Weighted Spaces of holomorphic functions on the upper halfplane

Dissertation

submitted by

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Paderborn 2010

Dedicated to my ***Wife***

Zusammenfassung

In dieser Dissertation studieren wir gewichtete Räume holomorpher Funktionen auf der offenen oberen komplexen Halbebene $\mathbf{G}$ für zwei Arten von Gewichten, die wir Typ(I)- und Typ(II)-Gewichte nennen.

Ein Typ(I)-Gewicht ist eine Gewichtsfunktion v, die nur von den Imaginärteilen der Elemente aus $\mathbf{G}$ abhängt. Ferner ist $v(it)$ monoton aufsteigend in t und erfüllt $\lim_{t\to 0} v(it) = 0$.
Dagegen ist v ein Typ(II)-Gewicht, wenn es
1. mit einem Typ(I)-Gewicht übereinstimmt auf allen $\omega \in \mathbf{G}$ mit $|\omega| \leq 1$ und
2. die Symmetriebedingung $v(\omega) = v(-\frac{1}{\omega})$ erfüllt für alle $\omega \in \mathbf{G}$.
Ferner arbeiten wir mit einer Bedingung, die die Wachstumsrate dieser Gewichte kontrolliert. Unsere Gewichte sollen nicht zu schnell wachsen oder fallen.

Im Mittelpunkt stehen folgende Banachräume

$$\mathbf{H}_v(\mathbf{G}) := \{\ f \mid f : \mathbf{G} \to \mathbb{C} \text{ holomorph und } ||f||_v < \infty\ \} \quad \text{und}$$

$$\mathbf{H}_{v_0}(\mathbf{G}) := \{\ f \in \mathbf{H}_v(\mathbf{G}) \mid fv \text{ verschwindet im Unendlichen }\}.$$

Dabei sei $||f||_v = \sup_{\omega\in\mathbf{G}} |f(\omega)|v(\omega)$.

Für viele unserer Resultate verwenden wir die Möbiustransformation $\alpha : \mathbb{D} \to \mathbf{G}$ definiert durch $\alpha(z) = \frac{1+z}{1-z}i$. ($\mathbb{D}$ ist dabei die Einheitskreisscheibe.) Wenn v ein Typ(II)-Gewicht ist, so zeigt sich, dass $v \circ \alpha$ äquivalent zu einem radialen Gewicht auf $\mathbb{D}$ ist. Dies ermöglicht uns, die wohlbekannten Resultate bezüglich der isomorphen Klassifizierung gewichteter Räume holomorpher Funktionen auf $\mathbb{D}$ zu übertragen auf $\mathbf{H}_v(\mathbf{G})$ und $\mathbf{H}_{v_0}(\mathbf{G})$. Deshalb erhalten wir eine vollständige isomorphe Klassifizierung für $\mathbf{H}_v(\mathbf{G})$ und $\mathbf{H}_{v_0}(\mathbf{G})$ im Falle von Typ(II)-Gewichten v. Unter unseren Voraussetzungen ist dann z.B. $\mathbf{H}_v(\mathbf{G})$ immer isomorph zu l_∞ oder $\mathbf{H}_\infty(\mathbb{D})$.

Leider kann man nicht dieselbe Methode für Typ(I)-Gewichte verwenden, denn in diesem Fall existiert $\lim_{z\to 1}(v\circ\alpha)(z)$ im Allgemeinen nicht und $v\circ\alpha$ ist nicht äquivalent zu einem radialen Gewicht auf $\mathbb{D}$. Deswegen beschränken wir uns bei Typ(I)-Gewichten

auf die folgenden Teilräume von $\mathbf{H}_v(\mathbf{G})$ und $\mathbf{H}_{v_0}(\mathbf{G})$:

$$U_{\pm}^{\beta} := \{\ f \in \mathbf{H}_v(\mathbf{G}) \mid \omega^{2\beta} f(\omega) = \pm f(-\frac{1}{\omega}),\ \omega \in \mathbf{G}\ \},\quad U_{\pm,0}^{\beta} := U_{\pm}^{\beta} \cap \mathbf{H}_{v_0}(\mathbf{G}),$$

$$\mathbf{H}_v^{2\pi}(\mathbf{G}) := \{f \in \mathbf{H}_v(\mathbf{G}) |\ f \text{ ist } 2\pi - \text{periodisch }\}$$

und $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}) := \mathbf{H}_v^{2\pi}(\mathbf{G}) \cap \mathbf{H}_{v_0}(\mathbf{G})$. Wir erhalten eine vollständige isomorphe Klassifizierung dieser Räume. Wiederum gilt, dass z.B. $\mathbf{H}_v^{2\pi}(\mathbf{G})$ und $U_{\pm}^{\beta}$ entweder isomorph zu l_∞ oder $\mathbf{H}_\infty(\mathbb{D})$ sind. Weiterhin zeigen wir, dass $U_{\pm}^{\beta}$ und $U_{\pm,0}^{\beta}$ komplementäre Teilräume von $\mathbf{H}_v(\mathbf{G})$ und $\mathbf{H}_{v_0}(\mathbf{G})$ sind.

Schliesslich studieren wir die Stetigkeit von Differential-, Kompositions- und Multiplikationsoperatoren zwischen gewichteten Räumen holomorpher Funktionen auf $\mathbf{G}$ und darüberhinaus zwischen gewichteten Räumen holomorpher 2π-periodischer Funktionen. Wir erhalten hinreichende (und manchmal notwendige) Bedingungen für die Stetigkeit dieser Operatoren, wenn unsere Gewichte den Typ(I) oder den Typ(II) haben.

Introduction

The concept of weight and weighted space of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}$ has been discussed by many authers, especially by Shileds and Willams in a series of papers. [19–21]
They studied weighted spaces :
$\mathbf{H}_v(\mathbb{D}) = \{f \mid f : \Omega \longrightarrow \mathbb{C}$ is holomorphic and $\|f\|_v < \infty\}$.
and
$\mathbf{H}_{v_0}(\mathbb{D}) = \{f \in \mathbf{H}_v(\mathbb{D}) : fv$ vanishes at infinity$\}$,
(Here $\|f\|_v := \sup_{z\in\mathbb{D}} \mid f(z) \mid v(\mid z \mid)$).
when the weight function v satiesfies certain properties which they called normality.
Their work opened a new field for research and many people studied these spaces from different aspects [1–3, 11, 14, 15]. Among these different aspects, two subjects seem to be particulary interesting. Firstly, finding isomorphic classifications of these weighted spaces as Banach spaces [10, 12, 13], secondly, studying operators such as composition, multiplication and differentiation operators between these spaces. [5–9, 17, 23].
Unlike the unit disc, the case of upper halfplane $\mathbf{G}$=$\{\omega \in \mathbb{C} : Im\omega > 0\}$ has not been studied too much.
In this Ph.D thesis, we study weighted spaces of holomorphic functions for two kinds of weights which we call type(I) and type(II) weights (see Definition 1.2.1).
We impose some conditions on our weights in order to control the rate of growth of these weights. Our conditions are :
$(*)_I, (*)_{II}$ and $(**)$. (See Definition 1.2.7).
In chapter one, firstly we collect some preliminary facts which we need in the next chapters. Secondly, we study equivalent properties in order to characterize $(*)_I, (*)_{II}$ and $(**)$. Moreover we present some examples and counterexamples of type(I) and type(II) weights with the above properties.
To obtain results about the Banach spaces
$\mathbf{H}_v(\mathbf{G})$:=$\{f \mid f : \mathbf{G} \longrightarrow \mathbb{C}$ is holomorphic and $\|f\|_v < \infty$ $\}$ and $\mathbf{H}_{v_0}(\mathbf{G})$=: {f$\in$ $\mathbf{H}_v(\mathbf{G})$:

fv vanishes at infinity} (when v is a weight of type(II) and has a moderate rate of growth) we apply the Moebius transform $\alpha : \mathbb{D} \longrightarrow \mathbf{G}$ defined by $\alpha(z) = \frac{1+z}{1-z}i$.
In this case the weight $v \circ \alpha$ is equivalent to a radial weight on $\mathbb{D}$ (see Theorem 2.2.1). This enables us to transfer wellknown results about isomorphic classification of weighted spaces of holomorphic functions on $\mathbb{D}$ to $\mathbf{H}_v(\mathbf{G})$ or $\mathbf{H}_{v_0}(\mathbf{G})$. Therefore we present a complete isomorphic classification for $\mathbf{H}_v(\mathbf{G})$ and $\mathbf{H}_{v_0}(\mathbf{G})$ in Theorem 2.2.3. For example under the certain assumptions on v, $\mathbf{H}_v(\mathbf{G})$ is either isomorphic to ℓ_∞ or $\mathbf{H}_\infty(\mathbb{D})$.
In chapter three, in a similar way(by applying Theorem 2.2.1) we use the wellknown results for differentiation, composition and multiplication operators on $\mathbf{H}_v(\mathbb{D})$ to obtain some results about operators between weighted spaces of holomorphic functions on upper halfplane $\mathbf{G}$ for type(II) weights.
Unfortunately, for type(I) weights, we cannot use the same method, because in this case $\lim_{z\to 1}(v \circ \alpha)(z)$ does not exist and $v \circ \alpha$ is not equivalent to a radial weight on $\mathbb{D}$. Therefore for type(I) weights, we restrict ourselves to some special subspaces of $\mathbf{H}_v(\mathbf{G})$ and $\mathbf{H}_{v_0}(\mathbf{G})$ such as:
$U_\pm^\beta := \{f \in \mathbf{H}_v(\mathbf{G}) : \omega^{2\beta} f(\omega) = \pm f(-\frac{1}{\omega})\ \forall \omega \in \mathbf{G}\}$, $U_{\pm,\ 0}^\beta := U_\pm^\beta \cap \mathbf{H}_{v_0}(\mathbf{G})$,
$\mathbf{H}_v^{2\pi}(\mathbf{G}) := \{f \in \mathbf{H}_v(\mathbf{G}) : f \text{ is } 2\pi\text{−periodic}\}$
and
$\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}) := \{f \in \mathbf{H}_{v_0}(\mathbf{G}) : f \text{ is } 2\pi\text{−periodic}\}$.
In Theorems 2.3.12 and 2.4.13 we obtain isomorphic classifications of $U_\pm^\beta$ & $U_{\pm,\ 0}^\beta$ and $\mathbf{H}_v^{2\pi}(\mathbf{G})$ & $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G})$ respectively. Again, we have, $\mathbf{H}_v^{2\pi}(\mathbf{G})$ and $U_\pm^\beta$ are isomorphic to ℓ_∞ or $\mathbf{H}_\infty(\mathbb{D})$.
In Theorem 2.5.4 we show that $U_\pm^\beta$ & $U_{\pm,\ 0}^\beta$ are complemented subspaces of $\mathbf{H}_v(\mathbf{G})$ and $\mathbf{H}_{v_0}(\mathbf{G})$ respectively. Unfortunately, the isomorphic classifications of the complements of $U_\pm^\beta$ and $U_{\pm,\ 0}^\beta$ are not known(see Remark 2.5.5).
Chapter four is devoted to studying operators between weighted spaces of holomorphic functions for type(I) weights. In this chapter, we study the continuity of differentiation and composition operators not only between weighted spaces of holomorphic

functions, but also between weighted spaces of 2π-periodic functions. Our results give sufficient(and sometimes necessary) conditions for continuity of these operators.

Acknowledgements

I am heartily thankful to my supervisor, Prof. Dr. Wolfgang Lusky, whose encouragement, guidance, assistance and support from the initial to the final enabled me to develop an understanding of the subject.
Last, but not least, I offer my regards and blessing to my ***Mother*** and all of those who supported me in any respect during the completion of the project.

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Contents

Chapter 1

Preliminaries

Introduction to chapter one: The goal of this chapter is to collect the definitions and lemmas which we need in the next chapters.
In section one, we define $\mathbf{H}_\upsilon(\Omega)$ and $\mathbf{H}_{\upsilon_0}(\Omega)$ which have the main role throughout this thesis. Also we introduce the open unit disk $\mathbb{D}$ and upper halfplane $\mathbf{G}$ of $\mathbb{C}$ and a Möbius transform α which maps $\mathbb{D}$ biholomorphically onto $\mathbf{G}$.
In section two, we define two types of weights on $\mathbf{G}$ and we investigate properties concerning the rate of growth for some weights and discuss some examples.

Section one : Basic definitions and lemmas.

Definition 1.1.1 : Let Ω be open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a function.

a) A weight on Ω is a function $\upsilon : \Omega \longrightarrow (0, +\infty)$.

b) We define $\|f\|_\upsilon := \sup_{\omega\in\Omega} \mid f(\omega) \mid \upsilon(\omega)$.

c) We define $\mathbf{H}_\upsilon(\Omega) := \{f \mid f : \Omega \longrightarrow \mathbb{C}$ is holomorphic and $\|f\|_\upsilon < \infty\}$.

d) We say $f\upsilon$ vanishes at infinity if for any $\epsilon > 0$ there is a compact set $K \subseteq \Omega$ such that$\mid f(\omega) \mid \upsilon(\omega) < \epsilon \quad \forall\omega \in \Omega \setminus K$.

e) We define $\mathbf{H}_{\upsilon_0}(\Omega) := \{f \in \mathbf{H}_\upsilon(\Omega) : f\upsilon$ vanishes at infinity$\}$.

Definition 1.1.2: a) The sets $\mathbb{D} = \{z \in \mathbb{C} : \mid z \mid< 1\}$ and $\mathbf{G} = \{\omega \in \mathbb{C} : \ Im\omega > 0\}$ are the unit disc and upper halfplane respectively.

b) For any $\delta > 0$, we define $\mathbf{G}_\delta := \{\omega \in \mathbb{C} : \ Im\omega \geq \delta\}$.

c) Suppose $x \in \mathbb{C}$ and $r \in \mathbb{R}(r > 0)$, then $x + r\partial\mathbb{D}$ denotes the circle with the center x and radius r in the complex plane.

In particular $r\mathbb{D} := \{rz : z \in \mathbb{D}\}$ denotes the disc with origin as center and radius r and $r\partial\mathbb{D} = \{z \in \mathbb{C} :\mid z \mid= r\}$.

d) Suppose $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is a complex function. we define $M_\infty(f, \Omega) := \sup_{\omega\in\ \Omega} \mid f(\omega) \mid$

Definition 1.1.3: For any $\delta,\ \delta \geq 0$ we define $L_\delta := \{\omega \in \mathbf{G} : Im\omega = \delta\}$.

In particular $L_0 := \{\omega \in \mathbf{G} : Im\omega = 0\}$ is the real line.

Definition 1.1.4: Define $\alpha : \mathbb{D} \longrightarrow \mathbb{C}$ by $\alpha(z) = \frac{1+z}{1-z}i$.

Remark 1.1.5: Suppose $z \in \mathbb{D}$, an easy computation shows that

$\alpha(z) = -2\frac{Imz}{|z|^2+1-2Rez} + \frac{1-|z|^2}{|z|^2+1-2Rez}i$.

Hence $\alpha(\mathbb{D}) \subseteq \mathbf{G}$. Put $\beta(\omega) = \frac{\omega-i}{\omega+i}\ \forall\omega \in \mathbf{G}$, then we have $\alpha \circ \beta = id_{|\mathbf{G}}$ and $\beta \circ \alpha = id_{|\mathbb{D}}$. Hence $\beta = \alpha^{-1}$ and $\alpha(\mathbb{D}) = \mathbf{G}$.

It is easily seen that $\alpha^{-1}(\omega) = \frac{|\omega|^2-1}{|\omega|^2+1+2Im\omega} - 2\frac{Re\omega}{|\omega|^2+1+2Im\omega}i$.

Lemma 1.1.6: *If* $z \in \mathbb{D}$ *then*

i) $Im\alpha(z) = \frac{1-|z|^2}{|1-z|^2}$.

ii) $\alpha(-z) = -\frac{1}{\alpha(z)}$.

iii) $\alpha(z^2) = \frac{\alpha(z)}{2} - \frac{1}{2\alpha(z)}$.

iv) $Rez \leq 0$ *if and only if* $|\ \alpha(z)\ | \leq 1$.

v) $\alpha(\frac{\delta}{1+\delta} + \frac{1}{1+\delta}\mathbb{D} \setminus \{1\}) = L_\delta \ \ \forall\ \delta > 0$.

Proof: (i), (ii) and (iii) are trivial.

iv) Since $|\ \alpha(z)\ |^2 = |\ \frac{1+z}{1-z}\ |^2 = \frac{1+|z|^2+2Rez}{1+|z|^2-2Rez}$ so $|\ \alpha(z)\ |^2 \leq 1 \ \Leftrightarrow\ Rez \leq 0$.

v) Put $\alpha(z) = \omega$. If $Im\alpha(z) = Im\omega = \delta$ then we have $\frac{1-|z|^2}{1+|z|^2-2Rez} = \delta \ \Leftrightarrow$
$\frac{1}{1+\delta} = |\ z\ |^2 + \frac{\delta}{1+\delta} - \frac{2\delta}{1+\delta}Rez$

$\Leftrightarrow\ \frac{1}{1+\delta} - \frac{\delta}{1+\delta} + \frac{\delta^2}{(1+\delta)^2} = |\ z - \frac{\delta}{1+\delta}\ |^2 \Leftrightarrow \frac{1}{(1+\delta)^2} = |\ z - \frac{\delta}{1+\delta}\ |^2 \Leftrightarrow |\ \alpha^{-1}(\omega) - \frac{\delta}{1+\delta}\ | = \frac{1}{1+\delta}$.

So α^{-1} maps the line $Im\omega = \delta$ to the circle $\frac{\delta}{1+\delta} + \frac{1}{1+\delta}\partial\mathbb{D}$.

Thus $\alpha^{-1}(L_\delta) = \frac{\delta}{1+\delta} + \frac{1}{1+\delta}\mathbb{D} \setminus \{1\}$ or equivalently $\alpha(\frac{\delta}{1+\delta} + \frac{1}{1+\delta}\mathbb{D} \setminus \{1\}) = L_\delta$. □

Lemma 1.1.7: *If* $f : \mathbf{G} \longrightarrow \mathbb{C}$ *is a holomorphic function, then there are* $\alpha_k \in \mathbb{C}$ *such that* $f(\omega) = \sum_{k=0}^{\infty} \alpha_k (\frac{\omega-i}{\omega+i})^k\ \forall\ \omega \in \mathbf{G}$, *where the series converges uniformly on compact subsets of* $\mathbf{G}$.

Proof: Since the function α is holomorphic on $\mathbb{D}$ so $f \circ \alpha$ is holomorphic on $\mathbb{D}$ and it has a Taylor series representation. Hence $(f \circ \alpha)(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ for some α_k and the series converges uniformly on compact subsets of $\mathbb{D}$. Now the lemma follows from the fact that if $\alpha(z) = \omega$ then $z = \alpha^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. □

Definition 1.1.8: a) We recall that $\partial\mathbb{D} = \{z \in \mathbb{C} :|\ z\ |= 1\}$. Let $\mathbf{L}^\infty(\partial\mathbb{D})$ be the space of all essentially bounded functions on $\partial\mathbb{D}$, normed by the essential supremum norm $\|.\|_\infty$.

b) Define $\mathbf{H}_\infty(\mathbb{D}) = \{g \mid g : \mathbb{D} \longrightarrow \ \mathbb{C}$ is holomorphic and $M_\infty(g, \mathbb{D}) < \infty\}$.

Theorem 1.1.9: *To every* $g \in \mathbf{H}_\infty(\mathbb{D})$ *there corresponds a function* $g^\star \in \mathbf{L}^\infty(\partial\mathbb{D})$, *defined* *almost everywhere by $g^\star(e^{it}) = \lim_{r\to 1} g(re^{it})$. Then equality $M_\infty(g,\mathbb{D}) = \|g^\star\|_\infty$ holds.*

Proof: See [18] Theorem 11.3.2 . □

Note that Theorem1.1.9 is also true for any translation of the unit disk $\mathbb{D}$.

Lemma 1.1.10: *Let $\tau > \delta > 0$ and let $f : \mathbf{G} \longrightarrow \mathbb{C}$ be holomorphic such that $f_{|\mathbf{G}_\delta}$ is a bounded function. Then $M_\infty(f, L_\tau) \le M_\infty(f, L_\delta)$.*

Proof: Put $\mathbb{D}_1 = \alpha^{-1}(\mathbf{G}_\delta)$ and $\mathbb{D}_2 = \alpha^{-1}(\mathbf{G}_\tau)$ then $\partial\mathbb{D}_1 \setminus \{1\} = \alpha^{-1}(L_\delta)$ and $\partial\mathbb{D}_2 \setminus \{1\} = \alpha^{-1}(L_\tau)$. Put $\widetilde{f}(z) = (f \circ \alpha)(z)$. Since $f_{|\mathbf{G}_\delta}$ is bounded so $\widetilde{f}$ is bounded on $\mathbb{D}_1$. So $\widetilde{f} \in \mathbf{H}_\infty(\mathbb{D}_1)$. Now using Theorem 1.1.9, there is a function $\widetilde{f}^* \in \mathbf{L}^\infty(\partial\mathbb{D})$ such that $M_\infty(\widetilde{f}, \mathbb{D}) = \|\widetilde{f}^*_{|\partial\mathbb{D}_1}\|_\infty$. Since one point has a Lebesgue measure zero so
$\|\widetilde{f}^* \cdot \chi_{\partial\mathbb{D}_1 \setminus \{1\}}\|_\infty = \|\widetilde{f}^*_{|\partial\mathbb{D}_1}\|_\infty$ and we have

$M_\infty(\widetilde{f}, \mathbb{D}_1) = \|\widetilde{f}^* \cdot \chi_{\partial\mathbb{D}_1 \setminus \{1\}}\|_\infty \qquad (1).$

where χ is the characteristic function. With the similar argument we have

$M_\infty(\widetilde{f}, \mathbb{D}_2) = \|\widetilde{f}^* \cdot \chi_{\partial\mathbb{D}_2 \setminus \{1\}}\|_\infty \qquad (2).$

Since $\mathbb{D}_2 \subseteq \mathbb{D}_1$ we have $M_\infty(\widetilde{f}, \mathbb{D}_2) \le M_\infty(\widetilde{f}, \mathbb{D}_1)$. Now (1),(2) and Theorem 1.1.9 yield $M_\infty(f, L_\tau) = \|\widetilde{f}^*_{|\partial\mathbb{D}_2}\|_\infty = M_\infty(\widetilde{f}, \mathbb{D}_2) \le M_\infty(\widetilde{f}, \mathbb{D}_1) = \|\widetilde{f}^*_{|\partial\mathbb{D}_1}\|_\infty = M_\infty(f, L_\delta)$. □

Remark 1.1.11: Notice that in Lemma 1.1.10 the assumption f is bounded on $\mathbf{G}_\delta$ is necessary. The lemma is not true if f is only bounded on $\mathbf{L}_\delta$ and $\mathbf{L}_\tau$. See the following example.

Example 1.1.12: Define $f : \mathbf{G} \longrightarrow \mathbb{C}$ by $f(\omega) = e^{i\omega}$. Then $| f_{|\mathbf{L}_\delta} | = e^\delta$ and $| f_{|\mathbf{L}_\tau} | = e^\tau$, but $M_\infty(f, \mathbf{L}_\delta) < M_\infty(f, \mathbf{L}_\tau)$ whenever $\tau > \delta > 0$.

Section two : Characterization of certain types of positive weight functions.

Definition 1.2.1: *Let* υ be a continuous weight on $\mathbf{G}$:

a) We say υ is of type(I) if $\lim_{r\to 0^+}\upsilon(ir)=0$ and there is a constant $C>0$ such that $\upsilon(\omega_1)\leq\ C\upsilon(\omega_2)$ whenever $Im\omega_1\leq\ Im\omega_2$.

b) We say υ is of type(II) if there is a type(I) weight υ_1 and a constant $C>0$ such that $\upsilon(\omega)=\upsilon_1(\omega)$ if $\mid\omega\mid\leq 1$ and $\frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})}\leq C$ for any $\omega\in\mathbf{G}$.

Example 1.2.2: Define $\upsilon_1,\upsilon_2,\upsilon_3:\mathbf{G}\longrightarrow(0,+\infty)$ by

$\upsilon_1(\omega)=(Im\omega)^\beta$ for some $\beta>0$.

$\upsilon_2(\omega)=\min((Im\omega)^\beta,1)$ for some $\beta>0$.

$$\upsilon_3(\omega)=\begin{cases}(1-\ln(Im\omega))^\gamma & \text{if } Im\omega\leq 1\\ 1 & \text{if } Im\omega\geq 1\end{cases}\quad\text{for some }\gamma,\ \gamma<0.$$

and $\upsilon_{k+3}(\omega)=\upsilon_k(\frac{Im\omega}{\max(|\omega|^2,1)}i)$ where $k\in\ \{1,2,3\}$.

Then υ_1,υ_2 and υ_3 are type(I) weights, but υ_{k+3} are type(II) weights which are not of type(I).

Proof: It is easy to see that υ_1,υ_2 and υ_3 are type(I) weights. Indeed for all of them the constant C is equal to 1.

For showing that υ_{k+3} ($k\in\ \{1,2,3\}$) are type(II) weights use the fact

$Im(\frac{-1}{\omega})=\frac{Im\omega}{|\omega|^2}\ \forall\omega\in\mathbf{G}$. Since $\sup\{\frac{\upsilon_{k+3}(\omega_2)}{\upsilon_{k+3}(\omega_1)}:\ \omega_1,\omega_2\in\mathbf{G}$ and $Im\omega_1<Im\omega_2\}=\infty$ for $k\in\{1,2,3\}$, so υ_{k+3} are not of type(I). For example put $k=1,\omega'_n=n^2i$ and $\omega_n=ni$ then $Im\omega_n<Im\omega'_n$ and $\frac{\upsilon_4(\omega_n)}{\upsilon_4(\omega'_n)}=n^\beta$. So $\sup\{\frac{\upsilon_4(\omega_n)}{\upsilon_4(\omega'_n)}:\ n\in\mathbb{N}\}=\infty$. □

In the Example 1.2.2 all the weights depend only on the imaginary part. Now we peresent weights which depend both on the real and imaginary parts.

Example 1.2.3: If υ is a type(I)(type(II)) weight, then $\upsilon_7(\omega)=\upsilon(Im\omega i)\arctan(\mid Re\omega\mid+\sqrt{3})$ and $\upsilon_8(\omega)=\upsilon(Im\omega i)(\sin\mid\omega\mid+2)$ are type(I)(type(II)) weights.

Remark 1.2.4: a) Notice that a type(I) weight υ is almost constant on the lines $\mathbf{L}_\delta$. Since $Im\omega_1 \leq Im\omega_2$ and $Im\omega_2 \leq Im\omega_1$ whenever $\omega_1, \omega_2 \in \mathbf{L}_\delta$ so we have
$\frac{1}{C}\ \upsilon(\omega_1) \leq \upsilon(\omega_2) \leq C\upsilon(\omega_1)$ if $\omega_1, \omega_2 \in \mathbf{L}_\delta$.
This means that the weight υ_1 with $\upsilon_1(\omega) = \upsilon(Im\omega i)$ satisfies
$\frac{1}{C}\upsilon(\omega) \leq \upsilon_1(\omega) \leq C\upsilon(\omega)$ for all $\omega \in \mathbf{G}$.
Therefore from now on we always assume a type(I) weight υ satiesfies the property $\upsilon(\omega) = \upsilon(Im\omega i)$. By the preceding argument this is no loss of generality.
b) If υ is a type(II) weight then by definition of type(II) weight there is a constant $C > 0$ such that $\frac{1}{C}\upsilon(\omega) \leq \upsilon(\frac{-1}{\omega}) \leq C\upsilon(\omega)$ for all $\omega \in \mathbf{G}$.

Lemma 1.2.5: *Let υ be a type(I) weight on $\mathbf{G}$. Define*
$\upsilon_1(\omega_0) = \inf\{\upsilon(\omega) : \omega \in \mathbf{G}, Im\omega \geq Im\omega_0\}$ *for any $\omega_0 \in \mathbf{G}$. Then $\upsilon_1(\omega)$ is an increasing function on the positive imaginary axis. Also there is a constant $C > 0$ such that* $\upsilon_1(\omega) \leq \upsilon(\omega) \leq C\upsilon_1(\omega)$.

Proof: Suppose $\omega_1, \omega_2 \in \mathbf{G}$ are such that $Im\omega_1 \leq Im\omega_2$. Then
$\upsilon_1(\omega_1) = \inf\{\upsilon(\omega) : Im\omega \geq Im\omega_1\} \leq \inf\{\upsilon(\omega) : Im\omega \geq Im\omega_2\} = \upsilon_1(\omega_2)$. So υ_1 is an increasing function on the positive imaginary axis. Clearly $\upsilon_1(\omega) \leq \upsilon(\omega)$. If $Im\omega_0 \leq Im\omega$ then $\upsilon(\omega_0) \leq C\upsilon(\omega)$, so $\frac{1}{C}\upsilon(\omega_0)$ is a lower bound for the set $\{\upsilon(\omega) : Im\omega \geq Im\omega_0\}$, therefore $\frac{1}{C}\upsilon(\omega_0) \leq \inf\{\upsilon(\omega) : Im\omega \geq Im\omega_0\} = \upsilon_1(\omega_0)$. So $\upsilon(\omega_0) \leq C\upsilon_1(\omega_0)$. Since $\omega_0 \in \mathbf{G}$ is arbitrary we are done. □

Remark 1.2.6: From now on we assume that a type(I) weight satisfies
$\upsilon(\omega) = \upsilon(Im\omega i), \omega \in \mathbf{G}$ and $\upsilon(\omega_1) \leq \upsilon(\omega_2)$ whenever $Im\omega_1 \leq Im\omega_2$.
In view of Remark 1.2.4(a) and Lemma 1.2.5, this is no loss of generality.

Definition 1.2.7: A weight υ on $\mathbf{G}$ satisfies
a) $(*)_I$(with respect to β) if there are constants $C > 0, \beta > 0$ such that
$\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \leq C(\frac{Im\omega_1}{Im\omega_2})^\beta$ whenever $Im\omega_1 \geq Im\omega_2$.
b) $(*)_{II}$ if there are constants $C > 0, \beta > 0$ such that $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \leq C(\frac{Im\omega_1}{Im\omega_2})^\beta$ whenever

$Im\omega_1 \geq Im\omega_2$ *and* $\mid \omega_1 \mid, \mid \omega_2 \mid \leq 1$.

c) $(**)$ if there are constants $C > 0, \gamma > 0$ such that $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \geq C(\frac{Im\omega_1}{Im\omega_2})^\gamma$ whenever $Im\omega_1 \geq Im\omega_2$ and $\mid \omega_1 \mid, \mid \omega_2 \mid \leq 1$.

It is clear that $(*)_I$ implies $(*)_{II}$. In the following lemmas we characterize the properties $(*)_I, (*)_{II}$ and $(**)$. These lemmas enable us to present examples and also counterexamples for these properties.

Lemma 1.2.8: i) : *If* υ *is of type(I), then* υ *satiesfies* $(*)_I \iff \sup_{n\in\mathbb{Z}} \frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)} < \infty$.

ii) *Let* υ *be a weight on* **G** *with the property, there exists a* $C > 0$ *such that* $\upsilon(\omega_1) \leq C\upsilon(\omega_2)$ *whenever* $Im\omega_1 \leq Im\omega_2 \leq 1$. *Then* υ *satisfies*

$(*)_{II} \iff \sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)} < \infty$.

Proof: (i): $\Longrightarrow$: Suppose $n \in \mathbb{Z}$, put $\omega_1 = 2^{n+1}i$ and $\omega_2 = 2^n i$. Since υ satisfies $(*)_I$ then $\frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)} \leq C(\frac{2^{n+1}}{2^n})^\beta = C2^\beta$ so $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)} < \infty$.

$\Longleftarrow$: Let $\omega_1, \omega_2 \in \mathbf{G}$ be such that $Im\omega_1 = t_1 \geq Im\omega_2 = t_2 > 0$. We can find $n \in \mathbb{Z}$ and $k \in \mathbb{N}\cup\{0\}$ such that $\quad 2^n < t_2 \leq 2^{n+1} \quad \& \quad 2^{n+k} < t_1 \leq 2^{n+k+1}$. Then

$$\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} = \frac{\upsilon(t_1 i)}{\upsilon(t_2 i)} \leq \frac{\upsilon(2^{n+k+1}i)}{\upsilon(2^n i)} = \prod_{j=0}^{k} \frac{\upsilon(2^{n+j+1}i)}{\upsilon(2^{n+j}i)} \leq C^{k+1} \text{ where } C = \sup_{n\in\mathbb{Z}} \frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)}.$$

Now with $\beta = \frac{\ln C}{\ln 2}$ we have $\frac{\upsilon(t_1 i)}{\upsilon(t_2 i)} \leq C^{k+1} = 2^{(k+1)\beta} = 4^\beta(\frac{2^{n+k}}{2^{n+1}})^\beta \leq 4^\beta(\frac{t_1}{t_2})^\beta$.

(ii): $\Longrightarrow$ **:** suppose $n \in \mathbb{N}\cup\{0\}$ is arbitrary. Put $\omega_1 = \frac{1}{2^n i}$ and $\omega_2 = \frac{1}{2^{n+1}i}$, so $Im\omega_1 \geq Im\omega_2$. Now since υ satisfies $(**)_I$ so there exist $C > 0$ and $\beta > 0$ such that

$\frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)} \leq C(\frac{2^{-n}}{2^{-n-1}})^\beta \leq C\ 2^\beta$. Therefore $\quad \sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)} < \infty$.

$\Longleftarrow$ **:** Let $\omega_1, \omega_2 \in \mathbf{G}$ with $Im\omega_1 = t_1 \geq Im\omega_2 = t_2$ and $\mid \omega_1 \mid, \mid \omega_2 \mid \leq 1$ be given. We can find n and k *in* $\mathbb{N}\cup\{0\}$ such that $\quad 2^{-n-k-1} < t_2 \leq 2^{-n-k}$ and $2^{-n-1} < t_1 \leq 2^{-n}$. Now an argument similar to what we have done in part(i)($\Longleftarrow$) completes the proof. □

Lemma 1.2.9: *Let υ be a weight on* $\mathbf{G}$ *with the following property:*
There exists a $C > 0$ *such that* $\upsilon(\omega_1) \leq C\upsilon(\omega_2)$ *whenever* $Im\omega_1 \leq Im\omega_2 \leq 1$.

Then υ $satisfies(**) \Longleftrightarrow \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(2^{-n-k}i)}{\upsilon(2^{-n}i)} < 1$.

Proof: $\Longrightarrow$ Since υ satisfies $(**)$ so there exist C and $\gamma (C, \gamma > 0)$ such that

$\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \geq C(\frac{Im\omega_1}{Im\omega_2})^\gamma$ whenever $|\omega_1|, |\omega_2| \leq 1$ and $Im\omega_1 \geq Im\omega_2 > 0$. Equivalently

$\frac{\upsilon(\omega_2)}{\upsilon(\omega_1)} \leq \frac{1}{C}(\frac{Im\omega_2}{Im\omega_1})^\gamma$. Put $\omega_1 = 2^{-n}i$ and $\omega_2 = 2^{-n-k_0}i$ where k_0 has been selected such

that $\frac{1}{C}\ 2^{-k_0\gamma} < 1$. So $\frac{\upsilon(2^{-n-k_0}i)}{\upsilon(2^{-n}i)} \leq \frac{1}{C}(\frac{2^{-n-k_0}}{2^{-n}})^\gamma = \frac{1}{C}\ 2^{-k_0\gamma} < 1$. Thus

$\limsup_{n\to\infty} \frac{\upsilon(2^{-n-k_0}i)}{\upsilon(2^{-n}i)} < 1$ and therefore $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(2^{-n-k}i)}{\upsilon(2^{-n}i)} < 1$.

$\Longleftarrow$: $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(2^{-n-k}i)}{\upsilon(2^{-n}i)} < 1$ implies that there are $k_0, n_0 \in \mathbb{N}$ and

$q < 1$ with $\frac{\upsilon(2^{-n-k_0}i)}{\upsilon(2^{-n}i)} \leq q$ for $n \geq n_0$.

Since $\upsilon(2^{-n-k}i) \leq \upsilon(2^{-n-k_0}i)$ for $k \geq k_0$ we also have $\frac{\upsilon(2^{-n-k}i)}{\upsilon(2^{-n}i)} \leq q\ for\ n \geq n_0$,

$k \geq k_0$. Fix $m_0 \geq k_0, n_0$. Then we obtain $\frac{\upsilon(2^{-m_0(j+1)}i)}{\upsilon(2^{-m_0 j}i)} \leq q$ for $j = 0, 1, ...$

Now let $\omega_1, \omega_2 \in \mathbf{G}$ such that $|\omega_1|, |\omega_2| \leq 1$, $0 < Im\omega_2 \leq Im\omega_1 \leq 1$, say

$2^{-m_0(j+1)} \leq Im\omega_1 \leq 2^{-m_0 j}$, $2^{-m_0(j+k+1)} \leq Im\omega_2 \leq 2^{-m_0(j+k)}$ for some j and k.

If $k \geq 2$ we have $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} = \frac{\upsilon(Im\omega_1 i)}{\upsilon(Im\omega_2 i)} \geq \frac{\upsilon(2^{-m_0(j+1)}i)}{\upsilon(2^{-m_0(j+k)}i)} = \prod_{l=j+k-1}^{j+1} \frac{\upsilon(2^{-m_0 l}i)}{\upsilon(2^{-m_0(l+1)}i)} \geq (\frac{1}{q})^{k-2} =$

$q^2 2^{\gamma m_0 k}$ where $\gamma = \frac{\ln(n^{-1})}{m_0 \ln 2}$.

If $k \in \{0, 1\}$ then there is a constant $C \leq 1$ such that $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \geq 1 \geq Cq^2 2^{\gamma m_0 k}$.

Hence $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \geq Cq^2 2^{\gamma m_0 k} = C\ 2^{-m_0\gamma} q^2 (\frac{2^{-m_0 j}}{2^{-m_0(j+k+1)}})^\gamma \geq C\ 2^{-m_0\gamma} q^2 (\frac{Im\omega_1}{Im\omega_2})^\gamma$. $\square$

Example 1.2.10: a) The weights $\upsilon_1, \upsilon_2, \upsilon_3$ and υ_4 of Example 1.2.2 satisfy $(*)_I$, while the weight υ_7 defined on $\mathbf{G}$ by

$$\upsilon_7(\omega) = \begin{cases} \frac{1}{1-\ln(Im\omega)} & Im\omega \leq 1 \\ e^{Im\omega - 1} & Im\omega > 1 \end{cases} \text{ only satisfies } (*)_{II}.$$

b) The weights υ_1, υ_2 of Example 1.2.2 satisfy$(**)$, while the weight υ_3 in this example does not satisfy $(**)$.

Proof: a) Clearly υ_1 and υ_2 satisfy$(*)_I$. Moreover we have

$\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_3(2^{-n}i)}{\upsilon_3(2^{-n-1}i)} = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{(1+n\ln 2)^\gamma}{(1+(n+1)\ln 2)^\gamma}$. Since $\lim_{n\to\infty}(\frac{1+n\ln 2}{1+(n+1)\ln 2})^\gamma = 1$ for any $\gamma < 0$ so $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_3(2^{-n}i)}{\upsilon_3(2^{-n-1}i)} = 1 < \infty$. Also $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_3(2^{n+1}i)}{\upsilon_3(2^n i)} = 1$ by definition of υ_3. Now part(ii) of the Lemma 1.2.8 implies that υ_3 satisfies$(*)_I$. A similar proof shows that υ_4 satisfies $(*)_I$. Finally we have $\upsilon_{7|Im\omega\leq 1} = \upsilon_3$ for $\gamma = -1$ so $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_7(2^{-n}i)}{\upsilon_7(2^{-n-1}i)} < \infty$. Now part(i) of the Lemma 1.2.8 implies that υ_7 satisfies $(*)_{II}$. But

$\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_7(2^{n+1}i)}{\upsilon_7(2^n i)} = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{e^{2^{n+1}-1}}{e^{2^n-1}} = \sup_{n\in\mathbb{N}\cup\{0\}} e^{2^n} = \infty.$

b) Let $k \in \mathbb{N}$ be arbitrary. $\frac{\upsilon_1(2^{-n-k}i)}{\upsilon_1(2^{-n}i)} = 2^{-\beta k} < 1$ So $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon_1(2^{-n-k}i)}{\upsilon_1(2^{-n}i)} = 0 < 1$. Now Lemma 1.2.9 implies that υ_1 satisfies $(**)$. Similary we can show that υ_2 satisfies$(**)$ too. But for υ_3, since $\frac{\upsilon_3(2^{-n-k}i)}{\upsilon_3(2^{-n}i)} = \frac{(1+(n+k)\ln 2)^\gamma}{(1+n\ln 2)^\gamma}$ and $\lim_{n\to\infty} \frac{(1+(n+k)\ln 2)^\gamma}{(1+n\ln 2)^\gamma} = 1 \ \forall\gamma, \gamma < 0$. we have $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon_3(2^{-n-k}i)}{\upsilon_3(2^{-n}i)} = 1$. Thus υ_3 does not satisfy $(**)$. $\square$

We conclude this chapter with the following remark.

Remark 1.2.11: υ_4 is a type(II) weight which is not a type(I) weight, but it satisfies $(*)_I$, so the assumption being of type(I) weight together with the following assumptions

$\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)} < \infty$ and $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)} < \infty$

is a sufficient condition for concluding that the weight υ satisfies $(*)_I$. Also note that only being of type(I) does not imply that the weight υ satisfies $(*)_I$. υ_7 is an example of such a weight.

Chapter 2

Isomorphic classification of weighted spaces

Introduction to chapter two: The aim of this chapter is to obtain isomorphic classification of weighted spaces of holomorphic functions on the upper halfplane $\mathbf{G}$ and some subspaces of them, by using the wellknown results for weighted spaces of holomorphic functions on the unit disc $\mathbb{D}$. This chapter is divided in to five sections:
Section one includes preliminary definitions and wellknown theorems which are necessary for obtaing our results in section two.
In section two we are dealing only with the weights of type(II) and at the end of this section, we present a complete isomorphic classification of weighted spaces of holomorphic function, whenever the weight v has a moderate rate of growth.
In the remainder of this chapter we study type(I) weights.
In the section three we define special subspaces of $\mathbf{H}_v(\mathbf{G})$ and $\mathbf{H}_{v_0}(\mathbf{G})$ namely, $\mathbf{U}^{\beta}_{\pm}, \mathbf{U}^{\beta}_{\pm,\ 0}$ (for definitions see 1.1.1 & 2.3.6) and we obtain their isomorphic classifications.
In section four we focus on the interesting subspaces of $\mathbf{H}_v(\mathbf{G})$ and $\mathbf{H}_{v_0}(\mathbf{G})$, consisting of 2π-periodic functions(see definition 2.4.3) and again we obtain their isomorphic claasifications (see Theorem 2.4.13).
In section five, we return to the subspaces $\mathbf{U}^{\beta}_{\pm}, \mathbf{U}^{\beta}_{\pm,\ 0}$ and we will show that $\mathbf{U}^{\beta}_{\pm}, (\mathbf{U}^{\beta}_{\pm,\ 0})$ are complemented subspaces in $\mathbf{H}_v(\mathbf{G})(\mathbf{H}_{v_0}(\mathbf{G}))$.

Sections one: Wellknown results about isomorphic classification of weighted spaces of holomorphic functions on the unit disc.

As we said before, in this section we mention the wellknown theorems which are necessary for the next section.

Remark 2.1.1: We mentioned the definitions of $\mathbf{H}_{\upsilon}(\Omega)$ and $\mathbf{H}_{\upsilon_0}(\Omega)$ in Definition 1.1.1. Note that in particular we are dealing with cases $\Omega = \mathbf{G}$ or $\Omega = \mathbb{D}$.
In particular $\mathbf{H}_{\upsilon}(\mathbf{G})(\mathbf{H}_{\upsilon}(\mathbb{D}))$ is the weighted space of holomorphic functions on the upper halfplane $\mathbf{G}$(unit disc $\mathbb{D}$) and $\mathbf{H}_{\upsilon_0}(\mathbf{G})(\mathbf{H}_{\upsilon_0}(\mathbb{D}))$ is the subspace of $\mathbf{H}_{\upsilon}(\mathbf{G})$ $(\mathbf{H}_{\upsilon}(\mathbb{D}))$ consisting of those functions which vanish at infinity.

Definition 2.1.2: Let υ be weight on $\mathbb{D}$. Then
a) we say υ is a radial weight on $\mathbb{D}$ if $\upsilon(z) = \upsilon(\mid z \mid)$ for all $z \in \mathbb{D}$.
b) we say υ is a standard weight on $\mathbb{D}$ if υ is a radial weight and it is a continuous, non increasing function from $[0, 1[$ into $[0, +\infty)$ such that $\lim_{|z|\to 1^-} \upsilon(\mid z \mid) = 0$.

Remark 2.1.3: a) By $\lim_{|z|\to 1^-} \mid f(z) \mid \upsilon(z) = 0$ uniformly, we mean that
$\forall \epsilon > 0 \; \exists \; r(\epsilon) > 0$ such that $\forall z \in \mathbb{D} \;\; \mid z \mid > r(\epsilon) \;\; \mid f(z) \mid \upsilon(z) < \epsilon$.
b) It is easy to see that $\mid f(z) \mid \upsilon(z)$ vanishes at infinity iff
$\lim_{|z|\to 1^-} \mid f(z) \mid \upsilon(z) = 0$ (uniformly).
c) Note that $f \in \mathbf{H}_{\upsilon}(\mathbb{D})$ implies that $\mid f(z) \mid \upsilon(z)$ is a bounded function on $\mathbb{D}$.

Definition 2.1.4: a) Let $I \subset \mathbb{R}$ be an interval. A function $f : I \longrightarrow [0, \infty)$ is called almost decreasing (resp. almost increasing) if there exsist a positive constant C, such that for any $x > y$(resp. $x > y$) it follows that $f(y) \leq C \; f(x)$.
b) Let υ be a radial weight on $\mathbb{D}$. We say υ is almost decreasing (resp. almost increasing) on $\mathbb{D}$, if $\upsilon_{|_{[0,1[}}$ is almost decreasing (resp. almost increasing) function.

Lemma 2.1.5: *Let υ be a continuous, radial and almost decreasing weight on $\mathbb{D}$. Then $\upsilon_1(r) := \sup_{t\geq r} \upsilon(t)$ is a standard weight on $\mathbb{D}$ and there is a constant $C > 0$ such that $\upsilon(r) \leq \upsilon_1(r) \leq C\upsilon(r)$.*

Proof: Similar to Lemma 1.2.5. □

Definition 2.1.6: We define

a) $\ell_\infty := \{(\alpha_k) : \alpha_k \in \mathbb{C} \quad \& \quad \sup_{k\in\mathbb{N}} \mid \alpha_k \mid < \infty\}$.

b) $c_0 := \{(\alpha_k) \in \ell_\infty : \quad \lim_{k\to\infty} \alpha_k = 0\}$.

c) $H_n := \{P \mid P : \partial\mathbb{D} \longrightarrow \mathbb{C}$ is a polynomial of degree$\leq n\}$ $\quad (n \in \mathbb{N})$, where H_n is endowed with the supremum norm on $\partial\mathbb{D}$, that is $\|f\|_\infty = \sup_{z\in\partial\mathbb{D}} \mid f(z) \mid$.

d) $(\sum_{n\in\mathbb{N}} \oplus H_n)_0 := \{(P_n) \mid P_n \in \quad H_n \quad \& \quad \lim_{n\to\infty} \|P_n\| = 0\}$, where $\|P_n\| := \sup_{n\in\mathbb{N}} \|P_n\|_\infty$.

Remark 2.1.7: It is wellknown that $(\sum_{n\in\mathbb{N}} \oplus H_n)_0^{**}$ is isomorphic to $\mathbf{H}_\infty(\mathbb{D})$. See [24]

Theorem 2.1.8: *Let υ be a standard weight on $\mathbb{D}$, which satisfies the condition* $\inf_{n\in\mathbb{N}} \frac{\upsilon(1-2^{-n-1})}{\upsilon(1-2^{-n})} > 0. \quad (*)'$.

Then

i) *If υ satisfies condition $(**)'$:* $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(1-2^{-n-k})}{\upsilon(1-2^{-n})} < 1$,

then $\mathbf{H}_{\upsilon_0}(\mathbb{D})$ is isomorphic to c_0.

ii) *If υ does not satisfy condition $(**)'$, that is* $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(1-2^{-n-k})}{\upsilon(1-2^{-n})} = 1$, *then $\mathbf{H}_{\upsilon_0}(\mathbb{D})$ is not isomorphic to c_0.*

iii) *If υ satisfies both conditions $(*)'$ and condition $(**)'$, then $\mathbf{H}_\upsilon(\mathbb{D})$ is isomorphic to ℓ_∞.*

iv) *If υ satisfies condition $(*)'$ but not condition $(**)'$, then $\mathbf{H}_\upsilon(\mathbb{D})$ is isomorphic to $\mathbf{H}_\infty(\mathbb{D})$.*

Proof: For part (i) and (ii) see [14]. For part (iii) and (iv) see [13] Corollary 1.3. □

Corollary 2.1.9: *Let υ be a standard weight on $\mathbb{D}$ satisfying $(*)'$. Then there are only two possibilities for isomorphism classes of $\mathbf{H}_\upsilon(\mathbb{D})$. Either*

i) *$\mathbf{H}_\upsilon(\mathbb{D})$ is isomorphic to ℓ_∞ and $\mathbf{H}_{\upsilon_0}(\mathbb{D})$ is isomorphic to c_0.*

or

ii) *$\mathbf{H}_\upsilon(\mathbb{D})$ is isomorphic to $\mathbf{H}_\infty(\mathbb{D})$ and $\mathbf{H}_{\upsilon_0}(\mathbb{D})$ is isomorphic to $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$.*

Remark 2.1.10: Note that Corollary 2.1.9 is also true without the assumption that the weight υ satisfies $(*)'$.See [12] Theorem 1.1

Section two: Isomorphic classification of weighted spaces of holomorphic functions on the upper halfplane for type(II) weights.

We begin this section with the following theorem which is a new result and has a main role in obtaining the isomorphic classification of weighted spaces of holomorphic functions for type(II) weights.

Theorem 2.2.1: *Let υ be a type(II) weight on $\mathbf{G}$ satisfying $(*)_{II}$. Put $\tilde{\upsilon}(z) = \upsilon(\alpha(- \mid z \mid)) = \upsilon(\frac{1-|z|}{1+|z|}i)$. Then $\tilde{\upsilon}(z)$ is a radial weight on $\mathbb{D}$. Moreover the map T, defined by $(Tf)(z) = f(\alpha(z)) \quad \forall z \in \mathbb{D}$ is an isomorphism from $\mathbf{H}_\upsilon(\mathbf{G})(\mathbf{H}_{\upsilon_0}(\mathbf{G}))$ onto $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})(\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$.*

Proof: Clearly $\tilde{\upsilon}(z) = \tilde{\upsilon}(\mid z \mid)$. Since υ is of type(II) there is a constant $C > 0$ such that

$\frac{1}{C}\upsilon(\alpha(z)) \leq \upsilon(\alpha(-z)) \leq C\upsilon(\alpha(z)).$ (1).

(See Lemma 1.1.6(ii) and Remark 1.2.4(b)). Also there is a constant $C' > 0$ such that $\upsilon(\omega_1) \leq C'\upsilon(\omega_2)$ whenever $\mid \omega_1 \mid, \mid \omega_2 \mid \leq 1$ and $Im\omega_1 \leq Im\omega_2$.

Consider a fixed $z \in \mathbb{D}$. Firstly assume $Rez \leq 0$. Since $\mid z \mid \geq -Rez$, so

$\frac{1-|z|}{1+|z|} = \frac{1-|z|}{1+|z|}\frac{1+|z|}{1+|z|} = \frac{1-|z|^2}{1+|z|^2+2|z|} \leq \frac{1-|z|^2}{1+|z|^2-2Rez} = \frac{1-|z|^2}{|1-z|^2} = Im(\alpha(z)) \leq \frac{1-|z|^2}{1+|z|^2} < 1$. Thus

$\tilde{\upsilon}(z) \leq C'\upsilon(\alpha(z)).$ (2)

Since $Im(\alpha(z)) \geq Im(\alpha(- \mid z \mid))$ and υ satisfies $(*)_{II}$, there exsist C'' and β such that $\frac{\upsilon(\alpha(z))}{\upsilon(\alpha(-|z|))} \leq C''(\frac{\frac{1-|z|^2}{|1-z|^2}}{\frac{1-|z|}{1+|z|}})^\beta$. So

$\upsilon(\alpha(z)) \leq C''2^{2\beta}\upsilon(\alpha(- \mid z \mid)) = C''2^{2\beta}\tilde{\upsilon}(z)$ (3)

The relations (2) and (3) give us

$\tilde{\upsilon}(z) \leq C'\upsilon(\alpha(z)) \leq C'C''2^{2\beta}\tilde{\upsilon}(z)$ whenever $z \in \mathbb{D}$ and $Rez \leq 0$.

Now if $Rez > 0$ then $Re(-z) < 0$. Using relations (1),(2) and (3) we have

$\tilde{\upsilon}(z) = \tilde{\upsilon}(-z) \leq C'\upsilon(\alpha(-z)) \leq C'C\upsilon(\alpha(z)) \leq C'C^2\upsilon(\alpha(z)) \leq C'C^2\upsilon(\alpha(-z)) \leq C'^2C^2C''2^{2\beta}\tilde{\upsilon}(-z) \leq C'^2C^2C''2^{2\beta}\tilde{\upsilon}(z).$

So the weights $\upsilon \circ \alpha$ and $\tilde{\upsilon}$ are equivalent on $\mathbb{D}$. Thus the map T is welldefined and $g \in \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})(\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$ if and only if $g \circ \alpha^{-1} \in \mathbf{H}_{\upsilon}(\mathbf{G})(\mathbf{H}_{\upsilon_0}(\mathbf{G}))$. This proves the theorem. $\square$

We continue this section with the following theorem which yields the goal of this section.

Remark 2.2.2: From now on the notation $\sim$ has two meanings :

1. Between two Banach spaces means is isomorphic to.

2. Between two norms means is equivalent to.

Theorem 2.2.3: *Let υ be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. Then*

a) *The follwing are equivalent.*

i) *υ satisfies $(**)$.*

ii) *$\mathbf{H}_{\upsilon_0}(\mathbf{G})$ is isomorphic to c_0.*

iii) *$\mathbf{H}_{\upsilon}(\mathbf{G})$ is isomorphic to ℓ_∞.*

b) *The follwing are equivalent.*

i) *υ does not satisfy $(**)$.*

ii) *$\mathbf{H}_{\upsilon}(\mathbf{G})$ is isomorphic to $\mathbf{H}_\infty(\mathbb{D})$.*

iii) *$\mathbf{H}_{\upsilon_0}(\mathbf{G})$ is isomorphic to $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$.*

Proof: By Theorem 2.2.1 we have

$\mathbf{H}_{\upsilon}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ & $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim (\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$ (1)

where $\tilde{\upsilon}(z) = \upsilon(\frac{1-|z|}{1+|z|}i)$.

Since υ is of type(II) there is a constant $C > 0$ such that if $| z_1 |<| z_2 | \Rightarrow \frac{1-|z_2|}{1+|z_2|} < \frac{1-|z_1|}{1+|z_1|} < 1 \Rightarrow \tilde{\upsilon}(| z_2 |) \le C\tilde{\upsilon}(| z_1 |)$. Thus $\tilde{\upsilon}$ is an almost decreasing weight on $\mathbb{D}$. Now define $\upsilon_1(z) := \upsilon_1(| z |) = \sup_{t\ge|z|} \tilde{\upsilon}(t)$. Lemma 2.1.5 implies that υ_1 is a decreasing weight on $\mathbb{D}$ and

$\upsilon_1(z) \ge \tilde{\upsilon}(z) \ge \frac{1}{C}\upsilon_1(z)$. (2)

Relations (1) and (2) imply that

$\mathbf{H}_{\upsilon_1}(\mathbb{D}) \sim \mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{(\upsilon_1)_0}(\mathbb{D}) \sim \mathbf{H}_{\upsilon_0}(\mathbf{G})$. (3)

We have $\lim_{|z|\to 1^-} \upsilon_1(| z |) = \lim_{|z|\to 1^-}(\sup_{t\ge|z|} \tilde{\upsilon}(t))$. Now use the definition of $\tilde{\upsilon}$ and note that $\frac{1+|z|}{1-|z|} \to \infty$ as $\mid z \mid \to 1^-$, so $\lim_{|z|\to 1^-} \upsilon_1(\mid z \mid) = \lim_{t\to\infty} \upsilon(ti)$. Since υ is of type(II) so there is a constant $C' > 0$ such that $\upsilon(ti) \le C'\upsilon(\frac{-1}{ti}) = C'\upsilon(\frac{1}{t}i)$.

Thus $\lim_{t\to\infty} \upsilon(ti) = \lim_{t\to\infty} \upsilon(\frac{1}{t}i) = 0$. Therefore υ_1 is a standard weight in the sense of Definition 2.1.2(b).

Since υ satisfies $(*)_{II}$, so Lemma 1.2.8 implies $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(\frac{1}{2^n}i)}{\upsilon(\frac{1}{2^{n+1}}i)} < \infty$.

Put $a = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(\frac{1}{2^n}i)}{\upsilon(\frac{1}{2^{n+1}}i)}$ then $\forall \in \mathbb{N}\cup\{0\}$ we have $\frac{\upsilon(\frac{1}{2^{n+1}}i)}{\upsilon(\frac{1}{2^n}i)} \ge \frac{1}{a}$. Using relation (2) so

$\upsilon_1(1-\frac{1}{2^{n+1}}) \ge \tilde{\upsilon}(1-\frac{1}{2^{n+1}}) = \upsilon(\frac{1}{2^{n+2}-1}i)$ and $\frac{1}{\upsilon_1(1-\frac{1}{2^n})} \ge \frac{1}{C\upsilon(\frac{1}{2^{n+1}-1}i)}$.

Thus $\frac{\upsilon_1(1-\frac{1}{2^{n+1}})}{\upsilon_1(1-\frac{1}{2^n})} \ge \frac{\upsilon(\frac{1}{2^{n+2}-1}i)}{C\upsilon(\frac{1}{2^{n+1}-1}i)}$. Since υ is of type(II) so $\upsilon(\frac{1}{2^{n+2}}i) \le C\upsilon(\frac{1}{2^{n+2}-1}i)$

and $\frac{1}{\upsilon(\frac{1}{2^{n+1}-1}i)} > \frac{1}{C\upsilon(\frac{1}{2^n}i)}$.

Thus $\frac{\upsilon_1(1-\frac{1}{2^{n+1}})}{\upsilon_1(1-\frac{1}{2^n})} \ge \frac{\upsilon(\frac{1}{2^{n+2}-1}i)}{C\upsilon(\frac{1}{2^{n+1}-1}i)} \ge \frac{\frac{1}{C}\upsilon(\frac{1}{2^{n+2}}i)}{C^2\upsilon(\frac{1}{2^n}i)} \ge \frac{1}{C^3}\frac{\upsilon(\frac{1}{2^{n+2}}i)}{\upsilon(\frac{1}{2^{n+1}}i)}\frac{\upsilon(\frac{1}{2^{n+1}}i)}{\upsilon(\frac{1}{2^n}i)} \ge \frac{1}{C^3a^2} > 0$.

Therefore $\inf_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon_1(1-\frac{1}{2^{n+1}})}{\upsilon_1(1-\frac{1}{2^n})} > 0$. This means that υ_1 satisfies $(*)'$ in the theorem 2.1.8.

Proof of (a): i) $\Rightarrow$ii) : If υ satisfies $(**)$ then Lemma 1.2.9 implies that

$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(\frac{1}{2^{n+k}}i)}{\upsilon(\frac{1}{2^n}i)} < 1$. With an argument similar to the above argument we can conclude that

$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon_1(1-2^{-n-k})}{\upsilon_1(1-2^{-n})} < 1. \quad (**)'$

Part (i) of the Theorem 2.1.8 implies $\mathbf{H}_{(\upsilon_1)_0}(\mathbb{D}) \sim c_0$. Now relation (3) implies $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim c_0$.

ii$\Rightarrow$iii**:** Assume $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim c_0$. Since $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D})$ so $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim c_0$. Now using Corollary 2.1.9 we have $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \ell_\infty$. By Theorem 2.2.1 $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \mathbf{H}_{\upsilon}(\mathbf{G})$ so $\mathbf{H}_{\upsilon}(\mathbf{G}) \sim \ell_\infty$.

iii$\Rightarrow$i**:** Assume $\mathbf{H}_{\upsilon}(\mathbf{G}) \sim \ell_\infty$. Using (3) we obtain $\mathbf{H}_{\upsilon_1}(\mathbb{D}) \sim \ell_\infty$. Now use part (iv) of Theorem 2.1.8 to see that υ_1 must satisfy $(**)'$ and it can be easily seen that υ must satisfy $(**)$.

Proof of (b): i$\Rightarrow$ii**:** Since υ does not satisfy $(**)$ so υ_1 does not satisfy $(**)'$.

So using (iv) of Theorem 2.1.8 yields $\mathbf{H}_{\upsilon_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Again use (3) to show

$\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$.

ii⇒iii: Assume $\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$. By relation (3) we have $\mathbf{H}_{v_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Now using part (ii) of the Corollary 2.1.9 we see that $\mathbf{H}_{(v_1)_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

Relation (3) gives us $\mathbf{H}_{v_0}(\mathbf{G}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

iii⇒i: If $\mathbf{H}_{v_0}(\mathbf{G}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$ then by realation (3) we have $\mathbf{H}_{(v_1)_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. Now using part (i) of the Corollary 2.1.9 we obtain $\mathbf{H}_{v_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. The weight v satisfies $(*)$. Assume v also satisfies $(**)$. Then v_1 satisfies both $(*)'$ and $(**)'$ and in this case Theorem 2.1.8 (iii) implies $\mathbf{H}_{v_1}(\mathbb{D}) \sim \ell_\infty$, which is a contradiction , so v does not satisfy $(**)$. □

Corollary 2.2.4: *Let v be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$.*

Then

i) $\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_{v_0}(\mathbf{G})^{**}$.

ii) *If v satisfies $(*)_I$ then $\mathbf{H}_{v_0}(\mathbf{G})$ has a Schauder basis.*

Proof: i) By Theorem 2.2.3 $\mathbf{H}_v(\mathbf{G}) \sim \ell_\infty$ or $\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$.

If $\mathbf{H}_v(\mathbf{G}) \sim \ell_\infty$ then by Theorem 2.2.1 $\mathbf{H}_{\tilde{v}}(\mathbb{D}) \sim \ell_\infty$. Now Corollary 2.1.9 implies $\mathbf{H}_{v_0}(\mathbb{D}) \sim c_0$. Again by Theorem 2.2.1 we have $\mathbf{H}_{v_0}(\mathbf{G}) \sim c_0$.

Therefore $\mathbf{H}_{v_0}(\mathbf{G})^{**} \sim c_0^{**} = \ell_\infty$. So $\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_{v_0}(\mathbf{G})^{**}$.

If $\mathbf{H}_v(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$ again by Theorem 2.2.1 we have $\mathbf{H}_{\tilde{v}}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Now Corollary 2.1.9(ii) implies $\mathbf{H}_{\tilde{v}_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. Again by Theorem 2.2.1 $\mathbf{H}_{v_0}(\mathbf{G}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. Therfore $\mathbf{H}_{v_0}(\mathbf{G})^{**} \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0^{**} \sim \mathbf{H}_\infty(\mathbb{D})$. (See Remark 2.1.7)

ii): Theorem 2.2.3 implies $\mathbf{H}_{v_0}(\mathbf{G}) \sim c_0$ or $\mathbf{H}_{v_0}(\mathbf{G}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. it is known that c_0 and $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$ have Schauder basis. See [4] □

Corollary 2.2.5: *Let v be a standard weight satisfying $(*)'$. Then*

a) *The following are equivalent.*

i)**:** *v satisfies $(**)'$.*

ii)**:** $\mathbf{H}_v(\mathbb{D}) \sim \ell_\infty$.

iii)**:** $\mathbf{H}_{v_0}(\mathbb{D}) \sim c_0$.

b) *The following are equivalent.*

i)**:** υ *does not satisfy* $(**)'$.

ii)**:** $\mathbf{H}_\upsilon(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$.

iii)**:** $\mathbf{H}_{\upsilon_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

Proof: Use Theorem 2.1.8, Corollary 2.1.9 and an argument similar to what has been done in the proof of the Theorem 2.2.3. □

Section three: Special subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and their isomorphism classification for type(I) weights.

In this section we introduce special subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$(for type(I) weights).
We will show under certain conditions that these subspaces are isomorphic to one of the following spaces: $\mathbf{H}_{\infty}(\mathbb{D}), \ell_{\infty}, c_0$ or $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$.
Before we begin study of subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ we show that for the type(I) weight $\upsilon(\omega) = (Im\omega)^{\beta}$(for some $\beta > 0$) $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ are isomorphic to ℓ_{∞} and c_0 respectively.

Lemma 2.3.1: *Let υ be a weight on $\mathbf{G}$. Then the maps $T' : \mathbf{H}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon\circ\alpha}(\mathbb{D})$ or $T_1' : \mathbf{H}_{\upsilon_0}(\mathbf{G}) \longrightarrow \mathbf{H}_{(\upsilon\circ\alpha)_0}(\mathbb{D})$ defined by $T'(f) = f \circ \alpha$ and $T_1'(f_1) = f_1 \circ \alpha$ are onto isometries.*

Proof: If $f \in \mathbf{H}_{\upsilon}(\mathbf{G})$ then $f\circ\alpha$ is holomorphic on $\mathbb{D}$. Since α is a one-to-one and onto map on $\mathbb{D}$ so
$\|T'(f)\|_{\upsilon\circ\alpha} = \sup_{z\in\mathbb{D}} \mid f(\alpha(z)) \mid \upsilon(\alpha(z)) = \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \upsilon(\omega) = \|f\|_{\upsilon}$ where $\alpha(z) = \omega \quad \forall z \in \mathbb{D}$. Thus T' is a welldefined map.
Conversely if $g \in \mathbf{H}_{\upsilon\circ\alpha}(\mathbb{D})$ then $g \circ \alpha^{-1}$ is holomorphic on $\mathbf{G}$ and we have
$\|g\circ\alpha^{-1}\|_{\upsilon} = \sup_{\omega\in\mathbf{G}} \mid g(\alpha^{-1}(\omega)) \mid \upsilon(\omega) = \sup_{z\in\mathbb{D}} \mid g(z) \mid \upsilon(\alpha(z)) = \|g\|_{\upsilon}$. Therefore T' is an onto map and $T'(g \circ \alpha^{-1}) = g$.
Proof for the map T_1' is the same. We have only to show that if $f\upsilon$ vanishes at infinity on $\mathbf{G}$, then $(f \circ \alpha)\upsilon \circ \alpha$ vanishes at infinity on $\mathbb{D}$ and if g vanishes at infinity on $\mathbb{D}$ then $g \circ \alpha^{-1}$ vanishes at infinity on $\mathbf{G}$. But the above assertions are true since α and α^{-1} are continuous maps. □

Lemma 2.3.2: *Define $\upsilon(\omega) = (Im\omega)^{\beta}$(for some $\beta > 0$). υ is of type(I) weight and for this weight we have $\mathbf{H}_{\upsilon}(\mathbf{G}) \sim \ell_{\infty}$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim c_0$.*

Proof: Clearly υ is of type(I). By definition of υ we have $\upsilon(\alpha(z)) = (\frac{1-|z|^2}{|1-z|^2})^{\beta}$. Now define $T : \mathbf{H}_{\upsilon\circ\alpha}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ and $T_1 : (\mathbf{H}_{\upsilon\circ\alpha})_0(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D})$ by $T(f) = \frac{f}{(1-z)^{2\beta}}$

and $T_1(f_1) = \frac{f_1}{(1-z)^{2\beta}}$ where $\tilde{\upsilon}(z) = (1- \mid z \mid^2)^\beta$

Claim: The map $T(T_1)$ is an onto isometry.

Now Lemma 2.3.1 and our claim imply that $\mathbf{H}_\upsilon(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D})$. But it can be easily seen that $\tilde{\upsilon}$ satisfies $(*)'$ and $(**)'$ therefore by Theorem 2.1.8 we are done.

Proof of claim: Let $f \in \mathbf{H}_{\upsilon\circ\alpha}(\mathbb{D})$ then $\frac{f}{(1-z)^{2\beta}}$ is a holomorphic function on $\mathbb{D}$. Now we have $\|Tf\|_{\tilde{\upsilon}} = \|\frac{f}{(1-z)^{2\beta}}\| = \sup_{z\in\mathbb{D}} \frac{f(z)}{|1-z|^{2\beta}}(1- \mid z \mid^2)^\beta = \sup_{z\in\mathbb{D}} \mid f(z) \mid (\frac{1-|z|^2}{|1-z|^2})^\beta =$ $\sup_{z\in\mathbb{D}} \mid f(z) \mid \upsilon(\alpha(z)) = \|f\|_{\upsilon\circ\alpha}$.So T is welldefined and isometry map.

Suppose $g \in \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$. Put $f_1(z) = (1-z)^{2\beta} g(z)$ then f_1 is a holomorphic function on $\mathbb{D}$ and clearly $\|f_1\|_{\upsilon\circ\alpha} = \|g\|_{\tilde{\upsilon}}$. Therefore T is an onto map. The proof for T_1 is strightforward. □

Lemma 2.3.3: *Let υ be a type(I) weight on $\mathbf{G}$. Suppose $a \in \mathbb{R}$ is arbitrary. Then the translation operators $T_a, T_a : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_\upsilon(\mathbf{G})$, $T_a : \mathbf{H}_{\upsilon_0}(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_0}(\mathbf{G})$ defined by $(T_a f)(\omega) = f(\omega + a)$ for all $\omega \in \mathbf{G}$, are uniformly bounded on $\mathbf{H}_\upsilon(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$.*

Proof: Clearly $(T_a f)$ is a holomorphic function on $\mathbf{G}$. Since υ is a type(I) weight and $Im\omega = Im(\omega + a)$ $\forall \omega \in \mathbf{G}$ $\forall\, a \in \mathbb{R}$, so from part (a) of Remark 1.2.4 there exists a constant $C > 0$ such that $\frac{1}{C}\ \upsilon(\omega) \leq \upsilon(\omega + a) \leq C\upsilon(\omega)$. Thus

$$\begin{aligned} \frac{1}{C} \mid f(\omega+a) \mid \upsilon(\omega) &\leq \mid f(\omega+a) \mid \upsilon(\omega+a) \\ &\leq C \mid f(\omega+a) \mid \upsilon(\omega). \end{aligned}$$

Therefore

$$\begin{aligned} \|T_a f\|_\upsilon &= \sup_{\omega\in\mathbf{G}} \mid f(\omega+a) \mid \upsilon(\omega) \\ &\leq C \sup_{\omega\in\mathbf{G}} \mid f(\omega+a) \mid \upsilon(\omega+a) \\ &= C \sup_{\omega'\in\mathbf{G}} \mid f(\omega') \mid \upsilon(\omega') \\ &= C\|f\|_\upsilon. \end{aligned}$$

So it remains to prove that if $f \in \mathbf{H}_{\upsilon_0}(\mathbf{G})$ then $T_a f \in \mathbf{H}_{\upsilon_0}(\mathbf{G})$. We must show that for given $\epsilon > 0$ there is a compact set $K,\ K \subset \mathbf{G}$ such that $\mid f(\omega' + a) \mid \upsilon(\omega') < \epsilon$ for all $\omega' \in \mathbf{G} \setminus K$. Since $f \in \mathbf{H}_{\upsilon_0}(\mathbf{G})$ for given $\epsilon > 0$, there is a compact set $K' \in \mathbf{G}$ such that $| f(\omega) | \upsilon(\omega) < \epsilon \ \ \forall \omega \in \mathbf{G} \setminus K'$.

Define $K = K' - a = \{\omega - a : \omega \in K'\}$. Clearly $K \subset \mathbf{G}$ and K is compact. We claim $| f(\omega' + a) | \upsilon(\omega') < \epsilon \ \ \forall \omega' \in \mathbf{G} \setminus K$. If there exist a $\omega' \in \mathbf{G} \setminus (K' - a)$ such that $| f(\omega' + a) | \upsilon(\omega') > \epsilon$ then, since $\omega' \notin (K' - a)$, we have $\omega' \neq \omega - a \quad \forall \omega \in K'$ or $\omega' + a \neq \omega \quad \forall \omega \in K'$. Thus $\omega' + a \in \mathbf{G} \setminus K'$, therefore $| f(\omega' + a) | \upsilon(\omega' + a) < \epsilon$. But by using Remark 1.2.6 we have

$\epsilon <| f(\omega' + a) | \upsilon(\omega') \leq | f(\omega' + a) | \upsilon(\omega' + a) < \epsilon$ which is a contradiction. $\square$

Lemma 2.3.4: *Let υ be a type(I) weight on* $\mathbf{G}$:

a) *Then for any $\delta_0 > 0$ there is a constant $C > 0$ (depending on δ_0) such that for any $f \in \mathbf{H}_\upsilon(\mathbf{G})$ we have*

$\frac{1}{C} \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0 \text{ or } \delta \geq \frac{1}{\delta_0}\} \leq \|f\|_\upsilon \leq C \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0 \text{ or } \delta \geq \frac{1}{\delta_0}\}$ (1)

b) *If υ is bounded then for any $\delta_0 > 0$, there is a universal constant $d > 0$ such that for any $f \in \mathbf{H}_\upsilon(\mathbf{G})$*

$\frac{1}{d} \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0\} \leq \|f\|_\upsilon \leq d \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0\}$.

Proof: a) Note that since υ is of type(I), there is a constant $C' > 0$ such that $\frac{1}{C'}\upsilon(\omega_1) \leq \upsilon(\omega_2) \leq C'\upsilon(\omega_1)$ whenever $\omega_1, \omega_2 \in L_\delta(\delta > 0)$. Since $\|f\|_\upsilon = \sup_{\omega \in \mathbf{G}} | f(\omega) | \upsilon(\omega)$ we have

$$\frac{1}{C'} \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : \delta > 0\} \leq \|f\|_\upsilon \leq C' \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : \delta > 0\}. \quad (2)$$

Thus, clearly for any $\delta_0 > 0$,

$\frac{1}{C'} \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0 \text{ or } \delta \geq \frac{1}{\delta_0}\} \leq \|f\|_\upsilon$. If $\delta_0 \geq 1$, then clearly

$\|f\|_\upsilon \leq C' \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \leq \delta_0 \text{ or } \delta \geq \frac{1}{\delta_0}\}$.

Now let $0 < \delta_0 < 1$ be given. Since $\delta_0 < 1$ we can divide the upper halfplane $\mathbf{G}$ as follows in to three parts.

Put $G_1 := \{\omega \in \mathbf{G} : 0 < Im\omega \leq \delta_0\}, \ G_2 := \{\omega \in \mathbf{G} : \delta_0 \leq Im\omega \leq \frac{1}{\delta_0}\}$ and $G_3 := \{\omega \in \mathbf{G} : Im\omega \geq \frac{1}{\delta_0}\}$ then

$\|f\|_\upsilon = \max\{ \ \|f_{|G_1}\|_\upsilon, \ \|f_{|G_2}\|_\upsilon, \ \|f_{|G_3}\|_\upsilon\}$. (3)

Suppose $\delta_0 \le \delta \le \frac{1}{\delta_0}$. We want to estimate $M_\infty(f, L_\delta)\upsilon(\delta i)$. Note since $\frac{\upsilon(\delta i)}{\upsilon(\delta_0 i)}$ and $\frac{\upsilon(\delta i)}{\upsilon(\frac{1}{\delta_0} i)}$ are continus functions on the compact set $[\delta_0, \frac{1}{\delta_0}]$ (with respect to δ) there exist constants $C_1, C_2 > 0$ such that

$$\upsilon(\delta i) \le C_1 \upsilon(\delta_0 i) \quad \text{and} \quad \upsilon(\delta i) \le C_2 \upsilon(\tfrac{1}{\delta_0} i). \qquad (4)$$

By the Phragmen-Lindelöf Theorem(See [18] Theorem 12.8) we have

$$\begin{aligned} M_\infty(f, L_\delta) &\le M_\infty(f, L_{\delta_0})^{\frac{\frac{1}{\delta_0}-\delta}{\frac{1}{\delta_0}-\delta_0}} \, M_\infty(f, L_{\frac{1}{\delta_0}})^{\frac{\delta-\delta_0}{\frac{1}{\delta_0}-\delta_0}} \\ &\le \max(M_\infty(f, L_{\delta_0}), M_\infty(f, L_{\frac{1}{\delta_0}}))^{\frac{\frac{1}{\delta_0}-\delta}{\frac{1}{\delta_0}-\delta_0}} \max(M_\infty(f, L_{\delta_0}), M_\infty(f, L_{\frac{1}{\delta_0}}))^{\frac{\delta-\delta_0}{\frac{1}{\delta_0}-\delta_0}} \\ &\le \max(M_\infty(f, L_{\delta_0}), M_\infty(f, L_{\frac{1}{\delta_0}})). \qquad (5) \end{aligned}$$

If $\max(M_\infty(f, L_{\delta_0}), M_\infty(f, L_{\frac{1}{\delta_0}})) = M_\infty(f, L_{\delta_0})$, then relations (4) & (5) imply that

$$M_\infty(f, L_\delta)\upsilon(\delta i) \le C_1 M_\infty(f, L_{\delta_0})\upsilon(\delta_0 i). \qquad (6)$$

If $\max(M_\infty(f, L_{\delta_0}), M_\infty(f, L_{\frac{1}{\delta_0}})) = M_\infty(f, L_{\frac{1}{\delta_0}})$, then relations (4) & (5) imply that

$$M_\infty(f, L_\delta)\upsilon(\delta i) \le C_2 M_\infty(f, L_{\frac{1}{\delta_0}})\upsilon(\tfrac{1}{\delta_0} i). \qquad (7)$$

Now relations (6) & (7) imply that

$$M_\infty(f, L_\delta)\upsilon(\delta i) \le C_3 \sup\{M_\infty(f, L_\tau)\upsilon(\tau i) : 0 < \tau \le \delta_0 \text{ or } \tau \ge \tfrac{1}{\delta_0}\}. \quad (8)$$

where $C_3 = \max\{C_1, C_2\}$. Now consider relations (3) & (8) and put $C = \max\{C', C_3\}$ we are done.

Proof of b): Let $\delta_0 > 0$ be given. By relation (1) we have

$\frac{1}{C} \sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta \le \delta_0\} \le \|f\|_\upsilon$

Since $f \in \mathbf{H}_\upsilon(\mathbf{G})$ and υ is bounded so $f_{|G_\delta}$ is bounded. Now Lemma 1.1.10 implies $M_\infty(f, L_\delta) \le M_\infty(f, L_{\delta_0})$. Thus $M_\infty(f, L_\delta)\upsilon(\delta i) \le M_\infty(f, L_{\delta_0})\ \upsilon(\delta_0 i)\ \frac{\upsilon(\delta i)}{\upsilon(\delta_0 i)}$

Since υ is bounded and it is almost increasing on the imaginary axis, there is a constant $C' > 0$ such that $\frac{\upsilon(\delta i)}{\upsilon(\delta_0 i)} \le C'\ \forall\ \delta > \delta_0$. So $\sup\{M_\infty(f, L_\delta)\upsilon(\delta i) : 0 < \delta < \delta_0\} \le C' M_\infty(f, L_{\delta_0})\upsilon(\delta_0 i)$. Put $d = \max\{C, C'\}$. We are done. □

Proposition 2.3.5: *Suppose υ is a bounded type(I) weight. Put*

$\tilde{\upsilon}(z) = \upsilon(\alpha(-\mid z\mid)) = \upsilon(\frac{1-|z|}{1+|z|}i)$. *Then there are constants* $C_1 > 0$ & $C_2 > 0$ *such that* $C_1\|f\|_\upsilon \le \sup_{a\in\mathbb{R}} \|(T_a f)\circ\alpha\|_{\tilde{\upsilon}} \le C_2\|f\|_\upsilon$ *where* T_a *is as in Proposition* 2.3.3.

Proof: By part(b) of Lemma 2.3.4, there is a constant $d > 0$ such that

$\frac{1}{d}\sup_{0<\delta<1} M_\infty(f, L_\delta)\upsilon(\delta i) \le \|f\|_\upsilon \le d\sup_{0<\delta<1} M_\infty(f, L_\delta)\upsilon(\delta i)$ (1)

Consider an arbitrarily fixed $\omega \in \mathbf{G}$ with $\delta := Im\omega < 1$. Put $a = Re\omega$. Then since υ is a type(I) weight so there is a universal constant $d_1 > 0$ such that

$\mid f(\omega)\mid \upsilon(\omega) \le d_1 \mid (T_a f)(i\delta)\mid \upsilon(\delta i)$

Put $z := \alpha^{-1}(i\delta) = \frac{\delta-1}{\delta+1}$. Then $\mid z\mid = -Rez = \frac{1-\delta}{\delta+1}$ (since $\delta < 1$) so $\delta = \frac{1-|z|}{1+|z|}$ and $\mid f(\omega)\mid \upsilon(\omega) \le d_1 \mid (T_a f)(\frac{1-|z|}{1+|z|}i)\mid \upsilon(\frac{1-|z|}{1+|z|}i)$. Therefore $\mid f(\omega)\mid \upsilon(\omega) \le d_1 \mid (T_a f)(\alpha(-\mid z\mid))\mid \upsilon(\alpha(-\mid z\mid))$. Since $-\mid z\mid = Rez = z$ the previous relation implies

$\mid f(\omega)\mid \upsilon(\omega) \le d_1 \sup_{z\in\mathbb{D}} \mid (T_a f)\circ\alpha(z)\mid \tilde{\upsilon}(z) = d_1\|T_a f\circ\alpha\|_{\tilde{\upsilon}}$

Therefore $\sup\{\mid f(\omega)\mid \upsilon(\omega) : \omega \in \mathbf{G}, Im\omega < 1\} \le d_1 \sup_{a\in\mathbb{R}} \|(T_a f)\circ\alpha\|_{\tilde{\upsilon}}$ (2)

The relations (1) and (2) imply that there is a $C_1 > 0$ such that $C_1\|f\|_\upsilon \le \sup_{a\in\mathbb{R}} \|(T_a f)\circ\alpha\|_{\tilde{\upsilon}}$. Now consider a fixed $z \in \mathbb{D}$ and a fixed $a \in \mathbb{R}$. We recall that $\frac{1-|z|}{1+|z|} \le \frac{1-|z|^2}{|1-z|^2} = Im\alpha(z)$. Since υ is of type(I) there is a constant $d_2 > 0$ such that

$\upsilon(\frac{1-|z|}{1+|z|}i) \le d_2\upsilon(\frac{1-|z|^2}{|1-z|^2}i)$. So $\mid (T_a f)\circ\alpha(z)\mid \tilde{\upsilon}(z) \le d_2 \mid (T_a f)\circ\alpha(z)\mid \upsilon(Im(\alpha(z))i)$.

Put $\alpha(z) = \omega$. Then $\mid (T_a f)\circ\alpha(z)\mid \tilde{\upsilon}(z) \le d_2 \mid (T_a f)(\omega)\mid \upsilon(\omega) \le d_2\|T_a f\|_\upsilon \le d_3\|f\|_\upsilon$

The last inequality is a consequence of Propositon 2.3.3 so $\sup_{a\in\mathbb{R}}\sup_{z\in\mathbb{D}} \mid (T_a f)\circ\alpha(z)\mid \tilde{\upsilon}(z) = \sup_{a\in\mathbb{R}} \|(T_a f)\circ\alpha(z)\|_{\tilde{\upsilon}} \le d_3\|f\|_\upsilon$. □

Definition 2.3.6: Let υ be a type(I) weight on $\mathbf{G}$. Asumme that β is an even integer. We define

$U^\beta_\pm := \{f \in \mathbf{H}_\upsilon(\mathbf{G}) : \omega^{2\beta}f(\omega) = \pm f(-\frac{1}{\omega})\ \forall\omega \in \mathbf{G}\}$ & $U^\beta_{\pm,\ 0} := U^\beta_\pm \cap \mathbf{H}_{\upsilon_0}(\mathbf{G})$.

Remark 2.3.7: a) Let $f \in \mathbf{H}_\upsilon(\mathbf{G})(\mathbf{H}_{\upsilon_0}(\mathbf{G}))$. Since $\frac{(f\circ\alpha)(z)}{(1-z)^{2\beta}}$ is a holomorphic function on $\mathbb{D}$ so there are $\gamma'_k \in \mathbb{C}$ such that $\frac{(f\circ\alpha)(z)}{(1-z)^{2\beta}} = \sum_{k=0}^\infty \gamma'_k z^k$. Put $\alpha(z) = \omega$ (so $z = \alpha^{-1}(\omega)$).

Therefore $f(\omega) = \sum_{k=0}^{\infty} \gamma_k \frac{1}{(\omega+i)^{2\beta}} (\frac{\omega-i}{\omega+i})^k$ where $\gamma_k = 2^{2\beta}\gamma'_k$

b) Note that in general $U_{\pm}^{\beta}$ and $U_{\pm,0}^{\beta}$ are not necessarily nontrivial subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ respectively. But we will show that if the weight υ satisfies $(*)_I$, then $U_{\pm}^{\beta}$ and $U_{\pm,0}^{\beta}$(for certain βs) are infinite dimensional subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ respectively.

Lemma 2.3.8: (i): *For any $f \in U_{+}^{\beta}$, there are $\gamma_k \in \mathbb{C}$ such that*
$f(\omega) = \sum_{k=0}^{\infty} \gamma_k (\omega+i)^{-2\beta} (\frac{\omega-i}{\omega+i})^{2k}$.
ii) *For any $f \in U_{-}^{\beta}$, there are $\gamma_k \in \mathbb{C}$ such that $f(\omega) = \sum_{k=0}^{\infty} \gamma_k (\omega+i)^{-2\beta} (\frac{\omega-i}{\omega+i})^{2k+1}$.*

Proof: we prove part (i). The proof of part (ii) is similar.
Suppose $f \in U_{+}^{\beta}$. Put $\alpha(z) = \frac{1+z}{1-z} i = \omega$, so $\alpha(-z) = -\frac{1}{\omega}$.
Since β is an even integer we obtain $\omega^{2\beta} = \frac{(1+z)^{2\beta}}{(1-z)^{2\beta}}$. Thus $\frac{f(\alpha(z))}{(1-z)^{2\beta}} = \frac{f(\alpha(-z))}{(1+z)^{2\beta}}$. Put $g(z) := \frac{f(\alpha(z))}{(1-z)^{2\beta}}$. Then g is holomorphic on $\mathbb{D}$ (but not necessarily bounded) so there are $\gamma'_k \in \mathbb{C}$ such that $g(z) = \sum_{k=0}^{\infty} \gamma'_k z^k$. Now suppose $n \in \mathbb{N}$ is arbitrary and fixed. Define $h_n(z) = \frac{1}{2\pi} \int_o^{2\pi} g(ze^{i\phi}) e^{-in\phi} d\phi$. Since $\sum_{k=0}^{\infty} \gamma'_k z^k$ is convergent on $\mathbb{D}$, we have $h_n(z) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \gamma'_k z^k \int_o^{2\pi} e^{i(k-n)\phi} d\phi = \gamma'_n z^n$. Since n is fixed and $g(z) = g(-z)$ we obtain $h_n(z) = h_n(-z)$. That is $\gamma'_n z^n = \gamma'_n (-z)^n \ \forall z \in \mathbb{D}$. This implies $\gamma'_n = 0$ if n is odd. Thus $g(z) = \sum_{k=0}^{\infty} \gamma'_k z^{2k}$. So we have $\frac{f(\alpha(z))}{(1-z)^{2\beta}} = \sum_{k=0}^{\infty} \gamma'_k z^{2k}$. But
$1 - z = 1 - \alpha^{-1}(\omega) = \frac{2i}{\omega+i}$, therefore $f(\omega) = \sum_{k=0}^{\infty} \gamma_k (\omega+i)^{-2\beta} (\frac{\omega-i}{\omega+i})^{2k}$, where
$\gamma_k = 2^{\beta}\gamma'_k$. □

Remark 2.3.9: If the weight υ on $\mathbf{G}$ satisfies $(*)_I$, we can choose a new β (perhaps by increasing β) which is an even integer. From now on we always assume that β is an even integer whenever υ satisfies $(*)_I$ with respect to β.

Lemma 2.3.10: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_I$ with respect to β. Then $\mathrm{U}_{\pm}^{\beta}$ and $U_{\pm,0}^{\beta}$ are infinite dimensional subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ respectively.*

Proof: Define $\mathrm{f}_k(\omega) = \frac{1}{(\omega+i)^{2\beta}} (\frac{\omega-i}{\omega+i})^{2k}$ for any $k \in \mathbb{N} \cup \{0\}$, clearly

$\omega^{2\beta} f_k(\omega) = f_k(-\frac{1}{\omega})$ for any $k \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}\|f_k\|_\upsilon &= \sup_{\omega\in\mathbf{G}} \frac{1}{\mid \omega+i \mid^{2\beta}} \mid \frac{\omega-i}{\omega+i} \mid^{2k} \upsilon(\omega) \\ &\le \sup_{\omega\in\mathbf{G}} \frac{\upsilon(\omega)}{\mid \omega+i \mid^{2\beta}} \\ &= \max(\sup\{\frac{\upsilon(\omega)}{\mid \omega+i \mid^{2\beta}} : \omega \in \mathbf{G} \textit{ and } Im\omega \le 1\},\end{aligned}$$

$\sup\{\dfrac{\upsilon(\omega)}{\mid \omega+i \mid^{2\beta}} : \omega \in \mathbf{G} \textit{ and } Im\omega > 1\})$.

But $\sup\{\frac{\upsilon(\omega)}{|\omega+i|^{2\beta}} : \omega \in \mathbf{G}$ and $Im\omega \le 1\} \le \sup\{\upsilon(\omega) : \omega \in \mathbf{G}$ and $Im\omega \le 1\} \le \upsilon(i)$.

Also $\sup\{\frac{\upsilon(\omega)}{|\omega+i|^{2\beta}} : \omega \in \mathbf{G}$ and $Im\omega > 1\} = \sup\{\frac{\upsilon(\omega)}{\upsilon(i)} \frac{\upsilon(i)}{|\omega+i|^{2\beta}} : \omega \in \mathbf{G}$ and $Im\omega > 1\}$.

Now $(*)_I$ implies that there is a constant $C > 0$ such that

$\sup\{\frac{\upsilon(\omega)}{|\omega+i|^{2\beta}} : \omega \in \mathbf{G}$ and $Im\omega > 1\} \le C\frac{(Im\omega)^\beta}{|\omega+i|^{2\beta}}\upsilon(i) \le C\upsilon(i)$.

Therefore for any $k \in \mathbb{N} \cup \{0\}$ we have

$\|f_k\|_\upsilon \le \max(\upsilon(i), C\upsilon(i)) < \infty$. (1)

Relation (1) implies $f_k \in \mathbf{H}_\upsilon(\mathbf{G})$ for any $k \in \mathbb{N} \cup \{0\}$.

Since $\mid \frac{\omega-i}{\omega+i} \mid \to 1$ as $\mid \omega \mid \to 1$ so $\frac{1}{(\omega+i)^{2\beta}}(\frac{\omega-i}{\omega+i})^{2k}(1-(\frac{\omega-i}{\omega+i})^2) = f_k(\omega) - f_{k+1}(\omega) \in U^\beta_{+,\,0}$ for any $k \in \mathbb{N} \cup \{0\}$.

Similarly define $g_k(\omega) := (\frac{\omega-i}{\omega+i})f_k(\omega)$ for any $k \in \mathbb{N} \cup \{0\}$, then $g_k(\omega) \in U^\beta_-$ and $g_k(\omega) - g_{k+1}(\omega) \in U^\beta_{-,0}$ for any $k \in \mathbb{N} \cup \{0\}$. □

Proposition 2.3.11: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_I$ with respect to β. Put $\tilde{\upsilon}(z) = \upsilon(\alpha(-\mid z \mid)) = \upsilon(\frac{1-|z|}{1+|z|}i)$ on $\mathbb{D}$. Then $U^\beta_\pm(U^\beta_{\pm,\,o})$ are isomorphic to $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ $(\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$.*

Proof: Firstly, note that without loss of generality we assume that the constants(C) which appear in the definition of type(I) weight and relation $(*)_I$ are the same.

Since υ satisfies $(*)_I$ and $Im(-\frac{1}{\omega}) = \frac{Im\omega}{|\omega|^2}$ so there is a constant $C > 0$ such that

$$\frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})} \le \begin{cases} C \mid \omega \mid^{2\beta} & \text{if } \mid \omega \mid \ge 1 \\ C & \text{if } \mid \omega \mid \le 1 \end{cases} \qquad (1)$$

Consider $f \in U^\beta_\pm$ and $\omega \in \mathbf{G}$ with $\mid \omega \mid \ge 1$. Then (1) implies that

$$\mid f(\omega) \mid \upsilon(\omega) = \mid f(-\tfrac{1}{\omega}) \mid \upsilon(-\tfrac{1}{\omega}) \frac{1}{|\omega|^{2\beta}} \frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})} \le C \mid f(-\tfrac{1}{\omega}) \mid \upsilon(-\tfrac{1}{\omega}). \qquad (2)$$

If $|\,\omega\,| \geq 1$ then $|\,-\frac{1}{\omega}\,| \leq 1$. So (2) implies that

$\sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \geq 1\}$

$\leq C\ \ \sup\{|\ f(\omega')\ |\ v(\omega') : \omega' \in \mathbf{G}, |\ \omega'\ | \leq 1\}.$ (3)

Clearly

$\sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \geq 1\} \leq \|f\|_v =$

$\max(\sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \leq 1\},$

$\sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \geq 1\})$ (4)

Put this in (3). We obtain

$\sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \leq 1\} \leq \|f\|_v \leq C \sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \leq 1\}$ (5)

We recall that with $z := \alpha^{-1}(\omega)$ we have $Rez \leq 0$ iff $|\ \omega\ | \leq 1$. Also if $z \in \mathbb{D}$ and $Rez \leq 0$ then $\frac{1-|z|}{1+|z|} \leq \frac{1-|z|^2}{|1-z|^2} = Im(\alpha(z)) \leq \frac{1-|z|^2}{1+|z|^2}$. v is of type(I) hence $\tilde{v}(z) \leq Cv(\alpha(z)) \leq C^2 v(\frac{1-|z|^2}{1+|z|^2} i)$ for some constant $C > 0$. Put $\omega_1 = \frac{1-|z|^2}{1+|z|^2} i$ and $\omega_2 = \frac{1-|z|}{1+|z|} i$.

Since we satisfies $(*)_I$ we have

$\tilde{v}(z) \leq Cv(\alpha(z)) \leq C^2 v(\frac{1-|z|^2}{1+|z|^2} i) \leq C^3 2^{2\beta} v(\frac{1-|z|}{1+|z|} i) = C^3 2^{2\beta} \tilde{v}(z).$ (6)

The above relation is true $\forall z \in \mathbb{D}$ with $Rez \leq 0$.

For an arbitrary $z \in \mathbb{D}$ we have $\frac{1-|z|}{1+|z|} \leq \frac{1-|z|^2}{1+|z|^2} = Im(\alpha(z))$ which yields $C_1 \|f \circ \alpha\|_{\tilde{v}} \leq \|f \circ \alpha\|_{v \circ \alpha}$ if $f \in U \pm^{\beta}$ *where* $C_1 = \frac{1}{C}$. Clearly $\|f \circ \alpha\|_{v \circ \alpha} = \|f\|_v$.

Now using relation (5) we have

$C_1 \|f \circ \alpha\|_{\tilde{v}} \leq \|f \circ \alpha\|_{v \circ \alpha} = \|f\|_v \leq C\ \ \sup\{|\ f(\omega)\ |\ v(\omega) : \omega \in \mathbf{G}, |\ \omega\ | \leq 1\}$. So

$C_1 \|f \circ \alpha\|_{\tilde{v}} \leq \|f \circ \alpha\|_{v \circ \alpha} \leq C \sup\{|\ (f \circ \alpha)(z)\ |\ v \circ \alpha(z) : z \in \mathbb{D}$ and $Rez \leq 0\}.$ (7)

Using (6) we have $v(\alpha(z)) \leq C^2 2^{2\beta} \tilde{v}(z)\ \ \forall z \in \mathbb{D}\ Rez \leq 0$. So

$\sup\{|\ (f \circ \alpha)(z)\ |\ v \circ \alpha(z) : z \in \mathbb{D}\ \&\ Rez \leq 0\} \leq C_2 \sup\{|\ (f \circ \alpha)(z)\ |\ \tilde{v}(z) : z \in \mathbb{D}\ \&\ Rez \leq 0\} \leq C_2\ \|f \circ \alpha\|_{\tilde{v}},$ (8)

where $C_2 = C^2 2^{2\beta}$. Relations (7) and (8) yield

$$\begin{aligned} C_1 \|f \circ \alpha\|_{\tilde{v}} &\leq \|f \circ \alpha\|_{v \circ \alpha} \\ &\leq \sup\{| f(\alpha(z)) | v(\alpha(z)) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &\leq C_2 \ \|f \circ \alpha\|_{\tilde{v}}. \qquad (9) \end{aligned}$$

By Lemma 2.3.8 there are $\gamma_k \in \mathbb{C}$ such that $\frac{f \circ \alpha(z)}{(1-z)^{2\beta}} = \sum_{k=0}^{\infty} \gamma_k z^{2k}$ if $f \in U_+^\beta$ and $\frac{f \circ \alpha(z)}{(1-z)^{2\beta}} = \sum_{k=0}^{\infty} \gamma_k z^{2k+1}$ if $f \in U_-^\beta$ (β is an even integer)

Put $g(z) = \frac{f \circ \alpha(z)}{(1-z)^{2\beta}}$. Then $| g(z) |=| g(-z) | \ \ \forall z \in \mathbb{D}$ ($\forall f \in \ U_\pm^\beta$). Since $Re(-z) = -Rez$ we obtain

$$\begin{aligned} \|g\|_{\tilde{v}} &= \sup\{| g(z) | \tilde{v}(z) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &= \sup\{| (f \circ \alpha)(z) | \tilde{v}(z) \frac{1}{| 1 - z |^{2\beta}} : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &\leq \sup\{| (f \circ \alpha)(z) | \tilde{v}(z) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &\leq \|f \circ \alpha\|_{\tilde{v}} \\ &\leq C\|f \circ \alpha\|_{v \circ \alpha} \qquad (By (9)) \\ &\leq C \ C_2 \|f \circ \alpha\|_{\tilde{v}} \\ &\leq C^3 \ C_2 \ \sup\{| f \circ \alpha(z) | v \circ \alpha(z) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \qquad (By (7)) \\ &\leq C^7 2^{4\beta} \sup\{| (f \circ \alpha)(z) | \tilde{v}(z) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \qquad (By (6)) \\ &\leq C^7 2^{4\beta} \sup\{| g(z) | \tilde{v}(z) | 1 - z |^{2\beta} : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &\leq C^7 2^{6\beta} \sup\{| g(z) | \tilde{v}(z) : z \in \mathbb{D} \ \& \ Rez \leq 0\} \\ &= C^7 2^{6\beta} \|g\|_{\tilde{v}}. \end{aligned}$$

(Note that in the above computation we have used the relation $g(z) = \pm g(-z)$).

So we have proved $\|g\|_{\tilde{v}} \leq \|f \circ \alpha\|_{\tilde{v}} \leq C\|f\|_v \leq C^7 2^{6\beta} \|g\|_{\tilde{v}}$. (10)

But (10) implies that the maps

$T : U_\pm^\beta \longrightarrow \{g \in \mathbf{H}_{\tilde{v}}(\mathbb{D}) : g(z) = \pm g(-z) \ \forall z \in \mathbb{D}\}$

or

$T : U_{\pm,\ 0}^\beta \longrightarrow \{g \in \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) : g(z) = \pm g(-z) \ \forall z \in \mathbb{D}\}$

defined by $T(f) = \frac{f \circ \alpha}{(1-z)^{2\beta}}$ are isomorphisms. These maps are onto isomorphisms too.

For example let

$g \in \{g \in \mathbf{H}_{\tilde{v}}(\mathbb{D}) : g(z) = g(-z)\ \forall z \in \mathbb{D}\}$ be given. Put $f(\omega) = 2^{2\beta} \frac{g \circ \alpha^{-1}(\omega)}{(\omega+i)^{2\beta}}$. Since

$\|f\|_v = \|f \circ \alpha\|_{v \circ \alpha}$ so $\|f\|_v \leq 2^{2\beta} \sup_{z \in \mathbb{D}} \mid g(z) \mid v(\alpha(z))$.

Again use the relation (6) and facts $g(z) = g(-z)$ & $Re(z) = Re(-z)$ for all $z \in \mathbb{D}$, there exsists a constant $d > 0$ such that $\|f\|_v \leq d\|g\|_{\tilde{v}}$. Since $\|g\|_{\tilde{v}} < \infty$ we obtain $f \in \mathbf{H}_v(\mathbf{G})$. Also we have $\omega^{2\beta} f(\omega) = f(-\frac{1}{\omega})$ which implies $f \in U_+^\beta$. Finally by definition we have $T(f) = g$. So for completing the proof we must show that

$A_\pm = \{g \in \mathbf{H}_{\tilde{v}}(\mathbb{D}) : g(z) = \pm g(-z)\ \forall z \in \mathbb{D}\}$ is isomorphic to $\mathbf{H}_{\tilde{v}}(\mathbb{D})$ and

$B_\pm = \{g \in \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) : g(z) = \pm g(-z)\ \forall z \in \mathbb{D}\}$ is isomorphic to $\mathbf{H}_{\tilde{v}_0}(\mathbb{D})$. Since v satisfies $(*)_I$ Lemma 1.2.8 (ii) implies that $\sup_{n \in \mathbb{N} \cup \{0\}} \frac{v(2^{-n} i)}{v(2^{-n-1} i)} < \infty$. As in the proof of the Theorem 2.2.3 we can conclude $\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\tilde{v}(1-2^{-n})}{\tilde{v}(1-2^{-n-1})} < \infty$. But this relation implies that $\sup_{r<1} \frac{\tilde{v}(r^2)}{\tilde{v}(r)} < \infty$. Put

$a = \sup_{r<1} \frac{\tilde{v}(r^2)}{\tilde{v}(r)} < \infty$. So for all r with $0 \leq r < 1$ we have $\tilde{v}(r^2) < a\, \tilde{v}(r)$. (11)

Define $S_+ : \mathbf{H}_{\tilde{v}}(\mathbb{D}) \longrightarrow A_+ = \{g \in \mathbf{H}_{\tilde{v}}(\mathbb{D}) : g(z) = g(-z)\ \forall z \in \mathbb{D}\}$ and

$S_+ : \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) \longrightarrow B_+ = \{g \in \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) : g(z) = g(-z)\ \forall z \in \mathbb{D}\}$ by

$(S_+ g)(z) = g(z^2)$. And define

$S_- : \mathbf{H}_{\tilde{v}}(\mathbb{D}) \longrightarrow A_- = \{g \in \mathbf{H}_{\tilde{v}}(\mathbb{D}) : g(z) = -g(-z)\ \forall z \in \mathbb{D}\}$

or

$S_- : \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) \longrightarrow B_- = \{g \in \mathbf{H}_{\tilde{v}_0}(\mathbb{D}) : g(z) = -g(-z)\ \forall z \in \mathbb{D}\}$ by $(S_- g)(z) = z g(z^2)$

If we prove that $S_+,\ S_-$ are isomorphisms then

$S_+^{-1} T : U_+^\beta (U_{+,\ 0}^\beta) \longrightarrow \mathbf{H}_{\tilde{v}}(\mathbb{D})(\mathbf{H}_{\tilde{v}_0}(\mathbb{D}))$ and $S_-^{-1} T : U_-^\beta (U_{-,\ 0}^\beta) \longrightarrow \mathbf{H}_{\tilde{v}}(\mathbb{D})(\mathbf{H}_{\tilde{v}_0}(\mathbb{D}))$ are isomorphisms and proof will be complete.

S_+ is a welldefined map and

$$\begin{aligned}\|(S_+g)\|_{\tilde{v}} &= \sup_{z\in\mathbb{D}} | g(z^2) | \tilde{v}(z) \\ &= \sup_{z\in\mathbb{D}} | g(z^2) | \tilde{v}(z^2)\frac{\tilde{v}(z)}{\tilde{v}(z^2)} \\ &\leq C\sup_{z\in\mathbb{D}} | g(z^2) | \tilde{v}(z^2) \\ &\leq C\|g\|_{\tilde{v}}.\end{aligned}$$

So S_+ is bounded.(Note that $\tilde{v}$ is almost decreasing)

Define $(S_+^{-1}g)(z) = g(z^{\frac{1}{2}})$ then $(S_+^{-1}S_+)(g) = g$ and

$\|(S_+^{-1}g)\|_{\tilde{v}} = \sup_{z\in\mathbb{D}} | g(z^{\frac{1}{2}}) | \tilde{v}(z) = \sup_{z\in\mathbb{D}} | g(z^{\frac{1}{2}}) | \tilde{v}(z^{\frac{1}{2}})\frac{\tilde{v}(z)}{\tilde{v}(z^{\frac{1}{2}})} \leq a\|g\|_{\tilde{v}}$ by(11)

So S_+ is an onto isomorphism. Clearly S_- is a welldefined map.

$\|(S_-g)\|_{\tilde{v}} = \sup_{z\in\mathbb{D}} | z | | g(z^2) | \tilde{v}(z) \leq \sup_{z\in\mathbb{D}} | g(z^2) | \tilde{v}(z^2)\frac{\tilde{v}(z)}{\tilde{v}(z^2)} \leq C\|g\|_{\tilde{v}}$

So S_- is bounded. If $g \in A_-$ or $g \in B_-$ then there are $\alpha_k \in \mathbb{C}$ such that $g(z) = \sum_{k=0}^{\infty} \alpha_k z^{2k+1} = z\sum_{k=0}^{\infty} \alpha_k z^{2k}$. Put $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$. Then $(S_-^{-1}g)(z) = f(z) = z^{-\frac{1}{2}}g(z^{\frac{1}{2}})$

S_-^{-1} is bounded because

$$\begin{aligned}\|(S_-^{-1}g)\|_{\tilde{v}} &= \sup_{z\in\mathbb{D}} | (S_-^{-1}g)(z) | \tilde{v}(z) \\ &= \max(\sup_{|z|\leq\frac{1}{2}} | (S_-^{-1}g)(z) | \tilde{v}(z), \sup_{|z|\geq\frac{1}{2}} | (S_-^{-1}g)(z) | \tilde{v}(z)).\end{aligned}$$

But by using maximum principle and the fact $\tilde{v}$ is almost decreasing we have

$$\begin{aligned}\sup_{|z|\leq\frac{1}{2}} | (S_-^{-1}g)(z) | \tilde{v}(z) &\leq C\sup_{|z|=\frac{1}{2}} | (S_-^{-1})g(z) | \tilde{v}(\frac{1}{2})\frac{\tilde{v}(0)}{\tilde{v}(\frac{1}{2})} \\ &\leq C_1\sup_{|z|=\frac{1}{2}} | (S_-^{-1}g)(z) | \tilde{v}(\frac{1}{2}).\end{aligned}$$

So $\quad \|(S_-^{-1}g)\|_{\tilde{v}} \leq C_1\sup_{|z|\geq\frac{1}{2}} | (S_-^{-1}g)(z) | \tilde{v}(z) \leq C_2\|g\|_{\tilde{v}}$

Thus S_-^{-1} is an onto isomorphism and we are done. □

Theorem 2.3.12: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_I$ with respect to β.*

a) *The following are equivalent.*

i) U_+^{β}, U_-^{β} *are isomorphic to* ℓ_∞.

ii) $U^\beta_{+,\,0}$, $U^\beta_{-,\,0}$ *are isomorphic to* c_0.

iii) υ *satisfies* $(**)$.

b) *The following are equivalent.*

i) U^β_+, U^β_- *are isomorphic to* $\mathbf{H}_\infty(\mathbb{D})$.

ii) $U^\beta_{+,\,0}$, $U^\beta_{-,\,0}$ *are isomorphic to* $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

iii) υ *does not satisfy* $(**)$.

Proof:(a): i) $\Rightarrow$ii) : By Proposition 2.3.11 we have $U^\beta_+ \sim \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ so $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \ell_\infty$. Using Corollary 2.1.9(i) we conclude that $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim c_0$. Now Proposition 2.3.11 implies that $U^\beta_{+,\,0} \sim c_0$ and $U^\beta_{-,\,0} \sim c_0$.

ii) $\Rightarrow$iii) : Our assumption and Proposition 2.3.11 imply $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim c_0$. Now Theorem 2.1.8(ii) implies $\tilde{\upsilon}$ satisfies $(**)'$, so υ satisfies $(**)$.

iii) $\Rightarrow$i) : υ satisfies $(**)$ so $\tilde{\upsilon}$ satisfies $(**)'$. Now Theorem 2.1.8(i) implies that $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim c_0$. Now Corollary 2.1.9 implies that $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \ell_\infty$. Using Proposition 2.3.11 we are done.

Proof of (b): i) $\Rightarrow$ii) By Proposition 2.3.11 we have $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Now Corollary 2.1.9(ii) implies that $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. But Proposition 2.3.11 implies that $U^\beta_{+,\,0} \sim \mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$ and $U^\beta_{-,\,0} \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

ii) $\Rightarrow$iii) : By Proposition 2.3.11 we have $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. If υ satisfies $(**)$ then $\tilde{\upsilon}$ satisfies $(**)'$. Now Theorem 2.1.8(i) imples $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \sim c_0$ which is a contradiction, so υ does not satisfy $(**)$.

iii) $\Rightarrow$i) : υ does not satisfy $(**)$, so $\tilde{\upsilon}$ does not satisfy $(**)'$. Thus by Theorem 2.1.8(ii) we have $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D})$ is not isomorphic to c_0. Now Corollary 2.1.9 implies that
$\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$ and Proposition 2.3.11 gives us the result. □

Section four: Isomorphism classification of weighted spaces of 2π-periodic holomorphic functions as subspace of $\mathbf{H}_{\upsilon}(\mathbf{G})$ or $\mathbf{H}_{\upsilon_0}(\mathbf{G})$

Definition 2.4.1: A function $f : \mathbf{G} \longrightarrow \mathbb{C}$ is called $r-$periodic for some $r > 0$ if $f(\omega + r) = f(\omega)$ for all $\omega \in \mathbf{G}$.

Proposition 2.4.2: *Suppose $a > 0$ is arbitrary and fixed. Put $f_a(\omega) = f(a\omega)$. If f is* r- *periodic then* f_a *is $\frac{r}{a}$ periodic.*

Proof: $f_a(\omega + \frac{r}{a}) = f(a\omega + r) = f(a\omega) = f_a(\omega) \quad \forall \omega \in \mathbf{G}$. □

Definition 2.4.3: For any $r \in \mathbb{R},\ r > 0$ we define
$\mathbf{H}^r_{\upsilon}(\mathbf{G}) := \{f \in \mathbf{H}_{\upsilon}(\mathbf{G}) : f \text{ is } r-\text{periodic}\}$
and
$\mathbf{H}^r_{\upsilon_0}(\mathbf{G}) := \{f \in \mathbf{H}_{\upsilon_0}(\mathbf{G}) : f \text{ is } r-\text{periodic}\}$.

Lemma 2.4.4: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_I$. Then the operator U_a defiend by $U_a f = f_a,\quad f \in \mathbf{H}^r_{\upsilon}(\mathbf{G})(f \in \mathbf{H}^r_{\upsilon_0}(\mathbf{G}))$, is an isomorphism between $\mathbf{H}^r_{\upsilon}(\mathbf{G})(\mathbf{H}^r_{\upsilon_0}(\mathbf{G}))$ and $\mathbf{H}^{\frac{r}{a}}_{\upsilon}(\mathbf{G})(\mathbf{H}^{\frac{r}{a}}_{\upsilon_0}(\mathbf{G}))$.*

Proof: We have $\mid (U_a f)(\omega) \mid \upsilon(\omega) = \mid f(a\omega) \mid \upsilon(a\omega)\ \ \frac{\upsilon(\omega)}{\upsilon(a\omega)} \leq \|f\|_{\upsilon} \frac{\upsilon(\omega)}{\upsilon(a\omega)}$.
If $a \geq 1$ then, since υ is of a type(I), there exists a constant $C_1 > 0$ such that $\upsilon(\omega) \leq C_1 \upsilon(a\omega)$.
If $a < 1$ then since υ is satisfies $(*)_I$ so there exist $C_2 > 0$ and $\beta > 0$ such that $\frac{\upsilon(\omega)}{\upsilon(a\omega)} \leq C_2(\frac{1}{a})^{\beta}$. Put $C = \max\{C_1, C_2\}$, so we have $\|U_a\| \leq C \max(1, (\frac{1}{a})^{\beta})$ for any $a > 0$. Clearly $U_a^{-1} = U_{\frac{1}{a}}$ and similarly we have $\|U_a^{-1}\| = U_{\frac{1}{a}} \leq C \max(1, a^{\beta})$. So U_a is a bounded operator with bounded inverse $U_{\frac{1}{a}}$ and we are done. □

Remark 2.4.5: (a): Lemma 2.4.4 shows that without loss of generality we can always assume $r = 2\pi$.

b) By Remark 1.2.4(a) we know that a type(I) weight υ depends only on the imaginary

part of ω(up to a universal constant). Now by a result of Stanev either $\mathbf{H}_\upsilon(\mathbf{G}) = \{0\}$ or there is an integer $b\,' > 0$ such that $\upsilon(\omega) \le e^{b'Im\omega}$ for all $\omega \in \mathbf{G}$.See [22]
For avoiding the triviality we assume $\mathbf{H}_\upsilon(\mathbf{G}) \neq \{0\}$. Hence there is a smallest integer b with $\upsilon(\omega) \le e^{bIm\omega}$ for all $\omega \in \mathbf{G}$. Thus $\mid e^{ib\omega} \mid \upsilon(\omega) = e^{-bIm\omega}\upsilon(\omega)$ is bounded but $\mid e^{ik\omega} \mid \upsilon(\omega)$ is unbounded for any integer $k < b$. Since υ is of type(I), $\upsilon(ti)$ is almost increasing for $t > 0$. This implies that $b \ge 0$.

Proposition 2.4.6: *Let υ be a type(I) weight on $\mathbf{G}$. Then for each $f \in \mathbf{H}^{2\pi}_\upsilon(\mathbf{G})$ there exsist $\gamma_k \in \mathbb{C}$ such that $f(\omega) = \sum_{k=b}^{\infty}\gamma_k e^{ik\omega}$. Here the series converges uniformly on the compact subsets of $\mathbf{G}$ and b is as in Remark* 2.4.5(b).

Proof: Define $\tau : \mathbf{G} \longrightarrow \mathbf{G}$ by $\tau(\omega) = \frac{1+e^{i\omega}}{1-e^{i\omega}}i$. Indeed $\tau(\omega) = \alpha(e^{i\omega})$. Also define $\eta : \mathbf{G} \setminus \{i\} \longrightarrow \mathbf{G}$ by $\eta(\omega) = -i\log(\frac{\omega-i}{\omega+i})$. Then $(\tau \circ \eta)(\omega) = \omega$ for all $\omega \in \mathbf{G} \setminus \{i\}$ and $(\eta \circ \tau)(\omega) = \omega$ if $-\pi < Re\omega < \pi$. Indeed
$\eta(\tau(\omega)) = (-i)\log(\frac{\tau(\omega)-i}{\tau(\omega)+i})$ and $\frac{\tau(\omega)-i}{\tau(\omega)+i} = e^{i\omega}$, so $\eta(\tau(\omega)) = (-i)\log(\frac{\tau(\omega)-i}{\tau(\omega)+i})$. Since $\arg e^{i\omega} = Re\omega$ and $-\pi < Re\omega < \pi$ we obtain for the main branch of the logarithm
$\eta(\tau(\omega)) = \omega$.

Claim: Suppose $f \in \mathbf{H}^{2\pi}_\upsilon(\mathbf{G})$. Then $f \circ \eta$ is holomorphic on $\mathbf{G} \setminus \{i\}$.

Proof of the claim : Put $g(z) = f \circ \eta \circ \alpha(z) = f((-i)\log z)$ for all $z \in \mathbb{D} \setminus \{0\}$. So g is holomorphic on $\mathbb{D} \setminus \{z \in \mathbb{D} : Imz = 0 \quad \& \quad Rez \le 0\}$. Now consider another branch of the logarithm, say, $\widetilde{\log} z, \widetilde{\log} z = \log \mid z \mid + i\widehat{\arg} z$ where $0 < \widehat{\arg} z < 2\pi$. For this branch of the logarithm, define $\tilde{\eta}(\omega) = (-i)\log\frac{\omega-i}{\omega+i}$ so we have

$$\widetilde{\log} z = \begin{cases} \log z & \text{if } \ 0 < \arg z < \pi \\ \log z + 2\pi & \text{if } \ -\pi < \arg z < 0 \end{cases}$$

Since $f \in \mathbf{H}^{2\pi}_\upsilon(\mathbf{G})$, hence $f((-i)\log z) = f((-i)\widetilde{\log} z)$. Therefore we have
$f \circ \eta \circ \alpha = f \circ \tilde{\eta} \circ \alpha. \qquad (1)$
But $\log z$ is holomorphic on $\mathbb{D} \setminus \{z \in \mathbb{D} : Imz = 0 \quad \& \quad Rez \le 0\}$ and $\widetilde{\log} z$ is holomorphic on $\mathbb{D} \setminus \{z \in \mathbb{D} : Imz = 0 \quad \& \quad Rez \ge 0\}$. So using (1) we see that $f \circ \eta \circ \alpha$ is holomorphic on $\mathbb{D} \setminus \{0\}$. Now since α is holomorphic on $\mathbb{D}$ so $f \circ \eta$ is holomorphic on $\mathbf{G}\setminus\{i\}$. Since $f \circ \eta$ is holomorphic on$\mathbf{G} \setminus \{i\}$ function g is holomorphic on $\mathbb{D} \setminus \{0\}$ so

there exist $\gamma_k \in \mathbb{C}$ such that

$g(z) = \sum_{k=-\infty}^{+\infty} \gamma_k z^k \qquad \forall z \in \mathbb{D} \setminus \{0\}$.

Thus $f(\omega) = f((\eta \circ \tau)(\omega)) = \sum_{k=-\infty}^{+\infty} \gamma_k (\frac{\tau(\omega)-i}{\tau(\omega)+i})^k = \sum_{k=-\infty}^{+\infty} \gamma_k e^{ik\omega}$ if $-\pi < Re\omega < \pi$.

Since f is $2\pi-$periodic so $f(\omega) = \sum_{k=-\infty}^{+\infty} \gamma_k e^{ik\omega} \quad \forall \omega \in \mathbf{G}$.

Using orthogonality we have $\frac{1}{2\pi} \int_0^{2\pi} f(\omega+\lambda) e^{-ik\lambda} dx = \gamma_k e^{-ik\omega}$. So

$\mid \gamma_k e^{ik\omega} \mid v(\omega) = \frac{1}{2\pi} \mid \int_0^{2\pi} f(\omega+\lambda)\ e^{-ik\lambda} dx \mid v(\omega) \leq \sup_{0<\lambda<2\pi} \mid f(\omega+\lambda) \mid v(\omega) \leq C \sup_{0<\lambda<2\pi} \mid f(\omega+\lambda) \mid v(\omega+\lambda) \leq C\|f\|_v < \infty$.

(Here C is the universal constant which appears in the definition of type(I) weight). So $\gamma_k e^{ik\omega} \in \mathbf{H}_v(\mathbf{G}) \quad \forall\ k \in \mathbb{Z}$.

If $k < b$ then $\mid e^{ik\omega} \mid v(\omega)$ is unbounded by the choice of b. If there is $k < b$ such that $\gamma_k \neq 0$ then $\gamma_k e^{ik\omega} \notin \mathbf{H}_v(\mathbf{G})$ which is a contradiction. So for all $k < b$ we have $\gamma_k = 0$. Therefore $f(\omega) = \sum_{k=b}^{+\infty} \gamma_k e^{ik\omega}$. □

Remark 2.4.7: a) Notice that in Proposition 2.4.6 we did not require the property $(*)_I$. we only assumed that v is a type(I) weight.

b) Take b as in Remark 2.4.5(b) and put $v_b(\omega) = e^{-bIm\omega} v(\omega) =\mid e^{ib\omega} \mid v(\omega) \quad \forall \omega \in \mathbf{G}$. We recall that v_b is a bounded weight on $\mathbf{G}$.

Proposition 2.4.8: *Define the operator* T, *by* $(Tf)(\omega) = e^{ib\omega} f(\omega)$. *Then* T *maps* $\mathbf{H}_v(\mathbf{G}), (\mathbf{H}_{v_0}(\mathbf{G}))$ *isometrically onto* $\mathbf{H}_{v_b}(\mathbf{G}), (\mathbf{H}_{(v_b)_0}(\mathbf{G}))$. T *also maps* $\mathbf{H}_v^{2\pi}(\mathbf{G}), (\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}))$ *isometrically onto* $\mathbf{H}_{v_b}^{2\pi}(\mathbf{G}), (\mathbf{H}_{(v_b)_0}^{2\pi}(\mathbf{G}))$.

Proof: Clearly if f is $2\pi-$periodic then $e^{-ib\omega} f(\omega)$ is $2\pi-$periodic and if $f \in \mathbf{H}_{v_0}(\mathbf{G})$ then $e^{-ib\omega} f(\omega) \in \mathbf{H}_{(v_b)_0}(\mathbf{G})$. Also we have

$\|Tf\|_{v_b} = \sup_{\omega \in \mathbf{G}} \mid e^{-ib\omega} \mid\mid f(\omega) \mid\mid e^{ib\omega} \mid v(\omega) = \sup_{\omega \in \mathbf{G}} \mid f(\omega) \mid v(\omega) = \|f\|_v$. □

Definition 2.4.9: Define $\tilde{v}(z) := v_b(\ln(\frac{1}{|z|})i) =\mid z \mid^b v(\ln(\frac{1}{|z|})i)$ for all $z \in \mathbb{D} \setminus \{0\}$ and put $\tilde{v}(0) = 0$. Clearly $\tilde{v}(z) = \tilde{v}(\mid z \mid)$ and $\lim_{|z| \to 1^-} \tilde{v}(\mid z \mid) = 0$.

Remark 2.4.10: We should notice that the weight $\tilde{v}$ in Definition 2.4.9 may be non-continuous at zero. But we replace it by an equivalent weight $\tilde{v_1}$ which is continuous on $[0,1]$. Define

$$\tilde{\upsilon_1}(z) := \begin{cases} \tilde{\upsilon}(z) & \text{if } \mid z \mid \geq \frac{1}{2} \\ \tilde{\upsilon}(\frac{1}{2}) & \text{if } \mid z \mid \leq \frac{1}{2} \end{cases}$$

Put $C = \frac{\sup_{|z|\leq\frac{1}{2}} \tilde{\upsilon}(z)}{\tilde{\upsilon}(2^{-1})}$. If $h \in \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ and $z_0 \in \mathbb{D}$ such that $\mid z_0 \mid \leq \frac{1}{2}$ then $\frac{\tilde{\upsilon}(z_0)}{\tilde{\upsilon}(2^{-1})} \leq C$. So $\tilde{\upsilon}(z_0) \leq C\tilde{\upsilon}(\frac{1}{2})$. Also by maximum modulus principle we have $\mid h(z_0) \mid \tilde{\upsilon}(z_0) \leq C \sup_{|z|=\frac{1}{2}} \mid h(z) \mid \tilde{\upsilon}(\frac{1}{2})$. Therefore we obtain

$\|h\|_{\tilde{\upsilon}_1} \leq \|h\|_{\tilde{\upsilon}} \leq C\|h\|_{\tilde{\upsilon}_1}$. So $\|.\|_{\tilde{\upsilon}}$ and $\|.\|_{\tilde{\upsilon}_1}$ are equivalent.

Theorem 2.4.11: *The operator* S, *defined by* $(Sf)(z) = f(-i\log z)\,\forall z \in \mathbb{D}\backslash\{0\}$ *maps* $\mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})(\mathbf{H}^{2\pi}_{(\upsilon_b)_0}(\mathbf{G}))$ *isometrically to* $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})(\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$.

Proof: υ is a type(I) weight, so $\|f\|_\upsilon$ is equivalent to $\sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \upsilon(Im\omega i)$ and $\|f\|_{\upsilon_b}$ is equivalent to $\sup_{\omega\in\mathbf{G}} \mid f(\omega)e^{ib\omega} \mid \upsilon(Im\omega i) = \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid e^{-bIm\omega}\upsilon(Im\omega i)$. For a $f \in \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$ put $g(\omega) = e^{ib\omega}f(\omega)$. Hence

$\|g\|_\upsilon = \sup_{\omega\in\mathbf{G}} \mid f(\omega) \|\, e^{ib\omega} \mid \upsilon(\omega) = \|f\|_{\upsilon_b} < \infty$. So $g \in \mathbf{H}^{2\pi}_\upsilon(\mathbf{G})$. Then by Proposition 2.4.6 there are $\gamma_k \in \mathbb{C}$ such that $g(\omega) = \sum_{k=b}^\infty \gamma_k e^{ik\omega}$ and therefore $f(\omega) = \sum_{k=b}^\infty \gamma_k e^{i(k-b)\omega}$. We have

$(Sf)(z) = f(-i\log z) = \sum_{k=b}^\infty \gamma_k e^{i(k-b)(-i)\log z} = \sum_{k=b}^\infty \gamma_k z^{k-b}$. Put $k-b = j$ so $(Sf)(z) = \sum_{j=0}^\infty \gamma_{j+b} z^j$.

$$\begin{aligned} \|f\|_{\upsilon_b} &= \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \upsilon_b(Im\omega i) \\ &= \sup_{\omega\in\mathbf{G}} \mid \sum_{k=b}^\infty \gamma_k e^{i(k-b)\omega} \|\, e^{ib\omega} \mid \upsilon(Im\omega i) \\ &= \sup_{\omega\in\mathbf{G}} \mid \sum_{k=b}^\infty \gamma_k e^{ik\omega} \mid \upsilon(Im\omega i). \end{aligned}$$

Put $e^{i\omega} = z$ so $\omega = \ln(\frac{1}{|z|})i + \arg z$. Therefore $\upsilon(Im\omega i) = \upsilon(\ln(\frac{1}{|z|})i)$. So

$$\begin{aligned} \|f\|_{\upsilon_b} &= \sup_{z\in\mathbb{D}} \mid \sum_{k=b}^\infty \gamma_k z^k \mid \mid z^{-b} \mid \mid z^b \mid \upsilon(\ln(\frac{1}{\mid z \mid})i) \\ &= \sup_{z\in\mathbb{D}} \mid \sum_{k=b}^\infty \gamma_k z^{k-b} \mid \tilde{\upsilon}(z) = \sup_{z\in\mathbb{D}} \mid (Sf)(z) \mid \tilde{\upsilon}(z). \end{aligned}$$

Therefore we have

$$\|f\|_{\upsilon_b} = \|(Sf)\|_{\tilde{\upsilon}}. \qquad (1)$$

Relation (1) implies that S is an isometry from $\mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$ to $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$. A similar argument shows that S is also an isometry from $\mathbf{H}^{2\pi}_{(\upsilon_b)_0}(\mathbf{G})$ in to $\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D})$.
Define $S^{-1} : \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$ or $S^{-1} : \mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}) \longrightarrow \mathbf{H}^{2\pi}_{(\upsilon_b)_0}(\mathbf{G})$ by $(S^{-1}h)(\omega) = h(e^{i\omega})$. Then $(S^{-1}S)(f)(z) = f(z)$ and $(SS^{-1})(h)(\omega) = h(\omega)$. Relation (1) also implies that S^{-1} is bounded on $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})(\mathbf{H}_{\tilde{\upsilon}_0}(\mathbb{D}))$ and we have $\|(S^{-1}h)(\omega)\|_{\upsilon} = \|h\|_{\tilde{\upsilon}}$. this proves the assertion of the theorem. $\square$

Proposition 2.4.12: *Let $\tilde{\upsilon}$ be as in Definition 2.4.9. Then $\tilde{\upsilon}$ is almost decreasing on the set $\{z \in \mathbb{D} :\mid z \mid \geq \frac{1}{2}\}$.*

Proof: We have $\tilde{\upsilon}(z) =\mid z \mid^b \upsilon(\ln(\frac{1}{|z|})i) \;\; \forall z \in \mathbb{D} \setminus \{0\}$. Suppose $\frac{1}{2} \leq t \leq s \leq 1$. Then $\ln \frac{1}{t} \geq \ln \frac{1}{s}$. Since υ is of type(I) there is constant $C > 0$ such that
$C\upsilon(\ln(\frac{1}{t})i) \geq \upsilon(\ln(\frac{1}{s})i)$, so

$$\begin{aligned} Ct^b\upsilon(\ln(\frac{1}{t})i) &\geq s^b\upsilon(\ln(\frac{1}{s})i)(\frac{t}{s})^b \\ &\geq (\frac{1}{2})^b s^b\upsilon(\ln(\frac{1}{s})i). \end{aligned}$$

Thus $\tilde{\upsilon}(s) \leq C2^b\tilde{\upsilon}(t)$ and we are done. $\square$

Theorem 2.4.13: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_{II}$. Then*
i) $\mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G})(\mathbf{H}^{2\pi}_{\upsilon_0}(\mathbf{G}))$ *is isomorphic to $\ell_\infty(c_0) \Leftrightarrow \upsilon$ satisfies $(**)$.*
ii) $\mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G})(\mathbf{H}^{2\pi}_{\upsilon_0}(\mathbf{G}))$ *is isomorphic to $\mathbf{H}_\infty(\mathbb{D})((\sum_{n\in\mathbb{N}} \oplus H_n)_0) \Leftrightarrow \upsilon$ does not satisfy $(**)$.*

Proof: Using Proposition 2.4.8, Theorem 2.4.11 and Remark 2.4.10 we have
$\mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}_1}(\mathbb{D}).$ (1)
and
$\mathbf{H}^{2\pi}_{\upsilon_0}(\mathbf{G}) \sim \mathbf{H}_{(\tilde{\upsilon}_1)_0}(\mathbb{D}).$ (2)
By Proposition 2.4.12 $\tilde{\upsilon}_1$ is almost decreasing. So using Lemma 2.1.5, there is a decreasing weight which is equivalent to $\tilde{\upsilon}_1$. We call it again $\tilde{\upsilon}_1$ (Note that $\tilde{\upsilon}_1$ is a standard weight). υ satisfies $(*)_{II}$, so Lemma 1.2.8 implies that $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{n+1}i)}{\upsilon(2^n i)} < \infty$.
An argument similar to what has been done in the Theorem 2.2.3 shows that
$\inf_{n\in\mathbb{N}\cup\{0\}} \frac{\tilde{\upsilon}_1((1-\frac{1}{2^{n+1}}))}{\tilde{\upsilon}_1((1-\frac{1}{2^n}))} > 0.$ $(*)'$
If υ satisfies $(**)$, then Lemma 1.2.9 implies that $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\upsilon(2^{-n-k}i)}{\upsilon(2^{-n}i)} < 1$.

So $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\tilde{v_1}((1-2^{-n-k}))}{\tilde{v_1}((1-2^{-n}))} < 1.$ $(**)'$

Proof of (i): $\Rightarrow$: If $\mathbf{H}_v^{2\pi}(\mathbf{G}) \sim \ell_\infty$ then by relation (1) $\mathbf{H}_{\tilde{v}_1}(\mathbb{D}) \sim \ell_\infty$. Now, by part (iii) and (iv) of Theorem 2.1.8, $\tilde{v}_1$ satisfies $(**)'$. So v satisfies $(**)$. If $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}) \sim c_0$ then, by relation (2), $\mathbf{H}_{(\tilde{v}_1)_0}(\mathbb{D}) \sim c_0$. Now part (ii) of Theorem 2.1.8 imples that $\tilde{v}_1$ satisfies $(**)'$. So v satisfies $(**)$.

$\Leftarrow$: If v satisfies $(**)$ then $\tilde{v}_1$ satisfies $(**)'$. Since $\tilde{v}_1$ satisfies both $(*)'$ and $(**)'$ so Theorem 2.1.8(iii) implies that $\mathbf{H}_{\tilde{v}_1}(\mathbb{D}) \sim \ell_\infty$. By relation (1), we have $\mathbf{H}_v^{2\pi}(\mathbf{G}) \sim \ell_\infty$. Also Theorem 2.1.8(i) implies $\mathbf{H}_{(\tilde{v}_1)_0}(\mathbb{D}) \sim c_0$. Relation (2) implies $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}) \sim c_0$.

Proof of (ii): $\Rightarrow$: We have $\mathbf{H}_v^{2\pi}(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$, so, by relation (1) $\mathbf{H}_{\tilde{v}_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Now Theorem 2.1.8(iii) implies that $\tilde{v}_1$ does not satisfy $(**)'$, so v does not satisfy $(**)$. If $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$ then, by realation(2), $\mathbf{H}_{(\tilde{v}_1)_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. Corollary 2.1.9(i) implies that $\mathbf{H}_{\tilde{v}_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. Now Theorem 2.1.8 implies that $\tilde{v}$ does not satisfy $(**)'$, therefore v does not satisfy $(**)$.

$\Leftarrow$: If v satisfies $(*)$ but not $(**)$, then $\tilde{v}_1$ satisfies $(*)'$ but not $(**)'$. Now part (iii) of Theorem 2.1.8 implies that $\mathbf{H}_{\tilde{v}_1}(\mathbb{D}) \sim \mathbf{H}_\infty(\mathbb{D})$. By using relation (1) we obtain $\mathbf{H}_v^{2\pi}(\mathbf{G}) \sim \mathbf{H}_\infty(\mathbb{D})$. Also Corollary 2.1.9(ii) implies that $\mathbf{H}_{(\tilde{v}_1)_0}(\mathbb{D}) \sim (\sum_{n\in\mathbb{N}} \oplus H_n)_0$. Now using relation (2) we are done. □

Theorem 2.4.13 implies:

Corollary 2.4.14: *Let v be a type(I) weight on $\mathbf{G}$ satisfying $(*)_{II}$. Then either $\mathbf{H}_v^{2\pi}(\mathbf{G})$ is isomorphic to ℓ_∞ and $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G})$ is isomorphic to c_0*

or

$\mathbf{H}_v^{2\pi}(\mathbf{G})$ *is isomorphic to* $\mathbf{H}_\infty(\mathbb{D})$ *and* $\mathbf{H}_{v_0}^{2\pi}(\mathbf{G})$ *is isomorphic to* $(\sum_{n\in\mathbb{N}} \oplus H_n)_0$.

Section five: Some complemented subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$.

In this section we will prove $U^{\beta}_{\pm}, U^{\beta}_{\pm,\ 0}$ are complemented subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})$ and $\mathbf{H}_{\upsilon_0}(\mathbf{G})$ respectively for a type(I) weight υ.
Before we come to the main result of this section we recall some elementary facts from the geometry of Banach spaces.

Lemma 2.5.1: *Suppose X is a vector space and A and Y are subspaces of X. If Y is finite codimensional in X, then $A \cap Y$ is finite codimensional in A.*

Proof: We have $\dim \frac{X}{Y} = n < \infty$, so there are $x_1, ..., x_n \in X$ such that $\{x_1 + Y, ..., x_n + Y\}$ is a basis for $\frac{X}{Y}$. Consider an arbitrary $x \in X$. Then there exist unique $\alpha_1, ..., \alpha_n \in \mathbb{C}$ such that
$x+Y = \alpha_1(x_1+Y)+...+\alpha_n(x_n+Y) = (\alpha_1 x_1+...+\alpha_n x_n)+Y$, so $x-(\alpha_1 x_1+...+\alpha_n x_n) \in Y$. Now define $\phi_1, ..., \phi_n : X \longrightarrow \mathbb{C}$ by $\phi_k(x) = \alpha_k$ for $1 \leq k \leq n$, so $Y = \bigcap_{k=1}^{n} \ker \phi_k$. Thus $Y \cap A = \bigcap_{k=1}^{n} \ker(\phi_{k|A})$. Therefore $\dim \frac{A}{A \cap Y} < \infty$. $\square$

Lemma 2.5.2: *Suppose A and B are closed subspaces of the Banach space X and B is subspace of A ($B \subseteq A$). If B is a complemented subspace in X and $\dim \frac{A}{B} < \infty$, then A is a complemented subspace in X, too.*

Proof: Since B is a complemented subspace in X, there is a bounded projection $P : X \longrightarrow B$ such that $X = B \oplus \ker P$. Now the assumption $B \subseteq A$ implies $A = B \oplus \ker(P_{|A})$. Since $\dim \frac{A}{B} = n < \infty$ there is a finite dimensional subspace of A, say N such that $A = B \oplus N$ and $N = \langle\{a_1, ..., a_n\}\rangle$ where $\{a_1 + B, ..., a_n + B\}$ is a basis for $\frac{A}{B}$. Comparing the relations $A = B \oplus N$ and $A = B \oplus \ker(P_{|A})$, we conclude that $\ker(P_{|A})$ is a finite dimensional subspace of X, so there is a bounded projection $Q : X \longrightarrow \ker(P_{|A})$.
Now we claim $P + Q(id - P) : X \longrightarrow A$ is a bounded projection. Hence A is complemented in X. Since P and Q are bounded operators so $P + Q(id - P)$ is a bounded

operator. We obtain

$$\begin{aligned}(P+Q(id-P))(P+Q(id-P)) &= P^2+PQ(id-P)+Q(id-P)P \\ &+ Q(id-P)Q(id-P) \\ &= P+0+0+Q(id-P)\end{aligned}$$

so $P+Q(id-P)$ is a projection. We have $P(X)=B\subseteq A$ and
$Q(id-P)(X)\subseteq \ker(P_{|_A})$ so
$P(X)+Q(id-P)(X)\subset A.$ (1)
Suppose $a\in A$. $P(a)+(Q(id-P))(a)=P(a)+(id-P)(a)=a$. So
$(P+Q(id-P))_{|_A}=id_{|_A}$ (2)
Relations (1) and (2) imply that $P+Q(id-P)$ maps X onto A. $\square$

Lemma 2.5.3: *Suppose A and Y are closed subspaces of the Banach space X. Also assume that $\dim\frac{X}{Y}<\infty$ and $P:Y\longrightarrow A$ is a bounded linear operator with $P_{|A\cap Y}=id$. Then $A\cap Y$ and A are complemented subspaces of X.*

Proof: Since $\dim\frac{X}{Y}<\infty$, by Lemma 2.5.1 we have $\dim(\frac{A}{A\cap Y})<\infty$. Thus there is a bounded projection $Q_1:A\longrightarrow A\cap Y$. Since

$$\begin{aligned}(Q_1PQ_1P)(y) &= (Q_1P)[(Q_1P)(y)] \\ &= Q_1(P(y)) \\ &= (Q_1P)(y)\end{aligned}$$

we obtain that $(Q_1P):Y\longrightarrow A\cap Y$ is a bounded projection.
$\dim\frac{X}{Y}<\infty$ implies that Y is complemented in X. So there is a bounded projection $Q_2:X\longrightarrow Y:=rangeQ_2$. Note that

$$\begin{aligned}(Q_1PQ_2)(Q_1PQ_2)(x) &= (Q_1PQ_2)(Q_1P(Q_2(x))) \\ &= (Q_1P)(Q_1P)Q_2(x) \\ &= (Q_1PQ_2)(x)\end{aligned}$$

Thus $Q_1PQ_2:X\longrightarrow A\cap Y$ is a bounded projection. Therefore $A\cap Y$ is a complemented subspace of X. Now Lemma 2.4.2 implies A is also a complemented subspace of X. □

Theorem 2.5.4: *Let υ be a type(I) weight on $\mathbf{G}$ satisfying $(*)_I$ with respect to β.(Where β is a positive even intiger).*
Then $U^{\beta}_{\pm}$ $(U^{\beta}_{\pm,\,0})$ are complemented subspaces of $\mathbf{H}_{\upsilon}(\mathbf{G})(\mathbf{H}_{\upsilon_0}(\mathbf{G}))$.

Proof: We prove the theorem for U^{β}_{+}. The proofs for $U^{\beta}_{-}, U^{\beta}_{\pm,\,0}$ are similar.
$\omega^{2\beta}+1$ has 2β zeros of the form $\cos(\frac{1}{2\beta}(\pi+2\pi k))+i\sin(\frac{1}{2\beta}(\pi+2\pi k))$ for $0 \le k \le 2\beta-1$. β of this zeros, say $\omega_1, ..., \omega_\beta$ are in $\mathbf{G}$. Put $h(\omega) = \omega^{2\beta}+1$ then $h(\omega)$ and $\frac{1}{\omega^{2\beta}}+1$ have the same zeros in $\mathbf{G}$. Also, if ω_k is a zero of $h(\omega)$, then $\frac{-1}{\omega_k}$ is a zero of $h(\omega)$ too. Define $X := \{f \in \mathbf{H}_{\upsilon}(\mathbf{G}) : f(\omega_k) = 0, \quad k = 1, 2, ..., \beta\}$.

Claim 1: X is closed and finite codimensional in $\mathbf{H}_{\upsilon}(\mathbf{G})$.

Proof of Claim 1: For each k, $1 \le k \le \beta$, define the evaluvation functional $\delta_{\omega_k} : \mathbf{H}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbb{C}$ by $\delta_{\omega_k}(f) = f(\omega_k)$. We have
$\mid \delta_{\omega_k}(f) \mid = \mid f(\omega_k) \mid \upsilon(\omega_k)\frac{1}{\upsilon(\omega_k)} \le \frac{1}{\upsilon(\omega_k)}\|f\|_{\upsilon}$. So for $1 \le k \le \beta$, δ_{ω_k} is a bounded linear functional. In fact $\|\delta_{\omega_k}\| \le \frac{1}{\upsilon(\omega_k)}$. Now since $X = \bigcap_{k=1}^{\beta} \ker \delta_{\omega_k}$, so X is closed and finite codimensional in $\mathbf{H}_{\upsilon}(\mathbf{G})$.
Define $P : X \longrightarrow \mathbf{H}_{\upsilon}(\mathbf{G})$ by $(Pf)(\omega) = \frac{1}{\omega^{2\beta}+1}(f(\omega) + f(\frac{-1}{\omega}))$.

Claim 2: P is welldefined.

Proof of Claim 2: Firstly note that for any $x \in \mathbb{C}$ and any $r > 0$ by $B(x, r)$ we mean the set $\{\omega \in \mathbb{C} : \mid \omega - x \mid < r\}$. Also $\overline{B}(x, r)$ is the closure of the set $B(x, r)$.
We must prove $Pf \in \mathbf{H}_{\upsilon}(\mathbf{G}) \quad \forall f \in X$. Suppose $f \in X$, then to any $\omega_k (1 \le k \le \beta)$ there coresponds a $r_k > 0$ such that $f(\omega) = \sum_{j=1}^{\infty} \alpha_{k_j}(\omega - \omega_k)^j$ in $B(\omega_k, r_k)$. Put $r = \frac{1}{3}\min\{r_k : 1 \le k \le \beta\}$, so $B(\omega_k, r) \cap B(\omega_{k'}, r) = \emptyset$ whenever $1 \le k, k' \le \beta$ and $k \neq k'$. Since $\frac{-1}{\omega_k} \in \{\omega_1, ..., \omega_\beta\}(1 \le k \le \beta)$ we have $\frac{-1}{\omega_k} \in \mathbf{G}$ and $f(\frac{-1}{\omega_k})$ is holomorphic on $\mathbf{G}$. So to any $\omega_k (1 \le k \le \beta)$ there coresponds a $r'_k > 0$ such that
$f(\frac{-1}{\omega_k}) = \sum_{j=1}^{\infty} \gamma_{k_j}(\omega - \omega_k)^j$ in $B(\omega_k, r'_k)$ for suitable $\gamma_{k_j} \in \mathbb{C}$.
Put $r' = \frac{1}{3}\min\{r'_k : 1 \le k \le \beta\}$, so $B(\omega_k, r') \cap B(\omega_{k'}, r') = \emptyset$ whenever $1 \le k, k' \le \beta$ and

$k \neq k'$. Put $r'' = \frac{1}{3}\min\{r, r'\}$ and $U_k = \bar{B}(\omega_k, r'')$ for $1 \leq k \leq \beta$. Clearly, for any $1 \leq k \leq \beta$, U_k is compact, $U_k \cap U_{k'} = \emptyset$ whenever $1 \leq k, k' \leq \beta$ and $k \neq k'$ and the Taylor series of $f(\omega)$ and $f(\frac{-1}{\omega})$ are convergent in U_k. Put $U = \bigcup_{k=1}^{\beta} U_k$. If $\omega \in \mathbf{G} \setminus U$, then

$$| (Pf)(\omega) | v(\omega) \leq | f(\omega) | \frac{v(\omega)}{|\omega^{2\beta}+1|} + | f(\tfrac{-1}{\omega}) | v(\tfrac{-1}{\omega}) \frac{1}{|\omega^{2\beta}+1|} \frac{v(\omega)}{v(\frac{-1}{\omega})}$$

Put $d = \inf_{\omega \in \mathbf{G} \setminus U} | \omega^{2\beta} + 1 | > 0$, so $\frac{1}{|\omega^{2\beta}+1|} < \frac{1}{d} \quad \forall \omega \in \mathbf{G} \setminus U$. Since v is a type(I) weight which satisfies $(*)_I$, there exsist constants C_1, C' such that

$$\sup_{\omega \in \mathbf{G} \setminus U} \frac{1}{|\omega^{2\beta}+1|} \frac{v(\omega)}{v(\frac{-1}{\omega})} \leq \sup_{\omega \in \mathbf{G} \setminus U} \begin{cases} C_1 \frac{|\omega|^{2\beta}}{|\omega^{2\beta}+1|} & \text{if } | \omega | \geq 1 \\ \frac{C}{|\omega^{2\beta}+1|} & \text{if } | \omega | \leq 1 \end{cases}$$

If $| \omega |$ is sufficently large then $\frac{|\omega|^{2\beta}}{|\omega^{2\beta}+1|} < 1$, if $1 < | \omega | < 2$ then $\frac{|\omega|^{2\beta}}{|\omega^{2\beta}+1|} < \frac{2^{2\beta}}{d}$. So $\frac{|\omega|^{2\beta}}{|\omega^{2\beta}+1|} < \max(1, \frac{2^{2\beta}}{d})$ for all $\omega \in \mathbf{G} \setminus U, | \omega | \geq 1$.

Therefore there is a universal constant $C > 0$ such that

$\sup_{\omega \in \mathbf{G} \setminus U} | (Pf)(\omega) | v(\omega) \leq C \|f\|_v. \qquad (1).$

Now let $\omega \in U$, say $\omega \in U_k$(Since $U_k \cap U_{k'} = \emptyset$ if $k \neq k'$, so we have that ω is only an element of one U_k).

Since $\omega^{2\beta} + 1 = (\omega - \omega_1)(\omega - \omega_2), ..., (\omega - \omega_{2\beta})$, where $\omega_k (1 \leq k \leq 2\beta)$ are the zeros of $\omega^{2\beta} + 1$ we have

$\frac{f(\omega)}{\omega^{2\beta}+1} = \frac{1}{\pi_{j \neq k}(\omega - \omega_j)} \sum_{j=1}^{\infty} \alpha_{k_j} (\omega - \omega_k)^{j-1}$ and $\frac{f(-\frac{1}{\omega})}{\omega^{2\beta}+1} = \frac{1}{\pi_{j \neq k}(\omega - \omega_j)} \sum_{j=1}^{\infty} \gamma_{k_j} (\omega - \omega_k)^{j-1}$.

Thus $\frac{f(\omega)}{\omega^{2\beta}+1}$ and $\frac{f(-\frac{1}{\omega})}{\omega^{2\beta}+1}$ are bounded on U_k. So

$\sup_{\omega \in U} | (Pf)(\omega) | v(\omega) < \infty. \qquad (2).$

Relations (1) and (2) imply $\sup_{\omega \in \mathbf{G}} | (Pf)(\omega) | v(\omega) = \|Pf\|_v < \infty$. So $Pf \in \mathbf{H}_v(\mathbf{G})$.

Clearly $\omega^{2\beta}(Pf)(\omega) = (Pf)(-\frac{1}{\omega})$, so $Pf \in U_+^{\beta}$. Also if $f \in X \cap U_+^{\beta}$, then

$(Pf)(\omega) = \frac{1}{\omega^{2\beta}+1}(f(\omega) + \omega^{2\beta} f(\omega)) = f(\omega)$. So $P_{|X \cap U_+^{\beta}} = id$.

Claim 3: P is a bounded map on X.

Proof of Claim 3: We use the closed graph theorem. Suppose $(f, g) \in \text{graph}(P)$ closure, then there exists a sequence $(f_n, Pf_n) \in \text{graph}(P)$ such that

$(f_n, Pf_n) \longrightarrow (f, g)$ so $\begin{cases} f_n \longrightarrow f \\ Pf_n \longrightarrow g \end{cases}$ (pointwise). If $Pf = g$ then graph(P) is closed and P is bounded. X is a closed subspase of $\mathbf{H}_v(\mathbf{G})$ so $f_n \longrightarrow f$ (pointwise) implies $f \in X$ and $Pf \in \mathbf{H}_v(\mathbf{G})$. Also $f_n \longrightarrow f$(pointwise) implies $\begin{cases} f_n(\omega) \longrightarrow f(\omega) \\ f_n(\frac{-1}{\omega}) \longrightarrow f(\frac{-1}{\omega}) \end{cases}$ (pointwise). Thus

$$\begin{aligned} g &= \lim_{n\to\infty} (Pf_n)(\omega) \\ &= \lim_{n\to\infty} \frac{1}{\omega^{2\beta}+1}(f_n(\omega) + f_n(\frac{-1}{\omega})) \\ &= \frac{1}{\omega^{2\beta}+1}(f(\omega) + f(\frac{-1}{\omega})) \\ &= Pf. \end{aligned}$$

We have U_+^β and X are closed subspace of $\mathbf{H}_v(\mathbf{G}), \dim \frac{\mathbf{H}_v(\mathbf{G})}{X} < \infty$, $P : X \longrightarrow U_+^\beta \subseteq \mathbf{H}_v(\mathbf{G})$ is bounded and $P_{|U_+^\beta \cap X} = id$. Now use Lemma 2.5.3 with $A = U_+^\beta$ and $Y = X$, we are done. □

Remark 2.5.5: Looking more closely, the proof of the Lemma 2.5.4 reveals that the complement of U_+^β in $\mathbf{H}_v(\mathbf{G})$ is $U_-^0 := \{f \in \mathbf{H}_v(\mathbf{G}) : f(\omega) = -f(-\frac{1}{\omega}), \omega \in \mathbf{G}\}$ up to a finite dimensional subspace. Unfortunately the isomorphic classification of U_-^0 is not known.

Chapter 3

Operators on weighted spaces for type(II) weights.

Introduction to chapter three: There are many wellknown results about differentiation, composition and multiplication operators on the weighted spaces of holomorphic functions on the unit disc $\mathbb{D}$. In this chapter we want to obtain similar results for weighted spaces of holomorphic functions on the upper halfplane, where our weights are of type(II) and satisfy certain properties. For reaching this aim Theorem 2.2.1 is a very useful tool. We divide this chapter in to three sections:
In section one we are dealing with the differentiation operator between two weighted spaces of holomorphic functions and we study boundedness and surjectivity of this operator.
In section two we obtain necessary and sufficient conditions for continuity and compactness of composition operators between two weighted spaces of holomorphic functions.
In section three, we study multiplication operators on a weighted space of holomorphic functions into itself. We want to know under which conditions such a multiplication operator is:
a) a Fredholm operator

b) a continuous operator

c) an isomorphism.

Section one: Differentiation operator between weighted spaces of holomorphic functions on the upper halfplane for type(II) weights.

We recall that a standard weight u on $\mathbb{D}$ satisfies:

$(*)'$ if $\inf_{n\in\mathbb{N}} \frac{u(1-2^{-n-1})}{u(1-2^{-n})} > 0$

$(**)'$ if $\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{u(1-2^{-n-k})}{u(1-2^{-n})} < 1$.

For arriving at the main theorem of this section we need the following wellknown result on the unit disc.

Definition 3.1.1: For any open set $O \subseteq \mathbb{C}$ and any holomorphic function $f : O \longrightarrow \mathbb{C}$ we put $Df = f'$.

Theorem 3.1.2: *Suppose u is a standard weight on* $\mathbb{D}$ *which satisfies* $(*)'$. *Then*

i) *The differentiation operator* $D, D : \mathbf{H}_u(\mathbb{D}) \longrightarrow \mathbf{H}_{u_1}(\mathbb{D})$, *is bounded where* $u_1(t) = (1-t)u(t)$

ii) *Let u satisfy* $(**)'$ *in addition. Then D is a surjective map.*

iii) *If u does not satisfy* $(**)'$, *then D is not a surjective map.*

Proof: See [16] Theorem 3.1.

Lemma 3.1.3: *Theorem* 3.1.2 *remains valid if u is an almost decreasing radial weight (i.e* $u(z) = u(\mid z \mid)$ *is an almost decreasing function) on the unit disc* $\mathbb{D}$.

Proof: Let u be an almost decreasing weight on $\mathbb{D}$, then $(1-t)u(t)$ is also an almost decreasing weight on $\mathbb{D}$. By Lemma 2.1.5 $u^*(z) = \sup_{t\geq|z|} u(t)$ is a decreasing weight on $\mathbb{D}$. Now consider the following diagram

$D : \mathbf{H}_u(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-t)u(t)}(\mathbb{D})$

$\downarrow id \qquad\qquad \downarrow id_1$

$D_1 : \mathbf{H}_{u^*}(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-t)u^*(t)}(\mathbb{D})$

where D and D_1 are differentiation operators and id_1 again is the identity map. Clearly $id_1 \circ D = D_1 \circ id$. Now we have if D_1 is bounded (surjective,nonsurjective), then D is bounded (surjective,nonsurjective), respectively. Now using Theorem 3.1.2

we are done. □

Proposition 3.1.4: *Let υ be a type(II) weight on $\mathbf{G}$ satisfying $(*)_{II}$. Define $\upsilon_2 : \mathbf{G} \longrightarrow (0, +\infty)$ by*

$$\upsilon_2(\omega) = \begin{cases} Im\omega\ \upsilon(Im\omega i) & \text{if } \mid \omega \mid \leq 1 \\ Im(-\frac{1}{\omega})\upsilon(Im(-\frac{1}{\omega})i) & \text{if } \mid \omega \mid \geq 1 \end{cases}$$

Then υ_2 is a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$.

Proof: Since $\lim_{\delta\to 0^+} \upsilon(x + i\delta) = 0$ for all $x \in \mathbb{R}$, so $\lim_{\delta\to 0^+} \upsilon_2(x + i\delta) = 0$ for all $x \in \mathbb{R}$. Let $\omega_0 \in \mathbf{G}$ with $\mid \omega_0 \mid= 1$ be given. We have $\lim_{\omega\to\omega_0\ |\omega|\leq 1} \upsilon_2(\omega) = \lim_{\omega\to\omega_0\ |\omega|\leq 1}(Im\omega\ \upsilon(Im\omega\ i))$ and

$$\begin{aligned} \lim_{\omega\to\omega_0\ |\omega|\geq 1} \upsilon_2(\omega) &= \lim_{\omega\to\omega_0\ |\omega|\geq 1}(Im(-\frac{1}{\omega})\upsilon(Im(-\frac{1}{\omega})i)) \\ &= \lim_{\omega\to\omega_0\ |\omega|\leq 1}(Im\omega\ \upsilon(Im\omega\ i)). \end{aligned}$$

So $\lim_{\omega\to\omega_0\ |\omega|\leq 1} \upsilon_2(\omega) = \lim_{\omega\to\omega_0\ |\omega|\geq 1} \upsilon_2(\omega)$. Since υ is continuous so υ_2 is continuous. If $\mid \omega \mid\leq 1$ then $\upsilon_2(\omega) = Im\omega\ \upsilon(Im\omega i)$. Since υ is a type(II) weight so there is a type(I) weight υ_1 such that $\upsilon_2(\omega) = Im\omega\ \upsilon_1(Im\omega i)$ whenever $\mid \omega \mid\leq 1$.
Now suppose $\mid \omega_1 \mid, \mid \omega_2 \mid\leq 1$ and $Im\omega_1 \leq Im\omega_2$, thus there exists a constant C such that $\upsilon_2(\omega_1) \leq C\upsilon_2(\omega_2)$. Also by definition of υ_2 we have $\frac{\upsilon_2(\omega)}{\upsilon_2(-\frac{1}{\omega})} = 1\ \forall\omega \in \mathbf{G}$. So υ_2 is a type(II) weight. Suppose $\mid \omega_1 \mid, \mid \omega_2 \mid\leq 1$ and $Im\omega_1 \geq Im\omega_2$. Since υ satisfies $(*)_{II}$, there exist $C > 0$ and $\beta > 0$ such that

$\frac{\upsilon_2(\omega_1)}{\upsilon_2(\omega_2)} = \frac{Im\omega_1\upsilon(Im\omega_1 i)}{Im\omega_2\upsilon(Im\omega_2 i)} \leq C\frac{Im\omega_1}{Im\omega_2}(\frac{Im\omega_1}{Im\omega_2})^\beta = C(\frac{Im\omega_1}{Im\omega_2})^{\beta+1}$.

So υ_2 satisfies $(*)_{II}$. □

Proposition 3.1.5: *Let υ be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. Then the weights $\tilde{\upsilon}(z) = \upsilon(\alpha(-\mid z \mid)) = \upsilon(\frac{1-|z|}{1+|z|}i)$ and $\tilde{\upsilon_3}(z) = \upsilon((1-\mid z \mid)i)$ are equivalent.*

Proof: For any $z \in \mathbb{D}$, we have $\frac{1-|z|}{1+|z|} \leq 1-\mid z \mid\leq 1$. Since υ is of type(II) there is a constant $C > 0$ such that
$\upsilon(\frac{1-|z|}{1+|z|}i) \leq C\upsilon((1-\mid z \mid)i)$. Since υ satisfies $(*)_{II}$, there exist $C_1 > 0$ and $\beta > 0$ such that $\frac{\upsilon((1-|z|)i)}{\upsilon(\frac{1-|z|}{1+|z|}i)} \leq C_1 2^\beta$. Thus we are done. □

Remark 3.1.6: Theorem 2.2.1 and Proposition 3.1.5 imply that the map $T : \mathbf{H}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D})$ defined by $T(f) = f \circ \alpha$ is an isomorphism.

Lemma 3.1.7: *Let υ be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. Define $\delta : \mathbf{H}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_2}(\mathbf{G})$ by $\delta(f) = \sum_{k=1}^{\infty} k\alpha_k(\frac{\omega-i}{\omega+i})^{k-1}$ where $f(\omega) = \sum_{k=0}^{\infty} \alpha_k(\frac{\omega-i}{\omega+i})^k$ and υ_2 is as in Proposition* 3.1.4. *Then δ is a bounded (welldefined) map.*

Proof: Define $\tilde{\delta} : \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$ by $\tilde{\delta}(\tilde{f}) = \delta(f) \circ \ \alpha$ where
$\tilde{\upsilon}_3(z) = \upsilon((1- \mid z \mid)i), \tilde{\upsilon}_2(z) = \upsilon_2((1- \mid z \mid)i)$ and $\tilde{f} = f \circ \alpha$ for all $f \in \mathbf{H}_{\upsilon}(\mathbf{G})$.
Clearly $\delta(f) \circ \alpha = D(\tilde{f})$. Also
$\tilde{\upsilon}_2(z) = \upsilon_2((1- \mid z \mid)i) = (1- \mid z \mid)\upsilon((1- \mid z \mid)i) = (1- \mid z \mid)\tilde{\upsilon}_3(z)$. Now Lemma 3.1.3 and Theorem 3.1.2 imply that there is a constant $C' > 0$ such that
$\|\tilde{\delta}(\tilde{f})\|_{\tilde{\upsilon}_2} \leq C'\|\tilde{f}\|_{\tilde{\upsilon}_3}$. (1)
But by Proposition 3.1.4, Proposition 3.1.5, Remark 3.1.6 and Theorem 2.2.1 we have $\mathbf{H}_{\upsilon}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D})$ and $\mathbf{H}_{\upsilon_2}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$. Now the relation equivalent with relation (1) on $\mathbf{G}$ is $\|\delta(f)\|_{\upsilon_2} \leq C\|f\|_{\upsilon}$ for some universal constant C. □

Lemma 3.1.8: *Let υ be as in the Lemma* 3.1.7. *If υ satisfies $(**)$, then $\delta : \mathbf{H}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_2}(\mathbf{G})$ is a surjective map, where υ_2 is as in Proposition* 3.1.4.

Proof: Let $\tilde{\delta} : \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$ be as in Lemma 3.1.7. Since υ satisfies $(*)_{II}$ and $(**)$ so $\tilde{\upsilon}_3$ satisfies $(*)'$ and $(**)'$. Now Theorem 3.1.2 (ii) implies that $\tilde{\delta}$ is surjective.
Suppose $g \in \mathbf{H}_{\upsilon_2}(\mathbf{G})$, then $\tilde{g} = g \circ \alpha \in \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$. (Use Theorem 2.2.1 and Proposition 3.1.5 for υ_2). Since $\tilde{\delta}$ is surjective there exists a
$\tilde{g}_1 \in \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D})$ such that $\tilde{\delta}(\tilde{g}_1) = \tilde{g}$ where $g_1 \in \mathbf{H}_{\upsilon}(\mathbf{G})$ and $\tilde{g}_1 = g \circ \alpha$.
So $\delta(g_1) \circ \alpha = g \circ \alpha$ or equivalently $\delta(g_1) = g$. Thus δ is a surjective map. □

Lemma 3.1.9: *Define $T : \mathbf{H}_{\upsilon_2}(\mathbf{G}) \longrightarrow \mathbf{H} := \{\frac{2i}{(\omega+i)^2}h \mid h \in \mathbf{H}_{\upsilon_2}(\mathbf{G})\}$ by $T(h) = \frac{2i}{(\omega+i)^2}h$. Then $\mathbf{H} \subseteq \mathbf{H}_{\upsilon_2}(\mathbf{G})$ and T is bounded.*

Proof: Firstly note that:
If $\omega \in \mathbf{G}$ and $\omega = x + iy$ then $\mid \omega + i \mid^2 = x^2 + (y + 1)^2$. Since $y > 0$ we have $\frac{1}{|\omega+i|^2} < 1$

for all $\omega \in \mathbf{G}$. Suppose $g \in \mathbf{H}$, then there is a $h \in \mathbf{H}_{\upsilon_2}(\mathbf{G})$ such that $g = \frac{2i}{(\omega+i)^2} h$.

$$\begin{aligned} \|g\|_{\upsilon_2} &= \|\frac{2i}{(\omega+i)^2} h\|_{\upsilon_2} \\ &= \sup_{\omega \in \mathbf{G}} \mid \frac{2i}{(\omega+i)^2} h(\omega) \mid \upsilon_2(\omega) \leq \sup_{\omega \in \mathbf{G}} \mid \frac{2i}{(\omega+i)^2} \mid \sup_{\omega \in \mathbf{G}} \mid h(\omega) \mid \upsilon_2(\omega) \\ &\leq 2\|h\|_{\upsilon_2}. \end{aligned}$$

Since $h \in \mathbf{H}_{\upsilon_2}(\mathbf{G})$ we obtain $\|g\|_{\upsilon_2} \leq 2\|h\|_{\upsilon_2} < \infty$. Hence $\mathbf{H} \subseteq \mathbf{H}_{\upsilon_2}(\mathbf{G})$ and $\|T(h)\|_{\upsilon_2} \leq 2\|h\|_{\upsilon_2}$. So $\|T\| \leq 2$. □

Corollary 3.1.10: *Let υ be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. Then the differentiation operator $D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_2}(\mathbf{G})$ is a bounded operator.*

Proof: Consider the composition map $T \circ \delta : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}$.
$(T \circ \delta)(f) = T(\delta(f)) = \frac{2i}{(\omega+i)^2}\delta(f) = D(f)$ for any $f \in \mathbf{H}_\upsilon(\mathbf{G})$ so $D = T \circ \delta$. Now using Lemma 3.1.8 and Lemma 3.1.9 we deduce D is bounded. □

Lemma 3.1.11: *Let υ be a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. Then the differentiation operator $D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}$ is a surjective map.*

Proof: Clearly T is a surjective map. Lemma 3.1.8 implies, δ is a surjective map. Therefore $T \circ \delta = D$ is a surjective map. □

Lemma 3.1.12: *If the differentiation operator $D, D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}$, is a surjective map, then υ satisfies $(**)$.*

Proof: If υ does not satisfy $(**)$ then $\tilde{\upsilon}_3$ does not satisfy $(**)'$. Since D is surjective, $D = T \circ \delta$ and T is surjective so δ is surjective.
Consider $\tilde{g} = g \circ \alpha \in \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$, then $g \in \mathbf{H}_{\upsilon_2}(\mathbf{G})$. Since δ is surjective there exists a $f \in \mathbf{H}_\upsilon(\mathbf{G})$ such that $\delta(f) = g$. Let $\tilde{D} : \mathbf{H}_{\tilde{\upsilon}_3}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_2}(\mathbb{D})$ be the differentiation operator. We have $\delta(f) \circ \alpha = g \circ \alpha$ or equivalently
$\tilde{D}(\tilde{f}) = \tilde{g}$, which means $\tilde{D}$ is surjective. Since $\tilde{\upsilon}$ does not satisfy $(**)'$, surjectivity of $\tilde{D}$ is a contradiction with the Theorem 3.1.2(iii). Therefore υ must satisfy $(**)$. □

Now we summarize what we have obtained in the following theorem.

Theorem 3.1.13: *Let* υ *be a type(II) weight on* $\mathbf{G}$ *which satisfies* $(*)_{II}$.

i) *The differentiation operator* $D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_2}(\mathbf{G})$ *is a bounded operator, where* υ_2 *is as in Proposition* 3.1.4.

ii) υ *satisfies* $(**) \Leftrightarrow\ D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}$ *is a surjective map.*

Proof:i: Corollary 3.1.10.

ii: $\Rightarrow$: Lemma 3.1.11.

$\Leftarrow$: Lemma 3.1.12. $\square$

Section two: Composition operators between two weighted spaces of holomorphic functions on the upper halfplane for type(II) weights

Remark 3.2.1: Throughout this section we assume that $u : \mathbb{D} \longrightarrow [0, +\infty)$ and $\eta : \mathbb{D} \longrightarrow [0, +\infty)$ are standard weights on $\mathbb{D}$.

Firstly we state wellknown results for composition operators between two weighted spaces of holomorphic functions on the unit disc.

Definition 3.2.2: Suppose Ω is an open subset of $\mathbb{C}$

a) We define $\mathbf{H}(\Omega) := \{f \mid f : \Omega \longrightarrow \mathbb{C} \text{ is holomorphic}\}$

b) Suppose $\varphi : \Omega \longrightarrow \Omega$ is a holomorphic function. We define the composition operator $C_\varphi : \mathbf{H}(\Omega) \longrightarrow \mathbf{H}(\Omega)$ by $C_\varphi(f) = f \circ \varphi$.

In the following we are dealing with the cases $\Omega = \mathbb{D}$ or $\Omega = \mathbf{G}$.

Theorem 3.2.3: *The following conditions are equivalent for the composition operator* $C_\varphi : \mathbf{H}_u(\mathbb{D}) \longrightarrow \mathbf{H}_\eta(\mathbb{D})$.

i) *The composition operator* C_φ *is continuous.*

ii) $\sup_{n\in\mathbb{N}} \frac{\|\varphi(z)^n\|_\eta}{\|z^n\|_u} < \infty$.

In this case $\|C_\varphi\| \leq \sup_{n\in\mathbb{N}} \frac{\|\varphi(z)^n\|_\eta}{\|z^n\|_u} < \infty$.

Proof: See Proposition 2.1 of [7] and Proposition 6 of [6]. □

Theorem 3.2.4: *The following conditions are equivalent for the composition operator* $C_\varphi : \mathbf{H}_u(\mathbb{D}) \longrightarrow \mathbf{H}_\eta(\mathbb{D})$.

i) *The composition operator* C_φ *is compact.*

ii) $\lim_{n\to\infty} \frac{\|\varphi(z)^n\|_\eta}{\|z^n\|_u} = 0$.

Proof: See Theorem 3.3 of [7]. □

Now we want to obtain results similar to Theorem 3.2.3 and Theorem 3.2.4 for the composition operators between two weighted spaces of holomorphic functions on the upper halfplane.

Remark 3.2.5: Consider the composition operator $C_\varphi : \mathbf{H}(\mathbf{G}) \longrightarrow \mathbf{H}(\mathbf{G})$ (Put $\Omega = \mathbf{G}$ in Definition 3.2.2). In the following we also deal with $C_{\alpha^{-1}\circ\varphi\circ\alpha} : \mathbf{H}(\mathbb{D}) \longrightarrow \mathbf{H}(\mathbb{D})$. Note that $C_{\alpha^{-1}\circ\varphi\circ\alpha}(\tilde{f}) = f \circ \varphi \circ \alpha$, where $\tilde{f} = f \circ \alpha$.

Remark 3.2.6: In the remainder of this section we assume υ and υ' are type(II) weights on $\mathbf{G}$ which satisfy $(*)_{II}$ and put $\tilde{\upsilon}(z) = \upsilon(\alpha(- \mid z \mid))$ &
$\tilde{\upsilon}'(z) = \upsilon'(\alpha(- \mid z \mid))$.
Now Theorem 2.2.1 implies that
$\mathbf{H}_\upsilon(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ (1)
and
$\mathbf{H}_{\upsilon'}(\mathbf{G}) \sim \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ (2)
Relations (1) and (2) are the main tools for proving our results.

Remark 3.2.7: Assume that C_φ maps $\mathbf{H}_\upsilon(\mathbf{G})$ into $\mathbf{H}_{\upsilon'}(\mathbf{G})$. Then $\mathrm{C}_{\alpha^{-1}\circ\varphi\circ\alpha}$ maps $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ into $\mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ and vice versa.

Proof: If $\tilde{f} = f\circ\alpha \in \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ then relation (1) implies that $f \in \mathbf{H}_\upsilon(\mathbf{G})$, so $C_\varphi(f) = f\circ\varphi$ belongs to $\mathbf{H}_{\upsilon'}(\mathbf{G})$ and now relation (2) implies that $f \circ \varphi \circ \alpha = C_{\alpha^{-1}\circ\varphi\circ\alpha}(\tilde{f}) \in \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$. The proof for the converse is similar. □

Lemma 3.2.8: *The composition operator* $C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon'}(\mathbf{G})$ *is continuous iff the composition operator* $C_{\alpha^{-1}\circ\varphi\circ\alpha} : \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ *is continuous.*

Proof: It is clear because of relations $(1), (2)$ and continuity of maps α, α^{-1} and φ. □

Theorem 3.2.9: *The following conditions are equivalent for the composition operator* $C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon'}(\mathbf{G})$.

i) *The operator* C_φ *is continuous.*

ii) $\sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{\upsilon'}}{\|(\frac{\omega-i}{\omega+i})^n\|_\upsilon} < \infty.$

In this case there is a constant $C_{\upsilon'}$ *depending on the weight* υ' *such that* $\|C_\varphi\| \leq C_{\upsilon'} \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{\upsilon'}}{\|(\frac{\omega-i}{\omega+i})^n\|_\upsilon}$

Proof: **i** $\Rightarrow$ **ii** : Using Lemma 3.2.8, we see that $C_{\alpha^{-1}\circ\varphi\circ\alpha}$ is continuous. Now using

Theorem 3.2.3 for $C_{\alpha^{-1}\circ\varphi\circ\alpha}$ we have $\sup_{n\in\mathbb{N}} \frac{\|[(\alpha^{-1}\circ\varphi\circ\alpha)(z)]^n\|_{\tilde{v}'}}{\|z^n\|_{\tilde{v}}} < \infty$. But because of the relations (1) and (2) (see Remark 3.2.6), we have

$C_1\|[(f\circ\alpha^{-1})(\omega)]^n\|_v \leq \|z^n\|_{\tilde{v}} \leq C_2\|[(fo\alpha^{-1})(\omega)]^n\|_v$

and

$C_3\|[(\varphi_1\circ\alpha^{-1})(\omega)]^n\|_{v'} \leq \|\varphi_1(z)^n\|_{\tilde{v}'} \leq C_4\|[(\varphi_1\circ\alpha^{-1})(\omega)]^n\|_{v'}$.

Here C_1, C_2, C_3 and C_4 are universal constants, $f(z) = z$ and $\varphi_1 = \alpha^{-1}\circ\varphi\circ\alpha$. Since $\|[(\varphi_1\circ\alpha^{-1})(\omega)]^n\|_{v'} = \|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{v'}$ and $\|[(f\circ\alpha^{-1})(\omega)]^n\|_v = \|(\frac{\omega-i}{\omega+i})^n\|_v$ so

$$\sup_{n\in\mathbb{N}} \frac{\|\varphi_1(z)^n\|_{\tilde{v}'}}{\|z^n\|_{\tilde{v}}} < \infty \Leftrightarrow \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{v'}}{\|(\frac{\omega-i}{\omega+i})^n\|_v} < \infty. \qquad (1)$$

ii $\Rightarrow$ i: Do the above proof in the reverse direction and again use Lemma 3.2.8.

For proving the last assertion of the theorem firstly note that by definition we have

$\|C_{\alpha^{-1}\circ\varphi\circ\alpha}\| = \sup\{\|C_{\alpha^{-1}\circ\varphi\circ\alpha}(f\circ\alpha)\|_{\tilde{v}'} : \ \|f\circ\alpha\|_{\tilde{v}} \leq 1\}$ &

$\|C_\varphi\| = \sup\{\|C_\varphi(f)\|_{v'} : \ \|f\|_v \leq 1\}$

Now our assumptions imply that $\|.\|_v \sim \|.\|_{\tilde{v}}$ & $\|.\|_{v'} \sim \|.\|_{\tilde{v}'}$.

Using relation (1), the above facts and Theorem 3.2.3 there is a constant $C_{v'}$ such that

$$\|C_\varphi\| \leq C_{v'} \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{v'}}{\|(\frac{\omega-i}{\omega+i})^n\|_v} \qquad \square$$

Consider the following diagram where $\varphi_1 = \alpha^{-1}\circ\varphi\circ\alpha$ and T_1 & T_2 are the maps of Theorem 2.2.1.

$$\begin{array}{ccc} C_\varphi : \mathbf{H}_v(\mathbf{G}) & \longrightarrow & \mathbf{H}_{v'}(\mathbf{G}). \\ T_1 \quad \downarrow & & \downarrow \ T_2 \qquad\qquad (1) \\ C_{\varphi_1} : \mathbf{H}_{\tilde{v}}(\mathbb{D}) & \longrightarrow & \mathbf{H}_{\tilde{v}'}(\mathbb{D}) \end{array}$$

Lemma 3.2.10: *The diagram (1) is commutative, that is*

$C_{\varphi_1} = T_2\circ C_\varphi\circ T_1^{-1} : \mathbf{H}_{\tilde{v}}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{v}'}(\mathbb{D})$.

Proof: Firstly note that $\tilde{f} = f \circ \alpha$. So

$$\begin{aligned}(T_2 \circ C_\varphi \circ T_1^{-1})(\tilde{f}) &= (T_2 \circ C_\varphi)(T_1^{-1}(\tilde{f})) \\ &= (T_2 \circ C_\varphi)(f) \\ &= T_2(f \circ \varphi) \\ &= f \circ \varphi \circ \alpha \\ &= C_{\varphi_1}(\tilde{f}). \qquad \square\end{aligned}$$

Corollary 3.2.11: *The composition operator C_φ is compact iff the composition operator $C_{\alpha^{-1}\circ\varphi\circ\alpha}$ is compact.*

Proof: Theorem 2.2.1 implies that the maps T_1 and T_2 in Lemma 3.2.10 are isomorphisms, so we are done. □

Theorem 3.2.12: *The following conditions are equivalent for the composition operator $C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon'}(\mathbf{G})$.*

i) *The composition operator C_φ is compact.*

ii) $\lim_{n\to\infty} \frac{\|(\frac{\varphi(\omega)-i}{\varphi(\omega)+i})^n\|_{\upsilon'}}{\|(\frac{\omega-i}{\omega+i})^n\|_\upsilon} = 0.$

Proof: Suppose (i) holds. Corollary 3.2.11 implies that the composition operator $C_{\alpha^{-1}\circ\varphi\circ\alpha} : \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ is compact. If we use Theorem 3.2.4 for $C_{\alpha^{-1}\circ\varphi\circ\alpha}$ and the argument of the Theorem 3.2.9[**i** $\Rightarrow$ **ii**], then the proof is complete. □

Remark 3.2.13: Theorem 1 of [6] states that the compactness and weak compactness of the composition operator $C_\varphi : \mathbf{H}_\gamma(G_1) \longrightarrow \mathbf{H}_\delta(G_2)$ are equivalent if $\mathbb{C}^* \setminus G_1$ has no one point component, where $\mathbb{C}^*$ is the extended complex plane, G_1 and G_2 are open conected domains in $\mathbb{C}, \gamma : G_1 \longrightarrow (0, +\infty)$ and $\delta : G_2 \longrightarrow (0, +\infty)$ are continuous bounded strictly positive weights.

Now since the sets $\mathbb{D}$ and $\mathbf{G}$ satisfy the assumption of theorem so compactness and weak compactness of the composition operator C_φ (see Theorem 3.2.12) for bounded type(II) weights are the same.

We finish this section with the following example.

Example 3.2.14: If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is an automorphism on $\mathbb{D}$, then $\alpha \circ \varphi \circ \alpha^{-1} : \mathbf{G} \longrightarrow \mathbf{G}$ is an automorphism on $\mathbf{G}$.

As an special case consider $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ defined by $\varphi(z) = iz$, then

$(\alpha \circ \varphi \circ \alpha^{-1})(\omega) = (\alpha \circ \varphi)(\frac{\omega - i}{\omega + i}) = \alpha(\frac{\omega - i}{\omega + i} i) = \frac{(i-1)\omega + i - 1}{(1-i)\omega + i - 1}$ or equivalently

$(\alpha \circ \varphi \circ \alpha^{-1})(\omega) = \frac{\frac{\sqrt{2}}{2}\omega + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}\omega + \frac{\sqrt{2}}{2}}$.

The above relation is also an automorphism on $\mathbf{G}$ which is obtained by rotation 90° counterclockwise on $\mathbb{D}$.

Section three: Pointwise multiplication operators on weighted spaces of holomorphic functions on the upper halfplane for type(II) weights

Definition: 3.3.1 Let Ω be an open subset of $\mathbb{C}$. Suppose $\varphi \in \mathbf{H}\ (\Omega)$ and $\varphi \neq 0$ ($i.e\ \exists\ \omega \in \Omega$ such that $\varphi(\omega) \neq 0$). We define the pointwise multiplication operator $M_\varphi : \mathbf{H}(\Omega) \longrightarrow \mathbf{H}(\Omega)$ by $M_\varphi(f) = f\varphi$.
Note that in the following we deal with the cases $\Omega = \mathbb{D}$ or $\Omega = \mathbf{G}$.

Suppose $u : \mathbb{D} \longrightarrow (0, +\infty)$ is a continuous and strictly positive weight function. The following is wellknown :

Theorem 3.3.2: i) $M_\varphi(\mathbf{H}_u(\mathbb{D})) \subseteq \mathbf{H}_u(\mathbb{D})$ *iff* M_φ *is continuous iff* $M_\varphi \in \mathbf{H}_\infty(\mathbb{D})$ *and in this case*
$\|\varphi\|_\infty = \|M_\varphi\|$, *where* $\|\ .\|_\infty$ *is the supremum norm on* $\mathbb{D}$.
ii) M_φ *is injective.*
If $\varphi \in \mathbf{H}_\infty(\mathbb{D})$, *then*
iii) M_φ *is an isomorphism on* $\mathbf{H}_u(\mathbb{D})$ *iff* $\frac{1}{\varphi} \in \mathbf{H}_\infty(\mathbb{D})$.
iv) M_φ *is a Fredholm operator iff there is a* $\epsilon > 0$ *such that* $\mid \varphi(z) \mid \geq \epsilon$ *for all* z *with* $1 - \epsilon \leq \mid (z) \mid < 1$.

Proof: See [5, 8] □

Definition 3.3.3: Suppose $\varphi \in \mathbf{H}(\mathbf{G})$ and $\varphi \neq 0$ ($i.e\ \exists\ \omega \in \mathbf{G}$ such that $\varphi(\omega) \neq 0$). Consider the pointwise multiplication operator M_φ from $\mathbf{H}(\mathbf{G})$ into $\mathbf{H}(\mathbf{G})$. Corresponding to M_φ, define $\tilde{\varphi} = \varphi \circ \alpha \in \mathbf{H}(\mathbb{D})$. Then we have
$M_{\tilde{\varphi}} : \mathbf{H}(\mathbb{D}) \longrightarrow \mathbf{H}(\mathbb{D})$ where $\quad M_{\tilde{\varphi}}(\tilde{f}) = \tilde{f}\tilde{\varphi} = (f \circ \alpha)(\varphi \circ \alpha)$.
We call $\tilde{M_\varphi}$ the pointwise multiplication operator corresponding to M_φ.

Remark 3.3.4: Throughout this section we assume υ is a type(II) weight on $\mathbf{G}$ which satisfies $(*)_{II}$. So by Theorem 2.2.1 we have that $\mathbf{H}_\upsilon(\mathbf{G})$ is isomorphic to $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$, where $\tilde{\upsilon}(z) = \upsilon(\alpha(- \mid z \mid))$.
From the definition of M_φ and $M_{\tilde{\varphi}}$ the following proposition is clear:

Proposition 3.3.5: *The pointwise multiplication operator M_φ on $\mathbf{H}_v(\mathbf{G})$ is continuous iff the pointwise multiplication operator $M_{\tilde{\varphi}}$ on $\mathbf{H}_{\tilde{v}}(\mathbf{G})$ is continuous.*

Theorem 3.3.6: *Let the pointwise multiplication operator $M_\varphi : \mathbf{H}(\mathbf{G}) \longrightarrow \mathbf{H}(\mathbf{G})$ be given. Then*

i) *$M_\varphi(\mathbf{H}_v(\mathbf{G})) \subseteq \mathbf{H}_v(\mathbf{G})$ iff M_φ is continuous on $(\mathbf{H}_v(\mathbf{G}))$ iff $\varphi \in \mathbf{H}_\infty(\mathbf{G})$, and in this case $\|\varphi\|_\infty = \|M_\varphi\|$.*

ii) *M_φ is injective.*

If $\varphi \in \mathbf{H}_\infty(\mathbf{G})$, then

iii) *M_φ is an isomorphism on $\mathbf{H}_v(\mathbf{G})$ iff $\frac{1}{\varphi} \in \mathbf{H}_\infty(\mathbf{G})$.*

iv) *M_φ is a Fredholm operator iff there exists $\epsilon > 0$ such that $| \varphi(\omega) | \geq \epsilon$ for all $\omega \in \mathbf{G}$ such that $1 - \epsilon \leq | \frac{\omega - i}{\omega + i} | < 1$.*

Proof: i) Since $\mathbf{H}_v(\mathbf{G})$ is isomorphic to $\mathbf{H}_{\tilde{v}}(\mathbb{D})$ so $M_\varphi(\mathbf{H}_v(\mathbf{G})) \subseteq \mathbf{H}_v(\mathbf{G})$ iff $M_\varphi(\mathbf{H}_{\tilde{v}}(\mathbb{D})) \subseteq \mathbf{H}_{\tilde{v}}(\mathbb{D})$. Using Theorem 3.3.2, we obtain $M_{\tilde{\varphi}}(\mathbf{H}_{\tilde{v}}(\mathbb{D})) \subseteq \mathbf{H}_{\tilde{v}}(\mathbb{D})$ iff $M_{\tilde{\varphi}}$ is continuous on $(\mathbf{H}\tilde{v}(\mathbb{D}))$ iff $\tilde{\varphi} \in \mathbf{H}_\infty(\mathbb{D})$ and in this case $\|\tilde{\varphi}\|_\infty = \|M_{\tilde{\varphi}}\|$.
Now using Proposition 3.3.5, we conclude $M_{\tilde{\varphi}}$ is continuous iff M_φ is continuous. If $\tilde{\varphi} = (\varphi \circ \alpha) \in \mathbf{H}_\infty(\mathbb{D})$, since $\alpha^{-1} \in \mathbf{H}_\infty(\mathbf{G})$, so $\tilde{\varphi} \circ \alpha^{-1} = \varphi \in \mathbf{H}_\infty(\mathbf{G})$ and if $\varphi \in \mathbf{H}_\infty(\mathbf{G})$ then $\tilde{\varphi} = (\varphi \circ \alpha) \in \mathbf{H}_\infty(\mathbb{D})$.

ii) Since φ is holomorphic on $\mathbf{G}$ and $\varphi \neq 0$ we are done.

iii) From the definition of M_φ and $M_{\tilde{\varphi}}$, part(ii) of this theorem and part(ii) of Theorem 3.3.2, it is clear that M_φ and $M_{\tilde{\varphi}}$ are one-to-one linear maps. Also it is easy to see that M_φ is onto iff $M_{\tilde{\varphi}}$ is onto. Since $\mathbf{H}_v(\mathbf{G})$ is isomorphic to $\mathbf{H}_{\tilde{v}}(\mathbb{D})$, so M_φ is an isomorphism on $(\mathbf{H}_v(\mathbf{G}))$ iff $M_{\tilde{\varphi}}$ is an isomorphism on $\mathbf{H}_{\tilde{v}}(\mathbb{D})$ and (by Theorem 3.3.2 (ii)) iff $\frac{1}{\tilde{\varphi}} \in \mathbf{H}_\infty(\mathbb{D})$. But
$\frac{1}{\tilde{\varphi}} \circ \alpha^{-1} = \frac{1}{\varphi}$. Since $\alpha^{-1} \in \mathbf{H}_\infty(\mathbf{G})$, so $\frac{1}{\varphi} \in \mathbf{H}_\infty(\mathbf{G})$. If $\frac{1}{\varphi} \in \mathbf{H}_\infty(\mathbf{G})$, then
$\frac{1}{\varphi} \circ \alpha = \frac{1}{\tilde{\varphi}} \in \mathbf{H}_\infty(\mathbb{D})$.

iv) $M_\varphi(\mathbf{H}_v(\mathbf{G}))$ is closed and is finite codimensional in $\mathbf{H}_v(\mathbf{G})$ iff $M_{\tilde{\varphi}}(\mathbf{H}_{\tilde{v}}(\mathbb{D}))$ is closed and is finite codimensional in $\mathbf{H}_{\tilde{v}}(\mathbb{D})$. Also $\ker M_\varphi = \ker M_{\tilde{\varphi}} = \{0\}$. So M_φ is a Fredholm operator iff $M_{\tilde{\varphi}}$ is a Fredholm operator (now using Theorem 3.3.2(iv)) iff there

is a $\epsilon > 0$ such that $|\ \tilde{\varphi}(z)\ | \geq \epsilon$ for all $z \in \mathbb{D}$ with $1 - \epsilon \leq |\ z\ |$ (and transfering this to $\mathbf{G}$) iff there is a $\epsilon > 0$ such that $|\ \varphi(\omega)\ | \geq \epsilon$ for all $\omega \in \mathbf{G}$ with
$1 - \epsilon \leq |\ \frac{\omega - i}{\omega + i}\ | < 1$. $\square$

Chapter 4

Operators on weighted spaces for type(I) weights

Introduction to chapter four: In this chapter we investigate differentiation and composition operators on weighted spaces of holomorhpic functions for type(I) weights. We begin section one by studying the differentiation operator. We continue this section by obtaining some various sufficient conditions for continuity of composition operators where at least one of the weights are bounded.

In section two, by using Theorem 3.2.9, we will obtain another sufficient condition for the continuity of the composition operator when our weights are not necessarily bounded.

In section three we find a sufficient condition for the continuity of the composition operator C_φ (see Definition 3.2.2) between two weighted spaces of $2\pi-$periodic functions, where $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is a $2\pi-$periodic and holomorphic function.

In section four we study the differentiation operator between two weighted spaces of $2\pi-$periodic functions. We obtain necessary and sufficient condition for contiuity of the differentiation operator.

Also we want to know when this operator will be surjective.

Section one: Differentiation and composition operators between weighted spaces of holomorphic functions on the upper halfplane for type(I) weights.

We begin this section by studing the differentiation operator. For this aim we need the following.

Note that throughout this section the weight υ is of type(I).

Remark 4.1.1: Using Lemma 1.2.8 (i), if υ satisfies$(*)_I$, then

$\inf_{n\in\mathbb{N}} \frac{\tilde{\upsilon}(1-2^{-n+1})}{\tilde{\upsilon}(1-2^{-n})} > 0$

where $\tilde{\upsilon}(z) = \upsilon(\alpha(-\mid z\mid)) = \upsilon((\frac{1-|z|}{1+|z|})i)$.

As a consequence of Theorem 3.1.2 we obtain:

Corollary 4.1.2: *Let υ be a type(I) weight satisfying $(*)_I$.Consider the standard weight $\tilde{\upsilon}(z) = \upsilon((\frac{1-|z|}{1+|z|})i)$. Then the differentiation operator $D : \mathbf{H}_{\tilde{\upsilon}(z)}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}(z)(1-|z|)}(\mathbb{D})$ is a bounded operator.*

Proof: Since $\tilde{\upsilon}$ satisfies $(*)'$ (see begining of chapter three) and it is a standard weight so this follows from Theorem 3.1.2. □

Theorem 4.1.3: *Let υ be a type(I) weight on $\mathbf{G}$. If υ is bounded and satisfies $(*)_I$(for a bounded weight $(*)_I$ and $(*)_{II}$ are equivalent), then the differentiation operator $D : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon_1}(\mathbf{G})$ is a bounded operator where $\upsilon_1(\omega) = \min(Im\omega, 1)\upsilon(Im\omega i)$.*

Proof: Firstly note that it is easy to see that υ_1 is a bounded weight of type(I). Now Corollary 4.1.2 and Proposition 2.3.5 imply that there exists a constant $C'' > 0$ such that

$\sup_{a\in\mathbb{R}} \|D_z((T_a f)\circ\alpha)\|_{\tilde{\upsilon}(z)(1-|z|)} \leq C''\|f\|_\upsilon, \qquad (1)$

where D_z denotes the differentiation operator on the unit disc $\mathbb{D}$ and T_a is as in Lemma 2.3.3. We have

$\|D_z((T_a f)\circ\alpha)\|_{\tilde{\upsilon}(z)(1-|z|)} = \|(T_a f')\circ\alpha\cdot\alpha'\|_{\tilde{\upsilon}(z)(1-|z|)} = \|T_a f'\circ\alpha\|_{|\alpha'(z)|\tilde{\upsilon}(z)(1-|z|)}$

and $\upsilon_1(\omega) = \min(Im\omega, 1)\upsilon(Im\omega i)$. If $Im\omega \leq 1$, then $\upsilon_1(\omega) = Im\omega\upsilon(Im\omega i) =$

$v_1(Im\omega i)$. Suppose $\omega \in \mathbf{G}$ is such that $Im\omega \leq 1$. So

$| f'(\omega) | v_1(\omega) =| (T_{Re\omega}f')(Im\omega i) | v_1(Im\omega i) =| (T_a f')(Im\omega i) | v_1(Im\omega i),\ where$

$a = Re\omega$. Put $z \in \mathbb{D}$ such that

$\alpha^{-1}(Im\omega i) = z$. So $\frac{Im\omega-1}{Im\omega+1} = z := -r$. Thus $| z |= r, | z |= -z$ and $z = - | z |$.

$$\begin{aligned} | f'(\omega) | v_1(\omega) &= | Df(\omega) | v_1(Im\omega i) \\ &= | (T_a f')(Im\omega i) | v_1(Im\omega i) \\ &= | (T_a f')(Im\omega i) | Im\omega v(Im\omega i) \\ &= | (T_a f') \circ \alpha(-r) | Im\alpha(-r)v(\alpha(z)) \\ &= | (T_a f') \circ \alpha(z) | Im\alpha(z)v(\alpha(z)) \\ &= | (T_a f)' \circ \alpha(-r) | Im\alpha(- | z |)v(\alpha(- | z |)). \end{aligned}$$

Since $Im\alpha(- | z |) = \frac{1}{2} | \alpha'(- | z |) | (1- | z |^2)$ so

$| Df(\omega) | v_1(Im\omega i) = \frac{1}{2} | (T_a f') \circ \alpha(z) || \alpha'(z) | \tilde{v}(z)(1- | z |^2)$. (2)

By Lemma 2.3.4 (b) there is a constant $d > 0$ such that

$\|Df\|_{v_1} \leq d \sup_{\omega\in\mathbf{G}, Im\omega\leq 1} | Df(\omega) | v_1(Im\omega i)$. (3)

By relation (2) we have

$$\begin{aligned} \sup_{\omega\in\mathbf{G}, Im\omega\leq 1} | Df(\omega) | v_1(Im\omega i) &\leq \frac{1}{2} \sup_{a\in\mathbb{R}} \sup_{z\in\mathbb{D}} \sup_{Rez\leq 0\ \ Imz=0} | (T_a f') \circ \alpha(z) || \alpha'(z) | \tilde{v}(z)(1- | z |^2) \\ &\leq \sup_{a\in\mathbb{R}} \|(T_a f') \circ \alpha(z) \cdot \alpha'(z)\|_{\tilde{v}(z)(1-|z|)} \\ &\leq \sup_{a\in\mathbb{R}} \|D_z((T_a f) \circ \alpha)\|_{\tilde{v}(z)(1-|z|)}. \end{aligned}$$

Therefore

$\sup_{\omega\in\mathbf{G}, Im\omega\leq 1} | Df(\omega) | v_1(Im\omega i) \leq \sup_{a\in\mathbb{R}} \|D_z((T_a f)\circ\alpha)\|_{\tilde{v}(z)(1-|z|)}$. (4)

Now relations (3),(4) and (1) imply that there exists a constant $C > 0$ such that

$\|Df\|_{v_1} \leq C\|f\|_v$ and the proof is complete. □.

We continue this section with studying composition operators (See Remark 3.2.2).

We present sufficient conditions for boundedness of composition operators.

Proposition 4.1.4: *Let v and $\hat{v}$ be type(I) weights on $\mathbf{G}$ and assume that $\hat{v}$ is bounded. Moreover, put $\tilde{v}(z) = v(\alpha(- \mid z \mid))$ and $\tilde{\hat{v}}(z) = \hat{v}(\alpha(- \mid z \mid))$, $z \in \mathbb{D}$. Let $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ be a holomorphic function such that*

$\sup_{a\in\mathbb{R}} \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\alpha(z)+a)-i}{\varphi(\alpha(z)+a)+i})^n\|_{\tilde{\hat{v}}}}{\|z^n\|_{\tilde{v}}} < \infty$. *Then the composition operator*

$C_\varphi : \mathbf{H}_v(\mathbf{G}) \longrightarrow \mathbf{H}_{\hat{v}}(\mathbf{G})$ *is bounded.*

Proof: Using Proposition 2.3.5 there are constants $C_1, C_2 > 0$ such that for all $f \in \mathbf{H}_{\hat{v}}(\mathbf{G})$ we have

$C_1\|C_\varphi f\|_{\hat{v}} \leq \sup_{a\in\mathbb{R}} \|T_a(C_\varphi f) \circ \alpha)\|_{\tilde{\hat{v}}} \leq C_2\|C_\varphi f\|_{\hat{v}}$. (1)

Moreover, $(T_a(C_\varphi f))(\alpha(z)) = (C_\varphi f)(\alpha(z) + a) = f(\varphi(\alpha(z) + a))$. Put $\varphi_a(z) = \alpha^{-1}(\varphi(\alpha(z) + a))$. Then φ_a is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}$ and we obtain, with $\tilde{f} = f \circ \alpha$,

$(C_{\varphi_a}\tilde{f})(z) = (T_aC_\varphi f)(\alpha(z))$ (2)

By our assumption we have

$\sup_{a\in\mathbb{R}} \sup_{n\in\mathbb{N}} \frac{\|(\varphi_a(z))^n\|_{\tilde{\hat{v}}}}{\|z^n\|_{\tilde{v}}} < \infty.$

The above relation implies that the operators $C_{\varphi_a}(a \in \mathbb{R})$ from $\mathbf{H}_{\tilde{v}}(\mathbb{D})$ into $\mathbf{H}_{\tilde{\hat{v}}}(\mathbb{D})$ are uniformly bounded.(see Theorem 3.2.3). Therefore $C' := \sup_{a\in\mathbb{R}} \|C_{\varphi_a}\| < \infty$.

Now using relations (1)& (2) and the fact that $Im\alpha(- \mid z \mid) \leq Im\alpha(z)$ we have

$$\begin{aligned}
C_1\|C_\varphi f\|_{\hat{v}} &\leq \sup_{a\in\mathbb{R}} \|T_a(C_\varphi f) \circ \alpha)\|_{\tilde{\hat{v}}} \\
&= \sup_{a\in\mathbb{R}} \|C_{\varphi_a}\tilde{f}\|_{\tilde{\hat{v}}} \\
&\leq C'\|\tilde{f}\|_{\tilde{v}} \\
&= C' \sup_{z\in\mathbb{D}} \mid f(\alpha(z)) \mid v(\alpha(- \mid z \mid)) \\
&\leq C' \sup_{z\in\mathbb{D}} \mid f(\alpha(z)) \mid v(\alpha(z)) \\
&= C'\|f\|_v
\end{aligned}$$

Therefore there is a universal constant C such that for any $f \in \mathbf{H}_v(\mathbf{G})$

$\|C_\varphi f\|_{\hat{v}} \leq C\|f\|_v$. □

Corollary 4.1.5: *Let υ and $\hat{\upsilon}$ be type(I) weights on* $\mathbf{G}$ *such that $\hat{\upsilon}$ is bounded and υ satisfies $(*)_I$. Put $\upsilon_1(\omega) = \upsilon(\frac{2Im\omega}{|\omega|^2+1}i)$. Assume that $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is a holomorphic function and satisfies*

$\sup_{a\in\mathbb{R}} \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\omega+a)-i}{\varphi(\omega+a)+i})^n\|_{\hat{\upsilon}}}{\|(\frac{\omega-i}{\omega+i})^n\|_{\upsilon_1}} < \infty$. *Then $C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\hat{\upsilon}}(\mathbf{G})$ is bounded.*

Proof: Let $\omega = \alpha(z),\ z \in \mathbb{D}$. Since $\tilde{\hat{\upsilon}}(z) = \hat{\upsilon}(\alpha(-\mid z\mid)) \leq \hat{\upsilon}(\alpha(z))$ we obtain

$$\|(\frac{\varphi(\alpha(z)+a)-i}{\varphi(\alpha(z)+a)+i})^n\|_{\tilde{\hat{\upsilon}}} \leq \|(\frac{\varphi(\omega+a)-i}{\varphi(\omega+a)+i})^n\|_{\hat{\upsilon}} \qquad (1)$$

Since $\alpha(-\mid z\mid) = \frac{1-|z|}{1+|z|} \leq \frac{1-|z|^2}{1+|z|^2}$ and υ satisfies $(*)_I$ we obtain

$\upsilon(\frac{1-|z|^2}{1+|z|^2}i) \leq C\upsilon(\alpha(-\mid z\mid)) = C\tilde{\upsilon}(z)$ for some universal constant $C > 0$. But

$\frac{1-|\frac{\omega-i}{\omega+i}|^2}{1+|\frac{\omega-i}{\omega+i}|^2} = \frac{|\omega+i|^2-|\omega-i|^2}{|\omega+i|^2+|\omega-i|^2} = \frac{2Im\omega}{|\omega|^2+1}$.

Therefore $\tilde{\upsilon}(z) \geq \frac{1}{C}\upsilon_1(\omega)$. This implies

$$\|z^n\|_{\tilde{\upsilon}} \geq \frac{1}{C}\|(\frac{\omega-i}{\omega+i})^n\|_{\upsilon_1} \qquad (2)$$

Our assumption, (1) and (2) yield

$\sup_{a\in\mathbb{R}} \sup_{n\in\mathbb{N}} \frac{\|(\frac{\varphi(\alpha(z)+a)-i}{\varphi(\alpha(z)+a)+i})^n\|_{\tilde{\hat{\upsilon}}}}{\|z^n\|_{\tilde{\upsilon}}} < \infty.$

Now Proposition 4.1.4 concludes the proof. $\square$

Section two: Continuity of the composition operators for type(I) weights which satisfy$(*)_I$

In Proposition 4.1.4 we presented a sufficient condition for boundedness of the composition operator $C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\hat{\upsilon}}(\mathbf{G})$ where υ and $\hat{\upsilon}$ are type(I) weights and $\hat{\upsilon}$ is also bounded.
Now we want to find a suficient condition for boundedness of the composition operator C_φ, when our weights are not necessarily bounded, but satisfy condition $(*)_I$.
For proving our result we define special type(II) weights and we use Lemma 3.2.9.

Definition 4.2.1: Let υ be a type(I) weight on $\mathbf{G}$. For any $n \in \mathbb{N}$ we define $\upsilon_n(\omega) = \upsilon(n\omega)$

Lemma 4.2.2: *υ_n is a type(I) weight on $\mathbf{G}$ for all $n \in \mathbb{N}$.*

Proof: υ is of type(I). So by definition there exists a $C > 0$ such that
$\upsilon(\omega_1) \le C\upsilon(\omega_2)$ whenever $Im\omega_1 \le Im\omega_2$. Note that $Imn\omega = nIm\omega \ \forall \omega \in \mathbf{G}$.
If $Im\omega_1 \le Im\omega_2 \Rightarrow nIm\omega_1 = Im(n\omega_1) \le nIm\omega_2 = Im(n\omega_2)$. Therefore
$\upsilon(n\omega_1) = \upsilon_n(\omega_1) \le C\upsilon(n\omega_2) = \upsilon_n(\omega_2)$. □

Remark 4.2.3: Note that the weights υ_n are of type(I) with the same constant which appears in the definition of type(I) weight υ.

Definition 4.2.4: For all $\omega \in \mathbf{G}$ define $\hat{\upsilon}(\omega) = \min(\upsilon(\omega), C\upsilon(-\frac{1}{\omega}))$ and for all $n \in \mathbb{N}$ and all $\omega \in \mathbf{G}$ define $\hat{\upsilon}_n(\omega) = \min(\upsilon_n(\omega), C\upsilon_n(-\frac{1}{\omega}))$.
Here C is the constant which appears in the definition of type(I) weights υ_n.

Lemma 4.2.5: *$\hat{\upsilon}$ and $\hat{\upsilon}_n (n \in \mathbb{N})$ are type(II) weights on $\mathbf{G}$.*

Proof: We prove the lemma for $\hat{\upsilon}$. The proof for $\hat{\upsilon}_n (n \in \mathbb{N})$ is similar.
If $|\ \omega \ | \le 1$ then $Im(-\frac{1}{\omega}) = \frac{Im\omega}{|\omega|^2} \ge Im\omega$. Therefore
$\upsilon(\omega) \le C\upsilon(-\frac{1}{\omega})$. (1)
Relation (1) implies that $\hat{\upsilon}(\omega) = \upsilon(\omega)$ whenever $|\ \omega \ | \le 1$. Now we prove that there exists a $d > 0$ such that $\frac{\hat{\upsilon}(\omega)}{\hat{\upsilon}(-\frac{1}{\omega})} \le d \ \ \forall \omega \in \mathbf{G}$. By definition of $\hat{\upsilon}$,

$\hat{\upsilon}(-\frac{1}{\omega}) = \min(\upsilon(-\frac{1}{\omega}), C\upsilon(\omega))$.

Using relation (1), if $\mid \omega \mid \leq 1$ then

$\frac{\hat{\upsilon}(\omega)}{\hat{\upsilon}(-\frac{1}{\omega})} \leq \max(\frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})}, \frac{\upsilon(\omega)}{C\upsilon(\omega)}) \leq \max(C, \frac{1}{C})$.

If $\mid \omega \mid > 1$ then $Im(-\frac{1}{\omega}) = \frac{Im\omega}{|\omega|^2} \leq Im\omega$. Therefore

$\upsilon(-\frac{1}{\omega}) \leq C\upsilon(\omega)$. (2)

Now considering that $\hat{\upsilon}(\omega) = \upsilon(\omega) \;\Leftrightarrow\; \leq C\upsilon(-\frac{1}{\omega}) \Leftrightarrow \mid \omega \mid \leq 1$ and relation (2), if

$\mid \omega \mid > 1$, then

$\frac{\hat{\upsilon}(\omega)}{\hat{\upsilon}(-\frac{1}{\omega})} \leq \max(\frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})}, \frac{C\upsilon(-\frac{1}{\omega})}{\upsilon(-\frac{1}{\omega})}) \leq C$.

Now put $d = \max\{C, \frac{1}{C}\}$. So we are done. $\square$

Definition 4.2.6: For any $n \in \mathbb{N}$ we define $T_n : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\hat{\upsilon}_n}(\mathbf{G})$ by $T_n(f)(\omega) = f(n\omega)$ for all $f \in \mathbf{H}_\upsilon(\mathbf{G})$.

Lemma 4.2.7: *For all $n \in \mathbb{N}, T_n$ is a contractive map.*

Proof:

$$\begin{aligned}
\|(T_n f)\|_{\hat{\upsilon}_n} &= \sup_{\omega \in \mathbf{G}} \mid (T_n f)(\omega) \mid \hat{\upsilon}_n(\omega) \\
&= \sup_{\omega \in \mathbf{G}} \mid f(n\omega) \mid \min(\upsilon_n(\omega), C\upsilon_n(-\frac{1}{\omega})) \\
&\leq \sup_{\omega \in \mathbf{G}} \mid f(n\omega) \mid \upsilon_n(\omega) \\
&= \sup_{\omega \in \mathbf{G}} \mid f(n\frac{\omega}{n}) \mid \upsilon(n\frac{\omega}{n}) \\
&= \|f\|_\upsilon.
\end{aligned}$$
$\square$

Lemma 4.2.8: *If the weight υ satisfies $(*)_I$, then $\hat{\upsilon}_n (n \in \mathbb{N})$ satisfies $(*)_{II}$.*

Proof: Since υ satisfies $(*)_I$, so $\exists\ C > 0,\ \exists\ \beta > 0$ such that $\frac{\upsilon(\omega_1)}{\upsilon(\omega_2)} \leq C(\frac{Im\omega_1}{Im\omega_2})^\beta$ whenever $Im\omega_1 \geq Im\omega_2$. Let $n \in \mathbb{N}$ be arbitrary.
If $Im\omega_1 \geq Im\omega_2$ then $Im(n\omega_1) \geq Im(n\omega_2)$. So $\frac{\upsilon_n(\omega_1)}{\upsilon_n(\omega_2)} = \frac{\upsilon(n\omega_1)}{\upsilon(n\omega_2)} \leq C(\frac{Im\omega_1}{Im\omega_2})^\beta$.
So we have proved that υ_n satisfies $(*)_I$ for all $n \in \mathbb{N}$. Now since for all $n \in \mathbb{N}$ $\hat{\upsilon}_n(\omega) = \upsilon_n(\omega)$ whenever $\mid \omega \mid \leq 1$ (see Lemma 4.2.5) so $\hat{\upsilon}_n$ satisfies $(*)_{II}$ for all $n \in$

$\mathbb{N}$. □

Lemma 4.2.9: *Let $f : \mathbf{G} \longrightarrow \mathbb{C}$ be a holomorphic function. If* $\sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n} < \infty$ *then $f \in \mathbf{H}_\upsilon(\mathbf{G})$ and* $\|f\|_\upsilon = \sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n}$.

Proof:

$$\begin{aligned}
\|T_n f\|_{\hat{\upsilon}_n} &= \sup_{\omega\in\mathbf{G}} \mid (T_n f)(\omega) \mid \hat{\upsilon}_n(\omega) \\
&= \sup_{\omega\in\mathbf{G}} \mid (T_n f)(\frac{\omega}{n}) \mid \hat{\upsilon}_n(\frac{\omega}{n}) \\
&= \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \min(\upsilon_n(\frac{\omega}{n}), C\upsilon_n(\frac{-n}{\omega})) \\
&= \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \min(\upsilon(\omega), C\upsilon(\frac{-n^2}{\omega})). \qquad (1)
\end{aligned}$$

So we have $\|T_n f\|_{\hat{\upsilon}_n} \leq \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \upsilon(\omega) = \|f\|_\upsilon$. Since $n \in \mathbb{N}$ is arbitrary so $\sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n} \leq \|f\|_\upsilon$ (2)
Consider an arbitrary and fixed $\omega_0 \in \mathbf{G}$. Since $Im(-\frac{n^2}{\omega_0}) = n^2\frac{Im\omega_0}{|\omega_0|^2}$ so for large enough $n \in \mathbb{N}$ we have $Im(-\frac{n^2}{\omega_0}) \geq Im\omega_0$. Therefore $\upsilon(\omega_0) \leq C\upsilon(-\frac{n^2}{\omega_0})$ for large enough $n \in \mathbb{N}$. So
$\mid f(\omega_0) \mid \upsilon(\omega_0) = \mid f(\omega_0) \mid \min(\upsilon(\omega_0), C\upsilon(-\frac{n^2}{\omega_0})) \leq \sup_{\omega\in\mathbf{G}} \mid f(\omega) \mid \min(\upsilon(\omega), C\upsilon(-\frac{n^2}{\omega}))$
Now using relation (1) we obtain $\mid f(\omega_0) \mid \upsilon(\omega_0) \leq \|T_n f\|_{\hat{\upsilon}_n}$ for large enough $n \in \mathbb{N}$. Thus
$\|f\|_\upsilon = \sup_{\omega_0\in\mathbf{G}} \mid f(\omega_0) \mid \upsilon(\omega_0) \leq \sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n}$. (3)
Now relations (2), (3) and our assumption imply that $\|f\|_\upsilon = \sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n} < \infty$. Therefore we are done. □

Theorem 4.2.10: *Let υ and υ' be type(I) weights on $\mathbf{G}$ which satisfy $(*)_I$. If*

$a = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \frac{\|(\frac{\frac{1}{n}\varphi(n\omega)-i}{\frac{1}{n}\varphi(n\omega)+i})^m\|_{\hat{\upsilon'}_n}}{\|(\frac{\omega-i}{\omega+i})^m\|_{\hat{\upsilon}_n}} < \infty$, *then the composition operator*

$C_\varphi : \mathbf{H}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\upsilon'}(\mathbf{G})$ *is welldefined and bounded. Here $\hat{\upsilon}_n$ and $\hat{\upsilon'}_n$ are as in Definition* 4.2.4.

Proof: Let $n \in \mathbb{N}$ be arbitrary and fixed. Consider the following diagram, where T_n is as in Definition 4.2.6 and T'_n is defined by $T'_n(f) = f(n\omega)$ for all $f \in \mathbf{H}_{\upsilon'}(\mathbf{G})$.

$$\begin{array}{llcl} C_\varphi & : \mathbf{H}_\upsilon(\mathbf{G}) & \longrightarrow & \mathbf{H}_{\upsilon'}(\mathbf{G}) \\ T_n & \quad\downarrow & & \quad\downarrow \quad T'_n \\ T'_n \circ C_\varphi \circ T_{\frac{1}{n}} & : \mathbf{H}_{\hat{\upsilon}_n}(\mathbf{G}) & \longrightarrow & \mathbf{H}_{\hat{\upsilon'}_n}(\mathbf{G}) \end{array}$$

It is easy to see that $(\mathrm{T}'_n \circ C_\varphi \circ T_{\frac{1}{n}})(f) = f(\frac{1}{n}\varphi(n\omega)) = C_{\varphi_n}(f)$ where $\varphi_n(\omega) = \frac{\varphi(n\omega)}{n}$. So $T'_n \circ C_\varphi \circ T_{\frac{1}{n}}$ is really a composition operator on $\mathbf{H}_{\hat{\upsilon}_n}(\mathbf{G})$. $\hat{\upsilon}_n$ and $\hat{\upsilon'}_n$ are type(II) weights which satisfy $(*)_{II}$(see Lemma 4.2.5 and Lemma 4.2.8) also by our assumption for any $n \in \mathbb{N}$

$\sup_{m\in\mathbb{N}} \frac{\|(\frac{\frac{1}{n}\varphi(n\omega)-i}{\frac{1}{n}\varphi(n\omega)+i})^m\|_{\hat{\upsilon'}_n}}{\|(\frac{\omega-i}{\omega+i})^m\|_{\hat{\upsilon}_n}} < \infty$. Therefore Theorem 3.2.9 implies that $T'_n \circ C_\varphi \circ T_{\frac{1}{n}}$ is bounded as an operator from $\mathbf{H}_{\hat{\upsilon}_n}(\mathbf{G})$ into $\mathbf{H}_{\hat{\upsilon'}_n}(\mathbf{G})$ and in this case

$\|C_{\varphi_n}\| \leq C_{\hat{\upsilon'}_n} \sup_{m\in\mathbb{N}} \frac{\|(\frac{n\varphi(\frac{\omega}{n})-i}{n\varphi(\frac{\omega}{n})+i})^m\|_{\hat{\upsilon'}_n}}{\|(\frac{\omega-i}{\omega+i})^m\|_{\hat{\upsilon}_n}}$. So $\sup_{n\in\mathbb{N}} \|C_{\varphi_n}\| \leq a \sup_{n\in\mathbb{N}} C_{\hat{\upsilon'}_n}$.

Here $C_{\hat{\upsilon'}_n}$ are the constants of Theorem 3.2.9.

Now looking at the proof of the Theorem 2.2.1 we see that the constants $C_{\hat{\upsilon'}_n}$ depend on the constants which appear in the definition of the type(II) weights $\hat{\upsilon'}_n$ $(n \in \mathbb{N})$ and property $(*)_{II}$.

But the definition of the type(II) weights $\hat{\upsilon'}_n (n \in \mathbb{N})$, Remark 4.2.3, Lemma 4.2.5 and Lemma 4.2.8 show that the constants does not depend on $n \in \mathbb{N}$. Therefore $\sup_{n\in\mathbb{N}} \|C_{\varphi_n}\| \leq \infty$. This means that the family of the composition operators $\{T'_n \circ C_\varphi \circ T_{\frac{1}{n}}\}$ is uniformly bounded. Therefore by using the relation $\|C_{\varphi_n}\| = \sup\{\|C_{\varphi_n}(h)\|_{\hat{\upsilon'}_n} : \|h\|_{\hat{\upsilon}_n} \leq 1\}$ we deduce that

$\exists C > 0$ such that $\forall n \in \mathbb{N}$ $\forall h \in \mathbf{H}_{\hat{\upsilon}_n}(\mathbf{G})$

$$\|(T'_n \circ C_\varphi \circ T_{\frac{1}{n}})(h)\|_{\hat{\upsilon'}_n} \leq C\|h\|_{\hat{\upsilon}_n}. \tag{2}$$

From Lemma 4.2.9 we have

$$\|f\|_\upsilon = \sup_{n\in\mathbb{N}} \|T_n f\|_{\hat{\upsilon}_n}. \qquad \forall f \in \mathbf{H}_\upsilon(\mathbf{G}) \tag{3}$$

and

$$\|g\|_{\upsilon'} = \sup_{n\in\mathbb{N}} \|T'_n f\|_{\hat{\upsilon'}_n}. \qquad \forall g \in \mathbf{H}_{\upsilon'}(\mathbf{G}) \tag{4}$$

Let $f \in \mathbf{H}_\upsilon(\mathbf{G})$ be arbitrary. Then

$\sup_{n\in\mathbb{N}} \|T'_n \circ C_\varphi f\|_{\hat{v'}_n} = \sup_{n\in\mathbb{N}} \|T'_n \circ C_\varphi \circ T_{\frac{1}{n}} \circ (T_n f)\|_{\hat{v'}_n}$

Now relation (2) implies that $\sup_{n\in\mathbb{N}} \|T'_n \circ C_\varphi \circ T_{\frac{1}{n}}(T_n f)\|_{\hat{v'}_n} \leq C\|T_n f\|_{\hat{v}_n}$

Using relation (3) we have

$\sup_{n\in\mathbb{N}} \|T'_n \circ C_\varphi(f)\|_{\hat{v'}_n} \leq C\|f\|_v < \infty$ (since $f \in \mathbf{H}_v(\mathbf{G})$)

Again Lemma 4.2.9 implies that $C_\varphi(f) \in \mathbf{H}_{v'}(\mathbf{G})$ and $\|C_\varphi f\|_{v'} = \sup_{n\in\mathbb{N}} \|T'_n \circ C_\varphi f\|_{\hat{v'}_n}$.

Therefore $\|C_\varphi f\|_{v'} \leq C\|f\|_v$ for all $f \in \mathbf{H}_v(\mathbf{G})$ □

Section three: Composition operator between weighted spaces of 2π-periodic holomorphic functions

As we said before we study in this section the continuity of the composition operator C_φ (see Remark 3.2.5) between two weighted spaces of $2\pi-$periodic functions where $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is a $2\pi-$periodic and holomorphic function. Before arriving at our result we present an example of a $2\pi-$periodic and holomorphic function from $\mathbf{G}$ into $\mathbf{G}$.

Example 4.3.1: Define $\varphi(\omega) = e^{i\omega} + i$ for any $\omega \in \mathbf{G}$. Clearly $Im(e^{i\omega} + i) = (e^{-Im\omega} \sin Re\omega + 1) > 0$ and $\varphi(\omega + 2\pi) = \varphi(\omega)$. So $\varphi(\mathbf{G}) \subseteq \mathbf{G}$ and φ is $2\pi-$periodic.

Let υ and υ' be type(I) weights on $\mathbf{G}$. Also let b_1 and b_2 be the smallest integers such that $\mathrm{e}^{-b_1 Im\omega}\upsilon(\omega)$ and $e^{-b_2 Im\omega}\upsilon'(\omega)$ are bounded (see Remark 2.4.5(b)). Put $b = \max\{b_1, b_2\}$ then b is the smallest integer such that $\upsilon_b(\omega) = e^{-bIm\omega}\upsilon(\omega)$ and $\upsilon'_b = e^{-bIm\omega}\upsilon'(\omega)$ both are bounded weights. Also we define $\tilde{\upsilon}(z) := \upsilon_b(\ln(\frac{1}{|z|})i)$ and $\tilde{\upsilon}'(z) := \upsilon'_{b'}(\ln(\frac{1}{|z|})i)$. Hence $\tilde{\upsilon}(z) =| z |^b \upsilon(-i \ln | z |)$ and $\tilde{\upsilon}'(z) =| z |^b \upsilon'(-i \ln | z |)$. Proposition 2.4.8 implies that the maps $T : \mathbf{H}^{2\pi}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$ defined by $T(g) = e^{ib\omega}g(\omega)$ and $T_1 : \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G})$ defined by $T_1(g_1) = e^{ib\omega}g_1(\omega)$ are isometries. Also Theorem 2.4.11 implies that the maps $S : \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ defined by $S(h)(z) = h(-i \log z)$ and $S_1 : \mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ defined by $S_1(h_1)(z) = h_1(-i \log z)$ are isometries. So the maps $ST : \mathbf{H}^{2\pi}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ and $S_1T_1 : \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ are isometries.

Remark 4.3.2: Note that $\tilde{\upsilon}$ and $\tilde{\upsilon'}$ are not necessary continuous at zero. But there are radial weights $\tilde{\upsilon}_1$ and $\tilde{\upsilon'_1}$ which are continuous and $\|.\|_{\tilde{\upsilon}} \sim \|.\|_{\tilde{\upsilon}_1}$ and $\|.\|_{\tilde{\upsilon}'} \sim \|.\|_{\tilde{\upsilon'_1}}$. (see Remark 2.4.10). Using Remark 1.2.6, without loss of generality we can assume $\tilde{\upsilon}_1$ and $\tilde{\upsilon'_1}$ are also decreasing weights on the unit disk $\mathbb{D}$. Therefore the maps $ST : \mathbf{H}^{2\pi}_\upsilon(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}_1}(\mathbb{D})$ and $S_1T_1 : \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G}) \longrightarrow \mathbf{H}_{\tilde{\upsilon'_1}}(\mathbb{D})$ are isomorphisms.

Suppose $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is a $2\pi-$periodic and holomorphic function. Consider the composition operator C_φ and the following diagram

$C_\varphi : \mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G}).$

$ST \quad \downarrow \qquad \downarrow \; S_1T_1 \qquad (1)$

$\tilde{C_\varphi} : \mathbf{H}_{\tilde{\upsilon}_1}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon'_1}}(\mathbb{D})$

Here $\tilde{C_\varphi}$ is defined as follows $\tilde{C_\varphi} := (S_1T_1) \circ C_\varphi \circ (ST)^{-1}$. Since ST and S_1T_1 are isomorphisms (see Remark 4.3.2) so C_φ is continuous iff $\tilde{C_\varphi}$ is continuous. So it is enough to study the operator $\tilde{C_\varphi}$. Suppose $f \in \mathbf{H}_{\tilde{\upsilon}_1}(\mathbb{D})$ then $T^{-1}S^{-1}(f) = e^{-ib\omega} f(e^{i\omega})$ and $(\tilde{C_\varphi} f)(z) = (S_1T_1) \circ C_\varphi \circ (ST)^{-1}(f) = z^b e^{-ib\varphi(-i\log z)} f(e^{i\varphi(-i\log z)})$.
We can consider $\tilde{C_\varphi}$ as a weighted composition operator. There is a wellknown result for continuity of weighted composition operators on the unit disk $\mathbb{D}$.

We formulate it in the following way.

Theorem 4.3.3: *Let υ_1 and υ_2 be a standard weights on the unit disk $\mathbb{D}$. Also suppose $\varphi_1, \psi_1 \in \mathbf{H}(\mathbb{D})$ and $\varphi_1(\mathbb{D}) \subseteq \mathbb{D}$. Then the weighted composition operator $C_{\varphi_1,\psi_1} : \mathbf{H}_{\upsilon_1}(\mathbb{D}) \longrightarrow \mathbf{H}_{\upsilon_2}(\mathbb{D})$ defined by $C_{\varphi_1,\psi_1}(f) = \psi_1 \cdot (f \circ \varphi_1)$ is continuous iff $\sup_{n\in\mathbb{N}} \frac{\|\varphi_1(z)^n \psi_1(z)\|_{\upsilon_2}}{\|z^n\|_{\upsilon_1}} < \infty$.*

Proof: Use Proposition 3.1 of [9] and Proposition 6 of [6] □

Now we can obtain the following necessary and sufficient condition for the continuity of the composition operator C_φ in diagram (1).

Theorem 4.3.4: *Let υ and υ' be type(I) weights on $\mathbf{G}$. The composition operator $C_\varphi : \mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G})$ defined by $C_\varphi(f) = f \circ \varphi$ is continuous iff $\sup_{n\in\mathbb{N}} \frac{\|e^{in\varphi(\omega)}\|_{\upsilon'}}{\|e^{in\varphi(\omega)}\|_{\upsilon}} < \infty$.*
(Here $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is $2\pi-$periodic and holomorphic)

Proof: Use Theorem 4.3.3 for the operator $\tilde{C_\varphi}$ in diagram (1) where $\upsilon_1 = \tilde{\upsilon}_1, \upsilon_2 = \tilde{\upsilon'_1}, \psi_1(z) = z^b e^{-ib\varphi(-i\log z)}$ and $\varphi_1(z) = e^{i\varphi(-i\log z)}$.
Also note that since $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$ is holomophic so $\psi_1, \varphi_1 \in \mathbf{H}(\mathbb{D})$. we also have $| \varphi_1(z) |=| e^{i\varphi(-i\log z)} |=| e^{-Im\varphi(-i\log z)} |< 1 \quad \forall z \in \mathbb{D}$ since $Im\varphi(\omega) > 0$ for all $\omega \in \mathbf{G}$. Therefore

$\sup_{n\in\mathbb{N}} \frac{\|e^{in\varphi(-i\log z)}z^b e^{-ib\varphi(-i\log z)}\|_{\tilde{v_1'}}}{\|z^n\|_{\tilde{v}_1}} < \infty. \quad (1).$

Use the definition of $\tilde{v}$ and $\tilde{v'}$ to see that relation (1) is equivalent to

$\sup_{n\in\mathbb{N}} \frac{\|e^{i(n-b)\varphi(-i\log z)}z^{2b}\|_{u'}}{\|z^{n+b}\|_u} < \infty \quad (2).$

Here $u'(z) = v'(\ln(\frac{1}{|z|})i)$ and $u(z) = v(\ln(\frac{1}{|z|})i)$.

Put $m = n - b$. Since m is non-positive for only finitely many $n \in \mathbb{N}$, (exactly $n = 1, 2, ..., b$) so relation (2) is equivalent to

$\sup_{m\in\mathbb{N}} \frac{\|e^{im\varphi(-i\log z)}z^{2b}\|_{u'}}{\|z^{n+b}\|_u} < \infty. \qquad (3).$

It is wellknown that if $M \in \mathbb{N}$ then the map $T_M : \mathbf{H}_v(\mathbb{D}) \longrightarrow \mathbf{H}_v(\mathbb{D})$ defined by $T_M(f) = z^M f(z)$ is an isomorphism onto a finite codimensional subspace of $\mathbf{H}_v(\mathbb{D})$ for any standard weight v on the unit disc $\mathbb{D}$. See [5, 8]

Consider the relations $T_{2b}(1) = z^{2b}$, $T_{2b}(z^{n-b}) = z^{n+b}$ and $m = n-b$. So relation (3) is equivalent to $\sup_{m\in\mathbb{N}} \frac{\|e^{im\varphi(-i\log z)}\|_{u'}}{\|z^m\|_u} < \infty$. Now put $\omega = -i\log z$ or equivalently $z = e^{i\omega}$ we have $\sup_{m\in\mathbb{N}} \frac{\|e^{im\varphi(\omega)}\|_{v'}}{\|e^{im\omega}\|_v} < \infty$. Then Theorem 4.3.3 completes the proof. □

Section four: Differentiation operator between weighted spaces of 2π-periodic holomorphic functions

In this section we obtain a suficient condition for boundedness of the differentiation operator between weighted spaces of $2\pi-$periodic holomorphic functions.

Lemma 4.4.1: *Let υ be a type(I) weight on* $\mathbf{G}$ *satisfying the property* $\sup_{n\in\mathbb{N}} \frac{\upsilon(\frac{1}{2^n}i)}{\upsilon(\frac{1}{2^{n+1}}i)} < \infty$. *Then*

i) $\tilde{\upsilon}(z)$ *and* $\tilde{\upsilon}_1(z)$ *satisfy* $(*)'$.*(See Theorem* 2.1.8, *Definition* 2.4.9 *and Remark* 2.4.10 *for definitions of* $(*)'$, $\tilde{\upsilon}(z)$ *and* $\tilde{\upsilon}_1(z)$ *resp.)*

ii) *Put* $\upsilon'(\omega) = (1 - e^{-Im\omega})\upsilon(\omega)$. *Then* υ' *is a type(I) weight on* $\mathbf{G}$.

Proof: i) $\frac{\tilde{\upsilon}(1-2^{-n})}{\tilde{\upsilon}(1-2^{-n-1})} = \frac{(1-2^{-n})^b\upsilon(-i\ln(1-2^{-n}))}{(1-2^{-n-1})^b\upsilon(-i\ln(1-2^{-n-1}))}$. Clearly $\frac{(1-2^{-n})^b}{(1-2^{-n-1})^b} < 1$. Using the mean value theorem for the function $\ln x$ on the interval $[1-2^{-n}, 1]$, there exists a $C' \in (1-2^{-n}, 1)$ such that $\ln 1 - \ln(1-2^{-n}) = \frac{1}{C'2^n}$. Therefore
$-\ln(1-2^{-n}) \leq \frac{1}{(1-2^{-n})2^n} < \frac{1}{2^{n-2}}$ for any $n \geq 3$.
Similarly there exists a $C'' \in (1-2^{-n-1}, 1)$ such that $-\ln(1-2^{-n-1}) = \frac{1}{C''}\frac{1}{2^{n+1}}$. So
$-\ln(1-2^{-n-1}) \geq \frac{1}{2^{n+1}}$.
Now since υ is a type(I) weight there exists a $C > 0$ such that

$$\upsilon(-i\ln(1-2^{-n})) \leq C\upsilon(\tfrac{1}{2^{n-2}}i) \ \&\ \upsilon(\tfrac{1}{2^{n+1}}i) \leq C\upsilon(-i\ln(1-2^{-n-1})).$$

Therefore $\sup_{n\in\mathbb{N}} \frac{\tilde{\upsilon}(1-2^{-n})}{\tilde{\upsilon}(1-2^{-n-1})} \leq C^2 \sup_{n\geq 3} \frac{\upsilon(2^{-n+2}i)}{\upsilon(2^{-n+1}i)} \sup_{n\geq 3} \frac{\upsilon(2^{-n+1}i)}{\upsilon(2^{-n}i)} \sup_{n\geq 3} \frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)}$.

Now our assumption implies $\sup_{n\in\mathbb{N}} \frac{\tilde{\upsilon}(1-2^{-n})}{\tilde{\upsilon}(1-2^{-n-1})} < \infty$ So $\tilde{\upsilon}$ satisfies $(*)'$. We also have $\tilde{\upsilon}_1(z) = \tilde{\upsilon}(z)$ for all z with $| z | \geq \frac{1}{2}$. So $\tilde{\upsilon}_1(z)$ satisfies $(*)'$.

ii) It is clear. □

Remark 4.4.2: Let υ be a type(I) weight on $\mathbf{G}$.

i) If we define $\upsilon'(\omega) = (1 - e^{-Im\omega})\upsilon(\omega)$, then $\tilde{\upsilon'}(z) = (1- | z |)\tilde{\upsilon}(z)$. Also we have $\|.\|_{\tilde{\upsilon}'_1} \sim \|.\|_{\tilde{\upsilon'}}$.

ii) $\|.\|_{\tilde{\upsilon}} \sim \|.\|_{\tilde{\upsilon}_1}$ implies that $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \sim \mathbf{H}_{\tilde{\upsilon}_1}(\mathbb{D})$. Similarly we have $\mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D}) \sim \mathbf{H}_{\tilde{\upsilon}'_1}(\mathbb{D})$.

Lemma 4.4.3: *Let b and b' be the smallest integers k such that*

$\sup_{\omega\in\mathbf{G}} e^{-kIm\omega}\upsilon(\omega) < \infty$ & $\sup_{\omega\in\mathbf{G}} e^{-kIm\omega}\upsilon'(\omega) < \infty$ *respectively, where* υ *is of type(I) and* υ' *is as in Lemma* 4.4.1.

i) *Let* $a > 0$ *be given. There is a constant* $d > 0$ (d *depends on* a *and* υ') *such that* $A := \sup\{e^{-b'Im\omega}\upsilon'(\omega) : \omega \in \mathbf{G}, Im\omega \geq a\} \leq \sup_{\omega\in\mathbf{G}} e^{-b'Im\omega}\upsilon'(\omega) \leq dA$.

ii) $b = b'$.

Proof: i) $\sup_{\omega\in\mathbf{G}} e^{-b'Im\omega}\upsilon'(\omega) = \max(A, \sup\{e^{-b'Im\omega}\upsilon'(\omega) : \omega \in \mathbf{G},\ 0 < Im\omega < a\})$. Since υ' is of type(I) so there is a constant $d_0 > 0$ such that

$\sup\{e^{-b'Im\omega}\upsilon'(\omega) : \omega \in \mathbf{G},\ 0 < Im\omega < a\} \leq \sup\{\upsilon'(\omega) : \omega \in \mathbf{G},\ 0 < Im\omega < a\} \leq d_0$.

Therefore $\sup_{\omega\in\mathbf{G}} e^{-b'Im\omega}\upsilon'(\omega) \leq dA$ with $d = \frac{\max(d_0, A)}{A}$.

ii) If $b' > b$ then by definition we have

$\infty = \sup_{\omega\in\mathbf{G}} e^{-bIm\omega}(1-e^{-Im\omega})\upsilon(\omega) < \sup_{\omega\in\mathbf{G}} e^{-bIm\omega}\upsilon(\omega) < \infty$ which is a contradiction.

If $b' < b$ then by definition

$$\sup_{\omega\in\mathbf{G}} e^{-b'Im\omega}\upsilon(\omega) = \infty \qquad (1)$$

and

$$\sup_{\omega\in\mathbf{G}} e^{-b'Im\omega}\upsilon'(\omega) < \infty \qquad (2)$$

Now relation (2) implies that

$$\sup\{e^{-b'Im\omega}\upsilon'(\omega) : \omega \in \mathbf{G}, Im\omega \geq a\} < \infty. \qquad (3)$$

Put $a = \ln 2$ in relation (3). Then we have

$$\sup\{e^{-b'Im\omega}\upsilon(\omega) : \omega \in \mathbf{G}, Im\omega \geq \ln 2\} < \infty \qquad (4)$$

$$\sup\{e^{-b'Im\omega}\upsilon(\omega) : \omega \in \mathbf{G}, 0 < Im\omega \leq \ln 2\} \leq \sup\{\upsilon(\omega) : \omega \in \mathbf{G}, 0 < Im\omega \leq \ln 2\} \leq C_{\ln 2} < \infty. \qquad (5)$$

Now relations (4) and (5) contradict relation (1). $\square$

Now we prove the following theorem.

Theorem 4.4.4: *Let* υ *be a type(I) weight on* $\mathbf{G}$ *satisfying* $(*)_{II}$. *Put* $\upsilon'(\omega) = (1 - e^{-Im\omega})\upsilon(\omega)$. *Then the differentation operator* $D : \mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G})$ *is a welldefined and bounded map.*

Proof: Consider the following diagram.

D: $\mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G})$

$S \downarrow \qquad\qquad\qquad \downarrow \quad S_1$

$D_1 : \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$

Here D is differentation operator, $D_1 = S_1 \circ D \circ S^{-1}$and $S, S_1, \tilde{\upsilon}, \tilde{\upsilon}'$ are defined as in Theorem 2.4.11 and Definition 2.4.9 respectively.

So $S_1 \circ D \circ S^{-1}$ is welldefined and bounded iff D is welldefined and bounded. Since $(S_1 \circ D \circ S^{-1})(h) = izh'(z)$ so $S_1 \circ D \circ S^{-1}$ is welldefined and bounded iff the differentiation operator $D_z : \mathbf{H}_{\tilde{\upsilon}}(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ is welldefined and bounded.

Now consider the following diagram.

$D : \mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G})$

$T \quad \downarrow \qquad\qquad \downarrow \; T_1$

$D_2 : \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G}) \longrightarrow \mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G})$

Here D is differentiation operator and

$D_2 := (T_1 \circ D \circ T^{-1})$ (1).

Also T and T_1 are as in Proposition 2.4.8. Clearly

$D_2(g) := (-ibe^{i\omega(b'-b)}id + e^{i\omega(b'-b)}D)(g)$ for any $g \in \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$. Since $b' = b$ (see Lemma 4.4.3(ii)) so the above relation reduces to

$D_2(g) = (id(-bi) + D)(g) \; \forall g \in \mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G}).$ (2)

Note that since $\upsilon'_b(\omega) \leq \upsilon_b(\omega)$ so $\|g\|_{\upsilon'_b} \leq \|g\|_{\upsilon_b}$. This implies $\mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G}) \subseteq \mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G})$. Therefore relation (2) is welldefined.

Since T and T_1 are isometries, relation (1) implies D_2 is welldefined and bounded iff D is welldefined and bounded from $\mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G})$ into $\mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G})$. Relation (2) implies that D_2 is welldefined and bounded iff D is welldefined and bounded from $\mathbf{H}^{2\pi}_{\upsilon_b}(\mathbf{G})$ into $\mathbf{H}^{2\pi}_{\upsilon'_b}(\mathbf{G})$. Therefore we have the following:

D from $\mathbf{H}^{2\pi}_{\upsilon}(\mathbf{G})$ into $\mathbf{H}^{2\pi}_{\upsilon'}(\mathbf{G})$ is welldefined and bounded iff D_z from $\mathbf{H}_{\tilde{\upsilon}}(\mathbb{D})$ into $\mathbf{H}_{\tilde{\upsilon}'}(\mathbb{D})$ is welldefined and bounded.

Lemma 4.4.1(i) and Theorem 3.1.2 imply that $D_z : \mathbf{H}_{\tilde{\upsilon}_1(z)}(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-|z|)\tilde{\upsilon}_1(z)}(\mathbb{D})$ is welldefined and bounded. Also Part(ii) of Remark 4.4.2, implies that

$id : \mathbf{H}_{\tilde{v}_1(z)}(z)(\mathbb{D}) \longrightarrow \mathbf{H}_{\tilde{v}}(\mathbb{D})$ is welldefined and bounded.

The above facts and the following diagram imply that the differentiation operator D_z from $\mathbf{H}_{\tilde{v}}(\mathbb{D})$ into $\mathbf{H}_{(1-|z|)\tilde{v}(z)}(\mathbb{D})$ is welldefined and bounded iff

$id : \mathbf{H}_{(1-|z|)\tilde{v}_1(z)}(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-|z|)\tilde{v}(z)}(\mathbb{D})$ is welldefined and bounded.

$D : \mathbf{H}_{\tilde{v}_1(z)}(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-|z|)\tilde{v}_1(z)}(\mathbb{D})$

$id \quad \downarrow \qquad \downarrow id$

$D : \mathbf{H}_{\tilde{v}}(\mathbb{D}) \longrightarrow \mathbf{H}_{(1-|z|)\tilde{v}(z)}(\mathbb{D})$

But we are done if we prove $\|.\|_{(1-|z|)\tilde{v}_1(z)} \sim \|.\|_{(1-|z|)\tilde{v}(z)}$. (3)

Since $\tilde{v}_1$ and $\tilde{v}$ are decreasing and bounded weights such that $\tilde{v}_1(\frac{1}{2}), \tilde{v}(\frac{1}{2}) > 0$ so

$\|f\|_{(1-|z|)\tilde{v}(z)} \sim \sup_{|z|\geq\frac{1}{2}} | f(z) | (1- | z |)\tilde{v}(z).$ (4)

$\|f\|_{(1-|z|)\tilde{v}_1(z)} \sim \sup_{|z|\geq\frac{1}{2}} | f(z) | (1- | z |)\tilde{v}_1(z)$ (5)

The definition of $\tilde{v}_1$ and $\tilde{v}$ imply $\tilde{v}_1|_{[2^{-1},1]} \sim \tilde{v}|_{[2^{-1},1]}$.

Therefore $\sup_{|z|\geq\frac{1}{2}} | f(z) | (1- | z |)\tilde{v}_1(z) \sim \sup_{|z|\geq\frac{1}{2}} | f(z) | (1- | z |)\tilde{v}(z).$ (6)

Now relations (4),(5) and (6) give relation (3) and the proof is complete. □

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