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Representation Theory of EI-categories

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Abstract

In this thesis we investigate the category of finite-dimensional modules over an EI-category algebra. More precisely, we analyze the representation type for this class of algebras in the first part. It will be shown that a representation-finite EI-category is an amalgam of a representation-finite poset and a collection of representation-finite groups. We will then see that the representation type depends on the characteristic of the ground field. Furthermore, we give a necessary criterion for an EI-category with two objects to be representation-finite. Under additional assumptions on the automorphism groups of the objects we give a full classification of the representation-finite EI-categories with two objects. In the second part we present a new proof for the existence of an upper bound for the finitistic dimension of an EI-category algebra. Inspired by this proof we define a new class of algebras, which we call *algebras with a directed stratification.* We prove a result on the finitistic dimension of these algebras. This reduces the finitistic dimension conjecture to a class of algebras which we can describe combinatorially in terms of their Gabriel-quiver.

Zusammenfassung

In dieser Arbeit untersuchen wir die Kategorie der endlichdimensionalen Moduln über einer EI-Kategorienalgebra. Genauer analysieren wir im ersten Teil den Darstellungstyp dieser Klasse von Algebren. Es wird gezeigt, dass eine darstellungsendliche EI-Kategorie sich aus einer darstellungsendlichen halbgeordneten Menge und darstellungsendlichen Gruppen zusammensetzt und dass der Darstellungstyp von der Charakteristik des zugrundeliegenden Körpers abhängt. Darüber hinaus beweisen wir ein notwendiges Kriterium für die Endlichkeit des Darstellungstyps einer EI-Kategorie mit zwei Objekten. Unter zusätzlichen Voraussetzungen an die Automorphismengruppen, geben wir eine vollständige Klassifikation der darstellungsendlichen EI-Kategorien mit zwei Objekten an. Im zweiten Teil der Arbeit präsentieren wir einen neuen Beweis für die Existenz einer oberen Schranke für die finitistische Dimension einer EI-Kategorienalgebra. Dieser Beweis motiviert die Definition einer neuen Klasse von Algebren, die wir *Algebren mit einer gerichteten Stratifizierung* nennen. Für diese Algebren beweisen wir ein Resultat über die finitistische Dimension. Dieses Resultat reduziert die finitistische Dimensionsvermutung auf eine Klasse von Algebren, die wir kombinatorisch mithilfe ihres Gabriel-Köchers beschreiben können.

Contents

1 Introduction

The study of EI-categories and their representations has its origin in the work of tom Dieck [31] and Lück [26] in the late 1980s. These authors used representations of EI-categories in algebraic K-theory and it took some years before EI-categories attracted attention from mathematicians working with representations of finite-dimensional algebras. Webb [32] investigated the question under which circumstances EI-categories are standardly stratified (in the sense of [10, 11]) and his student Xu recently worked on the cohomology theory of EI-categories and related aspects of their representation theory, compare [33–35]. Also fusion systems, transporter categories and other categories constructed from the set of subgroups of a given group are EI-categories and the object of recent research in the framework of p-local finite groups, see for instance [8].

An EI-category $\mathcal{C}$ is a category in which every endomorphism is an isomorphism. For a fixed base ring k the associated category algebra is denoted by $k\mathcal{C}$. It has as basis the set of morphisms in $\mathcal{C}$ with multiplication induced by composition of morphisms. Hence, the category algebra of a finite EI-category is a simultaneous generalization of several important constructions in representation theory, such as the group algebra of a finite group, the path algebra of a finite quiver without oriented cycles or the incidence algebra of a finite poset.

Although representations of finite groups and representations of quivers are, up to a certain degree, well-understood and lie at the heart of modern representation theory, the two theories are in some sense orthogonal to each other. It is therefore a natural question to ask for a general theory which contains representations of finite groups and finite quivers as special cases. One way to obtain this goal is the analysis of representations of finite EI-categories.

In this thesis we will focus on two central questions that always arise naturally for a new class of algebras, namely: What are the algebras of finite representation type and does the finitistic dimension conjecture hold for this class of algebras? The latter is also motivated by work of Grodal and Smith [19], where the projective dimensions of certain modules over EI-category algebras play a role for the description of algebraic models for finite G-spaces that appear in algebraic topology. Furthermore, the finitistic dimension of standardly stratified algebras is always finite, while for stratified algebras the finitistic dimension conjecture is still open. By work of Webb [32] we know that an EI-category algebra is always stratified but not standardly stratified in general. Here, we follow Cline, Parshall and Scott [10, 11] for the definition of stratified and standardly stratified algebras.

A classification of the representation-finite EI-category algebras would be a new way of generalizing the results for groups and quivers. Bautista, Gabriel, Roiter, Salmeron and others developed a theory for representation-finite algebras in general in the 1980s, but they always assumed that the Gabriel-quiver of the algebra in question is known. In particular, they assumed that every simple module has dimension one. The computation of the quiver of an algebra is in general a non-trivial task. Therefore, the results of the mentioned authors are often not applicable for the treatment of algebras which are not given in terms of quivers with relations.

Throughout this thesis k will denote a commutative ring with unit and often an algebraically closed field. For a finite category $\mathcal{C}$ it is a well-known fact due to Mitchell [27] that the category of modules over the category algebra $k\mathcal{C}$ is equivalent to the category of k-linear representations of $\mathcal{C}$, i.e. the functor category $\operatorname{Fun}(\mathcal{C}, \operatorname{Mod} k)$. Thus, we can work with representations of $\mathcal{C}$ to derive results for the module category of $k\mathcal{C}$, which is often an advantage for explicit computations.

The characterizations of representation-finite path algebras, incidence algebras and group algebras of finite quivers, finite groups and finite posets respectively are classical results in the representation theory of finite-dimensional algebras. Since all these classes of algebras appear as special cases of EI-category algebras, a representation-finite EI-category algebra has to satisfy the conditions for representation-finiteness for these three classes of algebras simultaneously.

For a finite EI-category $\mathcal{C}$ we introduce a new category $\widehat{\mathcal{C}}$ which is the 'endotrivialization' of $\mathcal{C}$, i.e. all endomorphisms are made trivial and morphisms $x \to y$ that lie in the same $(\operatorname{Aut}(x) \times \operatorname{Aut}(y))$-orbit are identified. As a first step we show that if $k\mathcal{C}$ is a representation-finite EI-category algebra, then $k\widehat{\mathcal{C}}$ is a representation-finite incidence algebra. Furthermore, it is an easy observation that for a representation-finite EI-category algebra $k\mathcal{C}$ every group algebra $k \operatorname{Aut}(x)$ for $x \in \operatorname{Ob} \mathcal{C}$ is representation-finite. The question that needs to be answered for a characterization of all representation-finite EI-category algebras is the following: Which combinations of representation-finite groups and representation-finite posets give representation-finite EI-categories (always with respect to the ground field k)? It turns out that this question is not easy to answer in general. Therefore, we will restrict ourselves to EI-categories with two objects, which is the easiest class of EI-categories that has not yet been investigated. The analysis of the representation type of EI-categories with two objects provides us with necessary criteria for finite representation type.

The first observation is that the representation type of an EI-category algebra $k\mathcal{C}$ significantly depends on the characteristic of the ground field. Despite this dependence on the characteristic we will show that the category algebra $k\mathcal{C}$ of an EI-category $\mathcal{C}$ with two non-isomorphic objects x and y, where $\operatorname{Aut}(x) \times \operatorname{Aut}(y)$ acts freely on the morphism set $\mathcal{C}(x, y)$, is of infinite representation type for any algebraically closed field k. This result gives a necessary criterion for EI-category algebras to be of finite representation type. Unfortunately, we are not able to give a full classification of these algebras, even under the assumption that the underlying category $\mathcal{C}$ has only two objects. Under additional assumptions on the automorphism groups of the objects of the EI-category $\mathcal{C}$, one can compute the Gabriel-quiver of the associated category algebra explicitly and then use results about quivers with relations to determine the representation type. In that way we can for instance characterize all representation-finite EI-category algebras with two simple modules. It turns out that for EI-categories with two objects the representation type is governed by the group action of the automorphism groups of the two objects on the set of morphisms between the two objects.

The (little) finitistic dimension of an algebra A defined as

$$\operatorname{fin.dim}(A) = \sup \{ \operatorname{proj.dim} M \mid M \in \operatorname{mod} A,\ \operatorname{proj.dim} M < \infty \}$$

has been introduced by Bass [5] in 1960 and is an important invariant of the module category, which, roughly speaking, measures the complexity of $\operatorname{mod} A$. Bass proposed the question whether this finitistic dimension is always finite as a 'problem' and for finite-dimensional algebras it is nowadays known as the finitistic dimension conjecture (while for commutative rings it is easily seen to fail). Up to now there is no proof for this conjecture, but also no counterexample. The finitistic dimension has been calculated for several classes of algebras and turned out to be finite in these cases.

Lück [26] gave an upper bound for the finitistic dimension of an EI-category algebra $k\mathcal{C}$ (k a field), namely he showed $\operatorname{fin.dim} k\mathcal{C} \leq \ell(\mathcal{C})$, where $\ell(\mathcal{C})$ denotes the length of the category $\mathcal{C}$, i.e. the maximal length of a chain of non-isomorphisms in $\mathcal{C}$. We will present a new proof for this upper bound using recent results of Xu [33], that describe the structure of projective resolutions of modules over EI-category algebras. The intrinsic structure of an EI-category $\mathcal{C}$ that guarantees the finiteness of $\operatorname{fin.dim} k\mathcal{C}$ is the natural poset structure on the set of isomorphism classes of objects defined by $[x] \leq [y] \Leftrightarrow \mathcal{C}(x, y) \neq \emptyset$ together with the finiteness of the finitistic dimension of all automorphism groups of objects in $\mathcal{C}$, which are group algebras.

Inspired by this observation we define a new class of finite-dimensional algebras which we call *algebras with a directed stratification*. According to our definition an algebra A has a directed stratification of length n if there exist idempotents $e_1, \ldots, e_n$ in A with $1 = e_1 + \cdots + e_n$ and $e_j A e_i = 0$ for all $i > j$. This class of algebras contains EI-category algebras as a special case and we can describe the structure of projective resolutions of modules in the same way as Xu did for EI-category algebras and finally prove that an algebra A with a directed stratification given by $e_1, \ldots, e_n$ has finite finitistic dimension if and only if all algebras $e_i A e_i$ for $i = 1, \ldots, n$ have finite finitistic dimension. As a matter of fact, this result could also be obtained by induction from a result of Fossum, Griffith, Reiten [14] and Fuller, Saorin [15], but these authors used trivial extensions of abelian categories for the proof, which is a rather abstract machinery. With our proof we gain a deeper insight into the structure of projective resolutions of modules over algebras with a directed stratification. Furthermore, we give a combinatorial description of the algebras that do not admit a non-trivial directed stratification (i.e. of length > 1) in terms of their Gabriel-quiver. Finally, we construct examples which show that this reduction technique for the finitistic dimension conjecture is not a special case of other well-known results for the conjecture.

Outline

In Chapter 2 we collect the fundamental definitions of category algebras and EI-categories together with basic facts about their representation theory. We also recall the description of all simple and projective modules over EI-category algebras and introduce induction and restriction functors for representations of small categories. The third chapter contains our results on the representation type of EI-category algebras that have partly been mentioned above as well as short surveys on techniques we need for our proofs like covering theory and the Gabriel-quiver of a finite-dimensional algebra. In Chapter 4 we give a new proof for Lück's upper bound for the finitistic dimension of EI-category algebras. This motivates the fifth chapter, where we develop the whole theory of projective resolutions for modules over algebras with a directed stratification and prove the result on the finitistic dimension for these algebras. Finally, we conclude this thesis with chapter 6 in which we name open problems and some ideas how one could attack them in the future.

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2 Preliminaries

In this chapter we collect the definitions of category algebras, EI-categories and basic facts about their representation theory. Throughout this thesis all modules will be left modules.

2.1 Definition of EI-categories and examples

We begin with the classical definition of a category algebra.

Definition 2.1. Let $\mathcal{C}$ be any category and k a commutative ring with identity. Then we define the *category algebra* $k\mathcal{C}$ to be the free k-module whose basis is the set of morphisms in $\mathcal{C}$. The multiplication on two basis elements f and g of $k\mathcal{C}$ is defined as follows

$$f \cdot g = \begin{cases} f \circ g & \text{if } f \text{ and } g \text{ can be composed in } \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

We are mostly interested in the category $\operatorname{mod} k\mathcal{C}$ of finitely generated left modules over category algebras. Under certain assumptions, this category can be identified with the category of k-linear representations of $\mathcal{C}$ which is defined as follows.

Definition 2.2. Let $\mathcal{C}$ be a small category and k a commutative ring. A *representation of $\mathcal{C}$ over k* is a covariant functor $M : \mathcal{C} \to \operatorname{Mod} k$ from $\mathcal{C}$ into the category of k-modules. Together with natural transformations of functors this gives an abelian category with enough projective and injective objects. This category will be denoted by $\operatorname{Rep}_k \mathcal{C}$.

The following elementary observation relates the concepts of representations of $\mathcal{C}$ and modules over $k\mathcal{C}$.

Proposition 2.3 (Mitchell, [27])**.** *Let $\mathcal{C}$ be a category with finitely many objects and k a commutative ring. Then the categories $\operatorname{Rep}_k \mathcal{C}$ and $\operatorname{Mod} k\mathcal{C}$ are equivalent.*

Later on we will deal with category algebras over a field k. In this case the equivalence in the Proposition restricts to an equivalence $\operatorname{mod} k\mathcal{C} \to \operatorname{rep}_k \mathcal{C} = \operatorname{Fun}(\mathcal{C}, \operatorname{mod} k)$ from the category of finite-dimensional modules to the category of finite-dimensional representations.

We are particularly interested in a very special class of small categories, namely in the class of finite EI-categories.

Definition 2.4. An *EI-category* is a category $\mathcal{C}$ in which every endomorphism is an isomorphism. If $\mathcal{C}$ is a finite EI-category, the associated k-algebra

$$k\mathcal{C} = \left\{ \sum_{f \in \operatorname{Mor} \mathcal{C}} \lambda_f f \,\middle|\, \lambda_f \in k \right\}$$

is a finitely generated unital k-algebra, sometimes called the associated EI-algebra. The unit element is $\sum_{x \in \operatorname{Ob} \mathcal{C}} 1_x$ and obviously the elements $\{ 1_x \mid x \in \operatorname{Ob} \mathcal{C} \}$ form a set of pairwise mutually orthogonal idempotents in $k\mathcal{C}$. These idempotents are in general not primitive.

Example 2.5. EI-categories arise as important examples in at least in two branches of mathematics, namely representation theory of finite dimensional algebras and algebraic topology. We will present examples from both branches, starting with representation theory.

(1) Let G be a finite group and let $\mathcal{G}$ be the category with one object x and $\operatorname{End}(x) = G$. Then $\mathcal{G}$ is an EI-category and $k\mathcal{G} = kG$.
(2) Let Q be a finite quiver without oriented cycles and $\underline{Q}$ its path category. Then $\underline{Q}$ is an EI-category with $k\underline{Q} = kQ$.

Another important class of EI-algebras is the class of incidence algebras associated to finite partially ordered sets (short: finite posets).

Definition 2.6. Let $(X, \leq)$ be a finite poset, i.e. a finite set equipped with a binary relation $\leq$ which is reflexive, antisymmetric and transitive. The *incidence algebra* $A(X)$ (over k) of X consists of all incidence functions

$$A(X) = \{ f : X \times X \to k \mid f(x, y) = 0 \; if \; x \not\leq y \}$$

with pointwise summation and scalar multiplication. Moreover, we define the product of two such functions f and g as

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y).$$

The Kronecker delta function $\delta(x, y)$ is the two-sided identity of $A(X)$.

To every finite poset X we can also associate a finite category $\mathcal{C}_X$ in the following way:

- $\operatorname{Ob} \mathcal{C}_X = X$;
- For $x, y \in X$ there is a morphism $x \to y$ in $\mathcal{C}_X$ if and only if $x \leq y$ and we require that for any two objects x, y all morphisms $x \to y$ are equal.

In other words, $\mathcal{C}_X$ is the bound path category of a quiver with relations and we could alternatively define a poset to be a finite category $\mathcal{P}$ with the property that $|\mathcal{P}(x, y)| \leq 1$ for all $x, y \in \operatorname{Ob} \mathcal{P}$. The incidence algebra defined above is then isomorphic to $k\mathcal{P}$.

Example 2.7. Let $X = \{ a, b, c, d \}$ be equipped with the following partial order

$$a \leq \left\{ \begin{array}{c} b \\ c \end{array} \right\} \leq d.$$

Then the category $\mathcal{C}_X$ is the path category of the quiver

$$\begin{tikzcd}
 & d & \\
b \arrow[ur, "\gamma"] & & c \arrow[ul, "\delta"'] \\
 & a \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

bound by the relation $\gamma\alpha = \delta\beta$.

The three examples mentioned above are the most important and classical examples of EI-categories in representation theory of finite-dimensional algebras. Now we present some examples from algebraic topology such as fusion systems that have recently been studied by Broto, Levi and Oliver [8] in the context of p-local finite groups, orbit categories that play a prominent role in the theory of finite G-spaces, or transporter categories. All these categories are constructed from a set of subgroups of a given group in the following way.

Definition 2.8. Let $\mathcal{S}$ be a set of subgroups of a finite group G.

(1) The transporter category $\mathcal{T}_{\mathcal{S}}$ has as objects the elements of $\mathcal{S}$ and the morphisms are $\mathrm{Hom}(H, K) = N_G(H, K) = \{ g \in G \mid gHg^{-1} \subseteq K \}$. For the case $H = K$ the set of endomorphisms is the normalizer subgroup of H in G and therefore all the endomorphisms are isomorphisms.

(2) The orbit category $\mathcal{O}_{\mathcal{S}}$ is the category whose objects are the coset spaces G/H for $H \in \mathcal{S}$ and the morphisms in $\mathcal{O}_{\mathcal{S}}$ are the G-equivariant mappings. Those are all epimorphisms and hence, every endomorphism is an isomorphism.

(3) The Frobenius category $\mathcal{F}_{\mathcal{S}}$ associated to $\mathcal{S}$ (or the fusion system) plays an important role for the definition of a p-local finite group. Its objects are the elements of $\mathcal{S}$ and the morphisms are group homomorphisms $H \to K$ that are given by conjugation with an element of G. In this category every morphism is a monomorphism, which implies that $\mathcal{F}_{\mathcal{S}}$ is an EI-category.

To an arbitrary EI-category $\mathcal{C}$ we will associate another category $\widehat{\mathcal{C}}$ with only identity endomorphisms, which reflects the global structure of $\mathcal{C}$. This category $\widehat{\mathcal{C}}$ will play an important role in the analysis of the representation type of EI-categories later on.

Definition 2.9. Let f and g be two morphisms in a finite, skeletal EI-category $\mathcal{C}$. Then we define a relation $\sim$ on the set of morphisms of $\mathcal{C}$ as follows.

$$f \sim g :\Leftrightarrow f = f''h_1f' \text{ and } g = f''h_2f' \text{ for some } f'', f' \in \operatorname{Mor}\mathcal{C} \text{ and endomorphisms } h_1, h_2$$

This is clearly a reflexive and symmetric relation. We will consider the transitive hull of this relation and denote it again by $\sim$. This relation is also compatible with the composition of morphisms in $\mathcal{C}$. Therefore, we get a new category $\widehat{\mathcal{C}} := \mathcal{C}/\sim$ which is by construction an endotrivial category (in particular EI).

Roughly speaking, $\widehat{\mathcal{C}}$ is constructed from $\mathcal{C}$ by making all endomorphisms trivial and identifying all morphisms $x \to y$ in the same $(\operatorname{Aut}(x) \times \operatorname{Aut}(y))$-orbit. By construction it has the following important universal property. Suppose that $\mathcal{C}$ is an EI-category and $F : \mathcal{C} \to \mathcal{D}$ any functor to an endotrivial category $\mathcal{D}$. Then this functor F factors via a unique functor through the quotient functor $G : \mathcal{C} \to \widehat{\mathcal{C}}$, i.e. the following diagram is commutative.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "F"'] & \widehat{\mathcal{C}} \arrow[dl, dashed, "\exists!"] \\ \mathcal{D} & \end{tikzcd}$$

Example 2.10. (1) If G is a finite group and $\mathcal{C} = \mathcal{G}$ the associated EI-category, then $\widehat{\mathcal{G}}$ consists of one object x and the only morphism is the identity 1_x.

(2) If Q is a finite quiver without oriented cycles and $\mathcal{C} = \underline{Q}$ the path category, then $\widehat{\mathcal{C}} = \mathcal{C}$.

(3) If $\mathcal{C}$ is the EI-category associated to a finite poset $(X, \leq)$, we get $\widehat{\mathcal{C}} = \mathcal{C}$.

Remark 2.11. (1) An EI-category $\mathcal{C}$ is not uniquely determined by the category $\widehat{\mathcal{C}}$ together with its automorphism groups. To recover the entire structure of $\mathcal{C}$ one needs to know the composition of morphisms which is the same as the whole structure of $\mathcal{C}$. Nevertheless, the category $\widehat{\mathcal{C}}$ is of great importance for us. As an example consider the EI-category

$$\mathcal{C} : \begin{tikzcd} x \arrow[loop left, "f"] \arrow[r, bend left=50, "i_1"] \arrow[r, bend left=15, "i_2"] \arrow[r, bend right=15, "i_3"] \arrow[r, bend right=50, "i_4"] & y, \end{tikzcd}$$

satisfying the relations $f^4 = 1_x$ and $i_1 f = i_2$, $i_2 f = i_3$, $i_3 f = i_4$, $i_4 f = i_1$ and the EI-category

$$\mathcal{C}' : \begin{tikzcd} a \arrow[loop left, "g"] \arrow[r, "h"] & b, \end{tikzcd}$$

with the relations $g^4 = 1_a$ and $gh = h$. Then both $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$ are the path category of A_2 and $\mathcal{C}$ and $\mathcal{C}'$ have the same automorphism groups. However, they are not equivalent and the associated category algebras have completely different representation-theoretic properties. We will later see that $\mathcal{C}$ is representation-infinite and $\mathcal{C}'$ has finite representation type over any algebraically closed field k.

(2) By construction of $\widehat{\mathcal{C}}$ we have the quotient functor $G : \mathcal{C} \to \widehat{\mathcal{C}}$ which is the identity on objects and surjective on morphisms. This functor induces a fully faithful embedding of $\operatorname{mod} k\widehat{\mathcal{C}}$ into $\operatorname{mod} k\mathcal{C}$.

2.2 Simple and projective modules

In this part we give an explicit description of the projective modules and the simple modules over an EI-category algebra $k\mathcal{C}$. From now on we assume that k is a field (or a complete discrete valuation ring in order to have the Krull-Schmidt property), if not stated otherwise. We start with an important observation: If $\mathcal{C}$ is an EI-category, then one has a natural preorder defined on the set of objects $\operatorname{Ob}\mathcal{C}$, given by

$$x \leq y \Leftrightarrow \mathcal{C}(x, y) \neq 0.$$

This preorder clearly induces a partial order on the set of isomorphism classes of objects of $\mathcal{C}$.

In [26] Lück gave a characterization of all indecomposable projective and simple modules over EI-category algebras. These results can also be obtained using work of Auslander [4], but in the following formulation they are due to Lück.

Proposition 2.12 (Lück, [26]). *Let $\mathcal{C}$ be an EI-category. Then any finitely generated projective $k\mathcal{C}$-module is isomorphic to a direct sum of indecomposable projectives of the form $k\mathcal{C} \cdot e$, with $e \in kAut(x)$ being a primitive idempotent for some $x \in \operatorname{Ob}\mathcal{C}$.*

For an object x in $\mathcal{C}$ we denote by $[x]$ the isomorphism class of x. With this notation there is the following theorem.

Theorem 2.13 (Lück, [26]). *Let $\mathcal{C}$ be an EI-category. For each object $x \in \operatorname{Ob}\mathcal{C}$ and every simple $k\operatorname{Aut}(x)$-module V there is a simple $k\mathcal{C}$-module M such that $[x]$ is exactly the set of objects on which M is non-zero and $M(x) = V$. Conversely, if M is a simple $k\mathcal{C}$-module, then there is a unique isomorphism class of objects $[x]$ on which M is non-zero and each $M(x)$ is a simple $k\operatorname{Aut}(x)$-module. Thus, there is a bijection between the isomorphism classes of simple $k\mathcal{C}$-modules and the pairs $([x], V)$ where x is an object in $\mathcal{C}$ and V a simple $k\operatorname{Aut}(x)$-module.*

With this theorem it is natural to denote a simple $k\mathcal{C}$-module by $S_{x,V}$ if it corresponds to the pair $([x], V)$ and to write $P_{x,V}$ for its projective cover. Note that the structure of $P_{x,V}$ is determined by its value at x.

Remark 2.14. If $\mathcal{C}$ and $\mathcal{D}$ are equivalent EI-categories, then their associated module-categories $\operatorname{mod} k\mathcal{C}$ and $\operatorname{mod} k\mathcal{D}$ are equivalent as well.

Thus, we may throughout assume without loss of generality, that all EI-categories are skeletal and therefore the set of objects (not only the isomorphism classes) carries the natural structure of a finite poset.

2.3 Induction and restriction

Induction and restriction functors play an important role in modular representation theory of finite groups, for example to classify the group algebras of finite representation type. This concept can be carried over to the context of category algebras where we replace subgroups of a given finite group by subcategories of a finite category. In the general setting the definition of the restriction is the following.

Definition 2.15. Let k be a commutative ring and $\mathcal{C}$ and $\mathcal{D}$ be two small categories and $\mu : \mathcal{D} \to \mathcal{C}$ a covariant functor. Then we define the *restriction along* μ to be the functor $\operatorname{Res}_\mu : \operatorname{rep}_k \mathcal{C} \to \operatorname{rep}_k \mathcal{D}$ which sends a representation M of $\mathcal{C}$ to the representation $M \circ \mu$ of $\mathcal{D}$.

This functor also has its counterpart on the level of modules over the category algebras, which we also denote by $\operatorname{Res}_\mu : \operatorname{mod} k\mathcal{C} \to \operatorname{mod} k\mathcal{D}$ and it sends a module $M = \bigoplus_{x \in \operatorname{Ob} \mathcal{C}} M(x)$ to the module $\operatorname{Res}_\mu M = \bigoplus_{y \in \operatorname{Ob} \mathcal{D}} M(\mu(y))$.

A functor μ as in the definition extends naturally to a k-module homomorphism $\overline{\mu} : k\mathcal{D} \to k\mathcal{C}$, but this map is in general not an algebra homomorphism. The cases when this happens are characterized in the following proposition.

Proposition 2.16. *A functor $\mu : \mathcal{D} \to \mathcal{C}$ extends linearly to an algebra homomorphism $\overline{\mu} : k\mathcal{D} \to k\mathcal{C}$ if and only if μ is injective on* $\operatorname{Ob} \mathcal{D}$*. In this case, the induced functor followed by $1_{k\mathcal{D}}$, i.e. $1_{k\mathcal{D}} \cdot \downarrow^{k\mathcal{C}}_{k\mathcal{D}} : \operatorname{mod} k\mathcal{C} \to \operatorname{mod} k\mathcal{D}$ is exactly* Res_μ.

In this work we want to use this concept for the case where $\mathcal{D}$ is a (full) subcategory of a finite category $\mathcal{C}$ and we take for $\mu = \iota$ the inclusion functor. In this case the restriction $\operatorname{Res}_\iota : \operatorname{mod} k\mathcal{C} \to \operatorname{mod} k\mathcal{D}$ is determined by the algebra homomorphism $\overline{\iota} : k\mathcal{D} \to k\mathcal{C}$, hence by $\iota : \mathcal{D} \to \mathcal{C}$. Therefore, we will not distinguish Res_ι and $\downarrow^{k\mathcal{C}}_{k\mathcal{D}}$ and write $\downarrow^{k\mathcal{C}}_{k\mathcal{D}}$ and Res_ι as $\downarrow^{\mathcal{C}}_{\mathcal{D}}$, which is the usual notation in representation theory, for example in the articles of Xu. In the representation-theoretic setting the restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ has a left adjoint, which is the induction $\uparrow^{\mathcal{C}}_{\mathcal{D}} = k\mathcal{C} \otimes_{k\mathcal{D}} - : \operatorname{mod} k\mathcal{D} \to \operatorname{mod} k\mathcal{C}$. These induction and restriction functors will play a crucial role in almost all situations we will consider.

In the framework of restriction and induction functors we will later need the following definitions.

Definition 2.17. Let $\mathcal{C}$ be an EI-category.

(1) Let x be an object in $\mathcal{C}$. Then we define $\mathcal{C}_{\leq x}$ to be the full subcategory of $\mathcal{C}$ consisting of all objects $y \in \operatorname{Ob}\mathcal{C}$ with $\mathcal{C}(y,x) \neq \emptyset$. Similarly we define $\mathcal{C}_{\geq x}$.

(2) An *ideal* in $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ such that for any object x in $\mathcal{D}$ we have that $\mathcal{C}_{\leq x} \subseteq \mathcal{D}$. A coideal in $\mathcal{C}$ is a full subcategory $\mathcal{E}$ of $\mathcal{C}$ such that $\mathcal{C}_{\geq x} \subseteq \mathcal{E}$ for any $x \in \mathcal{E}$.

(3) Let M be a $k\mathcal{C}$-module. The *M-minimal objects* are the objects $x \in \operatorname{Ob}\mathcal{C}$ such that $M(x) \neq 0$ and for any $y \in \operatorname{Ob}\mathcal{C}$ with $y \not\cong x$ and $\mathcal{C}(y,x) \neq \emptyset$ one has $M(y) = 0$. Analogously one defines *M-maximal objects.*

(4) Let again M be a $k\mathcal{C}$-module. We put $\mathcal{C}_M$ to be the full subcategory consisting of all $y \in \operatorname{Ob}\mathcal{C}$ with $\mathcal{C}(x,y) \neq \emptyset$ for some M-minimal object x in $\mathcal{C}$.

It is clear by definition that any $k\mathcal{C}$-module M is determined by its values on $\mathcal{C}_M$. We are now going to point out a nice property of ideals, namely that in this case the restriction preserves projectives.

Proposition 2.18 (Xu, [33] Lemma 3.1.6)**.** *Let $\mathcal{D}$ be an ideal in an EI-category $\mathcal{C}$. Then the restriction functor $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projective (left-)modules*

If one would deal with right modules instead of left modules, then the restriction to coideals would preserve projectives.

3 EI-categories of finite representation type

Since the concept of EI algebras generalizes the concept of group algebras of finite groups and of path algebras of finite quivers or more generally of finite posets, it is a natural question to ask for a classification of the EI-category algebras of finite representation type. In 3.24 we will classify all endotrivial representation-finite EI-categories, which, roughly speaking, gives us the global shape of representation-finite EI-categories. Afterwards, we will turn our attention to EI-categories with two objects, since they are the easiest categories which are not group algebras. Even for this class it turns out that a classification of finite representation type is not easy. Nevertheless, we will derive some necessary criteria for finite representation type and compute various examples. In section 3.7 and section 3.8 we will put stronger conditions on the automorphism groups of the objects in our EI-categories and are then able to classify the representation-finite categories under this additional assumptions. It will become clear how involved this classification gets, even for very small categories. In the beginning of this chapter we will recall the classification of representation-finite group algebras, quivers and posets and afterwards present basic definitions and results on the Gabriel-quiver of a finite-dimensional algebra as well as a quick survey about covering theory for bound path algebras of quivers. In the introductory sections there will be no proofs in order to keep this part short and streamlined. Let us start with the most important definition for this chapter.

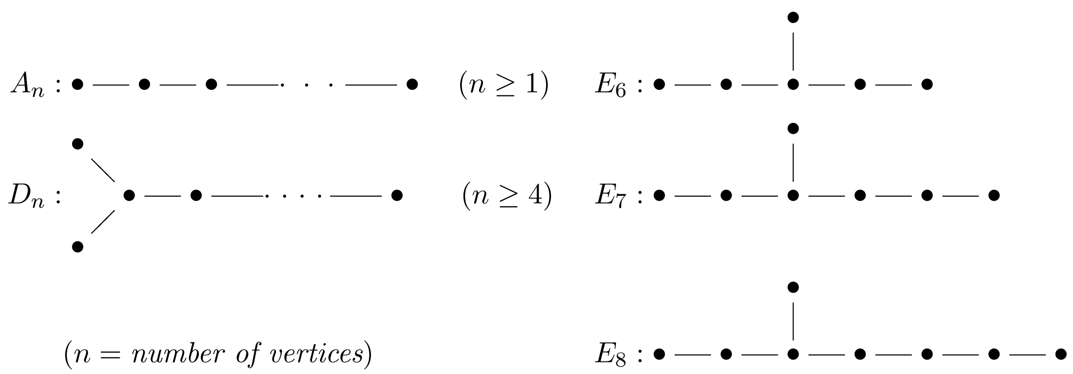
Definition 3.1. Let A be a finite-dimensional algebra over any field k. We say that A is *representation-finite* or *of finite representation type* if there are only finitely many isomorphism classes of indecomposable A-modules.

3.1 Finite representation type for finite groups, quivers and posets

The classification of representation-finite quivers and group algebras of finite groups is given by the following theorems.

Theorem 3.2 (Gabriel [18]). *Let Q be a finite and connected quiver and k any field. Then the path algebra kQ is of finite representation type if and only if the underlying graph of Q*

is a simply laced Dynkin diagram, i.e. one of the following graphs.

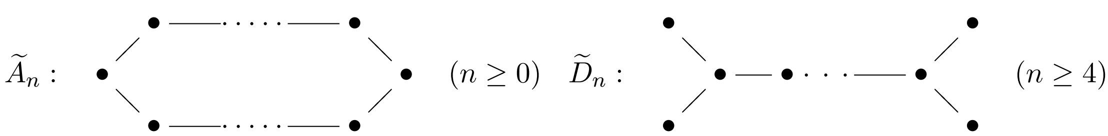


Theorem 3.3 (Highman [22], Kasch, Kneser and Kupisch [24])**.** *Let G be a finite group and let k be an algebraically closed field of characteristic p dividing the order of G. Then the group algebra kG is of finite representation type if and only if the Sylow p-subgroups G_p of G are cyclic.*

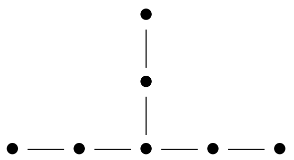
If the characteristic does not divide the order of G, a theorem of Maschke states that A is semisimple, in particular representation-finite. A good reference for both theorems is for example the book of Assem, Simson and Skowronski [3]. Also the representation-finite posets have been classified by Loupias (and independently by some russian mathematicians) and we want to recall the results here. A complete list of all representation-finite posets may be created using the criterion we will present. One can find such a list in the diploma thesis of H. Küchenhoff, who was a student of Gabriel, from 1982. Another good reference for representations of posets in general is the book of Simson [30], but one should note that he has a different definition of representation-finite posets. Therefore, we will mainly stick to the notations of Loupias since they are convenient for our purposes. From now on assume that $(I, \leq)$ is a connected finite poset.

Definition 3.4. Let I and J be two finite partially ordered sets and $f : I \to J$ a surjective morphism of ordered sets, i.e. $f(x) \leq f(y)$ if $x \leq y$. If $f^{-1}(j)$ is connected for every $j \in J$, then we call J a *contracted set of I*.

Theorem 3.5 (Loupias 1974, [25])**.** *Let I be a finite partially ordered set. Then I is of finite representation type if and only if it has no subset and no contracted set, which is given by one of the Hasse-diagrams (or their duals) in the list below. If there is a just a line between two points the orientation is arbitrary.*

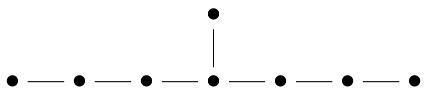


$\widetilde{E}_6$:

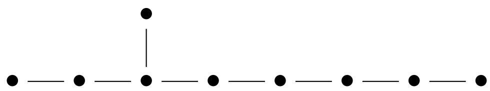


(*$n+1 =$ number of vertices*)

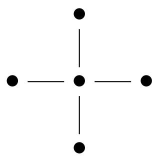
$\widetilde{E}_7$:



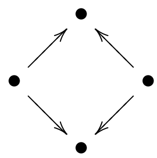
$\widetilde{E}_8$:



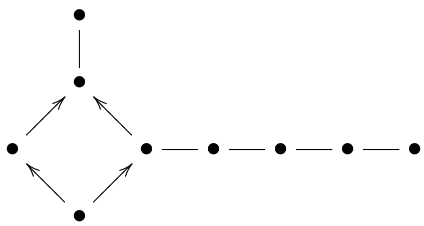
$\widetilde{D}_1$:



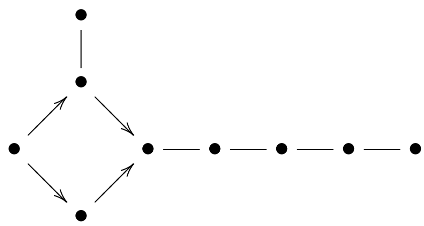
$\widetilde{A}_4$:



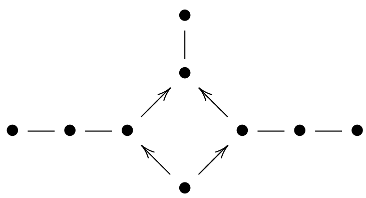
R_1 :



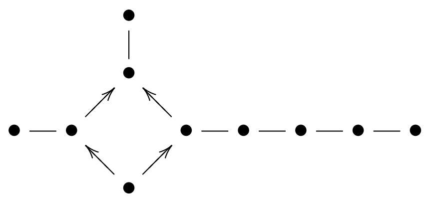
R_2 :



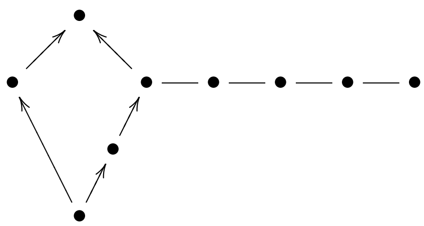
R_3 :



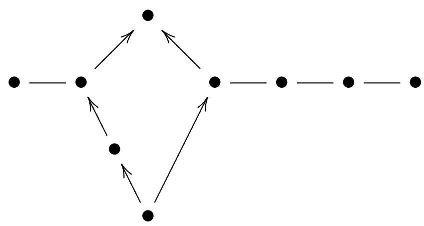
R_4 :



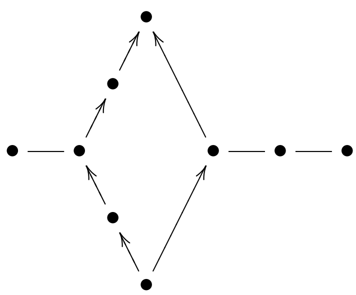
R_5 :



R_6 :



R_7 :



From this result it is possible to produce the Hasse-diagrams of all representation-finite partially ordered sets as it was done by Küchenhoff.

The leading question throughout this chapter is the following: Is there any general concept which contains the classification results from above as special cases and which gives a characterization of all representation-finite EI-category algebras?

3.2 The Gabriel-quiver of a finite dimensional algebra

To each finite-dimensional connected algebra A over a field k one can associate a finite quiver $\Gamma(A)$ and, in case the field k is algebraically closed, a theorem of Gabriel yields that A is Morita-equivalent to $k\Gamma(A)/I$ for some admissible ideal I. In this section we will briefly recall the most important definitions and results in this context, because we will use them intensively in our treatment of finite representation type for EI-categories. For more details for this whole section one may consult the book of Assem, Simson and Skowronski [3, Chapter II].

Definition 3.6. Let Q be a finite quiver and $\mathfrak{R}$ the ideal generated by all arrows in the path algebra kQ. A two-sided ideal $\mathfrak{I}$ in kQ is called *admissible* if there exists $n \geq 2$ such that $\mathfrak{R}^n \subseteq \mathfrak{I} \subseteq \mathfrak{R}^2$. If $\mathfrak{I}$ is an admissible ideal, then the pair $(Q, \mathfrak{I})$ is called a *bound quiver* and $kQ/\mathfrak{I}$ is said to be a *bound path algebra*.

Definition 3.7. Let A be a basic connected finite-dimensional algebra over a field k and $e_1, \dots, e_n$ a complete set of primitive orthogonal idempotents of A. The *Gabriel-quiver* of A, denoted by $\Gamma(A)$, is defined in the following way:

(i) The vertices of $\Gamma(A)$ correspond bijectively to the idempotents $e_1, \dots, e_n$;

(ii) For two vertices $a, b \in \Gamma(A)_0$, the arrows $\alpha : a \to b$ are in bijective correspondence with the vectors of a k-basis of the k-vector space $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.

One can easily verify that this quiver does not depend on the choice of the primitive idempotents in the definition. One should note here that every finite-dimensional k-algebra A is Morita-equivalent to a basic algebra A' and from the representation-theoretic point of view it makes no difference whether we deal with A or A'. One can also define $\Gamma(A)$ in a way that only depends on the category $\operatorname{mod} A$ and not on the structure of A itself as follows: The vertices $1, \dots, n$ of $\Gamma(A)$ are in bijective correspondence with the isomorphism classes $S_1, \dots, S_n$ of simple A-modules with $\dim_k \operatorname{Ext}^1_A(S_i, S_j)$ vertices from i to j. For explicit computations the first definition in terms of idempotents is often more applicable.

The central result in this context is the following theorem of Gabriel.

Theorem 3.8 (Gabriel)**.** *Let A be a basic and connected finite-dimensional algebra over an algebraically closed field k. Then there exists an admissible ideal $\mathfrak{I}$ in kQ such that $A \cong k\Gamma(A)/\mathfrak{I}$.*

Note that the ideal $\mathfrak{I}$ is in general not unique. See [3, II.2.2] for an example of an algebra A with two admissible ideals satisfying the conditions of the theorem. We will now illustrate how one can compute the quiver of an EI-category with one example.

Example 3.9. We consider the EI-category $\mathcal{C}$ which will also be treated in Proposition 3.27. That is: Let $\mathcal{C}$ be an EI-category with two objects x and y such that $\operatorname{End}(x) = \langle f \rangle \cong \mathbb{Z}_2$ and $\operatorname{End}(y) = \langle g \rangle \cong \mathbb{Z}_2$. Furthermore, we require that $\mathcal{C}(x, y) = \{ i_1, i_2, i_3, i_4 \}$ with $i_1 \circ f = i_2$, $i_3 \circ f = i_4$, $g \circ i_1 = i_3$ and $g \circ i_2 = i_4$. $\mathcal{C}$ may be illustrated as follows.

$$\mathcal{C}:\ \begin{tikzcd} x \arrow[loop left, "f"] \arrow[r, bend left=60, "i_1"] \arrow[r, bend left=20, "i_2"] \arrow[r, bend right=20, "i_3"'] \arrow[r, bend right=60, "i_4"'] & y \arrow[loop right, "g"] \end{tikzcd}$$

To compute the quiver of the (basic) algebra $k\mathcal{C}$ we have to distinguish between the case where $\operatorname{char}(k) = 2$ and the case $\operatorname{char}(k) \neq 2$.

Suppose $\operatorname{char}(k) = 2$. Then 1_x and 1_y form a complete list of orthogonal, primitive idempotents of $A := k\mathcal{C}$. We compute the radical to be $\operatorname{rad}(A) = \langle i_1, 1_x + f, 1_y + g \rangle$ and therefore $\operatorname{rad}^2(A) = \{ i_s + i_t \mid s \neq t \}$. Hence, the quiver $\Gamma(A)$ has two vertices corresponding to 1_x and 1_y each of this vertices having a loop corresponding to $1_x + f$ and $1_y + g$ respectively. Finally, there is one arrow from 1_x to 1_y corresponding to the class of i_1 in $1_y(\operatorname{rad}(A)/\operatorname{rad}^2(A))1_x$ and the quiver of A is

$$\Gamma(A):\ \begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & \circ \arrow[loop right, "\gamma"] \end{tikzcd}.$$

With the assignments from above, we get an algebra epimorphism $k\Gamma(A) \to A$ whose kernel is the admissible ideal (α^2, γ^2) and hence $A \cong k\Gamma(A)/(\alpha^2, \gamma^2)$.

If $\operatorname{char}(k) \neq 2$, then $k\mathcal{C}$ is hereditary as the path algebra of

$$\begin{tikzcd} \circ \arrow[r] \arrow[dr] & \circ \\ \circ \arrow[r] \arrow[ur] & \circ, \end{tikzcd}$$

which is a quiver with underlying Eucledian graph $\tilde{A}_3$.

3.3 Covering theory

In [7] Bongartz and Gabriel introduced covering theory for finite-dimensional algebras. Their work had been inspired by work of Riedtmann [28]. We will briefly recall some definitions, but mainly try to explain the use of covering theory for our purposes with an example.

Definition 3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be two k-linear categories. A k-linear functor $F : \mathcal{A} \to \mathcal{B}$ is called *covering functor* if the induced maps

$$\coprod_{Fy=b} \mathcal{A}(a,y) \to \mathcal{B}(Fa,b) \text{ and } \coprod_{Fy=b} \mathcal{A}(y,a) \to \mathcal{B}(b,Fx)$$

are bijective for every $a \in \mathcal{A}$ and $b \in B$.
Every covering functor gives rise to a *push-down functor* $F_\lambda : \mathrm{Rep}_k(\mathcal{A}) \to \mathrm{Rep}_k(\mathcal{B})$, where $F_\lambda(M)(b) = \coprod_{Fa=b} M(a)$. The definition on morphisms is fairly obvious.

A theorem of Gabriel [16] states that, under certain assumptions, the push-down functor preserves indecomposability. We will use this method for quivers with relations and loops at some vertices. We will explain how this procedure works in the following example, which, in our opinion, explains sufficiently well how a cover of the given quiver is constructed for all cases we are interested in.

Consider the Jordan-quiver

$$Q : \; \alpha \circlearrowright a \; ,$$

subject to the relation $\alpha^2 = 0$. This is the quiver of the group algebra of the cyclic group of order 2 in characteristic 2. For the infinite quiver

$$\overline{Q} : \; \cdots \xrightarrow{\gamma} x_{-1} \xrightarrow{\gamma} x_0 \xrightarrow{\gamma} x_1 \xrightarrow{\gamma} \cdots,$$

bound by $\gamma^2 = 0$, we get a functor $F : \overline{Q} \to Q$ defined by $F(x_i) = a$ for all i and $F(\gamma) = \alpha$, which is a covering functor (here we take for Q and $\overline{Q}$ the k-linear category spanned by their associated path categories). In this case the push-down functor preserves indecomposability (and also sends almost-split sequences to almost-split sequences). To get all indecomposable representations of Q/α^2, we pick a finite piece of $\overline{Q}/\gamma^2$ and knit its Auslander-Reiten quiver. Then we use the push-down to get indecomposable representations of Q/α^2 and see, that this procedure is 2-periodic and we immediately get all indecomposable representations "downstairs", namely two representations of dimension 1 and 2, respectively.

Analogously, one constructs the cover for all quivers with relations that appear in this thesis. The example we will see most frequently is the one of a quiver of the form

$$\Gamma : \; \alpha \circlearrowright \circ \xrightarrow{\beta} \circ \circlearrowleft \gamma \; ,$$

subject to some admissible relations. The universal cover of this quiver is then the infinite quiver which is displayed below, bound by the same relations as the original quiver. The covering basically coincides with the one in the last example and the push-down functor again preserves indecomposability. In the language of Gabriel, this is a Galois-covering.

$$\begin{tikzcd}
\vdots \arrow[d, "\alpha"'] & \vdots \arrow[d, "\gamma"] \\
\circ \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \circ \arrow[d, "\gamma"] \\
\circ \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \circ \arrow[d, "\gamma"] \\
\circ \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \circ \arrow[d, "\gamma"] \\
\vdots & \vdots
\end{tikzcd}$$

Figure 3.1: The universal cover of Γ

The fact we will heavily use is that, if the covering quiver of Q/I contains a relation-free part which is representation-infinite, then Q/I is representation-infinite since we can push-down the infinite family of indecomposable representations. Once again we refer to the literature for a more detailed treatment of this fact.

3.4 Relative projectivity, vertices and sources

The main tools for the classification of group algebras of finite representation type are restriction and induction functors as well as the concepts of relative projectivity, vertices, sources and defect groups. For EI-category algebras Xu [33] developed a theory of vertices and sources, which is very similar to the one for finite groups. In this subsection we will first briefly present the main definitions and results of this theory and then give an abstract classification of EI-category algebras of finite representation type in terms of vertices, which is a more or less immediate Corollary of Xu's results.

Definition 3.11. Let $\mathcal{C}$ be an EI-category and $\mathcal{D}$ a subcategory of $\mathcal{C}$. We call a $k\mathcal{C}$-module M *relatively $\mathcal{D}$-projective* (or *projective relative to $\mathcal{D}$*) if the canonical surjective $k\mathcal{C}$-module homomorphism $\varepsilon = \varepsilon_M : M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} = k\mathcal{C} \otimes_{k\mathcal{D}} M \to M,\ a \otimes m \mapsto a \cdot m$ splits.

Again as in the group case, we have several equivalent characterizations of relative projectivity.

Proposition 3.12 (Xu [33, Proposition 3.2.2])**.** *Let $\mathcal{C}$ be a finite EI-category, $\mathcal{D} \subseteq \mathcal{C}$ a subcategory and M a $k\mathcal{C}$-module. Then the following conditions are equivalent:*

(1) *M is relatively $\mathcal{D}$-projective;*
(2) *M is a direct summand of $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$;*

(3) *M is a direct summand of $N\uparrow_{\mathcal{D}}^{\mathcal{C}}$ for some $k\mathcal{D}$-module N;*
(4) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $k\mathcal{C}$-modules which splits in* $\operatorname{mod} k\mathcal{D}$*, then it splits in* $\operatorname{mod} k\mathcal{C}$*.*

Relative projectivity of a module M is closely related to M-minimal objects in the following way.

Lemma 3.13 (Xu [33, Lemma 3.2.5])**.** *Let M be a $k\mathcal{C}$-module and $\mathcal{D} \subseteq \mathcal{C}$ a subcategory.*

(1) *If M is relatively $\mathcal{D}$-projective, then* $\operatorname{Ob}\mathcal{D}$ *contains all M-minimal objects.*
(2) *If M is relatively $\mathcal{D}$-projective, then the module $M(x)$ is relatively* $\operatorname{Aut}_{\mathcal{D}}(x)$*-projective as a $k\operatorname{Aut}_{\mathcal{C}}(x)$-module for any M-minimal object x.*

For the case of full subcategories of an EI-category $\mathcal{C}$, relative projectivity of a module yields that this module is already generated by its values on the full subcategory relative to which it is projective in the following sense.

Proposition 3.14 (Xu [33, Proposition 3.3.1])**.** *Let $\mathcal{D}$ be a full subcategory of an EI-category $\mathcal{C}$ and M a $k\mathcal{C}$-module which is relatively $\mathcal{D}$-projective. Then M is generated by its values on $\mathcal{D}$, i.e. $M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}} \cong M$.*

The next theorem guarantees that for a representation-finite EI-category every connected, full subcategory has to be of finite representation type.

Theorem 3.15 (Xu [33, Theorem 3.3.2])**.** *Let $\mathcal{C}$ be an EI-category, $\mathcal{D}$ a connected, full subcategory and N an indecomposable $k\mathcal{D}$-module. Then the $k\mathcal{C}$-module $N\uparrow_{\mathcal{D}}^{\mathcal{C}}$ is indecomposable and relatively $\mathcal{D}$-projective.*

This theorem can be compared with Green's indecomposability theorem for representations of finite groups (see [2]). Its inverse holds as well:

Theorem 3.16 (Xu [33, Theorem 3.3.4])**.** *Let M be an indecomposable $k\mathcal{C}$-module which is relatively $\mathcal{D}$-projective for a connected, full subcategory $\mathcal{D} \subseteq \mathcal{C}$. Then $M\downarrow_{\mathcal{D}}^{\mathcal{C}}$ is an indecomposable $k\mathcal{D}$-module.*

The two previous theorems give an equivalence of categories by means of the following definition.

Definition 3.17. Let $\mathcal{D} \subseteq \mathcal{C}$ be a connected, full subcategory of an EI-category $\mathcal{C}$. We denote by $\operatorname{mod} k\mathcal{C}_{\mathcal{D}}$ the full subcategory of $\operatorname{mod} k\mathcal{C}$ consisting of those $k\mathcal{C}$-modules which are relatively $\mathcal{D}$-projective.

Proposition 3.18 (Xu [33, Proposition 3.3.6])**.** *The functors $\downarrow_{\mathcal{D}}^{\mathcal{C}}$ and $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ induce quasi-inverse equivalences*

$$\operatorname{mod} k\mathcal{C}_{D} \underset{\uparrow_{\mathcal{D}}^{\mathcal{C}}}{\overset{\downarrow_{\mathcal{D}}^{\mathcal{C}}}{\rightleftarrows}} \operatorname{mod} k\mathcal{D}\,.$$

The next result is necessary to develop the theory of vertices and sources in our framework.

Proposition 3.19 (Xu [33, Corollary 3.3.8]). *For any indecomposable $k\mathcal{C}$-module M there exists a smallest ideal $\widetilde{\mathcal{V}_M}$ in $\mathcal{C}$ relative to which M is projective.*

Definition 3.20. The full subcategory $\mathcal{V}_M = \widetilde{\mathcal{V}_M} \cap \mathcal{C}_M$ is called the *vertex of M*.

It is clear by definition that the vertex of an indecomposable $k\mathcal{C}$-module is always a connected, full subcategory of $\mathcal{C}$. We say that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is *convex* if, whenever there is a sequence of morphisms $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathcal{C}$ with $x, z \in \operatorname{Ob}\mathcal{D}$, then both α and β are in $\operatorname{Mor}\mathcal{D}$. With this definition there are three equivalent characterizations of the vertex of a $k\mathcal{C}$-module.

Proposition 3.21 (Xu [33, Proposition 3.3.12]). *Let M be an indecomposable $k\mathcal{C}$-module and $\mathcal{D} \subseteq \mathcal{C}$ a connected and full subcategory of $\mathcal{C}$. Then the following statements are equivalent:*

(1) *$\mathcal{D}$ is the vertex of M;*
(2) *$\mathcal{D}$ is the smallest ideal in $\mathcal{C}_M$ relative to which M is projective;*
(3) *$\mathcal{D}$ is the smallest full and convex subcategory of $\mathcal{C}$ relative to which M is projective.*

If M is again an indecomposable $k\mathcal{C}$-module and $\mathcal{V}_M$ its vertex, then $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}\uparrow^{\mathcal{C}}_{\mathcal{V}_M} \cong M$ and $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}$ is indecomposable. Therefore, M is (up to isomorphism) determined by the indecomposable $k\mathcal{V}_M$-module $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}$. For that reason, we call $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}$ the *source* for M.

Using the whole theory Xu developed, we derive the following easy Proposition which is an abstract characterization of EI-categories of finite representation type.

Proposition 3.22. *Let $\mathcal{C}$ be a finite, connected EI-category. Then the algebra $k\mathcal{C}$ is of finite representation type if and only if $k\mathcal{V}_M$ is of finite representation type for any indecomposable module M in* $\operatorname{mod} k\mathcal{C}$.

Proof. If $\mathcal{C}$ is of finite representation type, then every full, connected subcategory is of finite representation type by Theorem 3.15. Conversely, suppose that every $k\mathcal{V}_M$ is of finite representation type. Since $\mathcal{C}$ is finite, there can only be finitely many vertices of indecomposable $k\mathcal{C}$-modules, each having only finitely many indecomposable representations up to isomorphism. An indecomposable $k\mathcal{C}$-module M is (up to isomorphism) determined by its source $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}$, which is an indecomposable $k\mathcal{V}_M$-module. Together, this yields that there are only finitely many isomorphism classes of indecomposable $k\mathcal{C}$-modules. □

Remark 3.23. (i) In representation theory of finite groups it is known that for a finite group G and a field k of characteristic $p > 0$ dividing $|G|$ every kG-module is projective relative to the Sylow p-subgroup P of G. For that reason, the representation type of kG is governed by the representation type of kP. In contrast to that, for an EI-category algebra $k\mathcal{C}$ it almost always happens that the smallest full subcategory relative to which a $k\mathcal{C}$-module M is projective is the category $\mathcal{C}$ itself. Thus, the theory of vertices and sources for EI-categories does not help much for an explicit characterization of finite representation type.

With existing methods it is also not possible to generalize the concept of a defect group to an equivalent concept for EI-category algebras since the conjugation with group elements of a group G (as basis vectors in kG) is essential to obtain a reasonable theory for defect groups. For EI-categories there is no concept similar to conjugation.

(ii) Restriction and induction are defined for an arbitrary subcategory $\mathcal{D}$ of an EI-category $\mathcal{C}$. Most of the results of Xu only work for the case where $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. The problem with arbitrary subcategories is, that the computation of the induction of a module is in general very complicated. If $\mathcal{D}$ is a full subcategory, then the induction of every indecomposable $k\mathcal{D}$-module to $k\mathcal{C}$ is indecomposable. This is not true for arbitrary subcategories (e.g. subgroups of a finite group) and for that reason it is not clear how one can get a reduction technique for the representation type of an EI-category in terms of arbitrary subcategories.

3.5 The endotrivial case

For any EI-category $\mathcal{C}$ we have seen that there is a fully faithful embedding $\operatorname{mod} k\widehat{\mathcal{C}} \to \operatorname{mod} k\mathcal{C}$, where $\widehat{C}$ is the 'endotrivialization' of $\mathcal{C}$ which has been defined above. The existence of this embedding implies that, if $\mathcal{C}$ is representation-finite, then $\widehat{\mathcal{C}}$ is representation-finite as well. Therefore, it is natural to ask for a description of all representation-finite EI-categories with only trivial endomorphisms. This description characterizes the global structure of any representation-finite EI-category.

Theorem 3.24. *Let $\mathcal{C}$ be an EI-category with only trivial endomorphisms. Then $\mathcal{C}$ is of finite representation type if and only if $k\mathcal{C}$ is Morita-equivalent to an incidence algebra of finite representation type.*

Proof. Let $\mathcal{C}$ be endotrivial and of finite representation type. We have to show that $k\mathcal{C}$ is Morita-equivalent to an incidence algebra of finite type. The other direction in the theorem is trivial. Again, we may (up to Morita-equivalence of $k\mathcal{C}$) assume that $\mathcal{C}$ is skeletal.
Since $\mathcal{C}$ is representation-finite, there are no objects $x, y \in \mathcal{C}$ with two distinct morphisms $f, g : x \to y$. Otherwise one gets a fully faithful embedding of the category of representations of the 2-Kronecker into $\operatorname{mod} k\mathcal{C}$. Since $\mathcal{C}$ is skeletal its set of objects carries a natural structure of a finite poset defined by

$$x \leq y :\Leftrightarrow \exists f \in \mathcal{C}(x, y).$$

Using, that $\mathcal{C}(x, y) \leq 1$ for all $x, y \in \operatorname{Ob}\mathcal{C}$, we get that $k\mathcal{C}$ is the incidence algebra associated to the finite poset $(\operatorname{Ob}\mathcal{C}, \leq)$ and the claim follows. $\square$

Corollary 3.25. *Let $\mathcal{C}$ be a finite and skeletal EI-category and k a field such that $k\mathcal{C}$ is representation-finite. Then $\widehat{\mathcal{C}}$ is a finite poset of finite representation type.*

Remark 3.26. Suppose $\mathcal{C}$ is an endotrivial EI-category. Then its set of objects naturally carries the structure of a finite partially ordered set, but in general it will not be the case that $\mathcal{C}$ is the category associated to this poset. For instance, take $\mathcal{C}$ to be the category

$$\begin{tikzcd} & d & \\ b \arrow[ur, "\gamma"] & & c, \arrow[ul, "\delta"'] \\ & a \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

without any relations. Then the category associated to $(\operatorname{Ob}\mathcal{C}, \leq)$ is $\mathcal{C}$ modulo the relation $\gamma\alpha = \delta\beta$.

3.6 EI-categories with two objects

The easiest class of EI-categories for which the representation type has not yet been investigated is the class of EI-categories with two non-isomorphic objects. Clearly, a representation-finite EI-category $\mathcal{C}$ with two objects has to satisfy $\widehat{\mathcal{C}} = A_2$ and both group algebras attached to the two objects have to be of finite representation type. Therefore, we may illustrate a possibly representation-finite EI-category $\mathcal{C}$ with two non-isomorphic objects as follows.

$$x \xrightarrow{{}^{G_x}f^{G_y}} y$$

Here f is one representative of the class of morphisms from x to y with respect to our relation $\sim$ and G_x, G_y denote the endomorphism groups of x and y respectively. In other words, f is one representative of the unique orbit of the group action of $G_x \times G_y$ on $\mathcal{C}(x, y)$. One should note that the representation type of an endotrivial category does not depend on the characteristic of the ground field k. In contrast to that, the representation type of group algebras depends on $\operatorname{char}(k)$ and the same is true for EI-category algebras which are not group algebras, as we will see later.

An arbitrary EI-category algebra $k\mathcal{C}$ is representation-infinite if there exists at least one full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that the algebra $k\mathcal{D}$ is representation-infinite. Therefore, the treatment of EI-categories with two objects yields necessary criteria for an EI-category algebra to be representation-finite.

We assume that k denotes an algebraically closed field.

3.6.1 The easiest example

The following example, which we will treat in every detail, illustrates how involved the characterization of representation-finite EI-category algebras might get. We consider an EI-category $\mathcal{C}$ with two objects x and y such that $\mathrm{End}(x) = \langle f \rangle \cong \mathbb{Z}_2$ and $\mathrm{End}(y) = \langle g \rangle \cong \mathbb{Z}_2$. Under these assumptions there are 5 different EI-categories that can appear, namely:

(1) $\mathcal{C}(x, y) = \{ i \}$ with $i \circ f = i$ and $g \circ i = i$;

(2) $\mathcal{C}(x, y) = \{ i_1, i_2 \}$ with f acting trivially on $\{ i_1, i_2 \}$ and g permuting the two morphisms;

(3) $\mathcal{C}(x, y) = \{ i_1, i_2 \}$ with f permuting i_1 and i_2 and g acting trivially;

(4) $\mathcal{C}(x, y) = \{ i_1, i_2 \}$ with f and g permuting i_1 and i_2;

(5) $\mathcal{C}(x, y) = \{ i_1, i_2, i_3, i_4 \}$ with $i_1 \circ f = i_2$, $i_3 \circ f = i_4$, $g \circ i_1 = i_3$ and $g \circ i_2 = i_4$.

The second case has briefly been studied by Xu in [33], where he claims that this category is of infinite representation type in characteristic 2, but the representations he constructed turn out to be decomposable and (as we will see later) and the category is of finite representation type. It will turn out that the representation type of an EI-category with two objects is mainly governed by the group action of the endomorphism groups on the set of morphisms between the two non-isomorphic objects.

The investigation of case (5), which is obviously the one with the most complicated representation theory, leads to the following result.

Proposition 3.27. *Let $\mathcal{C}$ be an EI-category with two objects x and y such that $\mathrm{End}(x) = \langle f \rangle \cong \mathbb{Z}_2$ and $\mathrm{End}(y) = \langle g \rangle \cong \mathbb{Z}_2$. Furthermore, we require that $\mathcal{C}(x, y) = \{ i_1, i_2, i_3, i_4 \}$ with $i_1 \circ f = i_2$, $i_3 \circ f = i_4$, $g \circ i_1 = i_3$ and $g \circ i_2 = i_4$.*

$$\mathcal{C}: \begin{tikzcd} x \arrow[loop left, "f"] \arrow[r, bend left=60, "i_1"] \arrow[r, bend left=20, "i_2"] \arrow[r, bend right=20, "i_3"] \arrow[r, bend right=60, "i_4"'] & y \arrow[loop right, "g"] \end{tikzcd}$$

Then $k\mathcal{C}$ is of infinite representation type, no matter which base field k we choose.

Proof. We prove the theorem by constructing an infinite family of indecomposable representations of $\mathcal{C}$:

For $n \in \mathbb{N}$ let $V_n \in \mathrm{rep}_k\, \mathcal{C}$ be defined by $V_n(x) = k^{2n}$, $V_n(y) = k^{2n-1}$ and

$$V_n(f) = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}, V_n(g) = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & & \\ & \ddots & & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & & 1 \end{bmatrix}, V_n(i_1) = \begin{bmatrix} 0 & 0 & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

This gives (together with the compositions of the maps we defined above) a representation of $\mathcal{C}$. We prove the indecomposability of V_n (which we will for simplicity denote by V) by computing its endomorphism ring.

Let ϕ be an endomorphism of V, that means $\phi = (A, B)$ where $A \in M_{2n}(k)$ and $B \in M_{2n-1}(k)$. First of all, $A = (a_{i,j})$ has to commute with $V(f)$ which is equivalent to the condition that $a_{2l-1,2t-1} = a_{2l,2t}$ and $a_{2l-1,2t} = a_{2l,2t-1}$ for any $1 \leq l, t \leq n$. Therefore, A is of the following form.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{1,2} & a_{1,1} & a_{1,4} & a_{1,3} & \cdots & a_{1,2n} & a_{1,2n-1} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,2n-1} & a_{3,2n} \\ a_{3,2} & a_{3,1} & a_{3,4} & a_{3,3} & \cdots & a_{3,2n} & a_{3,2n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & a_{2n-1,3} & a_{2n-1,4} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n-1,2} & a_{2n-1,1} & a_{2n-1,4} & a_{2n-1,3} & \cdots & a_{2n-1,2n} & a_{2n-1,2n-1} \end{bmatrix}$$

Analogously, B has to commute with $V(g)$ which is equivalent to the conditions $b_{2l-1,2t-1} = b_{2l,2t}$, $b_{2l-1,2t} = b_{2l,2t-1}$, $b_{2l,2n-1} = b_{2l-1,2n-1}$ and $b_{2n-1,2t} = b_{2n-1,2t-1}$ for any $1 \leq l, t \leq n-1$, i.e. B is of the following form.

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,2n-3} & b_{1,2n-2} & b_{1,2n-1} \\ b_{1,2} & b_{1,1} & \cdots & b_{1,2n-2} & b_{1,2n-3} & b_{1,2n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{2n-3,1} & b_{2n-3,2} & \cdots & b_{2n-3,2n-3} & b_{2n-3,2n-2} & b_{2n-3,2n-1} \\ b_{2n-3,2} & b_{2n-3,1} & \cdots & b_{2n-3,2n-2} & b_{2n-3,2n-3} & b_{2n-3,2n-1} \\ b_{2n-1,1} & b_{2n-1,1} & \cdots & b_{2n-1,2n-3} & b_{2n-1,2n-3} & b_{2n-1,2n-1} \end{bmatrix}$$

Additionally, A and B have to satisfy the relation

$$BV(i_1) = V(i_1)A. \tag{3.1}$$

The left hand side of this equation equals $(0, B)$ where 0 denotes the zero vector in k^{2n-1}. The right hand side is easily seen to be $V(i_1)A = \widehat{A}$, where we want $\widehat{A}$ to denote the matrix one obtains from A by erasing the first line.

Now we claim, that (3.1) implies $A = \lambda \cdot E_{2n}$ and $B = \lambda \cdot E_{2n-1}$ with $\lambda := a_{1,1} = b_{1,1}$.

First of all, we note that both A and B only have entries whose first index is odd and that all diagonal entries of A and B are equal to λ. Now we compare the left hand side and the right hand side of (3.1) columnwise, starting with the first column. This yields $a_{2i-1,j} = 0$

for all $i = 1, \dots, n,\ j = 1, \dots 2n$ with $2i - 1 \neq j$ and $j \neq 2i + 1$ as well as $b_{2l-1,t} = 0$ for all $l = 1, \dots, n,\ t = 1, \dots 2n - 1$ with $t \neq 2l - 1$ and $t \neq 2l + 1$. Furthermore, we infer that $a_{2i-1,2i+1} = a_{2l-1,2l+1} = b_{2j-1,2j+1} = b_{2t-1,2t+1}$ for all $i, j, l, t = 1, \dots, n - 1$.

In other words the entries of A and B are zero if they do not lie on the diagonal or have indices of the form $2i - 1, 2i + 1$, and the latter are all equal. The last column of (3.1) gives $0 = a_{2n-1,2n} = b_{2n-3,2n-1}$ which implies $0 = a_{2i-1,2i+1} = b_{2j-1,2j+1}$ for all $i, j = 1, \dots, n-1$ and the claim follows. □

This is the first non-trivial example of an EI-category of infinite representation type, where non-trivial means, that neither $\widehat{\mathcal{C}}$ nor $k\,\mathrm{End}(x)$ (for some $x \in \mathrm{Ob}\,\mathcal{C}$) is of infinite representation type. Indeed, even the characteristic of the ground field is arbitrary in the proposition and therefore the group algebras associated to the two objects may be semisimple, e.g. for $\mathrm{char}(k) = 0$.

We will now dicuss all five cases from above by computing their Gabriel-quivers and deducing their representation type from this quiver (with relations). Surprisingly, it will turn out, that the characteristic of the ground field does not play any role in this special cases. We will also get the result from above again by this discussion, but the construction of an infinite family of non-isomorphic indecomposables is interesting on its own.

Proposition 3.28. *Let $\mathcal{C}$ be the EI-category from case (1). Then $k\mathcal{C}$ is of finite representation type.*

Proof. The computation of the Gabriel quiver of $k\mathcal{C}$ in characteristic 2 yields the quiver

$$Q:\ \begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & \circ \arrow[loop right, "\gamma"] \end{tikzcd}\ . \tag{3.2}$$

An easy calculation shows that $k\mathcal{C}$ is isomorphic to kQ/I, where I is the admissible ideal generated by the zero relations $0 = \alpha^2 = \gamma^2 = \beta\alpha = \gamma\beta$. Therefore, $k\mathcal{C}$ is a string algebra without any bands and those algebras are known to be of finite representation type. Alternatively one can consult the list at the end of [7] and see that this algebra is of finite representation type and also find its Auslander-Reiten quiver.

If $\mathrm{char}(k) \neq 2$, then the algebra $k\mathcal{C}$ is isomorphic to the path algebra kQ where Q is the quiver

$$\begin{tikzcd} \circ & \circ \\ \circ \arrow[r] & \circ, \end{tikzcd}$$

which is obviously of finite representation type. □

The other cases are discussed analogously and we deduce the following results (with respect to the numbering from the beginning of this subsection). Fix the quiver Q from

(3.2).

(2) $k\mathcal{C} \simeq kQ/\langle \alpha^2, \gamma^2, \beta\alpha \rangle$ if $\mathrm{char}(k) = 2$ and this algebra is of finite representation type, again as a string algebra without any bands. If the characteristic is different from 2, then $k\mathcal{C}$ is the path algebra of the representation-finite quiver

$$\begin{tikzcd} \circ & \circ \\ \circ \arrow[r] \arrow[ur] & \circ. \end{tikzcd}$$

(3) This situation is dual to the one above. In $\mathrm{char}(k) = 2$ we have $k\mathcal{C} \simeq kQ/\langle \alpha^2, \gamma^2, \gamma\beta \rangle$, which again is a string algebra without bands, and in $\mathrm{char}(k) \neq 2$ the algebra $k\mathcal{C}$ is hereditary of finite representation type as the path algebra of the following quiver

$$\begin{tikzcd} \circ \arrow[dr] & \circ \\ \circ \arrow[r] & \circ. \end{tikzcd}$$

(4) If $\mathrm{char}(k) = 2$, then $k\mathcal{C} \simeq kQ/\langle \alpha^2, \gamma^2, \beta\alpha - \gamma\beta \rangle$. This algebra is representation-finite, which can be seen by knitting the AR-quiver using covering theory as we explained before (or see for example [16]). If the characteristic is different from 2, then the algebra $k\mathcal{C}$ is isomorphic to the path algebra of the quiver

$$\begin{tikzcd} \circ \arrow[r] & \circ \\ \circ \arrow[r] & \circ, \end{tikzcd}$$

which is of finite representation type.

(5) For this case we have already seen that $k\mathcal{C}$ is of infinite representation type in any characteristic. This also follows from the computation of the Gabriel quiver and the attached relations. If $\mathrm{char}(k) = 2$, then we get that $k\mathcal{C} \simeq kQ/\langle \alpha^2, \gamma^2 \rangle$ which is a string algebra with infinitely many bands and therefore it is of infinite representation type. In any characteristic different from 2 the algebra $k\mathcal{C}$ is hereditary (as we know from work of Xu) and in this particular case isomorphic to the path algebra of the following quiver

$$\begin{tikzcd} \circ \arrow[r] \arrow[dr] & \circ \\ \circ \arrow[r] \arrow[ur] & \circ, \end{tikzcd}$$

which is a quiver with underlying Eucledian graph $\tilde{A}_3$.

3.6.2 The characteristic plays a role

In this subsection we present an example, which shows that the representation type of an EI-category algebra $k\mathcal{C}$, which does not come from a group, depends on the characteristic of the field k. The reason for this is that the same is true for group algebras. The easiest example where different characteristics of k yield different representation types is the following one.

Example 3.29. Consider the EI-category

$$\begin{tikzcd} \mathcal{C}: x \arrow[loop left, "g"] \arrow[r, bend left, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right, "f_3"'] & y \arrow[loop right, "h"] \end{tikzcd},$$

with the relations $g^2 = 1_x$, $f_i g = f_i$, for $i = 1, 2, 3$, $h^3 = 1_y$ and $hf_1 = f_2$, $hf_2 = f_3$, $hf_3 = f_1$.
We will now show that the associated k-algebra $k\mathcal{C}$ is of finite representation type if and only if $\mathrm{char}(k) \neq 2$. First of all, we suppose that $\mathrm{char}(k) \neq 2, 3$. Then $k\mathcal{C}$ is hereditary (and basic) and we compute the Gabriel quiver to be

$$\begin{tikzcd} \circ \arrow[r] \arrow[dr] \arrow[ddr] & \circ \\ \circ & \circ \\ & \circ, \end{tikzcd}$$

which is of finite representation type as the union of quivers with underlying graphs A_1 and D_4.

Now we assume that the characteristic of k is 3. Then the radical of $k\mathcal{C}$ is $\langle 1_y - h, f_1, f_2, f_3\rangle$ and $\mathrm{rad}^2 k\mathcal{C} = \langle (1_y - h)^2, f_1 - f_2, f_2 - f_3\rangle$ where $\langle - \rangle$ denotes the k-span. A complete list of primitive, orthogonal idempotents is given by $1_y, \frac{1}{2}(1_x + g), \frac{1}{2}(1_x - g)$. Therefore, $k\mathcal{C}$ is isomorphic to the path algebra of the quiver

$$\begin{tikzcd} \circ \arrow[r, "\beta"] & \circ \arrow[loop right, "\gamma"] , \\ \circ & \end{tikzcd}$$

bound by the relation $\gamma^3 = 0$. This bound path algebra is of finite representation type since it is the union of A_1 with a bound path algebra whose module category can be embedded into the module category of $k(\circ \underset{\delta}{\overset{\nu}{\rightleftarrows}} \circ \circlearrowright \rho)/\langle \rho^3, \delta\rho, \delta\nu, \delta\rho\nu\rangle$. The latter is known to be of finite type by work of Bautista, Gabriel, Roiter and Salmeron [6] and work of Bongartz and Gabriel [7].

The last case is the one where $\mathrm{char}(k) = 2$. Here the radical of $k\mathcal{C}$ is $\langle 1_x + g, f_1, f_2, f_3\rangle$ and its square is zero. A complete list of primitive, orthogonal idempotents is given by $1_x, \frac{1}{3}(1_y + h + h^2), \frac{1}{3}(1_y + \varepsilon h + \varepsilon^2 h^2), \frac{1}{3}(1_y + \varepsilon^2 h + \varepsilon h^2)$, where ε denotes a primitive third root of unity in k. Therefore, we deduce for the Gabriel quiver of $k\mathcal{C}$ the quiver

$$\Gamma:\ \begin{tikzcd} & \circ \\ \circ \arrow[loop left, "\alpha"] \arrow[ur, "\beta_1"] \arrow[r, "\beta_2"] \arrow[dr, "\beta_3"'] & \circ, \\ & \circ \end{tikzcd}$$

and $k\mathcal{C}$ is isomorphic to $k\Gamma/\langle\alpha^2, \beta_1\alpha, \beta_2\alpha, \beta_3\alpha\rangle$. This bound quiver is of infinite representation type since its universal cover contains the 4-subspace quiver without relations. It is also not difficult to write down a 1-parameter family of indecomposables of dimension vector $(3, 1, 1, 1)$.

The discussion of the first case in the example can be generalized to the following Proposition, which provides us with a situation where we can prove that the EI-category algebras in question always have finite representation type.

Proposition 3.30. *Let $\mathcal{C}$ be a finite, skeletal EI-category with 2 objects x and y such that $\mathcal{C}(x, y) = \{ f \}$ and the two groups $\mathrm{End}(x)$ and $\mathrm{End}(y)$ are abelian. Let k be an algebraically closed field whose characteristic neither divides the order of $\mathrm{End}(x)$ nor the order of $\mathrm{End}(y)$. Then $k\mathcal{C}$ is of finite representation type.*

Proof. Let $n := |\mathrm{End}(X)|$ and $m := |\mathrm{End}(Y)|$.

First of all we note that the algebra $k\mathcal{C}$ is hereditary (see Theorem 4.2.4 in [33]) and basic since the two groups are abelian. Therefore, it is isomorphic to the path algebra of its Gabriel quiver.

Since the groups $\mathrm{End}(x)$ and $\mathrm{End}(y)$ are assumed to be abelian of order not divisible by the characteristic of our field, it is known that $k\mathcal{C}$ has exactly $m + n$ isomorphism classes of simple modules (see [29] and [26]) and all the simples do not have self-extensions. This implies, that the Gabriel quiver $\Gamma(k\mathcal{C})$ has $m + n$ vertices. Furthermore, we know that the k-dimension of $k\mathcal{C}$ is $m + n + 1$. This yields that we have exactly one arrow in $\Gamma(k\mathcal{C})$ and the claim follows. □

3.6.3 Free action implies infinite type

In this subsection we prove the fact that free action of the automorphism groups of an EI-category $\mathcal{C}$ with two objects x and y on $\mathcal{C}(x, y)$ implies infinite representation type in any characteristic. This is the most general result we will achieve in our treatment of EI-categories with two objects. The proof is carried out by considering various cases. The most interesting cases will be presented as Lemmata starting with the following one.

Lemma 3.31. *Let $\mathcal{C}$ be an EI-category with two non-isomorphic objects x, y and abelian endomorphism groups* $\mathrm{End}(x)$ *and* $\mathrm{End}(y)$ *of order ≥ 2 such that the group action of* $\mathrm{End}(x) \times \mathrm{End}(y)$ *on $\mathcal{C}(x, y)$ is free and transitive. Let k be an algebraically closed field which characteristic neither divides the order of* $\mathrm{End}(x)$ *nor the order of* $\mathrm{End}(y)$. *Then $k\mathcal{C}$ is of infinite representation type.*

Proof. Analogous to the proof in the last subsection, we have that $k\mathcal{C}$ is isomorphic to the path algebra of its Gabriel-quiver which has $m+n$ vertices. Since $\dim_k k\mathcal{C} = m + n + m \cdot n$ there are $m \cdot n$ arrows in $\Gamma(k\mathcal{C})$. Therefore, the underlying graph of the Gabriel quiver is not Dynkin. □

If both group algebras are not semisimple, we can prove the assertion without constructing representations or computing the Gabriel quiver.

Lemma 3.32. *Let $\mathcal{C}$ be an EI-category with two objects x and y and let k be an algebraically closed field of positive characteristic p dividing both* $|\mathrm{End}(x)|$ *and* $|\mathrm{End}(y)|$. *Further, we assume that* $\mathrm{End}(x) \times \mathrm{End}(y)$ *acts freely on $\mathcal{C}(X, Y)$. Then $k\mathcal{C}$ is of infinite representation type.*

Proof. For simplicity we write $G := \mathrm{End}(x)$ and $H := \mathrm{End}(y)$. We will prove the theorem by constructing a fully faithful embedding $F : \mathrm{mod}\, k(G \times H) \to \mathrm{mod}\, k\mathcal{C}$. The construction of this functor is rather obvious. For $M \in \mathrm{mod}\, k(G \times H)$ let $F(M)(x) = M = F(M)(y)$ together with the natural actions of G and H on M given by $G \times \mathrm{id}_Y$ and $\mathrm{id}_X \times H$, respectively. Furthermore, we let $F(M)(\mathcal{C}(x, y)) = G \times H$. This is indeed a representation of $k\mathcal{C}$. Now let $\mu : M \to M'$ be a morphism in $\mathrm{mod}\, k(G \times H)$. We put $F(\mu) = (\mu, \mu)$ which gives a morphism $F(M) \to F(M')$ of $k\mathcal{C}$ modules. Finally this functor is fully faithful and k-linear by construction and it is known that $k(G \times H)$ (in this particular framework) is of infinite representation type. □

Remark 3.33. If $\mathcal{C}$ is a skeletal EI-category with two objects x, y such that $\mathrm{Aut}(x) \times \mathrm{Aut}(y)$ acts freely on $\mathcal{C}(x, y)$, then we can also localize the category $\mathcal{C}$ with respect to the set of morphisms $S := \mathcal{C}(x, y)$ in the set of Gabriel and Zisman [17]. This gives a new category $\mathcal{C}[S]$ and the category of representations of $\mathcal{C}[S]$ is equivalent to the subcategory of representations of $\mathcal{C}$ consisting of all representations V for which $V(f)$ is invertible for all f in $\mathcal{C}(x, y)$. Then, it is easy to see that this category contains $\mathrm{mod}(\mathrm{Aut}(x) \times \mathrm{Aut}(y))$ as a full subcategory as we have seen in the previous proof.

One should note that the localization of an EI-category is not again an EI-category in general, but if we assume that the EI-category has no parallel morphisms (i.e. $|\mathcal{C}(x,y)| \leq 1$ for all $x, y \in \mathrm{Ob}\,\mathcal{C}$), then every localization is again EI.

For cyclic groups the computation of the Gabriel quiver is rather easy, which gives the next lemma.

Lemma 3.34. *Let $\mathcal{C}$ be an EI-category with two non-isomorphic objects x and y such that the action of $\mathrm{End}(x) \times \mathrm{End}(y)$ on $\mathcal{C}(x,y)$ is free and $\mathrm{End}(x)$ and $\mathrm{End}(y)$ are cyclic of order ≥ 2. Then $k\mathcal{C}$ is of infinite representation type for any algebraically closed field k.*

Proof. In the case of cyclic endomorphism groups one can easily compute the Gabriel quiver of $k\mathcal{C}$ and derive the relations on it such that $k\mathcal{C}$ is isomorphic to this bound path algebra. One has to distinguish between the cases where the orders of both groups are divided by the characteristic, only one of them or none. In all the three cases one ends up with infinite representation type. We are not going to present the details here, the computations are exactly the same as in the examples we discussed above. □

Up to now we have seen several special cases of EI-categories with two objects and two non-trivial endomorphism groups with free action which are representation-infinite. We are now in the position to prove the general statement

Theorem 3.35. *Let $\mathcal{C}$ be a skeletal EI-category with two objects x and y such that the groups $\mathrm{End}(x)$ and $\mathrm{End}(y)$ are non-trivial and their product $\mathrm{End}(x) \times \mathrm{End}(y)$ acts freely on $\mathcal{C}(x,y)$. Then $k\mathcal{C}$ is of infinite representation type for any algebraically closed field k.*

Proof. The claim has already been proven for the case where both group algebras are non-semisimple, for the case where both groups are cyclic and for the case of two semisimple abelian group algebras. To prove the theorem we still have to distinguish between different cases. For simplicity denote $G := \mathrm{End}(X)$ and $H := \mathrm{End}(Y)$.

(a) Suppose that kG and kH are semisimple. We are going to construct an infinite family $(V_\lambda)_{\lambda \in k^\star}$ of pairwise non-isomorphic indecomposable representations of $\mathcal{C}$. Let $\lambda \in k^\star$ and define $V_\lambda(X) = M$, $V_\lambda(Y) = N$ where M is a kG-module, N a kH-module, both having dimension at least 2. Furthermore we have to choose one linear map $M \to N$ which we want, for some fixed basis, to be given by the matrix

$$A_\lambda := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \lambda & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

where all the dots stand for zeros. For the choice of M and N we again have to distinguish different cases.

(i) Suppose that neither G nor H is abelian. Then we can choose M to be a simple kG-module and N to be a simple kH-module, both of dimension ≥ 2. Then, since M and N are indecomposable, it is clear that every V_λ is indecomposable. We will now show that for $\lambda \neq \mu$ we have $V_\lambda \ncong V_\mu$. To see that, suppose that $(\phi, \psi) : V_\lambda \to V_\mu$ is an isomorphism of $k\mathcal{C}$-modules. This implies that $\phi \in \mathrm{End}_{kG}(M) \cong k$ and $\psi \in \mathrm{End}_{kH}(N) \cong k$, which means $\phi = \alpha \cdot 1_M$ and $\psi = \beta \cdot 1_N$. In addition, ϕ and ψ have to be compatible with the action of the matrix A_λ (defined above). This gives the equations $\alpha = \beta$ and $\alpha\lambda = \beta\mu$ and hence $\alpha = \beta = 0$ which contradicts the assumption that (ϕ, ψ) is an isomorphism.

(ii) Suppose that H is non-abelian and G is abelian (the other way around is dual). Choose N as above and put $M = k^2$ with G-action given by the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where $a \neq b$ and both are non-zero. In other words we want M to be the direct sum of two non-isomorphic one-dimensional simple kG-modules. In this case we have that $\mathrm{End}_{kG}(M) = \left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ and we deduce that V_λ is indecomposable. As above we see that $V_\lambda \ncong V_\mu$ for $\lambda \neq \mu$.

(b) The last case that has to be treated (again by subdivision into different cases) is the case where one of the group algebras kG and kH is semisimple while the other is not and not both of them are abelian. We will deal with the case where kG is not semisimple and kH is semisimple, the other case can be proven analogously.

We may assume, that the Sylow p-subgroup D of G is cyclic ($p = \mathrm{char}(k)$), since otherwise kG and hence $k\mathcal{C}$ is of infinite representation type and we have nothing to prove. By standard results from representation theory of finite groups we can then choose an indecomposable kG-module M such that its restriction $M \downarrow_D$ has a p-dimensional direct summand on which D acts by the matrix

$$S = \begin{pmatrix} 1 & 1 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ & & & & & 1 \end{pmatrix}.$$

If now H is not abelian we choose, as above, a simple kH-module N with $\dim N \geq 2$ and for $\lambda \in k^\star$ we denote by A_λ the same matrix as above. Then $M \xrightarrow{A_\lambda} N$ is an indecomposable representation of $\mathcal{C}$. We should now show that $V_\lambda \ncong V_\mu$ for $\lambda \neq \mu$. Suppose that $((b_{i,j}), (c_{i,j}))$ is an isomorhpism of representations $V_\lambda \to V_\mu$. Then the matrix $(b_{i,j})$ has to commute with the G-action on M, in particular with a matrix of the shape $\left(\begin{smallmatrix} S & 0 \\ 0 & \star \end{smallmatrix}\right)$, where $\star$ is any matrix. This gives the conditions $b_{2,1} = 0$ and $b_{1,1} = b_{2,2}$. An endomorphism of N as a kH-module is just a scalar multiple of the identity, i.e. $(c_{i,j}) = c \cdot 1_N$. Finally, the following diagram has to commute.

$$\begin{tikzcd} M \arrow[r, "A_\lambda"] \arrow[d, "(b_{i,j})"'] & N \arrow[d, "c"] \\ M \arrow[r, "A_\mu"] & N \end{tikzcd}$$

This yields the conditions $b_{1,2} = 0$, $c = b_{1,1} = b_{2,2}$ and $\lambda \cdot c = \mu \cdot c$, which give that $c = 0$ and hence $V_\lambda \ncong V_\mu$. If the group H is abelian we replace the module N from above by a 2-dimensional kH-module which is the direct sum of two non-isomorphic one-dimensional simple kH-modules and get the claim by the same computations as we have just done.

To finish the proof we should consider the case where kG is semisimple and kH is not and not both are abelian. In this case the argument is the same as in the case we have treated above, only the computations are a little bit different. ☐

3.7 EI-category algebras with two simple modules

As we have seen, the classification of EI-categories of finite representation type gets very complicated, even with the assumption that the category has only two objects. The distinguishing mark for finite or infinite representation type seems to be the nature of the group actions of the automorphism groups on the morphism sets between distinct objects. In this section we will give a classification of all representation-finite EI-category algebras with only two simple modules. This work is motivated by work of Bongartz and Gabriel [7] who classified all representation-finite k-categories with two simples and radical of codimension 2. We will compute the Gabriel quiver of a given EI-category algebra with two objects and then use the list of Bongartz and Gabriel. For the convenience of the reader we collect those representation-finite and representation-infinite bound path algebras of quivers with two vertices that we will need later on. For a complete list one may consult [6, page 242].

Every algebra (given by quiver and relations), that is a quotient or dual to a quotient of an algebra in the following list is of finite representation type.

$$\begin{tikzcd} \circ \arrow[r, "\nu"] & \circ \arrow[loop right, "\rho"] \end{tikzcd} \qquad (1) \quad 0 = \rho^2\nu = \rho^5,$$

$$\begin{tikzcd} \circ \arrow[loop left, "\sigma"] \arrow[r, "\nu"] & \circ \arrow[loop right, "\rho"] \end{tikzcd} \qquad (2) \quad 0 = \nu\sigma = \rho\nu = \sigma^t = \rho^t,\ t \geq 2,$$

(3) $\quad 0 = \nu\sigma = \sigma^t = \rho^2,\ t \geq 2,$

(4) $\nu\sigma = \rho^2\nu,\ 0 = \sigma^2 = \rho^3,$

(5) $\nu\sigma = \rho\nu,\ 0 = \sigma^2 = \rho^t,\ t = 2r \geq 2,$

(6) $\nu\sigma = \rho\nu,\ 0 = \sigma^2 = \rho^5,$

$$\begin{tikzcd} \circ \arrow[r, shift left, "\nu"] & \circ \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho"] \end{tikzcd} \qquad \begin{array}{l} (7)\ 0 = \rho^3 = \gamma\rho = \gamma\nu = \nu\gamma, \\[1em] (8)\ 0 = \rho^2 = \nu\gamma = \gamma\rho\nu. \end{array}$$

As mentioned above, this list is not a complete list of representation-finite k-categories with two simples, but it is sufficient for our purposes. For every algebra from this list, one can prove representation-finiteness using covering theory in the way we have explained it in the beginning of this chapter.
Conversely, the algebras in the following list are all representation-infinite and they are in some sense minimal with that property. For details we refer to [6].

$$\begin{tikzcd} \circ \arrow[r, "\nu"] & \circ \arrow[loop right, "\rho"] \end{tikzcd} \qquad \begin{array}{l} (9)\ \ 0 = \rho^2\nu = \rho^6, \\ (10)\ 0 = \rho^3\nu = \rho^4, \end{array}$$

$$\begin{tikzcd} \circ \arrow[loop left, "\sigma"] \arrow[r, "\nu"] & \circ \arrow[loop right, "\rho"] \end{tikzcd} \qquad \begin{array}{l} (11)\ 0 = \nu\sigma = \sigma^2 = \rho^2\nu = \rho^4, \\ (12)\ 0 = \nu\sigma = \sigma^3 = \rho^2\nu = \rho^3. \end{array}$$

Since we want to decide how many simples an EI-category algebra $k\mathcal{C}$ has and the simples of $k\mathcal{C}$ are given by the simple modules over the group algebras $k\operatorname{Aut}(x)$ for $x \in \operatorname{Ob}\mathcal{C}$, the following well-known result from representation theory of finite groups is useful.

Lemma 3.36 (see for example [2])**.** *Let G be a finite group and k an algebraically closed field of characteristic p. Then the number of simple kG-modules equals the number of conjugacy classes of elements in G whose order is not divisible by p.*

The elements of a finite group G of order not divisible by some prime p are called *p-regular*, the remaining ones are called *p-singular*. Every element of G can be written as a product of a p-singular and a p-regular element. Using this fact, we observe the following easy but important statement.

Corollary 3.37. *Let G be a finite group such that the group algebra kG has only one simple module. Then G is either the trivial group or a p-group for $p = \operatorname{char}(k)$.*

Proof. From the lemma we know that G has only one p-regular element, namely the unit 1_G. If p does not divide the order of G, then if follows that $G = \{ 1_G \}$. Suppose that $p \mid |G|$ and let $x \in G$ be any element. Then we write $x = zy$ as a product of a p-regular element z and a p-singular element y. By assumption we have $z = 1_G$ and therefore x is p-singular. Hence, every element has order divisible by p and G is a p-group. $\square$

This corollary together with the list from above implies the following result.

Theorem 3.38. *Let $\mathcal{C}$ be a skeletal EI-category and k an algebraically closed field such that $k\mathcal{C}$ has two simple modules. Then $k\mathcal{C}$ is of finite representation type if and only if it satisfies one of the following conditions.*

(1) *$\mathcal{C}$ has one object x, the group $\mathrm{Aut}(x) = \mathrm{Mor}\,\mathcal{C}$ has two conjugacy classes of p-regular elements and the Sylow p-subgroup of G is cyclic.*
(2) *$\mathcal{C}$ has two objects x and y, the natural action of the group $\mathrm{Aut}(x) \times \mathrm{Aut}(y)$ on $\mathcal{C}(x, y)$ has at most one orbit, the Sylow p-subgroups of $\mathrm{Aut}(x)$ and $\mathrm{Aut}(y)$ are cyclic or the groups are trivial and one of the follwing conditions holds.*
    (*a*) *$\mathcal{C}(x, y) = \emptyset$;*
    (*b*) *$|\mathrm{Aut}(x)| \cdot |\mathrm{Aut}(y)| \leq 3$;*
    (*c*) *$|\mathrm{Aut}(x)| \cdot |\mathrm{Aut}(y)| = 4$ and $|\mathcal{C}(x, y)| \leq 2$;*
    (*d*) *$|\mathrm{Aut}(x)| \cdot |\mathrm{Aut}(y)| \geq 5$ and $|\mathcal{C}(x, y)| = 1$.*

Proof. $\mathcal{C}$ has at most two objects since every object gives at least one simple $k\mathcal{C}$-module. If $\mathcal{C}$ has only one object, then $k\mathcal{C}$ is a group algebra and, if it is representation-finite with only two simples, it has to satisfy condition (1).

Suppose that $\mathcal{C}$ has two objects x and y. The assumption that $k\mathcal{C}$ has two simples implies that both $k\,\mathrm{Aut}(x)$ and $k\,\mathrm{Aut}(y)$ have one simple module. Hence, they are either trivial or p-groups. In case of two trivial automorphism groups, $\mathcal{C}$ is the path category of the Dynkin quiver A_2 which is representation-finite. If one of $\mathrm{Aut}(x)$ and $\mathrm{Aut}(y)$ is a p-group it has to be representation-finite which means that it is a cyclic p-group. Assume that $\mathrm{Aut}(x)$ is a cyclic p-group and $\mathrm{Aut}(y)$ is trivial. Then the computation of the Gabriel-quiver of $k\mathcal{C}$ yields, that $k\mathcal{C}$ is isomorphic to the following path algebra with relations or its dual:

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & \circ \end{tikzcd}, \qquad \alpha^m = 0 = \beta\alpha^n,\ m = p^r,\ n \mid m.$$

According to the Bongartz-Gabriel list, this algebra is representation-finite only for the following values of m and n

- $m = 2$, $n = 1, 2$;
- $m = 3$, $n = 1, 3$;
- $m = 4$, $n = 1, 2$;
- $m \geq 5$, $n = 1$.

Any of these cases fulfils one of the conditions from (2).
Analogously, we assume that both $\mathrm{Aut}(x)$ and $\mathrm{Aut}(y)$ are cyclic p-groups. In this case $k\mathcal{C}$ is isomorphic to the following path algebra with relations or its dual:

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & \circ \arrow[loop right, "\gamma"] \end{tikzcd}, \qquad \gamma^t = \alpha^m = 0 = \gamma^s\beta = \beta\alpha^n,\ m = p^r,\ t = p^l,\ n \mid m,\ s \mid t.$$

Again we consult the list of Bongartz-Gabriel and find that (up to duality) only the following values for m, n, s and t give a representation-finite algebra:

- $m = 2 = s$ and $n = 1, t = 2$ or $n = 2, t = 1$ or $n = t = 1$;
- $m, s \geq 3$ and $n = 1 = t$.

Again this fits into our assertion and no other cases can occur, which finishes the proof. □

Remark 3.39. For the representation-finite EI-categories with two simples one can compute the Auslander-Reiten quiver, since they are either hereditary of Dynkin type or occur in the list of Bongartz-Gabriel. For the latter case one again uses covering-theory to knit the Auslander-Reiten quiver of the covering and then pushes everything down to the algebra itself.

3.8 Two objects and cyclic automorphism groups

The last special case of EI-categories we will consider is the one of EI-categories $\mathcal{C}$ with two objects x and y such that $\operatorname{Aut}(x)$ is a cyclic p-group and $\operatorname{Aut}(y)$ is a cyclic q-group for two distinct primes p and q. For this class of EI-categories the characterization of finite representation type can be obtained in the same way as for EI-category algebras with two simples modules. We will only treat the case where one group algebra is semisimple and the other is not. The case with two semisimple group algebras is rather trivial.

Proposition 3.40. *Let $\mathcal{C}$ be an EI-category with two non-isomorphic objects x and y such that $G := \operatorname{Aut}(x)$ is a cylic p-group and $H := \operatorname{Aut}(y)$ is a cyclic q-group for two distinct primes p and q. Let k be an algebraically closed field of characteristic p. Then $k\mathcal{C}$ is representation-finite if and only if $\mathcal{C}(x, y)$ (or $\mathcal{C}(y, x)$) consists of one $G \times H$-orbit and $\mathcal{C}$ or its dual satisfies one of the following conditions.*

1. *H acts trivially on $\mathcal{C}(x, y)$ and the category $\mathcal{C}' = \mathcal{C}/\operatorname{Aut}(y)$ ('trivialize' all endomorphisms of y) is representation-finite with two simples;*
2. *$q = 2$, $\mathcal{C}(x, y) = \{ f_1, f_2 \}$ and G acts trivially on $\mathcal{C}(x, y)$;*
3. *$|\mathcal{C}(x, y)| = 1$, i.e. both G and H act trivially on $\mathcal{C}(x, y)$.*

Proof. The proof works in the same fashion as the proof in the last section. Therefore, we will not provide many details here. First of all, if H acts trivially, one computes the quiver of $k\mathcal{C}$ to be the union of

$$Q : \alpha \circlearrowright \circ \xrightarrow{\beta} \circ$$

with $|H| - 1$ isolated points. Hence, the representation type depends on Q and the relations for which this quiver is representation-finite have been listed in the previous section.

If H does not act trivially we get more arrows from left to right in the quiver of $k\mathcal{C}$. If this are more than three, the universal cover contains the 4-subspace quiver as a relation-free part, hence $k\mathcal{C}$ is representation-infinite. The remaining representation-finite cases are easily seen to be conditions (2) and (3). □

Remark 3.41. In the case of EI-categories with only two simple representations the representation type was really governed by the size of the group algebras. If the product of their orders is big enough, only trivial actions yield finite type. In contrast to this, we could take H to be arbitrarily big and still have that condition (2) from the proposition is satisfied.

4 The finitistic dimension of EI-category algebras

The finitistic dimensions of a ring Λ provide a measure for the complexity of the module category of Λ. They are defined as

$$\begin{aligned}
\operatorname{fin.dim}(\Lambda) &= \sup\{\operatorname{proj.dim} M \mid M \in \operatorname{mod}(\Lambda), \operatorname{proj.dim} M < \infty\}, \\
\operatorname{Fin.dim}(\Lambda) &= \sup\{\operatorname{proj.dim} M \mid M \in \operatorname{Mod}(\Lambda), \operatorname{proj.dim} M < \infty\}.
\end{aligned}$$

There are at least two canonical questions that arise in studying these invariants, namely: Are these two dimensions finite for any ring Λ and do they coincide? For noetherian rings both questions have to be answered in the negative, but for finite-dimensional algebras there is no counterexample up to now. In 1960 Bass published the two questions for finite-dimensional algebras as "problems" and they are nowadays known as the finitistic dimension conjectures.

The little finitistic dimension, fin. dim, is known to be finite for certain classes of algebras, for example for algebras with representation dimension at most 3, monomial algebras or algebras with radical cube zero. One may consult [37] for a survey on this conjecture and other homological conjectures (not including the result of Igusa and Todorov from [23] concerning the relation of the representation dimension and the finitistic dimension).

In [26] Lück gave an explicit upper bound for the finitistic dimension of an EI-category algebra. This result seems not to be well-known even among specialists and we will give a new and fairly elementary proof of this result in the language of representation theory. To be able to do this, we use results of Xu [33] on the structure of projective resolutions of modules over EI-category algebras, which we will briefly recall and prove in the beginning of this chapter.

The main theorem of this chapter is the following.

Theorem 4.1 (Lück, [26, Proposition 17.31]). *Let $\mathcal{C}$ be an EI-category and $k\mathcal{C}$ its associated unital k-algebra. Then the global dimension of $k\mathcal{C}$ is finite if and only if $|\operatorname{Aut}(x)|$ is invertible in k for any $x \in \operatorname{Ob}\mathcal{C}$ and*

$$\operatorname{Fin.dim}(k\mathcal{C}) \leq \ell(\mathcal{C}),$$

where $\ell(\mathcal{C})$ is the maximal length of a chain of non-isomorphisms in $\mathcal{C}$.

For the proof we need some preparations. The following characterization of projective resolutions of $k\mathcal{C}$-modules is the main tool to compute the finitistic dimension of EI-category algebras.

Lemma 4.2 (Xu, [33] Lemma 4.1.1)**.** *Let M be a $k\mathcal{C}$-module and P_M its projective cover. Then P_M is supported on $\mathcal{C}_M$ and for any M-minimal object x the module $P(x)$ is the projective cover of $M(x)$.*

Proof. By the characterization of the indecomposable projective modules and by definition of $\mathcal{C}_M$ it is clear that the support of P_M is contained in $\mathcal{C}_M$. Now let x be an M-minimal object in $\mathcal{C}$. The full subcategory $\mathcal{C}_{\leq x}$ is an ideal in $\mathcal{C}$ and its intersection with $\mathcal{C}_M$ is just x. Now by Proposition 2.18 the $k\,\mathrm{Aut}(x)$-module $P_M(x)$ is a projective module and admits a surjection onto $M(x)$. Thus, what remains to be shown is the minimality of $P_M(x)$. If $P_M(x)$ would not be the projective cover of $M(x)$, then by the universal property of the projective cover, there would exist projective modules P_1 and P_2 such that $P_M = P_1 \oplus P_2$ with $P_1(x)$ being the projective cover of $M(x)$ and $P_2(x) \neq 0$ such that, if $\pi : P_M \to M$ is the defining essential epimorphism, then $\pi\downarrow^{\mathcal{C}}_{\mathcal{D}}$ sends $P_2(x)$ to zero. The module P_2 has an indecomposable projective direct summand P_2'. Now, since $P_2'(x) = k\,\mathrm{Aut}(x) \cdot e$ for some primitive idempotent $e \in k\,\mathrm{Aut}(x)$ and $P_2' = k\mathcal{C} \cdot e$, it follows that π sends P_2' to zero. This is a contradiction to the minimality of P_M. □

This Lemma gives us the following description of the minimal projective resolution of a $k\mathcal{C}$-module M.

Corollary 4.3. *Let M be a $k\mathcal{C}$-module and $\mathcal{P}_M$ a minimal projective resolution. Then $\mathcal{P}_M$ is supported on $\mathcal{C}_M$ and for any M-minimal $x \in \mathrm{Ob}\,\mathcal{C}$ we have that $\mathcal{P}_M(x)$ is a minimal projective resolution of $M(x)$.*

With these preparations we are now in the position to prove Theorem 4.1.

Proof of Theorem 4.1. Let M be a $k\mathcal{C}$-module which is of finite projective dimension and consider a minimal projective resolution

$$\mathcal{P}_M : \quad 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Then $\mathcal{P}_M$ is supported on $\mathcal{C}_M$ as we have seen above. Now let x be an M-minimal object in $\mathcal{C}$. By Corollary 4.3 $\mathcal{P}_M(x)$ is a minimal projective resolution of $M(x)$ as $k\,\mathrm{Aut}(x)$-module. As a module over a group algebra of a finite group, the module $M(x)$ is either projective or of infinite projective dimension. The latter case is impossible since $\mathcal{P}_M$ is a finite resolution. This implies that $P_1(x) = 0$ and P_1 is supported on $\mathcal{C}_M \setminus \{M\text{-minimal objects}\}$. Applying this argument inductively to any of the P_i, we get that $n \leq \ell(\mathcal{C})$ and hence the claim. □

Finally we will present some examples to illustrate this result.

Example 4.4. (1) Let $\mathcal{C}$ be an EI category with only one object. Then $k\mathcal{C}$ is the group algebra of a finite group G. For this case it is well known that the finitistic dimension is zero and $\ell(\mathcal{C})$ equals zero as well.

(2) Suppose that $\mathcal{C}$ is the path category of a finite quiver without oriented cycles, then $k\mathcal{C}$ is hereditary and therefore $\operatorname{fin.dim}(k\mathcal{C}) = \operatorname{gl.dim}(k\mathcal{C}) \leq 1$. Thus, in this case the given bound is not optimal.

(3) Let

$$\mathcal{C}: \begin{tikzcd} X \arrow[loop left, "g"] \arrow[r, "f"] & Y \arrow[loop right, "1_Y"] \end{tikzcd}$$

be the EI-category given by the relations $g^2 = 1_X$ and $fg = f$. Further, suppose that k is a field of characteristic 2. Then the indecomposable projective representations of $\mathcal{C}$ are exactly

$$P_X: \begin{tikzcd} k^2 \arrow[loop left, "A"] \arrow[r, "(1\ 1)"] & k \arrow[loop right, "(1)"] \end{tikzcd} \text{ where } A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \text{ and}$$

$$P_Y: \begin{tikzcd} 0 \arrow[loop left, "(0)"] \arrow[r, "(0)"] & k \arrow[loop right, "(1)"] \end{tikzcd}.$$

Now it is clear that this algebra is of infinite global dimension since the group algebra $k\operatorname{Aut}(x)$ is not semisimple. Its finitistic dimension equals 1 since it is easy to see, that the representation

$$M: \begin{tikzcd} k^2 \arrow[loop left, "A"] \arrow[r, "(0)"] & 0 \arrow[loop right, "(0)"] \end{tikzcd} \text{ with } A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

has projective dimension 1 and there is no module with projective dimension greater than one (if the projective dimension is finite) by our theorem.

5 The finitistic dimension of algebras with a directed stratification

In this chapter we introduce the notion of a directed stratification for a finite-dimensional algebra A. This definition is inspired by the study of EI-category algebras, where we have already seen that the finitistic dimension is always finite. The proof of this finiteness of fin. dim for EI-category algebras reduces the problem to the finitistic dimension of the group algebras of the automorphism groups. This concept will be generalized to algebras with a directed stratification in the second subsection, where we show that finiteness of the finitistic dimension of such an algebra only depends on the finitistic dimensions of the strata. As a matter of fact, our reduction technique is a corollary (using induction) of a much more general theorem of Fossum, Griffith and Reiten in [14], which they obtained in the context of trivial extensions of abelian categories. We will briefly present their result in the first section of this chapter. Nevertheless, the approach we will present in terms of representations of a category which is associated to an algebra with a directed stratification gives a very convenient combinatorial description of the projective resolutions of the modules. Furthermore, our reduction technique will prove to be of broader applicability for the computation of concrete examples. This reduction reduces the finitistic dimension conjecture to the class of algebras, which are minimal in the sense that they do not admit a non-trivial directed stratification. We will give a combinatorial description of these algebras in terms of their Gabriel-quiver. Finally, in the last section, we relate our result to other known results for the finitistic dimension, for example to results of Happel [20], Cline, Parshall and Scott [9–11] and Huisgen-Zimmermann [36].

5.1 Trivial extensions of abelian categories and finitistic dimension

As mentioned in the introduction of this chapter, Fossum, Griffith and Reiten developed the theory of trivial extensions of abelian categories and derived remarkably beautiful and general results on the finitistic dimension. Let us begin with the definition of trivial extensions.

Definition 5.1. Let $\mathcal{A}$ be an abelian category and $F : \mathcal{A} \to \mathcal{A}$ an additive endofunctor of $\mathcal{A}$. We construct new additive categories $F \rtimes \mathcal{A}$ and $\mathcal{A} \ltimes F$ as follows.

The objects of $\mathcal{A} \ltimes F$ are morphisms $\alpha : FA \to A$ for some $A \in \operatorname{Ob}\mathcal{A}$ such that $\alpha \circ F\alpha = 0$. Let $\alpha : FA \to A$ and $\beta : FB \to B$ be objects in $\mathcal{A} \ltimes F$, then a morphism $\gamma : \alpha \to \beta$ is a morphism $\gamma : A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} FA \arrow[r, "F\gamma"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\ A \arrow[r, "\gamma"] & B \end{tikzcd}$$

The composition in $\mathcal{A} \ltimes F$ is just composition in $\mathcal{A}$.
Similarly we define the category $F \rtimes \mathcal{A}$ with objects $\alpha : A \to FA$ and morphisms defined in an analogous way as for $\mathcal{A} \ltimes F$.

Example 5.2. (i) Suppose R is a ring and M an R-bimodule. The category $\mathcal{A} = \operatorname{Mod} R$ is abelian and we have at least two natural functors associated with M, namely the tensor product $F = M \otimes_R -$ and the internal Hom $G = \operatorname{Hom}_R(M, -)$. These two functors give possibilities to define trivial extensions of R by M. One can also define it to be the ring whose additive group is the direct sum $R \oplus M$ with multiplication

$$(r, m) \cdot (r', m') = (rr', mr' + rm').$$

Denote this ring by $R \ltimes M$ (or $M \rtimes R$). One can show that $G \rtimes \mathcal{A}$, $\mathcal{A} \ltimes F$ and $\operatorname{Mod}(R \ltimes M)$ are isomorphic.

(ii) Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. The category $\operatorname{Map}(F\mathcal{A}, \mathcal{B})$ is the category whose objects are triples (A, f, B) where $A \in \operatorname{Ob}\mathcal{A}, B \in \operatorname{Ob}\mathcal{B}$ and $f : FA \to B$. The morphisms are pairs (α, β) of morphisms in $\mathcal{A} \times \mathcal{B}$ such that the following diagram commutes.

$$\begin{tikzcd} FA \arrow[r, "F\alpha"] \arrow[d, "f"'] & FA' \arrow[d, "f'"] \\ B \arrow[r, "\beta"] & B' \end{tikzcd}$$

The functor F induces a functor $\widetilde{F} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B}$ by $\widetilde{F}(A, B) = (0, FA)$ and $\widetilde{F}(\alpha, \beta) = (0, F\alpha)$ and the categories $\operatorname{Map}(F\mathcal{A}, \mathcal{B})$ and $(\mathcal{A} \times \mathcal{B}) \ltimes \widetilde{F}$ are isomorphic.

In the case of the second example Fossum, Griffith and Reiten obtained the following result.

Theorem 5.3 (Fossum, Griffith, Reiten, [14])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ a right exact functor. Let $\mathcal{M} := \operatorname{Map}(F\mathcal{A}, \mathcal{B})$. Then the following inequalities hold.*

(1) $\operatorname{Fin.dim}\mathcal{B} \leq \operatorname{Fin.dim}\mathcal{M} \leq 1 + \operatorname{Fin.dim}\mathcal{A} + \operatorname{Fin.dim}\mathcal{B}$,

(2) *If F is exact, then* $\operatorname{Fin.dim}\mathcal{M} \geq \operatorname{Fin.dim}\mathcal{A}$,
(3) $\max(\operatorname{gl.dim}\mathcal{A}, \operatorname{gl.dim}\mathcal{B}) \leq \operatorname{gl.dim}\mathcal{M} \leq 1 + \operatorname{gl.dim}\mathcal{A} + \operatorname{gl.dim}\mathcal{B}$.

In particular, this result applies to the setting of triangular matrix algebras: Let R and S be rings and M an R-S-bimodule. The associated triangular matrix algebra is defined as $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$. This is related to the trivial extensions and the result above, since here we have $\operatorname{Mod}\Lambda \cong \operatorname{Map}(F \operatorname{Mod} R, \operatorname{Mod} S)$, where $F = M \otimes_R -$. The corollary is the following.

Corollary 5.4. *Let R, S, M and Λ be as above, $M \neq 0$. Then*
(1) $\operatorname{Fin.dim} S \leq \operatorname{Fin.dim}\Lambda \leq 1 + \operatorname{Fin.dim} R + \operatorname{Fin.dim} S$,
(2) *If M is a flat R-module, then* $\operatorname{Fin.dim}\Lambda \geq \operatorname{Fin.dim} R$,
(3) $\max(\operatorname{gl.dim} R, \operatorname{gl.dim} S, \operatorname{proj.dim}_S M + 1) \leq \operatorname{gl.dim}\Lambda$
and $\operatorname{gl.dim}\Lambda \leq \max(\operatorname{gl.dim} R + \operatorname{proj.dim}_S M + 1, \operatorname{gl.dim} S)$.

Now, as another special case of this corollary, we get the following lemma.

Lemma 5.5 (Fossum, Griffith, Reiten [14] and Fuller, Saorin [15])**.** *Let A be any ring and e, f two non-zero idempotents in A with $1 = e + f$ and $eAf = 0$. Then*

$$\max(\operatorname{gl.dim} eAe, \operatorname{gl.dim} fAf) \leq \operatorname{gl.dim} A \leq \operatorname{gl.dim} eAe + \operatorname{gl.dim} fAf + 1,$$

and the same inequalities hold for $\operatorname{Fin.dim}$.

This result can be used inductively to obtain our result for algebras which admit what we call a directed stratification. We will see in the following sections that our approach will have the advantage that we gain new insight into the structure of the projective resolutions.

5.2 Basic notions and properties

In [11] Cline, Parshall and Scott introduced the notion of a stratifying ideal as well as the notion of stratified and standardly stratified algebras. Since our class of algebras fits into this framework we will recall their definitions.

Definition 5.6. An ideal J in an algebra A is called *stratifying* if the following conditions are satisfied.
(i) $J = AeA$ for some idempotent $e \in A$,
(ii) Multiplication induces an isomorphism $Ae \otimes_{eAe} eA \xrightarrow{\sim} J$,
(iii) $\operatorname{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n > 0$.

It was also observed by Cline, Parshall and Scott that an ideal J in A is stratifying if and only if the derived functor $i_\star : \mathbf{D}^+(A/J) \to \mathbf{D}^+(A)$ induced by the exact inflation functor $i_\star : \operatorname{mod} A/J \to \operatorname{mod} A$ is a full embedding.

A *stratification of A of length n* is a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

of ideals with the property that J_i/J_{i-1} is a stratifying ideal in A/J_{i-1}. The stratification is called (left-)*standard* if J_i/J_{i-1} is projective (as left A/J_{i-1}-module).

The class of algebras which we will define and deal with in this chapter somehow sits between stratified algebras and standardly stratified algebras. We will make this precise in Remark 5.10.

Definition 5.7. Let A be a finite-dimensional algebra over some field k. Then we say that A has a *directed stratification of length n* if there exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ in A with $\sum_{i=1}^{n} e_i = 1_A$ such that $e_i A e_j = 0$ for all $i < j$.

One should note that we do not require the idempotents to be primitive. It is clear that every algebra admits a directed stratification of length 1 given by its identity element, but in this case the theory we will develop will give us nothing new.

Example 5.8. (1) Let $\mathcal{C}$ be a finite and skeletal EI-category with n objects and $A = k\mathcal{C}$ its category algebra. Then we have a partial order defined on the set of objects of $\mathcal{C}$ which gives us a directed stratification of length n given by the idempotents 1_{X_i}, where $X_i, i = 1, \ldots, n$ are the objects of $\mathcal{C}$ and the numbering respects the partial order.

(2) Let Q be any finite quiver without oriented cycles and I any admissible ideal in kQ. Then $A = kQ/I$ admits a directed stratification of length $|Q_0|$ given by the primitive idempotents ε_i with a suitable numbering.

We will see more examples in the last part of this chapter.

As a matter of fact, one can identify the module category of an algebra A with a directed stratification given by $e_1, \ldots, e_n$ with the category of representations of a certain category $\mathcal{A}$ which we will define now.

Definition 5.9. Let A be as above. Then the associated category $\mathcal{A}$ is defined as follows. The objects $x_1, \ldots, x_n$ of $\mathcal{A}$ are in bijective correspondence with the idempotents $e_1, \ldots, e_n$ that define our stratification and the morphisms $x_i \to x_j$ are in bijective correspondence with a k-basis of $e_j A e_i$. Under the assumption that A is a finite-dimensional algebra, the category $\mathcal{A}$ is finite.

By means of this definition, A is the category algebra $k\mathcal{A}$ of $\mathcal{A}$.

In this setting we have seen, that the categories $\operatorname{rep}_k(\mathcal{A})$ and $\operatorname{mod} A$ are equivalent. For this reason, we will switch frequently between the concepts of representations and modules without any further explanation. For instance for an A-module M we write $M(x_i)$ for its evaluation at the object x_i as a functor.

Remark 5.10. With the above characterization of A as the category algebra of $\mathcal{A}$, we may apply a result of Webb [32, Proposition 2.2] which almost immediately gives that any algebra with a directed stratification is also stratified in the sense of Cline, Parshall and Scott. Precisely, if A has a directed stratification given by $e_1, \dots, e_n$, then we take $J_i = A(\sum_{l=n-i}^{n} e_l)A$. These are indeed stratifying ideals by Webbs theorem and therefore give a stratification of A of length n. Another result of Webb [32, Theorem 2.5] characterizes the standardly stratified EI-category algebras to be exactly those, that are given by an EI category $\mathcal{C}$ in which for every morphism $\alpha : x \to y$ the group $\mathrm{Stab}_{\mathrm{Aut}(y)}(\alpha) = \{ \theta \in \mathrm{Aut}(y) \mid \theta\alpha = \alpha \}$ has order invertible in k. Therefore, we may for instance take the category algebra $k\mathcal{C}$ of the following EI category

$$\mathcal{C}:\ \begin{tikzcd} y \arrow[loop left, "1_y"] \arrow[r, "\alpha"] & x \arrow[loop above, "1_x"] \arrow[loop below, "h"] \end{tikzcd},\quad h^2 = 1_x,\ h\alpha = \alpha.$$

If k has characteristic 2, then the category algebra $k\mathcal{C}$ is not standardly stratified but it clearly admits a directed stratification given by the idempotents $1_x, 1_y$.

Hence, an algebra with a directed stratification is always stratified but in general not standardly stratified. This is an interesting point since the finitistic dimension conjecture is known to hold for standardly stratified algebras by work of Ágoston, Happel, Lukács and Unger [1] while it is still open for stratified algebras.

We can describe the algebras A which do not admit a non-trivial directed stratification (i.e. of length ≥ 2) in a very convenient way by conditions that should be rather easy to check in concrete examples. This characterization is given by the following Proposition.

Proposition 5.11. *Let A be a finite-dimensional k-algebra, where k is any field. Denote by Q its Gabriel-quiver. Then A does admit a non-trivial directed stratification if and only if there exist disjoint subsets Q_0' and Q_0'' of Q_0 satisfying the following conditions.*

(1) $Q_0 = Q_0' \cup Q_0''$,
(2) *For any $i \in Q_0'$ and $j \in Q_0''$ there is no path from i to j in Q.*

Proof. First assume that A admits a directed stratification. We may without loss of generality assume that it has length 2 and hence is given by two idempotents e and f with $eAf = 0$. Denote by Γ the Gabriel-quiver of eAe and by Γ' the Gabriel quiver of fAf. Then we set $Q_0' = \Gamma_0$ and $Q_0'' = \Gamma_0'$. Since we have $1 = e + f$ condition (1) is satisfied and $eAf = 0$ implies the second condition.
For the converse implication assume that conditions (1) and (2) hold. Then put $e = \sum_{i \in Q_0'} e_i$ and $f = 1 - e = \sum_{j \in Q_0''} e_j$. By definition we have $1 = e + f$ and $eAf = 0$ follows from condition (2). □

Our main result in this chapter will then reduce the finitistic dimension conjecture to exactly the class of algebras mentioned in the proposition above.

The following result describes the simple and the projective A-modules if A admits a directed stratification. It is completely analogous to the one given by Lück in [26] for EI-category algebras and would again also follow from work of Auslander in [4].

Proposition 5.12. *Let A be an algebra with a directed stratification given by idempotents $e_1, \ldots, e_n$. Then for every simple A-module S one has $e_i S \neq 0$ for exactly one e_i. In other words as a representation S is supported on exactly one object X_i and $S(X_i)$ is a simple e_iAe_i-module. Their projective covers (i.e. all the indecomposable projective A-modules) are of the form Ae for some primitive idempotent $e \in e_iAe_i$ for some $i \in \{ 1, \ldots, n \}$.*

Proof. The assertion on the indecomposable projective modules is obvious since $1 = \sum_{i=1}^{n} e_i$. Then we decompose every e_i and infer that the summands have to be in e_iAe_i.

Let S be a simple A-module and choose e_i with $e_iS \neq 0$. Then consider the submodule U of S generated by e_iS. Since the idempotents $e_1, \ldots, e_n$ define a directed stratification we have $e_jU = 0$ whenever $j < i$. Let N be the submodule of S generated by all the e_jS with $j > i$. Again, since A has a directed stratification, it follows that $e_iN = 0$, which, together with the fact that S is simple, implies that $N = 0$ and therefore $e_jS = 0$ for every $j \neq i$. If e_iS would not be a simple e_iAe_i-module, then $S = e_iS$ would have a non-trivial submodule since it is itself an e_iAe_i-module. $\square$

With this Proposition it is natural to use the same notation as for EI-category algebras and denote the simple A-modules by $S_{x,V}$ where x is an object of $\mathcal{A}$ and V a simple e_xAe_x-module (here the idempotent e_x corresponds to the object x) and to let $P_{x,V}$ denote the projective cover of $S_{x,V}$.

The main tool to understand the structure of projective resolutions of modules over algebras with a directed stratifications will be the use of restriction functors as introduced in Chapter 2. We will use the same definitions as for EI-category algebras like ideals etc. and see that the whole theory can be carried over with slightly more complicated proofs. The following definition is completely analogous to the one for EI-categories.

Definition 5.13. Let A be an algebra with a directed stratification and let $\mathcal{A}$ be the associated finite category.

(1) Let x be an object in $\mathcal{A}$. Then we define $\mathcal{A}_{\leq x}$ to be the full subcategory of $\mathcal{A}$ consisting of all objects $y \in \operatorname{Ob} \mathcal{A}$ with $\mathcal{A}(y, x) \neq \emptyset$. Similarly we define $\mathcal{A}_{\geq x}$.

(2) An ideal in $\mathcal{A}$ is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that for any object x in $\mathcal{B}$ we have that $\mathcal{A}_{\leq x} \subseteq \mathcal{B}$.

(3) Let M be an A-module. The M-minimal objects are the objects $x \in \operatorname{Ob} \mathcal{A}$ such that $M(x) \neq \emptyset$ and for any $y \in \operatorname{Ob} \mathcal{A}$ with $\mathcal{A}(y, x) \neq \emptyset$ one has $M(y) = 0$.

(4) Let M again be an A-module. We put $\mathcal{A}_M$ to be the full subcategory consisting of all $y \in \operatorname{Ob}\mathcal{A}$ with $\mathcal{A}(x,y) \neq \emptyset$ for some M-minimal object x in $\mathcal{A}$.

Proposition 5.14. *Let A be an algebra with a directed stratification, $\mathcal{A}$ the associated category and $\mathcal{B}$ an ideal in $\mathcal{A}$. Then the restriction $\downarrow^{\mathcal{A}}_{\mathcal{B}}$ preserves projectives.*

Proof. We construct an exact right adjoint $F : \operatorname{rep}\mathcal{B} \to \operatorname{rep}\mathcal{A}$ of the restriction functor in the following way. For $\mathcal{M} \in \operatorname{rep}\mathcal{B}$ and any morphism $f : \mathcal{M} \to \mathcal{N}$ in $\operatorname{rep}\mathcal{B}$ let

$$F\mathcal{M}(x) = \begin{cases} \mathcal{M}(x) & \text{if } x \in \operatorname{Ob}\mathcal{B}, \\ 0 & \text{otherwise,} \end{cases} \qquad F(f)_x = \begin{cases} f_x & \text{if } x \in \operatorname{Ob}\mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

This defines an exact functor. Now let $\mathcal{M} \in \operatorname{rep}\mathcal{A}$ and $\mathcal{N} \in \operatorname{rep}\mathcal{B}$. Then we define a morphism $\Psi : \operatorname{Hom}_{\operatorname{rep}\mathcal{B}}(\mathcal{M}\downarrow^{\mathcal{A}}_{\mathcal{B}}, \mathcal{N}) \to \operatorname{Hom}_{\operatorname{rep}\mathcal{A}}(\mathcal{M}, F\mathcal{N})$ via

$$\Psi(f)_x = \begin{cases} f_x & \text{for } x \in \operatorname{Ob}\mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to $\mathcal{B}$ being an ideal, this gives a k-linear map which is easily seen to be an isomorphism. Therefore, we get that F is the desired exact right adjoint of the restriction. $\square$

5.3 Projective resolutions and the main result

In this section we will analyze the structure of projective resolutions for modules over algebras with a directed stratification. It will turn out, that they can be described in the same fashion as the ones for modules over EI-category algebras, only the proofs become a little bit more involved.

Theorem 5.15. *Let A be an algebra with a directed stratification and $\mathcal{A}$ the associated category. Let M be an A-module and $P = P_M$ its projective cover. Then P is supported on $\mathcal{A}_M$ and for any M-minimal object x in $\mathcal{A}$ the module $P(x)$ is a projective cover of $M(x)$ as an e_xAe_x-module.*

Proof. (i) Clearly, we have $P = \bigoplus_{y,U} P_{y,U}$ for some objects y in $\mathcal{A}$ and simple e_yAe_y-modules U. What we have to show is that no y' with $y' \notin \mathcal{A}_M$ appears in that direct sum. Let us assume the contrary and suppose that there is an object y' with $\mathcal{A}(y',x) \neq 0$ for an M-minimal object x that appears in the direct sum decomposition of P. Then, for any x with $\mathcal{A}(y',x) \neq 0$ and any $f \in \mathcal{A}(y',x)$ the following diagram has to commute

$$\begin{tikzcd} P_{y',U'}(x) \arrow[r, "\pi_x"] & M(x) \\ P_{y',U'}(y') \arrow[u, "P_{y',U'}(f)"] \arrow[r, "\pi_{y'}"] & M(y') = 0, \arrow[u] \end{tikzcd}$$

where $\pi : P \to M$ is the defining essential epimorphism. By the characterization of the projective A-modules we have $\sum_f \operatorname{Im} P_{y',U'}(f) = P_{y',U'}(x)$, which gives that $\pi_x = 0$ (for any such x). This is a contradiction to the minimality of P and we have proven the first assertion.

(ii) Since $\mathcal{A}_{\leq x}$ is an ideal and $\mathcal{A}_{\leq x} \cap \mathcal{A}_{\mathcal{M}} = \{x\}$, it follows from Proposition 5.14 that $P(x)$ is projective. Thus, we only have to show that $P(x)$ is the projective cover of $M(x)$.
Suppose $P(x)$ would not be the projective cover of $M(x)$. Then $P(x) = Q' \oplus Q''$ where Q', Q'' are projective and Q' is the projective cover of $M(x)$, whereas $\pi_x(Q'') = 0$. Since x is M-minimal, we have that $Q' = P'(x)$ and $Q'' = P''(x)$ for some projective A-modules P' and P''. Denote again by π the defining essential epimorphism $P \to M$. Now, using that P is supported on $\mathcal{A}_M$ and a similar diagram as in the first part of our proof, we get that $\pi(P'') = 0$ which contradicts the minimality of P. □

Corollary 5.16. *Let A be an algebra with a directed stratification and $\mathcal{A}$ the associated category. Suppose that M is an A-module and $\mathcal{P}$ a minimal projective resolution of M. Then for any M-minimal object x in $\mathcal{A}$ we have that $\mathcal{P}(x)$ is a minimal projective resolution of $M(x)$ as an e_xAe_x module.*

With this characterization of projective resolutions of A-modules we get the following theorem, which, roughly speaking, states that the finitistic dimension of an algebra with a directed stratification is determined by the finitistic dimension of the strata.

Theorem 5.17. *Let A be an algebra with a directed stratification and $\mathcal{A}$ the associated category.*

(1) *A has finite finitistic dimension if and only if e_xAe_x has finite finitistic dimension for any object x of $\mathcal{A}$. In this case* $\operatorname{fin.dim} A \leq \sum_{x \in \operatorname{Ob} \mathcal{A}} \operatorname{fin.dim} e_xAe_x + |\operatorname{Ob} \mathcal{A}| - 1$.

(2) *A is of finite global dimension if and only if e_xAe_x is of finite global dimension for any object x of $\mathcal{A}$. In this case* $\operatorname{gl.dim} A \leq \sum_{x \in \operatorname{Ob} \mathcal{A}} \operatorname{gl.dim} e_xAe_x + |\operatorname{Ob} \mathcal{A}| - 1$.

Proof. We only prove part (1). The proof of (2) is completely analogous.

Let $e_1, \ldots, e_n$ be the idempotents, that give the directed stratification of A. First suppose that the algebra e_iAe_i has infinite finitistic dimension for some $i = 1, \ldots, n$. Then there exists an indecomposable e_iAe_i-module N of projective dimension at least d for any natural number $d \geq 1$. Let $\mathcal{P}$ be a minimal projective resolution of N as an A-module. By definition, the object x of $\mathcal{A}$ corresponding to e_i is N-minimal. Hence, $\mathcal{P}(x)$ is a minimal projective resolution of N as an e_iAe_i-module, which therefore is of length at least d. Thus, we infer that N regarded as an A-module has projective dimension at least d and A has infinite finitistic dimension.

Now assume that for $i = 1, \ldots, n$ every e_iAe_i has finite finitistic dimension and denote by x_i the object in $\mathcal{A}$ corresponding to e_i. Let M be an A-module of finite projective dimension. We consider a minimal projective resolution of M:

$$\mathcal{P} : 0 \to P^m \to P^{m-1} \to \cdots \to P^1 \to P^0 \to M \to 0.$$

The object $x_1 \in \operatorname{Ob} \mathcal{A}$ is M-minimal for every A-module M. Therefore, $\mathcal{P}(x_1)$ is a minimal projective resolution of $M(x_1)$ as e_1Ae_1-module. Hence, $P^d(x_1) = 0$ for all $d > \operatorname{fin.dim} e_1Ae_1$. Denote by s the largest integer for which $P^s(x) \neq 0$. Then, for any $d > s$, the module P^d is supported on $\operatorname{Ob} \mathcal{A} \setminus \{x_1\}$ and the object x_2 is N-minimal, where $N = \ker(P^s \to P^{s-1})$. With the same argument as above we infer that $P^{s+d}(x_2) = 0$ for all $d > \operatorname{fin.dim} e_2Ae_2$. Now the claimed inequality follows by induction. □

The following corollary is equivalent to the theorem from above.

Corollary 5.18. *Let A be a finite-dimensional k-algebra with a directed stratification of length 2 given by idempotents $e, f \in A$ (i.e. $1 = e + f$ and $eAf = 0$). Then the following statements hold.*

(1) *A has finite finitistic dimension if and only if eAe and fAf have finite finitistic dimension. In this case* $\operatorname{fin.dim} A \leq \operatorname{fin.dim} eAe + \operatorname{fin.dim} fAf + 1$.

(2) *A has finite global dimension if and only if eAe and fAf have finite global dimension. In this case* $\operatorname{gl.dim} A \leq \operatorname{gl.dim} eAe + \operatorname{gl.dim} fAf + 1$.

Remark 5.19. (i) As we have seen in the first part of this chapter, Corollary 5.18 has already been obtained by Fossum, Griffith and Reiten and it also implies our theorem by using induction. Nevertheless, our proof is different and provides us with interesting new information about the structure of projective resolutions of A-modules, if A has a directed stratification. Furthermore we will see that the iterated version of Corollary 5.18 can easily be applied in examples that have not been studied so far.

(ii) Another immediate corollary of the theorem is the following well-known fact: Let Q be a finite quiver without oriented cycles and I any admissible ideal in kQ. Then the algebra kQ/I has finite global dimension. The theorem applies to this setting in the way that we take the natural directed stratification given by the primitive idempotents ε_i in a suitable numbering. Then every stratum is just the ground field k which has finite global dimension.

One interpretation of our result from above is to understand it as a technique to reduce the finitistic dimension conjecture to a smaller class of algebras, namely those that do not admit a non-trivial directed stratification. These algebras have been characterized in Proposition 5.11.

5.4 Relation to known results and examples

5.4.1 Relation to recollements

In [20] Happel developed a reduction technique for the finitistic dimension conjecture (and other homological conjectures) using recollements of the bounded derived categories. To explain the relation of this result to our situation we first recall the definition of a recollement.

Definition 5.20. Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be triangulated categories. Then a recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is given by six exact functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_\star = i_!" description] & \mathcal{C} \arrow[l, bend right, "i^\star"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^\star = j^!" description] & \mathcal{C}'', \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_\star"] \end{tikzcd}$$

satisfying the following conditions.

(R1) $(i^\star, i_\star), (i_!, i^!), (j_!, j^!)$ and $(j^\star, j_\star)$ are adjoint pairs of exact functors,

(R2) $j^\star i_\star = 0$,

(R3) $i^\star i_\star \cong \mathrm{id}, \mathrm{id} \cong i^! i_!, j^\star j_\star \cong \mathrm{id}$ and $\mathrm{id} \cong j^! j_!$,

(R4) for any $X \in \mathcal{C}$ there exist triangles

$$j_! j^! X \to X \to i_\star i^\star X \to j_! j^! X[1]$$

$$i_! i^! X \to X \to j_\star j^\star X \to i_! i^! X[1].$$

With this concept of recollements Happel obtained the following result.

Theorem 5.21 (Happel, [20])**.** *Let A be a finite-dimensional algebra and assume that $\mathbf{D}^b(A)$ has a recollement relative to $\mathbf{D}^b(A')$ and $\mathbf{D}^b(A'')$ for some finite-dimensional algebras A' and A''. Then* $\operatorname{fin.dim} A < \infty$ *if and only if* $\operatorname{fin.dim} A' < \infty$ *and* $\operatorname{fin.dim} A'' < \infty$.

The structure of this result is similar to that of Corollary 5.18. Therefore, it is a natural question to ask if the two reduction techniques are equivalent. To see that this is not the case we have to translate the setting of algebras with a directed stratification into the language of triangulated categories. Clearly, the triangulated categories that will appear are the (bounded) derived module categories of the algebra itself and the algebras $e_i A e_i$.

Let A be a finite-dimensional algebra with a directed stratification of length 2 given by idempotents e and f with $eAf = 0$. Then $J := AfA$ is a stratifying ideal of A. With $B = A/J$ it is clear that $B \cong eAe$. Following [11], we have partial recollement diagrams

$$\begin{tikzcd} \mathbf{D}^+(eAe) \arrow[r, shift left, "i_\star"] & \mathbf{D}^+(A) \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^\star"] & D\mathbf{D}^+(fAf) \arrow[l, shift left, "j_\star"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{D}^-(eAe) \arrow[r, shift right, "i_\star"'] & \mathbf{D}^-(A) \arrow[l, shift right, "i^\star"'] \arrow[r, shift right, "j^\star"'] & \mathbf{D}^-(fAf). \arrow[l, shift right, "j_!"'] \end{tikzcd}$$

If all the algebras involved have finite global dimension we get a full recollement of the bounded derived categories. Cline, Parshall and Scott also proved that A has finite global dimension if and only if both fAf and eAe have finite global dimension in this situation, which is exactly the result we obtained with our characterization of the projective resolutions of modules for A. The situation for the finitistic dimension is more complicated. The following theorem provides a criterion for the above diagrams to become full recollement diagrams for the bounded derived categories.

Theorem 5.22 (Cline, Parshall, Scott [9])**.** *Let A be a ring, J an ideal in A and $B = A/J$. The functor $i_! = i_\star : \mathbf{D}^b(B) \to \mathbf{D}^b(A)$ has a right adjoint $i^!$ satisfying $i_! i_! = \mathrm{id}_{\mathbf{D}^b(B)}$ if and only if*

(1) $\mathrm{Ext}^n_A(B_A, F) = 0$ *for all $n > 0$ and all free right B-modules F and*
(2) $\mathrm{proj.\,dim}\, B_A < \infty$.

We will now present an example of an algebra with a directed stratification in which condition (b) is not satisfied. Consider the following EI category (and its category algebra $A = k\mathcal{C}$) in characteristic 2:

$$\mathcal{C} : \begin{tikzcd} y \arrow[loop above, "1_y"] \arrow[loop below, "g"'] \arrow[r, "\alpha"] & x \arrow[loop above, "1_x"] \arrow[loop below, "h"'] \end{tikzcd}, \quad g^2 = 1_y,\ h^2 = 1_x,\ h\alpha = \alpha = \alpha g.$$

In this example we choose $f = 1_x$ and $e = 1_y$. The stratifying ideal is $J = AfA = \langle 1_x, h, \alpha \rangle_k$. Thus, the algebra eAe is just the group algebra $k\,\mathrm{Aut}(y)$. As an A-module, or as a representation of $\mathcal{C}$, the A-module B is

$$\begin{tikzcd} k^2 \arrow[loop above, "(1)"] \arrow[loop below, "M"'] \arrow[r, "0"] & 0 \arrow[loop above, "0"] \arrow[loop below, "0"'] \end{tikzcd}, \text{ where } M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

The projective cover $P_1 = P_B$ of B as an A module is

$$\begin{tikzcd} k^2 \arrow[loop above, "(1)"] \arrow[loop below, "M"] \arrow[r, "0"] & k \arrow[loop above, "(1)"] \arrow[loop below, "(1)"] \end{tikzcd}, \text{ again with } M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

The kernel K of the essential epimorphism $P_1 \to B$ is the representation

$$\begin{tikzcd} 0 \arrow[loop above, "(0)"] \arrow[loop below, "(0)"] \arrow[r, "0"] & k \arrow[loop above, "(1)"] \arrow[loop below, "(1)"] \end{tikzcd}$$

and we denote by P_2 the projective module

$$\begin{tikzcd} 0 \arrow[loop above, "(0)"] \arrow[loop below, "(0)"] \arrow[r, "0"] & k^2 \arrow[loop above, "1"] \arrow[loop below, "M"] \end{tikzcd}, \text{ with } M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

Finally, the minimal projective resolution of B as an A module looks as follows

$$\cdots \to P_2 \to P_2 \to P_2 \to P_1 \to B \to 0.$$

In particular, the projective dimension of B as an A module is infinite. Therefore we do not have a recollement of A relative to eAe and fAf, which means that we are not in the position to apply Happels result. However, with our result on algebras with a directed stratification it is obvious that A has finite finitistic dimension since the group algebras $k\,\mathrm{Aut}(x) = fAf$ and $k\,\mathrm{Aut}(y) = eAe$ have this property.

5.4.2 A non-trivial example

To prove the finiteness of the finitistic dimension of an algebra A in concrete examples, there are at the moment (to the best knowledge of the author) four important classes of algebras where the finiteness of fin. dim is known. First of all for algebras with radical cube zero and for monomial relation algebras Huisgen-Zimmermann showed that fin. dim is always finite. Those classes are very easy to detect, if the algebra in question is given by its quiver and relations. Igusa and Todorov proved that the finitistic dimension of an algebra Λ is finite if Λ has representation dimension at most 3. Here the representation dimension of Λ is defined as

$$\operatorname{rep.dim}\Lambda = \inf\{\, \operatorname{gl.dim}\operatorname{End}(M) \mid M \text{ is a generator cogenerator in } \operatorname{mod}\Lambda \,\}.$$

This dimension is very hard to calculate in general and therefore this result is often difficult to apply in concrete examples or it is at least not applicable without tedious calculations. For instance, it is not known how to calculate (or give an upper bound) for the representation dimension of a group algebra of an arbitrary finite group.

The fourth class of algebras where the finitistic dimension conjecture is known to hold true is the class of algebras Λ for which the category $\mathcal{P}^\infty(\operatorname{mod}\Lambda)$ of modules of finite projective dimension is contravariantly finite in $\operatorname{mod}\Lambda$. The notion of contravariant finiteness goes back to Auslander and Smalø while the result for the finitistic dimension is due to Auslander and Reiten. For the convenience of the reader we recall the definition here.

Definition 5.23. A full subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is called contravariantly finite if each module M in $\operatorname{mod}\Lambda$ has an $\mathcal{A}$-approximation in the following sense: there exists a homomorphism $f : A \to M$ for some $A \in \mathcal{A}$ such that every $g \in \operatorname{Hom}_\Lambda(B, M)$ with $B \in \mathcal{A}$ factors through f. This property may be illustrated by the following diagram.

$$\begin{tikzcd} A \arrow[r, "f"] & M \\ & B \arrow[u, "g"'] \arrow[ul, dotted, "\exists"] \end{tikzcd}$$

If the category $\mathcal{P}^\infty(\operatorname{mod}\Lambda)$ is contravariantly finite in $\operatorname{mod}\Lambda$, then Auslander and Reiten proved that $\operatorname{fin.dim}\Lambda < \infty$. The property of $\mathcal{P}^\infty$ being contravariantly finite has been investigated by Happel and Huisgen-Zimmermann in [21] and they observed that this property is rather 'unstable'. They also give an elementary criterion for $P^\infty(\operatorname{mod}\Lambda)$ not being contravariantly finite in $\operatorname{mod}\Lambda$ for a bound path algebra $\Lambda = kQ/I$. To state the theorem we need the following notation. Let $\Lambda = kQ/I$ be a bound path algebra and $p : e_1 \to e_2$ some path in kQ. If M is a Λ-module we write $f_p : e_1M \to e_2M$ for the linear map corresponding to p. With this convention the theorem of Happel and Huisgen-Zimmermann goes as follows.

Theorem 5.24 ([21])**.** *Let $\Lambda = kQ/I$. Suppose that e_1 and e_2 are vertices of Q and $p, q \in kQ \setminus I$ paths from e_1 to e_2 with $\Lambda p \cap \Lambda q = 0$. Moreover, suppose that*

(1) *the cyclic module $\Lambda(p, q)$ generated by $(p, q) \in \Lambda^2$ has finite projective dimension and that one of the following conditions is satisfied: either*

(2) *whenever $M \in \mathcal{P}^\infty(\operatorname{mod}\Lambda)$, then $f_p(e_1M \setminus \operatorname{rad}(\Lambda)M) \cap f_q(e_1 \operatorname{rad}(\Lambda)M) = \emptyset$; or,*

(2′) *whenever $M \in \mathcal{P}^\infty(\operatorname{mod}\Lambda)$, then $\operatorname{Ker}(f_p) \subseteq \operatorname{Ker}(f_q)$ and $\operatorname{Ker}(f_p) \subseteq e_1 \operatorname{rad}(\Lambda)M$.*

An easily recognizable situation in which the hypothesis of the criterion as well as both conditions are satisfied is the following one: p is an arrow $e_1 \to e_2$, $q \in kQ \setminus I$ a path from e_1 to e_2 of positive length, different from p such that $\operatorname{rad}(\Lambda)p = 0 = q \operatorname{rad}(\Lambda)$, and $\operatorname{proj.dim}(\Lambda q) < \infty$ whereas $\operatorname{proj.dim}(\Lambda e_2/\operatorname{rad}(\Lambda)e_2) = \infty$.

After this definitions and preparations, we can now present an example to which none of the above methods applies (perhaps besides calculation of rep. dim).

Example 5.25. Let Q be the following quiver:

$$\begin{tikzcd}
 & & 3 \arrow[dr, "\delta_2"] & \\
1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[loop above, "\gamma"] \arrow[ur, "\delta_1"] \arrow[dr, "\varepsilon_1"'] & & 5 \arrow[loop right, "\rho"] \\
 & & 4 \arrow[ur, "\varepsilon_2"'] &
\end{tikzcd}$$

Then let $A = kQ/I$ where I is the ideal generated by

$$\left\{ \gamma^2, \gamma\beta, \varepsilon_1\beta, \delta_1\beta, \varepsilon_1\gamma, \delta_1\gamma, \varepsilon_2\varepsilon_1 - \delta_2\delta_1, \rho\varepsilon_2, \rho\delta_2, \rho^5 \right\}.$$

A is by definition not monomial and does not satisfy $\operatorname{rad}^3(A) = 0$. Furthermore this algebra doesn't have the property that $\mathcal{P}^\infty(\operatorname{mod} A)$ is contravariantly finite in $\operatorname{mod} A$. To see this, we put $q = \alpha$ and $p = \beta$. Then $\operatorname{rad}(A)p = 0 = q \operatorname{rad}(A)$ and $Aq = A\alpha = Ae_2$ is a projective module. The module $Ae_2/\operatorname{rad}(A)e_2$ is of infinite projective dimension, since it is (as a representation) non-zero only on the vertex 2 and as an e_2Ae_2-module it has infinite projective dimension. Here we use that the idempotents $e_1, \ldots, e_5$ give a directed stratification of A and our results from the previous section.

The finiteness of fin. dim for A follows on the one hand very easily from the fact that $e_1, \ldots, e_5$ give a directed stratification of A and on the other hand by considering $e = e_1 + \cdots + e_4$ and $f = e_5$. These two idempotents give a directed stratification of A of length 2 and eAe as well as fAf have finite finitistic dimension, since they are monomial relation algebras. Here we notice that the iterated version of the result of Fossum, Griffith and Reiten, which we deduced with our methods, gives the finiteness of fin. dim A almost immediately, while for a directed stratification of length two one has to be more careful and use non-trivial results of other authors.

It is also interesting to point out that, for this example, there is no reasonable recollement-situation in sight which would give us the finiteness of fin. dim A immediately. For instance, if we take e and f as above, then the stratifying ideal is $J = AfA$ and $B := A/J = eAe$. In this case we don not get a recollement of $\mathbf{D}^b(A)$ relative to $\mathbf{D}^b(eAe)$ and $\mathbf{D}^b(fAf)$ because B as an A-module has the simple module $Ae_2/\operatorname{rad}(A)e_2$ as a summand and is therefore of infinite projective dimension.

To sum up, we have constructed an example of an algebra A which is not monomial, does not satisfy $\operatorname{rad}^3(A) = 0$, does not have the property that $\mathcal{P}^\infty(\operatorname{mod} A)$ is contravariantly finite in $\operatorname{mod} A$ and does not give a recollement-situation which gives the finiteness of fin. dim. Nevertheless, the finiteness of $\operatorname{fin.dim} A$ follows immediately from our theorem because we have a directed stratification with strata that are of finite finitistic dimension. What we have not done is to calculate the representation dimension of A in this example, but in general the calculation of rep. dim is a very hard task. Here it is (at least for the author) not obvious that A has representation dimension at most 3 and if it would be the case, then one could imagine how one could construct arbitrarily complicated examples to which our theorem applies and where one cannot calculate the representation dimension.

6 Outlook

In this final chapter we want to discuss some problems which we think are interesting for further investigation and which we could not solve within this thesis. We also mention techniques that might be used to attack the open problems.

The most interesting question that is still not answered is the following: What are the representation-finite EI-category algebras?

We proved a necessary criterion for finite representation type, namely we showed that if $k\mathcal{C}$ is representation-finite then, for any two objects x and y of $\mathcal{C}$ for which $\mathrm{Aut}(x)$ and $\mathrm{Aut}(y)$ are non-trivial, the group $\mathrm{Aut}(x) \times \mathrm{Aut}(y)$ does not act freely on $\mathcal{C}(x, y)$. A sufficient criterion for finite representation type, which may be applied to a large class of EI-category algebras, is still missing. However, the computation of various examples and the results for special classes of EI-categories that have been presented in this thesis lead us to the following conjecture.

Conjecture 6.1. *Let $\mathcal{C}$ be a finite EI-category and k an algebraically closed field such that the group algebra $k\,\mathrm{Aut}(x)$ is representation-finite for any object x of $\mathcal{C}$ and with $|\mathcal{C}(x, y)| \leq 1$ for any two distinct objects x and y of $\mathcal{C}$. Then the category algebra $k\mathcal{C}$ is representation-finite.*

EI-category algebras which satisfy the conditions of this conjecture are the easiest EI-categories which are not group algebras or incidence algebras.

For the group algebra of a finite group G it is well-known that the group algebra kG is semisimple if and only if $\mathrm{char}(k)$ does not divide $|G|$. In other words, the representation theory gets more complicated if the characteristic of the ground field divides the group order. Therefore, it is natural to expect a similar behaviour for EI-category algebras. An evidence for this expectation is the fact that an EI-category algebra has finite global dimension if and only if the characteristic of the field does not divide any of $|\mathrm{Aut}(x)|$ for $x \in \mathrm{Ob}\,\mathcal{C}$. Again, together with our experience from the examples we computed, one may conjecture the following:

Conjecture 6.2. *Let $\mathcal{C}$ be a finite EI-category and k an algebraically closed field whose characteristic does not divide any of $|\mathrm{Aut}(x)|$ for $x \in \mathrm{Ob}\,\mathcal{C}$. If the algebra $k\mathcal{C}$ has infinite representation type, then $k'\mathcal{C}$ has infinite representation type for any algebraically closed field k'.*

Roughly speaking, this conjecture states that, if every group algebra $k\,\mathrm{Aut}(x)$ for $x \in \mathrm{Ob}\,\mathcal{C}$ is semisimple, then we have the smallest number of indecomposable modules over $k\mathcal{C}$.

Unfortunately, there are, up to now, only few techniques that one can apply to study representations of EI-categories. Xu's theory of vertices and sources looks promising for a characterization of finite representation type at first sight. The problem is that one cannot easily imitate the definition of a defect group for a finite EI-category, because there is no analogue for the conjugation in a finite group. Nevertheless, it may be worth a try to work in this direction to derive some interesting results. Another idea to use Xu's results is to drop the assumption that one always restricts to full subcategories of an EI-category. The problem that arises here is that the computation of the induction is rather difficult if the subcategory is not full.

For quivers with relations the construction of universal covers is often an easy way to decide whether the given algebra is representation-finite or not. Covering theory as developed by Bongartz and Gabriel works for arbitrary representation-finite algebras in principle, but for the treatment of examples which are not given by quivers with relations it is often not applicable. For instance, it is not known how one constructs the universal cover of a finite group in general. It might therefore be interesting to develop a 'new' covering theory for EI-categories or more generally for small categories.

Finally, there is nothing known about the structure of the Auslander-Reiten quiver of an EI-category algebra. Recently, several authors proved results on the shape of certain components in the Auslander-Reiten quiver of a selfinjective algebra. Since the representation theory of EI-categories is somehow related to selfinjective algebras one might expect similar results in this framework.

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