Analysis of doubly degenerate chemotaxis-consumption systems

Duan Wu

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Herr Univ.-Prof. Dr. Michael Winkler

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Summary

This thesis undertakes a systematic analytical study of qualitative properties of solutions to a family of doubly degenerate chemotaxis-consumption systems. Global solvability and large time behavior are investigated for several variants of the system, reflecting different biological constellations. The analysis is based on various types of variational approaches, suitably adapted to the respective particular scenarios under consideration. Firstly, for systems involving certain signal-dependent bilinear growth terms the asymptotic behavior is described in one-dimensional boundary value problems, inter alia revealing that the limiting profile may exhibit spatial heterogeneity in the associated population density. A suitably refined analysis based on various types of apparently novel new functional inequalities, similar nontrivial patterning features are then detected also in some higher-dimensional problems. In contrast to this, for systems influenced by so-called Allee-type growth mechanisms, the large time dynamics are shown to be exclusively characterized by stabilization toward constant steady states.

Zusammenfassung

Diese Dissertation unternimmt eine systematische analytische Untersuchung der qualitativen Eigenschaften von Lösungen einer Familie doppelt degenerierter Chemotaxis-Absorptions–Systeme. Dabei werden für mehrere Varianten des Systems, die jeweils an unterschiedliche biologische Konstellationen angepasst sind, die globale Existenz von Lösungen nachgewiesen sowie deren Langzeitverhalten beschrieben. Die Analysis fußt auf verschiedenen Arten variationeller Ansätze, die geeignet an die jeweils betrachteten speziellen Situationen angepasst sind. Zunächst wird das asymptotische Verhalten in eindimensionalen Randwertproblemen für Systeme unter Einbezug gewisser signalabhängiger bilinearer Produktionsterme untersucht, wobei sich unter anderem herausstellt, dass entsprechende Grenzprofile räumliche Inhomogenitäten in der zugehörigen Populationsdichte zeigen. Auf Basis verschiedener Arten von offenbar neuen Funktionalungleichungen enthüllt eine geeignet verfeinerte Analysis ähnliche nichttriviale Musterbildungen auch in einigen höherdimensionalen Problemen. Im Gegensatz hierzu wird die Langzeitasymptotik in Systemen, die durch so genannte Alleeartige Wachstumsmechanismen beeinflusst werden, als ausschließlich durch Stabilisierung hin zu konstanten stationären Zuständen charakterisiert nachgewiesen.

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Statement of contributions

Chapter 1: I wrote the introductory chapter.

Chapter 2: I wrote the whole chapter.

Chapter 3: I wrote the whole chapter.

Chapter 4: This chapter is based on a joint work with Tobias Black and Shohei Kohatsu. Each author contributed equally.

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1 Introduction

1.1 Motivations

Beyond random diffusion, motile agents are capable of regulating their motion in response to chemical cues they perceive, exhibiting a widespread phenomenon in living systems. This type of directed migration, known as chemotaxis, can even be detected in allegedly simple unicellular species.

Identifying sensible models to describe the self-organized collective behavior in various subtle scenarios constitutes a major challenge. Not only can rigorous analytical studies of these models be used to evaluate whether the description is adequate, but they also provide predictive insights into the associated dynamics and deepen our understanding of the underlying mechanisms, thereby stimulating sustained interest among mathematicians.

A prominent example in this direction is provided by the minimal Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1.1}$$

which models the pattern formation of *Dictyostelium discoideum* (with u denoting the cell density) under the influence of a chemoattractant (with v representing its concentration). Since its proposal, a wide range of experimentally observed phenomena have been well captured by a growing body of mathematical studies on system (1.1.1) and its closely related variants ([88], [24], [20], [23], [51], [75]). In particular, the existence of heterogeneous equilibria of (1.1.1) is remarkable ([63], [71], [19], [1], [54], [55]).

By contrast, for signal consumption systems with a substantially different evolution mechanism–namely, that the signal is consumed rather than produced by cells–such as the one originally introduced to model the oxygenotactic motion of the bacterium *Bacillus subtilis* ([70]) (in the version neglecting interactions with liquid environments),

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{1.1.2}$$

the corresponding knowledge remains much more limited. In fact, within the realm of bounded solutions to (1.1.2) and its generalizations, the existing literature on asymptotic behavior is largely restricted to spatial stabilization toward constant steady states ([66], [78], [11]), even when couplings to the surrounding fluid are taken into account ([77], [35], [79]). Together with the complex structures observed in experiments, this indicates the limitations of system (1.1.2) in fully capturing the subtle mechanisms underlying the observed dynamics within prescribed environmental conditions.

This motivates an investigation into a class of more intricate models in the present work, inspired by the striking discoveries of fractal-like patterns that have been reported to spontaneously arise in colonies of *Bacillus subtilis* when exposed to nutrient-poor solid agars: Initially confined populations may temporarily spread due to random diffusion and chemotaxis upward nutrient gradients, but approach a stable state which can well remain spatially localized and exhibit sharply structured boundaries ([48]).

In order to appropriately describe especially this latter class of phenomena, the authors in [39] (see also [29] for a precedent) proposed taxis-type models of the form

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi\nabla \cdot (u^2 v\nabla v) + \ell uv, \\ v_t = \Delta v - uv, \end{cases} \tag{1.1.3}$$

for the bacterial population density $u = u(x,t)$ and the concentration $v = v(x,t)$ of the surrounding nutrient that partially directs the microbial motion according to chemoattraction. While the appearance of the factor v both in the diffusive and the cross-diffusive mechanisms reflects the experimentally well-endorsed property of individuals to reduce their speed of motion near sites of poor food resources ([49], [9]), the additional inclusion of the factors u and u^2 in the rates of migration can similarly be understood as quantifying a corresponding reduction of motilities at small population densities. A rigorous derivation of (1.1.3) is detailed in [61].

To incorporate broader biological motivations within this class of models, we do not restrict our attention to a single specific realization of the above model, but rather focus on a more general class of related variants. In particular, we consider the doubly degenerate system

$$\begin{cases} u_t = \nabla \cdot (D_0(u,v)\nabla u) - \nabla \cdot (S_0(u,v)\nabla v) + f(u,v), \\ v_t = \Delta v - g(u)v, \end{cases} \tag{1.1.4}$$

under no-flux boundary conditions in bounded domains, where the source term $f(u,v)$ is introduced to represent various possible growth mechanisms and $g(u)$ is used to characterize different potential modes of nutrient consumption. Of particular salience herein is the fact that the diffusion coefficient $D_0(u,v)$ and the chemotactic sensitivity function $S_0(u,v)$ not only degenerate at small values of u but also as v approaches the level zero.

1.2 Main difficulties

Core challenges for any mathematical analysis are linked to precisely this type of double degeneracy especially in the cell diffusion process induced by the choice of $D_0(u,v)$, which sharply distinguishes this problem class from quite well-understood Keller-Segel systems ([5], [10], [17], [14], [24], [52], [53]), or to diffusion mechanisms influenced by exclusively density-dependent degeneracies of standard porous medium type ([6], [31], [13], [15], [30], [36]). Studies on taxis-type evolution systems involving diffusion degeneracies near vanishing signal concentrations to date seem to have mainly been concentrating on cases in which, unlike in (1.1.4), a mathematical analysis can draw on favorable structural properties going along with the assumption that the undirected and the directed migration mechanisms can jointly be modeled by a single second-order operator, such as in

$$u_t = \Delta(u^m v^\alpha) + f(u,v)$$

with $m > 0, \alpha > 0$ and appropriate f ([41], [38], [98], [101]).

To overcome the difficulties arising in particular from the cross-degeneracy, it turns out that in the analysis of the evolution of $\int_\Omega u^p$, the coupling with a weighted energy functional of the form

$$\int_\Omega \frac{|\nabla v|^q}{v^r}$$

with parameters q and r chosen appropriately at different stages plays a fundamental role; a central ingredient in this process is the construction of delicate functional inequalities tailored to different scenarios (for instance Lemmata 2.2.6, 3.3.2, 3.3.3, 3.4.1, 3.3.8, 4.3.2, 4.4.1, 5.2.3, 6.2.4, 6.4.2), which allow for an effective exploitation of the signal-weighted (and hence weakened) dissipative contributions.

This thesis ultimately sets up appropriate analytical approaches to cope with the simultaneous presence of density- and signal-dependent diffusion degeneracies in (1.1.4), across a range of settings corresponding to various choices of $D_0(u,v)$, $S_0(u,v)$, $f(u,v)$ and $g(u)$, thereby facilitating the development of a global existence and asymptotic behavior theory for the system (1.1.4). The main results can be summarized in terms of the following initial-boundary value problem

$$\begin{cases} u_t = \nabla\cdot(u^{m-1}v\nabla u) - \nabla\cdot(S(u)v\nabla v) + f(u,v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - g(u)v, & x\in\Omega,\ t>0,\\ (u^{m-1}v\nabla u - S(u)v\nabla v)\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.2.1}$$

as associated with the choice $D_0(u,v) = u^{m-1}v$ in (1.1.4).

1.3 Main results

Chapter 2 focuses on the study of system (1.2.1) in the case where the migration operators depend nonlinearly on the population density u and linearly on the nutrient concentration v, in a one-dimensional setting. More precisely, with $m \ge 1$, $S \in C^1([0,\infty))$ satisfying

$$S(u) \le C_S u(u+1)^{\alpha-1} \qquad \text{for all} \quad u \ge 1$$

with $\alpha \ge 0$ and $C_S > 0$, $f(u,v) = \ell uv$ with $\ell \ge 0$, and $g(u) = u$, we prove the global existence of weak solutions for $1 \le m < 3$ as well as classical solutions for $m \ge 3$ in bounded intervals, when α lies in appropriate ranges. In addition, their large time behavior is shown, in which the limiting profile of the first component can be spatially heterogeneous, whereas the second component v converges to 0.

In Chapter 3, under the same choices of $S(u)$, $f(u,v)$, and $g(u)$ as considered in Chapter 2, we turn to the study of system (1.2.1) in the two-dimensional case. The global existence of weak or classical solutions, exhibiting properties analogous to those obtained in one spatial dimension, is established in bounded convex domains. Moreover, by deriving a Harnack-type inequality, we are able to uncover the nontrivial large time behavior of these solutions, which was previously constrained to one dimension.

The following three chapters are devoted to the prototypical setting of system (1.2.1) with $m = 2$ and $S(u) = u^2$, for various choices of $f(u,v)$ and $g(u)$.

Specifically, in Chapter 4 we investigate the case in the absence of production, by assuming $f(u,v) = 0$, together with a sublinear consumption mechanism characterized by $g \in C^1([0,\infty))$ satisfying

$$g(u) \geq c_g u^\alpha \qquad \text{for all } u \geq 1 \quad \text{and} \quad g(u) \leq C_g u^\alpha \qquad \text{for all } u > 0$$

for $\alpha \in (0,1)$ and constants $c_g, C_g > 0$. The global existence of weak solutions and their asymptotic behavior are proved in bounded domains of arbitrary space dimension $N \geq 2$ when $\alpha \in (0, \frac{2}{N})$. One of the highlights of this chapter is that the convexity restriction on the domain is not required.

Chapter 5 presents results for the physically most relevant two-dimensional case of (1.1.3). It is shown that bounded weak solutions exist in bounded convex domains, and additionally the large time behavior of these solutions is described. In comparison to results in existing literature, this removes any assumption on the strict positivity of u_0 and on the size of v_0, and thereby indicates that this original biologically motivated version can indeed be used to model phenomena of pattern formation and nontrivial evolution of positivity, as observed in experiments on bacterial population dynamics.

Chapter 6 considers system (1.2.1) with a typical population growth saturation effect and linear signal consumption, specified by $g(u) = u$ and the growth term $f(u,v)$ satisfying

$$f(u,v) \leq \kappa u - \mu u^\gamma \qquad \text{for all } u > 0$$

with $\kappa \geq 0$, $\mu > 0$ and $\gamma > 1$. We reveal that global weak solutions to (1.2.1) exist and remain bounded in bounded domains of space dimension $N \geq 2$, under suitable assumptions on the parameter γ. Furthermore, we establish the large time stabilization of these solutions: the solutions converge to constant states in an appropriate topology, in contrast to the behavior captured in Chapters 2-5.

1.4 Definition of weak solutions

To substantiate the goal of our considerations in this thesis, let us first specify the following notion of weak solvability that seems to form a fairly natural generalization of classical concepts when concretized to (1.2.1), and that has already been underlying precedent studies (see [81] and [85], for instance).

Definition 1.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $m \geq 1$, $0 \leq u_0 \in L^1(\Omega)$ and $0 \leq v_0 \in L^\infty(\Omega)$, and suppose that*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} \\ v \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{1.4.1}$$

are such that

$$\begin{cases} u^m \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ S(u)\nabla v \in L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^N) \qquad \textit{and} \\ f(u,v), \ g(u) \in L^1_{loc}(\overline{\Omega} \times [0,\infty)). \end{cases} \tag{1.4.2}$$

Then (u,v) will be called a global weak solution *of (1.2.1) if for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ we have*

$$\begin{aligned} & -\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) \\ = & -\frac{1}{m}\int_0^\infty \int_\Omega v\nabla u^m \cdot \nabla\varphi + \int_0^\infty \int_\Omega S(u)v\nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega f(u,v)\varphi \end{aligned} \tag{1.4.3}$$

and

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) \;=\; \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega g(u)v\varphi. \tag{1.4.4}$$

At the end of this chapter, we emphasize that the notation introduced in each subsequent chapter is specific to that chapter, unless stated otherwise. For further details on the related systems and the relevant literature, we refer the reader to the introductory sections at the beginning of Chapters 2 and 6.

2 Global solvability and nontrivial large time behavior in a one-dimensional problem

2.1 Introduction

The reaction-diffusion continuous system

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi\nabla \cdot (u^2 v\nabla v) + \ell uv, \\ v_t = \Delta v - uv, \end{cases} \tag{2.1.1}$$

was first introduced by Kawasaki et al. in the taxis–free case corresponding to $\chi = 0$ to describe the pattern formation of *Bacillus subtilis* grown on thin agar plates ([29]). Subsequently, to comply with experimental phenomena discovered in [22] and [8], Leyva et al. proposed in [39] that the chemotactic flux term should be included by assuming a positive chemotactic sensitivity $\chi > 0$.

The numerical simulations performed in [39, Sections 4 and 5] indicate that the behavior of numerical solutions of the refined system is indeed in better accordance with the experimentally observed complex population distributions than the taxis-free situation. These numerical findings thus sparked the interest in the mathematical analysis of system (2.1.1).

However, as mentioned before, due to the troubles caused by the degeneracies especially in the diffusion operator, virtually no mathematical findings regarding basic solution theory touched on it until the pioneering work [81] revealed the existence of global weak solutions for arbitrarily large initial data in one-dimensional space under the condition

$$\int_\Omega \ln u_0 > -\infty, \tag{2.1.2}$$

where the most stunning feature is that the first component of the solution asymptotically stabilizes toward a nontrivial function coinciding with the spatial profile of a solution to a scalar porous medium-type parabolic equation in stark contrast to the majority of previous results concerning the large time behavior. The integrability assumption (2.1.2) was removed in a subsequent work [42], whereas the uniqueness of one-dimensional weak solutions requires the initial data to satisfy $\frac{1}{u_0} \in L^1(\Omega)$ ([16]). In the corresponding taxis-free framework, namely $\chi = 0$, the global existence of weak solutions and the stabilization results were asserted in convex domains of any dimension ([85]).

In the current chapter, we take into account nonlinear crowding effects in the motility terms by considering a system of the form

$$\begin{cases} u_t = \nabla \cdot (u^{m-1} v \nabla u) - \nabla \cdot (S(u) v \nabla v) + \ell u v, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (u^{m-1} v \nabla u - S(u) v \nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.1.3}$$

where $m \geq 1$, $\ell \geq 0$, and $S \in C^1([0,\infty))$ satisfies

$$S(s) \leq C_S s (s+1)^{\alpha-1} \qquad \text{for all} \quad s \geq 1$$

with $\alpha \geq 0$ and $C_S > 0$. Here, the mechanical stress among densely packed cells is assumed to intensify in a porous medium-type manner, increasing nonlinearly with the cell density, which is biologically meaningful especially under resource-limited conditions (see [32], [26], [25], [7]).

Under the assumption that the initial data (u_0, v_0) satisfy

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0 & \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}, \end{cases} \tag{2.1.4}$$

we investigate how the interplay between the parameters m and α governs the global well-posedness of system (2.1.3). The main results obtained in this chapter are stated as follows.

The first result concerns the global solvability.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}$ be an open bounded interval, $m \geq 1$, and $\ell \geq 0$. Suppose that the initial data u_0 and v_0 satisfy* (2.1.4) *and that*

$$S \in C^1([0,\infty)) \ \ \text{is nonnegative.} \tag{2.1.5}$$

(i) If $1 \leq m < 2$ and S fulfills

$$S(s) \leq C_S s (s+1)^{\alpha-1} \quad \text{for all } s \geq 0 \tag{2.1.6}$$

with $\alpha \in [m-1, m]$ and $C_S > 0$, then system (2.1.3) *admits a global weak solution (u,v) in the sense of Definition 1.4.1, which is such that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$, that*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) & \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases} \tag{2.1.7}$$

and that

$$\sup_{t>0} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{2.1.8}$$

(ii) If $2 \leq m < 3$ and S fulfills

$$S(s) \leq C_S s^\alpha \quad \text{for all } s \geq 0 \tag{2.1.9}$$

with $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*, then there exists a global weak solution of* (2.1.3) *in the sense of Definition 1.4.1, which satisfies* $u \geq 0$ *and* $v > 0$ *in* $\overline{\Omega} \times [0,\infty)$*,* (2.1.7) *and* (2.1.8)*.*

(iii) In the case $m \geq 3$ *and* (2.1.9) *holds with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*, and when additionally* $u_0 > 0$ *in* $\overline{\Omega}$*, one can find a global classical solution* (u,v) *of* (2.1.3) *fulfilling*

$$\begin{cases} u \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \\ v \in \cap_{q>1} C^0\left([0,\infty); W^{1,q}(\Omega)\right) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases} \tag{2.1.10}$$

as well as $u > 0$ *and* $v > 0$ *in* $\overline{\Omega} \times [0,\infty)$*. Moreover,* (2.1.8) *holds.*

For the large time behavior of these solutions, we have the following result.

Corollary 2.1.2. *Let* $\Omega \subset \mathbb{R}$ *be an open bounded interval,* $m \geq 1$*, and* $\ell \geq 0$*. Suppose that besides satisfying* (2.1.5)*, the function* S *and the initial data* (u_0, v_0) *comply with one of the hypotheses in Theorem 2.1.1 (i), (ii) or (iii). Then there exists* $u_\infty \in C^0(\overline{\Omega})$ *such that the solution of* (2.1.3) *found in Theorem 2.1.1 has the property that*

$$u(\cdot,t) \to u_\infty \quad \text{and} \quad v(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty.$$

Moreover, the limit function satisfies $u_\infty = w(\cdot,1)$*, with* $w \in C^0(\overline{\Omega} \times [0,1])$ *being a nonnegative weak solution of*

$$\begin{cases} w_\tau = \left(a(x,\tau) w^{m-1} w_x\right)_x - \left(b(x,\tau) S(w)\right)_x + \ell a(x,\tau) w, & x \in \Omega,\ \tau \in (0,1), \\ w_x = 0, & x \in \partial\Omega,\ \tau \in (0,1) \\ w(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

in the sense that $w^m \in L^1_{loc}([0,1); W^{1,1}(\Omega))$ *and* $S(w) \in L^1_{loc}(\Omega \times [0,1))$*, and that for all* $\varphi \in C_0^\infty(\overline{\Omega} \times [0,1))$ *we have*

$$\begin{aligned} -\int_0^1 \int_\Omega w\varphi_\tau - \int_\Omega w_0 \varphi(\cdot,0) \quad = \quad & -\frac{1}{m} \int_0^1 \int_\Omega a(x,\tau)(w^m)_x \varphi_x \\ & + \int_0^1 \int_\Omega b(x,\tau) S(w) \varphi_x + \ell \int_0^1 \int_\Omega a(x,\tau) w\varphi \end{aligned}$$

where for $(x,\tau) \in \Omega \times (0,1)$ *and* $t = \phi^{-1}(\tau)$*,*

$$a(x,\tau) := L \cdot \frac{v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \quad \text{and} \quad b(x,\tau) := L \cdot \frac{v(x,t) v_x(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}},$$

with

$$L := \int_0^\infty \|v(\cdot,s)\|_{L^\infty(\Omega)} ds \quad \text{and} \quad \phi(t) := \frac{1}{L} \cdot \int_0^t \|v(\cdot,s)\|_{L^\infty(\Omega)} ds, \quad t \geq 0.$$

Moreover, there exists $C > 0$ *satisfying*

$$\frac{1}{C} \leq a(x,\tau) \leq C \quad \text{and} \quad |b(x,\tau)| \leq C \quad \text{for all } (x,\tau) \in \Omega \times (0,1).$$

Main ideas. The foundation of our method relies closely on a key functional of the form $\int_\Omega \frac{v_x^2}{v}$, which enjoys the favorable property that

$$\frac{d}{dt}\int_\Omega \frac{v_x^2}{v} + 2\int_\Omega \frac{u}{v}v_x^2 \le 2\int_\Omega u^2 v. \tag{2.1.11}$$

In fact, we start with a coupled functional expressed by

$$\int_\Omega u^p + \int_\Omega \frac{v_x^2}{v}, \tag{2.1.12}$$

where p is suitably chosen as $p = 3 - m$, roughly speaking. For the special cases $m = 2$ and $m = 3$, we substitute u^0 with the function $\ln u$ and accordingly replace u by $u \ln u$, which are dimensionally consistent with u^0 and u, respectively (Lemma 2.3.1).

Then, on the basis of an L^4 bound on v_x (Lemma 2.2.2), by means of functional inequality

$$\begin{aligned} \|\varphi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)} \quad \le \quad & \eta\int_\Omega \varphi^{p-1}\psi\varphi_x^2 + \eta\cdot\Big\{\int_\Omega \varphi^p\Big\}\cdot\int_\Omega \frac{\varphi}{\psi}\psi_x^2 \\ & + C(\eta,p,r)\cdot\Big\{\int_\Omega \varphi\Big\}^p\cdot\int_\Omega \varphi\psi \end{aligned}$$

(Lemma 2.2.6), the ill-signed contributions of (2.1.12) can be controlled by the dissipative integral $\int_\Omega \frac{u}{v}v_x^2$ appearing in (2.1.11), as well as by the favorable term $\int_\Omega v u_x^2$ arising from the evolution of $\int_\Omega u^p$. This furnishes some boundedness properties presented in Lemma 2.3.2, which are then utilized to derive bounds for u in L^p for any $p > 1$ (Lemma 2.3.3).

With these estimates at hand, we are able to obtain higher regularity properties (Lemma 2.4.1, Lemma 2.4.3) and thereby establish global existence. The asymptotic behavior can be shown in a manner similar to that in [42] thanks to the well-established Harnack-type inequality proved in [92].

2.2 Preliminaries

2.2.1 Regularized problems, local existence and basic properties

Similar to the approximating procedure used in [42], we consider the following regularized version of system (2.1.3) given by

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(u_\varepsilon^{m-1}v_\varepsilon\nabla u_\varepsilon) - \nabla\cdot(S(u_\varepsilon)v_\varepsilon\nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v(x,0) = v_{0\varepsilon}(x) := v_0(x), & x\in\Omega \end{cases} \tag{2.2.1}$$

with $\varepsilon \in (0,1)$, where $u_{0\varepsilon}(x)$, depending on m, is defined by

$$u_{0\varepsilon}(x) := \begin{cases} u_0(x) + \varepsilon, & 1 \le m < 3,\\ u_0(x) & m \ge 3. \end{cases} \tag{2.2.2}$$

Owing to the positivity of $u_{0\varepsilon}$, this problem admits the following local existence and extensibility criterion, which we state in arbitrary spatial dimension for later convenient reference in Chapters 2-6.

Lemma 2.2.1. *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $m \geq 1$, and $\ell \geq 0$. Suppose that besides fulfilling (2.1.4) and (2.1.5), the function S and the initial data (u_0, v_0) comply with one of the following hypotheses:*
(i) $1 \leq m < 2$ and S fulfills (2.1.6) with $\alpha \in [m-1, m]$ and $C_S > 0$;
(ii) $2 \leq m < 3$ and S fulfills (2.1.9) with $\alpha \in [m-1, m]$ and $C_S > 0$;
(iii) $m \geq 3$, S fulfills (2.1.9) with $\alpha \in [m-1, m]$ and $C_S > 0$, and additionally $u_0 > 0$ in $\overline{\Omega}$.
Then for each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0, \infty]$ and at least one pair $(u_\varepsilon, v_\varepsilon)$ of functions

$$\begin{cases} u_\varepsilon \in C^0\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \\ v_\varepsilon \in \cap_{q>1} C^0\left([0, T_{max,\varepsilon}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \end{cases} \tag{2.2.3}$$

such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{max,\varepsilon})$, that $(u_\varepsilon, v_\varepsilon)$ solves (2.2.1) in the classical sense in $\Omega \times (0, T_{max,\varepsilon})$, and that

$$\text{if } T_{max,\varepsilon} < \infty, \quad \text{then} \quad \limsup_{t \to T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2.4}$$

In addition, this solution satisfies

$$\int_\Omega u_{0\varepsilon} \leq \int_\Omega u_\varepsilon(\cdot,t) \leq \int_\Omega u_{0\varepsilon} + \ell \int_\Omega v_{0\varepsilon} \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{2.2.5}$$

as well as for all $t_0 \in [0, T_{max,\varepsilon})$,

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{2.2.6}$$

and

$$\int_{t_0}^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_\varepsilon(\cdot,t_0) \qquad \text{for all } \varepsilon \in (0,1). \tag{2.2.7}$$

Proof. By means of Theorem 14.4, Theorem 14.6 and Theorem 15.5 in [3], we can obtain the local existence and extensibility criterion in the following form

$$\begin{aligned} &\text{if } T_{max,\varepsilon} < \infty, \quad \text{then} \\ &\qquad \limsup_{t \to T_{max,\varepsilon}} \Big\{ \|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \\ &\qquad\qquad + \left\| \frac{1}{u_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} + \left\| \frac{1}{v_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} \Big\} = \infty. \end{aligned} \tag{2.2.8}$$

We claim that (2.2.8) and (2.2.4) are equivalent within this framework. It is clear that (2.2.4) implies (2.2.8). On the other hand, after assuming that for some $\varepsilon \in (0,1)$ there exists $c_1 = c_1(\varepsilon) > 0$ such that $\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1(\varepsilon)$ for all $t \in (0, T_{max,\varepsilon})$ with $T_{max,\varepsilon} < \infty$, we can find positive constants $c_2 = c_2(\varepsilon)$ and $c_3 = c_3(\varepsilon)$ such that

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq c_2 \qquad \text{for all } \; t \in (0, T_{max,\varepsilon}) \tag{2.2.9}$$

and

$$v_\varepsilon \geq c_3 \qquad \text{in} \ \ \Omega \times (0, T_{max,\varepsilon}) \tag{2.2.10}$$

by using the same argument as the first part of [84, Lemma 2.1]. Under the assumptions in (i), (ii) or (iii), it is easy to verify that there exists $c_4 = c_4(\varepsilon) > 0$ fulfilling

$$S(u_\varepsilon(x,t)) \leq c_4 u_\varepsilon^{\frac{m-1}{2}}(x,t) \qquad \text{for all} \ \ (x,t) \in \Omega \times (0, T_{max,\varepsilon}). \tag{2.2.11}$$

Thus, rewriting the first equation in (2.2.1) in the following form

$$u_{\varepsilon t} = \nabla \cdot a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) + b_\varepsilon(x,t,u_\varepsilon,), \qquad x \in \Omega,\ t \in (0, T_{max,\varepsilon})$$

with

$$\begin{aligned} &a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) := v_\varepsilon(x,t) u_\varepsilon^{m-1}(x,t)\nabla u_\varepsilon(x,t) - S(u_\varepsilon(x,t)) v_\varepsilon(x,t)\nabla v_\varepsilon(x,t) \qquad \text{and} \\ &b_\varepsilon(x,t) = \ell u_\varepsilon(x,t) v_\varepsilon(x,t), \qquad\qquad (x,t) \in \Omega \times (0, T_{max,\varepsilon}), \end{aligned}$$

we see from Young's inequality combined with (2.2.9)-(2.2.11) that

$$\begin{aligned} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon &= v_\varepsilon u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - S(u_\varepsilon) v_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon \\ &\geq c_3 u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - c_4 u_\varepsilon^{\frac{m-1}{2}} v_\varepsilon |\nabla v_\varepsilon||\nabla u_\varepsilon| \\ &\geq c_3 u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - c_2^2 c_4 u_\varepsilon^{\frac{m-1}{2}}|\nabla u_\varepsilon| \\ &\geq \frac{c_3}{2} u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - \frac{c_2^4 c_4^2}{2c_3} \qquad \text{for all} \ \ (x,t) \in \Omega \times (0, T_{max,\varepsilon}) \end{aligned}$$

as well as

$$\begin{aligned} &|a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)| \leq c_2 u_\varepsilon^{m-1}|\nabla u_\varepsilon| + c_2^2 c_4 u_\varepsilon^{\frac{m-1}{2}}, \qquad \text{and} \\ &|b_\varepsilon(x,t)| \leq \ell c_1 c_2 \qquad\qquad \text{for all } (x,t) \in \Omega \times (0, T_{max,\varepsilon}), \end{aligned}$$

which together with the Hölder estimates in [62] guarantee the existence of $\theta_1 = \theta_1(\varepsilon) \in (0,1)$ such that $u_\varepsilon \in C^{\theta_1, \frac{\theta_1}{2}}\left(\overline{\Omega} \times [0, T_{max,\varepsilon}]\right)$. Moreover, the parabolic Schauder theory in [33] yields the existence of $\theta_2 = \theta_2(\varepsilon) \in (0,1)$ such that $v_\varepsilon \in C^{2+\theta_2, 1+\frac{\theta_2}{2}}\left(\overline{\Omega} \times [\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}]\right)$.

Now we arrange the first equation in (2.2.1) to the following form

$$u_{\varepsilon t} = A_\varepsilon(x,t)\Delta u_\varepsilon + B_\varepsilon(x,t)\cdot\nabla u_\varepsilon + D_\varepsilon(x,t) u_\varepsilon \qquad x \in \Omega,\ t \in (0, T_{max,\varepsilon}),$$

where

$$\begin{aligned} A_\varepsilon(x,t) &:= u_\varepsilon^{m-1}(x,t) v_\varepsilon(x,t), \\ B_\varepsilon(x,t) &:= (m-1) u_\varepsilon^{m-2}(x,t) v_\varepsilon(x,t)\nabla u_\varepsilon(x,t) \\ &\quad + u_\varepsilon^{m-1}(x,t)\nabla v_\varepsilon(x,t) - S'(u_\varepsilon) v_\varepsilon(x,t)\nabla v_\varepsilon(x,t) \qquad \text{and} \\ D_\varepsilon(x,t) &:= -\frac{S\left(u_\varepsilon(x,t)\right)}{u_\varepsilon(x,t)} v_\varepsilon(x,t)\Delta v_\varepsilon(x,t) \\ &\quad - \frac{S\left(u_\varepsilon(x,t)\right)}{u_\varepsilon(x,t)} |\nabla v_\varepsilon(x,t)|^2 + \ell v_\varepsilon(x,t), \qquad (x,t) \in \Omega \times (0, T_{max,\varepsilon}). \end{aligned}$$

From the L^∞-boundedness of u_ε, one can see that

$$S\left(u_\varepsilon(x,t)\right) \le (1+(1+c_1)^{\alpha-1})u_\varepsilon \qquad \text{for all} \quad (x,t) \in \Omega \times (0, T_{max,\varepsilon}),$$

which together with the boundedness of v_ε, ∇v_ε and Δv_ε entails that there is $c_5 = c_5(\varepsilon) > 0$ such that

$$D_\varepsilon \ge -c_5 \qquad \text{in} \quad \Omega \times \left(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\right).$$

Then, by the comparison principle, we have

$$u_\varepsilon \ge \inf_{x\in\Omega} u_\varepsilon \left(x, \frac{1}{4}T_{max,\varepsilon}\right) e^{-c_5 \cdot \frac{3}{4}T_{max,\varepsilon}} \qquad \text{in} \quad \Omega \times \left(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\right). \tag{2.2.12}$$

Furthermore, applying the first order parabolic Hölder regularity theory ([45]), there exists $\theta_3 = \theta_3(\varepsilon) \in (0,1)$ such that $u_\varepsilon \in C^{1+\theta_3, \frac{1+\theta_3}{2}}\left(\overline{\Omega} \times [\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}]\right)$. In particular, we have

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_6 \qquad \text{for all} \quad t \in \left(\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}\right)$$

with $c_6 = c_6(\varepsilon) > 0$. This together with (2.2.9), (2.2.10) and (2.2.12) shows that (2.2.8) fails, so that our assertion is proved. We thus obtain (2.2.4).

Finally, the inequality in (2.2.6) results from an application of the maximum principle to the second equation, while the properties in (2.2.5) and (2.2.7) can be obtained by some basic integration arguments in the two equations of (2.2.1). For more details, one can refer to [81, Lemma 2.2]. □

Frow now on, we shall tacitly assume that $\Omega \subset \mathbb{R}$ is an open bounded interval, that $\ell \ge 0$ is fixed, and that S satisfies (2.1.5). Without further explicit mentioning, we assume u_0 and v_0 always fulfill (2.1.4), and accordingly let $(u_\varepsilon, v_\varepsilon)$ and $T_{max,\varepsilon}$ be as yielded by Lemma 2.2.1.

The following two lemmas provide gradient bounds and entropy-type estimates for v_ε.

Lemma 2.2.2. *There exists $C > 0$ such that*

$$\|v_{\varepsilon x}(\cdot,t)\|_{L^4(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{2.2.13}$$

Proof. According to (2.2.5), there exists $c_1 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le c_1 \qquad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Then the variation-of-constants representation and well-known smoothing properties of Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ ([73, Lemma 1.3]) imply that there exist $\lambda_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}
\|v_{\varepsilon x}(\cdot,t)\|_{L^4(\Omega)} &= \left\|\partial_x e^{t\Delta} v_0 - \int_0^t \partial_x e^{(t-s)\Delta} u_\varepsilon v_\varepsilon ds\right\|_{L^4(\Omega)} \\
&\le c_2\|v_0\|_{W^{1,\infty}(\Omega)} + c_2 \int_0^t \left(1+(t-s)^{-\frac{7}{8}}\right) e^{-\lambda_1(t-s)} \|u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)\|_{L^1(\Omega)} ds \\
&\le c_2\|v_0\|_{W^{1,\infty}(\Omega)} + c_1c_2\|v_0\|_{L^\infty(\Omega)} \int_0^t \left(1+(t-s)^{-\frac{7}{8}}\right) e^{-\lambda_1(t-s)} ds
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. This together with (2.1.4) entails (2.2.13). □

Lemma 2.2.3. *There exists $C > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon v_{\varepsilon x}^2 \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{2.2.14}$$

Proof. Testing the second equation in (2.2.1) by v_ε^2 and integrating it on Ω imply this conclusion, where the boundedness of $\int_\Omega v_{0\varepsilon}^3$ ensured by (2.1.4) is necessary. □

2.2.2 Testing procedures and a crucial functional inequality

In the next, we first present some outcomes of the solution (2.2.1) resulting from basic testing procedures.

Lemma 2.2.4. *For all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, the following estimations hold:*

(i) If $p > 1$, then we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 \\ \le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-m-1} S^2(u_\varepsilon) v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon;\end{aligned} \tag{2.2.15}$$

(ii) if $0 < p < 1$, then we have

$$\begin{aligned}-\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 \\ \le \frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p-m-1} S^2(u_\varepsilon) v_\varepsilon v_{\varepsilon x}^2;\end{aligned} \tag{2.2.16}$$

(iii) if $p < 0$, then we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 \\ \le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-m-1} S^2(u_\varepsilon) v_\varepsilon v_{\varepsilon x}^2;\end{aligned} \tag{2.2.17}$$

(iv) if $m = 2$, then we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 \\ \le \frac{1}{2}\int_\Omega \frac{S^2(u_\varepsilon)}{u_\varepsilon^2} v_\varepsilon v_{\varepsilon x}^2 + \ell\int_\Omega u_\varepsilon v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon;\end{aligned} \tag{2.2.18}$$

(v) if $m = 3$, then we have

$$-\frac{d}{dt}\int_\Omega \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 \le \frac{1}{2}\int_\Omega \frac{S^2(u_\varepsilon)}{u_\varepsilon^4} v_\varepsilon v_{\varepsilon x}^2. \tag{2.2.19}$$

Proof. If $p > 1$ or $p < 0$, using integration by parts and Young's inequality, we infer from the first equation in (2.2.1) along with the boundary conditions that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p &= p\int_\Omega u_\varepsilon^{p-1}\left(u_\varepsilon^{m-1}v_\varepsilon u_{\varepsilon x}\right)_x - p\int_\Omega u_\varepsilon^{p-1}\left(S(u_\varepsilon)v_\varepsilon v_{\varepsilon x}\right)_x + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\ &= -p(p-1)\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon u_{\varepsilon x}^2 + p(p-1)\int_\Omega u_\varepsilon^{p-2}S(u_\varepsilon)v_\varepsilon u_{\varepsilon x}v_{\varepsilon x} + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\ &\le -\frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon u_{\varepsilon x}^2 + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-m-1}S^2(u_\varepsilon)v_\varepsilon v_{\varepsilon x}^2 \\ &\quad + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \end{aligned} \tag{2.2.20}$$

Then, (2.2.15) follows. When $p < 0$, the negativity of the last term in (2.2.20) yields (2.2.17). When $0 < p < 1$, proceeding as in the derivation of (2.2.20) and noting that $p(1-p) > 0$, we have

$$\begin{aligned}-\frac{d}{dt}\int_\Omega u_\varepsilon^p &= -p(1-p)\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon u_{\varepsilon x}^2 + p(1-p)\int_\Omega u_\varepsilon^{p-2}S(u_\varepsilon)v_\varepsilon u_{\varepsilon x}v_{\varepsilon x} - p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\ &\le -\frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon u_{\varepsilon x}^2 + \frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p-m-1}S^2(u_\varepsilon)v_\varepsilon v_{\varepsilon x}^2\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, and hence (2.2.16) holds.

The proofs of (2.2.18) and (2.2.19) are analogous to those of (2.2.15) and (2.2.16), respectively, and are therefore omitted. □

The following inequality can be found in (3.9) of [42].

Lemma 2.2.5. *For all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, we have*

$$\frac{d}{dt}\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon x}^2 \le 2\int_\Omega u_\varepsilon^2 v_\varepsilon. \tag{2.2.21}$$

At the end of this section, we recall an important functional inequality from [42], which plays a key role in Lemma 2.3.2 to address ill-signed terms of the form $\int_\Omega u_\varepsilon^\beta v_\varepsilon$ and $\int_\Omega u_\varepsilon^\gamma v_\varepsilon v_{\varepsilon x}^2$ when analysing the functional shown in (2.1.12) along the trajectory of the solution $(u_\varepsilon\ v_\varepsilon)$.

Lemma 2.2.6. *Let $p \ge 1$ and $r > 1$. Then for all $\eta > 0$ there is $C(\eta,p,r) > 0$ such that*

$$\begin{aligned}\|\varphi^{\frac{p+1}{2}}\sqrt{\psi}\|_{L^r(\Omega)}^2 &\le \eta\int_\Omega \varphi^{p-1}\psi\varphi_x^2 + \eta\cdot\left\{\int_\Omega\varphi^p\right\}\cdot\int_\Omega\frac{\varphi}{\psi}\psi_x^2 \\ &\quad + C(\eta,p,r)\cdot\left\{\int_\Omega\varphi\right\}^p\cdot\int_\Omega\varphi\psi\end{aligned} \tag{2.2.22}$$

is valid for arbitrary nonnegative function $\varphi \in C^1(\overline{\Omega})$ and positive function $\psi \in C^1(\overline{\Omega})$.

2.3 Uniform L^p-boundedness of u_ε

In the course of utilizing Lemma 2.2.6 to handle the integral $\int_\Omega u_\varepsilon^2 v_\varepsilon$ arising on the right-hand side in (2.2.21), another unfavorable term $\int_\Omega v_\varepsilon u_{\varepsilon x}^2$ appears, which suggests how to choose appropriate p in (2.1.12) at a first stage. The following is a more concrete outcome based on Lemma 2.2.4.

Lemma 2.3.1. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (2.1.6) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*;*
(ii) $m \ge 2$ *and* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$.
Then there exist positive constants c *and* C *such that for each* $\varepsilon \in (0,1)$*, the function* $\mathcal{F}_\varepsilon$ *defined on* $t \in (0, T_{max,\varepsilon})$ *by letting*

$$\mathcal{F}_\varepsilon(t) := \begin{cases} \int_\Omega u_\varepsilon^{3-m} & \text{when} \quad 1 \le m < 2 \quad \text{or} \quad m > 3, \\ \int_\Omega u_\varepsilon \ln u_\varepsilon & \text{when} \quad m = 2, \\ -\int_\Omega \ln u_\varepsilon & \text{when} \quad m = 3, \\ -\int_\Omega u_\varepsilon^{3-m} & \text{when} \quad 2 < m < 3 \end{cases}$$

satisfies

$$\mathcal{F}_\varepsilon'(t) + c\int_\Omega v_\varepsilon u_{\varepsilon x}^2 \le C\int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + C\int_\Omega u_\varepsilon^2 v_\varepsilon + C\int_\Omega v_\varepsilon v_{\varepsilon x}^2 + C\int_\Omega u_\varepsilon v_\varepsilon \tag{2.3.1}$$

for all $t \in (0, T_{max,\varepsilon})$.

Proof. We prove this assertion in five cases.

Case 1: $1 \le m < 2$**.** Taking $p := 3 - m \in (1, 2]$ in (2.2.15), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^{3-m} + \frac{(2-m)(3-m)}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 \\ \le \quad & \frac{(2-m)(3-m)}{2}\int_\Omega u_\varepsilon^{2-2m} S^2(u_\varepsilon) v_\varepsilon v_{\varepsilon x}^2 + (3-m)\ell\int_\Omega u_\varepsilon^{3-m} v_\varepsilon \\ \le \quad & 2^{2\alpha-2} C_S^2 \int_\Omega u_\varepsilon^{4-2m} v_\varepsilon v_{\varepsilon x}^2 + 2^{2\alpha-2} C_S^2 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon v_{\varepsilon x}^2 \\ & +2\ell\int_\Omega u_\varepsilon^{3-m} v_\varepsilon \qquad \text{for all} \;\; t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \end{aligned} \tag{2.3.2}$$

where we use the fact that

$$S^2(s) \le C_S^2 s^2 (s+1)^{2\alpha-2} \le 2^{2\alpha-2} C_S^2 (s^{2\alpha} + s^2) \qquad \text{for all} \;\; s \ge 0.$$

Since $1 \le m < 2$ implies $0 < 4 - 2m \le 2$, and since $m - 1 \le \alpha \le m$ yields $0 \le 2 - 2m + 2\alpha \le 2$, the first two terms on the right-hand side of (2.3.2) can be estimated by Young's inequality as

$$\begin{aligned} 2^{2\alpha-2} C_S^2 \int_\Omega u_\varepsilon^{4-2m} v_\varepsilon v_{\varepsilon x}^2 + 2^{2\alpha-2} C_S^2 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon v_{\varepsilon x}^2 \\ \le \quad 2^{2\alpha-1} C_S^2 \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + 2^{2\alpha-1} C_S^2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Another application of Young's inequality together with the fact $1 < 3-m \le 2$ shows that the last term in (2.3.2) has the following estimation

$$2\ell \int_\Omega u_\varepsilon^{3-m} v_\varepsilon \le 2\ell \int_\Omega u_\varepsilon^2 v_\varepsilon + 2\ell \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Inserting the above two inequalities into (2.3.2) shows that if we take $c := \frac{(2-m)(3-m)}{2}$ and $C := \max\{2^{2\alpha-1} C_S^2, 2\ell\}$, then (2.3.1) holds.

Case 2: $m = 2$**.** Recalling (2.2.18) and using the fact $\ln u_\varepsilon \le u_\varepsilon$, we can invoke Young's inequality along with the fact that $0 \le 2\alpha - 2 \le 2$ to conclude that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_\Omega v_\varepsilon u_{\varepsilon x}^2 \\
&\quad \le \frac{C_S^2}{2} \int_\Omega u_\varepsilon^{2\alpha-2} v_\varepsilon v_{\varepsilon x}^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon \\
&\quad \le \frac{C_S^2}{2} \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + \frac{C_S^2}{2} \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon + \ell \int_\Omega u_\varepsilon^2 v_\varepsilon,
\end{aligned}$$

which leads to (2.3.1) by letting $c := \frac{1}{2}$ and $C := \max\left\{\frac{C_S^2}{2}, \ell\right\}$.

Case 3: $2 < m < 3$**.** Taking $p := 3-m \in (0,1)$ in (2.2.16) and making use of Young's inequality once more along with the fact $0 \le 2 - 2m + 2\alpha \le 2$ because of $m - 1 \le \alpha \le m$, we get

$$\begin{aligned}
&-\frac{d}{dt} \int_\Omega u_\varepsilon^{3-m} + \frac{(m-2)(3-m)}{2} \int_\Omega v_\varepsilon u_{\varepsilon x}^2 \\
&\quad \le \frac{(m-2)(3-m)}{2} C_S^2 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon v_{\varepsilon x}^2 \\
&\quad \le C_S^2 \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + C_S^2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Thus, (2.3.1) is obtained by taking $c := \frac{(2-m)(3-m)}{2}$ and $C := C_S^2$.

Case 4: $m = 3$**.** It is easy to verify from (2.2.19) that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
-\frac{d}{dt} \int_\Omega \ln u_\varepsilon + \frac{1}{2} \int_\Omega v_\varepsilon u_{\varepsilon x}^2 &\le \frac{C_S^2}{2} \int_\Omega u_\varepsilon^{2\alpha-4} v_\varepsilon v_{\varepsilon x}^2 \\
&\le \frac{C_S^2}{2} \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + \frac{C_S^2}{2} \int_\Omega v_\varepsilon v_{\varepsilon x}^2,
\end{aligned}$$

where the last inequality is provided by Young's inequality and the fact $0 \le 2\alpha - 4 \le 2$ guaranteed by $2 \le \alpha \le 3$. This thus implies (2.3.1) by taking $c := \frac{1}{2}$ and $C := \frac{C_S^2}{2}$.

Case 5: $m > 3$**.** The proof of this case can be derived in a very similar, even simpler, way as in Case 1. We point out that the only difference is "taking $p = 3 - m < 0$ in (2.2.17)" instead of "taking $p = 3 - m \in (1,2]$ in (2.2.15)" when applying Lemma 2.2.4. □

Based on Lemma 2.2.6, we are now able to derive the following conclusion by making use of an energy functional of the form (2.1.12).

Lemma 2.3.2. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (2.1.6) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*.*
(ii) $m \ge 3$*,* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*, and additionally* $u_0 > 0$ *in* $\overline{\Omega}$*.*
Then there exists $C > 0$ *such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon u_{\varepsilon x}^2 \le C \qquad \textit{for all } \varepsilon \in (0,1). \tag{2.3.3}$$

Proof. Let $\mathcal{F}_\varepsilon$ be as in Lemma 2.3.1. Then from (2.2.21) and (2.3.1), there exist $c_1 > 0$ and $c_2 > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \mathcal{F}_\varepsilon'(t) + \frac{d}{dt}\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} + c_1 \int_\Omega v_\varepsilon u_{\varepsilon x}^2 + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 \\ \le \ c_2 \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon^2 v_\varepsilon + c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon v_\varepsilon, \end{aligned} \tag{2.3.4}$$

where the Cauchy-Schwarz inequality and (2.2.13) imply the existence of $c_3 > 0$ fulfilling

$$\begin{aligned} c_2 \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon^2 v_\varepsilon \\ \le \ c_2 \|v_{\varepsilon x}\|_{L^4(\Omega)}^2 \|u_\varepsilon v_\varepsilon^{\frac{1}{2}}\|_{L^4(\Omega)}^2 + c_2 |\Omega|^{\frac{1}{2}} \|u_\varepsilon v_\varepsilon^{\frac{1}{2}}\|_{L^4(\Omega)}^2 \\ \le \ c_3 \|u_\varepsilon v_\varepsilon^{\frac{1}{2}}\|_{L^4(\Omega)}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{2.3.5}$$

From (2.2.5) and (2.1.4), we are able to pick $c_4 > 0$ such that

$$\int_\Omega u_\varepsilon \le c_4 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \tag{2.3.6}$$

and thus we can further apply Lemma 2.2.6 with $p := 1$ and $r := 4$ to find $c_5 > 0$ satisfying

$$\begin{aligned} c_3 \|u_\varepsilon v_\varepsilon^{\frac{1}{2}}\|_{L^4(\Omega)}^2 \ &\le \ \frac{c_1}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 + \frac{1}{2c_4} \cdot \left\{\int_\Omega u_\varepsilon\right\} \cdot \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + c_5 \cdot \left\{\int_\Omega u_\varepsilon\right\} \cdot \int_\Omega u_\varepsilon v_\varepsilon \\ &\le \ \frac{c_1}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + c_4 c_5 \int_\Omega u_\varepsilon v_\varepsilon \end{aligned} \tag{2.3.7}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Substituting (2.3.5) and (2.3.7) into (2.3.4), we have

$$\begin{aligned} \mathcal{F}_\varepsilon'(t) + \frac{d}{dt}\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} + \frac{c_1}{2}\int_\Omega v_\varepsilon u_{\varepsilon x}^2 + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 \\ \le \ c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + (c_2 + c_4 c_5) \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{2.3.8}$$

A combination of (2.1.4) with Lemma 2.2.3 and (2.2.7) entails that there is $c_6 > 0$ satisfying

$$c_2 \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + (c_2 + c_4 c_5) \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \le c_6 \qquad \text{for all } \varepsilon \in (0,1),$$

as a consequence of which, upon an integration of (2.3.8), we have

$$\frac{c_1}{2}\int_0^t \int_\Omega v_\varepsilon u_{\varepsilon x}^2 + \frac{1}{2}\int_0^t \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2$$

$$\begin{aligned}
&\leq c_2\int_0^t\int_\Omega v_\varepsilon v_{\varepsilon x}^2+(c_2+c_4c_5)\int_0^t\int_\Omega u_\varepsilon v_\varepsilon+\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)\\
&\leq c_6+\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)\qquad\text{for all } t\in(0,T_{max,\varepsilon})\text{ and }\varepsilon\in(0,1).
\end{aligned}$$

Thus, to complete the proof, it suffices to show that $\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)$ always has an upper bound. Indeed, (2.1.4) implies the existence of $c_7>0$ such that $\int_\Omega\frac{v_{0x}^2}{v_0}+\|u_0\|_{L^\infty(\Omega)}\leq c_7$. Then, when $1\leq m<2$, since $3-m>0$, it follows that

$$\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)\leq\int_\Omega\frac{v_{0x}^2}{v_0}+\int_\Omega u_{0\varepsilon}^{3-m}\leq c_7+(c_7+1)^{3-m}|\Omega|.$$

When $m=2$, the pointwise estimate $-u_\varepsilon\ln u_\varepsilon\leq\frac{1}{e}$ entails that

$$\begin{aligned}
\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)&=\int_\Omega\frac{v_{0x}^2}{v_0}+\int_\Omega u_{0\varepsilon}\ln u_{0\varepsilon}-\int_\Omega u_\varepsilon\ln u_\varepsilon\\
&\leq c_7+(c_7+1)\ln(c_7+1)|\Omega|+\frac{|\Omega|}{e}.
\end{aligned}$$

When $2<m<3$, since $0<3-m<1$, it follows from (2.3.6) and Young's inequality that

$$\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)\leq\int_\Omega\frac{v_{0x}^2}{v_0}+\int_\Omega u_\varepsilon^{3-m}\leq c_7+\int_\Omega u_\varepsilon+|\Omega|\leq c_7+c_4+|\Omega|.$$

When $m\geq 3$, the definition of $u_{0\varepsilon}$ and the strict positivity of u_0 entail the existence of $c_8>0$ independent of ε fulfilling $u_{0\varepsilon}>c_8$. Thus, when $m=3$, from the fact $\ln u_\varepsilon\leq u_\varepsilon$,

$$\begin{aligned}
\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)&=\int_\Omega\frac{v_{0x}^2}{v_0}-\int_\Omega\ln u_{0\varepsilon}+\int_\Omega\ln u_\varepsilon\\
&\leq c_7-\int_\Omega\ln u_{0\varepsilon}+\int_\Omega u_\varepsilon\leq c_7-\ln c_8|\Omega|+c_4,
\end{aligned}$$

and when $m>3$, since $3-m$ is negative, we have

$$\int_\Omega\frac{v_{0x}^2}{v_0}+\mathcal{F}_\varepsilon(0)-\mathcal{F}_\varepsilon(t)\leq c_7+\int_\Omega u_{0\varepsilon}^{3-m}\leq c_7+c_8^{3-m}|\Omega|.$$

The proof is finished. □

We are now prepared to derive the main result of this section.

Lemma 2.3.3. *Let $p>2$. Suppose that one of the following assumptions holds:*
(i) $1\leq m<2$ and S fulfills (2.1.6) with $\alpha\in[m-1,m]$ and $C_S>0$;
(ii) $2\leq m<3$ and S fulfills (2.1.9) with $\alpha\in[m-1,\frac{m}{2}+1]$ and $C_S>0$;
(iii) $m\geq 3$, S fulfills (2.1.9) with $\alpha\in[m-1,\frac{m}{2}+1]$ and $C_S>0$, and additionally $u_0>0$ in $\overline{\Omega}$. Then for any choice of $\varepsilon\in(0,1)$, there exists $C(p)>0$ such that

$$\int_\Omega u_\varepsilon^p(\cdot,t)\leq C(p)\qquad\textit{for all } t\in(0,T_{max,\varepsilon}).\tag{2.3.9}$$

Proof. According to Lemma 2.2.2, (2.2.5) and (2.1.4), we can fix $c_1 > 0$ such that

$$\int_\Omega v_{\varepsilon x}^4 + \int_\Omega u_\varepsilon \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{2.3.10}$$

Now we focus on the case $1 \le m < 2$. From (2.2.15), we see the existence of $c_2 = c_2(p) > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 \\
&\quad\le \frac{p(p-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p-m+1}(u_\varepsilon+1)^{2\alpha-2} v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad\le c_2 \int_\Omega u_\varepsilon^{p-m-1+2\alpha} v_\varepsilon v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon^{p-m+1} v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon
\end{aligned} \tag{2.3.11}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. It is clear that $0 < p-m+1 \le p \le p+m-1$ for $p > 2$ and $m \ge 1$, and that $0 < p-m-1+2\alpha \le p+m-1$ holds whenever $m-1 \le \alpha \le m$. These allow us to use Young's inequality and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&c_2 \int_\Omega u_\varepsilon^{p-m-1+2\alpha} v_\varepsilon v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon^{p-m+1} v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad\le 2c_2 \int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon v_{\varepsilon x}^2 + 2c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon + p\ell \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le \left(2c_2 \cdot \left\{\int_\Omega v_{\varepsilon x}^4\right\}^{\frac12} + p\ell|\Omega|^{\frac12}\right) \cdot \left\{\int_\Omega u_\varepsilon^{2(p+m-1)} v_\varepsilon^2\right\}^{\frac12} + 2c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le \left(2c_2 c_1^{\frac12} + p\ell|\Omega|^{\frac12}\right) \left\| u_\varepsilon^{\frac{p+m-1}{2}} v_\varepsilon^{\frac12} \right\|_{L^4(\Omega)}^2 + 2c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, where Lemma 2.2.6 is available to proceed to entail the existence of $c_3 = c_3(p) > 0$ such that

$$\begin{aligned}
&\left(2c_2 c_1^{\frac12} + p\ell|\Omega|^{\frac12}\right) \left\| u_\varepsilon^{\frac{p+m-1}{2}} v_\varepsilon^{\frac12} \right\|_{L^4(\Omega)}^2 \\
&\quad\le \frac{p(p-1)}{2} \int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 + \left\{\int_\Omega u_\varepsilon^{p+m-2}\right\} \cdot \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 \\
&\qquad + c_3 \cdot \left\{\int_\Omega u_\varepsilon\right\}^{p+m-2} \cdot \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

which together with (2.3.11) implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p \le \left\{\int_\Omega u_\varepsilon^{p+m-2}\right\} \cdot \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + 2c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + \left(c_3 c_1^{p+m-2} + p\ell\right) \int_\Omega u_\varepsilon v_\varepsilon.$$

Here since $0 < p+m-2 < p$ holds automatically, we can see from Young's inequality that

$$\int_\Omega u_\varepsilon^{p+m-2} \le \int_\Omega u_\varepsilon^p + |\Omega| \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

That is, with some $c_4 = c_4(p) > 0$ we hence have

$$\frac{d}{dt}\left\{\int_\Omega u_\varepsilon^p + 1\right\} \le c_4 \left\{\int_\Omega u_\varepsilon^p + 1\right\} \cdot \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + 2c_2 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + c_4 \int_\Omega u_\varepsilon v_\varepsilon \tag{2.3.12}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. If we write

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon^p(\cdot,t) + 1, \quad t \in [0, T_{max,\varepsilon})$$

and

$$g_\varepsilon(t) := c_4 \int_\Omega \frac{u_\varepsilon(\cdot,t)}{v_\varepsilon(\cdot,t)} v_{\varepsilon x}^2(\cdot,t), \quad t \in (0, T_{max,\varepsilon})$$

as well as

$$h_\varepsilon(t) := 2c_2 \int_\Omega v_\varepsilon(\cdot,t) v_{\varepsilon x}^2(\cdot,t) + c_4 \int_\Omega u_\varepsilon(\cdot,t) v_\varepsilon(\cdot,t), \quad t \in (0, T_{max,\varepsilon}),$$

then we get

$$y_\varepsilon'(t) \le g_\varepsilon(t) y_\varepsilon(t) + h_\varepsilon(t) \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

with the property that there is $c_5 = c_5(p) > 0$ satisfying

$$\int_0^t g_\varepsilon(s)ds \le c_5 \quad \text{and} \quad \int_0^t h_\varepsilon(s)ds \le c_5 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

provided by Lemma 2.2.3, Lemma 2.3.2, (2.2.7) and (2.1.4). An ODE comparison argument together with the fact that $y_\varepsilon(0) = \int_\Omega (u_0 + \varepsilon)^p \le c_6$ with some $c_6 = c_6(p) > 0$ yields (2.3.9).

For the case $m \ge 2$, invoking $S(u_\varepsilon) \le C_S u_\varepsilon^\alpha$ into (2.2.15), we see from the Cauchy-Schwarz inequality and Young's inequality along with the fact that $0 < p - m - 1 + 2\alpha \le p + 1$ ensured by $m - 1 \le \alpha \le \frac{m}{2} + 1$ that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon u_{\varepsilon x}^2 \\
&\quad\le \frac{p(p-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p-m-1+2\alpha} v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad\le c_7 \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + c_7 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + p\ell \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le \left(c_7 \cdot \left\{ \int_\Omega v_{\varepsilon x}^4 \right\}^{\frac{1}{2}} + p\ell |\Omega|^{\frac{1}{2}} \right) \cdot \left\{ \int_\Omega u_\varepsilon^{2(p+1)} v_\varepsilon^2 \right\}^{\frac{1}{2}} + c_7 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le \left(c_7 c_1^{\frac{1}{2}} + p\ell |\Omega|^{\frac{1}{2}} \right) \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^4(\Omega)}^2 + c_7 \int_\Omega v_\varepsilon v_{\varepsilon x}^2 + p\ell \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{2.3.13}$$

with $c_7 := \frac{p(p-1)}{2} C_S^2$ for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Since $0 < p - 1 \le p + m - 3$ when $m \ge 2$, applying Lemma 2.2.6 once more, we infer from Young's inequality that with some $c_8 = c_8(p) > 0$ we have

$$\begin{aligned}
&\left(c_7 c_1^{\frac{1}{2}} + p\ell |\Omega|^{\frac{1}{2}} \right) \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^4(\Omega)}^2 \\
&\quad\le \frac{p(p-1)}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \left\{ \int_\Omega u_\varepsilon^p \right\} \cdot \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + c_8 \cdot \left\{ \int_\Omega u_\varepsilon \right\}^p \cdot \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

$$\leq \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon u_{\varepsilon x}^2 + \left\{\int_\Omega u_\varepsilon^p\right\}\cdot\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon x}^2 + \int_\Omega v_\varepsilon u_{\varepsilon x}^2 + c_8 c_1^p \int_\Omega u_\varepsilon v_\varepsilon$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which in conjunction with (2.3.13) implies that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p \leq \left\{\int_\Omega u_\varepsilon^p\right\}\cdot\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon x}^2 + \int_\Omega v_\varepsilon u_{\varepsilon x}^2 + c_7\int_\Omega v_\varepsilon v_{\varepsilon x}^2 + (c_8 c_1^p + p\ell)\int_\Omega u_\varepsilon v_\varepsilon$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Then trivially applying the procedures after (2.3.12) thereby proves (2.3.9) for the case $m \geq 2$. □

2.4 Uniform L^∞-boundedness of u_ε

Now, by making use of Lemma 2.3.3, we can derive the $W^{1,\infty}(\Omega)$-boundedness of v_ε from the standard well-known smoothing properties of Neumann heat semigroup.

Lemma 2.4.1. *Suppose that one of the following assumptions holds:*
(i) $1 \leq m < 2$ *and* S *fulfills* (2.1.6) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*;*
(ii) $2 \leq m < 3$ *and* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*;*
(iii) $m \geq 3$, S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*, and additionally* $u_0 > 0$ *in* $\overline{\Omega}$*. Then there exists* $C > 0$ *satisfying*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{2.4.1}$$

Proof. According to Lemma 2.3.3, there is $c_1 > 0$ satisfying

$$\|u_\varepsilon(\cdot,t)\|_{L^3(\Omega)} \leq c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Employing known smoothing estimates for the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ ([73]), we can fix $\lambda_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}
\|v_{\varepsilon x}(\cdot,t)\|_{L^\infty(\Omega)} &= \left\|\partial_x e^{t\Delta}v_0 - \int_0^t \partial_x e^{(t-s)\Delta}u_\varepsilon v_\varepsilon ds\right\|_{L^\infty(\Omega)} \\
&\leq c_2\|v_0\|_{W^{1,\infty}(\Omega)} \\
&\quad + c_2\|v_0\|_{L^\infty(\Omega)}\int_0^t \left(1+(t-s)^{-\frac{2}{3}}\right)e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^3(\Omega)}\,ds \\
&\leq c_2\|v_0\|_{W^{1,\infty}(\Omega)} + c_1c_2\|v_0\|_{L^\infty(\Omega)}\int_0^t \left(1+(t-s)^{-\frac{2}{3}}\right)e^{-\lambda_1(t-s)}ds,
\end{aligned}$$

which together with (2.1.4) and (2.2.6) implies (2.4.1). □

Having Lemmas 2.3.3 and 2.4.1 at hand, we can conclude by using a method very similar to [84, Lemma 4.1] that the approximate solutions obtained in Lemma 2.2.1 are global.

Lemma 2.4.2. *Suppose that one of the following assumptions holds:*
(i) $1 \leq m < 2$ *and* S *fulfills* (2.1.6) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*;*
(ii) $2 \leq m < 3$ *and* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*;*
(iii) $m \geq 3$, S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*, and additionally* $u_0 > 0$ *in* $\overline{\Omega}$*. Then for all* $\varepsilon \in (0,1)$*,*

$$T_{max,\varepsilon} = \infty.$$

In the following, we show that in light of the elliptic Harnack-type inequality documented in [81] and [92], the L^∞-boundedness of u_ε can be derived by adapting an analogous strategy in [42].

Lemma 2.4.3. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (2.1.6) *with* $\alpha \in [m-1, m]$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*;*
(iii) $m \ge 3$*,* S *fulfills* (2.1.9) *with* $\alpha \in [m-1, \frac{m}{2}+1]$ *and* $C_S > 0$*, and additionally* $u_0 > 0$ *in* $\overline{\Omega}$*. Then there exists* $C > 0$ *such that for all* $\varepsilon \in (0,1)$*,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0. \tag{2.4.2}$$

Proof. This result can be proved in much the same way as [42, Lemma 5.3, Lemma 5.4]. We therefore only sketch the proof. First, thanks to the Harnack-type inequality ([81], [92])

$$v_\varepsilon(x,t) \ge c_1 \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } \ x \in \Omega, \ \ t > 0 \ \text{ and } \ \varepsilon \in (0,1) \tag{2.4.3}$$

with $c_1 > 0$, it is easy to verify from (2.2.5) and (2.2.7) that

$$L_\varepsilon := \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt, \qquad \varepsilon \in (0,1)$$

is well-defined. Setting

$$\tau := \phi_\varepsilon(t) := \frac{1}{L_\varepsilon} \int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds, \qquad t \ge 0$$

and

$$w_\varepsilon(x,\tau) := u_\varepsilon(x, \phi_\varepsilon^{-1}(\tau)), \qquad x \in \overline{\Omega}, \ \tau \in [0,1),$$

we infer from (2.2.1) that for each $\varepsilon \in (0,1)$,

$$\begin{cases} w_{\varepsilon\tau} = \big(a_\varepsilon(x,\tau) w_\varepsilon^{m-1} w_{\varepsilon x}\big)_x - \big(b_\varepsilon(x,\tau) S(w_\varepsilon)\big)_x + \ell a_\varepsilon(x,\tau) w_\varepsilon, & x \in \Omega, \ \tau \in (0,1), \\ w_{\varepsilon x} = 0, & x \in \partial\Omega, \ \tau \in (0,1), \\ w_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega \end{cases}$$

holds with

$$a_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}} \qquad \text{and} \qquad b_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x,t) v_{\varepsilon x}(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}}.$$

Additionally, there exists $c_2 > 0$ such that

$$\frac{1}{c_2} \le a_\varepsilon(x,\tau) \le c_2 \text{ and } |b_\varepsilon(x,\tau)| \le c_2 \qquad \text{for all } \ (x,\tau) \in \Omega \times (0,1) \text{ and } \varepsilon \in (0,1),$$

which together with Lemma 2.3.3 yields that for any $p > 1$ there is $c_3 = c_3(p) > 0$ satisfying

$$\|w_\varepsilon(\cdot,\tau)\|_{L^p(\Omega)} + \|b_\varepsilon(\cdot,\tau)\|_{L^p(\Omega)} \le c_3 \qquad \text{for all } \ \tau \in (0,1) \text{ and } \varepsilon \in (0,1).$$

From this, the Moser-type result ([65, Lemma A.1]) is available to assert that with some $c_4 > 0$ we have

$$\|w_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \le c_4 \qquad \text{for all } \ \tau \in (0,1) \text{ and } \varepsilon \in (0,1).$$

Rescaling back to u_ε, we finish the proof. □

2.5 Proofs of Theorem 2.1.1 and Corollary 2.1.2

Now we are able to give the proofs of our main results.

Proof of Theorem 2.1.1. When $1 \leq m < 3$, based on the L^∞-boundedness of w_ε and the upper bounds for a_ε, we can invoke the standard Hölder theory ([62]) to derive the uniform Hölder bounds for w_ε (thus for u_ε) on finite time intervals. Then the Schauder regularity for v_ε can be obtained by the classical parabolic Schauder estimate ([33]). Thereafter, in light of Lemmas 2.4.1, 2.3.3 and 2.4.3, by applying limiting arguments quite similar to those used in [81, Lemma 8.1] (see also [42, Lemmas 4.3 and 5.7]), we can assert that there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$ satisfying

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases}$$

such that (u,v) indeed forms a global weak solution of (2.1.3) in the sense of Definition 1.4.1 and enjoys the properties announced in (2.1.8).

When $m \geq 3$, we recall how the regularized form behaves in (2.2.1) and (2.2.2) and find that $(u_\varepsilon, v_\varepsilon)$ in this case (actually independent of ε) is a classical solution of (2.1.3). The property in (2.1.10) is a consequence of (2.2.3). □

Proof of Corollary 2.1.2. With the aid of the Harnack-type inequality valid in one-dimensional setting ([81], [92]), the large time behavior can be obtained by arguing similarly to that in [81, Lemmas 10.1-10.5]. □

3 Enhanced migration mechanisms in two-dimensional models

3.1 Introduction

In parabolic models proposed to describe biological phenomena in living systems, the existence of solutions is a fundamental prerequisite for their meaningful use. Once existence is ensured, the study of the large time behavior of solutions can serve as one of the criteria for assessing whether or not the model is capable of accurately capturing the emergence and evolution of structures in certain biological contexts.

However, for the doubly degenerate system arising as a model for capturing structure-supporting features of *Bacillus subtilis* in nutrient-limited environments

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (S(u)v\nabla v) + \ell uv, \\ v_t = \Delta v - uv, \end{cases} \tag{3.1.1}$$

rigorous studies concerning the asymptotic behavior are currently available only for specific one-dimensional versions ([81], Corollary 2.1.2 in Chapter 2). In the physically most relevant two-dimensional setting, only numerical simulations have been performed in [39], providing evidence that the inclusion of a chemotactic flux term (i.e. $S(u) \not\equiv 0$) is potentially more suitable for capturing the formation of complex patterning structures.

Regarding global solvability in two spatial dimensions, weak solutions to (3.1.1) in the case $S(u) = u^2$ were constructed in convex domains under a suitable smallness assumption on v_0 ([84]), whereas, without imposing any smallness restrictions on the initial data, weak solutions are only known to exist for sensitivities of the form $S(u) = \chi u^\alpha$ with $1 < \alpha < \frac{3}{2}$. It appears that there is a gap between these two results. Moreover, compared to the one-dimensional case, the boundedness of the first component obtained in the above two precedents is weak, in the sense that uniform-in-time L^∞ bounds for u can be established in one dimension, whereas the existing two-dimensional results are limited to L^p-boundedness for finite $p > 1$.

In this chapter, we continue our investigation of the global existence and asymptotic behavior of solutions to the following initial-boundary value problem in two-dimensional settings

$$\begin{cases} u_t = \nabla \cdot (u^{m-1}v\nabla u) - \nabla \cdot (S(u)v\nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (u^{m-1}v\nabla u - S(u)v\nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1.2}$$

with $m \geq 1$ and $\ell \geq 0$, under the overall hypotheses that the chemotactic sensitivity S, as in Chapter 2, satisfying

$$S \in C^1([0,\infty)) \quad \text{is nonnegative} \tag{3.1.3}$$

is such that either

$$S(s) \leq C_S s(s+1)^{\alpha-1} \qquad \text{for all} \quad s \geq 0 \tag{3.1.4}$$

or

$$S(s) \leq C_S s^{\alpha} \qquad \text{for all} \quad s \geq 0 \tag{3.1.5}$$

with $\alpha \geq 0$ and $C_S > 0$.

We finally arrive at the following results. For the global existence and boundedness, we have

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\ell \geq 0$. Suppose that the initial data (u_0, v_0) satisfy*

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{is nonnegative with } u_0 \not\equiv 0, \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{is positive in } \overline{\Omega}. \end{cases} \tag{3.1.6}$$

Assume that one of the following conditions holds:
(i) $1 \leq m < 2$, S fulfills (3.1.3) and (3.1.4) with $m-1 < \alpha < m$ and $C_S > 0$;
(ii) $2 \leq m < 3$, S fulfills (3.1.3) and (3.1.5) with $m-1 < \alpha < \frac{m}{2}+1$ and $C_S > 0$.
Then the system (3.1.2) admits a global weak solution (u,v) in the sense of Definition 1.4.1 satisfying

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)). \end{cases} \tag{3.1.7}$$

Moreover, if (iii) $m \geq 3$, S fulfills (3.1.3) and (3.1.5) with $m-1 < \alpha < \frac{m}{2}+1$ and $C_S > 0$, and additionally $u_0 > 0$ in $\overline{\Omega}$, then there exists a global classical solution (u,v) of (3.1.2) satisfying

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \qquad \text{and} \\ v \in \cap_{q>1} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)). \end{cases} \tag{3.1.8}$$

In addition, all solutions enjoy the properties that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$ and

$$\sup_{t>0} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{3.1.9}$$

Remark 3.1.2. When $m = 2$, the upper bound on α obtained in this paper seems natural and reasonable, in view of the fact that if $\alpha = 2$, solvability requires imposing smallness conditions on the initial data, as indicated in [84].

Remark 3.1.3. In a manner rather similar to proving Theorem 3.1.1, the corresponding results in higher-dimensional space are able to be derived: at least in the context of three dimensions with the case $m = 2$, we can check that the global solvability holds within the range $1 < \alpha < \frac{3}{2}$, which together with Theorem 3.1.1 extends the previous results in [40].

The L^∞-boundedness of u obtained in Theorem 3.1.1 essentially provides the possibility for the following result on asymptotic behavior.

Theorem 3.1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, $m \geq 1$, and $\ell \geq 0$. Suppose that S and the initial data (u_0, v_0) satisfy one of the hypotheses in Theorem 3.1.1 (i), (ii) or (iii). Then there exists $u_\infty \in C^0(\overline{\Omega})$ such that the solution of* (3.1.2) *found in Theorem 3.1.1 has the property that*

$$u(\cdot,t) \to u_\infty \quad \text{and} \quad v(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{3.1.10}$$

Here $u_\infty = w(\cdot,1)$ with $w \in C^0(\overline{\Omega} \times [0,1])$ being a weak solution of

$$\begin{cases} w_\tau = \nabla\cdot(a(x,\tau)w^{m-1}\nabla w) - \nabla\cdot(b(x,\tau)S(w)) + \ell a(x,\tau)w, & x \in \Omega,\ \tau \in (0,1), \\ \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ \tau \in (0,1) \\ w(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

in the sense that

$$\begin{cases} w^m \in L^1_{loc}([0,1); W^{1,1}(\Omega)) \qquad \text{and} \\ S(w) \in L^1_{loc}(\bar{\Omega} \times [0,1)), \end{cases}$$

and for all $\varphi \in C_0^\infty\left(\overline{\Omega} \times [0,1)\right)$,

$$\begin{aligned} -\int_0^1\int_\Omega w\varphi_\tau - \int_\Omega w_0\varphi(\cdot,0) \quad = \quad & -\frac{1}{m}\int_0^1\int_\Omega a(x,\tau)\nabla w^m \cdot \nabla\varphi \\ & + \int_0^1\int_\Omega b(x,\tau)S(w)\cdot\nabla\varphi + \ell\int_0^1\int_\Omega a(x,\tau)w\varphi, \end{aligned} \tag{3.1.11}$$

where for $(x,\tau) \in \Omega \times (0,1)$ and $t = \phi^{-1}(\tau)$,

$$a(x,\tau) := L \cdot \frac{v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \quad \text{and} \quad b(x,\tau) := L \cdot \frac{v(x,t)\nabla v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}},$$

with

$$L := \int_0^\infty \|v(\cdot,s)\|_{L^\infty(\Omega)}ds \quad \text{and} \quad \phi(t) := \frac{1}{L}\cdot\int_0^t \|v(\cdot,s)\|_{L^\infty(\Omega)}ds, \quad t \geq 0$$

are such that there exists $C > 0$ satisfying

$$\frac{1}{C} \leq a(x,\tau) \leq C \quad \text{and} \quad |b(x,\tau)| \leq C \qquad \text{for all } (x,\tau) \in \Omega \times (0,1). \tag{3.1.12}$$

Main ideas. It is clear from Lemma 3.3.9 that if an L^{p_0} bound for u is available for some $p_0 > 1$, then one can employ an energy-like functional of the form

$$\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^q}{v^{q-1}} \tag{3.1.13}$$

with suitably chosen

$$q \in \left(\frac{2(p+m-1)}{p_0},\ 2(p_0+p+m-2)\right) =: I$$

as a direct and effective approach to derive L^p bounds for u with arbitrary $p > 2$. This strategy, however, is not applicable if one merely relies on the basic L^1-boundedness of u, since the interval I becomes empty when $p_0 = 1$.

Therefore, the main difficulty lies in obtaining an L^{p_0} bound for u with some $p_0 > 1$. To achieve this, we commence by considering the following quasi-energy functional

$$c\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^4}{v^3} + \int_\Omega u^{p_0}, \tag{3.1.14}$$

where $c > 0$, and p is chosen as "$3-m$" in a rough sense. Some basic inequalities for the first two summands in (3.1.14) are established in Lemma 3.3.4, while a key role in addressing the unfavorable expressions therein is played by the crucial observation that

$$\int_\Omega u^\beta v \le \eta \int_\Omega u^\kappa v |\nabla u|^2 + \eta \int_\Omega \frac{|\nabla v|^2}{v} |D^2 \ln v|^2 + C_3(\beta, \eta) \int_\Omega uv \tag{3.1.15}$$

holds for $\beta \in [1, \kappa + 3)$ with $\kappa \in (-1, 0)$ (Lemma 3.3.2).

The second integral on the right-hand side of (3.1.15) can be dissipated through the evolution of $\int_\Omega \frac{|\nabla v|^4}{v^3}$, as shown in Lemma 3.2.4, while the first integral on the right-hand side will be absorbed by the diffusion-induced quantity $\int_\Omega u^{p^*+m-3} v |\nabla u|^2$ arising from testing the first equation in (3.1.2) with u^{p^*} for appropriate $p^* := 3 - m + \kappa \in (2 - m, 3 - m)$. To this end, combining the functional in (3.1.14) with the contribution $\int_\Omega u^{p^*}$, that is, considering the coupled functional

$$c\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^4}{v^3} + \int_\Omega u^{p_0} + \int_\Omega u^{p^*}, \tag{3.1.16}$$

we can complete the derivation of an L^{p_0} bound for u (Lemmas 3.3.5-3.3.7).

Finally, by means of a Moser iteration argument based on a functional inequality shown in Lemma 3.4.1, we obtain the L^∞-boundedness of u. This allows us to derive higher regularity results for the components (Lemma 3.5.3), which are instrumental in proving global existence, and also facilitate the establishment of a Harnack-type inequality (Lemma 3.6.1) to describe the asymptotic behavior.

3.2 Preliminaries

3.2.1 Regularized problems and basic information

As in the approximating procedure used in Chapter 2 (see also [42]), we consider the regularized variant of (3.1.2) given by

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot (u_\varepsilon^{m-1} v_\varepsilon \nabla u_\varepsilon) - \nabla \cdot (S(u_\varepsilon) v_\varepsilon \nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad v(x, 0) = v_{0\varepsilon}(x) := v_0(x), & x \in \Omega \end{cases} \tag{3.2.1}$$

with $\varepsilon \in (0,1)$, where $u_{0\varepsilon}(x)$, depending on m, is defined by

$$u_{0\varepsilon}(x) := \begin{cases} u_0(x) + \varepsilon, & 1 \leq m < 3, \\ u_0(x) & m \geq 3. \end{cases} \tag{3.2.2}$$

Then the following result concerning the local existence, extensibility criterion, and some basic boundedness information follows directly from Lemma 2.2.1 in Chapter 2.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, $m \geq 1$, and $\ell \geq 0$. Suppose that besides fulfilling* (3.1.3) *and* (3.1.6)*, the function S and the initial data (u_0, v_0) comply with one of the following hypotheses:*
(i) $1 \leq m < 2$ and S fulfills (3.1.4) *with $\alpha \in (m-1, m)$ and $C_S > 0$;*
(ii) $2 \leq m < 3$ and S fulfills (3.1.5) *with $\alpha \in (m-1, m)$ and $C_S > 0$;*
(iii) $m \geq 3$, S fulfills (3.1.5) *with $\alpha \in (m-1, m)$ and $C_S > 0$, and $u_0 > 0$ in $\overline{\Omega}$.*
Then for each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0, \infty]$ and at least one pair $(u_\varepsilon, v_\varepsilon)$ of functions

$$\begin{cases} u_\varepsilon \in C^0\left(\overline{\Omega} \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \\ v_\varepsilon \in \cap_{q>1} C^0\left([0, T_{max,\varepsilon}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \end{cases} \tag{3.2.3}$$

such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{max,\varepsilon})$, that $(u_\varepsilon, v_\varepsilon)$ solves (3.2.1) *in the classical sense in $\Omega \times (0, T_{max,\varepsilon})$, and that*

$$if\ T_{max,\varepsilon} < \infty, \quad then \quad \limsup_{t \to T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{3.2.4}$$

In addition, this solution satisfies

$$\int_\Omega u_{0\varepsilon} \leq \int_\Omega u_\varepsilon(\cdot,t) \leq \int_\Omega u_{0\varepsilon} + \ell \int_\Omega v_{0\varepsilon} \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{3.2.5}$$

as well as for all $t_0 \in [0, T_{max,\varepsilon})$,

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{3.2.6}$$

and

$$\int_{t_0}^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_\varepsilon(\cdot,t_0) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.2.7}$$

Unless otherwise stated, we assume throughout that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary, that the initial data (u_0, v_0) fulfill (3.1.6), and that S satisfies (3.1.3). From now on, we fix $\ell \geq 0$, and let $(u_\varepsilon, v_\varepsilon)$ and $T_{max,\varepsilon}$ be as given by Lemma 3.2.1.

We next show a boundedness estimate for a space-time integral involving the second component v_ε and its gradient.

Lemma 3.2.2. *For any $\varepsilon \in (0,1)$, with some $C > 0$ we have*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \leq C. \tag{3.2.8}$$

Proof. This follows by arguments analogous to those in Lemma 2.2.3 of Chapter 2. □

3.2.2 Basic testing procedures

Now we present two crucial inequalities from [82, Lemma 3.4], which will be used frequently in the sequel.

Lemma 3.2.3. *Let $N \geq 2$ and $q \geq 2$. Then every $\varphi \in C^2(\overline{\Omega})$ fulfilling $\varphi > 0$ in Ω and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ satisfies*

$$\int_\Omega \frac{|\nabla \varphi|^{q+2}}{\varphi^{q+1}} \leq (q+\sqrt{N})^2 \int_\Omega \frac{|\nabla \varphi|^{q-2}}{\varphi^{q-3}} |D^2 \ln \varphi|^2 \tag{3.2.9}$$

and

$$\int_\Omega \frac{|\nabla \varphi|^{q-2}}{\varphi^{q-1}} |D^2 \varphi|^2 \leq (q+\sqrt{N}+1)^2 \int_\Omega \frac{|\nabla \varphi|^{q-2}}{\varphi^{q-3}} |D^2 \ln \varphi|^2. \tag{3.2.10}$$

The next lemma concerns functionals of $\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}$ for $q \geq 2$, which were first introduced in [82, Lemma 3.3] for general domains with boundary integral $\int_{\partial\Omega} \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}} \cdot \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu}$ arising. It is not possible to handle this boundary integral term as in [82, Lemma 3.5] because the term $\int_0^\infty \int_\Omega v_\varepsilon$ is not intended to appear later in our case. As previously mentioned in [40], the convexity of Ω in two-dimensional settings is imperative to allow us to disregard the boundary integral.

Lemma 3.2.4. *For all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, there exists $C_1 > 0$ such that*

$$\frac{d}{dt} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq C_1 \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2, \tag{3.2.11}$$

and for $q \geq 2$, there exists $C_2(q) > 0$ such that

$$\frac{d}{dt} \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + \frac{q}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} |D^2 \ln v_\varepsilon|^2 + (q-1)^2 \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} \leq C_2(q) \int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon. \tag{3.2.12}$$

Proof. From Lemma 3.3 in [82], we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} &+ q \int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} |D^2 \ln v_\varepsilon|^2 + (q-1)^2 \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} \\ &\leq q(q-2+\sqrt{2}) \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-2}} |D^2 v_\varepsilon| + \frac{q}{2} \int_{\partial\Omega} \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}} \cdot \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} \end{aligned} \tag{3.2.13}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Because of $\frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} \leq 0$ on $\partial\Omega$ by convexity of Ω ([47]), we can disregard the last term in (3.2.13). For the penultimate term in (3.2.13), Young's inequality, (3.2.9) and (3.2.10) enable us to infer that there exist $c_1 = c_1(q) > 0$ and $c_2 = c_2(q) > 0$ such that

$$q(q-2+\sqrt{2}) \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-2}} |D^2 v_\varepsilon|$$

$$
\begin{aligned}
&\leq \frac{q}{4(q+\sqrt{2}+1)^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}}|D^2 v_\varepsilon|^2 + c_1\int_\Omega u_\varepsilon^2\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}\\
&\leq \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + c_1\int_\Omega u_\varepsilon^2\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}\\
&\leq \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + \frac{q}{4(q+\sqrt{2})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + c_2\int_\Omega u_\varepsilon^{\frac{q+2}{2}}v_\varepsilon\\
&\leq \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + c_2\int_\Omega u_\varepsilon^{\frac{q+2}{2}}v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),
\end{aligned}
$$

which combined (3.2.13) gives (3.2.12). According to (2.12) in [40], one has

$$
\frac{d}{dt}\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 4\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq -4\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}(\nabla u_\varepsilon\cdot\nabla v_\varepsilon) \tag{3.2.14}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Again using Young's inequality and (3.2.9) with $q:=4$, we see the existence of $c_3>0$ such that

$$
\begin{aligned}
-4\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}(\nabla u_\varepsilon\cdot\nabla v_\varepsilon) &\leq \frac{3}{(4+\sqrt{2})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_3\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\
&\leq 3\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + c_3\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2
\end{aligned}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Inserting this into (3.2.14) yields (3.2.11). □

We close this section by presenting some basic properties regarding the first component u_ε, which is quite similar to Lemma 2.2.4 in Chapter 2.

Lemma 3.2.5. *For all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, the following estimations hold:*

(i) If $p>1$, then we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2\\
&\qquad\leq \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-m-1}S^2(u_\varepsilon)v_\varepsilon|\nabla v_\varepsilon|^2 + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon;
\end{aligned} \tag{3.2.15}
$$

(ii) if $0<p<1$, then we have

$$
\begin{aligned}
&-\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2\\
&\qquad\leq \frac{p(1-p)}{2}\int_\Omega u_\varepsilon^{p-m-1}S^2(u_\varepsilon)v_\varepsilon|\nabla v_\varepsilon|^2;
\end{aligned} \tag{3.2.16}
$$

(iii) if $p<0$, then we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2\\
&\qquad\leq \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-m-1}S^2(u_\varepsilon)v_\varepsilon|\nabla v_\varepsilon|^2;
\end{aligned} \tag{3.2.17}
$$

(iv) if $m = 2$, then we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \le \frac{1}{2}\int_\Omega \frac{S^2(u_\varepsilon)}{u_\varepsilon^2} v_\varepsilon |\nabla v_\varepsilon|^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon; \tag{3.2.18}$$

(v) if $m = 3$, then we have

$$-\frac{d}{dt}\int_\Omega \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \le \frac{1}{2}\int_\Omega \frac{S^2(u_\varepsilon)}{u_\varepsilon^4} v_\varepsilon |\nabla v_\varepsilon|^2. \tag{3.2.19}$$

Proof. These inequalities constitute the two-dimensional analogue of Lemma 2.2.4, and therefore the details are omitted. □

3.3 Uniform L^p-boundedness of u_ε

3.3.1 Bounds for u_ε in L^{p_0} with some $p_0 > 1$

As shown in Chapter 2, the differential inequality (2.2.22) plays a critical role in exploiting the degenerate diffusive structure of the first equation in (2.2.1). However, this approach fails in two-dimensional settings, since the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ is only continuous but not compact. To ensure that a term of the form $\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2$ involves an arbitrarily small coefficient, we adopt the following functional inequality ([40, Lemma 3.1]).

Lemma 3.3.1. *Let $p > 0$ and $r \ge 2$. Then for all $\eta > 0$ there exists $C(\eta, p, r) > 0$ such that*

$$\left\| \varphi^{\frac{p+1}{2}} \sqrt{\psi} \right\|_{L^r(\Omega)}^2 \le \eta \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta\int_\Omega \varphi^{p+1}\frac{|\nabla\psi|^2}{\psi} + C(\eta,p,r)\cdot\left\{\int_\Omega \varphi\right\}^p \cdot \int_\Omega \varphi\psi \tag{3.3.1}$$

is valid for arbitrary nonnegative function $\varphi \in C^1(\overline{\Omega})$ and positive function $\psi \in C^1(\overline{\Omega})$.

The key to our analysis in what follows is to control the unfavorable terms $\int_\Omega u_\varepsilon^\beta v_\varepsilon$ and $\int_\Omega u_\varepsilon^\gamma v_\varepsilon |\nabla v_\varepsilon|^2$. Fortunately, it follows from Lemma 3.3.1 that this objective can be achieved provided that the exponents β and γ fall within appropriate intermediate ranges, respectively. Additionally, this consideration provides insight into why the exponent $p*$ in (3.1.16) is taken arbitrarily close to $3 - m$, but not equal to the critical value.

Lemma 3.3.2. *If $\kappa \in (-1, 0)$, then for any $\beta \in [1, \kappa+3)$ and $\eta > 0$, there exists $C(\beta,\eta) > 0$ such that*

$$\int_\Omega u_\varepsilon^\beta v_\varepsilon \le \eta \int_\Omega u_\varepsilon^\kappa v_\varepsilon |\nabla u_\varepsilon|^2 + \eta\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + C(\beta,\eta)\int_\Omega u_\varepsilon v_\varepsilon \tag{3.3.2}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. The proof will be divided into two cases according to the range of β. We begin by establishing the validity of (3.3.2) for $\beta \in [\kappa+2, \kappa+3)$. From (3.2.5) and (3.1.6), it follows that there exists $c_1 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le c_1 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{3.3.3}$$

Taking

$$\vartheta := \frac{1}{3+\kappa-\beta} \quad \text{and} \quad \vartheta^* := \frac{\vartheta}{\vartheta-1} = \frac{1}{\beta-\kappa-2},$$

we conclude from $\beta \in [\kappa+2, \kappa+3)$ that

$$\vartheta \ge 1 \quad \text{and} \quad \vartheta^* > 1.$$

Therefore, for given $\eta > 0$, the Hölder inequality together with Lemma 3.3.1 and (3.3.3) ensures the existence of $c_2 = c_2(\beta,\eta) > 0$ satisfying

$$\begin{aligned}
\int_\Omega u_\varepsilon^\beta v_\varepsilon &= \int_\Omega \left(u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac12}\right)^2 u_\varepsilon^{\beta-\kappa-2} \\
&\le \left\|u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac12}\right\|^2_{L^{2\vartheta}(\Omega)} \cdot \left\|u_\varepsilon{}^{\beta-\kappa-2}\right\|_{L^{\vartheta^*}(\Omega)} \\
&\le (c_1^{\frac{1}{\vartheta^*}} + 1) \left\|u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac12}\right\|^2_{L^{2\vartheta}(\Omega)} \\
&\le \frac{\eta}{2}\int_\Omega u_\varepsilon^\kappa v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{\kappa+2}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + c_2\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.3.4}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. In treating the second term on the right-hand side of (3.3.4), we first apply Lemma 3.2.3 with $q := 4$ to see the existence of $c_3 > 0$ such that

$$\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le c_3 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Then let

$$\chi := -\frac{2}{\kappa} \quad \text{and} \quad \chi^* := \frac{\chi}{\chi-1} = \frac{2}{\kappa+2},$$

where our assumption $\kappa \in (-1,0)$ entails that

$$\chi > 1 \quad \text{and} \quad \chi^* > 1,$$

whence by means of Young's inequality and the Hölder inequality, Lemma 3.3.1 and (3.3.3) are applied again so as to imply that with $c_4 = c_4(\beta,\eta) > 0$ and $c_5 = c_5(\beta,\eta) > 0$, we have

$$\begin{aligned}
\int_\Omega u_\varepsilon^{\kappa+2}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &\le \frac{\eta}{2c_3}\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_4\int_\Omega u_\varepsilon^{\frac32\kappa+3} v_\varepsilon \\
&\le \frac{\eta}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + c_4\int_\Omega \left(u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac12}\right)^2 u_\varepsilon^{\frac12\kappa+1}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\eta}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + c_4 \left\| u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^{2\chi}(\Omega)}^2 \cdot \left\| u_\varepsilon^{\frac{1}{2}\kappa+1} \right\|_{L^{\chi^*}(\Omega)} \\
&\le \frac{\eta}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + c_1^{\frac{1}{\chi^*}} c_4 \left\| u_\varepsilon^{\frac{\kappa+2}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^{2\chi}(\Omega)}^2 \\
&\le \frac{\eta}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + \frac{\eta}{4}\int_\Omega u_\varepsilon^\kappa v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon^{\kappa+2}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + c_5\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which implies that

$$
\int_\Omega u_\varepsilon^{\kappa+2}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le \eta \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + \frac{\eta}{2}\int_\Omega u_\varepsilon^\kappa v_\varepsilon|\nabla u_\varepsilon|^2 + 2c_5\int_\Omega u_\varepsilon v_\varepsilon \tag{3.3.5}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Inserting (3.3.5) into (3.3.4) will yield (3.3.2).

We next illustrate that (3.3.2) holds for $\beta \in [1, \kappa+2)$. Using Young's inequality, we see that

$$
\int_\Omega u_\varepsilon^\beta v_\varepsilon \le \int_\Omega u_\varepsilon^{\kappa+2} v_\varepsilon + \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
$$

which in conjunction with the fact that (3.3.2) holds for $\kappa+2$ completes the proof. □

The following conclusion is a byproduct of the above lemma.

Lemma 3.3.3. *If $\kappa \in (-1,0)$, then for any $\gamma \in [0, \frac{\kappa}{2}+2)$ and $\eta > 0$, there exists $C(\gamma,\eta) > 0$ such that*

$$
\begin{aligned}
\int_\Omega u_\varepsilon^\gamma v_\varepsilon |\nabla v_\varepsilon|^2 \ \le\ & \eta\int_\Omega u_\varepsilon^\kappa v_\varepsilon|\nabla u_\varepsilon|^2 + \eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \eta\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 \\
&+ \int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + C(\gamma,\eta)\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.3.6}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. If $\gamma \in [0,1]$, using Young's inequality twice and relying on (3.2.6), we obtain

$$
\begin{aligned}
\int_\Omega u_\varepsilon^\gamma v_\varepsilon|\nabla v_\varepsilon|^2 &\le \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \\
&\le \eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4\eta}\int_\Omega u_\varepsilon v_\varepsilon^5 + \int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \\
&\le \eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{c_1^4}{4\eta}\int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2
\end{aligned}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $c_1 := \|v_0\|_{L^\infty(\Omega)}$. This shows that (3.3.6) holds for $\gamma \in [0,1]$. When $\gamma \in (1, \frac{\kappa}{2}+2)$, it is evident that $2\gamma - 1 \in (1, \kappa+3)$. Hence, it is available to use Young's inequality, (3.2.6) and Lemma 3.3.2 to see that with $c_2 = c_2(\gamma,\eta) > 0$, we have

$$
\begin{aligned}
\int_\Omega u_\varepsilon^\gamma v_\varepsilon|\nabla v_\varepsilon|^2 &\le \eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{c_1^4}{4\eta}\int_\Omega u_\varepsilon^{2\gamma-1}v_\varepsilon \\
&\le \eta\int_\Omega u_\varepsilon^\kappa v_\varepsilon|\nabla u_\varepsilon|^2 + \eta\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + \eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. This completes the proof. □

In the following, we first analyse the time evolution of the first two summands in (3.1.14).

Lemma 3.3.4. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1<\alpha<m$ *and* $C_S>0$*;*
(ii) $m \ge 2$ *and* S *fulfills* (3.1.5) *with* $m-1<\alpha<m$ *and* $C_S>0$.
Then one can find positive constants c *and* C *such that for each* $\varepsilon \in (0,1)$*, the function* $\mathcal{G}_\varepsilon$ *defined on* $t \in (0, T_{max,\varepsilon})$ *by letting*

$$\mathcal{G}_\varepsilon(t) := \begin{cases} c\int_\Omega u_\varepsilon^{3-m} + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} & \text{when} \quad 1 \le m < 2 \quad \text{or} \quad m > 3, \\ c\int_\Omega u_\varepsilon \ln u_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}, & \text{when} \quad m = 2, \\ -c\int_\Omega \ln u_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} & \text{when} \quad m = 3, \\ -c\int_\Omega u_\varepsilon^{3-m} + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} & \text{when} \quad 2 < m < 3 \end{cases}$$

satisfies that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned} &\mathcal{G}_\varepsilon'(t) + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\quad \le \; C\int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + C\int_\Omega u_\varepsilon^2 v_\varepsilon + C\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{when } m \in \{1,2\}, \end{aligned} \tag{3.3.7}$$

$$\begin{aligned} &\mathcal{G}_\varepsilon'(t) + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\quad \le \; C\int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + C\int_\Omega u_\varepsilon^{4-2m} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &\qquad + C\int_\Omega u_\varepsilon^{3-m} v_\varepsilon \qquad \text{when } 1 < m < 2 \end{aligned} \tag{3.3.8}$$

and

$$\begin{aligned} &\mathcal{G}_\varepsilon'(t) + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\quad \le \; C\int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 \qquad \text{when } m > 2. \end{aligned} \tag{3.3.9}$$

Proof. From (3.2.11), it follows that there exists $c_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le c_1 \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \tag{3.3.10}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. In the case $1 \le m < 2$ and $m > 3$, we define

$$\mathcal{G}_\varepsilon(t) := \frac{2(c_1+1)}{(2-m)(3-m)} \int_\Omega u_\varepsilon^{3-m} + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.$$

When $1 \le m < 2$, we see from (3.2.15) that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{2(c_1+1)}{(2-m)(3-m)} \frac{d}{dt}\int_\Omega u_\varepsilon^{3-m} + (c_1+1)\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2$$

$$\leq (c_1+1)C_S^2\int_\Omega u_\varepsilon^{4-2m}(u_\varepsilon+1)^{2\alpha-2}v_\varepsilon|\nabla v_\varepsilon|^2+\frac{2\ell(c_1+1)}{2-m}\int_\Omega u_\varepsilon^{3-m}v_\varepsilon,$$

which together with (3.3.10) implies that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}&\mathcal{G}_\varepsilon'(t)+\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2+\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\ &\quad\leq (c_1+1)C_S^2\int_\Omega u_\varepsilon^{4-2m}(u_\varepsilon+1)^{2\alpha-2}v_\varepsilon|\nabla v_\varepsilon|^2+\frac{2\ell(c_1+1)}{2-m}\int_\Omega u_\varepsilon^{3-m}v_\varepsilon.\end{aligned}\tag{3.3.11}$$

Obviously, when $1<m<2$, (3.3.8) can be derived directly by relying on the fact that $(s+1)^{2\alpha-2}\leq c_2s^{2\alpha-2}+c_2$ holds for any $s\geq 0$ with some $c_2>0$. For the case $m=1$, the inequality $(s+1)^{2\alpha-2}\leq s^{2\alpha-2}$ holds for any $s\geq0$ because of the assumption $0<\alpha<1$, which in conjunction with (3.3.11) proves (3.3.7) for $m=1$.

When $m>3$, it is easy to verify from (3.2.17) that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\frac{2(c_1+1)}{(2-m)(3-m)}\frac{d}{dt}\int_\Omega u_\varepsilon^{3-m}+(c_1+1)\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\leq(c_1+1)C_S^2\int_\Omega u_\varepsilon^{2-2m+2\alpha}v_\varepsilon|\nabla v_\varepsilon|^2,$$

which combined with (3.3.10) implies (3.3.9) for the case $m>3$.

When $m=2$, for fixed c_1, we get from (3.2.18) that

$$\begin{aligned}&2(c_1+1)\frac{d}{dt}\int_\Omega u_\varepsilon\ln u_\varepsilon+(c_1+1)\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\ &\quad\leq (c_1+1)C_S^2\int_\Omega u_\varepsilon^{2\alpha-2}v_\varepsilon|\nabla v_\varepsilon|^2+2(c_1+1)\ell\int_\Omega u_\varepsilon v_\varepsilon+2(c_1+1)\ell\int_\Omega u_\varepsilon^2v_\varepsilon\end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, which together with (3.3.10) implies (3.3.7) for the case $m=2$ by taking

$$\mathcal{G}_\varepsilon(t):=2(c_1+1)\int_\Omega u_\varepsilon\ln u_\varepsilon+\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.$$

Analogously, for the case $2<m<3$, based on (3.3.10), we can prove (3.3.9) by letting

$$\mathcal{G}_\varepsilon(t):=-\frac{2(c_1+1)}{(m-2)(3-m)}\int_\Omega u_\varepsilon^{3-m}+\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3},$$

taking $p:=3-m\in(0,1)$ in (3.2.16) and dividing both sides by $\frac{(m-2)(3-m)}{2(c_1+1)}$.

For the case $m=3$, (3.3.9) can be established by (3.2.19) and (3.3.10) with

$$\mathcal{G}_\varepsilon(t):=-2(c_1+1)\int_\Omega\ln u_\varepsilon+\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.$$

The proof is completed. □

Based on the above lemma, we can readily derive an L^{p_0} bound for u_ε with some $p_0>1$ when $1\leq m<2$. We note that, in this case, one can choose p_0 satisfying $p_0>1$ and $p_0\in(2-m,3-m)$ simultaneously. As a consequence, the term $\int_\Omega u^{p_0}$ appearing in (3.1.14) already provides the type of control required from $\int_\Omega u^{p^*}$ in (3.1.16). Therefore, in the following lemma, we work with the functional defined in (3.1.14).

Lemma 3.3.5. *Suppose that* $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$. *Then there exist* $p_0 > 1$ *and* $C_1(p_0) > 0$ *such that*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t) \le C_1(p_0) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).$$

Moreover, there exists $C_2 > 0$ *such that for all* $\varepsilon \in (0,1)$,

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_0^{T_{max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \le C_2.$$

Proof. Let $\mathcal{G}_\varepsilon$ be defined as in Lemma 3.3.4. It follows that there exists $c_1 > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, when $m = 1$,

$$\begin{aligned} \mathcal{G}_\varepsilon'(t) &+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + c_1 \int_\Omega u_\varepsilon^{3-m} v_\varepsilon + c_1 \int_\Omega u_\varepsilon v_\varepsilon, \end{aligned} \tag{3.3.12}$$

and when $1 < m < 2$,

$$\begin{aligned} \mathcal{G}_\varepsilon'(t) &+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + c_1 \int_\Omega u_\varepsilon^{4-2m} v_\varepsilon |\nabla v_\varepsilon|^2 + c_1 \int_\Omega u_\varepsilon^{3-m} v_\varepsilon. \end{aligned} \tag{3.3.13}$$

Under the assumptions $m - 1 < \alpha < m$ and $m < 2$, we observe that $\max\{1, m+1-2\alpha, 3-5m+4\alpha\} < 3 - m$. Hence, it is possible to pick $p_0 > 1$ satisfying

$$\max\{m+1-2\alpha, 3-5m+4\alpha\} < p_0 < 3 - m, \tag{3.3.14}$$

which ensures that

$$0 < p_0 - m + 2\alpha - 1 < 2 - 2m + 2\alpha < \frac{p_0}{2} + \frac{m}{2} + \frac{1}{2} \tag{3.3.15}$$

and moreover, since $m \ge 1$ implies $3 - m \le 3m - 1$, we also have

$$0 < p_0 - m + 1 < \frac{p_0}{2} + \frac{m}{2} + \frac{1}{2}. \tag{3.3.16}$$

For the fixed $p_0 > 1$, we employ (3.2.15) to find $c_2 = c_2(p_0) > 0$ fulfilling

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_\varepsilon^{p_0} &+ \frac{p_0(p_0-1)}{2} \int_\Omega u_\varepsilon^{p_0+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le \frac{p_0(p_0-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_0-m+1} (u_\varepsilon+1)^{2\alpha-2} v_\varepsilon |\nabla v_\varepsilon|^2 + p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon \\ &\le c_2 \int_\Omega u_\varepsilon^{p_0-m+2\alpha-1} v_\varepsilon |\nabla v_\varepsilon|^2 + c_2 \int_\Omega u_\varepsilon^{p_0-m+1} v_\varepsilon |\nabla v_\varepsilon|^2 + p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon \end{aligned} \tag{3.3.17}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Observing that our restrictions $1 < p_0 < 3-m$ and $m \geq 1$ warrant $\kappa := p_0 + m - 3 \in (-1, 0)$, we may invoke Lemma 3.3.3 along with (3.3.15) and (3.3.16) to infer the existence of $c_3 = c_3(p_0) > 0$ such that

$$\begin{aligned} c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + c_2 \int_\Omega u_\varepsilon^{p_0-m+2\alpha-1} v_\varepsilon |\nabla v_\varepsilon|^2 + c_2 \int_\Omega u_\varepsilon^{p_0-m+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\ \leq \frac{p_0(p_0-1)}{8} \int_\Omega u_\varepsilon^{p_0+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 \\ + \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_3 \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{3.3.18}$$

For the taken κ, it is clear that $1 < p_0 < \kappa + 3$ and $1 < 3 - m < \kappa + 3$, which allows us to apply Lemma 3.3.2 to show the existence of $c_4 = c_4(p_0) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon + c_1 \int_\Omega u_\varepsilon^{3-m} v_\varepsilon \\ \leq \frac{p_0(p_0-1)}{8} \int_\Omega u_\varepsilon^{p_0+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + c_4 \int_\Omega u_\varepsilon v_\varepsilon. \end{aligned} \tag{3.3.19}$$

When $m = 1$, combining (3.3.12) and (3.3.17)-(3.3.19) implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \mathcal{G}_\varepsilon'(t) + \frac{d}{dt} \int_\Omega u_\varepsilon^{p_0} + \frac{1}{2} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ \leq \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + (c_1 + c_3 + c_4) \int_\Omega u_\varepsilon v_\varepsilon. \end{aligned} \tag{3.3.20}$$

We note that when $1 < m < 2$, the fact $7 - 5m < 3 - m$ ensures the possibility to pick $p_0 > 1$ not only satisfying (3.3.14), but also fulfilling $p_0 > 7 - 5m$, which yields

$$0 < 4 - 2m < \frac{p_0}{2} + \frac{m}{2} + \frac{1}{2}.$$

Another application of Lemma 3.3.3 implies $c_5 = c_5(p_0) > 0$ such that

$$\begin{aligned} c_1 \int_\Omega u_\varepsilon^{4-2m} v_\varepsilon |\nabla v_\varepsilon|^2 \\ \leq \frac{p_0(p_0-1)}{8} \int_\Omega u_\varepsilon^{p_0+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{4} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\ + \frac{1}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_5 \int_\Omega u_\varepsilon v_\varepsilon \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which combined with (3.3.13) and (3.3.17)-(3.3.19) gives that when $1 < m < 2$, for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned} \mathcal{G}_\varepsilon'(t) + \frac{d}{dt} \int_\Omega u_\varepsilon^{p_0} + \frac{1}{4} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} |D^2 \ln v_\varepsilon|^2 + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ \leq 2 \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + (c_3 + c_4 + c_5) \int_\Omega u_\varepsilon v_\varepsilon. \end{aligned} \tag{3.3.21}$$

From (3.1.6), we can conclude that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\mathcal{G}_\varepsilon(0) + \int_\Omega u_{0\varepsilon}^{p_0} \le c_6 \int_\Omega (u_0+1)^{3-m} + \int_\Omega \frac{|\nabla v_0|^4}{v_0^3} + \int_\Omega (u_0+1)^{p_0} \le c_7$$

with $c_6 > 0$ and $c_7 > 0$. Then, integrating (3.3.20) and (3.3.21) in time respectively, we infer the existence of $c_8 > 0$ from (3.2.7), (3.1.6) and Lemma 3.2.2 such that whenever $1 \le m < 2$,

$$\begin{aligned} \int_\Omega u_\varepsilon^{p_0} + \frac{1}{4}\int_0^t\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_0^t\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ \le\ 2\int_0^t\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + (c_1+c_3+c_4+c_5)\int_0^t\int_\Omega u_\varepsilon v_\varepsilon + \mathcal{G}_\varepsilon(0) + \int_\Omega u_{0\varepsilon}^{p_0} \\ \le\ c_8 + c_7 \qquad \text{for all } \ t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Finally, combining this estimate with the inequality

$$\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le (4+\sqrt{2})^2 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \qquad \text{for all } \ t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

which follows from (3.2.9) with $q := 4$, completes the proof. □

When $m \ge 2$, in contrast to the above lemma, it is no longer possible to pick $p_0 > 1$ satisfying $p_0 \in (2-m, 3-m)$. Therefore, relying solely on the functional in (3.1.14) is insufficient to derive the desired estimate directly. Hence, we consider the functional in (3.1.16) in the following two lemmas.

Lemma 3.3.6. *Suppose that $2 \le m < 3$ and S fulfills (3.1.5) with $m-1 < \alpha < \frac{m}{2}+1$ and $C_S > 0$. Then there exist $p_0 > 1$ and $C_1(p_0) > 0$ such that*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t) \le C_1(p_0) \qquad \textit{for all } t \in (0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).$$

Moreover, there exists $C_2 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_0^{T_{max,\varepsilon}}\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \int_0^{T_{max,\varepsilon}}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \le C_2.$$

Proof. We first note that, under the assumption $m-1 < \alpha < \frac{m}{2}+1$, it is possible to find $p_* > 0$ satisfying

$$\max\{m+1-2\alpha, 4\alpha-3m-1\} < p_* < 3-m \le 1. \tag{3.3.22}$$

Evidently, we have $\frac{p_*}{2} + \frac{3m}{2} + \frac{3}{2} - 2\alpha > 1$ and $p_* + m > 1$, and hence it is also possible to fix $p_0 > 1$ such that

$$p_0 < \min\left\{\frac{p_*}{2} + \frac{3m}{2} + \frac{3}{2} - 2\alpha, p_* + m\right\}. \tag{3.3.23}$$

Taking $\kappa := p_* + m - 3$, we infer from (3.3.22) that

$$\kappa \in (-1, 0). \tag{3.3.24}$$

Moreover, in view of $\alpha > m-1$, it follows from (3.3.23) that

$$0 < p_0 - m - 1 + 2\alpha < \frac{p_*}{2} + \frac{m}{2} + \frac{1}{2} =: \frac{\kappa}{2} + 2. \tag{3.3.25}$$

Furthermore, by the choice of $p*$ in (3.3.22), we obtain

$$0 < p_* - m - 1 + 2\alpha < 2 - 2m + 2\alpha < \frac{p_*}{2} + \frac{m}{2} + \frac{1}{2} =: \frac{\kappa}{2} + 2, \tag{3.3.26}$$

where the last inequality follows from the fact that $p_* > 4\alpha - 3m - 1 \geq 3 - 5m + 4\alpha$ because of $m \geq 2$.

Next, let $\mathcal{G}_\varepsilon$ be defined as in Lemma 3.3.4. According to (3.3.7) and (3.3.9), there exists $c_1 > 0$ such that

$$\begin{aligned} \mathcal{G}'_\varepsilon(t) &+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\leq c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + c_1 \int_\Omega u_\varepsilon^2 v_\varepsilon + c_1 \int_\Omega u_\varepsilon v_\varepsilon \end{aligned} \tag{3.3.27}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. For the above choices of $p*$ and p_0, we draw on (3.2.16) and (3.2.15), respectively, to see that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, we have

$$-\frac{d}{dt}\int_\Omega u_\varepsilon^{p_*} + \frac{p_*(1-p_*)}{2}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \leq \frac{p_*(1-p_*)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_*-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2$$

and

$$\frac{d}{dt}\int_\Omega u_\varepsilon^{p_0} \leq \frac{p_0(p_0-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_0-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon,$$

which in conjunction with (3.3.27) yield that

$$\begin{aligned} \mathcal{G}'_\varepsilon(t) &+ \frac{d}{dt}\int_\Omega u_\varepsilon^{p_0} - \frac{d}{dt}\int_\Omega u_\varepsilon^{p_*} + \frac{p_*(1-p_*)}{2}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ \leq\ & \frac{p_*(1-p_*)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_*-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{p_0(p_0-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_0-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &+ c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon + c_1 \int_\Omega u_\varepsilon^2 v_\varepsilon + c_1 \int_\Omega u_\varepsilon v_\varepsilon \end{aligned} \tag{3.3.28}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. In view of (3.3.24)-(3.3.26), Lemma 3.3.3 becomes applicable to entail the existence of $c_2 = c_2(p_0) > 0$ fulfilling

$$\begin{aligned} \frac{p_*(1-p_*)}{2} C_S^2 &\int_\Omega u_\varepsilon^{p_*-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{p_0(p_0-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_0-m-1+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &+ c_1 \int_\Omega u_\varepsilon^{2-2m+2\alpha} v_\varepsilon |\nabla v_\varepsilon|^2 \\ \leq\ & \frac{p_*(1-p_*)}{4} \int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \end{aligned}$$

$$+\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + c_2\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \qquad (3.3.29)$$

On the other hand, it is obvious from (3.3.23) that $1<p_0<p_*+m:=\kappa+3$, which together with (3.3.24) enables us to employ Lemma 3.3.2 to find $c_3=c_3(p_0)>0$ such that

$$\begin{aligned}
&p_0\ell\int_\Omega u_\varepsilon^{p_0}v_\varepsilon + c_1\int_\Omega u_\varepsilon^2 v_\varepsilon\\
&\qquad\le \frac{p_*(1-p_*)}{4}\int_\Omega u_\varepsilon^{p_*+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac14\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2\\
&\qquad\quad +c_3\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \qquad (3.3.30)
\end{aligned}$$

Inserting (3.3.29) and (3.3.30) into (3.3.28) implies that with $c_4:=c_1+c_2+c_3$,

$$\begin{aligned}
&\mathcal{G}_\varepsilon'(t)+\frac{d}{dt}\int_\Omega u_\varepsilon^{p_0}-\frac{d}{dt}\int_\Omega u_\varepsilon^{p_*}+\frac12\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\frac12\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2+\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\
&\quad\le \int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + c_4\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \qquad (3.3.31)
\end{aligned}$$

Before proceeding, we first claim that $\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)+\int_\Omega u_\varepsilon^{p_*}+\int_\Omega u_{0\varepsilon}^{p_0}$ is bounded for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Indeed, from (3.1.6) there exists $c_5>0$ such that

$$\|u_0\|_{L^\infty(\Omega)}+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}\le c_5,$$

and from (3.2.5), by Young's inequality along with the facts $0<p_*<1$ and $0\le 3-m<1$, with some $c_6>0$ we have

$$\int_\Omega u_\varepsilon^{3-m}+\int_\Omega u_\varepsilon^{p_*}+\int_\Omega u_{0\varepsilon}^{p_0}\le 2\left(\int_\Omega u_\varepsilon+|\Omega|\right)+(c_5+1)^{p_0}|\Omega|\le c_6.$$

Thus, when $m=2$, with some $c_7>0$ we have

$$\begin{aligned}
\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)+\int_\Omega u_\varepsilon^{p_*}+\int_\Omega u_{0\varepsilon}^{p_0} &\le c_7\int_\Omega u_{0\varepsilon}\ln u_{0\varepsilon}-c_7\int_\Omega u_\varepsilon\ln u_\varepsilon+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}+c_6\\
&\le c_7(c_5+1)^2|\Omega|+\frac{c_7|\Omega|}{e}+c_5+c_6,
\end{aligned}$$

and when $2<m<3$, with some $c_8>0$ we have

$$\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)+\int_\Omega u_\varepsilon^{p_*}+\int_\Omega u_{0\varepsilon}^{p_0} \le c_8\int_\Omega u_\varepsilon^{3-m}+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}+c_6\le(c_8+1)c_6+c_5.$$

Hence, by integrating (3.3.31) from 0 to t, in light of (3.2.7) and Lemma 3.2.2, we can conclude from (3.1.6) that

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p_0}+\frac12\int_0^t\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\frac12\int_0^t\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2+\int_0^t\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\
&\quad\le \int_0^t\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2+c_4\int_0^t\int_\Omega u_\varepsilon v_\varepsilon+\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)+\int_\Omega u_\varepsilon^{p_*}+\int_\Omega u_{0\varepsilon}^{p_0}\\
&\quad\le c_9 \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1)
\end{aligned}$$

with some $c_9=c_9(p_0)>0$. This together with (3.2.9) completes the proof. □

Lemma 3.3.7. *Suppose that $m \geq 3$, S fulfills (3.1.5) with $m-1<\alpha<\frac{m}{2}+1$ and $C_S>0$, and additionally $u_0>0$ in $\overline{\Omega}$. Then there exist $p_0>1$ and $C_1(p_0)>0$ such that*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t) \leq C_1(p_0) \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).$$

Moreover, there exists $C_2>0$ such that for all $\varepsilon\in(0,1)$,

$$\int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\int_0^{T_{max,\varepsilon}}\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5}+\int_0^{T_{max,\varepsilon}}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\leq C_2.$$

Proof. Let $\mathcal{G}_\varepsilon$ be defined as in Lemma 3.3.4 with $c_1>0$ for $m=3$, and with $c_2>0$ for $m>3$. We first show that there exists some positive constant c such that

$$\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)\leq c \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \tag{3.3.32}$$

Indeed, by (3.1.6), there is $c_3>0$ such that

$$\|u_0\|_{L^\infty(\Omega)}+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}\leq c_3.$$

In addition, the strict positivity of u_0 in $\overline{\Omega}$ warrants the existence of $c_4>0$ independent of ε fulfilling $u_{0\varepsilon}\geq c_4$, and (3.2.5) guarantees $c_5>0$ such that $\int_\Omega u_\varepsilon\leq c_5$ for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Thus, when $m=3$, using the fact that $\ln s\leq s$ for all $s>0$, we have

$$\begin{aligned}\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t) &\leq -c_1\int_\Omega\ln u_{0\varepsilon}+c_1\int_\Omega\ln u_\varepsilon+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}\\ &\leq -c_1\int_\Omega\ln u_{0\varepsilon}+c_1\int_\Omega u_\varepsilon+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}\\ &\leq -c_1\ln c_4|\Omega|+c_1c_5+c_3 \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),\end{aligned}$$

and when $m>3$, since $3-m$ is negative, for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$ we have

$$\mathcal{G}_\varepsilon(0)-\mathcal{G}_\varepsilon(t)\leq c_2\int_\Omega u_{0\varepsilon}^{3-m}+\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}\leq c_2c_4^{3-m}|\Omega|+c_3.$$

Now, from (3.3.9), we infer the existence of $c_6>0$ satisfying

$$\begin{aligned}&\mathcal{G}_\varepsilon'(t)+\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2+\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\ &\quad\leq c_6\int_\Omega u_\varepsilon^{2+2\alpha-2m}v_\varepsilon|\nabla v_\varepsilon|^2 \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).\end{aligned} \tag{3.3.33}$$

Similar to the discussions in the proof of Lemma 3.3.6, we can pick $p_*\in(2-m,3-m)$ and $p_0>1$ such that the following three conditions

$$0<p_*+2\alpha-m-1<2+2\alpha-2m<\frac{p_*}{2}+\frac{m}{2}+\frac{1}{2}, \tag{3.3.34}$$

$$0<p_0+2\alpha-m-1<\frac{p_*}{2}+\frac{m}{2}+\frac{1}{2} \tag{3.3.35}$$

and

$$p_0 < p_* + m \tag{3.3.36}$$

are fulfilled simultaneously. For the fixed p_*, recalling (3.2.17), we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon^{p_*} + \frac{p_*(p_*-1)}{2}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \le \frac{p_*(p_*-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_*+2\alpha-m-1} v_\varepsilon |\nabla v_\varepsilon|^2$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. This in conjunction with (3.3.33) yields that

$$\begin{aligned}\mathcal{G}_\varepsilon'(t) &+ \frac{d}{dt}\int_\Omega u_\varepsilon^{p_*} + \frac{p_*(p_*-1)}{2}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &+ \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le c_6 \int_\Omega u_\varepsilon^{2+2\alpha-2m} v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{p_*(p_*-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_*+2\alpha-m-1} v_\varepsilon |\nabla v_\varepsilon|^2\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, whereas thanks to (3.3.34), applying Lemma 3.3.3 with $\kappa := p^* + m - 3$ results in that there is $c_7 > 0$ fulfilling

$$\begin{aligned}c_8 &\int_\Omega u_\varepsilon^{2+2\alpha-2m} v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{p_*(p_*-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_*+2\alpha-m-1} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &\le \frac{p_*(p_*-1)}{4}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \\ &\quad + \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_7 \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),\end{aligned}$$

so that

$$\begin{aligned}\mathcal{G}_\varepsilon'(t) &+ \frac{d}{dt}\int_\Omega u_\varepsilon^{p_*} + \frac{p_*(p_*-1)}{4}\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &+ \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_7 \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).\end{aligned}$$

Upon integration in time, in view of (3.2.7) and Lemma 3.2.2, this together with (3.3.32) and the fact that $u_{0\varepsilon} \ge c_4$ and p_* is negative, this implies that there is $c_8 > 0$ satisfying

$$\begin{aligned}&\frac{p_*(p_*-1)}{4}\int_0^t\int_\Omega u_\varepsilon^{p_*+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \\ &\quad + \frac{1}{2}\int_0^t\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_0^t\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\le \int_0^t\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_7 \int_0^t\int_\Omega u_\varepsilon v_\varepsilon + \mathcal{G}_\varepsilon(0) - \mathcal{G}_\varepsilon(t) + \int_\Omega u_{0\varepsilon}^{p_*} \\ &\le c_8 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).\end{aligned} \tag{3.3.37}$$

For the above fixed p_0, it follows from (3.2.15) that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^{p_0} \le \frac{p_0(p_0-1)}{2} C_S^2 \int_\Omega u_\varepsilon^{p_0+2\alpha-m-1} v_\varepsilon |\nabla v_\varepsilon|^2 + p_0 \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon,$$

where in light of (3.3.35) and Lemma 3.3.3, we get

$$\begin{aligned}\frac{p_0(p_0-1)}{2}C_S^2\int_\Omega u_\varepsilon^{p_0+2\alpha-m-1}v_\varepsilon|\nabla v_\varepsilon|^2\\ \le\ \int_\Omega u_\varepsilon^{p_*+m-3}v_\varepsilon|\nabla u_\varepsilon|^2+\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2\\ +\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2+c_9\int_\Omega u_\varepsilon v_\varepsilon\qquad\text{for all}\ \ t\in(0,T_{max,\varepsilon})\text{ and }\varepsilon\in(0,1)\end{aligned}$$

with $c_9=c_9(p_0)>0$, and in view of (3.3.36), an application of Lemma 3.3.2 implies that

$$p_0\ell\int_\Omega u_\varepsilon^{p_0}v_\varepsilon\le\int_\Omega u_\varepsilon^{p_*+m-3}v_\varepsilon|\nabla u_\varepsilon|^2+\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2+c_{10}\int_\Omega u_\varepsilon v_\varepsilon$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$ with $c_{10}=c_{10}(p_0)>0$. We therefore obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^{p_0}\ \le\ 2\int_\Omega u_\varepsilon^{p_*+m-3}v_\varepsilon|\nabla u_\varepsilon|^2+\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}+2\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2\\ +\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2+(c_9+c_{10})\int_\Omega u_\varepsilon v_\varepsilon\qquad\text{for all}\ \ t\in(0,T_{max,\varepsilon})\text{ and }\varepsilon\in(0,1),\end{aligned}$$

while once integrated in time, this together with (3.3.37) and (3.2.9) completes the lemma. □

3.3.2 Bounds for u_ε in L^p for any $p>1$

Now we are ready to derive the L^p bounds for u_ε for any $p>1$. Before doing this, we first state the following conclusion, which allows us to control some ill-contributions arising in Lemma 3.3.9.

Lemma 3.3.8. *Let $m\ge1$ and $p>1$. Suppose that with some $p_0>1$ there exists $c(p_0)>0$ satisfying*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t)\le c(p_0)\qquad\textit{for all } t\in(0,T_{max,\varepsilon})\textit{ and }\varepsilon\in(0,1).\tag{3.3.38}$$

Then for any $\beta\in[p+m-1,p_0+p+m-1)$ and $\eta>0$, there is $C(p_0,q,\beta,\eta)>0$ such that

$$\int_\Omega u_\varepsilon^\beta v_\varepsilon\le\eta\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2+\eta\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2+C(p_0,q,\beta,\eta)\int_\Omega u_\varepsilon v_\varepsilon\tag{3.3.39}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, where $q>\frac{2(p+m-1)}{p_0}$.

Proof. We observe that the assumptions $q>\frac{2(p+m-1)}{p_0}$, $m\ge1$ and $p>1$ entail

$$p+m-1<\frac{(p+m-1)(q+2)}{q}<p_0+p+m-1.$$

Since for any $\beta \in \left[p+m-1, \frac{(p+m-1)(q+2)}{q}\right)$, Young's inequality implies that

$$\int_\Omega u_\varepsilon^\beta v_\varepsilon \le \int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon + \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

we claim that to prove this lemma, it is sufficient to concentrate on the case

$$\beta \in \left[\frac{(p+m-1)(q+2)}{q}, p_0 + p + m - 1\right), \tag{3.3.40}$$

from which, we have

$$\vartheta^* := \frac{p_0}{\beta - p - m + 1} > 1 \quad \text{and} \quad \vartheta := \frac{\vartheta^*}{\vartheta^* - 1} > 1.$$

For given $\eta > 0$, we see from the Hölder inequality, Lemma 3.3.1 and (3.3.38) that there is $c_1 = c_1(p_0, \beta, \eta) > 0$ fulfilling

$$\begin{aligned}
\int_\Omega u_\varepsilon^\beta v_\varepsilon &= \int_\Omega \left(u_\varepsilon^{\frac{p+m-1}{2}} v_\varepsilon^{\frac{1}{2}}\right)^2 u_\varepsilon^{\beta-(p+m-1)} \\
&\le \left\| u_\varepsilon^{\frac{p+m-1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^{2\vartheta}(\Omega)}^2 \cdot \left\| u_\varepsilon^{\beta-(p+m-1)} \right\|_{L^{\vartheta^*}(\Omega)} \\
&\le \frac{\eta}{2} \int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{p+m-1} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + c_1 \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.3.41}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Here Lemma 3.2.3 and two applications of Young's inequality show that there exist $c_2 = c_2(q, \eta) > 0$ and $c_3 = c_3(q, \eta, \beta) > 0$ such that

$$\begin{aligned}
\int_\Omega u_\varepsilon^{p+m-1} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &\le \frac{\eta}{2(q+\sqrt{2})^2} \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + c_2 \int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon \\
&\le \frac{\eta}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} |D^2 \ln v_\varepsilon|^2 + \frac{1}{2} \int_\Omega u_\varepsilon^\beta v_\varepsilon + c_3 \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which when inserted into (3.3.41) yields

$$\int_\Omega u_\varepsilon^\beta v_\varepsilon \le \eta \int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 + \eta \int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} |D^2 \ln v_\varepsilon|^2 + 2(c_1 + c_3) \int_\Omega u_\varepsilon v_\varepsilon$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. □

With the improved integrability of u_ε (Lemmas 3.3.5-3.3.7) at hand, the strategy for further deriving the desired boundedness of u_ε relies on analysing the energy functional shown in (3.1.13) for conveniently large $p > 1$ and suitably chosen $q > 2$. To control the unfavorable summands of the type $\int_\Omega u_\varepsilon^\beta v_\varepsilon$ appearing on the right-hand sides of (3.3.48) and (3.3.49), we require that the associated parameter β satisfy the assumptions of Lemma 3.3.8, which necessitates choosing q within an appropriate intermediate range. Throughout this procedure, the condition $p_0 > 1$ is of critical importance.

Lemma 3.3.9. *Let $p > 3$. Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ and S fulfills (3.1.4) with $m-1 < \alpha < m$ and $C_S > 0$;
(ii) $2 \le m < 3$ and S fulfills (3.1.5) with $m-1 < \alpha < \frac{m}{2}+1$ and $C_S > 0$;
(iii) $m \ge 3$, S fulfills (3.1.5) with $m-1 < \alpha < \frac{m}{2}+1$ and $C_S > 0$, and $u_0 > 0$ in $\overline{\Omega}$.
Then for any choice of $\varepsilon \in (0,1)$, there exists $C(p) > 0$ such that

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{3.3.42}$$

and

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \le C(p) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.3.43}$$

Proof. According to Lemmas 3.3.5-3.3.7, there exist $p_0 > 1$ and $c_1 = c_1(p_0) > 0$ satisfying

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t) \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{3.3.44}$$

Moreover, recalling Lemma 3.2.2 and (3.2.7), with some $c_2 > 0$ we have

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \le c_2 \qquad \text{for all } \varepsilon \in (0,1). \tag{3.3.45}$$

As $\frac{2(p+m-1)}{p_0} < 2(p_0+p+m-2)$, we are able to pick $q > 2(p+m-2)$ in a way that

$$q \in \left(\frac{2(p+m-1)}{p_0},\ 2(p_0+p+m-2) \right),$$

which clearly warrants

$$p+m-1 < \frac{(p+m-1)(q+2)}{q} < p_0+p+m-1 \tag{3.3.46}$$

and

$$p+m-1 < \frac{q+2}{2} < p_0+p+m-1. \tag{3.3.47}$$

Our assumptions on α and m entail $0 < p-m+1 \le p+m-1$ and $0 < p+2\alpha-m-1 < p+m-1$. Thus, we may invoke Young's inequality several times to deduce from (3.2.15), (3.2.9) and (3.1.6) that there are $c_3 = c_3(p) > 0$ and $c_4 = c_4(p) > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3} v_\varepsilon |\nabla u_\varepsilon|^2 \\
\le\ & c_3 \int_\Omega u_\varepsilon^{p+2\alpha-m-1} v_\varepsilon |\nabla v_\varepsilon|^2 + c_3 \int_\Omega u_\varepsilon^{p-m+1} v_\varepsilon |\nabla v_\varepsilon|^2 + p\ell \int_\Omega u_\varepsilon^p v_\varepsilon \\
\le\ & 2c_3 \int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon |\nabla v_\varepsilon|^2 + 2c_3 \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + p\ell \int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon + p\ell \int_\Omega u_\varepsilon v_\varepsilon \\
\le\ & \frac{q}{2(q+\sqrt{2})^2} \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + c_4 \int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon + 2c_3 \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \\
& + p\ell \int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon + p\ell \int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + c_4\int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon + 2c_3\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 \\
&\quad + p\ell\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon + p\ell\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \quad (3.3.48)
\end{aligned}
$$

This in conjunction with

$$
\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 \le c_5\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon \qquad (3.3.49)
$$

with $c_5 = c_5(p) > 0$ identified from Lemma 3.2.4, implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}\right\} + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 \\
&\le c_4\int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon + c_5\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon + p\ell\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon \\
&\quad + 2c_3\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + p\ell\int_\Omega u_\varepsilon v_\varepsilon. \qquad (3.3.50)
\end{aligned}
$$

Owing to (3.3.44), (3.3.46) and (3.3.47), Lemma 3.3.8 becomes applicable so as to ensure that there exists $c_6 = c_6(p) > 0$ such that

$$
\begin{aligned}
&c_4\int_\Omega u_\varepsilon^{\frac{(p+m-1)(q+2)}{q}} v_\varepsilon + c_5\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon + p\ell\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon \\
&\le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + c_6\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}
$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Inserting this into (3.3.50), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}\right\} + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 \\
&\le 2c_3\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + (p\ell + c_6)\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),
\end{aligned}
$$

which together with (3.3.45) and (3.1.6), upon an integration in time, entails that with some $c_7 = c_7(p) > 0$, we have

$$
\begin{aligned}
&\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{4}\int_0^t\int_\Omega u_\varepsilon^{p+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{q}{4}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 \\
&\le 2c_3\int_0^t\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + (p\ell + c_6)\int_0^t\int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega (u_0+1)^p + \int_\Omega \frac{|\nabla v_0|^q}{v_0^{q-1}} \\
&\le c_7 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned}
$$

Thus (3.3.42) and (3.3.43) are included. □

3.4 Uniform L^∞-boundedness of u_ε

The next inequality is taken from [88, Lemma 6.2], which plays a critical role in the iterative argument leading to the L^∞ bound for u_ε. For completeness, we include the proof here.

Lemma 3.4.1. *Let $p_* > 2$. Then there exist $\kappa = \kappa(p_*) > 0$ and $C(p_*) > 0$ such that for any choice of $p \geq p_*$ and $\eta \in (0,1]$,*

$$\int_\Omega \varphi^{p+1}\psi \leq \eta \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^6}{\psi^5} + C(p_*)\eta^{-\kappa}p^{2\kappa} \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^2 \cdot \int_\Omega \varphi\psi \tag{3.4.1}$$

is valid for arbitrary positive functions $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$.

Proof. As $p_* > 2$, we have

$$q \equiv q(p_*) := \frac{6p_*}{5p_*+2} > 1, \tag{3.4.2}$$

so that the Gagliardo-Nirenberg inequality in the two-dimensional domain Ω provides $c_1 > 0$ such that

$$\|\rho\|^2_{L^2(\Omega)} \leq c_1\|\nabla\rho\|^{\frac{2q}{2q-1}}_{L^q(\Omega)}\|\rho\|^{\frac{2q-2}{2q-1}}_{L^{\frac{2}{3}}(\Omega)} + c_1\|\rho\|^2_{L^{\frac{2}{3}}(\Omega)} \qquad \text{for all} \quad \rho \in C^1(\overline{\Omega}).$$

Given $p \geq p_*$, $\eta \in (0,1]$ as well as $0 < \varphi \in C^1(\overline{\Omega})$ and $0 < \psi \in C^1(\overline{\Omega})$, we thus obtain that

$$\begin{aligned}\int_\Omega \varphi^{p+1}\psi &= \|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^2(\Omega)} \\ &\leq c_1\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^{\frac{2q}{2q-1}}_{L^q(\Omega)}\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^{\frac{2q-2}{2q-1}}_{L^{\frac{2}{3}}(\Omega)} + c_1\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{\frac{2}{3}}(\Omega)},\end{aligned} \tag{3.4.3}$$

where writing

$$c_2 := \max\left\{1,\ |\Omega|^{\frac{2}{q}}\right\} \quad \text{and} \quad \delta \equiv \delta(p,\eta) := \min\left\{\frac{\eta}{(p+1)^2|\Omega|^{\frac{2-q}{q}}},\ \frac{\eta^{\frac{1}{3}}}{c_2}\right\}, \tag{3.4.4}$$

using Young's inequality we find that

$$\begin{aligned}&c_1\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^{\frac{2q}{2q-1}}_{L^q(\Omega)}\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^{\frac{2q-2}{2q-1}}_{L^{\frac{2}{3}}(\Omega)} \\ &= \left\{\delta\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^2_{L^q(\Omega)}\right\}^{\frac{q}{2q-1}} \cdot c_1\delta^{-\frac{q}{2q-1}}\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^{\frac{2q-2}{2q-1}}_{L^{\frac{2}{3}}(\Omega)} \\ &\leq \delta\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^2_{L^q(\Omega)} + c_1^{\frac{2q-1}{q-1}}\delta^{-\frac{q}{q-1}}\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{\frac{2}{3}}(\Omega)}.\end{aligned} \tag{3.4.5}$$

Here, once more by Young's inequality,

$$\delta\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^2_{L^q(\Omega)} = \delta\left\|\frac{p+1}{2}\varphi^{\frac{p-1}{2}}\psi^{\frac{1}{2}}\nabla\varphi + \frac{1}{2}\varphi^{\frac{p+1}{2}}\psi^{-\frac{1}{2}}\nabla\psi\right\|^2_{L^q(\Omega)}$$

$$
\begin{aligned}
&\le \delta \cdot \left\{ \frac{p+1}{2} \left\| \varphi^{\frac{p-1}{2}} \psi^{\frac{1}{2}} \nabla\varphi \right\|_{L^q(\Omega)} + \frac{1}{2} \left\| \varphi^{\frac{p+1}{2}} \psi^{-\frac{1}{2}} \nabla\psi \right\|_{L^q(\Omega)} \right\}^2 \\
&\le \frac{(p+1)^2\delta}{2} \left\| \varphi^{\frac{p-1}{2}} \psi^{\frac{1}{2}} \nabla\varphi \right\|_{L^q(\Omega)}^2 + \frac{\delta}{2} \left\| \varphi^{\frac{p+1}{2}} \psi^{-\frac{1}{2}} \nabla\psi \right\|_{L^q(\Omega)}^2, \qquad (3.4.6)
\end{aligned}
$$

and observing that $q < 2$ we may reply on the Hölder inequality to estimate

$$
\begin{aligned}
\frac{(p+1)^2\delta}{2} \left\| \varphi^{\frac{p-1}{2}} \psi^{\frac{1}{2}} \nabla\varphi \right\|_{L^q(\Omega)}^2 &\le \frac{(p+1)^2\delta}{2} \cdot |\Omega|^{\frac{2-q}{q}} \left\| \varphi^{\frac{p-1}{2}} \psi^{\frac{1}{2}} \nabla\varphi \right\|_{L^2(\Omega)}^2 \\
&\le \frac{\eta}{2} \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 \qquad (3.4.7)
\end{aligned}
$$

according to the first restriction on δ contained in (3.4.4). Apart from that, again by means of the Hölder inequality we see that

$$
\begin{aligned}
\frac{\delta}{2} \left\| \varphi^{\frac{p+1}{2}} \psi^{-\frac{1}{2}} \nabla\psi \right\|_{L^q(\Omega)}^2 &= \frac{\delta}{2} \cdot \left\{ \int_\Omega \varphi^{\frac{(p+1)q}{2}} \psi^{-\frac{q}{2}} |\nabla\psi|^q \right\}^{\frac{2}{q}} \\
&= \frac{\delta}{2} \cdot \left\{ \int_\Omega \left(\frac{|\nabla\psi|^6}{\psi^5} \right)^{\frac{q}{6}} \cdot \varphi^{\frac{(p+1)q}{2}} \psi^{\frac{q}{3}} \right\}^{\frac{2}{q}} \\
&\le \frac{\delta}{2} \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^6}{\psi^5} \right\}^{\frac{1}{3}} \cdot \left\{ \int_\Omega \varphi^{\frac{3(p+1)q}{6-q}} \psi^{\frac{2q}{6-q}} \right\}^{\frac{6-q}{3q}}, \qquad (3.4.8)
\end{aligned}
$$

and that here

$$
\begin{aligned}
\left\{ \int_\Omega \varphi^{\frac{3(p+1)q}{6-q}} \psi^{\frac{2q}{6-q}} \right\}^{\frac{6-q}{3q}} &= \left\{ \int_\Omega (\varphi^{p+1}\psi)^{\frac{2q}{6-q}} \cdot \psi^{\frac{(p+1)q}{6-q}} \right\}^{\frac{6-q}{3q}} \\
&\le \left\{ \int_\Omega \varphi^{p+1}\psi \right\}^{\frac{2}{3}} \cdot \left\{ \int_\Omega \varphi^{\frac{(p+1)q}{6-3q}} \right\}^{\frac{2-q}{q}}. \qquad (3.4.9)
\end{aligned}
$$

Now the definition (3.4.2) applies in its full strength so as to assert that since $\frac{d}{d\xi}\frac{6\xi}{5\xi+2} \ge 0$ for all $\xi > 0$, the inequality $p \ge p_*$ ensures that $q \le \frac{6p}{5p+2}$ and hence $\frac{(p+1)q}{6-3q} = \frac{p+1}{\frac{6}{q}-3} \le \frac{p+1}{\frac{5p+2}{p}-3} = \frac{p}{2}$, so that a final application of the Hölder inequality shows that

$$
\left\{ \int_\Omega \varphi^{\frac{(p+1)q}{6-3q}} \right\}^{\frac{2-q}{q}} \le |\Omega|^{\frac{6p-5pq-2q}{3pq}} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{3p}} \le c_2 \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{3p}} \qquad (3.4.10)
$$

with c_2 as in (3.4.4), because clearly $0 \le \frac{6p-5pq-2q}{3pq} \le \frac{2}{q}$.
From (3.4.8)-(3.4.10) we now obtain, employing Young's inequality once again, that

$$
\begin{aligned}
\frac{\delta}{2} \left\| \varphi^{\frac{p+1}{2}} \psi^{-\frac{1}{2}} \nabla\psi \right\|_{L^q(\Omega)}^2 &\le \frac{c_2\delta}{2} \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^6}{\psi^5} \right\}^{\frac{1}{3}} \cdot \left\{ \int_\Omega \varphi^{p+1}\psi \right\}^{\frac{2}{3}} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{3p}} \\
&= \left\{ \frac{1}{2} \int_\Omega \varphi^{p+1}\psi \right\}^{\frac{2}{3}} \cdot \frac{c_2\delta}{2^{\frac{1}{3}}} \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^6}{\psi^5} \right\}^{\frac{1}{3}} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{3p}} \\
&\le \frac{1}{2} \int_\Omega \varphi^{p+1}\psi + \frac{c_2^3\delta^3}{2} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^6}{\psi^5},
\end{aligned}
$$

whence collecting (3.4.5)-(3.4.7) and (3.4.3), we conclude from (3.4.4) that

$$\begin{aligned}\int_\Omega \varphi^{p+1}\psi &\le \frac{\eta}{2}\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \frac{1}{2}\int_\Omega \varphi^{p+1}\psi + \frac{\eta}{2}\cdot\left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}}\cdot\int_\Omega \frac{|\nabla\psi|^6}{\psi^5}\\ &\quad + \left(c_1^{\frac{2q-1}{q-1}}\delta^{-\frac{q}{q-1}} + c_1\right)\cdot\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right\|^2_{L^{\frac{2}{3}}(\Omega)}.\end{aligned}$$

As

$$\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right\|^2_{L^{\frac{2}{3}}(\Omega)} = \left\{\int_\Omega \varphi^{\frac{p+1}{3}}\psi^{\frac{1}{3}}\right\}^3 = \left\{\int_\Omega (\varphi\psi)^{\frac{1}{3}}\cdot\varphi^{\frac{p}{3}}\right\}^3 \le \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^2\cdot\int_\Omega \varphi\psi,$$

this entails that

$$\begin{aligned}\int_\Omega \varphi^{p+1}\psi &\le \eta\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta\cdot\left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}}\cdot\int_\Omega \frac{|\nabla\psi|^6}{\psi^5}\\ &\quad +2\left(c_1^{\frac{2q-1}{q-1}}\delta^{-\frac{q}{q-1}} + c_1\right)\cdot\left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^2\cdot\int_\Omega \varphi\psi,\end{aligned}$$

and thereby establishes (3.4.1) with

$$\kappa \equiv \kappa(p_*) := \frac{q}{q-1} \quad \text{and} \quad C(p_*) := 2c_1^{\frac{2q-1}{q-1}}\cdot\max\left\{\left(4|\Omega|^{\frac{2-q}{q}}\right)^{\frac{q}{q-1}}, \ c_2^{\frac{q}{q-1}}\right\} + 2c_1,$$

because the inequalities $p \ge 1$ and $\eta \le 1$ warrant that, by (3.4.4),

$$\begin{aligned}2c_1^{\frac{2q-1}{q-1}}\delta^{-\frac{q}{q-1}} &= 2c_1^{\frac{2q-1}{q-1}}\cdot\max\left\{\left(\frac{(p+1)^2|\Omega|^{\frac{2-q}{q}}}{\eta}\right)^{\frac{q}{q-1}}, \ \left(\frac{c_2}{\eta^{\frac{1}{3}}}\right)^{\frac{q}{q-1}}\right\}\\ &\le 2c_1^{\frac{2q-1}{q-1}}\cdot\max\left\{\left(\frac{(2p)^2|\Omega|^{\frac{2-q}{q}}}{\eta}\right)^{\frac{q}{q-1}}, \ \left(\frac{c_2p^2}{\eta}\right)^{\frac{q}{q-1}}\right\}\\ &\le 2c_1^{\frac{2q-1}{q-1}}\eta^{-\frac{q}{q-1}}p^{\frac{2q}{q-1}}\cdot\max\left\{\left(4|\Omega|^{\frac{2-q}{q}}\right)^{\frac{q}{q-1}}, \ c_2^{\frac{q}{q-1}}\right\},\end{aligned}$$

and that $2c_1 \le 2c_1\eta^{-\frac{q}{q-1}}p^{\frac{2q}{q-1}}$. □

Another auxiliary tool is the following lemma, which can be found in [88, Lemma 6.3].

Lemma 3.4.2. *Let $a \ge 1, b \ge 1, d \ge 0$ and $(M_k)_{k\in\mathbb{N}} \subset [1,\infty)$ be such that*

$$M_k \le a^k M_{k-1}^{2+d\cdot 2^{-k}} + b^{2^k} \qquad \textit{for all } k \ge 1.$$

Then

$$\liminf_{k\to\infty} M_k^{\frac{1}{2^k}} \le \left(2\sqrt{2}a^3 b^{1+\frac{d}{2}} M_0\right)^{e^{\frac{d}{2}}}.$$

Based on Lemma 3.3.9, the $W^{1,\infty}$ bounds for v_ε can be established, which in turn play an important role in proving the L^∞-boundedness of u_ε.

Lemma 3.4.3. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*;*
(iii) $m \ge 3$*,* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*, and* $u_0 > 0$ *in* $\overline{\Omega}$*.*
Then for any choice of $\varepsilon \in (0,1)$*, there exists* $C > 0$ *such that*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \textit{for all } t \in (0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{3.4.11}$$

Proof. The proof is identical to that of Lemma 2.4.1 in Chapter 2. □

We are now able to establish the L^∞ bounds for u_ε by using the Moser iteration technique.

Lemma 3.4.4. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*;*
(iii) $m \ge 3$*,* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*, and* $u_0 > 0$ *in* $\overline{\Omega}$*.*
Then for any choice of $\varepsilon \in (0,1)$*, there exists* $C > 0$ *such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0,T_{max,\varepsilon}).$$

Proof. Take $p_0 = m$, and define

$$p_k := 2p_{k-1} + 2 - m, \qquad k \in \{1,2,3,...\}. \tag{3.4.12}$$

Then the sequence $(p_k)_{k\in\mathbb{N}}$ is increasing and satisfies

$$2^k \le p_k \le c_1 \cdot 2^k \qquad \text{for all } \ k \ge 1 \tag{3.4.13}$$

with $c_1 := 2 + m$. Setting

$$M_{k,\varepsilon}(T) := 1 + \sup_{t\in(0,T)} \int_\Omega u_\varepsilon^{p_k}(\cdot,t), \qquad T \in (0,T_{max,\varepsilon}),\ k \in \mathbb{N} \text{ and } \varepsilon \in (0,1), \tag{3.4.14}$$

we know from (3.3.42) that each $M_{k,\varepsilon}(T)$ is finite, and there exists $c_2 > 0$ independent of T satisfying

$$M_{0,\varepsilon}(T) \le c_2 \qquad \text{for all } \ T \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{3.4.15}$$

Now we try to estimate $M_{k,\varepsilon}(T)$ for $T \in (0,T_{max,\varepsilon})$, $k \ge 1$ and $\varepsilon \in (0,1)$. According to Lemma 3.4.3, there exists $c_3 > 0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \qquad \text{for all } \ t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Moreover, by recalling (3.1.4) and (3.1.5), we see the existence of $c_4 > 0$ fulfilling

$$S^2(u_\varepsilon) \le c_4 u_\varepsilon^2 \left(u_\varepsilon^{2\alpha-2} + 1\right) \qquad \text{for all } \ \varepsilon \in (0,1).$$

Therefore, using Young's inequality along with the facts that $1 < p_k+2\alpha-m-1 < p_k+m-1$ and $1 < p_k - m + 1 \le p_k \le p_k + m - 1$ guaranteed by our restrictions $m - 1 < \alpha < m$ and $m \ge 1$, we can infer from (3.2.5) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} \le\ & -\frac{p_k(p_k-1)}{2}\int_\Omega u_\varepsilon^{p_k+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p_k(p_k-1)}{2}\int_\Omega u_\varepsilon^{p_k-m-1}S^2(u_\varepsilon)v_\varepsilon|\nabla v_\varepsilon|^2 \\
& +p_k\ell\int_\Omega u_\varepsilon^{p_k}v_\varepsilon \\
\le\ & -\frac{p_k^2}{4}\int_\Omega u_\varepsilon^{p_k+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + c_3^2c_4p_k^2\left\{\int_\Omega u_\varepsilon^{p_k-m+1}v_\varepsilon + \int_\Omega u_\varepsilon^{p_k+2\alpha-m-1}v_\varepsilon\right\} \\
& +p_k\ell\int_\Omega u_\varepsilon^{p_k}v_\varepsilon \\
\le\ & -\frac{p_k^2}{4}\int_\Omega u_\varepsilon^{p_k+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + (2c_3^2c_4+\ell)p_k^2\int_\Omega u_\varepsilon^{p_k+m-1}v_\varepsilon \\
& +(2c_3^2c_4+\ell)p_k^2\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all} \quad t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{3.4.16}$$

Since $p_k \ge m+2$ for all $k \ge 1$, we have $p_k + m - 1 \ge 3$, which allows us to apply Lemma 3.4.1 with $p_* := 3$ to infer the existence of $\kappa > 0$ and $c_5 > 0$ satisfying

$$\begin{aligned}
& (2c_3^2c_4+\ell)p_k^2\int_\Omega u_\varepsilon^{p_k+m-1}v_\varepsilon \\
& \quad \le \frac{p_k^2}{4}\int_\Omega u_\varepsilon^{p_k+m-3}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p_k^2}{4}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k+m-2}{2}}\right\}^{\frac{2(p_k+m-1)}{p_k+m-2}}\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \\
& \qquad +4^\kappa(2c_3^2c_4+\ell)^{\kappa+1}c_5p_k^2(p_k+m-2)^{2\kappa}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k+m-2}{2}}\right\}^2\cdot\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.4.17}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0, 1)$. Combining (3.4.16) and (3.4.17), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} \le\ & \frac{p_k^2}{4}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k+m-2}{2}}\right\}^{\frac{2(p_k+m-1)}{p_k+m-2}}\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_6p_k^2\int_\Omega u_\varepsilon v_\varepsilon \\
& +c_7p_k^2(p_k+m-2)^{2\kappa}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k+m-2}{2}}\right\}^2\cdot\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.4.18}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0, 1)$ with $c_6 := 2c_3^2c_4 + \ell$ and $c_7 := 4^\kappa c_6^{\kappa+1}c_5$. From (3.4.12) and (3.4.13), it follows that

$$2^k \le p_k + m - 2 = 2p_{k-1} \le c_1 \cdot 2^k \qquad \text{for all} \quad k \ge 1,$$

which together with (3.4.13), (3.4.14) and (3.4.18) implies that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} \le\ & \frac{(2^kc_1)^2}{4}\cdot\left\{\int_\Omega u_\varepsilon^{p_{k-1}}\right\}^{2+\frac{2}{p_k+m-2}}\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \\
& +c_7(2^kc_1)^{2\kappa+2}\cdot\left\{\int_\Omega u_\varepsilon^{p_{k-1}}\right\}^2\cdot\int_\Omega u_\varepsilon v_\varepsilon + c_6(2^kc_1)^2\int_\Omega u_\varepsilon v_\varepsilon \\
\le\ & c_1^2 4^{k-1}M_{k-1,\varepsilon}^{2+2\cdot2^{-k}}(T)\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_7c_1^{2\kappa+2}4^{(\kappa+1)k}M_{k-1,\varepsilon}^2(T)\cdot\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&+c_6 c_1^2 4^k \int_\Omega u_\varepsilon v_\varepsilon \\
\leq\ & c_8 4^{(\kappa+1)k} M_{k-1,\varepsilon}^{2+2\cdot 2^{-k}}(T) \cdot \left\{ \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \int_\Omega u_\varepsilon v_\varepsilon \right\}
\end{aligned}
$$

for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $c_8 := c_1^2 + c_7 c_2^{2\kappa+2} + c_6 c_1^2$. Integrating this in time, we obtain that for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$
\int_\Omega u_\varepsilon^{p_k} \leq c_8 c_9 4^{(\kappa+1)k} M_{k-1,\varepsilon}^{2+2\cdot 2^{-k}}(T) + \int_\Omega (u_0+1)^{p_k}, \tag{3.4.19}
$$

where

$$
c_9 := \sup_{\varepsilon \in (0,1)} \left\{ \int_0^{T_{max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \right\} < \infty
$$

warranted by (3.2.7) and Lemmas 3.3.5-3.3.7. If we write

$$
a := (c_8 c_9 + 1) \cdot 4^{\kappa+1} \quad \text{and} \quad b := 1 + (|\Omega|+1) \|u_0+1\|_{L^\infty(\Omega)}^{c_1},
$$

then it is easy to verify that

$$
\|u_0+1\|_{L^\infty(\Omega)}^{p_k} \cdot |\Omega| + 1 \leq 1 + (|\Omega|+1)^{2^k} \|u_0+1\|_{L^\infty(\Omega)}^{c_1\cdot 2^k} \leq b^{2^k}
$$

and

$$
c_8 c_9 4^{(\kappa+1)k} \leq (c_8 c_9 + 1)^k 4^{(\kappa+1)k} = a^k.
$$

Consequently, we can further conclude from (3.4.19) and (3.4.14) that

$$
\begin{aligned}
M_{k,\varepsilon}(T) &\leq c_8 c_9 4^{(\kappa+1)k} M_{k-1,\varepsilon}^{2+2\cdot 2^{-k}}(T) + \|u_0+1\|_{L^\infty(\Omega)}^{p_k} \cdot |\Omega| + 1 \\
&\leq a^k M_{k-1,\varepsilon}^{2+2\cdot 2^{-k}}(T) + b^{2^k} \qquad \text{for all} \ \ T \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).
\end{aligned} \tag{3.4.20}
$$

Since $k \geq 1$ is arbitrary here, together with (3.4.13) and (3.4.15), we may use Lemma 3.4.2 to claim that for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \liminf_{k\to\infty} \left\{ \int_\Omega u_\varepsilon^{p_k}(\cdot,t) \right\}^{\frac{1}{p_k}} \leq \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{p_k}}(T) \\
&\leq \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{2^k}}(T) \leq (2\sqrt{2} a^3 b^2 c_3)^e.
\end{aligned}
$$

This clearly proves the lemma. □

3.5 Global solvability. Proof of Theorem 3.1.1

We can now make sure that all approximate solutions in fact exist globally.

Lemma 3.5.1. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*;*
(iii) $m \ge 3$*,* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*, and* $u_0 > 0$ *in* $\overline{\Omega}$*.*
Then $T_{max,\varepsilon} = \infty$ *for all* $\varepsilon \in (0,1)$*.*

Proof. This is an immediate result of Lemma 3.4.4 in conjunction with Lemma 3.2.1. □

We note that the properties established so far are already sufficient to complete the proof of the global existence of classical solutions in the case $m \ge 3$. In the following, we therefore restrict our attention to the case $1 \le m < 3$. Firstly, based on the L^∞-boundedness of u_ε, we can derive a time-dependent lower bound for v_ε.

Lemma 3.5.2. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*.*
Then for any $T > 0$*, there exist* $c > 0$ *and* $C > 0$ *such that*

$$v_\varepsilon(x,t) \ge Ce^{-ct} \qquad \textit{for all } x \in \Omega,\ t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{3.5.1}$$

Proof. From Lemma 3.4.4, we see the existence of a positive constant c_1 such that

$$u_\varepsilon(x,t) \le c_1 \qquad \text{for all } \ x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1),$$

which, by the second equation in (3.2.1), yields that for all $\varepsilon \in (0,1)$,

$$v_{\varepsilon t} \ge \Delta v_\varepsilon - c_1 v_\varepsilon \qquad \text{in } \Omega \times (0,\infty).$$

Then by the comparison principle, we have

$$v_\varepsilon(x,t) \ge c_2 e^{-c_1 t} \qquad \text{for all } \ x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1)$$

with $c_2 = \inf_\Omega v_0 > 0$ due to the strict positivity of v_0 asserted by (3.1.6). □

We next show that u_ε and v_ε enjoy higher regularities.

Lemma 3.5.3. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*.*
Then for any $T > 0$*, there exist* $\theta = \theta(T) \in (0,1)$ *and* $C(T) > 0$ *such that*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \qquad \textit{for all } \varepsilon \in (0,1) \tag{3.5.2}$$

and

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.5.3}$$

Moreover, for any $\tau > 0$ *and any* $T' > \tau$*, there exist* $\theta' = \theta'(\tau,T') \in (0,1)$ *and* $C'(\tau,T') > 0$ *such that*

$$\|v_\varepsilon\|_{C^{2+\theta',1+\frac{\theta'}{2}}(\overline{\Omega}\times[\tau,T'])} \le C'(\tau,T') \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.5.4}$$

Proof. Recalling Lemma 3.4.4, we can conclude from (3.1.4) and (3.1.5) that there exists $c_1 > 0$ such that

$$S^2(u_\varepsilon) \le c_1 u_\varepsilon^{m-1} \qquad \text{for all } \ \varepsilon \in (0,1).$$

Moreover, according to Lemma 3.5.2, there exists $c_2 = c_2(T) > 0$ such that

$$v_\varepsilon \ge c_2 \qquad \text{for all } \ \varepsilon \in (0,1).$$

Therefore, rewriting the first equation in (3.2.1) in the form

$$u_{\varepsilon t} = \nabla \cdot A_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) + B_\varepsilon(x,t,u_\varepsilon), \qquad x \in \Omega,\ t > 0$$

with

$$\begin{aligned} &A_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) := v_\varepsilon(x,t)u_\varepsilon^{m-1}(x,t)\nabla u_\varepsilon(x,t) - S(u_\varepsilon(x,t))v_\varepsilon(x,t)\nabla v_\varepsilon(x,t) \qquad \text{and} \\ &B_\varepsilon(x,t) := \ell u_\varepsilon(x,t)v_\varepsilon(x,t), \qquad\qquad (x,t) \in \Omega \times (0,\infty), \end{aligned}$$

in light of (3.2.6), Lemmas 3.4.3 and 3.4.4, we see from Young's inequality that with $c_3 > 0$,

$$\begin{aligned} A_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon &= v_\varepsilon u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - S(u_\varepsilon)v_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon \\ &\ge \frac{1}{2}v_\varepsilon u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - \frac{S^2(u_\varepsilon)}{u_\varepsilon^{m-1}}v_\varepsilon|\nabla v_\varepsilon|^2 \\ &\ge \frac{c_2}{2}u_\varepsilon^{m-1}|\nabla u_\varepsilon|^2 - c_3 \qquad \text{for all } \ (x,t)\in\Omega\times(0,T) \end{aligned}$$

as well as

$$\begin{aligned} &|A_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)| \le c_3 u_\varepsilon^{m-1}|\nabla u_\varepsilon| + c_3 u_\varepsilon^{\frac{m-1}{2}}, \qquad \text{and} \\ &|B_\varepsilon(x,t)| \le c_3 \qquad\qquad \text{for all } \ (x,t)\in\Omega\times(0,T). \end{aligned}$$

We may invoke the Hölder estimates in [62] to obtain (3.5.2). The property in (3.5.3) follows by a similar but simpler argument applied to the second equation in (3.1.2). Furthermore, the parabolic Schauder theory in [33] is applicable to obtain (3.5.4). □

The following result is about the $L^2(\Omega\times(0,T))$ bounds for $|\nabla u_\varepsilon^m|$.

Lemma 3.5.4. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m-1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m-1 < \alpha < \frac{m}{2}+1$ *and* $C_S > 0$*.*
Then for any $T > 0$*, there exists* $C(T) > 0$ *such that*

$$\int_0^T\int_\Omega |\nabla u_\varepsilon^m|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.5.5}$$

Proof. According to Lemma 3.5.2, for any $T > 0$, there exits $c_1 = c_1(T) > 0$ such that

$$v_\varepsilon(x,t) \ge c_1 \qquad \text{for all } \ x\in\Omega \text{ and } t\in(0,T).$$

By (3.3.43) and Lemma 3.5.1, with some $c_2 > 0$ we have

$$\int_0^\infty \int_\Omega u_\varepsilon^{2m-2} v_\varepsilon |\nabla u_\varepsilon|^2 < c_2 \qquad \text{for all } \ \varepsilon \in (0,1).$$

As a consequence, for all $\varepsilon \in (0,1)$ we obtain

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^m|^2 = m^2 \int_0^T \int_\Omega u_\varepsilon^{2m-2} |\nabla u_\varepsilon|^2 = m^2 \int_0^T \int_\Omega u_\varepsilon^{2m-2} v_\varepsilon |\nabla u_\varepsilon|^2 \cdot \frac{1}{v_\varepsilon} \le \frac{c_2}{c_1}.$$

This completes the proof. □

Now we are close to completing the global existence. With all the preparations in place, the following lemma can be proved in a manner similar to that of [88, Proposition 3.1]. For the reader's convenience, we provide the details of the proof.

Lemma 3.5.5. *Suppose that one of the following assumptions holds:*
(i) $1 \le m < 2$ *and* S *fulfills* (3.1.4) *with* $m - 1 < \alpha < m$ *and* $C_S > 0$*;*
(ii) $2 \le m < 3$ *and* S *fulfills* (3.1.5) *with* $m - 1 < \alpha < \frac{m}{2} + 1$ *and* $C_S > 0$.
Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *and functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)) \qquad \textit{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap W^{1,\infty}(\Omega \times (0,\infty)) \end{cases} \tag{3.5.6}$$

such that $\varepsilon_j \searrow 0$ *as* $j \to \infty$*, that*

$$\nabla u^m \in L^2_{loc}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^2\right), \tag{3.5.7}$$

and that as $\varepsilon = \varepsilon_j \searrow 0$ *we have*

$$u_\varepsilon \to u \quad \textit{in} \ \ C^0_{loc}\left(\overline{\Omega} \times [0,\infty)\right), \tag{3.5.8}$$

$$\nabla u_\varepsilon^m \rightharpoonup \nabla u^m \quad \textit{in} \ \ L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.5.9}$$

$$v_\varepsilon \to v \quad \textit{in} \ \ C^0_{loc}\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}_{loc}\left(\overline{\Omega} \times (0,\infty)\right) \qquad \textit{and} \tag{3.5.10}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \qquad \textit{in } L^\infty(\Omega \times (0,\infty)). \tag{3.5.11}$$

Moreover, $u \ge 0$ *and* $v > 0$ *a.e. in* $\Omega \times (0,\infty)$*, and* (u,v) *forms a global weak solution of* (3.1.2) *in the sense of Definition 1.4.1.*

Proof. According to Arzelà-Ascoli theorem, we infer from Lemma 3.5.3 that there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and nonnegative functions u and v satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$ and

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases}$$

such that for $\varepsilon = \varepsilon_j \searrow 0$ we have (3.5.8) and (3.5.10). From Lemma 3.4.3, we know that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{and} \quad (\nabla v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{are bounded in } L^\infty\left(\Omega \times (0,\infty)\right),$$

which yields $v \in W^{1,\infty}(\Omega \times (0,\infty))$, and after possibly passing to a further subsequence, (3.5.11) holds. Moreover, it follows from Lemma 3.5.4 that

$$(\nabla u_\varepsilon^m)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^2\right),$$

thus, taking another subsequence if necessary, we obtain (3.5.7) and (3.5.9). Additionally, Lemma 3.4.4 and (3.5.8) entail $u \in L^\infty(\Omega \times (0,\infty))$. Thus, the regularities in (3.5.6) are derived.

The identities in (1.4.3) (1.4.4) can be accomplished by some basic convergence analysis based on (3.5.8)-(3.5.11) and (3.2.1). Finally, the strict positivity of v results from a strong maximum principle. □

Proof of Theorem 3.1.1. In the case $1 \le m < 3$, it is an immediate consequence of Lemma 3.5.5. For $m \ge 3$, the regularity in (3.1.8) follows directly from (3.2.3), due to the construction of the initial data $u_{0\varepsilon}(x)$ in (3.2.2) within the perturbation system (3.2.1). Finally, the uniform boundedness in (3.1.9) is ensured by Lemmas 3.4.3 and 3.4.4. □

3.6 Harnack-type inequality for v_ε

Based on the L^∞-boundedness of the first component u, we can proceed to derive a Harnack-type inequality for the second component v, which is of essential importance to the subsequent outcome regarding asymptotic behavior.

Lemma 3.6.1. *Suppose that one of the hypotheses in Theorem 3.1.1 is satisfied. Then there exists $\lambda > 0$ such that*

$$v_\varepsilon(x,t) \ge \lambda \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \textit{for all } x \in \Omega,\ t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.6.1}$$

Proof. According to Lemma 3.4.4, there exists $c_1 > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \le c_1 \qquad \text{for all } \ \varepsilon \in (0,1),$$

which together with the second equation in (3.2.1) makes [27, Lemma 2.5] become applicable so as to show that with $\lambda_* > 0$ we have

$$v_\varepsilon(x,t) \ge \lambda_* \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } \ x \in \Omega,\ t > 1 \text{ and } \varepsilon \in (0,1). \tag{3.6.2}$$

For $0 < t \le 1$, by (3.5.1), there exists $c_2 > 0$ such that

$$v_\varepsilon(x,t) \ge c_2 \qquad \text{for all } \ x \in \Omega \text{ and } \varepsilon \in (0,1),$$

whereas (3.2.6) shows

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } \ \varepsilon \in (0,1).$$

It follows from the above two estimates that

$$v_\varepsilon(x,t) \ge \frac{c_2}{\|v_0\|_{L^\infty(\Omega)}} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } \ x \in \Omega,\ 0 < t \le 1 \text{ and } \varepsilon \in (0,1). \tag{3.6.3}$$

Combining (3.6.2) and (3.6.3), we establish (3.6.1) by taking $\lambda = \min\left\{\lambda_*, \frac{c_2}{\|v_0\|_{L^\infty(\Omega)}}\right\}$. □

3.7 Large time behavior. Proof of Theorem 3.1.4

With the above elliptic Harnack-type inequality at hand, we can immediately derive the following result, which is similar to that in [42, Lemma 5.2].

Lemma 3.7.1. *Suppose that one of the hypotheses in Theorem 3.1.1 is satisfied. Then with* λ *taken from Lemma 3.6.1, we have*

$$\int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt \le \frac{\int_\Omega v_0}{\lambda \int_\Omega u_0} \qquad \text{for all } \varepsilon \in (0,1). \tag{3.7.1}$$

Proof. Making use of (3.2.5)-(3.2.7) and (3.6.1), we have

$$\begin{aligned} \int_\Omega v_0 &\ge \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \ge \lambda \int_0^\infty \left\{ \int_\Omega u_\varepsilon \right\} \cdot \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} ds \\ &\ge \lambda \cdot \left\{ \int_\Omega u_0 \right\} \cdot \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} ds \qquad \text{for all } \varepsilon \in (0,1), \end{aligned}$$

which shows (3.7.1). □

The time integrability of $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ allows for a rescaling of the time variable in system (3.2.1). Moreover, the Harnack-type inequality established in Lemma 3.6.1 ensures that the resulting transformed system reduces to a non-degenerate parabolic problem of porous medium type. This leads to the following result.

Lemma 3.7.2. *Suppose that one of the hypotheses in Theorem 3.1.1 is satisfied. With* $(\varepsilon_j)_{j\in\mathbb{N}}$ *taken from Lemma 3.5.5, let*

$$L_\varepsilon := \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt, \qquad \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},$$

$$\tau := \phi_\varepsilon(t) := \frac{1}{L_\varepsilon}\int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds, \qquad t \ge 0$$

and

$$w_\varepsilon(x,\tau) := u_\varepsilon(x,\phi_\varepsilon^{-1}(\tau)), \qquad x \in \overline{\Omega},\ \tau \in [0,1).$$

Then we have

$$\begin{cases} w_{\varepsilon\tau} = \nabla\cdot\left(a_\varepsilon(x,\tau)w_\varepsilon^{m-1}\nabla w_\varepsilon\right) - \nabla\cdot\left(b_\varepsilon(x,\tau)S(w_\varepsilon)\right) + \ell a_\varepsilon(x,\tau)w_\varepsilon, & x\in\Omega,\ \tau\in(0,1), \\ \nabla w_\varepsilon\cdot\nu = 0, & x\in\partial\Omega,\ \tau\in(0,1), \\ w_\varepsilon(x,0) = u_0(x)+\varepsilon & x\in\Omega \end{cases}$$

with

$$a_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}} \quad \text{and} \quad b_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x,t)\nabla v_\varepsilon(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}}.$$

Additionally, there exists $C > 0$ such that

$$\frac{1}{C} \le a_\varepsilon(x,\tau) \le C \quad \textit{and} \quad |b_\varepsilon(x,\tau)| \le C \quad \textit{for all } (x,\tau) \in \Omega \times (0,1) \ \textit{ and } \ \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}, \tag{3.7.2}$$

and

$$L_\varepsilon \to L := \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} \qquad \textit{as} \quad \varepsilon = \varepsilon_j \searrow 0. \tag{3.7.3}$$

Proof. Let λ be taken from Lemma 3.6.1. According to (3.5.1), there exist $c_1 > 0$ and $c_2 > 0$ fulfilling

$$a_\varepsilon(x,\tau) > \lambda L_\varepsilon = \lambda \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \ge \lambda c_1 \int_0^\infty e^{-c_2 t} dt = \frac{\lambda c_1}{c_2}$$

for all $(x,\tau) \in \Omega \times (0,1)$ and $\varepsilon \in (0,1)$. The upper bounds for $a_\varepsilon(x,\tau)$ and $|b_\varepsilon(x,\tau)|$ in (3.7.2) follow from Lemma 3.4.3 and Lemma 3.7.1. On the other hand, by (3.7.1), we infer from Fatou's lemma and (3.5.10) that

$$\begin{aligned}\int_0^\infty \|v(\cdot,t)\|_{L^1(\Omega)} dt \quad &\le \quad |\Omega| \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \\ &\le \quad |\Omega| \cdot \lim_{\varepsilon\to 0} \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \le \frac{|\Omega| \int_\Omega v_0}{\lambda \int_\Omega u_0},\end{aligned} \tag{3.7.4}$$

and additionally, we know that $\|v(\cdot,t)\|_{L^1(\Omega)}$ is uniformly continuous with respect to $t > 0$. Therefore, an application of [4, Lemma 3.1] together with (3.7.4) shows that for any $\eta > 0$, there exists $t_0 > 0$ such that

$$\|v(\cdot,t_0)\|_{L^1(\Omega)} \le \frac{\eta\lambda \int_\Omega u_0}{6},$$

which in conjunction with (3.5.10) implies that there exists $\varepsilon_* \in (0,1)$ satisfying

$$\|v_\varepsilon(\cdot,t_0)\|_{L^1(\Omega)} \le \frac{\eta\lambda \int_\Omega u_0}{3} \qquad \text{for all } \ \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon < \varepsilon_*,$$

whence similar to the proof of Lemma 3.7.1, we have

$$\int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \le \frac{\int_\Omega v_\varepsilon(\cdot,t_0)}{\lambda \int_\Omega u_0} \le \frac{\eta}{3}. \tag{3.7.5}$$

Now we apply (3.5.10) and Fatou's lemma once more to see that

$$\int_{t_0}^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \le \lim_{\varepsilon\to 0} \int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \le \frac{\eta}{3}, \tag{3.7.6}$$

and moreover, we can pick $\varepsilon_{**} \in (0,1)$ fulfilling

$$\|v(\cdot,t) - v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\eta}{3t_0} \qquad \text{for all } \ t \in (0,t_0) \text{ and } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon < \varepsilon_{**}. \tag{3.7.7}$$

Thus, combining (3.7.5)-(3.7.7) gives

$$|L_\varepsilon - L| \quad = \quad \left| \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt - \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \right|$$

$$\begin{aligned}
&\le \int_0^{t_0} \|v_\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} dt + \int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt + \int_{t_0}^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \\
&\le \eta \qquad \text{for all } \ \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon < \min\{\varepsilon_*, \varepsilon_{**}\}.
\end{aligned}$$

This thereby proves (3.7.3). □

Proof of Theorem 3.1.4. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as in Lemma 3.5.5, and L_ε, ϕ_ε, a_ε, b_ε be as in Lemma 3.7.2. Then by (3.5.10) and (3.7.3), as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$\phi_\varepsilon(t) \to \phi(t) \qquad \text{for all } \ t > 0,$$

and for all $(x,\tau) \in \Omega \times (0,1)$,

$$a_\varepsilon(x,\tau) \to a(x,\tau) \ \text{ and } \ b_\varepsilon(x,\tau) \to b(x,\tau). \tag{3.7.8}$$

Therefore, it follows from (3.5.8) that

$$w_\varepsilon(x,\tau) \to u(x,\phi^{-1}(\tau)) \qquad \text{for all } \ (x,\tau) \in \Omega \times (0,1) \qquad \text{as } \varepsilon = \varepsilon_{\mathrm{j}} \searrow 0. \tag{3.7.9}$$

Owing to the boundedness information stated in (3.7.2), we may rely on the Hölder regularity in quasilinear degenerate parabolic equations ([62]) to conclude that there exist $\theta \in (0,1)$ and $C > 0$ such that

$$\|w_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}\left(\overline{\Omega}\times[0,1]\right)} \le C \qquad \text{for all } \ \varepsilon \in (0,1)$$

in a manner similar to that presented in Lemma 3.5.3. Then, by the Arzelà-Ascoli theorem,

$$w_\varepsilon(x,\tau) \to w(x,\tau) \quad \text{in} \quad C^0\left(\overline{\Omega} \times [0,1]\right) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

for some $w \in C^0\left(\overline{\Omega} \times [0,1]\right)$. Together with (3.7.9), this implies that

$$w(x,\tau) = u(x,\phi^{-1}(\tau)) \qquad \text{for all } \ (x,\tau) \in \Omega \times (0,1),$$

which along with the continuity of $w(\cdot,1)$ in $\overline{\Omega}$ yields

$$u(\cdot,t) \to u_\infty := w(\cdot,1) \ \text{ in } \ L^\infty(\Omega) \qquad \text{as } t \to \infty.$$

On the other hand, we can conclude from (3.2.6) and (3.5.10) that

$$\begin{aligned}
\|v(\cdot,t)\|_{L^\infty(\Omega)} &= \lim_{\varepsilon\to 0} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le \lim_{\varepsilon\to 0} \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} = \|v(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } \ t_0 \ge 0 \ \text{ and } \ t > t_0,
\end{aligned}$$

which together with (3.7.4) gives that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty.$$

Consequently, (3.1.10) is proved. Finally, (3.1.11) is a consequence of (1.4.3), and (3.1.12) can be derived from (3.7.8) and (3.7.2). □

4 Effects of sublinear signal consumption rates

4.1 Introduction

The search for nutrients is probably the most universal feature of life. This principle may lead to complex population distributions, such as the fractal-like structures observed in primitive species *Bacillus subtilis*, as discussed in previous chapters.

In fact, the rigorous nontrivial stabilization results, established in Chapters 2 and 3 as well as in the existing literature ([85], [81]), exhibit strong agreement with the experimental observations reported in [21], [48] and [56]. This provides compelling evidence that nonlinear cross-degenerate reaction-diffusion systems

$$\begin{cases} u_t = \nabla \cdot \left(u^{m-1} v \nabla u\right) - \nabla \cdot (S(u) v \nabla v) + \ell uv \\ v_t = \Delta v - uv, \end{cases} \tag{4.1.1}$$

are highly effective in generating the intricate bacterial patterning arising from movement-driven food acquisition in *Bacillus subtilis* under nutrient-limited environments.

To incorporate more subtle biological considerations, such as resource limitation and the resulting competition within the population, this chapter is devoted to the study of a nearby variant of (4.1.1) with a nonlinear consumption mechanism. In fact, we are going to consider a system of parabolic equations of the form

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - g(u)v, & x \in \Omega,\ t > 0, \\ (uv\nabla u - u^2 v \nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1.2}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^N$ with arbitrary dimension $N \geq 2$, in which the function g is intended to account for certain saturation effects in the process of nutrient uptake, and is assumed to satisfy

$$g \in C^1([0,\infty)) \quad \text{is such that} \quad g(0) = 0,\ g'(0) > 0 \quad \text{and} \quad g > 0 \quad \text{on} \quad (0,\infty) \tag{4.1.3}$$

and

$$g(s) \geq c_g s^\alpha \qquad \text{for all } s \geq 1 \quad \text{and} \quad g(s) \leq C_g s^\alpha \qquad \text{for all } s > 0. \tag{4.1.4}$$

A typical example satisfying (4.1.3) and (4.1.4) is

$$g(s) = \frac{s}{(1+s)^{1-\alpha}}, \qquad s \geq 0,$$

which reflects the classical consumption behavior at low population density, where the competition is not yet intense, while exhibiting a fairly natural nonlinear saturation effect at large population density due to limited resource availability. We also emphasize that the choice of $g(s) = s^\alpha$ for all $s \ge 0$, which has already been investigated in various contexts ([2], [67], [99]), falls within the framework of (4.1.3) and (4.1.4).

In the following, we center our investigation on the influence of the nutrient consumption rate on the existence and qualitative behavior of the solutions. We will first show that a sublinear rate $\alpha \in (0, \frac{2}{N})$ of the nutrients is always sufficient for the existence of a global weak solution, which additionally is also continuous and remains bounded.

Theorem 4.1.1. *Let $N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then for any choice of initial data (u_0, v_0) satisfying*

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0 \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}, \end{cases} \tag{4.1.5}$$

there exist functions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases} \tag{4.1.6}$$

such that $u \ge 0$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$, and that (u,v) forms a global weak solution of (4.1.2) in the sense of Definition 1.4.1. Moreover, there is $C(N) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C(N) \qquad \text{for all } t > 0. \tag{4.1.7}$$

In a second part, we establish the following result on large time behavior of the weak solution obtained above.

Theorem 4.1.2. *Let $N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume (4.1.5), and suppose that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then the solution (u,v) of (4.1.2) from Theorem 4.1.1 has the property that with some $u_\infty \in C^0(\overline{\Omega})$, we have*

$$u(\cdot,t) \to u_\infty \ \text{ in } \ L^\infty(\Omega) \ \text{ and } \ v(\cdot,t) \to 0 \ \text{ in } \ L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{4.1.8}$$

Here the limit function satisfies $u_\infty = w(\cdot,1)$ with $w \in C^0(\overline{\Omega} \times [0,1])$ being a weak solution of

$$\begin{cases} w_\tau = \nabla \cdot (a(x,\tau) w \nabla w) - \nabla \cdot (b(x,\tau) w^2), & x \in \Omega,\ \tau \in (0,1), \\ \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ \tau \in (0,1), \\ w(x,0) = u_0(x), & x \in \Omega \end{cases}$$

in the sense that $w^2 \in L^1_{loc}([0,1); W^{1,1}(\Omega))$ and that

$$-\int_0^1 \int_\Omega w \varphi_\tau - \int_\Omega w_0 \varphi(\cdot,0) = -\frac{1}{2} \int_0^1 \int_\Omega a(x,\tau) \nabla w^2 \cdot \nabla \varphi + \int_0^1 \int_\Omega b(x,\tau) w^2 \cdot \nabla \varphi \tag{4.1.9}$$

for all $\varphi \in C_0^\infty\left(\overline{\Omega} \times [0,\infty)\right)$*, where for* $(x,\tau) \in \Omega \times (0,1)$

$$a(x,\tau) := L \cdot \frac{v(x,\phi^{-1}(\tau))}{\|v(\cdot,\phi^{-1}(\tau))\|_{L^\infty(\Omega)}} \quad \text{and} \quad b(x,\tau) := L \cdot \frac{v(x,\phi^{-1}(\tau))\nabla v(x,\phi^{-1}(\tau))}{\|v(\cdot,\phi^{-1}(\tau))\|_{L^\infty(\Omega)}},$$

with

$$L := \int_0^\infty \|v(\cdot,s)\|_{L^\infty(\Omega)} ds \quad \text{and} \quad \phi(t) := \frac{1}{L} \cdot \int_0^t \|v(\cdot,s)\|_{L^\infty(\Omega)} ds, \quad t \geq 0 \text{ and } \tau = \phi(t)$$

are such that there exists $C(N) > 0$ *satisfying*

$$\frac{1}{C(N)} \leq a(x,\tau) \leq C(N) \quad \text{and} \quad |b(x,\tau)| \leq C(N) \qquad \text{for all } (x,\tau) \in \Omega \times (0,1). \tag{4.1.10}$$

Main ideas. In the previous works ([82], Chapters 2 and 3), a functional of the form

$$\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^q}{v^{q-1}}$$

with suitably chosen p and q is the key ingredient in their analysis. In addition, in order to estimate ill-signed terms appearing in a corresponding differential inequality, it was essential to exploit the control on $\int_0^\infty \int_\Omega uv$ provided by linear consumption rate featured in the second equation of the systems considered there.

In our framework, however, due to the constraints on g, only estimates of the form

$$\int_0^\infty \int_{\{u>1\}} u^\alpha v < \int_\Omega v_0$$

are obtainable from directly integrating the second equation in (4.1.2) (Lemma 4.2.2). This, in turn, necessitates additional arguments when dealing with combined quantities of the form $\int_\Omega u^\gamma v^\kappa$. Additionally, to drop the convexity restriction on the domain Ω required in previous works, we incorporate

$$\int_\Omega \frac{|\nabla v|^q}{v^r}$$

with $r \in [\frac{q}{2}, q-1)$ rather than the previously utilized $\int_\Omega \frac{|\nabla v|^q}{v^{q-1}}$. The boundary integral arising in the corresponding testing procedure (Lemma 4.2.5) can be estimated by the favorably signed term $\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2$, which is well-controlled in Lemma 4.2.3.

Finally, considering the energy-like functional

$$\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^q}{v^r} \tag{4.1.11}$$

with suitable q, in light of the functional inequality

$$\int_{\{u_\varepsilon>2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa \;\leq\; \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \eta \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}$$

$$+C(p,q,r,\alpha,N,\gamma,\kappa,\eta)\int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon$$

(Lemma 4.3.2), we establish the quintessential L^p bounds on u for any $p>1$ in Lemma 4.3.3. Using these estimates to establish a gradient estimate for v (Lemma 4.4.2), we can then employ an iteration procedure to obtain the L^∞-boundedness of u (Lemma 4.4.3) and conclude Theorem 4.1.1 in a straightforward manner.

For Theorem 4.1.2 we draw on an elliptic-type pointwise Harnack inequality (Lemma 4.6.1) to obtain a bound for v in $L^1((0,\infty);L^\infty(\Omega))$ (Lemma 4.6.2), which enables the reformulation of the first equation in (4.1.2) by an associated parabolic PDE (Lemma 4.6.3) and thus is accessible to our arguments regarding large time behavior.

4.2 Preliminaries

4.2.1 Regularized problems and basic a priori information

As in Chapters 2 and 3, for better control over the degeneracy, we are going to consider the following family of approximate problems

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(u_\varepsilon v_\varepsilon \nabla u_\varepsilon) - \nabla\cdot(u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon), & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - g(u_\varepsilon)v_\varepsilon, & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_0(x)+\varepsilon,\quad v_\varepsilon(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{4.2.1}$$

where $\varepsilon\in(0,1)$. Straightforward adaptation of the arguments used in Lemma 2.2.1 of Chapter 2 provides time-local solutions to the approximate problems.

Lemma 4.2.1. *Let $N\geq 2$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. Assume (4.1.5) and (4.1.3). Then for each $\varepsilon\in(0,1)$ there exist $T_{max,\varepsilon}\in(0,\infty]$ and a pair $(u_\varepsilon,v_\varepsilon)$ of functions*

$$\begin{cases} u_\varepsilon \in C^0\left(\overline{\Omega}\times(0,T_{max,\varepsilon}))\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T_{max,\varepsilon})\right)\\ v_\varepsilon\in\cap_{q>1}C^0\left([0,T_{max,\varepsilon});W^{1,q}(\Omega)\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T_{max,\varepsilon})\right) \end{cases} \tag{4.2.2}$$

such that $u_\varepsilon>0$ and $v_\varepsilon>0$ in $\overline{\Omega}\times(0,T_{max,\varepsilon})$, that $(u_\varepsilon,v_\varepsilon)$ solves (4.2.1) classically in $\Omega\times(0,T_{max,\varepsilon})$, and that

$$\textit{either}\quad T_{max,\varepsilon}=\infty\quad\textit{or}\quad \limsup_{t\to T_{max,\varepsilon}}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{4.2.3}$$

From now on, we assume that $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, that the initial data (u_0,v_0) saisfy (4.1.5), and that $(u_\varepsilon,v_\varepsilon)$ and $T_{max,\varepsilon}$ are given by Lemma 4.2.1.

Exploiting the structure of the equations and our assumptions on the function g, we can easily obtain a first collection of a priori estimates for u_ε and v_ε.

Lemma 4.2.2. *Let $N \geq 2$. Assume that (4.1.3) and (4.1.4) hold with $\alpha > 0$ and some constants $0 < c_g < C_g$. Then,*

$$\int_\Omega u_\varepsilon = \int_\Omega u_{0\varepsilon} \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{4.2.4}$$

as well as for all $t_0 \in [0, T_{max,\varepsilon})$,

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{4.2.5}$$

and

$$c_g \int_{t_0}^{T_{max,\varepsilon}} \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon \leq \int_{t_0}^{T_{max,\varepsilon}} \int_\Omega g(u_\varepsilon) v_\varepsilon \leq \int_\Omega v_\varepsilon(\cdot, t_0) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.2.6}$$

Proof. The mass conservation property in (4.2.4) follows directly from integrating the first equation of (4.2.1) due to the prescribed homogeneous boundary conditions. The estimate in (4.2.5) is a consequence of the maximum principle employed to the second equation of (4.2.1) and the non-negativity of $u_\varepsilon, v_\varepsilon$ and g. Finally, the inequalities in (4.2.6) can be obtained by integrating the second equation of (4.2.1) in time and space and drawing on the lower bound for g as prescribed in (4.1.4). □

Testing the second equation against v_ε^{s-1} with $s > 0$ provides a first rather mild estimate for a quantity containing the gradient of v_ε.

Lemma 4.2.3. *Let $N \geq 2$ and assume that (4.1.3) holds. For all $s > 0$, there exists $C_1(s, N) > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon^{s-1} |\nabla v_\varepsilon|^2 \leq C_1(s,N) \qquad \text{for all } \varepsilon \in (0,1).$$

In particular, for $q > 2$ and $r > 1$ satisfying $r < q - 1$, there is $C_2(q, r, N) > 0$ such that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon^{q-r-2} |\nabla v_\varepsilon|^2 \leq C_2(q,r,N) \qquad \text{for all } \varepsilon \in (0,1), \tag{4.2.7}$$

and there is $C_3(N) > 0$ such that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \leq C_3(N) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.2.8}$$

Proof. Testing the second equation in (4.2.1) against v_ε^s yields

$$\frac{d}{dt} \int_\Omega v_\varepsilon^{s+1} \leq -s(s+1) \int_\Omega v_\varepsilon^{s-1} |\nabla v_\varepsilon|^2 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Integrating this in time results in

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon^{s-1} |\nabla v_\varepsilon|^2 \leq \frac{1}{s(s+1)} \int_\Omega v_0^{s+1} \leq \frac{|\Omega|}{s(s+1)} \|v_0\|_{L^\infty(\Omega)}^{s+1},$$

which together with (4.1.5) completes the proof upon choosing s appropriately. □

4.2.2 Testing procedures for $\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}$ and $\int_\Omega u_\varepsilon^p$

In this section, we present some preliminary results for our energy-like functional. We start by stating a version of [82, Lemma 3.4], which provides crucial estimates for the terms arising in the differential inequality associated with $\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}$.

Lemma 4.2.4. *Let $N \geq 2$, $q \geq 2$ and $1 < r \leq q-1$. Then every positive $\varphi \in C^2(\overline{\Omega})$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ satisfies*

$$\int_\Omega \frac{|\nabla\varphi|^{q+2}}{\varphi^{r+2}} \leq \left(\frac{q+\sqrt{N}}{q-r}\right)^2 \int_\Omega \frac{|\nabla\varphi|^{q-2}}{\varphi^{r-2}}|D^2\ln\varphi|^2 \tag{4.2.9}$$

and

$$\int_\Omega \frac{|\nabla\varphi|^{q-2}}{\varphi^{r}}|D^2\varphi|^2 \leq \left(\frac{q+\sqrt{N}}{q-r}+1\right)^2 \int_\Omega \frac{|\nabla\varphi|^{q-2}}{\varphi^{r-2}}|D^2\ln\varphi|^2. \tag{4.2.10}$$

With this lemma at hand, we can now derive the following differential inequality for $\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}$. Note that since we do not assume Ω to be convex, we have to treat the boundary integral in a more intricate way. Moreover, as we do not have control over the time-space interval of $u_\varepsilon^\alpha v_\varepsilon$ on the whole domain Ω, but only on the set where $u_\varepsilon > 1$, we will have to prepare the term containing mixed powers of u_ε and v_ε in a way suitable for subsequent treatment.

Lemma 4.2.5. *Let $N \geq 2$, $q > 2$ and $r \in \left[\frac{q}{2}, q-1\right]$. Assume that (4.1.3) and (4.1.4) hold with $\alpha > 0$ and some constants $0 < c_g < C_g$. Then there exists $C(q,N) > 0$ such that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r} + \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2\ln v_\varepsilon|^2 \\
&\quad\leq \ C(q,N)\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 + C(q,N)\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{q+2}{2}\alpha} v_\varepsilon^{q-r}
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. In light of the identities $\nabla v_\varepsilon \cdot \nabla\Delta v_\varepsilon = \frac{1}{2}\Delta|\nabla v_\varepsilon|^2 - |D^2 v_\varepsilon|^2$ and $\nabla|\nabla v_\varepsilon|^2 = 2D^2 v_\varepsilon \cdot \nabla v_\varepsilon$, a direct computation shows that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r} &= q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\nabla v_\varepsilon\cdot\nabla(\Delta v_\varepsilon - g(u_\varepsilon)v_\varepsilon) - r\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{r+1}}(\Delta v_\varepsilon - g(u_\varepsilon)v_\varepsilon) \\
&= \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\Delta|\nabla v_\varepsilon|^2 - q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\nabla v_\varepsilon\cdot\nabla(g(u_\varepsilon)v_\varepsilon) \\
&\quad -q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}|D^2 v_\varepsilon|^2 - r\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{r+1}}\Delta v_\varepsilon + r\int_\Omega g(u_\varepsilon)\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r} \\
&= -\frac{q(q-2)}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-4}}{v_\varepsilon^r}\left|\nabla|\nabla v_\varepsilon|^2\right|^2 + qr\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r+1}}\nabla v_\varepsilon\cdot\nabla|\nabla v_\varepsilon|^2 \\
&\quad -q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}|D^2 v_\varepsilon|^2 - r(r+1)\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} + \frac{q}{2}\int_{\partial\Omega} \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}
\end{aligned}$$

$$
\begin{aligned}
&+q(q-2)\int_\Omega g(u_\varepsilon)\frac{|\nabla v_\varepsilon|^{q-4}}{v_\varepsilon^{r-1}}\nabla v_\varepsilon\cdot(D^2v_\varepsilon\cdot\nabla v_\varepsilon)+q\int_\Omega g(u_\varepsilon)\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-1}}\Delta v_\varepsilon\\
&-(q-1)r\int_\Omega g(u_\varepsilon)\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\\
=:\;& I_1+\cdots+I_8 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned}
$$

Thanks to the fact that

$$
|D^2v_\varepsilon|^2=v_\varepsilon^2|D^2\ln v_\varepsilon|^2+\frac{1}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla|\nabla v_\varepsilon|^2-\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^2} \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),
$$

we can rewrite the first four integrals in the way that

$$
\begin{aligned}
I_1+I_2+I_3+I_4 \;=\;& -q\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2\ln v_\varepsilon|^2-\frac{q(q-2)}{4}\int_\Omega\frac{|\nabla v_\varepsilon|^{q-4}}{v_\varepsilon^r}\Big|\nabla|\nabla v_\varepsilon|^2\Big|^2\\
&+q(r-1)\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r+1}}\nabla v_\varepsilon\cdot\nabla|\nabla v_\varepsilon|^2-(r(r+1)-q)\int_\Omega\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}
\end{aligned}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, where Young's inequality implies that

$$
q(r-1)\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r+1}}\nabla v_\varepsilon\cdot\nabla|\nabla v_\varepsilon|^2\le\frac{q(q-2)}{4}\int_\Omega\frac{|\nabla v_\varepsilon|^{q-4}}{v_\varepsilon^r}\Big|\nabla|\nabla v_\varepsilon|^2\Big|^2+\frac{q(r-1)^2}{q-2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Since

$$
\frac{q(r-1)^2}{q-2}-(r(r+1)-q)=\frac{(2r-q)(r-(q-1))}{q-2}
$$

is nonpositive due to our restriction $\frac{q}{2}\le r\le q-1$, we have

$$
I_1+I_2+I_3+I_4\le -q\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2\ln v_\varepsilon|^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
$$

Similar to the arguments in [28, Proposition 3.2], we utilize the fact that $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}\le c_1|\nabla v_\varepsilon|^2$ holds with some $c_1>0$ and the embedding $W^{1,1}(\Omega)\hookrightarrow L^1(\partial\Omega)$ to see the existence of $c_2=c_2(q,N)>0$ such that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$
\begin{aligned}
I_5 \;\le\;& \frac{q}{2}c_1\int_{\partial\Omega}\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\le c_2\int_\Omega\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}+c_2\int_\Omega\left|\nabla\left(\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\right)\right|\\
\le\;& c_2q\int_\Omega\frac{|\nabla v_\varepsilon|^{q-1}}{v_\varepsilon^r}|D^2v_\varepsilon|+c_2r\int_\Omega\frac{|\nabla v_\varepsilon|^{q+1}}{v_\varepsilon^{r+1}}+c_2\int_\Omega\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r},
\end{aligned}
$$

herein Young's inequality further ensures that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$
\begin{aligned}
I_5 \;\le\;& \frac{q(q-r)^2}{32(q+\sqrt{N})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\Big|D^2v_\varepsilon\Big|^2+\frac{q(q-r)^2}{16(q+\sqrt{N})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}+c_3\int_\Omega\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\\
\le\;& \frac{q(q-r)^2}{32(q+\sqrt{N})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^r}\Big|D^2v_\varepsilon\Big|^2+\frac{q(q-r)^2}{8(q+\sqrt{N})^2}\int_\Omega\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}+c_4\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2
\end{aligned}
$$

with $c_3 = c_3(q,N) > 0$ and $c_4 = c_4(q,N) > 0$, so that, by Lemma 4.2.4 we obtain

$$I_5 \le \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 + c_4\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).$$

Similarly, we find from (4.1.4) and the pointwise inequality $|\Delta v_\varepsilon| \le \sqrt{N}|D^2 v_\varepsilon|$ that, by Young's inequality, there is $c_5(q,N) > 0$ such that

$$\begin{aligned} I_6 + I_7 &\le q(q-2+\sqrt{N})C_g\int_\Omega u_\varepsilon^\alpha \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-1}}|D^2 v_\varepsilon| \\ &\le \frac{q(q-r)^2}{8(2q+\sqrt{N}-r)^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r}}|D^2 v_\varepsilon|^2 + c_5\int_\Omega u_\varepsilon^{2\alpha}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} \end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Accordingly, drawing on Lemma 4.2.4 for the first term and splitting the second term, we conclude that

$$\begin{aligned} I_6 + I_7 &\le \frac{q}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 + c_5\int_{\{u_\varepsilon\le 2\}} u_\varepsilon^{2\alpha}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} + c_5\int_{\{u_\varepsilon> 2\}} u_\varepsilon^{2\alpha}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} \\ &\le \frac{q}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 + 4^\alpha c_5\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} + c_5\int_{\{u_\varepsilon> 2\}} u_\varepsilon^{2\alpha}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} \end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, so that two additional applications of Young's inequality and Lemma 4.2.4 entail the existence of $c_6 = c_6(q,N) > 0$ and $c_7 = c_7(q,N) > 0$ such that

$$\begin{aligned} I_6 + I_7 &\le \frac{q}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 + \frac{q(q-r)^2}{16(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} + c_6\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 \\ &\quad + \frac{q(q-r)^2}{16(q+\sqrt{N})^2}\int_{\{u_\varepsilon>2\}} \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} + c_7\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{q+2}{2}\alpha} v_\varepsilon^{q-r} \\ &\le \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 + c_6\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 + c_7\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{q+2}{2}\alpha} v_\varepsilon^{q-r} \end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. An amalgamation of the estimates above together with the non-positivity of I_8 concludes the proof. □

The differential inequality for $\int_\Omega u_\varepsilon^p$ is more straightforward. Nevertheless, we still have to prepare the ill-signed terms in a way that is compatible with the superlevel-set estimate from Lemma 4.2.2.

Lemma 4.2.6. *Let $N\ge 2$, and assume $p>1$. Then for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,*

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \\ &\qquad \le 2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + \frac{p-1}{2}\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2. \end{aligned}$$

Proof. Testing the first equation in (4.2.1) against u_ε^{p-1}, we obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p = -(p-1)\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + (p-1)\int_\Omega u_\varepsilon^p v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon$$

$$\begin{aligned}
&\leq -\frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2+\frac{p-1}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2\\
&= -\frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2+\frac{p-1}{2}\int_{\{u_\varepsilon\leq 2\}} u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2\\
&\quad +\frac{p-1}{2}\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2\\
&\leq -\frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2+2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2\\
&\quad +\frac{p-1}{2}\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2 \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

The proof is complete. □

4.3 Uniform L^p-boundedness of u_ε

To prepare estimates for combined quantities of the form $\int_\Omega u_\varepsilon^\gamma v_\varepsilon^\kappa$, we fix a smooth cutoff function $\xi\in C^\infty([0,\infty))$ with the properties

$$\begin{cases} \xi(s)\equiv 0, & s\in[0,1],\\ 0\leq\xi(s)\leq 1, & s\in(1,2),\\ \xi(s)\equiv 1, & s\in[2,\infty), \end{cases} \tag{4.3.1}$$

and derive the following estimate.

Lemma 4.3.1. *Let $N\geq 2$ and $s\in[2,\frac{2N}{N-2})$. Assume $\alpha>0$ and that ξ fulfills (4.3.1). Then for all $p>\max\{1,\alpha\}$, $\kappa>1$ and $\eta>0$, there exists $C(p,\alpha,N,s,\eta,\kappa)>0$ such that*

$$\begin{aligned}
\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|_{L^s(\Omega)}^2 &\leq \eta\|\psi\|_{L^\infty(\Omega)}^{\kappa-1}\cdot\int_\Omega\varphi^{p-1}\psi|\nabla\varphi|^2+\eta\int_\Omega\xi^2(\varphi)\varphi^{p+1}\psi^{\kappa-2}|\nabla\psi|^2\\
&\quad +C(p,\alpha,N,s,\eta,\kappa)\|\psi\|_{L^\infty(\Omega)}^{\kappa-1}\cdot\left\{\int_\Omega\varphi\right\}^{p+1-\alpha}\cdot\int_\Omega\xi^2(\varphi)\varphi^\alpha\psi
\end{aligned}\tag{4.3.2}$$

is valid for arbitrary nonnegative function $\varphi\in C^1(\overline{\Omega})$ and positive function $\psi\in C^1(\overline{\Omega})$.

Proof. Set $c_1:=\|\xi'\|_{L^\infty((0,\infty))}$, fix $\eta>0$ and introduce

$$\eta_1\equiv\eta_1(p,\eta,\kappa):=\min\left\{\frac{\eta}{(p+1)^2+8c_1^2},\frac{\eta}{\kappa^2}\right\}.$$

Since $p>\alpha$ implies $0<\frac{2}{p+2-\alpha}<1$ and we assume $2\leq s<\frac{2N}{N-2}$, we can employ the compact embedding $W^{1,2}(\Omega)\hookrightarrow L^s(\Omega)$ and the continuous embeddings $L^s(\Omega)\hookrightarrow L^1(\Omega)\hookrightarrow L^{\frac{2}{p+2-\alpha}}(\Omega)$ to conclude from Ehrling's lemma and the Hölder inequality that there exists $c_2=c_2(p,\alpha,N,s,\eta,\kappa)>0$ such that

$$\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|_{L^s(\Omega)}^2\leq\eta_1\left\|\nabla\left(\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right)\right\|_{L^2(\Omega)}^2+c_2\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|_{L^{\frac{2}{p+2-\alpha}}(\Omega)}^2. \tag{4.3.3}$$

Then direct computations utilizing $\xi \le 1$ show that

$$
\begin{aligned}
\eta_1 \left\| \nabla \left(\xi(\varphi) \varphi^{\frac{p+1}{2}} \psi^{\frac{\kappa}{2}} \right) \right\|_{L^2(\Omega)}^2 \\
\le\ & (p+1)^2 \eta_1 \int_\Omega \xi^2(\varphi) \varphi^{p-1} \psi^\kappa |\nabla\varphi|^2 + \kappa^2 \eta_1 \int_\Omega \xi^2(\varphi) \varphi^{p+1} \psi^{\kappa-2} |\nabla\psi|^2 \\
& + 2\eta_1 \int_\Omega \xi'^2(\varphi) \varphi^{p+1} \psi^\kappa |\nabla\varphi|^2 \\
\le\ & (p+1)^2 \eta_1 \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \int_\Omega \varphi^{p-1} \psi |\nabla\varphi|^2 + \kappa^2 \eta_1 \int_\Omega \xi^2(\varphi) \varphi^{p+1} \psi^{\kappa-2} |\nabla\psi|^2 \\
& + 8c_1^2 \eta_1 \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \int_{\{1 \le \varphi \le 2\}} \varphi^{p-1} \psi |\nabla\varphi|^2 \\
\le\ & \eta \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \int_\Omega \varphi^{p-1} \psi |\nabla\varphi|^2 + \eta \int_\Omega \xi^2(\varphi) \varphi^{p+1} \psi^{\kappa-2} |\nabla\psi|^2, \qquad (4.3.4)
\end{aligned}
$$

whereas an application of the Hölder inequality yields

$$
\begin{aligned}
c_2 \left\| \xi(\varphi) \varphi^{\frac{p+1}{2}} \psi^{\frac{\kappa}{2}} \right\|_{L^{\frac{2}{p+2-\alpha}}(\Omega)}^2 &= c_2 \left\{ \int_\Omega \varphi^{\frac{p+1-\alpha}{p+2-\alpha}} \cdot \xi^{\frac{2}{p+2-\alpha}}(\varphi) \varphi^{\frac{\alpha}{p+2-\alpha}} \psi^{\frac{\kappa}{p+2-\alpha}} \right\}^{p+2-\alpha} \\
&\le c_2 \left\{ \int_\Omega \varphi \right\}^{p+1-\alpha} \cdot \int_\Omega \xi^2(\varphi) \varphi^\alpha \psi^\kappa \\
&\le c_2 \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \left\{ \int_\Omega \varphi \right\}^{p+1-\alpha} \cdot \int_\Omega \xi^2(\varphi) \varphi^\alpha \psi. \qquad (4.3.5)
\end{aligned}
$$

Plugging (4.3.4) and (4.3.5) back into (4.3.3) completes the proof. □

Carefully exploiting the lemma above, we can now show the following functional inequality.

Lemma 4.3.2. *Let $N \ge 2$, $\alpha > 0$ and $p > \max\{1, \alpha\}$. Assume that ξ fulfills (4.3.1). Then for any $\gamma \in [\alpha, p+1+\frac{2}{N})$, $\kappa > 1$ and $\eta > 0$, there exists $C(p,q,\alpha,N,\gamma,\kappa,\eta) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,*

$$
\begin{aligned}
\int_{\{u_\varepsilon > 2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa \le\ & \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \eta \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \\
& + C(p,q,\alpha,N,\gamma,\kappa,\eta) \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon, \qquad (4.3.6)
\end{aligned}
$$

where $q > N(p+1)$ and $\max\{\frac{q}{2}, q-\kappa\} \le r \le q-1$.

Proof. According to Lemma 4.2.2, with $c_1 := \int_\Omega u_0 + |\Omega|$ and $c_2 := \|v_0\|_{L^\infty(\Omega)}$ we have

$$
\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \qquad (4.3.7)
$$

and

$$
\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \qquad (4.3.8)
$$

Now, distinguish the following two cases for γ. If $\gamma \in [p+1, p+1+\frac{2}{N})$, we set

$$
\vartheta \equiv \vartheta(p,\gamma) := \frac{1}{\gamma - p - 1} \quad \text{and} \quad \vartheta^* \equiv \vartheta^*(p,\gamma) := \frac{\vartheta}{\vartheta - 1} = \frac{1}{2 + p - \gamma}.
$$

It can easily be verified that then

$$\vartheta > \frac{N}{2} \quad \text{and} \quad 1 \le \vartheta^* < \frac{N}{N-2}.$$

Thus, using (4.3.1), (4.3.7) and the Hölder inequality we find that

$$\begin{aligned}
\int_{\{u_\varepsilon>2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa &\le \int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^\gamma v_\varepsilon^\kappa \\
&= \int_\Omega \Big(\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big)^2 u_\varepsilon^{\gamma-p-1} \\
&\le \Big\|\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big\|_{L^{2\vartheta^*}(\Omega)}^2 \cdot \Big\|u_\varepsilon{}^{\gamma-p-1}\Big\|_{L^\vartheta(\Omega)} \\
&\le (c_1^{\frac{1}{\vartheta}}+1) \Big\|\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big\|_{L^{2\vartheta^*}(\Omega)}^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

Here, for given $\eta > 0$ we may employ Lemma 4.3.1 (with $s := 2\theta^*$, $\varphi := u_\varepsilon$ and $\psi := v_\varepsilon$) to find $c_3 = c_3(p, \alpha, N, \gamma, \kappa, \eta) > 0$ satisfying

$$\begin{aligned}
\int_{\{u_\varepsilon>2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa &\le \frac{\eta}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{\eta}{2}\int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^{p+1} v_\varepsilon^{\kappa-2}|\nabla v_\varepsilon|^2 \\
&\quad + c_3\int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^\alpha v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \qquad (4.3.9)
\end{aligned}$$

To further estimate the second term on the right-hand side, we introduce

$$\chi \equiv \chi(p,q) := \frac{q}{2p+2} \quad \text{and} \quad \chi^* \equiv \chi^*(p,q) := \frac{\chi}{\chi-1} = \frac{q}{q-2p-2},$$

where our assumptions $q > N(p+1)$ and $r \ge q-\kappa$ ensure that

$$\chi > \frac{N}{2}, \quad 1 < \chi^* < \frac{N}{N-2} \quad \text{and} \quad \kappa + r - q \ge 0.$$

Drawing on (4.3.1), (4.3.7) and (4.3.8), we deduce from Young's inequality and the Hölder inequality that

$$\begin{aligned}
\int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^{p+1} v_\varepsilon^{\kappa-2}|\nabla v_\varepsilon|^2 &\le \int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\kappa+\frac{2}{q}(\kappa+r-q)} + \int_\Omega \xi^2(u_\varepsilon)\frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \\
&\le c_2^{\frac{2}{q}(\kappa+r-q)}\int_\Omega \Big(\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big)^2 u_\varepsilon^{\frac{2p+2}{q}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \\
&\le c_2^{\frac{2}{q}(\kappa+r-q)}\Big\|\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big\|_{L^{2\chi^*}(\Omega)}^2 \cdot \|u_\varepsilon\|_{L^1(\Omega)}^{\frac{1}{\chi}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \\
&\le c_2^{\frac{2}{q}(\kappa+r-q)} c_1^{\frac{1}{\chi}}\Big\|\xi(u_\varepsilon) u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{\kappa}{2}}\Big\|_{L^{2\chi^*}(\Omega)}^2 + \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0, 1)$. Herein, we may employ Lemma 4.3.1 once more (this time with $s := 2\chi^*$) to obtain $c_4 = c_4(p, q, \alpha, N, \kappa) > 0$ satisfying

$$\int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^{p+1} v_\varepsilon^{\kappa-2}|\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^{p+1} v_\varepsilon^{\kappa-2}|\nabla v_\varepsilon|^2$$

$$+c_4 \int_\Omega \xi^2(u_\varepsilon) u_\varepsilon^\alpha v_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Rearranging and plugging this back into (4.3.9) yields

$$\int_{\{u_\varepsilon > 2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa \le \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \eta \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} + (c_3 + c_4\eta) \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, i.e. (4.3.6) holds for $\gamma \in [p+1, p+1+\frac{2}{N})$.

In the case of $\gamma \in [\alpha, p+1)$, we employ Young's inequality and (4.3.8) to see that

$$\begin{aligned} \int_{\{u_\varepsilon > 2\}} u_\varepsilon^\gamma v_\varepsilon^\kappa &\le \int_{\{u_\varepsilon > 2\}} u_\varepsilon^{p+1} v_\varepsilon^\kappa + \int_{\{u_\varepsilon > 2\}} u_\varepsilon^\alpha v_\varepsilon^\kappa \\ &\le \int_{\{u_\varepsilon > 2\}} u_\varepsilon^{p+1} v_\varepsilon^\kappa + c_2^{\kappa-1} \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which in conjunction with the fact that (4.3.6) holds for $p+1$ completes the proof. □

The lemma above was the last jigsaw piece missing for the combination of the prepared differential inequalities. In particular, we note that choosing the parameter r strictly less than (rather than equal to) $q-1$, we may additionally draw on (4.2.7) from Lemma 4.2.3 to effectively control the unfavorable gradient term present in Lemma 4.2.5. This step is moreover essential for relaxing the convexity assumption on Ω. In fact, we are now able to obtain the following two essential bounds.

Lemma 4.3.3. *Let $N \ge 2$. Assume that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then for any $p > 1$, there exists $C(p,N) > 0$ such that*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p,N) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1) \tag{4.3.10}$$

and

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le C(p,N) \qquad \textit{for all } \varepsilon \in (0,1). \tag{4.3.11}$$

Furthermore, for any $q > 2N$ and $\max\{\frac{q}{2}, \frac{q^2-q-6}{q+2}\} \le r < q-1$, there exists $C'(q,N) > 0$ such that

$$\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^q}{v_\varepsilon^r(\cdot,t)} \le C'(q,N) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1) \tag{4.3.12}$$

and

$$\int_0^{T_{max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \le C'(q,N) \qquad \textit{for all } \varepsilon \in (0,1). \tag{4.3.13}$$

Proof. First, by Lemma 4.2.2, there exists $c_1 > 0$ such that

$$\int_0^{T_{max,\varepsilon}} \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon \le c_1 \qquad \text{for all } \varepsilon \in (0,1). \tag{4.3.14}$$

Then, since $\alpha \in (0, \frac{2}{N})$ entails $\frac{2(p+1+\frac{2}{N})}{\alpha} - 2 > N(p+1)$, we are able to pick q satisfying

$$q \in \left(N(p+1),\ \frac{2(p+1+\frac{2}{N})}{\alpha} - 2 \right),$$

which clearly warrants

$$\alpha < \frac{(p+1)(q+2)}{q} < p+1+\frac{2}{N} \tag{4.3.15}$$

and

$$\alpha < \frac{q+2}{2}\alpha < p+1+\frac{2}{N}. \tag{4.3.16}$$

Additionally, it is easy to verify from the assumption $r \geq \frac{q^2-q-6}{q+2}$ that

$$r \geq q - \left(1 + \frac{6+2r}{q}\right). \tag{4.3.17}$$

Accordingly, for q taken above and the given r, drawing on Lemma 4.2.3 we see the existence of $c_2 = c_2(q,N) > 0$ such that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon^{q-r-2} |\nabla v_\varepsilon|^2 \leq c_2 \qquad \text{for all } \ \varepsilon \in (0,1). \tag{4.3.18}$$

Moreover, invoking Young's inequality in the differential inequality provided by Lemma 4.2.6, we obtain from (4.2.9) that there is $c_3 = c_3(p,q,N) > 0$ such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\quad\leq\ 2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + \frac{p-1}{2}\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\quad\leq\ 2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + \frac{q(q-r)^2}{4(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} \\
&\qquad +c_3\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{1+\frac{6+2r}{q}} \\
&\quad\leq\ 2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 \\
&\qquad +c_3\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{1+\frac{6+2r}{q}} \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{4.3.19}$$

This in conjunction with Lemma 4.2.5 implies the existence of $c_4 = c_4(q,N) > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\right\} + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2\ln v_\varepsilon|^2 \\
&\quad\leq\ 2^p(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 + c_3\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{1+\frac{6+2r}{q}} \\
&\qquad +c_4\int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{q+2}{2}\alpha} v_\varepsilon^{q-r} + c_4\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2
\end{aligned} \tag{4.3.20}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Then, owing to the conditions imposed in (4.3.15)-(4.3.17) and the fact $r < q-1$, Lemma 4.3.2 can be applied to ensure that there exists $c_5 = c_5(p,q,N) > 0$ such that

$$\begin{aligned} c_3 \int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{1+\frac{6+2r}{q}} + c_4 \int_{\{u_\varepsilon>2\}} u_\varepsilon^{\frac{q+2}{2}\alpha} v_\varepsilon^{q-r} &\le \frac{p-1}{4}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{q(q-r)^2}{8(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{r+2}} + c_5 \int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon \\ &\le \frac{p-1}{4}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{q}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}} |D^2 \ln v_\varepsilon|^2 + c_5 \int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon \end{aligned} \tag{4.3.21}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, where the last inequality is due to (4.2.9). Inserting (4.3.21) into (4.3.20), we obtain that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r}\right\} + \frac{p-1}{4}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{q}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 \\ \le 2^p(p-1)\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_4 \int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 + c_5 \int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon. \end{aligned}$$

This together with (4.3.14) and (4.3.18), upon an integration in time, entails that

$$\begin{aligned} \frac{1}{p}\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^r} + \frac{p-1}{4}\int_0^t\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{q}{8}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{r-2}}|D^2 \ln v_\varepsilon|^2 \\ \le 2^p(p-1)\int_0^t\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + c_4 \int_0^t\int_\Omega v_\varepsilon^{q-r-2}|\nabla v_\varepsilon|^2 + c_5 \int_0^t\int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon \\ + \frac{1}{p}\int_\Omega u_{0\varepsilon}^p + \int_\Omega \frac{|\nabla v_0|^q}{v_0^r} \\ \le 2^p(p-1)c_2 + c_4 c_2 + c_5 c_1 + \frac{1}{p}\int_\Omega (u_0+1)^p + \int_\Omega \frac{|\nabla v_0|^q}{v_0^r} \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. This in conjunction with (4.1.5) gives (4.3.10) and (4.3.11), and also yields (4.3.12) and (4.3.13) in light of the selection of q and the arbitrariness of p. □

4.4 Uniform L^∞-boundedness of u_ε

The following functional inequality is similar to [88, Lemma 6.2] (see also Lemma 3.4.1 in Chapter 3) but accounts for the superlevel-set limitations.

Lemma 4.4.1. *Let $N \ge 2$, $\alpha > 0$ and $p_* > 2$. Then for any $p \ge p_*$, $\kappa > 1$ and $\eta > 0$, there exist $\delta = \delta(N, p_*) > 0$ and $C(N, p_*) > 0$ such that*

$$\begin{aligned} \int_{\{\varphi>2\}} \varphi^{p+1}\psi^\kappa \le \eta \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \\ + C(N,p_*)p^{2\delta}\eta^{-\delta}\kappa^{2\delta} \cdot \|\psi\|_{L^\infty(\Omega)}^{\kappa-1} \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1-\alpha)}{p}} \cdot \int_{\{\varphi>1\}} \varphi^\alpha \psi \end{aligned} \tag{4.4.1}$$

is valid for arbitrary positive functions $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$.

Proof. Taking ξ as in (4.3.1) and

$$\mu \equiv \mu(N,p_*) := \frac{2Np_*}{Np_*+p_*+1} > \frac{2N}{N+2}, \tag{4.4.2}$$

we can apply the Gagliardo-Nirenberg inequality to find $c_1 = c_1(N,p_*) > 0$ such that for any positive functions $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$,

$$\begin{aligned} \|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\|^2_{L^2(\Omega)} \le\ & c_1\|\nabla\left(\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right)\|^{2\theta}_{L^\mu(\Omega)}\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\|^{2(1-\theta)}_{L^{\frac12}(\Omega)} \\ & +c_1\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\|^2_{L^{\frac12}(\Omega)} \end{aligned}$$

with

$$\theta \equiv \theta(N,p_*) := \frac{2-\frac12}{2-\frac{1}{\mu}+\frac{1}{N}} = \frac{3Np_*}{3Np_*+p_*-1} \in (0,1).$$

For any given $\eta > 0$, write

$$c_2 := \max\{|\Omega|,\ 1\}, \quad c_3 := \|\xi'\|_{L^\infty((0,\infty))},$$

and introduce

$$\eta_1 \equiv \eta_1(\eta,p,N,\kappa) := \min\left\{\frac{\eta}{(2(p+1)^2+2^5c_3^2)c_2},\ \frac{\eta^{\frac{1}{2N}}}{\kappa^2 c_2}\right\}.$$

By Young's inequality, we obtain

$$\begin{aligned} &\int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa \\ &\quad= \left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|^2_{L^2(\Omega)} \\ &\quad\le \eta_1\left\|\nabla\left(\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right)\right\|^2_{L^\mu(\Omega)} + \left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|^2_{L^{\frac12}(\Omega)} \\ &\quad\le (p+1)^2\eta_1\left\|\xi(\varphi)\varphi^{\frac{p-1}{2}}\psi^{\frac{\kappa}{2}}\nabla\varphi\right\|^2_{L^\mu(\Omega)} + 4\eta_1\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\xi'(\varphi)\nabla\varphi\right\|^2_{L^\mu(\Omega)} \\ &\quad\quad+\frac{\kappa^2}{2}\eta_1\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa-2}{2}}\nabla\psi\right\|^2_{L^\mu(\Omega)} + \left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|^2_{L^{\frac12}(\Omega)}, \end{aligned} \tag{4.4.3}$$

where in view of the Hölder inequality, direct computations show that

$$\begin{aligned} &(p+1)^2\eta_1\left\|\xi(\varphi)\varphi^{\frac{p-1}{2}}\psi^{\frac{\kappa}{2}}\nabla\varphi\right\|^2_{L^\mu(\Omega)} + 4\eta_1\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\xi'(\varphi)\nabla\varphi\right\|^2_{L^\mu(\Omega)} \\ &\quad\le (p+1)^2\eta_1\cdot\left\{\int_\Omega \varphi^{\frac{(p-1)\mu}{2}}\psi^{\frac{\kappa\mu}{2}}|\nabla\varphi|^\mu\right\}^{\frac{2}{\mu}} + 2^4c_3^2\eta_1\cdot\left\{\int_{\{1\le\varphi\le 2\}}\varphi^{\frac{(p-1)\mu}{2}}\psi^{\frac{\kappa\mu}{2}}|\nabla\varphi|^\mu\right\}^{\frac{2}{\mu}} \\ &\quad\le \left((p+1)^2+2^4c_3^2\right)|\Omega|^{\frac{2-\mu}{\mu}}\eta_1\cdot\|\psi\|^{\kappa-1}_{L^\infty(\Omega)}\cdot\int_\Omega\varphi^{p-1}\psi|\nabla\varphi|^2 \\ &\quad\le \left((p+1)^2+2^4c_3^2\right)c_2\eta_1\cdot\|\psi\|^{\kappa-1}_{L^\infty(\Omega)}\cdot\int_\Omega\varphi^{p-1}\psi|\nabla\varphi|^2 \\ &\quad\le \frac{\eta}{2}\|\psi\|^{\kappa-1}_{L^\infty(\Omega)}\cdot\int_\Omega\varphi^{p-1}\psi|\nabla\varphi|^2 \end{aligned} \tag{4.4.4}$$

and

$$
\begin{aligned}
&\frac{\kappa^2}{2}\eta_1 \left\| \xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa-2}{2}}\nabla\psi \right\|^2_{L^\mu(\Omega)} \\
&= \frac{\kappa^2}{2}\eta_1 \cdot \left\{ \int_\Omega \xi^\mu(\varphi)\varphi^{\frac{(p+1)\mu}{2}}\psi^{\frac{(\kappa-2)\mu}{2}}|\nabla\psi|^\mu \right\}^{\frac{2}{\mu}} \\
&= \frac{\kappa^2}{2}\eta_1 \cdot \left\{ \int_\Omega \left(\xi^2(\varphi)\frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \right)^{\frac{\mu}{4N}} \cdot \xi^{\frac{(2N-1)\mu}{2N}}(\varphi)\varphi^{\frac{(p+1)\mu}{2}}\psi^{(\frac{\kappa}{2}-\frac{\kappa}{4N})\mu} \right\}^{\frac{2}{\mu}} \\
&\le \frac{\kappa^2}{2}\eta_1 \cdot \left\{ \int_\Omega \xi^2(\varphi)\frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \right\}^{\frac{1}{2N}} \cdot \left\{ \int_\Omega \xi^{\frac{2(2N-1)\mu}{4N-\mu}}(\varphi)\varphi^{\frac{2N(p+1)\mu}{4N-\mu}}\psi^{\frac{(2N-1)\kappa\mu}{4N-\mu}} \right\}^{\frac{4N-\mu}{2N\mu}} \\
&\le \frac{\kappa^2}{2}\eta_1 \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \right\}^{\frac{1}{2N}} \cdot \left\{ \int_\Omega (\xi^2(\varphi)\varphi^{p+1}\psi^\kappa)^{\frac{(2N-1)\mu}{4N-\mu}} \cdot \varphi^{\frac{\mu(p+1)}{4N-\mu}} \right\}^{\frac{4N-\mu}{2N\mu}} \\
&\le \frac{\kappa^2}{2}\eta_1 \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \right\}^{\frac{1}{2N}} \cdot \left\{ \int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa \right\}^{\frac{2N-1}{2N}} \cdot \left\{ \int_\Omega \varphi^{\frac{\mu(p+1)}{2N(2-\mu)}} \right\}^{\frac{2-\mu}{\mu}}.
\end{aligned}
$$

Since $p \ge p_*$, it follows from (4.4.2) that

$$
\mu \le \frac{2Np}{Np+p+1}.
$$

Thus

$$
\frac{\mu(p+1)}{2N(2-\mu)} \le \frac{\frac{2Np}{Np+p+1}(p+1)}{2N(2-\frac{2Np}{Np+p+1})} = \frac{p}{2},
$$

which allows us to use the Hölder inequality to deduce that

$$
\left\{ \int_\Omega \varphi^{\frac{\mu(p+1)}{2N(2-\mu)}} \right\}^{\frac{2-\mu}{\mu}} \le |\Omega|^{\frac{2-\mu}{\mu}-\frac{p+1}{Np}} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{p+1}{Np}} \le c_2 \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{p+1}{Np}}
$$

because of $0 \le \frac{2-\mu}{\mu} - \frac{p+1}{Np} < 1$. Then, we can proceed to use Young's inequality to see

$$
\begin{aligned}
&\frac{\kappa^2}{2}\eta_1 \left\| \xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa-2}{2}}\nabla\psi \right\|^2_{L^\mu(\Omega)} \\
&\le \frac{\kappa^2}{2}\eta_1 c_2 \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \right\}^{\frac{1}{2N}} \cdot \left\{ \int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa \right\}^{\frac{2N-1}{2N}} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{p+1}{Np}} \\
&\le \frac{1}{2}\int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa + \frac{\kappa^{4N}\eta_1^{2N}c_2^{2N}}{2} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}} \\
&\le \frac{1}{2}\int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa + \frac{\eta}{2} \cdot \left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}}. \qquad (4.4.5)
\end{aligned}
$$

Combining (4.4.3)-(4.4.5), we get that

$$
\int_\Omega \xi^2(\varphi)\varphi^{p+1}\psi^\kappa \le \eta\|\psi\|^{\kappa-1}_{L^\infty(\Omega)} \cdot \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta\left\{ \int_\Omega \varphi^{\frac{p}{2}} \right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^{4N}}{\psi^{4N-\kappa}}
$$

$$+2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|^2_{L^{\frac12}(\Omega)}, \tag{4.4.6}$$

where

$$\begin{aligned}
&2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\|\xi(\varphi)\varphi^{\frac{p+1}{2}}\psi^{\frac{\kappa}{2}}\right\|^2_{L^{\frac12}(\Omega)}\\
&= 2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\{\int_\Omega \varphi^{\frac{p+1-\alpha}{4}}\cdot(\xi^2(\varphi)\varphi^\alpha\psi^\kappa)^{\frac14}\right\}^4\\
&\le 2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\{\int_\Omega \varphi^{\frac{p+1-\alpha}{3}}\right\}^3\cdot\int_\Omega\xi^2(\varphi)\varphi^\alpha\psi^\kappa\\
&\le 2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)|\Omega|^{\frac{p+2\alpha-2}{p}}\cdot\|\psi\|^{\kappa-1}_{L^\infty(\Omega)}\cdot\left\{\int_\Omega\varphi^{\frac p2}\right\}^{\frac{2(p+1-\alpha)}{p}}\cdot\int_\Omega\xi^2(\varphi)\varphi^\alpha\psi\\
&\le 2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)c_2\cdot\|\psi\|^{\kappa-1}_{L^\infty(\Omega)}\cdot\left\{\int_\Omega\varphi^{\frac p2}\right\}^{\frac{2(p+1-\alpha)}{p}}\cdot\int_{\{\varphi>1\}}\varphi^\alpha\psi
\end{aligned} \tag{4.4.7}$$

because of $0<\frac{p+2\alpha-2}{p}<1$. Here, using the facts $p>2$, $\kappa>1$ and $\eta<1$ we obtain

$$\eta_1^{-\frac{\theta}{1-\theta}}=\max\left\{\left(\frac{(2(p+1)^2+2^5c_3^2)c_2}{\eta}\right)^{\frac{\theta}{1-\theta}},\ \left(\frac{\kappa^2c_2}{\eta^{\frac{1}{2N}}}\right)^{\frac{\theta}{1-\theta}}\right\}\ \le\ \left(\frac{(4+2^5c_3^2)p^2\kappa^2c_2}{\eta}\right)^{\frac{\theta}{1-\theta}}.$$

Writing

$$\delta\equiv\delta(N,p_*):=\frac{\theta}{1-\theta}\quad\text{and}\quad C(N,p_*):=2(4+2^5c_3^2)^\delta c_1^{\frac{1}{1-\theta}}c_2^{\delta+1}+2c_1c_2,$$

we find that

$$\begin{aligned}
2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)c_2 &\le 2(4+2^5c_3^2)^\delta c_1^{\frac{1}{1-\theta}}c_2^{\delta+1}p^{2\delta}\eta^{-\delta}\kappa^{2\delta}+2c_1c_2\\
&\le \left(2(4+2^5c_3^2)^\delta c_1^{\frac{1}{1-\theta}}c_2^{\delta+1}+2c_1c_2\right)p^{2\delta}\eta^{-\delta}\kappa^{2\delta}\\
&= C(N,p_*)p^{2\delta}\eta^{-\delta}\kappa^{2\delta}.
\end{aligned} \tag{4.4.8}$$

In conjunction with (4.4.6)-(4.4.8), we complete the proof. □

The final ingredient for the L^∞ bound on u_ε is the following pointwise estimate for ∇v_ε.

Lemma 4.4.2. *Let $N\ge 2$. Assume that* (4.1.3) *and* (4.1.4) *hold with $\alpha\in(0,\frac2N)$ and some constants $0<c_g<C_g$. Then for any $0<\sigma<1$, there exists $C(\sigma,N)>0$ such that*

$$|\nabla v_\varepsilon(x,t)|\le C(\sigma,N)v_\varepsilon^{1-\sigma}(x,t)\qquad \text{for all } (x,t)\in\Omega\times(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \tag{4.4.9}$$

Proof. For any $\varepsilon\in(0,1)$, defining

$$z_\varepsilon(x,t):=v_\varepsilon^\sigma(x,t)\qquad\text{for all }\ (x,t)\in\Omega\times(0,T_{max,\varepsilon}),$$

we see from (4.2.1) that z_ε satisfies

$$z_{\varepsilon t}=\Delta z_\varepsilon-\sigma g(u_\varepsilon)v_\varepsilon^\sigma+\sigma(1-\sigma)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^{2-\sigma}},\qquad (x,t)\in\Omega\times(0,T_{max,\varepsilon}).$$

An application of the standard L^p-L^q estimates for the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ ([73, Lemma 1.3]) ensures the existence of $c_1 > 0$ such that

$$\begin{aligned}
&\|\nabla z_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \\
= \quad & \left\| \nabla e^{t\Delta} z_{0\varepsilon} + \int_0^t \nabla e^{(t-s)\Delta} \left\{ -\sigma g(u_\varepsilon(\cdot,s)) v_\varepsilon^\sigma(\cdot,s) + \sigma(1-\sigma) \frac{|\nabla v_\varepsilon(\cdot,s)|^2}{v_\varepsilon^{2-\sigma}(\cdot,s)} \right\} ds \right\|_{L^\infty(\Omega)} \\
\leq \quad & \|\nabla z_{0\varepsilon}\|_{L^\infty(\Omega)} + c_1 \int_0^t (1+(t-s)^{-\frac{3}{4}}) e^{-\lambda_1(t-s)} \|u_\varepsilon^\alpha(\cdot,s) v_\varepsilon^\sigma(\cdot,s)\|_{L^{2N}(\Omega)} ds \\
& + c_1 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2\varsigma}}) e^{-\lambda_1(t-s)} \left\| \frac{|\nabla v_\varepsilon(\cdot,s)|^2}{v_\varepsilon^{2-\sigma}(\cdot,s)} \right\|_{L^\varsigma(\Omega)} ds \\
=: \quad & J_1 + J_2 + J_3 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
\end{aligned} \tag{4.4.10}$$

where

$$\varsigma \equiv \varsigma(\sigma, N) := \max\left\{ \frac{3}{\sigma},\ 2N \right\}$$

satisfies

$$\sigma - \frac{1}{\varsigma} - \frac{3}{\varsigma^2} > 0, \quad 2\varsigma - 1 - \frac{3}{\varsigma} > \frac{4\varsigma^2 - 2\varsigma - 6}{2\varsigma + 2} \quad \text{and} \quad \frac{1}{2} + \frac{N}{2\varsigma} < 1. \tag{4.4.11}$$

Herein,

$$J_1 = \sigma \, \|\nabla \ln v_0\|_{L^\infty(\Omega)} \, \|v_0\|_{L^\infty(\Omega)}^\sigma \tag{4.4.12}$$

is finite due to (4.1.5). For all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, one can see the existence of $c_2 = c_2(\sigma, N) > 0$ from Lemma 4.2.2 and Lemma 4.3.3 such that

$$J_2 \leq c_1 \int_0^t (1+(t-s)^{-\frac{3}{4}}) e^{-\lambda_1(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^{2N\alpha}(\Omega)}^\alpha \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^\sigma ds \leq c_2. \tag{4.4.13}$$

In view of (4.4.11), according to (4.2.5) and (4.3.12), there is $c_3 = c_3(\sigma, N) > 0$ such that

$$\left\| \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^{2-\sigma}} \right\|_{L^\varsigma(\Omega)} = \left\{ \int_\Omega \frac{|\nabla v_\varepsilon|^{2\varsigma}}{v_\varepsilon^{2\varsigma-1-\frac{3}{\varsigma}}} \cdot v_\varepsilon^{\sigma\varsigma-1-\frac{3}{\varsigma}} \right\}^{\frac{1}{\varsigma}} \leq \|v_\varepsilon\|_{L^\infty(\Omega)}^{\sigma-\frac{1}{\varsigma}-\frac{3}{\varsigma^2}} \cdot \left\{ \int_\Omega \frac{|\nabla v_\varepsilon|^{2\varsigma}}{v_\varepsilon^{2\varsigma-1-\frac{3}{\varsigma}}} \right\}^{\frac{1}{\varsigma}} \leq c_3$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$J_3 \leq c_1 c_3 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2\varsigma}}) e^{-\lambda_1(t-s)} ds \leq c_4 \tag{4.4.14}$$

with some $c_4 = c_4(\sigma, N) > 0$. Finally, (4.4.9) follows from (4.4.10), (4.4.12)-(4.4.14) and the definition of z_ε. □

With all necessary boundedness information prepared, we turn to proving uniform boundedness of u_ε in $L^\infty(\Omega)$.

Lemma 4.4.3. *Let $N \geq 2$. Assume that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then there exists $C(N) > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C(N) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{4.4.15}$$

Proof. Let $p_0 = 2$, and recursively define

$$p_k := 2^k p_0, \qquad k \geq 1. \tag{4.4.16}$$

Setting

$$M_{k,\varepsilon}(T) := 1 + \sup_{t\in(0,T)} \int_\Omega u_\varepsilon^{p_k}(\cdot,t), \quad T \in (0,T_{max,\varepsilon}), \quad k \in \mathbb{N} \text{ and } \varepsilon \in (0,1), \tag{4.4.17}$$

according to (4.3.10) we can find $c_1 = c_1(N) > 0$ such that

$$M_{0,\varepsilon}(T) \leq c_1 \qquad \text{for all } T \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{4.4.18}$$

Next, we proceed with estimating $M_{k,\varepsilon}(T)$ for $T \in (0,T_{max,\varepsilon})$, $k \geq 1$ and $\varepsilon \in (0,1)$. For this, we obtain from Lemma 4.4.2 a constant $c_2 = c_2(N) > 0$ fulfilling

$$|\nabla v_\varepsilon(x,t)| \leq c_2 v_\varepsilon^{\frac{3}{4N}}(x,t) \qquad \text{for all } (x,t) \in \Omega \times (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

and then in virtue of Lemma 4.2.6, for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} + \frac{p_k^2}{4}\int_\Omega u_\varepsilon^{p_k-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\quad\leq 2^{p_k} p_k(p_k-1)\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{p_k(p_k-1)}{2}\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p_k+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\quad\leq 2^{p_k} p_k^2 \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{c_2^2}{2} p_k^2 \int_{\{u_\varepsilon>2\}} u_\varepsilon^{p_k+1} v_\varepsilon^{1+\frac{3}{2N}}.
\end{aligned} \tag{4.4.19}$$

Since Lemma 4.2.2 entails the existence of $c_3 > 1$ satisfying

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_3 \qquad \text{for all } t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

taking $p_* := 3$ in Lemma 4.4.1, we may conclude that with $\delta = \delta(N) > 0$ and $c_4 = c_4(N) > 0$ provided by said lemma, we have

$$\begin{aligned}
\int_{\{u_\varepsilon>2\}} u_\varepsilon^{p_k+1} v_\varepsilon^{1+\frac{3}{2N}} \leq\ & \frac{1}{2c_2^2}\int_\Omega u_\varepsilon^{p_k-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2c_2^2}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1)}{p_k}} \cdot \int_\Omega \frac{|\nabla v_\varepsilon|^{4N}}{v_\varepsilon^{4N-(1+\frac{3}{2N})}} \\
& + c_4 p_k^{2\delta} (2c_2^2 c_3^{\frac{3}{2N}})^\delta \left(1+\frac{3}{2N}\right)^{2\delta} c_3^{\frac{3}{2N}} \cdot \left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1-\alpha)}{p_k}} \cdot \int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon
\end{aligned}$$

for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Inserting this back into (4.4.19) hence implies that for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, with $c_5 := 2^{3\delta} c_4 c_2^{2\delta+2} c_3^{\frac{3}{2N}(1+\delta)}$ we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} \leq\ & \frac{p_k^2}{4}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1)}{p_k}} \cdot \int_\Omega \frac{|\nabla v_\varepsilon|^{4N}}{v_\varepsilon^{4N-(1+\frac{3}{2N})}} + 2^{p_k} p_k^2 \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \\
& + c_5 p_k^{2\delta+2}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1-\alpha)}{p_k}} \cdot \int_{\{u_\varepsilon>1\}} u_\varepsilon^\alpha v_\varepsilon.
\end{aligned} \tag{4.4.20}$$

Now, take $q := 4N-2$, $r := 4N-(3+\frac{3}{2N})$. It follows that

$$\max\left\{\frac{q}{2}, \frac{q^2-q-6}{q+2}\right\} \le r < q-1,$$

and then Lemma 4.3.3 becomes applicable. Accordingly, there is $c_6 = c_6(N) > 0$ such that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^{4N}}{v_\varepsilon^{4N-(1+\frac{3}{2N})}} \le c_6 \qquad \text{for all} \quad \varepsilon \in (0,1). \tag{4.4.21}$$

Moreover, invoking Lemmas 4.2.2 and 4.2.3, with $c_7 > 0$ and $c_8 = c_8(N) > 0$ we have

$$\int_0^{T_{max,\varepsilon}} \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha v_\varepsilon \le c_7 \qquad \text{for all} \ \ \varepsilon \in (0,1) \tag{4.4.22}$$

and

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \le c_8 \qquad \text{for all} \ \ \varepsilon \in (0,1). \tag{4.4.23}$$

Integrating (4.4.20) from 0 to t for $t \in (0, T_{max,\varepsilon})$ and combining the resulting inequality with (4.4.21)-(4.4.23), we hence get that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon^{p_k} \le c_6 p_k^2 \cdot \left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1)}{p_k}} + c_8 2^{p_k} p_k^2 + c_5 c_7 p_k^{2\delta+2} \cdot \left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1-\alpha)}{p_k}} + \int_\Omega (u_0+1)^{p_k}.$$

Recalling the definitions of p_0, p_k and $M_{k,\varepsilon}(T)$, we obtain

$$\begin{aligned} M_{k,\varepsilon}(T) &\le c_9 p_k^{2\delta+2} M_{k-1,\varepsilon}^{\frac{2(p_k+1)}{p_k}}(T) + \|u_0+1\|_{L^\infty(\Omega)}^{p_k} \cdot |\Omega| + c_8 2^{p_k} p_k^2 + 1 \\ &= 2^{(k+1)(2\delta+2)} c_9 M_{k-1,\varepsilon}^{2+2^{-k}}(T) + \|u_0+1\|_{L^\infty(\Omega)}^{2^{k+1}} \cdot |\Omega| + c_8 4^{2^k+k+1} + 1 \end{aligned} \tag{4.4.24}$$

for all $\varepsilon \in (0,1)$, $k \ge 1$ and each $T \in (0, T_{max,\varepsilon})$, where $c_9 := c_6 + c_5 c_7 + 1$. Write

$$a := 4^{2\delta+2} c_9 \ \text{ and } \ b := (|\Omega|+1)\,\|u_0+1\|_{L^\infty(\Omega)}^2 + 16(c_8+1) + 1,$$

then it is easy to verify that

$$2^{(k+1)(2\delta+2)} c_9 \le (4^{2\delta+2})^k c_9^k = a^k$$

and

$$\begin{aligned} &\|u_0+1\|_{L^\infty(\Omega)}^{2^{k+1}} \cdot |\Omega| + c_8 4^{2^k+k+1} + 1 \\ &\quad \le (|\Omega|+1)^{2^k} \left(\|u_0+1\|_{L^\infty(\Omega)}^2\right)^{2^k} + (c_8+1)^{2^k} 16^{2^k} + 1 \\ &\quad \le b^{2^k}. \end{aligned}$$

Therefore, we can further conclude from (4.4.24) that

$$M_{k,\varepsilon}(T) \le a^k M_{k-1,\varepsilon}^{2+2^{-k}}(T) + b^{2^k} \qquad \text{for all} \ \ \varepsilon \in (0,1),\ k \ge 1 \text{ and each } T \in (0, T_{max,\varepsilon}).$$

Since $k \geq 1$ and $T \in (0, T_{max,\varepsilon})$ are arbitrary here, together with (4.4.18), we may use Lemma 3.4.2 to find that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \|u_\varepsilon(\cdot,t)\|^2_{L^\infty(\Omega)} &= \liminf_{k\to\infty} \left\{ \int_\Omega u_\varepsilon^{p_k}(\cdot,t) \right\}^{\frac{2}{p_k}} \leq \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{2}{p_k}}(T) \\ &= \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{2^k}}(T) \leq (2\sqrt{2}a^3 b^{\frac{3}{2}} c_1)^{e^{\frac{1}{2}}}, \end{aligned}$$

which clearly proves the assertion. □

4.5 Global weak solutions. Proof of Theorem 4.1.1

As a direct consequence of Lemma 4.4.3, we find that all approximate solutions are global in time.

Lemma 4.5.1. *Let $N \geq 2$. Assume that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then $T_{max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$.*

Proof. This is an immediate result of Lemma 4.4.3 in conjunction with (4.2.3). □

We next show that u_ε and v_ε possess higher regularity properties. The proof is quite similar to that in Lemma 3.5.3 of Chapter 3. Therefore, we only sketch the main ideas here.

Lemma 4.5.2. *Let $N \geq 2$. Assume that (4.1.3) and (4.1.4) hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then for all $T > 0$, there exist $\theta = \theta(N,T) \in (0,1)$ and $C(N,T) > 0$ such that*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \leq C(N,T) \qquad \text{for all } \varepsilon \in (0,1) \tag{4.5.1}$$

and

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \leq C(N,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.5.2}$$

Moreover, for any $\tau > 0$ and any $T' > \tau$, one can find $\theta' = \theta'(N,\tau,T') \in (0,1)$ and $C'(N,\tau,T') > 0$ such that

$$\|v_\varepsilon\|_{C^{2+\theta',1+\frac{\theta'}{2}}(\overline{\Omega}\times[\tau,T'])} \leq C'(N,\tau,T') \qquad \text{for all } \varepsilon \in (0,1). \tag{4.5.3}$$

Proof. According to (4.1.4) and Lemma 4.4.3, there exists $c_1 = c_1(N) > 0$ such that

$$g(u_\varepsilon) \leq c_1 \qquad \text{for all } \ x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1).$$

Hence, applying the comparison principle to

$$v_{\varepsilon t} \geq \Delta v_\varepsilon - c_1 v_\varepsilon \qquad \text{in } \Omega \times (0,\infty),$$

and combining the strict positivity of v_0, we can claim that with $c_2 := \inf_\Omega v_0 > 0$, we have

$$v_\varepsilon(x,t) \geq c_2 e^{-c_1 t} \qquad \text{for all } \ x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1). \tag{4.5.4}$$

From Lemmata 4.2.2, 4.4.2 and 4.4.3, the functions u_ε, $g(u_\varepsilon)$, v_ε and ∇v_ε are all bounded in $L^\infty(\Omega \times (0,\infty))$ (uniformly in ε), meaning that in light of the lower estimate for v_ε, one can invoke the well-known Hölder estimates established in [62] to obtain both (4.5.1) and (4.5.2). The property in (4.5.3) is an immediate consequence of the standard parabolic Schauder theory in [33, Theorem 5.3] along with (4.5.1). □

Gathering the above properties, we can conclude the existence of limit functions, which in fact form a global weak solution as claimed in Theorem 4.1.1.

Lemma 4.5.3. *Let $N \geq 2$. Assume that* (4.1.3) *and* (4.1.4) *hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)) \qquad \textit{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap W^{1,\infty}(\Omega \times (0,\infty)) \end{cases} \tag{4.5.5}$$

satisfying $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$ and

$$\nabla u^2 \in L^2_{loc}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^N\right), \tag{4.5.6}$$

which are such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$u_\varepsilon \to u \quad \textit{in} \quad C^0_{loc}\left(\overline{\Omega} \times [0,\infty)\right), \tag{4.5.7}$$

$$\nabla u_\varepsilon^2 \rightharpoonup \nabla u^2 \quad \textit{in} \quad L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{4.5.8}$$

$$v_\varepsilon \to v \quad \textit{in} \quad C^0_{loc}\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}_{loc}\left(\overline{\Omega} \times (0,\infty)\right) \qquad \textit{and} \tag{4.5.9}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \qquad \textit{in } L^\infty(\Omega \times (0,\infty)). \tag{4.5.10}$$

Moreover, (u,v) forms a global weak solution of (4.1.2) *in the sense of Definition 1.4.1.*

Proof. The Arzelà-Ascoli theorem together with (4.5.1) and (4.5.3) entails the existence of $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$ satisfying

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases}$$

such that $\varepsilon = \varepsilon_j \searrow 0$ as $j \to \infty$, and that (4.5.7) and (4.5.9) hold as $\varepsilon = \varepsilon_j \searrow 0$. Then, Lemma 4.4.3 along with (4.5.7) yields $u \in L^\infty(\Omega \times (0,\infty))$. In addition, for any $T > 0$ and $\varepsilon \in (0,1)$, a direct computation shows that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^2|^2 = 4 \int_0^T \int_\Omega u_\varepsilon^2 |\nabla u_\varepsilon|^2 = 4 \int_0^T \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla u_\varepsilon|^2 \cdot \frac{1}{v_\varepsilon},$$

which combined with (4.3.11) and (4.5.4) implies

$$(\nabla u_\varepsilon^2)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}\left(\overline{\Omega} \times [0,\infty); \mathbb{R}^N\right).$$

This in conjunction with (4.5.7) implies that there exists a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ (still denoted by $(\varepsilon_j)_{j\in\mathbb{N}}$) such that (4.5.8) holds as $\varepsilon = \varepsilon_j \searrow 0$, and hence (4.5.6) follows. From Lemmas 4.4.2 and 4.2.2, we find that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } W^{1,\infty}(\Omega \times (0,\infty)).$$

This combined with (4.5.9) implies $v \in W^{1,\infty}(\Omega \times (0,\infty))$ and (4.5.10) (if necessary going to a subsequence).

Finally, the identities in (1.4.3) and (1.4.4) result from (4.5.7)-(4.5.10) and (4.2.1) in a standard manner. □

Proof of Theorem 4.1.1. This is a direct consequence of Lemma 4.5.3. □

4.6 Large time behavior. Proof of Theorem 4.1.2

Based on the uniform L^∞ bound for u_ε in Lemma 4.4.3, we can derive the following elliptic-type pointwise Harnack inequality in quite a similar way as stated in Lemma 3.6.1.

Lemma 4.6.1. *Let $N \geq 2$. Assume that* (4.1.3) *and* (4.1.4) *hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then there exists $\lambda_*(N) > 0$ such that*

$$v_\varepsilon(x,t) \geq \lambda_*(N)\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1). \tag{4.6.1}$$

With this at hand, we immediately have the following result.

Lemma 4.6.2. *Let $N \geq 2$. Assume that* (4.1.3) *and* (4.1.4) *hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. Then there exists $\lambda(N) > 0$ such that*

$$\int_{t_0}^{\infty} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \leq \frac{\int_\Omega v_\varepsilon(\cdot,t_0)}{\lambda(N)\int_\Omega u_0} \qquad \text{for all } t_0 \geq 0 \text{ and } \varepsilon \in (0,1). \tag{4.6.2}$$

Proof. From Lemma 4.4.3, there exists $c_1 = c_1(N) > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

which together with the fact $0 < \alpha < 1$ leads to

$$u_\varepsilon^\alpha = u_\varepsilon \cdot u_\varepsilon^{\alpha-1} \geq c_1^{\alpha-1} u_\varepsilon \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1). \tag{4.6.3}$$

Recalling (4.1.3), one can see the existence of $c_2 > 0$ fulfilling

$$g(s) \geq c_2 s \qquad \text{for all } s \in [0,1]. \tag{4.6.4}$$

Then combining (4.6.4) with (4.1.4), (4.6.3) and Lemma 4.2.2, we have

$$\begin{aligned}
\int_\Omega g(u_\varepsilon) &= \int_{\{u_\varepsilon > 1\}} g(u_\varepsilon) + \int_{\{u_\varepsilon \leq 1\}} g(u_\varepsilon) \geq c_g \int_{\{u_\varepsilon > 1\}} u_\varepsilon^\alpha + c_2 \int_{\{u_\varepsilon \leq 1\}} u_\varepsilon \\
&\geq c_g c_1^{\alpha-1} \int_{\{u_\varepsilon > 1\}} u_\varepsilon + c_2 \int_{\{u_\varepsilon \leq 1\}} u_\varepsilon \geq c_3 \int_\Omega u_\varepsilon \\
&> c_3 \int_\Omega u_0 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)
\end{aligned}$$

with $c_3 := \min\{c_g c_1^{\alpha-1}, c_2\}$. Letting $\lambda_* = \lambda_*(N)$ be provided by Lemma 4.6.1, we make use of (4.2.6) and (4.6.1) to find that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}\int_\Omega v_\varepsilon(\cdot,t_0) \ &\geq\ \int_{t_0}^\infty \int_\Omega g(u_\varepsilon)v_\varepsilon \geq \lambda_* \int_{t_0}^\infty \left\{\int_\Omega g(u_\varepsilon)\right\} \cdot \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \\ &\geq\ \lambda_* c_3 \cdot \left\{\int_\Omega u_0\right\} \cdot \int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt,\end{aligned}$$

which concludes the proof by taking $\lambda(N) := \lambda_* c_3$. □

In fact, with Lemmas 4.6.1, 4.6.2 and 4.5.3 at hand, by a quite similar approach to Lemma 3.7.2 in Chapter 3, we can now show that $v_\varepsilon \to v$ in $L^1((0,\infty); L^\infty(\Omega))$ along the sequence obtained in Lemma 4.5.3.

Lemma 4.6.3. *Let $N \geq 2$. Assume that* (4.1.3) *and* (4.1.4) *hold with $\alpha \in (0, \frac{2}{N})$ and some constants $0 < c_g < C_g$. For $\varepsilon \in (0,1)$ define*

$$L_\varepsilon := \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt$$

and

$$\tau := \phi_\varepsilon(t) := \frac{1}{L_\varepsilon} \int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds, \qquad t \geq 0.$$

Then,

$$w_\varepsilon(x,\tau) := u_\varepsilon(x, \phi_\varepsilon^{-1}(\tau)), \qquad x \in \overline{\Omega},\ \tau \in [0,1) \tag{4.6.5}$$

solves

$$\begin{cases} w_{\varepsilon\tau} = \nabla\cdot\left(a_\varepsilon(x,\tau) w_\varepsilon \nabla w_\varepsilon\right) - \nabla\cdot\left(b_\varepsilon(x,\tau) w_\varepsilon^2\right), & x \in \Omega,\ \tau \in (0,1), \\ \nabla w_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ \tau \in (0,1) \\ w_\varepsilon(x,0) = u_0(x) + \varepsilon, & x \in \Omega \end{cases}$$

with

$$a_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x, \phi_\varepsilon^{-1}(\tau))}{\left\|v_\varepsilon(\cdot, \phi_\varepsilon^{-1}(\tau))\right\|_{L^\infty(\Omega)}} \quad \text{and} \quad b_\varepsilon(x,\tau) := L_\varepsilon \cdot \frac{v_\varepsilon(x, \phi_\varepsilon^{-1}(\tau)) \nabla v_\varepsilon(x, \phi_\varepsilon^{-1}(\tau))}{\left\|v_\varepsilon(\cdot, \phi_\varepsilon^{-1}(\tau))\right\|_{L^\infty(\Omega)}}.$$

Additionally, there exists $C(N) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\frac{1}{C(N)} \leq a_\varepsilon(x,\tau) \leq C(N) \quad \text{and} \quad |b_\varepsilon(x,\tau)| \leq C(N) \qquad \text{for all } (x,\tau) \in \Omega \times (0,1), \tag{4.6.6}$$

and that

$$L := \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \leq C(N). \tag{4.6.7}$$

Furthermore, denoting by $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ the sequence obtained in Lemma 4.5.3, we have

$$L_\varepsilon \to L \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.6.8}$$

Proof of Theorem 4.1.2. With Lemma 4.6.3 at hand, the proof of the large time behavior follows along the same lines as that of Theorem 3.1.4. To avoid redundancy, we omit the details here. □

5 Nonnegative solutions in the critical case on planar domains

5.1 Introduction

In view of the results presented in the preceding three chapters (Chapters 2-4) and the related works cited therein, it becomes apparent that, ever since its introduction, the model

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v) + \ell uv, \\ v_t = \Delta v - uv \end{cases} \tag{5.1.1}$$

has attracted considerable attention from mathematicians working on biologically motivated evolutionary PDE systems. Nevertheless, a comprehensive theory of global bounded nonnegative weak solutions seems available only in spatially one-dimensional versions ([42], [81], Chapter 2).

While findings on global solvability in higher-dimensional boundary value problems for (5.1.1) so far seem either restricted to settings of initial data which are essentially positive in their population density component, or to rely on suitable smallness assumptions on the initial nutrient concentration ([100], [88]); more complete results to date seem available only for some modified variants in which the present dissipative mechanisms are suitably enhanced ([40], [88], [89], Chapters 3 and 4).

In particular, it yet appears unknown whether in the physically most relevant planar case, the original system (5.1.1) can be viewed eligible as a meaningful model for the evolution of the free boundaries $\partial\{u(\cdot,t) > 0\}$ associated with arbitrary solutions, in the sense of admitting global solutions emanating from initial data of arbitrary size which are merely required to be nonnegative in their first component.

The purpose of the present chapter is to develop the global solution theory for the problem

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (uv\nabla u - u^2 v \nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{5.1.2}$$

with parameter $\ell \geq 0$, under mild assumptions on the initial data $u_0 \geq 0$ and $v_0 > 0$ which are compatible with the requirements described above. Apart from this motivation specifically aiming at (5.1.1), in two key places our analysis will involve strategies that might be of some more general relevance beyond.

In consequence, we establish the following result on global existence of bounded solutions to (5.1.2), widely unconditional with respect to the choice of the initial data, and especially for any choice of reasonably regular but merely nonnegative initial data.

Theorem 5.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\ell \geq 0$. Then whenever*

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is such that } u_0 \geq 0 \text{ and } u_0 \not\equiv 0, \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ satisfies } v_0 > 0 \text{ in } \overline{\Omega}, \end{cases} \tag{5.1.3}$$

one can find functions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases} \tag{5.1.4}$$

such that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$, and that the pair (u,v) solves (5.1.2) in the weak sense specified in Definition 1.4.1. This solution is bounded in the sense that

$$\sup_{t>0} \Big\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \Big\} < \infty. \tag{5.1.5}$$

We note that the boundedness property in (5.1.5) particularly guarantees that thanks to a Harnack-type inequality thus satisfied by v ([27], [95]), the above solutions also stabilize in the large time limit, with the corresponding asymptotic profiles known to be nonconstant in some cases.

Assuming (5.1.3), we let (u,v) be as in Theorem 5.1.1, and define

$$L := \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt < \infty \qquad \text{and} \qquad \varrho(t) := \frac{1}{L} \cdot \int_0^t \|v(\cdot,s)\|_{L^\infty(\Omega)} ds, \quad t \geq 0, \tag{5.1.6}$$

as well as

$$\begin{aligned} a(x,\tau) &:= L \cdot \frac{v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \qquad \text{and} \\ b(x,\tau) &:= L \cdot \frac{v(x,t)\nabla v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \qquad \text{for } x \in \Omega,\ \tau \in (0,1) \text{ and } t = \varrho^{-1}(\tau). \end{aligned} \tag{5.1.7}$$

Then a straightforward adaptation of similar statements from [95], [42] and [87] (see also Lemma 3.1.4 in Chapter 3 and Lemma 4.1.2 in Chapter 4) will, inter alia, relate the large time behavior in (5.1.2) to the evaluations of some weak solutions to

$$\begin{cases} w_\tau = \nabla \cdot \big(a(x,\tau) w \nabla w\big) - \nabla \cdot \big(b(x,\tau) w^2\big) + \ell a(x,\tau) w, & x \in \Omega,\ \tau \in (0,1), \\ \big(a(x,\tau) w \nabla w - b(x,\tau) w^2\big) \cdot \nu = 0, & x \in \partial\Omega,\ \tau \in (0,1), \\ w(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.1.8}$$

at the finite time $\tau = 1$.

Corollary 5.1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\ell \geq 0$. Assume that* (5.1.3) *holds.*

i) Let u, v, L, a and b be as in Theorem 5.1.1, (5.1.6) and (5.1.7). Then with some $C > 0$ we have $\frac{1}{C} \leq a(x,\tau) \leq C$ and $|b(x,\tau)| \leq C$, and there exists $w \in C^0(\overline{\Omega} \times [0,1])$ which solves (5.1.8) in the sense that $w^2 \in L^1_{loc}([0,1); W^{1,1}(\Omega))$ and

$$-\int_0^1 \int_\Omega w\varphi_\tau - \int_\Omega u_0 \varphi(\cdot,0) = -\frac{1}{2}\int_0^1 \int_\Omega a(x,\tau)\nabla w^2 \cdot \nabla\varphi + \int_0^1 \int_\Omega b(x,\tau) w^2 \cdot \nabla\varphi + \ell \int_0^1 \int_\Omega a(x,\tau) w \varphi$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,1))$. Moreover,

$$u(\cdot,t) \to u_\infty := w(\cdot,1) \quad \textit{and} \quad v(\cdot,t) \to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty. \tag{5.1.9}$$

ii) If additionally $u_0 \not\equiv const.$, then there exists $\eta > 0$ with the property that whenever

$$\|v_0\|_{W^{1,\infty}(\Omega)} + \|\nabla\sqrt{v_0}\|_{L^2(\Omega)} \leq \eta,$$

the limit function u_∞ in (5.1.9) has the property that

$$u_\infty \not\equiv const.$$

Main ideas. Objective #1: Browsing the vicinity of an ill-defined Lyapunov functional. A first challenge to be dealt with, namely, is related to the circumstance that while (5.1.2) does possess a favorable gradient-like structure formally expressed in the energy inequality

$$\frac{d}{dt}\Big\{-\int_\Omega \ln u + \frac{1}{2}\int_\Omega |\nabla v|^2\Big\} + \int_\Omega \frac{v}{u}|\nabla u|^2 + \int_\Omega |\Delta v|^2 = -\int_\Omega u|\nabla v|^2 - \ell\int_\Omega v, \tag{5.1.10}$$

the Lyapunov functional herein evidently is constantly infinite and hence meaningless along trajectories involving compactly supported components u. In order to nevertheless make a theory for (5.1.2) accessible to some essential parts of this structure, in a first pivotal step of our analysis we will trace the evolution of an *explicitly time-dependent regularization* of this functional, and thereby build our subsequent consideration on a quasi-energy inequality of the form

$$\frac{d}{dt}\Big\{-\int_\Omega \ln\big(u(\cdot,t)+\delta(t)\big) + \frac{1}{2}\int_\Omega |\nabla v|^2\Big\} + \frac{1}{2}\int_\Omega \frac{uv}{(u+\delta(t))^2}|\nabla u|^2 + \frac{1}{2}\int_\Omega |\Delta v|^2 \leq C\delta(t)\int_\Omega |\nabla v|^2 + C\cdot\frac{|\delta'(t)|}{\delta(t)} + C\delta^2(t) \tag{5.1.11}$$

(Lemma 5.3.2). Upon a simple choice of the positive free parameter function δ herein, in Lemma 5.3.3 we will thereby gain that even in the presence of compactly supported u_0, the quantities which on the left of (5.1.10) have appeared as genuine dissipation rates do, in the context of (5.1.2) and a regularized variant thereof (see (5.2.1)), retain some relaxation feature by satisfying an inequality of type

$$\int_0^t \int_\Omega \frac{uv}{(u+1)^2} |\nabla u|^2 + \int_0^t \int_\Omega |\Delta v|^2 \le C \cdot \ln(t+e) \tag{5.1.12}$$

which, as $C > 0$ is independent both of the initial data and t, quantifies some temporal decay of corresponding averages for any such u_0.

Objective #2: Deriving and exploiting bounds in $L \log^2 L$. A second crucial task will subsequently be linked to the question how far the fundamental regularity information can be used as a starting point for an appropriate bootstrap procedure. Yet remaining at moderate levels of integrability, in an intermediate step we may ignore the particular sublinear time dependence of the right-hand side in (5.1.12) and use the estimate thus provided on time intervals of fixed length to successively control the growth first of the rather standard type of functional given by

$$t \mapsto \int_\Omega u(\ln u - 1) - \int_\Omega uv \tag{5.1.13}$$

(Lemma 5.4.4), and then of the presumably less established quantities

$$t \mapsto \int_\Omega \Phi(z) e^v, \quad z := ue^{-v}, \qquad \Phi(\xi) := \xi \ln^2(\xi + e), \ \xi \ge 0 \tag{5.1.14}$$

(Lemma 5.4.6). For fixed $T > 0$, this will lead to an estimate of the form

$$\int_\Omega u \ln^2(u+e) \le C(T), \qquad t \in (0,T), \tag{5.1.15}$$

and the announced second cornerstone of our approach will consist in making appropriate use of the fact that here the logarithmic factor quantifying deviation from linear growth itself appears at a superlinear power. According to a logarithmically refined smoothing property of heat semigroups in the considered spatially planar scenarios (Lemma 5.4.7), this slight but crucial difference to classical situations in which estimates in $L \log L$ result from bounds for logarithmic entropies such as that in (5.1.13), namely, enables us to control solutions to the Neumann problem for $w_t = \Delta w + u$ with respect to L^∞ norms. In Lemma 5.4.8, upon a Hopf-Cole transformation we will thereby see that for any fixed $T > 0$ and some $C(T) > 0$,

$$v(x,t) \ge C(T) \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T), \tag{5.1.16}$$

confirming that, in fact, the cross-degeneracy in the diffusion process in (5.1.2) does not come into effect within any finite time interval. Accordingly, standard methods from the analysis of chemotaxis systems involving degeneracies of porous medium type become applicable so as to assert suitable further regularity properties (Lemma 5.4.9).

In a last step, we will return to (5.1.12) and now make use of the decay information encrypted therein. In convex domains, namely, this will entail that $v(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$ (Lemma 5.5.2), which in turn implies that ill-signed contributions to the evolution of

$$t \mapsto \int_\Omega u(\ln u - 1) + a \int_\Omega \frac{|\nabla v|^4}{v^3} \qquad \text{and} \qquad t \mapsto \int_\Omega u^2 \tag{5.1.17}$$

can conveniently be estimated after some suitably large waiting time (Lemma 5.5.3 and Lemma 5.5.4).

5.2 Preliminaries

5.2.1 Regularized problems and basic properties

Following [42] and [88], and in the same spirit as Chapters 2-4, we consider the regularized variants of (5.1.2)

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot (u_\varepsilon v_\varepsilon \nabla u_\varepsilon) - \nabla \cdot (u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{5.2.1}$$

for $\varepsilon \in (0,1)$, and may then rely on essentially well-known arguments to obtain the following statement on local-in-time existence and extensibility of classical solutions, and on some basic properties thereof.

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\ell \geq 0$. Assume that (5.1.3) holds. Then for each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \quad \text{and} \\ v_\varepsilon \in \bigcap_{q>1} C^0([0, T_{max,\varepsilon}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \end{cases} \tag{5.2.2}$$

such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0, T_{max,\varepsilon})$, that $(u_\varepsilon, v_\varepsilon)$ forms a classical solution of (5.2.1) in $\Omega \times (0, T_{max,\varepsilon})$, and that

$$\text{if } T_{max,\varepsilon} < \infty, \quad \text{then} \quad \limsup_{t \to T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{5.2.3}$$

This solution satisfies

$$\int_\Omega u_\varepsilon(\cdot,t) \leq m := \int_\Omega (u_0 + 1) + \ell \int_\Omega v_0 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{5.2.4}$$

and for all $t_0 \in (0, T_{max,\varepsilon})$, $t \in (t_0, T_{max,\varepsilon})$ and $p \in [1,\infty]$,

$$\|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq \|v_\varepsilon(\cdot,t_0)\|_{L^p(\Omega)} \leq \|v_0\|_{L^p(\Omega)} \qquad \text{for all } \varepsilon \in (0,1) \tag{5.2.5}$$

as well as

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_0 \qquad \text{for all } \varepsilon \in (0,1). \tag{5.2.6}$$

Proof. This readily follows from a copy of the reasoning detailed in Lemma 2.2.1 of Chapter 2 (see also [42, Lemma 2.2]). □

Throughout the sequel, without explicit further mentioning we will always assume that Ω is a smoothly bounded planar domain, that $\ell \geq 0$ and that (5.1.3) holds, and for $\varepsilon \in (0,1)$ we will let $T_{max,\varepsilon}$ as well as u_ε and v_ε be as obtained in Lemma 5.2.1.

5.2.2 Two functional inequalities resulting from a planar Sobolev embedding

For frequent later use, in this section we draw two consequences of the following well-known embedding inequality.

Lemma 5.2.2. *Let $r \in (0,2)$. Then there exists $\Lambda(r) > 0$ such that*

$$\|\phi\|_{L^2(\Omega)} \le \Lambda(r)\|\nabla\phi\|_{L^1(\Omega)} + \Lambda(r)\|\phi\|_{L^r(\Omega)} \qquad \text{for all } \phi \in W^{1,1}(\Omega). \tag{5.2.7}$$

Proof. This is an immediate consequence of the Sobolev inequality in the two-dimensional domain Ω. □

Indeed, Lemma 5.2.2 implies two functional inequalities that will play crucial roles in Lemmata 5.4.5, 5.5.3, 5.5.4 and 5.5.5.

Lemma 5.2.3. *Let $\alpha \ge 1$, $0 < \beta \le 1$ and $\gamma \ge 2$, and $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ be such that $\varphi \ge 0$ and $\psi > 0$ in $\overline{\Omega}$. Then*

$$\begin{aligned}\int_\Omega \varphi^{2\alpha}\psi \ \le\ & 8\alpha^2\Lambda^2 \cdot \Big\{\int_\Omega \varphi^{\alpha-1}\sqrt{\psi}|\nabla\varphi|\Big\}^2 + 2^{\frac{\gamma}{2}}\Lambda^\gamma \cdot \Big\{\int_\Omega \varphi^{\frac{4\alpha}{\gamma}}\Big\}^{\frac{\gamma}{2}} \cdot \int_\Omega \frac{|\nabla\psi|^\gamma}{\psi^{\gamma-1}} \\ & +4\Lambda^2 \cdot \Big\{\int_\Omega \varphi\Big\}^{2\alpha-1} \cdot \int_\Omega \varphi\psi \end{aligned}\tag{5.2.8}$$

as well as

$$\begin{aligned}\Big\{\int_\Omega \varphi^{2\alpha}\psi^4\Big\}^\beta \ \le\ & (4\alpha\Lambda)^{2\beta}\|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \Big\{\int_\Omega \varphi^{\frac{2\alpha}{3}}\Big\}^\beta \cdot \Big\{\int_\Omega \varphi^{\frac{4\alpha-6}{3}}\psi|\nabla\varphi|^2\Big\}^\beta \\ & +(8\Lambda)^{2\beta}\|\psi\|_{L^\infty(\Omega)}^{\beta} \cdot \Big\{\int_\Omega \varphi^{\frac{2\alpha}{3}}\Big\}^\beta \cdot \Big\{\int_\Omega \varphi^{\frac{4\alpha}{3}}\psi|\nabla\psi|^2\Big\}^\beta \\ & +(2\Lambda)^{2\beta}\|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \Big\{\int_\Omega \varphi\Big\}^{(2\alpha-1)\beta} \cdot \Big\{\int_\Omega \varphi\psi\Big\}^\beta, \end{aligned}\tag{5.2.9}$$

where $\Lambda = \Lambda(\frac{1}{\alpha})$ is taken from Lemma 5.2.2.

Proof. Using (5.2.7), we see that

$$\|\phi\|_{L^2(\Omega)} \le \Lambda\|\nabla\phi\|_{L^1(\Omega)} + \Lambda\|\phi\|_{L^{\frac{1}{\alpha}}(\Omega)} \qquad \text{for all } \phi \in W^{1,1}(\Omega), \tag{5.2.10}$$

which when applied to $\phi := \varphi^\alpha\sqrt{\psi}$ entails that

$$\begin{aligned}\int_\Omega \varphi^{2\alpha}\psi \ =\ & \|\varphi^\alpha\sqrt{\psi}\|_{L^2(\Omega)}^2 \\ \le\ & 2\Lambda^2 \cdot \Big\{\int_\Omega \alpha\varphi^{\alpha-1}\sqrt{\psi}|\nabla\varphi| + \int_\Omega \frac{\varphi^\alpha}{2\sqrt{\psi}}|\nabla\psi|\Big\}^2 + 2\Lambda^2 \cdot \Big\{\int_\Omega \varphi\psi^{\frac{1}{2\alpha}}\Big\}^{2\alpha} \\ \le\ & 4\alpha^2\Lambda^2 \cdot \Big\{\int_\Omega \varphi^{\alpha-1}\sqrt{\psi}|\nabla\varphi|\Big\}^2 + \Lambda^2 \cdot \Big\{\int_\Omega \frac{\varphi^\alpha}{\sqrt{\psi}}|\nabla\psi|\Big\}^2 + 2\Lambda^2 \cdot \Big\{\int_\Omega \varphi\psi^{\frac{1}{2\alpha}}\Big\}^{2\alpha}. \end{aligned}$$

As the Hölder inequality and Young's inequality show that

$$2\Lambda^2 \cdot \left\{ \int_\Omega \varphi \psi^{\frac{1}{2\alpha}} \right\}^{2\alpha} = 2\Lambda^2 \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha-1}{2\alpha}} \cdot (\varphi\psi)^{\frac{1}{2\alpha}} \right\}^{2\alpha} \le 2\Lambda^2 \cdot \left\{ \int_\Omega \varphi \right\}^{2\alpha-1} \cdot \int_\Omega \varphi\psi$$

as well as

$$\begin{aligned} \Lambda^2 \cdot \left\{ \int_\Omega \frac{\varphi^\alpha}{\sqrt{\psi}} |\nabla\psi| \right\}^2 &= \Lambda^2 \cdot \left\{ \int_\Omega \left(\frac{|\nabla\psi|^\gamma}{\psi^{\gamma-1}} \right)^{\frac{1}{\gamma}} \cdot (\varphi^{2\alpha}\psi)^{\frac{\gamma-2}{2\gamma}} \cdot \varphi^{\frac{2\alpha}{\gamma}} \right\}^2 \\ &\le \Lambda^2 \cdot \left\{ \int_\Omega \frac{|\nabla\psi|^\gamma}{\psi^{\gamma-1}} \right\}^{\frac{2}{\gamma}} \cdot \left\{ \int_\Omega \varphi^{2\alpha}\psi \right\}^{\frac{\gamma-2}{\gamma}} \cdot \int_\Omega \varphi^{\frac{4\alpha}{\gamma}} \\ &\le \frac{1}{2} \int_\Omega \varphi^{2\alpha}\psi + 2^{\frac{\gamma-2}{2}} \Lambda^\gamma \cdot \left\{ \int_\Omega \varphi^{\frac{4\alpha}{\gamma}} \right\}^{\frac{\gamma}{2}} \cdot \int_\Omega \frac{|\nabla\psi|^\gamma}{\psi^{\gamma-1}}, \end{aligned}$$

this implies (5.2.8). We next apply (5.2.10) to $\phi := \varphi^\alpha \psi^2$ to find that

$$\begin{aligned} \left\{ \int_\Omega \varphi^{2\alpha}\psi^4 \right\}^\beta &= \|\varphi^\alpha\psi^2\|_{L^2(\Omega)}^{2\beta} \\ &\le (2\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \alpha\varphi^{\alpha-1}\psi^2|\nabla\varphi| + \int_\Omega 2\varphi^\alpha\psi|\nabla\psi| \right\}^{2\beta} + (2\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi\psi^{\frac{2}{\alpha}} \right\}^{2\alpha\beta} \\ &\le (4\alpha\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi^{\alpha-1}\psi^2|\nabla\varphi| \right\}^{2\beta} + (8\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi^\alpha\psi|\nabla\psi| \right\}^{2\beta} \\ &\quad + (2\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi\psi^{\frac{2}{\alpha}} \right\}^{2\alpha\beta}. \end{aligned} \tag{5.2.11}$$

Here, the Hölder inequality entails that

$$\begin{aligned} (4\alpha\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi^{\alpha-1}\psi^2|\nabla\varphi| \right\}^{2\beta} &\le (4\alpha\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha-3}{3}} \sqrt{\psi}|\nabla\varphi| \cdot \varphi^{\frac{\alpha}{3}} \right\}^{2\beta} \\ &\le (4\alpha\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \left\{ \int_\Omega \varphi^{\frac{4\alpha-6}{3}} \psi|\nabla\varphi|^2 \right\}^\beta \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha}{3}} \right\}^\beta, \end{aligned}$$

that similarly

$$\begin{aligned} (8\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi^\alpha\psi|\nabla\psi| \right\}^{2\beta} &\le (8\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{\beta} \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha}{3}} \sqrt{\psi}|\nabla\psi| \cdot \varphi^{\frac{\alpha}{3}} \right\}^{2\beta} \\ &\le (8\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{\beta} \cdot \left\{ \int_\Omega \varphi^{\frac{4\alpha}{3}} \psi|\nabla\psi|^2 \right\}^\beta \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha}{3}} \right\}^\beta, \end{aligned}$$

and that

$$\begin{aligned} (2\Lambda)^{2\beta} \cdot \left\{ \int_\Omega \varphi\psi^{\frac{2}{\alpha}} \right\}^{2\alpha\beta} &\le (2\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \left\{ \int_\Omega \varphi^{\frac{2\alpha-1}{2\alpha}} \cdot (\varphi\psi)^{\frac{1}{2\alpha}} \right\}^{2\alpha\beta} \\ &\le (2\Lambda)^{2\beta} \|\psi\|_{L^\infty(\Omega)}^{3\beta} \cdot \left\{ \int_\Omega \varphi \right\}^{(2\alpha-1)\beta} \cdot \left\{ \int_\Omega \varphi\psi \right\}^\beta, \end{aligned}$$

so that (5.2.9) results from (5.2.11). □

5.2.3 Basic testing procedures

Let us next collect some basic properties of the solutions to (5.2.1) which either result from rather straightforward testing procedures, or even have fully been covered by previous literature. Some first information on the evolution of $\int_\Omega u_\varepsilon \ln u_\varepsilon$ and $\int_\Omega u_\varepsilon^p$ for $p > 1$ can readily be obtained.

Lemma 5.2.4. *Let $\varepsilon \in (0,1)$. Then for all $t \in (0, T_{max,\varepsilon})$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon(\ln u_\varepsilon - 1) = -\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon \tag{5.2.12}$$

and for each $p > 1$, we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon. \tag{5.2.13}$$

Proof. Testing the first equation in (5.2.1) by $\ln u_\varepsilon$, we see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon(\ln u_\varepsilon - 1) &= \int_\Omega \ln u_\varepsilon \nabla \cdot \{u_\varepsilon v_\varepsilon \nabla u_\varepsilon - u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon\} + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon \\ &= -\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Moreover, for any $p > 1$ we may multiply the first equation in (5.2.1) by u_ε^{p-1} and integrate by parts to find that according to Young's inequality, indeed we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p &= p\int_\Omega u_\varepsilon^{p-1}\nabla \cdot \Big\{u_\varepsilon v_\varepsilon \nabla u_\varepsilon - u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon\Big\} + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\ &= -p(p-1)\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + p(p-1)\int_\Omega u_\varepsilon^p v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\ &\le -\frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. □

Also some basic evolution features of the signal gradients can be shown by well-established methods. We highlight that the following first statement in this regard yet applies to arbitrary domains which are not necessarily convex, and that according to the subsequent proof, the additive constant appearing on the right of (5.2.14) could actually be suppressed on assuming convexity of Ω. In its more general version included here, however, Lemma 5.2.5 will be sufficient for our later purposes (see Lemma 5.4.2).

Lemma 5.2.5. *There exists $C > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and any $\varepsilon \in (0,1)$,*

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{C}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{C}\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \le -2\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon + C. \tag{5.2.14}$$

Proof. Following the approach outlined in [88, Lemma 2.3], particularly with $q := 2$, we obtain that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + 2\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \\ &\quad = \; -2\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon}\cdot\nabla(u_\varepsilon v_\varepsilon) + \int_{\partial\Omega} \frac{1}{v_\varepsilon}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2, \end{aligned} \tag{5.2.15}$$

where by simple differentiation,

$$-2\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon}\cdot\nabla(u_\varepsilon v_\varepsilon) = -2\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon - 2\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2. \tag{5.2.16}$$

The second to last summand in (5.2.15) can be controlled by using that according to an argument detailed in [82, Lemma 3.5], thanks to a standard boundary trace embedding estimate we can find $c_1 > 0$ such that

$$\int_{\partial\Omega} \frac{1}{\varphi}\cdot\frac{\partial|\nabla\varphi|^2}{\partial\nu} \le \frac{1}{2(3+\sqrt{2})^2}\int_\Omega \frac{|D^2\varphi|^2}{\varphi} + \frac{1}{2(2+\sqrt{2})^2}\int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} + c_1\int_\Omega \varphi \tag{5.2.17}$$

for all $\varphi \in C^2(\overline{\Omega})$ such that $\varphi > 0$ in $\overline{\Omega}$. We moreover invoke Lemma 3.2.3 to see that

$$\begin{aligned} &\frac{1}{4(3+\sqrt{2})^2}\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2(2+\sqrt{2})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\ &\quad \le \; \frac{1}{2(3+\sqrt{2})^2}\int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2(2+\sqrt{2})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\ &\quad \le \; \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned}$$

When combined with (5.2.15)-(5.2.17) and (5.2.5), this shows that (5.2.14) is satisfied if we let $C := \max\left\{c_1\|v_0\|_{L^1(\Omega)}, 4(3+\sqrt{2})^2\right\}$. □

In contrast to the above, the following second basic statement on signal evolution, to be used in our large time analysis in Lemma 5.5.3, will now crucially rely on the assumption that Ω be convex. We remark that including general non-convex domains here would lead to an additional appearance of a summand of the form $C\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}$ on the right-hand side of (5.2.18), and it seems unclear how far the influence of such contributions can be controlled on large time scales.

Lemma 5.2.6. *Let Ω be convex. Then there exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{C}\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le C\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2. \tag{5.2.18}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. This is part of an actually more statement recorded in [88, Lemma 2.3]. □

In order to reduce our overall goal to the derivation of regularity properties in a comparatively moderate topological framework accessible to variational arguments, let us recall the following criterion for global existence and boundedness in (5.1.2), as obtained in [88, Proposition 3.1] by means of a Moser-type iterative reasoning.

Proposition 5.2.7. *Let Ω be convex, assume (5.1.3), and suppose that there exists $p_0 \in (1,3]$ such that the solutions of (5.2.1) from Lemma 5.2.1 satisfy*

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{max,\varepsilon})} \int_\Omega u_\varepsilon^{p_0}(\cdot,t) < \infty \tag{5.2.19}$$

and

$$\sup_{\varepsilon\in(0,1)} \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{p_0+2} v_\varepsilon < \infty \tag{5.2.20}$$

as well as

$$\sup_{\varepsilon\in(0,1)} \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon (u_\varepsilon+1)^{p_0-2} v_\varepsilon |\nabla u_\varepsilon|^2 < \infty. \tag{5.2.21}$$

Then there exist functions u and v satisfying (5.1.4) and (5.1.5), which are such that $u \geq 0$ and $v > 0$ in $\overline{\Omega}\times[0,\infty)$, and that (u,v) forms a global weak solution of (5.1.2) in the sense of Defintion 1.4.1. Moreover, we then have $T_{max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$, and there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that

$$u_\varepsilon \to u \qquad \text{in } C^0_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.2.22}$$

$$v_\varepsilon \to v \qquad \text{in } C^0_{loc}(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega}\times(0,\infty)) \qquad \text{and} \tag{5.2.23}$$

$$\nabla v_\varepsilon \stackrel{\star}{\rightharpoonup} \nabla v \qquad \text{in } L^\infty(\Omega\times(0,\infty)) \tag{5.2.24}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

5.3 An asymptotic energy structure

Now the first crucial part of our analysis will be concerned with the derivation and an exploitation of the quasi-energy structure announced near (5.1.11). Our considerations in this direction will be launched by a fairly simple observation.

Lemma 5.3.1. *Let $\delta \in C^1([0,\infty))$ be such that*

$$\delta(t) > 0 \quad \text{and} \quad \delta'(t) \leq 0 \qquad \text{for all } t \geq 0. \tag{5.3.1}$$

Then

$$\begin{aligned} -\frac{d}{dt}\int_\Omega \ln\left(u_\varepsilon(\cdot,t)+\delta(t)\right) &+ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 \\ &\leq |\Omega|\cdot\frac{|\delta'(t)|}{\delta(t)} + \int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \end{aligned} \tag{5.3.2}$$

for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$.

Proof. For $\varepsilon \in (0,1)$, we use the first equation in (5.2.1) to see that

$$-\frac{d}{dt}\int_\Omega \ln\left(u_\varepsilon+\delta(t)\right) = -\int_\Omega \frac{1}{u_\varepsilon+\delta(t)}\nabla\cdot\left\{u_\varepsilon v_\varepsilon\nabla u_\varepsilon - u_\varepsilon^2 v_\varepsilon\nabla v_\varepsilon\right\} - \ell\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+\delta(t)}$$

$$
\begin{aligned}
& -\delta'(t)\int_\Omega \frac{1}{u_\varepsilon+\delta(t)} \\
= & -\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \ell\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+\delta(t)} \\
& -\delta'(t)\int_\Omega \frac{1}{u_\varepsilon+\delta(t)} \\
\le & -\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
& -\delta'(t)\int_\Omega \frac{1}{u_\varepsilon+\delta(t)} \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),
\end{aligned}
$$

because $\ell \ge 0$. Since for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$
-\delta'(t)\int_\Omega \frac{1}{u_\varepsilon+\delta(t)} = |\delta'(t)|\int_\Omega \frac{1}{u_\varepsilon+\delta(t)} \le |\delta'(t)|\int_\Omega \frac{1}{\delta(t)} = |\Omega|\cdot\frac{|\delta'(t)|}{\delta(t)}
$$

by (5.3.1), this yields (5.3.2). □

When combined with the outcome of a standard evolution feature of the ∇v_ε which is yet markedly simpler than those recorded in Lemma 5.2.5 and Lemma 5.2.6, the previous lemma indeed implies the following rigorous counterpart of (5.1.11) when concretized in the setting of our approximation scheme, yet allowing for fairly general functions δ.

Lemma 5.3.2. *If $\delta\in C^1([0,\infty))$ satisfies (5.3.1), then for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,*

$$
\begin{aligned}
& \frac{d}{dt}\Big\{-\int_\Omega \ln\big(u_\varepsilon(\cdot,t)+\delta(t)\big)+\frac12\int_\Omega|\nabla v_\varepsilon|^2\Big\} + \frac12\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \frac12\int_\Omega|\Delta v_\varepsilon|^2 \\
& \quad \le \frac12\|v_0\|_{L^\infty(\Omega)}\delta(t)\int_\Omega|\nabla v_\varepsilon|^2 + |\Omega|\cdot\frac{|\delta'(t)|}{\delta(t)} + \frac12\delta^2(t)\int_\Omega v_0^2. \qquad (5.3.3)
\end{aligned}
$$

Proof. From the second equation in (5.2.1) it follows in a standard manner that

$$
\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega|\Delta v_\varepsilon|^2 &= \int_\Omega u_\varepsilon v_\varepsilon\Delta v_\varepsilon \\
&= -\int_\Omega u_\varepsilon|\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon
\end{aligned}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, whence dropping a nonpositive summand here, due to Lemma 5.3.1 we obtain that

$$
\begin{aligned}
& \frac{d}{dt}\Big\{-\int_\Omega \ln\big(u_\varepsilon+\delta(t)\big)+\frac12\int_\Omega|\nabla v_\varepsilon|^2\Big\} + \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \int_\Omega|\Delta v_\varepsilon|^2 \\
& \quad \le |\Omega|\cdot\frac{|\delta'(t)|}{\delta(t)} + \int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \qquad (5.3.4)
\end{aligned}
$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Here, using that

$$
\frac{\xi^2}{(\xi+\delta_0)^2}-1 = -\frac{2\delta_0\xi}{(\xi+\delta_0)^2} - \frac{\delta_0^2}{(\xi+\delta_0)^2} \qquad \text{for all } \xi\ge0 \text{ and } \delta_0>0,
$$

we can rewrite

$$\begin{aligned}
&\int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega v_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\qquad = -2\int_\Omega \frac{\delta(t)u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \delta^2(t)\int_\Omega \frac{v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon
\end{aligned} \tag{5.3.5}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, and in the first integral on the right-hand side herein we can combine Young's inequality with (5.2.5) to estimate

$$\begin{aligned}
-2&\int_\Omega \frac{\delta(t)u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\le \frac12\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + 2\int_\Omega \frac{\delta^2(t)u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla v_\varepsilon|^2 \\
&\le \frac12\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \frac12\delta(t)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 \\
&\le \frac12\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \frac12\|v_0\|_{L^\infty(\Omega)}\delta(t)\int_\Omega |\nabla v_\varepsilon|^2
\end{aligned} \tag{5.3.6}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, because an elementary maximization shows that

$$\frac{\delta_0^2\xi}{(\xi+\delta_0)^2}\le\frac{\delta_0}{4} \qquad \text{for all } \xi\ge0 \text{ and } \delta_0>0.$$

In the second summand on the right of (5.3.5), however, we rather integrate by parts first, thereby verifying that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
-\delta^2(t)&\int_\Omega \frac{v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&= \delta^2(t)\int_\Omega v_\varepsilon\nabla\frac{1}{u_\varepsilon+\delta(t)}\cdot\nabla v_\varepsilon \\
&= -\delta^2(t)\int_\Omega \frac{v_\varepsilon}{u_\varepsilon+\delta(t)}\Delta v_\varepsilon - \delta^2(t)\int_\Omega\frac{1}{u_\varepsilon+\delta(t)}|\nabla v_\varepsilon|^2,
\end{aligned}$$

and only then we invoke Young's inequality to find on trivially estimating the rightmost here that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
-\delta^2(t)\int_\Omega \frac{v_\varepsilon}{(u_\varepsilon+\delta(t))^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon &\le -\delta^2(t)\int_\Omega\frac{v_\varepsilon}{u_\varepsilon+\delta(t)}\Delta v_\varepsilon \\
&\le \frac12\int_\Omega|\Delta v_\varepsilon|^2 + \frac12\delta^4(t)\int_\Omega\frac{v_\varepsilon^2}{(u_\varepsilon+\delta(t))^2} \\
&\le \frac12\int_\Omega|\Delta v_\varepsilon|^2 + \frac12\delta^2(t)\int_\Omega v_\varepsilon^2 \\
&\le \frac12\int_\Omega|\Delta v_\varepsilon|^2 + \frac12\delta^2(t)\int_\Omega v_0^2,
\end{aligned} \tag{5.3.7}$$

because clearly $\frac{1}{(u_\varepsilon+\delta(t))^2}\le\frac{1}{\delta^2(t)}$ in $\Omega\times(0,T_{max,\varepsilon})$, and because $\int_\Omega v_\varepsilon^2(\cdot,t)\le\int_\Omega v_0^2$ for all $t\in(0,T_{max,\varepsilon})$ by (5.2.5). Combining (5.3.6) with (5.3.7) yields (5.3.3). □

Now suitably choosing the function δ turns (5.3.3) into the following version of (5.1.12) that forms the nucleus for all our subsequent arguments.

Lemma 5.3.3. *There exists $\Gamma_1 > 0$ with the property that*

$$\int_0^t \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 \le \Gamma_1 \cdot \ln(t+e) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1), \tag{5.3.8}$$

and that

$$\int_0^t \int_\Omega |\Delta v_\varepsilon|^2 \le \Gamma_1 \cdot \ln(t+e) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{5.3.9}$$

Proof. We fix any $\kappa > 1$ and apply Lemma 5.3.2 to

$$\delta(t) := (t+1)^{-\kappa}, \qquad t \ge 0. \tag{5.3.10}$$

Since

$$\frac{\delta'(t)}{\delta(t)} = -\kappa(t+1)^{-1} \qquad \text{for all } t \ge 0,$$

and since the inequality $\delta \le 1$ together with (5.2.4) and the fact that $\ln(\xi+1) \le \xi$ for all $\xi \ge 0$ ensures that

$$\int_\Omega \ln\left(u_\varepsilon + \delta(t)\right) \le \int_\Omega \ln(u_\varepsilon+1) \le \int_\Omega u_\varepsilon \le m \qquad \text{for all } t \in [0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

we thereby obtain on writing

$$y_\varepsilon(t) := m - \int_\Omega \ln\left(u_\varepsilon + \delta(t)\right) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2, \qquad t \in [0, T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

and

$$g_\varepsilon(t) := \frac{1}{2}\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2}|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2, \qquad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

that

$$y_\varepsilon \ge \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 \ge 0 \qquad \text{for all } t \in [0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{5.3.11}$$

and

$$\begin{aligned} y_\varepsilon'(t) + g_\varepsilon(t) \quad &\le \quad \frac{1}{2}c_1(t+1)^{-\kappa}\int_\Omega |\nabla v_\varepsilon|^2 + \kappa|\Omega| \cdot (t+1)^{-1} \\ &\quad + \frac{1}{2}(t+1)^{-2\kappa}\int_\Omega v_0^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \end{aligned}$$

with $c_1 := \|v_0\|_{L^\infty(\Omega)}$. Using the first inequality in (5.3.11), we thus infer by estimating $(t+1)^{-2\kappa} \le (t+1)^{-1}$ for $t \ge 0$ that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$y_\varepsilon'(t) + g_\varepsilon(t) \le c_1(t+1)^{-\kappa}y_\varepsilon(t) + c_2(t+1)^{-1}, \tag{5.3.12}$$

where $c_2 := \kappa|\Omega| + \frac{1}{2}\int_\Omega v_0^2$. Thanks to the nonnegativity of the g_ε, through an ODE comparison argument this firstly entails that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$y_\varepsilon(t) \quad \le \quad y_\varepsilon(0) \cdot \exp\left\{c_1 \int_0^t (s+1)^{-\kappa} ds\right\}$$

$$+c_2 \int_0^t \exp\left\{c_1 \int_s^t (\sigma+1)^{-\kappa} d\sigma\right\} \cdot (s+1)^{-1} ds,$$

and hence, as

$$\begin{aligned} y_\varepsilon(0) &= m - \int_\Omega \ln(u_0+\varepsilon+1) + \frac{1}{2}\int_\Omega |\nabla v_0|^2 \\ &\le c_3 := m + \frac{1}{2}\int_\Omega |\nabla v_0|^2 \qquad \text{for all } \varepsilon \in (0,1) \end{aligned} \tag{5.3.13}$$

and

$$\int_s^t (\sigma+1)^{-\kappa} d\sigma \le \int_0^\infty (\sigma+1)^{-\kappa} d\sigma = \frac{1}{\kappa-1} \qquad \text{for all } s \ge 0 \text{ and } t > s$$

due to the restriction that $\kappa > 1$, that

$$\begin{aligned} y_\varepsilon(t) &\le c_3 e^{\frac{c_1}{\kappa-1}} + c_2 e^{\frac{c_1}{\kappa-1}} \int_0^t (s+1)^{-1} ds \\ &= c_4 + c_5 \ln(t+1) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \end{aligned} \tag{5.3.14}$$

with $c_4 := c_3 e^{\frac{c_1}{\kappa-1}}$ and $c_5 := c_2 e^{\frac{c_1}{\kappa-1}}$. Thereupon, inserting (5.3.14) into (5.3.12) shows that by (5.3.11), and again by (5.3.13),

$$\begin{aligned} \int_0^t g_\varepsilon(s) ds &\le y_\varepsilon(0) + c_1 \int_0^t (s+1)^{-\kappa} y_\varepsilon(s) ds + c_2 \int_0^t (s+1)^{-1} ds \\ &\le c_3 + c_1 c_4 \int_0^t (s+1)^{-\kappa} ds + c_1 c_5 \int_0^t (s+1)^{-\kappa} \ln(s+1) ds \\ &\quad + c_2 \int_0^t (s+1)^{-1} ds \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Since $\xi^{1-\kappa} \ln \xi \le \frac{1}{(\kappa-1)e}$ for all $\xi > 0$ and thus

$$\ln(t+1) \le \frac{1}{(\kappa-1)e} \cdot (t+1)^{\kappa-1} \qquad \text{for all } t \ge 0,$$

this implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \int_0^t g_\varepsilon(s) ds &\le c_3 + \left\{c_1 c_4 + \frac{c_1 c_5}{(\kappa-1)e} + c_2\right\} \cdot \int_0^t (s+1)^{-1} ds \\ &= c_3 + \left\{c_1 c_4 + \frac{c_1 c_5}{(\kappa-1)e} + c_2\right\} \cdot \ln(t+1), \end{aligned}$$

from which (5.3.8) follows if we let $\Gamma_1 := 2 \cdot \{c_3 + c_1 c_4 + \frac{c_1 c_5}{(\kappa-1)e} + c_2\}$, because $\frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+\delta(t))^2} \ge \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}$ in $\Omega \times (0, T_{max,\varepsilon})$ for all $\varepsilon \in (0,1)$ by (5.3.10). □

5.4 Short time analysis

5.4.1 Space-time bounds for $\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}$, $\frac{|\Delta v_\varepsilon|^2}{v_\varepsilon}$ and $\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2$ as well as $u_\varepsilon^2 v_\varepsilon$

A first application of Lemma 5.3.3 will concentrate on the derivation of further estimates valid on fixed time intervals of finite length. This will be achieved in Lemma 5.4.2, where the integral on the right-hand side of (5.2.14) will be controlled by means of the following functional inequality.

Lemma 5.4.1. *Let $\eta > 0$. Then whenever $\phi \in C^1(\overline{\Omega})$ and $\psi \in C^2(\overline{\Omega})$ are such that $\phi \geq 0$ in Ω, $\psi > 0$ in $\overline{\Omega}$ and $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$,*

$$\Big|\int_\Omega \nabla\phi\cdot\nabla\psi\Big| \leq \eta\int_\Omega \frac{\phi}{\psi}|\nabla\psi|^2 + \eta\int_\Omega |\Delta\psi|^2 + \frac{1}{\eta}\int_\Omega \frac{\phi\psi}{(\phi+1)^2}|\nabla\phi|^2 + \frac{|\Omega|}{\eta}. \tag{5.4.1}$$

Proof. We fix $\rho \in C_0^\infty([0,\infty))$ such that $\rho \equiv 1$ on $[0,1]$, $\rho \equiv 0$ on $[2,\infty)$ and $0 \leq \rho \leq 1$, and let $P(\xi) := \int_0^\xi \rho(\sigma)d\sigma$ for $\xi \geq 0$. The decomposing

$$\int_\Omega \nabla\phi\cdot\nabla\psi = \int_\Omega \rho(\phi)\nabla\phi\cdot\nabla\psi + \int_\Omega (1-\rho(\phi))\nabla\phi\cdot\nabla\psi, \tag{5.4.2}$$

we can integrate by parts and use that $\frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0$ as well as $0 \leq P(\xi) \leq 2$ for all $\xi \geq 0$ to see that due to Young's inequality,

$$\begin{aligned}\Big|\int_\Omega \rho(\phi)\nabla\phi\cdot\nabla\psi\Big| &= \Big|\int_\Omega \nabla P(\phi)\cdot\nabla\psi\Big| = \Big|-\int_\Omega P(\phi)\Delta\psi\Big| \\ &\leq \eta\int_\Omega |\Delta\psi|^2 + \frac{1}{4\eta}\int_\Omega P^2(\phi) \leq \eta\int_\Omega |\Delta\psi|^2 + \frac{|\Omega|}{\eta}.\end{aligned} \tag{5.4.3}$$

In the last summand in (5.4.2), using that $1-\rho(\xi) = 0$ for all $\xi \leq 1$ we may directly employ Young's inequality to find that

$$\Big|\int_\Omega (1-\rho(\phi))\nabla\phi\cdot\nabla\psi\Big| \leq \eta\int_\Omega \frac{\phi}{\psi}|\nabla\psi|^2 + \frac{1}{4\eta}\int_{\{\phi>1\}} \frac{(1-\rho(\phi))^2\psi}{\phi}|\nabla\phi|^2, \tag{5.4.4}$$

so that since the inequality $0 \leq 1-\rho \leq 1$ ensures that

$$\frac{\frac{(1-\rho(\phi))^2}{\phi}}{\frac{\phi}{(\phi+1)^2}} = \big(1-\rho(\phi)\big)^2\cdot\frac{(\phi+1)^2}{\phi^2} \leq \frac{(\phi+1)^2}{\phi^2} = \Big(1+\frac{1}{\phi}\Big)^2 \leq 4 \qquad \text{in } \{\phi > 1\},$$

it follows that

$$\Big|\int_\Omega (1-\rho(\phi))\nabla\phi\cdot\nabla\psi\Big| \leq \eta\int_\Omega \frac{\phi}{\psi}|\nabla\psi|^2 + \frac{1}{\eta}\int_\Omega \frac{\phi\psi}{(\phi+1)^2}|\nabla\phi|^2.$$

In conjunction with (5.4.3) and (5.4.2), this establishes (5.4.1). □

Indeed, we can thereby turn Lemma 5.3.3 into further bounds for the signal v_ε. We may here refrain from tracing particular functional dependencies of the constants in (5.4.5)-(5.4.7) on the length T of the time intervals under consideration, as these will not be of relevance for our subsequent purposes.

Lemma 5.4.2. *For all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq C(T) \qquad \textit{for all } t \in (0,T)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1), \tag{5.4.5}$$

that

$$\int_0^t \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} \le C(T) \qquad \textit{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1), \tag{5.4.6}$$

and that

$$\int_0^t \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \le C(T) \qquad \textit{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{5.4.7}$$

Proof. An application of Lemma 5.4.1 yields $c_1 > 0$ such that

$$\begin{aligned} -2\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon &\le \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 \\ &\quad + c_1 \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 + c_1 \qquad \text{for all } t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

whence using Lemma 5.2.5 we can find $c_2 > 0$ and $c_3 > 0$ fulfilling

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + c_2 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2 \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\quad \le \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + c_1 \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 + c_3 \qquad \text{for all } t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned}$$

After an integration, in view of Lemma 5.3.3 this shows that with Γ_1 as provided there,

$$\begin{aligned} &\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} + c_2 \int_0^t \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2 \int_0^t \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2}\int_0^t \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\quad \le \int_\Omega \frac{|\nabla v_0|^2}{v_0} + \frac{1}{2}\int_0^t \int_\Omega |\Delta v_\varepsilon|^2 + c_1 \int_0^t \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 + c_3 t \\ &\quad \le \int_\Omega \frac{|\nabla v_0|^2}{v_0} + \Big(\frac{1}{2} + c_1\Big)\Gamma_1 \ln(t+e) + c_3 t \\ &\quad \le \int_\Omega \frac{|\nabla v_0|^2}{v_0} + \Big(\frac{1}{2} + c_1\Big)\Gamma_1 \ln(T+e) + c_3 T \end{aligned}$$

for all $t \in (0,T) \cap (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which establishes (5.4.5)-(5.4.7) with an evident choice of $C(T)$. □

By an appropriate interpolation using the Sobolev inequality (5.2.7), we can develop (5.3.8) and (5.4.7) into a weighted space-time L^2 bound for u_ε.

Lemma 5.4.3. *For all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^t \int_\Omega u_\varepsilon^2 v_\varepsilon \le C(T) \qquad \textit{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{5.4.8}$$

Proof. We let $\rho \in C^\infty([0,\infty))$ be such that $\rho \equiv 0$ on $[0,1]$, $\rho \equiv 1$ on $[2,\infty)$ and $0 \le \rho' \le 2$, and combine (5.2.7) with Young's inequality to see that with $\Lambda = \Lambda(1)$ taken from Lemma 5.2.2,

$$\int_\Omega \rho^2(u_\varepsilon) u_\varepsilon^2 v_\varepsilon = \|\rho(u_\varepsilon) u_\varepsilon \sqrt{v_\varepsilon}\|_{L^2(\Omega)}^2$$

$$\begin{aligned}
&\le 2\Lambda^2\|\nabla(\rho(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon})\|_{L^1(\Omega)}^2 + 2\Lambda^2\|\rho(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}\|_{L^1(\Omega)}^2 \\
&= 2\Lambda^2 \cdot \Big\{ \int_\Omega \Big|\rho(u_\varepsilon)\sqrt{v_\varepsilon}\nabla u_\varepsilon + \rho'(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}\nabla u_\varepsilon + \frac{\rho(u_\varepsilon)u_\varepsilon}{2\sqrt{v_\varepsilon}}\nabla v_\varepsilon\Big|\Big\}^2 \\
&\quad +2\Lambda^2 \cdot \Big\{ \int_\Omega \rho(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}\Big\}^2 \\
&\le 6\Lambda^2 \cdot \Big\{ \int_\Omega \rho(u_\varepsilon)\sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 + 6\Lambda^2 \cdot \Big\{ \int_\Omega \rho'(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 \\
&\quad +\frac{3\Lambda^2}{2} \cdot \Big\{ \int_\Omega \frac{\rho(u_\varepsilon)u_\varepsilon}{\sqrt{v_\varepsilon}}|\nabla v_\varepsilon|\Big\}^2 + 2\Lambda^2 \cdot \Big\{ \int_\Omega \rho(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}\Big\}^2 \qquad (5.4.9)
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Here, the Cauchy-Schwarz inequality implies that since $\operatorname{supp}\rho \subset [1,\infty)$ and $0 \le \rho \le 1$,

$$\begin{aligned}
\Big\{ \int_\Omega \rho(u_\varepsilon)\sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 &\le \Big\{ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\Big\} \cdot \int_\Omega \frac{\rho^2(u_\varepsilon)(u_\varepsilon+1)^2}{u_\varepsilon} \\
&\le \Big\{ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\Big\} \cdot \int_{\{u_\varepsilon \ge 1\}} \frac{(u_\varepsilon+1)^2}{u_\varepsilon} \\
&\le 4 \cdot \Big\{ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\Big\} \cdot \int_\Omega u_\varepsilon \qquad (5.4.10)
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, because

$$\frac{(\xi+1)^2}{\xi} = \Big(1+\frac{1}{\xi}\Big)^2 \cdot \xi \le 4\xi \qquad \text{for all } \xi \ge 1.$$

Next, again by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\Big\{ \int_\Omega \rho'(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 \\
&\quad \le \Big\{ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\Big\} \cdot \int_\Omega \rho'^2(u_\varepsilon)u_\varepsilon(u_\varepsilon+1)^2 \\
&\quad \le 4 \cdot \Big\{ \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\Big\} \cdot \int_{\{u_\varepsilon \le 2\}} u_\varepsilon(u_\varepsilon+1)^2 \\
&\quad \le 72 \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \qquad (5.4.11)
\end{aligned}$$

as $\rho' \equiv 0$ on $(2,\infty)$ and $\xi(\xi+1)^2 \le 18$ for all $\xi \in [0,2]$. Again since $0 \le \rho \le 1$, one final application of the Cauchy-Schwarz inequality reveals that

$$\Big\{ \int_\Omega \frac{\rho(u_\varepsilon)u_\varepsilon}{\sqrt{v_\varepsilon}}|\nabla v_\varepsilon|\Big\}^2 \le \Big\{ \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2\Big\} \cdot \int_\Omega \rho^2(u_\varepsilon)u_\varepsilon \le \Big\{ \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2\Big\} \cdot \int_\Omega u_\varepsilon \qquad (5.4.12)$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, whereas due to (5.2.5),

$$\Big\{ \int_\Omega \rho(u_\varepsilon)u_\varepsilon\sqrt{v_\varepsilon}\Big\}^2 \le \|v_0\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega u_\varepsilon\Big\}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and} \varepsilon \in (0,1). \qquad (5.4.13)$$

Now (5.4.10)-(5.4.13) in conjunction with (5.2.4) show that (5.4.9) entails the inequality

$$\begin{aligned}\int_0^t\int_\Omega \rho^2(u_\varepsilon)u_\varepsilon^2 v_\varepsilon \quad &\le \quad (24\Lambda^2 m + 432\Lambda^2)\int_0^t\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2 + \frac{3\Lambda^2 m}{2}\int_0^t\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\quad +2\Lambda^2\|v_0\|_{L^\infty(\Omega)}m^2 t \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),\end{aligned}$$

whence noting that furthermore for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\int_0^t\int_\Omega (1-\rho^2(u_\varepsilon))u_\varepsilon^2 v_\varepsilon \le \int_0^t\int_{\{u_\varepsilon\le 2\}} u_\varepsilon^2 v_\varepsilon \le 4\cdot\Big\{\int_\Omega v_0\Big\}\cdot t$$

according to (5.2.5), we obtain (5.4.8) as a consequence of Lemmas 5.3.3 and 5.4.2. □

5.4.2 Bounds for u_ε in $L\log L$ and $L\log^2 L$

We next approach the second core part of our analysis, by namely striving for an estimate of the form in (5.1.15) as a sufficient prerequisite for the pointwise lower bound in (5.1.16). The first step in this direction is yet fairly straightforward and uses Lemma 5.4.2 and Lemma 5.4.3 to control the growth of the functional in (5.1.13) along trajectories of (5.2.1).

Lemma 5.4.4. *For all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_\Omega u_\varepsilon(\cdot,t)|\ln u_\varepsilon(\cdot,t)| \le C(T) \qquad \textit{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1) \tag{5.4.14}$$

as well as

$$\int_0^t\int_\Omega v_\varepsilon|\nabla u_\varepsilon - u_\varepsilon\nabla v_\varepsilon|^2 \le C(T) \qquad \textit{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{5.4.15}$$

Proof. For all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, from (5.2.12) it follows that

$$\frac{d}{dt}\int_\Omega u_\varepsilon(\ln u_\varepsilon - 1) = -\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon, \tag{5.4.16}$$

and that

$$\begin{aligned}-\frac{d}{dt}\int_\Omega u_\varepsilon v_\varepsilon \quad &= \quad -\int_\Omega v_\varepsilon\cdot\{\nabla\cdot(u_\varepsilon v_\varepsilon\nabla u_\varepsilon) - \nabla\cdot(u_\varepsilon^2 v_\varepsilon\nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon\} \\ &\quad -\int_\Omega u_\varepsilon\cdot\{\Delta v_\varepsilon - u_\varepsilon v_\varepsilon\} \\ &= \quad \int_\Omega u_\varepsilon v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon|^2 - \ell\int_\Omega u_\varepsilon v_\varepsilon^2 \\ &\quad -\int_\Omega u_\varepsilon\Delta v_\varepsilon + \int_\Omega u_\varepsilon^2 v_\varepsilon.\end{aligned} \tag{5.4.17}$$

Since $\ln\xi \le \xi$ for all $\xi > 0$ and thus for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\ell\int_\Omega u_\varepsilon v_\varepsilon\ln u_\varepsilon - \int_\Omega u_\varepsilon\Delta v_\varepsilon + \int_\Omega u_\varepsilon^2 v_\varepsilon \le (\ell+2)\int_\Omega u_\varepsilon^2 v_\varepsilon + \frac14\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon}$$

by Young's inequality, and since

$$-\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + 2\int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon - \int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon|^2 = -\int_\Omega v_\varepsilon|\nabla u_\varepsilon - u_\varepsilon\nabla v_\varepsilon|^2$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, by trivially estimating $-\ell\int_\Omega u_\varepsilon v_\varepsilon^2\le 0$ we obtain on adding (5.4.17) to (5.4.16) that

$$\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon(\ln u_\varepsilon-1) - \int_\Omega u_\varepsilon v_\varepsilon\Big\} + \int_\Omega v_\varepsilon|\nabla u_\varepsilon - u_\varepsilon\nabla v_\varepsilon|^2 \le (\ell+2)\int_\Omega u_\varepsilon^2 v_\varepsilon + \frac14\int_\Omega\frac{|\Delta v_\varepsilon|^2}{v_\varepsilon}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Therefore,

$$\begin{aligned}
&\int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + \int_0^t\int_\Omega v_\varepsilon|\nabla u_\varepsilon - u_\varepsilon\nabla v_\varepsilon|^2 \\
&\quad\le \int_\Omega(u_0+\varepsilon)\ln(u_0+\varepsilon) + \int_\Omega u_\varepsilon - \int_\Omega(u_0+\varepsilon) + \int_\Omega u_\varepsilon v_\varepsilon - \int_\Omega(u_0+\varepsilon)v_0 \\
&\qquad+(\ell+2)\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon + \frac14\int_0^t\int_\Omega\frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

As (5.2.4) and (5.2.5) ensure that

$$\begin{aligned}
&\int_\Omega(u_0+\varepsilon)\ln(u_0+\varepsilon) + \int_\Omega u_\varepsilon(\cdot,t) - \int_\Omega(u_0+\varepsilon) + \int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) - \int_\Omega(u_0+\varepsilon)v_0 \\
&\quad\le \int_\Omega(u_0+1)\ln(u_0+1) + m + m\|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

the claim thus results from Lemma 5.4.2 and Lemma 5.4.3 due to the fact that $\xi\ln\xi\ge-\frac1e$ for all $\xi>0$, and that hence

$$\int_\Omega u_\varepsilon\ln u_\varepsilon = \int_\Omega u_\varepsilon|\ln u_\varepsilon| + 2\int_{\{u_\varepsilon<1\}} u_\varepsilon\ln u_\varepsilon \ge \int_\Omega u_\varepsilon|\ln u_\varepsilon| - \frac{2|\Omega|}{e}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. □

Again by interpolation, the $L\log L$ bound implied by (5.4.14) can be combined with Lemma 5.4.2 to introduce a logarithmic refinement in the estimate from Lemma 5.4.3 as follows.

Lemma 5.4.5. *Let $T>0$. Then one can find $C(T)>0$ such that*

$$\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon\ln^2 u_\varepsilon \le C(T) \qquad \textit{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{5.4.18}$$

Proof. We let $\rho\in C^\infty([0,\infty))$ be such that $\rho\equiv0$ on $[0,1]$, $\rho\equiv1$ on $[2,\infty)$ and $0\le\rho'\le2$, and take $\Lambda=\Lambda(1)$ as in Lemma 5.2.2. For $\varepsilon\in(0,1)$ writing $z_\varepsilon:=u_\varepsilon e^{-v_\varepsilon}$, we apply (5.2.8) to $\alpha:=1$ and $\gamma:=4$ and thereby obtain that

$$\begin{aligned}
\int_\Omega\rho^2(z_\varepsilon)v_\varepsilon z_\varepsilon^2\ln^2 z_\varepsilon &\le 8\Lambda^2\cdot\Big\{\int_\Omega\sqrt{v_\varepsilon}|\nabla(\rho(z_\varepsilon)z_\varepsilon\ln z_\varepsilon)|\Big\}^2 \\
&\quad+4\Lambda^4\cdot\Big\{\int_\Omega\rho(z_\varepsilon)z_\varepsilon|\ln z_\varepsilon|\Big\}^2\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}
\end{aligned}$$

$$+4\Lambda^2 \cdot \left\{ \int_\Omega \rho(z_\varepsilon) z_\varepsilon |\ln z_\varepsilon| \right\} \cdot \int_\Omega \rho(z_\varepsilon) v_\varepsilon z_\varepsilon |\ln z_\varepsilon| \qquad (5.4.19)$$

for all $t \in (0, T_{max,\varepsilon})$, where since $0 \le \rho \le 1$, according to (5.2.5) we have

$$\begin{aligned} 4\Lambda^4 \cdot \left\{ \int_\Omega \rho(z_\varepsilon) z_\varepsilon |\ln z_\varepsilon| \right\}^2 \cdot \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 4\Lambda^2 \cdot \left\{ \int_\Omega \rho(z_\varepsilon) z_\varepsilon |\ln z_\varepsilon| \right\} \cdot \int_\Omega \rho(z_\varepsilon) v_\varepsilon z_\varepsilon |\ln z_\varepsilon| \\ \le \left\{ 4\Lambda^4 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 4\Lambda^2 \|v_0\|_{L^\infty(\Omega)} \right\} \cdot \left\{ \int_\Omega z_\varepsilon |\ln z_\varepsilon| \right\}^2 \qquad (5.4.20) \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$. Moreover, as $\operatorname{supp}\rho \subset [1,\infty)$, $\operatorname{supp}\rho' \subset (1,2)$ and $0 \le \rho' \le 2$, using the Hölder inequality and the Cauchy-Schwarz inequality shows that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} & 8\Lambda^2 \cdot \left\{ \int_\Omega \sqrt{v_\varepsilon} |\nabla(\rho(z_\varepsilon) z_\varepsilon \ln z_\varepsilon)| \right\}^2 \\ &= 8\Lambda^2 \cdot \left\{ \int_\Omega \sqrt{v_\varepsilon} |\rho(z_\varepsilon)\nabla z_\varepsilon + \rho(z_\varepsilon)\ln z_\varepsilon \nabla z_\varepsilon + \rho'(z_\varepsilon) z_\varepsilon \ln z_\varepsilon \nabla z_\varepsilon| \right\}^2 \\ &\le 8\Lambda^2 \cdot \left\{ \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 \right\} \cdot \int_\Omega \Big(\rho(z_\varepsilon) + \rho(z_\varepsilon)\ln z_\varepsilon + \rho'(z_\varepsilon) z_\varepsilon \ln z_\varepsilon \Big)^2 \\ &\le 24\Lambda^2 \cdot \left\{ \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 \right\} \cdot \left\{ \int_\Omega \rho^2(z_\varepsilon)(1 + \ln^2 z_\varepsilon) + \int_\Omega \rho'^2(z_\varepsilon) z_\varepsilon^2 \ln^2 z_\varepsilon \right\} \\ &\le 24\Lambda^2 \cdot \left\{ \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 \right\} \cdot \left\{ \int_{\{z_\varepsilon \ge 1\}} (1 + \ln^2 z_\varepsilon) + 4 \int_{\{1 \le z_\varepsilon \le 2\}} z_\varepsilon^2 \ln^2 z_\varepsilon \right\} \\ &\le 24\Lambda^2 \cdot \left\{ \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 \right\} \cdot \left\{ \int_{\{z_\varepsilon \ge 1\}} \ln^2 z_\varepsilon + 17|\Omega| \right\}, \qquad (5.4.21) \end{aligned}$$

whence inserting (5.4.20) and (5.4.21) into (5.4.19) we find that

$$\begin{aligned} \int_\Omega \rho^2(z_\varepsilon) v_\varepsilon z_\varepsilon^2 \ln^2 z_\varepsilon &\le 24\Lambda^2 \cdot \left\{ \int_{\{z_\varepsilon \ge 1\}} \ln^2 z_\varepsilon + 17|\Omega| \right\} \cdot \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 \\ &\quad + \left\{ 4\Lambda^4 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 4\Lambda^2 \|v_0\|_{L^\infty(\Omega)} \right\} \cdot \left\{ \int_\Omega z_\varepsilon |\ln z_\varepsilon| \right\}^2 \qquad (5.4.22) \end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Since from the definition of the z_ε it follows that

$$\begin{aligned} \int_\Omega v_\varepsilon |\nabla z_\varepsilon|^2 &= \int_\Omega v_\varepsilon e^{-2v_\varepsilon} |\nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon|^2 \\ &\le \int_\Omega v_\varepsilon |\nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon|^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

and that for any such t we moreover have

$$\begin{aligned} \int_{\{z_\varepsilon \ge 1\}} \ln^2 z_\varepsilon &\le \int_\Omega z_\varepsilon |\ln z_\varepsilon| = \int_\Omega u_\varepsilon e^{-v_\varepsilon} |\ln u_\varepsilon - v_\varepsilon| \\ &\le \int_\Omega u_\varepsilon |\ln u_\varepsilon| + \int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega u_\varepsilon |\ln u_\varepsilon| + m\|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } \varepsilon \in (0,1) \end{aligned}$$

because of (5.2.4) and (5.2.5), for each $T>0$ using that $\xi^2\ln^2\xi \le 4$ for $\xi\in(0,2)$ and thus, again by (5.2.5),

$$\int_0^t \int_{\{z_\varepsilon<2\}} v_\varepsilon z_\varepsilon^2 \ln^2 z_\varepsilon \le 4\|v_0\|_{L^\infty(\Omega)}|\Omega|T \qquad \text{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1)$$

from Lemma 5.4.4, Lemma 5.4.2 and the fact that $\rho\equiv 1$ on $[2,\infty)$ we infer that there exists $c_1=c_1(T)>0$ such that

$$\begin{aligned}\int_0^t\int_\Omega v_\varepsilon z_\varepsilon^2\ln^2 z_\varepsilon &\le \int_0^t\int_\Omega \rho^2(z_\varepsilon)v_\varepsilon z_\varepsilon^2\ln^2 z_\varepsilon + \int_0^t\int_{\{z_\varepsilon<2\}} v_\varepsilon z_\varepsilon^2\ln^2 z_\varepsilon \\ &\le c_1 \qquad \text{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).\end{aligned}$$

Since from the definition of $(z_\varepsilon)_{\varepsilon\in(0,1)}$ and (5.2.5) it furthermore follows that for all $t\in(0,T_{max,\varepsilon})$ and any $\varepsilon\in(0,1)$ we have

$$\begin{aligned}\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2 u_\varepsilon &= \int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon(\ln z_\varepsilon + v_\varepsilon)^2 \\ &\le 2\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon\ln^2 z_\varepsilon + 2\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon^3 \\ &= 2\int_0^t\int_\Omega v_\varepsilon e^{2v_\varepsilon} z_\varepsilon^2\ln^2 z_\varepsilon + 2\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon^3 \\ &\le 2e^{2\|v_0\|_{L^\infty(\Omega)}}\int_0^t\int_\Omega v_\varepsilon z_\varepsilon^2\ln^2 z_\varepsilon + 2\|v_0\|_{L^\infty(\Omega)}^2\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon,\end{aligned}$$

when combined with Lemma 5.4.3 this yields the claim. □

According to the previous lemma, we are now in the position to control ill-signed contributions to the evolution of the functional in (5.1.14), and to thereby improve (5.4.14) in the following sense.

Lemma 5.4.6. *For any $T>0$, there exists $C(T)>0$ such that*

$$\int_\Omega u_\varepsilon(\cdot,t)\ln^2\big(u_\varepsilon(\cdot,t)+e\big) \le C(T) \qquad \text{for all } t\in(0,T)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \tag{5.4.23}$$

Proof. For $\varepsilon\in(0,1)$, we again let $z_\varepsilon := u_\varepsilon e^{-v_\varepsilon}$ and now use that according to (5.2.1) we have

$$\begin{aligned} z_{\varepsilon t} &= e^{-v_\varepsilon}u_{\varepsilon t} - u_\varepsilon e^{-v_\varepsilon}v_{\varepsilon t} \\ &= e^{-v_\varepsilon}\nabla\cdot\big\{u_\varepsilon v_\varepsilon e^{v_\varepsilon}e^{-v_\varepsilon}(\nabla u_\varepsilon - u_\varepsilon\nabla v_\varepsilon)\big\} + \ell u_\varepsilon v_\varepsilon e^{-v_\varepsilon} - u_\varepsilon e^{-v_\varepsilon}\Delta v_\varepsilon + u_\varepsilon^2 v_\varepsilon e^{-v_\varepsilon} \\ &= e^{-v_\varepsilon}\nabla\cdot(u_\varepsilon v_\varepsilon e^{v_\varepsilon}\nabla z_\varepsilon) + \ell u_\varepsilon v_\varepsilon e^{-v_\varepsilon} - u_\varepsilon e^{-v_\varepsilon}\Delta v_\varepsilon + u_\varepsilon^2 v_\varepsilon e^{-v_\varepsilon} \end{aligned} \tag{5.4.24}$$

in $\Omega\times(0,T_{max,\varepsilon})$. We moreover note that

$$\Phi(\xi) := \xi\ln^2(\xi+e), \qquad \xi\ge 0$$

satisfies

$$\Phi'(\xi) = \ln^2(\xi+e) + \frac{2\xi}{\xi+e}\ln(\xi+e) \qquad \text{for all } \ \xi\ge 0$$

and

$$\Phi''(\xi) = \frac{2\ln(\xi+e)}{\xi+e} + \frac{2\xi}{(\xi+e)^2} + \frac{2e}{(\xi+e)^2}\ln(\xi+e) \qquad \text{for all } \xi \geq 0,$$

whence it particularly follows that Φ is nonnegative and convex, and that

$$-\xi\Phi'(\xi) + \Phi(\xi) = -\frac{2\xi^2}{\xi+e}\ln(\xi+e) \qquad \text{for all } \xi \geq 0$$

and

$$\Phi'(\xi) \leq \ln^2(\xi+e) + 2\ln(\xi+e) \leq 3\ln^2(\xi+e) \qquad \text{for all } \xi > 0,$$

and thus

$$\Phi'(\xi) \leq 3(1+\ln 2)^2 \ln^2 \xi \qquad \text{for all } \xi \geq e, \tag{5.4.25}$$

because

$$\frac{\ln(\xi+e)}{\ln\xi} \leq \frac{\ln(e+e)}{\ln e} = 1 + \ln 2 \qquad \text{for all } \xi \geq e. \tag{5.4.26}$$

From (5.4.24) and the second equation in (5.2.1) we therefore obtain that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon} &= \int_\Omega \Phi'(z_\varepsilon)e^{v_\varepsilon}z_{\varepsilon t} + \int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon}v_{\varepsilon t} \\
&= -\int_\Omega \Phi''(z_\varepsilon)u_\varepsilon v_\varepsilon e^{v_\varepsilon}|\nabla z_\varepsilon|^2 + \ell\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon v_\varepsilon \\
&\quad -\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon\Delta v_\varepsilon + \int_\Omega \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon \\
&\quad +\int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon}\Delta v_\varepsilon - \int_\Omega \Phi(z_\varepsilon)u_\varepsilon v_\varepsilon e^{v_\varepsilon} \\
&\leq \int_\Omega \{-\Phi'(z_\varepsilon)z_\varepsilon + \Phi(z_\varepsilon)\}e^{v_\varepsilon}\Delta v_\varepsilon + \ell\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon v_\varepsilon + \int_\Omega \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon \\
&= -2\int_\Omega \frac{z_\varepsilon^2}{z_\varepsilon+e}\ln(z_\varepsilon+e)e^{v_\varepsilon}\Delta v_\varepsilon + \ell\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon v_\varepsilon + \int_\Omega \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon,
\end{aligned} \tag{5.4.27}$$

and that here for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\ell\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon v_\varepsilon \leq 3\ell\int_\Omega u_\varepsilon v_\varepsilon \ln^2(u_\varepsilon+e) \leq \frac{12\ell}{e^2}\int_\Omega u_\varepsilon^2 v_\varepsilon + \frac{12\ell}{e}\int_\Omega u_\varepsilon v_\varepsilon \tag{5.4.28}$$

and

$$\begin{aligned}
\int_\Omega \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon &= \int_{\{z_\varepsilon<e\}} \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon + \int_{\{z_\varepsilon\geq e\}} \Phi'(z_\varepsilon)u_\varepsilon^2 v_\varepsilon \\
&\leq 3(1+\ln 2)^2\int_\Omega u_\varepsilon^2 v_\varepsilon + 3(1+\ln 2)^2\int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2 u_\varepsilon,
\end{aligned} \tag{5.4.29}$$

because $\ln^2\xi \leq \frac{4\xi}{e^2}$ for all $\xi \geq 1$, because $z_\varepsilon \leq u_\varepsilon$, and because the convexity of Φ ensures that $\Phi'(\xi) \leq \Phi'(e) \leq 3(1+\ln 2)^2$ for all $\xi \in (0,e)$. Apart from that, Young's inequality together with (5.4.26) guarantees that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$-2\int_\Omega \frac{z_\varepsilon^2}{z_\varepsilon+e}\ln(z_\varepsilon+e)e^{v_\varepsilon}\Delta v_\varepsilon$$

$$\begin{aligned}
&\le \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega \Big|\frac{z_\varepsilon^2}{z_\varepsilon + e}\ln(z_\varepsilon+e)e^{v_\varepsilon}\Big|^2 v_\varepsilon \\
&\le \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2(z_\varepsilon+e) \\
&= \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_{\{z_\varepsilon<e\}} u_\varepsilon^2 v_\varepsilon \ln^2(z_\varepsilon+e) + \int_{\{z_\varepsilon\ge e\}} u_\varepsilon^2 v_\varepsilon \ln^2(z_\varepsilon+e) \\
&\le \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + (1+\ln 2)^2 \int_\Omega u_\varepsilon^2 v_\varepsilon + (1+\ln 2)^2 \int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2 u_\varepsilon. \qquad (5.4.30)
\end{aligned}$$

Now inserting (5.4.28)-(5.4.30) into (5.4.27) shows that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon} \le \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + c_1\int_\Omega u_\varepsilon^2 v_\varepsilon + c_2\int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2 u_\varepsilon + c_3\int_\Omega u_\varepsilon v_\varepsilon$$

with $c_1 := \frac{12\ell}{e^2} + 4(1+\ln 2)^2$, $c_2 := 4(1+\ln 2)^2$ and $c_3 := \frac{12\ell}{e}$, so that

$$\begin{aligned}
\int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon} \le\; & \int_\Omega \Phi((u_0+\varepsilon)e^{-v_0})e^{v_0} + \int_0^t\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} \\
&+ c_1\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon + c_2\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon \ln^2 u_\varepsilon \\
&+ c_3\int_0^t\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1). \qquad (5.4.31)
\end{aligned}$$

As, on the other hand, for any $\varepsilon \in (0,1)$,

$$\ln(z_\varepsilon+e) = \ln\big(u_\varepsilon e^{-v_\varepsilon}+e\big) \ge \ln(u_\varepsilon e^{-v_\varepsilon} + e\cdot e^{-v_\varepsilon}) = \ln(u_\varepsilon+e) - v_\varepsilon$$

and hence

$$\ln^2(z_\varepsilon+e) \ge \frac{1}{2}\ln^2(u_\varepsilon+e) - v_\varepsilon^2$$

in $\Omega\times(0,T_{max,\varepsilon})$ by Young's inequality, we can use (5.2.4) and (5.2.5) to estimate

$$\begin{aligned}
\int_\Omega \Phi(z_\varepsilon)e^{v_\varepsilon} &= \int_\Omega u_\varepsilon \ln^2(z_\varepsilon+e) \ge \frac{1}{2}\int_\Omega u_\varepsilon\ln^2(u_\varepsilon+e) - \int_\Omega u_\varepsilon v_\varepsilon^2 \\
&\ge \frac{1}{2}\int_\Omega u_\varepsilon\ln^2(u_\varepsilon+e) - m\|v_0\|^2_{L^\infty(\Omega)}
\end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Therefore, (5.4.31) together with (5.2.6), Lemma 5.4.2, Lemma 5.4.3 and Lemma 5.4.5 yields (5.4.23). □

5.4.3 A pointwise lower bound for v_ε. Estimates for u_ε in L^p

In the two-dimensional setting considered here, the above can indeed be used to obtain pointwise lower bounds for v_ε on the basis of the following regularization feature of the Neumann heat semigroup $e^{t\Delta}$ in Ω that is implied by [90, Corollary 1.5].

Lemma 5.4.7. *Let $\alpha > 0$, $K > 0$ and $T > 0$. Then there exists $C(\alpha, K, T) > 0$ such that whenever $\varphi \in C^0(\overline{\Omega})$ is a nonnegative function fulfilling*

$$\int_\Omega \varphi \ln^\alpha(\varphi + e) \le K,$$

we have

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \le C(\alpha, K, T) t^{-1} \ln^{-\alpha}\left(\frac{1}{t} + e\right) \qquad \text{for all } t \in (0, T).$$

In fact, relying on the fact that the power carried by the logarithmic expression in (5.4.23) exceeds the value 1, from Lemma 5.4.6 and Lemma 5.4.7 we obtain the following.

Lemma 5.4.8. *For all $T > 0$ there exists $C(T) > 0$ such that*

$$v_\varepsilon(x,t) \ge C(T) \qquad \text{for all } x \in \Omega,\ t \in (0,T) \cap (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{5.4.32}$$

Proof. Setting $w_\varepsilon := \ln \frac{1}{v_\varepsilon}$ for $\varepsilon \in (0,1)$, from the second equation in (5.2.1) we know that w_ε satisfies

$$w_{\varepsilon t} = \Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + u_\varepsilon \le \Delta w_\varepsilon + u_\varepsilon \quad \text{in } \Omega \times (0, T_{max,\varepsilon}).$$

In view of Lemma 5.4.6, Lemma 5.4.7 becomes applicable so as to assert that there exists $c_1 = c_1(T) > 0$ fulfilling

$$\|e^{t\Delta}u_\varepsilon\|_{L^\infty(\Omega)} \le c_1 t^{-1} \ln^{-2}\left(\frac{1}{t} + e\right) \qquad \text{for all } t \in (0,T) \cap (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Consequently, using the comparison principle we infer that for all $t \in (0,T) \cap (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
w_\varepsilon(\cdot, t) &\le e^{t\Delta} \ln \frac{1}{v_0} + \int_0^t e^{(t-s)\Delta} u_\varepsilon(\cdot, s) ds \\
&\le \sup_{x\in\Omega} \ln \frac{1}{v_0(x)} + c_1 \int_0^t \frac{1}{(t-s)\ln^2(\frac{1}{t-s} + e)} ds \\
&\le \sup_{x\in\Omega} \ln \frac{1}{v_0(x)} + c_1 \int_{\ln\left(\frac{1}{t}+e\right)}^\infty \frac{e^y}{e^y - e} \cdot \frac{1}{y^2} dy \\
&\le \sup_{x\in\Omega} \ln \frac{1}{v_0(x)} + c_1(et+1) \int_1^\infty \frac{1}{y^2} dy \\
&\le \sup_{x\in\Omega} \ln \frac{1}{v_0(x)} + c_1(eT+1) \quad \text{in } \Omega,
\end{aligned}$$

because $\frac{e^y}{e^y - e} \le et + 1$ holds for $y \ge \ln\left(\frac{1}{t} + e\right)$. In conjunction with the uniform positivity of v_0, this implies (5.4.32). □

Having thus ruled out the effective appearance of cross-degeneracies in (5.2.1) within any bounded time interval, for arbitrary $p > 1$ we can now return to (5.2.13) and control the right-hand side therein in a straightforward manner.

Lemma 5.4.9. *If $p > 1$ and $T > 0$, then one can fix $C(p,T) > 0$ in such a way that*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p,T) \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \tag{5.4.33}$$

and that

$$\int_0^t \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le C(p,T) \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{5.4.34}$$

Proof. Let $T > 0$. Then in light of Lemma 5.4.8, from Lemma 5.4.4 we now obtain $c_1 = c_1(T) > 0$ such that the functions $z_\varepsilon := u_\varepsilon e^{-v_\varepsilon}$, $\varepsilon \in (0,1)$, satisfy

$$\int_0^t \int_\Omega |\nabla z_\varepsilon|^2 \le c_1 \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

By means of the Gagliardo-Nirenberg inequality, (5.2.4) and the fact that $z_\varepsilon \le u_\varepsilon$ for $\varepsilon \in (0,1)$, we thus infer that with some $c_2 > 0$ and $c_3 = c_3(T) > 0$,

$$\begin{aligned}
\int_0^t \int_\Omega z_\varepsilon^3 &\le c_2 \int_0^t \|\nabla z_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^2 \|z_\varepsilon(\cdot,s)\|_{L^1(\Omega)} ds + c_2 \int_0^t \|z_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^3 ds \\
&\le c_2 m \int_0^t \int_\Omega |\nabla z_\varepsilon|^2 + c_2 m^3 t \\
&\le c_3 \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

so that again by (5.2.5),

$$\int_0^t \int_\Omega u_\varepsilon^3 = \int_0^t \int_\Omega z_\varepsilon^3 e^{3v_\varepsilon} \le c_3 e^{3\|v_0\|_{L^\infty(\Omega)}} \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Therefore, relying on Young's inequality, (5.2.5) and a known smoothing property of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ we see that with some $c_4 > 0$ and $c_5 = c_5(T) > 0$,

$$\begin{aligned}
\|\nabla v_\varepsilon(\cdot,t)\|_{L^4(\Omega)} &= \left\| \nabla e^{t\Delta} v_0 - \int_0^t \nabla e^{(t-s)\Delta} u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s) ds \right\|_{L^4(\Omega)} \\
&\le c_4 \|v_0\|_{W^{1,\infty}(\Omega)} + c_4 \int_0^t \left(1 + (t-s)^{-\frac{7}{12}}\right) \|u_\varepsilon(\cdot,s)\|_{L^3(\Omega)} \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\le c_4 \|v_0\|_{W^{1,\infty}(\Omega)} + c_4 \int_0^t \int_\Omega u_\varepsilon^3 + c_4 \|v_0\|_{L^\infty(\Omega)}^{\frac{3}{2}} \int_0^t \left(1 + (t-s)^{-\frac{7}{12}}\right)^{\frac{3}{2}} ds \\
&\le c_5 \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
\end{aligned} \tag{5.4.35}$$

because $\frac{7}{12} \cdot \frac{3}{2} = \frac{7}{8} < 1$.

Now fixing an arbitrary $p > 1$, we rely on (5.2.13) to see again using Lemma 5.4.8 and (5.2.5), that there exist $c_6 = c_6(p,T) > 0$ and $c_7 = c_7(p) > 0$ fulfilling

$$\frac{d}{dt} \int_\Omega u_\varepsilon^p + c_6 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + c_6 \int_\Omega |\nabla u_\varepsilon^{\frac{p+1}{2}}|^2 \le c_7 \int_\Omega u_\varepsilon^{p+1} |\nabla v_\varepsilon|^2 + c_7 \int_\Omega u_\varepsilon^p \tag{5.4.36}$$

for all $t \in (0,T) \cap (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Here, the Cauchy-Schwarz inequality and (5.4.35), followed by applications of the Gagliardo-Nirenberg inequality, (5.2.4) and Young's inequality, show that we can find $c_8 = c_8(p,T) > 0$ and $c_9 = c_9(p,T) > 0$ such that

$$\begin{aligned} c_7 \int_\Omega u_\varepsilon^{p+1} |\nabla v_\varepsilon|^2 &\le c_7 \|\nabla v_\varepsilon\|_{L^4(\Omega)}^2 \|u_\varepsilon^{\frac{p+1}{2}}\|_{L^4(\Omega)}^2 \\ &\le c_8 \|\nabla u_\varepsilon^{\frac{p+1}{2}}\|_{L^2(\Omega)}^{\frac{2p+1}{p+1}} \|u_\varepsilon^{\frac{p+1}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{1}{p+1}} + c_8 \|u_\varepsilon^{\frac{p+1}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^2 \\ &\le c_6 \|\nabla u_\varepsilon^{\frac{p+1}{2}}\|_{L^2(\Omega)}^2 + c_9 \qquad \text{for all } t \in (0,T) \cap (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Therefore, (5.4.36) implies that writing $c_{10} := \max\{c_7, c_9\}$, for all $t \in (0,T) \cap (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ we have

$$\frac{d}{dt}\Big\{1 + \int_\Omega u_\varepsilon^p\Big\} + c_6 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le c_{10} \cdot \Big\{1 + \int_\Omega u_\varepsilon^p\Big\},$$

which upon an integration readily yields the claim. □

5.5 Large time regularity implied by eventual smallness of v_ε

By going back to (5.3.9) and relying on the sublinear growth of the expression on the right-hand side therein, we can derive a statement on uniform decay of the v_ε in the large time limit, which independently of the results obtained in the previous section will imply regularity and especially boundedness beyond some suitably large waiting time.

In order to provide accessibility of spatially uniform estimates to (5.1.2), let us record a simple consequence of the fact that $W^{2,2}(\Omega)$ continuously embeds into $L^\infty(\Omega)$.

Lemma 5.5.1. *There exists $\Gamma_2 > 0$ such that for all nonnegative $\phi \in C^2(\overline{\Omega})$ such that $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$,*

$$\Big\| \phi - \|\phi\|_{L^\infty(\Omega)} \Big\|_{L^\infty(\Omega)} \le \Gamma_2 \|\Delta\phi\|_{L^2(\Omega)}. \tag{5.5.1}$$

Proof. We fix any $q \in (2,\infty)$, and then obtain from two Sobolev inequalities that there exist $c_1 > 0$ and $c_2 > 0$ fulfilling

$$\|\nabla\psi\|_{L^q(\Omega)} \le c_1 \|\psi\|_{W^{2,2}(\Omega)} \qquad \text{for all } \psi \in C^2(\overline{\Omega}) \tag{5.5.2}$$

and

$$|\psi(x) - \psi(x_0)| \le c_2 \|\nabla\psi\|_{L^q(\Omega)} \qquad \text{for all } \psi \in C^1(\overline{\Omega}) \text{ and any } x \in \overline{\Omega} \text{ and } x_0 \in \overline{\Omega}. \tag{5.5.3}$$

Moreover, using that $-\Delta$ acts as a homeomorphism from $W^{2,2}_{N,\perp}(\Omega) := \{\phi \in W^{2,2}(\Omega) \mid \frac{\partial\phi}{\partial\nu}|_{\partial\Omega} = 0 \text{ and } \int_\Omega \phi = 0\}$ onto $L^2_\perp(\Omega) := \{\phi \in L^2(\Omega) \mid \int_\Omega \phi = 0\}$ we can find $c_3 > 0$ such that

$$\|\psi\|_{W^{2,2}(\Omega)} \le c_3 \|\Delta\psi\|_{L^2(\Omega)} \qquad \text{for all } \psi \in W^{2,2}_{N,\perp}(\Omega). \tag{5.5.4}$$

Now given $0 \le \phi \in C^2(\overline{\Omega})$ such that $\frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$, we fix $x_0 \in \overline{\Omega}$ in such a way that $\phi(x_0) = \|\phi\|_{L^\infty(\Omega)}$, and applying (5.5.2)-(5.5.4) to $\psi := \phi - \frac{1}{|\Omega|}\int_\Omega \phi$ we obtain that

$$\Big|\phi(x) - \|\phi\|_{L^\infty(\Omega)}\Big| = |\psi(x) - \psi(x_0)| \le c_1c_2c_3\|\Delta\psi\|_{L^2(\Omega)} = c_1c_2c_3\|\Delta\phi\|_{L^2(\Omega)} \qquad \text{for all } x \in \Omega,$$

from which (5.5.1) follows with $\Gamma_2 := c_1c_2c_3$. □

In fact, the information on decay of temporal averages contained in (5.3.9) can thereby be seen to entail the key observation of this section.

Lemma 5.5.2. *Let $\eta > 0$. Then there exists $T_0 = T_0(\eta) > 0$ with the property that*

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \eta \qquad \textit{for all } t \in [T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{5.5.5}$$

Proof. We let Γ_1, Γ_2 and m be as in Lemma 5.3.3, Lemma 5.5.1 and (5.2.4), and given $\eta > 0$ we may then use that $\frac{\ln(t+e)}{t} \to 0$ as $t \to \infty$ to fix $T_0 = T_0(\eta) > 0$ large enough fulfilling

$$\sqrt{\Gamma_1}\Gamma_2 m \cdot \sqrt{\frac{\ln(t+e)}{t}} \le \frac{m_0}{2}\cdot\eta \quad \text{and} \quad \frac{1}{t}\int_\Omega v_0 \le \frac{m_0}{2}\cdot\eta \qquad \text{for all } t \ge T_0, \tag{5.5.6}$$

where $m_0 := \int_\Omega u_0$ is positive by (5.1.3). Then whenever $\varepsilon \in (0,1)$ is such that $T_{max,\varepsilon} > T_0$, we can use that $\int_\Omega u_\varepsilon \ge \int_\Omega (u_0+\varepsilon) \ge m_0$ for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ to estimate

$$\begin{aligned}
&\frac{m_0}{t}\int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\quad\le \frac{1}{t}\int_0^t \Big\{\int_\Omega u_\varepsilon(\cdot,s)\Big\}\cdot\|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\quad= \frac{1}{t}\int_0^t\int_\Omega u_\varepsilon v_\varepsilon - \frac{1}{t}\int_0^t\int_\Omega u_\varepsilon(\cdot,s)\cdot\Big\{v_\varepsilon(\cdot,s) - \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\Big\}ds
\end{aligned} \tag{5.5.7}$$

for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, where by (5.2.4), Lemma 5.5.1, the Cauchy-Schwarz inequality and Lemma 5.3.3,

$$\begin{aligned}
&-\frac{1}{t}\int_0^t\int_\Omega u_\varepsilon(\cdot,s)\cdot\Big\{v_\varepsilon(\cdot,s) - \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\Big\}ds \\
&\quad\le \frac{1}{t}\int_0^t\Big\{\int_\Omega u_\varepsilon(\cdot,s)\Big\}\cdot\Big\|v_\varepsilon(\cdot,s) - \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\Big\|_{L^\infty(\Omega)}ds \\
&\quad\le \frac{\Gamma_2 m}{t}\int_0^t \|\Delta v_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds \\
&\quad\le \frac{\Gamma_2 m}{\sqrt{t}}\cdot\Big\{\int_0^t\int_\Omega |\Delta v_\varepsilon|^2\Big\}^{\frac{1}{2}} \\
&\quad\le \sqrt{\Gamma_1}\Gamma_2 m\cdot\sqrt{\frac{\ln(t+e)}{t}} \\
&\quad\le \frac{m_0}{2}\cdot\eta \qquad \text{for all } t \in [T_0,\infty)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1)
\end{aligned}$$

according to the first restriction in (5.5.6). As moreover

$$\frac{1}{t}\int_0^t\int_\Omega u_\varepsilon v_\varepsilon \le \frac{1}{t}\int_\Omega v_0 \le \frac{m_0}{2}\cdot\eta \qquad \text{for all } t\in[T_0,\infty)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1)$$

by (5.2.6) and the second inequality in (5.5.6), from (5.5.7) we obtain that

$$\frac{m_0}{t}\int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \le m_0\eta \qquad \text{for all } t\in[T_0,\infty)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

so that, in particular, we can find $t_0 = t_0(\eta,\varepsilon)\in[0,T_0]$ such that $\|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \le \eta$. In view of (5.2.5), this entails (5.5.5). □

Thanks to the smallness property of v_ε hence obtained, we can conveniently control the growth of the functional in (5.1.17) after some suitably large relaxation period. By drawing not only on (5.2.12) but also on Lemma 5.2.6 here, in the following we crucially rely on the assumption that Ω be convex.

Lemma 5.5.3. *Let Ω be convex. Then there exist $T_0>0$ and $C>0$ such that*

$$\int_{T_0}^t\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le C \qquad \textit{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1) \tag{5.5.8}$$

and that moreover

$$\int_{T_0}^t\int_\Omega u_\varepsilon^2 v_\varepsilon \le C \qquad \textit{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{5.5.9}$$

Proof. According to Lemma 5.2.6, we can find $c_1>0$ and $c_2>0$ such that

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le c_2\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \quad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1), \tag{5.5.10}$$

whence recalling (5.2.12) we obtain that for $a := \frac{1}{2c_2}$ and

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon(\ln u_\varepsilon - 1) + a\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}, \qquad t\in[0,T_{max,\varepsilon}),\ \varepsilon\in(0,1), \tag{5.5.11}$$

we have

$$y_\varepsilon'(t) + \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + c_1 a\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \le \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. Since

$$\int_\Omega u_\varepsilon v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \le \frac{1}{4}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon|^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

this implies that

$$y_\varepsilon'(t) + \frac{1}{4}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + c_1 a\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5}$$

$$\leq \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{5.5.12}$$

To proceed from this, we abbreviate $c_3 := (\frac{4}{c_1 a})^{\frac{1}{2}}$, and taking m as well as $\Lambda_1 = \Lambda(\frac{2}{3})$ and $\Lambda_2 = \Lambda(1)$ from (5.2.4) and Lemma 5.2.2 we choose $\eta > 0$ and $\delta > 0$ in such a way that

$$8\Lambda_2^2 |\Omega| \delta \leq \frac{1}{8} \qquad \text{and} \qquad 8\Lambda_2^6 (m + |\Omega|)^3 \delta \leq \frac{c_1 a}{4} \tag{5.5.13}$$

as well as

$$36 c_3 \Lambda_1^2 m \eta^3 \leq \frac{1}{16} \qquad \text{and} \qquad 64 c_3 \Lambda_1^2 m \eta \leq \frac{1}{2}, \tag{5.5.14}$$

and thereafter rely on Lemma 5.5.2 to find $T_1 > 0$ such that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \eta \qquad \text{for all } t \in (T_1,\infty) \cap (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{5.5.15}$$

Now fixing any $\varepsilon \in (0,1)$, we note that the above selection of c_3 ensures that due to Young's inequality,

$$\begin{aligned} \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 &= \int_\Omega \Big\{ \frac{c_1 a}{4} \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \Big\}^{\frac{1}{3}} \cdot \Big(\frac{4}{c_1 a}\Big)^{\frac{1}{3}} u_\varepsilon^2 v_\varepsilon^{\frac{8}{3}} \\ &\leq \frac{c_1 a}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_3 \int_\Omega u_\varepsilon^3 v_\varepsilon^4 \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \end{aligned} \tag{5.5.16}$$

where from (5.2.9) it follows upon taking $\alpha := \frac{3}{2}$ and $\beta := 1$ there that due to (5.5.15) and (5.5.14),

$$\begin{aligned} c_3 \int_\Omega u_\varepsilon^3 v_\varepsilon^4 &\leq 36 c_3 \Lambda_1^2 m \eta^3 \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + 64 c_3 \Lambda_1^2 m \eta \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 + 4 c_3 \Lambda_1^2 m^2 \eta^3 \int_\Omega u_\varepsilon v_\varepsilon \\ &\leq \frac{1}{16} \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 + 4 c_3 \Lambda_1^2 m^2 \eta^3 \int_\Omega u_\varepsilon v_\varepsilon \end{aligned}$$

for all $t \in (T_1,\infty) \cap (0, T_{max,\varepsilon})$, so that from (5.5.16) we obtain that

$$\int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 \leq \frac{1}{8} \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{c_1 a}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + c_4 \int_\Omega u_\varepsilon v_\varepsilon \tag{5.5.17}$$

for all $t \in (T_1,\infty) \cap (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $c_4 := 8 c_3 \Lambda_1^2 m^2 \eta^3$.

We next use that $\frac{\ln \xi}{\xi} \to 0$ as $\xi \to \infty$ to fix $c_5 > 0$ such that

$$\ell \xi \ln \xi \leq \frac{\delta}{2} \xi^2 + c_5 \xi \qquad \text{for all } \xi > 0,$$

and can then estimate the rightmost summand in (5.5.12) according to

$$\begin{aligned} &\frac{\delta}{2} \int_\Omega u_\varepsilon^2 v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon \\ &\qquad \leq \delta \int_\Omega u_\varepsilon^2 v_\varepsilon + c_5 \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{5.5.18}$$

Here, an application of (5.2.8) to $\alpha := 1$ and $\gamma := 6$ shows that thanks to Young's inequality and (5.5.13) we have

$$\begin{aligned}
\delta\int_\Omega u_\varepsilon^2 v_\varepsilon &\le 8\Lambda_2^2\delta\cdot\Big\{\int_\Omega \sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 + 8\Lambda_2^6\delta\cdot\Big\{\int_\Omega u_\varepsilon^{\frac{2}{3}}\Big\}^3\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + 4\Lambda_2^2\delta\cdot\int_\Omega u_\varepsilon\cdot\int_\Omega u_\varepsilon v_\varepsilon \\
&\le 8\Lambda_2^2|\Omega|\delta\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + 8\Lambda_2^6(m+|\Omega|)^3\delta\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + 4\Lambda_2^2 m\delta\int_\Omega u_\varepsilon v_\varepsilon \\
&\le \frac{1}{8}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{c_1 a}{4}\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + 4\Lambda_2^2 m\delta\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned}$$

for all $t\in(T_1,\infty)\cap(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, which together with (5.5.18) and (5.5.17) shows that (5.5.12) implies the inequality

$$y_\varepsilon'(t) + \frac{c_1 a}{4}\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \frac{\delta}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon \le c_6\int_\Omega u_\varepsilon v_\varepsilon$$

for all $t\in(T_1,\infty)\cap(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$ with $c_6 := c_4 + 4\Lambda_2^2 m\delta + c_5$. Therefore, (5.2.6) ensures that for all $t\in(T_1+1,\infty)\cap(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
&y_\varepsilon(t) + \frac{c_1 a}{4}\int_{T_1+1}^t\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \frac{\delta}{2}\int_{T_1+1}^t\int_\Omega u_\varepsilon^2 v_\varepsilon \\
&\quad\le \inf_{s\in(T_1,T_1+1)} y_\varepsilon(s) + c_6\int_{T_1}^t\int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le \inf_{s\in(T_1,T_1+1)} y_\varepsilon(s) + c_6\int_\Omega v_0,
\end{aligned}$$

so that the claim results with $T_0 := T_1 + 1$ upon estimating

$$\int_\Omega u_\varepsilon(\ln u_\varepsilon - 1) \ge -\frac{|\Omega|}{e} - m \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

and noting that $\sup_{\varepsilon\in(0,1)}\inf_{s\in(T_1,T_1+1)} y_\varepsilon(s)$ is finite thanks to (5.4.14) and (5.4.5). □

Once more starting from (5.2.13), we can use the above to derive bounds in the flavor of those in (5.2.19) and (5.2.21) with $p_0 := 2$, again valid for suitably large times.

Lemma 5.5.4. *If Ω is convex, then there exist $T_0 > 0$ and $C > 0$ such that*

$$\int_\Omega u_\varepsilon^2(\cdot,t) \le C \qquad \textit{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1), \tag{5.5.19}$$

and that moreover

$$\int_{T_0}^t\int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2 \le C \qquad \textit{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{5.5.20}$$

Proof. Let $\varepsilon\in(0,1)$. Then by Lemma 5.2.4, $y_\varepsilon(t) := 1 + \int_\Omega u_\varepsilon^2$, $t\in[0,T_{max,\varepsilon})$, satisfies

$$y_\varepsilon'(t) + \int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2 \le \int_\Omega u_\varepsilon^3 v_\varepsilon|\nabla v_\varepsilon|^2 + 2\ell\int_\Omega u_\varepsilon^2 v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{5.5.21}$$

and here the Hölder inequality ensures that

$$\begin{aligned}\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 &= \int_\Omega \Big\{\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5}\Big\}^{\frac{1}{3}} \cdot u_\varepsilon^3 v_\varepsilon^{\frac{8}{3}} \\ &\le h_\varepsilon^{\frac{1}{3}}(t) \cdot \bigg\{\int_\Omega u_\varepsilon^{\frac{9}{2}} v_\varepsilon^4\bigg\}^{\frac{2}{3}} \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \qquad (5.5.22)\end{aligned}$$

where we have set

$$h_\varepsilon(t) := \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + \int_\Omega u_\varepsilon^2 v_\varepsilon + \int_\Omega u_\varepsilon v_\varepsilon, \qquad t \in (0, T_{max,\varepsilon}). \qquad (5.5.23)$$

Now one more application of (5.2.9), now for $\alpha := \frac{9}{4}$ and $\beta := \frac{2}{3}$, reveals that with $\Lambda = \Lambda(\frac{4}{9})$ as provided there,

$$\begin{aligned}\bigg\{\int_\Omega u_\varepsilon^{\frac{9}{2}} v_\varepsilon^4\bigg\}^{\frac{2}{3}} &\le (9\Lambda)^{\frac{4}{3}} \|v_0\|_{L^\infty(\Omega)}^2 \cdot \bigg\{\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot \bigg\{\int_\Omega u_\varepsilon^{\frac{3}{2}}\bigg\}^{\frac{2}{3}} \\ &\quad +(8\Lambda)^{\frac{4}{3}} \|v_0\|_{L^\infty(\Omega)}^{\frac{2}{3}} \cdot \bigg\{\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot \bigg\{\int_\Omega u_\varepsilon^{\frac{3}{2}}\bigg\}^{\frac{2}{3}} \\ &\quad +(2\Lambda)^{\frac{4}{3}} \|v_0\|_{L^\infty(\Omega)}^2 \cdot \bigg\{\int_\Omega u_\varepsilon v_\varepsilon\bigg\}^{\frac{2}{3}} \cdot \bigg\{\int_\Omega u_\varepsilon\bigg\}^{\frac{7}{3}} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),\end{aligned}$$

where the Hölder inequality guarantees that

$$\bigg\{\int_\Omega u_\varepsilon^{\frac{3}{2}}\bigg\}^{\frac{2}{3}} \le \bigg\{\int_\Omega u_\varepsilon^2\bigg\}^{\frac{1}{3}} \cdot \bigg\{\int_\Omega u_\varepsilon\bigg\}^{\frac{1}{3}} \le m^{\frac{1}{3}} \cdot y_\varepsilon^{\frac{1}{3}}(t) \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Combined with (5.5.22), this shows that if we abbreviate

$$c_1 := (9\Lambda)^{\frac{4}{3}} m^{\frac{1}{3}} \|v_0\|_{L^\infty(\Omega)}^2, \quad c_2 := (8\Lambda)^{\frac{4}{3}} m^{\frac{1}{3}} \|v_0\|_{L^\infty(\Omega)}^{\frac{2}{3}} \quad \text{and} \quad c_3 := (2\Lambda)^{\frac{4}{3}} m^{\frac{7}{3}} \|v_0\|_{L^\infty(\Omega)}^2,$$

then due to Young's inequality and (5.5.23),

$$\begin{aligned}\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 &\le c_1 h_\varepsilon^{\frac{1}{3}}(t) \cdot \bigg\{\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot y_\varepsilon^{\frac{1}{3}}(t) \\ &\quad +c_2 h_\varepsilon^{\frac{1}{3}}(t) \cdot \bigg\{\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot y_\varepsilon^{\frac{1}{3}}(t) + c_3 h_\varepsilon(t) \\ &= \bigg\{\frac{1}{4}\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot 4^{\frac{2}{3}} c_1 h_\varepsilon^{\frac{1}{3}}(t) y_\varepsilon^{\frac{1}{3}}(t) \\ &\quad +\bigg\{\frac{1}{2}\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2\bigg\}^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} c_2 h_\varepsilon^{\frac{1}{3}}(t) y_\varepsilon^{\frac{1}{3}}(t) + c_3 h_\varepsilon(t) \\ &\le \frac{1}{4}\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + 16 c_1^3 h_\varepsilon(t) y_\varepsilon(t) \\ &\quad +\frac{1}{2}\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 + 4 c_2^3 h_\varepsilon(t) y_\varepsilon(t) + c_3 h_\varepsilon(t) \qquad \text{for all } t \in (0, T_{max,\varepsilon}).\end{aligned}$$

Thus,

$$\int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + (32c_1^3+8c_2^3)h_\varepsilon(t)y_\varepsilon(t) + 2c_3 h_\varepsilon(t) \quad \text{for all } t\in(0,T_{max,\varepsilon}),$$

so that since $1 \le y_\varepsilon(t)$ for all $t\in(0,T_{max,\varepsilon})$, from (5.5.21) and again (5.5.23) it follows that

$$y_\varepsilon'(t) + g_\varepsilon(t) \le c_4 h_\varepsilon(t) y_\varepsilon(t) \qquad \text{for all } t\in(0,T_{max,\varepsilon})$$

with $g_\varepsilon(t) := \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2$, $t\in(0,T_{max,\varepsilon})$, and $c_4 := 32c_1^3+8c_2^3+2c_3+2\ell$. Since Lemma 5.5.3 along with (5.2.6) provides $T_0>0$ and $c_5>0$ such that

$$\int_{T_0}^t h_\varepsilon(s)ds \le c_5 \qquad \text{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

and since Lemma 5.4.9 ensures the existence of $c_6>0$ fulfilling

$$y_\varepsilon(t) \le c_6 \qquad \text{for all } t\in[0,T_0]\cap(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

this implies that in the nontrivial case when $\varepsilon\in(0,1)$ is such that $T_{max,\varepsilon}>T_0$,

$$y_\varepsilon(t) + \int_{T_0}^t e^{c_4\int_s^t h_\varepsilon(\sigma)d\sigma} g_\varepsilon(s)ds \le y_\varepsilon(T_0) e^{c_4\int_{T_0}^t h_\varepsilon(s)ds} \qquad \text{for all } t\in(T_0,T_{max,\varepsilon})$$

and hence

$$\int_\Omega u_\varepsilon^2(\cdot,t) + \frac{1}{2}\int_{T_0}^t\int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2 \le c_6 e^{c_4c_5} \qquad \text{for all } t\in(T_0,T_{max,\varepsilon}),$$

because h_ε is nonnegative. □

One further interpolation based on (5.2.8) and the previous two lemmata, finally, shows that also an expression resembling that in (5.2.20) with $p_0 := 2$ can eventually be controlled.

Lemma 5.5.5. *Let Ω be convex. Then with some $T_0>0$ and some $C>0$, we have*

$$\int_{T_0}^t\int_\Omega u_\varepsilon^4 v_\varepsilon \le C \qquad \textit{for all } t\in(T_0,\infty)\cap(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{5.5.24}$$

Proof. Let $\varepsilon\in(0,1)$. Then taking $\alpha:=2$ and $\gamma:=6$ in (5.2.8), with $\Lambda=\Lambda(\frac{1}{2})$ as in Lemma 5.2.2 from (5.2.4) we infer that

$$\begin{aligned}
\int_\Omega u_\varepsilon^4 v_\varepsilon &\le 32\Lambda^2\cdot\Big\{\int_\Omega u_\varepsilon\sqrt{v_\varepsilon}|\nabla u_\varepsilon|\Big\}^2 + 8\Lambda^6\cdot\Big\{\int_\Omega u_\varepsilon^{\frac{4}{3}}\Big\}^3\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \\
&\quad +4\Lambda^2\cdot\Big\{\int_\Omega u_\varepsilon\Big\}^3\cdot\int_\Omega u_\varepsilon v_\varepsilon \\
&\le 32\Lambda^2 m\int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2 + 8\Lambda^6\cdot\Big\{\int_\Omega u_\varepsilon^2+|\Omega|\Big\}^3\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \\
&\quad +4\Lambda^2 m^3\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } \ t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Upon an integration in time and applying Lemma 5.5.4, Lemma 5.5.3 and (5.2.6) we may conclude as intended. □

5.6 Proofs of Theorem 5.1.1 and Corollary 5.1.2

The derivation of our main result on global existence and boundedness in (5.1.2) thus reduces to collecting and merging:

Proof of Theorem 5.1.1. A combination of Lemma 5.4.9 with Lemma 5.5.4 and Lemma 5.5.5 shows that (5.2.19), (5.2.20) and (5.2.21) are satisfied with $p_0 := 2$. The claim therefore is a consequence of Proposition 5.2.7. □

The announced statement on large time behavior can readily be obtained as a by-product of our analysis:

Proof of Corollary 5.1.2. On the basis of (5.1.5), Part i) can be obtained by straightforward adaptation of the reasoning in [95, Theorem 1.2] (see also Lemma 3.1.4 in Chapter 3), while Part ii) can be concluded from [87, Lemma 5.4] in exactly the same way as soon as we have obtained that for any $\delta > 0$ there exists $\eta > 0$ such that if

$$\|v_0\|_{W^{1,\infty}(\Omega)} + \|\nabla\sqrt{v_0}\|_{L^2(\Omega)} \le \eta,$$

then

$$\big\|u_\infty - u_0\big\|_{(W^{1,\infty}(\Omega))^*} \le \delta.$$

To achieve this, we assume on the contrary that there exists $\delta_0 > 0$ such that for any $\eta > 0$ one can find $v_0 \in W^{1,\infty}(\Omega)$ which satisfies $v_0 > 0$ in $\overline{\Omega}$ and

$$\|v_0\|_{W^{1,\infty}(\Omega)} + \|\nabla\sqrt{v_0}\|_{L^2(\Omega)} \le \eta, \tag{5.6.1}$$

but

$$\big\|u_\infty - u_0\big\|_{(W^{1,\infty}(\Omega))^*} \ge \delta_0, \tag{5.6.2}$$

where for definiteness we choose the norm in $W^{1,\infty}(\Omega)$ as being given by $\|\psi\|_{W^{1,\infty}(\Omega)} := \|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)}$ for $\psi \in W^{1,\infty}(\Omega)$. To derive a contradiction from this, we first invoke Proposition 5.2.7 to fix $c_1 > 0$ such that whenever $v_0 \in W^{1,\infty}(\Omega)$ is positive in $\overline{\Omega}$ and satisfies $\|v_0\|_{W^{1,\infty}(\Omega)} + \|\nabla\sqrt{v_0}\|_{L^2(\Omega)} \le 1$, with $(\varepsilon_j)_{j\in\mathbb{N}}$ as provided there we have

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{5.6.3}$$

We moreover employ Lemma 5.4.9, Lemma 5.5.4 and Lemma 5.5.5 along with Young's inequality, (5.2.6) and (5.6.3) to find $c_2 > 0$ such that for any v_0 with these properties,

$$\begin{aligned}
&\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^\infty \int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\quad\le \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + c_1^2 \int_0^\infty \int_\Omega u_\varepsilon^4 v_\varepsilon + c_1^2 \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le c_2 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}.
\end{aligned} \tag{5.6.4}$$

With δ_0 as fixed above, we now take $\eta \in (0,1]$ small enough fulfilling

$$2c_2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}\eta^{\frac{1}{2}} + \ell|\Omega|\eta < \delta_0, \tag{5.6.5}$$

and suppose that $0 < v_0 \in W^{1,\infty}(\Omega)$ satisfies (5.6.1). Then since for any $\psi \in W^{1,\infty}(\Omega)$ satisfying $\|\psi\|_{W^{1,\infty}(\Omega)} \le 1$ we know from (5.2.1) that for all $t > 0$ and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\begin{aligned}\int_\Omega u_{\varepsilon t} \cdot \psi &= \int_\Omega \nabla \cdot (u_\varepsilon v_\varepsilon \nabla u_\varepsilon)\psi - \int_\Omega \nabla \cdot (u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon)\psi + \ell \int_\Omega u_\varepsilon v_\varepsilon \psi \\ &= \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon \cdot \nabla\psi + \int_\Omega u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon \cdot \nabla \psi + \ell \int_\Omega u_\varepsilon v_\varepsilon \psi,\end{aligned}$$

and that thus, according to our selection of $\|\cdot\|_{W^{1,\infty}(\Omega)}$,

$$\|u_{\varepsilon t}\|_{(W^{1,\infty}(\Omega))^*} \le \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon| + \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon| + \ell \int_\Omega u_\varepsilon v_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},$$

in view of the Hölder inequality we may then draw on (5.6.4) to see upon an integration in time that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\begin{aligned}\int_0^t \|u_{\varepsilon s}(\cdot,s)\|_{(W^{1,\infty}(\Omega))^*} ds &\le \left\{ \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 \right\}^{\frac12} \cdot \left\{ \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \right\}^{\frac12} \\ &\quad + \left\{ \int_0^\infty \int_\Omega u_\varepsilon^3 v_\varepsilon |\nabla v_\varepsilon|^2 \right\}^{\frac12} \cdot \left\{ \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \right\}^{\frac12} + \ell \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \\ &\le 2c_2^{\frac12} \cdot \left\{ \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \right\}^{\frac12} + \ell \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t > 0.\end{aligned}$$

By means of (5.2.6) and (5.6.1), we thus deduce that for all $t > 0$ and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\|u_\varepsilon(\cdot,t) - u_0 - \varepsilon\|_{(W^{1,\infty}(\Omega))^*} \le 2c_2^{\frac12} c_3 |\Omega|^{\frac12} \eta^{\frac12} + c_3|\Omega|\eta,$$

which upon recalling Proposition 5.2.7 and (5.1.9) leads to a contradiction to (5.6.5). □

6 Global boundedness and large time behavior in systems involving growth saturation

6.1 Introduction

The Allee effect characterizes a nonlinear population growth mechanism in which the per capita growth rate declines at low population density due to weakened interaction ability (such as mating and cooperative foraging) or environmental regulation; additionally, the mortality effects caused by resource limitations at high density, as featured in well-known logistic growth, are also taken into account. This phenomenon has been widely observed in numerous evolutionary dynamics of interacting populations within ecological systems ([18], [64]), and has been investigated through both numerical simulations and theoretical analyses ([50], [68], [69]).

In this chapter, we aim to incorporate the Allee effect into a reaction-diffusion framework by considering the doubly degenerate parabolic system

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v) + f(u), \\ v_t = \Delta v - uv, \end{cases} \tag{6.1.1}$$

where the growth function $f(u)$ is chosen so as to encompass various growth behaviors, notably including the Allee effect and the logistic law.

If the source function $f(u)$ is replaced by the proliferation term ℓuv with $\ell > 0$, the existing literature on globally bounded weak solutions has been reviewed and discussed in the previous four chapters (Chapters 2-5).

In general, the presence of absorption mechanisms, such as the logistic source, appears to be more amenable to the suppression of blow-up phenomena–not only in nutrient-consumption models governed by the second equation in (6.1.1) ([72], [37], [35], [97]), but also in models exhibiting self-enhancing effects through signal production terms as featured in the second equation of (1.1.1) ([57], [74], [76], [34], [46], [80])). However, this situation is not readily apparent in the system (6.1.1). At first glance, this may be attributed to the rough intuition that the degeneracy in v not only tends to diminish the overall dissipative effect of the signal-incorporated diffusion term, but may also allow for the possibility that the term ℓuv, as studied in the previous literature, exerts a weaker adverse influence than the absorption mechanism itself.

The main objective of the present chapter is to investigate the extent to which the damping effects of the absorption mechanism in (6.1.1) retain the capacity to counteract the occurrence of blow-up. Results for this class of systems remain relatively scarce. For the case $f(u) = \kappa u - \mu u^\gamma$, the global solvability is available only in [58] and [43]; in [58], the argument was launched by setting γ conveniently large ($\gamma > \frac{N+2}{N}$) to ensure a positive lower bound for v within any finite time interval so that the first equation in (6.1.1) can be treated as a porous medium-type equation. Subsequently, the two-dimensional case was improved in [43], where the threshold was lowered to $\gamma = 2$ through a subtle exploitation of a favorable dissipative structure inherent in the system. However, neither of these works provides any uniform boundedness information. The corresponding results concerning large time behavior can be found in [59] and [44].

Specifically, we are concerned with the problem

$$\begin{cases} u_t = \nabla\cdot(uv\nabla u) - \nabla\cdot(u^2 v\nabla v) + f(u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0,\\ (uv\nabla u - u^2 v\nabla v)\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\quad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{6.1.2}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where f is assumed to satisfy

$$f(s) \leq \kappa s - \mu s^\gamma, \qquad \text{for all } \ s > 0 \tag{6.1.3}$$

with $\kappa \geq 0$, $\mu > 0$ and $\gamma > 1$. We develop an approach that allows us to establish not only global existence, but also boundedness and even asymptotic stabilization under weaker assumptions on γ.

We obtain the following result on global existence and boundedness.

Theorem 6.1.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then there exists $\mu_0 = \mu_0(N)$ with the property that if f satisfies* (6.1.3) *with $\kappa \geq 0$ and*

$$\begin{cases} \gamma = 2 & \text{and } \ \mu > 0, & N = 2,\\ \gamma = \frac{N+8}{4} & \text{and } \ \mu > 0, & N = \{3,4\},\\ \gamma = 3 & \text{and } \ \mu \geq \mu_0, & N \geq 5, \end{cases} \tag{6.1.4}$$

then for any initial data (u_0, v_0) fulfilling

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0 \qquad \text{and}\\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}, \end{cases} \tag{6.1.5}$$

one can find functions

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \qquad \text{and}\\ v \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{6.1.6}$$

such that the pair (u, v) forms a global weak solution of (6.1.2) *in the sense of Definition 1.4.1. Moreover, $u \geq 0$ and $v > 0$ in $\overline{\Omega}\times[0,\infty)$, and with some $C(N) > 0$ we have*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C(N) \qquad \text{for all } \ t > 0. \tag{6.1.7}$$

Remark 6.1.2. It is worth pointing out that in the two-dimensional case, the quadratic degradation is strong enough to ensure the existence and boundedness of global weak solutions, even without imposing any largeness condition on the factor μ. The approach employed here is expected to allow for an improvement of the corresponding result in [37].

Remark 6.1.3. In the case of high spatial dimensions ($N \geq 5$), stronger nonlinear damping can compensate for small values of μ. In other words, the conclusion of Theorem 6.1.1 for $N \geq 5$ remains valid for any $\mu > 0$, provided that $\gamma > 3$. Indeed, fixing $\mu_0 = \mu_0(N)$ as in Theorem 6.1.1, one can see from Young's inequality that

$$\mu_0 s^3 = \mu^{\frac{2}{\gamma-1}} s^{\frac{2\gamma}{\gamma-1}} \cdot \mu_0 \mu^{-\frac{2}{\gamma-1}} s^{\frac{\gamma-3}{\gamma-1}} \leq \mu s^\gamma + \mu_0^{\frac{\gamma-1}{\gamma-3}} \mu^{-\frac{2}{\gamma-3}} s \qquad \text{for all} \ \ s > 0.$$

Consequently, it follows that

$$f(s) \leq \left(\kappa + \mu_0^{\frac{\gamma-1}{\gamma-3}} \mu^{-\frac{2}{\gamma-3}} \right) s - \mu_0 s^3 =: \tilde{\kappa} s - \mu_0 s^3 \qquad \text{for all} \ \ s > 0$$

with $\tilde{\kappa} \geq 0$. Theorem 6.1.1 can thus be applied to verify this claim.

The second purpose of the present chapter is to shed light on the asymptotic behavior of the solutions obtained in Theorem 6.1.1. This relies on an additional assumption on the initial data u_0 (see (6.1.8) below), and reads as follows.

Theorem 6.1.4. *In addition to the assumptions of Theorem 6.1.1, suppose that*

$$\int_\Omega \ln u_0 > -\infty. \tag{6.1.8}$$

Let (u, v) be as accordingly given by Theorem 6.1.1. Then,

$$u(\cdot, t) \overset{*}{\rightharpoonup} \left(\frac{\kappa}{\mu} \right)^{\frac{1}{\gamma-1}} \quad \text{in} \ \ L^\infty(\Omega) \qquad \text{as } t \to \infty \tag{6.1.9}$$

and

$$v(\cdot, t) \to 0 \quad \text{in} \ \ L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{6.1.10}$$

Remark 6.1.5. This differs from the results for doubly degenerate systems with proliferation term ℓuv ([81], [95] and Chapters 2-5), in which the first solution component u may stabilize toward heterogeneous steady states for a broad class of initial data. In contrast, our result asserts that the solutions converge to a constant state in the appropriate topology for arbitrary sufficiently regular initial data.

Main ideas. As is already well known, both the nonlinear growth term $f(u)$ and the diffusion term $\nabla \cdot (uv\nabla u)$ in the first equation contribute favorably. The main challenge lies in identifying how to effectively exploit these contributions, along with the dissipative effects from the second equation, to control the adverse effects stemming from cross-diffusion when analysing the evolution of the functional $\int_\Omega u^p$ (Lemma 6.2.2).

To fully exploit the signal-weighted dissipation rates, we develop a functional inequality (Lemma 6.2.4) valid in all spatial dimensions $N \geq 2$

$$\begin{aligned}\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2 \ &\le\ \eta\int_\Omega \varphi^{\alpha-2}\psi|\nabla\varphi|^2+\eta\int_\Omega\varphi\psi \\ &\quad +C(\alpha,N,\eta)\cdot\left\{\int_\Omega\varphi^\alpha+\left\{\int_\Omega\varphi\right\}^{\frac{(N+2)\alpha-N}{2}}\right\}\cdot\int_\Omega\frac{|\nabla\psi|^{N+2}}{\psi^{N+1}},\end{aligned}\tag{6.1.11}$$

which was inspired by [87, Lemma 3.5]. The subsequent analysis focuses on the coupled energy-like functional

$$c\int_\Omega u^p+\int_\Omega\frac{|\nabla v|^{N+2}}{v^{N+1}}\tag{6.1.12}$$

with suitable $c > 0$ and $p > 0$, and makes careful observations on the second quantity (Lemma 6.2.3) to derive the bounds for

$$\int_\Omega\frac{|\nabla v|^{N+2}}{v^{N+1}}$$

(Lemmas 6.3.1, 6.3.3 and 6.3.4). When applied to (6.1.11), this together with the degradation actions ultimately leads to L^p bounds for u (Lemma 6.3.2 and Lemma 6.3.5). In the course of our analysis across different scenarios, it turns out that as γ gets larger the contributions from absorption may become stronger than those from diffusion.

By means of these and further deduced higher-order estimates (Lemma 6.4.1, Lemma 6.4.3 and Lemma 6.5.2), the global bounded weak solutions can be constructed. Finally, the asymptotic behavior is established by considering the functional

$$\int_\Omega u-u_\infty\int_\Omega\ln\frac{u}{u_\infty}+\frac{M_1M_2}{4}u_\infty\int_\Omega v^2,$$

which serves as a starting point, and by using the estimates obtained in the previous steps.

6.2 Preliminaries

6.2.1 Regularized problems and basic properties

Similar to previous studies [42] and [88] (see also Chapters 2-5), our investigation begins with the following perturbed systems

$$\begin{cases} u_{\varepsilon t}=\nabla\cdot\left(u_\varepsilon v_\varepsilon\nabla u_\varepsilon\right)-\nabla\cdot\left(u_\varepsilon^2v_\varepsilon\nabla v_\varepsilon\right)+f(u_\varepsilon), & x\in\Omega,\ t>0,\\ v_{\varepsilon t}=\Delta v_\varepsilon-u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu}=\frac{\partial v_\varepsilon}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x)+\varepsilon,\quad v_\varepsilon(x,0)=v_0(x), & x\in\Omega,\end{cases}\tag{6.2.1}$$

which possess local classical solutions subject to an extensibility criterion.

Lemma 6.2.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that f satisfies* (6.1.3) *with $\kappa \geq 0$, $\mu > 0$ and $\gamma > 1$, and that (u_0, v_0) fulfill* (6.1.5)*. Then for each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and functions*

$$\begin{cases} u_\varepsilon \in C^0\left(\overline{\Omega} \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max,\varepsilon})\right) \qquad \textit{and} \\ v_\varepsilon \in \bigcap_{q>1} C^0([0, T_{max,\varepsilon}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \end{cases} \tag{6.2.2}$$

such that $u_\varepsilon, v_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{max,\varepsilon})$, that $(u_\varepsilon, v_\varepsilon)$ solves (6.2.1) *in the classical sense in $\Omega \times (0, T_{max,\varepsilon})$, and that*

$$\textit{if } T_{max,\varepsilon} < \infty, \quad \textit{then} \quad \limsup_{t \to T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{6.2.3}$$

In addition, for all $t_0 \in [0, T_{max,\varepsilon})$ this solution satisfies

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \textit{for all } t \in (t_0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1) \tag{6.2.4}$$

and

$$\int_{t_0}^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_\varepsilon(\cdot,t_0) \qquad \textit{for all } \varepsilon \in (0,1). \tag{6.2.5}$$

Furthermore, for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon(\cdot,t) \leq m := \max\left\{(\kappa+1)^{\frac{\gamma}{\gamma-1}}|\Omega|\left(\frac{2}{\mu}\right)^{\frac{1}{\gamma-1}}, \int_\Omega (u_0+1)\right\} \tag{6.2.6}$$

as well as for any $\tau \in \left(0, \min\left\{1, \frac{T_{max,\varepsilon}}{2}\right\}\right)$,

$$\int_t^{t+\tau} \int_\Omega u_\varepsilon^\gamma \leq \frac{4m}{\mu} \qquad \textit{for all } t \in (0, T_{max,\varepsilon} - \tau) \textit{ and } \varepsilon \in (0,1). \tag{6.2.7}$$

Proof. A slightly modified version of the arguments in Lemma 2.2.1 of Chapter 2 can be used to prove (6.2.2)-(6.2.5). To show (6.2.6) and (6.2.7), we integrate the first equation in (6.2.1) on Ω and use Young's inequality to find that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon + \int_\Omega u_\varepsilon \leq (\kappa+1)\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma \leq -\frac{\mu}{2}\int_\Omega u_\varepsilon^\gamma + (\kappa+1)^{\frac{\gamma}{\gamma-1}}|\Omega|\left(\frac{2}{\mu}\right)^{\frac{1}{\gamma-1}}. \tag{6.2.8}$$

A comparison argument then implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon \leq \max\left\{(\kappa+1)^{\frac{\gamma}{\gamma-1}}|\Omega|\left(\frac{2}{\mu}\right)^{\frac{1}{\gamma-1}}, \int_\Omega (u_0+1)\right\},$$

which yields (6.2.6), and in turn entails (6.2.7) by integrating (6.2.8) from t to $t+\tau$ for $t \in (0, T_{max,\varepsilon} - \tau)$. □

Throughout the sequel, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and that the initial data $(u_0,\ v_0)$ fulfilling (6.1.5) are fixed. With $T_{max,\varepsilon}$ given as in Lemma 6.2.1 we denote by $(u_\varepsilon, v_\varepsilon)$ the solution of (6.2.1) provided there. Furthermore, unless otherwise specified, the parameters κ and μ are assumed to satisfy $\kappa \geq 0$ and $\mu > 0$.

6.2.2 Basic testing procedures, a functional inequality and an ODI result

To avoid redundant computations, several fundamental results concerning the time evolution of certain quantities are first presented in this section. We begin by performing standard testing procedures for the first sub-problem of (6.2.1).

Lemma 6.2.2. *Let $N \geq 2$. Assume that (6.1.3) holds with $\gamma > 1$. Then for any $K > 0$, there exists $C(K) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + K\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla v_\varepsilon|^2 + C(K), \tag{6.2.9}$$

and for each $p > 1$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_\varepsilon^p + K\int_\Omega u_\varepsilon^p + \frac{p\mu}{2}\int_\Omega u_\varepsilon^{p+\gamma-1} + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\quad\leq \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon|\nabla v_\varepsilon|^2 + p(\kappa+K)^{\frac{p+\gamma-1}{\gamma-1}}\left(\frac{2}{\mu}\right)^{\frac{p}{\gamma-1}}|\Omega|. \end{aligned}\tag{6.2.10}$$

Proof. For $K > 0$ we define

$$c_0 := \max\left\{1, \left(\frac{\kappa+K}{\mu}\right)^{\frac{1}{\gamma-1}}\right\},$$

so that

$$(\kappa+K)u_\varepsilon \ln u_\varepsilon \leq \mu u_\varepsilon^\gamma \ln u_\varepsilon \quad \text{in} \quad \{u_\varepsilon \geq c_0\}.$$

Hence, by Young's inequality, for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned} &\kappa\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma + (\kappa+K)\int_\Omega u_\varepsilon \ln u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma \ln u_\varepsilon \\ &= \kappa\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma + \int_{\{u_\varepsilon \geq c_0\}}\Big\{(\kappa+K)u_\varepsilon \ln u_\varepsilon - \mu u_\varepsilon^\gamma \ln u_\varepsilon\Big\} \\ &\quad +(\kappa+K)\int_{\{u_\varepsilon<c_0\}} u_\varepsilon \ln u_\varepsilon - \mu\int_{\{u_\varepsilon<c_0\}} u_\varepsilon^\gamma \ln u_\varepsilon\Big\} \\ &\leq \frac{\kappa^{\frac{\gamma}{\gamma-1}}}{\mu^{\frac{1}{\gamma-1}}} + (\kappa+K)c_0\ln c_0|\Omega| + \frac{\mu}{\gamma e}|\Omega| \\ &=: c_1. \end{aligned}$$

Now, testing the first equation in (6.2.1) by $1+\ln u_\varepsilon$ and applying integration by parts twice, we obtain by using Young's inequality that for any $K > 0$, $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + K\int_\Omega u_\varepsilon \ln u_\varepsilon &= \int_\Omega (1+\ln u_\varepsilon)\,\nabla\cdot(u_\varepsilon v_\varepsilon \nabla u_\varepsilon) - \int_\Omega (1+\ln u_\varepsilon)\,\nabla\cdot(u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon) \\ &\quad +\kappa\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma + (\kappa+K)\int_\Omega u_\varepsilon \ln u_\varepsilon - \mu\int_\Omega u_\varepsilon^\gamma \ln u_\varepsilon \\ &\leq -\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon\cdot\nabla v_\varepsilon + c_1 \end{aligned}$$

$$\leq \quad -\frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon|^2 + c_1.$$

Next, for each $p > 1$ and $K > 0$, testing the first equation in (6.2.1) by u_ε^{p-1} and using Young's inequality several times, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^p + K\int_\Omega u_\varepsilon^p \quad &= \quad p\int_\Omega u_\varepsilon^{p-1}\nabla\cdot\left\{u_\varepsilon v_\varepsilon\nabla u_\varepsilon - u_\varepsilon^2 v_\varepsilon\nabla v_\varepsilon\right\} + (p\kappa + K)\int_\Omega u_\varepsilon^p - p\mu\int_\Omega u_\varepsilon^{p+\gamma-1} \\
&= \quad -p(p-1)\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + p(p-1)\int_\Omega u_\varepsilon^p v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\quad\quad +p(\kappa + K)\int_\Omega u_\varepsilon^p - p\mu\int_\Omega u_\varepsilon^{p+\gamma-1} \\
&\leq \quad -\frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2 \\
&\quad\quad -\frac{p\mu}{2}\int_\Omega u_\varepsilon^{p+\gamma-1} + p(\kappa+K)^{\frac{p+\gamma-1}{\gamma-1}}\left(\frac{2}{\mu}\right)^{\frac{p}{\gamma-1}}|\Omega|
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. The proof is finished. □

We now turn our attention to the time evolution of signal-related quantities involving spatial gradients of v_ε, which serves as a powerful analytical tool in a wide range of pertinent nutrient-consumption systems (for instance [66], [83], [86], [41]). We point out that the following lemma is established in more general domains compared to the convexity condition imposed in [93] (see also Lemma 3.2.4 in Chapter 3). This is because, by leveraging the approach in [82] to handle the boundary integral arising in the evolution process, the presence of the term $\int_\Omega v_\varepsilon$ in (6.2.11) and (6.2.12) can be handled in our subsequent analysis.

Lemma 6.2.3. *Let $N \geq 2$ and $q \geq 2$. Then for any $K > 0$, there exist positive constants $C_1(q,K,N)$ and $C_2(K,N)$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + K\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + \frac{q}{4(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} \\
&\quad \leq \quad 2^q q(q+\sqrt{N}+1)^{2q+2}\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon + C_1(q,K,N)\int_\Omega v_\varepsilon
\end{aligned} \tag{6.2.11}$$

and

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + K\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq 8(5+\sqrt{N})^4\int_\Omega \frac{u_\varepsilon^2}{v_\varepsilon}|\nabla v_\varepsilon|^2 + C_2(K,N)\int_\Omega v_\varepsilon. \tag{6.2.12}$$

Proof. From Lemma 3.3 in [82], for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + q\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 \\
&\quad \leq \quad q(q-2+\sqrt{N})\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-2}}|D^2 v_\varepsilon| + \frac{q}{2}\int_{\partial\Omega}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}.
\end{aligned} \tag{6.2.13}$$

For the boundary integral, we apply the same argument detailed in [82, Lemma 3.5] to see the existence of $c_1 = c_1(q,N) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{q}{2}\int_{\partial\Omega}\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}$$

$$
\begin{aligned}
&\le \frac{q}{8(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + \frac{q}{8(q+\sqrt{N}+1)^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}}|D^2 v_\varepsilon|^2 + c_1\int_\Omega v_\varepsilon \\
&\le \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{{v_\varepsilon}^{q-3}}|D^2\ln v_\varepsilon|^2 + c_1\int_\Omega v_\varepsilon
\end{aligned}
\tag{6.2.14}
$$

by Lemma 3.2.3, and then employ Young's inequality to obtain that

$$
\begin{aligned}
&K\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + q(q-2+\sqrt{N})\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-2}}|D^2 v_\varepsilon| \\
&\quad\le \frac{q}{8(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + \frac{q}{8(q+\sqrt{N}+1)^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-1}}|D^2 v_\varepsilon|^2 \\
&\quad\quad + c_2\int_\Omega v_\varepsilon + 2q(q+\sqrt{N}+1)^2(q-2+\sqrt{N})^2\int_\Omega u_\varepsilon^2\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} \\
&\quad\le \frac{q}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + 2q(q+\sqrt{N}+1)^4\int_\Omega u_\varepsilon^2\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} + c_2\int_\Omega v_\varepsilon
\end{aligned}
\tag{6.2.15}
$$

with $c_2 = c_2(q,K,N) > 0$. Now taking $q := 4$ and combining (6.2.13)-(6.2.15), we deduce (6.2.12). To derive (6.2.11), we further estimate the second summand on the right-hand side of (6.2.15) by applying Young's inequality to verify that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&\frac{q}{4(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + 2q(q+\sqrt{N}+1)^4\int_\Omega u_\varepsilon^2\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}} \\
&\le \frac{q}{2(q+\sqrt{N})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q+2}}{v_\varepsilon^{q+1}} + \left\{\left(\frac{4(q+\sqrt{N})^2}{q}\right)^{\frac{q-2}{q+2}}\cdot 2q(q+\sqrt{N}+1)^4\right\}^{\frac{q+2}{4}}\int_\Omega {u_\varepsilon}^{\frac{q+2}{2}} v_\varepsilon \\
&\le \frac{q}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + 2^q q(q+\sqrt{N}+1)^{2q+2}\int_\Omega {u_\varepsilon}^{\frac{q+2}{2}} v_\varepsilon.
\end{aligned}
$$

When combined with (6.2.13)-(6.2.15), this shows (6.2.11). □

The next lemma extends the result of [87, Lemma 3.5], originally restricted to spatially one- and two-dimensional settings, to arbitrary space dimensions. The proof relies on fundamental Sobolev embedding inequalities, where the choice of the embedding index therein plays a critical role in ensuring that the resulting estimate is optimally structured to facilitate the effective use of the weighted dissipative quantity $\int_\Omega u_\varepsilon^p v_\varepsilon |\nabla u_\varepsilon|^2$.

Lemma 6.2.4. *Let $N \ge 2$ and $\alpha \ge \frac{2N}{N+2}$. For any $\eta > 0$, with $C(\alpha, N) > 0$ we have*

$$
\begin{aligned}
\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2 &\le \eta\int_\Omega \varphi^{\alpha-2}\psi|\nabla\varphi|^2 + \eta\int_\Omega \varphi\psi \\
&+ C(\alpha,N)\cdot\left(1+\frac{1}{\eta^{\frac{N}{2}}}\right)\cdot\left\{\int_\Omega \varphi^\alpha + \left\{\int_\Omega \varphi\right\}^{\frac{(N+2)\alpha-N}{2}}\right\}\cdot\int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}}
\end{aligned}
\tag{6.2.16}
$$

is valid for arbitrary nonnegative functions $\varphi \in C^1(\overline{\Omega})$ and positive $\psi \in C^1(\overline{\Omega})$.

Proof. From Young's inequality, we get

$$\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2 \le \left\{\int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}}\right\}^{\frac{2}{N+2}} \cdot \left\{\int_\Omega \varphi^{\frac{(N+2)\alpha}{N}}\psi\right\}^{\frac{N}{N+2}}. \tag{6.2.17}$$

Since $\alpha \ge \frac{2N}{N+2}$, according to the Sobolev embedding $W^{1,\frac{2N}{N+2}}(\Omega) \hookrightarrow L^2(\Omega)$ and the continuous embeddings $L^2(\Omega) \hookrightarrow L^{\frac{2N}{N+2}}(\Omega) \hookrightarrow L^{\frac{2N}{(N+2)\alpha}}(\Omega)$, with $c_1 = c_1(\alpha, N) > 0$ we have

$$\|\phi\|_{L^2(\Omega)} \le c_1 \|\nabla\phi\|_{L^{\frac{2N}{N+2}}(\Omega)} + c_1 \|\phi\|_{L^{\frac{2N}{(N+2)\alpha}}(\Omega)} \qquad \text{for all } \phi \in W^{1,\frac{2N}{N+2}}(\Omega). \tag{6.2.18}$$

When applying (6.2.18) to $\phi := \varphi^{\frac{(N+2)\alpha}{2N}}\psi^{\frac{1}{2}}$, we see the existence of $c_2 = c_2(N) > 0$ and $c_3 = c_3(\alpha, N) > 0$ fulfilling

$$\begin{aligned}
\left\{\int_\Omega \varphi^{\frac{(N+2)\alpha}{N}}\psi\right\}^{\frac{N}{N+2}} &= \left\|\varphi^{\frac{(N+2)\alpha}{2N}}\psi^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{\frac{2N}{N+2}} \\
&\le c_2 \left\|\nabla\left(\varphi^{\frac{(N+2)\alpha}{2N}}\psi^{\frac{1}{2}}\right)\right\|_{L^{\frac{2N}{N+2}}(\Omega)}^{\frac{2N}{N+2}} + c_2 \left\|\varphi^{\frac{(N+2)\alpha}{2N}}\psi^{\frac{1}{2}}\right\|_{L^{\frac{2N}{(N+2)\alpha}}(\Omega)}^{\frac{2N}{N+2}} \\
&\le c_3 \int_\Omega \left(\varphi^{\frac{(N+2)\alpha}{2N}-1}\psi^{\frac{1}{2}}|\nabla\varphi|\right)^{\frac{2N}{N+2}} + c_2 \int_\Omega \left(\frac{\varphi^{\frac{(N+2)\alpha}{2N}}}{\sqrt{\psi}}|\nabla\psi|\right)^{\frac{2N}{N+2}} \\
&\quad + c_3 \cdot \left\{\int_\Omega \varphi\psi^{\frac{N}{(N+2)\alpha}}\right\}^\alpha \\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{6.2.19}$$

where three applications of the Hölder inequality show that

$$I_1 = c_3 \int_\Omega \left(\varphi^{\alpha-2}\psi|\nabla\varphi|^2\right)^{\frac{N}{N+2}} \cdot \varphi^{\frac{2\alpha}{N+2}} \le c_3 \cdot \left\{\int_\Omega \varphi^{\alpha-2}\psi|\nabla\varphi|^2\right\}^{\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\}^{\frac{2}{N+2}},$$

$$I_2 = c_3 \int_\Omega \left(\frac{\varphi^\alpha}{\psi}|\nabla\psi|^2\right)^{\frac{N}{N+2}} \cdot \varphi^{\frac{2\alpha}{N+2}} \le c_3 \cdot \left\{\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2\right\}^{\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\}^{\frac{2}{N+2}}$$

and

$$I_3 = c_2 \cdot \left\{\int_\Omega \varphi^{1-\frac{N}{(N+2)\alpha}} \cdot (\varphi\psi)^{\frac{N}{(N+2)\alpha}}\right\}^\alpha \le c_2 \cdot \left\{\int_\Omega \varphi\right\}^{\alpha-\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi\psi\right\}^{\frac{N}{N+2}}.$$

Inserting (6.2.19) into (6.2.17) and employing Young's inequality reveal that for each $\eta > 0$,

$$\begin{aligned}
\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2 &\le c_3 \cdot \left\{\int_\Omega \varphi^{\alpha-2}\psi|\nabla\varphi|^2\right\}^{\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\}^{\frac{2}{N+2}} \cdot \left\{\int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}}\right\}^{\frac{2}{N+2}} \\
&\quad + c_3 \cdot \left\{\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2\right\}^{\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\}^{\frac{2}{N+2}} \cdot \left\{\int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}}\right\}^{\frac{2}{N+2}}
\end{aligned}$$

$$
\begin{aligned}
&+c_2 \cdot \left\{\int_\Omega \varphi\psi\right\}^{\frac{N}{N+2}} \cdot \left\{\int_\Omega \varphi\right\}^{\alpha-\frac{N}{N+2}} \cdot \left\{\int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}}\right\}^{\frac{2}{N+2}} \\
\leq\ & \frac{\eta}{2}\int_\Omega \varphi^{\alpha-2}\psi|\nabla\varphi|^2 + c_3^{\frac{N+2}{2}}\left(\frac{2}{\eta}\right)^{\frac{N}{2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\} \cdot \int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}} \\
&+\frac{1}{2}\int_\Omega \frac{\varphi^\alpha}{\psi}|\nabla\psi|^2 + c_3^{\frac{N+2}{2}} 2^{\frac{N}{2}} \cdot \left\{\int_\Omega \varphi^\alpha\right\} \cdot \int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}} \\
&+\frac{\eta}{2}\int_\Omega \varphi\psi + c_2^{\frac{N+2}{2}}\left(\frac{2}{\eta}\right)^{\frac{N}{2}} \cdot \left\{\int_\Omega \varphi\right\}^{\frac{(N+2)\alpha-N}{2}} \cdot \int_\Omega \frac{|\nabla\psi|^{N+2}}{\psi^{N+1}},
\end{aligned}
$$

which implies (6.2.16) by choosing $C(\alpha, N) := \max\{(2c_2)^{\frac{N+2}{2}}, (2c_3)^{\frac{N+2}{2}}\}$. □

Finally, we introduce the following statement on the ODI, which is a copy of [60, Lemma 2.3] and will be frequently used in Section 6.3.

Lemma 6.2.5. *Let $T \in (0, \infty]$ and $0 < \tau < T$. Suppose that y is a nonnegative absolutely continuous function and satisfies*

$$
y'(t) + a(t)y(t) \leq b(t)y(t) + c(t) \qquad \textit{for all } t \in (0, T)
$$

with functions $0 < a \in L^1_{loc}([0, T))$ and $0 \leq b, c \in L^1_{loc}([0, T))$ fulfilling

$$
\int_t^{t+\tau} b(s)ds \leq b_1, \quad \int_t^{t+\tau} c(s)ds \leq c_1 \qquad \textit{for all } t \in (0, T-\tau)
$$

and

$$
\int_t^{t+\tau} a(s)ds - \int_t^{t+\tau} b(s)ds \geq \varrho \qquad \textit{for all } t \in (0, T-\tau),
$$

where b_1, c_1 and ϱ are positive constants. Then

$$
y(t) \quad \leq \quad y_0 e^{b_1} + \frac{c_1 e^{2b_1}}{1 - e^{-\varrho}} + c_1 e^{b_1} \qquad \textit{for all } t \in (0, T).
$$

6.3 Uniform L^p-boundedness of u_ε

With the preparations in the previous section, we are now in a position to derive L^p bounds for u_ε for any $p > 1$ by initiating an analysis of the energy-like functional

$$
c\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} \tag{6.3.1}
$$

for suitably chosen values of p and $c = c(p)$. The treatment of the respective unfavorable terms appearing in (6.2.9)-(6.2.12) slightly differs depending on the spatial dimension. From now on, for convenience, we fix

$$
\tau := \min\left\{1, \frac{T_{max,\varepsilon}}{2}\right\}.
$$

6.3.1 Uniform L^p bounds for u_ε when $N = 2$

In the two-dimensional case, since the ill-signed term $\int_\Omega \frac{u_\varepsilon^2}{v_\varepsilon}|\nabla v_\varepsilon|^2$ arising on the right-hand side of (6.2.12) can potentially be estimated by means of the functional inequality (6.2.16) with the choice $\alpha := 2$ therein, we substitute the first quantity in (6.3.1) by $\int_\Omega u_\varepsilon \ln u_\varepsilon$, which exhibits the same scaling as $\int_\Omega u_\varepsilon$ and the ability to transform $\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2$ into a favorable one. Consequently, by combining this with the space-time integrability of u_ε^2, we obtain the following boundedness result.

Lemma 6.3.1. *Let $N = 2$. Assume that f satisfies* (6.1.3) *with $\gamma = 2$. Then there exists $C > 0$ such that*

$$\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^4}{v_\varepsilon^3(\cdot,t)} \le C \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{6.3.2}$$

Proof. Let $c_1 := 8(5+\sqrt{2})^4$ and $c_2 := \|v_0\|_{L^\infty(\Omega)}$. Employing (6.2.6) and Lemma 6.2.4, we can fix $c_3 > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\left(c_1 + \frac{c_2^2}{2}\right)\int_\Omega \frac{u_\varepsilon^2}{v_\varepsilon}|\nabla v_\varepsilon|^2 \;&\le\; \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon \\
&\quad + c_3\cdot\left\{\int_\Omega u_\varepsilon^2 + \left\{\int_\Omega u_\varepsilon\right\}^3\right\}\cdot\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\le\; \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + c_3\cdot\left\{\int_\Omega u_\varepsilon^2 + m^3\right\}\cdot\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{mc_2}{2},
\end{aligned} \tag{6.3.3}$$

where due to (6.2.7) there exists $c_4 > 0$ fulfilling

$$c_3\int_t^{t+\tau}\left(\int_\Omega u_\varepsilon^2 + m^3\right) \le c_4 \qquad \text{for all} \quad t \in (0, T_{max,\varepsilon} - \tau) \text{ and } \varepsilon \in (0,1). \tag{6.3.4}$$

Now, taking $K := 2c_4$ in (6.2.12) and (6.2.9), one can see the existence of $c_5 > 0$ and $c_6 > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 2c_4\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le c_1\int_\Omega \frac{u_\varepsilon^2}{v_\varepsilon}|\nabla v_\varepsilon|^2 + c_5\int_\Omega v_\varepsilon$$

and

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + 2c_4\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \le \frac{1}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon|^2 + c_6.$$

Trivially adding the above two inequalities, we obtain that for

$$y_\varepsilon(t) = \frac{|\Omega|}{e} + \int_\Omega u_\varepsilon \ln u_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}, \quad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1),$$

we have

$$y_\varepsilon'(t) + 2c_4 y_\varepsilon(t) + \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 \le \left(c_1 + \frac{c_2^2}{2}\right)\int_\Omega \frac{u_\varepsilon^2}{v_\varepsilon}|\nabla v_\varepsilon|^2 + c_2c_5|\Omega| + \frac{2c_4|\Omega|}{e} + c_6$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. This combined with (6.3.3) and the nonnegativity of the y_ε implies that

$$y_\varepsilon'(t) + 2c_4 y_\varepsilon(t) \le b_\varepsilon(t) y_\varepsilon(t) + c_\varepsilon(t) \qquad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

with

$$b_\varepsilon(t) := c_3 \cdot \left\{ \int_\Omega u_\varepsilon^2 + m^3 \right\} \quad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

and

$$c_\varepsilon(t) := \frac{mc_2}{2} + c_2 c_5 \cdot |\Omega| + \frac{2c_4}{e} \cdot |\Omega| + c_6 \quad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1).$$

Due to (6.3.4), Lemma 6.2.5 becomes applicable so as to entail that there is $c_9 > 0$ satisfying

$$y_\varepsilon(t) \le c_9 \qquad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Since $\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{|\Omega|}{e} \ge 0$ for any $\varepsilon \in (0,1)$, this directly yields the intended (6.3.2). □

We are now able to derive L^p estimates for u_ε for any $p > 1$ in the two-dimensional setting. Notably, if attempting to control the unfavorable term $\int_\Omega u_\varepsilon^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2$ in (6.3.7) directly via (6.2.16) alongside the L^∞-boundedness of v_ε, the parameter μ would be required to be sufficiently large. To avoid imposing such an unnecessarily extra condition, we adopt a more delicate treatment on this term.

Lemma 6.3.2. *Let $N = 2$. Assume that f satisfies* (6.1.3) *with $\gamma = 2$. Then for any $p > 1$, there exists $C(p) > 0$ such that*

$$\int_\Omega u_\varepsilon^p(\cdot, t) \le C(p) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1) \tag{6.3.5}$$

and

$$\int_t^{t+\tau} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le C(p) \qquad \textit{for all } t \in (0, T_{max,\varepsilon} - \tau) \textit{ and } \varepsilon \in (0,1). \tag{6.3.6}$$

Proof. In light of Young's inequality and Lemma 6.3.1, it follows from (6.2.10) that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u_\varepsilon^p + \int_\Omega u_\varepsilon^p + \frac{p\mu}{2} \int_\Omega u_\varepsilon^{p+1} + \frac{p(p-1)}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\
\le \ & \frac{p(p-1)}{2} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 + c_1 \\
\le \ & \frac{p(p-1)}{2} \left\{ \int_\Omega u_\varepsilon^{2(p+1)} v_\varepsilon^5 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \right\}^{\frac{1}{2}} + c_1 \\
\le \ & \frac{p(p-1)}{2} c_2^{\frac{3}{2}} c_3^{\frac{1}{2}} \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^4(\Omega)}^2 + c_1
\end{aligned} \tag{6.3.7}$$

with $c_1 = c_1(p) > 0$, $c_2 := \|v_0\|_{L^\infty(\Omega)}$ and

$$c_3 := \sup_{\substack{t\in(0,T_{max,\varepsilon})\\ \varepsilon\in(0,1)}} \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^4}{v_\varepsilon^3(\cdot,t)} < \infty.$$

Due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and the continuity of the embeddings $L^4(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$, a two-dimensional version of the Sobolev inequality implies that for any $\delta > 0$, there exists $c_4 = c_4(p,\delta) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\frac{p(p-1)}{2} c_2^{\frac{3}{2}} c_3^{\frac{1}{2}} \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^4(\Omega)}^2 \leq \delta \left\| \nabla \left(u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right) \right\|_{L^2(\Omega)}^2 + c_4 \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^1(\Omega)}^2, \tag{6.3.8}$$

where by Lemma 6.2.4, with some $c_5 = c_5(p) > 0$ and $c_6 := \frac{mc_2\delta}{2} + c_3 c_5 m^{2p+1}\delta$ we have

$$\begin{aligned}
\delta \left\| \nabla \left(u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right) \right\|_{L^2(\Omega)}^2 &\leq \frac{(p+1)^2\delta}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{\delta}{2} \int_\Omega \frac{u_\varepsilon^{p+1}}{v_\varepsilon} |\nabla v_\varepsilon|^2 \\
&\leq \frac{(p+1)^2\delta + \delta}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{\delta}{2} \int_\Omega u_\varepsilon v_\varepsilon \\
&\quad + c_5\delta \cdot \left\{ \int_\Omega u_\varepsilon^{p+1} + \left\{ \int_\Omega u_\varepsilon \right\}^{2p+1} \right\} \cdot \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\leq (p+1)^2\delta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + c_3 c_5 \delta \int_\Omega u_\varepsilon^{p+1} + c_6,
\end{aligned} \tag{6.3.9}$$

and moreover, using Young's inequality with (6.2.6) and (6.2.4) we can see the existence of $c_7 = c_7(p,\delta) > 0$ such that

$$c_4 \left\| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac{1}{2}} \right\|_{L^1(\Omega)}^2 \leq c_4 \int_\Omega u_\varepsilon^p \cdot \int_\Omega u_\varepsilon v_\varepsilon \leq c_2 c_4 m \int_\Omega u_\varepsilon^p \leq \frac{p\mu}{4} \int_\Omega u_\varepsilon^{p+1} + c_7. \tag{6.3.10}$$

Now taking

$$\delta := \min\left\{ \frac{p(p-1)}{4(p+1)^2},\ \frac{p\mu}{2c_3c_5} \right\},$$

we then deduce from (6.3.7)-(6.3.10) that

$$\frac{d}{dt} \int_\Omega u_\varepsilon^p + \int_\Omega u_\varepsilon^p + \frac{p-1}{4} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \leq c_1 + c_6 + c_7$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$, which in conjunction with Lemma 6.2.5 yields (6.3.5). Finally, integrating this in time implies (6.3.6). □

6.3.2 Uniform L^p bounds for u_ε when $N \geq 3$

The favorable structural alignment that allows interaction-driven term $\int_\Omega u_\varepsilon^2 \frac{|\nabla v_\varepsilon|^N}{v_\varepsilon^{N-1}}$ to be directly estimated by (6.2.16) in the case $N = 2$ breaks down when $N > 2$, primarily due to the superquadratic growth of ∇v_ε.

In the intermediate-dimensional cases ($N = \{3,4\}$), we first apply Young's inequality to this ill-signed term and subsequently exploit the damping effect to absorb it, which necessitates choosing a relatively large exponent $p := \frac{N+4}{4}$ in (6.3.1). However, to ensure that (6.2.16) remains effective in controlling the resulting taxis-induced term, a more stringent condition on γ is required in this setting.

Lemma 6.3.3. *Let $N = \{3,4\}$. Assume that f satisfies* (6.1.3) *with $\gamma = \frac{N+8}{4}$. Then there exists $C(N) > 0$ such that*

$$\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^{N+2}}{v_\varepsilon^{N+1}(\cdot,t)} \le C(N) \qquad \text{for all } t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{6.3.11}$$

Proof. Let $c_1 := 2^{N+2}(N+2)(N+3+\sqrt{N})^{2N+6}$, $c_2 := \|v_0\|_{L^\infty(\Omega)}$, and let m be as in (6.2.6). Invoking Lemma 6.2.4, we can find $c_3 = c_3(N) > 0$ such that for all $t \in (0,T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}\frac{Nc_1c_2^3}{4\mu}\int_\Omega \frac{u_\varepsilon^{\frac{N+8}{4}}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \le\ & \frac{Nc_1c_2}{4\mu}\int_\Omega u_\varepsilon^{\frac{N}{4}} v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{Nc_1c_2}{4\mu}\int_\Omega u_\varepsilon v_\varepsilon \\ & + c_3\cdot\left\{\int_\Omega u_\varepsilon^{\frac{N+8}{4}} + \left\{\int_\Omega u_\varepsilon\right\}^{\frac{N^2+6N+16}{8}}\right\}\cdot\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} \\ \le\ & \frac{Nc_1c_2}{4\mu}\int_\Omega u_\varepsilon^{\frac{N}{4}} v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{Nmc_1c_2^2}{4\mu}\cdot|\Omega| \\ & + b_\varepsilon(t)\cdot\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}},\end{aligned} \tag{6.3.12}$$

where

$$b_\varepsilon(t) := c_3\cdot\left\{\int_\Omega u_\varepsilon^{\frac{N+8}{4}} + m^{\frac{N^2+6N+16}{8}}\right\}, \qquad t \in (0,T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

satisfies

$$\int_t^{t+\tau} b_\varepsilon(s)ds \le c_4 \qquad \text{for all} \quad t \in (0,T_{max,\varepsilon}-\tau) \text{ and } \varepsilon \in (0,1) \tag{6.3.13}$$

with some $c_4 = c_4(N) > 0$ by (6.2.7). We next apply (6.2.11) with $q := N+2$ and $K := 2c_4$ to see the existence of $c_5 = c_5(N) > 0$ satisfying

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} + 2c_4\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} \\ &\quad\le c_1\int_\Omega u_\varepsilon^{\frac{N+4}{2}} v_\varepsilon + c_5\int_\Omega v_\varepsilon \\ &\quad\le c_1c_2\int_\Omega u_\varepsilon^{\frac{N+4}{2}} + c_5c_2\cdot|\Omega| \qquad \text{for all} \quad t \in (0,T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),\end{aligned} \tag{6.3.14}$$

and then invoke (6.2.10) with $p := \frac{N+4}{4}$ and $K := 2c_4$ to find $c_6 = c_6(N) > 0$ such that

$$\frac{8c_1c_2}{(N+4)\mu}\frac{d}{dt}\int_\Omega u_\varepsilon^{\frac{N+4}{4}} + \frac{16c_1c_2c_4}{(N+4)\mu}\int_\Omega u_\varepsilon^{\frac{N+4}{4}} + c_1c_2\int_\Omega u_\varepsilon^{\frac{N+4}{2}} + \frac{Nc_1c_2}{4\mu}\int_\Omega u_\varepsilon^{\frac{N}{4}} v_\varepsilon|\nabla u_\varepsilon|^2$$

$$\leq \frac{Nc_1c_2}{4\mu}\int_\Omega u_\varepsilon^{\frac{N+8}{4}} v_\varepsilon|\nabla v_\varepsilon|^2 + c_6$$
$$\leq \frac{Nc_1c_2^3}{4\mu}\int_\Omega \frac{u_\varepsilon^{\frac{N+8}{4}}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + c_6 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \qquad (6.3.15)$$

Combining (6.3.12), (6.3.14) and (6.3.15), we thereby obtain on writing

$$y_\varepsilon(t) = \frac{8c_1c_2}{(N+4)\mu}\int_\Omega u_\varepsilon^{\frac{N+4}{4}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}}, \quad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

and

$$c_\varepsilon(t) := \frac{Nmc_1c_2^2}{4\mu}\cdot|\Omega| + c_5c_2\cdot|\Omega| + c_6, \qquad t \in (0, T_{max,\varepsilon}),\ \varepsilon \in (0,1)$$

that

$$y_\varepsilon'(t) + 2c_4y_\varepsilon(t) \leq b_\varepsilon(t)y_\varepsilon(t) + c_\varepsilon(t) \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Thanks to (6.3.13), we can again apply Lemma 6.2.5 to deduce the existence of $c_7 = c_7(N) > 0$ satisfying

$$y_\varepsilon(t) \leq c_7 \qquad \text{for all } \quad t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

This proves (6.3.11). □

From Lemma 6.3.1 and Lemma 6.3.3, we observe that as the spatial dimension increases, a correspondingly larger value of γ is required. This accordingly enhances the dissipative effect of the absorption term, which may even dominate that of the diffusion term. It turns out that if one employs similar techniques as those in the lower-dimensional cases–namely leveraging (6.2.16) to estimate the cross-diffusive contributions in the standard $\int_\Omega u_\varepsilon^p$ testing procedure –then stricter conditions on γ are required.

Instead, in higher dimensions, we utilize the absorption mechanism to control all the troublesome integrals on the right-hand side of the evolution of (6.3.1). In truth, for any $p > 1$, we can directly derive L^p-boundedness of u_ε by considering the energy functional

$$\int_\Omega u_\varepsilon^p + \int_\Omega \frac{|\nabla v_\varepsilon|^{2p+2}}{v_\varepsilon^{2p+1}},$$

provided that μ depending on p is conveniently large. To eliminate this dependence, we first choose the specific value $p := \frac{N}{2}$ to derive bounds for $\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}}$ in the next lemma.

Lemma 6.3.4. *Let $N \geq 5$. Assume that f satisfies* (6.1.3) *with $\gamma = 3$. Then there exists $\mu_0 = \mu_0(N) > 0$ with the property that if $\mu \geq \mu_0$, one can find $C(N) > 0$ such that*

$$\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^{N+2}}{v_\varepsilon^{N+1}(\cdot,t)} \leq C(N) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).$$

Proof. Applying (6.2.11) with $q := N+2$ and $K := 1$, we find $c_1 = c_1(N) > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} + \frac{1}{c_1}\int_\Omega \frac{|\nabla v_\varepsilon|^{N+4}}{v_\varepsilon^{N+3}} \le c_1 \int_\Omega u_\varepsilon^{\frac{N+4}{2}} v_\varepsilon + c_1 \int_\Omega v_\varepsilon \quad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{6.3.16}$$

On the other hand, in view of (6.2.10), we may further invoke Young's inequality to see the existence of $c_2 = c_2(N) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^{\frac{N}{2}} + \int_\Omega u_\varepsilon^{\frac{N}{2}} + \frac{\mu N}{4}\int_\Omega u_\varepsilon^{\frac{N+4}{2}} &\le \frac{N(N-2)}{8}\int_\Omega u_\varepsilon^{\frac{N+2}{2}} v_\varepsilon |\nabla v_\varepsilon|^2 + c_2 \\ &\le \frac{1}{c_1}\int_\Omega \frac{|\nabla v_\varepsilon|^{N+4}}{v_\varepsilon^{N+3}} + \left(\frac{N(N-2)}{8}\right)^{\frac{N+4}{N+2}} c_1^{\frac{2}{N+2}} \int_\Omega u_\varepsilon^{\frac{N+4}{2}} v_\varepsilon^{\frac{3N+10}{N+2}} + c_2,\end{aligned}$$

which together with (6.3.16) implies that

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega u_\varepsilon^{\frac{N}{2}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}}\right\} + \left\{\int_\Omega u_\varepsilon^{\frac{N}{2}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}}\right\} \\ &\le c_1 \int_\Omega u_\varepsilon^{\frac{N+4}{2}} v_\varepsilon + \left(\frac{N(N-2)}{8}\right)^{\frac{N+4}{N+2}} c_1^{\frac{2}{N+2}} \int_\Omega u_\varepsilon^{\frac{N+4}{2}} v_\varepsilon^{\frac{3N+10}{N+2}} - \frac{\mu N}{4}\int_\Omega u_\varepsilon^{\frac{N+4}{2}} \\ &\quad + c_1 \int_\Omega v_\varepsilon + c_2 \\ &\le \left(c_1 c_3 + \left(\frac{N(N-2)}{8}\right)^{\frac{N+4}{N+2}} c_1^{\frac{2}{N+2}} c_3^{\frac{3N+10}{N+2}} - \frac{\mu N}{4}\right)\int_\Omega u_\varepsilon^{\frac{N+4}{2}} + c_1 c_3 \cdot |\Omega| + c_2\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$ with $c_3 := \|v_0\|_{L^\infty(\Omega)}$. Thus, by choosing

$$\mu_0 \equiv \mu_0(N) := \frac{4}{N}\left(c_1 c_3 + \left(\frac{N(N-2)}{8}\right)^{\frac{N+4}{N+2}} c_1^{\frac{2}{N+2}} c_3^{\frac{3N+10}{N+2}}\right),$$

we can apply Lemma 6.2.5 to finish the proof. □

After establishing the crucial estimates for $\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}}$, we can once again apply the functional inequality (6.2.16) to derive the desired L^p bounds for u_ε.

Lemma 6.3.5. *Let $N \ge 3$. Assume that f satisfies* (6.1.3) *with*

$$\begin{cases} \gamma = \frac{N+8}{4} \quad \text{and} \quad \mu > 0, & N = \{3,4\}, \\ \gamma = 3 \quad \text{and} \quad \mu \ge \mu_0, & N \ge 5, \end{cases}$$

where μ_0 is taken from in Lemma 6.3.4. Then for any $p > 1$, there exists $C(p,N) > 0$ such that

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p,N) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{6.3.17}$$

and

$$\int_t^{t+\tau}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \le C(p,N) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}-\tau) \textit{ and } \varepsilon\in(0,1). \tag{6.3.18}$$

Proof. In view of (6.2.10), taking $K := 1$ therein provides the existence of $c_1 = c_1(p,N) > 0$ satisfying

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \int_\Omega u_\varepsilon^p + \frac{p\mu}{2}\int_\Omega u_\varepsilon^{p+\gamma-1} + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \\ \le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2 + c_1 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1).\end{aligned} \tag{6.3.19}$$

Because Lemma 6.3.3 and Lemma 6.3.4 entail that there exists $c_2 = c_2(N) > 0$ such that

$$\int_\Omega \frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} \le c_2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1),$$

it follows from Lemma 6.2.4, (6.2.4) and (6.2.6) that for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}\frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2 &\le \frac{p(p-1)c_3^2}{2}\int_\Omega \frac{u_\varepsilon^{p+1}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon v_\varepsilon \\ &\quad + c_4\cdot\left\{\int_\Omega u_\varepsilon^{p+1} + \left\{\int_\Omega u_\varepsilon\right\}^{\frac{(N+2)(p+1)-N}{2}}\right\}\cdot\int_\Omega\frac{|\nabla v_\varepsilon|^{N+2}}{v_\varepsilon^{N+1}} \\ &\le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \\ &\quad + c_2c_4\int_\Omega u_\varepsilon^{p+1} + \frac{p(p-1)c_3m}{4} + c_2c_4m^{\frac{p(N+2)+2}{2}} \\ &\le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p\mu}{2}\int_\Omega u_\varepsilon^{p+\gamma-1} + c_5\end{aligned}$$

with $c_3 := \|v_0\|_{L^\infty(\Omega)}$, $c_4 = c_4(p,N) > 0$ and $c_5 := \frac{p^2c_3m}{4} + c_2c_4m^{\frac{p(N+2)+2}{2}} + (c_2c_4)^{\frac{p+\gamma-1}{\gamma-2}}\left(\frac{2}{p\mu}\right)^{\frac{p+1}{\gamma-2}}$, where the last inequality is deduced by Young's inequality. Inserting this into (6.3.19) yields

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \int_\Omega u_\varepsilon^p + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \le c_1 + c_5$$

for all $t\in(0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$, which combined with a simplified version of Lemma 6.2.5 implies (6.3.17). Finally, integrating from t to $t+\tau$ gives (6.3.18), as desired. □

6.4 Uniform L^∞-boundedness of u_ε

In this section, we first show that based on the boundedness of $\|u_\varepsilon\|_{L^p(\Omega)}$ established in Lemma 6.3.2 and Lemma 6.3.5, the regularity of v_ε can be improved to obtain uniform bounds on its gradient in L^∞ norm, as well as control over the quantity $\int_\Omega \frac{|\nabla v_\varepsilon|^{3N}}{v_\varepsilon^{3N-1}}$.

Lemma 6.4.1. *Let $N \geq 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4)*. Then there exists $C(N) > 0$ such that*

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C(N) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1) \tag{6.4.1}$$

and

$$\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^{3N}}{v_\varepsilon^{3N-1}(\cdot,t)} \leq C(N) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \tag{6.4.2}$$

Proof. The proof of (6.4.1) is analogous to the proof of Lemma 2.4.1. To verify (6.4.2), we recall (6.2.11) and (6.2.4) to see the existence of $c_2 = c_2(N) > 0$ satisfying

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^{3N}}{v_\varepsilon^{3N-1}} + \int_\Omega \frac{|\nabla v_\varepsilon|^{3N}}{v_\varepsilon^{3N-1}} \leq c_2 \int_\Omega u_\varepsilon^{\frac{3N+2}{2}} + c_2 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1).$$

Then in virtue of Lemma 6.3.2 and Lemma 6.3.5, an ODE argument gives (6.4.2). □

This, in turn, plays a key role in deducing the L^∞-boundedness of u_ε. Before proceeding further, we introduce the next functional inequality, which generalizes a two-dimensional statement presented in [88, Lemma 6.2] to arbitrary spatial dimensions.

Lemma 6.4.2. *Let $N \geq 2$. Assume that $p_* > 2$. Then for any $p \geq p*$ and $0 < \eta < 1$, there exist $\delta = \delta(N, p_*) > 0$ and $C(N, p_*) > 0$ such that*

$$\begin{aligned} \int_\Omega \varphi^{p+1}\psi \ \ &\leq \ \ \eta \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}} \cdot \int_\Omega \frac{|\nabla\psi|^{3N}}{\psi^{3N-1}} \\ &\quad + C(N,p_*)p^{2\delta}\eta^{-\delta} \cdot \left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^2 \cdot \int_\Omega \varphi\psi \end{aligned} \tag{6.4.3}$$

is valid for arbitrary positive functions $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$.

Proof. Letting

$$\lambda \equiv \lambda(N, p_*) := \frac{6Np_*}{3Np_* + 4p_* + 4},$$

since $p \geq p_* > 2$, we have

$$1 \leq \frac{2N}{N+2} < \lambda \leq \frac{6Np}{3Np + 4p + 4} < 2,$$

and therefore

$$\theta \equiv \theta(N, p_*) := \frac{2 - \frac{1}{2}}{2 - \frac{1}{\lambda} + \frac{1}{N}} \in (0,1)$$

as well as

$$\frac{2\lambda(p+1)}{3N(2-\lambda)} \leq \frac{\frac{12Np}{3Np+4p+4}(p+1)}{3N(2 - \frac{6Np}{3Np+4p+4})} = \frac{p}{2}. \tag{6.4.4}$$

We now fix $\eta > 0$ and define

$$\eta_1 \equiv \eta_1(p, N, \eta) := \min\left\{\frac{\eta}{(p+1)^2 c_1},\ \frac{\eta^{\frac{2}{3N}}}{c_1}\right\} \tag{6.4.5}$$

with

$$c_1 := \max\{|\Omega|,\ 1\}.$$

Then for any positive functions $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$, we see from the Gagliardo-Nirenberg inequality and Young's inequality that with $c_2 = c_2(N, p_*) > 0$, we have

$$\begin{aligned}
\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^2(\Omega)} &\le c_2\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\|^{2\theta}_{L^\lambda(\Omega)}\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^{2(1-\theta)}_{L^{\frac{1}{2}}(\Omega)} + c_2\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{\frac{1}{2}}(\Omega)} \\
&\le \eta_1\left\|\nabla\left(\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right)\right\|^2_{L^\lambda(\Omega)} + \left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}} + c_2\right)\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right\|^2_{L^{\frac{1}{2}}(\Omega)} \\
&\le \frac{(p+1)^2\eta_1}{2}\left\|\varphi^{\frac{p-1}{2}}\psi^{\frac{1}{2}}\nabla\varphi\right\|^2_{L^\lambda(\Omega)} + \frac{\eta_1}{2}\left\|\varphi^{\frac{p+1}{2}}\psi^{-\frac{1}{2}}\nabla\psi\right\|^2_{L^\lambda(\Omega)} \\
&\quad + \left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}} + c_2\right)\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\right\|^2_{L^{\frac{1}{2}}(\Omega)}.
\end{aligned} \tag{6.4.6}$$

Here, due to $0 < \frac{2-\lambda}{\lambda} < 1$, several applications of the Hölder inequality show that

$$\begin{aligned}
\frac{(p+1)^2\eta_1}{2}\left\|\varphi^{\frac{p-1}{2}}\psi^{\frac{1}{2}}\nabla\varphi\right\|^2_{L^\lambda(\Omega)} &= \frac{(p+1)^2\eta_1}{2}\cdot\left\{\int_\Omega \varphi^{\frac{(p-1)\lambda}{2}}\psi^{\frac{\lambda}{2}}|\nabla\varphi|^\lambda\right\}^{\frac{2}{\lambda}} \\
&\le \frac{(p+1)^2\eta_1}{2}|\Omega|^{\frac{2-\lambda}{\lambda}}\cdot\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 \\
&\le \frac{\eta}{2}\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2
\end{aligned} \tag{6.4.7}$$

and

$$\begin{aligned}
\frac{\eta_1}{2}\left\|\varphi^{\frac{p+1}{2}}\psi^{-\frac{1}{2}}\nabla\psi\right\|^2_{L^\lambda(\Omega)} &= \frac{\eta_1}{2}\cdot\left\{\int_\Omega\left(\frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}\right)^{\frac{\lambda}{3N}}\cdot\left(\varphi^{p+1}\psi\right)^{\frac{(3N-2)\lambda}{6N}}\cdot\varphi^{\frac{\lambda(p+1)}{3N}}\right\}^{\frac{2}{\lambda}} \\
&\le \frac{\eta_1}{2}\cdot\left\{\int_\Omega\frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}\right\}^{\frac{2}{3N}}\cdot\left\{\int_\Omega\varphi^{p+1}\psi\right\}^{\frac{3N-2}{3N}}\cdot\left\{\int_\Omega\varphi^{\frac{2\lambda(p+1)}{3N(2-\lambda)}}\right\}^{\frac{2-\lambda}{\lambda}} \\
&\le \frac{\eta_1 c_1}{2}\cdot\left\{\int_\Omega\frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}\right\}^{\frac{2}{3N}}\cdot\left\{\int_\Omega\varphi^{p+1}\psi\right\}^{\frac{3N-2}{3N}}\cdot\left\{\int_\Omega\varphi^{\frac{p}{2}}\right\}^{\frac{4(p+1)}{3Np}},
\end{aligned}$$

where the last inequality is guaranteed by (6.4.4). From (6.4.5), we proceed to employ Young's inequality to verify

$$\begin{aligned}
\frac{\eta_1}{2}\left\|\varphi^{\frac{p+1}{2}}\psi^{-\frac{1}{2}}\nabla\psi\right\|^2_{L^\mu(\Omega)} &\le \frac{\eta^{\frac{2}{3N}}}{2}\cdot\left\{\int_\Omega\frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}\right\}^{\frac{2}{3N}}\cdot\left\{\int_\Omega\varphi^{p+1}\psi\right\}^{\frac{3N-2}{3N}}\cdot\left\{\int_\Omega\varphi^{\frac{p}{2}}\right\}^{\frac{4(p+1)}{3Np}} \\
&\le \frac{1}{2}\int_\Omega\varphi^{p+1}\psi + \frac{\eta}{2}\cdot\left\{\int_\Omega\varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}}\cdot\int_\Omega\frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}.
\end{aligned} \tag{6.4.8}$$

Combining (6.4.6)-(6.4.8), we get

$$\begin{aligned}\int_\Omega \varphi^{p+1}\psi &\le \eta\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2+\eta\cdot\left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^{\frac{2(p+1)}{p}}\cdot\int_\Omega \frac{|\nabla\psi|^{3N}}{\psi^{3N-1}}\\ &\quad+2\left(c_1^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_1\right)\left\|\varphi^{\frac{p+2}{2}}\psi^{\frac12}\right\|^2_{L^{\frac12}(\Omega)},\end{aligned} \tag{6.4.9}$$

where

$$\begin{aligned}&2\left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_2\right)\left\|\varphi^{\frac{p+1}{2}}\psi^{\frac12}\right\|^2_{L^{\frac12}(\Omega)}\\ &\quad= 2\left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_2\right)\left\{\int_\Omega \varphi^{\frac{p}{4}}\cdot(\varphi\psi)^{\frac14}\cdot 1\right\}^4\\ &\quad\le 2\left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_2\right)c_1\cdot\left\{\int_\Omega \varphi^{\frac{p}{2}}\right\}^2\cdot\int_\Omega \varphi\psi.\end{aligned} \tag{6.4.10}$$

Here, using $p>2$ and $\eta<1$, we obtain from (6.4.5) that

$$\eta_1^{-\frac{\theta}{1-\theta}}=\max\left\{\left(\frac{(p+1)^2c_1}{\eta}\right)^{\frac{\theta}{1-\theta}},\ \left(\frac{c_1}{\eta^{\frac{2}{3N}}}\right)^{\frac{\theta}{1-\theta}}\right\}\le\left(\frac{4p^2c_1}{\eta}\right)^{\frac{\theta}{1-\theta}},$$

which implies

$$\begin{aligned}2\left(c_2^{\frac{1}{1-\theta}}\eta_1^{-\frac{\theta}{1-\theta}}+c_2\right)c_1 &= 2c_1c_2+2^{\frac{2\theta}{1-\theta}+1}c_1^{\frac{1}{1-\theta}}c_2^{\frac{1}{1-\theta}}p^{\frac{2\theta}{1-\theta}}\eta^{-\frac{\theta}{1-\theta}}\\ &\le \left(2c_1c_2+2^{\frac{2\theta}{1-\theta}+1}c_1^{\frac{1}{1-\theta}}c_2^{\frac{1}{1-\theta}}\right)p^{\frac{2\theta}{1-\theta}}\eta^{-\frac{\theta}{1-\theta}}\\ &= C(N,p_*)p^{2\delta}\eta^{-\delta}\end{aligned} \tag{6.4.11}$$

by writing

$$\delta\equiv\delta(N,p_*):=\frac{\theta}{1-\theta}\quad\text{and}\quad C(N,p_*):=2c_1c_2+2^{2\delta+1}c_1^{\delta+1}c_2^{\delta+1}.$$

In conjunction with (6.4.9)-(6.4.11), this completes the proof. □

We are now ready to derive uniform bounds on $\|u_\varepsilon\|_{L^\infty(\Omega)}$ by employing a Moser-type iteration scheme, inspired by [88]. Crucially, the structure of (6.4.3), which involves a lower integrability exponent on the right-hand side, paves the way to make Lemma 3.4.2 applicable in this framework.

Lemma 6.4.3. *Let $N\ge 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4). *Then there exists $C(N)>0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C(N)\qquad \textit{for all } t\in(0,T_{max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{6.4.12}$$

Proof. Setting $p_0=2$, and defining

$$p_k:=2^kp_0,\qquad k\ge 1, \tag{6.4.13}$$

we see from Lemma 6.3.2 and Lemma 6.3.5 that each of the numbers defined by

$$M_{k,\varepsilon}(T) := 1 + \sup_{t\in(0,T)} \int_\Omega u_\varepsilon^{p_k}(\cdot,t), \quad T \in (0, T_{max,\varepsilon}), \quad k \in \mathbb{N} \ \text{ and } \ \varepsilon \in (0,1) \qquad (6.4.14)$$

is finite, and that there exists $c_1 > 0$ satisfying

$$M_{0,\varepsilon}(T) \le c_1 \qquad \text{for all } \ T \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1). \qquad (6.4.15)$$

In the following, we aim to estimate $M_{k,\varepsilon}(T)$ for $k \ge 1$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. Firstly, according to (6.4.1), there exists $c_2 = c_2(N) > 0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

which together with (6.2.10) implies that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} + \int_\Omega u_\varepsilon^{p_k} + \frac{p_k^2}{4}\int_\Omega u_\varepsilon^{p_k-1} v_\varepsilon |\nabla u_\varepsilon|^2 & \\ \le \ \frac{c_2^2 p_k^2}{2}\int_\Omega u_\varepsilon^{p_k+1} v_\varepsilon + p_k(\kappa+1)^{\frac{p_k+\gamma-1}{\gamma-1}} \left(\frac{2}{\mu}\right)^{\frac{p_k}{\gamma-1}} |\Omega|. & \qquad (6.4.16) \end{aligned}$$

Since (6.4.2) provides the existence of $c_3 = c_3(N) > 0$ such that

$$\int_\Omega \frac{|\nabla v_\varepsilon|^{3N}}{v_\varepsilon^{3N-1}} \le c_3 \qquad \text{for all } \ t \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

we may invoke Lemma 6.4.2 with $p_* := 3$ to find $\delta = \delta(N) > 0$ and $c_4 = c_4(N) > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \int_\Omega u_\varepsilon^{p_k+1} v_\varepsilon \ \le \ & \frac{1}{2c_2^2}\int_\Omega u_\varepsilon^{p_k-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{c_3}{2c_2^2}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1)}{p_k}} \\ & + c_4 p_k^{2\delta}(2c_2^2)^\delta \cdot \left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^2 \cdot \int_\Omega u_\varepsilon v_\varepsilon, \end{aligned}$$

which together with (6.4.16), (6.2.4) and (6.2.6) implies that for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^{p_k} + \int_\Omega u_\varepsilon^{p_k} \ \le \ & \frac{c_3 p_k^2}{4}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^{\frac{2(p_k+1)}{p_k}} + 2^\delta c_2^{2\delta+2} c_4 p_k^{2\delta+2}\cdot\left\{\int_\Omega u_\varepsilon^{\frac{p_k}{2}}\right\}^2 \cdot \int_\Omega u_\varepsilon v_\varepsilon \\ & + p_k(\kappa+1)^{\frac{p_k+\gamma-1}{\gamma-1}} \left(\frac{2}{\mu}\right)^{\frac{p_k}{\gamma-1}} |\Omega| \\ \le \ & c_5 p_k^{2\delta+2} M_{k-1,\varepsilon}^{\frac{2(p_k+1)}{p_k}}(T) + p_k(\kappa+1)^{\frac{p_k+\gamma-1}{\gamma-1}} \left(\frac{2}{\mu}\right)^{\frac{p_k}{\gamma-1}} |\Omega| \end{aligned}$$

with $c_5 := \frac{c_3}{4} + 2^\delta c_2^{2\delta+2} c_4 m \|v_0\|_{L^\infty(\Omega)}$. Then a comparison principle immediately shows that for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon^{p_k} \le \max\left\{\int_\Omega (u_0+1)^{p_k}, \ c_5 p_k^{2\delta+2} M_{k-1,\varepsilon}^{\frac{2(p_k+1)}{p_k}}(T) + p_k(\kappa+1)^{\frac{p_k+\gamma-1}{\gamma-1}} \left(\frac{2}{\mu}\right)^{\frac{p_k}{\gamma-1}} |\Omega|\right\}. \ (6.4.17)$$

From the definition of p_k, it is clear that

$$c_5 p_k^{2\delta+2} \le (c_5+1)^k (4^{2\delta+2})^k,$$

and that due to $\gamma \ge 2$,

$$\begin{aligned}
&\int_\Omega (u_0+1)^{p_k} + p_k(\kappa+1)^{\frac{p_k+\gamma-1}{\gamma-1}} \left(\frac{2}{\mu}\right)^{\frac{p_k}{\gamma-1}} |\Omega| + 1 \\
&\quad \le \left\{ \|u_0+1\|_{L^\infty(\Omega)}^{p_k} + p_k(\kappa+1)^{p_k+1} \left(\frac{2}{\mu}\right)^{p_k} \right\} \cdot |\Omega| + 1 \\
&\quad \le \left\{ \|u_0+1\|_{L^\infty(\Omega)}^{2^{k+1}} + 2^{k+1}(\kappa+1)^{2^{k+2}} \left(\frac{2}{\mu}\right)^{2^{k+1}} \right\} \cdot (|\Omega|+1) \\
&\quad \le \left\{ \|u_0+1\|_{L^\infty(\Omega)}^{2} + \frac{8(\kappa+1)^4}{\mu^2} \right\}^{2^k} \cdot (|\Omega|+1)^{2^k},
\end{aligned}$$

thus if writing

$$a := (c_5+1)4^{2\delta+2} \quad \text{and} \quad b := \left\{ \|u_0+1\|_{L^\infty(\Omega)}^{2} + \frac{8(\kappa+1)^4}{\mu^2} \right\} \cdot (|\Omega|+1),$$

from (6.4.17) we have

$$M_{k,\varepsilon}(T) \le a^k M_{k-1,\varepsilon}^{2+2^{-k}}(T) + b^{2^k} \qquad \text{for all } k \ge 1,\ T \in (0, T_{max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

which in conjunction with Lemma 3.4.2 and (6.4.15) implies that

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^2 &= \liminf_{k\to\infty} \left\{ \int_\Omega u_\varepsilon^{p_k}(\cdot,t) \right\}^{\frac{2}{p_k}} \le \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{2}{p_k}}(T) \\
&= \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{2^k}}(T) \le (2\sqrt{2}a^3 b^{\frac{3}{2}} c_1)^{e^{\frac{1}{2}}}
\end{aligned}$$

for all $t \in (0,T)$, $T \in (0, T_{max,\varepsilon})$ and $\varepsilon \in (0,1)$. The proof is completed. □

6.5 Global weak solution. Proof of Theorem 6.1.1

The first implication of Lemma 6.4.3 is that the solution $(u_\varepsilon, v_\varepsilon)$ obtained in Lemma 6.2.1 is, in fact, global in time.

Lemma 6.5.1. *Let $N \ge 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4). *Then $T_{max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$.*

Proof. This follows immediately by combining Lemma 6.4.3 with (6.2.3). □

The second application of Lemma 6.4.3 is to build a bound for v_ε from below, which subsequently yields improved regularity properties of the solution.

Lemma 6.5.2. *Let $N \geq 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4)*. Then for any $T > 0$, there exists $C(N,T) > 0$ such that*

$$v_\varepsilon(x,t) \geq C(N,T) \qquad \text{for all } x \in \Omega,\ t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{6.5.1}$$

Moreover, for any $T_1 > 0$, there exist $\theta_1 = \theta_1(T_1) \in (0,1)$ and $C_1(N,T_1) > 0$ such that

$$\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T_1])} \leq C_1(N,T_1) \qquad \text{for all } \varepsilon \in (0,1) \tag{6.5.2}$$

and

$$\|v_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T_1])} \leq C_1(N,T_1) \qquad \text{for all } \varepsilon \in (0,1). \tag{6.5.3}$$

In addition, for each $\hat{\tau} > 0$ and any $T_2 > \hat{\tau}$, there exist $\theta_2 = \theta_2(\hat{\tau},T_2) \in (0,1)$ and $C_2(N,\hat{\tau},T_2) > 0$ such that

$$\|v_\varepsilon\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[\hat{\tau},T_2])} \leq C_2(N,\hat{\tau},T_2) \qquad \text{for all } \varepsilon \in (0,1). \tag{6.5.4}$$

Proof. We only sketch the main idea of the proof. In fact, due to the boundedness information on u_ε provided by Lemma 6.4.3, we can deduce (6.5.1) by employing a comparison argument on the second equation of (6.2.1). Thereafter, the first equation in (6.2.1) can be treated like a classical porous-medium type equation within any finite time intervals. Consequently, standard parabolic Hölder estimates [62] and Schauder theory [33] become applicable to obtain the desired regularities as intended. For more detailed arguments, one can refer to [43, Lemmas 6.3-6.4] (see also Lemma 3.5.3 in Chapter 3). □

With all the preparations, we are now close to concluding our first main result.

Lemma 6.5.3. *Let $N \geq 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4)*. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty)) \qquad \text{and} \\ v \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \cap W^{1,\infty}(\Omega\times(0,\infty)) \end{cases} \tag{6.5.5}$$

satisfying $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$ and

$$\nabla u^2 \in L^2_{loc}\left(\overline{\Omega}\times[0,\infty);\mathbb{R}^N\right) \tag{6.5.6}$$

are such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$u_\varepsilon \to u \quad \text{in} \quad C^0_{loc}\left(\overline{\Omega}\times[0,\infty)\right), \tag{6.5.7}$$

$$\nabla u_\varepsilon^2 \rightharpoonup \nabla u^2 \quad \text{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{6.5.8}$$

$$v_\varepsilon \to v \quad \text{in} \quad C^0_{loc}\left(\overline{\Omega}\times[0,\infty)\right) \cap C^{2,1}_{loc}\left(\overline{\Omega}\times(0,\infty)\right) \qquad \text{and} \tag{6.5.9}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \qquad \text{in } L^\infty(\Omega\times(0,\infty)), \tag{6.5.10}$$

and that (u,v) forms a global weak solution of (6.1.2) *in the sense of Definition 1.4.1.*

Proof. For any $T > 0$, from (6.3.6) and (6.3.18) we see the existence of $c_1 = c_1(N) > 0$ such that

$$\int_0^T \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla u_\varepsilon|^2 \le c_1 \qquad \text{for all } \varepsilon \in (0,1),$$

which together with (6.5.1) implies that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^2|^2 = 4\int_0^T \int_\Omega u_\varepsilon^2 |\nabla u_\varepsilon|^2 = 4\int_0^T \int_\Omega u_\varepsilon^2 v_\varepsilon |\nabla u_\varepsilon|^2 \cdot \frac{1}{v_\varepsilon} \le 4c_1c_2$$

for all $\varepsilon \in (0,1)$ with some $c_2 = c_2(N,T) > 0$. Thus, in view of Lemma 6.4.3 we obtain that

$$(u_\varepsilon^2)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2((0,T);W^{1,2}(\Omega)) \qquad \text{for all } T > 0.$$

By means of the conclusions obtained in Lemmas 6.4.1, 6.4.3 and 6.5.2, the rest of the proof can be carried out similarly to the argument in Lemma 3.5.5 of Chapter 3. □

Proof of Theorem 6.1.1. This is an immediate consequence of Lemma 6.5.3. □

6.6 Large time behavior. Proof of Theorem 6.1.4

In this section, we are devoted to studying the stabilization of the solutions constructed in Theorem 6.1.1. Firstly, building upon the boundedness information on u_ε and v_ε, an analysis based on a functional well-explored in [4] leads to the following properties.

Lemma 6.6.1. *Let $N \ge 2$. Assume that f satisfies* (6.1.3) *with* (6.1.4)*. Then if* (6.1.8) *holds, with $\lambda(N) > 0$ and $C(N) > 0$ we have*

$$\int_\Omega u_\varepsilon(\cdot,t) > \lambda(N) \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1) \tag{6.6.1}$$

and

$$\int_0^\infty \int_\Omega (u_\varepsilon - u_\infty)^2 \le C(N) \qquad \textit{for all } \varepsilon \in (0,1), \tag{6.6.2}$$

where $u_\infty := \left(\frac{\kappa}{\mu}\right)^{\frac{1}{\gamma-1}}$.

Proof. By means of Lemma 6.4.3 and (6.2.4), we can find that

$$M_1 := \sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$$

and

$$M_2 := \sup_{\varepsilon\in(0,1)} \|v_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$$

are both finite. Then, the evolution of the functional defined by

$$E_\varepsilon(t) := \int_\Omega u_\varepsilon - u_\infty \int_\Omega \ln\frac{u_\varepsilon}{u_\infty} + \frac{M_1M_2}{4}u_\infty \int_\Omega v_\varepsilon^2, \qquad t \ge 0$$

satisfies

$$\begin{aligned} E'_\varepsilon(t) &= \int_\Omega u_{\varepsilon t} - u_\infty \int_\Omega \frac{u_{\varepsilon t}}{u_\varepsilon} + \frac{M_1M_2}{2} u_\infty \int_\Omega v_\varepsilon v_{\varepsilon t} \\ &= -u_\infty \int_\Omega \frac{1}{u_\varepsilon} \nabla \cdot (u_\varepsilon v_\varepsilon \nabla u_\varepsilon) + u_\infty \int_\Omega \frac{1}{u_\varepsilon} \nabla \cdot \left(u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon \right) + \frac{M_1M_2}{2} u_\infty \int_\Omega v_\varepsilon \Delta v_\varepsilon \\ &\quad + \kappa \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^\gamma - \kappa u_\infty \int_\Omega 1 + \mu u_\infty \int_\Omega u_\varepsilon^{\gamma-1} - \frac{M_1M_2}{2} u_\infty \int_\Omega u_\varepsilon v_\varepsilon^2 \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, where integration by parts and Young's inequality can further show that

$$\begin{aligned} & E'_\varepsilon(t) + \int_\Omega (u_\infty - u_\varepsilon) \left(\kappa - \mu u_\varepsilon^{\gamma-1} \right) \\ &\quad = -u_\infty \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} |\nabla u_\varepsilon|^2 + u_\infty \int_\Omega v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \frac{M_1M_2}{2} u_\infty \int_\Omega |\nabla v_\varepsilon|^2 \\ &\quad \le -\frac{u_\infty}{2} \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} |\nabla u_\varepsilon|^2 + \frac{u_\infty}{2} \int_\Omega u_\varepsilon v_\varepsilon |\nabla v_\varepsilon|^2 - \frac{M_1M_2}{2} u_\infty \int_\Omega |\nabla v_\varepsilon|^2 \\ &\quad \le 0 \qquad \text{for all } \ t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Since $\gamma \ge 2$, we have

$$\int_\Omega (u_\infty - u_\varepsilon) \left(\kappa - \mu u_\varepsilon^{\gamma-1} \right) = \mu \int_\Omega (u_\infty - u_\varepsilon) \left(u_\infty^{\gamma-1} - u_\varepsilon^{\gamma-1} \right) \ge \mu u_\infty^{\gamma-2} \int_\Omega (u_\varepsilon - u_\infty)^2 ,$$

which implies that

$$E'_\varepsilon(t) + \mu u_\infty^{\gamma-2} \int_\Omega (u_\varepsilon - u_\infty)^2 \le 0 \qquad \text{for all } \ t > 0 \text{ and } \varepsilon \in (0,1). \tag{6.6.3}$$

On the other side, for any $\varepsilon \in (0,1)$, it is easy to verify that

$$E_\varepsilon(t) > 0 \qquad \text{for all } \ t > 0,$$

and that because of (6.1.8),

$$E_\varepsilon(0) \le \int_\Omega (u_0 + 1) + u_\infty \int_\Omega \ln u_\infty - u_\infty \int_\Omega \ln u_0 + \frac{M_1M_2}{4} u_\infty \int_\Omega v_0^2 =: \bar{\lambda} < \infty.$$

Thus, integrating (6.6.3) in time not only yields (6.6.2), but also leads to

$$u_\infty \int_\Omega \ln \frac{u_\varepsilon}{u_\infty} \ge \int_\Omega u_\varepsilon + \frac{u_\infty}{4} M_1M_2 \int_\Omega v_\varepsilon^2 - E_\varepsilon(0) \ge -\bar{\lambda} \qquad \text{for all } \ t > 0 \text{ and } \varepsilon \in (0,1),$$

which in virtue of Jensen's inequality implies that

$$\frac{1}{u_\infty |\Omega|} \int_\Omega u_\varepsilon = \frac{1}{|\Omega|} \int_\Omega e^{\ln \frac{u_\varepsilon}{u_\infty}} \ge e^{\frac{1}{|\Omega|} \int_\Omega \ln \frac{u_\varepsilon}{u_\infty}} \ge e^{-\frac{\bar{\lambda}}{u_\infty |\Omega|}} \qquad \text{for all } \ t > 0 \text{ and } \varepsilon \in (0,1).$$

Hence, (6.6.1) is obtained. □

As indicated by (6.5.2) in Lemma 6.5.2, since only time-dependent Hölder continuity of u_ε is available, an approach similar to that in [37] enables us to extract strong convergence of u_ε in L^∞ along certain sequences $t_k \to \infty$, rather than establishing convergence for all sufficiently large times. Therefore, we next shift our attention to a weaker dual space.

Lemma 6.6.2. *Let $N \geq 2$, and let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as in Lemma 6.5.3. Assume that f satisfies* (6.1.3) *with* (6.1.4). *Then if* (6.1.8) *holds, we have*

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \int_t^{t+1} \|u_{\varepsilon t}(\cdot,s)\|_{(W^{1,\infty}(\Omega))^*}\, ds \to 0 \qquad \text{as } t\to\infty. \tag{6.6.4}$$

Proof. In view of (6.2.4) and (6.4.1), we can claim from Lemmas 6.3.2 and 6.3.5 that there is $c_1 = c_1(N) > 0$ such that for all $t > 0$,

$$\sup_{\varepsilon\in(0,1)} \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2 + \sup_{\varepsilon\in(0,1)} \int_t^{t+1}\int_\Omega u_\varepsilon^3 v_\varepsilon|\nabla v_\varepsilon|^2 + \sup_{\varepsilon\in(0,1)} \int_t^{t+1}\int_\Omega u_\varepsilon^2 \leq c_1. \tag{6.6.5}$$

For any $\psi \in W^{1,\infty}(\Omega)$ with $\|\psi\|_{W^{1,\infty}(\Omega)} \leq 1$, direct computations and integration by parts imply that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}\left|\int_\Omega u_{\varepsilon t}\cdot\psi\right| &= \left|\int_\Omega \nabla\cdot(u_\varepsilon v_\varepsilon\nabla u_\varepsilon)\psi - \int_\Omega \nabla\cdot(u_\varepsilon^2 v_\varepsilon \nabla v_\varepsilon)\psi + \kappa\int_\Omega u_\varepsilon\psi - \mu\int_\Omega u_\varepsilon^\gamma\psi\right| \\ &\leq \int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon| + \int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon| + \int_\Omega |\kappa u_\varepsilon - \mu u_\varepsilon^\gamma|,\end{aligned}$$

from which, it follows by denoting $u_\infty := \left(\frac{\kappa}{\mu}\right)^{\frac{1}{\gamma-1}}$ as in Lemma 6.6.1 that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\|u_{\varepsilon t}\|_{(W^{1,\infty}(\Omega))^*} \leq \int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon| + \int_\Omega u_\varepsilon^2 v_\varepsilon|\nabla v_\varepsilon| + \mu\int_\Omega u_\varepsilon|u_\infty^{\gamma-1} - u_\varepsilon^{\gamma-1}|.$$

By the Hölder inequality, for all $t > 0$ and $\varepsilon \in (0,1)$ we obtain

$$\begin{aligned}\int_t^{t+1}\|u_{\varepsilon s}(\cdot,s)\|_{(W^{1,\infty}(\Omega))^*}\,ds &\leq \left\{\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon|\nabla u_\varepsilon|^2\right\}^{\frac12}\cdot\left\{\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon\right\}^{\frac12} \\ &\quad + \left\{\int_t^{t+1}\int_\Omega u_\varepsilon^3 v_\varepsilon|\nabla v_\varepsilon|^2\right\}^{\frac12}\cdot\left\{\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon\right\}^{\frac12} \\ &\quad + \mu\cdot\left\{\int_t^{t+1}\int_\Omega u_\varepsilon^2\right\}^{\frac12}\cdot\left\{\int_t^{t+1}\int_\Omega \left(u_\infty^{\gamma-1} - u_\varepsilon^{\gamma-1}\right)^2\right\}^{\frac12},\end{aligned}$$

where using the mean value theorem, we see from Lemma 6.4.3 that

$$|u_\infty^{\gamma-1} - u_\varepsilon^{\gamma-1}| \leq (\gamma-1)\max\left\{u_\infty^{\gamma-2},\ \sup_{\varepsilon\in(0,1)}\|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}^{\gamma-2}\right\}|u_\infty - u_\varepsilon| =: c_2|u_\infty - u_\varepsilon|,$$

so that by (6.6.5), it follows that

$$\begin{aligned}&\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}\int_t^{t+1}\|u_{\varepsilon s}(\cdot,s)\|_{(W^{1,\infty}(\Omega))^*}\,ds \\ &\quad\leq 2c_1^{\frac12}\cdot\left\{\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon\right\}^{\frac12}\end{aligned}$$

$$+\mu c_1^{\frac{1}{2}} c_2 \cdot \left\{ \sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \int_t^{t+1} \int_\Omega (u_\infty - u_\varepsilon)^2 \right\}^{\frac{1}{2}} \qquad \text{for all} \quad t > 0. \tag{6.6.6}$$

In the following, we aim to show

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon \to 0 \qquad \text{as} \quad t \to \infty. \tag{6.6.7}$$

To see this, by Lemma 6.5.3 we assert that for any given $\eta > 0$, one can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfying $\varepsilon < \varepsilon_0$,

$$\|u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) - u(\cdot,t)v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\eta}{2|\Omega|} \qquad \text{for all} \quad t > 0,$$

and this implies that for all such ε,

$$\int_t^{t+1} \int_\Omega (u_\varepsilon v_\varepsilon - uv) \le \frac{\eta}{2} \qquad \text{for all} \quad t > 0. \tag{6.6.8}$$

Furthermore, since (6.2.5) entails

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon < \infty \qquad \text{for all} \quad \varepsilon \in (0,1), \tag{6.6.9}$$

we can deduce from Lemma 6.5.3 that

$$\int_0^\infty \int_\Omega uv < \infty,$$

and thus there exists $t_0 = t_0(\eta) > 0$ such that

$$\int_t^{t+1} \int_\Omega uv < \frac{\eta}{2} \qquad \text{for all} \quad t > t_0,$$

which combined with (6.6.8) implies that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfying $\varepsilon < \varepsilon_0$,

$$\int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon \le \int_t^{t+1} \int_\Omega (u_\varepsilon v_\varepsilon - uv) + \int_t^{t+1} \int_\Omega uv \le \eta \qquad \text{for all} \quad t > t_0. \tag{6.6.10}$$

On the other hand, for each $\varepsilon \in (0,1)$, from (6.6.9) we may find $t_\varepsilon = t_\varepsilon(\eta) > 0$, decreasing in ε, satisfying

$$\int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon < \eta \qquad \text{for all} \quad t > t_\varepsilon,$$

and so this enables us to pick

$$t_1 \equiv t_1(\eta) := \sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}\cap[\varepsilon_0,1]} t_\varepsilon(\eta) < \infty$$

such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfying $\varepsilon \in [\varepsilon_0, 1]$, we have

$$\int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon < \eta \qquad \text{for all} \quad t > t_1. \tag{6.6.11}$$

Collecting (6.6.11) and (6.6.10), we can obtain that

$$\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon < \eta \qquad \text{for all } \ t > \max\{t_0, t_1\} \text{ and } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}.$$

This immediately yields (6.6.7). Finally, in a quite similar manner, we can deduce

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \int_t^{t+1}\int_\Omega (u_\infty - u_\varepsilon)^2 \to 0 \qquad \text{as } \ t\to\infty \tag{6.6.12}$$

from (6.6.2). Consequently, (6.6.4) can be derived by an amalgamation of (6.6.6), (6.6.7) and (6.6.12). □

A combination of Lemma 6.6.1 and Lemma 6.6.2 reveals the following convergence property.

Lemma 6.6.3. *Let $N \geq 2$, and let $(\varepsilon_j)_{j\in\mathbb{N}}$ and u_∞ be as in Lemma 6.5.3 and Lemma 6.6.1, respectively. Assume that f satisfies* (6.1.3) *with* (6.1.4). *Then if* (6.1.8) *holds, we have*

$$\|u(\cdot,t) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \to 0 \qquad \text{as } t\to\infty. \tag{6.6.13}$$

Proof. We prove this by using the contradiction argument. If (6.6.13) were false, then there would exist $c_1 > 0$ and sequence $(\tilde{t}_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\tilde{t}_k \to \infty$ as $k\to\infty$ such that

$$\|u(\cdot,\tilde{t}_k) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \geq c_1 \qquad \text{for all } \ k\in\mathbb{N}. \tag{6.6.14}$$

According to Lemma 6.6.2, it follows that for all $h\in(0,1)$,

$$\begin{aligned}
&\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \|u_\varepsilon(\cdot,\tilde{t}_k+h) - u_\varepsilon(\cdot,\tilde{t}_k)\|_{(W^{1,\infty}(\Omega))^*} \\
&\quad = \sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \left\| \int_{\tilde{t}_k}^{\tilde{t}_k+h} u_{\varepsilon s}(\cdot,s) \right\|_{(W^{1,\infty}(\Omega))^*} ds \\
&\quad \leq \sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \int_{\tilde{t}_k}^{\tilde{t}_k+1} \|u_{\varepsilon s}(\cdot,s)\|_{(W^{1,\infty}(\Omega))^*} ds \\
&\quad \to 0 \qquad \text{as } k\to\infty,
\end{aligned}$$

which in conjunction with Lemma 6.5.3 implies that for all $h\in(0,1)$,

$$\|u(\cdot,\tilde{t}_k+h) - u(\cdot,\tilde{t}_k)\|_{(W^{1,\infty}(\Omega))^*} \to 0 \qquad \text{as } k\to\infty.$$

Therefore, there exists $k_0 > 0$ such that

$$\|u(\cdot,\tilde{t}_k+h) - u(\cdot,\tilde{t}_k)\|_{(W^{1,\infty}(\Omega))^*} \leq \frac{c_1}{2} \qquad \text{for all } \ k\geq k_0.$$

Combined with (6.6.14), this entails that

$$\begin{aligned}
&\|u(\cdot,\tilde{t}_k+h) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \\
&\quad \geq \|u(\cdot,\tilde{t}_k) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} - \|u(\cdot,\tilde{t}_k+h) - u(\cdot,\tilde{t}_k)\|_{(W^{1,\infty}(\Omega))^*} \\
&\quad \geq \frac{c_1}{2} \qquad \text{for all } \ k\geq k_0 \text{ and } h\in(0,1).
\end{aligned}$$

Since $L^2(\Omega)$ is continuously embedded into $(W^{1,\infty}(\Omega))^*$, there exists $c_2 > 0$ such that

$$\begin{aligned} \|u(\cdot,t) - u_\infty\|_{L^2(\Omega)} &\geq c_2 \|u(\cdot,t) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \\ &\geq \frac{c_1 c_2}{2} \qquad \text{for all } t \in (\tilde{t}_k, \tilde{t}_k + 1) \text{ with } k \geq k_0. \end{aligned}$$

Hence,

$$\int_0^\infty \int_\Omega (u - u_\infty)^2 \geq \sum_{k=k_0}^\infty \int_{\tilde{t}_k}^{\tilde{t}_k+1} \|u(\cdot,t) - u_\infty\|_{L^2(\Omega)}^2 = \infty. \tag{6.6.15}$$

However, from (6.5.7), (6.6.2) and Fatou's lemma, we have

$$\int_0^\infty \int_\Omega (u - u_\infty)^2 < \infty,$$

which contradicts (6.6.15), and thereby shows (6.6.13). □

We can now prove our second result.

Proof of Theorem 6.1.4. Let $u_\infty := \left(\frac{\kappa}{\mu}\right)^{\frac{1}{\gamma-1}}$ be as defined in Lemma 6.6.1. We employ (6.1.7) to infer the existence of $c_1 = c_1(N) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + u_\infty \leq c_1 \qquad \text{for all} \quad t > 0.$$

Since $W^{1,\infty}(\Omega)$ is dense in $L^1(\Omega)$, for any given $\eta > 0$ and $\varphi \in L^1(\Omega)$ we can find $\psi \in W^{1,\infty}(\Omega)$ fulfilling

$$\|\varphi - \psi\|_{L^1(\Omega)} \leq \frac{\eta}{2c_1}.$$

Moreover, by Lemma 6.6.3 there exists $t_0 = t_0(\eta) > 0$ such that

$$\|u(\cdot,t) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \leq \frac{\eta}{2\|\psi\|_{W^{1,\infty}(\Omega)}} \qquad \text{for all} \quad t \geq t_0.$$

We thereupon obtain

$$\begin{aligned} \int_\Omega (u(\cdot,t) - u_\infty)\varphi &= \int_\Omega (u(\cdot,t) - u_\infty)\psi + \int_\Omega (u(\cdot,t) - u_\infty)(\varphi - \psi) \\ &\leq \|u(\cdot,t) - u_\infty\|_{(W^{1,\infty}(\Omega))^*} \|\psi\|_{W^{1,\infty}(\Omega)} \\ &\quad + (\|u(\cdot,t)\|_{L^\infty(\Omega)} + u_\infty)\|\varphi - \psi\|_{L^1(\Omega)} \\ &\leq \eta \qquad \text{for all} \quad t \geq t_0. \end{aligned}$$

This gives (6.1.9). Next, we turn to the proof of (6.1.10). Similar to Lemma 3.6.1, the uniform L^∞-boundedness of u_ε obtained in Lemma 6.4.3, together with the upper and lower bounds for v_ε established in Lemmas 6.2.1 and 6.5.2, ensures an elliptic-type pointwise Harnack inequality. That is, there exists $\lambda_* = \lambda_*(N) > 0$ such that

$$v_\varepsilon(x,t) \geq \lambda_* \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1).$$

Combining this with (6.6.1), we obtain that with $\lambda = \lambda(N) > 0$ provided therein,

$$\begin{aligned} \int_\Omega v_0 &\geq \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \geq \lambda_* \int_0^\infty \left\{ \int_\Omega u_\varepsilon \right\} \cdot \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \\ &\geq \lambda_* \lambda \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \qquad \text{for all} \quad \varepsilon \in (0,1). \end{aligned}$$

Applying Fatou's lemma and (6.5.9), we deduce that

$$\int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt \leq \liminf_{\varepsilon \to 0} \int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt < \frac{\int_\Omega v_0}{\lambda_* \lambda}.$$

This completes the proof by combining the fact that $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ is monotonically decreasing with respect to $t > 0$ indicated by (6.2.4) and (6.5.9). $\square$

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