\begin{tikzcd} G_R \arrow[rr, "\phi"] \arrow[dr, "\varphi_R"'] & & G_{R,F} \arrow[dl, "\varphi_{R,F}"] \\ & \Omega_R & \end{tikzcd}$$

Let U be a cylinder set (which form a base for the topology of $G_{R,F}$), that is, a set so that there are finite sets $S \subset \mathbb{N}$ and $U_i \subset \mathcal{O}_K/F(R_i)$ for $i \in S$ such that

$$U = \{g \in G_{R,F} : g_i \in U_i \text{ for } i \in S\}.$$

By Lagrange's Theorem we have that $|\phi_i^{-1}(U_i)| = |U_i||F(R_i)|$ and $|\mathcal{O}_K/F(R_i)| = N(\mathfrak{b}_i)/|F(R_i)|$, therefore

$$\mathbb{P}(\phi^{-1}(U)) = \prod_{i \in S} \frac{|U_i||F(R_i)|}{N(\mathfrak{b}_i)} = \prod_{i \in S} \frac{|U_i|}{|\mathcal{O}_K/F(R_i)|} = \mathbb{P}^F(U).$$

As desired, we obtain

$$\nu_R(U) = \mathbb{P}(\varphi_R^{-1}(U)) = \mathbb{P}(\phi^{-1}(\varphi_{R,F}^{-1}(U))) = \mathbb{P}^F(\varphi_{R,F}^{-1}(U)).$$

It remains to prove that $S \circ \varphi_{R,F} = \varphi_{R,F} \circ T^F$. Since $\phi(T_a(g)) = T_a^F(\phi(g))$, and ϕ is surjective, we get that for any $a \in \mathcal{O}_K$, $g \in G_{R,F}$, there is some $g' \in G_R$ such that $\phi(g') = g$. Then we have that $S_a(\varphi_{R,F}(g))$ is equal to (using Lemma 3.12 in the second equality)

$$S_a(\varphi_{R,F}(\phi(g'))) = S_a(\varphi_R(g')) = \varphi_R(T_a(g')) = \varphi_{R,F}(\phi(T_a(g'))) = \varphi_{R,F}(T_a^F(\phi(g'))) = \varphi_{R,F}(T_a^F(g)).$$

It follows that $S \circ \varphi_{R,F} = \varphi_{R,F} \circ T^F$ which concludes our proof. $\square$

Remark 7.12. The map ϕ used in the proof of Lemma 7.11 is surjective and continuous, since the pre-image of cylinder sets in $G_{R,F}$ will be cylinder sets in G_R. We have that $G_R = \overline{\{T_a(\mathbf{0}) : a \in \mathcal{O}_K\}}$. Write $\mathbf{0}^F$ for the identity element of $G_{R,F}$. Using continuity and surjectivity of ϕ, we get

$$\overline{\{T_a^F(\mathbf{0}^F) : a \in \mathcal{O}_K\}} = \overline{\{\phi(T_a(\mathbf{0})) : a \in \mathcal{O}_K\}} \supset \phi\left(\overline{\{T_a(\mathbf{0}) : a \in \mathcal{O}_K\}}\right) = \phi(G_R) = G_{R,F}.$$

Consequently, we have that $G_{R,F} = \overline{\{T_a^F(\mathbf{0}^F) : a \in \mathcal{O}_K\}}$, that is, $(G_{R,F}, T^F)$ is a minimal rotation of a compact group.

It remains to show Equation (30). Given a sieve R we define

$$\mathcal{S}_R := \{R' : \Omega_R = \Omega_{R'} \text{ and } \mathcal{B}_R = \mathcal{B}_{R'}\} \tag{33}$$

to be the set of all sieves supported on the same set as R and with the same admissible sets. By Lemma 7.3, there is a bijection $\Phi : G_{R,F} \to \mathcal{S}_R$ that is the map that sends $g \in G_{R,F}$ to the sieve $\Phi(g)$ defined by

$$\Phi(g)_i = -g_i + R_i.$$

Let $\Psi : \mathcal{S}_R \to \Omega_R$ be the map that sends R' to $\mathcal{F}_{R'}$. We have that $\varphi_{R,F} = \Psi \circ \Phi$, since

$$a \in \varphi_{R,F}(g) \Leftrightarrow \forall_i\, a + g_i \notin R_i \Leftrightarrow \forall_i\, a \notin \Phi(g)_i \Leftrightarrow a \in \mathcal{F}_{\Phi(g)},$$

that is, we have the following commutative diagram.

$$\begin{tikzcd} G_{R,F} \arrow[rr, "\Phi"] \arrow[dr, "\varphi_{R,F}"'] & & \mathcal{S}_R \arrow[dl, "\Psi"] \\ & \Omega_R & \end{tikzcd}$$

We now define

$$\sigma_R = \Phi_* \mathbb{P}^F \tag{34}$$

to be the pushforward of $\mathbb{P}^F$ in $\mathcal{S}_R$. Theorem 2.28 (the Ergodic Theorem) together with Theorem 3.21 give us the following result.

Lemma 7.13. *Let R be an Erdős sieve and I_N a tempered Følner sequence. We have*

$$\sigma_R(\{R' \in \mathcal{S}_R : R' \textit{ has weak light tails with respect to } I_N\}) = 1.$$

Proof. Since $\varphi_{R,F} = \Psi \circ \Phi$, we have that

$$\Psi_* \sigma_R = (\Psi \circ \Phi)_* \mathbb{P}^F = (\varphi_{R,F})_* \mathbb{P}^F = \nu_R,$$

where the last equality was shown in Lemma 7.11. It follows that

$$\nu_R(\Psi(\mathcal{S}_R)) = \sigma_R(\Psi^{-1}(\Psi(\mathcal{S}_R))) = \sigma_R(\mathcal{S}_R) = 1.$$

By Theorem 3.21, we have that

$$\{R' \in \mathcal{S}_R : R' \text{ has weak light tails with respect to } I_N\} = \Psi^{-1}(\Psi(\mathcal{S}_R) \cap \operatorname{Gen}(\nu_R, I_N)),$$

so the result follows from showing that $\nu_R(\Psi(\mathcal{S}_R) \cap \operatorname{Gen}(\nu_R, I_N)) = 1$. But since I_N is tempered and ν_R is ergodic, Theorem 2.28 implies that $\nu_R(\operatorname{Gen}(\nu_R, I_N)) = 1$, which concludes the proof. $\square$

The following theorem together with the fact that $\Psi_* \sigma_R = \nu_R$ implies that Equation (30) holds.

Theorem 7.14. *Let R be an Erdős sieve, and I_N a tempered Følner sequence. We have*

$$\sigma_R(\{R' \in \mathcal{S}_R : R' \textit{ has strong light tails with respect to } I_N\}) = 1.$$

Proof. Let

$$H_R = \bigoplus_i \mathcal{O}_K/F(R_i).$$

We can write elements $h \in H_R$ as sequences $h = (h_1, h_2, \dots)$ such that $h_i = 0$ except for a finite number of indices. Let V be the action of H_R in $\mathcal{S}_R$ given by $V_h(R')_i = R'_i + h_i$. Theorem 5.29 can be restated as saying that

$$\{R' \in \mathcal{S}_R : R' \text{ has strong light tails for } I_N\} = \bigcap_{h \in H_R} V_h(\{R' \in \mathcal{S}_R : R' \text{ has weak light tails for } I_N\}).$$

Clearly σ_R is V invariant, so, by Lemma 7.13, the right hand side is a countable intersection of sets of measure 1. Therefore, the left hand side also has measure 1, as we wanted to show. □

Remark 7.15. In particular, for every sieve R there is some sieve R' with strong light tails with respect to B_N such that $\Omega_R = \Omega_{R'}$.

With this, it is now easy to show the desired isomorphism.

Theorem 7.16. *Let R be an Erdős sieve. Then (Ω_R, S, ν_R) is isomorphic to $(G_{R,F}, T^F, \mathbb{P}^F)$.*

Proof. Let $G'_{R,F}$ be the set defined in Equation (32) and

$$L_R := \{R' \in \mathcal{S}_R : R' \text{ has strong light tails with respect to } B_N\}.$$

We have to show that the map $\varphi_{R,F}$ that sends $g \in G_{R,F}$ to $\mathcal{F}_{R(g)}$ is injective when restricted $G'_{R,F}$, and that $\mathbb{P}_F(G'_{R,F}) = 1$. Since $\sigma_R = \Phi_* \mathbb{P}_F$, and $G'_{R,F} = \Phi^{-1}(L_R)$, the fact that $\mathbb{P}_F(G'_{R,F}) = 1$ follows directly from Theorem 7.14, which states that $\sigma_R(L_R) = 1$.

It remains to show that $\varphi_{R,F}$ when restricted to $G'_{R,F}$ is injective. Since $\varphi_{R,F} = \Psi \circ \Phi$, and Φ is a bijection, it remains to show that Ψ restricted to L_R is injective. As we have pointed out before, by Lemma 7.5, we are free to assume that R is minimal. We now claim that this implies that every $R(g)$ is also minimal. Indeed, if $R(g)$ was not minimal, then there would be some i and some finite set A of proper divisors of $\mathfrak{b}_i$ such that $R(g)_i$ can be written as the union of congruence classes modulo the elements of A. But if $R(g)_i = \bigcup_{\mathfrak{b} \in A} E_i + \mathfrak{b}$, then $R_i = \bigcup_{\mathfrak{b} \in A}(g_i + E_i) + \mathfrak{b}$. Hence, we see that $R(g)$ is minimal if and only if R also is. Theorem 6.21 implies that Ψ restricted to L_R is injective, which completes the proof. □

In the particular case where R is a sieve over $\mathbb{Q}$, we know by Lemma 7.5 that by successively contracting R, we will obtain an equivalent minimal sieve R' such that (Ω_R, S, ν_R) is isomorphic to $(\Omega_{R'}, S, \nu_{R'})$. In this case, we have seen that $F(R'_i) = \mathfrak{b}_i$ for every i. Therefore, we get the following corollary[4] of Theorem 7.16.

[4]We point out that Theorem 7.16 was already known for sieves over $\mathbb{Q}$, see Lemma 2.2.21 of [52].

Corollary 7.17. *Let R be an Erdős sieve over $\mathbb{Q}$. Let R' be the minimal sieve equivalent to R obtained by successively contracting R, and let $\mathcal{B}_{R'}$ be the set on which R' is supported. Then (Ω_R, S, ν_R) is isomorphic to the system $(G_{R'}, T, \mathbb{P})$ where $G_{R'}$ is the group*

$$G_{R'} = \prod_{b \in \mathcal{B}_{R'}} \mathbb{Z}/b\mathbb{Z}.$$

This means that over $\mathbb{Z}$, if R is a minimal Erdős sieve, then the dynamical system associated to Ω_R does not depend on the congruence classes being sieved, only on the support $\mathcal{B}_R$ of R. This is rather surprising since by Lemma 7.3 we would expect two random sieves supported on the same set to have very different sets of admissible sets. For example, let R be the squarefree sieve $R_p = p^2\mathbb{Z}$, and R' the sieve defined by $R'_p = \{0,1\} + p^2\mathbb{Z}$. Then, $\Omega_{R'} \subset \Omega_R$, but have $\nu_R(\Omega_{R'}) = 0$ (because $\Omega_{R'} \cap Y_R = \emptyset$ where Y_R will be defined in Equation (36)). Still, the associated measure theoretical dynamical systems will be isomorphic.

Indeed, not even the support $\mathcal{B}_R$ characterizes (Ω_R, S, ν_R), as we now show. Given a set of pairwise coprime ideals $\mathcal{B}$, let $\mathcal{P}(\mathcal{B})$ be the set

$$\mathcal{P}(\mathcal{B}) := \{\mathfrak{p}^{\max_{\mathfrak{b} \in \mathcal{B}} v_{\mathfrak{p}}(\mathfrak{b})} : \mathfrak{p} \text{ prime ideal of } \mathcal{O}_K\}, \tag{35}$$

where $v_{\mathfrak{p}}$ is the $\mathfrak{p}-$adic valuation. For example, if $\mathcal{B} = \{p_{2i}^2 p_{2i+1}^2 \mathbb{Z}\}$, then $\mathcal{P}(\mathcal{B})$ is the set of all the primes squared. Let $G(\mathcal{B}) = \prod_{\mathfrak{b} \in \mathcal{B}} \mathcal{O}_K/\mathfrak{b}$. We will now show that $(G(\mathcal{B}), T)$ and $(G(\mathcal{P}(\mathcal{B})), T)$ are always topologically conjugate.

By the Chinese Remainder Theorem, we have for any $\mathfrak{b} \in \mathcal{B}$ isomorphisms

$$V_{\mathfrak{b}} : \mathcal{O}_K/\mathfrak{b} \to \prod_{\mathfrak{p} | \mathfrak{b}} \mathcal{O}_K/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{b})}.$$

The product of all these maps gives a map $V : G(\mathcal{B}) \to G(\mathcal{P}(\mathcal{B}))$, which, since every $V_{\mathfrak{b}}$ is a bijection and satisfies $V_{\mathfrak{b}}(x+y) = x + V_{\mathfrak{b}}(y)$ for every $x \in \mathcal{O}_K$, is our desired isomorphism. Since both systems $(G(\mathcal{B}), T)$ and $(G(\mathcal{P}(\mathcal{B})), T)$ are uniquely ergodic, this implies that $(G(\mathcal{B}), T, \mathbb{P})$ and $(G(\mathcal{P}(\mathcal{B})), T, \mathbb{P})$ are also isomorphic.

In the next subsection we will compute the spectrum of these dynamical systems, which will show that for minimal sieves over $\mathbb{Q}$, the set $\mathcal{P}(\mathcal{B}_R)$ characterizes these dynamical systems.

Before that, we give an alternative proof of Theorem 7.16, which is much closer to the proofs of less general results as presented in for example [1], [2] or [26]. The idea here is that we will define a map θ_R that will be an inverse of $\varphi_{R,F}$.

To define this map, we need a set to take as his domain, so we define the set $Y_R \subset \Omega_R$ by

$$Y_R := \{A \in \Omega_R : \forall_i \ |A + \mathfrak{b}_i| = |R_i^c|\}. \tag{36}$$

For example, we have that if R satisfies the local global principle (or has weak light tails for some Følner sequence), then $\mathcal{F}_R \in Y_R$ (see Definition 5.22). It is clear that Y_R is S invariant, therefore, by ergodicity of ν_R, we have that $\nu_R(Y_R) \in \{0,1\}$. We will later show that it is 1 whenever R is Erdős, so it is an appropriate set to take as domain of θ_R.

We define $\theta_R : Y_R \to G_{R,F}$ as the map characterized by $\theta_R(A)_i$ being the unique element of $\mathcal{O}_K/F(R_i)$ such that

$$(\theta_R(A)_i + A) \cap R_i = \emptyset \tag{37}$$

for every i. This is indeed well defined, since if $(x+A)\cap R_i = (x'+A)\cap R_i = \emptyset$ for $A \in Y_R$, then because $|A + \mathfrak{b}_i| = |R_i^c|$, we must have $(x + A) = (x' + A) = R_i^c$, and so we see that $(x - x') + R_i = R_i$, which shows that x is congruent to x' mod $F(R_i)$.

We want to show that θ_R is a factor map from (Ω_R, S, ν_R) to $(G_{R,F}, T^F, \mathbb{P}^F)$. To show this, we need to prove that $\nu_R(Y_R) = 1$. One way of doing this, is by noticing that if $R(g)$ has strong light tails for B_N then, by Lemma 5.19, $R(g)$ satisfies the local global principle, which implies that $\mathcal{F}_{R(g)} \in Y_R$. Hence, we have that

$$\{\mathcal{F}_{R(g)} \in \Omega_R : R(g) \text{ has strong light tails for } B_N\} \subset Y_R,$$

so Equation (30), which is a consequence of Theorem 7.14, implies that $\nu_R(Y_R) = 1$.

Alternatively, we can show that $\nu_R(Y_R) = 1$ using the following results. The first one generalizes Proposition 3.17 in [2], and has basically the same proof.

Lemma 7.18. *Let R be an Erdős sieve. Any measure ν of maximal entropy on (Ω_R, S) is such that $\nu(Y_R) = 1$.*

Proof. As pointed out in [2], by considering the ergodic decomposition of ν (Theorem 2.18), it is enough to assume that it is ergodic. For a sequence $\underline{s} = (s_1, s_2, \dots)$ of integers s_i such that $0 \le s_i \le |R_i^c|$, let (as in Section 4)

$$Y_{R,\underline{s}} := \{A \in \Omega_R : |A + \mathfrak{b}_i| = |R_i^c| - s_i\}.$$

It is clear that $Y_{R,\underline{s}} \subset Y_{R,\geq \underline{s}}$ for every $\underline{s}$ and $Y_{R,\underline{0}} = Y_R$. Let $Y_{R,k}(\mathfrak{b}_j) = \{A \in \Omega_R : |A + \mathfrak{b}_j| = |R_j^c| - k\}$. Then, the sets $Y_{R,k}(\mathfrak{b}_j)$ with $0 \le k \le |R_j^c|$ form a partition of Ω_R, and since these are disjoint and $S-$invariant, ergodicity implies that there is a unique s_j such that $\nu(Y_{R,s_j}(\mathfrak{b}_j)) = 1$. Writing $\underline{s}$ as $(s_1, s_2, \dots)$ with the s_i so chosen, we have that $\bigcap_j Y_{R,s_j}(\mathfrak{b}_j) = Y_{R,\underline{s}}$, and so $1 = \nu(Y_{R,\underline{s}}) = \nu(Y_{R,\geq\underline{s}})$ (with $Y_{R,\geq\underline{s}}$ defined as in Equation (19)).

Therefore, the systems (Ω_R, S, ν) and $(Y_{R,\geq\underline{s}}, S, \nu)$ are isomorphic, and using Theorem 4.1 together with the variational principle (Theorem 2.26), we have

$$h(\Omega_R, S, \nu) = h(Y_{R,\geq\underline{s}}, S, \nu) \le \mathrm{h_{top}}(Y_{R,\geq\underline{s}}, S) = \prod_i \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)}\right).$$

Since ν is a measure of maximal entropy, we get by Theorem 4.1 that

$$\prod_i \left(\frac{|R_i^c|}{N(\mathfrak{b}_i)}\right) \le \prod_i \left(\frac{|R_i^c| - s_i}{N(\mathfrak{b}_i)}\right),$$

which, since R is Erdős, can only happen if $s_i = 0$ for every i. Therefore, we get $\nu(Y_{R,\underline{0}}) = \nu(Y_R) = 1$. □

We also need the following result, that generalizes Equations (3.25) and (3.26) in [2].

Lemma 7.19. *Let R be an Erdős sieve. We have that*

(1) $\theta_R \circ \varphi_{R,F}$ *is the identity map on* $\varphi_{R,F}^{-1}(Y_R) \subset G_{R,F}$.
(2) $\varphi_R(\theta_R(Y_R)) \subset Y_R$.

Proof. To show point (1), take any $g \in \varphi_{R,F}^{-1}(Y_R)$. By definition of θ_R, $[\theta_R(\varphi_{R,F}(g))_i + \varphi_{R,F}(g)] \cap R_i = \emptyset$. By definition of $\varphi_{R,F}$, we have that for every $a \in \varphi_{R,F}(g)$, $g_i + a \notin R_i$, and so $(g_i + \varphi_{R,F}(g)) \cap R_i = \emptyset$. By uniqueness, it follows that $\theta_R(\varphi_{R,F}(g))_i = g_i$, as we wanted to show.

On the other hand, given any $a \in \mathcal{O}_K$, and $A \in Y_R$, we have that $a \in \varphi_{R,F}(\theta_R(A))$ is equivalent to $\theta_R(A)_i + a \notin R_i$ for every i. By definition of θ_R, this holds for every $a \in A$. Consequently, we have that $|\varphi_{R,F}(\theta_R(A)) + \mathfrak{b}_i| = |A + \mathfrak{b}_i| = |R_i^c|$ for every i, and so we conclude that $\varphi_R(\theta_R(Y_R)) \subset Y_R$. □

Example 7.20. We write down an explicit example. Take R to be the squarefree sieve $R_p = p^2\mathbb{Z}$ and $A = \mathcal{F}_R \setminus 1$. For any p, there is some $x \in \mathcal{F}_R$ different from 1 that is congruent to $1 \mod p^2$ (by Lemma 5.19, such numbers have positive density for any p), so $A \in Y_R$. This is a minimal sieve over $\mathbb{Q}$, so $\varphi_{R,F} = \varphi_R$. We have that $A + p^2\mathbb{Z} = \mathcal{F}_R + p^2\mathbb{Z} = R_p^c$ for every p, so $\theta_R(A) = \theta_R(\mathcal{F}_R) = \mathbf{0}$. We get $A \subsetneq \varphi_R(\theta_R(A)) = \mathcal{F}_R$, but we do have $\theta_R(\varphi_R(\mathbf{0})) = \theta_R(\mathcal{F}_R) = \mathbf{0}$.

It is now easy to show that $\nu_R(Y_R) = 1$. Let R be an Erdős sieve, and let ν be a measure of maximal entropy for the system (Ω_R, S). Notice that $(\theta_R)_*\nu = \mathbb{P}^F$, since the system $(G_{R,F}, T^F)$ is uniquely ergodic. Because $\varphi_{R,F}(\theta_R(Y_R)) \subset Y_R$ (as shown in Lemma 7.19), we have that

$$\nu_R(Y_R) \geq \nu_R(\varphi_{R,F}(\theta_R(Y_R))) = \mathbb{P}^F(\varphi_{R,F}^{-1}(\varphi_{R,F}(\theta_R(Y_R)))) \geq \mathbb{P}^F(\theta_R(Y_R)) = \nu(\theta_R^{-1}(\theta_R(Y_R))) \geq \nu(Y_R) = 1, \tag{38}$$

where in the last step we are using Lemma 7.18. With this, we now show that θ_R is a factor map.

Lemma 7.21. *Let R be an Erdős sieve. The map θ_R is a factor map from (Ω_R, S, ν_R) to $(G_{R,F}, T^F, \mathbb{P}^F)$.*

Proof. We start by showing that $T^F \circ \theta_R = \theta_R \circ S$. To do this we have to show that for any $a \in \mathcal{O}_K$, $A \in Y_R$ and $i \geq 1$, we have $T_a^F(\theta_R(A))_i = \theta_R(S_a(A))_i$. This can be done by showing that

$$[T_a^F(\theta_R(A))_i + S_a(A)] \cap R_i = \emptyset,$$

since $\theta_R(S_a(A))_i$ is the unique element of $\mathcal{O}_K/F(R_i)$ for which this equality holds. Since $S_a(A) = A - a$ and $T_a^F(\theta_R(A))_i = \theta_R(A)_i + a$, we see that this is equivalent to $(\theta_R(A)_i + A) \cap R_i = \emptyset$, which holds by definition of θ_R. Since $(G_{R,F}, T^F)$ is a minimal rotation of a group (by Remark 7.12), it is uniquely ergodic, so $(\theta_R)_*\nu_R = \mathbb{P}^F$, as we wanted to show. □

We now conclude with the alternative proof of Theorem 7.16

Proof. By Lemma 7.21, we have a well defined factor map $\theta_R : Y_R \to G_{R,F}$. We claim that $\varphi_{R,F}$ restricted to $\theta_R(Y_R)$ is a bijection between this set and $\varphi_{R,F}(\theta_R(Y_R))$. Surjectivity is clear, and injectivity follows from $\theta_R(Y_R) \subset \varphi_{R,F}^{-1}(Y_R)$, together with the fact that $\theta_R \circ \varphi_{R,F}$ is the identity map on $\varphi_{R,F}^{-1}(Y_R)$.

Therefore, all it remains to show is that $\mathbb{P}^F(\theta_R(Y_R)) = 1$, as well as $\nu_R(\varphi_{R,F}(\theta_R(Y_R))) = 1$. The first equality follows from

$$\mathbb{P}^F(\theta_R(Y_R)) = \nu_R(\theta_R^{-1}(\theta_R(Y_R))) \geq \nu_R(Y_R) = 1.$$

The second equality then follows from the first, since

$$\nu_R(\varphi_{R,F}(\theta_R(Y_R))) = \mathbb{P}^F(\varphi_{R,F}^{-1}(\varphi_{R,F}(\theta_R(Y_R)))) \geq \mathbb{P}^F(\theta_R(Y_R)) = 1. \qquad \square$$

Remark 7.22. Note that there always exist a measure of maximal entropy in (Ω_R, S). Indeed, this system is expansive, since for any $\epsilon > 0$, and $A \neq B$ in Ω_R, there is some $a \in \mathcal{O}_K$ such that $d(S_a(A), S_a(B)) > 1-\epsilon$ (simply taking some $a \in A\Delta B$). Every $\mathbb{Z}^n$ action on an expansive system is such that there is a measure of maximum entropy (see Theorem 1.1 in [67]). This is because for any compact metric X, the space $\mathcal{M}^T(X)$ of $T-$invariant probability measures is compact, and if X is expansive, the map $\mu \to h(X,T,\mu)$ is upper semi-continuous. The supremum of $h(X,T,\mu)$ is $\mathrm{h_{top}}(X,T)$ by the Variational Principle (see Theorem 2.26), and it must be attained, since an upper continuous map from a compact topological space always attains its supremum.

When R is a sieve over $\mathbb{Q}$, Theorem 2.2.25 in [52] shows that there is a unique invariant measure of (Ω_R, S) that has maximum entropy. For sieves over an étale $\mathbb{Q}-$algebra K of degree greater than 1, it is currently not known whether this is the case.

7.3. **Spectrum Computation.** We now compute the spectrum of (Ω_R, S, ν_R). This is relevant to us since by the Halmos-von Neumman Theorem (Theorem 2.20), this is an invariant of measure theoretical dynamical systems.

Theorem 7.23. *Let R be an Erdős sieve. Then, we have*

$$\sigma_p(\Omega_R, S, \nu_R) = \{\chi \in \widehat{\mathcal{O}_K} : \text{ there exists some } C \subset \mathbb{N} \text{ finite such that } \chi|_{\bigcap_{j\in C} F(R_j)} = 1\}.$$

Proof. By Theorem 7.16, our problem reduces to computing the spectrum of $(G_{R,F}, T^F, \mathbb{P}^F)$. For any finite $C \subset \mathbb{N}$, we have a surjection

$$\mathcal{O}_K \twoheadrightarrow \prod_{j\in C} \mathcal{O}_K/F(R_j),$$

obtained from composing the surjections from $\mathcal{O}_K$ to $\prod_{j\in C} \mathcal{O}_K/\mathfrak{b}_j$ and from this to $\prod_{j\in C} \mathcal{O}_K/F(R_j)$. Hence, we get an isomorphism

$$\mathcal{O}_K \bigcap_{j\in C} F(R_j) \to \prod_{j\in C} \mathcal{O}_K/F(R_j)$$

which takes x and sends it to $(x_j + F(R_j))_{j\in S}$. Note that $\mathfrak{b}_j \subset F(R_j)$, so the product of the $\mathfrak{b}_j$ is contained in $\bigcap_{j\in C} F(R_j)$. Since these are finite abelian groups, we get an isomorphism between the character groups

$$\widehat{\mathcal{O}_K/\bigcap_{j\in C} F(R_j)} \to \prod_{j\in C} \widehat{\mathcal{O}_K/F(R_j)}.$$

Hence, given any character χ of $\mathcal{O}_K$ such that $\chi|_{\bigcap_{j\in C} F(R_j)} = 1$ for some finite $C \subset \mathbb{N}$, there are characters χ_j of $\mathcal{O}_K$ such that $\chi_j|_{F(R_j)} = 1$ and $\chi = \prod_{j\in C} \chi_j$. Consequently, by considering the map $\zeta_\chi : G_{R,F} \to \mathbb{C}$ given by $\zeta_\chi(g) = \prod_{j\in C} \chi_j(g_j)$, we have

$$\zeta_\chi(T_a(g)) = \prod_{j\in C} \chi_j(a+g_j) = \prod_{j\in C} \chi_j(a)\chi_j(g_j) = \chi(a)\zeta_\chi(g).$$

Therefore we have that

$$\{\chi \in \widehat{\mathcal{O}_K} : \text{ there exists some } C \subset \mathbb{N} \text{ finite such that } \chi|_{\bigcap_{j\in C} F(R_j)} = 1\} \subset \sigma_p(G_{R,F}, T, \mathbb{P}^F).$$

Let $\chi' \in \sigma_p(G_{R,F}, T, \mathbb{P}^F)$. We want to show that there is some finite S such that $\chi'|_{\bigcap_{j\in C} F(R_j)} = 1$. Note that the functions ζ_χ with χ some character such that $\chi|_{\bigcap_{j\in C} F(R_j)} = 1$ correspond exactly to the characters of $G_{R,F}$, as we have

$$\widehat{G_{R,F}} = \bigoplus_i \mathcal{O}_K/F(R_i).$$

By Parseval's Theorem, these form an orthonormal basis of $L^2(G_{R,F})$. Let $f \in L^2(G_{R,F})$ be a non-zero eigenfunction of the Koopman representation with eigenvalue χ', that is, we have $f(T_a(g)) = \chi'(a)f(g)$ for all $a \in \mathcal{O}_K$ and $g \in G_{R,F}$. Writing $f = \sum_{\zeta_\chi \in \widehat{G_{R,F}}} c_\chi \zeta_\chi(g)$, we get

$$\sum_{\zeta_\chi \in \widehat{G_{R,F}}} c_\chi(\chi(a) - \chi'(a))\zeta_\chi(g) = 0.$$

By linear independence, this means that for every $a \in \mathcal{O}_K$, and for every χ such that $\zeta_\chi \in \widehat{G_{R,F}}$ and $c_\chi \neq 0$, we have $\chi(a) = \chi'(a)$. Consequently, there must be a unique χ such that $\zeta_\chi \in \widehat{G_{R,F}}$ and $c_\chi \neq 0$ for which we have $\chi = \chi'$. This concludes the proof. ☐

Example 7.24. Let R be a minimal Erdős sieve over $\mathbb{Q}$, supported on the set $\mathcal{B}_R = \{b_1, b_2, b_3, \dots\}$. We can identify a character χ such that $\chi|_{b_i\mathbb{Z}} = 1$ with $\chi(1)$ which will be a root of unity of degree b_i. By Theorem 7.23, we conclude that the spectrum of (Ω_R, S, ν_R) is the union of all roots of unity of degree $\prod_{j\in S} b_j$ over all finite $S \subset \mathbb{N}$. As we pointed out, this shows that (Ω_R, S, ν_R) is not determined by $\mathcal{B}_R$, but rather by $\mathcal{P}(\mathcal{B}_R)$ (as defined in Equation (35)).

8. X_R and Ω_R

The objective of this section is to generalize point (3) of Sarnak's Program for sieves. As proven in [2], if R is an Erdős $\mathcal{B}-$free system, then $X_R = \Omega_R$. The following example shows that this will not hold for general sieves, even if they are Erdős with strong light tails for some Følner sequence.

Example 8.1. Let R be the sieve supported on the set $\mathcal{B}_R = \{p^2\mathbb{Z} : p \text{ prime}\}$ defined by $R_p = \{0, 1\} + p^2\mathbb{Z}$ for $p \geq 3$ and $R_2 = \emptyset$. The set $A = \{2, 4\}$ is admissible, since $A \cap R_p = \emptyset$ for every $p \geq 3$. However, we see that if $x \notin \mathcal{F}_R$, then either one of $x - 1$ or $x + 1$ are not in $\mathcal{F}_R$. It follows that $d(A, \mathcal{F}_R + a) = 1$ for every a, so $A \notin X_R$.

This leads us to a number of distinct questions. First, given a sieve R when does a sieve R satisfy $X_R = \Omega_R$? We answer this question by characterizing when a set A is in X_R, with two different conditions, shown in Lemma 8.2 and Theorem 8.6.

Secondly, note how from the point of view of measure theoretical dynamics, it is not relevant that $X_R \neq \Omega_R$, as long as $\nu_R(X_R) = 1$. In Theorem 8.4 we show that this happens if and only if R has weak light tails with respect to at least one Følner sequence.

In the context of general $\mathcal{B}-$free systems, there is a space $\widetilde{X}_R$ contained in Ω_R that is sometimes considered (see [26]), which corresponds to the smallest hereditary system that contains X_R (see Definition 8.7). In Theorem 8.9 we show that $\widetilde{X}_R$ must equal Ω_R whenever R has weak light tails for some Følner sequence I_N.

Finally, in the previous section we showed that if R and R' are minimal Erdős sieves, then $\Omega_R = \Omega_{R'}$ implies that $\mathcal{B}_R = \mathcal{B}_{R'}$, and for every $\mathfrak{b} \in \mathcal{B}_R$, there is some $\delta_\mathfrak{b} \in \mathcal{O}_K$ such that $R_\mathfrak{b} = \delta_\mathfrak{b} + R'_\mathfrak{b}$. In Theorem 8.12, we show that if R and R' have weak light tails for some (not necessarily common) Følner sequence, then $X_R = X_{R'}$ if and only if $\Omega_R = \Omega_{R'}$. We conclude by providing in Proposition 8.13 an easy to check condition that is sufficient but not necessary in order for X_R to equal Ω_R.

We start with the following lemma, which gives an answer to the first question.

Lemma 8.2. *Let R be an Erdős sieve that has weak light tails for some Følner sequence I_N. An $R-$admissible set A belongs to X_R if and only if for every of its finite subsets A' and sequences $b_1, \ldots, b_l$ of elements of $\mathcal{O}_K$ not in A, there are indexes $i_1, \ldots, i_l$ such that $-b_j + R_{i_j} \not\subset -A' + R_{i_j}$ and if $i_j = i$ for every j in some finite set Q, then $\bigcap_{j \in Q}(-b_j + R_i) \setminus (-A' + R_i) \neq \emptyset$, .*

Proof. To show that $A \in X_R$, we have to show that for any finite set M, there is some $x \in \mathcal{O}_K$ such that $S_x(\mathcal{F}_R) \cap M = A \cap M$. Fix M, and write $A' = A \cap M$, $B = M \setminus A'$. The equality $S_x(\mathcal{F}_R) \cap M = A \cap M$ is equivalent to $A' \subset S_x(\mathcal{F}_R)$ and $b \notin S_x(\mathcal{F}_R)$ for every $b \in B$. We have seen (in the proof of Theorem 5.2) that the first condition $A' \subset S_x(\mathcal{F}_R)$ is equivalent to $x \in \mathcal{F}_{R'}$, where R' is the sieve defined by the condition $R'_i = -A' + R_i$ for all i. This sieve can be written as the union of $|A'|$ sieves, and since A' is admissible (given that it is a subset of A), $-A' + R_i$ is always distinct from $\mathcal{O}_K$. It follows that R' is an Erdős sieve with weak light tails for some Følner sequence, and therefore satisfies the local global principle.

The second condition $b \notin S_x(\mathcal{F}_R)$, which can be written as $x + b \notin \mathcal{F}_R$, is equivalent to there being some i such that $x + b \in R_i$. By our hypothesis, there is for every $b \in B$, some index i_b and $x_{i_b} \in \mathcal{O}_K$ such that $x_{i_b} \in -b + R_{i_b}$, but $x_{i_b} \notin R'_{i_b}$. We now want to apply the local global principle of R' for the congruence relations $x_{i_b} \mod \mathfrak{b}_{i_b}$. If the i_b are all unique, this is well defined, and we will get some x that satisfies both of our desired conditions, so we will get $S_x(\mathcal{F}_R) \cap M = A \cap M$.

If the i_b are not unique, then we proceed as follows. Order B so that we can write $B = \{b_1, b_2, \ldots, b_l\}$. We define an equivalence relation on the set of numbers from 1 to l such that $j \sim k$ if and only if $i_{b_j} = i_{b_k}$. For any congruence class C, we associate to it i_C, which equals i_{b_j} for any $j \in C$. By hypothesis, there is some $x_C \in \mathcal{O}_K$ such that $x_C \in \bigcap_{j \in C}(-b_j + R_{i_C}) \setminus (R'_{i_C}) \neq \emptyset$. Now, using the local global principle for R', we can find some $x \in \mathcal{F}_{R'}$ such that $x \equiv x_C \mod \mathfrak{b}_{i_C}$ for every C, which means that for any $j \in C$, $x + b_j \in R_{i_C}$. Again, these two conditions together imply that $S_x(\mathcal{F}_R) \cap M = A \cap M$.

On the other hand, suppose that there is a sequence $b_1, \ldots, b_l$ of elements not in A, and some finite subset A' of A, such that there is no choice of i_j for which $-b_j + R_{i_j} \not\subset -A' + R_{i_j}$ for every j, with $\bigcap_{j \in Q}(-b_j + R_i) \setminus (-A' + R_i) \neq \emptyset$, if $i_j = i$ for every j in some finite set Q. We have two cases. First, it might be that there is some j such that $-b_j + R_i \subset -A' + R_i$ for every i. Then, there is no $x \in \mathcal{F}_{R'}$ (that is, for which $A' \subset S_x(\mathcal{F}_R)$) such that $b_j \notin S_x(\mathcal{F}_R)$, as such an x would have to be in $-b_j + R_i$ for some i, while simultaneously not being in $-A + R_i$ for every i. Therefore, for any finite set $M \subset \mathcal{O}_K$ containing b_j, there is no x such that $S_x(\mathcal{F}_R) \cap M = A'$, and so A (along with any other admissible set containing A') cannot be in X_R.

Alternatively, we could have that for every j, there is a finite positive number of indexes i_j such that $-b_j + R_i \not\subset -A + R_i$. Write $M = A' \cup \{b_1, \ldots, b_l\}$. We will show that there is no $x \in \mathcal{F}_{R'}$ such that $S_x(\mathcal{F}_R) \cap M = A'$. We will do this by contradiction, assuming that this is the case, and concluding that there is a set of indexes i_j contradicting our hypothesis.

Assume that such an $x \in \mathcal{F}_{R'}$ exists. Then, for each j, there is some i_j such that $b_j \in -x + R_{i_j}$, that is, $x \in -b_j + R_{i_j}$. Since $x \notin R'_{i_j}$ (as it is an element of $\mathcal{F}_{R'}$), this means that for each i_j we have $x \in (-b_j + R_{i_j}) \setminus R'_{i_j}$, that must therefore be a non-empty set for every i_j. We are left with showing that if we have $i_j = i$ for j in a finite set $Q \subset \{1, \ldots, l\}$, then

$$\bigcap_{j \in Q}(-b_j + R_i) \setminus R'_i \neq \emptyset.$$

But this is clear, since x must belong to this set.

We conclude that if there is no choice of i_j for which $-b_j + R_{i_j} \not\subset -A' + R_{i_j}$ for every j, with $\bigcap_{j \in Q}(-b_j + R_i) \setminus (-A' + R_i) \neq \emptyset$, if $i_j = i$ for every j in some finite set Q, then $A' \notin X_R$, and indeed, there is no x such that $S_x(\mathcal{F}_R) \in C^R_{A', \{b_1, \ldots, b_l\}}$. □

Remark 8.3. Let R be a sieve satisfying the hypothesis of Lemma 8.2. If we can show that for every finite admissible set A' and $x \notin A'$, there are infinitely many i such that $-x + R_i \not\subset -A' + R_i$, then for any finite collection of $x_1, \ldots, x_k$ not in some admissible set A, we can find indexes $i_1, \ldots, i_k$ all distinct such that $-x_j + R_{i_j} \not\subset -A + R_{i_j}$. By Lemma 8.2, it follows that this is a sufficient condition to show that $X_R = \Omega_R$. We will sometimes use this in what follows. Yet, this condition is too strong, as it may happen that we have some x such that $-x + R_i \not\subset -A + R_i$ for only finitely many i, but $A \in X_R$.

Take the example of the sieve R defined by $R_1 = \{3, 4\} + 8\mathbb{Z}$ and $R_i = \{4, 5, 6\} + p_i^2\mathbb{Z}$, whenever $i \geq 2$. The set $A = \{0, 3\}$ is clearly an admissible set for R, and we have $-A + R_1 = \{0, 1, 3, 4\} + 8\mathbb{Z}$, $-A + R_i = \{1, 2, 3, 4, 5, 6\} + p_i^2\mathbb{Z}$ when $i \geq 2$. We now consider the numbers 1 and 2, which are not in A. The set $-1 + R_1$ equals $\{2, 3\} + 8\mathbb{Z}$ which is not contained in $-A + R_1$, and neither is $-2 + R_1 = \{1, 2\} + 8\mathbb{Z}$. But for any other i, both $-1 + R_i = \{3, 4, 5\} + p_i^2\mathbb{Z}$ and $-2 + R_i = \{2, 3, 4\} + p_i^2\mathbb{Z}$ are contained in $-A + R_i$. Since $2 \in (-1 + R_1) \cap (-2 + R_2)$, but it is not in $-A + R_1$, and for every other $j \notin \{0, 1, 2, 3\}$, we have $-j + R_i = \{-j + 4, -j + 5, -j + 6\} + p_i^2\mathbb{Z} \not\subset \{1, 2, 3, 4, 5, 6\} + p_i^2\mathbb{Z}$ if i is big enough, Lemma 8.2 implies that $A \in X_R$.

We can tweak the previous example, to show that it is not enough to just assume that for every $b \notin A$, there is some i such that $-b + R_i \not\subset -A + R_i$, to show that $A \in X_R$. Let R be now the sieve defined by $R_1 = 0 + 8\mathbb{Z}$, and $R_i = \{4, 5, 6\} + p_i^2\mathbb{Z}$, whenever $i \geq 2$. We again consider the admissible set $A = \{0, 3\}$, which satisfies $-A + R_1 = \{-3, 0\} + 8\mathbb{Z}$, and the elements 1 and 2, which are not in A. Clearly both $-1 + R_1$ and $-2 + R_1$ are not contained in $-A + R_1$, but this fails when we take $i \geq 2$. Notice how A cannot be in X_R, since if $\mathcal{F}_R$ has a "hole" (a sequence $x, x + 1, \ldots, x + k$ all of which are not in $\mathcal{F}_R$), then it must either be of size 1, or of size ≥ 3, but A has a hole of size 2.

Theorem 8.4. *Let R be an Erdős sieve. Then, there is a Følner sequence I_N with respect to which R has weak light tails if and only if*

$$\nu_R(X_R) = 1.$$

Proof. If R has weak light tails with respect to some I_N, then $\mathcal{F}_R$ is generic with respect to I_N by Theorem 3.21, which implies that $\nu(X_R) = 1$ by Lemma 2.31.

On the other hand, assume that $\nu_R(X_R) = 1$. By Theorem 7.14, it follows that there is some sieve R' such that $\Omega_R = \Omega_{R'}$, R' has strong light tails for B_N and $\mathcal{F}_{R'} \in X_R$. This means that for every N, there are a_N such that

$$S_{a_N}(\mathcal{F}_R) \cap B_N = \mathcal{F}_{R'} \cap B_N.$$

Let I_N be the Følner sequence

$$I_N := a_N + B_N.$$

Then we have

$$\mathcal{F}_R \cap I_N = \mathcal{F}_R \cap (a_N + B_N) = a_N + (S_{a_N}(\mathcal{F}_R) \cap B_N) = a_N + (\mathcal{F}_{R'} \cap B_N).$$

By Theorem 7.8, we know that since $\Omega_R = \Omega_{R'}$ we have $\nu_R = \nu_{R'}$. Consequently, we have

$$d_I(\mathcal{F}_R) = d(\mathcal{F}_{R'}) = \nu_{R'}(C^{R'}_{\{0\},\emptyset}) = \nu_R(C^R_{\{0\},\emptyset}).$$

Therefore R has weak light tails for I_N by Theorem 3.21. $\square$

We provide some extra intuition to the fact that if R has weak light tails for some I_N then $\nu(X_R) = 1$. When R has weak light tails for some Følner sequence I_N, the Mirsky measure ν_R quantifies the prevalence of patterns in $\mathcal{F}_R$ (in the sense that $\nu_R(C^R_{A,B}) = d_I(\{x \in \mathcal{O}_K : x + A \subset \mathcal{F}_R \text{ and } (x + B) \cap \mathcal{F}_R = \emptyset\})$). Therefore any set of the form $C^R_{A,B}$, where A, B describe a finite pattern that does not appear in $\mathcal{F}_R$, should have measure 0. Consequently, the admissible sets that contain a finite pattern that does not appear in $\mathcal{F}_R$ should all be contained in a set of measure 0. By Lemma 8.2, any admissible set A that contains any of these patterns, does not belong to X_R, hence we should expect for $\nu_R(X_R)$ to be 1. This same line of thinking was used in [26] to show that an analogue of $\nu_R(X_R) = 1$ holds for every pseudosieve over $\mathbb{Z}$ of the form $R_i = b_i\mathbb{Z}$ (see Corollary 4.3 in [26]).

A close analysis of the proof of Lemma 8.2 will show that we never really require that R is Erdős and with weak light tails, but rather that for every finite admissible A, the sieve R' defined by $R'_i = -A + R_i$ satisfies the local global principle. For this to hold, it is not enough to assume that R has the local global principle, since sieves may lose this property upon having new congruence classes added to them (see Section 5.3). This same phenomenon does not apply for sieves with weak light tails as shown in Lemma 5.17.

Example 8.5. Let W and W' be the sieves defined in Example 5.14, that is, W is the sieve defined by

$$W_1 = 1 + 4\mathbb{Z} \quad W_{2i} = 1 + 4(i-1) + p_{2i}^2\mathbb{Z} \quad W_{2i+1} = 1 - 4(i-1) + p_{2i+1}^2\mathbb{Z},$$

which has weak light tails for $I_N = [0, N]$, and W' is the sieve defined by $W_1 = \emptyset$ and $W_i = R_i$ for $i > 1$. W' does not have weak light tails for any Følner sequence, but we have $\mathcal{F}_W = \mathcal{F}_{W'}$. Since the local global principle is preserved after removing ideals, we have that W' does satisfy the local global principle. We see that $\{0, 1, 2, 3\}$ is not admissible for W, but it is for W', and so $\Omega_W = X_W = X_{W'} \subsetneq \Omega_{W'}$. Additionally, notice that for any $b \notin \{0, 1, 2, 3\}$, there are infinitely many i such that $-b + W_i \not\subset -A + W_i$. This shows that although W' satisfies the local global principle, Lemma 8.2 does not hold for W

By Lemma 8.2, if a finite admissible set A is not in X_R, then there is some B such that $\nu_R(C^R_{A,B}) = 0$. The following theorem, which generalizes a result in [2], elucidates the relation between these implications.

Theorem 8.6. *Let R be an Erdős sieve with weak light tails for any Følner sequence I_N. Then a finite set A is $R-$admissible, if and only if $\nu_R(C^R_{A,\emptyset}) > 0$.*

Additionally, an $R-$admissible set A, we have $A \in X_R$ if and only if for any finite $A' \subset A$ and $B \subset \mathcal{O}_K$ disjoint from A, we have $\nu_R(C^R_{A,B}) > 0$.

In particular, we have that $X_R = \Omega_R$ if and only if for every finite $R-$admissible set A and finite B disjoint from A, we have $\nu_R(C^R_{A,B}) > 0$.

Proof. If A is a finite set and $\nu_R(C^R_{A,\emptyset}) > 0$, then there is some $Q \in C^R_{A,\emptyset}$ which by definition is in Ω_R and $A \subset Q$. Consequently, A must also be in Ω_R.

On the other hand, using Equation (12), we have that $\nu_R(C^R_{A,\emptyset}) > 0$ is equivalent to

$$\prod_i \left(1 - \frac{|-A + R_i|}{N(\mathfrak{b}_i)}\right) > 0.$$

Since R is Erdős, this is equivalent to $-A + R_i \neq \mathcal{O}_K$ for all i, which is equivalent to $A \in \Omega_R$ by definition.

We now show that if $X_R = \Omega_R$, then for any finite admissible set A and some B disjoint from A we have $\nu_R(C^R_{A,B}) > 0$. Take any finite $A \in \Omega_R$, and B a finite set disjoint from A. Let us write $B = \{b_1, \cdots, b_r\}$. Since A is admissible we have that $\nu_R(C^R_{A,\emptyset}) > 0$. We will use this to prove that $\nu_R(C^R_{A,B}) > 0$.

By hypothesis, since $A \in \Omega_R$, it is in X_R. By Lemma 8.2 there is a set $\mathcal{S} = \{s_1, \ldots, s_r\}$ of indexes, such that $-b_j + R_{s_j} \not\subset -A + R_{s_j}$ for every j, with $\bigcap_{k\in T}(-b_k + R_i) \setminus (-A + R_i) \neq \emptyset$, if $s_k = i$ for every k in some finite set T. We partition $\{1, 2, \ldots, r\}$ by an equivalence relation where i and j are equivalent if $s_i = s_j$. For j in an equivalence class C where $s_j = i$, we choose x_j that satisfy $x_j \in \bigcap_{k\in C}(-b_k + R_i) \setminus (-A + R_i)$.

Given any $h \in \varphi_R^{-1}(C^R_{A,\emptyset})$, we define

$$g_i = \begin{cases} x_j, & \text{if } i = s_j \in \mathcal{S}, \\ h_i, & \text{otherwise.} \end{cases}$$

At most a finite number of distinct h can produce the same g by this process (they can only be distinct for indexes in $\mathcal{S}$), so by showing that every such g belongs to $\varphi_R^{-1}(C^R_{A,B})$, we will get that $\nu_R(C^R_{A,B}) > 0$. This corresponds to showing that $g_i \notin -A + R_i$ for every i, and that for every $1 \leq j \leq r$, there is some k_j such that $b_j + g_{k_j} \in R_{k_j}$.

If $i = s_j \in \mathcal{S}$, then $b_j + g_{s_j} \in R_{s_j}$ since $x_j \in -b_j + R_{s_j}$, so we see that g is contained in $\varphi_R^{-1}(C^R_{\emptyset,B})$. It remains to show that $g_i \notin -A + R_i$ for every i. By the definition of x_j, this is immediate for $i \in S$. If $i \notin S$, then $g_i = h_i$ for some $h \in \varphi_R^{-1}(C^R_{A,\emptyset})$, which is equivalent to $h_i \notin -A + R_i$ for every i. This conclude the proof that $\nu_R(C^R_{A,B}) > 0$.

We now show the converse. Take any $A \in \Omega_R$. To show that A is in X_R, we will show that for any finite set M, there is some $x \in \mathcal{O}_K$ such that $A \cap M = S_x(\mathcal{F}_R) \cap M$. Let $A' := A \cap M$ and $B := M \setminus A'$. Then A' is finite admissible and disjoint from B, so $\nu_R(C^R_{A',B}) > 0$ by hypothesis. Since R has weak light tails for I_N, we have by Theorem 3.21 that

$$\frac{1}{|I_N|} \sum_{a \in I_N} \mathbb{1}_{C^R_{A',B}}(S_a(\mathcal{F}_R)) \to \nu_R(C^R_{A',B}) > 0,$$

and so we conclude that there must exist some $x \in \mathcal{O}_K$ such that $S_x(\mathcal{F}_R) \cap M = A' = A \cap M$. $\square$

In [26], pseudo sieves of the form $R_i = b_i\mathbb{Z}$ are considered. In this case, not only can we have $X_R \subsetneq \Omega_R$, but there can also exist a third system $\widetilde{X}_R$ between these two, of the smallest hereditary subshift containing X_R.

Definition 8.7. *We say a set $X \subset \{0,1\}^{\mathcal{O}_K}$ is hereditary if for $A \in X$, if $B \subset A$, then $B \in X$.*

We have that

$$\widetilde{X}_R = \overline{\{A : A \subset B \text{ for some } B \in X_R\}}, \tag{39}$$

given that this set contains X_R, is closed, hereditary, and it must be contained in any hereditary closed system containing X_R.

Example 8.8. Consider the sieve R defined by

$$R_1 = 0 + 4\mathbb{Z} \qquad R_{2i} = (i+1) + p_{2i}^2\mathbb{Z} \qquad R_{2i+1} = -(i+1) + p_{2i+1}^2\mathbb{Z}.$$

We have that $\mathcal{F}_R = \{-1, 1\}$ since $0 \in R_1$, $i \in R_{2(i-1)}$ if $i \geq 2$, and $i \in R_{2(-i)-1}$ if $i \leq -2$. The set $\mathcal{F}_R$ has arbitrarily large holes so $\{\emptyset\} \cup (\mathcal{O}_K + \mathcal{F}_R) \subset X_R$. We show this is an equality. Take any Y not in $\{\emptyset\} \cup (\mathcal{O}_K + \mathcal{F}_R)$. If $|Y| > 2$, it is clear that for any N such that $|B_N \cap Y| > 2$, we cannot have $S_a(\mathcal{F}_R) \cap B_N = Y \cap B_N$, so $Y \notin X_R$. If $Y = \{y_1, y_2\}$ with $|y_1 - y_2| \neq 2$, taking $N \geq 5$, we again see that it is impossible to have $S_a(\mathcal{F}_R) \cap B_N = Y \cap B_N$. Finally, taking $|Y| = \{y\}$, we see that there cannot be closer elements to Y in $\mathcal{O}_K + \mathcal{F}_R$ than $S_{-y+1}(\mathcal{F}_R)$ and $S_{-y-1}(\mathcal{F}_R)$ but they don't equal it.

Because X_R does not contain any set with one element, we have that X_R is not hereditary. Taking $\widetilde{X}_R$ to be the set that contains X_R and all singletons (sets of the form $\{x\}$ with $x \in \mathcal{O}_K$), we get an hereditary system that contains X_R. Note that this is a very different set from Ω_R, since it is countable, in opposition to Ω_R which is uncountable.

This example shows that we can have $X_R \subsetneq \widetilde{X}_R \subsetneq \Omega_R$ if R does not have weak light tails for any Følner sequence. Yet, as a consequence of Theorem 8.6, this cannot happen if R has weak light tails for some Følner sequence I_N.

Theorem 8.9. *Let R be an Erdős sieve with weak light tails for some Følner sequence I_N. Then,*

$$\widetilde{X}_R = \Omega_R.$$

Proof. We start by showing that any finite $R-$admissible set A is in $\widetilde{X}_R$. By Theorem 8.6, if A is finite then $\nu_R(C_{A,\emptyset}^R) > 0$. Because R has weak light tails for some Følner sequence I_N, this implies (by Theorem 3.21) that there is some $x \in \mathcal{O}_K$ such that $S_x(\mathcal{F}_R) \in C_{A,\emptyset}^R$, that is, such that $A \subset S_x(\mathcal{F}_R)$. Since $\widetilde{X}_R$ contains X_R, we must have $S_x(\mathcal{F}_R) \in \widetilde{X}_R$, and by the hereditary property, we necessarily have $A \in \widetilde{X}_R$.

Now, take any arbitrary $A \in \Omega_R$, and let $A_1 \subset A_2 \subset \dots$ be a sequence of finite subsets of A such that $\bigcup_i A_i = A$. As we have seen, all of these belong to $\widetilde{X}_R$. The sequence A_i is Cauchy, since for any N, taking the smallest index m such that $\bigcup_{i=1}^{\infty} A_i \cap B_N = \bigcup_{i=1}^{m} A_i \cap B_N$, we have that $d(A_i, A_j) \leq 1/N$ if $i, j \geq m$. Since $\widetilde{X}_R$ is closed, the sequence A_i converges to some $Y \in \widetilde{X}_R$. But this sequence converges in Ω_R to A, so we must have $Y = A$, that is, $A \in \widetilde{X}_R$. $\square$

In particular, this shows that if R is Erdős and has weak light tails from some Følner sequence, then X_R is hereditary if and only if it is equal to Ω_R.

Given two sieves R and R', both with weak light tails with respect to some Følner sequences, we would now expect that if $X_R = X_{R'}$, then $\widetilde{X}_R = \widetilde{X}_{R'}$ which by Theorem 8.9 will imply that $\Omega_R = \Omega_{R'}$. Indeed, this is an equivalence, as we now show.

Corollary 8.10. *Let R and R' be two Erdős sieves with weak light tails for some (not necessarily common) Følner sequence. Then $X_R = X_{R'}$ if and only if $\Omega_R = \Omega_{R'}$.*

Proof. Suppose that $X_R = X_{R'}$. Then, the smallest hereditary set that contains X_R and the smallest hereditary set that contains $X_{R'}$ must agree as can be seen from Equation (39). This means that $\widetilde{X}_R = \widetilde{X}_{R'}$, which by Theorem 8.9 implies that $\Omega_R = \Omega_{R'}$.

We now show that if $\Omega_R = \Omega_{R'}$, then $X_R = X_{R'}$. Since $\Omega_R = \Omega_{R'}$, we always have that $C^R_{A,B} = C^{R'}_{A,B}$. By Theorem 8.6, we know that $A \in X_R$ if and only if for every finite $A' \subset A$, and $B \cap A = \emptyset$, we have $\nu_R(C^R_{A',B}) > 0$. Since $\Omega_R = \Omega_{R'}$ implies that $\nu_R = \nu_{R'}$ (Theorem 7.8), this holds if and only if we have $\nu_{R'}(C^{R'}_{A',B}) > 0$, so $X_R = X_{R'}$.

□

Remark 8.11. There is another proof of the fact that if $X_R = X_{R'}$, then $\Omega_R = \Omega_{R'}$. Indeed, suppose that we have sieves R and R' such that $X_R = X_{R'}$ but $\Omega_R \neq \Omega_{R'}$. Then, there is a set $A \in \Omega_{R'}$ that is not $R-$admissible. Consequently, this set has a finite subset A' that is also not $R-$admissible (there must be some i such that $-A + R_i = \mathcal{O}_K$, so we just take a finite subset A' such that $-A' + R_i = \mathcal{O}_K$). Since R' is Erdős, we will have $\nu_{R'}(C^{R'}_{A',\emptyset}) > 0$. But $X_R \cap C^{R'}_{A',\emptyset} = \emptyset$, since $A' \not\subset S_a(\mathcal{F}_R)$ for any $a \in \mathcal{O}_K$. Consequently, we must have $\nu_{R'}(X_{R'}) = \nu_{R'}(X_R) < 1$. This contradicts Theorem 8.4, so we must have that $\Omega_R = \Omega_{R'}$.

Let R and R' be minimal Erdős sieves with weak light tails for some (not necessarily common) Følner sequences. Using Theorem 7.7 together with Corollary 8.10 we get the following result.

Theorem 8.12. *Let R and R' be minimal Erdős sieves with weak light tails for some (not necessarily common) Følner sequences. The following are equivalent.*

(1) $\mathcal{B}_R = \mathcal{B}_{R'}$*, and for every* $\mathfrak{b} \in \mathcal{B}_R$*, there is some* $\delta_{\mathfrak{b}} \in \mathcal{O}_K$ *such that* $R_{\mathfrak{b}} = \delta_{\mathfrak{b}} + R'_{\mathfrak{b}}$,
(2) $X_R = X_{R'}$,
(3) $\Omega_R = \Omega_{R'}$.

By using Lemma 8.2, we can give a full characterization of those sieves R for which $X_R = \Omega_R$. Yet, it is not very easy to use this condition to say if a given sieve R satisfies $X_R = \Omega_R$ or not. We now provide an example of a condition that is easy to verify, and that is sufficient (but not necessary) for a sieve R to satisfy $X_R = \Omega_R$.

To do this, we start by defining a function λ on the powerset of $\mathcal{O}_K$ such that for $S \subset \mathcal{O}_K$ we have

$$\lambda(S) := \min_{x,y \in S} |x - y|. \tag{40}$$

We now want to consider sieves R such that $\limsup_i \lambda(R_i) = \infty$. For any $\mathcal{B}$-free system this clearly holds, since $\lambda_1(\mathfrak{b}_i)$ goes to infinity as the norm of the ideal $\mathfrak{b}_i$ grows. Meanwhile, for the sieve R defined by $R_p = \{0, 1\} + p^2\mathbb{Z}$, we have $\limsup_i \lambda(R_i) = 2$.

Proposition 8.13. *Let R be an Erdős sieve with weak light tails for some Følner sequence, such that* $\limsup_i \lambda(R_i) = \infty$. *Then* $X_R = \Omega_R$.

Proof. As pointed out in Remark 8.3, it is enough to show that for any finite admissible A and $x \notin A$ there are infinitely many indexes i such that $-x + R_i \not\subset -A' + R_i$.

To show this, take any N so big that $A - x \subset B_N$. The condition $\limsup_i \lambda(R_i) = \infty$ implies that there are infinitely many i's such that $R_i \cap (r_i + B_N) = \{r_i\}$ for any $r_i \in R_i$. Since $A - x \subset B_N$ and $x \notin A$, the condition $(r_i + B_N) \cap R_i = \{r_i\}$ implies that $r_i + (A - x) \cap R_i = \emptyset$ (since $A - x$ does not contain 0), and so $(-x + r_i) \notin (-A + R_i)$. Since r_i was an arbitrary element of R_i, it follows that $(-x + R_i) \cap (-A + R_i) = \emptyset$ holds for infinitely many i, as we wanted to show. □

Remark 8.14. The condition $\limsup_i \lambda(S_i) = \infty$ is too powerful. Indeed, consider the sieve R defined by $R_p = \{0, 1\} + p^2\mathbb{Z}$, with $\mathcal{B}_R = \{p^2\mathbb{Z} : p \text{ prime}\}$. The set $\{0, 2\}$ is not admissible, and indeed, if A is any admissible set, $x \in A$ implies $x + 2 \notin A$. We now show that $X_R = \Omega_R$, by showing that for every finite admissible set A, $x \notin A$, and i large enough, we have

$$-x + R_i = \{-x, -x + 1\} + p_i^2\mathbb{Z} \not\subset -A + R_i.$$

If i is big enough, computations in $\mathbb{Z}/p_i^2\mathbb{Z}$ are the same as in $\mathbb{Z}$, and we have the equality

$$-A + R_i = (-A + p_i^2\mathbb{Z}) \cup (-A + 1 + p_i^2\mathbb{Z}).$$

Therefore, $\{-x, -x+1\} + p_i^2\mathbb{Z} \subset -A + R_i$ implies $-x \in -A$ or $-x \in -A + 1$. The first case can't happen since $x \notin A$, so we get $-x \in -A + 1$, that is $x + 1 \in A$. Similarly, we can't have $-x + 1 \in -A + 1$, so we get $-x + 1 \in -A$, that is $x - 1 \in A$. But then, $x - 1$ and $x + 1$ must both be in A, which cannot happen since A is admissible and $(x + 1) - (x - 1) = 2$.

Comparing the examples of the sieves in Remark 8.14 and Example 8.1, we see that small changes in a sieve can change whether $X_R = \Omega_R$, since in the remark we get a sieve R for which this holds, but in Example 8.1 we provide a sieve R' obtained from R by removing just one ideal, such that $X_{R'} \neq \Omega_{R'}$.

Consider a case where we have two sieves R and R' which are Erdős $\mathcal{B}-$free systems. They are both minimal, so Lemma 7.6 implies that $\Omega_R = \Omega_{R'}$ if and only if $\mathcal{B}_R = \mathcal{B}_{R'}$. Since Erdős $\mathcal{B}-$free systems have strong light tails for B_N (as shown in Theorem 5.7), Theorem 8.12 together with Corollary 6.22 give the following result, which generalizes Proposition 2.3.1 of [52] for Erdős $\mathcal{B}-$free systems over étale $\mathbb{Q}-$algebras.

Corollary 8.15. *Let R and R' be Erdős $\mathcal{B}-$free systems over an étale $\mathbb{Q}-$algebra K. Then the following are equivalent*

(1) $\mathcal{B}_R = \mathcal{B}_{R'}$,
(2) $\mathcal{F}_R = \mathcal{F}_{R'}$,
(3) $X_R = X_{R'}$,
(4) $\Omega_R = \Omega_{R'}$.

If R is an Erdős $\mathcal{B}-$free system over a number field K of degree n, then Proposition 8.13 implies that $X_R = \Omega_R$, using the fact that $\lambda(R_i) = \lambda_1(\mathfrak{b}_i) \asymp N(\mathfrak{b}_i)^{1/n}$ (by Lemma 2.4). Hence, the equivalence between (3) and (4) in Corollary 8.15 becomes 'trivial' in the sense that it is stating the same thing twice. Yet, it may happen that this is not the case if R is a sieve over an étale-$\mathbb{Q}$ algebra.

Example 8.16. Let R be the sieve over $\mathbb{Q} \times \mathbb{Q}$ given by $R_p = p^2\mathbb{Z} \times \mathbb{Z}$ for all primes $p \geq 2$. Writing $\mathcal{S}$ to be the squarefree numbers, we have $\mathcal{F}_R = \mathcal{S} \times \mathbb{Z}$. The set $A = \{(0,0),(0,2)\}$ is admissible, since $|-A + R_p| = 2 < p^2$ for all primes p. Yet, it is impossible for $A \cap B_N$ to be equal to $S_a(\mathcal{F}_R) \cap B_N$ for any $a \in \mathcal{O}_K$ and $N \geq 3$, given that $(x,y) \in S_a(\mathcal{F}_R)$ implies that $(x, y+1) \in S_a(\mathcal{F}_R)$, but $(0,0) \in A$ while $(0,1) \notin A$. It follows that $A \in \Omega_R \setminus X_R$.

9. Applications to Number Theory

In this section we provide number theoretic applications of the theory of Erdős sieves. In subsection 9.1, we investigate sets C such that there are infinite $A, B \subset \mathbb{Z}$, with $d(B) > 0$ and $A + B \subset C$. We show in Corollary 9.4 that many sets of the form $\mathcal{F}_R$ for some Erdős sieve R with strong light tails for B_N have this property, and we make Conjecture 9.5 that in fact, this holds for all such sets. In Proposition 9.7 we show that if $\mathcal{F}_R \subset \mathcal{F}_{R'}$ for sieves R and R' such that R has strong light tails, then R' also has strong light tails. This shows that it is unlikely one could use Conjecture 9.5 to show that a larger class of sets has the property of containing $A + B$ with A infinite and $d(B) > 0$, since even within the class of sets of the form $\mathcal{F}_R$ for some Erdős sieve R, all of those such that R does not have strong light tails for B_N does not contain $\mathcal{F}_{R'}$ as a subset, for any Erdős sieve R' with strong light tails for B_N.

In Subsection 9.2, we show that the set of squarefree values of an irreducible polynomial $f \in \mathbb{Z}[X]$ can be realized as the R^f-free numbers for some sieve R^f. We give an overview of the history of the problem in number theory of determining the density of such a set, and in particular, show that it is equivalent to showing that the sieve R^f has weak light tails. Using the results of the previous sections, we show in Corollary 9.11 and Corollary 9.12 that whenever R^f has weak light tails, we have that the set of squarefree values of f contains repeated finite patterns with positive density (corresponding to finite $A \in \Omega_{R^f}$), and if $f_1, \dots, f_m$ is a set of irreducible polynomials such that every R^{f_i} has weak light tails, then the set of values that are simultaneously squarefree for all f_i also has positive density. We conclude by using the union of sieves to show in Remark 9.16 how these results generalize when f is the product of distinct irreducible polynomials f_i such that every R^{f_i} has weak light tails. We also show in Proposition 9.17 that if f is an irreducible polynomial and R^f has weak light tails, then $X_{R^f} = \Omega_{R^f}$.

In the final subsection we show an Ergodic Prime Number Theorem (we explain this nomenclature in Remark 9.21) for the set $\mathcal{F}_R$ when R is a sieve over $\mathbb{Q}$ that has weak light tails for B_N. We show how this theorem generalizes the results of the recent preprint [72], which inspired our investigation of this result.

9.1. **Infinite Patterns in $\mathcal{F}_R$.** We again consider the measure space $(\mathcal{S}_R, \sigma_R)$ as defined in Equations (33) and (34). Given a sieve R and a set $A \subset \mathcal{O}_K$, we will write $A + R$ to be the sieve defined by

$$(A + R)_i = A + R_i$$

for every $i \in \mathbb{N}$. We have the following result.

Theorem 9.1. *Let R be an Erdős sieve and A an infinite $R-$admissible set. If*

$$\prod_i \left(1 - \frac{|-A + R_i|}{N(\mathfrak{b}_i)}\right) > 0,$$

then

$$\sigma_R(\{R' \in \mathcal{S}_R : (-A + R') \text{ has strong light tails with respect to } I_N\}) = 1$$

for any tempered Følner sequence I_N.

Proof. When A is finite, note that if R has strong light tails for I_N, then so does $-A+R$, given that $-A+R = \bigcup_{a\in A}(-a+R)$, and the union of sieves with strong light tails has strong light tails by Lemma 6.26. Hence, for finite A the result follows directly from Theorem 7.14.

To show the result for infinite A, we start by showing that

$$\sigma_R(\{R' \in \mathcal{S}_R : (-A+R') \text{ has weak light tails with respect to } I_N\}) = 1$$

and then proceed as in the proof of Theorem 7.14.

Let A be an infinite $R-$admissible set, and let us write $\mathrm{Gen}_A(\Omega_R, I_N)$ to be the set of points of Ω_R such that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{a\in I_N} \mathbb{1}_{C^R_{A,\emptyset}}(S_a(x)) = \nu_R(C^R_{A,\emptyset}).$$

The function $\mathbb{1}_{C^R_{A,\emptyset}}$ is not continuous, but it is in $L^1(\Omega_R)$, given that it is a bounded function in a compact space. Hence, the Pointwise Ergodic Theorem (Theorem 2.28) implies that for any tempered Følner sequence I_N, we have

$$\nu_R(\mathrm{Gen}_A(\Omega_R, I_N)) = 1. \tag{41}$$

By our hypothesis that

$$\prod_i \left(1 - \frac{|-A+R_i|}{N(\mathfrak{b}_i)}\right) > 0,$$

we have that $\nu_R(C^R_{A,\emptyset}) > 0$ and that $-A+R'$ is an Erdős sieve for any $R' \in \mathcal{S}_R$.

Let $\Psi_R : \mathcal{S}_R \to \Omega_R$ be the map that sends R' to $\mathcal{F}_{R'}$. For $\mathcal{F}_{R'}$ to be in $\Psi_R(\mathcal{S}_R) \cap \mathrm{Gen}_A(\Omega_R, I_N)$ is equivalent to

$$\nu_R(C^R_{A,\emptyset}) = \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{a\in I_N} \mathbb{1}_{C^R_{A,\emptyset}}(S_a(\mathcal{F}_{R'})) = d_I(\{x \in \mathcal{O}_K : x + A \subset \mathcal{F}_{R'}\}) = d_I(\mathcal{F}_{(-A+R')}).$$

Using $\nu_R(C^R_{A,\emptyset}) = \nu_{(-A+R)}\left(C^{(-A+R)}_{\{0\},\emptyset}\right)$, this is equivalent to $(-A+R')$ having weak light tails for I_N by Theorem 5.2.

Therefore showing that

$$\sigma_R(\{R' \in \mathcal{S}_R : (-A+R') \text{ has weak light tails with respect to } I_N\}) = 1$$

follows from showing that $\nu_R(\Psi_R(\mathcal{S}_R) \cap \mathrm{Gen}_A(\Omega_R, I_N)) = 1$. From the proof of Lemma 7.13 we know that $\nu_R(\Psi_R(\mathcal{S}_R)) = 1$. Together with Equation (41) the result follows.

Let

$$H_R = \bigoplus_i \mathcal{O}_K/F(R_i)$$

and V be the action of H_R is $\mathcal{S}_R$ given by $V_h(R)_i = h_i + R_i$. By Theorem 5.29, it is equivalent for the sieve $-A+R$ to have strong light tails for I_N, and for $-A+V_h(R)$ to have weak light tails with respect to I_N for every $h \in H_R$. Hence, the set

$$\{R' \in \mathcal{S}_R : (-A+R') \text{ has strong light tails with respect to } I_N\}$$

is equal to

$$\bigcap_{h\in H_R} V_h^{-1}(\{R'\in\mathcal{S}_R : (-A+R') \text{ has weak light tails with respect to } I_N\}).$$

Since σ_R is invariant under V, this is a countable union of sets of measure 1, consequently so is their intersection, as we wanted to show. $\square$

Of course, this does not imply that if R has strong light tails for some I_N, then $-A+R$ will also have strong light tails, just that there will be some $g_i \in G_{R,F}$ and a sieve $R = R(g_i)'$ such that $-A+R'$ has strong light tails for I_N.

Example 9.2. Let R be the cubefree sieve given by $R_i = p_i^3\mathbb{Z}$ and define a set $A = \{a_1, a_2, \dots\}$ by choosing the a_i in such a way that

(1) $a_1 = -1$,
(2) $a_i \equiv -i \mod p_i^3$ for all i,
(3) $a_i \equiv a_j \mod p_j^3$ for all $j < i$.

The Chinese Remainder Theorem guarantees us that we can build such a set. Additionally, we have that $|-A+p_i^3\mathbb{Z}| \le i$ for all i, and so $-A+R$ will be an Erdős sieve. Yet, it is clear that it does not have weak light tails for $I_N := [1,N]$, since $\mathcal{F}_{-A+R} \cap I_N = \emptyset$, given that $i \in -A+R_i$ for all $i \in \mathbb{N}$.

When R is a sieve with weak/strong light tails, then for any admissible finite A we will have that $-A+R$ will have weak/strong light tails, as implied by Lemma 6.26. On the other hand, if A is any admissible set, we have that if $-A+R$ has strong light tails, then R has strong light tails. This is because if $-A+R$ has strong light tails, then for any $a\in A$, we get that $a-A+R$ has strong light tails. Since $R_i \subset a-A+R_i$ for every i, this implies that R will have strong light tails.

Yet, the same does not happen when we replace strong by weak light tails, even assuming that A is finite.

Example 9.3. Let $A = \{-2,-1,0\}$ and R be the sieve defined by

$$R_1 = \{0,2\} + 8\mathbb{Z} \qquad R_i = 8(i-1) + \{0,1\} + p_i^2\mathbb{Z}.$$

As in Example 5.14, we can show that R does not have weak light tails for any Følner sequence. But $-A+R_1 = \{0,1,2,3,4\} + 8\mathbb{Z}$, and

$$-A+R_i = 8(i-1) + \{0,1,2,3\} + p_i^2\mathbb{Z},$$

so $-A+R$ does have weak light tails for $I_N = [0,N]$.

As a corollary of Theorem 9.1, we get the following result.

Corollary 9.4. *Let A be a subset of $\mathcal{O}_K$ and R an Erdős sieve such that*

$$\prod_i \left(1 - \frac{|-A+R_i|}{N(\mathfrak{b}_i)}\right) > 0.$$

Then, there exists a $g = (g_i)_{i\in\mathbb{N}}$ in $G_{R,F}$ and some B with $d(B) > 0$ such that

$$A + B \subset \mathcal{F}_{R(g)},$$

where $R(g)$ is defined as in Equation (29).

Proof. By Theorem 9.1, there exists some $g \in G_{R,F}$ such that $-A + R(g)$ has strong light tails. Taking B to equal $\mathcal{F}_{-A+R(g)}$, we have that $d(B) > 0$ by hypothesis, and $A + B \subset \mathcal{F}_{R(g)}$. $\square$

In [61], it was shown that for any $C \subset \mathbb{N}$, if $\overline{d}(C) > 0$, then there are infinite $A, B \subset \mathbb{N}$ such that $A + B \subset C$. Yet, Host has shown in Proposition 2 of [48], that there are sets C with $\overline{d}(C) > 0$ such that if there are infinite A and B are such that $A + B \subset C$, then we must have $\overline{d}(A) = \overline{d}(B) = 0$. It would be interesting to know if such an example can appear with $C = \mathcal{F}_R$ for some sieve R with strong light tails for B_N. We conjecture that this is not the case.

Conjecture 9.5. *For every Erdős sieve R with strong light tails for B_N, there exists some infinite $A \subset \mathcal{O}_K$ such that $-A + R$ has strong light tails for B_N.*

Recently the case where R is the squarefree sieve over $\mathbb{Q}$ was solved by van Doorn and Tao in [25] for the Følner sequence $I_N = [0, N]$ (see Theorem 8 of this paper). Although their construction does not directly apply for B_N, we adapt their argument for completeness, as it is not immediate that the constructed object is indeed a sieve with strong light tails for I_N. In order to do this, we will use the following lemma (see Lemma 13 in [25]).

Lemma 9.6. *There exists an infinite sequence of positive integers m_j such that*

(1) $m_j \equiv 0 \mod p^2$ *for* $p \leq 3 \exp\exp j$, *and*
(2) $m_j + a \not\equiv 0 \mod p^2$ *if* $p > 3\exp\exp j$ *and* $1 \leq a \leq \frac{p}{(\log\log p)^2}$.

Let $A = \{m_j : j \in \mathbb{N}\}$ be a set such that each m_j satisifes the properties in Lemma 9.6. We claim that if R is the squarefree sieve defined by $R_p = p^2\mathbb{Z}$, then the sieve $-A + R$ has strong light tails for I_N. Let W be the sieve defined by $W_p = (-A + R_p) \setminus R_p$. Since $-A + R = W \cup R$, and R has strong light tails for I_N, Lemma 6.26 implies that it is enough to show that W has strong light tails for I_N.

Assume that $a \in W_p \cap I_N = I_N \cap (-A + p^2\mathbb{Z}) \setminus p^2\mathbb{Z}$ for some p. This means that there is some $m_j \in A$ such that $a \equiv -m_j \not\equiv 0 \mod p^2$. It follows that $m_j \not\equiv 0 \mod p^2$, and so p cannot be smaller than or equal to $3\exp\exp j$, as this would contradict point (1) in Lemma 9.6. We must therefore have that $p > 3\exp\exp j$, and since $m_j + a \equiv 0 \mod p^2$, we must have that

$$N \geq a > \frac{p}{(\log\log p)^2}. \tag{42}$$

Note that here we are using the fact that a is positive, and that is why this argument does not immediately apply with I_N replaced by B_N. Taking the logarithm on both sides of Equation (42) gives that

$$\log p < \log(N) + 2\log\log\log p,$$

and so $\log p \ll \log N$. Replacing this in Equation (42) gives

$$p \ll N(\log\log N)^2.$$

We conclude that for any fixed $L \geq 1$,

$$\bigcup_{L<p} W_p \cap I_N \subset \bigcup_{L<p\ll N(\log\log N)^2} W_p \cap I_N.$$

For any fixed p, we have that $m_j \equiv 0 \mod p^2$ for all j such that $p \leq 3\exp\exp j$. All j that are bigger than $\log\log p$ satisfy this, so we conclude that $|W_p| \leq \log\log p$. Since each congruence class modulo p^2 intersects I_N in up to $N/p^2 + 1$ points, we get

$$\begin{aligned}\left|\bigcup_{L<p} W_p \cap I_N\right| &\leq \sum_{L<p\ll N(\log\log N)^2} (\log\log p)\left(\frac{N}{p^2}+1\right)\\ &\ll N \sum_{L<p\ll N(\log\log N)^2} \frac{\log\log p}{p^2} + \sum_{L<p\ll N(\log\log N)^2} \log\log p.\end{aligned}$$

Since the series $\sum(\log\log p)/p^2$ converges, the first term will go to zero once we divide by N and take L to infinity. By using the prime number theorem to count primes up to $N(\log\log N)^2$, it is then easy to see that the second sum is in $o(N)$, which completes the proof.

If Conjecture 9.5 holds, then an obvious follow up question is which sorts of sets A are such that $\mathcal{F}_R \subset A$ for some Erdős sieve R with strong light tails for B_N. First, this would surely imply that $\underline{d}(A) > 0$. But it is not true that all such sets contain some $\mathcal{F}_R$ as a subset, with R an Erdős sieve with strong light tails. This follows quickly from the following observation: if R and R' are Erdős sieves, R with strong light tails for B_N, and $\mathcal{F}_R \subset \mathcal{F}_{R'}$, then R' must also have strong light tails for B_N. This shows that simply taking $A = \mathcal{F}_{R'}$ for some sieve R' which only has weak light tails for B_N, or does not have weak light tail at all, already gives us the example of some A such that $\mathcal{F}_R \not\subset A$ for every R with strong light tails for B_N.

Proposition 9.7. *Let R be an Erdős sieve with strong light tails with respect to some Følner sequence I_N. If R' is an Erdős sieve such that*

$$\mathcal{F}_R \subset \mathcal{F}_{R'},$$

then R' has strong light tails with respect to I_N.

Proof. Our hypothesis is equivalent to the fact that

$$\bigcup_i R_i \supset \bigcup_i R'_i.$$

Take any $i \in \mathbb{N}$. As part of the proof of Lemma 6.14, we showed that if $R_i' \subset \bigcup_j R_j$, then we have

$$R_i' \subset \bigcup_{j:(\mathfrak{b}_j,\mathfrak{b}_i')\neq 1} R_j.$$

The result now follows by the same argument used in the proof of Theorem 6.19. $\square$

Example 9.8. Take the sieve W defined by $W_1 = 4\mathbb{Z}$ and $W_i = 1 + 4i + p_i^2\mathbb{Z}$ for $i > 1$. This sieve does not have weak light tails for any Følner sequence I_N (as seen in Example 5.14), but we have $d(\mathcal{F}_W) > 0$. By Proposition 9.7, there is no sieve R with strong light tails for some Følner sequence I_N such that $\mathcal{F}_R \subset \mathcal{F}_W$.

9.2. **Squarefree Values of Polynomials.** Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial. The set of squarefree values of f, that is

$$\Sigma_f = \{x \in \mathbb{Z} : f(x) \text{ is squarefree}\}$$

is an important object of study in number theory. Consider the sieve R^f such that $\mathcal{B}_{R^f} = \{p^2\mathbb{Z} : p \text{ prime}\}$ and defined by

$$R_p^f = \{m \in \mathbb{Z}/p^2\mathbb{Z} : f(m) \equiv 0 \mod p^2\} + p^2\mathbb{Z}.$$

Noting that $f(y + p^2k) \equiv f(y) \mod p^2$ for any $k \in \mathbb{Z}$, we see that for y to be in $\mathcal{F}_{R^f}$ is equivalent to $f(y) \not\equiv 0 \mod p^2$ for every p, which is equivalent to $f(y)$ being squarefree. That is, for any irreducible polynomial $f \in \mathbb{Z}[X]$ we have

$$\Sigma_f = \mathcal{F}_{R^f}. \tag{43}$$

Let ρ_f be the function given by

$$\rho_f(x) = |\{m \in \mathbb{Z}/x\mathbb{Z} : f(m) \equiv 0 \mod x\}|.$$

When $x = p^2$ for some prime, it is clear we have $\rho_f(p^2) = |R_p^f|$. We have the following result (see for example Lemma 2.2 in [17]).

Lemma 9.9. *If f is an irreducible polynomial of degree d we have*

$$\rho_f(p^2) \leq d$$

for all but a finite number of primes p.

As a consequence of Lemma 9.9, we see that if f is irreducible, then R^f is Erdős. Notice that this is not the case for the polynomial $f(X) = X^2$, that has only -1 and 1 as squarefree values.

When studying Σ_f, we usually want to show that it is infinite, by proving that, for $I_N = [0, N]$,

$$|\Sigma_f \cap I_N| = N \prod_{p \text{ prime}} \left(1 - \frac{\rho_f(p^2)}{p^2}\right) + o(N). \tag{44}$$

Since R^f is Erdős, Theorem 3.21 together with Equation (43), imply that Equation (44) will hold for some polynomial f if and only if R^f has weak light tails for I_N. Many authors have worked on this problem, usually also providing an explicit bound for the $o(N)$ error term. Estermann showed in [34] that Equation (44) holds for the polynomial $f(X) = X^2 + l$ for any non-zero integer l.

We can easily show this, by showing that R^f has strong light tails for I_N if $f(X) = aX^2 + bX + C$ is a degree 2 polynomial such that R^f is Erdős. Indeed, if R_p^f intersects I_N in some point x, then $p^2 \mid ax^2 + bx + c$ which implies that $p \ll_f N$. Hence, there is some C such that

$$\left| I_N \cap \bigcup_{p>L} R_p^f \right| \leq \sum_{L<p<CN} \rho_f(p^2)\left(1 + \frac{N}{p^2}\right) \leq N \sum_{L<p<CN} \frac{\rho_f(p^2)}{p^2} + \pi(CN).$$

Dividing both sides by N and letting N go to infinity will make the right hand side go to 0 if R^f is Erdős, showing that R^f has strong light tails.

The same approach cannot be used to show that any polynomial of degree 3 has strong light tails, since if $p^2 \mid x^3$ with $|x| \leq N$, the best bound we can provide is $p \ll N^{3/2}$, and then we would be bounding $\left| I_N \cap \bigcup_{i>L} R_i^f \right|$ by $\pi(N^{3/2})$, which is worse than the trivial bound. When f is an irreducible polynomial of degree 3, Hooley showed in [47] that R^f has strong light tails. More generally, he showed the following result.

Theorem 9.10. *Let f be an irreducible polynomial of degree $d \geq 3$. The sieve $R^{f,d-1}$ defined by*

$$R_p^{f,d-1} = \{m \in \mathbb{Z}/p^{d-1}\mathbb{Z} : f(m) \equiv 0 \mod p^{d-1}\mathbb{Z}\} + p^{d-1}\mathbb{Z}$$

has strong light tails.

More generally, taking any integer $l \geq 2$, we write $R^{f,l}$ to be the sieve defined by

$$R_p^{f,l} = \{m \in \mathbb{Z}/p^l\mathbb{Z} : f(m) \equiv 0 \mod p^l\mathbb{Z}\} + p^l\mathbb{Z}. \tag{45}$$

Because $p^{l+1} \mid f(m)$ implies that $p^l \mid f(m)$, we have that $R_p^{f,l+1} \subset R_p^{f,l}$ for every $l \geq 2$. Consequently, if f is an irreducible polynomial of degree $d \geq 3$, Theorem 9.10 implies that $R^{f,l}$ has strong light tails for all $l \geq d-1$. In the particular case where $f(X) = X^d + c$ is an irreducible polynomial, Heath-Brown showed in Theorem 1 of [44] that if $l \geq (5d+3)/9$, then $R^{f,l}$ has strong light tails.

There is no specific irreducible polynomial of degree $d \geq 4$ for which it is currently known that $R^{f,2}$ has weak light tails for $I_N = [0, N]$. Yet, Filaseta showed in [35] that almost all irreducible polynomials (with respect to their height) have infinitely many squarefree values. Browning and Shparlinski further improved this, by showing that for almost all irreducible polynomials Equation (44) holds (for the specific result, see Theorem 1.1 in [17]).

Finally, we remark that Granville has shown (see Theorem 1 of [41]) that Equation (44) holds for all irreducible polynomials, independently of degree, if we assume the abc-conjecture. A more extensive review of the history of this problem can be found in [51].

The squarefree value of multivariate polynomials is also a problem of interest (see [13] for an application to counting number fields with Galois group S_n for $n \leq 5$). For this reason, in what will follow, we will work in the following level of generality. Given a number field K of degree n, integers $v, l \geq 1$ and a polynomial $f \in \mathcal{O}_K[X_1, \ldots, X_v]$, we write $R^{f,l}$ to be the sieve supported on

$$\mathcal{B}_R = \{\mathfrak{p}^l \times \cdots \times \mathfrak{p}^l : \mathfrak{p} \text{ prime ideal of } \mathcal{O}_K\}$$

and defined by

$$R_{\mathfrak{p}}^{f,l} = \{m \in \mathcal{O}_K/\mathfrak{p}^l \times \cdots \times \mathcal{O}_K/\mathfrak{p}^l : f(m) \in \mathfrak{p}^l\} + \mathfrak{p}^l \times \cdots \times \mathfrak{p}^l,$$

where $\mathfrak{p}^l \times \cdots \times \mathfrak{p}^l$ is the Cartesian product of v copies of $\mathfrak{p}^l$. Note that there may be $\mathfrak{p}$ such that $R_{\mathfrak{p}}^{f,l} = \emptyset$.

Note that $R^{f,l}$ is always Erdős when f is an irreducible polynomial. Over $\mathbb{Z}$ with $l = 2$ this is Lemma 9.9. For $\mathbb{Z}[X_1, \ldots, X_v]$, with $l = 2$, see the proof of Theorem 3.2 in [66]. The same methods show that

$$\left|R_{\mathfrak{p}}^{f,l}\right| = O_K(N(\mathfrak{p})^{lv-2})$$

for any arbitrary number field K. Using Theorem 5.1 we automatically get the following result.

Corollary 9.11. *Let $f \in \mathcal{O}_K[X_1, \ldots, X_v]$ be an irreducible polynomial such that $R^{f,l}$ is a sieve with weak light tails with respect to a Følner sequence I_N, and A a finite $R^{f,l}-$admissible set. Then,*

$$d_I(\{x \in \mathcal{O}_K/\mathfrak{p}^l \times \cdots \times \mathcal{O}_K/\mathfrak{p}^l : \forall_{a\in A}\, f(x+a) \text{ is } l-\text{free}\}) = \prod_{\mathfrak{p}} \left(1 - \frac{|-A + R_{\mathfrak{p}}^{f,l}|}{N(\mathfrak{p})^{lv}}\right).$$

Similarly, let $f_1, \ldots, f_m \in \mathcal{O}_K[X_1, \ldots, X_v]$ be a collection of irreducible polynomials. If we want to study those x such that $f_i(x)$ is $l-$free for every i, we then just have to consider the sieve $R = \bigcup_{i=1}^m R^{f_i,l}$, since $\mathcal{F}_R = \bigcap_{i=1}^m \mathcal{F}_{R^{f_i,l}}$. Given that weak light tails are preserved under union of sieves by Lemma 6.26, we get the following corollary.

Corollary 9.12. *Let $f_1, \ldots, f_m \in \mathcal{O}_K[X_1, \ldots, X_v]$ be irreducible polynomials such that $R^{f_i,l}$ has weak light tails for every i. Then,*

$$d_I(\{x \in \mathcal{O}_K/\mathfrak{p}^l \times \cdots \times \mathcal{O}_K/\mathfrak{p}^l : \forall_{1\leq i\leq m}\, f_i(x) \text{ is } l-\text{free}\}) = \prod_{\mathfrak{p}} \left(1 - \frac{|\bigcup_{i=1}^m R_{\mathfrak{p}}^{f_i,l}|}{N(\mathfrak{p})^{lv}}\right).$$

Example 9.13. In [24], Dimitrov showed that there are infinitely many x such that x^2+1 and x^2+2 are squarefree, and provides the density of such x. We can use Corollary 9.12 to obtain this density easily.

Let $f(X) = X^2+1$ and $g(X) = X^2+2$. Since both are polynomials of degree 2, we have that R^f and R^g have strong light tails with respect to $I_N = [0, N]$. We also have that $R_2^f = R_2^g = \emptyset$ and that $R_p^f \cap R_p^g = \emptyset$ for all p. Since $|R_p^f| = \left(\frac{-1}{p}\right) + 1$ and $|R_p^g| = \left(\frac{-2}{p}\right) + 1$ for $p > 2$ (where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol), it follows that the union $R^f \cup R^g$ is well defined and

$$|R_p^f \cup R_p^g| = \left(\frac{-1}{p}\right) + \left(\frac{-2}{p}\right) + 2$$

for all $p \geq 2$, given that $f(x) = g(x) + 1$. Corollary 9.12 now shows that

$$d_I\left(\{x \in \mathbb{Z} : x^2+1 \text{ and } x^2+2 \text{ are squarefree}\}\right) = \prod_{p>2}\left(1 - \frac{\left(\frac{-1}{p}\right) + \left(\frac{-2}{p}\right) + 2}{p^2}\right).$$

In what remains of this section, we will work over $\mathbb{Q}$ with $l = 2$, although similar results could be obtained more generally. A first question that is very natural, is of when do we have $R^f \sim R^g$, for some arbitrary polynomials $f, g \in \mathbb{Z}[X]$. Since both sieves are supported on the same set, Lemma 6.15 shows that this is equivalent to $R^f = R^g$, if at least one of the sieves has strong light tails for some I_N.

Consequently, we need to answer when $R^f = R^g$. Note that this does not require that $f = g$, if one of these polynomials is not irreducible (over $\mathbb{Z}$).

Example 9.14. Let $f(X) = 2X^2 + 1$ and $g(X) = 2f(X)$. If $p = 2$, it is easy to see that $R^f_2 = R^g_2 = \emptyset$. On the other hand, for any $p > 2$, we have that $p^2 \mid g(m)$ is equivalent to $p^2 \mid f(m)$. Consequently, it follows that $R^f = R^g$.

More generally, if f is an irreducible polynomial, and $\mathcal{P}_f$ is the set of primes p such that $R^f_p = \emptyset$, then for any $m = p_1 \dots p_k$ with $p_i \in \mathcal{P}_f$, we will have that $R^f = R^{mf}$. If we take both f and g to be irreducible, then $R^f = R^g$ implies that $f = g$.

Theorem 9.15. *Let $f, g \in \mathbb{Z}[X]$ be distinct irreducible polynomials. We have $R^f \neq R^g$.*

For this proof, we will make use of the resultant $\operatorname{Res}(f, g) \in \mathbb{Z}$ (see Section 3 in [21] for the definition). If f and g are irreducible polynomials in $\mathbb{Z}[X]$, then there are polynomials $a, b \in \mathbb{Z}[X]$ such that

$$a(X)f(X) + b(X)g(X) = \operatorname{Res}(f, g).$$

Consequently, if $f(x) \equiv g(x) \equiv 0 \mod p$ for some prime p, we must have that $p \mid \operatorname{Res}(f, g)$, and so there are only finitely many such primes p.

Proof. We start by noticing that there are infinitely many primes p such that $f(x) \equiv 0 \mod p$. These correspond to primes that split completely in the splitting field of f, so there are infinitely many by the Cheboratev Density Theorem (see for example Theorem 13.4 in Chapter 7 of [62])[5].

Taking one such prime p that does not divide the discriminant of f, nor the resultant of f and g, we will have that there is some x such that $f(x) \equiv 0 \mod p$, and by Hensel's Lemma (see for example (4.6) in Chapter 2 of [62]) there is some s such that $f(s) \equiv 0 \mod p^2$ and $s \equiv x \mod p$. Given that $g(x)$ is not congruent to 0 modulo p^2, we have $g(s) \not\equiv 0 \mod p^2$, and so $s \in R^f_p$ while $s \notin R^g_p$. ☐

Remark 9.16. Take f and g to be two irreducible polynomials. We have pointed out that the there are only finitely many p such that

$$W_p = \{x \in \mathbb{Z}/p^2\mathbb{Z} : f(x) \equiv g(x) \equiv 0 \mod p\} + p^2\mathbb{Z}$$

is non-empty. Noting that if p^2 divides $(fg)(x)$, then either p^2 divides one of $f(x)$ or $g(x)$, or alternatively, p divides both $f(x)$ and $g(x)$, we get

$$R^{fg}_p = R^f_p \cup R^g_p \cup W_p.$$

It follows that, assuming that this union is well defined, the sieve R^{fg} will have weak light tails for B_N if the same holds for R^f and R^g (by Lemma 6.26) and $\mathcal{F}_{R^{fg}} = \Sigma_{fg}$ (as pointed out in Equation (43)). Consequently, if f is a polynomial that can be written as the product of distinct irreducible polynomials

[5]This result was apparently first shown by Schur in [70], although we were not able to access this source.

f_i all of which satisfying that R^{f_i} has weak light tails for B_N, we will have that Corollary 9.11 also holds for f.

We conclude this section by showing that if f is irreducible and R^f has weak light tails for some I_N, then $X_{R^f} = \Omega_{R^f}$.

Proposition 9.17. *Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial such that R^f has weak light tails for some I_N. Then*

$$X_{R^f} = \Omega_{R^f}.$$

Proof. Let f be an irreducible polynomial in $\mathbb{Z}[X]$. We want to show that $\limsup_{p\to\infty} \lambda(R_p^f) = \infty$ (with λ defined as in Equation (40)) and then the result will follow from Proposition 8.13.

Since f is irreducible, so is $g(X) := f(X+y)$ for any $y \in \mathbb{Z}$, given that the map that sends $f(X)$ to $f(X+y)$ is a ring automorphism of $\mathbb{Z}[X]$. Consequently, there are only finitely many primes p for which $f(x) \equiv g(x) \equiv 0 \mod p$ has a solution (at most those that divide the resultant of f and g). Since any solution to $f(x) \equiv g(x) \equiv 0 \mod p^2$ would be a solution modulo p, it follows that for any y, there are only finitely many primes p such that $f(x) \equiv f(x+y) \equiv 0 \mod p^2$ has a solution.

Consequently, for any $N \geq 1$, we can always find some big enough p such that $f(x) \equiv 0 \mod p^2$ has a solution, but $f(x+y) \not\equiv 0 \mod p^2$ for any non-zero $y \in [-N, N]$. It follows that $\limsup_{p\to\infty} \lambda(R_p^f) = \infty$. which concludes the proof. $\square$

Remark 9.18. If f is the product of irreducible polynomials, this might not hold. Take

$$f(X) = (2X+1)(2(X-1)+1).$$

We have that $R_2^f = \emptyset$. Additionally, we have that the resultant of $(2X+1)$ and $(2(X-1)+1)$ is -4 (a power of 2), so $R_p^f = R_p^{(2X+1)} \cup R_p^{(2(X-1)+1)}$ for every $p \geq 3$. It follows that R_p^f is always of the form $R_p^f = \{x_p, x_p+1\} + p^2\mathbb{Z}$, where x_p is the unique solution to $2x_p + 1 \equiv 0 \mod p^2$. By the same argument as in Example 8.1, it follows that $X_{R^f} \neq \Omega_{R^f}$.

9.3. **Ergodic Prime Number Theorem for R-free Numbers.** Inspired by the work in [72], we will show in this section an ergodic Prime Number Theorem for $R-$free numbers.

Given a function $a : \mathbb{N} \to \mathbb{N}$, we say it is *Besicovitch almost periodic* (see [11]), if for every $\epsilon > 0$, there is some trigonometric polynomial $P_\epsilon(x) = \sum_{j=1}^k c_j e(x\alpha_j)$ (where $e(x) = e^{2\pi i x}$) such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} |a(m) - P_\epsilon(m)| < \epsilon.$$

We will show that if R is an Erdős sieve with weak light tails for the Følner sequence $I_N = [1, N]$, then $\mathbb{1}_{\mathcal{F}_R}$ is a Besicovitch almost periodic function, in order to apply Corollary 1.26 from [11]. Let v_p denote the $p-$adic valuation, and

$$\Omega(m) = \sum_{p \text{ prime}} v_p(m). \tag{46}$$

This result states the following.

Lemma 9.19. *Let (X,T) be a uniquely ergodic dynamical system with unique $T-$invariant measure μ. Let $a(n):\mathbb{N}\to\mathbb{C}$ be a Besicovitch almost periodic function and let $M(a) := \lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} a(m)$ be its mean value. Then for every function $f\in C(X)$ and $x\in X$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} a(m)f(T^{\Omega(m)}x) = M(a)\int_X f\,d\mu.$$

We get the following result.

Theorem 9.20. *Let R be an Erdős sieve with weak light tails for $I_N=[1,N]$. Let (X,T) be a uniquely ergodic dynamical system, and μ its unique invariant measure. Then for every function $f\in C(X)$ and $x\in X$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m\in\mathcal{F}_R\cap I_N} f(T^{\Omega(m)}x) = d_I(\mathcal{F}_R)\int_X f\,d\mu.$$

Proof. By Lemma 9.19 it is enough to show that $\mathbb{1}_{\mathcal{F}_R}$ is Besicovitch almost periodic. We start by pointing out that for any arithmetic sequence $a+b\mathbb{Z}$, we have

$$\mathbb{1}_{a+b\mathbb{Z}}(x) = \frac{1}{b}\sum_{j=0}^{b-1} e\left(j\frac{(x-a)}{b}\right),$$

which follows from character orthogonality for the cyclic group $\mathbb{Z}/b\mathbb{Z}$. Consequently, denoting by $\overline{R}_i$ the image of R_i in $\mathbb{Z}/b_i\mathbb{Z}$, we have that for any L the function

$$P_L(x) = 1-\prod_{i\le L}\prod_{y\in\overline{R}_i}(1-\mathbb{1}_{y+b_i\mathbb{Z}}(x))$$

is a trigonometric polynomial, which is 0 if $x\in R_i$ for some $i\le L$, and 1 otherwise.

We can partition the set $I_N=[1,N]$ like $I_N=(\mathcal{F}_R\cap[1,N])\cup S_1\cup S_2$, where

$$S_1 = I_N\cap\bigcup_{i\le L}R_i \text{ and } S_2 = I_N\cap\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\le L}R_j\right).$$

If $m\in\mathcal{F}_R$, then for all L, $P_L(m)=1$, and so $|\mathbb{1}_{\mathcal{F}_R}(m)-P_L(m)|=0$. If $m\in S_1$, both values are 0, and we get $|\mathbb{1}_{\mathcal{F}_R}(m)-P_L(m)|=0$. Finally, if $m\in S_2$, we get $|\mathbb{1}_{\mathcal{F}_R}(m)-P_L(m)|=1$, so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}|\mathbb{1}_{\mathcal{F}_R}(m)-P_L(m)| = \lim_{N\to\infty}\frac{1}{N}\sum_{m\in S_2}1 = d_I\left(\bigcup_{i>L}R_i\setminus\bigcup_{j\le L}R_j\right).$$

Since R has weak light tails if and only if this density goes to 0 as L goes to infinity, the result follows. ☐

Remark 9.21. Let us explain why we call this result a 'Prime Number Theorem'. It is well known (see [11]) that the Prime Number Theorem is equivalent to the fact that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}(-1)^{\Omega(n)} = 0.$$

When considering the system $X = \{-1, 1\}$ with $T(x) = -x$ and $f(x) = x$, the result implies the Prime Number Theorem, as we get

$$\frac{1}{N}\sum_{m=1}^{N} f(T^{\Omega(m)x}) \to \frac{1}{2}(f(1) + f(-1)) = 0.$$

We point out that Theorem 9.20 implies all results in [72], since each of the sets the authors consider in this paper can be realized as R-free numbers for some sieve R with proven strong light tails. As previously mentioned, Heath-Brown showed in Theorem 1 of [44] that if $f(X) = X^d + c$ is an irreducible polynomial and $l \geq (5d + 3)/9$, then $R^{f,l}$ (see Equation (45)) has strong light tails for I_N. Applying Theorem 9.20 gives the following result, which is Theorem 1.1 in [72].

Corollary 9.22. *Let $f(X) = X^d + c$ be an irreducible polynomial, l an integer such that $l \geq (5d+3)/9$, and (X, T) a uniquely ergodic dynamical system, with μ its unique invariant measure. Then for every function $g \in C(X)$ and $x \in X$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq m\leq N: m^d+c \text{ is squarefree}} g(T^{\Omega(m)}x) = \prod_p \left(1 - \frac{|R_p^{f,l}|}{p^l}\right) \int_X g\, d\mu.$$

Theorem 9.10 together with Remark 9.16 imply that if f is a polynomial that can be written as the product of distinct irreducible polynomials all of degree smaller than or equal to 3, then R^f has strong light tails for I_N. Applying Theorem 9.20 for this sieve gives the following result, which corresponds to Theorem 4.1 in [72].

Corollary 9.23. *Let $f \in \mathbb{Z}[X]$ be a polynomial that can be written as the product of distinct irreducible polynomials all of degree smaller than or equal to 3. Let (X, T) be a uniquely ergodic dynamical system, with μ its unique invariant measure. Then for every function $g \in C(X)$ and $x \in X$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq m\leq N: g(m) \text{ is squarefree}} g(T^{\Omega(m)}x) = \prod_p \left(1 - \frac{|R_p^f|}{p^2}\right) \int_X g\, d\mu.$$

Finally, using Example 9.13, we get the following result, which corresponds to Corollary 4.2 in [72].

Corollary 9.24. *Let (X, T) be a uniquely ergodic dynamical system, with μ its unique invariant measure. Then for every function $g \in C(X)$ and $x \in X$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\leq m\leq N \\ m^2+1, m^2+2 \text{ squarefree}}} g(T^{\Omega(m)}x) = \prod_{p>2} \left(1 - \frac{\left(\frac{-1}{p}\right) + \left(\frac{-2}{p}\right) + 2}{p^2}\right) \int_X g\, d\mu.$$

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