

Linear Spaces, Buildings and Principal Bundles on Curves in Tropical Geometry



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Abstract

This dissertation establishes new connections between tropical linear spaces, Bruhat–Tits buildings, and valuated matroids. Secondly, it investigates these structures from the perspective of real tropical geometry. In addition, it develops a tropical analogue of reductive groups, along with a theory of tropical principal bundles on tropical curves, i.e., on metric graphs.

The first part formulates a tropical interpretation of the Bruhat–Tits building of the projective general linear group $\mathrm{PGL}_{r+1}(K)$. We prove that this building is the inverse limit of all tropicalized linear spaces, obtaining a linear version of Payne’s limit theorem, and we prove a faithful tropicalization result for compactified linear spaces. We further extend the construction of tropical linear spaces to valuated matroids on infinite ground sets and show that the tropical linear space of the universal realizable matroid coincides with the building itself, extending a result of Dress and Terhalle.

The second part introduces the signed Goldman–Iwahori space for real closed valued fields, a signed analogue of the building studied in the first part. Using methods from real tropical geometry, we show that this space arises as the inverse limit of all real tropicalized linear embeddings and identify it with the real tropical linear space of the universal realizable oriented matroid. While many constructions from the first part extend naturally, the signed case exhibits new phenomena, including non-diagonalizable signed seminorms. In the constant coefficient case, for $K = \mathbb{R}$, we describe this space explicitly and relate it to real Bergman fans.

The final part introduces an elementary tropical analogue of a reductive group that combines the datum of a Weyl group and the tropicalization of a fixed maximal torus. For the classical groups, as well as G_2 , these tropical reductive groups admit concrete realizations as tropical matrix groups that resemble their classical algebraic counterparts. Employing this perspective, we introduce tropical principal bundles on metric graphs and study their explicit presentations as pushforwards of line bundles along covers with symmetries and extra data. We classify tropical principal bundles on metric graphs of genus zero in analogy with the classical result of Grothendieck and Harder, and, in parallel with the classical results of Frățișăwe describe the tropical moduli spaces of stable and semistable bundles on a tropical elliptic curve. Our main result identifies the essential skeleton of the moduli space of semistable principal bundles on a Tate curve with its tropical analogue.

Zusammenfassung

Diese Dissertation stellt neue Verbindungen zwischen tropischen linearen Räumen, Bruhat–Tits-Gebäuden und bewerteten Matroiden her. Im zweiten Teil untersucht sie diese Strukturen aus der Perspektive der reellen tropischen Geometrie. Darüber hinaus entwickelt sie tropische Analoga reductiver Gruppen sowie eine Theorie tropischer Hauptfaserbündel auf tropischen Kurven, das heißt, auf metrischen Graphen

Der erste Teil entwickelt eine tropische Interpretation des Gebäudes der projektiven allgemeinen linearen Gruppe $\mathrm{PGL}_{r+1}(K)$. Wir zeigen, dass dieses Gebäude als inverser Limes aller tropikalisierten linearen Räume entsteht und erhalten damit eine lineare Version des Satzes von Payne. Zudem beweisen wir in diesem Zusammenhang ein Resultat zur treuen Tropikalisierung für kompaktifizierte lineare Räume. Die Konstruktion tropischer linearer Räume wird anschließend auf bewertete Matroide mit unendlicher Grundmenge erweitert und wir zeigen, dass der zu dem universellen realisierbaren Matroid gehörige tropische lineare Raum mit dem Gebäude selbst übereinstimmt. Dies ist eine Verallgemeinerung eines Resultats von Dress und Terhalle.

Der zweite Teil führt den signierten Goldman–Iwahori–Raum für reell abgeschlossene bewertete Körper ein, ein signiertes Analogon des im ersten Teil untersuchten Gebäudes. Unter Verwendung von Methoden der reellen tropischen Geometrie zeigen wir, dass dieser Raum als inverser Limes aller reell tropikalisierten linearen Räume entsteht und mit dem reell-tropischen linearen Raum des universellen realisierbaren orientierten Matroids übereinstimmt. Während sich viele Konstruktionen aus dem ersten Teil erwartungsgemäß übertragen lassen, treten im signierten Fall zugleich neue Phänomene auf, darunter signierte Seminormen, die nicht diagonalisierbar sind. Im Fall konstanter Koeffizienten, also für $K = \mathbb{R}$, beschreiben wir diesen Raum explizit und stellen Verbindungen zu reellen Bergman–Fächern her.

Der letzte Teil entwickelt ein elementares tropisches Analogon einer reductiven Gruppe, das die Daten einer Weyl-Gruppe und der Tropikalisierung eines maximalen Torus kombiniert. Für klassische Gruppen sowie für G_2 lassen sich diese tropischen Gruppen als tropische Matrixgruppen realisieren, die ihre klassischen algebraischen Versionen widerspiegeln. In diesem Rahmen führen wir tropische Hauptfaserbündel auf metrischen Graphen ein und untersuchen deren explizite Darstellungen als Pushforwards von Geradenbündeln entlang Überlagerungen mit Symmetrien und zusätzlicher Struktur. Wir klassifizieren tropische Hauptfaserbündel auf metrischen Graphen vom Geschlecht null in Analogie zu klassischen Resultaten von Grothendieck und Harder und beschreiben – entsprechend den klassischen Resultaten von Frăţilă – die tropischen Modulräume stabiler und semistabiler Bündel auf einer tropischen elliptischen Kurve. Das Hauptresultat identifiziert schließlich das essentielle Skelett des Modulraums semistabiler Bündel auf einer Tate-Kurve mit seinem tropischen Gegenstück.

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Introduction

Tropical geometry has emerged as a bridge between algebraic geometry and combinatorics, providing a powerful dictionary that translates algebro-geometric objects into combinatorial, polyhedral ones. The general philosophy lies in the tropicalization process, which assigns to an algebraic variety X a piecewise linear tropical object $\text{Trop}(X)$. This new object retains many of the essential information about X , while at the same time greatly simplifying its structure. The tropical objects obtained through this procedure are fundamentally different from the original varieties: While algebraic varieties are defined by systems of polynomial equations, their tropical counterparts are discrete, piecewise linear, polyhedral structures. They can be viewed as combinatorial shadows, or as non-Archimedean skeleta, of the original spaces.

A feature of the tropicalization procedure is that it does not behave functorially. This means that the same variety X can have vastly differing tropicalizations $\text{Trop}(X)$ depending on the way X is embedded. An influential theorem of Payne states that, by taking the inverse limit of all embedded tropicalizations, one recovers the Berkovich analytification X^{an} , establishing a strong connection between Berkovich analytic spaces and embedded tropicalizations.

Another key aspect is that, although tropicalization constitutes a drastic degeneration, many fundamental notions from algebraic geometry, such as curves, divisors, line bundles, Jacobians, varieties embedded in toric varieties, linear spaces, and even certain moduli spaces, admit meaningful tropical analogues. These tropical counterparts retain enough essential structure to support a rich theory, one that both mirrors the classical theory and provides new insights into questions in algebraic geometry.

A central philosophy in tropical geometry is to develop its theory intrinsically, in a way that is parallel to that of algebraic geometry. One particularly fruitful direction is the tropical theory of line bundles on curves. Metric graphs serve as tropical analogues of smooth projective curves, and there exists a well-developed theory of divisors and line bundles on metric graphs. Since its appearance in the mathematical world in [MZ08], the theory of divisors and linear systems on metric graphs has sparked extensive research in the past decades. Key foundational achievements include the tropical Riemann–Roch theorem for metric graphs [BN07, MZ08, GK08] and Baker’s specialization method [Bak08b], which has produced major applications both to classical Brill–Noether theory for algebraic curves [CDPR12, JR21] and to the study of the birational geometry of moduli spaces [FJP24].

This dissertation enriches the existing “dictionary” of tropical geometry in several directions. First, we build bridges between Bruhat–Tits buildings and tropical linear spaces, thereby generalizing the tropicalization process to new settings. In Chapter 1, we show that the inverse limit of all linear tropicalizations yields the compactified Bruhat–Tits building of the group $\text{PGL}_{r+1}(K)$. Using the theory of tropical linear spaces, encoded via valuated matroids, we further show that this building arises as a universal realizable tropical linear space.

Real algebraic geometry studies algebraic varieties over the real numbers or, more generally,

over real closed fields. The presence of an ordering allows for a refined version of tropicalization: the ambient space decomposes into orthants, each of which can be tropicalized separately and then glued together. This approach plays a key role in Viro’s patchworking [Vir84], which constructs real hypersurfaces with prescribed topological invariants. In Chapter 2, we combine methods from real tropical geometry, the theory of oriented matroids, and the theory of buildings to introduce a real tropical analogue of the building of PGL.

Bruhat–Tits buildings are rich polyhedral objects encoding the structure of a reductive group G over a valued field. Their building blocks are apartments which are affine spaces tiled according to the combinatorics of the Weyl group W associated with G . In this way, a building reflects the “combinatorial essence” of the group G . From a different yet related perspective within \mathbb{F}_1 -geometry, Weyl groups can actually be interpreted as analogues of the classical groups defined over the so-called field \mathbb{F}_1 with one element. These ideas play a central role in Chapter 3, where we introduce an elementary tropical analogue of a reductive group over a valued field. By combining the tropicalization of a maximal torus in a given reductive group G with the combinatorial structure of its Weyl group, we construct a tropical reductive group G^{trop} . For all classical groups and for G_2 , these tropical groups admit concrete realizations via tropical matrix groups, closely mirroring their classical counterparts.

Within this framework, we develop a theory of tropical principal bundles on metric graphs that serve as tropical analogues of principal G -bundles on algebraic curves. We classify such bundles in genus zero and describe the moduli spaces of stable and semistable bundles on a metric circle. Finally, we prove that for a Tate curve X , the essential skeleton of the moduli space of semistable principal G -bundles on X naturally identifies with its tropical analogue.

Before turning to the new material developed in Chapters 1, 2, and 3, we first present in the following sections the fundamental ideas of tropical geometry and the related fields that form the basis of this work.

1 Embedded Tropical Geometry

One approach to tropical geometry originates in theoretical computer science and is based on tropical algebra. From this point of view, tropical geometry can be regarded as algebraic geometry over the *tropical semifield* \mathbb{T} . This algebraic structure was introduced in the work of Imre Simon [Sim88], who was working in Brazil, hence the term *tropical*, and is defined as the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ together with *tropical addition*

$$x \oplus y := \min\{x, y\}$$

and *tropical multiplication*

$$x \odot y := x + y.$$

With these operations \mathbb{T} is a semifield, i.e., tropical addition and multiplication satisfy all the axioms of a field, except for the existence of additive inverses. Although this definition may appear ad hoc at first glance, these operations arise naturally by applying a valuation to ordinary addition and multiplication in a non-Archimedean field.

Recall that a non-Archimedean *valuation* on a field K is a map

$$\text{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$$

such that

- (i) $\text{val}(a) = \infty$ if and only if $a = 0$,
- (ii) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$, and
- (iii) $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$ for all $a, b \in K$.

Example 1.1. A guiding example is the field of *Puiseux series* over \mathbb{C} which is defined as

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n})),$$

where $\mathbb{C}((t^{1/n}))$ is the field of Laurent series in a formal variable $t^{1/n}$. A Puiseux series is a formal power series

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots,$$

where $c_i \in \mathbb{C}^*$ for all i , and $a_1 < a_2 < a_3 < \dots$ are rational numbers that share a common denominator. The valuation is given by

$$\text{val} : \mathbb{C}\{\{t\}\} \longrightarrow \mathbb{R} \cup \{\infty\},$$

defined by sending a non-zero Puiseux series $c(t) \in \mathbb{C}\{\{t\}\}^*$ to the smallest exponent a_1 that appears in its expansion.

Let $\iota : X \hookrightarrow \mathbb{A}^n$ be a closed embedding of an affine variety X . There are two ways to tropicalize the embedded variety X : one can either tropicalize the points of X , or tropicalize the equations that define X . We begin with the first method. The embedded *tropicalization* $\text{Trop}(X, \iota)$ of X with respect to the embedding ι is the Euclidean closure of

$$\{(\text{val}(x_1), \dots, \text{val}(x_n)) \mid (x_1, \dots, x_n) \in X(L)\} \subseteq \mathbb{T}^n,$$

where L/K is an extension of valued fields such that L is algebraically closed and non-trivially valued. This construction also works well in projective coordinates: For a projective variety $\iota : X \hookrightarrow \mathbb{P}_K^n$ its tropicalization $\text{Trop}(X, \iota)$ then lives in tropical projective space

$$\mathbb{TP}^n := (\mathbb{T}^{n+1} \setminus (\infty, \dots, \infty)) / \mathbb{R}(1, \dots, 1).$$

A characteristic – viewed either as a feature or a bug – of tropical geometry is its dependence on the embedding: the same variety can have very different tropicalizations, depending on the chosen coordinates. For example, Figure 1 illustrates this phenomenon in its simplest form. It shows on the left the tropicalization of a line in the plane \mathbb{P}^2 , and on the right the tropicalization of a line embedded in the space \mathbb{P}^3 that has an additional ray.

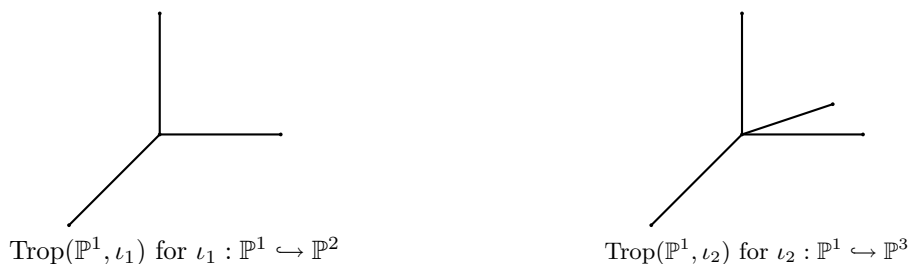


Figure 1: Tropicalizations of different embeddings of \mathbb{P}^1 .

Therefore, in our notation we highlight the importance of the choice of the embedding. In Chapters 1 and 2, we will fix $X = \mathbb{P}^n$ and let the embedding vary.

By work of Bieri–Groves [BG84] and Speyer–Sturmfels [SS04], the tropicalization $\text{Trop}(X, \iota)$ can be given the structure of a balanced, weighted, *rational polyhedral complex* of pure dimension $\dim X$. In particular, $\text{Trop}(X, \iota)$ can be written as a union of polyhedra intersecting nicely, making it an inherently combinatorial object.

The philosophy of tropical geometry is that in many cases the combinatorics of $\text{Trop}(X, \iota)$ reflects geometric properties of X . Figure 2 shows the tropicalization of a smooth plane cubic curve. Such tropical plane cubics are called *symmetric honeycomb forms* by Chan and Sturmfels [CS13]. Here a fundamental topological invariant of the algebraic curve, its genus, is preserved under tropicalization and appears combinatorially as a cycle in the tropical curve.

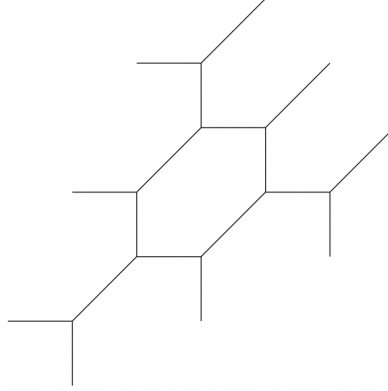


Figure 2: The tropicalization of a plane cubic curve. Adapted from [Jel20].

On the other hand, one can also tropicalize the polynomials defining our variety X given by a homogeneous ideal $I(X) \subseteq K[x_0, \dots, x_n]$. For each homogeneous polynomial $f = \sum_J a_J x_J \in I(X)$, we define its tropicalization $\text{trop}(f)$ to be the tropical polynomial obtained by replacing addition and multiplication in the expression for f with tropical addition and tropical multiplication, respectively, and replacing the coefficients in K with their valuations. This tropical polynomial $\text{trop}(f)$ together with the semifield operations induces a piecewise linear function

$$\begin{aligned} \mathbb{T}^{n+1} &\longrightarrow \mathbb{T}, \\ (x_0, \dots, x_n) &\longmapsto \min_{J=(j_0, \dots, j_n) \in \mathbb{N}^{n+1}} \{\text{val}(a_J) + j_0 x_0 + \dots + j_n x_n\}. \end{aligned} \quad (1)$$

The *tropical zero locus* of $\text{trop}(f)$ is then defined as

$$V^{\text{trop}}(\text{trop } f) := \{x \in \mathbb{TP}^n \mid \text{the minimum in (1) is attained at least twice}\}.$$

The *Fundamental Theorem* of tropical geometry tells us that whether we tropicalize the equations or the solutions, we obtain the same object:

Theorem 1.2 ([MS15, 3.2.5], [EKL06, 2.1.1]). *Let $\iota : X \hookrightarrow \mathbb{P}^n$ be a projective variety.*

(a) *The tropicalization of the vanishing ideal $I(X) \subseteq K[x_0, \dots, x_n]$ cuts out the tropicalization of X :*

$$\text{Trop}(X, \iota) = \bigcap_{f \in I(X)} V^{\text{trop}}(\text{trop } f).$$

(b) *If $X = V(f)$ is a hypersurface, then even*

$$\text{Trop}(X, \iota) = V^{\text{trop}}(\text{trop } f).$$

Example 1.3. We aim to illustrate Theorem 1.2 in the example of the line $X = V(f) \subseteq \mathbb{P}^2$ where $f(x_0, x_1, x_2) = x_0 + x_1 - x_2$. Suppose $x_0, x_1, x_2 \in K$, think of $K = \mathbb{C}\{\{t\}\}$, satisfy $x_0 + x_1 - x_2 = 0$. Let

$$a = \text{val}(x_0), \quad b = \text{val}(x_1), \quad c = \text{val}(-x_2) = \text{val}(x_2).$$

If the minimum among a, b, c were attained only once, say at a , then the lowest-order term of x_0 would have valuation a , while the lowest-order terms of x_1 and $-x_2$ would have strictly larger valuations. Thus the lowest term of the sum $x_0 + x_1 - x_2$ could not cancel, contradicting that it equals zero. Hence the minimum of

$$\text{val}(x_0), \text{val}(x_1), \text{val}(-x_2)$$

must be attained at least twice. This shows one inclusion of the aforementioned theorem.

In order to visualize this, we consider the chart $x_2 = 1$, such that the tropicalization becomes the set

$$\{(z, w) \in \mathbb{T}^2 \mid \min\{z, w, 0\} \text{ is attained at least twice}\}. \quad (2)$$

Here we identify $(a, b, c) \sim (a - c, b - c, 0)$ in \mathbb{TP}^2 to obtain a picture in the plane (see the left part of Figure 1). We can also read off the polyhedral structure from the description (2): There are three rays emanating from the origin, in the directions of the vectors $(1, 0)$, $(0, 1)$, and $(-1, -1)$.

2 Non-Archimedean Geometry and Berkovich Spaces

The preceding discussion can be elaborated on further and placed into a broader framework using Berkovich spaces. In particular, when passing to Berkovich analytic spaces, embedded tropicalizations arise elegantly as continuous images of Berkovich spaces. In this section, we describe the non-Archimedean analytic approach to tropical geometry, which relies essentially on Berkovich's theory of non-Archimedean analytic spaces [Ber90].

Throughout this section, we fix a field K . A non-Archimedean *absolute value* is a map $|\cdot|_K : K \rightarrow \mathbb{R}$ such that

- (i) $|\lambda|_K \geq 0$ for all $\lambda \in K$ and $|\lambda|_K = 0$ if and only if $\lambda = 0$,
- (ii) $|\lambda \cdot \mu|_K = |\lambda|_K \cdot |\mu|_K$ for all $\lambda, \mu \in K$,
- (iii) $|\lambda + \mu|_K \leq \max\{|\lambda|_K, |\mu|_K\}$ for all $\lambda, \mu \in K$.

A non-Archimedean field is a field endowed with a non-Archimedean absolute value. One can translate between valuations and non-Archimedean absolute values by taking logarithms or exponentials, i.e. the valuation is given by $\text{val}(\cdot) = -\log |\cdot|_K$.

In Example 1.1, the absolute value of a power series $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots$ in the field $K = \mathbb{C}\{\{t\}\}$ is given by $|c(t)|_K = \exp(-a_1)$. One can view the element t as an infinitesimally small element in the sense that $|nt| < 1$ for all $n \in \mathbb{N}$. This illustrates the adjective “non-Archimedean” in the absolute value.

Other typical examples are the fields \mathbb{Q}_p , which is the completion of \mathbb{Q} with respect to the p -adic absolute value, as well as the p -adic counterpart \mathbb{C}_p of the complex numbers, defined as the completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p . Moreover, any field can be equipped with the trivial absolute value, which takes the value 1 on all non-zero elements. This example may sound unimportant, but it is a key feature of Berkovich's theory and of tropical geometry that they also yield a rich and unified theory in the trivially valued case.

From now on, we assume that K is complete with respect to a non-Archimedean absolute value $|\cdot|_K$. This means that K is complete as a metric space, with respect to $|\cdot|_K$. With this

extra datum, K -varieties have an additional topological structure that is very different from the Zariski topology. The topology on K induced by the given absolute value is Hausdorff, but it is also totally disconnected and not locally compact. This makes it difficult to define a good notion of an analytic function on K and, more generally, on the K -points of a variety X . Roughly speaking, *Berkovich spaces* add “a lot” of new points to fill in the gaps so that one obtains spaces with nice topological properties. We now give a definition.

Let A be a finitely generated K -algebra and let $X = \text{Spec } A$ be an affine scheme of finite type over K . A multiplicative seminorm

$$|\cdot| : A \rightarrow \mathbb{R}$$

satisfies the following axioms:

- (i) $|a| = |a|_K$ for all $a \in K$,
- (ii) $|f + g| \leq \max\{|f|, |g|\}$ for all $f, g \in A$,
- (iii) $|fg| = |f||g|$ for all $f, g \in A$.

The *Berkovich analytification* X^{an} is defined as the set of multiplicative seminorms on A , endowed with the coarsest topology such that for all $f \in A$ the evaluation maps

$$X^{\text{an}} \rightarrow \mathbb{R}, \quad |\cdot| \mapsto |f|$$

are continuous. We have a natural inclusion $X(K) \rightarrow X^{\text{an}}$ via $x \mapsto [f \mapsto |f(x)|_K]$. If X is a finite type scheme over K that is not necessarily affine, the topological space X^{an} is obtained by taking an affine open cover of X , analytifying everything separately and then gluing. Berkovich also defined a structure sheaf on X^{an} , though it will not be relevant for this thesis. More importantly for us, the Berkovich analytification X^{an} has desirable topological properties: It is path-connected if and only if X is connected and compact if and only if X is proper.

Of particular interest to us is the Berkovich projective space $(\mathbb{P}^n)^{\text{an}}$ which can be described as the set of equivalence classes of elements in $(\mathbb{A}^{n+1})^{\text{an}} \setminus \{0\}$ with respect to a certain equivalence relation, which will be explained in Remark 1.2.4. In particular, a point $x \in (\mathbb{P}^n)^{\text{an}}$ is represented by a seminorm $|\cdot|_x$ on $K[t_0, \dots, t_n]$.

The key relation between tropicalization and analytification is the following: There is a natural tropicalization map

$$\begin{aligned} \text{trop}_{\mathbb{P}^n} : (\mathbb{P}^n)^{\text{an}} &\longrightarrow \mathbb{TP}^n \\ x &\longmapsto [-\log |t_0|_x : \dots : -\log |t_n|_x]. \end{aligned} \tag{3}$$

By construction, this commutes with the inclusion of the K -points of \mathbb{P}^n and its tropicalization from Section 1, i.e.,

$$\begin{array}{ccc} \mathbb{P}^n(K) & \hookrightarrow & (\mathbb{P}^n)^{\text{an}} \\ & \searrow \text{trop} & \downarrow \text{trop}_{\mathbb{P}^n} \\ & & \mathbb{TP}^n \end{array}$$

where $\text{trop} : \mathbb{P}^n(K) \rightarrow \mathbb{TP}^n, (x_0 : \dots : x_n) \mapsto (\text{val}(x_0) : \dots : \text{val}(x_n))$.

Let $\iota : X \hookrightarrow \mathbb{P}^n$ be an embedded projective variety. Then the embedded tropicalization of X equals the projection of X^{an} under $\text{trop}_{\mathbb{P}^n}$ into \mathbb{TP}^n (see [Pay09, Gub13]), i.e.,

$$\text{Trop}(X, \iota) = \text{trop}_{\mathbb{P}^n}(X^{\text{an}}).$$

Considering a tropical variety as the image of a Berkovich space makes some topological considerations easier. For example, we see that the tropicalization of a connected subvariety

$X \subseteq \mathbb{P}^n$ is connected since it is a continuous image of a connected space. The tropicalization construction for \mathbb{P}^n and its subvarieties is, in fact, an incarnation of a more general construction for arbitrary toric varieties (see [Pay09] or [Kaj08]). One reason tropicalizations are so valuable is that they can provide a finite and combinatorial piece of the far more intricate and more complicated space X^{an} . The precise formulation of this fact is among the most influential theorems connecting tropical geometry with non-Archimedean geometry, and is due to Payne:

Theorem 2.1 ([Pay09]). *For a quasi-projective variety X , the analytification is homeomorphic to the limit of all tropicalizations:*

$$X^{\text{an}} \xrightarrow{\sim} \varprojlim_{\iota \in I} \text{Trop}(X, \iota),$$

where I denotes the category of all embeddings of X into quasi-projective toric varieties.

Consequently, taking the inverse limit over all embedded tropicalizations recovers the entire Berkovich space. This theorem thus offers a perspective on the Berkovich analytification as a “universal” tropicalization. It serves as the central motivation for one of the main theorems in Chapter 1. There, we replace X^{an} with the Bruhat–Tits building of $\text{PGL}_{r+1}(K)$ – which can be regarded as less complicated than the Berkovich space – and take the limit over linear embeddings rather than over all embeddings.

3 Tropical Linear Spaces and Valuated Matroids

In Section 1 we explained how to tropicalize varieties which result in interesting polyhedral objects. Although linear subspaces are among the simplest algebraic varieties, their tropicalizations already exhibit a surprisingly rich and subtle combinatorial structure. Understanding this structure naturally leads to matroid theory and, more precisely, to its valuated refinement.

Let $\iota = [f_0 : \dots : f_n] : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ with $f_0, \dots, f_n \in (K^{r+1})^*$ be a linear embedding. The subspace $\iota(\mathbb{P}^r) \subseteq \mathbb{P}^n$ is the row space of the $(r+1) \times (n+1)$ -matrix $A_\iota = (f_0 \mid \dots \mid f_n)$. The *Plücker coordinates* of $\iota(\mathbb{P}^r)$ are given by the maximal minors of A_ι , and they depend only on the subspace $\iota(\mathbb{P}^r)$ and not on the chosen matrix.

Applying the valuation to these minors produces a combinatorial object. More precisely, we obtain a *realizable valuated matroid* M_ι of rank $r+1$ on $[n] = \{0, \dots, n\}$. It is given by the map

$$M_\iota : \binom{[n]}{r+1} \longrightarrow \overline{\mathbb{R}},$$

$$\{a_0, \dots, a_r\} \longmapsto \text{val} \left(\det [f_{a_0} \ \dots \ f_{a_r}] \right)$$

that fulfills the following axioms:

- (i) There exists $A \in \binom{[n]}{r+1}$ with $M_\iota(A) \neq \infty$.
- (ii) For all $A, B \in \binom{[n]}{r+1}$ and $a \in A - B$ we have the *valuated basis exchange property*

$$M_\iota(A) + M_\iota(B) \geq \min_{b \in B} \{M_\iota(b \cup A \setminus a) + M_\iota(a \cup B \setminus b)\}.$$

When the valuation on K is trivial, this construction reduces to the classical notion of a *matroid*. It remembers which minors are zero or non-zero and the combinatorial object is

called a *matroid*. It encodes the linear independencies among the column vectors in A_ι . A collection of $r + 1$ columns of A_ι is linearly independent if and only if the corresponding Plücker coordinate of $\iota(\mathbb{P}^r)$ is nonzero. Thus, the underlying matroid captures exactly the zero pattern of the Plücker coordinates.

For a nontrivial valuation, the map M_ι retains more refined information: it records not only whether a minor vanishes, but also the valuation of each nonzero minor. These valuated matroids, introduced by Dress and Wenzel [DW92b], abstract the quadratic Grassmann–Plücker relations satisfied by these valuations. They provide the correct combinatorial framework for studying tropicalizations of linear spaces as we now aim to describe.

Recall that by Theorem 1.2, computing the tropicalization of $\iota(\mathbb{P}^r)$ a priori requires intersecting the tropical hypersurfaces associated to all polynomials in the ideal defining the linear space. Remarkably, for linear spaces this is simpler: Given a $(r + 2)$ -subset $C \subseteq [n]$, we define the tropical polynomial

$$f_C = \min_{e \in C} \{x_e + M_\iota(C \setminus e)\}$$

in $\mathbb{T}[x_0, \dots, x_n]$.

By [Spe08], the tropicalization of the linear space $\iota(\mathbb{P}^r)$ is given by

$$\text{Trop}(\mathbb{P}^r, \iota) = \bigcap_C V^{\text{trop}}(f_C),$$

where C runs over those $(n + 2)$ -subsets of $[n]$ that come from minimal dependencies among the column vectors f_i . They are called *valuated circuits* in matroid theory and thus suffice to compute the linear tropicalization $\text{Trop}(\mathbb{P}^r, \iota)$. In particular, the tropicalization of a linear space is determined by its valuated matroid.

In Chapter 1, we extend this construction and associate valuated matroids on infinite ground sets with a tropical linear space. Of particular interest is the case where the ground set is an entire vector space, which will appear very naturally and lead to a close connection with Bruhat–Tits buildings.

4 Tropical Real Algebraic Geometry

An early success of what became tropical geometry is *Viro’s patchworking* [Vir84], in which he constructed real plane curves with specific topological constraints. This takes place in the world of real algebraic geometry, where varieties are defined over \mathbb{R} , carrying additional topological structure.

Analogously to how tropical geometers frequently consider nontrivially valued extensions of \mathbb{C} , such as the field of Puiseux series, one can likewise enlarge \mathbb{R} to obtain a nontrivially valued real closed field K . Our guiding example is the field of real Puiseux series

$$K = \mathbb{R}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{R}(t^{1/n}).$$

In addition to having a non-Archimedean valuation, K has a (unique) ordering. The positive elements are those Puiseux series $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots$ with $c_1 > 0$. This field is a prototypical example of a real closed field: it is not algebraically closed, and in K every positive number is a square. From now on, let K be a real closed field.

In the world of tropical real algebraic geometry, the absolute value is replaced by the signed absolute value $K \rightarrow \mathbb{R}, x \rightarrow \text{sgn}(x)|x|_K$. In the example of $K = \mathbb{R}\{\{t\}\}$ this is given by

$$\begin{aligned} K &\longrightarrow \mathbb{R}, \\ c_1 t^{a_1} + c_2 t^{a_2} + \dots &\longmapsto \text{sgn}(c_1) \cdot \exp(-a_1). \end{aligned}$$

The additional sign data distinguishes points that are invisible to ordinary tropicalization. As a consequence, for any projective variety $\iota : X \hookrightarrow \mathbb{P}_K^n$, the additional structure of an ordering on K enables a refined version of tropicalization, called the *real tropicalization* $\text{Trop}_r(X, \iota)$ of X . Roughly speaking, one tropicalizes all orthants separately and glues the resulting spaces together. Equivalently, under some tropical smoothness assumption, one can equip the ordinary tropicalization of X with a *real phase structure* [RRS23]. The real tropicalization of X is exactly the patchwork of its (ordinary) tropicalization with respect to this real phase structure.

We illustrate the difference between ordinary and real tropicalization with a simple example. Consider the linear embedding

$$\iota : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2, \quad (s : t) \mapsto (s : t : s + t).$$

The ordinary tropicalization $\text{Trop}(\mathbb{P}^1, \iota)$ is a tropical line in \mathbb{TP}^2 . When working over a real closed valued field, the resulting *real tropicalization* $\text{Trop}_r(\mathbb{P}^1, \iota)$ carries additional combinatorial information, reflecting the signs of the defining linear forms, which is invisible in the ordinary tropicalization. Figure 3 shows both ordinary tropicalization and its real analogue.

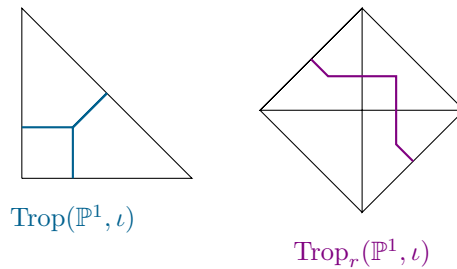


Figure 3: Left: the tropicalization $\text{Trop}(\mathbb{P}^1, \iota)$. Right: the real tropicalization $\text{Trop}_r(\mathbb{P}^1, \iota)$.

A systematic non-Archimedean framework for real tropical geometry was developed in [JSY22]. There, the authors introduce the *real analytification* X_r^{an} of a variety X , which may be viewed as a refinement of the Berkovich analytification X^{an} that incorporates the ordering on K . The tropicalization map (3) is correspondingly refined to a real tropicalization map

$$\text{trop}_r : \mathbb{P}_r^{n, \text{an}} \longrightarrow \mathbb{RP}^n.$$

In [JSY22], the authors prove a theorem similar to Payne’s Limit Theorem 2.1 with real analytification and real tropicalization replacing their “classical” counterparts. Combining this theory with the story developed in Chapter 1 and employing the theory of real tropical linear spaces is the starting point of Chapter 2.

5 Bruhat–Tits Buildings

Tropical geometry provides a powerful framework for translating algebraic geometry into combinatorial and polyhedral terms. While tropical varieties arise by degenerating algebraic varieties over valued fields, there exists a much older and remarkably rich polyhedral structure attached directly to algebraic groups over valued fields: the *Bruhat–Tits building*. These spaces arise independently of tropical geometry and play a central role in the structure theory of reductive groups over non-Archimedean fields.

Buildings were introduced by Tits in the 1960s [TJ75] and further developed by Bruhat and Tits [BT72, BT84], who showed that to each reductive group over a non-Archimedean field one can associate a suitable building on which the group acts with strong symmetry properties. These Bruhat–Tits buildings encode the structure of maximal compact subgroups of the group and have found applications in arithmetic geometry and representation theory. From a geometric perspective, they may be viewed as non-Archimedean analogues of the symmetric spaces associated to real Lie groups.

In this section, our aim is to illustrate some features of the theory of buildings using the example of the projective general linear group $\mathrm{PGL}_{r+1}(K)$ for a local field K . Let k be the residue field. The Bruhat–Tits building of $\mathrm{PGL}_{r+1}(K)$ is the space of homothety classes of norms on the vector space K^{r+1} (see Definition 1.1.8) and has been first introduced by Goldman and Iwahori in [GI63] over the field $K = \mathbb{Q}_p$. We denote the Bruhat–Tits building of $\mathrm{PGL}_{r+1}(K)$ by $\mathcal{B}(\mathrm{PGL}_{r+1}(K))$. Every maximal k -split torus $T \subset \mathrm{PGL}_{r+1}(K)$, or equivalently a choice of a basis B of K^{r+1} , determines an apartment $\mathcal{A}(T)$. In this case, it arises concretely as a real extension of the cocharacter lattice of T , that is,

$$\mathcal{A}(T) = \mathrm{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r+1}/\mathbb{R}.$$

Hence, each apartment is homeomorphic to the r -dimensional Euclidean space $\mathbb{R}^{r+1}/\mathbb{R}$. The collection of all apartments covers the building and they are glued in a certain way.

Each apartment carries additional combinatorial structure: The Weyl group $W = S_{r+1}$ of $\mathrm{PGL}_{r+1}(K)$ acts on an apartment $\mathcal{A}(T)$ by permuting coordinates. This action is faithful, and together with the affine structure we obtain in this way an action of $S_{r+1} \ltimes (\mathbb{R}^{r+1}/\mathbb{R})$ on $\mathcal{A}(T)$. The roots in the root datum of PGL_{r+1} determine reflecting hyperplanes, which induce a tiling of each apartment by simplices. Figure 4 shows an apartment of $\mathcal{B}(\mathrm{PGL}_3(K))$, which is a tessellation of the Euclidean plane by regular triangles.

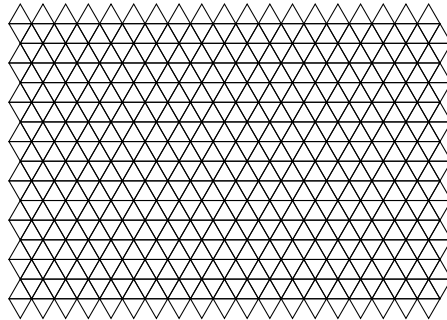


Figure 4: An Euclidean tiling of an apartment of $\mathcal{B}(\mathrm{PGL}_3(K))$.

Werner [Wer04], who constructed a compactification of the Bruhat–Tits building of PGL by adding boundary points corresponding to seminorms, established a decisive link to non-Archimedean geometry. More precisely, there is a close connection with the Berkovich analytification of projective space (see for example [RTW15]), which will play a central role in the study of tropicalizations of linear embeddings in Chapter 1.

6 Tropical Reductive Groups

A natural question arising in tropical geometry is whether there exists a meaningful tropical analogue of a reductive algebraic group. As discussed in the previous section, the Bruhat–Tits building of a reductive group over a valued field is a highly structured polyhedral object that encodes much of the group’s geometry. In particular, the structure of the building shows that two pieces of data play a distinguished role: maximal tori and the Weyl group.

More precisely, the apartments are copies of real cocharacter spaces of maximal tori, while the Weyl group acts on these spaces by reflections arising from the root datum of the group. In Chapter 3, we use exactly these ingredients to formulate a tropical analogue of a reductive group. In this section, we explain the construction in the basic and motivating example of

the general linear group GL_n .

For this exposition, let K be a field with a surjective valuation. We consider the diagonal torus $T = \mathbb{G}_m^n \subseteq \mathrm{GL}_n(K)$ and the Weyl group $N_G(T)/T = S_n$ of $\mathrm{GL}_n(K)$. The tropicalization of the torus T is a real extension of the cocharacter lattice

$$\mathrm{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n.$$

Concretely, it arises as the image of the coordinatewise valuation map

$$T \rightarrow \mathbb{R}^n, (t_1, \dots, t_n) \mapsto (\mathrm{val}(t_1), \dots, \mathrm{val}(t_n)).$$

The Weyl group S_n of GL_n is its combinatorial counterpart and therefore should play an important role in its tropicalization. Its appearance in the tropical setting is also expected from the viewpoint of \mathbb{F}_1 -geometry, where the \mathbb{F}_1 -points of a suitable model of a reductive group correspond canonically to its Weyl group (see, for example, [LT23]). The Weyl group S_n acts naturally on the additive group \mathbb{R}^n by permuting coordinates, and therefore we obtain the group

$$\mathrm{GL}_n^{\mathrm{trop}} = \mathbb{R}^n \rtimes S_n$$

which we call the *tropical reductive group* associated to GL_n . This construction is an instance of our central Definition 3.1.2.

The group $\mathrm{GL}_n^{\mathrm{trop}}$ arises naturally in tropical linear algebra. Indeed, one observes that it admits a concrete realization as a group of generalized permutation matrices. Allermann [All12] showed that this group coincides precisely with the group of tropical invertible matrices $\mathrm{GL}_n(\mathbb{T})$, where $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ is the tropical semifield. Explicitly,

$$\mathrm{GL}_n(\mathbb{T}) = \{A \in \mathbb{T}^{n \times n} \mid A \odot A^{-1} = A^{-1} \odot A = I_n \text{ for some } A^{-1} \in \mathbb{T}^{n \times n}\}.$$

Here, the identity matrix I_n is the tropical matrix with zeros on the diagonal and ∞ elsewhere. The semifield operations on \mathbb{T} extend naturally to matrix multiplication such that $\mathbb{T}^{n \times n}$ becomes a semigroup, and $\mathrm{GL}_n(\mathbb{T})$ consists precisely of its invertible elements.

This identification shows that $\mathrm{GL}_n^{\mathrm{trop}}$ is not an artificial construction, but rather an intrinsic group governing tropical linear algebra. Building on this observation, Gross, Ulirsch, and Zakharov developed a theory of tropical $\mathrm{GL}_n(\mathbb{T})$ -bundles on metric graphs, providing a tropical analogue of vector bundles on algebraic curves [GUZ22].

In Chapter 3 we expand on the perspective $\mathrm{GL}_n^{\mathrm{trop}} = \mathrm{GL}_n(\mathbb{T})$ and introduce an elementary theory of tropical reductive groups in other Dynkin–Lie types and find their natural realizations in tropical linear algebra.

7 Tropical Line Bundles on Metric Graphs

Our main motivation for constructing tropical analogues of reductive groups is that they naturally serve as the structure group for tropical principal bundles on tropical curves. In tropical geometry, the counterparts of smooth projective curves or Riemann surfaces are metric graphs, which arise as combinatorial skeleta of degenerating algebraic curves.

A *metric graph* Γ is a metric space obtained by identifying the edges of a finite graph with real intervals $[0, l(e)]$ of given positive lengths $l(e)$ for the edges e of the graph. A *piecewise linear function* on an open subset $U \subseteq \Gamma$ is a continuous function $f: U \rightarrow \mathbb{R}$ whose restriction to any line segment is piecewise linear with integer slopes. This defines a sheaf $\mathcal{P}\mathcal{L}_\Gamma$ of abelian groups on Γ . A global section of $\mathcal{P}\mathcal{L}_\Gamma$ is called a *rational function* on Γ , and we write

$$\mathrm{Rat}(\Gamma) := \mathcal{P}\mathcal{L}_\Gamma(\Gamma).$$

A *divisor* on an open subset $U \subset \Gamma$ is a formal sum

$$D = \sum_{p \in U} D(p) \cdot p$$

over the points p of U , such that the set $\{p \in U \mid D(p) \neq 0\}$ is finite. We denote by Div the sheaf of abelian groups on Γ . There is a natural sheaf homomorphism $\text{div}: \mathcal{P}\mathcal{L}_\Gamma \rightarrow \text{Div}$ given on open subsets $U \subseteq \Gamma$ by

$$\begin{aligned} \mathcal{P}\mathcal{L}_\Gamma(U) &\longrightarrow \text{Div}(U), \\ f &\longmapsto \sum_{p \in U} \text{ord}_p(f) \cdot p, \end{aligned}$$

where $\text{ord}_p(f)$ denotes the sum of the outgoing slopes of f at the point p .

The image of the group of rational functions $\text{Rat}(\Gamma)$ under div is the subgroup of *principal divisors*, which we denote by $\text{PDiv}(\Gamma) \subseteq \text{Div}(\Gamma)$. Two divisors $D, D' \in \text{Div}(\Gamma)$ are said to be *linearly equivalent*, written $D \sim D'$, if their difference is a principal divisor, i.e.,

$$D - D' = \text{div}(f)$$

for a rational function $f \in \text{Rat}(\Gamma)$. The *Picard group* of Γ is defined as the quotient

$$\text{Pic}(\Gamma) := \text{Div}(\Gamma) / \text{PDiv}(\Gamma).$$

We define the sheaf \mathcal{H}_Γ of harmonic functions on Γ via the following short exact sequence of sheaves of abelian groups on Γ :

$$0 \longrightarrow \mathcal{H}_\Gamma \longrightarrow \mathcal{P}\mathcal{L}_\Gamma \xrightarrow{\text{div}} \text{Div} \longrightarrow 0. \quad (4)$$

That is, for an open subset $U \subseteq \Gamma$, the group $\mathcal{H}_\Gamma(U)$ consists of those piecewise linear functions whose outgoing slopes at every point sum to zero.

This exact sequence is the tropical analogue of the classical sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^* / \mathcal{O}_X^* \longrightarrow 0. \quad (5)$$

where \mathcal{O}_X^* and \mathcal{M}_X^* are the sheaves of invertible holomorphic and meromorphic functions, respectively, on a Riemann surface X .

In this sense, harmonic functions on Γ serve as tropical transition functions for line bundles, just as invertible holomorphic functions do in the classical setting:

Definition 7.1 ([MZ08]). A *tropical line bundle* on Γ is an \mathcal{H}_Γ -torsor.

Therefore, the set of isomorphism classes of tropical line bundles on Γ is given by $H^1(\Gamma, \mathcal{H}_\Gamma)$. In order to compute this, we see that the short exact sequence (4) gives rise to a long exact sequence in cohomology

$$\text{Rat}(\Gamma) \xrightarrow{\text{div}} \text{Div}(\Gamma) \longrightarrow H^1(\Gamma, \mathcal{H}_\Gamma) \longrightarrow 0. \quad (6)$$

The exactness can be proved similarly to the case of Riemann surfaces (see for example [GUZ22, §1]).

It follows that

$$H^1(\Gamma, \mathcal{H}_\Gamma) \cong \text{Pic}(\Gamma),$$

so that isomorphism classes of tropical line bundles correspond precisely to linear equivalence classes of divisors, exactly as in complex algebraic geometry.

Many foundational results of algebraic geometry admit natural tropical analogues in this setting. In particular, the Riemann–Roch theorem [GK08, BN07] and the Abel–Jacobi theorem [MZ08, BF11] have tropical counterparts for metric graphs. From this perspective, the theory of line bundles and divisors is not merely a combinatorial curiosity, but an intrinsic and rich geometric theory in its own right.

At the same time, this tropical theory is tightly connected to algebraic geometry through the process of tropicalization. In the next section, we explain how tropical line bundles arise as degenerations of algebraic line bundles on curves.

8 Tropicalization of Line Bundles on Curves

So far, we have focused on tropicalizations of varieties embedded in projective space. However, algebraic varieties, e.g., *curves*, can also be studied intrinsically, without reference to a particular embedding. This raises the question of whether there exists an *abstract* tropicalization of a curve, one that depends only on the curve itself and not on how it is embedded. This problem is particularly well understood for curves, and we now illustrate the basic ideas through two fundamental examples.

Example 8.1 (adapted from Example 3.1 [Cha17]). Let $K = \mathbb{C}((t))$ and let $R = \mathbb{C}[[t]]$ be its valuation ring with residue field $k = \mathbb{C}$. Fix a positive integer ℓ , and let X/K be the projective plane curve defined by

$$xy = t^\ell z^2 \tag{7}$$

in \mathbb{P}_K^2 .

The curve X is a smooth conic over K , but we regard it as a family of curves with parameter $t \neq 0$. Each general member X_t for $t \neq 0$ is a smooth plane conic in $\mathbb{P}_{\mathbb{C}}^2$. Equation (7) also defines a scheme X/R . Its special fiber $X_k = X \times_R k$ has the equation $xy = 0$ in $\mathbb{P}_{\mathbb{C}}^2$, and therefore consists of two rational curves that meet transversely at a node (see the left side of Figure 5).

The associated abstract tropical curve is obtained by forming the *dual metric graph* of the special fiber. We assign a vertex to each irreducible component of X_k and an edge to each node where the two components intersect. Étale locally around a node p of the special fiber, X can be written as

$$R[x, y]/\langle xy - \lambda \rangle$$

for some $\lambda \in R$. If e is the edge corresponding to p , its length is defined by $\ell(e) := \text{val}(\lambda) \in \mathbb{R}_{>0}$, measuring the speed of the degeneration. In this example, the abstract tropical curve associated with X is the metric graph consisting of two vertices joined by an edge of length ℓ (see the right side of Figure 5).

The previous example illustrates how the degeneration of a curve gives rise to a metric graph encoding the combinatorics of the special fiber. Our next example shows how Berkovich geometry provides a canonical and intrinsic framework for this construction.

Example 8.2. Let X be an elliptic curve with multiplicative reduction, i.e., $\text{val}(j(X)) < 0$. Such an elliptic curve is also called a *Tate curve*, since Tate’s theory provides a non-Archimedean uniformization

$$X^{\text{an}} = \mathbb{G}_m^{\text{an}}/q^{\mathbb{Z}}$$

for a uniquely determined element $q \in K^*$ that satisfies $\text{val}(q) = -\text{val}(j(X))$ (see [Sil13, Chapter V]).



Figure 5: Left: the special fiber X_k given by $xy = 0$. Right: the associated abstract tropical curve, an edge of length ℓ joining two vertices.

From a degeneration-theoretic perspective, as in the previous example, the Tate curve X admits a semistable model over the valuation ring R whose special fiber is a cycle of projective lines. The dual graph of this special fiber is therefore a metric circle.

Berkovich geometry recovers this object intrinsically. The Berkovich analytification X^{an} contains a finite metric graph $\Gamma_X \subset X^{\text{an}}$, called the *minimal skeleton*, onto which X^{an} admits a strong deformation retraction $\rho : X^{\text{an}} \rightarrow \Gamma_X$. In the case of a Tate curve, this minimal skeleton is precisely a metric circle of length $\text{val}(q) = -\text{val}(j(X))$ (see Figure 6).

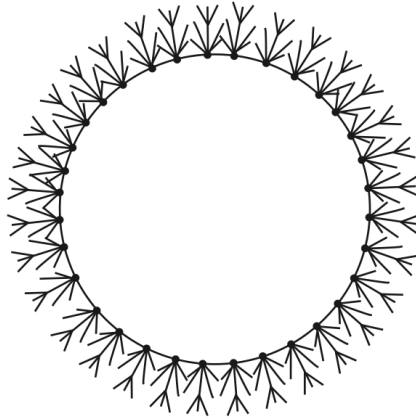


Figure 6: The Berkovich analytification of a Tate curve, adapted from [Bak08a].

These examples show that for curves, tropicalization can be understood intrinsically via metric graphs arising as skeleta of non-Archimedean analytic spaces. We now explain how line bundles behave under this correspondence.

Let X be the Tate curve of Example 8.2 and let $L = \mathcal{O}_X(D) \in \text{Pic}(X)$ be a line bundle associated to a divisor D . The tropicalization of D is defined as $D^{\text{trop}} = \rho_* D$, which means that the divisor D is pushed down to Γ_X . As explained in Section 7, divisors on Γ_X determine tropical line bundles up to isomorphism. Thus, the divisor D^{trop} gives rise to a tropical line bundle

$$L^{\text{trop}} := \mathcal{H}_{\Gamma_X}(D^{\text{trop}})$$

on the metric graph Γ_X .

A fundamental result of Baker and Rabinoff shows that this tropicalization process behaves as nice as it can.

Theorem 8.3 ([BR15]). *The essential skeleton of $\text{Pic}(X)^{\text{an}}$ is naturally isomorphic to $\text{Pic}(\Gamma_X)$.*

This theorem establishes a precise correspondence between algebraic line bundles on X and tropical line bundles on its skeleton, demonstrating that tropicalization preserves the intrinsic geometry of line bundles in a highly non-trivial way. The theory of tropical line bundles and divisors has led to significant applications in algebraic geometry, notably to the classical subject of Brill–Noether theory on algebraic curves [CDPR12, JR21], as well as to the study of the birational geometry of moduli spaces [FJP24].

In Chapter 3, we expand on the tropicalization of line bundles on curves to address a classical and natural question: What is a principal bundle in tropical geometry? We hereby employ our developed theory of tropical reductive groups and generalize the role of \mathbb{G}_m to tropical reductive groups G^{trop} . This generalizes the theory of tropical vector bundles, i.e. the GL_n -case, of Gross, Ulirsch, and Zakharov [GUZ22].

9 Contributions of this Thesis

Chapter 1: Buildings, Valuated Matroids, and Tropical Linear Spaces

We construct a natural tropicalization map from a compactification of the Bruhat–Tits building of $\text{PGL}_{r+1}(K)$ to the tropicalized linear space $\text{Trop}(\mathbb{P}^r, \iota)$ for any linear embedding ι into projective space. Our construction relies on the strong connections between the Berkovich analytification and Bruhat–Tits buildings [RTW15]. To be precise, in the non-spherically complete case, we replace the building with the *Goldman–Iwahori space*. The tropicalization maps induce a continuous map from the building into the inverse limit of all tropicalized linear spaces, which we prove to be a homeomorphism. The crucial difference from Payne’s limit theorem [Pay09] is that we only consider *linear* embeddings. Furthermore, we prove a faithful tropicalization result by constructing a continuous section of the tropicalization map from any tropicalized linear space $\text{Trop}(\mathbb{P}^n, \iota)$ into the building.

We then generalize the known construction of tropical linear spaces from valuated matroids to the setting of possibly infinite ground sets, while still assuming that the rank is finite. In the realizable case, the resulting tropical linear spaces are tropicalizations of embeddings of a finite-dimensional vector space into an infinite-dimensional one. We prove that the tropical linear space associated with the *universal realizable matroid* coincides with the building of PGL over K , which extends a result of Dress and Terhalle [DT98]. This chapter was written in collaboration with Luca Battistella, Kevin Kühn, Martin Ulirsch, and Alejandro Vargas. It has been published in the Journal of the LMS in 2024 [BKK⁺24].

Chapter 2: The Signed Goldman–Iwahori Space and Real Tropical Linear Spaces

We extend the results from Chapter 1 to the case of real closed valued fields. We introduce the *signed Goldman–Iwahori space* as the space of *signed seminorms* on a vector space. This new space can be seen as the linear algebraic version of the real analytification of projective space, the latter having been introduced in [JSY22]. We study this space using methods from real tropical geometry by constructing natural real tropicalization maps from the signed Goldman–Iwahori space to all real tropicalized linear spaces. We prove that this space is the limit of all real tropicalized linear embeddings. While many constructions from the first part

extend naturally, new phenomena arise in the signed case, including signed seminorms that are not diagonalizable.

We give a combinatorial interpretation of this result by showing that the signed Goldman–Iwahori space is the real tropical linear space associated to the universal realizable oriented matroid. In the constant coefficient case for $K = \mathbb{R}$, we describe this space explicitly and relate it to real Bergman fans. This chapter was written in collaboration with Kevin Kühn and is available as a preprint [KK24].

Chapter 3: Tropical Reductive Groups and Principal Bundles on Metric Graphs

We propose an elementary tropical analogue of a reductive group that combines the datum of a Weyl group and the tropicalization of a fixed maximal torus. For the classical groups, as well as G_2 , these tropical reductive groups admit descriptions as tropical matrix groups that resemble their classical counterparts. Employing this perspective, we introduce *tropical principal bundles* on metric graphs and study their explicit presentations as pushforwards of line bundles along covers with symmetries and extra data. We give a classification of principal bundles on metric graphs of genus zero in analogy with the classical theorems of Grothendieck [Gro57] Harder [Har68]. In analogy to the classification of semistable and stable principal G -bundles on an elliptic curve given by Frățilă [Fră16, Fră21], we prove tropical statements describing the moduli spaces of semistable and stable G^{trop} -bundles on a metric circle.

Then we give a tropicalization procedure for stable G -bundles on a Tate curve by reducing them to $N_G(T)$ -bundles, where $N_G(T) \subseteq G$ is the normalizer of a fixed maximal torus T of G , and semistable bundles by passing to a Levi subgroup. Our main result identifies the essential skeleton of the moduli space of semistable principal bundles on a Tate curve with its tropical analogue. This chapter was written in collaboration with Andreas Gross, Martin Ulirsch and Dmitry Zakharov and is available as a preprint [GKU25].

Chapter 1

Buildings, Valuated Matroids, and Tropical Linear Spaces

Introduction

Let K be a complete non-Archimedean field (possibly carrying the trivial absolute value). The non-Archimedean analytic approach to tropical geometry (see e.g. [BPR16, EKL06, Gub13, Pay09]) makes crucial use of non-Archimedean analytic spaces in the sense of Berkovich [Ber90]; the particular topological properties of these Berkovich spaces let us think of the analytification X^{an} of an algebraic variety X as a form of universal tropicalization that is independent of the chosen coordinate system. In this regard, the central result presented in [Pay09] tells us that the analytification X^{an} of a quasi-projective algebraic variety X over K is the projective limit of all tropicalizations with respect to all the embeddings in toric varieties (also see [FGP14, GG22, KSU21] for generalizations of this result).

In this chapter we argue that when considering only the tropicalization of projective spaces linearly embedded into higher-dimensional projective spaces, the role of Berkovich analytic space is played by the Goldman–Iwahori space $\mathcal{X}_r(K)$ of homothety classes of non-trivial seminorms on $(K^{r+1})^*$ (see [GI63] but also [RTW15, Wer04, RTW12]). The locus $\overline{\mathcal{B}}_r(K)$ of diagonalizable seminorms in $\mathcal{X}_r(K)$ is (a compactification of) the affine Bruhat–Tits building of $\text{PGL}_{r+1}(K)$, when K carries a non-trivial valuation, and the cone over the spherical building of $\text{PGL}_{r+1}(K)$, when the valuation on K is trivial. When K is spherically complete, we have $\overline{\mathcal{B}}_r(K) = \mathcal{X}_r(K)$.

Let $\iota: \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ be a linear closed immersion. The tropicalization $\text{Trop}(\mathbb{P}^r, \iota)$ of \mathbb{P}^r with respect to the embedding ι is the projection of $\iota(\mathbb{P}^r)^{\text{an}} \subseteq (\mathbb{P}^n)^{\text{an}}$ under the natural tropicalization map

$$\text{trop}_{\mathbb{P}^n}: (\mathbb{P}^n)^{\text{an}} \longrightarrow \mathbb{TP}^n$$

to $\mathbb{TP}^n = \overline{\mathbb{R}}^{n+1} - \{(\infty, \dots, \infty)\} / \mathbb{R} \cdot \mathbb{1}$ that is essentially given by taking coordinate-wise valuations (see Section 1.2 below for details). The tropicalization $\text{Trop}(\mathbb{P}^r, \iota)$ is a projective tropical linear space in \mathbb{TP}^n and is associated to a realizable valuated matroid (see Section 1.4 below for details).

We shall see in Section 1.2 below that there is a natural continuous and surjective restriction map $\tau: (\mathbb{P}^n)^{\text{an}} \rightarrow \overline{\mathcal{X}}_n(K)$ that factors the tropicalization map as

$$(\mathbb{P}^n)^{\text{an}} \xrightarrow{\tau} \overline{\mathcal{X}}_n(K) \xrightarrow{\text{trop}_{\overline{\mathcal{X}}_n}} \mathbb{TP}^n$$

such that the tropicalization $\text{Trop}(\mathbb{P}^r, \iota)$ is given as the projection of $\overline{\mathcal{X}}_r(K) \subseteq \overline{\mathcal{X}}_n(K)$ under $\text{trop}_{\overline{\mathcal{X}}_n}$. The connection between affine buildings and Berkovich analytic spaces is well-established in the literature. We refer the reader in particular to [Ber90] as well as to [RTW10, RTW12, RTW15].

Denote by I the category whose objects are linear closed immersions $\iota: \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$ into a torus-invariant open subset $U \subseteq \mathbb{P}^n$; a morphism between $\iota: \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$ and $\iota': \mathbb{P}^r \hookrightarrow U' \subseteq \mathbb{P}^n$ is given by a linear toric morphism $\varphi: U \rightarrow U'$ such that $\varphi \circ \iota = \iota'$. The tropicalization with respect to $\iota: \mathbb{P}^r \hookrightarrow U$ is naturally homeomorphic to the tropicalization with respect to the composition $\mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$. For a toric morphism $\varphi: U \rightarrow U'$ such that $\varphi \circ \iota = \iota'$, we have a natural induced map $\varphi^{\text{trop}}: U^{\text{trop}} \rightarrow U'^{\text{trop}}$ such that $\varphi^{\text{trop}}(\text{Trop}(\mathbb{P}^r, \iota)) \subseteq \text{Trop}(\mathbb{P}^r, \iota')$, making I into a cofiltered category. Requiring morphisms only to be defined on an open subset of projective space provides us with the added flexibility that we need in the following.

Theorem A. *The tropicalization maps induce a natural homeomorphism*

$$\overline{\mathcal{X}}_r(K) \xrightarrow{\sim} \varprojlim_{\iota \in I} \text{Trop}(\mathbb{P}^r, \iota),$$

where the projective limit is taken over the category I .

Let I' be the full subcategory of I , whose objects are linear embeddings $\iota: \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$ whose images meet the big torus $\mathbb{G}_m^n \subseteq \mathbb{P}^n$. Then Theorem A provides us with a homeomorphism between the space of norms $\mathcal{X}_r(K)$ and the projective limit of all non-compactified tropicalizations $\text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^n)$. When K is spherically complete, the Goldman–Iwahori space $\mathcal{X}_r(K)$ is equal to $\mathcal{B}_r(K)$, and therefore we have a natural homeomorphism

$$\mathcal{B}_r(K) \xrightarrow{\sim} \varprojlim_{\iota \in I'} \text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^n).$$

In addition to Theorem A we also prove the following, which one may think of as a new addition to the literature on faithful tropicalization (see e.g. [BPR16, CHW14, GRW16, GRW17] for other instances).

Theorem B. *Let $\iota: \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ be a linear closed immersion. Then there is a natural piecewise linear embedding $J: \text{Trop}(\mathbb{P}^r, \iota) \rightarrow \overline{\mathcal{B}}_r(K)$ that makes the following diagram commute*

$$\begin{array}{ccc} \overline{\mathcal{B}}_r(K) & \xleftarrow{J} & \text{Trop}(\mathbb{P}^r, \iota) \\ & \searrow \overline{\mathcal{B}}(\iota) & \searrow \subseteq \\ & \overline{\mathcal{B}}_n(K) & \xrightarrow{\text{trop}} \mathbb{TP}^n. \end{array}$$

It is worth emphasizing that in Theorem B the compactified tropical linear space $\text{Trop}(\mathbb{P}^r, \iota)$, which may be thought of as a finite polyhedral approximation of the affine building $\overline{\mathcal{B}}_r(K)$, naturally embeds into $\overline{\mathcal{B}}_r(K)$ and not just into the Goldman–Iwahori space $\overline{\mathcal{X}}_r(K)$, even when K is not spherically complete. This curious fact is, at least heuristically, explained by the observation that the difference between $\overline{\mathcal{B}}_r(K)$ and $\overline{\mathcal{X}}_r(K)$ is only visible in non-diagonalizable seminorms on $(K^{r+1})^*$ that are defined via an asymptotic process (e.g. by the projective limit in Theorem A).

Suppose now that K is discretely valued. Then, for one, the section in Theorem B recovers the so-called *membrane* of a realization of $\text{Trop}(\mathbb{P}^r, \iota)$ from [JSY07, Lemma 4]. If K is also local, there is a natural embedding of $\overline{\mathcal{B}}_r(K) = \overline{\mathcal{X}}_r(K)$ into the Berkovich analytic space $(\mathbb{P}^r)^{\text{an}}$ (see [RTW12, Section 3] for details). Theorem A then tells us that the collection of all linear re-embeddings $\mathbb{P}^r \hookrightarrow \mathbb{P}^n$ recovers exactly $\overline{\mathcal{B}}_r(K) = \overline{\mathcal{X}}_r(K)$. The main result of [Pay09],

on the other hand, tells us that, once we also allow non-linear algebraic re-embeddings of \mathbb{P}^r into suitable toric varieties, we recover the whole Berkovich analytification of \mathbb{P}^r .

Theorem A and Theorem B together provide us with a heuristic saying that $\overline{\mathcal{X}}_r(K)$ is in some sense a *universal (realizable) tropical linear space*. This goes hand in hand with the work of Dress and Terhalle [DT98], in which the authors realize the building $\mathcal{B}_r(K)$ as the tight span of a suitable infinite valuated matroid, see Section 1.7.2 below.

In Section 1.7.1 we build on this observation and construct a tropical linear space for any infinite valuated matroid (expanding on the finite case, see [Spe08, SS04] as well as [MS15, Chapter 4]). This allows us to make precise the idea that the Goldman-Iwahori space is the tropical linear space $\text{Trop}(w_{\text{univ}})$ of the *universal realizable valuated matroid* w_{univ} that is given by the map

$$w_{\text{univ}} : \binom{(K^{r+1})^*}{r+1} \longrightarrow \overline{\mathbb{R}}$$

induced by the permutation-invariant map $\text{val} \circ \det : K^{(r+1) \times (r+1)} \rightarrow \overline{\mathbb{R}}$.

Theorem C. *Let w_{univ} be the universal realizable valuated matroid. Then the Goldman-Iwahori space is the tropical linear space associated to the universal realizable matroid w_{univ} , i.e.,*

$$\overline{\mathcal{X}}_r(K) = \text{Trop}(w_{\text{univ}}).$$

Our approach, in particular, provides us with a notion of tropicalization for a finite-dimensional linear space embedded into an infinite-dimensional space, see again Section 1.7.1 for details. Theorem C is also, in spirit, very similar to the universal (possibly non-linear) tropicalization of [GG22], which may be identified with Berkovich analytic space. An order-theoretic approach to the problem has been developed by Hirai: generalising the cryptomorphic characterization of matroids in terms of their lattice of flats, he proved that tropical linear spaces are characterized by their integer points forming a *uniform semimodular lattice* [Hir19]. From this point of view, affine buildings of type A correspond to uniform modular lattices [Hir20]. Thus, roughly speaking, they are identified with tropical linear spaces with many symmetries.

It is well-known that Bruhat–Tits buildings also admit a simplicial structure which can be described in terms of filtrations by linear subspaces, when K carries the trivial absolute value, and by sublattices, when K carries a non-trivial discrete absolute value. We incorporate those perspectives into our story in Examples 1.1.10 and 1.1.12, while, in Section 1.4.5, we recall the analogous story of how valuated matroids over a field with trivial absolute value are nothing but matroids. In Section 1.6 we illustrate our main results using this alternative point of view on buildings.

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Conventions

We write $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{\infty\}$ with tropical operations \min as well as $+$, and \mathbb{TP}^n for $\overline{\mathbb{R}}^{n+1} \setminus \{(\infty, \dots, \infty)\} / \sim$, where \sim is the equivalence relation generated by translation by a real multiple of the vector $\mathbb{1} = (1, \dots, 1)$. We write $[n]$ for the set $\{0, \dots, n\}$ and $\binom{E}{r}$ for the set of subsets of cardinality r of a given set E .

1.1 The Affine Building and its Compactification

Let K be a complete non-Archimedean field. In this section, we recall some fundamentals about the space of seminorms on a finite-dimensional vector space V over K . The (compactification of the) affine building of PGL_n is the a priori proper subset of diagonalizable (semi)norms. If K is spherically complete these two spaces will turn out to be the same.

Let V be a vector space over K of dimension n .

Definition 1.1.1. A *norm* on V is a map $\|\cdot\|: V \rightarrow \mathbb{R}$ that fulfills the following axioms:

- (i) For all $v \in V$ we have $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$;
- (ii) For all $v \in V$ and $\lambda \in K$ we have

$$\|\lambda \cdot v\| = |\lambda| \cdot \|v\| .$$

- (iii) For all $v, w \in V$ the strong triangle inequality

$$\|v + w\| \leq \max \{ \|v\|, \|w\| \}$$

holds.

If in (i) we only require $\|v\| \geq 0$ and allow vectors $v \in V - \{0\}$ with $\|v\| = 0$ we say that $\|\cdot\|$ is a *seminorm*. A seminorm $\|\cdot\|$ is said to be *non-trivial* if there is a $v \in V$ such that $\|v\| \neq 0$.

Example 1.1.2. Pick a basis $\mathbf{e} = (e_1, \dots, e_n)$ of V and $\vec{a} = (a_1, \dots, a_n) \in \overline{\mathbb{R}}^n$. We may associate to this datum a seminorm $\|\cdot\|_{\mathbf{e}, \vec{a}}$ on V given by associating to $v = \sum_{i=1}^n \lambda_i e_i$ the value $\max_{i=1, \dots, n} \{ |\lambda_i| e^{-a_i} \}$. The seminorm is non-trivial if and only if at least one $a_i \neq \infty$ and it is a norm if and only if all $a_i \neq \infty$.

A seminorm of the form $\|\cdot\|_{\mathbf{e}, \vec{a}}$ for a basis \mathbf{e} and coefficients $\vec{a} \in \overline{\mathbb{R}}^n$ is said to be *diagonalizable*.

Definition 1.1.3. A non-Archimedean field K is said to be *spherically complete* if any decreasing sequence of closed balls has non-empty intersection.

A valued field extension L/K is said to be *immediate* if it has the same value group and the same residue field. A field is called *maximally complete* if it does not admit any proper immediate extension. By a classical result from Kaplansky (see e.g. [Sch50], II.6, Theorem 8), it turns out that this concept is equivalent to spherical completeness. We thus have the following:

Theorem 1.1.4 ([Kru32] Satz 24). *Every non-Archimedean field admits a (non-unique) immediate extension that is spherically complete.*

The following observation will be central for our result:

Proposition 1.1.5. *Let K be a non-Archimedean field. Then K is spherically complete if and only if every seminorm on a finite-dimensional K -vector space is diagonalizable.*

Proof. By [BE21, Lemma 1.12] the non-Archimedean field K is spherically complete if and only if all norms on a finite-dimensional vector space over K are diagonalizable. Our

statement follows from the fact that a seminorm is diagonalizable if and only if the induced norm on the quotient modulo the kernel is diagonalizable. \square

- Example 1.1.6.** (i) Every trivially valued field is obviously maximally (hence spherically) complete.
- (ii) Every discretely valued complete field is spherically complete, as by [RTW12, Proposition 3.1] every norm on a finite-dimensional vector space is diagonalizable.
- (iii) In particular, for any prime number p the field of p -adic numbers \mathbb{Q}_p is local and spherically complete.
- (iv) The algebraic closure $\overline{\mathbb{Q}_p}$ of the p -adic numbers is not complete (see [Rob00, Section 3.1.4]).
- (v) The completion \mathbb{C}_p of $\overline{\mathbb{Q}_p}$ is still algebraically closed, but not spherically complete.
- (vi) Over \mathbb{C} , the field of rational functions $\mathbb{C}(t)$ is not Cauchy complete, as the Cauchy sequence $(\sum_{i=0}^n \frac{1}{i!} t^i)_{n \in \mathbb{N}}$ has no limit. Its completion is the field of (formal) Laurent series $\mathbb{C}((t))$, which is spherically complete, because the valuation is discrete.
- (vii) For any field k let $k\{\{t\}\} = \bigcup_{n \geq 1} k((t^{1/n}))$ be the field of Puiseux series. It is not complete. Its completion is the Levi-Civita field [BCS18, Theorem 4.10], in which the exponents of a series need not have a common denominator, but for any upper bound, there are only finitely many exponents with non-zero coefficients. The Levi-Civita field is not spherically complete, its spherical completion is the Malcev-Neumann field $k((t^{\mathbb{Q}}))$ of power series with rational exponents and well-ordered support (see [BE21, Example 1.1]).

Let $\mathcal{N}(V)$ and $\overline{\mathcal{N}}(V)$ be the set of norms and respectively non-trivial seminorms on the dual vector space V^* (note the dualization!). For $x \in \overline{\mathcal{N}}(V)$ we denote by $\|\cdot\|_x$ the associated seminorm. We endow $\overline{\mathcal{N}}(V)$ with the coarsest topology that makes the natural evaluation maps

$$\overline{\mathcal{N}}(V) \ni x \longmapsto \|v\|_x \in \mathbb{R}$$

for all $v \in V^*$ continuous. The space $\mathcal{N}(V)$ has been first introduced by Goldman and Iwahori in [GI63] over $K = \mathbb{Q}_p$.

Remark 1.1.7 (Topology of $\overline{\mathcal{N}}(V)$). (i) Equivalently, one may define the topology on $\overline{\mathcal{N}}(V)$ as the topology of pointwise convergence: a net (x_α) in $\overline{\mathcal{N}}(V)$ converges to a seminorm $x \in \overline{\mathcal{N}}(V)$ if and only if $\|v\|_{x_\alpha}$ converges to $\|v\|_x$ in \mathbb{R} for all $v \in V^*$.

(ii) A basis of the topology on $\overline{\mathcal{N}}(V)$ is given by open subsets of the form

$$\begin{aligned} U &= \{x \in \overline{\mathcal{N}}(V) \mid \|v_i\|_x \in (a_i, b_i) \text{ for all } i = 1, \dots, l\} \\ &= ev_{v_1}^{-1}((a_1, b_1)) \cap \dots \cap ev_{v_l}^{-1}((a_l, b_l)) \end{aligned}$$

for some $v_1, \dots, v_l \in V^*$ and open intervals $(a_1, b_1), \dots, (a_l, b_l) \subseteq \mathbb{R}$, where ev denotes the natural evaluation map.

Two points $x, y \in \overline{\mathcal{N}}(V)$ are said to be *homothetic*, written as $x \sim y$, if there is a constant $c > 0$ such $\|\cdot\|_x = c \cdot \|\cdot\|_y$. Homothety defines an equivalence relation on $\overline{\mathcal{N}}(V)$ that restricts to an equivalence relation on $\mathcal{N}(V)$.

We denote by $\mathcal{N}^{\text{diag}}(V) \subseteq \mathcal{N}(V)$ the subspace of diagonalizable norms and $\overline{\mathcal{N}}^{\text{diag}}(V) \subseteq \overline{\mathcal{N}}(V)$ the subspace of diagonalizable seminorms. Note that seminorms that are homothetic to a diagonalizable one are themselves diagonalizable. Let $\mathcal{X}(V) = \mathcal{N}(V)/\sim$.

Definition 1.1.8. The *affine Bruhat–Tits building* of $\text{PGL}(V)$ is defined to be the quotient

space of diagonalizable norms by homothety:

$$\mathcal{B}(V) = \mathcal{N}^{\text{diag}}(V)/\sim.$$

The quotient $\overline{\mathcal{B}}(V) = \overline{\mathcal{N}}^{\text{diag}}(V)/\sim \subseteq \overline{\mathcal{X}}(V) = \overline{\mathcal{N}}(V)/\sim$ forms a natural bordification of $\mathcal{B}(V)$ (in fact, by Corollary 1.2.8 below $\overline{\mathcal{X}}(V)$ is compact). In particular, for a spherically complete field K the quotient $\overline{\mathcal{B}}(V) = \overline{\mathcal{X}}(V)$ is a natural compactification of $\mathcal{B}(V)$. This expands on the construction in [Wer04].

When $V = K^{n+1}$, we write $\mathcal{X}_n(K)$ and $\overline{\mathcal{X}}_n(K)$ for $\mathcal{X}(K^{n+1})$ and $\overline{\mathcal{X}}(K^{n+1})$ as well as $\mathcal{B}_n(K)$ and $\overline{\mathcal{B}}_n(K)$ for $\mathcal{B}(K^{n+1})$ and $\overline{\mathcal{B}}(K^{n+1})$, respectively.

Remark 1.1.9. (Topological structure)

- (i) By [BE21, Theorem 1.19] the locus $\mathcal{N}^{\text{diag}}(V)$ of diagonalizable seminorms is dense in $\mathcal{N}(V)$ with respect to the Goldman-Iwahori metric

$$d(\|\cdot\|_x, \|\cdot\|_y) = \sup_{v \in V^*} (\log \|v\|_x - \log \|v\|_y).$$

The finiteness of the supremum follows from the fact that any two norms are equivalent (see [BE21, Page 10]). The topology induced by this metric is finer than the one defined above since uniform convergence implies pointwise convergence. It follows that $\mathcal{B}(V)$ is dense in $\mathcal{X}(V)$. Hence also $\overline{\mathcal{B}}(V)$ is dense in $\overline{\mathcal{X}}(V)$. Indeed, a stratum of $\overline{\mathcal{B}}(V)$ of seminorms with a fixed kernel $W \subset V$ can be identified with $\mathcal{B}(V/W)$.

- (ii) Despite the notation suggesting that $\mathcal{N}(V) \subset \overline{\mathcal{N}}(V)$ is open, this need not be true in general, as we will show in Remark 1.2.10. On the other hand, it is when K is local. Indeed, let $\{x_\alpha\}_\alpha$ be a net of seminorms, converging to \bar{x} , and let $\{v_\alpha\}_\alpha$ be a net of unit vectors (with respect to some fixed norm on V) such that $\|v_\alpha\|_{x_\alpha} = 0$. Since the field is local, the unit sphere is compact and so we can find a convergent subnet converging to $\bar{v} \neq 0$. We conclude that $\|\bar{v}\|_{\bar{x}} = 0$ by continuity and thus \bar{x} is also a proper seminorm.

Example 1.1.10. Let K have trivial valuation. Then we can give an explicit description of the space of seminorms up to homothety on a vector space V of dimension $r+1$. First recall that by Example 1.1.6 Item (i) the field K is spherically complete and hence by Proposition 1.1.5 we have $\overline{\mathcal{X}}(V) = \overline{\mathcal{B}}(V)$. We claim that there is a bijection:

$$\overline{\mathcal{B}}(V) \xrightarrow{1:1} \left\{ (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V^*, c_1 > \cdots > c_{l-1}) \mid c_1, \dots, c_{l-1} \in \overline{\mathbb{R}}_{>0} \right\}_{l=1, \dots, r+1},$$

where $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V^*$ is a flag of subspaces of V^* . We allow the special case of $l=1$ where we just have the flag $(0 \subsetneq V^*)$ without any coordinates.

Proof. Let $\|\cdot\| \in \overline{\mathcal{N}}(V)$ be a representative of $x \in \overline{\mathcal{B}}(V)$, i.e. a non-trivial seminorm on V^* . Then all balls around 0 in V^* are subspaces. By letting the radius vary, we obtain a unique filtration $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V^*$ such that for all $i = 1, \dots, l$ the restriction of $\|\cdot\|$ to $V_i \setminus V_{i-1}$ is a constant function and the values are strictly increasing.

Let d_i denote the constant value on $V_i \setminus V_{i-1}$ and set for $i = 1, \dots, l-1$

$$c_i := -\log \frac{d_i}{d_l}.$$

Note that any seminorm homothetic to $\|\cdot\|$ yields the same filtration of subspaces and only multiplies all the d_i simultaneously by a common scalar. This does not change the c_i and thus we obtain a well-defined map.

Vice versa, every flag of subspaces $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l = V^*$ together with coordinates $c_1 > \dots > c_{l-1} \in \overline{\mathbb{R}}_{>0}$ gives rise to coordinates d_i via the formula above by setting $d_l = 1$, and thus we obtain a well-defined diagonalizable seminorm with

$$\|\cdot\|_{V_i \setminus V_{i-1}} = d_i$$

and generic value 1. By construction, these maps are inverses of each other. \square

Remark 1.1.11. The bijection in Example 1.1.10 restricts to a bijection

$$\mathcal{B}(V) \xrightarrow{1:1} \{(0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l = V^*, c_1 > \dots > c_{l-1}) \mid c_1, \dots, c_{l-1} \in \overline{\mathbb{R}}_{>0}\}.$$

This can be identified with the cone over the order complex of the lattice of non-trivial subspaces of V^* . The latter comes with a natural weak topology, where a set is open if and only if its intersection with each cone is relatively open in that cone. This turns this set into the colimit of all cones corresponding to a fixed filtration. However, the topology on the building is much coarser than the weak topology of the cone complex, as the following example will show.

Example 1.1.12. We consider the case where K is any infinite field with trivial valuation. Then we can identify $\overline{\mathcal{B}}_1(K)$ with the set:

$$\{(0 \subsetneq V_1 \subsetneq (K^2)^*, c) \mid c \in \overline{\mathbb{R}}_{\geq 0}\} \cup \{(0 \subsetneq (K^2)^*)\}.$$

Here $\eta := (0 \subsetneq (K^2)^*)$ corresponds to the homothety class of a norm that is constant away from 0. Each cone corresponds to a one-dimensional subspace and the point at infinity of each cone corresponds to the homothety class of a proper seminorm. In rank one, the homothety class of a proper seminorm is uniquely determined by its kernel, and thus we can identify $\overline{\mathcal{B}}_1(K) \setminus \mathcal{B}_1(K)$ with $\mathbb{P}^1(K)$.

A sketch of $\overline{\mathcal{B}}_1(K)$ can be found in Figure 1.1. Here \mathcal{F} is shorthand for “ $0 \subsetneq V_1 \subsetneq (K^2)^*$ ” for any subspace V_1 of dimension 1. The coordinate c is a positive real number. A norm in the homothety class corresponding to (\mathcal{F}, c) has generic value 1, and value e^{-c} on $V_1 \setminus \{0\}$. In the case of $c = \infty$, we have a proper seminorm with kernel V_1 .

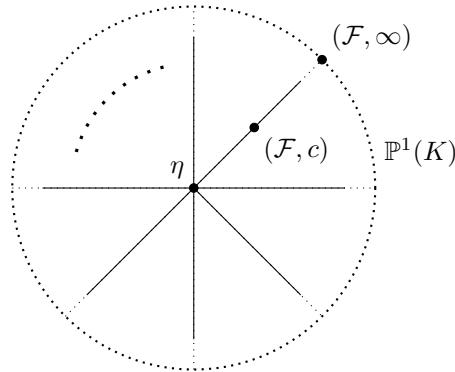


Figure 1.1: The building $\overline{\mathcal{B}}_1(K)$ of a trivially valued field.

A basis of the topology of $\overline{\mathcal{B}}_1(K)$ is given as follows: choose finitely many vectors $v_1, \dots, v_l \in (K^2)^*$ and intervals $(a_1, b_1), \dots, (a_l, b_l) \subset \mathbb{R}$ and define

$$U = \{x \in \overline{\mathcal{B}}_1(K) \mid \exists \text{ a representative } \|\cdot\| \in x \text{ with } \|v_i\| \in (a_i, b_i) \text{ for all } i = 1, \dots, l\}.$$

In particular, if such a set is a neighbourhood of η , then it contains all cones corresponding to subspaces which do not contain any vector v_1, \dots, v_l , i.e. all but finitely many. This is of course not necessarily true for open neighborhoods of η in the weak topology of the cone complex.

The space $\mathcal{B}(V)$ has the structure of an affine building in the sense of [RTW15, Definition 1.9] (see e.g. [Par00, III.1.2] for a proof). We refer the reader also to [BS14] for a discussion of the various axiom systems for affine buildings over possibly non-discretely valued fields. For us, however, the most important notion is the following:

Definition 1.1.13. An *apartment* $\mathcal{A}(\mathbf{e})$ in $\mathcal{B}(V)$ (associated to a basis \mathbf{e} of V^*) is given by the image in $\mathcal{B}(V)$ of

$$\begin{aligned} \mathbb{R}^{n+1} &\hookrightarrow \mathcal{N}^{\text{diag}}(V), \\ \vec{a} &\longmapsto \|\cdot\|_{\mathbf{e}, \vec{a}}. \end{aligned}$$

Every apartment is a closed subset homeomorphic to $\mathbb{R}^{n+1}/\mathbb{R} \simeq \mathbb{R}^n$. The above parametrization can be extended to $\overline{\mathbb{R}}^{n+1} - \{(\infty, \dots, \infty)\}$. Its image, denoted by $\overline{\mathcal{A}}(e)$, is the closure of $\mathcal{A}(e)$ and may be naturally identified with $\mathbb{TP}^n = \overline{\mathbb{R}}^{n+1} - \{(\infty, \dots, \infty)\} / \mathbb{R} \cdot \mathbb{1}$. We refer to $\overline{\mathcal{A}}(e)$ as a *compactified apartment*.

Example 1.1.14. In Example 1.1.12, every apartment is the union of exactly two one-dimensional cones: a (homothety class of a) seminorm is diagonalizable by a basis $\{v_1, v_2\} \subset (K^2)^*$ if and only if it lies in the cone corresponding to $\langle v_1 \rangle$ or $\langle v_2 \rangle$.

Our conventions are chosen so that the associations $V \mapsto \overline{\mathcal{X}}(V)$ and $V \mapsto \overline{\mathcal{B}}(V)$ are covariant functors from finite-dimensional K -vector spaces to topological spaces. In particular, any embedding $\iota : V \hookrightarrow W$ induces an embedding $\overline{\mathcal{B}}(\iota) : \overline{\mathcal{B}}(V) \hookrightarrow \overline{\mathcal{B}}(W)$. We have a natural operation of $\text{PGL}(V)$ on $\overline{\mathcal{X}}(V)$, which respects diagonalizability and non-degeneracy. It is easy to show, that for $\dim V > 1$ this operation of $\text{PGL}(V)$ on $\mathcal{B}(V)$ is transitive if and only if the valuation on K is surjective.

Example 1.1.15. In the case of a discretely valued field K , the affine building $\mathcal{B}_r(K)$ is a flag simplicial complex whose vertices correspond to equivalence classes of lattices (see Section 1.6.2). Let \mathcal{O}_K denote the valuation ring, π a uniformiser, and $k = \mathcal{O}_K/(\pi)$ the residue field. A *lattice* in K^{r+1} is a free \mathcal{O}_K -submodule of rank $r+1$. Two lattices L_1, L_2 are *homothetic* if $L_1 = cL_2$ for some $c \in K^*$. Two homothety equivalence classes Λ_1, Λ_2 of lattices are *adjacent* if there are representatives L_1 and L_2 such that $\pi L_1 \subseteq L_2 \subseteq L_1$. The simplices of $\mathcal{B}_r(K)$ correspond to flags of adjacent lattices $\pi L_1 \subsetneq L_k \subsetneq \dots \subsetneq L_2 \subsetneq L_1$. A lattice L corresponds to an integral norm $|\cdot|_L$ given by

$$|x|_L := \inf \{ |\lambda| \mid \lambda \in K^*, \lambda^{-1}x \in L \}$$

for $x \in K^{r+1}$, and one recovers the lattice as the closed unit ball of the norm. Given a lattice L , the star of $[L]$ in $\mathcal{B}_r(K)$ can be identified with the spherical building $\mathcal{B}_r(k)$ (see Example 1.1.10) by sending a flag of lattices $\pi L \subsetneq L_k \subsetneq \dots \subsetneq L_2 \subsetneq L$ to its image in $L/\pi L$, which is a flag of k -linear subspaces.

Example 1.1.16. Let $r = 1$ and $K = \mathbb{Q}_p$. Then the affine Bruhat–Tits building $\mathcal{B}_1(\mathbb{Q}_p)$ is an infinite tree whose vertices have valency $p+1$. Let $\mathbf{e} = (e_1, e_2)$ be a basis of $(\mathbb{Q}_p^2)^*$. The apartment $\mathcal{A}(\mathbf{e})$ is an infinite path in the tree which uses all \mathbb{Z}_p -lattices with basis $(p^{u_1}e_1, p^{u_2}e_2)$ where $(u_1, u_2) \in \mathbb{Z}^2$. See Figure 1.2 for the compactified Bruhat–Tits tree of \mathbb{Q}_2 . The open part is the usual trivalent infinite tree $\mathcal{B}_1(\mathbb{Q}_2)$. The boundary $\overline{\mathcal{B}}_1(\mathbb{Q}_2) \setminus \mathcal{B}_1(\mathbb{Q}_2)$ (illustrated by the circle) can be identified with $\mathbb{P}^1(\mathbb{Q}_2)$ since any homothety class of a non-

trivial proper seminorm on $(\mathbb{Q}_2^2)^*$ can be identified with its kernel, which is a 1-dimensional subspace of $(\mathbb{Q}_2^2)^*$.

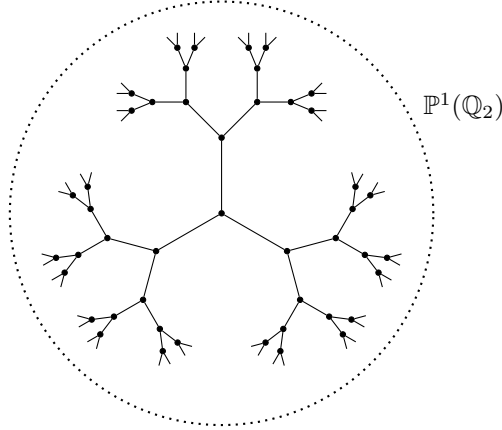


Figure 1.2: The affine Bruhat–Tits building $\overline{\mathcal{B}}_1(\mathbb{Q}_2)$.

1.2 Analytification and Tropicalization

For a complete non-Archimedean field K we recall the Berkovich analytification for any locally finite type scheme over K . We describe the relations between the Berkovich space, the space of seminorms on $(K^{r+1})^*$, and the compactified building. This allows us to define a tropicalization map from the space of seminorms on $(K^{r+1})^*$ to tropical projective space. Let A be a finitely generated K -algebra and write $U = \text{Spec } A$.

Definition 1.2.1. A (multiplicative) seminorm on A is a map $|\cdot|: A \rightarrow \mathbb{R}$ such that

- (i) $|f| \geq 0$ for all $f \in A$ as well as $|a| = |a|_K$ for all $a \in K$;
- (ii) $|f \cdot g| = |f| \cdot |g|$ for all $f, g \in A$; and
- (iii) $|f + g| \leq \max\{|f|, |g|\}$ for all $f, g \in A$.

We think of the set U^{an} of multiplicative seminorms on A as a space, the *analytification* of U in the sense of Berkovich [Ber90]; we write $x \in U^{\text{an}}$ for a point in U^{an} as well as $|\cdot|_x$ for the associated seminorm. The (analytic) topology of U^{an} is the coarsest that makes all evaluation maps

$$\begin{aligned} U^{\text{an}} &\longrightarrow \mathbb{R}, \\ x &\longmapsto |f|_x \end{aligned}$$

for $f \in A$ continuous. For a scheme X that is locally of finite type over K , we define its *analytification* X^{an} locally as above, and globally by gluing over affine open covers. See [Ber90, Chapter 3] for details.

Remark 1.2.2. For an affine scheme U of finite type over K , we have a natural inclusion $U(K) \rightarrow U^{\text{an}}$ via $x \mapsto [f \mapsto |f(x)|]$. For any scheme X , locally of finite type over K , these inclusions on affine open subsets glue to an inclusion $X(K) \rightarrow X^{\text{an}}$. If the valuation on K is non-trivial, the image of $X(\overline{K})$ is dense in X^{an} for an algebraic closure \overline{K} of K .

The association $X \mapsto X^{\text{an}}$ is a covariant functor that commutes with the inclusion of the K -points of a scheme into its analytification. See [Gub13, Section 2.6] for details.

Remark 1.2.3. To distinguish between multiplicative seminorms on a K -algebra and seminorms on a finite-dimensional K -vector space, we denote the former by $|\cdot|_x$ and the latter by $\|\cdot\|_x$.

Let V be a vector space over K of dimension $r + 1$.

Remark 1.2.4 (Analytification of projective spaces). As explained in [RTW15, Section 2.1.1], the analytification of the projective space $\mathbb{P}(V)$ can be identified with the quotient of $\mathbb{A}(V)^{\text{an}} - \{0\}$ modulo homothety.

Let $S^\bullet V^*$ be the symmetric algebra of the dual vector space V^* . This is a finitely generated graded K -algebra. Every choice of a basis (e_0, \dots, e_r) of V^* induces an isomorphism of $S^\bullet V^*$ with the polynomial ring over K in $r + 1$ indeterminates. The affine space $\mathbb{A}(V)$ and projective space $\mathbb{P}(V)$ are defined as

$$\mathbb{A}(V) = \text{Spec}(S^\bullet V^*) \quad \text{and} \quad \mathbb{P}(V) = \text{Proj}(S^\bullet V^*).$$

As said above, $\mathbb{A}(V)^{\text{an}}$ consists of all multiplicative seminorms on $S^\bullet V^*$. Then the analytification $\mathbb{P}(V)^{\text{an}}$ is the quotient of $\mathbb{A}(V)^{\text{an}} - \{0\}$ by homothety: we define $x \sim y$ if and only if there exists a constant $c > 0$ such that for all $f \in S^n V^*$ we have $|f|_x = c^n |f|_y$. Here the analytic topology on $\mathbb{P}(V)^{\text{an}}$ is equal to the quotient topology.

A multiplicative seminorm on $S^\bullet V^*$ induces a seminorm on $V^* = S^1 V^*$ by restriction, hence we have a natural continuous map $\tau : \mathbb{A}(V)^{\text{an}} \rightarrow \overline{\mathcal{N}}(V)$ such that $\tau(x) = 0$ if and only if $x = 0$. Since this map is compatible with the equivalence relations, it descends to a continuous *restriction map*

$$\tau : \mathbb{P}(V)^{\text{an}} \rightarrow \overline{\mathcal{X}}(V).$$

Proposition 1.2.5. *Let K be spherically complete. Then the restriction map admits a section*

$$J : \overline{\mathcal{X}}(V) = \overline{\mathcal{B}}(V) \rightarrow \mathbb{P}(V)^{\text{an}}.$$

Given a diagonalizable seminorm $\|\cdot\|$, we may choose a basis e_0, \dots, e_r of V^* and $c_0, \dots, c_r \in \mathbb{R}_{\geq 0}$ such that

$$\left\| \sum \lambda_i e_i \right\| = \max_i \{c_i |\lambda_i|\}.$$

The multiplicative seminorm $J(\|\cdot\|)$ on $S^\bullet V^*$ is defined by

$$J(\|\cdot\|) \left(\sum_{I=(i_0, \dots, i_r)} a_I e_0^{i_0} \dots e_r^{i_r} \right) = \max_I \{ |a_I| c_0^{i_0} \dots c_r^{i_r} \}.$$

When K is a local field, the section J is continuous.

Proof. We refer the reader to [RTW12, Section 3] for details on this construction. Note that in [RTW12] the authors assume that K is local, although everything but the continuity of J goes through when K is spherically complete. \square

Remark 1.2.6. We do not know whether J is continuous, when K is not local. Luckily this statement is not needed in the remainder of this chapter.

Proposition 1.2.7. *Let K be any complete non-Archimedean field. Then the restriction map*

$$\tau : \mathbb{P}(V)^{\text{an}} \rightarrow \overline{\mathcal{X}}(V)$$

is surjective.

Proof. We want to construct a section $\overline{\mathcal{X}}(V) \rightarrow \mathbb{P}(V)^{\text{an}}$ in the general case. We pick any spherically complete extension L/K and denote $V \otimes_K L$ by V_L . Then for any seminorm $\|\cdot\|$ on V^* we have an induced seminorm $\|\cdot\|_L$ on V_L^* given by

$$\|w\|_L = \inf \max_i \{ |\lambda_i| \cdot \|v_i\| \},$$

where the infimum runs over all possible decompositions $w = \sum_i \lambda_i v_i$ with $\lambda_i \in L$ and $v_i \in V^*$. It has been shown in [BE21, Proposition 1.25] that the map $\overline{\mathcal{X}}(V) \rightarrow \overline{\mathcal{X}}(V_L)$ given by $\|\cdot\| \mapsto \|\cdot\|_L$ is injective and that this map is a section of the natural restriction map $\overline{\mathcal{X}}(V_L) \rightarrow \overline{\mathcal{X}}(V)$. Now, we define a section of τ by composing

$$\overline{\mathcal{X}}(V) \longrightarrow \overline{\mathcal{X}}(V_L) = \overline{\mathcal{B}}(V_L) \xrightarrow{J} \mathbb{P}(V_L)^{\text{an}} \longrightarrow \mathbb{P}(V)^{\text{an}},$$

where the last map is the restriction of a multiplicative seminorm on the symmetric algebra $S^\bullet V^* \subseteq S^\bullet V_L^*$. \square

Corollary 1.2.8. *The space $\overline{\mathcal{X}}(V)$ is compact.*

Proof. This follows immediately from surjectivity of $\tau : \mathbb{P}(V)^{\text{an}} \rightarrow \overline{\mathcal{X}}(V)$ and compactness of $\mathbb{P}(V)^{\text{an}}$ (see e.g. [Ber90, Theorem 3.4.8 and 3.5.3]). \square

Remark 1.2.9. We do not know if the section constructed in the proof of Proposition 1.2.7 is independent of the choice of the spherical completion (which need not be unique in some cases in positive characteristic, see [BCS18, Theorem 6.17]), or if the section is continuous.

Remark 1.2.10. We can now show that $\mathcal{N}(V) \subseteq \overline{\mathcal{N}}(V)$ need not be open. Let K be algebraically closed. The set of K -points of $\mathbb{P}(V)$ is dense in $\mathbb{P}(V)^{\text{an}}$ and gets mapped to homothety classes of proper diagonalizable seminorms in $\overline{\mathcal{X}}(V)$, thus its image is also dense. Consequently, neither inclusion $\mathcal{X}(V) \subseteq \overline{\mathcal{X}}(V)$, nor $\mathcal{B}(V) \subseteq \overline{\mathcal{B}}(V)$, nor $\mathcal{N}(V) \subseteq \overline{\mathcal{N}}(V)$ can be open in this case.

From now on we consider the vector space $V = K^{n+1}$ together with its standard basis $\mathbf{e} = (e_0, \dots, e_n)$ and the associated dual basis $\mathbf{e}^* = (e_0^*, \dots, e_n^*)$ of V^* . This identifies $\mathbb{A}(V)$ and $\mathbb{P}(V)$ with $\mathbb{A}^{n+1} = \text{Spec } K[t_0, \dots, t_n]$ and $\mathbb{P}^n = \text{Proj } K[t_0, \dots, t_n]$ respectively. As explained e.g. in [Pay09], there is a natural continuous tropicalization map

$$\begin{aligned} \text{trop}_{\mathbb{A}^{n+1}} : (\mathbb{A}^{n+1})^{\text{an}} &\longrightarrow \overline{\mathbb{R}}^{n+1}, \\ x &\longmapsto (-\log |t_0|_x, \dots, -\log |t_n|_x) \end{aligned}$$

that is compatible with the diagonal \mathbb{G}_m -operation. Therefore, this induces a tropicalization map

$$\begin{aligned} \text{trop}_{\mathbb{P}^n} : (\mathbb{P}^n)^{\text{an}} &\longrightarrow \mathbb{TP}^n \\ x &\longmapsto [-\log |t_0|_x : \dots : -\log |t_n|_x]. \end{aligned}$$

Tropicalizing the analytification of the big torus in \mathbb{P}^n yields only elements with finite coordinates:

$$\text{trop}_{\mathbb{P}^n}(\mathbb{G}_m^n)^{\text{an}} = \mathbb{R}^{n+1}/\mathbb{1}.$$

Proposition 1.2.11. *The tropicalization map $\text{trop}_{\mathbb{P}^n}$ factors as*

$$(\mathbb{P}^n)^{\text{an}} \xrightarrow{\tau} \overline{\mathcal{X}}_n(K) \xrightarrow{\text{trop}_{\overline{\mathcal{X}}_n}} \mathbb{TP}^n, \quad (1.1)$$

where $\text{trop}_{\overline{\mathcal{X}}_n} : \overline{\mathcal{X}}_n(K) \rightarrow \mathbb{TP}^n$ is a continuous and surjective map given by associating to a seminorm $\|\cdot\|_x : V^* \rightarrow \mathbb{R}$ the tuple

$$\text{trop}_{\overline{\mathcal{X}}_n}(x) = [-\log \|e_0^*\|_x : \dots : -\log \|e_n^*\|_x] \in \mathbb{TP}^n.$$

Proof. The tropicalization map $\text{trop}_{\overline{\mathcal{X}}_n}$ is well-defined, since, for two seminorms $\|\cdot\|_x, \|\cdot\|_y$ on V^* together with a homothety $x \sim y$ we have a $c > 0$ such that $\|\cdot\|_x = c \cdot \|\cdot\|_y$ and thus

$$\begin{aligned} \text{trop}_{\overline{\mathcal{X}}_n}(x) &= [-\log \|e_0^*\|_x : \cdots : -\log \|e_n^*\|_x] \\ &= [-\log c - \log \|e_0^*\|_y : \cdots : -\log c - \log \|e_n^*\|_y] \\ &= [-\log \|e_0^*\|_y : \cdots : -\log \|e_n^*\|_y] \\ &= \text{trop}_{\overline{\mathcal{X}}_n}(y). \end{aligned}$$

It is continuous, since it is given by evaluation maps in each coordinate, and surjective, since the compactified apartment map

$$\begin{aligned} \overline{\mathbb{R}}^{n+1} - \{\infty^{n+1}\} &\longrightarrow \overline{\mathcal{B}}_n(K), \\ (a_0, \dots, a_n) &\longmapsto \|\cdot\|_{\mathbf{e}^*, (a_0, \dots, a_n)} \end{aligned}$$

induces a continuous section $\mathbb{T}\mathbb{P}^n \rightarrow \overline{\mathcal{B}}_n(K) \subseteq \overline{\mathcal{X}}_n(K)$ of $\text{trop}_{\overline{\mathcal{X}}_n}$. The factorization (1.1) follows from the observation that, under the identification $S^\bullet V^* \simeq K[t_0, \dots, t_n]$ the linear one-form e_i^* is naturally identified with the linear polynomial t_i . Therefore, we have $|t_i|_x = \|e_i^*\|_{\tau(x)}$ for all $x \in (\mathbb{P}^n)^{\text{an}}$ and $i = 0, \dots, n$. This implies

$$\begin{aligned} \text{trop}_{\overline{\mathcal{X}}_n}(\tau(x)) &= [-\log \|e_0^*\|_{\tau(x)} : \cdots : -\log \|e_n^*\|_{\tau(x)}] \\ &= [-\log |t_0|_x : \cdots : -\log |t_n|_x] \\ &= \text{trop}_{\mathbb{P}^n}(x) \end{aligned}$$

for all $x \in (\mathbb{P}^n)^{\text{an}}$. □

Similarly, tropicalizing the non-compactified space of norms, one obtains all finite points:

$$\text{trop}_{\overline{\mathcal{X}}_n}(\mathcal{X}_n(K)) = \mathbb{R}^{n+1}/\mathbb{1}.$$

Let $Y \subseteq \mathbb{P}^n$ be a Zariski-closed subscheme. Then, following [Pay09, Gub13], the *tropicalization* of Y is defined to be the projection

$$\text{Trop}_{\mathbb{P}^n}(Y) = \text{trop}_{\mathbb{P}^n}(Y^{\text{an}})$$

of Y^{an} under $\text{trop}_{\mathbb{P}^n}$ into $\mathbb{T}\mathbb{P}^n$. Let L be an algebraically closed extension of K with non-trivial absolute value $|\cdot|_L$. By [Gub13, Proposition 3.8] the tropicalization $\text{Trop}_{\mathbb{P}^n}(Y)$ is equal to the closure of

$$\left\{ [-\log(|t_0|_L) : \cdots : -\log(|t_n|_L)] \mid [t_0 : \cdots : t_n] \in Y(L) \subseteq \mathbb{P}^n(L) \right\} \subseteq \mathbb{T}\mathbb{P}^n.$$

Proposition 1.2.12. *For a linear embedding $\iota: \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ the tropicalization $\text{Trop}(\mathbb{P}^r, \iota) = \text{trop}_{\mathbb{P}^n}(\iota(\mathbb{P}^r)^{\text{an}})$ is equal to the projection of $\overline{\mathcal{X}}(\iota)(\overline{\mathcal{X}}_r(K)) \subseteq \overline{\mathcal{X}}_n(K)$ under $\text{trop}_{\overline{\mathcal{X}}_n}$ into $\mathbb{T}\mathbb{P}^n$.*

Proof. Since the restriction map τ is surjective, the commutativity of

$$\begin{array}{ccc} (\mathbb{P}^r)^{\text{an}} & \xrightarrow{\tau} & \overline{\mathcal{X}}_r(K) \\ \iota^{\text{an}} \downarrow & & \downarrow \overline{\mathcal{X}}(\iota) \\ (\mathbb{P}^n)^{\text{an}} & \xrightarrow{\tau} & \overline{\mathcal{X}}_n(K) \end{array}$$

implies that $\overline{\mathcal{X}}(\iota)(\overline{\mathcal{X}}_r(K)) = \tau(\iota^{\text{an}}(\mathbb{P}^r)^{\text{an}})$. The factorization of the tropicalization map in Proposition 1.2.11 then yields the claim. □

Henceforth, we will define for any linear embedding $\iota : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ the composition π_ι by

$$\pi_\iota := \text{trop}_{\overline{\mathcal{X}}_n} \circ \overline{\mathcal{X}}(\iota) : \overline{\mathcal{X}}_r(K) \longrightarrow \text{Trop}(\mathbb{P}^r, \iota).$$

Note that π_ι is continuous and surjective. A direct computation shows that if $\iota = [f_0 : \dots : f_n]$ for $f_0, \dots, f_n \in (K^{r+1})^*$ we have

$$\pi_\iota(x) = [-\log(\|f_0\|_x) : \dots : -\log(\|f_n\|_x)]$$

for all $x \in \overline{\mathcal{X}}_r(K)$.

1.3 Limits of Linear Tropicalizations

In this section we will set up and prove Theorem A. We first set up a category of linear embeddings such that tropicalization yields a covariant functor into the category of topological spaces.

Definition 1.3.1. Let I be the category of linear embeddings $\mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$, where U is a torus-invariant open subset of \mathbb{P}^n , with morphisms given by commutative triangles

$$\begin{array}{ccc} \mathbb{P}^r & \xrightarrow{\iota} & U \\ & \searrow \iota' & \downarrow \\ & & U' \end{array}$$

where $U \rightarrow U'$ is a toric morphism.

Remark 1.3.2. The codimension of the complement of $U \subseteq \mathbb{P}^n$ in Definition 1.3.1 must be at least $r+1$. Thus, for all $j = 0, \dots, r$, the Chow group of U of codimension j is isomorphic to that of \mathbb{P}^n . The degree of ι' being 1 can be measured by intersecting its image with a generic linear subspace of codimension r . Since the same is true for ι , we see that hyperplanes must pull back to hyperplanes along $U \rightarrow U'$, i.e. this morphism is forced to be linear.

Remark 1.3.3. Allowing the toric morphism to be defined on a smaller torus-invariant open subset instead of the whole projective space endows our index category with many more morphisms, most notably coordinate projections. This makes I into a cofiltered category: indeed, for any two objects $\iota_i : \mathbb{P}^r \hookrightarrow U_i \subseteq \mathbb{P}^{n_i}$, $i = 1, 2$, we can find a third one dominating both, namely $\iota : \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^N$, where $N = n_1 + n_2 + 1$ with projective coordinates z_0, \dots, z_N and

$$U = (\mathbb{P}^N \setminus V(z_0, \dots, z_{n_1}) \cap \text{pr}_1^{-1}(U_1)) \cap (\mathbb{P}^N \setminus V(z_{n_1+1}, \dots, z_{n_1+n_2+1}) \cap \text{pr}_2^{-1}(U_2)).$$

And for any two morphisms $f, g : U_1 \rightarrow U_2$ commuting with ι_i as above, we can equalize them by defining $U_0 = \{f = g\} \subseteq U_1$, and noticing that ι_1 factors through U_0 , and the closure of U_0 in \mathbb{P}^{n_1} is a linear subspace $\mathbb{P}^{n_0} \subseteq \mathbb{P}^{n_1}$.

The tropicalization U^{trop} of a torus-invariant open subset of \mathbb{P}^n is a special case of the tropicalization of toric varieties, as introduced in [Pay09, Section 3]. In our case this means removing those tropical torus-orbits from $\mathbb{T}\mathbb{P}^n$, for which the corresponding algebraic torus orbit is not contained in U . For a toric morphism $\varphi : U \rightarrow U'$ such that $\varphi \circ \iota = \iota'$, we have a natural induced map $\varphi^{\text{trop}} : U^{\text{trop}} \rightarrow U'^{\text{trop}}$ such that $\varphi^{\text{trop}}(\text{Trop}(\mathbb{P}^r, \iota)) \subseteq \text{Trop}(\mathbb{P}^r, \iota')$. In particular, if ϕ is a coordinate projection, then ϕ^{trop} is the analogous tropical coordinate projection. We refer the reader to [Pay09], to [Rab12, Section 3 and 5], and to [MS15, Section 6.2] for more details on the tropical geometry of toric varieties.

Let $\iota : \mathbb{P}^r \hookrightarrow U$ be a linear closed immersion, where U is a torus-invariant open subset of \mathbb{P}^n , and let $j : U \rightarrow \mathbb{P}^n$ be the inclusion. Then we have a homeomorphism of tropicalizations

$$\text{Trop}(\mathbb{P}^r, \iota) \xrightarrow{j^{\text{trop}}} \text{Trop}(\mathbb{P}^r, j \circ \iota) \subseteq \mathbb{T}\mathbb{P}^n.$$

Using this observation we will henceforth identify any tropicalization arising from a linear morphism $\mathbb{P}^r \rightarrow U$ with a subset of \mathbb{TP}^n .

Lemma 1.3.4. *Let $\iota : \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$ and $\iota' : \mathbb{P}^r \hookrightarrow U' \subseteq \mathbb{P}^{n'}$ be linear embeddings and $\varphi : U \rightarrow U'$ be a toric morphism with $\varphi \circ \iota = \iota'$. Then the diagram*

$$\begin{array}{ccc} \overline{\mathcal{X}}_r(K) & \xrightarrow{\pi_\iota} & \text{Trop}(\mathbb{P}^r, \iota) \\ & \searrow \pi_{\iota'} & \downarrow \varphi^{\text{trop}} \\ & & \text{Trop}(\mathbb{P}^r, \iota') \end{array}$$

commutes.

Proof. This follows immediately from Proposition 1.2.11 and the fact that the same holds for the Berkovich analytification [Pay09]. \square

This lemma yields a well-defined continuous map

$$\overline{\mathcal{X}}_r(K) \xrightarrow{\varprojlim_{\iota \in I} \pi_\iota} \varprojlim_{\iota \in I} \text{Trop}(\mathbb{P}^r, \iota),$$

where the limit runs over all linear embeddings $\iota : \mathbb{P}^r \hookrightarrow U \subseteq \mathbb{P}^n$ into torus-invariant open subsets. The limit is endowed with the coarsest topology making all projections continuous; in particular, the limit topology is generated by (the preimage under projection of) all opens in all (finite) tropicalized linear spaces. The right-hand side is thus a pro-object in the category of topological spaces.

Theorem 1.3.5 (Theorem A). *The map*

$$\varprojlim_{\iota \in I} \pi_\iota : \overline{\mathcal{X}}_r(K) \longrightarrow \varprojlim_{\iota \in I} \text{Trop}(\mathbb{P}^r, \iota)$$

is a homeomorphism.

Proof. As $\varprojlim_{\iota \in I} \text{Trop}(\mathbb{P}^r, \iota)$ is a Hausdorff space and $\overline{\mathcal{X}}_r(K)$ is compact by Proposition 1.2.7, it suffices to show bijectivity.

We first show the **injectivity** of $\varprojlim_{\iota \in I} \pi_\iota$. Assume that $\varprojlim_{\iota \in I} \pi_\iota(x) = \varprojlim_{\iota \in I} \pi_\iota(x')$ for $x, x' \in \overline{\mathcal{X}}_r(K)$. We choose representatives y and y' of the homothety classes x and x' respectively.

Claim: The seminorms y and y' have the same kernel.

By symmetry, it suffices to show one inclusion. Let $0 \neq f \in (K^{r+1})^*$ with $\|f\|_y = 0$. Then by extending f to a generating set f, f_1, \dots, f_n of $(K^{r+1})^*$ and looking at the corresponding embedding $\iota = [f : f_1 : \dots : f_n] : \mathbb{P}^r \rightarrow \mathbb{P}^n$, we obtain that the first coordinate of $\pi_\iota(x) = \pi_\iota(x')$ equals $-\log \|f\|_y = \infty$. By assumption, also $-\log \|f\|_{y'} = \infty$ and thus also $\|f\|_{y'} = 0$.

Let now $f_0, f_1 \in (K^{r+1})^*$, with $\|f_0\|_y \neq 0$ and $\|f_1\|_y \neq 0$. Then also $\|f_0\|_{y'} \neq 0$ and $\|f_1\|_{y'} \neq 0$ and we need to show that $\frac{\|f_0\|_y}{\|f_0\|_{y'}} = \frac{\|f_1\|_y}{\|f_1\|_{y'}}$, as this immediately implies that y, y' are homothetic and thus $x = x'$. We extend f_0, f_1 to a generating set f_0, f_1, \dots, f_n of $(K^{r+1})^*$. Let $\iota = [f_0 : \dots : f_n]$ be the corresponding embedding from \mathbb{P}^r into \mathbb{P}^n , then $\pi_\iota(y) = \pi_\iota(y')$ and thus their difference is a multiple of $\mathbb{1}$, in particular

$$-\log \frac{\|f_0\|_y}{\|f_0\|_{y'}} = -(\log \|f_0\|_y - \log \|f_0\|_{y'}) = -(\log \|f_1\|_y - \log \|f_1\|_{y'}) = -\log \frac{\|f_1\|_y}{\|f_1\|_{y'}}.$$

We now show the **surjectivity** of $\varprojlim_{\iota \in I} \pi_\iota$. Let $(y_j)_{j \in I} \in \varprojlim_{j \in I} \text{Trop}(\mathbb{P}^r, j)$. First, we consider the identity $\text{id} = [e_0^* : \dots : e_r^*] : \mathbb{P}^r \rightarrow \mathbb{P}^r$. After a permutation of coordinates we may assume that the first coordinate $y_{\text{id},0}$ of $y_{\text{id}} \in \mathbb{T}\mathbb{P}^r$ is not ∞ . We will construct a seminorm $\|\cdot\|$ with $\|e_0^*\| = 1$ and $\pi_j(\|\cdot\|) = y_j$ for all $j \in I$.

Claim: For all linear embeddings $\iota = [e_0^* : f_1 : \dots : f_n]$, the first coordinate $y_{\iota,0}$ is not ∞ . After composing with the corresponding embedding into projective space we can assume that the codomain of ι is \mathbb{P}^n . We consider the linear embedding

$$[e_0^* : \dots : e_r^* : f_1 : \dots : f_n] : \mathbb{P}^r \rightarrow U \subset \mathbb{P}^{r+n}$$

where $U = \mathbb{P}^{r+n} \setminus V(x_0, \dots, x_r) \cup V(x_0, x_{r+1}, \dots, x_{r+n})$. Then we have a projection $U \rightarrow \mathbb{P}^r$ onto the first $r+1$ coordinates and a projection $U \rightarrow \mathbb{P}^n$ given by $[x_0 : \dots : x_{r+n}] \mapsto [x_0 : x_{r+1} : \dots : x_{r+n}]$. Since $(y_j)_{j \in J}$ is an inverse system, this shows that the first coordinate of y_ι cannot be ∞ .

Construction of the seminorm.

Let $f \in (K^{r+1})^*$. We choose an embedding $j = [e_0^* : f : f_2 : \dots : f_n] : \mathbb{P}^r \rightarrow \mathbb{P}^n$ and define

$$\|f\| := \exp(y_{j,0} - y_{j,1})$$

where we set $\exp(-\infty) = 0$. Note that $y_{j,0} \neq \infty$ and that this does not depend on the representative of $y_j \in \mathbb{T}\mathbb{P}^n$.

We show that this is independent of the choice of j . Let $j' = [e_0^* : f : f'_2 : \dots : f'_{n'}]$. Similarly to before, consider the embedding

$$[e_0^* : f : f_2 : \dots : f_n : f'_2 : \dots : f'_{n'}] : \mathbb{P}^r \rightarrow U \subseteq \mathbb{P}^{n+n'-1},$$

where $U = \mathbb{P}^{n+n'-1} \setminus V(x_0, \dots, x_n) \cup V(x_0, x_1, x_{n+1}, \dots, x_{n+n'-1})$ as before. By the same argument, applying the projections to \mathbb{P}^n and $\mathbb{P}^{n'}$ shows that $y_{j,0} - y_{j,1} = y_{j',0} - y_{j',1}$.

By a permutation automorphism one can show that for any linear embedding $\iota = [e_0^* : \dots : f : \dots]$, where f is in the i -th entry, we have $\|f\| := \exp(y_{j,0} - y_{j,i})$.

We check that the constructed map is indeed a seminorm. For $f \in (K^{r+1})^*$ and $\lambda \in K$ consider any embedding $j = [e_0^* : f : \lambda f : \dots]$. Then, by Proposition 1.2.12, for every $y_j \in \text{Trop}(\mathbb{P}^r, j)$ there is a class of a seminorm $[\|\cdot\|'] \in \overline{\mathcal{X}}_r(K)$ with

$$\begin{aligned} y_j &= \pi_j([\|\cdot\|']) \\ &= (\text{trop}_{\overline{\mathcal{X}}_n} \circ \overline{\mathcal{X}}(j))([\|\cdot\|']) \\ &= [-\log \|e_0^*\|' : -\log \|f\|' : -\log \|\lambda f\|' : \dots] \end{aligned}$$

and thus $y_{j,1} + \text{val}(\lambda) = y_{j,2}$, where val is the valuation on K . Therefore

$$\|\lambda f\| = \exp(y_{j,0} - y_{j,2}) = \exp(y_{j,0} - (y_{j,1} + \text{val}(\lambda))) = |\lambda| \cdot \|f\|.$$

For $f, g \in (K^{r+1})^*$, the inequality $\|f + g\| \leq \max\{\|f\|, \|g\|\}$ follows similarly by considering an embedding containing f, g and $f + g$.

By construction, the seminorm $\|\cdot\|$ is an inverse image of $(y_j)_{j \in I}$: Let $j = [f_1 : \dots : f_n] : \mathbb{P}^r \rightarrow U \subseteq \mathbb{P}^{n-1}$ be a linear embedding into a torus-invariant open subset $U \subseteq \mathbb{P}^{n-1}$. Consider the embedding $j' = [e_0^* : f_1 : \dots : f_n] : \mathbb{P}^r \rightarrow V \subseteq \mathbb{P}^n$ where V is the complement of the intersection of the last n coordinate hyperplanes. Since the projection of V onto the last n coordinates is a toric morphism, we have an induced projection map

$\text{Trop}(\mathbb{P}^r, j') \longrightarrow \text{Trop}(\mathbb{P}^r, j)$. The same permutation argument as before shows

$$\begin{aligned} [-\log(\|f_1\|) : \cdots : -\log(\|f_n\|)] &= [-\log(\exp(y_{j',0} - y_{j',1})) : \cdots : -\log(\exp(y_{j',0} - y_{j',n}))] \\ &= [y_{j',1} - y_{j',0} : \cdots : y_{j',n} - y_{j',0}] \\ &= [y_{j',1} : \cdots : y_{j',n}] \\ &= [y_{j,0} : \cdots : y_{j,n-1}]. \end{aligned} \quad \square$$

Remark 1.3.6. Instead of the category I used above, several subcategories would yield the same limit:

- (a) The full subcategory I' of I of non-degenerate embeddings where no coordinate equals 0. Then I' is cofinal in I , and thus the respective limits of tropicalizations are naturally isomorphic.
- (b) The full subcategory of I of non-degenerate embeddings which are different in every coordinate.
- (c) The wide subcategory of I , where instead of all (linear) toric morphisms we only allow coordinate projections. Note that these morphisms are the only ones used in the proof of 1.3.5.

Following the proof of Theorem 1.3.5 one can similarly show that the restriction of the map $\varprojlim_{\iota \in I'} \pi_\iota$ to the non-compactified space $\mathcal{X}_r(K)$ induces a homeomorphism

$$\mathcal{X}_r(K) \xrightarrow{\sim} \varprojlim_{\iota \in I'} \text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^{n_\iota}) = \varprojlim_{\iota \in I'} (\text{Trop}(\mathbb{P}^r, \iota) \cap \mathbb{R}^{n_\iota+1} / \mathbb{R} \cdot \mathbf{1}),$$

where n_ι is the dimension of projective space that is the codomain of ι .

Remark 1.3.7. In [GK15] the authors consider a projective limit of tropicalizations with respect to all linear re-embeddings of a fixed affine variety. They, in particular, show that this construction recovers the whole Berkovich analytification in the case of an affine smooth algebraic curve. Theorem A may be thought of as a natural linear-algebraic incarnation of the authors' ideas.

1.4 Valuated Matroids and Tropical Linear Spaces

In this section, we recall some of the basic definitions and results on valuated matroids in the sense of Dress and Wenzel [DW92a], in particular how to associate a (projective) tropical linear space to a valuated matroid and to describe its local structure.

1.4.1 Essentials of Matroids

A *matroid* M of rank r is given by an arbitrary set E , called the *ground set*, and a set \mathcal{I} of subsets of E , called *independent sets*, such that the following axioms are satisfied:

- (I1) The empty set is independent.
- (I2) Subsets of independent sets are independent.
- (I3) If $A, B \in \mathcal{I}$ and $|A| > |B|$, then there is $a \in A$ such that $B \cup \{a\} \in \mathcal{I}$.
- (I4) If A is an inclusion-wise maximal independent set, then $|A| = r$.

Note that if the ground set E is finite, then axiom (I4) follows from (I3), but if E is infinite then (I4) is a necessary axiom. Clearly these axioms are modeled after linear independence of a set of vectors whose span is r -dimensional; e.g. we get a matroid by considering the linearly independent subsets of a subset E of a vector space K^n . The *rank* of $A \subseteq E$ is the

cardinality of a maximal independent in A . The family $\mathcal{B}(M)$ of inclusion-wise maximal sets of \mathcal{I} are called the *bases* of M , and by axiom (I2) they determine \mathcal{I} . A *circuit* $C \subseteq E$ is a minimal dependent set, and a *flat* is a subset $F \subset E$ such that $|C \setminus F| \neq 1$ for all circuits C . A set is a flat if and only if adding any other element to it increases its rank. A *loop* is an element in E that is contained in no basis; equivalently the singleton with only this element is dependent.

1.4.2 Valuated Matroids

We begin with the following definition due to Dress and Wenzel [DW92a].

Definition 1.4.1. A *valuated matroid* of rank r on a ground set E is a function

$$v: \binom{E}{r} \longrightarrow \overline{\mathbb{R}}$$

that fulfils the following axioms:

- (i) There exists $A \in \binom{E}{r}$ with $v(A) \neq \infty$.
- (ii) For all $A, B \in \binom{E}{r}$ and $a \in A - B$ we have the *valuated basis exchange property*

$$v(A) + v(B) \geq \min_{b \in B} \{v(b \cup A \setminus a) + v(a \cup B \setminus b)\} .$$

The elements $A \in \binom{E}{r}$ with $v(A) < \infty$ form the set of bases of a matroid, called the *underlying matroid* of v . We explicitly do not require the underlying set E to be finite.

Remark 1.4.2. Valuated matroids on E of rank r are parameterized by the *Dressian* $\text{Dr}(E, r) \subseteq \overline{\mathbb{R}}^{\binom{E}{r}}$. This is the tropical prevariety of points $v \in \overline{\mathbb{R}}^{\binom{E}{r}}$ such that for all $\tau \in \binom{E}{r+1}$ and all $\sigma \in \binom{E}{r-1}$ the expression

$$\min_{j \in \tau} (v(\tau \setminus j) + v(\sigma \cup j)) \tag{1.2}$$

attains the minimum at least twice. It is straightforward to verify that the basis exchange axiom of valuated matroids is equivalent to the minimum in Equation (1.2) being attained at least twice for all $\tau \in \binom{E}{r+1}$ and all $\sigma \in \binom{E}{r-1}$. If v is a point in the interior $\text{Dr}(E, r)^\circ = \text{Dr}(E, r) \cap \mathbb{R}^{\binom{E}{r}}$, then the underlying matroid of v is the *uniform matroid on E of rank r* , i.e. the bases are all subsets of cardinality r . Valuated matroids with different underlying matroids lie at the boundary. That is, if M is a matroid on E of rank r , we obtain the *Dressian with underlying matroid M* by intersecting with hyperplanes at infinity:

$$\text{Dr}(M) = \text{Dr}(E, r) \cap \bigcap_{\substack{\sigma \in \binom{E}{r} \\ \sigma \text{ not a base}}} \{v \in \overline{\mathbb{R}}^{\binom{E}{r}} \mid v(\sigma) = \infty\}.$$

Example 1.4.3 (Realizable valuated matroids). Let K be a field with a non-Archimedean valuation $\text{val} : K \rightarrow \overline{\mathbb{R}}$.

(a) Let $\{f_0, \dots, f_n\}$ be a generating subset of K^{r+1} . The map

$$v: \binom{[n]}{r+1} \longrightarrow \overline{\mathbb{R}},$$

$$A = \{a_0, \dots, a_r\} \longmapsto \text{val} \left(\det [f_{a_0} \ \cdots \ f_{a_r}] \right)$$

defines a valuated matroid of rank $r + 1$. This follows from the *Grassmann–Plücker* identity:

$$\det(v_0, \dots, v_r) \cdot \det(w_0, \dots, w_r) = \sum_{i=0}^n \det(v_0, \dots, v_{i-1}, w_0, v_{i+1}, \dots, v_r) \cdot \det(v_i, w_0, \dots, w_r)$$

for all $v_0, \dots, v_r, w_0, \dots, w_r \in K^{r+1}$. All valuated matroids of this form are called *realizable*.

(b) Extending on (a), the map

$$w_{\text{univ}}: \binom{(K^{r+1})^*}{r+1} \longrightarrow \overline{\mathbb{R}}$$

induced by the permutation-invariant map $\text{val} \circ \det: K^{(r+1) \times (r+1)} \rightarrow \overline{\mathbb{R}}$ is a valuated matroid, called the *universal realizable matroid*.

(c) If the valuation on K is trivial, then in (a) we have

$$v(A) = \begin{cases} 0 & \text{if } A \text{ is a basis,} \\ \infty & \text{if } A \text{ is dependent.} \end{cases}$$

Thus, the notion of an ordinary matroid that is realizable over a fix trivially valued field corresponds exactly to that of a valuated matroid that is realizable over the same trivially valued field.

1.4.3 Tropical Linear Spaces and Matroid Polytopes

Valuated matroids v with ground set E and rank $r + 1$ have an associated polyhedral complex $\text{Trop}(v)$ of pure dimension r in \mathbb{TP}^E that is connected through codimension 1 and which satisfies intersection-theoretic properties analogous to linear spaces. Hence the $\text{Trop}(v)$ are called *tropical linear spaces*. We recall the definition, some properties, the associated Matroid polytope, and a regular subdivision induced by v that is dual to $\text{Trop}(v)$. When the matroid is realizable, this recovers the coordinate-wise tropicalization of any linear subspace associated to v .

In the following, let $E = \{0, \dots, n\}$ be the (finite) ground set of a valuated matroid v of rank $r + 1$ with underlying matroid M .

Definition 1.4.4. The *tropical linear space* $\text{Trop}(v) \subset \mathbb{TP}^n$ associated to v is the locus of those $(u_e)_{e \in E} \in \mathbb{TP}^n$ such that for any $\tau \in \binom{E}{r+2}$ the minimum in $v(\tau \setminus e) + u_e$ is attained at least twice.

If some $f \in E$ is a loop, by taking $\tau = B \cup \{f\}$ for any basis B , one can see that $u_f = \infty$ for each $(u_e)_{e \in E} \in \text{Trop}(v)$. Thus, adding or deleting loops only yields homeomorphic associated tropical linear spaces. Consequently, for simplicity, we now assume that M has no loops.

Now, we give a characterization of matroids in terms of polytopes. We use the following notation for the indicator vector in \mathbb{R}^E of a set $A \subset E$:

$$e_A = \sum_{i \in A} e_i \in \mathbb{R}^E.$$

Definition 1.4.5. The *matroid polytope* P_M of a matroid M is the convex hull of

$$\{e_B \mid B \in \mathcal{B}(M)\} \subseteq \mathbb{R}^{n+1}.$$

The valuated matroid v can be regarded as a height function on the vertices of the polytope P_M giving rise to the lifted polytope $\Gamma(v)$, which is defined to be the convex hull of

$$\{(e_B, v(B)) \in \mathbb{R}^{n+2} \mid B \in \mathcal{B}(M)\}.$$

Projecting the lower facets of $\Gamma(v)$ back to \mathbb{R}^{n+1} induces a polytopal subdivision \mathcal{D}_v of P_M , called the *regular subdivision* induced by v . By [Spe08, Proposition 2.2] (also see [MS15, Lemma 4.4.4] or [Jos21, Theorem 10.36]) a real-valued function from the vertices of P_M is a valuated matroid if and only if all the faces of the induced regular subdivision are matroid polytopes.

A vector $u \in \mathbb{R}^{n+1}$ selects a face of the regular subdivision induced by v by taking the convex hull of all vertices e_B of P_M such that $v(B) - u \cdot e_B$ is minimized. Such a face corresponds to the polytope of a matroid, the so-called *initial matroid* M_u of M at u . For a loopless valuated matroid v we have that:

Proposition 1.4.6 ([FO22] or [Spe08, Proposition 2.3]). *The interior of the tropical linear space $\text{Trop}(v)$, i.e. the points with finite coordinates and satisfying Equation (1.2), equals the set*

$$\text{Trop}(v)^\circ = \{u \in \mathbb{R}^{n+1} \mid M_u \text{ has no loops}\}.$$

The closure operation only adds points with infinite coordinates.

The condition of M_u not having loops is related to the minimum being achieved twice in Equation (1.2). The set $\text{Trop}(v)^\circ$ has a natural polyhedral structure labelled by the initial matroids, where a cell consists of all the points in $\text{Trop}(v)$ whose associated initial matroid is constant, a given M_u . By taking the closure, this polyhedral structure extends to $\text{Trop}(v)$.

1.4.4 Tropicalized Linear Spaces

Let $\iota = [f_0 : \dots : f_n] : \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ with $f_0, \dots, f_n \in (K^{r+1})^*$ be a linear embedding. Let v be the valuated matroid of rank $r+1$ on $E = \{0, \dots, n\}$ associated to the f_0, \dots, f_n , as in Example 1.4.3. Note that another representative f'_0, \dots, f'_n of $[f_0 : \dots : f_n]$ is related by multiplying with a scalar $\lambda \neq 0$. Hence its associated valuated matroid v' satisfies that $v' = v + (r+1) \cdot \text{val } \lambda$. So both v and v' define the same tropical linear space $\text{Trop}(v)$ and the same underlying matroid M . Moreover, we have:

Theorem 1.4.7 ([Spe08, Proposition 4.2], [BEZ21, Theorem B]). *The tropical linear space associated with a realizable valuated matroid coincides with the tropicalization of the corresponding linear embedding:*

$$\text{Trop}(v) = \text{Trop}(\mathbb{P}^r, \iota).$$

Without loss of generality, we may assume that $f_i \neq 0$ for all $i \in \{0, \dots, n\}$. Then $\text{Trop}(\mathbb{P}^r, \iota)$ equals the closure in \mathbb{TP}^n of the non-compactified tropicalization

$$\text{Trop}(\mathbb{P}^r, \iota) \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} = \text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^n).$$

Again, we can get an initial matroid M_u from $u = [u_0 : \dots : u_n] \in \text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^n)$ by considering the matroid of rank $r+1$ on E whose bases are the bases $B = \{b_0, \dots, b_r\}$ of M such that $v(B) - u_{b_0} - \dots - u_{b_r}$ is minimal. This definition is independent of the choice of representative of u .

In [Rin13] the author defines the notion of a *local tropical linear space* which can be extended to our compact setting, i.e. for tropical linear spaces in \mathbb{TP}^n .

Definition 1.4.8. Let $B = \{b_0, \dots, b_r\}$ be a basis of M . The *local tropical linear space* $\text{Trop}(\mathbb{P}^r, \iota)_B \subset \mathbb{TP}^n$ is defined as the closure of the set of vectors $u \in \text{Trop}(\iota(\mathbb{P}^r) \cap \mathbb{G}_m^n)$ such that M_u contains the basis B .

The tropical linear space $\text{Trop}(\mathbb{P}^r, \iota)$ is the union of all its local tropical linear spaces $\text{Trop}(\mathbb{P}^r, \iota)_B$.

Remark 1.4.9. In terms of polyhedral subdivisions, the (open part) of the local tropical linear space $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ is a polyhedral complex dual to the faces of the regular subdivision \mathcal{D}_v that contain the vertex e_B . For details, we refer the reader to [Rin13, Corollary 2.5].

1.4.5 The Trivial Valuation Case

Throughout this subsection, we assume that v is trivially valued, i.e. for all $A \in \binom{E}{r+1}$, we have that $v(A) = 0$ if and only if A is a basis. This way we may identify v with its underlying matroid M and the subdivision \mathcal{D}_v of P_M from Subsection 1.4.3 is trivial. So the polyhedral complex of $\text{Trop}(v)$ is a fan, known as the *Bergman fan* of M (cf. [MS15, Section 4.2]).

The following theorem refines the polyhedral structure of the Bergman fan, by realizing its support as the order complex associated to the poset of flats of M .

Theorem 1.4.10 ([MS15, Theorem 4.2.6]). *Let M be a loopless matroid. The cones $\langle e_{F_1}, \dots, e_{F_l} \rangle_{\mathbb{R}_{\geq 0}} + \mathbb{R}\mathbb{1}$ in $\mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ for every chain of flats $\emptyset \subsetneq F_1 \subsetneq \dots \subsetneq F_l$ form a fan with support $\text{Trop}(M)$. In particular, $\text{Trop}(M) \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ is homeomorphic to the cone over the order complex of the lattice of flats of M .*

With this polyhedral structure we can describe the boundary at infinity of the cones, and see that it differs from an usual coordinate-wise compactification at infinity. Namely, for any cone σ given by a chain of flats $\emptyset \subsetneq F_1 \subsetneq \dots \subsetneq F_l$, the closure in \mathbb{TP}^n is given by $\langle e_{F_1}, \dots, e_{F_l} \rangle_{\overline{\mathbb{R}_{\geq 0}}} + \mathbb{R}\mathbb{1}$. In particular, if the i -th coordinate of a point in σ is infinite, then we consider the minimal j such that $i \in F_j$, and we have that the i' -th coordinate is also infinity for any $i' \in F_j$. Thus, there is a single maximal stratum in the boundary of $\text{Trop}(v)$ for each cone, which can alternatively be explained by the fact that the cone structure above triangulates the Bergman fan of M (see [MS15, Section 4.2]).

Example 1.4.11. We consider the embedding $\mathbb{P}^2 \xrightarrow{\text{id}} \mathbb{P}^2$. Then the associated matroid has as ground set $\{e_0^*, e_1^*, e_2^*\}$, and every subset is independent, i.e. we have the uniform matroid $U_{3,3}$. The Bergman fan of $U_{3,3}$ consists of a single cone \mathbb{R}^2 . However, there are 6 non-trivial flats, namely all three one-dimensional subspaces and three two-dimensional subspaces. In Figure 1.3 we labelled all one-dimensional cones corresponding to non-trivial flats.

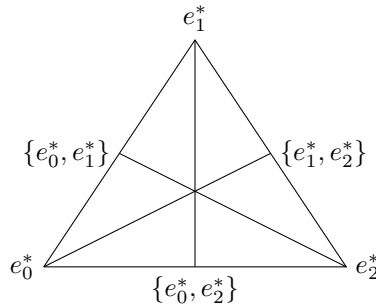


Figure 1.3: The compactified cones of $\text{Trop}(\mathbb{P}^2, \text{id}) = \mathbb{TP}^2$ given by flats of the uniform matroid $U_{3,3}$. This also represents the compactified apartment in the spherical building $\overline{B}_2(K)$.

1.5 Faithful Linear Tropicalization

The goal of this section is to show Theorem B from the introduction. We recall its statement:

Theorem 1.5.1 (Theorem B). *Let $\iota: \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ be a linear closed immersion. Then there is a natural piecewise linear embedding $J: \text{Trop}(\mathbb{P}^r, \iota) \rightarrow \overline{\mathcal{B}}_r(K)$ that makes the following diagram commute*

$$\begin{array}{ccc}
 \overline{\mathcal{B}}_r(K) & \xleftarrow{J} & \text{Trop}(\mathbb{P}^r, \iota) \\
 \searrow^{\overline{\mathcal{B}}(\iota)} & & \searrow^{\subseteq} \\
 \overline{\mathcal{B}}_n(K) & \xrightarrow{\text{trop}} & \mathbb{TP}^n.
 \end{array}$$

By a piecewise linear embedding J we mean that we have a finite covering of $\text{Trop}(\mathbb{P}^r, \iota)$ by subcomplexes such that the image of each subcomplex lies in an apartment and the restriction of J on each subcomplex is piecewise linear. In particular, π_ι induces a piecewise linear homeomorphism between the union of apartments $\bigcup_{B \in \mathcal{B}(M)} \overline{\mathcal{A}}(B)$ and the tropicalized linear subspace $\text{Trop}(\mathbb{P}^r, \iota)$.

Choose $f_0, \dots, f_n \in (K^{r+1})^*$ defining the embedding $\iota: \mathbb{P}^r \hookrightarrow \mathbb{P}^n$ and let v be the corresponding valuated matroid of rank $r+1$ on $E = \{0, \dots, n\}$ as in Example 1.4.3 (a). As above, we assume that $f_i \neq 0$ for all $i \in E$. Let $B \in \mathcal{B}(M)$ be a basis of the underlying matroid M . Recall that the compactified apartment $\overline{\mathcal{A}}(B)$ in $\overline{\mathcal{B}}_r(K)$ denotes the set of all seminorms on $(K^{r+1})^*$ diagonalized by B . It is homeomorphic to \mathbb{TP}^r : a parametrization $\mathbb{TP}^r \xrightarrow{\sim} \overline{\mathcal{A}}(B)$ is given by

$$v = [v_0 : \dots : v_r] \longmapsto \|\cdot\|_{B,v}.$$

This map is well defined because different projective representatives $v' = v + \lambda \mathbb{1}$ give rise to homothetic seminorms $\|\cdot\|_{B,v'} = \exp(-\lambda) \|\cdot\|_{B,v}$.

For the proof of Theorem B we need a couple of technical results first. We begin with a valuative version of Cramer's rule.

Lemma 1.5.2. *Let B be a basis of M , and $k \in E \setminus B$. Write $f_k = \sum_{b \in B} \lambda_b f_b$ with $\lambda_b \in K$. For all $b \in B$ we have*

$$\text{val}(\lambda_b) = v(k \cup B \setminus b) - v(B).$$

Proof. Label the elements of B as b_0, \dots, b_r , with b_0 equal to our chosen b . Using multilinearity of the determinant and properties of the valuation we find:

$$\begin{aligned}
 v(k \cup B \setminus b) &= \text{val} \left(\det [f_k \quad f_{b_1} \quad \dots \quad f_{b_r}] \right) \\
 &= \text{val} \left(\sum_{i \in B} \lambda_i \det [f_i \quad f_{b_1} \quad \dots \quad f_{b_r}] \right) \\
 &= \text{val} \left(\lambda_{b_0} \cdot \det [f_{b_0} \quad \dots \quad f_{b_r}] \right) \\
 &= \text{val}(\lambda_b) + v(B). \quad \square
 \end{aligned}$$

Lemma 1.5.3. *Let B be a basis of (E, v) . If u is in $\text{Trop}(v)$, then there is $k \in B$ such that $u_k \neq \infty$.*

Proof. If $B = E$, the statement follows from the definition of tropical projective space. Otherwise, assume there is $u \in \text{Trop}(v)$ with $u_i = \infty$ for all $i \in B$. Choose $k \in E \setminus B$ such that $u_k < \infty$ and set $\tau = k \cup B$. Since B is a basis, we have that $v(B) + u_b < \infty$. But this is the only finite term in the minimum for Definition 1.4.4 hence it is not attained twice for τ , which is a contradiction. \square

For the following statement we interpret [Rin13, Theorem 2.6] in our setting and extend to the compactifications. It gives a piecewise linear homeomorphism between the local tropical linear space $\text{Trop}(\mathbb{P}^r, \iota)_B$ (as in Definition 1.4.8) and the compactified apartment $\overline{\mathcal{A}}(B)$.

Proposition 1.5.4. *Let $B = \{b_0, \dots, b_r\}$ be a basis of M . The map J_B sending $u \in \text{Trop}(\mathbb{P}^r, \iota)_B$ to the seminorm $\|\cdot\|_{B, u_B}$, where $u_B = [u_{b_0} : \dots : u_{b_r}]$, is a piecewise linear homeomorphism between $\text{Trop}(\mathbb{P}^r, \iota)_B$ and $\overline{\mathcal{A}}(B)$. Its inverse is the restriction of π_ι . Explicitly, the seminorm $x = \|\cdot\|_{B, v} \in \overline{\mathcal{A}}(B)$ is mapped to $\pi_\iota(x) \in \text{Trop}(\mathbb{P}^r, \iota)_B$ with*

$$\pi_\iota(x)_k = \begin{cases} v_k & \text{if } k \in B, \\ \min_{i \in B} v(k \cup B \setminus i) - v(B) + v_k & \text{otherwise.} \end{cases}$$

Proof. First, we define J_B on the open dense subset $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ and show that this gives a piecewise linear homeomorphism to $\mathcal{A}(B)$. Clearly, the map π_ι is injective on $\mathcal{A}(B)$. So we show that $\pi_\iota \circ J_B$ is the identity on $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$, that is for u in the latter set we show

$$\pi_\iota(J_B(u)) = \pi_\iota(\|\cdot\|_{B, u_B}) = [-\log \|f_0\|_{B, u_B} : \dots : -\log \|f_n\|_{B, u_B}] = [u_0 : \dots : u_n].$$

Since $-\log \|f_b\|_{B, u_B} = u_b$ for $b \in B$, we only need to check the equality for k in $E \setminus B$. As $u \in \text{Trop}(\mathbb{P}^r, \iota)$, the condition in Definition 1.4.4 for $B \cup \{k\}$ says that the minimum is attained at least twice in

$$\min_{i \in k \cup B} \{v(k \cup B \setminus i) + u_i\}.$$

By subtracting $u \cdot e_{k \cup B}$ we see it is equivalent to the minimum being attained at least twice in

$$\min_{i \in k \cup B} \{v(k \cup B \setminus i) - u \cdot e_{k \cup B \setminus i}\}.$$

Since B is a basis of the initial matroid M_u , it minimizes the expression $v(\sigma) - u \cdot e_\sigma$ over all $\sigma \in \binom{E}{r+1}$. So the minimum is achieved at $i = k$ and some other $i = l$. That is, for all $i \in B$ we have

$$v(B) + u_k = v(k \cup B \setminus l) + u_l \leq v(k \cup B \setminus i) + u_i.$$

Writing $f_k = \sum_{i \in B} \lambda_i f_i$, by Lemma 1.5.2 this is equivalent to

$$u_k = \text{val}(\lambda_l) + u_l \leq \text{val}(\lambda_i) + u_i$$

for all $i \in B$. Hence, $-\log \|f_k\|_{B, u_B} = \min_{i \in B} (\text{val}(\lambda_i) + u_i) = u_k$.

This also gives the alternative description of π_ι . We are left to show that $\pi_\iota(\mathcal{A}(B))$ lies in $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ for which we refer to [Rin13, Theorem 2.6]. Hence, J_B and the restriction of π_ι are piecewise linear inverse homeomorphisms between $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ and $\mathcal{A}(B)$. We now extend J_B naturally to a piecewise linear map $\text{Trop}(\mathbb{P}^r, \iota)_B \rightarrow \overline{\mathcal{A}}(B)$ by sending $u = [u_0 : \dots : u_n]$ to $\|\cdot\|_{B, u_B}$. This is well-defined if $[u_{b_0} : \dots : u_{b_r}]$ lies in $\mathbb{T}\mathbb{P}^r$, namely if there is at least one finite coordinate in $[u_{b_0} : \dots : u_{b_r}]$; this is proven below in Lemma 1.5.3 and using the fact that $\text{Trop}(\mathbb{P}^r, \iota) = \text{Trop}(v)$. Then, $\pi_\iota \circ J_B$ is the identity on $\text{Trop}(\mathbb{P}^r, \iota)_B$ since we have seen it is the identity for a dense subset and $\text{Trop}(\mathbb{P}^r, \iota)_B$ is Hausdorff. Since $\mathcal{A}(B)$ is dense in $\overline{\mathcal{A}}(B)$, the image $\pi_\iota(\overline{\mathcal{A}}(B))$ is exactly the closure of $\text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ which is $\text{Trop}(\mathbb{P}^r, \iota)_B$. As before, $J_B \circ \pi_\iota|_{\overline{\mathcal{A}}(B)} = \text{id}$ on $\overline{\mathcal{A}}(B)$ which concludes the proof. \square

Proof of Theorem B. Recall that $\text{Trop}(\mathbb{P}^r, \iota)$ is the union of all its local tropical linear spaces $\text{Trop}(\mathbb{P}^r, \iota)_B$ where B runs over the bases of the matroid M associated to $\text{Trop}(\mathbb{P}^r, \iota)$. Thus,

we define J locally as J_B and show that the maps J_B glue. For $A, B \in \mathcal{B}(M)$, we show that J_A and J_B glue on the open part $\text{Trop}(\mathbb{P}^r, \iota)_A \cap \text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$. Choose $u = [u_0 : \dots : u_n]$ in the latter set and let $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ be a representative. As in Proposition 1.5.4, we write u_A, u_B for the vectors in \mathbb{R}^{r+1} with the coordinates of u indexed by A and B , respectively. If $\|\cdot\|_{A, u_A}$ equals $\|\cdot\|_{B, u_B}$, we get a piecewise linear homeomorphism between $\text{Trop}(\mathbb{P}^r, \iota)_A \cap \text{Trop}(\mathbb{P}^r, \iota)_B \cap \mathbb{R}^{n+1}/\mathbb{R}\mathbb{1}$ and $\mathcal{A}(A) \cap \mathcal{A}(B)$.

Claim: The seminorms $\|\cdot\|_{A, u_A}$ and $\|\cdot\|_{B, u_B}$ are equal.

First assume that A and B differ by two elements, i.e. there are $a \in A$ and $b \in B$ such that $A \setminus a = B \setminus b$. Choose $v \in (K^{r+1})^*$ and write it in terms of the bases A and B : let $v = \sum_{i \in A} \alpha_i f_i = \alpha_a f_a + \sum_{i \in A \setminus a} \alpha_i f_i$ and $f_a = \beta_b f_b + \sum_{j \in B \setminus b} \beta_j f_j$. Replacing, we get $v = \alpha_a \beta_b f_b + \alpha_a \sum_{j \in B \setminus b} \beta_j f_j + \sum_{i \in A \setminus a} \alpha_i f_i = \alpha_a \beta_b f_b + \sum_{j \in B \setminus b} (\alpha_a \beta_j + \alpha_j) f_j$. We want to show equality of the expressions

$$\begin{aligned} -\log \|v\|_{A, u_A} &= \min \left\{ \text{val}(\alpha_a) + u_a, \min_{i \in A \setminus a} \text{val}(\alpha_i) + u_i \right\}, \\ -\log \|v\|_{B, u_B} &= \min \left\{ \text{val}(\alpha_a) + \text{val}(\beta_b) + u_b, \min_{j \in B \setminus b} \{ \text{val}(\alpha_a \beta_j + \alpha_j) + u_j \} \right\}. \end{aligned} \quad (1.3)$$

By Lemma 1.5.2 we have for every $j \in B$ that

$$\text{val}(\beta_j) = v(a \cup B \setminus j) - v(B) = v(A \cup B \setminus j) - v(B). \quad (1.4)$$

As $u \in \text{Trop}(\mathbb{P}^r, \iota)_B$, the basis B is in the initial matroid M_u , so B minimizes the expression $v(\sigma) - u \cdot e_\sigma$ over all $\sigma \in \binom{E}{r+1}$, so $v(B) - u \cdot e_B \leq v(A \cup B \setminus j) - u \cdot e_{A \cup B \setminus j}$ for all $j \in B$. By Equation (1.4) we get

$$u_a - u_j = u \cdot (e_{A \cup B \setminus j} - e_B) \leq v(A \cup B \setminus j) - v(B) = \text{val}(\beta_j). \quad (1.5)$$

Also u is in $\text{Trop}(\mathbb{P}^r, \iota)_A$, so A is in the initial matroid M_u , thus $v(A) - u \cdot e_A \leq v(B) - u \cdot e_B$ and so

$$u_a - u_b \geq v(A) - v(B) = \text{val}(\beta_a). \quad (1.6)$$

If we set j equals b in Equation (1.5) and combine it with Equation (1.6) we get $\text{val}(\beta_b) + u_b = \text{val}(\beta_a) + u_a$, and furthermore $\text{val}(\alpha_a) + \text{val}(\beta_b) + u_b = \text{val}(\alpha_a) + u_a$. So equality in the first terms of Equation (1.3) happens.

It remains to show for $i \in A \setminus a$ that either $\text{val}(\alpha_a \beta_i + \alpha_i) = \text{val}(\alpha_i)$ or $\text{val}(\alpha_i) + u_i \geq \text{val}(\alpha_a \beta_i + \alpha_i) + u_i \geq \text{val}(\alpha_a) + \text{val}(\beta_b) + u_b$. By properties of valuations we have

$$\text{val}(\alpha_a \beta_i + \alpha_i) \geq \min(\text{val}(\alpha_a) + \text{val}(\beta_i), \text{val}(\alpha_i)),$$

and moreover if $\text{val}(\alpha_a) + \text{val}(\beta_i) > \text{val}(\alpha_i)$, then $\text{val}(\alpha_a \beta_i + \alpha_i) = \text{val}(\alpha_i)$ and we are done. Thus, assume that $\text{val}(\alpha_a) + \text{val}(\beta_i) \leq \text{val}(\alpha_i)$. In that case, we calculate using Equations (1.4) and (1.6), the following

$$\begin{aligned} (\text{val}(\alpha_a \beta_i + \alpha_i) + u_i) - (\text{val}(\alpha_a) + \text{val}(\beta_b) + u_b) &\geq \text{val}(\beta_i) - \text{val}(\beta_b) + u_i - u_b \\ &\geq u_a - u_b - \text{val}(\beta_b) \geq 0. \end{aligned}$$

Thus, both minima in Equation (1.3) coincide. For the general case where $A \Delta B$ has $2m$ elements, since both A and B are bases in the initial matroid M_u , by the basis exchange axiom there is a sequence of bases B_0, B_1, \dots, B_m such that $B_0 = A$ and $B_m = B$ and every pair B_q, B_{q+1} differs by two elements. Thus, we may apply our previous argument to every pair B_q, B_{q+1} to conclude the claim for inner points of the local maps. Again, extending to the compactifications concludes the proof. \square

1.6 The Trivially and the Discretely Valued Case

1.6.1 The Trivial Valuation Case

We can make Theorems A and B explicit when the valuation is trivial. Recall from Example 1.1.10, that there is a bijection

$$\overline{\mathcal{B}}(V) \xrightarrow{1:1} \left\{ (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V^*, c_1 > \cdots > c_{l-1}) \mid c_1, \dots, c_{l-1} \in \overline{\mathbb{R}}_{>0} \right\}_{l=1, \dots, r+1}.$$

In other words, a class of a seminorm is given by a flag of subspaces together with decreasing coordinates corresponding to logarithms of the (constant) values of the representative with generic value 1. We fix an embedding $\iota = [f_0 : \dots : f_n] : \mathbb{P}^r \rightarrow \mathbb{P}^n$ and obtain a realizable matroid M on $[n]$. Recall that we can compute $\text{Trop}(\mathbb{P}^r, \iota)$ as a compactification of the cone complex over the order complex of flats of M , cf. Theorem 1.4.7 and Theorem 1.4.10.

Theorem A

We can explicitly compute the maps π_ι in terms of both the coordinates of the building above and the description and coefficients for flats of M .

Let $x \in \overline{\mathcal{B}}_r(K)$ be given by a flag $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = (K^{r+1})^*$ and coordinates $c_1 > \cdots > c_{l-1}$ as in Example 1.1.10. We formally set $c_0 := \infty$, $c_l := 0$. Let $\|\cdot\| \in x$ be a representative with generic value 1 and $d_j := \exp(-c_j)$. Recall that then d_j is the constant value of $\|\cdot\|$ on V_j for $j = 0, \dots, l$. Fix a given coordinate $i \in [n]$ and let j be minimal such that $f_i \in V_j$. Then $-\log(\|f_i\|) = -\log(d_j) = c_j$. We consider the matroid on $[n]$ induced by ι and for $j = 0, \dots, l$ we define a flat $F_j = \{i \in [n] \mid f_i \in V_j\}$. Then the above computation shows

$$\begin{aligned} \pi_\iota(\|\cdot\|) &= [-\log(\|f_0\|) : \cdots : -\log(\|f_n\|)] \\ &= \sum_{j=0}^{l-1} (c_j - c_{j+1}) e_{F_j} \in \text{Trop}(\mathbb{P}^r, \iota). \end{aligned}$$

Theorem B

A point $u = [u_0 : \cdots : u_n] \in \text{Trop}(\mathbb{P}^r, \iota)$ can be written as

$$u = \sum_{j=0}^l a_j e_{F_j}$$

for $a_j \in \mathbb{R}_{\geq 0}$ and a chain of flats $F_0 \subseteq \cdots \subseteq F_l = [n]$. Note that each flat F_j of M yields a subspace V_j of $(K^{r+1})^*$ and that proper inclusions of flats yield proper inclusions of their corresponding subspaces.

Let $B \in \mathcal{B}(M)$ be any basis such that for all $j = 0, \dots, l$ the rank of $F_j \cap B$ equals the rank of F_j . Such a basis can be obtained by successively extending bases of the flats. Then u lies in the local tropical linear space $\text{Trop}(\mathbb{P}^r, \iota)_B$. We want to compute the map $J_B : \text{Trop}(\mathbb{P}^r, \iota)_B \rightarrow \overline{\mathcal{B}}_r(K)$ from Proposition 1.5.4.

Fix $i \in B$ and let j be minimal such that $i \in F_j$. Then we have

$$u_i = \sum_{k:i \in F_k} a_k = \sum_{k=j}^l a_k.$$

Then J_B sends u to the homothety class of the seminorm having generic constant value $\exp(-u_i) = \exp(-\sum_{k=j}^l a_k)$ on V_j . In the coordinates from Example 1.1.10, if we set

$c_j := \sum_{k=j}^{l-1} a_k$, we have

$$J(u) = J_B(u) = (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = (K^{r+1})^*, c_1 > \cdots > c_{l-1}).$$

1.6.2 Lattices and the Discrete Valuation Case

Let V be a vector space of dimension $r + 1$ over a discretely valued field K .

Definition 1.6.1. A *lattice* in V^* is an \mathcal{O}_K -submodule Λ such that

$$\Lambda \otimes_{\mathcal{O}_K} K \simeq V^*.$$

To a linear map $f: V \rightarrow W$ and a lattice $\Lambda \subseteq V^*$ we can associate the lattice $(f^*)^{-1}(\Lambda)$ in W^* , where $f^*: W^* \rightarrow V^*$ refers to the dual linear map.

Remark 1.6.2. That $\Lambda \otimes_{\mathcal{O}_K} K \simeq V^*$ is equivalent to the following: for every $f \in V^*$ there exists a c in K such that $c^{-1}f$ is in Λ .

Note that, despite K being generated over \mathcal{O}_K by infinitely many elements $e_{-\gamma}, \gamma \in \Gamma_{\geq 0}$ (where $\text{val}(e_{-\gamma}) = -\gamma$), these generators satisfy obvious relations $e_{-\gamma_1} = r_{12}e_{-\gamma_2}$ with $r_{12} \in \mathcal{O}_K$ for $\gamma_1 < \gamma_2$. When inverting all the r_{ij} , the relations show that the resulting module is freely generated by one element e_0 , which can be taken to be $1 \in K$, showing that $K \otimes_{\mathcal{O}_K} K \simeq K$. So a lattice could contain a vector subspace of V^* , in which case it would not be finitely generated as an \mathcal{O}_K -module.

Lemma 1.6.3. *If a lattice is finitely generated, then it is free on $r + 1$ generators.*

Proof. Since the lattice spans a vector space of dimension $r + 1$ over K , $r + 1$ is clearly the minimum number of generators. Suppose that there were $r + 2$. Then we would find a non-trivial K -linear dependence relation. Compare the denominators and multiply by the one (say it is the 0th) achieving the highest absolute value. We thus obtain an \mathcal{O}_K -linear dependence relation with invertible 0th coefficient, showing that the 0th generator is redundant. \square

In the following we present ways to go between seminorms and lattices. In general, they are not inverse to one another (see [Sch02, Lemma I.2.2]).

Definition 1.6.4. Let $\Lambda \subseteq V^*$ be a lattice. The seminorm associated to Λ is (called its *gauge*):

$$q_\Lambda(f) = \inf_{f \in c\Lambda} |c| \in \overline{\mathbb{R}}.$$

Definition 1.6.5. The lattice associated to a seminorm q is the closed unit ball $\Lambda_q = q^{-1}([0, 1])$.

Remark 1.6.6. If Λ contains a vector subspace of V^* , the associated seminorm is not a norm, and vice versa.

Note that all norms on a finite-dimensional vector space are equivalent. In particular, the space is complete with respect to any norm. Once a basis $V \simeq K^{r+1}$ is chosen, one such norm is given by $\|(\lambda_0, \dots, \lambda_r)\|_\infty = \max_i |\lambda_i|$, which is clearly diagonalizable. The *closed unit polydisc* is the lattice $\{v \in V : \|v\|_\infty \leq 1\}$. It is finitely generated by the elements of the chosen basis.

Proposition 1.6.7. *If K is spherically complete, the correspondences above induce an equivalence between closed unit polydiscs (with respect to some basis and $\|\cdot\|_\infty$) and Γ -valued norms.*

Proof. Let q be a Γ -valued norm. Since K is spherically complete, there exists a basis $\{v_0, \dots, v_r\}$ of V^* such that $q(\lambda_0 v_0 + \dots + \lambda_r v_r) = \max_i (|\lambda_i| \alpha_i)$. Note that $\alpha_i = q(e_i)$ is an element of the value group by assumption. Let $c_i \in K$ be any element such that $|c_i| = \alpha_i$. Then the associated lattice Λ_q is the closed unit polydisc with respect to the basis $\{c_0^{-1} v_0, \dots, c_r^{-1} v_r\}$.

Vice versa, if Λ is the closed unit polydisc with respect to a basis $\{v_0, \dots, v_r\}$, then $q_\Lambda = \|\cdot\|_\infty$ with respect to the same basis. Note that $v \in c\Lambda_q$ if and only if $q(v) \leq |c|$, therefore

$$q_{\Lambda_q}(v) = \inf \{|c| \mid v \in c\Lambda_q\} = q(v),$$

since we have assumed that q takes values in Γ . The inclusion $\Lambda \subseteq \Lambda_{q_\Lambda}$ is always true. On the other hand, $v \in \Lambda_{q_\Lambda}$ if and only if $q_\Lambda(v) \leq 1$; but since $q_\Lambda = \|\cdot\|_\infty$ with respect to some basis for which Λ is the closed unit polydisc, it is clear that $q_\Lambda(v) \leq 1$ if and only if $v \in \Lambda$. \square

Remark 1.6.8. In the spherically complete case, a description of all lattices can be found in [CM21, Theorem 3.6].

Corollary 1.6.9. *If K is a complete discretely valued field, the correspondences above induce an equivalence between finitely generated lattices and integer-valued norms.*

Since we are interested in the building of PGL , we consider (semi)norms up to homothety ($q \sim \gamma q$ for any $\gamma \in \exp(\Gamma)$), which correspond to lattices up to homothety ($\Lambda \sim c\Lambda$ for any $c \in K^*$).

Simplices In the discretely valued case, there is a way of reconstructing the simplicial structure of the building in terms of nested sequences of finitely generated lattices and collections of real numbers.

Indeed, to any norm q we can associate a nested sequence of lattices

$$\{\Lambda(c) = q^{-1}([0, c]) \mid c \in [1, e]\}.$$

Up to homothety, this list consists of $0 < k+1 \leq r+1$ lattices $\Lambda_0 \subseteq \dots \subseteq \Lambda_k \subseteq \pi^{-1}\Lambda_0$, where π is a uniformiser of \mathcal{O}_K . Moreover, we can associate to q the list of jumps:

$$\{c_i = \inf\{c \in (1, e] \mid \Lambda(c) = \Lambda_i\}\}_{i=1, \dots, k}.$$

Vice versa, given a nested sequence of lattices $\Lambda_0 \subseteq \dots \subseteq \Lambda_k \subseteq \pi^{-1}\Lambda_0$, we can find a basis $\{e_0, \dots, e_r\}$ of Λ_k such that

$$\Lambda_{h-1} = \langle \pi e_0, \dots, \pi e_{i_h}, e_{i_h+1}, \dots, e_r \rangle$$

for some $0 \leq i_k < \dots < i_1 < r =: i_0$. Given $c_1, \dots, c_k \in (1, e)$, we set $c_0 = 1$ and $\alpha_i = c_j$ if $i_j < i \leq i_{j-1}$. We then define the associated norm

$$q(\lambda_0 e_0 + \dots + \lambda_r e_r) = \max_i \{|\lambda_i| \alpha_i\}.$$

Remark 1.6.10. Possibly infinitely-generated lattices in V^* are dual to finitely generated submodules of possibly non-maximal rank in V , which provides another description of the compactification of the building, as explained in [Wer04, §3-4] in the case of a local field.

Convexity In the discretely-valued case, there are various notions of convexity in the building: Weyl convexity (see for instance [AB08, §4.11]) is the one that bears the closest resemblance to the metric approach to buildings. There are weaker notions of convexity that

have been studied in [JSY07] from a tropical perspective: a set of lattices up to homothety is $+$ -convex (resp. \cap -convex) if it is closed under rescaling and taking sums (resp. intersections) as submodules of K^{r+1} . In terms of norms, these operations correspond to taking pointwise maximum (resp. the largest norm that is bounded above by pointwise minimum).

Membranes, introduced in [KT06], are $+$ -convex unions of apartments, consisting of all lattices admitting a basis of the form $\{\pi^{a_0} f_{i_0}, \dots, \pi^{a_r} f_{i_r}\}$, where the a_i 's are integers and the f_i 's are chosen from a fixed set of $n+1$ vectors $\{f_0, \dots, f_n\}$ in K^{r+1} . In [KT06, Theorem 4.11] and [JSY07, Theorem 18], the authors show that the lattice points in the membrane $[M]$ correspond bijectively to integer points in $\text{Trop}(\mathbb{P}^r, \iota)$, where ι denotes the embedding $[f_0 : \dots : f_n] : \mathbb{P}^r \rightarrow \mathbb{P}^n$. Both the membrane and the tropical linear space are indeed the tropical convex hull of finitely many points (at infinity). The correspondence is based upon the *nearest point map* onto the tropical lattice polytope $\text{Trop}(\mathbb{P}^r, \iota)$ described in [JSY07, Lemma 21], which can be interpreted as the tropicalization map π_ι and the section J (see Section 1.5 for the definitions) restricted to the affine part $\text{Trop}(\mathbb{P}^r, \iota) \cap \mathbb{R}^n / \mathbb{R}\mathbf{1}$.

1.7 The Universal Realizable Valuated Matroid

1.7.1 Infinite Tropicalization

Let v be a valuated matroid on a, possibly infinite, ground set E , and let $E' \subset E$ be any subset containing a basis. Then the restriction of v to $\binom{E'}{r+1}$ yields again a valuated matroid v' . We associate a tropical linear space to v in general, by gluing together the usual construction for finite valuated matroids. We define the sets

$$\begin{aligned} \mathbb{TP}^E &:= (\{(u_e)_{e \in E} \mid u_e \in \overline{\mathbb{R}}\} \setminus \{(\infty)_{e \in E}\}) / \mathbb{R}\mathbf{1} \\ U_v &:= \{(u_e)_{e \in E} \in \mathbb{TP}^E \mid \text{for all bases } A \subset E \text{ there is } a \in A \text{ with } u_a \neq \infty\} \subset \mathbb{TP}^E. \end{aligned}$$

Definition 1.7.1. The *tropical linear space* $\text{Trop}(v) \subset \mathbb{TP}^E$ associated to v is the set of $(u_e)_{e \in E} \in \mathbb{TP}^E$ such that for any $\tau \in \binom{E}{r+2}$ the minimum in $v(\tau \setminus \{e\}) + u_e$ is attained at least twice.

Note that the proof of Lemma 1.5.3, does not use the finiteness condition of E , hence we have for a valuated matroid v and its associated tropical linear space that

$$\text{Trop}(v) \subseteq U_v \subseteq \mathbb{TP}^E.$$

Passing to the smaller set U_v allows us to endow it with a limit topology as follows. Let I be the category of finite subsets E' of E containing a basis, with inclusions as morphisms. Then we have a functor from I^{op} into the category of topological spaces, assigning to each E' the space $U_{v'} \subseteq \mathbb{TP}^{E'}$, where v' is the restriction of v to E' , and to every inclusion the corresponding coordinate projection. Note that these coordinate projections are well-defined by the construction of the $U_{v'}$. We see that

$$U_v = \varprojlim_{E' \in I} U_{v'},$$

and we can endow it with the limit topology. Since we also have an identification

$$\text{Trop}(v) \xrightarrow{\sim} \varprojlim_{E' \in I} \text{Trop}(v'),$$

and in particular we can endow $\text{Trop}(v)$ with the limit topology.

We now restrict our attention to realizable valuated matroids. Recall that the *universal realizable matroid* w_{univ} is given by $w_{\text{univ}}(A) = \text{val}(\det(A))$ for $A \in \binom{(K^{r+1})^*}{r+1}$. Theorem A and Speyer's result on the tropicalization of finite linear subspaces (Theorem 1.4.7) allow us to identify the space of seminorms on $(K^{r+1})^*$ up to homothety with the tropical linear space associated to the universal realizable matroid.

Theorem 1.7.2 (Theorem C). *The Goldman-Iwahori space is the tropical linear space associated to the universal realizable matroid w_{univ} , i.e.*

$$\overline{\mathcal{X}}_r(K) = \text{Trop}(w_{\text{univ}}).$$

Proof. By Theorem A and Remark 1.3.6 (c), we can write $\overline{\mathcal{X}}_r(K)$ as the limit of all linear tropicalizations with respect to the category of coordinate projections. The tropicalization functor from this category is naturally equivalent to the functor which associates to an embedding $\iota = [f_0 : \dots : f_n]$ the tropical linear space associated to the valuated matroid given by $\{f_0, \dots, f_n\}$, as repeating entries and permuting coordinates yields homeomorphic tropicalizations. \square

Let E denote the set of (non-zero) vectors in K^{r+1} . We obtain linear maps

$$\bigoplus_E K \longrightarrow K^{r+1}, \quad (1.7)$$

and dually

$$\iota_{\text{univ}} : K^{r+1} \hookrightarrow K^E.$$

As in Example 1.4.3 and Section 1.4.4, we may associate to ι_{univ} the realizable valuated matroid w_{univ} . Hence, we can interpret $\overline{\mathcal{X}}_r(K)$ as the tropicalization of the universal projective linear subspace of rank r .

In the following we will show that $\overline{\mathcal{X}}_r(K)$ is cut out by much simpler equations than the ones coming from the universal realizable valuated matroid. We can think of the map (1.7) as the $(r+1) \times E$ matrix whose e -th column vector represents e in the standard basis of K^{r+1} . It follows that the i -th row corresponds to the i -th coordinate projection as a function on E , that is:

Proposition 1.7.3. *The image of $\iota_{\text{univ}} : K^{r+1} \hookrightarrow K^E$ consists of (the restrictions of) all the linear maps from K^{r+1} (resp. E) to K . In particular, the equations of ι_{univ} involve only finitely many variables.*

Proof. Let x_e denote the coordinate on K^E such that $x_e(f) = f(e)$. The equations of ι_{univ} are

$$\begin{aligned} x_{\lambda e} &= \lambda x_e, & \text{for } \lambda \in K^\times, e, \lambda e \in E; \\ x_{e_1+e_2} &= x_{e_1} + x_{e_2}, & \text{for } e_1, e_2, e_1 + e_2 \in E. \end{aligned} \quad \square$$

Remark 1.7.4. These are the equations of $K^{r+1} \hookrightarrow K^{E'}$ for any subset $E' \subseteq E$ and the corresponding projection $K^E \rightarrow K^{E'}$ (restriction of functions).

Given a basis (e_1, \dots, e_{r+1}) of K^{r+1} (e.g. the standard one), these equations are equivalent to

$$x_e = \sum_{i=1}^{r+1} [e]_i x_{e_i}.$$

As a curiosity, we note that the *large* circuits of Definition 1.4.4 are equivalent to the tropicalization of the *small* circuits from Proposition 1.7.3.

Proposition 1.7.5. Write w for w_{univ} . The minimum is attained at least twice in all

$$\min(u_{\lambda e}, u_e + \text{val}(\lambda)) \quad \text{for } \lambda \in K^\times \text{ and } e, \lambda e \in E; \quad (1.8)$$

$$\min(u_{e_1+e_2}, u_{e_1}, u_{e_2}) \quad \text{for } e_1, e_2, e_1 + e_2 \in E. \quad (1.9)$$

if and only if the minimum is attained at least twice in all

$$\min_{i \in \tau} (u_i + w(\tau \setminus i)) \quad \text{for } \tau \in \binom{E}{r+2}. \quad (1.10)$$

Proof. (1.10) \Rightarrow (1.8): If e is not 0, consider a basis $B = \{e, e_1, \dots, e_r\}$ and apply (1.10) to $\{\lambda e\} \cup B$. Note that $w(\tau \setminus i) = \infty$ unless $i = e, \lambda e$. Equation (1.8) follows. Equation (1.9) follows similarly.

(1.8) and (1.9) \Rightarrow (1.10): From Equation (1.8) we get $u_{\lambda e} = u_e + \text{val}(\lambda)$, and by induction from Equation (1.9) we get that for $k \geq 2$ the minimum is attained at least twice in

$$\min(u_{e_1+\dots+e_k}, u_{e_1}, \dots, u_{e_k}). \quad (1.11)$$

Let $\tau \in \binom{E}{r+2}$. If τ contains no basis, then Equation (1.10) is trivially true. Thus, suppose $\tau = f \cup B$, with B a basis, and write $f = \sum_{e \in B} \lambda_e e$. By Lemma 1.5.2, we have that $\text{val}(\lambda_i) = w(\tau \setminus i) - w(B)$ for all $i \in B$. If we set $\lambda_f = 1$, so $\text{val}(\lambda_f) = 0$ and $\text{val}(\lambda_i) = v(\tau \setminus i) - w(B)$ also for $i = f$, we get

$$\begin{aligned} \min_{i \in \tau} (u_i + w(\tau \setminus i)) - w(B) &= \min_{i \in \tau} (u_i + w(\tau \setminus i) - w(B)) \\ &= \min_{i \in \tau} (u_i + \text{val}(\lambda_i)) = \min_{i \in \tau} (u_{\lambda_i i}), \end{aligned}$$

and the conclusion follows from Equation (1.11). \square

1.7.2 Tight Spans

Many of our results extend work by Dress and collaborators in T-theory [DT98]. A tight span is an isometric embedding of a metric space E into a hyperconvex metric space T_E . The motivation for these spaces is fitting phylogenetic data; see the cited work for a discussion. There are also applications to extending valuations in p -adic geometry [DT93]. A so-called four-point condition [DT98, Section 4.6] is necessary and sufficient for a tight span to be an \mathbb{R} -tree; see Figure 1.1 for an example of an \mathbb{R} -tree. This condition is essentially the basis exchange property for rank-2 valuated matroids. Hence, generalizing to higher dimensions, [DT93] introduces the tight span of a rank- r valuated matroid (E, v) as

$$T_{(E,v)} = \left\{ p \in \mathbb{R}^E \mid \forall e \in E : p(e) = \max_{e_2, \dots, e_r \in E} \left\{ v(e, e_2, \dots, e_r) - \sum_{i=2}^r p(e_i) \right\} \right\}. \quad (1.12)$$

The maximum in Equation (1.12) says that the functions p efficiently satisfy a triangle inequality. Their formulation using \max is dual to our work using \min . Another difference is that their space is in affine \mathbb{R}^E instead tropical projective space \mathbb{TP}^n .

Recall from Definition 1.4.8 the concept of local tropical linear space, and from Proposition 1.5.4 the homeomorphism between a local tropical linear space and a compactified apartment in the building. In the setting of tight spans an analogous description by local pieces holds: Given a basis $B = \{b_1, \dots, b_r\}$, the map Φ_B that sends a point (u_1, \dots, u_r) in the hyperplane $H_{v(B)} = \left\{ \sum_{i=1}^r u_i = v(B) \right\}$ to the linear map $\Phi_B(u) : E \rightarrow \mathbb{R}$ given by

$$e \longmapsto \max_{i \in B} \{v(e \cup B \setminus i) + u_i\} - v(B)$$

is injective [DT98, Proposition 1]. There is also a polyhedral description for intersections $\Phi_A(H) \cap \Phi_B(H)$ with A and B bases of (E, v) . Moreover, as B varies over all bases of (E, v) , the whole $T_{(E, v)}$ is covered. It can be shown via the theory of (B, N) -pairs that the $\Phi_B(H)$ form the apartments of what is sometimes called the *extended affine building* of GL_n .

Theorem 1.7.6 (Theorem 1 in [DT98]). *Let K be a non-Archimedean field with discrete valuation, and w_{univ} the universal realizable matroid of rank r as in Example 1.4.3. The space $T_{(K^r \setminus 0, w_{\mathrm{univ}})}$ is equal to the space $\mathcal{N}(V)$ for $V = K^r$.*

A point $p \in T_{(K^r \setminus 0, w_{\mathrm{univ}})}$ given by $p = \Phi_B(u)$ corresponds to seminorm $\|\cdot\|_p : K^r \setminus 0 \rightarrow \mathbb{R}$ given by $\|\cdot\|_p = \exp p(\cdot)$, which by Lemma 1.5.2 is diagonalizable by B and u . The expression $\Phi_B(u)(e) - v(B)$ attains the minimum at least twice for all $e \in E \setminus B$, i.e. once for e and once for some i in B . Hence, the equations from Definition 1.4.4 are satisfied for all $\tau = e \cup B$. It is straightforward to show that these equations imply the same result for arbitrary τ . Thus, as remarked by Speyer on [Spe08, p.6], the tight span $T_{(E, v)}$ is a lift of $\mathrm{Trop}(v)$ to \mathbb{R}^E .

Chapter 2

The Signed Goldman–Iwahori Space and Real Tropical Linear Spaces

Introduction

The Goldman–Iwahori space is the space of non-Archimedean seminorms on a finite-dimensional vector space over a non-Archimedean field. It was introduced in [GI63] to give a non-Archimedean analogue of symmetric spaces for the group PGL. If the ground field is spherically complete, this space is a compactification of a building in the sense of Bruhat and Tits. Bruhat–Tits buildings have proven to be an effective framework to study reductive groups over non-Archimedean fields (see [BT72, BT84]). Their close relation to Berkovich geometry is well-established in the literature (see [Ber90, Wer04, RTW10, RTW15]). Expanding on [DT98], in Chapter 1, we introduced a tropical approach to study the Goldman–Iwahori space by identifying it with the limit of tropicalized linear spaces and giving a matroidal interpretation.

Suppose the ground field is now also real closed such that the unique ordering is compatible with the absolute value. This chapter presents a signed analogue of the Goldman–Iwahori space, taking the order of the field into consideration. We approach this *signed Goldman–Iwahori space* by methods of real tropical geometry. This extends the non-Archimedean framework to real tropical geometry, which was introduced in [JSY22].

We recall some basics of tropical geometry: let K be an algebraically closed field with a non-trivial non-Archimedean absolute value $|\cdot|$. We denote the *tropicalization map* by

$$\text{trop} : K^n \rightarrow (\mathbb{R} \cup \{\infty\})^n, \quad (x_1, \dots, x_n) \mapsto (-\log |x_1|, \dots, -\log |x_n|).$$

The closure of the image of an algebraic subvariety $X \subseteq K^n$ under trop is called the *tropicalization* of X and usually denoted by $\text{Trop}(X)$. This space has the structure of a finite polyhedral complex of dimension $\dim X$ and its combinatorics carries rich information about the variety X , like its degree and Chow cohomology class.

It is a fundamental trait of tropical geometry that $\text{Trop}(X)$ not only depends on the variety X , but also on the chosen embedding into K^n . Hence, we will write the tropicalization as $\text{Trop}(X, \iota)$ to emphasize the dependence on the embedding $\iota : X \hookrightarrow K^n$. More generally, one can define tropicalizations of closed subvarieties of any toric variety, not just K^n [Pay09, §3]. One can also drop the conditions of K being algebraically closed or non-trivially valued

by choosing an algebraically closed, non-trivially field valued extension L/K and defining $\text{Trop}(X, \iota) := \text{Trop}(X \times_K L, \iota_L)$.

A natural question is, whether there is a universal tropicalization of a variety that is independent of the embedding. One way to construct this, is taking the projective limit of all tropicalizations with respect to all possible embeddings for a quasi-projective variety X . In the influential paper [Pay09], Payne showed that this limit space is homeomorphic to the *Berkovich analytification* X^{an} of X . This analytic space X^{an} , defined by Berkovich, is a connected Hausdorff topological space that contains the set of K -points $X(K)$ as a dense subset (see [Ber90] for details). Payne’s result led to ample research into limits of tropicalizations, for example in [FGP14, KSU21, GG22].

Recently, there have been two new developments that this chapter combines:

- (a) If one considers $X = \mathbb{P}^n$, in Chapter 1 it was shown that the *Goldman–Iwahori space* $\overline{\mathcal{X}}_n(K)$ is the limit of all tropicalizations with respect to *linear* embeddings of \mathbb{P}^n into higher-dimensional projective spaces. This space $\overline{\mathcal{X}}_n(K)$ consists of homothety classes of seminorms on $(K^{n+1})^*$ and has first been studied by Goldman and Iwahori as a piecewise linear analogue of symmetric spaces [GI63] for the group $\text{PGL}(K^{n+1})$. If K is spherically complete, it coincides with (a compactification of) the affine Bruhat–Tits building of $\text{PGL}(K^{n+1})$, *e.g.* studied by Werner in [Wer04]. The relation between affine buildings and Berkovich analytic spaces has been well established in [Ber90, RTW10, RTW12, RTW15].
- (b) In [JSY22], the authors define a “real” version of a Berkovich analytification: In this setup the field K is real closed instead of algebraically closed and a *real analytification* X_r^{an} of a variety X is constructed, which takes into account the order on K . The authors introduce a non-Archimedean approach to real tropicalizations using the real analytification. Roughly speaking, given an affine variety X over K , one restricts the tropicalization map to each orthant of K^n and glues the resulting 2^n tropicalizations together. The resulting space $\text{Trop}_r(X, \iota)$ is called the *real tropicalization* of X with respect to the embedding $\iota : X \hookrightarrow K^n$. In [JSY22], it was shown that X_r^{an} is the limit of all real tropicalizations. Real tropicalizations have garnered particular attention due to Viro’s patchworking [Vir84], which was one of the earliest achievements of what is now called tropical geometry. An equivalent perspective on real tropicalizations is given by real phase structures, and recently, in [RRS23] the authors defined the real part of a smooth tropical variety equipped with a real phase structure, which is locally (the topological realization of) an oriented matroid [RRS22].

We merge these two approaches: Consider a real closed field K equipped with a compatible non-Archimedean absolute value. An example is the field of real Puiseux series $\mathbb{R}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{\frac{1}{n}}))$, where the positive Puiseux series are those with positive leading coefficients. Let $X = \mathbb{P}^n$ be the projective space over K . We define the *signed Goldman–Iwahori space* $\mathfrak{R}\mathcal{X}_n(K)$ to be the space of homothety classes of non-trivial signed seminorms on $(K^{n+1})^*$ (*cf.* Definition 2.4.8) equipped with the topology of pointwise convergence. We study the geometry of this space via real tropical geometry, in particular via real tropical linear spaces: Let I be the cofiltered category of linear embeddings $\mathbb{P}^n \hookrightarrow U \subseteq \mathbb{P}^m$, where U is a torus-invariant open subset of \mathbb{P}^m with morphisms given by commutative triangles

$$\begin{array}{ccc} \mathbb{P}^n & \xhookrightarrow{\iota} & U \\ & \searrow \iota' & \downarrow \\ & & U' \end{array}$$

where $U \rightarrow U'$ is a composition of coordinate projections and permutations. For any linear embedding $\iota : \mathbb{P}^n \hookrightarrow U \subseteq \mathbb{P}^m$ as above, we construct a surjective and continuous map $\pi_\iota : \mathfrak{R}\mathcal{X}_n(K) \rightarrow \text{Trop}_r(\mathbb{P}^n, \iota)$. For any morphism in I , the coordinate projections and

permutations induce a continuous map between the respective real tropicalizations. This setup allows us to formulate and prove:

Theorem A (Theorem 2.5.3). *The canonical map*

$$\varinjlim_{\iota \in I} \pi_{\iota} : \mathfrak{R}\mathcal{X}_n(K) \longrightarrow \varinjlim_{\iota \in I} \mathrm{Trop}_r(\mathbb{P}^n, \iota)$$

is a homeomorphism.

Theorem A can both be viewed as a signed version of Theorem A or as a linear version of [JSY22, Theorem 6.14]. Even though the signed Goldman–Iwahori space $\mathfrak{R}\mathcal{X}_n(K)$ is not a building in the sense of Bruhat and Tits, it can be seen as a signed analogue of the building of PGL. From this perspective, it is a real analytic and non-Archimedean analogue of a symmetric space for PGL. It might be interesting to examine this space more closely in order to study reductive groups over real closed fields. Taking absolute values yields a natural, surjective, and continuous map $\mathfrak{R}\mathcal{X}_n(K) \rightarrow \overline{\mathcal{X}}_n(K)$ to the Goldman–Iwahori space (cf. Section 2.5.1).

Valuated matroids are the combinatorial objects that naturally arise with (ordinary) tropicalizations of linear spaces. To every linear embedding $\iota : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ we may associate a valuated matroid that uniquely determines the tropicalization $\mathrm{Trop}(\mathbb{P}^n, \iota)$. Even in the non-realizable case, one can associate a linear space to a valuated matroid, which agrees with the Bergman fan in the case of trivial valuation. In Chapter 1, the construction was generalized to allow us to describe finite rank matroids on infinite ground sets. In Theorem C we showed, that the Goldman–Iwahori space $\overline{\mathcal{X}}_n(K)$ is the tropical linear space associated to the *universal realizable valuated matroid* w_{univ} . The underlying matroid has as ground set $(K^{n+1})^*$ and the rank function is given by the dimension of the subspace spanned by a subset.

When the field K is real closed (or, more generally, ordered), the additional decoration on a matroid is that of an orientation, which has been studied extensively [BLVS⁺99, RGZ97]. As matroids capture the combinatorics of linear dependence over fields, oriented matroids capture the combinatorics of linear dependence over ordered fields, taking into account signs of linear dependencies. If the field K is in addition equipped with a compatible non-Archimedean value, we obtain an *oriented valuated matroid*, which is a matroid with both decorations that satisfy a compatibility condition.

Baker and Bowler introduce matroids with decorations in the language of hyperfields [BB16], where valuated/oriented/oriented valuated matroids arise as matroids over the tropical hyperfield \mathbb{T} /the sign hyperfield \mathbb{S} /the real tropical hyperfield \mathbb{RT} .

To every oriented valuated matroid \mathcal{M} on a ground set E , one can associate a *real tropical linear space* $\mathrm{Trop}_r(\mathcal{M})$ [Tab15, Jür18]. If \mathcal{M} is represented by a linear embedding over a real closed field, then the real tropicalization of the linear embedding coincides with the real tropical linear space associated to \mathcal{M} . This extends the trivial valuation case considered in [AKW06, Cel19] who shows that the real tropical linear space equals the real Bergman fan of the associated oriented matroid.

We generalize these ideas and construct a real tropical linear space for any oriented valuated matroid of finite rank, whose ground set explicitly need not be finite (see Section 2.6). We then show:

Theorem B (Theorem 2.6.3). *The signed Goldman–Iwahori space is the real tropical linear space associated to the universal realizable oriented valuated matroid $\mathcal{M}_{\mathrm{univ}}$, i.e.*

$$\mathfrak{R}\mathcal{X}_n(K) = \mathrm{Trop}_r(\mathcal{M}_{\mathrm{univ}}).$$

Here, the *universal realizable oriented valuated matroid* is realized by the ground set $E = (K^{n+1})^*$. This gives $\mathfrak{R}\mathcal{X}_n(K)$ an appealing interpretation as the real tropicalization of the

natural universal embedding

$$\iota_{\text{univ}} : \mathbb{P}^n \hookrightarrow \mathbb{P}(K^E).$$

In particular, this allows for a notion of a real tropicalization of an embedding of a finite-dimensional variety into an infinite-dimensional vector space. This gives an interpretation of the signed Goldman–Iwahori space as a universal realizable real tropical linear space.

In the special case $K = \mathbb{R}$ with trivial valuation signed seminorms are always diagonalizable, which need not even be true for other trivially valued fields (*cf.* Example 2.4.7). Therefore, in general, computing signed Goldman–Iwahori spaces turns out to be difficult. However, if $K = \mathbb{R}$, we can explicitly parameterize the signed Goldman–Iwahori space and determine the fibers of the map $\mathfrak{R}\mathcal{X}_n(\mathbb{R}) \rightarrow \overline{\mathcal{X}}_n(\mathbb{R})$.

In the case of trivial valuation, the real tropical linear space of an oriented matroid agrees with its *real Bergman fan* studied in [Cel19]. We show that for the universal realizable oriented matroid, this is still true if $K = \mathbb{R}$, but fails for other real closed fields (*cf.* Theorem 2.8.15 and Remark 2.8.16 (d)). Tropical linear spaces relate to tropical convexity (see [Jos21, §10] for details), in the same way that real tropical linear spaces relate to signed tropical convexity [LS22]. The real tropicalization of a linear space is the TC-convex hull of all \mathbb{RT} -cocircuits of the associated \mathbb{RT} -matroid [LS22, Theorem 7.8]. We give an interpretation of this result for the real tropicalization of the universal realizable oriented matroid $\mathcal{M}_{\text{univ}}$.

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Conventions

We write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For any set E , we write $\mathbb{TP}^E := (\overline{\mathbb{R}}^E \setminus \infty) / \mathbb{R}\mathbb{1}$, where $\mathbb{1}$ refers to the function with constant value 1. If $E = \{0, \dots, n\}$ for some natural number n , we write \mathbb{TP}^n for \mathbb{TP}^E . For a field K , which will be clear from the context, we write $\mathbb{A}^n = \text{Spec}(K[t_1, \dots, t_n])$ for affine space and $\mathbb{P}^n = \text{Proj}(K[t_0, \dots, t_n])$ for projective space. If $K = \mathbb{R}$, we write \mathbb{RP}^n for the set of \mathbb{R} -points of \mathbb{P}^n and consider this space with the Euclidean topology. We consider the ordering $+, - > 0$ of signs which we extend componentwise to a partial order on sign vectors. Given an ordered field K , we denote by $\text{sgn} : K \rightarrow \{0, +1, -1\}$ the sign function.

2.1 Real Tropical Geometry

We describe the *real part* of tropical projective space which is homeomorphic to the real projective space \mathbb{RP}^n . This space is the ambient space of the real tropicalizations of subvarieties of \mathbb{P}_K^n over a real closed field K with compatible absolute value. We describe this construction of a real tropicalization, which, in contrast to ordinary tropicalization, also takes the order on K into account. A more general and thorough treatment can be found for example in [RRS23].

2.1.1 Tropical Patchworking

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For $n \in \mathbb{N}$, the *tropical projective space* of dimension n is given by $\mathbb{TP}^n := \overline{\mathbb{R}}^{n+1} \setminus \{(\infty, \dots, \infty)\} / \mathbb{R}\mathbb{1}$, where $\mathbb{1} = (1, \dots, 1)$. Via the identification $\overline{\mathbb{R}} \cong \mathbb{R}_{\geq 0}$, $x \mapsto \exp(-x)$, we can identify $\mathbb{TP}^n \cong \mathbb{R}_{\geq 0}^{n+1} \setminus \{(0, \dots, 0)\} / (\mathbb{R}_{> 0})$. The space \mathbb{TP}^n is naturally stratified, where the closures of strata are given by $\{(x_0 : \dots : x_n) \in \mathbb{TP}^n \mid x_i = \infty \forall i \in I\}$, where $I \subsetneq \{0, \dots, n\}$.

We can glue together 2^n symmetric copies of \mathbb{TP}^n indexed by $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{n+1}/\mathbb{1}$ to obtain a space

$$\mathfrak{RTP}^n := \bigcup_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{n+1}/\mathbb{1}} \mathbb{TP}^n(\varepsilon) / \sim.$$

Here, \sim identifies the codimension-one strata $\{(x_0 : \dots : x_n) \in \mathbb{TP}^n(\varepsilon) \mid x_i = \infty\}$ and $\{(x_0 : \dots : x_n) \in \mathbb{TP}^n(\varepsilon') \mid x_i = \infty\}$ if $\varepsilon + \varepsilon' = e_i$ in $(\mathbb{Z}/2\mathbb{Z})^{n+1}/\mathbb{1}$. The identification $\mathbb{TP}^n \cong \mathbb{R}_{\geq 0}^{n+1} \setminus \{(0, \dots, 0)\} / (\mathbb{R}_{> 0})$ yields an identification

$$\begin{aligned} \mathfrak{RTP}^n &\cong \mathbb{RP}^n, \\ (\varepsilon, (x_0 : \dots : x_n)) &\mapsto ((-1)^{\varepsilon_0} \exp(-x_0) : \dots : (-1)^{\varepsilon_n} \exp(-x_n)). \end{aligned}$$

There is an obvious continuous retraction map $|\cdot|_{\mathbb{RP}^n} : \mathbb{RP}^n \rightarrow \mathbb{R}_{\geq 0}^{n+1} \setminus \{(0, \dots, 0)\} / (\mathbb{R}_{> 0}) \cong \mathbb{TP}^n$, sending each $(x_0 : \dots : x_n) \in \mathbb{RP}^n$ to $(|x_0| : \dots : |x_n|)$, where $|\cdot|$ denotes the absolute value on the real numbers.

Remark 2.1.1. This is a special case of the construction $\mathfrak{RT}\Sigma$ in [RRS23, GKZ08] associated to a pointed polyhedral fan Σ , which is a space homeomorphic to the real part of a complex toric variety (in our setting $\mathbb{P}_{\mathbb{C}}^n$) by gluing together multiple symmetric copies of the tropical toric variety (in our setting \mathbb{TP}^n). In [RRS23] this is the ambient space of the *patchworks* of tropical varieties.

Example 2.1.2. We make the space $\mathfrak{RTP}^2 \cong \mathbb{RP}^2$ explicit by gluing four triangles, each homeomorphic to \mathbb{TP}^2 , together as in Figure 2.1 and identifying antipodal points. The retraction map $|\cdot|_{\mathbb{RP}^2}$ folds up the four triangles to one.

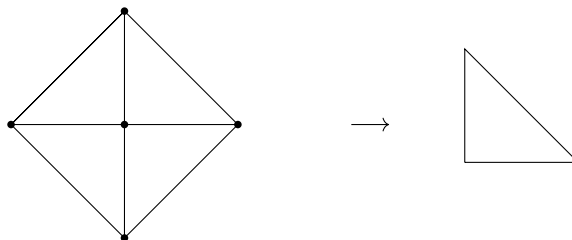


Figure 2.1: The map $\mathfrak{RTP}^2 \rightarrow \mathbb{TP}^2$.

For the sake of clarity, we will henceforth always use the coordinates of \mathbb{RP}^n and omit any logarithms. The downside is, that our tropicalizations are not polyhedral complexes in these coordinates. However, the formulas are much clearer since these coordinates make it easier to keep track of the signs.

2.1.2 Ordered and Real Closed Fields

We recall the definition of an *ordering* on a ring: For a commutative ring A , a subset $P \subseteq A$ is called an *ordering* of A if $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = A$, and $P \cap -P$ is a prime ideal

of A . The support of P is $\text{supp}(P) := P \cap -P$. To P we associate an order relation and a sign function in the natural way: Write $f >_P 0$ and $\text{sgn}_P(f) = +1$ if $-f \notin P$, write $f \geq_P 0$ if $f \in P$, and $\text{sgn}_P(f) = 0$ if $f \in \text{supp}(P)$. We have $\text{sgn}_P(-f) = -\text{sgn}_P(f)$. We may refer to the ordering as either P or $<_P$. An ordered field K is *real closed* if every non-negative element $x \geq 0$ has a square root in K . If K is real closed, there is a unique ordering on K defined by taking the positive elements to be precisely the non-zero squares. Every ordered field has a real closed extension, called the *real closure*. Whenever a real closed field is equipped with a non-Archimedean absolute value $|\cdot|_K$, we assume that the absolute value is *compatible* with the (unique) ordering. This means, if $0 \leq a \leq b$ then $|a|_K \leq |b|_K$, or equivalently, if $|a|_K > |b|_K$ then $\text{sgn}(a + b) = \text{sgn}(a) = \text{sgn}(a - b)$.

Example 2.1.3. Let $\mathbb{R}\{\{t\}\}$ be the field of *real Puiseux series*, i.e.,

$$\mathbb{R}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{\frac{1}{n}})).$$

Every element of $\mathbb{R}\{\{t\}\}$ is a power series with real coefficients and rational exponents with bounded denominator. This field is equipped with the absolute value that maps $f = \sum_{q \in \mathbb{Q}} a_q t^q$ to $\exp(-q_0)$, where q_0 is the minimal index such that $a_{q_0} \neq 0$. Moreover, let $\text{sgn}(f) := \text{sgn}(a_{q_0})$, i.e., the positive Puiseux series are those with positive leading coefficients. Then $\mathbb{R}\{\{t\}\}$ is a real closed field whose absolute value is compatible with the order.

2.1.3 Real Tropicalization

We will begin by providing a brief overview of ordinary tropicalization. Let K be an algebraically closed field with a non-trivial non-Archimedean absolute value $|\cdot|_K : K \rightarrow \mathbb{R}_{\geq 0}$. Denote the *tropicalization map* by

$$\begin{aligned} \text{trop} : \mathbb{P}^n(K) &\longrightarrow (\mathbb{R}_{\geq 0}^{n+1} \setminus \{0\}) / \mathbb{R}_{>0}, \\ (x_0 : \cdots : x_n) &\longmapsto (|x_0|_K : \cdots : |x_n|_K). \end{aligned}$$

For any closed subvariety $\iota : X \hookrightarrow \mathbb{P}^n$ the *tropicalization* $\text{Trop}(X, \iota)$ is defined to be the closure of the image of X under trop . If K is trivially valued or not algebraically closed, we pass to a non-trivially valued, algebraically closed extension L/K and define $\text{Trop}(X, \iota) := \text{Trop}(X_L, \iota_L) \subseteq (\mathbb{R}_{\geq 0}^{n+1} \setminus \{0\}) / \mathbb{R}_{>0}$, where $X_L = X \times_K L$. This is independent of the choice of L (e.g. by [Gub13, Proposition 3.8]).

Remark 2.1.4. It is more standard to write trop in the coordinates of \mathbb{TP}^n , for which we have

$$\text{trop}(x_0 : \cdots : x_n) = (-\log |x_0|_K : \cdots : -\log |x_n|_K).$$

In these coordinates, the Bieri–Groves theorem [BG84, Theorem A] and [EKL06, Theorem 2.2.3] states that, if X is irreducible, then $\text{Trop}(X, \iota)$ has the structure of a pure-dimensional rational polyhedral set of the same dimension as X .

Let now K be real closed instead of algebraically closed. We assume that $|\cdot|_K$ is compatible with the unique ordering on K . The *real tropicalization map* is given by

$$\begin{aligned} \text{trop}_r : \mathbb{P}^n(K) &\longrightarrow \mathbb{RP}^n, \\ (x_0 : \cdots : x_n) &\longmapsto (\text{sgn}(x_0)|x_0|_K : \cdots : \text{sgn}(x_n)|x_n|_K). \end{aligned}$$

Note that this map takes into account the order on K . This can be seen as an orthantwise tropicalization, where we restrict the tropicalization map to each orthant and glue the resulting tropicalizations together. The affine version of this map is considered by Jell,

Scheiderer, and Yu in [JSY22] and [AGS20], where the authors study properties of the images of semialgebraic sets under the real tropicalization map. The logarithmic version of this construction, without signs, was used by Alessandrini in [Ale13] who showed that the logarithmic limit of a real semialgebraic set is a polyhedral complex.

As before, for every closed subvariety (or, more generally, for every semialgebraic subset) $\iota : X \hookrightarrow \mathbb{P}^n$, we define the *real tropicalization* $\text{Trop}_r(X, \iota) \subseteq \mathbb{R}\mathbb{P}^n$ as the closure of the image of $X(K)$ under trop_r . If K is trivially valued, we choose a non-trivially valued, real closed field extension L/K such that the absolute value on L extends the one on K and define $\text{Trop}_r(X, \iota) := \text{Trop}_r(X_L, \iota_L) \subseteq \mathbb{R}\mathbb{P}^n$. The real tropicalization is independent of the choice of extension by [JSY22, Theorem 6.9]. By construction, the map $|\cdot|_{\mathbb{R}\mathbb{P}^n} : \mathbb{R}\mathbb{P}^n \rightarrow \mathbb{T}\mathbb{P}^n$ maps $\text{Trop}_r(X, \iota)$ to $\text{Trop}(X, \iota)$.

Example 2.1.5. Let $K = \mathbb{R}$ be equipped with the trivial valuation and let $X = V(x_0 + x_1 - x_2) \subseteq \mathbb{P}_{\mathbb{R}}^2$. We want to compute $\text{Trop}_r(X, \iota)$, where $\iota : X \hookrightarrow \mathbb{P}^2$ is the inclusion. Consider the base change to the field L of real Puiseux series. Let $(y_0 : y_1 : y_2) \in X_L(L) = V(x_0 + x_1 - x_2) \subseteq \mathbb{P}_L^2$, *i.e.*, $y_0 + y_1 = y_2$. Then

$$\text{sgn}((y_0, y_1, y_2)) \in \{(+, +, +), (-, -, -), (+, -, -), (-, +, +), (+, -, +), (-, +, -)\},$$

i.e., if $\text{sgn}(y_0) = \text{sgn}(y_1)$, then all signs are the same. By the axioms of a non-Archimedean valuation it is clear that $|y_2| \leq \max(|y_0|, |y_1|)$ and that, if $|y_0| \neq |y_1|$, we have equality. Moreover, \bullet we also have equality if $\text{sgn}(y_0) = \text{sgn}(y_1)$. Finally, if $\text{sgn}(y_0) \neq \text{sgn}(y_1)$ and $|y_0| = |y_1|$, then $\text{sgn}(y_2) \cdot |y_2|$ can have any value in $[-|y_0|, |y_0|] \cap \mathbb{Q}$. We may choose representatives in \mathbb{P}^2 such that $\max\{|y_0|, |y_1|, |y_2|\} = 1$ and $y_2 \geq 0$. In Figure 2.2 the real tropicalization $\text{Trop}_r(X, \iota)$ is now indicated in red and the (usual) tropicalization $\text{Trop}(X, \iota)$ in blue.

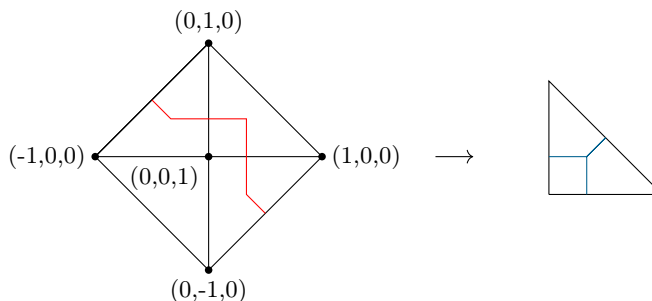


Figure 2.2: The map $\mathfrak{R}\mathbb{T}\mathbb{P}^2 \rightarrow \mathbb{T}\mathbb{P}^2$.

Remark 2.1.6. In [RRS23] the authors introduce *real phase structures* on rational polyhedral spaces and tropical varieties. For a rational polyhedral subspace X in \mathbb{R}^n , a real phase structure is an assignment of an affine subspace of $(\mathbb{Z}/2\mathbb{Z})^n$ to each facet of X . The assignment of a real phase structure is used to describe the *patchwork* of a tropical variety, which should be thought of as its real part. The authors show that a patchwork describes, up to homeomorphism, fibers of real analytic families with non-singular tropical limits. As explained in [RRS23, §4.8] their construction of a real tropicalization, *i.e.*, the patchwork, agrees with the logarithmic version of the real tropicalization from [JSY22] that we described above and use in this chapter.

2.2 Matroids over the Real Tropical Hyperfield

Matroids over hyperfields due to Baker and Bowler [BB16, BB19] simultaneously generalize linear subspaces, matroids, oriented matroids, and valuated matroids. We recall their notion as well as the hyperfields that are of our interest such as the sign hyperfield, the tropical hyperfield, and, most importantly, the real tropical hyperfield. Matroids over the sign hyperfield correspond to *oriented matroids*, while matroids over the tropical hyperfield correspond to *valuated matroids*. *Oriented valuated matroids* are hybrid objects combining oriented matroids and valuated matroids in a compatible way. From the perspective of matroids over hyperfields they arise naturally as matroids over the real tropical hyperfield \mathbb{RT} . We recall the definition of a real tropical linear space associated to an oriented valuated matroid defined via its \mathbb{RT} -circuits.

2.2.1 Hyperfields

In [Vir10] Viro introduced hyperfields as a convenient technique in tropical geometry. Especially in the last few years there has been a surge of interest and research in the realm of hyperfields and tropical geometry [Lor22, BB16, BL21, MS23, Max24]. A *hyperfield* \mathbb{H} is a set with a multiplication \cdot and addition \oplus , where addition may be multivalued, that satisfies axioms similar to those for a field. Several of the following hyperfields were first introduced in Viro's paper [Vir10], to which we refer for precise definitions.

Example 2.2.1.

- (a) Any field K can trivially be considered a hyperfield with its ordinary multiplication and addition, where we consider the sum of two elements as a singleton set.
- (b) The *Krasner hyperfield* \mathbb{K} as a set is $\mathbb{K} = \{0, 1\}$ with the usual multiplication and $-1 = 1$. For addition 0 is the neutral element and $1 \oplus 1 = \mathbb{K}$.
- (c) The *sign hyperfield* \mathbb{S} on elements $\{0, +1, -1\}$ has multiplicative group $(\{\pm 1\}, \cdot)$. The addition is given by $0 \oplus x = 0$ for all $x \in \mathbb{S}$, $1 \oplus 1 = 1$, $-1 \oplus -1 = -1$, and $-1 \oplus 1 = \mathbb{S}$.
- (d) The *tropical hyperfield* \mathbb{T} on elements $\mathbb{R}_{\geq 0}$ with multiplicative notation has as multiplicative group $(\mathbb{R}_{>0}, \cdot)$, and the addition is defined by

$$a \oplus b = \begin{cases} \max(a, b) & \text{if } a \neq b, \\ [0, a] & \text{if } a = b. \end{cases}$$

- (e) The *real tropical hyperfield* \mathbb{RT} on elements \mathbb{R} has multiplicative group (\mathbb{R}^*, \cdot) with addition

$$a \oplus b = \begin{cases} a & \text{if } |a| > |b|, \\ b & \text{if } |a| < |b|, \\ a & \text{if } a = b, \\ [a, b] & \text{if } a = -b \leq 0, \\ [b, a] & \text{if } a = -b \geq 0. \end{cases}$$

A *homomorphism of hyperfields* is a map $f : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ such that $f(0) = 0$, $f(1) = 1$, $f(x \cdot y) = f(x) \cdot f(y)$, and $f(x \oplus y) \subseteq f(x) \oplus f(y)$ for $x, y \in \mathbb{H}_1$.

Example 2.2.2.

- (a) Every hyperfield \mathbb{H} has a canonical homomorphism into \mathbb{K} by sending 0 to 0 and every non-zero element to 1.

- (b) The maps $\mathbb{R}\mathbb{T} \rightarrow \mathbb{T}, a \mapsto |a|$ taking the absolute value and $\mathbb{R}\mathbb{T} \rightarrow \mathbb{S}, a \mapsto \operatorname{sgn}(a)$ keeping only the sign information are hyperfield homomorphisms.
- (c) For a field K , a map $K \rightarrow \mathbb{S}$ a hyperfield homomorphism if and only if it is of the form sgn_P for an ordering $P \subset K$.
- (d) For a field K , a map $K \rightarrow \mathbb{T} = \mathbb{R}_{\geq 0}$ is a hyperfield homomorphism if and only if it is a non-Archimedean absolute value.
- (e) Let K be an ordered field with a compatible absolute value $|\cdot|_K$. The natural signed absolute value map $K \rightarrow \mathbb{R}\mathbb{T}, x \mapsto \operatorname{sgn}(x) \cdot |x|_K$ is a hyperfield homomorphism.

2.2.2 Matroids over Hyperfields

Definition 2.2.3 ([BB16]). Let E be a non-empty finite set, \mathbb{H} a hyperfield, and let n be a positive integer. A *Grassmann–Plücker function of rank n on E with coefficients in \mathbb{H}* is a function $\varphi : E^n \rightarrow \mathbb{H}$ such that:

- (a) φ is not identically zero,
- (b) φ is *alternating*, i.e.,

$$\begin{aligned} \varphi(x_1, \dots, x_i, \dots, x_j, \dots, x_n) &= -\varphi(x_1, \dots, x_j, \dots, x_i, \dots, x_n) \text{ and} \\ \varphi(x_1, \dots, x_n) &= 0 \quad \text{if } x_i = x_j \text{ for some } i, j. \end{aligned}$$

- (c) (Grassmann–Plücker relations) For any two subsets $\{x_1, \dots, x_{n+1}\}, \{y_1, \dots, y_{n-1}\} \subseteq E$,

$$0 \in \bigoplus_{k=1}^{n+1} \varphi(x_1, \dots, \hat{x}_k, \dots, x_{n+1}) \cdot \varphi(x_k, y_1, \dots, y_{n-1}).$$

Two Grassmann–Plücker functions φ_1 and φ_2 are *equivalent* if $\varphi_1 = \alpha \cdot \varphi_2$ for some $\alpha \in \mathbb{H}^\times$. A *(strong) matroid over a hyperfield \mathbb{H}* on E of rank r is an equivalence class of a Grassmann–Plücker function of rank r on E .

Remark 2.2.4. Baker and Bowler define strong and weak matroids over hyperfields, which coincide for doubly distributive hyperfields by [BB19, Theorem 5.4]. Since all the hyperfields we consider (see Example 2.2.1) are doubly distributive hyperfields, we will simply refer to matroids over hyperfields.

For the hyperfields of Example 2.2.1, we obtain the following (classical) notions:

- (a) A matroid over a field K is the same thing as a linear subspace of K^E of rank n . This is the classical representation of a subspace via its Plücker coordinates that satisfy the Grassmann–Plücker relations.
- (b) When $\mathbb{H} = \mathbb{K}$ is the Krasner hyperfield, a matroid over \mathbb{K} is the same as a usual matroid since the Grassmann–Plücker relations are equivalent to the basis exchange axiom for matroids.
- (c) A matroid over the sign hyperfield \mathbb{S} is the same as an *oriented* matroid, where a choice of a Grassmann–Plücker φ function is called a *chirotope* (note that $\varphi, -\varphi$ are the only chirotopes). This choice gives us then a notion of an *oriented basis*, where (b_1, \dots, b_n) is a *positively oriented basis*, if $\varphi(b_1, \dots, b_n) = 1$.
- (d) A matroid over the tropical hyperfield \mathbb{T} is the same as a *valuated* matroid in the sense of Dress–Wenzel [DW92a]. The Grassmann–Plücker function is usually called a tropical Plücker vector.
- (e) A matroid over the real tropical hyperfield $\mathbb{R}\mathbb{T}$ is an *oriented valuated* matroid. This particular case holds significant interest within this chapter.

Matroids over hyperfields admit a useful pushforward operation (cf. [BB16, §4.2]): Given a Grassmann–Plücker function $\varphi : E^n \rightarrow \mathbb{H}_1$ and a homomorphism of hyperfields $f : \mathbb{H}_1 \rightarrow \mathbb{H}_2$, the pushforward $f_*\varphi : E^n \rightarrow \mathbb{H}_2$ is defined by the formula

$$f_*\varphi(e_1, \dots, e_n) = f(\varphi(e_1, \dots, e_n)).$$

This is again a Grassmann–Plücker function. If \mathcal{M} is a matroid over \mathbb{H}_1 given by φ , the pushforward $f_*(\mathcal{M})$ is defined by $f_*\varphi$ and is a matroid over \mathbb{H}_2 .

Definition 2.2.5. Let $f : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be a homomorphism of hyperfields, and let \mathcal{M}_2 be a matroid on E with coefficients in \mathbb{H}_2 . We say that \mathcal{M}_2 is *realizable* with respect to f if there is a matroid \mathcal{M}_1 over \mathbb{H}_1 such that $f_*(\mathcal{M}_1) = \mathcal{M}_2$.

Example 2.2.6.

- (a) Let \mathbb{H} be a hyperfield and let $\omega : \mathbb{H} \rightarrow \mathbb{K}$ be the canonical hyperfield homomorphism. If \mathcal{M} is an \mathbb{H} -matroid, the pushforward $\omega_*(\mathcal{M})$ is called the *underlying matroid*.
- (b) Let K be an ordered field and let $\text{sgn} : K \rightarrow \mathbb{S}$ be the sign map. If $W \subseteq K^m$ is a linear subspace (considered in the natural way as a K -matroid), the pushforward $\text{sgn}_*(W)$ coincides with the oriented matroid which one traditionally associates to W .
- (c) Let K be a field with a non-Archimedean absolute value $|\cdot|_K : K \rightarrow \mathbb{T}$ (considered as a hyperfield homomorphism) and let $W \subseteq K^n$ be a linear subspace. Then the pushforward $|\cdot|_*(W)$ is the valuated matroid associated to W .
- (d) Let K be a real closed field with a compatible absolute value $|\cdot|_K$. If $W \subseteq K^m$ is a linear subspace, the pushforward of W under the signed absolute value map $K \rightarrow \mathbb{RT}, x \mapsto \text{sgn}(x) \cdot |x|_K$ is the oriented valuated matroid associated to W . Explicitly, let W be the row-space of the $n \times m$ -matrix $[v_1, \dots, v_m]$. Let $E = \{v_1, \dots, v_m\}$ be the set of column vectors of the matrix which form a spanning set of vectors of K^n . The associated Grassmann–Plücker function of the \mathbb{RT} -matroid associated to W is given by

$$E^n \rightarrow \mathbb{R}, \\ (v_{a_1}, \dots, v_{a_n}) \mapsto \text{sgn}(\det[v_{a_1}, \dots, v_{a_n}]) \cdot |\det[v_{a_1}, \dots, v_{a_n}]|_K.$$

- (e) Consider an \mathbb{RT} -matroid (i.e., an oriented valuated matroid) \mathcal{M} . The pushforwards of \mathcal{M} under the natural maps $\mathbb{RT} \rightarrow \mathbb{T}$ and $\mathbb{RT} \rightarrow \mathbb{S}$ yield the underlying valuated matroid and the underlying oriented matroid, respectively.

2.2.3 Oriented Valuated Matroids

Since oriented valuated matroids are not as well studied as for example oriented or valuated matroids, we describe them now in more detail. The following explicit description is taken from [Gia23, §1.5]:

Let \mathcal{M} be an oriented valuated matroid given by the Grassmann–Plücker function $\varphi : E^n \rightarrow \mathbb{R}$. Condition (3) in Definition 2.2.3 explicitly means: For each $\{x_1, \dots, x_{n+1}\} \in E^{n+1}$ and $\{y_1, \dots, y_{n-1}\} \in E^{n-1}$, either the numbers

$$\{(-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{n+1}) \cdot \varphi(x_k, y_1, \dots, y_{n-1})\}_{k=1, \dots, n+1}$$

are all zero, or the maximum modulus occurs with both signs. The group $\mathbb{RT}^* = \mathbb{R}^*$ acts on the set of such φ by multiplication, and an oriented valuated matroid is an orbit.

As in Example 2.2.6 (e) the pushforwards of an \mathbb{RT} -matroid \mathcal{M} yield an underlying valuated matroid given by a tropical Plücker vector $v = |\varphi| : E^n \rightarrow \mathbb{T} = \mathbb{R}_{\geq 0}$ and an underlying oriented matroid given by the chirotope (i.e., Grassmann–Plücker function) $\text{sgn} \circ \varphi : E^n \rightarrow \mathbb{S} = \{0, +1, -1\}$. An equivalent definition of an oriented valuated matroid is also given by a *compatible* pair of a tropical Plücker vector and a chirotope, see [Gia23, Definition 1.5.1].

Remark 2.2.7. The more common, but equivalent, notion of a valuated matroid arises by taking $v = -\log |\varphi|$, where one takes hyperfield addition using min and multiplication given by usual addition.

There is a cryptomorphic definition of \mathbb{H} -matroids via \mathbb{H} -circuits. For $\mathbb{H} \in \{\mathbb{K}, \mathbb{S}, \mathbb{T}\}$ these give the usual axioms for circuits, signed circuits and valuated circuits. The general cryptomorphic axiomatization of \mathbb{H} -circuits can be found in [BB16, Definition 3.7]. For $\mathbb{H} = \mathbb{RT}$, we will now state the axioms for the set of \mathbb{RT} -circuits. For any $C, C' \in \mathbb{RT}^E$ we define the *support* of C as $\text{Supp } C := \{e \in E \mid C_e \neq 0\}$, and we define the *composition*

$$(C \circ C')_e = \begin{cases} C_e & \text{if } |C_e| \geq |C'_e|, \\ C'_e & \text{otherwise.} \end{cases}$$

Proposition 2.2.8 ([BP19, Theorem 45]). *Let E be a finite set. An \mathbb{RT} -matroid on E is equivalent to a subset $\mathcal{C} \subseteq \mathbb{RT}^E$ that satisfies the following circuit axioms:*

- (C0) $0 \notin \mathcal{C}$,
- (C1) if $C \in \mathcal{C}$ and $\alpha \in \mathbb{RT}^* = \mathbb{R}^*$, then $\alpha \cdot C \in \mathcal{C}$,
- (C2) if $C, C' \in \mathcal{C}$ and $\text{Supp } C \subseteq \text{Supp } C'$, then there exists an $\alpha \in (\mathbb{RT})^* = \mathbb{R}^*$ such that $C' = \alpha \cdot C$,
- (C3) for any $C, C' \in \mathcal{C}$, $e, f \in E$ such that $C_e = -C'_e \neq 0$ and $|C_f| > |C'_f|$, there exists a $C'' \in \mathcal{C}$ such that $C''_e = 0$, $C''_f = C_f$, and $|C''_g| < |C_g \circ C'_g|$ or $C''_g \in C_g \oplus C'_g$ for all $g \in E$.

We call \mathbb{RT} -circuits *signed valuated circuits*.

Lemma 2.2.9 ([BB16, Lemma 4.5]). *If $f : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ is a homomorphism of hyperfields and \mathcal{M} is an \mathbb{H}_1 -matroid on E with set of circuits \mathcal{C} , then the set of circuits of the pushforward $f_*(\mathcal{M})$ is given by*

$$\{cf_*(C) \mid c \in \mathbb{H}_2^\times, C \in \mathcal{C}\}.$$

In particular, the set of supports of \mathcal{C} is the set of circuits of the underlying matroid.

Example 2.2.10. Let K be an ordered field with a compatible absolute value $|\cdot|_K$. An \mathbb{RT} -matroid \mathcal{M} is realizable over K if there is a K -matroid, *i.e.*, a linear space $W \subseteq K^E$, such that the pushforward of W under the signed absolute value map equals \mathcal{M} . As in Example 2.2.6 (d) let $E = \{v_1, \dots, v_m\}$ be a spanning set of vectors of K^n and W the row-space of the $n \times m$ -matrix $[v_1, \dots, v_m]$. The set of circuits of W is given by the vectors with minimal support in E . By Lemma 2.2.9 the set of signed valuated circuits, (*i.e.* \mathbb{RT} -circuits) of \mathcal{M} , is given by

$$\mathcal{C} = \left\{ (\text{sgn}(\lambda_1)|\lambda_1|_K, \dots, \text{sgn}(\lambda_m)|\lambda_m|_K) \in \mathbb{R}^m \mid \sum_i \lambda_i v_i = 0 \text{ is a minimal linear dependence} \right\}.$$

This generalizes signed circuits of an oriented matroid by considering the trivial valuation on K .

2.2.4 Real Tropical Linear Spaces

Analogous to the definition of the tropical linear space of a valuated matroid [Spe08], the real tropical linear space $\text{Trop}_r(\mathcal{M})$ associated to an oriented valuated matroid \mathcal{M} is defined as follows:

Definition 2.2.11 ([Jür18, 1.2.7]). Let \mathcal{M} be an oriented valuated matroid on the ground set E . For a signed valuated circuit $C \in \mathcal{C}$, we define the *real tropical hyperplane*

$$\begin{aligned} \text{Trop}_r(\mathcal{M}_C) &= \{(y_0 : \dots : y_m) \in \mathbb{R}\mathbb{P}^m \mid 0 \in \bigoplus_{e \in \text{Supp } C} y_e \cdot C_e\} \\ &= \{(y_0 : \dots : y_m) \in \mathbb{R}\mathbb{P}^m \mid \text{there exist indices } i \neq j \text{ such that} \\ &\quad \max_{e \in \text{Supp } C} (|y_e| \cdot |C_e|) \text{ is attained at } i, j \text{ and } y_i \cdot C_i = -y_j \cdot C_j\}. \end{aligned}$$

The *real tropical linear space* associated to \mathcal{M} is defined as

$$\text{Trop}_r(\mathcal{M}) := \bigcap_{C \in \mathcal{C}} \text{Trop}_r(\mathcal{M}_C).$$

Remark 2.2.12. This definition of the real tropical linear space is exactly the zero set of linear polynomials over the real tropical hyperfield \mathbb{RT} .

Remark 2.2.13. If K is trivially valued, $\text{Trop}_r(\mathcal{M})$ equals the *real Bergman fan* of [Cel19], see Section 2.8. In the general case, the real tropical linear space of a matroid over the real tropical hyperfield is the so-called *TC-convex hull* of the \mathbb{RT} -circuits of \mathcal{M} [LS22]. The relation to signed tropical convexity is elaborated further in Section 2.8.4.

The tropicalization of a linear space over a non-Archimedean field is determined by its associated valuated matroid [Spe08, Proposition 4.2]. If K is a real closed field with a compatible absolute value $|\cdot|_K$, then similarly the real tropicalization of a linear space over K is determined by the associated oriented valuated matroid.

Proposition 2.2.14 ([Tab15, Theorem 3.14], [Jür18, Theorem 1.2.11]). *Let K be a real closed field with a compatible absolute value $|\cdot|_K$. Let $\iota = (f_0 : \dots : f_m) : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ be a linear embedding, and \mathcal{M}_ι be the associated realizable oriented valuated matroid on $\{f_0, \dots, f_m\} \subset (K^{n+1})^*$ (cf. Example 2.2.10). Then*

$$\text{Trop}_r(\mathbb{P}^n, \iota) = \text{Trop}_r(\mathcal{M}_\iota).$$

In other words, the real tropicalization of a linear space only depends on the associated oriented valuated matroid.

2.3 Real Analytification and Tropicalization

Throughout this section, we fix a real closed, non-Archimedean valued field K with absolute value $|\cdot|_K$ that is compatible with the order on K . We recall the real analytification of a variety over K which was introduced in [JSY22]. We then show that, similarly to the Berkovich analytification, the real analytification of \mathbb{P}^n can be described as a quotient of $\mathbb{A}_r^{n+1, \text{an}} \setminus \{0\}$, and hence can be described in terms of signed multiplicative seminorms on a polynomial ring. This gives a real tropicalization map from $\mathbb{P}_r^{n, \text{an}}$ to $\mathbb{R}\mathbb{P}^n$ extending the real tropicalization map from affine varieties to closed subvarieties of \mathbb{P}^n .

2.3.1 The Real Analytification X_r^{an}

In [JSY22] the authors define a *real analytification* of a K -variety X . By a K -variety X we always mean a reduced, irreducible, separated scheme of finite type. For a point p in the scheme X , let $K(p)$ denote its residue field.

Definition 2.3.1. The *real analytification* of X is the set X_r^{an} consisting of all triples $x = (p_x, |\cdot|_x, \langle \cdot \rangle_x)$, where $p_x \in X$, $|\cdot|_x$ is an absolute value on $K(p_x)$ extending $|\cdot|_K$, and $\langle \cdot \rangle_x$ is

an order on $K(p_x)$ compatible with $|\cdot|_x$. We equip X_r^{an} with the coarsest topology such that the support map

$$\begin{aligned} \text{supp} : X_r^{\text{an}} &\longrightarrow X, \\ x &\longmapsto p_x \end{aligned}$$

is continuous and the map

$$\begin{aligned} \text{supp}^{-1}(U) &\longrightarrow \mathbb{R}, \\ x &\longmapsto \text{sgn}_x(f) \cdot |f|_x \end{aligned}$$

is continuous for every open $U \subseteq X$ and every regular function f on U .

Many properties that hold for the Berkovich analytification also hold for the real analytification. For example:

- For any morphism $f : X \rightarrow Y$ of varieties over K , there is a natural induced continuous map $f_r^{\text{an}} : X_r^{\text{an}} \rightarrow Y_r^{\text{an}}$ of the corresponding real analytifications. This assignment is functorial.
- The space X_r^{an} is a connected Hausdorff space and if X is proper over K , it is compact [JSY22, Proposition 3.6, Corollary 3.10].

There is a natural map $X_r^{\text{an}} \rightarrow X^{\text{an}}$ which forgets the last entry. This map is always continuous and proper, but in general neither injective nor surjective [JSY22, Example 3.12]. Just like for the classical Berkovich analytification, for an affine K -variety $X = \text{Spec}(A)$, the real analytification X_r^{an} has a description as the space of signed multiplicative seminorms on the K -algebra A .

Definition 2.3.2. A *signed multiplicative seminorm* on A is a map $|\cdot|^{\text{sgn}} : A \rightarrow \mathbb{R}$ such that:

- (i) $|a|^{\text{sgn}} = \text{sgn}(a) \cdot |a|_K$ if $a \in K$,
- (ii) $|f \cdot g|^{\text{sgn}} = |f|^{\text{sgn}} \cdot |g|^{\text{sgn}}$ if $f, g \in A$,
- (iii) $\min(|f|^{\text{sgn}}, |g|^{\text{sgn}}) \leq |f + g|^{\text{sgn}} \leq \max(|f|^{\text{sgn}}, |g|^{\text{sgn}})$ if $f, g \in A$.

The space of signed multiplicative seminorms is endowed with the coarsest topology that makes the natural evaluation maps $\text{ev}_f : |\cdot|^{\text{sgn}} \mapsto |f|^{\text{sgn}}$ for all $f \in A$ continuous. By [JSY22, Proposition 3.4] for $X = \text{Spec}(A)$, we have that X_r^{an} is the space of signed multiplicative seminorms on A . The map to the Berkovich analytification is given by $|\cdot|^{\text{sgn}} \mapsto ||\cdot|^{\text{sgn}}|$.

Remark 2.3.3. As explained in [JSY22], one can view $X_r^{\text{an}} = X(\mathbb{RT})$, *i.e.*, as the \mathbb{RT} -points of X . Namely, if $X = \text{Spec}(A)$ is an affine K -scheme, then $X_r^{\text{an}} = \text{Hom}_K(A, \mathbb{RT})$. The latter set is the set of all hyperfield homomorphisms $A \rightarrow \mathbb{RT}$ that factor the canonical hyperfield homomorphism $K \rightarrow \mathbb{RT}, x \mapsto \text{sgn}(x) \cdot |x|_K$ from Example 2.2.2 (e). This construction glues to varieties over K . In fact, from this hyperfield point of view, [Jun21] showed that $X = X(\mathbb{K})$ and $X^{\text{an}} = X(\mathbb{T})$.

In particular, for the affine space $\mathbb{A}^n = \text{Spec } K[t_1, \dots, t_n]$, the real Berkovich analytification $(\mathbb{A}^n)_r^{\text{an}}$ is given as the set of signed multiplicative seminorms on $K[t_1, \dots, t_n]$. Similar to the Berkovich Proj construction, there exists a corresponding construction for the real analytification, hence we can describe $\mathbb{P}_r^{n, \text{an}}$ in terms of signed multiplicative seminorms. Let $|\cdot|_1^{\text{sgn}}$ and $|\cdot|_2^{\text{sgn}}$ be points in $\mathbb{A}_r^{n+1, \text{an}} \setminus \{0\}$, *i.e.*, non-zero signed multiplicative seminorms on $K[t_0, \dots, t_n]$ extending the absolute value on K . We call $|\cdot|_1^{\text{sgn}}$ and $|\cdot|_2^{\text{sgn}}$ *homothetic*, if there exists a constant $0 \neq c \in \mathbb{R}$ such that for every homogeneous polynomial f of degree d , we have $|f|_1^{\text{sgn}} = c^d |f|_2^{\text{sgn}}$.

Proposition 2.3.4. *The real analytification $\mathbb{P}_r^{n, \text{an}}$ is the quotient of $\mathbb{A}_r^{n+1, \text{an}} \setminus \{0\}$ by homothety.*

Proof. This works the same way as for the Berkovich analytification: Let $x = (p_x, |\cdot|_x, \langle_x)$ be a point in $\mathbb{P}_r^{n, \text{an}}$. Then p_x is contained in an open affine $U \subseteq \mathbb{P}^n$, i.e., $x \in U_r^{\text{an}}$. Let $(z_0 : \dots : z_n)$ be homogeneous coordinates on \mathbb{P}^n . We assume that $U = U_0 = \{z \in \mathbb{P}^n \mid z_0 \neq 0\}$. Thus x corresponds to a signed multiplicative seminorm $|\cdot|_0^{\text{sgn}}$ on $K[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}]$. This induces a non-zero signed multiplicative seminorm on $K[t_0, \dots, t_n]$ by mapping $f(t_0, \dots, t_n) \in K[t_0, \dots, t_n]$ to $|f(1, \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0})|_0^{\text{sgn}}$, which we also call $|\cdot|_0^{\text{sgn}}$. Let $p_x \in U'$ for another open affine $U' \subseteq \mathbb{P}^n$, say $U' = U_1 = \{z \in \mathbb{P}^n \mid z_1 \neq 0\}$. Let $|\cdot|_1^{\text{sgn}}$ denote the corresponding signed multiplicative seminorm on $K[\frac{t_0}{t_1}, \frac{t_2}{t_1}, \dots, \frac{t_n}{t_1}]$ which induces a signed multiplicative seminorm on $K[t_0, \dots, t_n]$ via $f(t_0, \dots, t_n) \mapsto |f(\frac{t_0}{t_1}, 1, \frac{t_2}{t_1}, \dots, \frac{t_n}{t_1})|_1^{\text{sgn}}$. We now show that $|\cdot|_0^{\text{sgn}}$ and $|\cdot|_1^{\text{sgn}}$ are homothetic signed seminorms on $K[t_0, \dots, t_n]$. Let $f \in K[t_0, \dots, t_n]$ be homogeneous of degree d . Since

$$f\left(1, \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right) \cdot \left(\frac{t_0}{t_1}\right)^d = f\left(\frac{t_0}{t_1}, 1, \frac{t_2}{t_1}, \dots, \frac{t_n}{t_1}\right),$$

we have

$$\left|f\left(\frac{t_0}{t_1}, 1, \frac{t_2}{t_1}, \dots, \frac{t_n}{t_1}\right)\right|_1^{\text{sgn}} = c^d \left|f\left(1, \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right)\right|_0^{\text{sgn}}$$

for $c = |\frac{t_0}{t_1}|_1^{\text{sgn}} \neq 0$. This shows that a point $x \in \mathbb{P}_r^{n, \text{an}}$ defines a homothety class in $(\mathbb{A}_r^{n+1, \text{an}} \setminus \{0\}) / \sim$.

Conversely, let $[|\cdot|]^{\text{sgn}}$ be an homothety class of a non-zero signed multiplicative seminorm on $K[t_0, \dots, t_n]$. Since $|\cdot|^{\text{sgn}} \neq 0$, after a coordinate change we may assume that $|t_0|^{\text{sgn}} \neq 0$. Then we define a signed multiplicative seminorm on $K[\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}]$ by

$$f\left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right) \mapsto \left|t_0^d f\left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right)\right|^{\text{sgn}},$$

where d is the degree of f . This corresponds to a point in $(U_0)_r^{\text{an}} \subseteq \mathbb{P}_r^{n, \text{an}}$. If also $|t_1|^{\text{sgn}} \neq 0$, then this similarly defines a point in $(U_1)_r^{\text{an}}$. In order to see that this is the same point in the intersection $(U_0 \cap U_1)_r^{\text{an}}$, we observe that the signed multiplicative seminorm on $K[t_0, \dots, t_n]_{t_0, t_1}$ is naturally induced by $|\cdot|^{\text{sgn}}$, hence the two signed multiplicative seminorms agree. It is straightforward to check that the topologies agree, which concludes the proof. \square

There is a natural continuous surjective real tropicalization map

$$\begin{aligned} \text{trop}_r : \mathbb{A}_r^{n+1, \text{an}} &\longrightarrow \mathbb{R}^{n+1}, \\ |\cdot|^{\text{sgn}} &\longmapsto (|t_0|^{\text{sgn}}, \dots, |t_n|^{\text{sgn}}), \end{aligned}$$

which induces a real tropicalization map

$$\begin{aligned} \text{trop}_r : \mathbb{P}_r^{n, \text{an}} &\longrightarrow \mathbb{R}\mathbb{P}^n, \\ [|\cdot|^{\text{sgn}}] &\longmapsto (|t_0|^{\text{sgn}} : \dots : |t_n|^{\text{sgn}}). \end{aligned}$$

Let $\iota : X \hookrightarrow \mathbb{P}^n$ be a closed subvariety. By a projective version of the real analogue of the fundamental theorem [JSY22, Theorem 6.9], we have

$$\text{trop}_r(X_r^{\text{an}}) = \text{Trop}_r(X, \iota).$$

2.4 The Signed Goldman–Iwahori Space

The signed Goldman–Iwahori space associated to a vector space can be thought of as a linear analogue of the real analytification of \mathbb{P}^n . Alternatively, it is a version of the Goldman–Iwahori space in [GI63] that takes signs of the ground field into account. In this section, we

will examine its structure and its relation to the real analytification. Further, we describe a real tropicalization map from the signed Goldman–Iwahori space to real projective space. This allows us to define a real tropicalization map from the signed Goldman–Iwahori space to real tropicalizations of linear spaces. From now on, let K be a real closed field with compatible non-Archimedean absolute value $|\cdot|$ and let V be a finite-dimensional vector space over K .

2.4.1 Signed Seminorms

Definition 2.4.1. A *signed seminorm* on V is a map $\|\cdot\|^{\text{sgn}} : V \rightarrow \mathbb{R}$ such that:

- (i) For all $v \in V$ and $\lambda \in K$, we have

$$\|\lambda v\|^{\text{sgn}} = \text{sgn}(\lambda)|\lambda| \|v\|^{\text{sgn}}.$$

- (ii) For all $v, w \in V$ both strong triangle inequalities

$$\min(\|v\|^{\text{sgn}}, \|w\|^{\text{sgn}}) \leq \|v + w\|^{\text{sgn}} \leq \max(\|v\|^{\text{sgn}}, \|w\|^{\text{sgn}})$$

hold.

Remark 2.4.2. This definition is equivalent to $\|\cdot\|^{\text{sgn}} : V \rightarrow \mathbb{RT}$ being an \mathbb{RT} -hypervector space homomorphism.

Lemma 2.4.3. Let $\|\cdot\|^{\text{sgn}}$ be a signed seminorm on V and $v, w \in V$.

(a) The absolute value of a signed seminorm is a seminorm (cf. Definition 1.1.1).

(b) If $\left| \|v\|^{\text{sgn}} \right| > \left| \|w\|^{\text{sgn}} \right|$, then $\|v + w\|^{\text{sgn}} = \|v\|^{\text{sgn}}$.

(c) If $\left| \|v + w\|^{\text{sgn}} \right| < \max \left\{ \left| \|v\|^{\text{sgn}} \right|, \left| \|w\|^{\text{sgn}} \right| \right\}$, then $\text{sgn}(\|v\|^{\text{sgn}}) = -\text{sgn}(\|w\|^{\text{sgn}})$.

Proof. (a) Let $\lambda \in K, v, w \in V$. We have $\left| \|\lambda v\|^{\text{sgn}} \right| = |\lambda| \cdot \left| \|v\|^{\text{sgn}} \right|$. If $\|v + w\|^{\text{sgn}} \geq 0$, then

$$\left| \|v + w\|^{\text{sgn}} \right| = \|v + w\|^{\text{sgn}} \leq \max(\|v\|^{\text{sgn}}, \|w\|^{\text{sgn}}) \leq \max(\left| \|v\|^{\text{sgn}} \right|, \left| \|w\|^{\text{sgn}} \right|).$$

The other case is obtained by multiplying with -1 and applying the other strong triangle inequality.

- (b) By (a) and the corresponding well-known statement for non-Archimedean seminorms we have:

$$\left| \|v + w\|^{\text{sgn}} \right| = \left| \|v\|^{\text{sgn}} \right|.$$

Assume that $\text{sgn}(\|v + w\|^{\text{sgn}}) = -\text{sgn}(\|v\|^{\text{sgn}})$, and after possibly taking the negative, that $\|v\|^{\text{sgn}} > 0$. Then

$$\|w\|^{\text{sgn}} \leq \max(\|v + w\|^{\text{sgn}}, \|-v\|^{\text{sgn}}) = -\|v\|^{\text{sgn}}$$

which is a contradiction.

- (c) By (b), the case $\left| \|v + w\|^{\text{sgn}} \right| < \max \left\{ \left| \|v\|^{\text{sgn}} \right|, \left| \|w\|^{\text{sgn}} \right| \right\}$ can only occur if

$$= \left| \|v\|^{\text{sgn}} \right| = \left| \|w\|^{\text{sgn}} \right|.$$

If the signs were also equal, the minimum and maximum in the strong triangle inequalities would agree, giving a contradiction. \square

Proposition 2.4.4. Let $\|\cdot\|_1^{\text{sgn}}, \|\cdot\|_2^{\text{sgn}}$ be signed seminorms on a vector space V . Then the composition

$$\begin{aligned} \|\cdot\|_1^{\text{sgn}} \circ \|\cdot\|_2^{\text{sgn}} : V &\longrightarrow \mathbb{R}, \\ v &\longmapsto \begin{cases} \|v\|_1^{\text{sgn}} & \text{if } \left| \|v\|_1^{\text{sgn}} \right| \geq \left| \|v\|_2^{\text{sgn}} \right|, \\ \|v\|_2^{\text{sgn}} & \text{else} \end{cases} \end{aligned}$$

is again a signed seminorm.

Proof. We set $\|\cdot\|^{\text{sgn}} := \|\cdot\|_1^{\text{sgn}} \circ \|\cdot\|_2^{\text{sgn}}$. Surely, for $\lambda \in K, v \in V$, we have $\|\lambda v\|^{\text{sgn}} = |\lambda| \|v\|^{\text{sgn}}$. It remains to show that $\|v + w\|^{\text{sgn}} \leq \max\{\|v\|^{\text{sgn}}, \|w\|^{\text{sgn}}\}$ for all $v, w \in V$, because we can replace all involved signed seminorms by their negative. We can assume $\|v\|^{\text{sgn}} \geq \|w\|^{\text{sgn}}$ and $\|v\|^{\text{sgn}} \geq 0$ after possibly switching v, w and multiplying by -1 . If $|\|w\|^{\text{sgn}}| > |\|v\|^{\text{sgn}}|$, by Lemma 2.4.3 (b), we have $\|v + w\|^{\text{sgn}} = \|w\|^{\text{sgn}}$ and hence

$$\|v + w\|^{\text{sgn}} = \|w\|^{\text{sgn}} \leq \max\{\|v\|^{\text{sgn}}, \|w\|^{\text{sgn}}\}.$$

If $|\|w\|^{\text{sgn}}| \leq |\|v\|^{\text{sgn}}|$, then

$$\|v + w\|^{\text{sgn}} \leq \max\{|\|v\|^{\text{sgn}}|, |\|w\|^{\text{sgn}}|\} = |\|v\|^{\text{sgn}}| = \|v\|^{\text{sgn}},$$

where the first inequality follows from the fact, that the maximum of two seminorms is again a seminorm. This completes the proof. \square

Example 2.4.5 (Diagonalizable signed seminorms). Let $B = (b_1, \dots, b_n)$ be an ordered basis of V and $\vec{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$ parameters such that $c_1 \geq \dots \geq c_n$. We may associate to this datum a map

$$\begin{aligned} \|\cdot\|_{B, \vec{c}}^{\text{sgn}} : V &\longrightarrow \mathbb{R}, \\ v = \sum_{i=0}^n \lambda_i b_i &\longmapsto \text{sgn}(\lambda_j) |\lambda_j| c_j \quad \text{if } j \text{ is minimal with } |\lambda_j| c_j = \max_{i \in [n]} |\lambda_i| c_i. \end{aligned}$$

Indeed, $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ is a signed seminorm: Let $B_j = (b_j, b_1, \dots, \hat{b}_j, \dots, b_n)$ be a reordering of B and $\vec{c}_j = (c_j, 0, \dots, 0) \in \mathbb{R}_{\geq 0}^n$. It is easy to check that $\|\cdot\|_{B_j, \vec{c}_j}^{\text{sgn}}$ is a signed seminorm. We have

$$\|\cdot\|_{B, \vec{c}}^{\text{sgn}} = \|\cdot\|_{B_1, \vec{c}_1}^{\text{sgn}} \circ \dots \circ \|\cdot\|_{B_n, \vec{c}_n}^{\text{sgn}},$$

so by Proposition 2.4.4 we obtain the result. Seminorms of the form $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ for an ordered basis $B = (b_1, \dots, b_n)$ and parameters $\vec{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$ such that $c_1 \geq \dots \geq c_n$ are called *diagonalizable*.

Example 2.4.6. To illustrate that the order of the basis B really matters, we consider $V = K^2$, $B = (e_1, e_2)$, $B' = (e_2, e_1)$, and $\vec{c} = (1, 1)$. Then $\|e_1 - e_2\|_{B, \vec{c}}^{\text{sgn}} = 1$, but $\|e_1 - e_2\|_{B', \vec{c}}^{\text{sgn}} = -1$.

In the non-signed case, if the field is spherically complete (or equivalently, maximally complete), then every seminorm on a finite-dimensional vector space is diagonalizable Proposition 1.1.5. In particular, this is true for trivially or discretely valued fields, since they are spherically complete. In fact, the reverse implication also holds: If $\dim V \geq 2$ and K is not spherically complete, then there exists a seminorm on V that is not diagonalizable. In the signed world, the situation is more complicated, as the following example will show:

Example 2.4.7. Let $K = \mathbb{R}\{\{t\}\}$, $|\cdot|_K$ the t -adic non-Archimedean absolute value on K (cf. Example 2.1.3), and $|\cdot|_{\text{triv}}$ the trivial absolute value on K . Consider $V = K^2$ and the map

$$\begin{aligned} \|\cdot\|^{\text{sgn}} : K^2 &\rightarrow \mathbb{R}, \\ (x, y) &\mapsto \begin{cases} \text{sgn}(x) & \text{if } |x|_K \geq |y|_K, \\ \text{sgn}(y) & \text{else.} \end{cases} \end{aligned}$$

Then $\|\cdot\|^{\text{sgn}}$ is a signed seminorm over K with respect to $|\cdot|_{\text{triv}}$ (note that we choose the trivial absolute value!) that is not diagonalizable: Surely, $\|\lambda v\|^{\text{sgn}} = \text{sgn}(\lambda)|\lambda|_{\text{triv}}\|v\|^{\text{sgn}}$ for all $\lambda \in K, v \in V$. Checking the triangle inequalities involves checking a lot of different cases, but is entirely straightforward and will therefore be omitted.

To see that $\|\cdot\|^{\text{sgn}}$ is not diagonalizable, assume that there exists an ordered basis $B = (b_1, b_2)$ of K^2 and $c_1 \geq c_2 \geq 0$ that diagonalizes $\|\cdot\|^{\text{sgn}}$. Then $c_1 = c_2 = 1$, since this is the only positive value that $\|\cdot\|^{\text{sgn}}$ attains and $(0, 0)$ is the only vector being mapped to 0. For $\lambda \in K$ with $0 < |\lambda|_K \ll 1$, we have

$$\|\lambda b_1 + b_2\|^{\text{sgn}} = \|b_2\|^{\text{sgn}} = \|-\lambda b_1 + b_2\|^{\text{sgn}},$$

which contradicts the assumption that $B, (c_1, c_2)$ diagonalize $\|\cdot\|^{\text{sgn}}$.

We will show in Proposition 2.7.5, that if $K = \mathbb{R}$ with trivial valuation, then every signed seminorm is diagonalizable. It remains open, if there are other real closed fields (with trivial or non-trivial valuation), for which every signed seminorm on finite-dimensional vector spaces are diagonalizable. Our proof will make crucial use of general hyperplane separation, which is only true for \mathbb{R} , but fails for all other real closed fields [Rob91].

We endow the set of signed seminorms on V with the topology of pointwise convergence. This is the coarsest topology such that all evaluation maps $\text{ev}_v : \|\cdot\|^{\text{sgn}} \mapsto \|v\|^{\text{sgn}}$ for $v \in V$ are continuous. It agrees with the subspace topology of $\mathbb{R}\mathbb{P}^{(K^{n+1})^*}$. Two signed seminorms $\|\cdot\|_1^{\text{sgn}}, \|\cdot\|_2^{\text{sgn}}$ are said to be *homothetic*, written $\|\cdot\|_1^{\text{sgn}} \sim \|\cdot\|_2^{\text{sgn}}$, if there is a constant $c \neq 0$ such that $\|\cdot\|_1^{\text{sgn}} = c\|\cdot\|_2^{\text{sgn}}$. Homothety defines an equivalence relation on the space of signed seminorms.

Definition 2.4.8. The *signed Goldman–Iwahori space* $\mathfrak{A}\mathcal{X}(V)$ is defined to be the quotient of the space of non-trivial signed seminorms on the dual space V^* by homothety, *i.e.*,

$$\mathfrak{A}\mathcal{X}(V) = (\{\|\cdot\|^{\text{sgn}} : V^* \rightarrow \mathbb{R} \text{ signed seminorm}\} \setminus \{0\}) / \sim.$$

When $V = K^{n+1}$, we write $\mathfrak{A}\mathcal{X}_n(K)$ for $\mathfrak{A}\mathcal{X}(V)$.

In Chapter 1, the (non-signed) Goldman–Iwahori space is denoted by $\overline{\mathcal{X}}(V)$, which is a compactification of the space of norms modulo homothety $\mathcal{X}(V)$. Even though $\mathfrak{A}\mathcal{X}(V)$ will turn out to be compact (Corollary 2.5.4), we omit the bar to declutter notation.

Note the dualization in Definition 2.4.8. This makes the assignment $V \mapsto \mathfrak{A}\mathcal{X}(V)$ a covariant functor from the category of finite-dimensional vector spaces over K to the category of topological spaces via pulling back signed seminorms under linear maps. For a linear map $f : V \rightarrow W$, we write $\mathfrak{A}\mathcal{X}(f) : \mathfrak{A}\mathcal{X}(V) \rightarrow \mathfrak{A}\mathcal{X}(W)$ for the induced continuous map.

2.4.2 Tropicalization of $\mathfrak{A}\mathcal{X}_n(K)$

From now on, we consider the vector space $V = K^{n+1}$ together with its defined standard basis $\mathbf{e} = (e_0, \dots, e_n)$ and the associated dual basis $\mathbf{e}^* = (e_0^*, \dots, e_n^*)$ of V^* . This identifies $\mathfrak{A}(V)$ and $\mathbb{P}(V)$ with $\mathbb{A}^{n+1} = \text{Spec } K[t_0, \dots, t_n]$ and $\mathbb{P}^n = \text{Proj } K[t_0, \dots, t_n]$, respectively.

We have a natural continuous map $\tau : \mathbb{P}_r^{n,\text{an}} \rightarrow \mathfrak{R}\mathcal{X}_n(K)$ by restricting a non-zero signed multiplicative seminorm on $K[t_0, \dots, t_n]$ to its degree 1 part $K[t_0, \dots, t_n]_1 \cong (K^{n+1})^*$. Note that this is well-defined, taking into account the respective equivalence relations: if $|\cdot|_1^{\text{sgn}} \sim |\cdot|_2^{\text{sgn}} \in \mathbb{P}_r^{n,\text{an}}$, then there is $c \neq 0$ such that for any homogeneous linear polynomial $f \in K[t_0, \dots, t_n]_1$ we have $|f|_1^{\text{sgn}} = c|f|_2^{\text{sgn}}$, hence the restrictions of $|\cdot|_1^{\text{sgn}}$ and $|\cdot|_2^{\text{sgn}}$ are homothetic.

Remark 2.4.9. While in the non-signed case the natural restriction map $\mathbb{P}^{n,\text{an}} \rightarrow \overline{\mathcal{X}}_n(K)$ was proven to be surjective in Proposition 1.2.7, it is currently not clear to the author, if the same holds in the signed case. However, we will show surjectivity in Proposition 2.7.10, for $K = \mathbb{R}$ with trivial absolute value. In that case, every signed seminorm is diagonalizable and this allows us to construct a preimage. If one could show that the subspace of diagonalizable signed seminorms of $\mathfrak{R}\mathcal{X}_n(K)$ is dense, then topological arguments would imply surjectivity of τ .

Definition 2.4.10. The *real tropicalization map* $\text{trop}_{\mathfrak{R}\mathcal{X}_n} : \mathfrak{R}\mathcal{X}_n(K) \rightarrow \mathbb{R}\mathbb{P}^n$ is given by associating to a signed seminorm $\|\cdot\|^{\text{sgn}} : (K^{n+1})^* \rightarrow \mathbb{R}$ the tuple

$$\text{trop}_{\mathfrak{R}\mathcal{X}_n}(\|\cdot\|^{\text{sgn}}) = (\|e_0^*\|^{\text{sgn}} : \dots : \|e_n^*\|^{\text{sgn}}) \in \mathbb{R}\mathbb{P}^n.$$

Note that the association in Definition 2.4.10 only depends on the homothety class of $\|\cdot\|^{\text{sgn}}$, so it indeed descends to a map $\mathfrak{R}\mathcal{X}_n(K) \rightarrow \mathbb{R}\mathbb{P}^n$. Moreover, by the definition of the topology of $\mathfrak{R}\mathcal{X}_n(K)$, the real tropicalization map $\text{trop}_{\mathfrak{R}\mathcal{X}_n}$ is continuous. Using Example 2.4.5, one can construct an inverse image for each point in $\mathbb{R}\mathbb{P}^n$ after possibly replacing some basis vectors by their negatives, hence $\text{trop}_{\mathfrak{R}\mathcal{X}_n}$ is also surjective.

Recall that the real tropicalization map is given by

$$\begin{aligned} \text{trop}_r : \mathbb{P}_r^{n,\text{an}} &\longrightarrow \mathbb{R}\mathbb{P}^n, \\ [|\cdot|^{\text{sgn}}] &\longmapsto (|t_0|^{\text{sgn}} : \dots : |t_n|^{\text{sgn}}). \end{aligned}$$

By construction, $\text{trop}_r : \mathbb{P}_r^{n,\text{an}} \rightarrow \mathbb{R}\mathbb{P}^n$ factors as

$$\mathbb{P}_r^{n,\text{an}} \xrightarrow{\tau} \mathfrak{R}\mathcal{X}_n(K) \xrightarrow{\text{trop}_{\mathfrak{R}\mathcal{X}_n}} \mathbb{R}\mathbb{P}^n.$$

Let $\iota = (f_0 : \dots : f_m) : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ be a linear embedding, where $f_i \in (K^{n+1})^*$. We define

$$\pi_\iota := \text{trop}_{\mathfrak{R}\mathcal{X}_m} \circ \mathfrak{R}\mathcal{X}(\iota) : \mathfrak{R}\mathcal{X}_n(K) \longrightarrow \mathbb{R}\mathbb{P}^m.$$

A direct computation shows that for all $[|\cdot\|^{\text{sgn}}] \in \mathfrak{R}\mathcal{X}_n(K)$, we have

$$\pi_\iota([|\cdot\|^{\text{sgn}}]) = (\|f_0\|^{\text{sgn}} : \dots : \|f_m\|^{\text{sgn}}).$$

Proposition 2.4.11. *For a linear embedding $\iota : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$, we have*

$$\text{Trop}_r(\mathbb{P}^n, \iota) = \text{trop}_{\mathfrak{R}\mathcal{X}_m}(\mathfrak{R}\mathcal{X}(\iota)(\mathfrak{R}\mathcal{X}_n(K))).$$

In particular, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}_r^{n,\text{an}} & \xrightarrow{\iota_r^{\text{an}}} & \mathbb{P}_r^{m,\text{an}} \\ \downarrow \tau & & \downarrow \tau \\ \mathfrak{R}\mathcal{X}_n(K) & \xrightarrow{\mathfrak{R}\mathcal{X}(\iota)} & \mathfrak{R}\mathcal{X}_m(K) \\ \downarrow \pi_\iota & & \downarrow \text{trop}_{\mathfrak{R}\mathcal{X}_m} \\ \text{Trop}_r(\mathbb{P}^n, \iota) & \xrightarrow{\subseteq} & \mathbb{R}\mathbb{P}^m \end{array}$$

Proof. Let \mathcal{M} be the realizable oriented valuated matroid on $\{f_0, \dots, f_m\} \subset (K^{n+1})^*$ associated to the embedding ι . By Proposition 2.2.14, $\text{Trop}_r(\mathbb{P}^n, \iota)$ equals the intersection of the real tropical hypersurfaces $\text{Trop}_r(\mathcal{M}_C)$, where C ranges over the signed valuated circuits of \mathcal{M} . Let $\|\cdot\|^{\text{sgn}}$ be a signed seminorm on $(K^{n+1})^*$ and let C be a signed valuated circuit of \mathcal{M} . Then C is associated to a minimal linear dependence $\sum_i \lambda_i f_i = 0$ among the f_i , i.e., $C = \{(\text{sgn}(\lambda_1)|\lambda_1|, \dots, \text{sgn}(\lambda_m)|\lambda_m|)\}$. We need to show that $(\|f_0\|^{\text{sgn}} : \dots : \|f_m\|^{\text{sgn}})$ lies in

$$\text{Trop}_r(\mathcal{M}_C) = \{(y_0 : \dots : y_m) \in \mathbb{R}^m : \text{there exist indices } i \neq j \text{ such that} \\ \max_{e \in \text{Supp } C} (|y_e| \cdot |C_e|) \text{ is attained at } i, j \text{ and } y_i \cdot C_i = -y_j \cdot C_j\}.$$

The strong triangle inequalities for signed seminorms imply that

$$\left| \left\| \sum_i \lambda_i f_i \right\|^{\text{sgn}} \right| \leq \max_i (|\lambda_i| \|f_i\|^{\text{sgn}}) = \max_{e \in \text{Supp } C} (|C_e| \cdot \|f_e\|^{\text{sgn}}).$$

Since $\sum_i \lambda_i f_i = 0$, the maximum is attained twice. Let $I \subseteq \text{Supp } C$ be the set of indices where the maximum $\max_{e \in \text{Supp } C} (|C_e| \cdot \|f_e\|^{\text{sgn}})$ is attained. Again by the triangle inequalities applied to $\left\| \sum_{i \in I} \lambda_i f_i \right\|^{\text{sgn}}$ it follows from Lemma 2.4.3 that there must exist indices $i, j \in I$, $i \neq j$ with $\|f_i\|^{\text{sgn}} \cdot C_i = -\|f_j\|^{\text{sgn}} \cdot C_j$. \square

2.5 The Limit Theorem

In this section, we show a linear version of the limit theorem [JSY22, Theorem 6.14] which is a real analogue of the limit theorem A. We will have a similar setup as in Section 1.3 which we now recall. We first set up a category of linear embeddings such that real tropicalization yields a covariant functor into the category of topological spaces.

Definition 2.5.1. Let I be the cofiltered category of linear embeddings $\mathbb{P}^n \hookrightarrow U \subseteq \mathbb{P}^m$, where U is a torus-invariant open subset of \mathbb{P}^m with morphisms given by commutative triangles

$$\begin{array}{ccc} \mathbb{P}^m & \xhookrightarrow{\iota} & U \\ & \searrow \iota' & \downarrow \\ & & U' \end{array}$$

where $U \rightarrow U'$ is a composition of a coordinate projection and a coordinate permutation.

Lemma 2.5.2. Let $\iota : \mathbb{P}^n \hookrightarrow U \subseteq \mathbb{P}^m$ and $\iota' : \mathbb{P}^n \hookrightarrow U' \subseteq \mathbb{P}^{m'}$ be linear embeddings and $\varphi : U \rightarrow U'$ be a morphism in I with $\varphi \circ \iota = \iota'$.

(a) The morphism φ induces a natural composition of coordinate projections and permutations

$$\varphi_r^{\text{trop}} : \text{Trop}_r(\mathbb{P}^n, \iota) \rightarrow \text{Trop}_r(\mathbb{P}^n, \iota').$$

(b) The following diagram commutes:

$$\begin{array}{ccc} \mathfrak{R}\mathcal{X}_n(K) & \xrightarrow{\pi_\iota} & \text{Trop}_r(\mathbb{P}^n, \iota) \\ & \searrow \pi_{\iota'} & \downarrow \varphi_r^{\text{trop}} \\ & & \text{Trop}_r(\mathbb{P}^n, \iota') \end{array}$$

In particular, there is a natural map

$$\varprojlim_{\iota \in I} \pi_\iota : \varprojlim_{\iota \in I} \mathfrak{R}\mathcal{X}_n(K) \rightarrow \varprojlim_{\iota \in I} \text{Trop}_r(\mathbb{P}^n, \iota).$$

Theorem 2.5.3. *The natural map*

$$\varprojlim_{\iota \in I} \pi_\iota : \mathfrak{R}\mathcal{X}_n(K) \xrightarrow{\cong} \varprojlim_{\iota \in I} \text{Trop}_r(\mathbb{P}^n, \iota)$$

from Lemma 2.5.2 is a homeomorphism.

Proof. The proof is a combination of [Pay09, Theorem 1.1], [JSY22, Theorem 6.13] and Theorem 1.3.5.

To show injectivity of $\varprojlim_{\iota \in I} \pi_\iota$, let $[\|\cdot\|_1^{\text{sgn}}], [\|\cdot\|_2^{\text{sgn}}] \in \mathfrak{R}\mathcal{X}_n(K)$ be two homothety classes such that $\varprojlim_{\iota \in I} \pi_\iota([\|\cdot\|_1^{\text{sgn}}]) = \varprojlim_{\iota \in I} \pi_\iota([\|\cdot\|_2^{\text{sgn}}])$. By the same argument as in the proof of Theorem 1.3.5 the signed seminorms $\|\cdot\|_1^{\text{sgn}}, \|\cdot\|_2^{\text{sgn}}$ have the same kernel. Let $f, g \in (K^{n+1})^*$ be outside of this kernel. We extend f, g to a generating set f, g, f_2, \dots, f_m of $(K^{n+1})^*$ and we consider the corresponding linear embedding $\iota = [f : g : f_2 : \dots : f_m] : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$. Then we have $\pi_\iota([\|\cdot\|_1^{\text{sgn}}]) = \pi_\iota([\|\cdot\|_2^{\text{sgn}}])$ as elements in $\mathbb{R}\mathbb{P}^m$ and thus the quotient of the two first coordinates equals

$$\frac{\|f\|_1^{\text{sgn}}}{\|g\|_1^{\text{sgn}}} = \frac{\|f\|_2^{\text{sgn}}}{\|g\|_2^{\text{sgn}}}.$$

This implies that $\|\cdot\|_1^{\text{sgn}}$ and $\|\cdot\|_2^{\text{sgn}}$ are homothetic.

To show surjectivity of $\varprojlim_{\iota \in I} \pi_\iota$, let $(y_j)_{j \in I} \in \varprojlim_{j \in I} \text{Trop}_r(\mathbb{P}^n, j)$. First, we consider the identity $\text{id} = [e_0^* : \dots : e_n^*] : \mathbb{P}^n \rightarrow \mathbb{P}^n$. After a permutation of coordinates we may assume that the first coordinate $y_{\text{id},0}$ of $y_{\text{id}} \in \mathbb{R}\mathbb{P}^n$ is not 0. As in the proof of Theorem 1.3.5 it follows that for all linear embeddings $\iota = [e_0^* : f_1 : \dots : f_m]$, the first coordinate $y_{\iota,0}$ is not 0. We will construct a signed seminorm $\|\cdot\|^{\text{sgn}}$ with $\|e_0^*\|^{\text{sgn}} = 1$ and $\pi_j(\|\cdot\|^{\text{sgn}}) = y_j$ for all $j \in I$. Let $f \in (K^{n+1})^*$. We choose an embedding $j = [e_0^* : f : f_2 : \dots : f_m] : \mathbb{P}^n \rightarrow \mathbb{P}^m$ and define

$$\|f\|^{\text{sgn}} := \frac{y_{j,1}}{y_{j,0}}.$$

By the same argument as in the proof of Theorem 1.3.5 using coordinate projections and permutations, this definition does not depend on the choice of j .

We check that the constructed map is indeed a signed seminorm. For $f \in (K^{n+1})^*$ and $\lambda \in K$ consider any embedding $j = [e_0^* : f : \lambda f : \dots]$. Then, by Proposition 2.4.11, for every $y_j \in \text{Trop}_r(\mathbb{P}^n, j)$ there is a class of a signed seminorm $[\|\cdot\|_1^{\text{sgn}}] \in \mathfrak{R}\mathcal{X}_n(K)$ with

$$\begin{aligned} y_j &= \pi_j([\|\cdot\|_1^{\text{sgn}}]) \\ &= (\text{trop}_{\mathfrak{R}\mathcal{X}_m} \circ \mathfrak{R}\mathcal{X}(j))([\|\cdot\|_1^{\text{sgn}}]) \\ &= [\|e_0^*\|_1^{\text{sgn}} : \|f\|_1^{\text{sgn}} : \|\lambda f\|_1^{\text{sgn}} : \dots] \end{aligned}$$

and thus $\text{sgn}(\lambda)|\lambda|y_{j,1} = y_{j,2}$. Therefore,

$$\|\lambda f\|^{\text{sgn}} = \frac{y_{j,2}}{y_{j,0}} = \frac{\text{sgn}(\lambda)|\lambda|y_{j,1}}{y_{j,0}} = \text{sgn}(\lambda)|\lambda| \|f\|^{\text{sgn}}.$$

For $f, g \in (K^{n+1})^*$, the inequalities $\min\{\|f\|^{\text{sgn}}, \|g\|^{\text{sgn}}\} \leq \|f+g\|^{\text{sgn}} \leq \max\{\|f\|^{\text{sgn}}, \|g\|^{\text{sgn}}\}$ follow similarly by considering an embedding containing f, g and $f+g$. By construction, the signed seminorm $\|\cdot\|^{\text{sgn}}$ is an inverse image of $(y_j)_{j \in I}$.

Finally, as in the proof of [JSY22, Theorem 6.13], the map is a homeomorphism because the topology on the left is defined as the coarsest topology such that $\|\cdot\|^{\text{sgn}} \mapsto \|f\|^{\text{sgn}}$ is continuous for all $f \in (K^{n+1})^*$, while on the right the topology is defined such that all projection maps, that is, all maps $\varprojlim_{\iota \in I} \text{Trop}_r(\mathbb{P}^n, \iota) \rightarrow \text{Trop}_r(\mathbb{P}^n, \iota)$ to a particular ι are continuous. These conditions are equivalent. \square

Corollary 2.5.4. *The signed Goldman–Iwahori space $\mathfrak{R}\mathcal{X}_n(K)$ is compact and Hausdorff.*

Proof. This follows from Theorem 2.5.3 since all $\mathrm{Trop}_r(\mathbb{P}^n, \iota)$ are compact and Hausdorff, hence the inverse limit is a closed subspace of the compact product space and is thus itself compact and Hausdorff. \square

Remark 2.5.5. Let K be trivially valued. Then all real tropicalized linear spaces $\mathrm{Trop}_r(\mathbb{P}^n, \iota)$ are homeomorphic to $\mathbb{R}\mathbb{P}^n$, see Corollary 2.8.13 and Proposition 2.8.14. One might suspect that the limit space $\mathfrak{R}\mathcal{X}_n(K)$ is also homeomorphic to $\mathbb{R}\mathbb{P}^n$. However, in the category I there are coordinate projections, which do not induce homeomorphisms between the respective real tropicalizations. Hence, one cannot conclude that $\mathfrak{R}\mathcal{X}_n(K)$ is homeomorphic to $\mathbb{R}\mathbb{P}^n$.

2.5.1 Relation to the Goldman–Iwahori Space

Recall that the *Goldman–Iwahori space* $\overline{\mathcal{X}}_n(K)$ is the space of non-trivial (ordinary) seminorms on $(K^{n+1})^*$ modulo homothety. It has been studied first by Goldman and Iwahori [GI63] (also see for more recent accounts [Wer04, RTW12, RTW15]). However, our notation will follow Chapter 1, where we proved that $\overline{\mathcal{X}}_n(K)$ is the limit of all linear tropicalizations of \mathbb{P}^n . The *compactified affine Bruhat–Tits building* $\overline{\mathcal{B}}_n(K)$ of $\mathrm{PGL}((K^{n+1})^*)$ is the subspace of $\overline{\mathcal{X}}_n(K)$ of classes of *diagonalizable* seminorms. The inclusion $\overline{\mathcal{B}}_n(K) \subseteq \overline{\mathcal{X}}_n(K)$ is dense by Remark 1.1.9 and we have an equality if and only if K is *spherically complete* by [Kru32, Satz 24].

The association $\|\cdot\|^{\mathrm{sgn}} \mapsto |\|\cdot\|^{\mathrm{sgn}}|$ defines a natural map

$$\Phi : \mathfrak{R}\mathcal{X}_n(K) \longrightarrow \overline{\mathcal{X}}_n(K).$$

Since morphisms in I commute with taking pointwise absolute values, we obtain the following:

Theorem 2.5.6. *For any morphism $\varphi \in I$ between linear embeddings ι, ι' of \mathbb{P}^n , the following diagram commutes:*

$$\begin{array}{ccccc} & & \pi_{\iota'} & & \\ & & \curvearrowright & & \\ \mathfrak{R}\mathcal{X}_n(K) & \xrightarrow{\pi_{\iota}} & \mathrm{Trop}_r(\mathbb{P}^n, \iota) & \xrightarrow{\varphi_r^{\mathrm{trop}}} & \mathrm{Trop}_r(\mathbb{P}^n, \iota') \\ & \downarrow \Phi & \downarrow & & \downarrow \\ \overline{\mathcal{X}}_n(K) & \xrightarrow{\pi_{\iota}} & \mathrm{Trop}(\mathbb{P}^n, \iota) & \xrightarrow{\varphi^{\mathrm{trop}}} & \mathrm{Trop}(\mathbb{P}^n, \iota') \\ & & \pi_{\iota'} & & \end{array}$$

In particular, Φ is exactly the map induced by all $|\pi_{\iota}| : \mathfrak{R}\mathcal{X}_n(K) \rightarrow \mathrm{Trop}(\mathbb{P}^n, \iota)$.

Proposition 2.5.7. *The map $\Phi : \mathfrak{R}\mathcal{X}_n(K) \longrightarrow \overline{\mathcal{X}}_n(K)$ is surjective.*

Proof. The Goldman–Iwahori space is a compact Hausdorff space, since it is the limit of compact Hausdorff spaces. Moreover, so is the signed Goldman–Iwahori space by Corollary 2.5.4. Via Example 2.4.5, we construct preimages for all classes $[|\cdot|] \in \overline{\mathcal{B}}_n(K)$, hence the image of Φ contains $\overline{\mathcal{B}}_n(K)$. Since the image of Φ is compact, it needs to be closed, thus density of $\overline{\mathcal{B}}_n(K)$ implies surjectivity of Φ . \square

In general, the fibers of Φ , though non-empty, can vastly differ in complexity. In Section 2.7, we will consider the trivially valued case $K = \mathbb{R}$ and show that the fibers can be singletons, finite sets, or even infinite.

2.5.2 Towards a Generalization to Tropical Extensions of Hyperfields

Recently, there have been several developments that aim to generalize tropical geometry to hyperfields. In particular, we highlight the geometry of tropical extensions of hyperfields or, more generally, of tracts [MS23, Max24, Smi24]. Using this language, we can unify the Goldman–Iwahori space $\overline{\mathcal{X}}_n$ and the signed Goldman–Iwahori space $\mathfrak{R}\mathcal{X}_n$ such that those two spaces become special cases for the tropical hyperfield \mathbb{T} and the real tropical hyperfield \mathbb{RT} .

Recall that a non-Archimedean absolute value on a field K is the same as a hyperfield homomorphism $K \rightarrow \mathbb{T}$. In [MS23, §2.1], the authors study certain generalizations of such hyperfield homomorphisms, called *enriched valuations*. This is a morphism of hyperfields $v : K \rightarrow \mathbb{H}[\mathbb{R}]$ from a field K to the hyperfield $\mathbb{H}[\mathbb{R}]$ which is a *tropical extension* (see [MS23, Def. 2.6]) of a hyperfield \mathbb{H} that encodes additional \mathbb{H} -information about the field element. We give three natural examples to illustrate this concept:

- (i) For a non-Archimedean valued field, the absolute value $K \rightarrow \mathbb{T} = \mathbb{K}[\mathbb{R}]$ is an enriched valuation where $\mathbb{T} = \mathbb{K}[\mathbb{R}]$ is the tropical extension of the Krasner hyperfield \mathbb{K} .
- (ii) The signed absolute value $K \rightarrow \mathbb{RT} = \mathbb{S}[\mathbb{R}]$, $x \mapsto \text{sgn}(x) \cdot |x|$ also encodes the sign of an element of an ordered non-Archimedean valued field K .
- (iii) Let k be a field and let $k[[t^{\mathbb{R}}]]$ the field of Hahn series over k . We can enrich the usual valuation defining the *fine valuation*

$$\text{fval} : k[[t^{\mathbb{R}}]] \rightarrow k[\mathbb{R}],$$

that maps a Hahn series to its leading term. Hence, it remembers the leading coefficient as well as the valuation of a series.

Definition 2.5.8. Let K be a field and $v : K \rightarrow \mathbb{H}[\mathbb{R}]$ be an enriched valuation.

- (a) A *v*-*seminorm* on a K -vector space V is a map $\|\cdot\| : V \rightarrow \mathbb{H}[\mathbb{R}]$ such that for all $w, w' \in V, \lambda \in K$ we have:

$$\begin{aligned} \|w + w'\| &\in \|w\| \oplus \|w'\|, \\ \|\lambda w\| &= v(\lambda) \|w\|. \end{aligned}$$

- (b) The *v*-*Goldman–Iwahori space* $\mathcal{X}_n(K, v)$ is the space of non-trivial *v*-seminorms on $(K^{n+1})^*$ modulo $(\mathbb{H}[\mathbb{R}])^*$. If $\mathbb{H}[\mathbb{R}]$ has a topology, we consider $\mathcal{X}_n(K, v)$ as equipped with the topology of pointwise convergence.

We note that this generalizes the construction of the (signed) Goldman–Iwahori space, which use the enriched valuations (i) and (ii), respectively. We will now deduce a limit theorem for example (iii), *i.e.*, $K = k[[t^{\mathbb{R}}]]$ and $v = \text{fval}$, which will be fixed from now on.

Definition 2.5.9. Let $\iota : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ be a linear embedding. Then the *fval-tropicalization* is given by

$$\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota) = \text{fval}(\iota(\mathbb{P}^n)) \subseteq \mathbb{P}_{k[\mathbb{R}]}^m.$$

With a similar argument as in (i) and (ii) for valuated matroids and valuated oriented matroids, by [Smi24, Prop. 5.12], $\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota)$ is determined by the associated $k[\mathbb{R}]$ -matroid $\mathcal{M}_\iota := \text{fval}_*(\iota(\mathbb{P}^n))$. Explicitly, $\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota)$ is orthogonal to the $k[\mathbb{R}]$ -circuits of \mathcal{M}_ι . Therefore, $\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota)$ is an *enriched tropical linear space* under the fine valuation (see [Smi24, Definition 5.8]).

The key insight is that for each linear embedding $\iota = (f_0 : \dots : f_m) : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ we can

construct a natural map

$$\begin{aligned} \pi_\iota : \mathcal{X}_n(k[[t^{\mathbb{R}}]], \text{fval}) &\longrightarrow \text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota) \\ [|\cdot|] &\longmapsto (||f_0|| : \cdots : ||f_m||), \end{aligned}$$

as in Section 2.4.2. Well-definedness follows analogously to Proposition 2.4.11 by using the description of the enriched tropicalized linear space $\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota)$ via the $k[[\mathbb{R}]]$ -circuits of \mathcal{M}_ι . This makes crucial use of [Smi24, Proposition 5.12].

The following can be proven similarly to Theorems A and 2.5.3:

Theorem 2.5.10. *The natural maps π_ι induce a homeomorphism*

$$\varprojlim_{\iota \in I} \pi_\iota : \overline{\mathcal{X}}_n(k[[t^{\mathbb{R}}]], \text{fval}) \longrightarrow \varprojlim_{\iota \in I} \text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota).$$

In other words, the fval-Goldman–Iwahori space $\overline{\mathcal{X}}_n(k[[t^{\mathbb{R}}]], \text{fval})$ is the limit of all enriched tropicalized linear spaces $\text{Trop}_{\text{fval}}(\mathbb{P}^n, \iota)$ for linear embeddings $\iota : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$.

It would be interesting to study in a similar spirit this fval-Goldman–Iwahori space as a universal enriched tropical linear space for the fine valuation. Moreover, one could study limit theorems for Goldman–Iwahori spaces with other enriched valuations.

2.6 The Universal Realizable Oriented Valuated Matroid

We may extend the definition of a matroid over a hyperfield (Definition 2.2.3) to infinite ground sets. This generalization to infinite ground sets works as in Sections 1.4 and 1.7.1. There, this extension is done for the tropical hyperfield \mathbb{T} , *i.e.*, for valuated matroids of finite rank on possibly infinite ground sets. In this section, we will do the same for the real tropical hyperfield \mathbb{RT} .

Let E be now a possibly infinite ground set and \mathcal{M} be an oriented valuated matroid given by a Grassmann–Plücker function $\varphi : E^n \rightarrow \mathbb{RT}$. Note that the rank of \mathcal{M} is n , so it is in particular finite. We define $\mathbb{RP}^E := \{(y_e)_{e \in E} | y_e \in \mathbb{R}\} \setminus \{(0)_{e \in E}\} / \mathbb{R}^*$ and equip it with the topology of pointwise convergence. Analogous to Definition 2.2.11 we define:

Definition 2.6.1. The *real tropical linear space* $\text{Trop}_r(\mathcal{M}) \subseteq \mathbb{RP}^E$ associated to \mathcal{M} is the set of $(y_e)_{e \in E} \in \mathbb{RP}^E$ such that for any $C = (f_0, \dots, f_{n+1}) \in E^{n+2}$ we have that

$$0 \in \bigoplus_{i=0}^{n+1} (-1)^i y_{f_i} \cdot \varphi(C - f_i).$$

Recall that \oplus denotes the hyperfield sum of the real tropical hyperfield $\mathbb{RT} = \mathbb{R}$.

Let K be a real closed field with a compatible absolute value $|\cdot|_K$. We now extend the construction of realizable oriented valuated matroids from Example 2.2.10 to the ground set $E = (K^{n+1})^*$.

Definition 2.6.2. Let $E = (K^{n+1})^*$. The *universal realizable oriented valuated matroid* or *universal \mathbb{RT} -matroid* $\mathcal{M}_{\text{univ}}$ of rank $n + 1$ is given by the Grassmann–Plücker function

$$\begin{aligned} \varphi_{\text{univ}} : E^{n+1} &\longrightarrow \mathbb{RT}, \\ (f_0, \dots, f_n) &\longmapsto \text{sgn}(\det[f_0, \dots, f_n]) \cdot |\det[f_0, \dots, f_n]|. \end{aligned}$$

By considering $-\log |\varphi_{\text{univ}}|$, we obtain the universal realizable valuated matroid $w_{\text{univ}} = \text{val} \circ \det$ studied in Chapter 1. Hence, in contrast to w_{univ} , the map φ_{univ} also keeps track of the signs.

Note that both the signed Goldman–Iwahori space and the real tropical linear space associated to the universal realizable oriented valuated matroid are defined to be subsets of \mathbb{RP}^E .

Theorem 2.6.3 (Theorem B). *The signed Goldman–Iwahori space is the real tropical linear space associated to the universal realizable oriented valuated matroid $\mathcal{M}_{\text{univ}}$, i.e.*

$$\mathfrak{RX}_n(K) = \text{Trop}_r(\mathcal{M}_{\text{univ}}).$$

Proof. Consider a signed seminorm $\|\cdot\|^{\text{sgn}}$ on $(K^{n+1})^*$. To show that $[\|\cdot\|^{\text{sgn}}] \in \text{Trop}_r(\mathcal{M}_{\text{univ}})$, take $C = (f_0, \dots, f_{n+1}) \in E^{n+2}$. Then, by an application of Cramer’s rule

$$\sum_i (-1)^i \det(C - f_i) f_i = 0.$$

We denote by $\lambda_i := (-1)^i \det(C - f_i)$. Then

$$0 = \left\| \sum_i \lambda_i f_i \right\|^{\text{sgn}} \in \bigoplus_i \text{sgn}(\lambda_i) |\lambda_i| \cdot \|f_i\|^{\text{sgn}}.$$

By construction, $\text{sgn}(\lambda_i) |\lambda_i| = (-1)^i \varphi_{\text{univ}}(C - f_i)$. This shows that $[\|\cdot\|^{\text{sgn}}] \in \text{Trop}_r(\mathcal{M}_{\text{univ}})$. Conversely, using similar methods as in Section 1.7.1, one can see that the circuit conditions of $(y_e)_{e \in E}$ being in $\text{Trop}_r(\mathcal{M}_{\text{univ}})$ imply that $\|e\|^{\text{sgn}} = y_e$ is a signed seminorm. \square

Remark 2.6.4. (a) Theorem 2.6.3 is a signed analogue of Theorem C which gives an identification $\overline{\mathcal{X}}_n(K) = \text{Trop}(w_{\text{univ}})$.

(b) Theorem 2.6.3 has a compelling interpretation in light of the limit Theorem 2.5.3. For any finite subset $E' \subseteq E$ containing a basis, we can restrict $\mathcal{M}_{\text{univ}}$ to E' and we have a natural surjective map $\text{Trop}_r(\mathcal{M}_{\text{univ}}) \rightarrow \text{Trop}_r(\mathcal{M}_{\text{univ}}|_{E'})$. We have

$$\text{Trop}_r(\mathcal{M}_{\text{univ}}) \cong \varprojlim_{\substack{E' \subseteq E \\ |E'| < \infty}} \text{Trop}_r(\mathcal{M}_{\text{univ}}|_{E'})$$

where the limit is taken over all finite generating subsets E' of E . Hence, one could alternatively prove Theorem 2.6.3 via Theorem A and Proposition 2.2.14.

(c) Theorem 2.6.3 gives an interpretation of $\mathfrak{RX}_n(K)$ as a real tropicalized linear space in the following way: Consider for $E = (K^{n+1})^*$ the *universal embedding*

$$\iota_{\text{univ}} : \mathbb{P}^n \hookrightarrow \mathbb{P}(K^E).$$

Then the associated oriented valuated matroid is of course $\mathcal{M}_{\iota_{\text{univ}}} = \mathcal{M}_{\text{univ}}$. Hence, $\mathfrak{RX}_n(K)$ can be seen as the real tropicalization of \mathbb{P}^n with respect to ι_{univ} .

2.7 The Case of Real Numbers with Trivial Valuation

For the case that $K = \mathbb{R}$ with trivial valuation, it will turn out that all signed seminorms are diagonalizable. This lets us describe the signed Goldman–Iwahori space explicitly over this field. Moreover, it allows us to describe the real tropical linear space of $\mathcal{M}_{\text{univ}}$ as a *real Bergman fan*.

2.7.1 Diagonalizability of Signed Seminorms

We want to describe $\mathfrak{RX}(V)$ explicitly by making use of the map $\Phi : \mathfrak{RX}(V) \rightarrow \overline{\mathcal{B}}(V)$, which assigns to each signed seminorm its absolute value. Therefore, we quickly recall an explicit description of $\overline{\mathcal{B}}(V)$ in terms of flags of subspaces. The following holds for any trivially valued field K :

Proposition 2.7.1 (Example 1.1.10). *Let the dimension of V be n . There is a bijection*

$$\overline{\mathcal{B}}(V) \xrightarrow{1:1} \{(0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V^*, 0 < d_1 < \cdots < d_{l-1} < 1), d_i \in \mathbb{R}\}_{l=1, \dots, n}.$$

In other words, a seminorm is given by a flag of subspaces together with increasing real weights.

The bijection works the following way: for a homothety class of seminorms we choose the representative $\|\cdot\|$ that has maximal value 1 on V . Then one obtains the flag of subspaces as subsets $\|\cdot\|^{-1}(0, \varepsilon)$ by letting ε vary. The kernel of $\|\cdot\|$ then equals V_0 and the coordinates d_i are given by the constant value of $\|\cdot\|$ on $V_i \setminus V_{i-1}$ for $i = 1, \dots, l-1$. In particular, $\overline{\mathcal{B}}(V)$ can be identified with the compactified cone over the order complex of the lattice of non-trivial subspaces of V^* .

Example 2.7.2. In the building $\overline{\mathcal{B}}_1(K)$ homothety classes of seminorms correspond to flags of subspaces of $(K^2)^*$ together with a single coordinate $0 < d < 1$. Each cone corresponds to a one-dimensional subspace V_1 and the point at infinity of each cone corresponds to the homothety class of a proper seminorm, as visualized in Figure 2.3. A norm in the homothety class corresponding to (V_1, d) has generic value 1, and value d on $V_1 \setminus \{0\}$. A boundary point is given by a subspace V_0 which is the kernel of a proper seminorm. Hence in this particular case of rank one, the boundary equals $\mathbb{P}^1(K)$. The central point η corresponds to the class of the seminorm that takes value 1 everywhere except at 0.

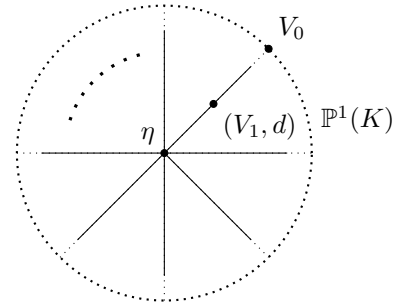


Figure 2.3: The building $\overline{\mathcal{B}}_1(K)$.

From now on, we again consider $K = \mathbb{R}$. Preimages of intervals in the signed case are no longer subspaces, but only convex cones:

Lemma 2.7.3. *Let $\|\cdot\|^{sgn}$ be a signed seminorm on V and $[a, b]$ be a closed interval in \mathbb{R} . Then $L = (\|\cdot\|^{sgn})^{-1}([a, b]) \cup \{0\}$ is a convex cone in V . Moreover, if $b \geq 0$ and $a = -b$, then L is a subspace.*

Proof. Surely for $\lambda \in \mathbb{R}_{>0}$ and $v \in V$, we have $\|\lambda v\|^{sgn} = \lambda \|v\|^{sgn}$, so the preimage of an interval is a cone. To see convexity, let $v, w \in L, t \in [0, 1]$. Then

$$\|(1-t)v + tw\| \leq \max\{\|(1-t)v\|, \|tw\|\} = \max\{\|v\|, \|w\|\} \leq b.$$

Similarly,

$$\|(1-t)v + tw\| \geq \min\{\|(1-t)v\|, \|tw\|\} = \min\{\|v\|, \|w\|\} \geq a.$$

If $b \geq 0$ and $a = -b$, then surely also $v \in L$ implies $-v \in L$, so with the above L is also a subspace. \square

Lemma 2.7.4. *Let $0 \neq \|\cdot\|^{sgn}$ be a signed seminorm on V .*

(a) *The image of $\|\cdot\|^{sgn}$ is finite.*

(b) *For $a = \max_{v \in V} \|v\|^{sgn}$, we have that $A := \{v \in V \mid \|v\|^{sgn} = a\}$ is a convex cone.*

(c) *There exists $w \in V^*$ such that*

$$\begin{aligned} A^\circ &= \{v \in V \mid \langle v, w \rangle > 0\}, \\ \overline{A} &= \{v \in V \mid \langle v, w \rangle \geq 0\} \text{ and} \\ \partial A &= w^\perp \end{aligned}$$

where $A^\circ/\bar{A}/\partial A$ denote the interior/closure/boundary of A in V with respect to the Euclidean topology.

- (d) A signed seminorm $\|\cdot\|^{sgn}$ induces a well-defined signed seminorm on the quotient $V/(\ker \|\cdot\|^{sgn}) \rightarrow \mathbb{R}$.
- (e) The restriction of $\|\cdot\|^{sgn}$ to any subspace is again a signed seminorm.

Proof. (a) Follows immediately from Proposition 2.7.1.

(b) Follows from Lemma 2.7.3 applied to the interval $[a - \varepsilon, a + \varepsilon]$ for a sufficiently small $\varepsilon > 0$.

(c) The second equality of sets follows from the first one. Since A is convex, so is $-A$. By the separation Lemma [SW12, 3.3.9], there is a hyperplane H separating A and $-A$. Let $w \in V^*$ such that $H = w^\perp$ and $A \subseteq \{v \in V \mid \langle v, w \rangle \geq 0\}$. Then $A^\circ \subseteq \{v \in V \mid \langle v, w \rangle > 0\}$. From Proposition 2.7.1, one can deduce the other inclusion.

We omit the proofs of (d) and (e) as these are straightforward. \square

We note that $K = \mathbb{R}$ is the only real closed field that admits the separation Lemma [SW12, 3.3.9]. Thus for an arbitrary real closed field K we cannot expect a hyperplane $\langle \cdot, w \rangle = 0$ separating A and $-A$ to exist.

This now allows us to prove, that every signed seminorm is diagonalizable:

Proposition 2.7.5. *Let $K = \mathbb{R}$. Every signed seminorm $\|\cdot\|^{sgn}$ on V is of the form $\|\cdot\|_{B, \vec{c}}^{sgn}$ for an ordered basis $B = (b_1, \dots, b_n)$ and parameters $c_1 \geq c_2 \geq \dots \geq c_n \in \mathbb{R}_{\geq 0}$ (cf. Example 2.4.5).*

Proof. The trivial signed seminorm is diagonalizable by any basis setting $\vec{c} = 0$. Let $\|\cdot\|^{sgn}$ be non-trivial. As in Lemma 2.7.4, let $a = \max_{v \in V} \|a\|^{sgn}$, $A := \{v \in V \mid \|v\|^{sgn} = a\}$, and $H = \partial A$. Set b_1 to be any vector in A° and $c_1 := a$. By Lemma 2.7.4 (d), the restriction $\|\cdot\|^{sgn}|_H$ is a signed seminorm and therefore, by induction, it is diagonalizable with $b_2, \dots, b_n \in H$, and $c_2, \dots, c_n \in \mathbb{R}$ with $c_2 \geq \dots \geq c_n$. Then (b_1, \dots, b_n) is a basis of V , by definition $c_1 \geq c_2 \geq \dots \geq c_n$, and for any $v = \sum_{i=1}^n \lambda_i b_i$ we have that $v \in H$ if and only if $\lambda_1 = 0$. Moreover, we have $v \in A$ if and only if $\lambda_1 > 0$, and $v \in -A$ if and only if $\lambda_1 < 0$. In all three cases we have $\|v\|^{sgn} = \|v\|_{B, \vec{c}}^{sgn}$. \square

Remark 2.7.6. The key ingredient for Proposition 2.7.5 is hyperplane separation for general convex sets, which, as noted above, only holds over \mathbb{R} . It is currently not clear to the author, if there are other real fields, trivially or non-trivially valued, for which every signed seminorm is diagonalizable.

Diagonalizability gives us a description of $\mathfrak{R}\mathcal{X}(V)$ in the flavor of Proposition 2.7.1.

Definition 2.7.7. A *signed flag* of subspaces is given by the data of

- a flag $0 \subseteq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l = V$ of subspaces such that for all $i = 1, \dots, l$ we have $\dim V_{i-1} = \dim V_i - 1$, and
- a choice of a convex region in $V_i \setminus V_{i-1}$ for all $i = 1, \dots, l$.

We consider two signed flags *equivalent*, if the underlying flag of subspaces is the same and the choice of region is exactly opposite for all i . The notion of signed flags helps us to formulate a signed analogue of Proposition 2.7.1.

Proposition 2.7.8. *Let V be of dimension n . There is a bijection*

$$\mathfrak{R}\mathcal{X}(V) \xrightarrow{1:1} \left\{ (0 \subseteq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l = V^*, 0 < d_1 \leq \dots \leq d_{l-1} \leq 1) \right\},$$

where the right hand side runs over all equivalence classes of signed flags.

Proof. For a point in $\mathfrak{R}\mathcal{X}(V)$, we first choose a representative $\|\cdot\|^{\text{sgn}}$ such that its maximal value is 1. Now, by Proposition 2.7.5 $\|\cdot\|^{\text{sgn}} = \|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ for an ordered basis $B = (b_1, \dots, b_n)$ and $\vec{c} = (c_1 = 1, c_2, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$ with $c_i \geq c_{i-1}$ for all i . Let $i_0 = \min\{i = 1, \dots, n+1 \mid c_i = 0\}$, where we formally set $c_{n+1} = 0$. We define a signed flag by setting $V_0 = \langle b_{i_0}, \dots, b_n \rangle$, $V_i = \langle b_{i_0-i}, \dots, b_n \rangle$, and chose the region of V_i that contains b_i . Note that both the choice of the other representative or of a different diagonalizing basis would yield an equivalent signed flag. Moreover, \vec{c} is independent of both choices. Finally, we set $d_i := c_{i_0-i}$ for $i = 1, \dots, i_0 - 2$ to obtain the desired data.

The above procedure is reversible by successively constructing a basis and the homothety class of the resulting signed seminorm does not depend on the choice of the constructed basis. \square

Together, Proposition 2.7.1 and Proposition 2.7.8 allow us to study the natural map $\Phi : \mathfrak{R}\mathcal{X}(V) \rightarrow \overline{\mathcal{B}}(V)$. We want to understand $\Phi([\|\cdot\|^{\text{sgn}}])$ for some $[\|\cdot\|^{\text{sgn}}] \in \mathfrak{R}\mathcal{X}(V)$. Let $0 \subseteq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l = V^*$ be the corresponding signed flag and $0 \leq d_1 \leq \dots \leq d_{l-1} \leq 1$ constructed as in Proposition 2.7.8. Let $i_0 := 0$ and $0 < i_1 \leq \dots \leq i_l < l$ be the indices such that there is a strict inequality $d_{i_j} < d_{i_{j+1}}$ (here we set $d_l := 1$). Set $V'_j := V_{i_j}$ for $j = 1, \dots, l$ and $d'_j = d_{i_j}$. Then $\Phi([\|\cdot\|^{\text{sgn}}])$ is given by the flag $0 = V'_0 \subsetneq V'_1 \subsetneq \dots \subsetneq V'_l \subsetneq V'_{l+1} := V^*$ and coordinates $0 \leq d'_1 < \dots < d'_l < 1$ as in Proposition 2.7.1.

A natural question to ask is: what do the fibers of Φ look like? This has an easy answer, if the flag that is induced by the homothety class of a seminorm is complete, *i.e.* jumps of subspaces are only by one dimension. In that case, all preimages are given by the same coordinates d_i and any choice of signature on the flag yields a signed seminorm, hence the fiber under Φ has exactly 2^{n-1} elements. The situation gets more complicated, if the flag is not complete, as the following example will show.

Example 2.7.9. Consider the case $V = \mathbb{R}^2$. As explained in Example 2.7.2, there are three types of points of $\overline{\mathcal{B}}(V)$:

- (a) the homothety class η of the constant norm,
- (b) classes of proper norms with 2 different non-zero values,
- (c) classes of proper seminorms (*i.e.*, seminorms with non-trivial kernel).

If K was algebraically closed, we would have $\overline{\mathcal{B}}_1(K) = \mathbb{P}^{1, \text{an}}$ and these would exactly correspond to Berkovich type II, type III, respectively type I points of the space. However, since \mathbb{R} not algebraically closed and thus $\overline{\mathcal{B}}_1(\mathbb{R}) \subsetneq \mathbb{P}^{1, \text{an}}$.

As discussed before, if a point in $\overline{\mathcal{B}}_1(\mathbb{R})$ is of type (b), then its fiber consists of two points. If the point is of type (c), then there is only one preimage, as the two choices of signed flags are equivalent. In contrast, the fiber of η is quite large. For every line $V_1 \subset \mathbb{R}^2$, there are two equivalence classes of signed flags with underlying flag $0 \subsetneq V_1 \subsetneq \mathbb{R}^2$. Hence, there is a (non-canonical) bijection

$$\Phi^{-1}(\eta) \cong \mathbb{P}^1(\mathbb{R}) \sqcup \mathbb{P}^1(\mathbb{R}).$$

Note that this bijection is not continuous, so it does not tell us anything about the topology. To find out more about the topology of this fiber we refer to [JSY22, Example 3.12], since in the special case of dimension 2, one can show that $\overline{\mathcal{B}}_1(\mathbb{R})$ is the set of real points of $\mathbb{P}^{1, \text{an}}$ and $\mathfrak{R}\mathcal{X}_1(\mathbb{R})$ equals $\mathbb{P}_r^{1, \text{an}}$. In Figure 2.4 we sketch the map Φ . Note that because of the infinite nature of the spaces, the topology on both $\mathfrak{R}\mathcal{X}_1(\mathbb{R})$ and $\overline{\mathcal{B}}_1(\mathbb{R})$ is coarser than the figure might lead us to believe. For example, the space $\mathfrak{R}\mathcal{X}_1(\mathbb{R})$ is connected (this follows from 2.7.10), even though as a set it is a disjoint union of closed intervals.

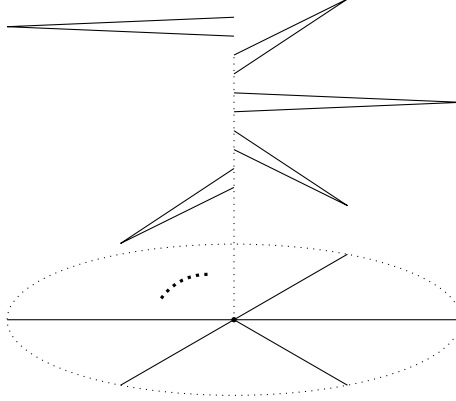


Figure 2.4: The map $\Phi : \mathfrak{R}\mathcal{X}_1(\mathbb{R}) \rightarrow \overline{\mathcal{B}}_1(\mathbb{R})$.

Proposition 2.7.10. *The restriction map $\tau : \mathbb{P}_r^{n,\text{an}} \rightarrow \mathfrak{R}\mathcal{X}_n(\mathbb{R})$ is surjective.*

Proof. Let $[\|\cdot\|^{\text{sgn}}]$ be the class of a non-trivial signed seminorm on $(\mathbb{R}^{n+1})^*$. Then by Proposition 2.7.5 there is an ordered basis $B = (b_0, \dots, b_n)$ and $\vec{c} \in \mathbb{R}^{n+1}$ with $c_0 \geq c_1 \geq \dots \geq c_n \in \mathbb{R}_{\geq 0}$ such that $\|\cdot\|^{\text{sgn}} = \|\cdot\|_{B, \vec{c}}^{\text{sgn}}$. After scaling \vec{c} , we can assume $c_n = 1$.

We define a signed multiplicative seminorm $|\cdot|^{\text{sgn}}$ on $\mathbb{R}[t_0, \dots, t_n]$ as follows. After a coordinate change we can write an element $f \in \mathbb{R}[t_0, \dots, t_n]$ as

$$f = \sum_{I=(i_0, \dots, i_n)} a_I b_0^{i_0} \cdot \dots \cdot b_n^{i_n}.$$

Using the lexicographic order \geq_{lex} we define a monomial order as follows. For $I, J \in \mathbb{N}^n$ we define $I \geq_{\vec{c}} J$ if the two conditions are satisfied:

- $c^I \geq c^J$
- $c^I = c^J$ implies $I \geq_{\text{lex}} J$.

Let I_0 be the leading monomial of f with respect to this order and $a_{I_0} b^{I_0}$ the corresponding leading term. We define $|f|^{\text{sgn}} = \text{sgn}(a_{I_0}) c^{I_0}$. This is a signed multiplicative seminorm on $\mathbb{R}[t_0, \dots, t_n]$. Indeed, if $f \in \mathbb{R}$, then $|f|^{\text{sgn}} = \text{sgn}(f) = \text{sgn}(f) \cdot |f|_{\text{triv}}$. Let $f, g \in \mathbb{R}[t_0, \dots, t_n]$ and write $f = \sum_{I=(i_0, \dots, i_n)} a_I b_0^{i_0} \cdot \dots \cdot b_n^{i_n}$ and $g = \sum_{I=(i_0, \dots, i_n)} a'_I b_0^{i_0} \cdot \dots \cdot b_n^{i_n}$. Let $a_{I_0} b^{I_0}$ and $a'_{I_1} b^{I_1}$ be the corresponding leading terms. Since $\geq_{\vec{c}}$ is a monomial order the leading term of fg is $a_{I_0} a'_{I_1} b^{I_0 + I_1}$. This shows $|fg|^{\text{sgn}} = |f|^{\text{sgn}} |g|^{\text{sgn}}$. We omit the verification of the strong triangle inequalities. Clearly, $|\cdot|^{\text{sgn}}$ restricts to $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ on $\mathbb{R}[t_0, \dots, t_n]_1 \cong (\mathbb{R}^{n+1})^*$ and the homothety class of $|\cdot|^{\text{sgn}}$ only depends on the homothety class of $\|\cdot\|^{\text{sgn}}$. Thus, we have $\tau[|\cdot|^{\text{sgn}}] = [\|\cdot\|^{\text{sgn}}]$. \square

Remark 2.7.11. With minor amendments, the above proof can be generalized to any non-Archimedean valued real closed field to show that each class of a diagonalizable signed seminorm has a preimage. In particular, the image of the natural map $\tau : \mathbb{P}_r^{n,\text{an}} \rightarrow \mathfrak{R}\mathcal{X}_n(K)$ always contains the diagonalizable locus.

2.8 Real Bergman Fans

The tropical linear space associated to a matroid has the structure of a fan, called the *Bergman fan*. Recall that to each oriented valuated matroid one can associate a real tropical

linear space. If the valuation is trivial, *i.e.*, we have an oriented matroid, the corresponding fan is called the *real Bergman fan*. This construction in the case of finite ground sets was introduced in [Cel19, Chapter 2.3]. We will first discuss circuit axiomatization of infinite oriented matroids and then recall the notion of *covectors* of an oriented matroid, which are then used to define its real Bergman fan. In this section, we consider K to be a real closed and trivially valued field.

2.8.1 Infinite Oriented Matroids

A *signed subset* is a function $C : E \rightarrow \{0, +1, -1\}$. We denote by $C^+ := \{e \in E \mid C(e) = +1\}$, $C^- := \{e \in E \mid C(e) = -1\}$, $\text{Supp}(C) := C^+ \cup C^-$, and $C^0 := E \setminus \text{Supp}(C)$. The *separation set* of two signed sets C, C' is defined by $S(C, C') := \{e \in E \mid C(e) = -C'(e) \neq 0\}$.

We quickly discuss the circuit axiomatization of infinite oriented matroids of finite rank. Fix a possibly infinite ground set E . Recall that an oriented matroid is given by a chirotope (*i.e.*, a Grassmann–Plücker function) $\varphi : E^n \rightarrow \mathbb{S} = \{0, +1, -1\}$. Equivalently, there is the same circuit axiomatization as in Proposition 2.2.8 (replacing \mathbb{RT} with \mathbb{S}), with one additional axiom, which ensures finite rank:

- (C4) There is a positive integer $n \in \mathbb{N}$ such that for all subsets $A \subseteq E$ with $|A| > n$ there exists $C \in \mathcal{C}$ with $\text{Supp}(C) \subseteq A$.

By restricting and applying the finite case (*e.g.*, [BLVS⁺99, Theorem 3.2.5]) one obtains the *strong circuit elimination*:

- (C3') For all $C \neq -C' \in \mathcal{C}$, $e \in S(C, C')$, and $f \in (C^+ \setminus C'^-) \cup (C^- \setminus C'^+)$, there is $C'' \in \mathcal{C}$ such that $C''(e) = 0$, $f \in \text{Supp}(C'')$, and

$$C''^+ \subseteq (C^+ \cup C'^+), \quad C''^- \subseteq (C^- \cup C'^-).$$

As a consequence, our notion of oriented matroids agrees with the characterization given in [BF88, Theorem 3] with the additional requirement of finite rank.

2.8.2 Covectors

Let \mathcal{M} be an oriented matroid given by a chirotope $\varphi : E^n \rightarrow \{0, +1, -1\}$.

Definition 2.8.1. The set of *signed cocircuits* \mathcal{C}^* of \mathcal{M} is given by the set of signed sets

$$\mathcal{C}^* = \{\pm[e \mapsto \varphi(\mu, e)] \mid \mu \in E^{n-1}\} \setminus \{0\}.$$

Note that the choice of the sign of the chirotope φ does not change the set of signed cocircuits.

Remark 2.8.2. In the case that E is finite, we have that \mathcal{C}^* is a set of signed circuits of an oriented matroid, which is called the *dual* oriented matroid. As expected, dualizing commutes with taking the underlying matroid. A crucial difference is, that the dual of an infinite (oriented) matroid is not of finite rank and hence behaves inherently different from our case. Hence, we will not consider any duals of infinite matroids.

The *composition* of two signed sets $C, C' : E \rightarrow \{0, +1, -1\}$ is defined by

$$(C \circ C')(e) = \begin{cases} C(e) & \text{if } C(e) \neq 0, \\ C'(e) & \text{if } C(e) = 0. \end{cases}$$

We write $C \leq C'$ if $C'^0 \subseteq C^0$ and $C'|_{\text{Supp } C} = C|_{\text{Supp } C}$. In other words, this partial order is induced by the partial order $0 < +1, -1$, considered coordinate-wise.

Definition 2.8.3.

- (a) A *vector* of \mathcal{M} is a finite composition of signed circuits.
- (b) Dually, a *covector* is a finite composition of cocircuits. We consider the set of covectors Cov as a partially ordered set with the usual ordering of signed sets.

If E is finite, the covectors of the oriented matroid are exactly the vectors of its dual [BLVS⁺99, Definition 3.7.1].

Example 2.8.4. Let \mathcal{M} be realizable and $\phi : K^{(E)} \rightarrow K^n$ be the corresponding realization. Then the set of vectors is given by

$$\{[e \mapsto \text{sgn}(\lambda_e)] \mid (\lambda_e)_{e \in E} \in \ker \phi\}.$$

For the covectors, consider the dual map $\phi^* : K^n \rightarrow K^E$. Then the set of covectors is given by compositions of elements of

$$\{\text{sgn} \circ \phi^*(v) \mid v \in K^n\}.$$

Proposition 2.8.5 (Covectors of $\mathcal{M}_{\text{univ}}$). *For $K = \mathbb{R}$ and the vector space $(\mathbb{R}^{n+1})^*$, the following sets of maps $(\mathbb{R}^{n+1})^* \rightarrow \{0, +1, -1\}$ are the same:*

- (a) The covectors of $\mathcal{M}_{\text{univ}}$.
- (b) Maps of the form $X : (\mathbb{R}^{n+1})^* \rightarrow \{0, +1, -1\}$, $e \mapsto \text{sgn}(\|e\|^{\text{sgn}})$ for some signed seminorm $\|\cdot\|^{\text{sgn}}$ on \mathbb{R}^{n+1} .
- (c) Signed seminorms with range $\{0, +1, -1\}$.
- (d) All maps such that X^+, X^- are strictly convex cones without 0 and $\emptyset \neq X^0 \subseteq (\mathbb{R}^{n+1})^*$ is a subspace.

Proof. We first show that the functions in (b) and (c) are exactly the covectors of $\mathcal{M}_{\text{univ}}$.

- (b) Let $X : (\mathbb{R}^{n+1})^* \rightarrow \{0, +1, -1\}$ be a cocircuit given by $\mu = (\mu_0, \dots, \mu_{n-1}) \in ((\mathbb{R}^{n+1})^*)^n$. Choose any $\mu_n \in (\mathbb{R}^{n+1})^*$ with $X(\mu_n) = +1$. Using multilinearity of the determinant, one can show

$$X = \text{sgn}(\|\cdot\|_{B, \vec{c}}^{\text{sgn}})$$

for either for $B = (\mu_n, \mu_0, \dots, \mu_{n-1})$ or $B = (-\mu_n, \mu_0, \dots, \mu_{n-1})$ and $\vec{c} = (1, 0, \dots, 0)$. Since the space of signed seminorms is closed under composition (cf. Proposition 2.4.4), we obtain (a) \subseteq (b).

Vice versa, let $\|\cdot\|^{\text{sgn}}$ be a signed seminorm. Then by Proposition 2.7.5 it is diagonalizable by an ordered basis $B = (b_0, \dots, b_n)$ with vector $\vec{c} = (c_0, \dots, c_n)$. Let $B_i = ((-1)^i b_i, b_1, \dots, b_n)$ and $\vec{c}_i := (c_i, 0, \dots, 0, \dots, 0)$. Then $\|\cdot\|_{B, \vec{c}}^{\text{sgn}} = \|\cdot\|_{B, \vec{c}_0}^{\text{sgn}} \circ \dots \circ \|\cdot\|_{B, \vec{c}_n}^{\text{sgn}}$. As before we have that $\text{sgn}(\|\cdot\|_{B, \vec{c}_i}^{\text{sgn}})$ is a cocircuit, hence the composition is a covector.

- (c) Follows immediately from part (b) by replacing $\|\cdot\|^{\text{sgn}}$ by the signed seminorm $\frac{\|\cdot\|^{\text{sgn}}}{\|\cdot\|_{\text{sgn}}}$.

Now we show (c)=(d): From Lemma 2.7.3 we immediately deduce \subseteq .

For “ \supseteq ”, let $X : (\mathbb{R}^{n+1})^* \rightarrow \{0, +1, -1\}$ be a function such that X^+, X^- are strictly convex cones without 0 and $\emptyset \neq X^0 \subseteq (\mathbb{R}^{n+1})^*$ is a subspace. Strict convexity of the cones immediately implies axiom (i) from 2.4.1. Let $v, w \in (\mathbb{R}^{n+1})^* \setminus 0$. By convexity and (i), if $X(v) = X(w)$ or $X(v) = -X(w)$, then axiom (ii) is fulfilled. By symmetry, the only case that remains to show is if $X(v) = +1$ and $X(w) = 0$. If $X(v+w) = -1$, then by convexity and (i) we have

$$X(w) = X\left(\frac{w}{2}\right) = X\left(-\frac{v}{2} + \frac{v+w}{2}\right) = -1,$$

which is a contradiction. □

Remark 2.8.6 (Covector Axioms). The covector poset uniquely determines the oriented matroid [BLVS⁺99, §3.7]. If E is finite, any poset of signed sets is the covector poset of an oriented matroid if and only if it fulfills:

(Cov1) $0 \in \text{Cov}$,

(Cov2) $X \in \text{Cov}$ if and only if $-X \in \text{Cov}$,

(Cov3) Cov is closed under composition,

(Cov4) For all $X, Y \in \text{Cov}$ and $e \in S(X, Y)$ there exists $Z \in \text{Cov}$ with $Z(e) = 0$ and

$$Z(e') = (X \circ Y)(e') \text{ for all } e' \notin S(X, Y).$$

In other words, the covector axioms are a cryptomorphic definition of an oriented matroid.

If E is infinite, the covector axiom (Cov4) need not be true. For example, one can show that the covector poset of the restriction of $\mathcal{M}_{\text{univ}}$ to E' in Example 2.8.9 does not fulfill (iv).

Proposition 2.8.7. *The covector poset Cov of $\mathcal{M}_{\text{univ}}$ fulfills the covector axioms for $K = \mathbb{R}$.*

Proof. Axioms (Cov1)-(Cov3) are obviously true, so it remains to show (Cov4). Let $X, Y \in \text{Cov}$ and $e \in S(X, Y)$ (in particular, neither $X = 0$ nor $Y = 0$). Without loss of generality, assume $X(e) = +1$. Set

$$V := \text{Span}(\{e\} \cup (X^0 \cap Y^0)) \subseteq (\mathbb{R}^{n+1})^* \text{ and } \pi : (\mathbb{R}^{n+1})^* \rightarrow (\mathbb{R}^{n+1})^*/V$$

the projection map. We define

$$A = (X \circ Y)^+ \setminus S(X, Y) = (X^+ \cap (Y^+ \cup Y^0)) \cup (X^0 \cap Y^+).$$

Then $(\mathbb{R}^{n+1})^* \setminus S(X, Y) = A \cup -A \cup (X^0 \cap Y^0)$ and $Z \in \text{Cov}$ fulfils axiom (iv) if and only if $Z|_V = 0$, $Z|_A = +1$, $Z|_{-A} = -1$. We claim that $\pi(A) \cap \pi(-A) = \emptyset$. Consider $a \in A, \lambda \in \mathbb{R}, w \in X^0 \cap Y^0$; we have to show that $a + \lambda e + w \notin -A$. If $\lambda = 0$, then $X \circ Y(a + \lambda e + w) = X \circ Y(a)$. If $\lambda > 0$, then

$$X(a + \lambda e + w) = X(a + \lambda e) = +1.$$

If $\lambda < 0$, then $Y(\lambda e) = +1$ and thus

$$Y(a + \lambda e + w) = Y(a + \lambda e) + 1.$$

In either case, $a + \lambda e + w \notin -A$. By symmetry, this implies $\pi(A) \cap \pi(-A) = \emptyset$. Since A is convex, so is $-A$ and hence also $\pi(A)$ and $\pi(-A)$. Therefore, we can apply the hyperplane separation Theorem [SW12, 3.3.9] in $(\mathbb{R}^{n+1})^*/V$ to the sets $\pi(A)$, $\pi(-A)$ to obtain a separating hyperplane $H' \subseteq (\mathbb{R}^{n+1})^*/V$. The pull-back $H = \pi^{-1}(H')$ then separates A , $-A$ and $V \subseteq H$. By induction, on H there is a covector fulfilling (iv), given by the sign of a diagonalizable seminorm, which in return is given by an ordered basis (b_1, \dots, b_n) of H and $(c_1 \geq \dots \geq c_n)$. Choosing any $b_0 \in A \setminus H$ and $c_0 \gg c_i$ yields a signed seminorm $\|\cdot\|^{\text{sgn}}$ on $(\mathbb{R}^{n+1})^*$ such that $Z = \text{sgn}(\|\cdot\|^{\text{sgn}})$ fulfills (iv). \square

The restriction $\mathcal{M}|_{E'}$ of an oriented matroid to a subset $E' \subseteq E$ is the oriented matroid whose signed circuits are the signed subsets of \mathcal{C} whose support is contained in E' . One can easily check that the axioms (C0)-(C4) hold and that restriction commutes with taking the underlying matroid. This is equivalent to restricting a chirotope.

Lemma 2.8.8. *Let \mathcal{M} be an oriented matroid on a (possibly infinite) ground set E with covector set Cov and let $E' \subseteq E$ a finite subset. Then the restriction of \mathcal{M} to E' has covector set $\text{Cov}' = \{X|_{E'} \mid X \in \text{Cov}\}$. In particular, there is an order-preserving, surjective restriction map $\text{Cov} \rightarrow \text{Cov}'$.*

Proof. If E is finite, this follows from [BLVS⁺99, Proposition 3.7.11]. If E be infinite, since composition of signed sets commutes with restriction, it suffices to consider cocircuits of \mathcal{M} and show that their restriction to E' is a covector. Choose a chirotope φ and X be a cocircuit given by $X(e) = \varphi(\mu_1, \dots, \mu_{n-1}, e)$ for $\mu_1, \dots, \mu_{n-1} \in E$. Consider $E'' = E' \cup \{\mu_1, \dots, \mu_{n-1}\}$. Then surely, $X|_{E''}$ is a covector (even a cocircuit), hence $X_{E'} = (X|_{E''})|_{E'}$ is a covector on E' by the finite case. \square

Consequences of the absence of duality leads to different behavior of the covector poset:

Example 2.8.9. The condition of E' being finite in Lemma 2.8.8 is indeed necessary. Consider the universal realizable matroid $\mathcal{M}_{\text{univ}}$ with ground set $E = \mathbb{R}^3$. Let $E' = E \setminus H \cup \{v\}$, where $H \subset \mathbb{R}^3$ is any hyperplane and $v \in H \setminus 0$. Let X be a cocircuit of $\mathcal{M}_{\text{univ}}$ which is given by a basis μ of H as in Definition 2.8.1 with any chirotope. Then one can show that $X|_{E'}$ is not a covector of the restriction to E' .

Proposition 2.8.10. *Let \mathcal{M} be an oriented matroid of rank n with covector poset Cov , and let \mathcal{F} be the lattice of flats of the underlying matroid. The assignment*

$$\begin{aligned} \text{Cov} &\longrightarrow \mathcal{F} \\ X &\longmapsto X^0 \end{aligned}$$

is well-defined, surjective, and strictly monotonic. In particular, any chain in Cov has length at most n .

Proof. Let $C^* \in \mathcal{C}^*$ be a cocircuit that is given by $\mu \in E^{n-1}$ as in Definition 2.8.3. Then $(C^*)^0$ equals the closure of μ in the underlying matroid, hence it is a flat. Now the zero set of a covector equals the intersection of the zero sets of the circuits which appear in the composition, hence it is also a flat.

Let $F \in \mathcal{F}$ be any flat of rank $d \leq n$. Choose a basis $f_1, \dots, f_d, b_1, \dots, b_{n-d}$ of M such that F is the closure of f_1, \dots, f_d . Define

$$C_i^*(e) = \varphi(f_1, \dots, f_d, b_1, \dots, \hat{b}_i, \dots, b_{n-d}, e),$$

then for $X := C_1^* \circ \dots \circ C_{n-d}^*$ we have $X^0 = F$. \square

2.8.3 Real Bergman Fans

Let \mathcal{M} be an oriented matroid of rank $n + 1$ on any ground set E with poset of covectors Cov . We define its *real Bergman fan* $\Sigma_{\mathcal{M}}^*$ to be given by the collection of cones

$$\langle X_1, \dots, X_l \rangle_{\mathbb{R}_{>0}} \subset \mathbb{RP}^E$$

for each chain of non-zero covectors $X_1 < \dots < X_l$. This is, as in the non-oriented case, a fan of pure dimension n . If E is infinite, $\Sigma_{\mathcal{M}}^*$ may have infinitely many cones and is embedded into an infinite-dimensional real projective space.

Remark 2.8.11. (a) In his thesis [Cel19], Celaya considered finite oriented matroids and showed that the real Bergman fan shares many of the same features of a Bergman fan. After taking the componentwise logarithm, the real Bergman fan restricted to the positive orthant coincides with the positive Bergman fan considered in [AKW06].

(b) The real Bergman fan arises naturally from the perspective of matroids over hyperfields. The oriented matroid \mathcal{M} can be interpreted as a matroid over the real tropical hyperfield \mathbb{RT} , with trivial valuation. In [And19], the author introduces the notion of vectors of a matroid over a hyperfield where the \mathbb{RT} -covectors are the signed covectors of the oriented

matroid \mathcal{M} . The set of \mathbb{RT} -vectors of \mathcal{M} coincides exactly with the support of $\Sigma_{\mathcal{M}}^*$. In unsigned tropical geometry, the Bergman fan of a matroid M arises by considering it as a matroid over the Krasner hyperfield \mathbb{K} and taking the set of \mathbb{K} -vectors of M .

- (c) In [RRS22], the authors showed that real phase structures on the Bergman fan of a finite matroid are in one to one correspondence with orientations of that matroid. In particular, the datum of the real Bergman fan is cryptomorphic to that of the Bergman fan and a choice of a real phase structure.

For finite oriented matroids, the notions of its real Bergman fan and its associated real tropical linear space agree:

Proposition 2.8.12 ([Cel19, Proposition 2.4.7, Corollary 2.4.8]). *Let \mathcal{M} be an oriented matroid on a finite ground set. Then*

$$\text{Trop}_r(\mathcal{M}) = \Sigma_{\mathcal{M}}^*$$

Corollary 2.8.13 ([Cel19, Proposition 2.5.4]). *Let $\iota = (f_0 : \dots : f_m) : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ be a linear embedding for any real closed and trivially valued field K . Let \mathcal{M}_ι be the associated realizable oriented matroid on $\{f_0, \dots, f_m\} \subset (K^{n+1})^*$. Then*

$$\text{Trop}_r(\mathbb{P}^n, \iota) = \Sigma_{\mathcal{M}_\iota}^*.$$

Proposition 2.8.14 (cf. [Cel19, 3.3.3], [BLVS+99, Theorem 5.2.1]). *The real Bergman fan $\text{Trop}_r(\mathcal{M})$ of a finite oriented matroid \mathcal{M} of rank n is homeomorphic to \mathbb{RP}^n .*

Theorem 2.8.15. *For $K = \mathbb{R}$ and the vector space $(\mathbb{R}^{n+1})^*$, there is a homeomorphism*

$$\mathfrak{RX}_n(\mathbb{R}) \cong \Sigma_{\mathcal{M}_{\text{univ}}}^*.$$

Proof. We can make the correspondence very explicit as follows. Let $x \in \Sigma_{\mathcal{M}_{\text{univ}}}^*$. Then there are $\alpha_i \geq 0$ and a flag of covectors $X_0 < \dots < X_n$ of $\mathcal{M}_{\text{univ}}$ such that $x = \sum_{i=0}^n \alpha_i e_{X_i}$. Hence there are signed cocircuits Y_0, \dots, Y_n such that $X_0 = Y_0, X_1 = Y_0 \circ Y_1, \dots, X_n = Y_0 \circ \dots \circ Y_n$. The signed cocircuits are given by $Y_i = \text{sgn}(\|\cdot\|_i^{\text{sgn}})$ for signed seminorms $\|\cdot\|_i^{\text{sgn}}$ on $(\mathbb{R}^{n+1})^*$. As in the proof of Proposition 2.8.5 these signed seminorms are diagonalized by an ordered basis $B = (b_0, \dots, b_n)$ and coordinate vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ such that $\|\cdot\|_i^{\text{sgn}} = \|\cdot\|_{B, e_i}^{\text{sgn}}$. Then, $x = (\alpha_1 + \dots + \alpha_n) e_{Y_1} \circ (\alpha_2 + \dots + \alpha_n) e_{Y_2} \circ \dots \circ \alpha_n e_{Y_n}$. Setting $c_0 = \alpha_0 + \dots + \alpha_n, \dots, c_n = \alpha_n$ we obtain $x = \|\cdot\|_{B, \vec{c}}^{\text{sgn}}$. Conversely, a signed seminorm $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ yields coordinates α_i and a flag of covectors of $\mathcal{M}_{\text{univ}}$ by reversing the construction. \square

Remark 2.8.16. (a) The interpretation of Remark 2.6.4 (b) becomes even more clear in the case of real numbers with trivial valuation. For any finite subset $\{f_0, \dots, f_m\} = E' \subset E = (\mathbb{R}^{n+1})^*$, there is a restriction map from the covector poset of $\mathcal{M}_{\text{univ}}$ to the one of $\mathcal{M}_{\text{univ}}|_{E'}$ by Lemma 2.8.8. The induced map of the respective order complexes is exactly the map $\mathfrak{RX}_n(\mathbb{R}) \rightarrow \text{Trop}_r(\mathbb{P}^n, \iota)$ for $\iota = (f_0 : \dots : f_m)$.

(b) Proposition 2.8.10 gives us a map from the covectors poset of $\mathcal{M}_{\text{univ}}$ to the lattice of flats of the underlying matroid, the universal realizable matroid. The continuous map on the corresponding order complexes is exactly the map $\Phi : \mathfrak{RX}_n(\mathbb{R}) \rightarrow \overline{\mathcal{X}}_n(\mathbb{R})$ induced by taking absolute values.

(c) Combining (a) and (b), we can reprove Theorem 2.5.6 for real numbers with trivial valuation. Let Cov be the covector lattice of $\mathcal{M}_{\text{univ}}$, \mathcal{F} the lattice of flats of the underlying matroid, and $E' \subset E$ a finite generating subset. We observe that the following diagram commutes:

$$\begin{array}{ccc}
\text{Cov} & \longrightarrow & \text{Cov}|_{E'} \\
\downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathcal{F}|_{E'}
\end{array}$$

If now $\iota : \mathbb{P}^n \rightarrow \mathbb{P}(\mathbb{R}^{E'})$ is the corresponding embedding, we obtain commutativity for

$$\begin{array}{ccc}
\mathfrak{R}\mathcal{X}_n(\mathbb{R}) & \longrightarrow & \text{Trop}_r(\mathbb{P}^n, \iota) \\
\downarrow & & \downarrow \\
\overline{\mathcal{X}}_n(\mathbb{R}) & \longrightarrow & \text{Trop}(\mathbb{P}^n, \iota).
\end{array}$$

- (d) Following the proof of Theorem 2.8.15, if we replace \mathbb{R} by any other trivially valued real closed field K , the real Bergman fan agrees with the diagonalizable locus in $\mathfrak{R}\mathcal{X}_n(K)$. The existence of non-diagonalizable signed seminorms (cf. 2.4.7) shows, that the real Bergman fan is then a proper subspace.

2.8.4 Tropical Convexity and Signed Tropical Convexity

In this section we describe the relationship of Goldman–Iwahori space $\overline{\mathcal{X}}_n(K)$ and the signed Goldman–Iwahori space $\mathfrak{R}\mathcal{X}_n(K)$ to tropical convexity (see [Jos21] for details) and signed tropical convexity [LS22].

Let K be a non-Archimedean field. Tropical linear spaces are tropically convex, *i.e.* closed under taking tropical addition and tropical multiplication. By [YY06, Theorem 16], a tropicalized linear space is the tropical convex hull of all valuated cocircuits. A similar result extends to the tropicalization of the universal realizable valuated matroid w_{univ} . The space $\overline{\mathcal{X}}_n(K) = \text{Trop}(w_{\text{univ}})$ is tropically convex, since the maximum of two seminorms is again a seminorm. Further, we can interpret the locus of diagonalizable seminorms $\overline{\mathcal{B}}_n(K)$ as the tropical convex hull of all valuated cocircuits. Explicitly, these valuated cocircuits arise as follows: Let μ be a n -subset of $E = (K^{n+1})^*$. This defines a seminorm $\|\cdot\|_{\mu}$ via $\|f\|_{\mu} = \exp(-w_{\text{univ}}(\mu \cup f)) = |\det(\mu \cup f)|_K$ for $f \in E$. Given a diagonalizable seminorm $\|\cdot\|_{B, \vec{c}}$ we can express $\|\cdot\|_{B, \vec{c}}$ as a tropical linear combination of $\|f\|_{\mu_i}$ where $\mu_i = B \setminus i$. By a small computation we obtain $\|f\|_{B, \vec{c}} = \max_i c_i |\det(B)|_K^{-1} \|f\|_{\mu_i}$. In particular, if K is spherically complete, then $\text{Trop}(w_{\text{univ}}) = \overline{\mathcal{X}}_n(K)$ is the tropical convex hull of the valuated cocircuits.

Let K be a real closed field with a compatible absolute value. In [LS22] the authors introduce TC-convexity as a notion of signed tropical convexity which is used to study oriented (valuated) matroids. The real tropicalization of a linear space is the TC-convex hull of all \mathbb{RT} -cocircuits of the associated \mathbb{RT} -matroid [LS22, Theorem 7.8]. As in the unsigned case, we give an interpretation of this result for the locus of diagonalizable signed seminorms in $\mathfrak{R}\mathcal{X}_n(K)$. A n -ordered subset $\mu = (\mu_1, \dots, \mu_n)$ of $E = (K^{n+1})^*$ defines a signed seminorm $\|\cdot\|_{\mu}^{\text{sgn}}$ via $\|f\|_{\mu}^{\text{sgn}} = \varphi_{\text{univ}}(f, \mu_1, \dots, \mu_n)$ for $f \in E$. Let $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ be a diagonalizable signed seminorm where $B = (b_0, \dots, b_n)$ is an ordered basis and $c_0 \geq \dots \geq c_n \geq 0$. As in the proof in 2.8.5, we can write $\|\cdot\|_{B, \vec{c}}^{\text{sgn}}$ as a composition of scalar multiples of $\|\cdot\|_{B, e_i}^{\text{sgn}}$ and hence as a composition of scalar multiples of the signed seminorms $\|f\|_{\mu_i}^{\text{sgn}}$ where $\mu_i = (b_0, \dots, \hat{b}_i, \dots, b_n)$. In particular, for $K = \mathbb{R}$ we have that $\text{Trop}_r(\mathcal{M}_{\text{univ}}) = \mathfrak{R}\mathcal{X}_n(K)$ is the TC-convex hull of the signed valuated cocircuits.

Chapter 3

Tropical Reductive Groups and Principal Bundles on Metric Graphs

Introduction

Denote by $\mathbb{T} = (\mathbb{R} \sqcup \{\infty\}, \min, +)$ the semifield of tropical numbers. It is an elementary fact (see for example [All12, Lemma 1.4]) that the group $\mathrm{GL}_n(\mathbb{T})$ of invertible $n \times n$ matrices over \mathbb{T} is the group of generalized tropical permutation matrices. In other words, it is isomorphic to the semidirect product $\mathbb{R}^n \rtimes S_n$. In [GUZ22] this observation was used to build a theory of tropical vector bundles on metric graphs, expanding on [All12], that is to say, principal bundles with structure group $\mathrm{GL}_n(\mathbb{T})$.

We observe that the two terms in the semidirect product $\mathrm{GL}_n(\mathbb{T}) \simeq \mathbb{R}^n \rtimes S_n$ have the following interpretation: S_n is the Weyl group of the reductive algebraic group \mathbf{GL}_n , while \mathbb{R}^n is the tropicalization of the diagonal torus $\mathbb{G}_m^n \subseteq \mathbf{GL}_n$. In this article, we expand on this observation and introduce an elementary theory of tropical reductive groups in other Dynkin–Lie types, and an associated theory of principal bundles on metric graphs.

Tropical reductive groups

Let \mathbf{G} be a reductive algebraic group over an algebraically closed field k with a maximal torus $\mathbf{T} \subseteq \mathbf{G}$. Then \mathbf{G} is uniquely determined (up to isomorphism) by its root datum

$$\Phi = (M, R, \check{M}, \check{R}),$$

where M and \check{M} are the character and cocharacter lattices of \mathbf{T} , and $R \subseteq M$ and $\check{R} \subseteq \check{M}$ are the sets of roots and coroots, respectively (see Definition 3.1.1). The Weyl group W_Φ is the group of automorphisms of M generated by the reflections corresponding to the roots in R and naturally acts on the dual space $\check{M}_{\mathbb{R}} = \check{M} \otimes_{\mathbb{Z}} \mathbb{R}$. We define the *tropical reductive group* associated to the root datum Φ as

$$\mathbf{G}^{\mathrm{trop}} = \check{M}_{\mathbb{R}} \rtimes W_\Phi.$$

This construction depends only on the root datum Φ and, in particular, does not depend on the ground field k .

For $\mathbf{G} = \mathbf{GL}_n$, this construction recovers the matrix group $\mathrm{GL}_n(\mathbb{T}) \simeq \mathbb{R}^n \rtimes S_n$ described above. In Section 3.1, we expand on this observation and describe tropical analogues of other classical groups in terms of tropical linear algebra, closely mirroring their classical counterparts.

Theorem A (Propositions 3.1.4, 3.1.9, 3.1.13, and 3.1.16). *The tropical reductive groups associated to the root data of $\mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}$ and G_2 admit natural descriptions as matrix groups over \mathbb{T} that are analogous to the matrix descriptions of the corresponding algebraic reductive groups.*

It would be interesting to determine matrix-theoretic descriptions of the tropical reductive groups associated to the remaining exceptional root systems.

Principal $\mathbf{G}^{\mathrm{trop}}$ -bundles on metric graphs

Let Γ be a compact metric graph and denote by \mathcal{H}_Γ the sheaf of continuous real-valued harmonic functions with integer slopes on Γ . We define a *tropical principal $\mathbf{G}^{\mathrm{trop}}$ -bundle* as a $(\check{M} \otimes_{\mathbb{Z}} \mathcal{H}_\Gamma) \rtimes W_\Phi$ -torsor on Γ . In [GUZ22] the authors described an equivalence of the category of tropical principal $\mathrm{GL}_n(\mathbb{T})$ -bundles on Γ to the category of free covers $\Gamma' \rightarrow \Gamma$ together with a tropical line bundle on Γ' . In Section 3.2.2, we provide an explicit description of tropical principal bundles in the classical Lie types as line bundles on covers of Γ with suitable Weyl group symmetries together with extra data, generalizing the $\mathrm{GL}_n(\mathbb{T})$ -case described in [GUZ22].

Theorem B (Corollaries 3.2.6, 3.2.7, 3.2.8, and Example 3.2.9). *Let $G = \check{M}_{\mathbb{R}} \rtimes W_\Phi$ be a tropical reductive group associated to a root datum Φ . Then the category of tropical principal G -bundles on Γ is equivalent to the category of data consisting of*

- (i) *a free cover $\Gamma' \rightarrow \Gamma$ determined by the associated W_Φ -torsor, and*
- (ii) *a tropical line bundle on Γ' equipped with additional structure reflecting the action of W_Φ .*

Specifically, for the classical Lie types this equivalence specializes to the following explicit descriptions:

GL_n : *a multi-line bundle $(\Gamma' \rightarrow \Gamma, L)$, consisting of a free degree n cover $\Gamma' \rightarrow \Gamma$ and a tropical line bundle L on Γ' ([GUZ22, Prop 3.2]);*

SL_n : *as for GL_n , with a trivialization of the determinant line bundle $\det(L)$;*

Sp_{2n} : *multi-line bundles $(\Gamma' \rightarrow \Gamma, L)$, where $\Gamma' \rightarrow \Gamma$ is a degree $2n$ cover and L a tropical line bundle on Γ' together with a fixed-point-free involution ι on the cover and a trivialization of $(L \otimes \iota^{-1}L)/\iota$;*

SO_{2n+1} : *as for Sp_{2n} ,*

SO_{2n} : *as for Sp_{2n} , with a trivialization of the orientation double cover;*

G_2 : *a degree 6 cover $\Gamma' \rightarrow \Gamma$ with a locally trivial identification of each fiber with the vertices of the Star of David, a tropical line bundle L on Γ' , and trivializations of $(L \otimes \iota^{-1}L)/\iota$ where the involution $\iota : \Gamma' \rightarrow \Gamma'$ exchanges the opposite vertices in each star, and a ι -invariant trivialization of the line bundle on the domain of the associated $\mathrm{Sp}_2(\mathbb{T})$ -cover whose fibers correspond to the two triangles.*

Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group associated to a root datum Φ , where $W = W_\Phi$ is the Weyl group. The moduli space $\mathcal{M}_G(\Gamma)$ of isomorphism classes of G -bundles on Γ decomposes as a finite disjoint union

$$\mathcal{M}_G(\Gamma) = \coprod_{\tau \in \mathcal{M}_W(\Gamma)} \mathcal{M}_{G,\tau}(\Gamma)$$

indexed by the isomorphism type of the associated W -torsor. For a W -torsor τ on Γ we show that $\mathcal{M}_{G,\tau}$ is the quotient of a disjoint union of torsors under tropical abelian varieties by the finite group $\text{Aut}(\tau)$ (see Theorem 3.4.12 below). In the case where $\tau = W_\Gamma$ is the trivial W -torsor on Γ , we obtain $\mathcal{M}_{G,W_\Gamma} \cong (\text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \tilde{M})/W$ (see Proposition 3.4.10 below), which allows classifying G -bundles on metric graphs of genus zero in analogy with the classical theorems of Grothendieck [Gro57] (for vector bundles) and Harder [Har68] (in general). We refer the reader to Example 3.4.11 below for details.

Tropicalization of G -bundles

In [GUZ22] we observed that $\text{GL}_n(\mathbb{T})$ should be viewed as an incomplete tropicalization of \mathbf{GL}_n , since, for example, the former has dimension n while the latter has dimension n^2 . For this reason, we cannot expect the moduli space of principal $\text{GL}_n(\mathbb{T})$ -bundles on the skeleton Γ_X of an algebraic curve X to be the tropicalization of the moduli space of \mathbf{GL}_n -bundles on X . The same problem exists for almost all other reductive groups. Nonetheless, it turns out that the tropicalization map is defined for semistable bundles on an elliptic curve.

Let X be a Tate elliptic curve over an algebraically closed and complete non-Archimedean field K of equicharacteristic 0. Frătilă [Fră16, Fră21] provides an explicit description of the moduli spaces $\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X)$ and $\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X)$ of stable and semistable \mathbf{G} -bundles of degree $\check{\lambda} \in \pi_1(\mathbf{G})$ on X , respectively. We recall this description, which generalizes work of Atiyah [Ati57], Tu [Tu93], and Laszlo [Las98], in Section 3.3. We prove an analogous tropical statement describing the moduli spaces of semistable and stable \mathbf{G}^{trop} -bundles on a metric circle, see Section 3.4. Finally, in Section 3.5, we tropicalize stable \mathbf{G} -bundles on X by reducing them to $N_{\mathbf{G}}(\mathbf{T})$ -bundles, where $N_{\mathbf{G}}(\mathbf{T}) \subseteq \mathbf{G}$ is the normalizer of a fixed maximal torus \mathbf{T} of \mathbf{G} , and semistable bundles by passing to a Levi subgroup. Our main result can be summarized as follows:

Theorem C (Theorem 3.5.18). *Let X be a Tate elliptic curve over an algebraically closed and complete non-Archimedean field K of equicharacteristic 0, so that we have a non-Archimedean uniformization $X^{\text{an}} = \mathbb{G}_m^{\text{an}}/q^{\mathbb{Z}}$ and the minimal skeleton of X^{an} is given by the metrized circle $\Gamma_X = \mathbb{R}/\text{val}(q)\mathbb{Z}$. Moreover, let \mathbf{G} be a reductive algebraic group over K and denote by $\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X)$ the moduli space of semistable principal \mathbf{G} -bundles of degree $\check{\lambda} \in \pi_1(\mathbf{G})$.*

There is a natural continuous tropicalization map $\text{Trop} : (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X))^{\text{an}} \rightarrow \mathcal{M}_{\mathbf{G}^{\text{trop}},\text{ind}}^{\check{\lambda}}(\Gamma_X)$ together with a homeomorphism between the moduli space $\mathcal{M}_{\mathbf{G}^{\text{trop}},\text{ind}}^{\check{\lambda}}(\Gamma_X)$ of indecomposable principal \mathbf{G}^{trop} -bundles on Γ_X of degree $\check{\lambda}$ and the essential skeleton $\Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X))$ of $(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X))^{\text{an}}$ that makes the diagram

$$\begin{array}{ccc}
 & & \Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X)) \\
 & \nearrow \rho & \downarrow \cong \\
 (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X))^{\text{an}} & & \mathcal{M}_{\mathbf{G}^{\text{trop}},\text{ind}}^{\check{\lambda}}(\Gamma_X) \\
 & \searrow \text{Trop} &
 \end{array}$$

commute.

Theorem C fits into a sequence of results establishing relationships between tropical moduli

spaces and non-Archimedean skeletons/tropicalizations of their algebraic counterparts that started with [BR15] in the case of the Jacobian of an algebraic curve and [ACP15] for the moduli space of curves. It generalizes [GUZ22, Theorem D], which covers the case of vector bundles, i.e. the case $\mathbf{G} = \mathbf{GL}_n$.

Further discussion and remarks

We expect that, in order to generalize Theorem C to moduli spaces of semistable bundles on Mumford curves of higher genus, we will need a more refined theory of tropical principal bundles than the one proposed in this article. The underlying deeper reason for this seems to be that the tropical reductive groups proposed here are relatively sparse matrix groups, so that there is no good way to directly tropicalize a reductive algebraic group \mathbf{G} onto its tropical counterpart \mathbf{G}^{trop} . For example, the dimension of \mathbf{G}^{trop} is usually strictly less than that of \mathbf{G} , and the same holds for the dimensions of the corresponding moduli spaces. On a Tate curve, strong classification results for algebraic principal bundles on elliptic curves allow us to circumvent this problem.

In [GKUZ25], the authors expand on the elementary framework of tropical vector bundles developed in [GUZ22] and show in [GKUZ25, Theorem B] how the essential skeletons of the moduli spaces of (semi-)homogeneous bundles (in the sense of [Muk78]) on abelian varieties with maximally degenerate reduction can be identified with suitable moduli spaces of tropical semi-homogeneous vector bundles on the tropicalized abelian variety. We believe that a common generalization of Theorem C and [GKUZ25, Theorem B] to homogeneous principal bundles on abelian varieties is, in principle, possible. Thanks to the comparative lack of moduli-theoretic classification results on the classical side, this could, however, turn out to be technically quite demanding.

In two parallel articles [KM24b] and [KM24a] Khan and Maclagan as well as Kaveh and Manon propose a seemingly quite different approach to the tropical geometry of vector bundles. Their central idea is a definition that abstracts the combinatorial data coming from Klyachko's classification of toric vector bundles. By its very nature, this approach leads to rather satisfying results when studying the tropicalization of toric vector bundles on toric varieties (and restrictions thereof to subvarieties of toric varieties). At this point, the framework proposed in [KM24a] and [KM24b] does not seem to be able to encode monodromy phenomena on tropical varieties with nontrivial fundamental group and, hence, seems to lead to results different from ours in the case of the Tate curve. The generalization of Klyachko's classification to the setting of torus-equivariant principal bundles on toric varieties (see e.g. [KM22]) could be the starting point of a satisfying theory of tropical toric principal bundles that generalizes [KM24b] and [KM24a].

An alternative approach to understand the nature of tropical (vector) bundles might arise from the work of Kennedy-Hunt and Ranganathan [KH25] on the construction of logarithmic Quot schemes, where the authors build upon ideas introduced in [MW22] for the logarithmic Picard variety. The central objects in [KH25] are coherent sheaves on suitable logarithmic modifications of a given logarithmic curve. The combinatorial shadow of a generalization of the chip-firing equivalence of line bundles (see [BJ16, Sect. 2 and 3]) would be another contender for an object that could be named "tropical vector bundle".

Essential skeletons of non-Archimedean analytic spaces were introduced and studied in [MN15, NX16, NXY19] in order to make precise ideas of Kontsevich and Soibelman [KS06] for a non-Archimedean approach to the SYZ-fibration in mirror symmetry. Our Theorem C may be seen as an explicit example of a non-Archimedean SYZ-fibration. Our approach is indebted to the results in [BM19], which allow us to study the behaviour of essential skeletons of finite group quotients.

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3.1 Tropical reductive groups

In this section, we describe tropical versions of the classical reductive groups by means of canonical real extension of the corresponding Weyl groups. We show that these groups have natural descriptions as matrix groups over the tropical semifield. The theory developed in this section may be seen as a generalization of an analogy proposed by Tits [Tit57], namely that Weyl groups should be seen as analogues of the classical groups over the field \mathbb{F}_1 with one element. In a certain sense, we obtain the corresponding tropical reductive groups by a base change to \mathbb{T} . While we do not explicitly make use of any of the various approaches to \mathbb{F}_1 -geometry, our treatment is informed and inspired by the work of Lorscheid in [Lor18], which provides a theoretical foundation for Tits' analogy using the perspective of blueprints, as introduced in [Lor12]. For more background on tropical matrix groups we also refer the reader to [LJK18].

3.1.1 Root systems and tropical reductive groups

According to a classical result of Chevalley, split reductive algebraic groups over a fixed field are classified by their root data (for example, see Theorems 9.6.2 and 10.1.1 in [Spr98]). We recall the definition.

Definition 3.1.1. A *root datum* is a quadruple $\Phi = (M, R, \check{M}, \check{R})$ consisting of

- free abelian groups M and \check{M} of finite rank with a duality pairing $\langle \cdot, \cdot \rangle : M \times \check{M} \rightarrow \mathbb{Z}$, and
- finite subsets of *roots* $R \subseteq M$ and *coroots* $\check{R} \subseteq \check{M}$ together with a bijection $(\check{\cdot}) : R \rightarrow \check{R}$

subject to the following two axioms:

- (i) For all $\alpha \in R$ we have $\langle \alpha, \check{\alpha} \rangle = 2$.
- (ii) The reflection homomorphisms $s_\alpha : M \rightarrow M$ and $s_{\check{\alpha}} : \check{M} \rightarrow \check{M}$ given by

$$u \mapsto u - \langle u, \check{\alpha} \rangle \alpha \quad \text{and} \quad v \mapsto v - \langle \alpha, v \rangle \check{\alpha}$$

satisfy

$$s_\alpha(R) = R \quad \text{and} \quad s_{\check{\alpha}}(\check{R}) = \check{R}$$

for all $\alpha \in R$.

A root datum $\Phi = (M, R, \check{M}, \check{R})$ is said to be *reduced*, if for all $\alpha \in R$ we have $2\alpha \notin R$. From now on, all of our root data will be assumed to be reduced and the term *root datum* will mean a *reduced root datum*.

The *Weyl group* W_Φ of the root datum Φ is the (necessarily finite) automorphism group of M generated by the reflections s_α for all $\alpha \in R$. The action of W_Φ on the lattice M defines a dual action on the dual lattice \check{M} , which extends to the vector space $\check{M}_\mathbb{R} = \check{M} \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 3.1.2. The *tropical reductive group* associated to the root datum $\Phi = (M, R, \check{M}, \check{R})$ is the semidirect product $G_\Phi = \check{M}_\mathbb{R} \rtimes W_\Phi$.

We emphasize that the Weyl group W_Φ of a root datum Φ is not defined as an abstract group, but is explicitly presented via its action on the lattice M . We use this presentation to construct our tropical reductive group, hence we do not simply associate a tropical object to an abstract group.

We now define homomorphisms of tropical reductive groups.

Definition 3.1.3. Let $G_{\Phi_1} = \check{M}_{1,\mathbb{R}} \rtimes W_{\Phi_1}$ and $G_{\Phi_2} = \check{M}_{2,\mathbb{R}} \rtimes W_{\Phi_2}$ be tropical reductive groups associated to root data $\Phi_1 = (M_1, R_1, \check{M}_1, \check{R}_1)$ and $\Phi_2 = (M_2, R_2, \check{M}_2, \check{R}_2)$, respectively. Let $f : \check{M}_1 \rightarrow \check{M}_2$ be a \mathbb{Z} -linear homomorphism and let $\phi : W_{\Phi_1} \rightarrow W_{\Phi_2}$ be a group homomorphism such that for any $m \in \check{M}_1$ and any $g \in W_{\Phi_1}$ we have $\phi(g)(f(m)) = f(g(m))$. The pair (f, ϕ) defines a *homomorphism of tropical reductive groups*

$$F : \check{M}_{1,\mathbb{R}} \rtimes W_{\Phi_1} \longrightarrow \check{M}_{2,\mathbb{R}} \rtimes W_{\Phi_2} \\ (m, g) \longmapsto (f(m), \phi(g)).$$

3.1.2 Type A_n : the tropical general, special, and projective linear groups

We now calculate the tropical reductive groups associated to the classical root data and show that they admit natural descriptions as matrix groups over the tropical semifield \mathbb{T} . We start with type A_n .

Recall that $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ with operations

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

The additive and multiplicative identities are ∞ and 0, respectively. We note that \mathbb{T} contains no nontrivial roots of unity,

$$\mu_n(\mathbb{T}) = \{x \in \mathbb{T} : x^{\odot n} = 0\} = \{x \in \mathbb{T} : nx = 0\} = \{0\},$$

hence the element 0 plays the role of both +1 and -1. The semifield operations on \mathbb{T} extend to a matrix product on the set $\text{Mat}(n \times n, \mathbb{T})$ of $(n \times n)$ -matrices with entries in \mathbb{T} :

$$(A \odot B)_{ij} = \bigoplus_{k=1}^n a_{ik} \odot b_{kj}.$$

The multiplicative identity in $\text{Mat}(n \times n, \mathbb{T})$ is the matrix

$$I_n = \begin{pmatrix} 0 & \infty & \cdots & \infty \\ \infty & 0 & \cdots & \infty \\ \cdots & \cdots & \cdots & \cdots \\ \infty & \infty & \cdots & 0 \end{pmatrix}.$$

We first describe the *tropical general linear group*, which is the group of invertible elements in $\text{Mat}(n \times n, \mathbb{T})$. Allermann shows (see Lemma 1.4 in [All12]) that these elements are the products of diagonal and permutation matrices:

$$\begin{aligned} \text{GL}_n(\mathbb{T}) &= \{A \in \text{Mat}(n \times n, \mathbb{T}) : A \odot A^{-1} = A^{-1} \odot A = I_n \text{ for some } A^{-1} \in \text{Mat}(n \times n, \mathbb{T})\} \\ &= \{D(y_1, \dots, y_n) \odot P_\sigma : y_1, \dots, y_n \in \mathbb{R}, \sigma \in S_n\} \\ &= \mathbb{R}^n \rtimes S_n. \end{aligned}$$

Here $D(y_1, \dots, y_n)$ is the tropical diagonal matrix with finite entries $y_1, \dots, y_n \in \mathbb{R}$ on the diagonal and ∞ everywhere else, and P_σ for $\sigma \in S_n$ is the tropical permutation matrix

$$(P_\sigma)_{ij} = \begin{cases} 0, & \text{if } i = \sigma(j), \\ \infty, & \text{otherwise.} \end{cases}$$

To define $\text{SL}_n(\mathbb{T})$, we recall that the *tropical determinant* [MS15] of a matrix $A \in \text{Mat}(n \times n, \mathbb{T})$ is

$$\det A = \bigoplus_{\sigma \in S_n} A_{1\sigma(1)} \odot \cdots \odot A_{n\sigma(n)}.$$

We note that the tropical determinant is the same as the tropical permanent, because both $+1$ and -1 tropicalize to 0 in \mathbb{T} . The determinant of an invertible matrix is finite (the converse is not true in general), is equal to the sum of the finite entries, and restricts to a homomorphism $\det : \text{GL}_n(\mathbb{T}) \rightarrow \mathbb{R} = \mathbb{T}^*$ given by

$$\det(D(y_1, \dots, y_n) \odot P_\sigma) = y_1 + \cdots + y_n.$$

We now define the *tropical special linear group* as

$$\begin{aligned} \text{SL}_n(\mathbb{T}) &= \{A \in \text{GL}_n(\mathbb{T}) : \det A = 0\} \\ &= \{D(y_1, \dots, y_n) \odot P_\sigma : y_i \in \mathbb{R}, y_1 + \cdots + y_n = 0, \sigma \in S_n\} \\ &= \mathbb{R}_0^n \rtimes S_n. \end{aligned}$$

Finally, we define the *tropical projective linear group* as the quotient of $\text{GL}_n(\mathbb{T})$ by its center, which is the subgroup of scalar matrices:

$$\text{PGL}_n(\mathbb{T}) = \text{GL}_n(\mathbb{T})/\mathbb{T}^* = (\mathbb{R}^n/\mathbb{R}) \rtimes S_n.$$

We now recall the root data of GL_n , SL_n , and PGL_n . The root datum $(M, R, \check{M}, \check{R})$ of GL_n has lattices $M = \check{M} = \mathbb{Z}^n$ with the standard pairing, and the roots and coroots are $R = \check{R} = \{e_i - e_j : i \neq j\}$, where the e_i are the standard basis vectors. The root datum of SL_n has the same roots and coroots, but the lattices are $M = \mathbb{Z}^n/(e_1 + \cdots + e_n)\mathbb{Z}$ and $\check{M} = \mathbb{Z}_0^n$, where $\mathbb{Z}_0^n \subseteq \mathbb{Z}^n$ is the set of vectors whose coordinates sum to zero. Finally, the root datum of PGL_n is the same as for SL_n , but with the lattices exchanged. Reflection through $e_i - e_j$ exchanges the i th and j th coordinates and fixes the rest, so the Weyl group in all three cases is the symmetric group S_n acting by permutation matrices.

We therefore obtain the following result.

Proposition 3.1.4. *The tropical reductive groups associated to the root data of GL_n , SL_n , and PGL_n are respectively $\text{GL}_n(\mathbb{T})$, $\text{SL}_n(\mathbb{T})$, and $\text{PGL}_n(\mathbb{T})$.*

This proposition explains why we define the tropical reductive group of a root datum $\Phi = (M, R, \check{M}, \check{R})$ as $\check{M}_{\mathbb{R}} \rtimes W_\Phi$ and not $M_{\mathbb{R}} \rtimes W_\Phi$: exchanging M and \check{M} would exchange $\text{SL}_n(\mathbb{T})$ and $\text{PGL}_n(\mathbb{T})$. We now consider two basic examples of homomorphisms of tropical reductive groups.

Example 3.1.5. Let $f : \mathbb{Z}^n \rightarrow \mathbb{Z}$ and $\phi : S_n \rightarrow S_1$ be respectively the sum map and the trivial map. Then the induced homomorphism of tropical reductive groups $F = (f, \phi) : \mathrm{GL}_n(\mathbb{T}) \rightarrow \mathrm{GL}_1(\mathbb{T}) = \mathbb{R}$ is the tropical determinant.

Example 3.1.6. Let \mathbb{Z}_0^n , \mathbb{Z}^n , and $\mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$ be the cocharacter lattices of SL_n , GL_n , and PGL_n , respectively. The canonical maps $\mathbb{Z}_0^n \rightarrow \mathbb{Z}^n \rightarrow \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$ and the trivial maps on S_n induce homomorphisms

$$\mathrm{SL}_n(\mathbb{T}) \rightarrow \mathrm{GL}_n(\mathbb{T}) \rightarrow \mathrm{PGL}_n(\mathbb{T})$$

of tropical reductive groups. We note that the composed map $\mathrm{SL}_n(\mathbb{T}) \rightarrow \mathrm{PGL}_n(\mathbb{T})$ is an isomorphism of abstract groups but not an isomorphism of tropical reductive groups, since the lattice map $\mathbb{Z}_0^n \rightarrow \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$ is not surjective. This may be seen as a characteristic one shadow of the fact that for an algebraically closed field k of characteristic p , the map $\mathrm{SL}_p(k) \rightarrow \mathrm{PGL}_p(k)$ is a bijection on the k -points, but not an isomorphism of schemes.

3.1.3 Type C_n : the tropical symplectic group

We now define the tropical symplectic group in complete analogy with the algebraic setting, which we now recall (see Section 5.3.3 in [Lor18]). Let J be the $2n \times 2n$ block matrix with off-diagonal blocks I_n and $-I_n$ and zero diagonal blocks. For any ring R , the set $\mathrm{Sp}_{2n}(R)$ of R -rational points of the symplectic group is the set of $2n \times 2n$ -matrices A with entries in R satisfying $A^t J A = J$.

In the semifield \mathbb{T} , 0 plays the role of both 1 and -1 , hence we replace J with the matrix

$$J = \begin{pmatrix} \infty & I_n \\ I_n & \infty \end{pmatrix}$$

and make the following definition.

Definition 3.1.7. The *tropical symplectic group* is

$$\mathrm{Sp}_{2n}(\mathbb{T}) = \{A \in \mathrm{GL}_{2n}(\mathbb{T}) : A^t \odot J \odot A = J\}.$$

We now give an explicit description of $\mathrm{Sp}_{2n}(\mathbb{T})$. We label the columns of a $2n \times 2n$ -matrix using the index set $[\pm n] = \{1, \dots, n, -1, \dots, -n\}$, which carries the fixed-point-free sign involution

$$\iota : [\pm n] \rightarrow [\pm n] \quad \text{given by} \quad \iota(k) = -k.$$

In terms of this identification, the matrix $J = P_\iota$ is the tropical permutation matrix associated to ι . We recall that the *signed permutation group* $S_n^B = S_n^C$ is the set of permutations of $[\pm n]$ commuting with ι :

$$S_n^B = \{\sigma \in S_{2n} : \sigma(-k) = -\sigma(k) \text{ for all } k \in [\pm n]\} \subseteq S_{2n}.$$

An element of S_n^B permutes the set of pairs $\{1, -1\}, \dots, \{n, -n\}$ and acts inside each pair, hence S_n^B is an extension of S_n by $(\mathbb{Z}/2\mathbb{Z})^n$.

Proposition 3.1.8. *The tropical symplectic group is the semidirect product*

$$\mathrm{Sp}_{2n}(\mathbb{T}) = \{D(y_1, \dots, y_n, -y_1, \dots, -y_n) \odot P_\sigma : \sigma \in S_n^B, y_1, \dots, y_n \in \mathbb{R}\} = \mathbb{R}^n \rtimes S_n^B.$$

We note that $\det A = 0$ for $A \in \mathrm{Sp}_{2n}(\mathbb{T})$, as one would expect.

Proof. Let $A = D(y_1, \dots, y_n, y_{-1}, \dots, y_{-n}) \odot P_\sigma \in \mathrm{GL}_{2n}(\mathbb{T})$ be an invertible matrix with $\sigma \in S_{2n}$ and $y_1, \dots, y_n, y_{-1}, \dots, y_{-n} \in \mathbb{R}$. Plugging this into $A^t \odot J \odot A = J$, we obtain

$$P_{\sigma^{-1}} \odot D(y_1, \dots, y_n, y_{-1}, \dots, y_{-n}) \odot P_\iota \odot D(y_1, \dots, y_n, y_{-1}, \dots, y_{-n}) \odot P_\sigma = P_\iota,$$

which is equivalent to

$$D(y_1, \dots, y_n, y_{-1}, \dots, y_{-n}) \odot D(y_{-1}, \dots, y_{-n}, y_1, \dots, y_n) \odot P_\iota \odot P_\sigma = P_\sigma \odot P_\iota.$$

This is satisfied if and only if $y_{-i} = -y_i$ for all $i = 1, \dots, n$ and furthermore $\iota\sigma = \sigma\iota$, so that $\sigma \in S_n^B$. \square

We now compute the tropical reductive group associated to the root datum $(M, R, \check{M}, \check{R})$ of Sp_{2n} . The lattices of this root datum are $M = \check{M} = \mathbb{Z}^n$ with the standard pairing. The roots $R \subseteq M$ are the vectors $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, while the coroots $\check{R} \subseteq \check{M}$ are $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$. Reflection in $e_i - e_j$ exchanges the i th and j th coordinates, while reflection in e_i changes the sign of the i th coordinate, so the Weyl group of this root datum is the signed permutation group S_n^B . Hence we have the following result.

Proposition 3.1.9. *The tropical reductive group associated to the root datum of Sp_{2n} is $\mathrm{Sp}_{2n}(\mathbb{T})$.*

We note that the embedding $\mathrm{Sp}_{2n}(\mathbb{T}) \rightarrow \mathrm{GL}_{2n}(\mathbb{T})$ described above is a homomorphism $F = (f, \phi)$ of tropical reductive groups, given by the \mathbb{Z} -linear homomorphism

$$f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{2n}, \quad f(x_1, \dots, x_n) = (x_1, \dots, x_n, -x_1, \dots, -x_n)$$

on the lattices that is compatible with the embedding $\phi : S_n^B \hookrightarrow S_{2n}$.

3.1.4 Types B_n and D_n : the tropical orthogonal and special orthogonal groups

Our description of the tropical orthogonal groups is likewise inspired by Lorscheid's integral models (see Section 4.3.4 in [Lor18]). In the algebraic setting, given a ring R , the orthogonal group ${}_m(R)$ is defined as the group of $m \times m$ invertible matrices over R preserving the standard split quadratic form, a notion that we can tropicalize directly. Defining the special orthogonal group in a characteristic-independent manner requires additional work. Namely, if m is odd, then the subgroup $\mathrm{SO}_m(R) \subseteq {}_m(R)$ is defined as the kernel of the determinant map. If m is even, however, then $\mathrm{SO}_m(R) \subseteq {}_m(R)$ is instead defined to be the kernel of the Dickson homomorphism $D_m : {}_m(R) \rightarrow \mathbb{Z}/2\mathbb{Z}$, which counts the number (mod 2) of terms in any factorization of an orthogonal matrix as a product of reflection matrices.

Let \mathbb{T}^m be a semimodule over \mathbb{T} of dimension m with coordinates $x = (x_1, \dots, x_n, x_{-1}, \dots, x_{-n})$ when $m = 2n$ and $x = (x_0, \dots, x_n, x_{-1}, \dots, x_{-n})$ when $m = 2n + 1$. We define the *standard split tropical quadratic form* $q_m : \mathbb{T}^m \rightarrow \mathbb{T}$ by the formulas

$$q_{2n}(x) = \bigoplus_{k=1}^n x_k \odot x_{-k} \quad \text{and} \quad q_{2n+1}(x) = x_0^{\odot 2} \oplus \bigoplus_{k=1}^n x_k \odot x_{-k}.$$

Definition 3.1.10. The *tropical orthogonal group* ${}_m(\mathbb{T})$ is

$${}_m(\mathbb{T}) = \{A \in \mathrm{GL}_m(\mathbb{T}) : q_m(A \odot x) = q_m(x) \text{ for all } x \in \mathbb{T}^m\}.$$

We defined the signed permutation group $S_n^B \subseteq S_{2n}$ as the group of permutations of the set $[\pm n]$ preserving the fixed-point-free sign involution. We also view S_n^B as a subgroup of S_{2n+1} , consisting of those permutations of the set $[\pm n] \cup \{0\}$ that preserve the sign involution (which now has the unique fixed point 0).

Proposition 3.1.11. *For $m = 2n + 1$, the tropical orthogonal group is the semidirect product*

$$\begin{aligned} {}_{2n+1}(\mathbb{T}) &= \{D(0, y_1, \dots, y_n, -y_1, \dots, -y_n) \odot P_\sigma : y_1, \dots, y_n \in \mathbb{R}, \sigma \in S_n^B \subseteq S_{2n+1}\} \\ &= \mathbb{R}^n \rtimes S_n^B. \end{aligned}$$

For $m = 2n$, the tropical orthogonal group is the semidirect product

$$\begin{aligned} {}_{2n}(\mathbb{T}) &= \{D(y_1, \dots, y_n, -y_1, \dots, -y_n) \odot P_\sigma : y_1, \dots, y_n \in \mathbb{R}, \sigma \in S_n^B \subseteq S_{2n}, \} \\ &= \mathbb{R}^n \rtimes S_n^B. \end{aligned}$$

Proof. We consider the case $m = 2n + 1$, the case of even m being similar. Let

$$x = (x_0, \dots, x_n, x_{-1}, \dots, x_{-n}) \in \mathbb{T}^m \quad \text{and} \quad A = D(y_0, \dots, y_n, y_{-1}, \dots, y_{-n}) \odot P_\sigma \in \text{GL}_m(\mathbb{T}),$$

then we have

$$q_m(A \odot x) = (y_0 \odot x_{\sigma^{-1}(0)})^{\odot 2} \oplus \bigoplus_{k=1}^n y_k \odot x_{\sigma^{-1}(k)} \odot y_{-k} \odot x_{\sigma^{-1}(-k)}.$$

Suppose that $\sigma \in S_n^B$, so that $\sigma(-k) = -\sigma(k)$ and also $\sigma^{-1}(-k) = -\sigma^{-1}(k)$ for all $k = 0, \dots, n$, and that $y_{-k} = -y_k$ for all $k = 0, \dots, n$. In particular, $\sigma(0) = 0$ and $y_0 = 0$. It follows that $q_m(A \odot x) = q_m(x)$ for all $x \in \mathbb{T}^m$, i.e. $A \in O_m(\mathbb{T})$. Conversely, assume that $A \in O_m(\mathbb{T})$. Choose $x \in \mathbb{T}^m$ such that $x_0 \in \mathbb{R}$ and $x_i = \infty$ for all $i \neq 0$. From $q_m(A \odot x) = q_m(x)$ we obtain $\sigma(0) = 0$ and $y_0 = 0$. Now let $j \neq 0$ and choose $x \in \mathbb{T}^m$ such that $x_j, x_{-j} \in \mathbb{R}$ and $x_i = \infty$ for all $i \in [\pm n] \cup \{0\} \setminus \{j, -j\}$. Since $q_m(A \odot x) = q_m(x) = x_j \odot x_{-j}$ it follows that $y_i \odot x_{\sigma^{-1}(i)} \odot y_{-i} \odot x_{\sigma^{-1}(-i)} = x_j \odot x_{-j}$ for i such that $\sigma^{-1}(i) = j$. Hence, $\sigma^{-1}(-i) = -\sigma^{-1}(i)$ and $y_{-i} = -y_i$. Since $j \neq 0$ was arbitrary, it follows that $\sigma^{-1} \in S_n^B$ and thus $\sigma \in S_n^B$ and $y_{-i} = -y_i$ for all $i \in [\pm n] \cup \{0\}$. \square

We now define the tropical special orthogonal groups, informed by the characteristic-independent algebraic definitions. First, we note that the determinant of a tropical orthogonal matrix is zero, reflecting the fact that \mathbb{T} has no nontrivial roots of unity. For $m = 2n + 1$ odd, we define $\text{SO}_{2n+1}(\mathbb{T})$ to be the kernel of the determinant on ${}_{2n+1}(\mathbb{T})$, which is all of ${}_{2n+1}(\mathbb{T})$. For $m = 2n$ even, we define $\text{SO}_{2n}(\mathbb{T})$ as the kernel of the *tropical Dickson invariant*,

$${}_{2n}(\mathbb{T}) \longrightarrow \{\pm 1\} \quad \text{given by} \quad D(y_i, -y_i) \odot P_\sigma \longmapsto \text{sgn}(\sigma),$$

which, in our setting, is simply the parity of the permutation.

Definition 3.1.12. For $m = 2n + 1$, the *tropical special orthogonal group* is

$$\text{SO}_{2n+1}(\mathbb{T}) = {}_{2n+1}(\mathbb{T}) = \mathbb{R}^n \rtimes S_n^B.$$

For $m = 2n$, the *tropical special orthogonal group* is

$$\text{SO}_{2n}(\mathbb{T}) = \{D(y_i, -y_i) \odot P_\sigma \in {}_{2n}(\mathbb{T}) : \sigma \in S_n^D = S_n^B \cap A_{2n}\} = \mathbb{R}^n \rtimes S_n^D,$$

where $S_n^D = S_n^B \cap A_{2n} \subseteq S_{2n}$ is the even signed permutation group.

We now compute the tropical reductive groups associated to the root data of SO_{2n+1} and SO_{2n} , respectively. The root datum $(M, R, \check{M}, \check{R})$ of SO_{2n+1} is dual to that of Sp_{2n} : the lattices are $M = \check{M} = \mathbb{Z}^n$ with the standard pairing, the roots $R \subseteq M$ are the vectors $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, and the Weyl group is the signed permutation group S_n^B . The root datum $(M, R, \check{M}, \check{R})$ of SO_{2n} has lattices $M = \check{M} = \mathbb{Z}^n$ with the standard pairing and roots $R = \check{R} = \{\pm e_i \pm e_j | i \neq j\}$. Reflection in $e_i - e_j$ exchanges the i th and j th coordinates, while reflection in $e_i + e_j$ switches e_i to $-e_j$ and e_j to $-e_i$. Hence the Weyl group consists of all permutations of n elements that switch an even number of their signs, hence it is isomorphic to the even signed permutation group S_n^D . Therefore, we have the following result.

Proposition 3.1.13. *The tropical reductive groups associated to the root systems of SO_{2n+1} and SO_{2n} are $\mathrm{SO}_{2n+1}(\mathbb{T})$ and $\mathrm{SO}_{2n}(\mathbb{T})$, respectively.*

3.1.5 Tropical G_2

As our last example, we define the tropical analogue of the group G_2 . We recall (see [CH88] and references therein) that the reductive group $G_2(F)$ over an algebraically closed field F (of characteristic $\neq 2, 3$) can be constructed as the isotropy group of a generic alternating trilinear form on a seven-dimensional vector space. Specifically, let $V = F^7$ with standard basis e_1, \dots, e_7 . The group $\mathbf{GL}(V)$ acts on the vector space $\bigwedge^3(F^*)$ with a unique open orbit, which contains the 3-form

$$\omega = e_1^* \wedge e_3^* \wedge e_5^* + e_2^* \wedge e_4^* \wedge e_6^* + e_1^* \wedge e_4^* \wedge e_7^* + e_2^* \wedge e_5^* \wedge e_7^* + e_3^* \wedge e_6^* \wedge e_7^*.$$

We then define

$$G_2(F) = \{A \in \mathbf{GL}(V) : \omega(Av_1, Av_2, Av_3) = \omega(v_1, v_2, v_3) \text{ for all } v_1, v_2, v_3 \in V\}.$$

We now translate this definition into the tropical setting. Since \mathbb{T} has no subtraction, we replace ω with a cubic form using the same formula, in the same manner that a symmetric bilinear form may be replaced with the associated quadratic form:

Definition 3.1.14. Define the tropical cubic form $c : \mathbb{T}^7 \rightarrow \mathbb{T}$ by the formula

$$c(x_1, \dots, x_7) = x_1 \odot x_3 \odot x_5 \oplus x_2 \odot x_4 \odot x_6 \oplus x_1 \odot x_4 \odot x_7 \oplus x_2 \odot x_5 \odot x_7 \oplus x_3 \odot x_6 \odot x_7.$$

We define

$$G_2(\mathbb{T}) = \{A \in \mathrm{GL}_7(\mathbb{T}) : c(Ax) = c(x) \text{ for all } x \in \mathbb{T}^7\}.$$

We first describe $G_2(\mathbb{T})$ explicitly. Let $D_6 \subseteq S_6$ be the group of symmetries of the regular hexagon, whose vertices are labeled 1 through 6 in order. The action of D_6 on \mathbb{R}^6 by permutation of coordinates preserves the two-dimensional subspace

$$U = \{(y_1, \dots, y_6) \in \mathbb{R}^6 : y_1 + y_3 + y_5 = y_2 + y_4 + y_6 = y_1 + y_4 = y_2 + y_5 = y_3 + y_6 = 0\} \subseteq \mathbb{R}^6,$$

where we note that either of the two relations $y_1 + y_3 + y_5 = 0$ and $y_2 + y_4 + y_6 = 0$ is redundant. We extend the embedding $D_6 \subseteq S_6$ to $D_6 \subseteq S_7$ by acting trivially on the 7.

Proposition 3.1.15. *The group $G_2(\mathbb{T})$ is isomorphic to*

$$\begin{aligned} G_2(\mathbb{T}) &= \{D(y_1, \dots, y_7) \odot P_\sigma \in \mathrm{GL}_7(\mathbb{T}) : (y_1, \dots, y_6) \in U, y_7 = 0, \sigma \in D_6 \subseteq S_7\} \\ &= \mathbb{R}^2 \rtimes D_6. \end{aligned}$$

Proof. Let $A = D(y_1, \dots, y_6, 0) \odot P_\sigma$ with $(y_1, \dots, y_6) \in U$ and $\sigma \in D_6$. The verification that $c(Ax) = c(x)$ for any $x \in \mathbb{T}^7$ is straightforward and left to the avid reader. For the converse implication, denote by T the set of three-element subsets of $\{1, \dots, 7\}$, and let $T_0 \subseteq T$ be the five-element subset indexing the monomials in c :

$$T_0 = \{\{1, 3, 5\}, \{2, 4, 6\}, \{1, 4, 7\}, \{2, 5, 7\}, \{3, 6, 7\}\}.$$

It is elementary to verify that a permutation $\sigma \in S_7$ lies in D_6 if and only if $\sigma(I) \in T_0$ for all $I \in T_0$.

Now let $A = D(y_i) \odot P_\sigma \in G_2(\mathbb{T})$. If $\sigma \notin D_6$, then there exists $I \in T_0$ such that $\sigma^{-1}(I) \notin T_0$. Define $x = (x_1, \dots, x_7)$ by $x_i = 0$ if $i \in I$ and ∞ otherwise, then $x_a \odot x_b \odot x_c = 0$ if

$\{a, b, c\} = I$ and ∞ otherwise. Hence $c(x) = 0$ but $c(Ax) = \infty \neq c(x)$, since each monomial in $c(Ax)$ has at least one infinite coordinate. Therefore $\sigma \in D_6$.

We similarly verify that $(y_1, \dots, y_6) \in U$ and $y_7 = 0$. Since we already know that $P_{\sigma^{-1}} \in G_2(\mathbb{T})$, we may replace A with $D(y_1, \dots, y_6, y_7)$. If $y_7 \neq 0$, then setting $x_1 = \dots = x_6 = 0$ and $x_7 = 2|y_7|$ we get $c(x) = 2|y_7|$ and $c(Ax) = 2|y_7| + y_7 \neq c(x)$. Similarly, the five linear expressions in the y_i defining U correspond to the five monomials in c . If any of these expressions are nonzero, we can pick x such that $c(x)$ is minimized at the corresponding monomial and such that $c(x) \neq c(Ax)$. This concludes the proof. \square

We now recall the root datum $(M, R, \check{M}, \check{R})$ of G_2 . The lattices M and \check{M} are the hexagonal lattices embedded in \mathbb{R}^2 with the standard Euclidean product, the roots are the 12 lattice points closest to the origin, and the Weyl group is D_6 , acting by symmetries of the lattice. Comparing with the description of $G_2(\mathbb{T})$ given above, we obtain the following result.

Proposition 3.1.16. *The tropical reductive group associated to the root system of G_2 is $G_2(\mathbb{T})$.*

This concludes our study of tropical reductive matrix groups.

3.2 Tropical principal bundles

We now define the tropical analogue of a principal bundle on an algebraic curve. A *metric graph* Γ is a metric space obtained by identifying the edges of a finite graph, called a *model* of Γ , with real intervals of given positive lengths. A *free cover* $\Gamma' \rightarrow \Gamma$ of metric graphs is a covering space in the topological sense that preserves the metric structure; equivalently, a free cover is a harmonic morphism having local degree one at all points of Γ' . Free covers are the only maps between metric graphs that we will consider (our restricted framework does not allow us to consider dilated harmonic morphisms).

3.2.1 Tropical G -covers and torsors over the Weyl group

Let Γ be a metric graph and let \mathcal{G} be a sheaf of (possibly non-abelian) groups on Γ . We recall that a \mathcal{G} -torsor on Γ is a sheaf of \mathcal{G} -sets F such that Γ can be covered by open subsets U for which $F|_U$ and $\mathcal{G}|_U \cong \mathcal{G}|_U$ are isomorphic as sheaves of \mathcal{G} -sets. Note that \mathcal{G} -torsors are classified up to isomorphism by the non-abelian cohomology set $H^1(\Gamma, \mathcal{G})$, which is a pointed set with a distinguished element given by the trivial torsor on Γ .

We now define principal bundles on Γ whose structure group $G = (\check{M} \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W_{\Phi}$ is the tropical reductive group associated to a root datum $\Phi = (M, R, \check{M}, \check{R})$. We recall that a metric graph Γ comes equipped with a sheaf of *harmonic functions* \mathcal{H}_{Γ} ; these are the continuous real-valued piecewise linear functions with integer slopes whose outgoing (or incoming) slopes at every point add up to zero. Taking the tensor product, we obtain the sheaf $G_{\Gamma} = (\check{M} \otimes_{\mathbb{Z}} \mathcal{H}_{\Gamma}) \rtimes W_{\Phi}$ of G -valued harmonic functions on Γ .

Definition 3.2.1. A *tropical G -bundle* on Γ is a torsor over the sheaf $G_{\Gamma} = (\check{M} \otimes_{\mathbb{Z}} \mathcal{H}_{\Gamma}) \rtimes W_{\Phi}$.

In Section 3.4 below, we investigate the set $H^1(\Gamma, G_{\Gamma})$ of isomorphism classes of G -bundles on Γ , which we interpret as the *moduli space of G -bundles on Γ* . The purpose of this section is to describe G -bundles on Γ in terms of line bundles on certain covers of Γ that are determined by their associated W_{Φ} -torsors.

We first recall from [GUZ22] this description for the vector bundle case $G = \mathrm{GL}_n(\mathbb{T}) = \mathbb{R}^n \rtimes S_n$. Let E be a $\mathrm{GL}_n(\mathbb{T})$ -bundle on a metric graph Γ . Projecting onto the second component defines

an S_n -torsor on Γ , which in turn defines a free cover $f : \Gamma' \rightarrow \Gamma$ of degree n . In [GUZ22], it was shown that the \mathbb{R}^n -part of the torsor E is canonically determined by a tropical line bundle L on Γ' , so that E is the direct image of L along f . We now extend this description to other tropical reductive groups.

First, we explain how to construct the covers. Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group corresponding to the root datum $\Phi = (M, R, \check{M}, \check{R})$, where $W = W_{\Phi}$ is the Weyl group. We choose a finite set T with n elements and an injective homomorphism $\rho : W \rightarrow S_T$, where S_T is the permutation group of T . For every metric graph Γ , we have an induced functor from the category of W -torsors on Γ to the category of S_T -torsors on Γ , which, in turn, is equivalent to the category of degree n covers of Γ . We now discuss what additional structure is necessary to put on a degree n cover to recover from it a W -torsor.

Definition 3.2.2. A ρ -cover of Γ is a free degree n cover $\Gamma' \rightarrow \Gamma$ together with an element

$$\xi_x \in \text{Bij}(T, \Gamma'_x)/W$$

for each $x \in \Gamma$, where Γ'_x is the fiber over $x \in \Gamma$ and W acts on the set $\text{Bij}(T, \Gamma'_x)$ of bijections between T and Γ'_x via ρ , such that each $x \in \Gamma$ has an open neighborhood U on which there is a trivialization

$$\phi : U \times T \xrightarrow{\cong} \Gamma'_U$$

for which ϕ_y represents ξ_y for every $y \in U$.

A morphism of two ρ -covers $(\Gamma' \rightarrow \Gamma, (\xi_x)_x)$ and $(\tilde{\Gamma} \rightarrow \Gamma, (\tilde{\xi}_x)_x)$ is a morphism $f : \Gamma' \rightarrow \tilde{\Gamma}$ of covers such that $f_x \circ \xi_x = \tilde{\xi}_x$ for all $x \in \Gamma$.

Proposition 3.2.3. *The category of ρ -covers of Γ is equivalent to the category of W -torsors on Γ .*

Proof. The trivial cover $I = \Gamma \times T$ becomes a ρ -cover by choosing $\xi_x = \text{id}_T$ for all $x \in \Gamma$. Since ρ -covers are locally isomorphic to I , the category of ρ -covers is equivalent to the category of torsors over the sheaf of automorphisms of I . Hence it suffices to prove that the morphism

$$\underline{W} \longrightarrow \underline{\text{Aut}}(I) \quad \text{given by} \quad w \longmapsto \text{id} \times \rho(w) ,$$

of sheaves on Γ is an isomorphism¹. The automorphism group of the cover I_U (ordinary cover, not ρ -cover for now) of a connected open subset U of Γ is precisely S_T . So it suffices to show that $\sigma \in S_T$ defines a morphism of ρ -covers if and only if $\sigma \in \rho(W)$. To see this, we observe that σ defines a morphism of ρ -covers if and only if σ is in the same W -orbit as the identity in $\text{Bij}(T, T)$, which happens if and only if there exists a $w \in W$ with

$$\sigma = \text{id}_T \circ \rho(w) = \rho(w) .$$

□

We now work out this correspondence for the tropical reductive groups that we considered in Section 3.1.

Example 3.2.4.

SL_n : Here we choose $T = [n]$ and an isomorphism $\rho : W \rightarrow S_n$, so that $\text{Bij}([n], \Gamma'_x)/W$ is a singleton for every fiber Γ'_x . Therefore, the category of W -torsors on Γ is equivalent to the category of degree n covers on Γ .

PGL_n : As for SL_n , the Weyl group W is isomorphic to S_n , and W -torsors are equivalent to degree n covers.

¹The two categories in question are neutral gerbes with global objects I and \underline{W} , respectively

Sp_{2n} : Here $T = [\pm n]$ is the $2n$ -element set with a fixed-point-free sign involution $\iota : [\pm n] \rightarrow [\pm n]$. The Weyl group W of Sp_{2n} is the signed permutation group S_n^B , which comes with a natural embedding $\rho : S_n^B \hookrightarrow S_T$. Let $\mathrm{Inv}(S)$ denote the set of fixed-point-free involutions of a set S of size $2n$. Then there is a natural isomorphism

$$\mathrm{Bij}([\pm n], S)/W \xrightarrow{\sim} \mathrm{Inv}(S) \quad \text{given by} \quad \bar{\xi} \longmapsto \xi \circ \iota \circ \xi^{-1} .$$

It follows that the category of W -torsors is equivalent to the category of degree $2n$ covers together with a fixed-point-free involution.

SO_{2n} : Here $T = [\pm n]$ as above, the Weyl group W is the even signed permutation group S_n^D , and the image of the embedding $\rho : S_n^D \hookrightarrow S_T$ lies in the alternating group A_T . Because we have

$$S_n^D = S_n^B \cap A_{2n} ,$$

we obtain, for every $2n$ -element set S , a natural bijection

$$\mathrm{Bij}(T, S)/W \xrightarrow{\cong} \mathrm{Inv}(S) \times \mathrm{Bij}(T, S)/A_T .$$

We recall that, given a degree m free cover $\Gamma' \rightarrow \Gamma$ defined by an S_m -torsor, the *orientation cover* $O(\Gamma') \rightarrow \Gamma$ is the degree 2 free cover defined by taking the quotient by A_m . In other words, if Γ'_x is the fiber of a cover $\Gamma' \rightarrow \Gamma$ over a point $x \in \Gamma$, then $\mathrm{Bij}([\pm n], \Gamma'_x)/A_{2n}$ is the fiber over x of the associated orientation cover. We thus obtain an equivalence of categories between the category of W -torsors and the category of degree $2n$ covers together with a fixed-point-free involution and a trivialization of the orientation cover.

SO_{2n+1} : In this case, $T = \{-n, \dots, n\}$ and the sign involution acts with a fixed point. The Weyl group W of SO_{2n+1} is S_n^B , the same as for $\mathrm{Sp}_{2n}(\mathbb{T})$ but now viewed as lying in the larger group S_T . The category of W -torsors on Γ is equivalent to the category of degree $2n + 1$ covers of Γ together with an involution having a unique fixed point in every fiber. Removing the fixed points (which form a copy of Γ), we obtain a degree $2n$ cover together with a fixed-point-free involution, as for Sp_{2n} .

G_2 : Here $T = \{1, \dots, 6\}$ and the image of $\rho : W \rightarrow S_6$ is the dihedral group D_6 . Therefore, for every 6-element set S , the set $\mathrm{Bij}(T, S)/W$ is identified with the possible arrangements of six distinct keys on a keychain, in other words the set of labelings of the vertices of a regular hexagon by elements of T , modulo rotations and reflections. Hence a W -torsor on Γ is a degree 6 free cover $\Gamma' \rightarrow \Gamma$ together with a locally trivial identification of the points of each fiber with the keys on a fixed keychain.

3.2.2 Tropical G -bundles via line bundles on covers

We now upgrade the injective homomorphism $\rho : W \rightarrow S_n$ to a *representation of tropical reductive groups*

$$F = (f, \rho) : G \longrightarrow \mathrm{GL}_n(\mathbb{T}) ,$$

where $G = \check{M}_{\mathbb{R}} \rtimes W$ is our tropical reductive group, $\mathrm{GL}_n(\mathbb{T}) = \mathbb{R}^n \rtimes S_n$, and the lattice map $f : \check{M} \rightarrow \mathbb{Z}^n$ is injective. We obtain, for any metric graph Γ , a morphism from the category of $(\check{M} \otimes \mathcal{H}_{\Gamma}) \rtimes W$ -torsors on Γ to the category of $(\mathbb{Z}^n \otimes \mathcal{H}_{\Gamma}) \rtimes S_n$ -torsors on Γ ; that is to say, tropical vector bundles on Γ . As described in [GUZ22], the category of tropical vector bundles on Γ is equivalent to the category of free covers $\Gamma' \rightarrow \Gamma$ together with a tropical line bundle on Γ' ; we refer to such a pair as a *multi-line bundle* on Γ . In particular, a $(\check{M} \otimes \mathcal{H}_{\Gamma}) \rtimes W$ -torsor on Γ induces a cover $\Gamma' \rightarrow \Gamma$ and a line bundle on Γ' . We now describe the extra structure needed on the multi-line bundle to recover the category of $(\check{M} \otimes \mathcal{H}_{\Gamma}) \rtimes W$ -torsors.

First, we temporarily consider a broader category of tropical groups. Let W be a finite group acting on a lattice \check{M} . We call the semidirect product $\check{M}_{\mathbb{R}} \rtimes W$ a *tropical linear group*. Similarly, a \mathbb{Z} -linear homomorphism $f : \check{M}_1 \rightarrow \check{M}_2$ and a group homomorphism $\phi : W_1 \rightarrow W_2$ satisfying $\phi(g)(f(m)) = f(g(m))$ define a *homomorphism of tropical linear groups*

$$F = (f, \phi) : \check{M}_{1, \mathbb{R}} \rtimes W_1 \rightarrow \check{M}_{2, \mathbb{R}} \rtimes W_2 \quad \text{given by} \quad F(m, g) = (f(m), \phi(g)) .$$

Given a metric graph Γ and a tropical linear group $G = \check{M}_{\mathbb{R}} \rtimes W$, we consider torsors over the sheaf $G_{\Gamma} = (\check{M} \otimes_{\mathbb{Z}} \mathcal{H}_{\Gamma}) \rtimes W$ as in Section 3.2.1.

Given a morphism $F : G \rightarrow H$ of tropical linear groups and a G_{Γ} -torsor E on Γ , we define the induced H_{Γ} -torsor, denoted by $F_*(E)$ or E_H if F is clear from the context, by

$$F_*(E) = E_H = (E \times H_{\Gamma}) / G_{\Gamma} ,$$

where G_{Γ} acts by the rule $g.(e, h) = (ge, hF(g)^{-1})$.

Proposition 3.2.5. *Let $G_i = \check{M}_{i, \mathbb{R}} \rtimes W_i$ for $i = 1, 2, 3$ be tropical linear groups and let*

$$G_1 \xrightarrow{F} G_2 \xrightarrow{H} G_3$$

be morphisms of tropical linear groups. Assume the following:

1. *F is injective.*
2. *There is a sublattice $\check{L} \subseteq \check{M}_3$ and a subgroup $Y \subseteq W_3$ such that $Y\check{L} \subseteq \check{L}$, the image of the map $\check{M}_1 \rtimes W_1 \rightarrow \check{M}_2 \rtimes W_2$ is the preimage of $\check{L} \rtimes Y$, and such that $\check{L} \rtimes Y$ and the image of $\check{M}_2 \rtimes W_2$ generate $\check{M}_3 \rtimes W_3$.*

Moreover, let Γ be a metric graph and let $K = \check{L}_{\mathbb{R}} \rtimes Y \subseteq G_3$ be the tropical linear group determined by \check{L} and Y . Then there is an equivalence of categories between the category of $G_{1, \Gamma}$ -torsors on Γ and the category of triples (T, T', ϕ) consisting of an $G_{2, \Gamma}$ -torsor T , a K_{Γ} -torsor T' and an isomorphism $(T')_{G_{3, \Gamma}} \xrightarrow{\phi} T_{G_{3, \Gamma}}$, where $(T')_{G_{3, \Gamma}}$ and $T_{G_{3, \Gamma}}$ are the $G_{3, \Gamma}$ -torsors induced by the homomorphisms $K \hookrightarrow G_3$ and $H : G_2 \rightarrow G_3$, respectively.

Proof. Let $I = (G_2, K, K_{G_3} \cong G_3 \xrightarrow{\text{id}} G_3 \cong (G_2)_{G_3})$ be the trivial element. By definition of the category, we have

$$\underline{\text{Aut}}(I) = \underline{\text{Aut}}(G_2) \times_{\underline{\text{Aut}}(G_3)} \underline{\text{Aut}}(K) .$$

As we have $\underline{\text{Aut}}(L_i) = L_i$ and $\underline{\text{Aut}}(K) = K$ it follows that

$$\underline{\text{Aut}}(I) = G_2 \times_{G_3} K \cong H^{-1}K \cong G_1 .$$

It thus suffices to prove that every object (T, T', ϕ) is locally isomorphic to I . Working locally, we may assume that we are given trivializations

$$G_2 \xrightarrow{\psi} T \quad \text{and} \quad K \xrightarrow{\chi} T' .$$

These induce trivializations

$$G_3 \cong (G_2)_{G_3} \xrightarrow{\psi_{G_3}} T_{G_3} \quad \text{and} \quad G_3 \cong K_{G_3} \xrightarrow{\chi_{G_3}} T'_{G_3} .$$

Let $\delta = \psi_{G_3}^{-1} \circ \phi \circ \chi_{G_3}$. This is a section of $\underline{\text{Aut}}(G_3) = G_3$, and by assumption we can locally decompose it as $\delta = H(\alpha) \cdot \beta^{-1}$ for some $\alpha \in G_2$ and $\beta \in K$. Denote by $r_{\alpha} : G_2 \rightarrow G_2$, $r_{H(\alpha)} : G_3 \rightarrow G_3$, and $r_{\beta} : K \rightarrow K$ right multiplication by α , $H(\alpha)$, and β , respectively. We claim that $(\psi \circ r_{\alpha}, \chi \circ r_{\beta})$ defines an isomorphism $I \rightarrow (T, T', \phi)$. And indeed, we have

$$\phi \circ (\chi \circ r_{\beta})_{G_3} = \phi \circ \chi_{G_3} \circ (r_{\beta})_{G_3} = \psi_{G_3} \circ \delta \circ (r_{\beta})_{G_3} = \psi_{G_3} \circ r_{H(\alpha)} = (\psi \circ r_{\alpha})_{G_3} \circ \text{id}_{G_3} .$$

□

Let $G_i = (\check{M}_i, W_i)$ for $i = 1, 2, 3$ be tropical linear groups. We say that a sequence of morphisms $F = (f, \phi) : G_1 \rightarrow G_2$ and $H = (h, \psi) : G_2 \rightarrow G_3$ is a *short exact sequence of tropical linear groups* if $0 \rightarrow \check{M}_1 \xrightarrow{f} \check{M}_2 \xrightarrow{h} \check{M}_3 \rightarrow 0$ and $1 \rightarrow W_1 \xrightarrow{\phi} W_2 \xrightarrow{\psi} W_3 \rightarrow 1$ are short exact sequences.

Corollary 3.2.6. *Let*

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be a short exact sequence of tropical linear groups and let Γ be a metric graph. Then there is an equivalence of categories between the category of $G_{1,\Gamma}$ -torsors on Γ and the category of pairs (T, ϕ) consisting of an $G_{2,\Gamma}$ -torsor T on Γ and a trivialization $G_3 \xrightarrow{\phi} T_{G_3}$.

Proof. This follows directly from Proposition 3.2.5 with $K = 1$. \square

We now describe G -covers on a metric graph Γ in terms of line bundles, in the case when the lattice map $f : \check{M} \rightarrow \mathbb{Z}^n$ associated to the chosen representation $F : G \rightarrow \mathrm{GL}_n(\mathbb{T})$ is the identity map.

Corollary 3.2.7. *Let $\rho : W \rightarrow S_n$ be an injective homomorphism and let Γ be a metric graph. Then there is an equivalence of categories of $\mathcal{H}_\Gamma^n \rtimes W$ -torsors on Γ and degree n multi-line bundles on Γ together with the structure of a ρ -cover on the underlying cover.*

Proof. This follows directly from Proposition 3.2.3 combined with Proposition 3.2.5 applied to the sequence

$$\mathbb{R}^n \rtimes W \rightarrow \mathbb{R}^n \rtimes S_n \rightarrow S_n$$

with $K = W$. \square

We also consider the more general setting where the map $f : \check{M} \rightarrow \mathbb{Z}^n$ associated to $F : G \rightarrow \mathrm{GL}_n(\mathbb{T})$ is injective. Let Γ be a metric graph and let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group. Since the inclusion $W \rightarrow G$ splits canonically, every G_Γ -torsor T has an associated W -torsor T_W , which in turn has an associated G_Γ -torsor $(T_W)_{G_\Gamma}$ that we denote by T^0 .

Corollary 3.2.8. *Let*

$$\check{M}_{1,\mathbb{R}} \rtimes W \xrightarrow{F=(f,\phi)} \check{M}_{2,\mathbb{R}} \rtimes W \xrightarrow{H=(h,\psi)} \check{M}_{3,\mathbb{R}} \rtimes W'$$

be a sequence of tropical reductive groups with the following properties:

1. *The kernel of ψ contains the image of ϕ .*

2. *The sequence $0 \rightarrow \check{M}_1 \xrightarrow{f} \check{M}_2 \xrightarrow{h} \check{M}_3 \rightarrow 0$ is exact.*

For any metric graph Γ , there is an equivalence of categories between the category of $(\check{M}_1 \otimes_{\mathbb{Z}} \mathcal{H}_\Gamma) \rtimes W$ -torsors on Γ and the category of pairs (T, ξ) consisting of an $(\check{M}_2 \otimes_{\mathbb{Z}} \mathcal{H}_\Gamma) \rtimes W$ -torsor T and an isomorphism $T_{(\check{M}_3 \otimes_{\mathbb{Z}} \mathcal{H}_\Gamma) \rtimes W'}^0 \xrightarrow{\xi} T_{(\check{M}_3 \otimes_{\mathbb{Z}} \mathcal{H}_\Gamma) \rtimes W'}$.

Proof. This follows directly from Proposition 3.2.5 when taking $K = W'$. \square

We now explicitly describe G -bundles on a metric graph Γ in terms of line bundles on covers of Γ' , in the case when G is one of the tropical reductive groups described in Section 3.1.

Example 3.2.9.

SL_n : The standard representation $\mathrm{SL}_n(\mathbb{T}) \rightarrow \mathrm{GL}_n(\mathbb{T})$, where the map $S_n \rightarrow S_n$ on the Weyl groups is the identity but the lattice map is $\mathbb{Z}_0^n \rightarrow \mathbb{Z}^n$, fits into the short exact sequence of tropical reductive groups

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{T}) \longrightarrow \mathrm{GL}_n(\mathbb{T}) \longrightarrow \mathbb{R} \longrightarrow 0$$

where the second map $\mathrm{GL}_n(\mathbb{T}) \rightarrow \mathbb{R} = \mathrm{GL}_1(\mathbb{T})$ is the tropical determinant. Denote by $\det(T)$ the tropical line bundle on Γ associated to a $\mathrm{GL}_n(\mathbb{T})$ -bundle T . By Corollary 3.2.6, there is an equivalence of categories between $\mathrm{SL}_n(\mathbb{T})$ -bundles on Γ and pairs (T, ϕ) consisting of a $\mathrm{GL}_n(\mathbb{T})$ -torsor T on Γ and a trivialization $\phi : \mathcal{H}_\Gamma \rightarrow \det(T)$. The $\mathrm{GL}_n(\mathbb{T})$ -bundle T may be represented by a multi-line bundle on a degree n cover $\Gamma' \rightarrow \Gamma$, whose fiber over $x \in \Gamma$ is the disjoint union of \mathbb{R} -torsors L_1, \dots, L_n . Under this identification, the fiber of $\det(T)$ over x can be naturally identified with $L_1 \otimes \dots \otimes L_n$.

Sp_{2n} : The description of $\mathrm{Sp}_{2n}(\mathbb{T})$ as a matrix group over \mathbb{T} gives a representation $(f, \rho) : \mathrm{Sp}_{2n}(\mathbb{T}) \rightarrow \mathrm{GL}_{[n] \sqcup -[n]}(\mathbb{T})$, which determines a sequence of tropical linear groups

$$\mathrm{Sp}_{2n}(\mathbb{T}) \longrightarrow \mathbb{R}^{[n] \sqcup -[n]} \rtimes S_n^B \longrightarrow \mathbb{R}^n \rtimes S_n = \mathrm{GL}_n(\mathbb{T}) .$$

Here the first morphism is the identity map $S_n^B \rightarrow S_n^B$ on the groups and the diagonal embedding $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{[n] \sqcup -[n]}$, $e_i \mapsto e_i - e_{-i}$ on the lattices, while the second morphism is the natural quotient map $S_n^B \rightarrow S_n$ on the groups and the map $(x_i)_{i \in [n] \sqcup -[n]} \mapsto (x_i + x_{-i})_{i \in [n]}$ on the lattices. By Corollary 3.2.8 the category of $\mathrm{Sp}_{2n}(\mathbb{T})$ -bundles on Γ is equivalent to the category of pairs (T, ϕ) consisting of a $\mathbb{R}^{[n] \sqcup -[n]} \rtimes S_n^B$ -bundle T on Γ and an isomorphism $\phi : T_{\mathrm{GL}_n}^0 \rightarrow T$. By Corollary 3.2.7, a $\mathbb{R}^{[n] \sqcup -[n]} \rtimes S_n^B$ -bundle T on Γ is equivalent to a line bundle on a ρ -cover $\Gamma' \rightarrow \Gamma$. We have seen in Example 3.2.4 that ρ -covers are in turn equivalent to pairs $(\Gamma' \rightarrow \Gamma, \iota)$ consisting of a degree $2n$ cover and a fixed-point-free involution ι of the cover. Hence the $\mathbb{R}^{[n] \sqcup -[n]} \rtimes S_n^B$ -bundle is equivalent to a multi-line bundle $(\Gamma' \rightarrow \Gamma, L)$ with a fixed-point-free involution ι of $\Gamma' \rightarrow \Gamma$. The associated $\mathrm{GL}_n(\mathbb{T})$ -torsor on Γ corresponds to the degree n cover $\Gamma'/\iota \rightarrow \Gamma$ equipped with the line bundle $(L \otimes \iota^{-1}L)/\iota$, whose fiber over $\bar{x} \in \Gamma'/\iota$ is $L_x \otimes L_{\iota(x)}$. In summary, the category of $\mathrm{Sp}_{2n}(\mathbb{T})$ -bundles on Γ is equivalent to the category of quadruples $(f : \Gamma' \rightarrow \Gamma, \iota, L, \phi)$ consisting of a degree $2n$ free cover f , a fixed-point-free involution ι of f , a tropical line bundle L on Γ' , and a trivialization $\phi : \mathcal{H}_{\Gamma'/\iota} \rightarrow (L \otimes \iota^{-1}L)/\iota$. Figure 3.1 below illustrates this in the case $n = 2$: an $\mathrm{Sp}_4(\mathbb{T})$ -bundle on a tropical elliptic curve Γ , represented by a degree 4 cover $\Gamma' \rightarrow \Gamma$ with a fixed-point-free involution $\iota : \Gamma' \rightarrow \Gamma'$ of the cover and a compatible tropical line bundle L on Γ' . Concretely, the line bundle L can be represented by a divisor D on Γ' such that $(D + \iota^{-1}D)/\iota \sim 0$.

SO_{2n+1} : Since $\mathrm{SO}_{2n+1}(\mathbb{T})$ and $\mathrm{Sp}_{2n}(\mathbb{T})$ are isomorphic as tropical linear groups, we obtain the same description as for $\mathrm{Sp}_{2n}(\mathbb{T})$ -bundles.

SO_{2n} : We have a short exact sequence of tropical linear groups

$$1 \longrightarrow \mathrm{SO}_{2n}(\mathbb{T}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{T}) \longrightarrow S_2 \longrightarrow 1 .$$

By Corollary 3.2.6, the category of $\mathrm{SO}_{2n}(\mathbb{T})$ -bundles on a metric graph Γ is equivalent to the category of quintuples $(f : \Gamma' \rightarrow \Gamma, \iota, L, \phi, \psi)$, where $(f : \Gamma' \rightarrow \Gamma, \iota, L, \phi)$ is as in the $\mathrm{Sp}_{2n}(\mathbb{T})$ -case and ψ is a trivialization of the orientation double cover $O(\Gamma') \rightarrow \Gamma$ associated to f (compare with the SO_{2n} -case in Example 3.2.4).

G_2 : In Subsection 3.1.5 we gave an explicit presentation of $G_2(\mathbb{T})$ inside $\mathrm{GL}_7(\mathbb{T})$, which restricts to a homomorphism $F = (f, \rho) : G_2(\mathbb{T}) \rightarrow \mathrm{GL}_6(\mathbb{T})$. Here $\rho : W = D_6 \rightarrow S_6$ is the standard embedding and the image of $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^6$ is the sublattice M given by

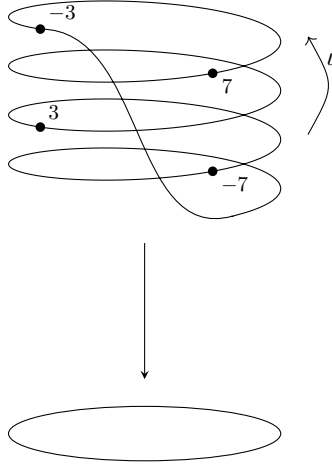


Figure 3.1: An $\mathrm{Sp}_4(\mathbb{T})$ -bundle on a tropical elliptic curve.

the four relations

$$x_1 + x_4 = 0, \quad x_2 + x_5 = 0, \quad x_3 + x_6 = 0, \quad x_1 + x_3 + x_5 = 0.$$

Let V be the lattice between M and \mathbb{Z}^6 given by the first three of these relations. Then there is a sequence of tropical linear groups

$$G_2(\mathbb{T}) \longrightarrow V_{\mathbb{R}} \rtimes D_6 \longrightarrow \mathrm{Sp}_2(\mathbb{T}),$$

where the second map sends $((x_i)_i, \sigma)$ to $(x_1 + x_3 + x_5, \mathrm{sgn}(\sigma))$. We note that an element $\sigma \in D_6$ is odd (as a permutation in S_6) if and only if it exchanges the sets $\{1, 3, 5\}$ and $\{2, 4, 6\}$, hence this map is a group homomorphism. Applying Corollary 3.2.8, we see that the category of $G_2(\mathbb{T})$ -bundles on a metric graph Γ is equivalent to the category of $V_{\mathbb{R}} \rtimes D_6$ -bundles on Γ together with an involution-invariant trivialization of the line bundle of the associated $\mathrm{Sp}_2(\mathbb{T})$ -cover.

To describe $V_{\mathbb{R}} \rtimes D_6$ -bundles in terms of covers, we note that the image of D_6 in S_6 is contained in S_3^B , the signed permutation group that preserves the involution corresponding to reflecting the hexagon through the origin. Hence we consider the sequence

$$V_{\mathbb{R}} \rtimes D_6 \rightarrow \mathrm{Sp}_6(\mathbb{T}) \rightarrow S_3^B$$

of tropical linear groups, where the first map on the lattices is the identity. We apply Proposition 3.2.5 with \check{L} trivial and $Y = D_6$ and obtain that the category of $V_{\mathbb{R}} \rtimes D_6$ -torsors is equivalent to the category of $\mathrm{Sp}_6(\mathbb{T})$ -covers, together with an order 6 cycle graph structure on the fibers such that opposite vertices in the cycle graph are interchanged by the involution of the $\mathrm{Sp}_6(\mathbb{T})$ -cover. The associated $\mathrm{Sp}_2(\mathbb{T})$ -cover has one branch for each of the distinguished triangles, which are interchanged by the involution, and the fibers of the line bundle on the domain of the $\mathrm{Sp}_2(\mathbb{T})$ -cover are the tensor products of the line bundles on the branches in each triangle.

In summary, the category of $G_2(\mathbb{T})$ -bundles on a metric graph Γ is equivalent to the category of quadruples $(\Gamma' \rightarrow \Gamma, L, \phi, \psi)$, where $\Gamma' \rightarrow \Gamma$ is a degree 6 free cover with a locally trivial identification of each fiber with the Star of David, L is a line bundle on Γ' , whereas ϕ is a trivialization of $(L \otimes \iota^{-1}L)/\iota$ on Γ'/ι , where the involution $\iota : \Gamma' \rightarrow \Gamma'$ exchanges the opposite vertices in each star, and ψ is an ι -invariant trivialization of the line bundle on the domain of the associated $\mathrm{Sp}_2(\mathbb{T})$ -cover whose fibers correspond to the two triangles.

3.3 Degree and stability in the algebraic setting

Let \mathbf{G} be a reductive linear algebraic group over an algebraically closed field k , and let X be a smooth projective curve over k . In this section, we recall the notions of degree and stability for \mathbf{G} -bundles on X and the associated stratification on the moduli space $\text{Bun}_{\mathbf{G}}(X)$ of \mathbf{G} -bundles on X , following [Sch15]. We also review the explicit description of the moduli spaces of stable and semistable \mathbf{G} -bundles on an elliptic curve, following [Frä16] and [Frä21].

3.3.1 Degree

We fix $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ a maximal torus and a Borel subgroup. We denote by $M = \mathbb{X}^*(\mathbf{T})$ and $\check{M} = \mathbb{X}_*(\mathbf{T})$ the character and cocharacter lattices of \mathbf{T} and denote by $R \subseteq M$ and $\check{R} \subseteq \check{M}$ the roots and coroots. The algebraic fundamental group of \mathbf{G} is defined as the quotient of the lattice of cocharacters by the lattice generated by the coroots: $\pi_1(\mathbf{G}) = \check{M} / \langle \check{\alpha} : \check{\alpha} \in \check{R} \rangle$. For $\check{\lambda} \in \check{M}$, we denote the corresponding element of the fundamental group by $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$. With these definitions, we have:

$$\begin{aligned} \pi_1(\mathbf{GL}_n) &= \mathbb{Z} , \\ \pi_1(\mathbf{SL}_n) &= 1 , \\ \pi_1(\mathbf{PGL}_n) &= \mathbb{Z}/n\mathbb{Z} , \\ \pi_1(\mathbf{Sp}_{2n}) &= 1 , \\ \pi_1(\mathbf{SO}_{2n}) &= \begin{cases} \mathbb{Z}/4\mathbb{Z} , & \text{if } n \text{ odd} , \\ (\mathbb{Z}/2\mathbb{Z})^2 , & \text{if } n \text{ even} . \end{cases} \end{aligned}$$

The choice of \mathbf{B} determines a partition $R = R^+ \sqcup R^-$ into positive and negative roots, as well as a set of simple positive roots $\{\alpha_i : i \in D\} \subseteq R^+$, where D is the set of vertices of the Dynkin diagram. Hence \mathbf{B} gives us a partial order on the cocharacter lattice \check{M} : we say that $\check{\lambda} \leq \check{\mu}$ if $\check{\mu} - \check{\lambda}$ is a nonnegative linear combination of positive coroots. This order extends naturally to real coefficients $\check{M}_{\mathbb{R}} = \check{M} \otimes_{\mathbb{Z}} \mathbb{R}$.

For a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, we denote by $\mathbf{L} = \mathbf{P}/U(\mathbf{P})$ its Levi quotient, where $U(\mathbf{P})$ is the unipotent radical. A parabolic subgroup \mathbf{P} of \mathbf{G} containing \mathbf{B} corresponds to a subset $D_{\mathbf{P}} \subseteq D$ of the simple roots (the Dynkin diagram of \mathbf{L}), in particular \mathbf{G} itself corresponds to D . We denote the algebraic fundamental group of \mathbf{P} by $\pi_1(\mathbf{P}) := \pi_1(\mathbf{L}) = \check{M} / \langle \check{\alpha}_i : i \in D_{\mathbf{P}} \rangle$, and for $\check{\lambda} \in \check{M}$ we denote by $\check{\lambda}_{\mathbf{P}}$ the corresponding element of $\pi_1(\mathbf{P})$.

We denote by $\text{Bun}_{\mathbf{G}}(X)$ the moduli stack of \mathbf{G} -bundles on X , that is to say, étale \mathbf{G} -torsors on X . It is well-known that the connected components of $\text{Bun}_{\mathbf{G}}(X)$ are in bijection with $\pi_1(\mathbf{G})$ (see [Hof10, Theorem 5.8] for a proof). We call elements of $\pi_1(\mathbf{G})$ *degrees*, and for $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$ we denote the corresponding connected component by $\text{Bun}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}}(X)$. For a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ with Levi quotient \mathbf{L} , the moduli spaces $\text{Bun}_{\mathbf{P}}(X)$ and $\text{Bun}_{\mathbf{L}}(X)$ have the same connected components and are in bijection with $\pi_1(\mathbf{P}) = \pi_1(\mathbf{L})$.

3.3.2 The slope map and semistability

We now define the *slope map* $\phi_{\mathbf{P}} : \pi_1(\mathbf{P}) \rightarrow \check{M}_{\mathbb{R}}$ for a parabolic subgroup $\mathbf{B} \subseteq \mathbf{P} \subseteq \mathbf{G}$, for details we refer to [Sch15]. We recall that the center of a reductive group \mathbf{L} with maximal torus \mathbf{T} is the intersection

$$Z(\mathbf{L}) = \bigcap_{\alpha \text{ root of } \mathbf{L}} \ker(\alpha) \subseteq \mathbf{T} .$$

The natural map $Z(\mathbf{L}) \rightarrow \mathbf{T}$ induces a map on the cocharacters. Taking the quotient by the coroots, we obtain a map

$$\mathbb{X}_*(Z(\mathbf{L})) \rightarrow \mathbb{X}_*(\mathbf{T}) = \check{M} \rightarrow \check{M}/\langle \check{\alpha} \text{ coroot of } \mathbf{L} \rangle = \pi_1(\mathbf{L}) = \pi_1(\mathbf{P}) ,$$

which becomes an isomorphism $\mathbb{X}_*(Z(\mathbf{L}))_{\mathbb{R}} \simeq \pi_1(\mathbf{P})_{\mathbb{R}}$ after tensoring with \mathbb{R} .

Definition 3.3.1. For a parabolic subgroup $\mathbf{B} \subseteq \mathbf{P} \subseteq \mathbf{G}$ with Levi subgroup \mathbf{L} we define the *slope map* $\phi_{\mathbf{P}} : \pi_1(\mathbf{P}) \rightarrow \check{M}_{\mathbb{R}}$ as

$$\pi_1(\mathbf{P}) \rightarrow \pi_1(\mathbf{P})_{\mathbb{R}} \cong \mathbb{X}_*(Z(\mathbf{L}))_{\mathbb{R}} \rightarrow \check{M}_{\mathbb{R}} .$$

The *slope* of a \mathbf{P} -bundle $F_{\mathbf{P}} \in \text{Bun}_{\mathbf{P}}^{\check{\lambda}_{\mathbf{P}}}(X)$ is the element $\phi_{\mathbf{P}}(\check{\lambda}_{\mathbf{P}})$. We say that a \mathbf{G} -bundle $F \in \text{Bun}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}}(X)$ on X is *(semi)stable* if for any proper parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ and for any reduction $F_{\mathbf{P}}$ of F to \mathbf{P} of degree $\check{\lambda}_{\mathbf{P}}$ we have

$$\phi_{\mathbf{P}}(\check{\lambda}_{\mathbf{P}}) \stackrel{(\leq)}{<} \phi_{\mathbf{G}}(\check{\lambda}_{\mathbf{G}}) .$$

For $\mathbf{G} = \mathbf{GL}_n$, these notions reduce to the standard notions of slope and stability for vector bundles on X (see [Sch15, Section 2.2.4]).

We will use the following result of Frătilă (see Lemma 2.12 in [Fră16]). First, we note that for a parabolic subgroup $\mathbf{B} \subseteq \mathbf{P} \subseteq \mathbf{G}$, taking the quotient by the remaining roots defines a natural quotient map $p : \pi_1(\mathbf{P}) \rightarrow \pi_1(\mathbf{G})$.

Theorem 3.3.2 ([Fră16, Lemma 2.12]). *Let \mathbf{G} be a reductive group and let $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$ be a degree. Then there exists a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ and a degree $\check{\lambda}_{\mathbf{P}} \in \pi_1(\mathbf{P})$ such that $\phi_{\mathbf{P}}(\check{\lambda}_{\mathbf{P}}) = \phi_{\mathbf{G}}(\check{\lambda}_{\mathbf{G}})$ and $p(\check{\lambda}_{\mathbf{P}}) = \check{\lambda}_{\mathbf{G}}$, and which is minimal with this property. The parabolic subgroup \mathbf{P} is unique up to conjugation.*

We note that \mathbf{P} is explicitly given by the set of roots $\{i \in D : \langle \omega_i, \phi_{\mathbf{G}}(\check{\lambda}_{\mathbf{G}}) - \check{\lambda} \rangle \notin \mathbb{Z}\}$, where $\check{\lambda} \in \check{M}$ is a lift of $\check{\lambda}_{\mathbf{G}}$ and the ω_i are the fundamental weights.

3.3.3 Stable and semistable \mathbf{G} -bundles over an elliptic curve

We now recall Frătilă's description of the moduli space of semistable \mathbf{G} -bundles on an elliptic curve X (see [Fră16] and [Fră21]). For $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$ we denote by $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \text{ss}}(X)$ and $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \text{st}}(X)$ the moduli spaces of semistable and stable \mathbf{G} -bundles on X of degree $\check{\lambda}_{\mathbf{G}}$, respectively, and we usually suppress the X from the notation.

We recall that the derived subgroup $\mathbf{G}^{\text{der}} = [\mathbf{G}, \mathbf{G}]$ of \mathbf{G} is semisimple with the same Weyl group W . The intersection $\mathbf{T} \cap \mathbf{G}^{\text{der}}$ is a maximal torus in \mathbf{G}^{der} and its character and cocharacter lattices are given by M/\check{R}^{\perp} and $\langle \check{R} \rangle^{\text{sat}}$, respectively (see [Spr98, Corollary 8.1.9]). The *cocenter* $Z^c(\mathbf{G})$ of \mathbf{G} is the quotient $\mathbf{G}/\mathbf{G}^{\text{der}}$, which is a torus with character and cocharacter lattices R^{\perp} and $\check{M}/\langle \check{R} \rangle^{\text{sat}}$. The quotient map $\mathbf{G} \rightarrow Z^c(\mathbf{G})$ is called the *determinant* (for $\mathbf{G} = \mathbf{GL}_n$, we have $Z^c(\mathbf{G}) = \mathbb{G}_m$ and this map is the usual matrix determinant).

First, we have the following explicit description of the moduli space of stable \mathbf{G} -bundles on X :

Theorem 3.3.3 ([Fră16, Corollary 4.3], [Fră21, Theorem 1.4]). *Let \mathbf{G} be a reductive group and let $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$.*

1. *The moduli space $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \text{st}}$ is nonempty only if \mathbf{G} is of type $\prod_i A_{n_i-1}$.*
2. *Suppose that \mathbf{G} is of type $\prod_i A_{n_i-1}$, so that*

$$\mathbf{G}^{\text{ad}} = \mathbf{G}/Z(\mathbf{G}) = \prod_i \mathbf{PGL}_{n_i} , \quad \pi_1(\mathbf{G}^{\text{ad}}) = \prod_i \mathbb{Z}/n_i\mathbb{Z} .$$

Then $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}},\text{st}}$ is nonempty if and only if the image of $\check{\lambda}_{\mathbf{G}}$ in $\pi_1(\mathbf{G}^{\text{ad}})$ is of the form $(d_i)_i$, where $\gcd(d_i, n_i) = 1$ for all i . Furthermore, in this case the determinant map

$$\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}},\text{st}} \rightarrow \mathcal{M}_{Z^c(\mathbf{G})}^{\det(\check{\lambda}_{\mathbf{G}}),\text{ss}}$$

is an isomorphism.

Definition 3.3.4. A degree $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$ satisfying condition (2) above is called *stable*.

This description generalizes to semistable bundles.

Theorem 3.3.5 ([Frä21, Theorem 1.2]). *Let $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$. Let $\mathbf{L} = \mathbf{L}_{\check{\lambda}_{\mathbf{G}}} \subseteq \mathbf{G}$ be the Levi subgroup corresponding to the parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ given by Theorem 3.3.2, and let $\check{\lambda}_{\mathbf{L}} = \check{\lambda}_{\mathbf{P}} \in \pi_1(\mathbf{L}) = \pi_1(\mathbf{P})$ be the corresponding degree. Then*

1. *the inclusion $\mathbf{L} \subseteq \mathbf{G}$ induces a map $\pi: \mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},\text{ss}} \rightarrow \mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}},\text{ss}}$, and all semistable \mathbf{L} -bundles in $\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},\text{ss}}$ are stable.*
2. *π is finite and generically Galois with Galois group $W_{\mathbf{L},\mathbf{G}} = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, where $N_{\mathbf{G}}(\mathbf{L})$ is the normalizer of \mathbf{L} in \mathbf{G} .*
3. *the quotient map*

$$\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},\text{st}}/W_{\mathbf{L},\mathbf{G}} \rightarrow \mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}},\text{ss}}$$

is an isomorphism.

3.4 Moduli spaces of tropical principal bundles

Let Γ be a metric graph and let G be a tropical reductive group. In this section, we define tropical notions of degree and stability for G -bundles on Γ and describe the connected components of the moduli space of tropical G -bundles on Γ . On a tropical elliptic curve (that is to say, a metric circle) we describe the main components of the moduli spaces of stable and semistable bundles, establishing tropical analogues of Theorems 3.3.3 and 3.3.5.

3.4.1 Degree and stability in the tropical setting

Let $\Phi = (M, R, \check{M}, \check{R})$ be a root datum and let $G = (\check{M} \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W_{\Phi}$ be the corresponding tropical reductive group. In analogy with the algebraic setting, we define the *fundamental group* of Φ as

$$\pi_1(\Phi) = \check{M}/\langle \check{R} \rangle.$$

In a slight abuse of notation we refer to $\pi_1(\Phi)$ as the *fundamental group of G* and denote it by $\pi_1(G)$. We denote the image of a cocharacter $\check{\lambda} \in \check{M}$ in the fundamental group by $\check{\lambda}_G \in \pi_1(G)$.

To define the degree of a tropical G -bundle, we first show that there is a natural projection from $\check{M} \rtimes W_{\Phi}$ to the fundamental group of G .

Lemma 3.4.1. *The group $\langle \check{R} \rangle \rtimes W_{\Phi}$ is normal in $\check{M} \rtimes W_{\Phi}$. Hence there exists a well-defined surjective homomorphism*

$$\check{M} \rtimes W_{\Phi} \rightarrow \check{M}/\langle \check{R} \rangle = \pi_1(G) \quad \text{given by } (m, w) \mapsto \overline{m}.$$

Proof. Let $(r, w) \in \langle \check{R} \rangle \rtimes W_{\Phi}$ with $r \in \langle \check{R} \rangle$ and $w \in W_{\Phi}$. Then for $(m, v) \in \check{M} \rtimes W_{\Phi}$ we have $(m, v) \cdot (r, w) \cdot (m, v)^{-1} = (m + v.r - v w v^{-1}.m, v w v^{-1})$. To show that $m + v.r - v w v^{-1}.m \in \langle \check{R} \rangle$ we observe that \check{R} is invariant under the W_{Φ} -action, i.e., $v.r \in \langle \check{R} \rangle$ and that $w.m - m \in \langle \check{R} \rangle$ for every $w \in W_{\Phi}$ and $m \in \check{M}$. The latter fact we show by induction on the length of w .

Indeed, if $w = s_\alpha$ for a root α , then $w.m - m = \langle \alpha, m \rangle \check{\alpha} \in \langle \check{R} \rangle$. Let $l(w) > 1$, and write $w = s_\alpha w'$ with $l(w') = l(w) - 1$. Then $w.m - m = s_\alpha(w'.m - m) + s_\alpha.m - m \in \langle \check{R} \rangle$. This shows that $\langle \check{R} \rangle \rtimes W_\Phi$ is normal in $\check{M} \rtimes W_\Phi$. Hence there is a homomorphism

$$\check{M} \rtimes W_\Phi \rightarrow (\check{M} \rtimes W_\Phi) / (\langle \check{R} \rangle \rtimes W_\Phi) \cong \check{M} / \langle \check{R} \rangle = \pi_1(G) . \quad \square$$

We recall that the set of isomorphism classes of G -bundles on Γ is the non-abelian cohomology set $H^1(\Gamma, G_\Gamma)$, where G_Γ is the sheaf $(\check{M} \otimes_{\mathbb{Z}} \mathcal{H}) \rtimes W_\Phi$ of G -valued harmonic functions on Γ . We define a degree map

$$H^1(\Gamma, G_\Gamma) \rightarrow \pi_1(G)$$

as follows. Recall that the sheaf Ω of *harmonic 1-forms* on Γ is the cokernel of the map $\mathbb{R} \rightarrow \mathcal{H}$, where \mathbb{R} is the constant sheaf. The quotient map $\mathcal{H} \rightarrow \Omega$ (which sends a harmonic function to its derivative) and the homomorphism of Lemma 3.4.1 induce maps of sheaves

$$G_\Gamma = (\check{M} \otimes_{\mathbb{Z}} \mathcal{H}) \rtimes W_\Phi \longrightarrow (\check{M} \otimes_{\mathbb{Z}} \Omega) \rtimes W_\Phi \longrightarrow \Omega \otimes_{\mathbb{Z}} \pi_1(G) ,$$

which in turn induce a map of pointed sets $H^1(\Gamma, G_\Gamma) \rightarrow H^1(\Gamma, \Omega \otimes_{\mathbb{Z}} \pi_1(G))$ in cohomology. Since $\Omega \otimes_{\mathbb{Z}} \pi_1(G)$ is a sheaf of abelian groups, its H^1 is a group that we can compute using the universal coefficient theorem. On a graph, all cohomology groups vanish in dimensions 2 and above. By Lemma 1.4 in [GUZ22] there is a natural isomorphism $H^1(\Gamma, \Omega) \cong \mathbb{Z}$, hence we obtain

$$H^1(\Gamma, \Omega \otimes_{\mathbb{Z}} \pi_1(G)) \cong H^1(\Gamma, \Omega) \otimes_{\mathbb{Z}} \pi_1(G) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \pi_1(G) \cong \pi_1(G) .$$

The degree map is obtained by composing all of the above maps.

Definition 3.4.2. Let $F \in H^1(\Gamma, G_\Gamma)$ be a G -bundle on a metric graph Γ . The *degree* $\check{\lambda}_G \in \pi_1(G)$ of F is the image of F in $\pi_1(G)$.

The degree is closely related to the *determinant map*, which is the quotient morphism $\det: G \rightarrow \check{M}_{\mathbb{R}} / \langle \check{R} \rangle_{\mathbb{R}}$, which is well-defined by Lemma 3.4.1. The induced morphism

$$\check{M} / \langle \check{R} \rangle = \pi_1(G) \xrightarrow{\det_*} \pi_1(\check{M}_{\mathbb{R}} / \langle \check{R} \rangle_{\mathbb{R}}) = \check{M} / \langle \check{R} \rangle^{\text{sat}}$$

is the quotient map that divides out the torsion of $\pi_1(G)$. It follows that if F is a G -bundle, then the degree of $\det(F)$ is the torsion-free part of $\deg(F)$. In particular, if $\pi_1(G)$ is torsion-free, as for example in the case $G = \text{GL}_n(\mathbb{T})$, the degree of F coincides with the degree of $\det(F)$. Consequently, Definition 3.4.2 agrees with the notion of degree for tropical vector bundles introduced in Section 2.4 of [GUZ22] in the case $G = \text{GL}_n(\mathbb{T})$.

To define stability, we first define parabolic subgroups in the tropical setting. Fix a splitting $R = R^+ \sqcup R^-$ and let $\{\alpha_i \in R^+ : i \in D\}$ be the set of simple roots. Let $D_P \subseteq D$ be a subset of the simple roots and let $W_{\Phi, P} \subseteq W_\Phi$ be the subgroup generated by the reflections in $\{\alpha_i : i \in D_P\}$.

Definition 3.4.3. The *standard parabolic subgroup* corresponding to $D_P \subseteq D$ is the tropical reductive group

$$P = (\check{M} \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W_{\Phi, P} \subseteq G = (\check{M} \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W_\Phi .$$

We note that there is no notion of unipotent groups in the tropical setting, hence a parabolic subgroup is the same as its associated Levi subgroup. In particular, parabolic subgroups are reductive, contrary to the algebraic situation.

We now define slope in analogy with the algebraic setting. First, we compute the center of a tropical reductive group.

Lemma 3.4.4. *The center of a tropical reductive group $G = \check{M}_{\mathbb{R}} \rtimes W_{\Phi}$ is*

$$Z(G) = R^{\perp} = \{(m, 1) \in \check{M}_{\mathbb{R}} \rtimes W_{\Phi} : \langle R, m \rangle = 0\} .$$

In particular, we have

$$Z(G) \cong \check{M}_{\mathbb{R}} / \langle \check{R} \rangle_{\mathbb{R}} = \pi_1(G)_{\mathbb{R}} .$$

Proof. Let $(k, w) \in Z(\check{M}_{\mathbb{R}} \rtimes W_{\Phi})$. Then for all $m \in \check{M}_{\mathbb{R}}$ we have $(k, w) \cdot (m, 1) = (m, 1) \cdot (k, w)$ which is equivalent to $(k + w.m, w) = (m + k, w)$. Now, $w.m = m$ for all $m \in \check{M}$ implies $w = 1$ since the action of W on \check{M} is free. To see that $(k, 1) \in Z(\check{M}_{\mathbb{R}} \rtimes W_{\Phi})$ if and only if $k \in R^{\perp} = \bigcap_{\alpha \in R} \ker(\alpha)$, let $\alpha \in R$ and $m \in \check{M}$. Then $(k, 1) \cdot (m, s_{\alpha}) = (m, s_{\alpha}) \cdot (k, 1)$ is equivalent to $k + m = m + s_{\alpha}.k$. The latter is equivalent to $k = s_{\alpha}.k = k - \langle \alpha, k \rangle \check{\alpha}$ which is equivalent to $k \in \ker(\langle \alpha, \cdot \rangle) = \ker \alpha$.

The “in particular” statement follows from the decomposition $\check{M}_{\mathbb{R}} = R^{\perp} \oplus \langle \check{R} \rangle_{\mathbb{R}}$. \square

We note that by definition, the fundamental group of a standard parabolic subgroup is

$$\pi_1(P) = \check{M} / \langle \check{\alpha}_i : i \in D_P \rangle .$$

Definition 3.4.5. Let P be a standard parabolic subgroup corresponding to $D_P \subseteq D$, and let F_P be a P -bundle on Γ of degree $\check{\lambda}_P \in \pi_1(P)$. The *slope* of F_P is the image of $\check{\lambda}_P$ under the map

$$\phi_P : \pi_1(P) = \check{M} / \langle \check{\alpha}_i : i \in D_P \rangle \longrightarrow \check{M}_{\mathbb{R}} / \langle \check{\alpha}_i : i \in D_P \rangle_{\mathbb{R}} \cong Z(\check{M}_{\mathbb{R}} \rtimes W_{\Phi, P}) \longrightarrow \check{M}_{\mathbb{R}} ,$$

where the second map is the isomorphism given in Lemma 3.4.4.

Let F be a G -bundle on Γ and let $P \subseteq G$ be a parabolic subgroup. We say that F *admits a reduction to P* if there exists a P -bundle F_P on Γ such that $i_*(F_P) = F$, where $i : P \rightarrow G$ is the inclusion.

Definition 3.4.6. Let $G = (\check{M} \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W_{\Phi}$ be a tropical reductive group and let F be a G -bundle on Γ of degree $\check{\lambda}_G \in \pi_1(G)$. We say that F is *(semi)-stable* if for every subset $D_P \subseteq D$ and for every element $\check{\lambda}_P \in \check{M}_P$ such that F admits a reduction F_P to P of degree $\check{\lambda}_P \in \pi_1(P)$ we have

$$\phi_P(\check{\lambda}_P) \preceq \phi_G(\check{\lambda}_G) .$$

For $G = \mathrm{GL}_n(\mathbb{T})$, this definition agrees with the notion of slope semi-stability for tropical vector bundles introduced in [GUZ22]. Recall that the slope of a tropical vector bundle E on a metric graph Γ of degree $\check{\lambda}_G \in \pi_1(G) \cong \mathbb{Z}$ is defined to be the quotient $\mu(E) = \frac{\check{\lambda}_G}{\mathrm{rk}(E)}$. The slope of E is given by the formula

$$\phi_G(\check{\lambda}_G) = (\mu(E), \dots, \mu(E)) \in \mathbb{R}^n$$

where $\check{M}_{\mathbb{R}}$ is canonically identified with \mathbb{R}^n (see Section 2.2.4 in [Sch15]). More generally, let F_P be a P -bundle of degree $\check{\lambda}_P \in \pi_1(P)$ for some parabolic $P \subseteq \mathrm{GL}_n(\mathbb{T})$. Let E_1, \dots, E_m be the summands of E . Then the slope $\phi_P(\check{\lambda}_P)$ is given as

$$\phi_P(\check{\lambda}_P) = (\mu(E_1), \dots, \mu(E_1), \dots, \mu(E_m), \dots, \mu(E_m)) \in \mathbb{R}^n ,$$

where each $\mu(E_i)$ is repeated $\mathrm{rk}(E_i)$ times. Using the same arguments as in the algebraic setting, it is then elementary to show that Definition 3.4.6 is equivalent to Definition 5.1 of [GUZ22].

3.4.2 Moduli of tropical G -bundles

We now interpret the set $H^1(\Gamma, G_\Gamma)$ of isomorphism classes of G -bundles on Γ as a moduli space.

Definition 3.4.7. Let Γ be a metric graph and let G be a tropical linear group. The *moduli space of G -bundles on Γ* is the set $\mathcal{M}_G(\Gamma) = H^1(\Gamma, G_\Gamma)$ of isomorphism classes of G -bundles on Γ . If G is reductive, then for a degree $\check{\lambda}_G \in \pi_1(G)$, we denote by $\mathcal{M}_G^{\check{\lambda}_G}(\Gamma)$ the set of isomorphism classes of G -bundles having degree $\check{\lambda}_G$. As in the algebraic setting, we normally suppress Γ from the notation.

We now describe \mathcal{M}_G as the non-abelian Čech cohomology set $\check{H}^1(\mathcal{U}(\Gamma), G_\Gamma)$, computed using a canonical (except when Γ is a circle) acyclic cover $\mathcal{U}(\Gamma)$ of Γ (in Theorem 3.4.12, we give a more explicit description of \mathcal{M}_G as a disjoint union of finite quotients of torsors under tropical abelian varieties). We fix an oriented simple loopless model for Γ , also denoted by Γ by abuse of notation, by placing a vertex at the midpoint of each loop, and similarly splitting all multiedges. For an edge $e \in E(\Gamma)$ denote by $U_e \subseteq \Gamma$ the corresponding open subset (not containing the root vertices of e). Similarly, for a vertex $v \in V(\Gamma)$, denote by U_v the *star* around v , which is the union of v and the U_e for all edges e incident to v . We call $\mathcal{U}(\Gamma) = \{U_v\}_{v \in V(\Gamma)}$ the *star cover* of Γ .

First, we explicitly describe the sections of G_Γ . For an oriented edge $e \in E(\Gamma)$, identify U_e with the interval $(0, \ell(e))$. A section $g_e \in G_\Gamma(U_e)$ is an affine linear function with integer slopes valued in G :

$$g_e : U_e = (0, \ell(e)) \longrightarrow G \quad \text{given by} \quad g_e(t) = A_e t + B_e ,$$

where $A_e \in \check{M} \rtimes W_\Phi$ and $B_e \in \check{M}_\mathbb{R} \rtimes W_\Phi$. Similarly, let $v \in V(\Gamma)$ be a vertex with incident edges e_1, \dots, e_k oriented outwards. Identifying each U_{e_j} with $(0, \ell(e_j))$, a section $f_v \in G_\Gamma(U_v)$ is a k -tuple of functions

$$f_{v, e_i}(t) = A_{v, e_i} t + B_v \quad \text{for} \quad i = 1, \dots, k ,$$

where $A_{v, e_i} \in \check{M} \rtimes W_\Phi$ and $B_v \in \check{M}_\mathbb{R} \rtimes W_\Phi$, such that the $A_{v, e_1} + \dots + A_{v, e_k} = 0$.

We now explicitly describe the set $\check{H}^1(\Gamma, G_\Gamma)$ using the star cover. For $v, w \in V(\Gamma)$, the intersection $U_v \cap U_w$ is U_{vw} if there is an edge $vw \in E(\Gamma)$, and empty otherwise. Furthermore, all triple intersections are empty. Since each U_{vw} is contractible, the star cover is acyclic for the sheaf G_Γ . Therefore

$$\check{H}^1(\Gamma, G_\Gamma) = \check{H}^1(\mathcal{U}(\Gamma), G_\Gamma) = \{(g_{vw}) \in \prod_{vw \in E(\Gamma)} G_\Gamma(U_{vw})\} / \sim ,$$

where $(g_{vw}) \sim (g'_{vw})$ if there exists a tuple $(f_v) \in \prod_{v \in V(\Gamma)} G_\Gamma(U_v)$ such that $g_{vw} = f_v g'_{vw} f_w^{-1}$. We note that all triple intersections are empty, hence the cocycle condition is trivially satisfied.

Example 3.4.8 (G -bundles on a metric circle). Let $j > 0$ be a real number, and let $\Gamma = \mathbb{R}/j\mathbb{Z}$ be a circle of length j . Let (G, l) be an oriented model with two vertices $v_1, v_2 \in V(G)$ and consider the associated cover $\mathcal{U}(G) = \{U_{v_1}, U_{v_2}\}$. Note that here $\mathcal{U}(G)$ is not the star cover since the intersection $U_{v_1} \cap U_{v_2} = e_1 \sqcup e_2$ is the disjoint union of the two open edges e_1, e_2 . An element in $\mathcal{H}(e_i)$ is an affine linear function with integer slope valued in G , hence $G_\Gamma(e_i) = (\check{M} \times \check{M}_\mathbb{R}) \rtimes W_\Phi$, where the Weyl group acts diagonally. Thus,

$$G_\Gamma(U_{v_1} \cap U_{v_2}) = G_\Gamma(e_1) \times G_\Gamma(e_2) = ((\check{M} \times \check{M}_\mathbb{R}) \rtimes W_\Phi) \times ((\check{M} \times \check{M}_\mathbb{R}) \rtimes W_\Phi) ,$$

and hence $\check{H}^1(\mathcal{U}(G), G_\Gamma)$ is the set of tuples

$$(a, b) \in ((\check{M} \times \check{M}_\mathbb{R}) \rtimes W_\Phi) \times ((\check{M} \times \check{M}_\mathbb{R}) \rtimes W_\Phi)$$

modulo the relation $(a, b) \sim (f_1|_{e_1} a f_2^{-1}|_{e_1}, f_1|_{e_2} b f_2^{-1}|_{e_2})$ for $f_i \in G_\Gamma(U_{v_i})$. Let

$$f_1 = (k, \beta, w) \in G_\Gamma(U_{v_1}) = (\check{M} \times M_{\mathbb{R}}) \rtimes W_\Phi$$

and denote the translated function by $f'_1 = (k, \beta + kj, w)$. By setting $f_2 = f'_1 b$, the map $(a, b) \mapsto ab^{-1}$ yields an isomorphism of pointed sets

$$H^1(\Gamma, G_\Gamma) = G_\Gamma(e_1) / \sim ,$$

where $c \sim f_1|_{e_1} c f_1'^{-1}|_{e_1}$ for $c \in G_\Gamma(e_1) = (\check{M} \times M_{\mathbb{R}}) \rtimes W_\Phi, f_1 \in G_\Gamma(U_{v_1})$.

Explicitly, let $(m, \alpha, w), (k, \beta, w) \in (\check{M} \times M_{\mathbb{R}}) \rtimes W_\Phi$. Then

$$\check{H}^1(\Gamma, G_\Gamma) = ((\check{M} \times M_{\mathbb{R}}) \rtimes W_\Phi) / \sim$$

where

$$(m, \alpha, w) \sim (k, \beta, v)(m, \alpha, w)(k, \beta + jk, v)^{-1} \sim (k + v.m - v w v^{-1}.k, \beta + v.\alpha - v w v^{-1}.(\beta + jk), v w v^{-1}).$$

The isomorphism class of a W -torsor τ on a metric circle Γ corresponds to the conjugacy class of $w \in W$. The automorphism group $\text{Aut}(\tau)$ can then be identified with the centralizer $C_W(w)$.

We recall that in Section 3.2.2 we defined the pushforward of a G -bundle along a homomorphism of tropical reductive groups.

Lemma 3.4.9. *Let $F = (f, \phi): G \rightarrow H$ be a morphism of tropical linear groups such that both f and ϕ are surjective. Then the induced morphism*

$$F_*: \mathcal{M}_G \longrightarrow \mathcal{M}_H$$

is surjective.

Proof. Let \mathcal{U} be the star cover of Γ . Then, since triple intersections of sets in \mathcal{U} are empty and the cocycle condition is trivially satisfied, the surjectivity of f implies that the map

$$\mathcal{M}_G \cong \check{H}^1(\mathcal{U}, G_\Gamma) \xrightarrow{F} \check{H}^1(\mathcal{U}, H_\Gamma) \cong \mathcal{M}_H ,$$

which agrees with F_* , is surjective as well. \square

Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group. The quotient morphism $q: G \rightarrow W$ yields a map

$$q_*: \mathcal{M}_G \longrightarrow \mathcal{M}_W .$$

For $\tau \in \mathcal{M}_W$, the fiber under this map

$$\mathcal{M}_{G, \tau} := q_*^{-1}\{\tau\}$$

is the set of isomorphism classes of G -bundles E on Γ whose associated W -torsor $q_*(E) = E_W$ is isomorphic to τ . We denote by $\widetilde{\mathcal{M}}_{G, \tau}$ the set of isomorphism classes of pairs (E, ϕ) , where E is a G -bundle and $\phi: E_W \rightarrow \tau$ is an isomorphism. For a degree $\check{\lambda}_G \in \pi_1(G)$, we denote by $\mathcal{M}_{G, \tau}^{\check{\lambda}_G}$ and $\widetilde{\mathcal{M}}_{G, \tau}^{\check{\lambda}_G}$ the corresponding moduli spaces of bundles with degree $\check{\lambda}_G$. There is a canonical right action of $\text{Aut}(\tau)$ on $\widetilde{\mathcal{M}}_{G, \tau}$ coming from postcomposing ϕ with an automorphism of τ , and by definition we have

$$\mathcal{M}_{G, \tau} = \widetilde{\mathcal{M}}_{G, \tau} / \text{Aut}(\tau) .$$

Proposition 3.4.10. *Denoting the trivial W -torsor on Γ by W_Γ , we have natural bijections*

$$\begin{aligned}\widetilde{\mathcal{M}}_{G,W_\Gamma} &= \text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \check{M}, \\ \mathcal{M}_{G,W_\Gamma} &= (\text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \check{M})/W .\end{aligned}$$

Proof. By Corollary 3.2.6, the short exact sequence

$$0 \longrightarrow \check{M}_{\mathbb{R}} \longrightarrow G \longrightarrow W \longrightarrow 1$$

yields an equivalence of categories between $\check{M}_{\mathbb{R}}$ -bundles and pairs (E, ϕ) consisting of a G -bundle E and an isomorphism $E_W \xrightarrow{\phi} W_\Gamma$. Therefore, there is a natural bijection

$$\mathcal{M}_{\check{M}_{\mathbb{R}}} \xrightarrow{\cong} \widetilde{\mathcal{M}}_{G,W_\Gamma} .$$

We note that $\mathbb{R} = \text{GL}_1(\mathbb{T})$ as a tropical reductive group, hence $\mathcal{M}_{\mathbb{R}}(\Gamma) = \text{Pic}(\Gamma)$. Since \check{M} is free, we have a canonical bijection

$$\mathcal{M}_{\check{M}_{\mathbb{R}}} \xrightarrow{\cong} \text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \check{M} ,$$

showing the first isomorphism. For the second, we note that $\text{Aut}(W_\Gamma) = W$ because Γ is connected. \square

Example 3.4.11. (*G -bundles on metric trees*) Let Γ be a compact and connected metric tree. Recall that up to isomorphism, there is exactly one line bundle $\mathcal{H}_\Gamma(d)$ of degree d on Γ , i.e., $\text{Pic}(\Gamma) \cong \mathbb{Z}$. Let $G = \check{M}_{\mathbb{R}} \rtimes W_\Phi$ be a tropical reductive group and let E be a G -bundle on Γ . Since the fundamental group of Γ is trivial, the associated W -torsor E_W is isomorphic to the trivial torsor W_Γ . Hence, by Proposition 3.4.10 we obtain a natural bijection

$$\mathcal{M}_G \cong (\text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \check{M})/W \cong \check{M}/W .$$

This is a tropical analogue of a theorem of Grothendieck which states that given a split reductive group \mathbf{G} and a maximal split torus \mathbf{T} , any \mathbf{G} -bundle on \mathbb{P}^1 has a reduction of structure group to the maximal torus \mathbf{T} unique up to the action of the Weyl group W (see [Gro57], [Har68], or [MT12, Theorem 0.3]). If $G = \text{GL}_n(\mathbb{T})$, this means that every vector bundle on a metric tree splits as a direct sum of line bundles (see [GUZ22, Example 3.3]).

Our next goal is to describe the moduli space of G -bundles on a metric graph Γ as a rational polyhedral space. We first note that the moduli space \mathcal{M}_G decomposes as a finite disjoint union by the isomorphism type of the associated W -torsor:

$$\mathcal{M}_G = \coprod_{\tau \in \mathcal{M}_W} \mathcal{M}_{G,\tau} .$$

We now describe these moduli spaces.

Theorem 3.4.12. *Let τ be a W -torsor on Γ . Then $\widetilde{\mathcal{M}}_{G,\tau}$ is a disjoint union of torsors under tropical abelian varieties. Therefore, $\mathcal{M}_{G,\tau}$ is the quotient of a disjoint union of torsors under tropical abelian varieties by the finite group $\text{Aut}(\tau)$.*

We note that the $\widetilde{\mathcal{M}}_{G,\tau}$ are rational polyhedral spaces, but the $\mathcal{M}_{G,\tau}$, and hence the moduli space \mathcal{M}_G , are only finite group quotients of rational polyhedral spaces. We first prove several preliminary lemmas.

Let τ be a W -torsor on Γ . Then its total space $\tau \rightarrow \Gamma$ is a free finite covering of graphs, and we can pull back τ itself and obtain a W -torsor on τ . Of course, the total space of τ has a

W -action and so it makes sense to talk about W -equivariant objects in a category of torsors over τ . Recall that for a tropical reductive group H , a W -equivariant H -bundle on τ is an H -bundle E together with morphisms $m_w: E \rightarrow l_w^* E$ of H -bundles, one for each $w \in W$, where l_w is multiplication by w on the left on τ and which satisfy the obvious compatibility.

Lemma 3.4.13. *There is a W -equivariant isomorphism $\tau \times_{\Gamma} \tau \cong W \times \tau$, where we equip $W \times \tau$ with the W -action as follows:*

$$w.(v, t) = (vw^{-1}, wt) .$$

Proof. The map

$$\tau \times_{\Gamma} \tau \rightarrow W \times \tau \quad \text{given by} \quad (t_1, t_2) \mapsto (t_1 t_2^{-1}, t_2)$$

on total spaces is clearly a morphism of torsors and W -equivariant. \square

Consider the short exact sequence

$$0 \rightarrow \check{M}_{\mathbb{R}} \rightarrow G \rightarrow W \rightarrow 1$$

By Corollary 3.2.6, there is an induced bijection between $\mathcal{M}_{\check{M}_{\mathbb{R}}}$ and $\widetilde{\mathcal{M}}_{G, W_{\Gamma}}$. In particular, the $\text{Aut}(W_{\Gamma})$ -action on $\widetilde{\mathcal{M}}_{G, W_{\Gamma}}$ induces a right $\text{Aut}(W_{\Gamma})$ -action on $\mathcal{M}_{\check{M}_{\mathbb{R}}}$. Embedding W into $\text{Aut}(W_{\Gamma})$ via right multiplication, we obtain a right W -action on $\mathcal{M}_{\check{M}_{\mathbb{R}}}$. On the other hand, W acts on $\check{M}_{\mathbb{R}}$ by conjugation, and thus there exists an induced action

$$\mathcal{M}_{\check{M}_{\mathbb{R}}} \times W \rightarrow \mathcal{M}_{\check{M}_{\mathbb{R}}} , \quad (L, w) \mapsto L^w := (c_w)_* L ,$$

where $c_w: \check{M}_{\mathbb{R}} \rightarrow \check{M}_{\mathbb{R}}$ is given by $c_w(m) = m^w = w^{-1} m w$. We now show that the two W -actions on $\mathcal{M}_{\check{M}_{\mathbb{R}}}$ coincide.

Lemma 3.4.14. *Let $w \in W$. Under the bijection $\widetilde{\mathcal{M}}_{G, W_{\Gamma}} \cong \mathcal{M}_{\check{M}_{\mathbb{R}}}$, if a pair (E, ϕ) corresponds to an $\check{M}_{\mathbb{R}}$ -bundle L , then the pair $(E, \phi \circ w^{-1})$ corresponds to L^w . Here, we embed W in $\text{Aut}(W_{\Gamma})$ via right multiplication.*

Proof. We consider the induced morphisms of sheaves of groups $\check{M}_{\mathbb{R}, \Gamma} \xrightarrow{i} G_{\Gamma}$ and $G_{\Gamma} \xrightarrow{\pi} W$. Let L be an $\check{M}_{\mathbb{R}}$ -bundle and let $E = L_{G_{\Gamma}}$ be the induced G -bundle under the bijection $\widetilde{\mathcal{M}}_{G, W_{\Gamma}} \cong \mathcal{M}_{\check{M}_{\mathbb{R}}}$. We recall that E is the sheaf associated to the presheaf

$$U \mapsto (G_{\Gamma}(U) \times L(U)) / \sim ,$$

where the equivalence relation is $(g \cdot i(m)^{-1}, m \cdot x) \sim (g, x)$ for $m \in \check{M}_{\mathbb{R}}(U)$, $g \in G_{\Gamma}(U)$, and $x \in L(U)$. Now, for $w \in W$, the $\check{M}_{\mathbb{R}}$ -bundle L^w as a sheaf is $L^w = L$ but with left $\check{M}_{\mathbb{R}, \Gamma}$ -action $m \cdot_w x = c_w^{-1}(m) \cdot x = w m w^{-1} \cdot x$. The induced G -bundle $E^w = (L^w)_{G_{\Gamma}}$ is the sheaf associated to

$$U \mapsto (G_{\Gamma}(U) \times L^w(U)) / \sim_w ,$$

but now the equivalence is $(g \cdot i(m)^{-1}, w m w^{-1} \cdot x) \sim_w (g, x)$ for $m \in \check{M}_{\mathbb{R}}(U)$.

For both E and E^w , the associated W -torsors E_W and E_W^w have canonical trivializations $\phi_{\text{can}}: W_{\Gamma} \xrightarrow{\sim} E_W$ and $\phi_{\text{can}}^w: W_{\Gamma} \xrightarrow{\sim} E_W^w$. Explicitly, ϕ_{can} is the unique map such that the composite

$$E \longrightarrow E / \check{M}_{\mathbb{R}, \Gamma} = E_W \xrightarrow{\phi_{\text{can}}^{-1}} W_{\Gamma}$$

maps the equivalence class $[(g, x)]$ to $\pi(g)$, and similarly for E^w .

In the bijection $\widetilde{\mathcal{M}}_{G,W_\Gamma} \cong \mathcal{M}_{\widetilde{M}_\mathbb{R}}$, the bundle L corresponds to (E, ϕ_{can}) and L^w corresponds to $(E^w, \phi_{\text{can}}^w)$. The following G_Γ -equivariant map is an isomorphism of G -bundles

$$\psi: E \xrightarrow{\sim} E^w \quad \text{and} \quad [(g, x)] \mapsto [(gw, x)] .$$

The isomorphism ψ induces an isomorphism of the associated W -torsors $\psi_W: E_W \xrightarrow{\sim} E_W^w$ such that $\phi_{\text{can}} \circ w^{-1} = \psi_W^{-1} \circ \phi_{\text{can}}^w$, i.e., the line bundle L^w corresponds to the pair $(E^w, \phi_{\text{can}}^w) \cong (E, \phi_{\text{can}} \circ w^{-1})$. \square

Lemma 3.4.15. *Let τ be a W -torsor and let $f: \Gamma' \rightarrow \Gamma$ be a free Galois cover, for which there exists an isomorphism $f^*\tau \xrightarrow{\chi} W_{\Gamma'}$. Then there is a natural bijection*

$$\widetilde{\mathcal{M}}_{G,\tau}(\Gamma) \cong \mathcal{M}_{\widetilde{M}_\mathbb{R}}(\Gamma')^{\text{Aut}(f)} ,$$

where on the right side we take $\text{Aut}(f)$ -invariants under the following action: every automorphism $a: \Gamma' \rightarrow \Gamma'$ of the cover f induces an automorphism

$$W_{\Gamma'} \xrightarrow{\chi^{-1}} f^*\tau \cong a^*(f^*\tau) \xrightarrow{a^*\chi} a^*W_{\Gamma'} \cong W_{\Gamma'} ,$$

so that we obtain a morphism $\sigma: \text{Aut}(f) \rightarrow \text{Aut}(W_{\Gamma'})$. The group $\text{Aut}(W_{\Gamma'})$ in turn acts by conjugation on $\mathcal{M}_{\widetilde{M}_\mathbb{R}}(\Gamma')$. The action on $\text{Aut}(f)$ on $\mathcal{M}_{\widetilde{M}_\mathbb{R}}(\Gamma')$ is given by $(a, E) \mapsto \sigma(a)(a^*E)$. Furthermore, there is a map $\delta: \text{Aut}(\tau) \rightarrow \text{Aut}(W_{\Gamma'})$ induced by the trivialization χ such that the natural $\text{Aut}(\tau)$ -action on $\widetilde{\mathcal{M}}_{G,\tau}(\Gamma)$ is given by

$$\begin{aligned} \mathcal{M}_{\widetilde{M}_\mathbb{R}}(\Gamma')^{\text{Aut}(f)} \times \text{Aut}(\tau) &\longrightarrow \mathcal{M}_{\widetilde{M}_\mathbb{R}}(\Gamma')^{\text{Aut}(f)} \\ (E, t) &\longmapsto E^{\delta(t)} . \end{aligned}$$

Proof. Let $\mathcal{B}_{G,\tau}(\Gamma)$ be the category of pairs (E, ϕ) consisting of a G -bundle E on Γ and an isomorphism $\phi: \tau \rightarrow E_W$. By definition, $\widetilde{\mathcal{M}}_{G,\tau}(\Gamma)$ is the set of isomorphism classes of the objects of $\mathcal{B}_{G,\tau}(\Gamma)$. Because f is a free cover, a function on Γ is harmonic if and only if its pullback to Γ' is harmonic. Therefore $f^*G_\Gamma \cong G_{\Gamma'}$, and hence the pullback f^*E of a G -bundle E on Γ is a G -bundle on Γ' . Moreover, this pull-back is naturally an $\text{Aut}(f)$ -equivariant bundle via the canonical morphisms

$$f^*E \xrightarrow{\cong} a^*f^*E$$

for $a \in \text{Aut}(f)$. Because f is Galois, the category of G -bundles on Γ is in fact equivalent to the category of $\text{Aut}(f)$ -equivariant G -bundles on Γ' (one recovers E as $f^*E / \text{Aut}(f)$). Arguing similarly for W -torsors, we obtain an equivalence of $\mathcal{B}_{G,\tau}(\Gamma)$ with the category $\mathcal{B}_{G,f^*\tau}^{\text{Aut}(f)}(\Gamma')$ of pairs (E', ϕ') consisting of an $\text{Aut}(f)$ -equivariant G -bundle E' on Γ' and an $\text{Aut}(f)$ -equivariant isomorphism $f^*\tau \xrightarrow{\phi'} E'_W$, where similarly $f^*\tau$ is a W -torsor on Γ' . Let $W_{\Gamma'}^\sigma$ denote the trivial W -torsor $W_{\Gamma'}$ together with the $\text{Aut}(f)$ -equivariant structure induced by σ . By construction of σ , the trivialization χ defines an equivariant isomorphism $f^*\tau \xrightarrow{\chi} W_{\Gamma'}^\sigma$, and therefore induces an equivalence of categories

$$\mathcal{B}_{G,f^*\tau}^{\text{Aut}(f)}(\Gamma') \longrightarrow \mathcal{B}_{G,W_{\Gamma'}^\sigma}^{\text{Aut}(f)}(\Gamma') .$$

The objects of the target category are objects (E, ϕ) of $\mathcal{B}_{G,W_{\Gamma'}^\sigma}(\Gamma')$, together with compatible morphisms

$$(E, \phi) \longrightarrow (a^*E, a^{-1}\phi \circ \sigma(a))$$

for $a \in \text{Aut}(f)$.

Let $\mathcal{B}_{\check{M}_{\mathbb{R}}}(\Gamma')$ be the category of $\check{M}_{\mathbb{R}}$ -bundles on Γ' (note that the corresponding Weyl group is trivial). By Lemma 3.4.14, the category $\mathcal{B}_{\check{M}_{\mathbb{R}}}(\Gamma')$ is equivalent to the category $\mathcal{B}_{G, W_{\Gamma'}}(\Gamma')$, in such a way that if $F \in \mathcal{B}_{\check{M}_{\mathbb{R}}}(\Gamma')$ corresponds to (E, ϕ) then $\sigma^{(a)}F$ corresponds to $(E, \phi \circ \sigma(a))$. Composing all of the equivalences above, we conclude that $\mathcal{B}_{G, \tau}(\Gamma)$ is equivalent to the category $\tilde{\mathcal{B}}_{\check{M}_{\mathbb{R}}}^{\text{Aut}(f)}(\Gamma')$ of *twisted Aut(f)-equivariant $\check{M}_{\mathbb{R}}$ -bundles*, that is to say, $\text{Aut}(f)$ -equivariant objects of $\mathcal{B}_{\check{M}_{\mathbb{R}}}(\Gamma')$ with respect to the $\text{Aut}(f)$ -action given by $a.F = \sigma^{(a)}(a^{-1}F)$. We can also track the action of $\text{Aut}(\tau)$ through this equivalence. The $\text{Aut}(\tau)$ -action on $\mathcal{B}_{G, \tau}(\Gamma)$ corresponds to the $\text{Aut}(\tau)$ -action on $\mathcal{B}_{G, W_{\Gamma'}^{\sigma}}^{\text{Aut}(f)}(\Gamma')$ induced by the morphism δ , which in turn corresponds to the action $F.t = F^{\delta(t)}$ on twisted equivariant $\text{Aut}(f)$ -bundles by Lemma 3.4.14.

Forgetting the twisted equivariant structure and taking isomorphism classes assigns to every object of $\tilde{\mathcal{B}}_{\check{M}_{\mathbb{R}}}^{\text{Aut}(f)}(\Gamma')$ an $\text{Aut}(f)$ -invariant element of $\mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma')$. To finish the proof, it suffices to show that every element $[\tilde{S}] \in \mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma')$ that is fixed by $\text{Aut}(f)$ determines a unique isomorphism class of twisted $\text{Aut}(f)$ -equivariant $\check{M}_{\mathbb{R}}$ -bundles on Γ' . Because the isomorphism class $[\tilde{S}]$ of \tilde{S} is $\text{Aut}(f)$ -invariant, there are isomorphisms $m_a^0: \tilde{S} \rightarrow \sigma^{(a)}(a^{-1}\tilde{S})$ for all $a \in \text{Aut}(f)$. Chosen at random, these will, in general, not define a twisted $\text{Aut}(f)$ -equivariant structure. The obstruction for this is the triviality of the automorphisms

$$\psi(a_1, a_2): \tilde{S} \xrightarrow{m_{a_1}^0} \sigma^{(a_1)}(a_1^{-1}\tilde{S}) \xrightarrow{\sigma^{(a_1)}(a_1^{-1}m_{a_2}^0)} \sigma^{(a_1 a_2)}((a_1 a_2)^{-1}\tilde{S}) \xrightarrow{(m_{a_1 a_2}^0)^{-1}} \tilde{S}. \quad (3.1)$$

Note that because \check{M} is abelian, we have $\text{Aut}(\tilde{S}) = H^0(\Gamma', \check{M}_{\mathbb{R}})$, and because Γ' is compact this is a finite-dimensional \mathbb{R} -vector space. Every other choice of isomorphism $\tilde{S} \rightarrow \sigma^{(a)}(a^{-1}\tilde{S})$ is of the form $m_a^0 \circ \eta(a)$ for some $\eta(a) \in \text{Aut}(\tilde{S})$. Replacing all m_a^0 by $m_a^0 \circ \eta(a)$ in Equation (3.1), we see that $(m_a^0 \circ \eta(a))_{w \in W}$ defines a twisted $\text{Aut}(f)$ -equivariant structure if and only if

$$(d\eta)(a_1, a_2) := a_1 \eta(a_2) - \eta(a_1 a_2) + \eta(w_1)$$

is equal to $-\psi(a_1, a_2)$ in $\text{Aut}(\tilde{S})$ for all (a_1, a_2) . The notation $d\eta$ is not an accident: η defines an inhomogeneous 1-cochain, which is an element in $C^1(\text{Aut}(f), \text{Aut}(\tilde{S}))$, and $d\eta$ is precisely its differential. One also checks that ψ (and hence $-\psi$), is an inhomogeneous 2-cocycle. Together, this shows that the obstruction for finding a twisted $\text{Aut}(f)$ -equivariant structure on \tilde{S} is the vanishing of ψ in the second group cohomology $H^2(\text{Aut}(f), \text{Aut}(\tilde{S}))$. We already pointed out that \tilde{S} is a finite-dimensional \mathbb{R} -vector space, so as a consequence of Maschke's theorem we have $H^2(\text{Aut}(f), \text{Aut}(\tilde{S})) = 0$. We conclude that a twisted $\text{Aut}(f)$ -equivariant structure exists. The vanishing of $H^1(\text{Aut}(f), \text{Aut}(\tilde{S}))$ tells us that if we are given two twisted $\text{Aut}(f)$ -equivariant structures on \tilde{S} , there exists an $a \in \text{Aut}(f)$ such that $\tilde{S} \xrightarrow{a} \tilde{S}$ is an $\text{Aut}(f)$ -equivariant isomorphism (domain and target being equipped with the two given twisted $\text{Aut}(f)$ -equivariant structures). \square

Example 3.4.16. Let $\Gamma = \mathbb{R}/l\mathbb{Z}$ be a metric circle of length l . Let $f: \Gamma' = \mathbb{R}/nl\mathbb{Z} \rightarrow \Gamma$ be the connected free Galois cover of degree n that corresponds to $1 \in \mathbb{Z}/n\mathbb{Z}$ under the identification $H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Let τ be the S_n -torsor on Γ that arises as the image of f under the map

$$H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^1(\Gamma, S_n).$$

induced by the morphism $\mathbb{Z}/n\mathbb{Z} \rightarrow S_n$ mapping 1 to $(12 \cdots n)$.

Note that this determines an isomorphism $\chi: f^*\tau \rightarrow (S_n)_{\Gamma'}$ of S_n -torsors on Γ' .

In this example, we compute $\tilde{\mathcal{M}}_{\text{GL}_n, \tau}(\Gamma)$, $\tilde{\mathcal{M}}_{\text{SL}_n, \tau}(\Gamma)$ and $\tilde{\mathcal{M}}_{\text{PGL}_n, \tau}(\Gamma)$.

- (a) We show that $\widetilde{\mathcal{M}}_{\mathrm{GL}_n, \tau}(\Gamma) \cong \mathrm{Pic}(\Gamma')$. Using the standard identification $\check{M} \cong \mathbb{Z}^n$, by the previous lemma there is a natural bijection

$$\widetilde{\mathcal{M}}_{\mathrm{GL}_n, \tau}(\Gamma) \cong \mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma')^{\mathrm{Aut}(f)} \cong (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\mathrm{Aut}(f)} \cong \left(\bigoplus_{i=1}^n \mathrm{Pic}(\Gamma') \right)^{\mathrm{Aut}(f)}$$

The group $\mathrm{Aut}(f)$ is the cyclic group of order n generated by the automorphism

$$g : \Gamma' \rightarrow \Gamma' \quad \text{given by} \quad x \rightarrow x + l .$$

The morphism $\sigma : \mathrm{Aut}(f) \rightarrow \mathrm{Aut}((S_n)_{\Gamma'}) \cong S_n$ is given by mapping g to the n -cycle $(12 \cdots n)$. Hence the action of $\mathrm{Aut}(f)$ on $\bigoplus_{i=1}^n \mathrm{Pic}(\Gamma')$ is given as follows: For $(L_1, \dots, L_n) \in \bigoplus_{i=1}^n \mathrm{Pic}(\Gamma')$ we have

$$g \cdot (L_1, \dots, L_n) = \sigma^{(g)}(g^*(L_1, \dots, L_n)) = \sigma^{(g)}(g^*L_1, \dots, g^*L_n) = (g^*L_2, \dots, g^*L_n, g^*L_1)$$

Thus, $(L_1, \dots, L_n) \in \left(\bigoplus_{i=1}^n \mathrm{Pic}(\Gamma') \right)^{\mathrm{Aut}(f)}$ if and only if $L_i = g^*L_{i-1} = (g^*)^{i+1}L_1$ for $i = 2, \dots, n$. We obtain a bijection

$$\mathrm{Pic}(\Gamma') \longrightarrow \left(\bigoplus_{i=1}^n \mathrm{Pic}(\Gamma') \right)^{\mathrm{Aut}(f)} \quad \text{given by} \quad L \mapsto (L, g^*L, \dots, (g^*)^{n-1}L) .$$

In particular, since $\mathrm{Aut}(\tau) = \mathrm{Aut}(f)$, we obtain

$$\mathcal{M}_{\mathrm{GL}_n, \tau}(\Gamma) \cong \mathrm{Pic}(\Gamma') / \mathrm{Aut}(f) .$$

- (b) We show that $\widetilde{\mathcal{M}}_{\mathrm{SL}_n, \tau}(\Gamma)$ consists of the n -torsion points of Γ' . The short exact sequence

$$0 \longrightarrow \mathbb{Z}_0^n \longrightarrow \mathbb{Z}^n \xrightarrow{\det} \mathbb{Z} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z}_0^n)^{\mathrm{Aut}(f)} \longrightarrow (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z}^n)^{\mathrm{Aut}(f)} \xrightarrow{\det} (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z})^{\mathrm{Aut}(f)}$$

We show that the kernel of the induced map \det is isomorphic to the n -torsion points of Γ' . Let $(L, g^*L, \dots, (g^*)^{n-1}L)$ be in the kernel of \det for $L \in \mathrm{Pic}(\Gamma')$. Then

$$L \otimes g^*L \otimes \cdots \otimes (g^*)^{n-1}L \cong \mathcal{H}_{\Gamma'} \tag{3.2}$$

which implies that $\deg L = 0$. We observe that for $L \in \mathrm{Pic}^0(\Gamma')$ we have $g^*L \cong L$. Therefore (3.2) shows that $L^{\otimes n} \cong \mathcal{H}_{\Gamma'}$. Conversely, if $L \in \mathrm{Pic}^0(\Gamma')$ is an n -torsion point, then $(L, g^*L, \dots) = (L, L, \dots)$ lies in $\ker(\det)$.

In summary,

$$\widetilde{\mathcal{M}}_{\mathrm{SL}_n, \tau}(\Gamma) \cong (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z}_0^n)^{\mathrm{Aut}(f)} \cong \mathrm{Pic}(\Gamma')[n] .$$

Since $\mathrm{Aut}(\tau) = \mathrm{Aut}(f)$ acts trivially, we obtain $\mathcal{M}_{\mathrm{SL}_n, \tau}(\Gamma) = \widetilde{\mathcal{M}}_{\mathrm{SL}_n, \tau}(\Gamma)$. In particular, we have

$$|\mathcal{M}_{\mathrm{SL}_n, \tau}(\Gamma)| = |\widetilde{\mathcal{M}}_{\mathrm{SL}_n, \tau}(\Gamma)| = |\mathrm{Pic}(\Gamma')[n]| = n .$$

- (c) We show that $\widetilde{\mathcal{M}}_{\mathrm{PGL}_n, \tau}(\Gamma) \cong \mathbb{Z}/n\mathbb{Z}$. The short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^n \rightarrow \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1) \rightarrow 0$$

induces a short exact sequence

$$0 \rightarrow (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z})^{\mathrm{Aut}(f)} \rightarrow (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z}^n)^{\mathrm{Aut}(f)} \rightarrow (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1))^{\mathrm{Aut}(f)} \rightarrow 0.$$

It is right exact because the last map $\widetilde{\mathcal{M}}_{\mathrm{GL}_n, \tau}(\Gamma) \rightarrow \widetilde{\mathcal{M}}_{\mathrm{PGL}_n, \tau}(\Gamma)$ is surjective. This follows from Lemma 3.4.9 since the underlying morphism of $\mathrm{GL}_n(\mathbb{T}) \rightarrow \mathrm{PGL}_n(\mathbb{T})$ on the lattices is surjective. Hence, $\widetilde{\mathcal{M}}_{\mathrm{PGL}_n, \tau}(\Gamma)$ is the cokernel of the map $\mathrm{Pic}(\Gamma')^{\mathrm{Aut}(f)} \rightarrow \mathrm{Pic}(\Gamma')$. Since $\mathrm{Pic}(\Gamma')^{\mathrm{Aut}(f)} = \mathrm{Pic}^{n\mathbb{Z}}(\Gamma')$, the cokernel is identified with $\mathbb{Z}/n\mathbb{Z}$, and likewise $\mathcal{M}_{\mathrm{PGL}_n, \tau}(\Gamma) = \widetilde{\mathcal{M}}_{\mathrm{PGL}_n, \tau}(\Gamma)$.

We are now ready to prove our main result.

Proof of Theorem 3.4.12. Let $f: \Gamma' \rightarrow \Gamma$ be the éspace étalé of τ , which is a Galois cover because τ is a torsor over a discrete group. We equip Γ' with the induced sheaf of harmonic functions to make f a free cover. By Lemma 3.4.13, the pull-back $f^*\tau$ has a canonical trivialization. We can thus apply Lemma 3.4.15 and obtain a bijection

$$\widetilde{\mathcal{M}}_{G, \tau} \cong (\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\mathrm{Aut}(f)}.$$

Since $\mathrm{Aut}(f)$ acts by pulling back and conjugation, the group action of $\mathrm{Aut}(f)$ on the components is by translates of morphisms of tropical abelian varieties. Therefore, $(\mathrm{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\mathrm{Aut}(f)}$ is a union of torsors over tropical abelian varieties as well.

The statement for $\mathcal{M}_{G, \tau}$ follows immediately, as it is the $\mathrm{Aut}(\tau)$ -quotient of $\widetilde{\mathcal{M}}_{G, \tau}$. \square

We note that the proof shows that each $\widetilde{\mathcal{M}}_{G, \tau}$ is in fact a group, not simply a torsor.

3.4.3 Stable bundles on tropical elliptic curves

In this section, we prove a tropical analogue of Theorem 3.3.3, which classifies stable G -bundles on an elliptic curve. We restrict our attention to tropical reductive groups of type $\prod A_{n_i-1}$, since in the algebraic setting there are no stable G -bundles of other types. Our main result is Theorem 3.4.20, which classifies stable tropical G -bundles of type $\prod A_{n_i-1}$ on a metric circle. The analogous tropical statement is not true on the nose, but one has to restrict to tropical bundles whose degree is stable and whose underlying W -torsor is indecomposable in the sense defined below.

Definition 3.4.17. Let $W = \prod_i S_{n_i}$ be a Weyl group of type $\prod_i A_{n_i-1}$. An element of W is called *indecomposable* if it is a product of n_i -cycles. A W -torsor on a metric circle is called *indecomposable* if it is defined by an indecomposable element of W .

We note that an S_n -torsor on a metric circle Γ is indecomposable if and only if the associated degree n cover is connected. Furthermore, all indecomposable elements of $W = \prod_i S_{n_i}$ are conjugate to each other, hence there is a unique (up to isomorphism) indecomposable W -torsor on Γ , which we denote by $\mathrm{ind} \in H^1(\Gamma, W)$.

Let $\Phi = (M, R, \check{M}, \check{R})$ be a root datum. Its associated *adjoint root datum* is given by $\Phi^{\mathrm{ad}} = (\langle R \rangle, R, \check{M}^{\mathrm{ad}}, \check{R})$, where

$$\check{M}^{\mathrm{ad}} = \{ \check{m} \in \check{M}_{\mathbb{Q}} : \langle r, \check{m} \rangle \in \mathbb{Z} \text{ for all } r \in R \}.$$

The tropical reductive group $G^{\mathrm{ad}} = G_{\Phi^{\mathrm{ad}}}$ associated to the adjoint root datum of a tropical reductive group $G = G_{\Phi}$ is the *adjoint group* of G , and the inclusion $\check{M} \rightarrow \check{M}^{\mathrm{ad}}$ induces a canonical morphism $G \rightarrow G^{\mathrm{ad}}$.

Lemma 3.4.18. *Let $G = \check{M}_{\mathbb{R}} \rtimes S_n$ be a tropical reductive group of type A_{n+1} and suppose that $\langle \check{R} \rangle$ is saturated in \check{M} . Let $\check{\lambda}_G \in \pi_1(G)$ be a stable degree (see Definition 3.3.4) and let $E \in \widetilde{\mathcal{M}}_{G, \text{ind}}^{\check{\lambda}_G}$. Then the stabilizer group $\text{Aut}(\text{ind})_E = 1$ is trivial.*

Proof. Let $Z_{\mathbb{R}} = Z(G)$ be the center, which is the vector space associated to the lattice $Z = Z(\check{M} \rtimes W)$. Then $G/Z_{\mathbb{R}}$ is simple of type A_{n-1} and the natural map $G/Z_{\mathbb{R}} \rightarrow G^{\text{ad}}$ induces an inclusion $\pi_1(G/Z_{\mathbb{R}}) \rightarrow \pi_1(G^{\text{ad}})$. The degree $\check{\lambda}_G$ can only be stable if its image generates $\pi_1(G^{\text{ad}})$, which implies that $G/Z_{\mathbb{R}} = G^{\text{ad}} = \text{PGL}_n$. We now reduce to the case where Z has rank 1. Note that because $\langle \check{R} \rangle$ is saturated, we cannot have $Z = 0$, because that would imply $\check{M} = \langle \check{R} \rangle$ and hence $G = \text{SL}_n$, which is a contradiction to $G/Z_{\mathbb{R}} = \text{PGL}_n$. Consider the morphism $\phi: Z \rightarrow \check{M}/\langle \check{R} \rangle$. It is injective and its cokernel equals $\pi_1(G) = \mathbb{Z}/n\mathbb{Z}$. Because ϕ has cyclic cokernel, all but one of the invariant factors in its Smith normal form are 1, so there is a rank $\text{rk } Z - 1$ sublattice K of Z such that $\phi(K)$ is saturated. In particular, the image of $\langle \check{R} \rangle$ is saturated in the quotient $H = G/K_{\mathbb{R}}$. If $f: G \rightarrow H$ denotes the quotient map, then f_*E has degree $\check{\lambda}_H = \check{\lambda}_G \in \pi_1(H) = \pi_1(G)$. As

$$f_*: \widetilde{\mathcal{M}}_{G, \text{ind}}^{\check{\lambda}_G} \longrightarrow \widetilde{\mathcal{M}}_{H, \text{ind}}^{\check{\lambda}_H}$$

is $\text{Aut}(\text{ind})$ -equivariant, it suffices to prove the statement for f_*E and H , which satisfy all the hypotheses of the assertion, and H has center equal to Z/K , which has rank 1.

Now assume Z has rank 1. We compute $\widetilde{\mathcal{M}}_{G, \text{ind}}$ similarly as in Example 3.4.16. Let $f: \Gamma' \rightarrow \Gamma$ be a connected cyclic degree n cover with W -torsor ind , then by Lemma 3.4.15 we have

$$\widetilde{\mathcal{M}}_{G, \text{ind}} = (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} .$$

As $\langle \check{R} \rangle$ is saturated, the degree map factors through the determinant map, which is given by the morphism

$$(\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} \rightarrow (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M}/\langle \check{R} \rangle)^{\text{Aut}(f)} = \text{Pic}(\Gamma) \otimes_{\mathbb{Z}} \check{M}/\langle \check{R} \rangle$$

The element $L \in (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)}$ corresponding to E satisfies $\sigma^{(a)}(a^*L) = L$ for all $a \in \text{Aut}(f)$, where $\sigma: \text{Aut}(f) \rightarrow \text{Aut}(W_{\Gamma'})$ as in Lemma 3.4.15. On the other hand, $t \in \text{Aut}(\text{ind})$ acts by $L.t = L^{\delta(t)}$, where $\delta: \text{Aut}(\text{ind}) \rightarrow \text{Aut}(W_{\Gamma'})$ is as in Lemma 3.4.15. But in our case, $\text{Aut}(\text{ind})$ and $\text{Aut}(f)$ are identified because Γ' is the total space of ind , hence $\sigma = \delta$. We conclude that for $a \in \text{Aut}(\text{ind}) = \text{Aut}(f)$ we have

$$L^{\sigma(a)} = a^*L$$

As pulling back along a leaves degrees invariant, it follows that the degree of L is invariant, so it is contained in $\check{M}^{\text{Aut}(\text{ind})} = Z$, where for the equality we use that $\langle \check{R} \rangle^{\text{Aut}(\text{ind})} = 0$. So we have

$$L \in \text{Pic}^0(\Gamma') \otimes_{\mathbb{Z}} \check{M} + \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} Z .$$

Let $z \in Z$ and $m \in \check{M}/\langle \check{R} \rangle$ be generators. As $Z + \langle \check{R} \rangle$ has index n in \check{M} , the map $\mathbb{Z} \cong Z \rightarrow \check{M}/\langle \check{R} \rangle \cong \mathbb{Z}$ maps z to $\pm nm$. So if $\deg(L) = k \cdot z$, then $\deg(\det(E)) = k \cdot m$. As the degree factors through the determinant, $\check{\lambda}_G$ can only be stable if $\gcd(k, n) = 1$.

It now suffices to prove that for $1 \neq a \in \text{Aut}(f)$, we have $a^*L \otimes L^{-1} \neq 0$. Since $a^*\check{L} \otimes \check{L}^{-1} \cong 0$ for all $\check{L} \in \text{Pic}^0(\Gamma') \otimes \check{M}$, we see that $a^*L \otimes L^{-1} \neq 0$ only depends on $\deg(L)$ and we may replace L by any element of $\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} Z$. With the chosen isomorphism

$$\mathbb{Z} \rightarrow Z, \quad 1 \mapsto z ,$$

we have $\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} Z \cong \text{Pic}(\Gamma')$ and the set of elements in $\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} Z$ of degree $k \cdot z$ corresponds to $\text{Pic}^k(\Gamma')$. But because $\gcd(k, n) = 1$, the $\text{Aut}(f)$ -action on $\text{Pic}^k(\Gamma')$ is free, concluding the proof. \square

Recall that by Lemma 3.4.1 there is a natural map $\check{M} \rtimes W \rightarrow \check{M}/\langle \check{R} \rangle = \pi_1(G)$. Tensoring with \mathbb{R} , we obtain the *determinant map*

$$\det : G \longrightarrow \check{M}_{\mathbb{R}}/\langle \check{R} \rangle_{\mathbb{R}} = \pi_1(G)_{\mathbb{R}} .$$

By abuse of notation, we also use \det to denote the corresponding map on degrees:

$$\det : \pi_1(G) \longrightarrow \pi_1(\check{M}_{\mathbb{R}}/\langle \check{R} \rangle_{\mathbb{R}}) = \check{M}/\langle \check{R} \rangle^{\text{sat}} = \pi_1(G)^{\text{tf}} .$$

Lemma 3.4.19. *Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group of type $\prod_{i=1}^k A_{n_i-1}$ such that $\langle \check{R} \rangle$ is saturated in \check{M} , and let $\check{\lambda}_G \in \pi_1(G)$ be a stable degree. Then the determinant induces a homeomorphism*

$$\det : \mathcal{M}_{G,\text{ind}}^{\check{\lambda}_G} \longrightarrow \mathcal{M}_T^{\det(\check{\lambda}_G)} ,$$

where $T = \check{M}_{\mathbb{R}}/\langle \check{R} \rangle_{\mathbb{R}}$.

Proof. Let ind be an indecomposable W -torsor and let $f: \Gamma' \rightarrow \Gamma$ be a cyclic degree $\text{lcm}(n_1, \dots, n_k)$ cover. Consider the short exact sequence

$$0 \longrightarrow \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \langle \check{R} \rangle \longrightarrow \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M} \longrightarrow \text{Pic}(\Gamma') \otimes \check{M}/\langle \check{R} \rangle \longrightarrow 0 .$$

Proposition 3.4.10 identifies each term with a moduli space of bundles on Γ' with trivial W -torsor $W_{\Gamma'} = f^*(\text{ind})$. Taking $\text{Aut}(f)$ -invariants and applying Lemma 3.4.15, we obtain a left exact sequence of abelian groups

$$0 \longrightarrow \widetilde{\mathcal{M}}_{G^{\text{sc}},\text{ind}}(\Gamma) \longrightarrow \widetilde{\mathcal{M}}_{G,\text{ind}}(\Gamma) \xrightarrow{\det} \mathcal{M}_T(\Gamma) \longrightarrow 0 ,$$

where we denote $G^{\text{sc}} = \prod \text{SL}_{n_i}$. By Lemma 3.4.9 applied to the surjective map $G \mapsto T \times W$, the second map is surjective and therefore the sequence is also exact on the right. Because the degree factors through the determinant, it follows that the determinant map induces a bijection

$$\widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma) / \widetilde{\mathcal{M}}_{G^{\text{sc}},\text{ind}}(\Gamma) \longrightarrow \mathcal{M}_T^{\det(\check{\lambda}_G)}(\Gamma) ,$$

where we do not write the degree in the quotient because $\pi_1(G^{\text{sc}})$ is trivial. The determinant map is also invariant under action of $\text{Aut}(\text{ind})$ and we need to show that we also have

$$\mathcal{M}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma) = \widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma) / \text{Aut}(\text{ind}) \cong \mathcal{M}_T^{\det(\check{\lambda}_G)}(\Gamma) .$$

As we have seen in part (2) of Example 3.4.16, we have

$$|\widetilde{\mathcal{M}}_{G^{\text{sc}},\text{ind}}(\Gamma)| = \prod |\widetilde{\mathcal{M}}_{\text{SL}_{n_i},\text{ind}}(\Gamma)| = \prod n_i ,$$

which coincides with $|\text{Aut}(\text{ind})|$. Therefore, it suffices to show that $\text{Aut}(\text{ind})$ acts freely on $\widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma)$. To show this, consider for $1 \leq i \leq k$ the quotient morphism

$$q^i : G \rightarrow G^i := \check{M}_{i,\mathbb{R}} \rtimes S_{n_i} \quad \text{given by} \quad \check{M}_i = \check{M} / \sum_{j \neq i} \langle \check{R}_j \rangle ,$$

where \check{R}_j is the set of coroots of the factor SL_{n_i} of G^{sc} . The induced push-forward

$$q_*^i : \widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma) \longrightarrow \widetilde{\mathcal{M}}_{G^i, q_*^i \text{ind}}^{q_*^i \check{\lambda}_G}(\Gamma)$$

respects the $\text{Aut}(\text{ind})$ -action. In particular, for

$$\mathbf{t} = (t_1, \dots, t_k) \in \prod \text{Aut}(q_*^i \text{ind}) = \text{Aut}(\text{ind}) ,$$

and $E \in \widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}(\Gamma)$, we have

$$q_*^i(E.\mathbf{t}) = (q_*^i E).t_i .$$

Because $q_*^i \check{\lambda}_G$ is stable, $\text{Aut}(q_*^i \text{ind})$ acts freely on $\widetilde{\mathcal{M}}_{G^i, q_*^i \text{ind}}^{\check{\lambda}_G}(\Gamma)$ by Lemma 3.4.18 and it follows that $E.\mathbf{t} \cong E$ if and only if $\mathbf{t} = (1, \dots, 1)$. \square

We are now ready to prove the tropical counterpart to Theorem 3.3.3.

Theorem 3.4.20. *Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group of type $\prod_{i=1}^k A_{n_i-1}$ and let $\check{\lambda}_G \in \pi_1(G)$ be a stable degree. Then the determinant induces a homeomorphism*

$$\det: \mathcal{M}_{G,\text{ind}}^{\check{\lambda}_G} \longrightarrow \mathcal{M}_T^{\det(\check{\lambda}_G)} ,$$

where $T = \check{M}_{\mathbb{R}} / \langle \check{R} \rangle_{\mathbb{R}}$.

We first prove the following algebraic fact.

Lemma 3.4.21. *Let \mathbf{G} be a reductive linear algebraic group. Then there is a morphism $p: \mathbf{G}' \rightarrow \mathbf{G}$ of reductive groups with the properties that*

1. $\ker(p) \subseteq Z(\mathbf{G}')$,
2. p is surjective,
3. $\ker(p)$ is connected,
4. $\pi_1(\mathbf{G}')$ is torsion-free.

Proof. Choose a maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and let $(M, R, \check{M}, \check{R})$ denote the root datum corresponding to (G, T) . Choose a family $(n_i)_{1 \leq i \leq k}$ generating \check{M} , let $\check{\Lambda} = \langle \check{R} \rangle \oplus \mathbb{Z}^k$, let $\Lambda = \check{\Lambda}^\vee$, and consider the morphism

$$\pi: \check{\Lambda} \longrightarrow \check{M}$$

that is the inclusion on $\langle \check{R} \rangle$ and maps the i th generator of \mathbb{Z}^k to n_i . Let $\check{\Phi} = \check{R} \subseteq \check{\Lambda}$ and let $\Phi = \pi^*(R)$. Then $(\Lambda, \Phi, \check{\Lambda}, \check{\Phi})$ is a root datum and π defines a morphism of root data. By the existence theorem for reductive groups [Spr98, Theorem 10.1.1] and the isogeny theorem [Spr98, Theorem 9.6.5], there is a reductive group \mathbf{G}' with maximal torus \mathbf{T}' corresponding to $(\Lambda, \Phi, \check{\Lambda}, \check{\Phi})$ and a central isogeny $p: \mathbf{G}' \rightarrow \mathbf{G}$ mapping \mathbf{T}' to \mathbf{T} and inducing π on the level of root data. As π is surjective, p is surjective and $\ker(p)$ is connected. Because $\langle \check{\Phi} \rangle$ is saturated in $\check{\Lambda}$, $\pi_1(\mathbf{G}')$ is torsion-free. We observe that \mathbf{G}' has the same Weyl group as \mathbf{G} . \square

Proof of Theorem 3.4.20. By Lemma 3.4.21, there exists a morphism

$$\phi = (\pi, \text{id}): G' = \check{\Lambda}_{\mathbb{R}} \rtimes W \longrightarrow \check{M}_{\mathbb{R}} \rtimes W$$

of tropical reductive groups (having the same Weyl group) such that the map $\pi: \check{\Lambda} \rightarrow \check{M}$ is surjective and $\langle \check{\Phi} \rangle$ is saturated in $\check{\Lambda}$, where $\check{\Phi}$ denotes the set of coroots in $\check{\Lambda}$. Denote by K the kernel of the morphism $\check{\Lambda} \rightarrow \check{M}$; we have $K \subseteq Z(\check{\Lambda}_{\mathbb{R}} \rtimes W)$. Let $f: \Gamma' \rightarrow \Gamma$ be the cyclic and connected degree $\text{lcm}(n_1, \dots, n_k)$ cover of Γ and consider the short exact sequence

$$0 \longrightarrow \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} K \longrightarrow \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{\Lambda} \longrightarrow \text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M} \longrightarrow 0 .$$

Because the pullback along f of an indecomposable W -torsor trivializes, Lemma 3.4.15 implies that taking $\text{Aut}(f)$ -invariants in the sequence above yields a left exact sequence

$$0 \longrightarrow \mathcal{M}_{K_{\mathbb{R}}} \longrightarrow \widetilde{\mathcal{M}}_{G',\text{ind}} \longrightarrow \widetilde{\mathcal{M}}_{G,\text{ind}} \longrightarrow 0 .$$

The sequence is also right exact by Lemma 3.4.9 applied to the surjective morphism $\phi : G' \rightarrow G$, which is also surjective on the cocharacter lattices. We also have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \check{\Lambda}/\langle\check{\Phi}\rangle & \longrightarrow & \check{M}/\langle\check{R}\rangle \longrightarrow 0 \\ & & \downarrow = & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & \pi_1(K_{\mathbb{R}}) & \longrightarrow & \pi_1(G') & \xrightarrow{\phi_*} & \pi_1(G) \longrightarrow 0 . \end{array}$$

The top row, and hence the bottom one as well, is exact because $\phi(\check{\Phi}) = \check{R}$ and $\langle\check{\Phi}\rangle \cap Z(G') = 0$. It follows that for every $\check{\lambda}_{G'} \in \pi_1(G')$ with $\phi_*\check{\lambda}_{G'} = \check{\lambda}_G$, the map ϕ induces a bijection

$$\widetilde{\mathcal{M}}_{G',\text{ind}}^{\check{\lambda}_{G'}} / \mathcal{M}_{K_{\mathbb{R}}}^0 \xrightarrow{\cong} \widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G} .$$

Now consider the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{G',\text{ind}}^{\check{\lambda}_{G'}} & \xrightarrow{\phi_*} & \mathcal{M}_{G,\text{ind}}^{\check{\lambda}_G} \\ \downarrow \det & & \downarrow \det \\ \mathcal{M}_{\check{\Lambda}_{\mathbb{R}}/\langle\check{\Phi}\rangle_{\mathbb{R}}}^{\det(\check{\lambda}_{G'})} & \xrightarrow{\phi_*} & \mathcal{M}_{\check{M}_{\mathbb{R}}/\langle\check{R}\rangle_{\mathbb{R}}}^{\det(\check{\lambda}_G)} . \end{array}$$

By Lemma 3.4.19, the determinant map on the left is a homeomorphism. The morphism

$$\check{\Lambda}/\langle\check{\Phi}\rangle \longrightarrow \check{M}/\langle\check{R}\rangle^{\text{sat}}$$

has finite index, and hence the lower morphism of the square is surjective. It follows that the determinant map on the right is surjective. It remains to show that the determinant on the right is injective. Let $E, E' \in \mathcal{M}_{G,\text{ind}}^{\check{\lambda}_G}$ with $\det(E) \cong \det(E')$. We can lift both E and E' first to $\widetilde{\mathcal{M}}_{G,\text{ind}}^{\check{\lambda}_G}$ and then to elements $F, F' \in \widetilde{\mathcal{M}}_{G',\text{ind}}^{\check{\lambda}_{G'}}$ with $\phi_*\det(F) \cong \phi_*\det(F')$. The map

$$\mathcal{M}_{K_{\mathbb{R}}}^0 \longrightarrow \mathcal{M}_{\check{\Lambda}_{\mathbb{R}}/(K+\langle\check{\Phi}\rangle)_{\mathbb{R}}}^0$$

is surjective because $K \rightarrow (K + \langle\check{\Phi}\rangle)^{\text{sat}}/\langle\check{\Phi}\rangle$ has finite index, and the sequence

$$0 \longrightarrow \mathcal{M}_{(K+\langle\check{\Phi}\rangle)_{\mathbb{R}}^{\text{sat}}/\langle\check{\Phi}\rangle_{\mathbb{R}}}^0 \longrightarrow \mathcal{M}_{\check{\Lambda}_{\mathbb{R}}/\langle\check{\Phi}\rangle_{\mathbb{R}}}^0 \longrightarrow \mathcal{M}_{\check{M}_{\mathbb{R}}/\langle\check{R}\rangle_{\mathbb{R}}}^0 \longrightarrow 0$$

is exact. Therefore, there exists $L \in \mathcal{M}_{K_{\mathbb{R}}}^0$ with $\det(F') = \det(L \otimes F)$. As $\phi_*(L \otimes F) = \phi_*F$ we may replace F by $L \otimes F$ and assume that $\det(F') = \det(F)$. By Lemma 3.4.19, there exists an automorphism t of the indecomposable W -torsor such that $F' = F.t$. It follows that

$$\phi_*(F') = \phi_*(F.t) = \phi_*(F).t$$

and hence $E \cong E'$, concluding the proof. \square

3.4.4 Semistable bundles on tropical elliptic curves

We now give a tropical analogue of Theorem 3.3.5 that describes semistable bundles on an elliptic curve.

Lemma 3.4.22. *Let W' be a Weyl group of a root datum and let W be a parabolic subgroup of type $\coprod A_{n_i-1}$. Let $w \in W$ be indecomposable. Then we have*

$$C_{W'}(w) \subseteq N_{W'}(W) .$$

Proof. Let $g \in C_{W'}(w)$, then we have $w \in W \cap g^{-1}Wg$. By [?, Lemma 2.25], the intersection $W \cap g^{-1}Wg$ is a parabolic subgroup of W . Now observe that no proper standard parabolic subgroup of $\prod_{i=1}^k S_{n_i}$ contains an indecomposable element. As all indecomposable elements are conjugate, it follows that no proper parabolic subgroup of $\prod_{i=1}^k S_{n_i}$, and hence of W , contains an indecomposable element. We conclude that $W \cap g^{-1}Wg = W$, that is $g^{-1}Wg \subseteq W$, which we needed to show. \square

For the next lemma, we note that an element $w \in W$ of a Weyl group W of type $\prod A_{n_i-1}$ is indecomposable if and only if it has maximal *reflection length*, which is the minimum number of reflections (not necessary simple) in a representation of w .

Lemma 3.4.23. *Let W be a parabolic subgroup of type $\prod A_{n_i-1}$ of a Weyl group W' of a root datum, and let $w \in W$ be indecomposable. Then the natural homomorphism*

$$C_{W'}(w)/C_W(w) \longrightarrow N_{W'}(W)/W ,$$

which exists by Lemma 3.4.22, is an isomorphism.

Proof. Injectivity is clear from the fact that

$$C_W(w) = C_{W'}(w) \cap W .$$

For surjectivity, let $g \in N_{W'}(W)$. The reflections in W are the reflections of W' that are contained in W (by [?, Lemma 2.25] applied with $|K| = 1$), so conjugation by $g \in N_{W'}(W)$ preserves reflection length of elements of W . As the indecomposable elements of W are precisely those of maximal reflection length, $g^{-1}wg$ is indecomposable as well. But all indecomposable elements of W are conjugate in W , that is $g^{-1}wg = h^{-1}wh$ for some $h \in W$. It follows that $gh^{-1} \in C_{W'}(w)$ and thus that $gW = (gh^{-1})W$ is in the image of $C_{W'}(W) \rightarrow N_{W'}(W)/W$. \square

Given a parabolic subgroup W of type $\prod A_{n_i-1}$ of a Weyl group W' and a metric circle Γ , we denote by $\text{ind}_{W'}$ the W' -torsor induced by the indecomposable W -torsor ind on Γ via the inclusion $W \rightarrow W'$.

Corollary 3.4.24. *Let Γ be a metric circle. Let W be a parabolic subgroup of type $\prod A_{n_i-1}$ of a Weyl group W' of a root datum Φ . Moreover, let $G' = \check{M}_{\mathbb{R}} \rtimes W'$ be the tropical reductive group corresponding to Φ and let $G = \check{M}_{\mathbb{R}} \rtimes W$. Then the action of $N_{W'}(W)/W$ on $\mathcal{M}_{G'}(\Gamma)$ induces an action on $\mathcal{M}_{G', \text{ind}_{W'}}(\Gamma)$.*

Proof. We have computed in Example 3.4.8 that isomorphism classes of W -torsors on Γ are in bijection with conjugacy classes of elements of W . The (unique) isomorphism class of indecomposable covers corresponds to the conjugacy class of indecomposable elements of W . By Lemma 3.4.23, this conjugacy class is fixed by conjugation by elements in $N_{W'}(W)/W$, which implies that the action of $N_{W'}(W)/W$ leaves the isomorphism class of the associated W -torsor of an element in $\mathcal{M}_{G', \text{ind}_{W'}}(\Gamma)$ invariant. \square

Theorem 3.4.25. *Let Γ be a metric circle. Let W be a parabolic subgroup of type $\prod_{i=1}^k A_{n_i-1}$ of a Weyl group W' of a root datum Φ . Moreover, let $G' = \check{M}_{\mathbb{R}} \rtimes W'$ be the tropical reductive group associated to Φ and let $G = \check{M}_{\mathbb{R}} \rtimes W$. Then the natural map*

$$\mathcal{M}_{G', \text{ind}_{W'}}(\Gamma)/(N_{W'}(W)/W) \longrightarrow \mathcal{M}_{G, \text{ind}}(\Gamma)$$

is a bijection.

Proof. Let $f: \Gamma' \rightarrow \Gamma$ be a cyclic cover of degree $\text{lcm}(n_1, \dots, n_k)$. Let ind be an indecomposable W -torsor. Then $f^*\text{ind}$ is trivial, and so is $f^*(\text{ind}_{W'})$. We can thus apply Lemma 3.4.15 and conclude that

$$\begin{aligned} \mathcal{M}_{G, \text{ind}}(\Gamma) &= (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} / \text{Aut}(\text{ind}) , \\ \mathcal{M}_{G', \text{ind}_{W'}}(\Gamma) &= (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} / \text{Aut}(\text{ind}_{W'}) . \end{aligned} \quad (3.3)$$

In this description, the automorphism group $\text{Aut}(\text{ind})$ (resp. $\text{Aut}(\text{ind}_{W'})$) acts by conjugation by elements in the image of a morphism

$$\delta: \text{Aut}(\text{ind}) \rightarrow \text{Aut}(f^*\text{ind}) \cong \text{Aut}(W_{\Gamma'}) = W ,$$

where the last morphism is induced by the trivialization of $f^*\text{ind}$ (and similarly, there is a morphism $\delta': \text{Aut}(\text{ind}_{W'}) \rightarrow W'$). We have seen in Example 3.4.8 that there is a trivialization of ind on an open subset on which $\text{Aut}(\text{ind})$ can be identified with the right action of the centralizer $C_W(\tilde{w})$ for some $\tilde{w} \in W$ whose conjugacy class determines the isomorphism class of ind . Because ind is indecomposable, the element w is indecomposable as well. The morphism δ is not necessarily defined using the same trivialization, so the image of δ equals $C_W(w)$ for some conjugate w of \tilde{w} , which is again indecomposable. Since we can choose the trivialization of $f^*(\text{ind}_{W'}) = (f^*\text{ind})_{W'}$ to be induced by the trivialization of $f^*\text{ind}$, we may assume that the image of δ' is given by $C_{W'}(w)$. Together with (3.3), we see that

$$\begin{aligned} \mathcal{M}_{G, \text{ind}}(\Gamma) &= (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} / C_W(w) , \\ \mathcal{M}_{G', \text{ind}_{W'}}(\Gamma) &= (\text{Pic}(\Gamma') \otimes_{\mathbb{Z}} \check{M})^{\text{Aut}(f)} / C_{W'}(w) , \end{aligned}$$

where $C_W(w)$ and $C_{W'}(w)$ act by conjugation. By Lemma 3.4.23, $C_W(w)$ is normal in $C_{W'}(w)$ so that there exists an induced $C_{W'}(w)/C_W(w)$ -conjugation action on the quotient $\mathcal{M}_{G, \text{ind}}(\Gamma)$, and

$$\mathcal{M}_{G', \text{ind}_{W'}}(\Gamma) = \mathcal{M}_{G, \text{ind}}(\Gamma) / (C_{W'}(w)/C_W(w)) .$$

Also by Lemma 3.4.23, we have $C_{W'}(w)/C_W(w) = N_{W'}(W)/W$, concluding the proof. \square

3.5 Tropicalization of principal bundles

Let K be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value $|\cdot|$ of equicharacteristic 0, let \mathbf{G} be a reductive group over K and let X be a Mumford curve over K . In this section, we consider the process of tropicalization for \mathbf{G} -bundles on X together with a reduction of structure group to the normalizer of a given maximal torus \mathbf{T} in \mathbf{G} . When X is a Tate curve, we show that every semistable \mathbf{G} -bundle on X is equivalent to one that admits such a reduction. This allows us to establish our main result, Theorem 3.5.14, which identifies the essential skeleton of the moduli space of semistable principal \mathbf{G} -bundles on X with a moduli space of tropical semistable principal \mathbf{G}^{trop} -bundles on the minimal skeleton Γ_X of X .

Let $\mathbf{T} \subseteq \mathbf{G}$ be a maximal torus with character lattice $M = \mathbb{X}^*(\mathbf{T})$ and cocharacter lattice $\check{M} = \mathbb{X}_*(\mathbf{T})$. We denote the associated root datum by $\Phi = (M, R, \check{M}, \check{R})$. Let $N_{\mathbf{G}}(\mathbf{T})$ be the normalizer of \mathbf{T} in \mathbf{G} and let $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group. The Weyl group acts on \mathbf{T} by conjugation and therefore there is an induced action of W on \check{M} . We define the *tropical reductive group* \mathbf{G}^{trop} associated to \mathbf{G} as

$$\mathbf{G}^{\text{trop}} = \check{M}_{\mathbb{R}} \rtimes W .$$

Note that the isomorphism type of \mathbf{G}^{trop} does not depend on the choice of \mathbf{T} as all maximal tori in \mathbf{G} are conjugate.

3.5.1 Tropicalizing \mathbf{T} -bundles over a Mumford curve

Let Γ_X be the minimal skeleton of the Berkovich analytic space X^{an} . Given a \mathbf{T} -bundle E on X , we obtain for every character $(\mathbf{T} \xrightarrow{m} \mathbb{G}_m) \in M$ an induced \mathbb{G}_m -bundle $m_*(E)$ on X , which we can tropicalize to a tropical line bundle $\text{Trop}(m_*(E))$ on Γ_X . Since tropicalization respects tensor products of line bundles, we obtain, for every non-Archimedean field extension L/K , a bilinear map

$$\mathcal{M}_{\mathbf{T}}(X)(L) \times M \longrightarrow \mathcal{M}_{\mathbb{R}}(\Gamma_X) \quad \text{given by} \quad (E, m) \longmapsto \text{Trop}(m_*(E)).$$

Equivalently, by the tensor-hom adjunction, there is a linear map

$$\mathcal{M}_{\mathbf{T}}(X)(L) \longrightarrow \text{Hom}(M, \mathcal{M}_{\mathbb{R}}(\Gamma_X)) \cong \mathcal{M}_{\mathbb{R}}(\Gamma_X) \otimes_{\mathbb{Z}} \check{M}.$$

If we choose a basis (m_1, \dots, m_n) of M , the map is given by $E \mapsto \sum_i \text{Trop}((m_i)_*E) \otimes m_i^\vee$. As we have defined this for an arbitrary non-Archimedean field extension of K , we have defined a map

$$(\mathcal{M}_{\mathbf{T}}(X))^{\text{an}} \longrightarrow \mathcal{M}_{\mathbb{R}}(\Gamma_X) \otimes_{\mathbb{Z}} \check{M}. \quad (3.4)$$

Similarly, there is a canonical bilinear map

$$\mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma_X) \times M \longrightarrow \mathcal{M}_{\mathbb{R}}(\Gamma_X) \quad \text{given by} \quad (F, m) \longmapsto m_*(F)$$

which induces an isomorphism

$$\mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma_X) \xrightarrow{\cong} \mathcal{M}_{\mathbb{R}}(\Gamma_X) \otimes_{\mathbb{Z}} \check{M}.$$

Composing the map in (3.4) with (the inverse of) this isomorphism yields a tropicalization map

$$\text{Trop}: (\mathcal{M}_{\mathbf{T}}(X))^{\text{an}} \longrightarrow \mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma_X).$$

Proposition 3.5.1. *Let \mathbf{T} be an algebraic torus with cocharacter lattice \check{M} . Then for every $\check{\lambda} \in \pi_1(\mathbf{T})$ there exists a homeomorphism*

$$\tau: \Sigma(\mathcal{M}_{\mathbf{T}}^{\check{\lambda}}(X)) \xrightarrow{\sim} \mathcal{M}_{\check{M}_{\mathbb{R}}}^{\check{\lambda}}(\Gamma_X),$$

where $\Sigma(\mathcal{M}_{\mathbf{T}}^{\check{\lambda}}(X))$ is the essential skeleton of $(\mathcal{M}_{\mathbf{T}}^{\check{\lambda}}(X))^{\text{an}}$, that fits into a commutative diagram

$$\begin{array}{ccc} & \Sigma(\mathcal{M}_{\mathbf{T}}^{\check{\lambda}}(X)) & \\ \nearrow \rho & \uparrow & \downarrow \cong \\ (\mathcal{M}_{\mathbf{T}}^{\check{\lambda}}(X))^{\text{an}} & & \mathcal{M}_{\check{M}_{\mathbb{R}}}^{\check{\lambda}}(\Gamma_X), \\ \searrow \text{Trop} & & \end{array}$$

where ρ is the retraction map.

Proof. When $\dim \mathbf{T} = 1$ and the degree is equal to zero, this is [BR15, Theorem 1.3]. This is generalized to an arbitrary degree in a special case of [GUZZ22, Theorem 6.2], by twisting by a base point and its tropicalization. When $\dim \mathbf{T} \geq 1$, we observe that $\mathcal{M}_{\mathbf{T}}(X) \cong \text{Pic}(X) \otimes_{\mathbb{Z}} \check{M}$ in order to deduce the general case from the one-dimensional situation. We also note that $\mathcal{M}_{\mathbf{T}}(X)$ is a Calabi–Yau variety and hence has an essential skeleton. \square

3.5.2 Tropicalizing $N_{\mathbf{G}}(\mathbf{T})$ -bundles over a Mumford curve

Let $\rho: X^{\text{an}} \rightarrow \Gamma_X$ denote the retraction map to the skeleton. Then pulling back along ρ defines a fully faithful functor from the category of W -torsors on Γ_X to the category of W -torsors on X^{an} . The espace étalé of any W -torsor over X^{an} is a covering space of X^{an} , which induces an analytic structure on the latter. In this way, we obtain a fully faithful functor from the category of W -torsors on X^{an} to the category of principal homogeneous spaces for W over X^{an} in the category of K -analytic spaces. Since W is finite, by the GAGA-principle, this category is in turn equivalent to the category of principal homogeneous spaces for W over X in the category of schemes. These in turn are all étale over X , and the category of principal homogeneous spaces for W over X is thus equivalent to the category of étale W -torsors on X [Mil80, III, Theorem 4.3]. Taking the composition of these embeddings and equivalences, we obtain a fully faithful functor from the category of W -torsors on Γ_X to the category of étale W -torsors on X . For a W -torsor τ over Γ_X , we denote the associated étale W -torsor on X by $\rho^*\tau$. For a W -torsor ψ on X , we denote by $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\psi}(X)$ the stack of $N_{\mathbf{G}}(\mathbf{T})$ -torsors on X whose associated W -torsor is ψ .

We can intrinsically characterize those W -torsors on X that are of the form $\rho^*\tau$ as follows. Consider the analytification of the total space of a W -torsor on X . This space comes with a W -action. If this action is free, then the quotient is a W -torsor τ on X^{an} , and the original torsor on X is $\rho^*\tau$.

Let τ be a W -torsor on Γ_X , let L/K be a non-Archimedean extension, and let $E \in \text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\rho^*\tau}(X)(L)$ be a bundle defined over L . To tropicalize E , we view $\rho^*\tau$ as a principal homogeneous space for W over X in the category of schemes. Let $\pi: \rho^*\tau \rightarrow X$ denote the structure map. Then, exactly as in Lemma 3.4.13, the W -torsor $\pi^*(\rho^*\tau)$ on $\rho^*\tau$ is canonically trivial. Therefore, exactly as in Lemma 3.4.15, the pullback π^*E is induced by a \mathbf{T} -bundle on $\rho^*\tau$ that is invariant under the $\text{Aut}(\pi)$ -action, and this \mathbf{T} -bundle is well-defined up to the $\text{Aut}(\tau)$ -action. More concisely, we obtain a map

$$\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\rho^*\tau}(X)(L) \longrightarrow \mathcal{M}_{\mathbf{T}}(\rho^*\tau)(L)^{\text{Aut}(\pi)} / \text{Aut}(\tau) .$$

The skeleton of $(\rho^*\tau)^{\text{an}}$ is the total space of τ . Because the tropicalization of \mathbf{T} -bundles on $\rho^*\tau$ respects the actions of both $\text{Aut}(\pi)$ and $\text{Aut}(\tau)$, we obtain a tropicalization map

$$\mathcal{M}_{\mathbf{T}}(\rho^*\tau)(L)^{\text{Aut}(\pi)} / \text{Aut}(\tau) \longrightarrow \mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\tau)^{\text{Aut}(\pi)} / \text{Aut}(\tau) .$$

As $\text{Aut}(\pi) = \text{Aut}(\tau/\Gamma)$, Lemma 3.4.15 yields an isomorphism

$$\mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\tau)^{\text{Aut}(\pi)} / \text{Aut}(\tau) \xrightarrow{\cong} \mathcal{M}_{\mathbf{G}^{\text{trop},\tau}}(\Gamma_X) .$$

Composing the three maps defines a tropicalization map for $N_{\mathbf{G}}(\mathbf{T})$ -bundles defined over the field extension L . Since the extension was arbitrary, we have in fact defined a tropicalization map

$$\text{Trop}: |\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\rho^*\tau}(X)^{\text{an}}| \longrightarrow \mathcal{M}_{\mathbf{G}^{\text{trop},\tau}}(\Gamma_X) ,$$

where $|\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\rho^*\tau}(X)^{\text{an}}|$ denotes the points of the stack $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\rho^*\tau}(X)^{\text{an}}$.

Example 3.5.2. In the case of $\mathbf{G} = \mathbf{GL}_n$, our construction of Trop differs from the one given in [GUZ22]. To compare the two constructions, suppose we are given an $\mathbb{G}_m^n \rtimes S_n$ -torsor E on X as a line bundle L on $\rho^*\Gamma'$, where $\Gamma' \xrightarrow{f} \Gamma_X$ is a free degree n cover of Γ_X . The tropicalization $\text{Trop}_{\text{GUZ}}(E)$ of E in the sense of [GUZ22] is given by the $\text{GL}_n(\mathbb{T})$ -bundle represented by the tropical line bundle $\text{Trop}(L)$ on the domain Γ' of the cover f .

Let τ be the W -torsor on Γ_X that corresponds to the cover f . Since the pull-back of τ to its espace étalé (which we also denote by τ) is canonically trivial, the cover

$$\tau \times_{\Gamma_X} \Gamma' \rightarrow \tau$$

is canonically trivial as well. The canonical trivialization determines n sections of this cover, or equivalently n morphisms $s_i: \tau \rightarrow \Gamma'$ over Γ_X . The twisted $\text{Aut}(\tau)$ -equivariant \mathbb{R}^n -bundle on τ corresponding to $\text{Trop}_{\text{GUZ}}(E)$ is given by

$$\bigoplus_{i=1}^n s_i^* \text{Trop}(L) .$$

Pulling the n sections back via the retraction ρ , we obtain n morphisms

$$t_i := \rho^* s_i: \rho^* \tau \longrightarrow \rho^* \Gamma' ,$$

which induce the canonical trivialization of the cover

$$\rho^* \tau \times_X \rho^* \Gamma' \longrightarrow \rho^* \tau .$$

The twisted $\text{Aut}(\tau)$ -equivariant \mathbb{G}_m^n -bundle on $\rho^* \tau$ corresponding to E is given by $\bigoplus_{i=1}^n t_i^* L$. Therefore, the tropicalization $\text{Trop}(E)$ in the sense of the present paper is the $\text{GL}_n(\mathbb{T})$ -bundle on Γ_X corresponding to the twisted $\text{Aut}(\tau)$ -equivariant \mathbb{R}^n -bundle

$$\bigoplus_{i=1}^n \text{Trop}(t_i^* L) .$$

As this equals $\bigoplus_{i=1}^n s_i^* \text{Trop}(L)$, we conclude that

$$\text{Trop}(E) = \text{Trop}_{\text{GUZ}}(E) .$$

Let us now return to the general situation. We have defined the tropicalization map using the techniques from Lemma 3.4.15 applied to the cover $\rho^* \tau \rightarrow X$, which can be canonically obtained from the $N_{\mathbf{G}}(\mathbf{T})$ -bundle we are tropicalizing. It is useful to also allow other covers. Let $f: \Gamma' \rightarrow \Gamma_X$ be a free Galois cover such that there exists a trivialization $f^* \tau \xrightarrow{\chi} W_{\Gamma'}$. Let $\rho^* f: X' \rightarrow X$ be the induced étale cover. Then, exactly as in our definition of tropicalization above, we obtain, for every field extension L/K , a sequence of maps

$$\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)(L) \longrightarrow \text{Bun}_{\mathbf{T}}(X')(L)^{\text{Aut}(f)} / \text{Aut}(\tau) \longrightarrow \mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma')^{\text{Aut}(f)} / \text{Aut}(\tau) \xrightarrow{\cong} \mathcal{M}_{\mathbf{G}^{\text{trop}}, \tau}(\Gamma_X) ,$$

and we denote by

$$\text{Trop}_{f, \chi}: \left| \text{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)^{\text{an}} \right| \longrightarrow \mathcal{M}_{\mathbf{G}^{\text{trop}}, \tau}(\Gamma_X)$$

the map induced by the composition. We note that, for τ a W -torsor on Γ_X , we have $\text{Trop} = \text{Trop}_{\pi_0, \chi_0}$, where $\pi_0: \tau \rightarrow \Gamma_X$ is the total space and $\chi_0: \pi_0^* \tau \rightarrow W_{\tau}$ is the canonical trivialization.

Lemma 3.5.3. *With notation as above, we have*

$$\text{Trop}_{f, \chi}(E) = \text{Trop}(E)$$

for every $E \in \left| \text{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)^{\text{an}} \right|$.

Proof. It suffices to show that $\text{Trop}_{f, \chi} = \text{Trop}_{g, \psi}$ for two covers f and g and trivializations χ and ψ . First we treat the case where $f = g$ are the same map $\Gamma' \rightarrow \Gamma_X$, but χ and ψ are allowed to differ. The two trivializations χ and ψ differ by

$$\psi \circ \chi^{-1}: W_{\Gamma'} \longrightarrow W_{\Gamma'} ,$$

which is given by right multiplication by some $w \in H^0(\Gamma', W_{\Gamma'})$. By the algebraic analogue of Lemma 3.4.14 (with analogous proof), if $E \in |\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)^{\text{an}}|$ is represented by L_{χ} (resp. L_{ψ}) in the invariants of $\mathcal{M}_{\mathbf{T}}(\rho^* f)$ with respect to the $\text{Aut}(f)$ -action induced by χ (resp. ψ), then L_{ψ} is obtained from L_{χ} by conjugating with w , up to the $\text{Aut}(\tau)$ -action. Therefore, the $\check{M}_{\mathbb{R}}$ -bundle $\text{Trop}(L_{\psi})$ on Γ' is obtained from $\text{Trop}(L_{\chi})$ by conjugating with w , up to the $\text{Aut}(\tau)$ -action. Now using Lemma 3.4.14 on the tropical side shows that $\text{Trop}_{f, \chi}(E) = \text{Trop}_{f, \psi}(E)$.

Now we treat the case where f and g differ. Any two covers can be dominated by a common cover, so we may assume that g factors through f , that is $g = f \circ h$ for some free Galois cover h . We already showed that $\text{Trop}_{g, \psi}$ does not depend on the choice of ψ , so we may assume that $\psi = h^* \chi$. Using the same notation as above, the \mathbf{T} -bundle L_{ψ} on the domain of $\rho^* g$ agrees with $(\rho^* h)^* L_{\chi}$ up to the $\text{Aut}(\tau)$ -action. Therefore, the $\check{M}_{\mathbb{R}}$ -bundle $\text{Trop}(L_{\psi})$ agrees with $h^* \text{Trop}(L_{\chi})$. Applying the algebraic argument backwards on the tropical side then shows the desired equality

$$\text{Trop}_{f, \chi}(E) = \text{Trop}_{g, \psi}(E) . \quad \square$$

Let $(M, R, \check{M}, \check{R})$ and $(\Lambda, \Phi, \check{\Lambda}, \check{\Phi})$ be root data and let (\mathbf{G}, \mathbf{T}) and $(\mathbf{G}', \mathbf{T}')$ be the corresponding reductive groups with maximal tori T and T' , respectively. Recall that a surjective morphism

$$\phi: \mathbf{G} \longrightarrow \mathbf{G}'$$

with $\phi(\mathbf{T}) \subseteq \mathbf{T}'$ and $\ker(\phi) \subseteq Z(\mathbf{G})$

induces a morphism

$$f: \check{M} \longrightarrow \check{\Lambda} ,$$

whose image has finite index and that defines bijections $\check{R} \rightarrow \check{\Phi}$ and, dually, $\Phi \rightarrow R$ (we note that the morphism $p: \mathbf{G}' \rightarrow \mathbf{G}$ in Lemma 3.4.21 is of this type). Conversely, given f , the morphism ϕ can be reconstructed up to conjugation by elements in \mathbf{T} by [Ste99].

The pair (f, id) induces a morphism of the tropical reductive groups \mathbf{G}^{trop} and $(\mathbf{G}')^{\text{trop}}$ corresponding to our root data, and we denote this morphism by

$$\phi^{\text{trop}} = (f, \text{id}): \mathbf{G}^{\text{trop}} \longrightarrow (\mathbf{G}')^{\text{trop}} .$$

Lemma 3.5.4. *Let $\phi: (\mathbf{G}, \mathbf{T}) \rightarrow (\mathbf{G}', \mathbf{T}')$ and $\phi^{\text{trop}}: \mathbf{G}^{\text{trop}} \rightarrow (\mathbf{G}')^{\text{trop}}$ be as above. Let E be a tropicalizable principal $N_{\mathbf{G}}(\mathbf{T})$ -bundle on X , in other words, assume that the associated W -torsor of E is of the form $\rho^* \tau$. Then we have*

$$\text{Trop}(\phi_* E) \cong \phi_*^{\text{trop}} \text{Trop}(E) .$$

Proof. We first treat the case where $\mathbf{G} = \mathbf{T}$ and $\mathbf{G}' = \mathbf{T}'$ are tori. Here, the statement follows from the commutativity, for every extension L/K of non-Archimedean fields, of the diagram

$$\begin{array}{ccccccc} \mathcal{M}_{\mathbf{T}}(X)(L) & \xrightarrow{\cong} & \text{Pic}(X)(L) \otimes_{\mathbb{Z}} \check{M} & \xrightarrow{\text{Trop} \otimes \text{id}} & \text{Pic}(\Gamma_X) \otimes_{\mathbb{Z}} \check{M} & \xrightarrow{\cong} & \mathcal{M}_{\check{M}_{\mathbb{R}}}(\Gamma_X) \\ \downarrow \phi_* & & \downarrow \text{id} \otimes f & & \downarrow \text{id} \otimes f & & \downarrow \phi_*^{\text{trop}} \\ \mathcal{M}_{\mathbf{T}'}(X)(L) & \xrightarrow{\cong} & \text{Pic}(X)(L) \otimes_{\mathbb{Z}} \check{\Lambda} & \xrightarrow{\text{Trop} \otimes \text{id}} & \text{Pic}(\Gamma_X) \otimes_{\mathbb{Z}} \check{\Lambda} & \xrightarrow{\cong} & \mathcal{M}_{\check{\Lambda}_{\mathbb{R}}}(\Gamma_X) . \end{array}$$

For general \mathbf{G} and \mathbf{G}' we denote by $\pi: \rho^* \tau \rightarrow X$ the projection from the total space. For an

extension L/K of non-Archimedean fields, consider the diagram

$$\begin{array}{ccccccc}
\mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)(L) & \longrightarrow & \mathcal{M}_{\mathbf{T}}(\rho^* \tau)^{\mathrm{Aut}(\pi)} / \mathrm{Aut}(\tau) & \xrightarrow{\mathrm{Trop}} & \mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\tau)^{\mathrm{Aut}(\pi)} / \mathrm{Aut}(\tau) & \xrightarrow{\cong} & \mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\Gamma_X) \\
\downarrow \phi_* & & \downarrow (\phi|_{\mathcal{T}})_* & & \downarrow (\phi^{\mathrm{trop}}|_{\tilde{M}_{\mathbb{R}}})_* & & \downarrow \phi_*^{\mathrm{trop}} \\
\mathrm{Bun}_{N_{\mathbf{G}'}(\mathbf{T}'), \rho^* \tau}(X)(L) & \longrightarrow & \mathcal{M}_{\mathbf{T}'}(\rho^* \tau)^{\mathrm{Aut}(\pi)} / \mathrm{Aut}(\tau) & \xrightarrow{\mathrm{Trop}} & \mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\tau)^{\mathrm{Aut}(\pi)} / \mathrm{Aut}(\tau) & \xrightarrow{\cong} & \mathcal{M}_{\tilde{M}_{\mathbb{R}}}(\Gamma_X) .
\end{array}$$

The compositions of the arrows in the two rows are the tropicalizations maps for $N_{\mathbf{G}}(\mathbf{T})$ - and $N_{\mathbf{G}'}(\mathbf{T}')$ -bundles, respectively. To complete the proof, it suffices to show that the diagram commutes. For the left and right square, commutativity follows from the compatibility of push-forward with pull-backs along covers. The square in the middle commutes by the case of tori that we treated first. \square

Lemma 3.5.5. *Let E be a tropicalizable principal $N_{\mathbf{G}}(\mathbf{T})$ -bundle. Then we have*

$$\mathrm{Trop}(\det(E)) \cong \det(\mathrm{Trop}(E)) .$$

Proof. Let τ be W torsor on Γ_X with $E \in \mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X)$, let $\pi_0: \tau \rightarrow \Gamma_X$ be the projection from the total space and let $\chi_0: \pi_0^* \tau \rightarrow W_\tau$ be the canonical trivialization. Then exactly as in Lemma 3.5.4, we see that

$$\det(\mathrm{Trop}(E)) = \mathrm{Trop}_{\pi_0}(\det(E)) ,$$

where Trop_{π_0} is missing the datum of the trivialization of the torsor because $\det(E)$ is a torus bundle and tori have trivial Weyl groups (and hence all torsors over those Weyl groups are canonically trivial). Applying Lemma 3.5.3 finishes the proof. \square

Recall that for a reductive group \mathbf{G} with fixed maximal torus \mathbf{T} , there is a natural identification $\pi_0(\mathrm{Bun}_{\mathbf{G}}(X)) \cong \pi_1(\mathbf{G})$ defined as follows: every principal \mathbf{G} -bundle can be degenerated to a principal \mathbf{T} -bundle, so the natural map

$$\mathbb{X}_*(\mathbf{T}) \cong \pi_0(\mathrm{Bun}_{\mathbf{T}}(X)) \longrightarrow \pi_0(\mathrm{Bun}_{\mathbf{G}}(X))$$

is a surjection. Moreover, two elements of $\mathbb{X}_*(\mathbf{T})$ have the same image precisely if they agree modulo the sublattice generated by the coroots. We refer to [Hof10] for details.

Recall that for a principal \mathbf{G} -bundle E we denote by $\deg(E)$ the element of $\pi_1(\mathbf{G})$ corresponding to the component of $\mathrm{Bun}_{\mathbf{G}}(X)$ containing E and call it the *degree* of E .

Proposition 3.5.6. *Let $E_{\mathbf{G}}$ be a \mathbf{G} -bundle induced by a tropicalizable principal $N_{\mathbf{G}}(\mathbf{T})$ -bundle E . Then*

$$\deg(E_{\mathbf{G}}) = \deg(\mathrm{Trop}(E)) .$$

Proof. For line bundles, that is if $\mathbf{G} = \mathbb{G}_m$, the statement is well-known. This implies it is also true for products of \mathbb{G}_m , that is for algebraic tori (note that all tori are split because the base field is algebraically closed).

Now assume that $\pi_1(\mathbf{G})$ is torsion-free. Then $\langle \check{R} \rangle$ is saturated and the determinant induces an isomorphism of $\pi_1(\mathbf{G})$ with the cocharacter lattice of the cocenter $Z^c(\mathbf{G})$ (see §3.3.3). As tropicalization commutes with determinants by Lemma 3.5.5 and taking degrees of tropical principal bundles is functorial, the assertion is reduced to the case $\mathbf{G} = Z^c(\mathbf{G})$, which we have already established because $Z^c(\mathbf{G})$ is an algebraic torus.

For \mathbf{G} arbitrary, let $p: \mathbf{G}' \rightarrow \mathbf{G}$ be as in Lemma 3.4.21 and let $\mathbf{T}' = p^{-1}(\mathbf{T})$. Since $\ker(p)$ is connected and central in \mathbf{G}' , it is an algebraic torus, and so is \mathbf{T}' . So by [Hof10, Proposition 3.1, Remark 3.3 ii)], the morphism $\mathrm{Bun}_{N_{\mathbf{G}'}(\mathbf{T}')} \rightarrow \mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T})}$ is surjective and we can lift

E to a principal $N_{\mathbf{G}'}(\mathbf{T}')$ -bundle E' . Of course, E' is tropicalizable as well. As $\pi_1(\mathbf{G}')$ is torsion-free, we have already seen that $\deg(E'_{\mathbf{G}'}) = \deg(\mathrm{Trop}(E'))$. Using Lemma 3.5.4, we conclude that

$$\deg(\mathrm{Trop}(E)) = \deg(\mathrm{Trop}(p_*E')) = \deg(\pi_* \mathrm{Trop}(E')) = \pi_* \deg(\mathrm{Trop}(E')) = \pi_* \deg(E'_{\mathbf{G}'}) = \deg(E_{\mathbf{G}}).$$

□

3.5.3 Tropicalizing stable \mathbf{G} -bundles over a Tate curve

We now assume the Mumford curve X is a *Tate curve* over K , i.e. a smooth projective curve of genus one, whose analytification is given by $X^{\mathrm{an}} = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$ with $\mathrm{val}(q) > 0$. Then the minimal skeleton Γ_X is isometric to a metric circle of circumference $\mathrm{val}(q)$. Theorems 3.3.3 and 3.3.5 describe the moduli spaces $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \mathrm{st}}(X)$ and $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \mathrm{ss}}(X)$ of stable and semistable \mathbf{G} -bundles on X , respectively. In this section and the next, we explain how to tropicalize these moduli spaces: stable bundles are tropicalized by reducing them to $N_{\mathbf{G}}(\mathbf{T})$ -bundles, and semistable bundles are reduced to stable bundles by passing to a Levi subgroup.

First, let $\check{\lambda}_{\mathbf{G}} \in \pi_1(\mathbf{G})$ be a degree for which the moduli space $\mathcal{M}_{\mathbf{G}}^{\check{\lambda}_{\mathbf{G}}, \mathrm{st}}(X)$ is nonempty. By Theorem 3.3.3, this only happens if $\mathbf{G}^{\mathrm{ad}} = \prod_i \mathbf{PGL}_{n_i}$, so that the Weyl group is $W = \prod_i S_{n_i}$. We recall from §3.4.3 that since Γ_X is a circle, up to isomorphism there is a unique indecomposable W -torsor ind on Γ_X , which induces a W -torsor on X that we also denote ind by abuse of notation.

Proposition 3.5.7. *Let E be a stable principal \mathbf{G} -bundle on X . Then E can only be in the image of*

$$\mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \rho^* \tau}(X) \longrightarrow \mathrm{Bun}_{\mathbf{G}}(X)$$

if τ is indecomposable.

Proof. For every stable \mathbf{G} -bundle the induced \mathbf{G}^{ad} -bundle is stable as well. This reduces to the case where $\mathbf{G} = \prod_i \mathbf{PGL}_{n_i}$. Treating each factor individually, we further reduce to the case $\mathbf{G} = \mathbf{PGL}_n$. Denote by $\mathbf{D}_n \subseteq \mathbf{GL}_n$ the diagonal torus and $\mathbf{D}_n/\mathbb{G}_m \subseteq \mathbf{PGL}_n$ the corresponding torus in \mathbf{PGL}_n . Let $E \in \mathrm{Bun}_{N_{\mathbf{PGL}_n}(\mathbf{D}_n/\mathbb{G}_m), \rho^* \tau}(X)$ be such that the associated \mathbf{PGL}_n -torsor $E_{\mathbf{PGL}_n}$ is stable, and assume that τ is decomposable. Then τ is induced by an $S_{k_1} \times \dots \times S_{k_d}$ -torsor for some nontrivial partition $\sum_{i=1}^d k_i = n$. The preimage of $S_{k_1} \times \dots \times S_{k_d}$ in $N_{\mathbf{PGL}_n}(\mathbf{D}_n/\mathbb{G}_m)$ is the normalizer N of the maximal torus $\mathbf{D}_n/\mathbb{G}_m$ in the Levi subgroup $(\prod_{i=1}^d \mathbf{GL}_{k_i})/\mathbb{G}_m$ of \mathbf{PGL}_n . By [BH10, Lemma 2.2.1] (see also [BH23]), the structure group of E can be reduced to N . In particular, the structure group of $E_{\mathbf{PGL}_n}$ can be reduced to $(\prod_{i=1}^d \mathbf{GL}_{k_i})/\mathbb{G}_m$. Lifting $E_{\mathbf{PGL}_n}$ to a \mathbf{GL}_n -bundle, which is possible by [Hof10, Corollary 3.4], we obtain a stable \mathbf{GL}_n -bundle whose structure group can be reduced to $\prod_{i=1}^d \mathbf{GL}_{k_i}$. But this is absurd, because stable vector bundles are indecomposable. □

We now study the morphism $\mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T}), \mathrm{ind}}(X) \rightarrow \mathrm{Bun}_{\mathbf{G}}(X)$.

Example 3.5.8. Let $\mathbf{G} = \mathbf{GL}_n$. Then $N_{\mathbf{G}}(\mathbf{T}) \cong \mathbb{G}_m^n \rtimes S_n$, $W = S_n$, and the stack $\mathrm{Bun}_{N_{\mathbf{G}}(\mathbf{T})}(X)$ is equivalent to the stack of pairs $(X' \xrightarrow{f} X, \mathcal{L})$, where f is a finite étale cover of degree n and \mathcal{L} is a line bundle on X' . The associated W -torsor of a pair (f, \mathcal{L}) is represented by f ; the stack of S_n -torsors is equivalent to the stack of finite étale covers of degree n . We can describe the cover f corresponding to the S_n -torsor ind on X defined by the unique indecomposable S_n -torsor ind on Γ_X explicitly by using a uniformization of X^{an} . Indeed, if $X^{\mathrm{an}} = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$ then $\rho^* \mathrm{ind}$ is represented by the quotient map

$$\pi_n: \mathbb{G}_m^{\mathrm{an}}/q^{n\mathbb{Z}} \longrightarrow \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}.$$

If X_n is the unique elliptic curve with $X_n^{\text{an}} = \mathbb{G}_m^{\text{an}}/q^{n\mathbb{Z}}$, we conclude that

$$\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\text{ind}}(X) \cong \mathcal{P}ic(X_n)/\text{Aut}(\pi_n) = \prod_{d \in \mathbb{Z}} \mathcal{P}ic^d(X_n)/\text{Aut}(\pi_n).$$

Note that π_n is a Galois cover and $\text{Aut}(\pi_n)$ is cyclic of degree n . We denote the connected component $\mathcal{P}ic^d(X_n)/\text{Aut}(\pi_n)$ of $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\text{ind}}(X)$ by $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\text{ind}}^d(X)$.

The morphism

$$\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\text{ind}}(X) \longrightarrow \text{Bun}_{\mathbf{G}}(X)$$

maps the $N_{\mathbf{G}}(\mathbf{T})$ -torsor corresponding to (π_n, \mathcal{L}) to the \mathbf{GL}_n -torsor corresponding to the vector bundle $(\pi_n)_*\mathcal{L}$. This vector bundle has degree $d = \deg(\mathcal{L})$. If d and n are coprime, then the argument in [Tu93, Appendix A] shows that $(\pi_n)_*\mathcal{L}$ is stable. In particular, for d coprime to n we obtain an induced map

$$\text{Bun}_{N_{\mathbf{GL}_n}(\mathbf{T}),\text{ind}}^d(X) \longrightarrow \text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X),$$

where $\text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X)$ denotes the open substack of $\text{Bun}_{\mathbf{GL}_n}(X)$ consisting of stable bundles of degree d .

Lemma 3.5.9. *Let d and n be coprime. Then the morphism*

$$\text{Bun}_{N_{\mathbf{GL}_n}(D_n),\text{ind}}^d(X) \longrightarrow \text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X)$$

is an isomorphism of algebraic stacks.

Proof. Consider the composition

$$\mathcal{P}ic^d(X_n) \longrightarrow \text{Bun}_{N_{\mathbf{GL}_n}(D_n),\text{ind}}^d(X) \longrightarrow \text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X), \quad (3.5)$$

which we denote by ϕ . Since $\mathcal{P}ic^d(X_n)$ is an $\text{Aut}(\pi_n)$ -torsor over $\text{Bun}_{N_{\mathbf{GL}_n}(D_n)}(X)$, it suffices to show that ϕ is an $\text{Aut}(\pi_n)$ -torsor. Given line bundles \mathcal{L} and \mathcal{L}' on $(X_n)_S = X_n \times_K S$ for some test K -scheme S and an isomorphism $(\pi_n)_*\mathcal{L} \xrightarrow{\psi} (\pi_n)_*\mathcal{L}'$ on X_S , there is an induced isomorphism

$$\bigoplus_{\sigma \in \text{Aut}(\pi_n)} \sigma^*\mathcal{L} \cong \pi_n^*(\pi_n)_*\mathcal{L} \xrightarrow{\pi_n^*\psi} \pi_n^*(\pi_n)_*\mathcal{L}' \cong \bigoplus_{\sigma \in \text{Aut}(\pi_n)} \sigma^*\mathcal{L}' \quad (3.6)$$

on $(X_n)_S$. Note that because $\text{Aut}(\pi_n)$ acts freely on $\text{Pic}^d(X_n)$, it follows that there is a unique $\sigma \in \text{Aut}(\pi_n)(S)$ such that the morphism $\mathcal{L} \rightarrow \sigma^*\mathcal{L}'$ induced from (3.6) is an isomorphism. This shows that the morphism

$$\begin{aligned} \text{Aut}(\pi_n) \times_K \mathcal{P}ic^d(X_n)(S) &\longrightarrow \mathcal{P}ic^d(X_n)(S) \times_{\text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X)(S)} \mathcal{P}ic^d(X_n)(S) \\ (\sigma, \mathcal{L}) &\longmapsto (\mathcal{L}, \sigma^*\mathcal{L}) \end{aligned}$$

is essentially surjective. It is also fully faithful, because ϕ_S is already fully faithful: as stable bundles are simple, we have, for \mathcal{L} a degree d line bundle on X_n , that

$$\text{Aut}((\pi_n)_*\mathcal{L}) = \mathbb{G}_m(S) = \text{Aut}(\mathcal{L}).$$

To conclude the proof, it suffices to show that ϕ is faithfully flat. By [GUZ22, Theorem 7.1], ϕ is surjective. Both factors of ϕ in (3.5) are representable and locally of finite type by [Hof10, Fact 2.3]. Moreover, we have already shown that the fibers of ϕ are finite (they are $\text{Aut}(\pi_n)$ -torsors), and both the target $\text{Bun}_{\mathbf{GL}_n}^{d,\text{st}}(X)$ and the source are smooth [Hof10, Proposition 4.1], so we are done by miracle flatness. \square

Definition 3.5.10. We define $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T})}^{\text{st}}(X)$ as the preimage of $\text{Bun}_{\mathbf{G}}^{\text{st}}(X)$ in $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T}),\text{ind}}(X)$ under the map $\text{Bun}_{N_{\mathbf{G}}(\mathbf{T})}(X) \rightarrow \text{Bun}_{\mathbf{G}}(X)$.

Theorem 3.5.11. *The morphism*

$$\text{Bun}_{N_{\mathbf{G}}(\mathbf{T})}^{\text{st}}(X) \longrightarrow \text{Bun}_{\mathbf{G}}^{\text{st}}(X)$$

is an isomorphism.

Proof. If there are stable \mathbf{G} -bundles, then $\mathbf{G}^{\text{ad}} \cong \prod_i \mathbf{PGL}_{n_i}$ by Theorem 3.3.3. Moreover, if $\check{\lambda} \in \pi_1(\mathbf{G})$ is such that $\text{Bun}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X)$ is nonempty, then $\check{\lambda}^{\text{ad}} \in \prod_i (\mathbb{Z}/n_i\mathbb{Z})^*$. The component $\text{Bun}_{\mathbf{G}^{\text{ad}}}^{\check{\lambda}^{\text{ad}}}(X)$ consists of a single point corresponding to a stable \mathbf{G}^{ad} -bundle. In particular, the map

$$\text{Bun}_{\mathbf{G}}(X) \longrightarrow \text{Bun}_{\mathbf{G}^{\text{ad}}}(X)$$

maps $\text{Bun}_{\mathbf{G}}^{\text{st}}(X)$ to $\text{Bun}_{\mathbf{G}^{\text{ad}}}^{\text{st}}(X)$. Let $\mathbf{T}^{\text{ad}} = \mathbf{T}/Z(\mathbf{G})$ denote the maximal torus of \mathbf{G}^{ad} induced by \mathbf{T} . Then $N_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}^{\text{ad}}}(\mathbf{T}^{\text{ad}}) \times_{\mathbf{G}^{\text{ad}}} \mathbf{G}$. Applying [BH10, Lemma 2.2.1] (see also [BH23]), we obtain a 2-cartesian diagram

$$\begin{array}{ccc} \text{Bun}_{N_{\mathbf{G}}(\mathbf{T})}^{\text{st}}(X) & \longrightarrow & \text{Bun}_{N_{\mathbf{G}^{\text{ad}}}(\mathbf{T}^{\text{ad}})}^{\text{st}}(X) \\ \downarrow & & \downarrow \\ \text{Bun}_{\mathbf{G}}^{\text{st}}(X) & \longrightarrow & \text{Bun}_{\mathbf{G}^{\text{ad}}}^{\text{st}}(X) . \end{array} \quad (3.7)$$

Therefore, we reduce to the case where \mathbf{G} is a product of \mathbf{PGL} 's. This in turn can be directly reduced to the case $\mathbf{G} = \mathbf{PGL}_n$. In that case, we again use the cartesian diagram (3.7), but with reversed roles: we set $\mathbf{G} = \mathbf{GL}_n$ in which case $\mathbf{G}^{\text{ad}} = \mathbf{PGL}_n$. Then the left vertical morphism is an isomorphism by Lemma 3.5.9. Moreover, the lower horizontal morphism is smooth by [Hof10, Corollary 4.2] and surjective. Since being an isomorphism is local on the target in the smooth topology and the vertical morphisms are representable by [Hof10, Fact 2.3], we are done. \square

Definition 3.5.12. Let E be a stable \mathbf{G} -bundle. Then we define the *tropicalization* of E by $\text{Trop}(E)$ (see Section 3.5.2), where E' is the unique indecomposable principal $N_{\mathbf{G}}(\mathbf{T})$ -bundle corresponding to E under the isomorphism of Theorem 3.5.11.

Corollary 3.5.13. *Let $E \in \text{Bun}_{\mathbf{G}}^{\text{st}}(X)$. Then we have*

$$\deg(E) = \deg(\text{Trop}(E)) .$$

Proof. This follows directly from Proposition 3.5.6. \square

Theorem 3.5.14. *Let \mathbf{G} be a reductive group of type $\prod A_{n_i}$ and let $\check{\lambda} \in \pi_1(\mathbf{G})$ be a stable degree. Then there exists a commutative diagram*

$$\begin{array}{ccc} & & \Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X)) \\ & \nearrow \rho & \downarrow \cong \\ (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X))^{\text{an}} & & \\ & \searrow \text{Trop} & \downarrow \\ & & \mathcal{M}_{\mathbf{G}^{\text{trop},\text{ind}}}^{\check{\lambda}}(\Gamma_X) , \end{array}$$

where ρ denotes the retraction map to the essential skeleton $\Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X))$ of $(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X))^{\text{an}}$.

Proof. Consider the diagram

$$\begin{array}{ccc} (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{st}}(X))^{\text{an}} & \xrightarrow{\det} & (\mathcal{M}_{Z^c(\mathbf{G})}^{\det(\check{\lambda})}(X))^{\text{an}} \\ \downarrow \text{Trop} & & \downarrow \text{Trop} \searrow \rho \\ \mathcal{M}_{\mathbf{G}^{\text{trop}},\text{ind}}^{\check{\lambda}}(\Gamma_X) & \xrightarrow{\det} & \mathcal{M}_{Z^c(\mathbf{G}^{\text{trop}})}^{\det(\check{\lambda})}(\Gamma_X) \xleftarrow{\cong} \Sigma(\mathcal{M}_{Z^c(\mathbf{G})}^{\det(\check{\lambda})}(X)) . \end{array}$$

The square on the left is commutative by Lemma 3.5.5. By Theorem 3.3.3, the algebraic determinant map on the top of the square is an isomorphism. By Theorem 3.4.20, the tropical determinant map on the bottom of the square is an isomorphism. It thus suffices to show the existence of the lower right isomorphism such that the triangle on the right commutes. But this is Proposition 3.5.1. \square

3.5.4 Tropicalizing semistable \mathbf{G} -bundles over a Tate curve

In this section, we continue our study of the tropicalization of principal bundles on a Tate curve X by reducing the semistable case to the case of stable bundles from the previous section.

Given a semistable bundle $F \in \mathcal{M}_{\mathbf{G}}^{\check{\lambda},\text{ss}}(X)$, let \mathbf{L} be the Levi subgroup determined by Theorem 3.3.5, corresponding to the degree $\check{\lambda}_{\mathbf{G}} = \deg(F) \in \pi_1(\mathbf{G})$ and chosen such that $\mathbf{T} \subseteq \mathbf{L}$. Reducing the structure group, we obtain an \mathbf{L} -bundle $F_{\mathbf{L}}$ on X of degree $\check{\lambda}_{\mathbf{L}}$, unique up to the $W_{\mathbf{L},\mathbf{G}}$ -action and stable by Theorem 3.3.5. We then use Definition 3.5.12 to tropicalize $F_{\mathbf{L}}$ to an \mathbf{L}^{trop} -bundle on Γ_X of degree $\check{\lambda}_{\mathbf{L}}$. The inclusion $\mathbf{L} \rightarrow \mathbf{G}$ induces a morphism $\mathbf{L}^{\text{trop}} \rightarrow \mathbf{G}^{\text{trop}}$ of tropical reductive groups and the induced map $\mathcal{M}_{\mathbf{L}^{\text{trop}}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X) \rightarrow \mathcal{M}_{\mathbf{G}^{\text{trop}}}^{\check{\lambda}_{\mathbf{G}}}(\Gamma_X)$ is $W_{\mathbf{L},\mathbf{G}}$ -equivariant. We then extend scalars to \mathbf{G}^{trop} to obtain the tropicalization of F .

Example 3.5.15. Let $F \in \mathcal{M}_{\mathbf{GL}_n}^{d,\text{ss}}(X)$ be a semistable \mathbf{GL}_n -bundle of degree $d \in \mathbb{Z}$. Note that in this case the Levi subgroup $\mathbf{L} \subseteq \mathbf{GL}_n$ from Theorem 3.3.5 is given by $\mathbf{L} = (\mathbf{GL}_{\frac{n}{h}})^h$ and $W_{\mathbf{L},\mathbf{G}} = S_h$, where $h = \gcd(n, d)$. Then one can show that F is equivalent to a direct sum $\bigoplus_{i=1}^h F_i$ of stable vector bundles of the same slope (see [GUZZ22, §7]) which is unique up to the S_h -action. In the more general framework, this is the same as a stable \mathbf{L} -bundle $F_{\mathbf{L}}$ on X of degree $(\frac{d}{h}, \dots, \frac{d}{h}) \in \mathbb{Z}^h$. Tropicalizing the stable \mathbf{L} -bundle, as explained in the previous section, corresponds to tropicalizing each summand F_i individually, which is precisely what is done in [GUZZ22]. In this sense, this section generalizes the tropicalization construction of semistable \mathbf{GL}_n -bundles on X of [GUZZ22].

First, we prove a lemma that bridges the gap between the algebraic structure group \mathbf{L} and its tropical counterpart \mathbf{L}^{trop} .

Lemma 3.5.16. *Let \mathbf{G} be a reductive group, and let \mathbf{L} be a Levi subgroup containing the (fixed) maximal torus \mathbf{T} with Weyl group $W_{\mathbf{L}} \subseteq W_{\mathbf{G}}$. Then there exists a natural isomorphism*

$$W_{\mathbf{L},\mathbf{G}} = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \xrightarrow{\cong} N_{W_{\mathbf{G}}}(W_{\mathbf{L}})/W_{\mathbf{L}} .$$

Moreover, if \mathbf{L} and $\check{\lambda}_{\mathbf{L}} \in \pi_1(\mathbf{L})$ are as in Theorem 3.3.5, then $N_{W_{\mathbf{G}}}(W_{\mathbf{L}})/W_{\mathbf{L}}$ acts on $\mathcal{M}_{\mathbf{L}^{\text{trop},\text{ind}}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X)$ and the tropicalization map

$$\text{Trop}: \mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},\text{st}}(X) \longrightarrow \mathcal{M}_{\mathbf{L}^{\text{trop},\text{ind}}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X)$$

is $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ -equivariant.

Proof. We first show that the quotient map

$$N_{\mathbf{G}}(\mathbf{L}) \cap N_{\mathbf{G}}(\mathbf{T}) \longrightarrow N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$$

is surjective. If $nL \in N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, then $n^{-1}\mathbf{T}n$ is a maximal torus of \mathbf{L} . As \mathbf{L} is reductive, all maximal tori in \mathbf{L} are conjugate, that is there exists $l \in \mathbf{L}$ with $(nl)^{-1}\mathbf{T}(nl) = \mathbf{T}$. Then we have $nl \in N_{\mathbf{G}}(\mathbf{L}) \cap N_{\mathbf{G}}(\mathbf{T})$ and $n\mathbf{L} = n\mathbf{L}$. Since $N_{\mathbf{G}}(\mathbf{L}) \cap N_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{L} = N_{\mathbf{L}}(\mathbf{T})$ we obtain a natural isomorphism

$$N_{\mathbf{G}}(\mathbf{L}) \cap N_{\mathbf{G}}(\mathbf{T})/N_{\mathbf{L}}(\mathbf{T}) \xrightarrow{\cong} N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} .$$

As $W_{\mathbf{L}} = N_{\mathbf{L}}(\mathbf{T})/\mathbf{T}$ and $N_{W_{\mathbf{G}}}(W_{\mathbf{L}}) = N_{N_{\mathbf{G}}(\mathbf{T})}(N_{\mathbf{L}}(\mathbf{T}))/\mathbf{T}$ we have

$$N_{W_{\mathbf{G}}}(W_{\mathbf{L}})/W_{\mathbf{L}} \cong N_{N_{\mathbf{G}}(\mathbf{T})}(N_{\mathbf{L}}(\mathbf{T}))/N_{\mathbf{L}}(\mathbf{T})$$

by the third isomorphism theorem. Therefore, it suffices to show that

$$N_{\mathbf{G}}(\mathbf{L}) \cap N_{\mathbf{G}}(\mathbf{T}) = N_{N_{\mathbf{G}}(\mathbf{T})}(N_{\mathbf{L}}(\mathbf{T})) .$$

The inclusion from left to right is clear, as any automorphism of \mathbf{L} that fixes \mathbf{T} also fixes the normalizer of \mathbf{T} . For the reverse inclusion let $n \in N_{N_{\mathbf{G}}(\mathbf{T})}(N_{\mathbf{L}}(\mathbf{T}))$. We need to show that n normalizes \mathbf{L} . Let Z be the connected component of the identity of $\bigcap_{\alpha \in \Psi} \ker(\alpha)$, where Ψ is the set of roots of \mathbf{L} . By [Hum81, Section 30.2], we have

$$\mathbf{L} = C_{\mathbf{G}}(Z) ,$$

so it suffices to show that n normalizes Z . As n normalizes $N_{\mathbf{L}}(\mathbf{T})$, it suffices to show that Z is the connected center of $N_{\mathbf{L}}(\mathbf{T})$. Because \mathbf{G} is reductive, we have $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ and hence $Z(N_{\mathbf{L}}(\mathbf{T})) \subseteq \mathbf{T}$. So $Z(N_{\mathbf{L}}(\mathbf{T}))$ is precisely the subset of \mathbf{T} fixed by all reflections in W , which is precisely $\bigcap_{\alpha \in \Psi} \ker(\alpha)$, the identity component of which is Z .

The normalizer $N_{W_{\mathbf{G}}}(W_{\mathbf{L}})$ is contained in $N_{\mathbf{G}^{\text{trop}}}(\mathbf{L}^{\text{trop}})$ and acts on $\mathcal{M}_{\mathbf{L}^{\text{trop}}, \text{ind}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X)$ by conjugation. Conjugation with inner automorphisms of \mathbf{L}^{trop} leaves \mathbf{L}^{trop} -bundles unchanged, so $W_{\mathbf{L}}$ is in the kernel of the action and we obtain an action of $N_{W_{\mathbf{G}}}(W_{\mathbf{L}})/W_{\mathbf{L}}$. Moreover, conjugation by an element in $N_{N_{\mathbf{G}}(\mathbf{T})}(N_{\mathbf{L}}(\mathbf{T}))$ tropicalizes to the conjugation by its image in $W_{\mathbf{G}}$. Therefore, the equivariance of Trop follows from Lemma 3.5.4. \square

Let $G = \check{M}_{\mathbb{R}} \rtimes W$ be a tropical reductive group and let $\check{\lambda} \in \pi_1(G)$. In general, it is not yet clear what it means for a principal G -bundle to have an indecomposable degree. But, by Theorem 3.3.2 there exists a parabolic subgroup $P = \check{M}_{\mathbb{R}} \rtimes W'$ in G such that there exists $\check{\lambda}_P \in \pi_1(P)$ with $\phi_P(\check{\lambda}_P) = \phi_G(\check{\lambda})$ and which is minimal with respect to that property. The parabolic P is unique up to conjugation and of type $\prod_i A_{n_i-1}$ by [Frä16, Cor. 4.2]. Let τ be an indecomposable W' -torsor. Then we denote

$$\mathcal{M}_{G, \text{ind}}^{\check{\lambda}}(\Gamma_X) := \mathcal{M}_{G, \tau W}^{\check{\lambda}}(\Gamma_X) .$$

Definition 3.5.17. Let \mathbf{G} be a reductive group and let E be a semi-stable principal \mathbf{G} -bundle of degree $\check{\lambda} \in \pi_1(\mathbf{G})$. By Theorem 3.3.5, there exists a Levi subgroup $\mathbf{L} \subseteq \mathbf{G}$, uniquely determined up to conjugation, and a degree $\check{\lambda}_{\mathbf{L}} \in \pi_1(\mathbf{L})$, such that \mathbf{L} is of type $\prod_i A_{n_i-1}$, the degree $\check{\lambda}_{\mathbf{L}}$ is stable, and E can be reduced to a stable \mathbf{L} -bundle $E_{\mathbf{L}}$ of degree $\check{\lambda}_{\mathbf{L}}$, uniquely up to the action of $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$. By Lemma 3.5.16, tropicalizing yields an object $\text{Trop}(E_{\mathbf{L}}) \in \mathcal{M}_{\mathbf{L}^{\text{trop}}, \text{ind}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X)$, well-defined up to the $N_{W_{\mathbf{G}}}(W_{\mathbf{L}})$ -action. Pushing forward to $\mathcal{M}_{\mathbf{G}^{\text{trop}}, \text{ind}}(\Gamma_X)$ then yields a uniquely determined element $\text{Trop}(E) \in \mathcal{M}_{\mathbf{G}^{\text{trop}}, \text{ind}}^{\check{\lambda}}(\Gamma_X)$.

Theorem 3.5.18. *Let X be a Tate elliptic curve over an algebraically closed complete nontrivially valued non-Archimedean field K of equicharacteristic 0 with minimal skeleton Γ_X , let \mathbf{G} be a reductive group and let $\check{\lambda} \in \pi_1(\mathbf{G})$. Denote by ρ the retraction map to the essential skeleton $\Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))$ of $(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))^{\text{an}}$. Then there exists a homeomorphism $\Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X)) \xrightarrow{\cong} \mathcal{M}_{\mathbf{G}^{\text{trop,ind}}}^{\check{\lambda}}(\Gamma_X)$ that makes the diagram*

$$\begin{array}{ccc}
 & & \Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X)) \\
 & \nearrow \rho & \downarrow \cong \\
 (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))^{\text{an}} & & \\
 & \searrow \text{Trop} & \\
 & & \mathcal{M}_{\mathbf{G}^{\text{trop,ind}}}^{\check{\lambda}}(\Gamma_X)
 \end{array}$$

commute.

Proof. Let \mathbf{L} and $\check{\lambda}_{\mathbf{L}}$ be as in Theorem 3.3.5. Consider the diagram of solid arrows

$$\begin{array}{ccccc}
 (\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X))^{\text{an}} & \xrightarrow{\hspace{10em}} & (\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))^{\text{an}} & & \\
 \downarrow \text{Trop} & \searrow \rho_{\mathbf{L}} & & \swarrow \rho & \downarrow \text{Trop} \\
 & \Sigma(\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X)) & \longrightarrow & \Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X)) & \\
 & \swarrow \cong & & \dashrightarrow & \\
 \mathcal{M}_{\mathbf{L}^{\text{trop,ind}}}^{\check{\lambda}_{\mathbf{L}}}(\Gamma_X) & \xrightarrow{\hspace{10em}} & \mathcal{M}_{\mathbf{G}^{\text{trop,ind}}}^{\check{\lambda}}(\Gamma_X) & , &
 \end{array}$$

where $\rho_{\mathbf{L}}$ is the retraction to the essential skeleton of $(\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X))^{\text{an}}$. The triangle to the left exists by Theorem 3.5.14. The retraction $\rho_{\mathbf{L}}$ is equivariant with respect to the action of $W_{\mathbf{L},\mathbf{G}} := N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} = N_{W_{\mathbf{G}}}(W_{\mathbf{L}})/W_{\mathbf{L}}$ by functoriality of the essential skeleton, and the map Trop in that triangle is $W_{\mathbf{L},\mathbf{G}}$ -equivariant by Lemma 3.5.16.

To show that the solid trapezoid on top exists, we first note that $\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}$ is isomorphic to a product of elliptic curves by Theorem 3.3.3, and hence its canonical bundle is trivial. Therefore, all pluricanonical forms on $\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X)$ define the same skeleton [MN15, Proposition 4.4.5 (5)], namely the essential skeleton $\Sigma(\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X))$. Let ω be a pluricanonical form on $\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X)$. The group $W_{\mathbf{L},\mathbf{G}}$ acts on ω by

$$w.\omega = \chi(w) \cdot \omega$$

for some character χ . If k is the order of χ , then $\omega^{\otimes k}$ is $W_{\mathbf{L},\mathbf{G}}$ -invariant, and hence there exist $W_{\mathbf{L},\mathbf{G}}$ -invariant pluricanonical forms. By [BM19, Proposition 6.1.9], every $W_{\mathbf{L},\mathbf{G}}$ -invariant pluricanonical form induces the same skeleton of $(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))^{\text{an}}$, namely the essential skeleton $\Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X))$, and we have

$$\Sigma(\mathcal{M}_{\mathbf{L}}^{\check{\lambda}_{\mathbf{L}},st}(X))/W_{\mathbf{L},\mathbf{G}} = \Sigma(\mathcal{M}_{\mathbf{G}}^{\check{\lambda},ss}(X)) .$$

The outer square is commutative by the construction of the tropicalization map for semistable bundles. Since we also have

$$\mathcal{M}_{\mathbf{L}^{\text{trop,ind}}}^{\check{\lambda}}(\Gamma_X)/W_{\mathbf{L},\mathbf{G}} = M_{\mathbf{G}^{\text{trop,ind}}}^{\check{\lambda}}(\Gamma_X).$$

by Theorem 3.4.25, it follows that the dashed arrow can be filled in uniquely by a homeomorphism that makes the whole diagram commutative. \square

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