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Mean values of multiplicative functions over multiplicative arithmetical semigroups

Dissertation

by

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Chapter 1

Introduction

Prime numbers play a central role in mathematics, due to their atomic nature. The justification for this is the Fundamental Theorem of Arithmetic, which says that each positive integer can be factorized into a product of prime numbers and this factorization is unique up to the order of the terms. This fact is not as obvious as it seems, which can be seen through examples given by particular ideals of number fields. Since prime numbers serve as a basic concept, it is natural to ask how many of them there are. More than 2000 years ago, Euclid proved that there are infinitely many primes among the naturals and Eratosthenes gave a method for recognizing prime numbers. Since the ancient Greeks worked with complicated objects such as "proportion" instead of numbers, the revolutionary work of Fibonacci (Liber Abaci, 1202) concerning the introduction of the Arabic Numeral System in Europe was principal to the theory of prime numbers as well as to the whole mathematics. The next important step in the investigation of the distribution of prime numbers was the work of Euler (early 18th century) who proved that

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges and who introduced the equation

$$\sum_{n\in\mathbb{N}} \frac{1}{n^\sigma} = \prod_{p \text{ prime}} (1 - \frac{1}{p^\sigma})^{-1}, \tag{1.1}$$

which is valid for $\sigma > 1$ and turned out to be essential for today's research. At the end of that century, Gauss and Legendre conjectured that the number of primes up to a positive integer n is asymptotically

$$\frac{n}{\log n}.$$

Moreover, Gauss later proposed the expression $\int_2^n \frac{1}{\log u} du$ instead of $\frac{n}{\log n}$. The conjecture of Gauss and Legendre was proved independently by Hadamard and de la Vallée Poussin in 1896. As a consequence of their works, we have for $x > 2$

$$(\pi(x) :=) \sum_{\substack{p \leq x \\ p \text{ prime}}} 1 = \frac{x}{\log x} + R(x), \tag{1.2}$$

where

$$R(x) = o(\frac{x}{\log x}) \quad (x \to \infty).$$

This result is now known as the Prime Number Theorem (PNT). In fact, Hadamard and de la Vallée Poussin proved more than what was conjectured by Legendre and Gauss, namely that

$$\pi(x) = \mathrm{Li(x)} + \mathrm{R(x)} \tag{1.3}$$

holds where

$$\mathrm{Li(x)} = \int_2^{\mathrm{x}} \frac{1}{\log \mathrm{u}} \mathrm{du},$$

and with some fixed $c > 0$

$$R(x) \ll x \exp(-c\sqrt{\log x}).$$

The implied constant in the Vinogradov symbol $\ll$ is uniform for all $x > 2$, but not for c.
Their result was based on the work of Riemann, who combined equation (1.1) with Fourier theoretic methods to obtain information on $\pi(x)$. He introduced the function

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}, \tag{1.4}$$

which is absolutely convergent for all complex numbers s with $\Re s > 1$, and he extended Euler's equation (1.1) for those s. Riemann showed how $\zeta(s)$ could be analytically continued to a meromorphic function on the whole complex plane, with a pole of order 1 at $s = 1$ with residue 1, i.e. that

$$\zeta(s) = \frac{1}{s-1} + h(s) \tag{1.5}$$

holds for all $\mathbb{C} \setminus \{1\}$ where $h(s)$ is an entire function. This uniquely defined meromorphic function is one of the most famous functions in mathematics, and it is called Riemann's zeta function. He established a functional equation for this function, from which one can see that the zeta function has zeros which occur for all negative even integers $s = -2, -4, -6 \ldots$. Those zeros are called "trivial" zeros while the other zeros are called "non-trivial" zeros. It was recognized by Riemann that the size of the function $R(x)$ in (1.3) depends on the zeros of this function and he made some conjectures about it, most of which seem to be unreachable even with today's tools of mathematics. One of these conjectures states that all of the "non-trivial" zeros of the zeta function lie on the vertical line with real part 1/2. It is equivalent to

$$R(x) \ll x^{1/2} \log x$$

in (1.3) (See for example [38]). Many of the properties of the zeta function depend on the fact that

$$(N(x) :=) \sum_{n \leq x} 1 = x + \mathcal{O}(1). \tag{1.6}$$

It is known that the Fundamental Theorem of Arithmetic is not valid in general for the algebraic integers in algebraic number fields (as for example in $\mathbb{Q}(\sqrt{-6})$), but the uniqueness can be obtained for the multiplicative structure of ideals. That is, every proper ideal in the ring of algebraic integers over an algebraic number field can be uniquely factorized into a product of prime ideals. In 1897, Weber proved that $N(x)$, the number of integral ideals with norm not exceeding x satisfies

$$N(x) = Ax + \mathcal{O}(x^{\theta}),$$

where $A > 0$ and $0 < \theta < 1$ are constants and depend on the base field. Landau proved in [34] that the prime ideals satisfy the Prime Number Theorem, i.e. the number of prime ideals with norm not exceeding x satisfies formula (1.2).

Knopfmacher, partially motivated by this and other results, introduced the terminology of the so-called arithmetical semigroups in [31, 32]. Let $\mathcal{P}$ be a nonempty set, and let $\mathcal{G} = (\mathcal{G}, \cdot)$ be the free commutative monoid generated by $\mathcal{P}$. Suppose that the mapping $|.| : \mathcal{G} \to \mathbb{R}_{\geq 1}$ satisfies

1. $|m \cdot n| = |m| \cdot |n|$ for all $m, n \in \mathcal{G}$,
2. $\#\{n \in \mathcal{G} : |n| \leq x\}$ is finite for each $x \geq 1$. (finiteness property)

In this case we say that G forms a multiplicative arithmetical semigroup (or simply multiplicative semigroup) with norm function $|.|$. The connection between the distribution of elements and prime elements of arithmetical semigroups is a central question in that topic, and was intensively studied by many authors. Let

$$(N_{\mathcal{G}}(x) =)N(x) := \sum_{\substack{n\in\mathcal{G} \\ |n|\le x}} 1,$$

and let

$$(\pi_{\mathcal{G}}(x) =)\pi(x) := \sum_{\substack{p\in\mathcal{P} \\ |p|\le x}} 1.$$

The assumptions of Knopfmacher, which he called Axiom A, considers all multiplicative semigroups satisfying

$$N(x) = Ax^{\delta} + \mathcal{O}(x^{\eta}),$$

where $A, \delta > 0$, $0 \le \eta < \delta$. The result of Landau concerning the distribution of Prime Ideals fits in this context and can be seen as an example of the following Abstract Prime Number Theorem, which was proved in its generality by Knopfmacher in [31]

Theorem (Knopfmacher, Landau). *Let $\mathcal{G}$ be a multiplicative semigroup which satisfies*

$$N(x) = Ax^{\delta} + \mathcal{O}(x^{\theta})$$

with some $\theta < \delta$, and $A > 0$ for all $x \ge 1$. Then

$$\pi(x) = \frac{x^{\delta}}{\delta\log x} + o(\frac{x^{\delta}}{\log x}) \quad (x \to \infty).$$

Under Axiom A Knopfmacher developed a broad of arithmetical semigroups paying attention to arithmetical functions which are of special interest. Subsequent authors investigated the stability of the error term in the theorem above, i.e. in which way the error term in the PNT depends on the error term of $N(x)$. (See [3, 5, 6, 2, 45]) Moreover Fainleb investigates in [13] the properties of $N(x)$ using asymptotics for $\psi(x)$, where

$$\psi(x) = \sum_{|p|^{\alpha}\le x} \log|p|.$$

One could call his result the "inverse PNT".
The set of natural numbers is the standard example for a multiplicative semigroup. Together with the above mentioned case with the integral ideals of algebraic number fields it would be an example for a multiplicative semigroup for which the norm function is integer valued. In [3] Beurling suggested the following general structure. Let $\mathcal{P}$ be a sequence of positive real numbers $(p_1, p_2, \ldots)$ such that

$$1 < p_1 \leq p_2 \leq \cdots, \quad p_j \to \infty \text{ as } j \to \infty.$$

He called the elements of $\mathcal{P}$ the (generalized)g-primes. The g-integers are then the real numbers $(n_0, n_1(= p_1), n_2, \ldots)$ of the form $\prod_{j=1}^{\infty} p_j^{\nu_j}$ where each ν_j is allowed to range over all nonnegative integers. Note that the values of g-integers are not necessarily distinct. In the case

$$n_{i-1} < n_i = n_{i+1} = \cdots = n_{i+m-1} < n_{i+m}$$

we say that n_i has multiplicity m, but if we take them as if they were distinct (as it is the case in the literature) then the set of g-integers is just a multiplicative arithmetical semigroup. Further examples and applications of arithmetical semigroups can be found in [32, 14, 1, 19].
A large number of scientific papers investigate the connection between $N(x)$, $\zeta(s)$ and $\pi(x)$. In order to understand better the behaviour of primes of natural numbers, there was a need to prove the PNT by elementary methods, i.e. without using the theory of complex functions. Another motivation was described by Hardy as quoted in [17]:

"No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent on the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem cannot be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say 'lie deep' and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten."

More than a half century after the result of Hadamard and de la Valée Poussin, A. Selberg was able to make the first step into this direction by developing his famous Symmetry Formula. Using this formula Erdős and Selberg gave the first elementary proofs of the PNT [12, 41] independently. Landau proved in [35] pp. 567-574 that

$$\sum_{n\le x} \mu(n) = o(x) \quad (x \to \infty)$$

is equivalent to the PNT, where μ is the Möbius function which is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 \cdots p_r, \quad p_1 < \cdots < p_r. \end{cases}$$

It is clear that the Möbius function is uniquely determined by the values on prime powers, and $\mu(nm) = \mu(n)\mu(m)$ whenever m and n do not have common prime divisors. In general, an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $f(nm) = f(n)f(m)$ for all $(m, n) = 1$.
The result of Landau was of special interest in producing alternative proofs of the PNT. In general, the asymptotic of the partial sums of f played an important role, where f is an arithmetical function, i.e. what can we say about the limit behaviour of $L(x)$ as x tends to infinity where

$$L(x) = \frac{1}{x} \sum_{n\le x} f(n) \ ?$$

The limit of $L(x)$ as $(x \to \infty)$, if it exists, is called the *mean value* of f and is denoted by $M(f)$. This question was intensively investigated by several authors, especially for multiplicative functions with values in the unit disc. For further references as well as for investigations of arithmetical functions on subsets of the natural numbers see [28, 29, 30, 36, 37, 42, 43].
Then the result of Landau says that the PNT is equivalent to the fact that the Möbius function possesses the mean value zero.
For multiplicative f with $|f(n)| \le 1$ Delange [9] proved that the mean value $M(f)$ exists and is different from zero if and only if the series

$$\sum_p \frac{1 - f(p)}{p} \tag{1.7}$$

converges, and for some positive r, $f(2^r) \neq -1$.
Assuming further that f is real-valued and the above series diverges, Wirsing [49] proved that f has mean-value $M(f) = 0$. In particular, this means that

the mean value $M(f)$ always exists for real-valued multiplicative functions of modulus ≤ 1, and that the PNT holds.
The result of Delange and Wirsing was extended by Halász in [20] to complex valued functions.

Theorem (Halász). *Let f be a multiplicative function with $|f(n)| \leq 1$, satisfying*

$$\sum_p \frac{1 - Ref(p)p^{-i\tau}}{p} < \infty$$

for some real τ. Then

$$\frac{1}{x}\sum_{n\leq x} f(n) = \frac{x^{i\tau}}{1+i\tau}\prod_{p\leq x}\left(1-\frac{1}{p}\right)\prod_{p\leq x}\left(1+\sum_{m\geq 1}\frac{f(p^m)}{p^{m(1+i\tau)}}\right) + o(1)$$

as x tends to infinity. On the other hand, if there is no such τ then

$$\frac{1}{x}\sum_{n\leq x} f(n) = o(1) \quad (x \to \infty).$$

In [8], Daboussi and Indlekofer proved this result by elementary methods, thus giving an alternative elementary proof of the PNT (compare [23],[43]). Indlekofer developed in [24, 25] ,[26] a simpler way to investigate the asymptotic behaviour of $L(x)$. His method turns out to have applications in generalizations of the questions mentioned just above as we will see in subsequent chapters (compare [27]).
Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function with values in the unit disc, and let the corresponding Dirichlet generating function formally defined by

$$F(s) = \sum_n \frac{f(n)}{n^s}$$

which is absolutely convergent for $\Re s > 1$. Let furthermore

$$M(x) := \sum_{n\leq x} f(n)$$

for $x > 1$.
Indlekofer's method could be described briefly as follows:
By using a convolution identity after applying twice the "differential like" logarithm operator, we obtain with a suitable weight function h

$$\log^2 xM(x) = \sum_{n\leq x} M(\frac{x}{n})h(n) + Error_1(x).$$

Then partial summation allows us to use the special behaviour of h (as for example Selberg's Symmetry Formula) to deduce

$$\sum_{n\leq x} |M(\frac{x}{n})h(n)| \leq cx\log x \int_1^x \frac{|M(u)|}{u^2}du + Error_2(x),$$

from which an application of Parseval's identity leads to

$$\log^2 xM(x) \leq cx\log^{3/2} x(\int_{-\infty}^{\infty} |\frac{F(1+\frac{1}{\log x}+i\tau)}{1+\frac{1}{\log x}+i\tau}|^2 d\tau)^{1/2} + Error_3(x).$$

Indlekofer obtained two other strong versions of the last inequality, one of which includes the square integral of the derivative of the generating function over the abscissa at $1/\log x$, and the other averages this last square integral over the abscissas between $1/\log x$ and 1. (See [26]) After some computation we can ensure that the contribution of $Error_3(x)$ is negligible and after using some elementary properties of $F(s)$ on the half plane right to 1 we arrive to an estimation of $M(x)$. This estimation lies at least as deep as the PNT, since applying for $f = \mu$,

$$\sum_{n\leq x} \mu(n) = o(x) \quad (x \to \infty)$$

would follow. Moreover, this method is suitable to give quantitative estimations for the partial sums, as it was shown by Germán, Indlekofer and Klesov in [16] for multiplicative functions f for which $|f(n)| \leq 1$ does not necessarily hold.

A generalization of Halász's result for arithmetical semigroups under Axiom A was done in [37] by Lucht and Reifenrat. They also show that their result implies the PNT for such semigroups. It was pointed out by Zhang in [45, 46] that in general arithmetical semigroups the Landau result concerning the equivalence of the PNT and the mean value zero for the Möbius function does not necessarily hold, thus disproving a conjecture of Hall in [21].

The aim of this work is to generalize the Halász theorem to arithmetical semigroups $\mathcal{G}$ under quite general conditions by using the Indlekofer method described above, to prove a quantitative version of Halász's theorem and to characterize the limit distribution of additive arithmetical functions defined on $\mathcal{G}$.

A general assumption on multiplicative semigroups used throughout in the present work, is that

$$N(x) = Ax^\delta \log^\beta x + R_1(x) \quad (x \to \infty) \tag{1.8}$$

with $A, \delta > 0$, $\beta \geq 0$ and

$$R_1(x) = o(x^\delta \log^\beta x) \quad (x \to \infty).$$

In general, relation (1.8) does not imply the PNT. The possible zeros of the generating zeta function play an important role here (See for example [3, 7] and for the stability of the Prime Number Theorem [3, 18]). Beurling has shown in [3] that if $c_0, \dots, c_r$ are constants, $c > 0$ and

$$N(x) = cx \log^\beta x + x \sum_{r=0}^{m} c_r \log^{\beta_r} x + \mathcal{O}(x \log^{-\eta} x) \tag{1.9}$$

with $\beta_0 < \beta_1 < \cdots < \beta_m < \beta$ such that $\beta \geq 0$, then the corresponding zeta function may have zeros on the vertical line with real part 1. Denoting the imaginary part of these possible zeros by t_n, $(n \in \mathbb{N})$, that is

$$\zeta(1 + it_n) = 0 \quad n \in \mathbb{N},$$

then

$$\sum_{|t_n|>0} \alpha(t_n) \leq (\beta + 1)/2$$

where $\alpha(t)$ - the degree of the corresponding zero - is defined by

$$\liminf_{\sigma \to 1+} \frac{\log |\zeta(\sigma + it)|}{\log(\sigma - 1)}.$$

Hence there are only finitely many zeros. Denoting them by $t_1, t_2, \dots, t_l$ Beurling obtained that if $\eta > 1 + (\beta+1)/2$ then the prime counting function $\pi(x)$ is such that

$$\pi(x) = (\beta + 1 - 2 \sum_{k=1}^{l} \frac{\alpha(t_k)}{1 + t_k^2} (t_k \sin(t_k \log x) + \cos(t_k \log x))) \frac{x}{\log x} + o(\frac{x}{\log x}) \quad (x \to \infty).$$

Thus in addition to (1.8) we shall suppose

$$\pi(x) = (\frac{\beta + 1}{\delta} - 2 \sum_{r=1}^{m} \frac{\alpha_r}{\sqrt{\delta^2 + t_r^2}} \cos(t_r \log x - \theta_r)) \frac{x^\delta}{\log x} + R_2(x) \tag{1.10}$$

with some positive integer m, where $\alpha_r \in \mathbb{N}_0$, $t_r > 0$, $r = 1, \dots, m$ such that

$$\sum_{r=1}^{m} \alpha_r \leq \frac{\beta + 1}{2},$$

and θ_r, $r = 1, \ldots, m$ are the angles such that

$$\sin\theta_r = \frac{t_r}{\sqrt{\delta^2 + t_r^2}} \quad \text{and } \cos\theta_r = \frac{\delta}{\sqrt{\delta^2 + t_r^2}}$$

and

$$R_2(x) = o(x^\delta) \quad (x \to \infty).$$

In Chapter 3 we prove that under these conditions Halász theorem remains valid, i.e. if f is a multiplicative function with $|f(n)| \leq 1$ such that the series

$$\sum_p \frac{1 - \Re f(p)|p|^{-ia}}{|p|^\delta} \tag{1.11}$$

converges for some real a, then

$$\sum_{|n|\leq x} f(n) = \frac{N(x)x^{ia}\delta}{\delta + ia} \prod_{|p|\leq x} (1 - \frac{1}{|p|^\delta})(1 + \sum_\alpha \frac{f(p^\alpha)}{|p|^{\alpha(\delta+ia)}}) + o(N(x)) \quad (x \to \infty). \tag{1.12}$$

On the other hand, if there is no such a then

$$\sum_{|n|\leq x} f(n) = o(N(x)) \quad (x \to \infty).$$

Note that it would be enough to consider only the case $\delta = 1$ and then by adjusting the value of a in (1.11) to deduce the case of an arbitrary δ. Nevertheless, for more transparency of the results we develop the general case. For a larger class of functions with $\delta = 1$ Zhang proved in [47] supposing (1.9) with $\eta > \eta_0$ and supposing (1.10) together with

$$R_2(x) = \mathcal{O}(x \log^{-M} x)$$

where $0 < \eta_0 \leq M$, that

$$(F(\sigma + i\tau) =)F(s) = \frac{c}{(s - 1 - i\alpha)^\xi} L(\frac{1}{\sigma - 1}) + o(\frac{|s|}{\sigma - 1}) \quad (\sigma \to 1+) \tag{1.13}$$

implies

$$\sum_{|n|\leq x} f(n) = \frac{cx^{1+i\alpha} \log x^{\xi-1}}{\Gamma(\xi)(1 + i\alpha)} L(\log x) + o(x \log^{\tau-1} x) \quad (x \to \infty),$$

where $\xi \geq 1$ and $L(u)$ is a slowly oscillating function with $|L(u)| = 1$ and Γ is the Gamma function. If $|f(n)| \leq 1$, then $F(s)$ satisfies (1.13), thus implying the results of Chapter 3 but under stronger conditions. Unfortunately, his method is not suitable to give quantitative results, which would have application to the distributions of additive functions (compare [11]). We deal with such estimations in Chapter 4. Assuming (1.9) we show how the error term in (1.12) depends on the values of f at prime powers, if f is in some sense close to a real $\kappa > \frac{1}{2(\beta+1)}$. In Chapter 5 we show that under the conditions of Chapter 4 for an additive arithmetical function $f : \mathcal{G} \to \mathbb{R}$ there exists a distribution function G for which

$$\lim_{x\to\infty} \frac{1}{N(x)} \#\{n \in \mathcal{G},\ |n| \leq x : f(n) \leq z\} = G(z)$$

holds in all continuity points z of G if and only if the three series

$$\sum_{|f(p)|\leq 1} \frac{f(p)}{|p|^\delta}, \quad \sum_{|f(p)|\leq 1} \frac{f^2(p)}{|p|^\delta}, \quad \sum_{|f(p)|>1} \frac{1}{|p|^\delta}$$

converge. If the limit law G exists then the characteristic function $\psi(t)$ of G is given by

$$\psi(t) = \prod_p (1 - \frac{1}{|p|^\delta})(1 + \sum_{\alpha\geq 1} \frac{e^{itf(p^\alpha)}}{|p|^{\delta\alpha}})$$

and G is continuous if and only if

$$\sum_{f(p)\neq 0} \frac{1}{|p|^\delta}$$

diverge. This result could be compared with Kolmogorov's three series theorem for the limit distribution of sums of independent random variables.

Chapter 2

Dirichlet convolution over arithmetical semigroups and generating zeta functions

The Dirichlet algebra $\mathcal{F}_{\mathcal{G}}$ of $\mathcal{G}$ consists of all arithmetical functions $f : \mathcal{G} \to \mathbb{C}$ with the usual linear operations and the Dirichlet convolution $* : \mathcal{F}_{\mathcal{G}} \times \mathcal{F}_{\mathcal{G}} \to \mathcal{F}_{\mathcal{G}}$ as multiplication, which for $f, g \in \mathcal{G}$ is defined by

$$f * g(a) = \sum_{mn=a} f(m)g(n) \quad (a \in \mathcal{G}).$$

The convolution is associative and commutative, thus $\mathcal{F}_{\mathcal{G}}$ is a commutative $\mathbb{C}$-algebra.
Let $f, g : \mathcal{G} \to \mathbb{C}$ be arithmetical functions on $\mathcal{G}$. The logarithm function on $\mathcal{G}$ will be denoted by L, i.e. $L(n) = \log |n|$, $n \in \mathcal{G}$.
It is clear that L is a differential operator in some sense, that is

$$L \cdot (f * g) = (L \cdot f) * g + f * (L \cdot g). \tag{2.1}$$

Let

$$\epsilon(n) = \begin{cases} 1 & \text{if |n|=1} \\ 0 & \text{otherwise,} \end{cases}$$

such that

$$f * \epsilon(n) = \sum_{d|n} f(d)\epsilon(\frac{n}{d}) = f(n).$$

If $f(1) \neq 0$ then the inverse of f is defined by $f^{-1}(1) = \frac{1}{f(1)}$ and

$$f^{-1}(n) = -\frac{\sum_{\substack{d|n \\ |d|<|n|}} f^{-1}(d) f(\frac{n}{d})}{f(1)}$$

for all $|n| > 1$. It is easy to see that with this definition

$$f^{-1} * f = \epsilon$$

holds. The corresponding von Mangoldt function Λ_f is defined by

$$\Lambda_f = Lf * f^{-1}.$$

The above defined f is called multiplicative if $f(nm) = f(n)f(m)$ if $(n,m) = 1$ and is called completely multiplicative if $f(nm) = f(n)f(m)$ holds for all $m, n \in \mathcal{G}$. Here $(n,m) = 1$ indicates that n, m does not have a common prime divisor. Since for such functions $f(1) = 1$, the inverse f^{-1} exists and it is easy to see that for completely multiplicative f we have

$$f^{-1}(n) = \mu(n)f(n) \quad (n \in \mathcal{G}).$$

Here μ is the inverse to the function $\mathbf{1}_0$, where with an $a \in \mathbb{R}$

$$\mathbf{1}_a(n) = |n|^{ia} \qquad (n \in \mathcal{G}).$$

For completely multiplicative f we clearly have

$$\Lambda_f = f\Lambda,$$

where $\Lambda = \Lambda_{\mathbf{1}_0}$. Furthermore, Λ is supported only on prime powers, and an easy computation shows

$$\Lambda(p^\alpha) = \log|p| \quad (p \in \mathcal{P}).$$

Note, that if $f,\ g$ are multiplicative then $f * g,\ f^{-1}$, is multiplicative as well. It is convenient to associate the summatory function to an arbitrary $f : \mathcal{G} \to \mathbb{C}$

$$(M(x) =)M_f(x) = \sum_{\substack{n\in\mathcal{G} \\ |n|\leq x}} f(n).$$

For further references on Dirichlet convolution see [31]. Let the generating Riemann Zeta function of $\mathcal{G}$ be defined formally by

$$(\zeta_{\mathcal{G}}(s) =)\zeta(s) := \sum_{n\in\mathcal{G}} \frac{1}{|n|^s}.$$

It may happen that the series on the right hand side converges for some $s \in \mathbb{C}$. Suppose that the abscissa of absolute convergence for ζ is finite, and denote it by δ. It is well known that the Euler product formula

$$\zeta(s) = \prod_{p \in \mathcal{P}} (1 - |p|^{-s})^{-1}$$

holds for all s with $\Re s > \delta$ and that $\zeta(s) \neq 0$ for such s. In this region we write

$$\zeta(s) = \frac{H(s)}{(s-\delta)^{\beta+1}} \tag{2.2}$$

for $\beta \in \mathbb{R}_0^+$ and appropriate $(H_\beta(s) =)H(s)$. Suppose that for $c_1 > 0$ there exist c_2, c_3 positive constants such that

$$c_2 < |H(s)| < c_3$$

holds for $\sigma > \delta$ and $|\tau| \leq c_1$. Then

$$\begin{aligned} -\frac{\zeta'}{\zeta}(s) &= -\sum_{n \in \mathcal{G}} \frac{\Lambda(n)}{|n|^s} \\ &= \oint \frac{\log H(s)}{(s-z)^2} dz + \alpha \oint \frac{\log(z-\delta)}{(s-z)^2} dz \\ &\ll \frac{1}{|s-\delta|} \end{aligned} \tag{2.3}$$

holds in the same neighbourhood of δ to the right. Let

$$\psi(x) := \sum_{|n| \leq x} \Lambda(n).$$

The two counting functions $\psi(x)$ and $\pi(x)$ are connected by the partial summation formula such that it is easy to compute the one from the other. By the absolute convergence of the generating zeta function we obtain by partial summation that

$$-\frac{\zeta'}{\zeta}(s) = \lim_{x \to \infty} x^{-s}\psi(x) + s \int_1^x \frac{\psi(u)}{u^{s+1}} du.$$

Since the series represented by the left hand side converges absolutely, and both terms on the right hand side are positive for real $\delta < s$, we obtain that

$$\psi(x) \ll_\epsilon x^{\delta+\epsilon}$$

for all $\epsilon > 0$. It follows immediately that

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(u)}{u^{s+1}}du.$$

This relation plays an important role in the theory of prime numbers. A common regularity assumption on $\psi(x)$ is

$$\psi(x) \asymp x^\delta$$

which Chebyshev originated. An example of Hall in [21] shows that this does not necessary holds, i.e. there are semigroups for which with

$$\liminf_{x\to\infty} \frac{\psi(x)}{N(x)} = a$$

and

$$\limsup_{x\to\infty} \frac{\psi(x)}{N(x)} = A$$

either $a = 0$ or $A = \infty$. It is an open question (see [48]) what values a and A can take.
Let

$$N(x) := \sum_{|n|\le x} 1.$$

Suppose that

$$N(x) = cx^\delta \log^\beta x + o(x^\delta \log^\beta x) \quad (x \to \infty) \tag{2.4}$$

where $\beta \ge 0$, $c, \delta > 0$. Then (2.2) holds. Generally, we have

Lemma 1. *Suppose that*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \ge 0$. *Then*

$$\zeta(s) = \frac{c\delta\Gamma(\beta+1)}{(s-\delta)^{\beta+1}} + o(\frac{1}{(\sigma-\delta)^{\beta+1}}) \quad (\sigma \to \delta+)$$

uniformly for all $|\tau| \ll 1$.

Proof. Using partial summation we obtain for all complex s with $\sigma > \delta$

$$\sum_{|n|\leq y} \frac{1}{|n|^s} = y^{-s}N(y) + s\int_1^y \frac{N(u)}{u^{s+1}}du. \tag{2.5}$$

Since

$$\int_1^\infty \frac{N(u)}{u^{s+1}}du$$

converges absolutely, letting y to infinity, the right hand side of (2.5) tends to a finite value. Thus

$$\sum_n \frac{1}{|n|^s}$$

converges absolutely for $\sigma > \delta$ and uniformly for $\sigma \geq \delta_1 > \delta$. Let $\epsilon > 0$ be fixed. Then there exists $1 < l(\epsilon)$, such that

$$\begin{aligned}\int_1^y \frac{N(u)}{u^{s+1}}du =& c\int_1^y \frac{\log^\beta u}{u^{(s-\delta+1)}}du \\ &+ \mathcal{O}(\int_1^{l(\epsilon)} \frac{\log^\beta u}{u^{(\sigma-\delta+1)}}du + \epsilon\int_{l(\epsilon)}^y \frac{\log^\beta u}{u^{(\sigma-\delta+1)}}du)\end{aligned} \tag{2.6}$$

holds if y is large enough. We compute

$$\begin{aligned}\int_1^y \frac{\log^\beta u}{u^{s-\delta+1}}du &= \int_1^y \exp(-(s-\delta+1)\log u + \beta\log\log u)du \\ &= \int_0^{\log y} \exp(-(s-\delta)t + \beta\log t)dt \\ &= \frac{1}{s-\delta}\int_0^{(s-\delta)\log y} \exp(-u)(\frac{u}{s-\delta})^\beta du \\ &= \frac{1}{(s-\delta)^{\beta+1}}\Gamma(\beta+1) + o(\frac{1}{|s-\delta|^{\beta+1}}) \quad (y\to\infty).\end{aligned} \tag{2.7}$$

Here we used that

$$\exp(-u)u^\beta$$

is holomorphic for $\Re u > 0$, such that by Cauchy's theorem

$$\oint \exp(-u)u^\beta du = 0$$

holds for every closed path contained in $\{z \in \mathbb{C} : \Re z > 0\}$. Since

$$\int_{(s-\delta)\log y}^{\log y} \exp(-u) u^{\beta} du \ll \log y \exp(-(\sigma - \delta)\log y) \log^{\beta} y$$

for all large enough y, (2.7) follows. Substituting into (2.6), then letting y to infinity, we deduce

$$|\zeta(s) - \frac{sc\Gamma(\beta+1)}{(s-\delta)^{\beta+1}}| \leq \epsilon \frac{|s|}{(\sigma-\delta)^{\beta+1}} \quad (\sigma \to \delta+),$$

which - since ϵ was arbitrary - proves the assertion. □

By supposing a substantially more precise error term we obtain the following

Lemma 2. *Suppose that*

$$N(x) = cx^{\delta} \log^{\beta} x + \mathcal{O}(x^{\delta} \log^{-\eta} x)$$

where $c, \delta > 0$, $\beta \geq 0$ and $\eta > 1$. Then there exists a function $g(s)$ which is continuous and bounded for $\Re s \geq \delta$, such that for $\Re s > \delta$

$$\zeta(s) = \frac{c\delta\Gamma(\beta+1)}{(s-\delta)^{\beta+1}} + \frac{c\Gamma(\beta+1)}{(s-\delta)^{\beta}} + sg(s).$$

Proof. As in the proof of the previous lemma

$$\begin{aligned}\zeta(s) =& s\int_1^{\infty} \frac{N(x)}{x^{s+1}} dx \\ =& s\int_1^{\infty} \frac{cx^{\delta}\log^{\beta} x}{x^{s+1}} dx + s\int_1^{\infty} \frac{N(x) - cx^{\delta}\log^{\beta} x}{x^{s+1}} dx.\end{aligned}$$

Putting

$$g(s) := \int_1^{\infty} \frac{N(x) - cx^{\delta}\log^{\beta} x}{x^{s+1}} dx,$$

we obtain that by the conditions g is uniformly bounded and continuous for $\Re s \geq 1$. □

Suppose that $c_0, \ldots, c_r$ are constants, $c > 0$ and

$$N(x) = cx\log^{\beta} x + x\sum_{r=0}^{m} c_r \log^{\beta_r} x + R_1(x)$$

with $-2 < \beta_0 < \beta_1 \cdots < \beta$ real numbers such that $\beta > -1$ where

$$R_1(x) = \mathcal{O}(x \log^{-\eta} x)$$

with $\eta > \max(3/2, 1 + (\beta+1)/2)$. In [3] Beurling has shown that there are certain non-zero real numbers $t_1, t_2 \ldots, t_l$, such that

$$\psi(x) = (\beta + 1 - 2\sum_{k=1}^{l} \frac{1}{\sqrt{1+t_k^2}}(\cos(t_k \log x - \arctan t_k)))x + R_2(x)$$

where

$$R_2(x) = o(x) \quad (x \to \infty).$$

In the following chapter we suppose

$$N(x) = cx^{\delta} \log^{\beta} x + R_3(x),$$

$$\psi(x) = (\beta + 1 - 2\sum_{k=1}^{l} \frac{1}{\sqrt{1+t_k^2}}(\cos(t_k \log x - \arctan t_k)))x^{\delta} + R_4(x),$$

where $c, \delta > 0$, $\beta \geq 0$ and

$$R_3(x) = o(x^{\delta} \log^{\beta} x) \quad (x \to \infty)$$

further, $t_1, t_2, \ldots, t_k$ are non-zero real numbers and

$$R_4(x) = o(x^{\delta}) \quad (x \to \infty).$$

Under these conditons we show for multiplicative functions with values in the unit disc that if

$$\sum_{p} \frac{1 - \Re f(p)|p|^{-ia}}{|p|^{\delta}}$$

converges for some real a ("convergent" case) then

$$\sum_{|n| \leq x} f(n) = \frac{N(x)x^{ia}\delta}{\delta + ia} \prod_{|p| \leq x} (1 - \frac{1}{|p|^{\delta}})(1 + \sum_{\alpha} \frac{f(p^{\alpha})}{|p|^{\alpha(\delta+ia)}}) + o(N(x)) \quad (x \to \infty).$$

On the other side, if there is no such a ("divergent" case) then we deduce

$$\sum_{|n| \leq x} f(n) = o(N(x)) \quad (x \to \infty).$$

Zhang proved a general result in [47], which as a consequence includes the above result for $\delta = 1$ with stronger conditions. His condition was that with appropriate positive η_0, M_0 in the above formulation $N(x)$ and $\psi(x)$ satisfies $\beta \geq 0$ and

$$R_1(x) = \mathcal{O}(x \log^{-\eta} x)$$

with $\eta > \eta_0 > 0$ and

$$R_2(x) = O(x \log^{-M} x) \quad (x \to \infty)$$

where $M > M_0 > 0$. Furthermore, under the conditions

$$\int_1^\infty \frac{|N(x) - Ax|}{x^2} dx < \infty$$

and either

$$\int_1^x \frac{(N(t) - At) \log t}{t} dt \ll x$$

or

$$\int_1^\infty \frac{|N(x) - Ax|^2 \log x}{x^3} dx < \infty$$

he proved the "divergent" case in [46].

Chapter 3

An analogue of the Theorem of Halász

We are interested in the limit behaviour of $\sum_{\substack{n\in\mathcal{G}\\|n|\le x}} f(n)$ as $x \to \infty$. In this chapter we generalize the Theorem of Halász mentioned in the introduction to arithmetical semigroups satisfying

$$N(x) = cx^{\delta}\log^{\beta} x + R(x),$$

$$\psi(x) = (\frac{\beta+1}{\delta} - 2\sum_{r=1}^{m}\frac{\alpha_r}{\sqrt{\delta^2+t_r^2}}\cos(t_r\log x - \theta_r))x^{\delta} + o(x^{\delta}) \quad (x \to \infty)$$

with $c, \delta > 0$, $\beta \ge 0$, and

$$R(x) = o(x^{\delta}\log^{\beta} x) \quad (x \to \infty).$$

Here $\alpha_r \in \mathbb{N}_0$ and t_r, $r = 1, \dots, m$ are real positive numbers such that

$$\sum_{r=1}^{m}\alpha_r \le \frac{\beta+1}{2},$$

and θ_r, $r = 1, \dots, m$ are the angles which satisfy

$$\sin\theta_r = \frac{t_r}{\sqrt{\delta^2+t_r^2}} \quad \text{and} \quad \cos\theta_r = \frac{\delta}{\sqrt{\delta^2+t_r^2}}.$$

The "divergent case" is entiled in Corollary 2 while the "convergent case" in Corollary 1.
The partial sums of f will be estimated first via convolution identity from

which we obtain a sum over the weighted partial sums of f. Then using the regularity properties of the weights (Selberg Symmetry Formula), this leads to an integral which avarages the partial sums of f. Using Parseval's identity we relate this to the $\mathbb{L}_2$ norm of $\frac{F(s)}{s}$ where F is the Dirichlet generating function of f. Then using elementary properties of the zeta function we estimate the Fourier transform.

3.1
A convolution identity

The following result was proved for $\mathcal{G} = \mathbb{N}$ in [25], and it remains true for general arithmetical semigroups.

Theorem 1 (Indlekofer). *Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative function, and put $(M(x) =)M_{f-A\mathbf{1}_a}(x) = \sum_{|n|\le x} f(n) - A\mathbf{1}_a(n)$, where $A \in \mathbb{C}$ and $a \in \mathbb{R}$. Define $\tilde{f}$ to be a completely multiplicative function with $\tilde{f}(p) = f(p)$, and let g be defined by $f = \tilde{f} * g$. Then for $x > 1$*

$$\begin{aligned}\log^2 xM(x) &= \sum_{|n|\le x} M(\frac{x}{|n|})\tilde{f}(n)\{\Lambda * \Lambda(n) + L(n)\Lambda(n)\} \\ &\quad + \sum_{|n|\le x}\{R_1 + R_2 + R_3\}(\frac{x}{|n|})\tilde{f}(n)\Lambda(n) \\ &\quad + (R_1 + R_2 + R_3)(x)\log x,\end{aligned}$$

where

$$R_1(x) = \sum_{|n|\le x} \log\frac{x}{|n|}(f(n) - A\mathbf{1}_a(n)),$$

$$R_2(x) = \sum_{|n|\le x}(\sum_{|m|\le \frac{x}{|n|}} \tilde{f}(m))g(n)\log|n|,$$

and

$$R_3(x) = -\sum_{|n|\le x}(\sum_{|m|\le \frac{x}{|n|}} A\mathbf{1}_a(m))\Lambda(n)(\mathbf{1}_a(n) - \tilde{f}(n)).$$

Proof. We have

$$\log xM(x) = \sum_{|n|\le x} L(n)(f - A\mathbf{1}_a)(n) + R_1(x),$$

where

$$R_1(x) = \sum_{|n|\le x} (f - A\mathbf{1}_a)(n) \log \frac{x}{|n|}.$$

Since by (2.1)

$$\begin{aligned} Lf = L(\tilde{f} * g) = L\tilde{f} * g + \tilde{f} * Lg =& \Lambda_{\tilde{f}} * \tilde{f} * g + \tilde{f} * Lg \\ =& \Lambda_{\tilde{f}} * f + \tilde{f} * Lg \end{aligned}$$

we obtain

$$\sum_{|n|\le x} L(f - A\mathbf{1}_a)(n) = \sum_{|n|\le x} \Lambda_{\tilde{f}} * f(n) - A \sum_{|n|\le x} \Lambda_{\mathbf{1}_a} * \mathbf{1}_a(n) + R_2(x),$$

where

$$R_2(x) = \sum_{|n|\le x} \tilde{f} * Lg(n).$$

By rearranging the terms, it follows

$$\log x M(x) = \sum_{|n|\le x} M(\frac{x}{|n|})\Lambda_{\tilde{f}}(n) + (R_1 + R_2 + R_3)(x), \tag{3.1}$$

where

$$R_3(x) = -A \sum_{|n|\le x} \mathbf{1}_a * (\Lambda_{\mathbf{1}_a} - \Lambda_{\tilde{f}})(n).$$

Thus,

$$\begin{aligned} \log^2 x M(x) =& \sum_{|n|\le x} \log \frac{x}{|n|} M(\frac{x}{|n|})\Lambda_{\tilde{f}}(n) \\ &+ \sum_{|n|\le x} M(\frac{x}{|n|})\Lambda_{\tilde{f}} L(n) + (R_1 + R_2 + R_3)(x) \log x. \end{aligned}$$

By (3.1) the right hand side equals

$$\begin{aligned} \sum_{|n|\le x} \sum_{|m|\le \frac{x}{|n|}} M(\frac{x}{|mn|})\Lambda_{\tilde{f}}(m)\Lambda_{\tilde{f}}(n) + \sum_{|n|\le x} (R_1 + R_2 + R_3)(\frac{x}{|n|})\Lambda_{\tilde{f}}(n) \\ + \sum_{|n|\le x} M(\frac{x}{|n|})\Lambda_{\tilde{f}} L(n) + (R_1 + R_2 + R_3)(x) \log x. \qquad \square \end{aligned}$$

By rearranging the summation in the first term, the assertion follows.

The set $|\mathcal{G}| := \{|n| : n \in \mathcal{G}\}$ is discrete. Its elements will be represented by n_ν with $\nu \in \mathbb{N}$ in a non-decreasing way. For an $x \in \mathbb{R}_{\geq 1}$ let $([x]_{\mathcal{G}} =)[x] := \max\{|n| \leq x : n \in \mathcal{G}\}$ and let $\lambda(x)$ be the uniquely defined integer with $n_{\lambda(x)} = [x]$.

Lemma 3 (Partial Summation). *Let $a, b : \mathbb{N} \to \mathbb{C}$, and $y > 1$. Then*

$$\sum_{\nu \leq y} a(\nu)b(\nu) = \sum_{\nu \leq y-1} A(\nu)\{b(\nu) - b(\nu+1)\} + A(y)b([y]_{\mathbb{N}}),$$

where

$$A(u) = \sum_{\mu \leq u} a(\mu).$$

Lemma 4. *Let*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \geq 0$. Then

$$n_\nu = n_{\nu-1}(1 + o(1))$$

as $\nu \to \infty$.

Proof. There exists a sequence $\epsilon(\nu) \to 0$ as $\nu \to \infty$ with

$$\begin{aligned} N(n_{\nu-1}) =& N(n_\nu - \epsilon(\nu)) \\ =& (c + o(1))(n_\nu - \epsilon(\nu))^\delta \log^\beta (n_\nu - \epsilon(\nu)) \quad (\nu \to \infty). \end{aligned}$$

On the other side

$$N(n_{\nu-1}) = (c + o(1))n_{\nu-1}^\delta \log^\beta n_{\nu-1}$$

as $\nu \to \infty$. Therefore,

$$(c + o(1))n_{\nu-1}^\delta \log^\beta n_{\nu-1} = (c + o(1))(n_\nu - \epsilon(\nu))^\delta \log^\beta (n_\nu - \epsilon(\nu)) \quad (\nu \to \infty).$$

Since

$$(n_\nu - \epsilon(\nu))^\delta \log^\beta (n_\nu - \epsilon(\nu)) = n_\nu^\delta \log^\beta n_\nu (1 + o(1))$$

as $\nu \to \infty$, a simple computation shows that

$$n_\nu = (1 + o(1))n_{\nu-1} \quad (\nu \to \infty). \qquad \square$$

Lemma 5. *Let $\eta > 0$. Suppose that*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \geq 0$ and that

$$\sum_{|n|\leq x} h(n) = (t(x) + \delta(x))x^\delta \log x \quad (x \to \infty) \tag{3.2}$$

for some $h \in \mathcal{F}_\mathcal{G}$. Suppose further that $\eta < t(x) \ll 1$ and that the derivative satisfies

$$|t'(x)| \ll x^{-1}$$

for $x > 1$ and that $\delta(x) \ll 1$ holds. Let $f \in \mathcal{F}_\mathcal{G}$. Then for all $x \geq 2$,

$$\begin{aligned} \sum_{|n|\leq x} |M(\frac{x}{|n|})| h(n) \ll \int_1^x |M(\frac{x}{t})| \log t dt^\delta t(t) \\ + (\max_{1<u<x} |\delta(u)| \log u + 1) x^\delta \sum_{|n|\leq x} \frac{|f(n)|}{|n|^\delta} \\ + N(x) \log x. \end{aligned} \tag{3.3}$$

Proof. At first note that under condition (3.2) we have $h(1) = 0$. For $u > 1$, let

$$T(u) = \frac{1}{\log u} \sum_{|n|\leq u} h(n). \tag{3.4}$$

Let

$$\tilde{h}(n_\nu) = \sum_{\substack{|n|=n_\nu \\ n\in\mathcal{G}}} h(n)$$

and let

$$r(n_\nu) := \tilde{h}(n_\nu) - \int_{n_{\nu-1}}^{n_\nu} \log u dT(u) \quad \text{for } n_\nu > 1.$$

By the definition of $\mathcal{G}$ and condition (3.2) there exists a fixed $\epsilon > 0$ such that $\sum_{|n|\leq u} h(n) = 0$ for $1 \leq u < 1 + \epsilon$. Since $\log u$ is of bounded variation in

$[a,b]$ if $1+\epsilon \le a < b < \infty$, it follows that $T(u)$ is of bounded variation here as well (See [39] Kap. VIII.3. Satz 4.). Furthermore, defining $T(u)=0$ for $1 \le u < 1+\epsilon$ we obtain that

$$\int_1^u \log t dT(t), \quad u > 1$$

exists. Therefore using (3.2)

$$\begin{aligned} \sum_{1<n_\nu \le u} r(n_\nu) = \sum_{|n|\le u} h(n) - \int_1^{[u]} \log t dT(t) = \int_1^{[u]} T(t) d\log t \\ \ll \int_1^u \frac{t^\delta}{t} du \\ \ll u^\delta. \end{aligned} \tag{3.5}$$

Setting $a(\nu) = r \circ n_\nu$ and $b(\nu) = |M| \circ \frac{x}{n_\nu}$, $y = \lambda(x)$ then using Lemma 3 and (3.5) we obtain that

$$\begin{aligned} |\sum_{1<n_\nu\le x} |M(\frac{x}{n_\nu})|\tilde{h}(n_\nu) - \sum_{1<n_\nu \le x} |M(\frac{x}{n_\nu})| \int_{n_{\nu-1}}^{n_\nu} \log u dT(u)| \\ =| \sum_{1<n_\nu\le x} |M(\frac{x}{n_\nu})| r(n_\nu)| \end{aligned}$$

does not exceed

$$\sum_{n_\nu \le n_{\lambda(x)-1}} n_\nu^\delta (||M(\frac{x}{n_\nu})| - |M(\frac{x}{n_{\nu+1}})||) + \mathcal{O}(|M(\frac{x}{n_{\lambda(x)}})|x^\delta).$$

Since $||a|-|b|| \le |a-b|$ for all real a and b, an application of Lemma 3 again leads to

$$\begin{aligned} \sum_{n_\nu \le n_{\lambda(x)-1}} n_\nu^\delta (||M(\frac{x}{n_\nu})| - |M(\frac{x}{n_{\nu+1}})||) \ll \sum_{n_\nu\le x} H(\frac{x}{n_\nu})(n_\nu^\delta - n_{\nu-1}^\delta) \\ + H(\frac{x}{n_{\lambda(x)}})x^\delta, \end{aligned} \tag{3.6}$$

where

$$H(x) = \sum_{|n|\le x} |f(n)|.$$

Furthermore, since

$$\begin{aligned}\sum_{n_\nu \le x} H(\frac{x}{n_\nu})(n_\nu^\delta - n_{\nu-1}^\delta) &= \sum_{n_\nu \le x} \sum_{|n| \le \frac{x}{n_\nu}} |f(n)|(n_\nu^\delta - n_{\nu-1}^\delta) \\ &= \sum_{|n|\le x} |f(n)| \sum_{n_\nu \le x/|n|} (n_\nu^\delta - n_{\nu-1}^\delta) \\ &\ll \sum_{|n|\le x} |f(n)|(x/|n|)^\delta \\ &\ll x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta},\end{aligned}$$

we deduce

$$\begin{aligned}\sum_{n_\nu \le x} |M(\frac{x}{n_\nu})|\tilde{h}(n_\nu) = &\sum_{1<n_\nu \le x} |M(\frac{x}{n_\nu})| \int_{n_{\nu-1}}^{n_\nu} \log u dT(u) \\ &+ \mathcal{O}(x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta}).\end{aligned}$$

Since $N(x) = (c + o(1))x^\delta \log^\beta x$, varying x at most by an amount of $\mathcal{O}(\frac{1}{x^\delta \log^{\beta+1} x})$ we can ensure that

$$\int_1^x |M(\frac{x}{t})| \log t dT(t)$$

exists. The contribution of the mistake we make in (3.3) is at most $x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta}$, which is acceptable. Due to the change in x,

$$\int_{n_{\nu-1}}^{n_\nu} ||M(\frac{x}{n_\nu})| - |M(\frac{x}{u})|| \log u dT(u) = \int_{n_{\nu-1}}^{n_\nu -} ||M(\frac{x}{n_\nu})| - |M(\frac{x}{u})|| \log u dT(u). \tag{3.7}$$

The right hand side of (3.7) equals zero if $n_{\nu-1} = 1$ or otherwise it equals the limit of

$$\sum_{k=1}^{m+1} ||M(\frac{x}{n_\nu})| - |M(\frac{x}{\xi_k})|| \log \xi_k (T(x_k) - T(x_{k-1})), \tag{3.8}$$

as we take $\max_{k=0,\ldots,m+1}(x_k - x_{k-1}) \to 0$, where $x_0 = n_{\nu-1} < x_1 < \ldots < x_m < n_\nu = x_{m+1}$ and $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, m+1$. Using (3.4), by the mean value theorem of calculus the above sum equals

$$-\sum_{|n|\leq n_{\nu-1}} h(n) \sum_{k=1}^{m+1} ||M(\frac{x}{n_\nu})| - |M(\frac{x}{\xi_k})|| \log \xi_k (\frac{1}{\varrho_k \log^2 \varrho_k})(x_k - x_{k-1}) \quad (3.9)$$

with appropriate $\varrho_k \in [x_k, x_{k-1}]$. Since $M(x)$ is continuous except of finitely many points, the limit in (3.9) does not depend on the choice of ξ_k. Therefore taking $\xi_k = \varrho_k$ the integral equals

$$-\sum_{|n|\leq n_{\nu-1}} h(n) \int_{n_{\nu-1}}^{n_\nu} ||M(\frac{x}{n_\nu})| - |M(\frac{x}{u})|| \frac{1}{u \log u} du.$$

Furthermore, we obtain

$$\sum_{1<n_\nu\leq x} |M(\frac{x}{n_\nu})| \int_{n_{\nu-1}}^{n_\nu} \log u dT(u) = \sum_{1<n_\nu\leq x} \int_{n_{\nu-1}}^{n_\nu} |M(\frac{x}{u})|) \log u dT(u) + Error,$$

where

$$Error = \mathcal{O}(\sum_{e<n_\nu\leq x} (\sum_{|n|\leq n_{\nu-1}} h(n))\{H(\frac{x}{n_{\nu-1}}) - H(\frac{x}{n_\nu})\}(\log\log n_\nu - \log\log n_{\nu-1}) + x^\delta \sum_{|n|\leq x} \frac{|f(n)|}{|n|^\delta}).$$

Then in view of condition (3.2)

$$Error = \mathcal{O}(\sum_{e<n_\nu\leq x} (H(\frac{x}{n_{\nu-1}}) - H(\frac{x}{n_\nu}))n_{\nu-1}^\delta).$$

Here we used that by Lemma 4

$$\log\log n_\nu - \log\log n_{\nu-1} \ll \frac{1}{\log n_\nu}$$

uniformly for $n_\nu > e$. A similar computation as in (3.6) above shows that *Error* is

$$\mathcal{O}(x^\delta \sum_{|n|\leq x} \frac{|f(n)|}{|n|^\delta}).$$

We arrive at

$$\sum_{|n|\le x} |M(x/|n|)|h(n) - \int_1^x |M(x/u)|\log u dT(u) \ll x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta}.$$

Setting

$$L(u) := T(u) - t(u)u^\delta,$$

we obtain

$$\int_1^x |M(x/u)|\log u dT(u) = \int_1^x |M(x/u)|\log u dt(u)u^\delta$$
$$+ |M(1)|\log x L(x) - \int_1^x L(u)d|M(x/u)|\log u.$$

We compute the second integral on the right hand side. It equals to the limit of

$$\sum_{k=0}^{m-1} L(\xi_k)\{|M(\frac{x}{x_{k+1}})|\log x_{k+1} - |M(\frac{x}{x_k})|\log x_k\}$$

as $\max_{0\le k\le m-1} |x_{k+1} - x_k| \to 0$, where $x_0 = 1 < x_1 < \ldots < x_m = x$ and $\xi_k \in [x_k, x_{k+1}]$. Furthermore, it is

$$\mathcal{O}\Big(\sum_{k=0}^{m-1} |L(\xi_k)|\log x_{k+1}||M(\frac{x}{x_{k+1}})| - |M(\frac{x}{x_k})||$$
$$+\sum_{k=0}^{m-1} |L(\xi_k)||M(\frac{x}{x_k})|(\log x_{k+1} - \log x_k)\Big).$$

We denote the first sum by Σ_1 and the second by Σ_2. Then Σ_1 does not exceed

$$\sum_{k=0}^{m-1} |L(\xi_k)|\log x_{k+1}(H(\frac{x}{x_k}) - H(\frac{x}{x_{k+1}}))$$
$$\ll \max_{1<u<x}|\delta(u)|\log u \sum_{x_k<x} \xi_k^\delta(H(\frac{x}{x_k}) - H(\frac{x}{x_{k+1}})).$$

Since $H(\frac{x}{u})$ is a step function with jumps at n_ν, we obtain by taking $\max_{0\le k\le m-1} |x_{k+1} - x_k| \to 0$ that Σ_1 is at most

$$c \max_{1<u<x} |\delta(u)|\log u \sum_{n_\nu\le x} \frac{x^\delta}{n_\nu^\delta}(H(n_\nu) - H(n_{\nu-1})).$$

Thus,

$$\Sigma_1 = \mathcal{O}(\max_{1<u<x} |\delta(u)| \log u) x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta}.$$

To compute Σ_2 we note that by taking $\max\limits_{0\le k\le m-1} |x_{k+1} - x_k| \to 0$ we have

$$|L(\xi_k)| H(\frac{x}{x_k}) \ll N(x),$$

such that applying the mean value theorem to $\log x_{k+1} - \log x_k$ we obtain

$$\begin{aligned}\Sigma_2 &\ll N(x) \int_1^x \frac{1}{u} du \\ &\ll \mathcal{O}(N(x) \log x).\end{aligned}$$

The proof is finished. □

3.2
An application of Parseval's identity

Analogous to [39] Kap.VIII Satz 2 we have the following

Lemma 6. *Let $f : [a, b] \to \mathbb{R}$ be an almost everywhere continuous function, and let $g : [a, b] \to \mathbb{R}$ be a function with an almost everywhere continuous and bounded derivative on $[a, b]$ such that f and g' does not have a common point of discontinuity. Then*

$$\int_a^b f(x) dg(x) = \int_a^b f(x) g'(x) dx.$$

Proof. Obvious. □

Lemma 7. *Suppose that*

$$N(x) \sim c x^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \ge 0$ and that $t(x)$ has an almost everywhere continuous derivative with $|t'(x)| \ll \frac{1}{x}$ and $t(x) \ll 1$. Let $f \in \mathcal{F}_{\mathcal{G}}$ with $|f| \ll 1$. Then *i*

$$\int_1^x |M(\frac{x}{t})| \log t dt^{\delta} t(t) \ll x^{\delta} \log^{3/2} x H(1/\log x), \tag{3.10}$$

ii

$$\int_1^x |M(\frac{x}{t})| \log t dt^{\delta} t(t) \ll x^{\delta} \log x T(1/\log x), \tag{3.11}$$

iii

$$\int_1^x |M(\frac{x}{t})| \log t dt^{\delta} t(t) \ll x^{\delta} \log x \{ \int_{1/\log x}^{1/2} \frac{T(u)}{u^{1/2}} du + \frac{1}{\log^{1/2} x} T(1/\log x) + 1 \}, \tag{3.12}$$

where

$$H(u) := (\int_{-\infty}^{\infty} |\frac{F(\delta + u + i\tau)}{\delta + u + i\tau}|^2 d\tau)^{1/2}$$

and

$$T(u) := (\int_{-\infty}^{\infty} |\frac{F'(\delta + u + i\tau)}{\delta + u + i\tau}|^2 d\tau)^{1/2}$$

and F is the Dirichlet generating function of f.

Proof. Using Lemma 6 we have

$$\begin{aligned} \int_1^x |M(\frac{x}{t})| \log t dt^{\delta} t(t) \ll & \log x \int_1^x |M(\frac{x}{t})| t^{\delta - 1} dt \\ = & x^{\delta} \log x \int_1^x \frac{|M(u)|}{u^{\delta+1}} du. \end{aligned}$$

To prove (3.10) we note that using the Cauchy-Schwarz inequality

$$\begin{aligned} (\int_1^x \frac{|M(u)|}{u^{\delta+1}} du)^2 \le & \int_1^x \frac{|M(u)|^2}{u^{2\delta+1}} du \int_1^x \frac{1}{u} du \\ = & \log x \int_0^{\log x} \frac{|M(e^{\omega})|^2}{e^{2\delta\omega}} d\omega. \end{aligned}$$

Since

$$1 \le \exp(2\omega/\log x) \le e^2 \quad \text{for} \quad 0 \le \omega \le \log x,$$

we deduce

$$\int_0^{\log x} \frac{|M(e^\omega)|^2}{e^{2\delta\omega}} d\omega \le e^2 \int_0^\infty |\frac{M(e^\omega)}{e^{(\delta+1/\log x)\omega}}|^2 d\omega. \tag{3.13}$$

Note that for $\sigma > \delta$

$$\begin{aligned}\sum_n \frac{f(n)}{|n|^s} =& \int_{1-}^\infty \frac{1}{u^s} dM(u) \\ =& s\int_1^\infty \frac{M(u)}{u^{s+1}} du \\ =& s\int_0^\infty \frac{M(e^\omega)}{e^{s\omega}} d\omega \\ =& s\int_0^\infty \frac{M(e^\omega)}{e^{\sigma\omega}} e^{-i\tau\omega} d\omega.\end{aligned}$$

Hence, the Fourier transform of

$$H(\omega) := M(e^\omega)e^{-\sigma\omega}$$

equals

$$\hat{H}(\tau) = (\sigma + i\tau)^{-1} F(\sigma + i\tau).$$

Since

$$M(e^\omega) \ll e^\omega \omega^\beta$$

and since by Lemma 1

$$F(\sigma + i\tau) \ll \frac{1}{(\sigma - \delta)^{\beta+1}},$$

both functions $H, \hat{H}$ belong to $\mathbb{L}^2(-\infty, \infty)$. Thus, by Parseval's identity $\|H\|_2 = \|\hat{H}\|_2$ such that with $\sigma = \delta + 1/\log x$ we obtain

$$\int_0^\infty |\frac{M(e^\omega)}{e^{\sigma\omega}}|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty |\frac{F(\sigma + i\tau)}{\sigma + i\tau}|^2 d\tau. \tag{3.14}$$

To prove (3.11) let

$$K(u) = \sum_{|n| \le u} f(n) \log |n|.$$

Integration by parts shows that for $u > 2$,

$$\begin{aligned} M(u) - 1 = & \int_{1+}^{u} \frac{1}{\log t} dK(t) \\ = & \frac{K(u)}{\log u} + \int_{1+}^{u} \frac{K(t)}{t \log^2 t} dt. \end{aligned}$$

We deduce

$$\int_2^x \frac{|M(u)|}{u^{\delta+1}} du \ll \int_2^x \frac{|K(u)|}{u^{\delta+1} \log u} du + \int_2^x \frac{1}{u^{\delta+1}} \int_{1+}^u \frac{|K(t)|}{t \log^2 t} dt du + 1.$$

Further,

$$\begin{aligned} \int_2^x \frac{1}{u^{\delta+1}} \int_{1+}^u \frac{|K(t)|}{t \log^2 t} dt du = & \int_{1+}^x \frac{|K(t)|}{t \log^2 t} \int_t^x \frac{1}{u^{\delta+1}} du dt \\ & - \int_{1+}^2 \frac{K(t)}{t \log^2 t} \int_t^2 \frac{1}{u^{\delta+1}} du dt \\ \ll & \int_2^x \frac{|K(t)|}{t^{\delta+1} \log^2 t} dt + 1. \end{aligned}$$

It follows

$$\int_2^x \frac{|M(u)|}{u^{\delta+1}} du \ll \int_2^x \frac{|K(t)|}{t^{\delta+1} \log t} dt + 1.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (\int_2^x \frac{|K(t)|}{t^{\delta+1} \log t} dt)^2 \ll & \int_1^x \frac{|K(t)|^2}{t^{2\delta+1}} dt \int_2^x \frac{1}{t \log^2 t} dt \\ \ll & \int_0^{\log x} \frac{|K(e^\omega)|^2}{e^{2\delta\omega}} d\omega. \end{aligned}$$

Since if $u > 1$, then

$$1 \leq \exp(2\omega / \log u) \leq e^2 \quad \text{for} \quad 0 \leq \omega \leq \log u$$

such that,

$$\int_0^{\log u} \frac{|K(e^\omega)|^2}{e^{2\delta\omega}} d\omega \leq e^2 \int_0^\infty |\frac{K(e^\omega)}{e^{(\delta + 1/\log u)\omega}}|^2 d\omega. \tag{3.15}$$

Note that for $\sigma > \delta$ we have

$$\begin{aligned}\sum_n \frac{f(n)\log n}{|n|^s} &= \int_{1-}^{\infty} \frac{1}{u^s} dK(u)\\ &= s\int_1^{\infty} \frac{K(u)}{u^{s+1}} du\\ &= s\int_0^{\infty} \frac{K(e^{\omega})}{e^{s\omega}} d\omega\\ &= s\int_0^{\infty} \frac{K(e^{\omega})}{e^{\sigma\omega}} e^{-i\tau\omega} d\omega.\end{aligned}$$

Hence, the Fourier transform of

$$L(\omega) := K(e^{\omega})e^{-\sigma\omega}$$

equals

$$\hat{L}(\tau) = -(\sigma + i\tau)^{-1}F'(\sigma + i\tau).$$

Since

$$K(e^{\omega}) \ll e^{\omega}\omega^{\beta+1}$$

and since by Lemma 1 using Cauchy's theorem

$$F'(\sigma + i\tau) \ll \frac{1}{(\sigma - \delta)^{\beta+2}},$$

both functions $L, \hat{L}$ belong to $\mathbb{L}^2(-\infty, \infty)$. Thus, by Parseval's identity $\|L\|_2 = \|\hat{L}\|_2$ such that with $\sigma > \delta$ we obtain

$$\int_0^{\infty} |\frac{K(e^{\omega})}{e^{\sigma\omega}}|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\frac{F'(\sigma + i\tau)}{\sigma + i\tau}|^2 d\tau. \tag{3.16}$$

To prove (3.12) we note that

$$\begin{aligned}\int_{e^2}^{x} \frac{|K(u)|}{u^{\delta+1}\log u} du \ll& \int_{e^2}^{x} \frac{|K(u)|}{u^{\delta+1}\log u} \int_{u^{1/2}}^{u} \frac{1}{t\log t} dt du\\ =& \int_{e}^{e^2} \frac{1}{t\log t} \int_{e^2}^{t^2} \frac{|K(u)|}{u^2 \log u} du dt\\ &+ \int_{e^2}^{x^{1/2}} \frac{1}{t\log t} \int_{t}^{t^2} \frac{|K(u)|}{u^{\delta+1}\log u} du dt\\ &+ \int_{x^{1/2}}^{x} \frac{1}{t\log t} \int_{t}^{x} \frac{|K(u)|}{u^{\delta+1}\log u} du dt.\end{aligned}$$

Furthermore, using (3.15) and (3.16) we obtain

$$\int_1^x \frac{|K(u)|}{u^{\delta+1}} du \ll \log^{1/2} x T(1/\log x).$$

Hence,

$$\int_{x^{1/2}}^x \frac{1}{t\log t} \int_t^x \frac{|K(u)|}{u^{\delta+1}\log u} dudt \ll \frac{1}{\log x} \int_1^x \frac{|K(u)|}{u^{\delta+1}} du$$
$$\ll \frac{1}{\log^{1/2} x} T(1/\log x).$$

Thus,

$$\begin{aligned}\int_2^x \frac{|K(u)|}{u^{\delta+1}\log u} du &\ll \int_e^{x^{1/2}} \frac{1}{t\log t} \int_t^{t^2} \frac{|K(u)|}{u^{\delta+1}\log u} dudt + \frac{1}{\log^{1/2} x} T(1/\log x) + 1 \\ &\ll \int_e^{x^{1/2}} \frac{1}{t\log^2 t} \int_1^{t^2} \frac{|K(u)|}{u^{\delta+1}} dudt + \frac{1}{\log^{1/2} x} T(1/\log x) + 1 \\ &\ll \int_e^{x^{1/2}} \frac{T(1/(2\log t))}{t\log^{3/2} t} dt + \frac{1}{\log^{1/2} x} T(1/\log x) + 1 \\ &\ll \int_{1/\log x}^2 \frac{T(u)}{u^{1/2}} du + \frac{1}{\log^{1/2} x} T(1/\log x) + 1,\end{aligned}$$

which proves (3.12). □

3.3 Estimation of the Fourier transform

Lemma 8. *Suppose that*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \geq 0$. *Let* $f : \mathcal{G} \to \mathbb{C}$ *be a multiplicative function such that* $|f(n)| \leq 1$. *Then*

$$F(s) = \prod_{|p|^\delta \leq 2} (1 + h(p,s)) \exp(\sum_{|p|^\delta > 2} \frac{f(p)}{|p|^s}) F_1(s)$$

holds for $\sigma > \delta$, *where*

$$h(p,s) = \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{|p|^{\alpha s}}$$

and $F_1(s)$ *is analytic for* $\sigma > \delta/2$, *and* $|F_1(s)| \asymp 1$ *for* $\sigma \geq \delta$.

Proof. Since

$$\sum_p \sum_\alpha \frac{f(p^\alpha)}{|p|^{\alpha s}}$$

converges absolutely for $\sigma > \delta$, the Euler product representation

$$F(s) = \prod_p (1 + h(p,s))$$

holds for $\sigma > \delta$. Defining

$$F_1(s) := \prod_{\substack{p \\ |p|^\delta > 2}} (1 + h(p,s)) \exp(-\frac{f(p)}{p^s}),$$

we have to show that $F_1(s)$ has the asserted properties. We have

$$|h(p,s)| \leq |p|^{-\sigma} \frac{1}{1 - \frac{1}{|p|^\sigma}} \tag{3.17}$$

for all $\sigma > 0$. Since for $0 < \epsilon < 1$

$$|\log(1+z) - z| \leq c(\epsilon)|z|^2$$

holds for all $|z| \leq 1 - \epsilon$, where $\log z$ is the principal value of the complex logarithm function,

$$|\log(1 + h(p,s)) - \frac{f(p)}{|p|^s}| \ll |p|^{-2\sigma} \tag{3.18}$$

holds uniformly for all large enough primes. Since

$$\sum_p \frac{1}{|p|^{2\sigma}} \ll \sum_n \frac{1}{|n|^{2\sigma}}$$

converges absolutely and uniformly for all $\sigma \geq \sigma_0 > \delta/2$, $F_1(s)$ is defined by a product, which converges uniformly and absolutely for such σ. Hence, $F_1(s)$ is analytic for $\sigma > \delta/2$. If $\sigma \geq \delta$ then by (3.17) with some $\epsilon' > 0$

$$|h(p,s)| \leq 1 - \epsilon'$$

holds uniformly for all primes p with $|p|^\delta > 2$. Therefore, (3.18) holds for all such primes. Consequently,

$$\log |F_1(s)| \ll 1,$$

which proves the assertion. Here we note that by the definition of $\mathcal{G}$, if $|p|^\delta > 2$, then there exists an $\epsilon > 0$ such that $|p|^\delta > 2 + \epsilon$. □

Remark. The lemma gives for the Zeta function

$$\zeta(s) = \prod_{|p|^\delta \le 2} (1 + h(p,s)) \exp(\sum_{|p|^\delta > 2} \frac{1}{|p|^s})\zeta_1(s), \quad \sigma > \delta,$$

where for $|p|^\delta \le 2$

$$\begin{aligned} |1 + h(p,s)| =& |(1 - \frac{1}{|p|^s})^{-1}| \\ \le& 1/\epsilon' \quad \sigma > \sigma_0 > 0 \end{aligned}$$

is valid for some $\epsilon' > 0$. This is the result of the fact that $1 + \epsilon < p$ for all primes p holds for some $\epsilon > 0$ by the definition of $\mathcal{G}$. Furthermore,

$$|1 + h(p,s)| \ge 1/2, \quad \sigma > 0$$

for all p. Thus,

$$\zeta(s) \asymp \exp(\sum_p \frac{1}{|p|^s})$$

uniformly for $\sigma > \delta$.

The following lemma appeared for Dirichlet series in [38] (Lemma II.3.) but we need it for Laplace-Stieltjes integrals.

Lemma 9 (Montgomery). *Let $A(\omega)$ be a function of bounded variation, and let $B(\omega)$ be a non-decreasing function such that $A(0) = B(0) = 0$ and*

$$|A(\omega_1) - A(\omega_2)| \le B(\omega_1) - B(\omega_2) \tag{3.19}$$

for all $\omega_1 > \omega_2$. Suppose that the Laplace-Stieltjes transform

$$\int_0^\infty e^{-\sigma t} dB(t)$$

converges for some $\sigma > 0$. Let

$$F_1(s) = \int_0^\infty e^{-st} dA(t), \quad \text{and} \quad F_2(s) = \int_0^\infty e^{-st} dB(t).$$

Then

$$\int_{-T}^{T} |F_1(s)|^2 d\tau \le 2 \int_{-2T}^{2T} |F_2(s)|^2 d\tau$$

holds for all $T > 0$.

Proof. First we note that the Fejér kernel is non-negative, that is

$$\int_{-1}^{1}(1-|t|)e^{ity}dt = (\frac{\sin(y/2)}{y/2})^2 \geq 0 \tag{3.20}$$

for all $y > 0$. Therefore,

$$\begin{aligned}\int_{-T}^{T}|F_1(s)|^2 d\tau \leq & 2\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})|F_1(s)|^2 d\tau \\ =&2\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})\int_0^\infty\int_0^\infty e^{-s\omega_1}e^{-\bar{s}\omega_2}dA(\omega_1)d\overline{A(\omega_2)}d\tau \\ =&2\int_0^\infty\int_0^\infty e^{-\sigma\omega_1}e^{-\sigma\omega_2}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})e^{i(-\omega_1+\omega_2)\tau}d\tau dA(\omega_1)d\overline{A(\omega_2)}.\end{aligned}$$

By (3.19) and (3.20), this last integral is at most

$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-\sigma\omega_1}e^{-\sigma\omega_2}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})e^{i(-\omega_1+\omega_2)\tau}d\tau dB(\omega_1)dB(\omega_2) \\ = \int_{-2T}^{2T}(1-\frac{|\tau|}{2T})\int_0^\infty\int_0^\infty e^{-s\omega_1}e^{-\bar{s}\omega_2}dB(\omega_1)dB(\omega_2)d\tau \\ \leq \int_{-2T}^{2T}|F_2(s)|^2 d\tau.\end{aligned}$$

□

The following well known lemma (see for example [25]) is useful in many cases and remains true for arithmetical semigroups.

Lemma 10. *Suppose that*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \geq 0$ *and that*

$$\psi(x) \asymp x^\delta.$$

Then if $f \in \mathcal{F}_\mathcal{G}$ *is a multiplicative function with* $|f| \ll 1$, *then*

$$\sum_{|n|\leq x} f(n) \ll \psi(x)\log^{-1} x \sum_{|n|\leq x}\frac{|f(n)|}{|n|^\delta}.$$

Proof. At first note that

$$\sum_{|n|\leq x}|f(n)|\log\frac{x}{|n|} \ll x^\delta \sum_{|n|\leq x}\frac{|f(n)|}{n^\delta}. \tag{3.21}$$

Since

$$\begin{aligned}
\log x \sum_{|n|\le x} f(n) &= \sum_{|n|\le x} f(n)\log|n| + \mathcal{O}(\sum_{|n|\le x} |f(n)|\log\frac{x}{|n|}) \\
&= \sum_{|n|\le x} f(n) \sum_{p^\alpha || n} \log|p|^\alpha + \mathcal{O}(x^\delta \sum_{|n|\le x} \frac{|f(n)|}{n^\delta}) \\
&= \sum_{|p^\alpha|\le x} f(p^\alpha)\log|p|^\alpha \sum_{\substack{|n|\le \frac{x}{|p|^\alpha} \\ p\nmid n}} f(n) + \mathcal{O}(x^\delta \sum_{|n|\le x} \frac{|f(n)|}{n^\delta}) \\
&\ll \sum_{|n|\le x} |f(n)| \sum_{|p|^\alpha \le \frac{x}{|n|}} |f(p^\alpha)|\log|p|^\alpha + x^\delta \sum_{|n|\le x}\frac{|f(n)|}{n^\delta} \\
&\ll \psi(x) \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta},
\end{aligned}$$

the assertion follows. □

Remark. 1. Since

$$\begin{aligned}
\sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta} &\ll \prod_{|p|\le x} (1 + \sum_{\alpha=1}^{\infty} \frac{|f(p^\alpha)|}{|p|^{\delta\alpha}}) \\
&\ll \exp(\sum_{|p|\le x} \frac{|f(p)|}{|p|^\delta}),
\end{aligned}$$

it follows, that

$$\sum_{|n|\le x} |f(n)| \ll \frac{\psi(x)}{\log x} \exp(\sum_{|p|\le x} \frac{|f(p)|}{|p|^\delta})$$

holds as well.

2. Since multiplicative functions are determined by their values on prime powers, it is reasonable to find an estimate for the partial sums of multiplicative functions by partial sums over their values on prime powers. Contrary to its simplicity, this lemma allows us to estimate partial sums quiet effectively even in non-trivial cases. For example, let $\mathcal{G} = \mathbb{N}$, furthermore, let $0 < z < 1$ and let $f(p^\alpha) = z$. The Sathe-Selberg method (see for example [44] Theorem II.6.1) gives us

$$\sum_{n\le x} f(n) = x\log^{z-1}(c_z + \mathcal{O}(\frac{1}{\log x}))$$

while using the above lemma we obtain

$$\sum_{n\le x} f(n) \ll x \log^{z-1}.$$

3.3.1
The "divergent" case

We will use (3.10) to obtain the following

Lemma 11. *Let $\eta > 0$. Let*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \ge 0$ and let $\eta < t(x) \ll 1$ which has an almost everywhere continuous derivative with $t'(x) \ll \frac{1}{x}$. Suppose further that for $x \ge 2$

$$\sum_{|p|\le x} \frac{1}{|p|^\delta} = (\beta+1)\log\log x + \mathcal{O}(1). \tag{3.22}$$

If $f \in \mathcal{F}_\mathcal{G}$ is multiplicative with $|f| \le 1$ and

$$\sum_p \frac{1 - \Re f(p)|p|^{i\tau}}{|p|^\delta}$$

diverges for all real τ, then

$$\int_1^x |M(\frac{x}{t})| \log t dt^\delta t(t) = o(x^\delta \log^{\beta+2} x) \quad (x \to \infty).$$

Proof. In the case of rational integers it is possible to obtain the analogous assertion with one of the inequalities (3.10), (3.11), (3.12). Using inequalities (3.11) and (3.12) the problem reduces to the investigation of the logarithmic derivative of the generating function F. Lemma 8 usually allows us to extract some knowledge about the logarithmic derivative, which is closely connected to prime number sums. Although we have some knowledge about prime number sums in our case, the existence of the logarithmic derivative is not always fulfilled. Therefore we proceed as in [25]. We use inequality (3.10) with $A = 0$ and $a = 0$, i.e.

$$\int_1^x |M(\frac{x}{t})| \log t dt^\delta t(t) \ll x^\delta \log^{3/2} x H(1/\log x), \tag{3.23}$$

where

$$H(u) := (\int_{-\infty}^{\infty} |\frac{F(\delta+u+i\tau)}{\delta+u+i\tau}|^2 d\tau)^{1/2}.$$

Let $M > 0$ be a large fixed number. Suppose that

$$\sum_p \frac{1-\Re f(p)|p|^{i\tau}}{|p|^\delta}$$

diverges for all real τ. Then by Lemma 8

$$\frac{F(s)}{\zeta(\sigma)} \ll \exp(-\sum_p \frac{1-\Re f(p)|p|^{i\tau}}{|p|^\sigma})$$

holds uniformly for all real τ. Therefore by a theorem of Dini

$$F(s) = o((\sigma-\delta)^{-\beta-1}) \quad \sigma \to \delta+ \tag{3.24}$$

uniformly for all $|\tau| < M$. Further

$$\int_{|\tau|\le M} |\frac{F(s)}{s}|^2 d\tau \ll \max_{|\tau|\le M} |F(s)|^{1/2} \int_{|\tau|\le M} \frac{|F(s)|^{3/2}}{|s|^2} d\tau.$$

Let $h(n)$ be a multiplicative function defined on prime powers by

$$h(p^\alpha) = \begin{cases} \frac{3f(p)}{4} & \text{if } \alpha = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the definition of h,

$$\prod_{|p|^\delta \le 2} (1 + \sum_\alpha \frac{h(p^\alpha)}{|p|^{s\alpha}}) \asymp 1$$

uniformly for $\sigma > \delta$. Thus using Lemma 8 and the remark concerning the Zeta function after that lemma we obtain

$$\begin{aligned} |F(s)|^{3/4} &\ll |\exp(\sum_p \frac{3f(p)}{4|p|^s})| \\ &\ll |\sum_n \frac{h(n)}{|n|^s}| \end{aligned}$$

for $\sigma > \delta$. Thus

$$\int_{|\tau|\leq M} \frac{|F(s)|^{3/2}}{|s|^2} d\tau \ll \int_{-\infty}^{\infty} \frac{|\exp(\sum_p \frac{h(p)}{|p|^s})|^2}{|s|^2} d\tau.$$

Then applying Lemma 10 and Parseval's identity to h we deduce

$$\int_{-\infty}^{\infty} \frac{|\exp(\sum_p \frac{h(p)}{|p|^s})|^2}{|s|^2} d\tau = \int_0^{\infty} |\sum_{|n|\leq e^\omega} h(n) e^{(\delta-\sigma)\omega}|^2 d\omega$$
$$\ll \int_e^{\infty} |\exp(\sum_{|p|\leq e^\omega} \frac{|h(p)|}{|p|^\delta})\omega^{-1} e^{(\delta-\sigma)\omega}|^2 d\omega + 1.$$

Using condition (3.22) it follows that the last integral is at most

$$\mathcal{O}((\sigma-\delta)^{-\frac{3}{2}\beta-1/2}),$$

and using (3.24) we arrive at

$$\int_{|\tau|\leq M} |\frac{F(s)}{s}|^2 d = o((\sigma-\delta)^{-2\beta-1}).$$

For $|\tau| > M$ we have

$$\int_{|\tau|>M} |\frac{F(s)}{s}|^2 d\tau \ll \sum_{|m|>M} \frac{1}{m^2} \int_{|\tau-m|\leq 1} |F(s)|^2 d\tau$$
$$\ll M^{-1}(\sigma-\delta)^{-2\beta-1}.$$

Here we used that since

$$\tilde{K}(e^\omega) := \sum_{|n|\leq e^\omega} f(n)|n|^{-im}$$
$$\ll N(e^\omega),$$

by (3.14)

$$\int_{-1}^{1} |F(\sigma+i(\tau+m))|^2 d\tau \ll \int_{-\infty}^{\infty} |\frac{F(\sigma+i(\tau+m))}{\sigma+i\tau}|^2 d\tau$$
$$=2\pi \int_0^{\infty} |\tilde{K}(e^\omega)e^{-\sigma\omega}|^2 d\tau$$
$$\ll (\sigma-\delta)^{-2\beta-1}.$$

Putting everything together,

$$\int_{-\infty}^{\infty} |\frac{F(\delta+u+i\tau)}{\delta+u+i\tau}|^2 d\tau = o(u^{-2\beta-1})$$

as $u \to 0$. Substituting into (3.23) the assertion follows. $\square$

Theorem 2. *Let*

$$N(x) \sim cx^{\delta} \log^{\beta} x \quad (x \to \infty)$$

where $c, \delta > 0$*,* $\beta \geq 0$ *and*

$$\psi(x) \asymp x^{\delta}.$$

Suppose further that for $x \geq 2$

$$\sum_{|p| \leq x} \frac{1}{|p|^{\delta}} = (\beta+1) \log\log x + \mathcal{O}(1)$$

and that the Selberg Symmetry Formula holds, i.e. that

$$\sum_{|n| \leq x} \Lambda * \Lambda(n) + \Lambda(n) \log |n| = t(x)x^{\delta} \log x + o(\psi(x) \log x) \quad (x \to \infty)$$

where $t(x)$ *is as in Lemma 5. Assume that* $f \in \mathcal{F}_{\mathcal{G}}$ *is multiplicative with* $|f| \leq 1$ *and*

$$\sum_{p} \frac{1 - \Re f(p)|p|^{i\tau}}{|p|^{\delta}}$$

diverges for all real τ*. Then*

$$\sum_{|n| \leq x} f(n) = o(N(x)) \quad (x \to \infty).$$

Proof. Choosing $A = 0$, $a = 0$ in Theorem 1 we obtain

$$\log^2 x \sum_{|n| \leq x} f(n) \ll \sum_{|n| \leq x} |M(\frac{x}{n})| \{\Lambda * \Lambda(n) + \Lambda L(n)\}$$
$$+ (R_1 + R_2)(x) \log x + \sum_{|n| \leq x} (R_1 + R_2)(\frac{x}{|n|}) \Lambda(n),$$

where

$$R_1(x) \ll \sum_{|n|\le x} \log \frac{x}{|n|}$$

and

$$R_2(x) \ll N(x) \sum_{|n|\le x} \frac{|g(n)| \log |n|}{|n|^\delta}.$$

Here g is defined by $f = \tilde{f} * g$, where $\tilde{f}$ is completely multiplicative with $\tilde{f}(p) = f$. By (3.21) we have

$$\begin{aligned} R_1(x) \ll& N(x) \log x - \int_1^x \log u dN(u) \\ =& \int_1^x \frac{N(u)}{u} \\ \ll& N(u). \end{aligned}$$

Further since

$$g = f * \tilde{f}^{-1} = f * \mu f,$$

we have

$$g(p^\alpha) = f(p^\alpha) - f(p) f(p^{\alpha-1}).$$

Therefore g is zero on primes and $|g(p^\alpha)| \le 2$. It follows that

$$\begin{aligned} R_2(x) \ll& N(x) \sum_{|n|\le x} \frac{|g(n)| \log |n|}{|n|^\delta} \\ \ll& N(x). \end{aligned}$$

Here we used that $|g(n^2)| \le 2^{\omega(n)}$, such that

$$\begin{aligned} \sum_{|n|\le x} \frac{|g(n)|}{|n|^{\frac{3\delta}{4}}} \ll& \prod_{|p|\le x} (1 + 2 \sum_{\alpha\ge 2} \frac{1}{|p|^{\alpha \frac{3\delta}{4}}}) \\ \ll& 1, \end{aligned}$$

where $\omega(n)$ counts the distinct prime divisors of n. Further since

$$\begin{aligned} \sum_{|n|\le x} \frac{\Lambda(n)}{|n|^\delta} =& \int_1^x \frac{1}{u^\delta} d\psi(u) \\ =& \frac{\psi(x)}{x^\delta} + \delta \int_1^x \frac{\psi(u)}{u^{\delta+1}} du \\ \ll& \log x, \end{aligned}$$

we obtain

$$\sum_{|n|\le x}(R_1+R_2)(\frac{x}{|n|})\Lambda(n) \ll N(x)\sum_{|n|\le x}\frac{\Lambda(n)}{|n|^\delta}$$
$$\ll N(x)\log x.$$

By Lemma 5 we have

$$\log^2 xM(x) \ll \int_1^x |M(\frac{x}{t})|\log t dt^\delta t(t) + o(N(x)\log^2 x) \quad (x\to\infty)$$

and by Lemma 11

$$\int_1^x |M(\frac{x}{t})|\log t dt^\delta t(t) = o(N(x)\log^2 x) \quad (x\to\infty),$$

so that the proof is finished. □

Corollary 1. *Let*

$$N(x)\sim cx^\delta \log^\beta x \quad (x\to\infty)$$

where $c,\delta>0$, $\beta\ge 0$ *and let*

$$\psi(x) = (\frac{\beta+1}{\delta} - 2\sum_{r=1}^m \frac{\alpha_r}{\sqrt{\delta^2+t_r^2}}\cos(t_r\log x-\theta_r))x^\delta + o(x^\delta) \quad (x\to\infty)$$

where $\alpha_r\in\mathbb{N}_0$, $t_r>0$, $r=1,\ldots,m$ *such that*

$$\sum_{r=1}^m \alpha_r \le \frac{\beta+1}{2},$$

and θ_r, $r=1,\ldots,m$ *are the angles which satisfy*

$$\sin\theta_r = \frac{t_r}{\sqrt{\delta^2+t_r^2}} \quad \text{and} \quad \cos\theta_r = \frac{\delta}{\sqrt{\delta^2+t_r^2}}.$$

Assume further that $f\in\mathcal{F}_\mathcal{G}$ *is multiplicative with* $|f|\le 1$ *and*

$$\sum_p \frac{1-\Re f(p)|p|^{i\tau}}{|p|^\delta}$$

diverges for all real τ. *Then*

$$\sum_{|n|\le x} f(n) = o(N(x)) \quad (x\to\infty).$$

Proof. By Theorem 2 it is enough to prove that

$$\sum_{|p|\le x} \frac{1}{|p|^\delta} = (\beta+1)\log\log x + \mathcal{O}(1) \tag{3.25}$$

and that the Selberg Symmetry Formula holds. First note that by the conditions using partial summation

$$\sum_{|p|\le x} \frac{\log|p|}{|p|^\delta} \ll \log x.$$

Then by Lemma 1 and by the remark concerning the zeta function after Lemma 8

$$(\beta+1)\log(\sigma-\delta) = \sum_p \frac{1}{|p|^\sigma} + \mathcal{O}(1)$$

for all $\delta+1>\sigma>\delta$. Choosing $\sigma=\delta+\frac{1}{\log x}$ we obtain

$$\begin{aligned}\sum_{|p|\le x} \frac{1}{|p|^\sigma} - \frac{1}{|p|^\delta} &\ll \sum_{|p|\le x} \frac{1}{|p|^\delta}(\exp((\sigma-\delta)\log|p|)-1)\\ &\ll \frac{1}{\log x}\sum_{|p|\le x}\frac{\log|p|}{|p|^\delta}\\ &\ll 1.\end{aligned}$$

Therefore

$$\sum_{|p|\le x}\frac{1}{|p|^\sigma} = \sum_{|p|\le x}\frac{1}{|p|^\delta} + \mathcal{O}(1),$$

for $\sigma=\delta+\frac{1}{\log x}$. Further, since

$$\begin{aligned}\sum_{|p|>x}\frac{1}{|p|^\sigma} &= \int_x^\infty \frac{1}{u^\sigma \log u}d\psi(u)\\ &\ll \frac{x^{\delta-\sigma}}{\log x} + \int_x^\infty \frac{\psi(u)}{u^{\sigma+1}\log u}du\\ &\ll 1 + \frac{1}{\log x}\int_x^\infty \frac{1}{u^{\sigma-\delta+1}}du\\ &\ll 1,\end{aligned}$$

(3.25) follows.
Concerning the Selberg Symmetry Formula we have to compute

$$\Sigma := \sum_{|n|\leq x} \Lambda * \Lambda(n) + \sum_{|n|\leq x} \Lambda(n) \log |n|. \tag{3.26}$$

We have

$$\begin{aligned}\sum_{|n|\leq x} \Lambda(n) \log |n| = & \int_1^x \log u d\psi(u) \\ =& \psi(x) \log x - \int_1^x \frac{\psi(u)}{u} du \\ =& \psi(x) \log x + \mathcal{O}(\psi(x)),\end{aligned}$$

and

$$\begin{aligned}\sum_{|n|\leq x} \Lambda * \Lambda(n) &= \sum_{|n|\leq x} \sum_{d|n} \Lambda(d)\Lambda(\frac{n}{d}) \\ &= \sum_{|d|\leq x} \Lambda(d) \sum_{|n|\leq \frac{x}{|d|}} \Lambda(d) \\ &= \sum_{|d|\leq x} \Lambda(d)\psi(\frac{x}{|d|}).\end{aligned}$$

We obtain that (3.26) equals

$$\begin{aligned}&[\sum_{|d|\leq x} \frac{\Lambda(d)}{|d|^{\delta}}(\frac{\beta+1}{\delta}) - 2\sum_{r=1}^{m} \frac{\alpha_r}{\sqrt{\delta^2+t_r^2}} \sum_{|d|\leq x} \frac{\Lambda(d)}{|d|^{\delta}} \cos(t_r \log \frac{x}{|d|} - \theta_r) \\ &\quad + \sum_{|d|\leq x} \frac{\Lambda(d)}{|d|^{\delta}} o_{\frac{x}{|d|}}(1)]x^{\delta} + \psi(x) \log x + \mathcal{O}(\psi(x)) \quad (x \to \infty).\end{aligned} \tag{3.27}$$

We have to compute

$$\Sigma_1 := \sum_{|d|\leq x} \frac{\Lambda(d)}{|d|^{\delta}}$$

and

$$\Sigma_2 := \sum_{|d|\leq x} \frac{\Lambda(d)}{|d|^{\delta}} \cos(t_r \log \frac{x}{|d|} - \theta_r)$$

and

$$\Sigma_3 := \sum_{|d|\le x} \frac{\Lambda(d)}{|d|^\delta} o_{\frac{x}{|d|}}(1) \quad (x \to \infty).$$

Concerning Σ_1 we have

$$\begin{aligned}\sum_{|d|\le x} \frac{\Lambda(d)}{|d|^\delta} &= \int_1^x \frac{1}{u^\delta} d\psi(u) \\ &= \frac{\psi(x)}{x^\delta} + \delta \int_1^x \frac{\psi(u)}{u^{\delta+1}} du.\end{aligned}$$

The integral on the right hand side equals

$$\begin{aligned}\frac{\beta+1}{\delta}\int_1^x \frac{1}{u}du - 2\sum_{r=1}^m \frac{\alpha_r}{\sqrt{\delta^2+t_r^2}}\int_1^x \frac{\cos(t_r\log u - \theta_r)}{u}du + \int_1^x \frac{o_u(1)}{u}du \\ = \frac{\beta+1}{\delta}\log x + o(\log x) \quad (x\to\infty).\end{aligned}$$

Further using the estimation above we have

$$\begin{aligned}\Sigma_3 =& o_x(1) \sum_{|d|\le \frac{x}{\log^{1/(2\delta)} x}} \frac{\Lambda(d)}{|d|^\delta} + \sum_{|d|> \frac{x}{\log^{1/(2\delta)} x}} \frac{\Lambda(d)}{|d|^\delta} \\ =& o(\log x) + \mathcal{O}(\frac{\sqrt{\log x}}{x^\delta}\psi(x)) \\ =& o(\log x) \quad (x\to\infty).\end{aligned}$$

It remains to estimate Σ_2. We have

$$\begin{aligned}\sum_{|d|\le x} \frac{\Lambda(d)}{|d|^\delta}\cos(t_r\log\frac{x}{|d|} - \theta_r) = \int_1^x \frac{\cos(t_r\log\frac{x}{u} - \theta_r)}{u^\delta} d\psi(u) \\ = \int_1^x \frac{\psi(u)(\delta\cos(t_r\log\frac{x}{u}-\theta_r) - t_r\sin(t_r\log\frac{x}{u}-\theta_r))}{u^{\delta+1}}du + \cos(\theta_r)\frac{\psi(x)}{x^\delta} \\ = \sqrt{\delta^2+t_r^2}\int_1^x \frac{\psi(u)\cos(t_r\log\frac{x}{u})}{u^{\delta+1}}du + \mathcal{O}(1).\end{aligned}$$

In the view of the above computations it is enough to estimate

$$\int_1^x \frac{\cos(t_j\log u)\cos(t_r\log\frac{x}{u})}{u}du. \tag{3.28}$$

Since

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2},$$

the integral in (3.28) above is $\mathcal{O}(1)$ except for those t_r, t_j, for which $t_r = t_j$, in which case it equals

$$\frac{\cos(t_r \log x)}{2}\log x + \mathcal{O}(1).$$

Putting it all together we deduce that Σ equals

$$[\frac{\beta+1}{\delta}(\beta+2) - 2\sum_{r=1}^{m}\frac{\alpha_r^2}{\sqrt{\delta^2+t_r^2}}\cos(t_r\log x) - 2\sum_{r=1}^{m}\frac{\alpha_r}{\sqrt{\delta^2+t_r^2}}\cos(t_r\log x - \theta_r)$$
$$+o(1)]x^\delta \log x \quad (x\to\infty),$$

which, noting that with an appropriate $0<\eta$

$$\eta < \frac{\beta+1}{\delta}(\beta+2) - 2\sum_{r=1}^{m}\frac{\alpha_r^2}{\sqrt{\delta^2+t_r^2}} - 2\sum_{r=1}^{m}\frac{\alpha_r}{\sqrt{\delta^2+t_r^2}},$$

proves the assertion. □

3.3.2
The "convergent" case

Before we continue with the rest we need some lemmas. For a real a let

$$F_a(s) := F(s+ia).$$

Note that by Lemma 8 we have

$$\frac{F_a(s)}{\zeta(s)} = Q(s)\exp(-\sum_{|p|^\delta>2}\frac{1-f(p)|p|^{-ia}}{|p|^s}),$$

for $\sigma>\delta$, where

$$Q(s) = \prod_{|p|^\delta\le 2}(1+\sum_{\alpha}\frac{f(p^\alpha)}{|p|^{\alpha(s+ia)}})\prod_{|p|^\delta>2}(1+\sum_{\alpha}\frac{f(p^\alpha)}{|p|^{\alpha(s+ia)}})\exp(-\frac{f(p)}{|p|^{s+ia}})$$
$$\times\prod_{|p|^\delta>2}(1-\frac{1}{|p|^s})\exp(\frac{1}{|p|^s})\prod_{|p|^\delta<2}(1-\frac{1}{|p|^s}). \quad (3.29)$$

Since for all large enough primes

$$|\log(1+\sum_{\alpha}\frac{f(p^{\alpha})}{|p|^{\alpha(s+ia)}})-\frac{f(p)}{|p|^{s+ia}}|\ll\frac{1}{|p|^{2\sigma}}$$

and since

$$|\prod_{|p|^{\delta}\leq 2}(1+\sum_{\alpha}\frac{f(p^{\alpha})}{|p|^{\alpha(s+ia)}})|\ll 1$$

for $\sigma>0$, we deduce that $|Q(s)|\ll 1$ uniformly for $\sigma>\sigma_0>\delta/2$. Furthermore each term in the product in (3.29) is continuous, therefore, by the same argument as above, $Q(s)$ is continuous as well in each rectangle $\delta\leq\sigma\leq\delta+1$, $|\tau|\leq M$. Setting

$$(A_x=)A=Q(1)\exp(-\sum_{\substack{|p|^{\delta}>2\\|p|\leq x}}\frac{1-f(p)|p|^{-ia}}{|p|^{\delta}}),\tag{3.30}$$

we obtain the following

Lemma 12. *Let*

$$N(x)\sim cx^{\delta}\log^{\beta}x\quad(x\to\infty)$$

where $c,\delta>0$, $\beta\geq 0$ and suppose that

$$\psi(x)\ll x^{\delta}.$$

Let $M>0$ be an arbitrary real number. Further assume that $f\in\mathcal{F}_{\mathcal{G}}$ is multiplicative with $|f|\leq 1$ and that

$$\sum_{p}\frac{1-\Re f(p)|p|^{-ia}}{|p|^{\delta}}$$

converges for some $a\in\mathbb{R}$. Then by setting $\sigma=\delta+\frac{1}{\log x}$,

$$F_a(s)-A\zeta(s)=o(|s-\delta|^{-\beta-1})\quad(\sigma\to\delta+)$$

uniformly for $|\tau|\leq M(\sigma-\delta)$.

Proof. We have

$$\sum_{|p|\leq y}|\frac{1}{|p|^{s}}-\frac{1}{|p|^{\delta}}|=\sum_{|p|\leq y}\frac{1}{|p|^{\delta}}|\exp((s-\delta)\log|p|)-1|$$
$$\ll|s-\delta|\log y,$$

for $y \leq x$. Furthermore, with $\eta \geq \delta$

$$\sum_{y<|p|<t} \frac{1}{|p|^\eta} \leq \int_y^t \frac{1}{u^\eta \log u} d\psi(u)$$

$$\ll \frac{\psi(y)}{y^\eta \log y} + \int_y^t \frac{\psi(u)}{u^{\eta+1} \log u} du.$$

The integral

$$\int_y^t \frac{\psi(u)}{u^{\eta+1} \log u} du$$

is at most a constant times

$$\frac{1}{\eta - \delta} \frac{1}{y^{\eta-\delta} \log y}$$

if $\eta > \delta$ and does not exceed $\mathcal{O}(\log \frac{\log t}{\log y})$ if $\eta = \delta$. Thus, using the Cauchy-Schwarz inequality

$$(\sum_{y<|p|} \frac{1 - f(p)|p|^{-ia}}{|p|^\sigma})^2 \ll \sum_{y<|p|} \frac{|1 - f(p)|p|^{-ia}|^2}{|p|^\sigma} \sum_{y<|p|} \frac{1}{|p|^\sigma}.$$

Further, since

$$|1 - f(p)|p|^{-ia}|^2 \leq 2(1 - Ref(p)p^{-ia}),$$

we obtain that

$$\sum_{\substack{|p|^\delta > 2 \\ |p| \leq x}} (1 - f(p)|p|^{-ia})(\frac{1}{|p|^s} - \frac{1}{|p|^\delta}) + \sum_{\substack{|p|^\delta > 2 \\ x<|p|}} |1 - f(p)|p|^{-ia}| \frac{1}{|p|^\sigma}$$

does not exceed

$$\sum_{\substack{|p|^\delta > 2 \\ |p| \leq y}} |\frac{1}{|p|^s} - \frac{1}{|p|^\delta}| + \sum_{\substack{|p|^\delta > 2 \\ y<|p| \leq x}} |1 - f(p)|p|^{-ia}| \frac{1}{|p|^\delta} + \sum_{\substack{|p|^\delta > 2 \\ x<|p|}} |1 - f(p)|p|^{-ia}| \frac{1}{|p|^\sigma},$$

which is at most

$$c\{\log y(\frac{1}{\log x} + |\tau|) + \delta^{1/2}(y) \log^{1/2} \frac{\log x}{\log y} + \delta^{1/2}(x)\}$$

where

$$\delta(z) = \sum_{z<|p|} \frac{1 - \Re f(p)|p|^{-ia}}{|p|^{\delta}}. \tag{3.31}$$

Choosing $y = \max\{x^{\delta(x)}, x^{1/\sqrt{\log x}}\}$, we deduce that $F_a(s)$ equals

$$\begin{aligned}
&Q(s)\exp(-\sum_{\substack{|p|^{\delta}>2\\|p|\le x}} \frac{1 - f(p)|p|^{-ia}}{|p|^{\delta}})\zeta(s)\\
&\quad\times\exp(-\sum_{\substack{|p|^{\delta}>2\\|p|\le x}} (1 - f(p)|p|^{-ia})(\frac{1}{|p|^{s}} - \frac{1}{|p|^{\delta}}) - \sum_{x<|p|} \frac{1 - f(p)|p|^{-ia}}{|p|^{s}})\\
&=(Q(1) + o(1))\exp(-\sum_{\substack{|p|^{\delta}>2\\|p|\le x}} \frac{1 - f(p)|p|^{-ia}}{|p|^{\delta}})\zeta(s)(1 + o(1))
\end{aligned}$$

uniformly for $|\tau| \le M(\sigma - \delta)$ as $\sigma \to \delta+$. Hence, in the view of Lemma 1 the assertion follows. □

Lemma 13. *Let*

$$N(x) \sim cx^{\delta}\log^{\beta} x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \ge 0$ and suppose that $f \in \mathcal{F}_{\mathcal{G}}$ is multiplicative with $|f| \le 1$ and that

$$\sum_{p} \frac{1 - \Re f(p)|p|^{-ia}}{|p|^{\delta}}$$

converges for some $a \in \mathbb{R}$. Then

$$F_a(s) \ll \frac{1}{K^{(\beta+1)/2}(\sigma - \delta)^{\beta+1}} + o(\frac{1}{(\sigma - \delta)^{\beta+1}}) \quad (\sigma \to \delta+)$$

holds uniformly for all $K(\sigma - \delta) \le |\tau| \le K$. The constant implied by the Vinogradov symbol does not depend on K.

Proof. We have

$$\begin{aligned}
2(1 - \Re|p|^{it}) =&|1 - |p|^{it}|^2\\
\le&2|1 - f(p)|p|^{-ia}|^2 + 2|f(p)|p|^{-ia} - |p|^{it}|^2\\
\le&2|1 - f(p)|p|^{-ia}|^2 + 4(1 - \Re f(p)|p|^{-ia}|p|^{-it})
\end{aligned}$$

thus

$$\exp(4\sum_p \frac{\Re f(p)|p|^{-it}|p|^{-ia}}{|p|^\sigma}) \le \exp(2\sum_p \frac{|1-f(p)|p|^{-ia}|^2}{|p|^\sigma}) \exp(2\sum_p \frac{1+\Re|p|^{it}}{|p|^\sigma}).$$

Since

$$\sum_p \frac{|1-f(p)|p|^{-ia}|^2}{|p|^\sigma} \ll 1$$

uniformly for $\sigma \ge 1$, using Lemma 8 we deduce

$$|F_a(s+it)|^2 \ll \zeta(\sigma)|\zeta(s-it)|.$$

Then using Lemma 1 the proof is finished. □

Lemma 14. *Let $\eta > 0$,*

$$N(x) \sim cx^\delta \log^\beta x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \ge 0$ and $t(x)$ is a real function with $\eta < t(x) \ll 1$ which has an almost everywhere continuous derivative with $t'(x) \ll \frac{1}{x}$. Suppose further that for $x \ge 2$

$$\sum_{|p|\le x} \frac{1}{|p|^\delta} = (\beta+1)\log\log x + \mathcal{O}(1). \tag{3.32}$$

Assume that $f \in \mathcal{F}_\mathcal{G}$ is multiplicative with $|f| \le 1$ and that

$$\sum_p \frac{1-\Re f(p)|p|^{-ia}}{|p|^\delta}$$

converges for some $a \in \mathbb{R}$. Then

$$\int_1^x |M_{f-A\mathbf{1}_a}(\frac{x}{t})| \log t dt^\delta t(t) = o(x^\delta \log^{\beta+2} x) \quad (x \to \infty)$$

where A is defined by (3.30).

Proof. We use inequality (3.10), i.e. that

$$\int_1^x |M(\frac{x}{t})| \log t dt^\delta t(t) \ll x^\delta \log^{3/2} x H(1/\log x)$$

where

$$H(u) := (\int_{-\infty}^{\infty} |\frac{F(\delta+u+i\tau) - A\zeta(\delta+u+i(\tau-a))}{\delta+u+i\tau}|^2 d\tau)^{1/2}.$$

We split the range of integration in $H(u)$ into three parts I_1, I_2, I_3. Let M be a fixed large number. In I_1 we integrate over $|\tau| \leq Mu$. Since

$$\int_{-\infty}^{\infty} |\frac{F(\delta+u+i\tau) - A\zeta(\delta+u+i(\tau-a))}{\delta+u+i\tau}|^2 d\tau \ll$$
$$\int_{-\infty}^{\infty} |\frac{F(\delta+u+i(\tau+a)) - A\zeta(\delta+u+i\tau)}{\delta+u+i\tau}|^2 d\tau = \int_{-\infty}^{\infty} |\frac{F_a(s) - A\zeta(s)}{s}|^2 d\tau,$$

by using Lemma 12 we obtain

$$\begin{aligned}\int_{|\tau|\leq M(\sigma-\delta)} |\frac{F_a(s) - A\zeta(s)}{s}|^2 d\tau =& o(\int_{|\tau|\leq M(\sigma-\delta)} \frac{1}{|s-\delta|^{2\beta+2}|s|^2} d\tau) \\ =& o((\sigma-\delta)^{-2\beta-1}) \quad (\sigma \to \delta+).\end{aligned}$$

In I_2 we integrate over $M(\sigma-\delta) \leq |\tau| \leq M$. By Lemma 13 and by Lemma 1

$$\begin{aligned}\int_{M(\sigma-\delta)\leq|\tau|\leq M} |\frac{F_a(s) - A\zeta(s)}{s}|^2 d\tau \ll& \max_{M(\sigma-\delta)\leq|\tau|\leq M} |F_a(s)|^{1/2} \int_{-\infty}^{\infty} \frac{|F_a(s)|^{3/2}}{|s|^2} d\tau \\ &+ |A| \max_{M(\sigma-\delta)\leq|\tau|\leq M} |\zeta(s)|^{1/2} \int_{-\infty}^{\infty} \frac{|\zeta(s)|^{3/2}}{|s|^2} d\tau.\end{aligned}$$

As we have seen in the proof of Lemma 11 here the two integrals do not exceed

$$c(\sigma-\delta)^{\frac{3\beta+1}{2}}.$$

Thus,

$$I_2 \ll (\frac{1}{M^{(\beta+1)/4}} + o(1))(\sigma-\delta)^{-2\beta-1}.$$

It remains to estimate I_3. In exactly the same way as in the proof of Lemma 11 we obtain that

$$I_3 \ll \frac{1}{M}(\sigma-\delta)^{-2\beta-1}$$

which - since M was arbitrary - proves the assertion. □

Theorem 3. *Let*

$$N(x) \sim cx^{\delta} \log^{\beta} x \quad (x \to \infty)$$

where $c, \delta > 0$, $\beta \geq 0$ *and*

$$\psi(x) \asymp x^{\delta}.$$

Suppose further that for $x \geq 2$

$$\sum_{|p| \leq x} \frac{1}{|p|^{\delta}} = (\beta + 1) \log \log x + \mathcal{O}(1).$$

Assume that $f \in \mathcal{F}_{\mathcal{G}}$ *is multiplicative with* $|f| \leq 1$ *and that*

$$\sum_{p} \frac{1 - \Re f(p)|p|^{-ia}}{|p|^{\delta}}$$

converges for some real a. *Suppose further that the Selberg Symmetry Formula holds, i.e. that*

$$\sum_{|n| \leq x} \Lambda * \Lambda(n) + \Lambda(n) \log |n| = t(x) x^{\delta} \log x + o(\psi(x) \log x) \quad (x \to \infty)$$

where $t(x)$ *is as in Lemma 5. Then*

$$\sum_{|n| \leq x} f(n) = \frac{N(x) x^{ia} \delta}{\delta + ia} \prod_{|p| \leq x} (1 - \frac{1}{|p|^{\delta}})(1 + \sum_{\alpha} \frac{f(p^{\alpha})}{|p|^{\alpha(\delta + ia)}}) + o(N(x)) \quad (x \to \infty).$$

Proof. We proceed as in the proof of Theorem 2. By Theorem 1 we obtain

$$\begin{aligned}\log^2 x |\sum_{|n| \leq x} f(n) - A|n|^{ia}| \ll \sum_{|n| \leq x} &|M(\frac{x}{n})| \{\Lambda * \Lambda(n) + \Lambda L(n)\} \\ &+ (R_1 + R_2 + R_3)(x) \log x \\ &+ \sum_{|n| \leq x} (R_1 + R_2 + R_3)(\frac{x}{|n|}) \Lambda(n), \qquad (3.33)\end{aligned}$$

where

$$R_1(x) \ll \sum_{|n| \leq x} \log \frac{x}{|n|}$$

and

$$R_2(x) \ll N(x) \sum_{|n| \le x} \frac{|g(n)| \log |n|}{|n|^\delta}.$$

Here g is defined by $f = \tilde{f} * g$ where $\tilde{f}$ is completely multiplicative with $\tilde{f}(p) = f$ and A is defined by (3.30). Further

$$R_3(x) = \sum_{|n| \le x} (\sum_{|m| \le \frac{x}{|n|}} A\mathbf{1}_a(m))\Lambda(n)|\mathbf{1}_a(n) - \tilde{f}(n)|.$$

Using Selberg's formula by Lemma 5 and Lemma 14 we have

$$\sum_{|n| \le x} |M(\frac{x}{|n|})|\{\Lambda * \Lambda(n) + \Lambda L(n)\} = o(N(x) \log^2 x) \quad (x \to \infty),$$

and in the proof of Theorem 2 we have seen that the error terms concerning R_1, R_2 are $\mathcal{O}(N(x) \log^2 x)$. Therefore, the right hand side of (3.33) equals

$$R_3(x) \log x + \sum_{|n| \le x} R_3(\frac{x}{|n|})\Lambda(n) + o(N(x) \log^2 x) \quad (x \to \infty).$$

Since

$$\begin{aligned}\sum_{|n| \le x} \frac{\log |p||f(p) - |p|^{ia}|}{|p|^\delta} &= \sum_{|n| \le y} \frac{\log |p||f(p) - |p|^{ia}|}{|p|^\delta} + \sum_{y < |n| \le x} \frac{\log |p||f(p) - |p|^{ia}|}{|p|^\delta} \\ &\ll \log y + \sqrt{\log x} (\sum_{|p| \le x} \frac{\log |p|}{|p|^\delta})^{1/2} (\sum_{y < |p|} \frac{|f(p)|p|^{-ia} - 1|^2}{|p|^\delta})^{1/2} \\ &\ll \log y + \log x (\sum_{y < |p|} \frac{1 - \Re f(p)|p|^{-ia}}{|p|^\delta})^{1/2},\end{aligned}$$

choosing $y = \max\{x^{\delta(x)}, x^{1/\sqrt{\log x}}\}$ where $\delta(x)$ is defined by (3.31) we obtain

$$\sum_{|n| \le x} \frac{\Lambda(n)|\tilde{f}(n) - |n|^{ia}|}{|n|^\delta} = o(\log x) \quad (x \to \infty).$$

Thus,

$$R_3(x) \ll |A| N(x) \sum_{|n| \le x} \frac{\Lambda(n)|\tilde{f}(n) - |n|^{ia}|}{|n|^\delta} = o(N(x) \log x) \quad (x \to \infty).$$

Furthermore, similarly

$$\begin{aligned}\sum_{|n|\le x} R_3(\frac{x}{|n|})\Lambda_{\tilde f}(n) \ll & A\sum_{|n|\le x}\sum_{|m|\le \frac{x}{|n|}} \mathbf{1}_a * |\Lambda_{\tilde f} - \Lambda_{\mathbf{1}_a}|(m)\Lambda_{\tilde f}(n)\\ =& A\sum_{|n|\le x} \mathbf{1}_a * |\Lambda_{\tilde f} - \Lambda_{\mathbf{1}_a}| * \Lambda_{\tilde f}(n)\\ =& A\sum_{|n|\le x} L * |\Lambda_{\tilde f} - \Lambda_{\mathbf{1}_a}|(n)\\ =& A\sum_{|n|\le x}(\sum_{|m|\le \frac{x}{|n|}} \log|m|)\Lambda(n)|\tilde f(n) - |n|^{ia}|\\ =& o(N(x)\log^2 x) \quad (x\to\infty).\end{aligned}$$

We have

$$\begin{aligned}\sum_{|n|\le x}|n|^{ia} =& \int_{1-}^{x} u^{ia} dN(u)\\ =& N(x)x^{ia} + O(1) - ia\int_1^x N(u)u^{ia-1}du.\end{aligned}$$

We distinguish two cases. For $\beta = 0$ we have

$$\int_1^x u^{\delta+ia-1}\log^\beta u du = \frac{x^{\delta+ia}}{\delta+ia} + \mathcal{O}(1),$$

while if $\beta > 0$ then

$$\int_1^x u^{\delta+ia-1}\log^\beta u du = \frac{x^{\delta+ia}\log^\beta x}{\delta+ia} - \frac{\beta}{\delta+ia}\int_1^x u^{\delta+ia-1}\log^{\beta-1} u du.$$

Since

$$\begin{aligned}\int_1^x u^{\delta+ia-1}\log^{\beta-1} u du =& \int_1^{\sqrt{x}} u^{\delta+ia-1}\log^{\beta-1} u du + \int_{\sqrt{x}}^x u^{\delta+ia-1}\log^{\beta-1} u du\\ \ll& x^{\delta/2}\log x + x^\delta \log^{\beta-1} x\\ =& o(N(x) \quad (x\to\infty),\end{aligned}$$

it remains to prove that

$$A = \prod_{|p|\le x}(1-\frac{1}{|p|^\delta})(1+\sum_\alpha \frac{f(p)}{|p|^{\alpha(\delta+ia)}}) + o(1) \quad (x\to\infty).$$

But since

$$\sum_{x<|p|}\log(1+\sum_{\alpha}\frac{f(p)}{|p|^{\alpha(\delta+ia)}})-\frac{f(p)}{|p|^{\delta+ia}}+\sum_{x<|p|}\log(1-\frac{1}{|p|^{\delta}})+\frac{1}{|p|^{\delta}}$$
$$\ll\sum_{x<|p|}\frac{1}{|p|^{2\delta}},$$

the assertion follows. □

In the view of the proof of Corollary 1 the following corollary is immediate.

Corollary 2. *Let*

$$N(x)\sim cx^{\delta}\log^{\beta}x \quad (x\to\infty),$$

where $c,\delta>0$, $\beta\geq 0$ *and let*

$$\psi(x)=(\frac{\beta+1}{\delta}-2\sum_{r=1}^{m}\frac{\alpha_r}{\sqrt{\delta^2+t_r^2}}\cos(t_r\log x-\theta_r))x^{\delta}+o(x^{\delta}) \quad (x\to\infty)$$

where $\alpha_r\in\mathbb{N}_0$, $t_r>0$, $r=1,\dots,m$ *such that*

$$\sum_{r=1}^{m}\alpha_r\leq\frac{\beta+1}{2},$$

and θ_r, $r=1,\dots,m$ *are the angles which satisfy*

$$\sin\theta_r=\frac{t_r}{\sqrt{\delta^2+t_r^2}} \quad \text{and} \quad \cos\theta_r=\frac{\delta}{\sqrt{\delta^2+t_r^2}}.$$

Assume further that $f\in\mathcal{F}_{\mathcal{G}}$ *is multiplicative with* $|f|\leq 1$ *and that*

$$\sum_{p}\frac{1-\Re f(p)|p|^{-ia}}{|p|^{\delta}}$$

converges for some real a. *Then*

$$\sum_{|n|\leq x}f(n)=\frac{N(x)x^{ia}\delta}{\delta+ia}\prod_{|p|\leq x}(1-\frac{1}{|p|^{\delta}})(1+\sum_{\alpha}\frac{f(p^{\alpha})}{|p|^{\alpha(\delta+ia)}})+o(N(x)) \quad (x\to\infty).$$

Chapter 4

Quantitative estimations

4.1 Introduction

Now we investigate a quantitative version of the results obtained in Chapter 3. Let f be a multiplicative function over the natural numbers.
Germán, Indlekofer and Klesov assuming that f is in some sense close to some positive real number larger then 1/2, developed quantitative results for the limit behavior of

$$\frac{1}{x}\sum_{n\le x} f(n)$$

in [16]. Their theorem is as follows:

Theorem. *Let $f \neq 0$ be multiplicative. Let $\kappa > 1/2$ and $0 \le \eta_0 < \kappa$, $0 < \lambda_0 \le 2$. Let $\tilde{f}$ be defined by the equation $\tilde{f}\Lambda = \Lambda_f$ (for the notation see Chapter 2) and let τ_κ be defined by*

$$\zeta^\kappa(s) = \sum_n \frac{\tau_\kappa(n)}{n^s} \quad (\sigma > 1).$$

Assume that

$$|\tilde{f}(p^\alpha) - \kappa| \le \eta\alpha(2-\lambda)^{\alpha-1} \text{ for all primes } p \text{ and all } \alpha \in \mathbb{N}, \tag{4.1}$$

where $0 \le \eta \le \eta_0$, and $\lambda_0 \le \lambda \le 2$. Put

$$A_x = \exp\Big(\sum_{p\le x} \frac{f(p)-\kappa}{p}\Big). \tag{4.2}$$

Then there exist positive constants c_1, c_2 which depend at most on $\kappa, \lambda_0, \eta_0$ such that, for $x \geq 2$,

$$|\sum_{n\leq x} f(n) - A_x \sum_{n\leq x} \tau_\kappa(n)| \leq c_1 \eta x \log^{\kappa-1} x |A_x| \qquad (4.3)$$
$$+ c_1 x \log^{\kappa-1} x \exp(\sum_{p\leq x} \frac{|f(p)|-\kappa}{p}) \{\exp(-\frac{c_2}{\eta}) + \log^{-c_2} x\}.$$

This theorem is a generalization of a result of Halász and Elliott [11] (Theorem 19.2). In this part of the work we release the conditions of this last theorem and we allow f to be a multiplicative function over a multiplicative semigroup the values at primes of which are close to an arbitrary $\kappa > \frac{1}{2(\beta+1)}$. For the sake of simplicity we assume that $\mathcal{G}$ is a multiplicative semigroup which satisfies the conditions mentioned at the beginning of Chapter 3 with

$$R(x) \ll x^\delta \log^{-\eta} x$$

where $0 < \eta_0 < \eta$. Remember that it follows that

$$\sum_{|p|\leq x} \frac{1}{|p|^\delta} = (\beta+1) \log\log x + \mathcal{O}(1)$$

and that the Selberg Symmetry formula

$$\sum_{|n|\leq x} \Lambda * \Lambda(n) + \Lambda(n) \log|n| = t(x) x^\delta \log x + \delta(x) \psi(x) \log x \qquad (4.4)$$

is valid, where $t(x)$ is as in Lemma 5 and $\delta(x) \to 0$ as x tends to infinity (for the proof see the proof of Corollary 1).

Lemma 15. *Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative function. The corresponding generating Dirichlet function F is formally defined by*

$$F(s) := \sum_n \frac{f(n)}{|n|^s}. \qquad (4.5)$$

Let $\kappa > \eta > 0$ and suppose

$$|\Lambda_f(n) - \kappa\Lambda(n)| \leq \eta\Lambda(n).$$

Then F is absolutely convergent in the halfplane $\sigma > \delta$ and $F(s) \neq 0$ there and for all such complex values the equation

$$F(s) = \exp(\sum_{|n|>1} \frac{\tilde{f}(n)\Lambda(n)}{|n|^s \log|n|}) \qquad (4.6)$$

holds with some multiplicative function $\tilde{f}$, such that $\Lambda_f = \tilde{f}\Lambda$.

Proof. Let g be the positive real valued multiplicative function, for which the generating Dirichlet function equals $\zeta(s)^{\kappa+\eta}$. Since $\zeta(s) \neq 0$ for $\sigma > \delta$, it is easy to see that

$$\zeta^{\kappa+\eta}(s) = \exp(\sum_{|n|>1} \frac{(\kappa+\eta)\Lambda(n)}{|n|^s \log |n|}),$$

and that

$$\Lambda_g = (\kappa+\eta)\Lambda.$$

Then by induction we have

$$\begin{aligned} |Lf| =& |\Lambda_f * f| \\ \leq& \Lambda_g * g \\ =& Lg. \end{aligned}$$

Thus, F converges absolutely for $\sigma > \delta$. By the absolute convergence we have

$$F(s) = 1 + u(s)$$

where

$$|u(s)| \to 0 \quad (\sigma \to \infty).$$

Therefore, there is a halfplane where $\log F(s)$ is a holomorphic function, where log is the principal value of the complex logarithm function. On the other hand, since

$$\begin{aligned} F^{-1}(s) =& (1+u(s))^{-1} \\ =& \sum_l (-1)^l u(s)^l, \end{aligned}$$

by rearranging the terms we obtain that

$$F^{-1}(s) = \sum_n \frac{h(n)}{|n|^s}$$

converges absolutely in the same halfplane. Using the unicity of the Dirichlet generating function, $h = f^{-1}$. Consequently,

$$\log' F(s) = \frac{F'}{F}(s) = -\sum_{|n|>1} \frac{Lf * f^{-1}(n)}{|n|^s}$$

in the same halfplane, and the right hand side converges absolutely. Integrating the left and right hand side of this equation we deduce that (4.6) holds by analytic continuation for all $\sigma > \delta$. From this exponential equation follows, that $F(s)$ is non-zero for $\sigma > \delta$. $\square$

Remark. A similar argument shows that the result remains true for multiplicative functions f wich satisfies

$$|\Lambda_f(p^\alpha) - \kappa\Lambda(p^\alpha)| \leq \eta\alpha\Lambda(p^\alpha)(n_2 - \lambda)^\alpha$$

for all primes $p \in \mathcal{P}$ and $\alpha \geq 1$ for some $0 < \lambda \leq n_2$. Remember that n_2 was the minimum of the norms of the primes (See the notations before Lemma 3).

Let $\kappa > \frac{1}{2(\beta+1)}$, and let τ_κ be the multiplicative function defined by

$$\sum_n^\infty \frac{\tau_\kappa(n)}{|n|^s} = \zeta^\kappa(s), \quad \Re(s) > \delta, \tag{4.7}$$

where $\zeta(s)$ is the Riemann's zeta function belonging to $\mathcal{G}$.

Remark. It is easy to see that Lemma 10 remains true if f satisfies

$$\sum_{|p|^\alpha \leq x} |f(p^\alpha)| \log |p|^\alpha \ll \psi(x)$$

instead of $|f| \ll 1$. Since $\exp(\frac{\log |n|}{\log x}) \asymp 1$ uniformly for all $1 \leq n \leq x$, using Lemma 1

$$\begin{aligned} \sum_{|n| \leq x} \tau_\kappa(n) &\ll \frac{x^\delta}{\log x} \zeta(\delta + \frac{1}{\log x})^\kappa \\ &\ll x^\delta \log^{(\beta+1)\kappa - 1} x. \end{aligned} \tag{4.8}$$

Under some strong assumptions on the analytic behaviour of $\zeta(s)$ it is possible to compute the right asymptotic properties of

$$\sum_{|n| \leq x} \tau_\kappa(n).$$

We have the following

Theorem 4. *Let $x > 2$, and let $f : \mathcal{G} \to \mathbb{C}$ be multiplicative. Let $\tilde{f}$ be defined by (4.6) and τ_κ by (4.7). Assume that $\kappa > \frac{1}{2(\beta+1)}$ and $0 \leq \eta_0 < \kappa$, $0 < \lambda_0 \leq n_2$. Suppose that $\tilde{f}(p^\alpha) = \kappa$ for all $p \in \mathcal{P}$ and $\alpha \geq 1$ such that $|p|^\alpha > x$, and that*

$$|\tilde{f}(p^\alpha) - \kappa| \leq \eta\alpha(n_2 - \lambda)^{\alpha-1} \text{ for all } p \in \mathcal{P},\ \alpha \geq 1, \tag{4.9}$$

where $0 \leq \eta \leq \eta_0$, and $\lambda_0 \leq \lambda \leq n_2$. Put

$$A = \exp\Big(\sum_{|p|\leq x} \frac{f(p) - \kappa}{|p|^\delta}\Big).$$

Then there exist positive constants c_1, c_2 which depend at most on $\kappa, \lambda_0, \eta_0, \mathcal{G}$ such that

$$\begin{aligned}
\Big|\sum_{|n|\leq x} f(n) - A \sum_{|n|\leq x} \tau_\kappa(n)\Big| \leq\, & c_1 x^\delta \log^{\kappa(\beta+1)-1} x |A| \eta \\
& + c_1 x^\delta \log^{\kappa(\beta+1)-1} x \exp\Big(\sum_{|p|\leq x} \frac{|f(p)| - \kappa}{|p|^\delta}\Big) \\
& \times \{\exp(-\frac{c_2}{\eta}) + \log^{-c_2} x\} \\
& + c_1 \frac{x^\delta}{\log^2 x} \sum_{|n|\leq x} \frac{|f(n)|}{|n|^\delta}\big(\max_{1<u<x} |\delta(u)| \log u + 1\big)
\end{aligned}$$

where $\delta(u)$ is defined by (4.4).

Remark. Since $Lf = \Lambda_f * f$, by using the method of Lemma 10 it is easy to see that under the conditions of the theorem above

$$\Big|\sum_{|n|\leq u} f(n)\Big| \ll \frac{u^\delta}{\log u} \sum_{|n|\leq u} \frac{|f(n)|}{|n|^\delta} \tag{4.10}$$

holds uniformly for $u \geq 2$.

We deduce the above theorem using

Theorem 5. *Let $x > 2$, and let $f : \mathcal{G} \to \mathbb{C}$ be multiplicative. Let $\tilde{f}$ be defined by (4.6) and τ_κ by (4.7). Assume that $\kappa > \frac{1}{2(\beta+1)}$ and $0 \leq \eta_0 < \kappa$, $0 < \lambda_0 \leq n_2$. Suppose that $\tilde{f}(p^\alpha) = \kappa$ for all $p \in \mathcal{P}$ and $\alpha \geq 1$ such that $|p|^\alpha > x$, and that (4.9) is satisfied. Putting*

$$M(x) = \sum_{|n|\leq x} (f - A\tau_\kappa)(n)$$

we have

$$\begin{aligned}\log^2 x|M(x)| \ll & x^\delta \log x \int_1^x \frac{|M\,(u)\,|}{u^2} du \\ & + x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta} (\max_{1<u<x} |\delta(u)| \log u + 1) \\ & + \log x(\eta + \frac{1}{\log x})|A| x^\delta \log^{(\beta+1)\kappa} x \end{aligned} \tag{4.11}$$

uniformly for all $A \in \mathbb{C}$ *where* $\delta(u)$ *is defined by* (4.4). *The implied constant depends at most on* $\kappa, \lambda_0, \eta_0, \mathcal{G}$.

Remark. These theorems are not uniform in $\mathcal{G}$. To be more precise, they are uniform for all multiplicative semigroups with $n_2 > 1 + \epsilon$ for some $\epsilon > 0$.

The integral appearing in the above theorems can be estimated with the help of Lemma 7.

4.2 A convolution identity

The quantitative estimation depends on a variant of a Theorem of Indlekofer in [25]. For the sake of completeness we give the proof of

Lemma 16. *Let the arithmetical function* $f, g \in \mathcal{F}_\mathcal{G}$ *satisfy* $f(1) \neq 0$ *and* $g(1) \neq 0$. *Putting* $M(x) = \sum_{|n|\le x} (f-g)(n)$ *we have*

$$\begin{aligned}\log^2(x) M(x) = & \sum_{|n|\le x} M(\frac{x}{|n|})(\Lambda_g * \Lambda_g(n) + \log|n| \Lambda_g(n)) \\ & + \sum_{|n|\le x} (R_1 + R_2)(\frac{x}{|n|}) \Lambda_g(n) \\ & + \log x (R_1 + R_2)(x),\end{aligned}$$

where

$$R_1 = \sum_{|n|\le x} \log \frac{x}{|n|} (f-g)(n)$$

and

$$R_2 = \sum_{|n|\le x} f * (\Lambda_f - \Lambda_g)(n).$$

Proof. We have

$$\log x M(x) = \sum_{|n|\le x} \log \frac{x}{|n|}(f-g)(n) + \sum_{|n|\le x} \log|n|(f-g)(n)$$

and, putting $R_1 = \sum_{|n|\le x} \log \frac{x}{|n|}(f-g)(n)$,

$$\begin{aligned}\log x M(x) &= \sum_{|n|\le x}(f*\Lambda_f)(n) - \sum_{|n|\le x}(g*\Lambda_g)(n) + R_1(x)\\ &= \sum_{|n|\le x}(f-g)*\Lambda_g(n) + \sum_{|n|\le x} f*(\Lambda_f-\Lambda_g)(n) + R_1(x)\\ &= \sum_{|n|\le x} M(\frac{x}{|n|})\Lambda_g(n) + R_1(x) + R_2(x), \qquad (4.12)\end{aligned}$$

where

$$R_2 = \sum_{|n|\le x} f*(\Lambda_f-\Lambda_g)(n).$$

We multiply (4.12) with $\log x$ and obtain

$$\begin{aligned}\log^2 x M(x) = &\sum_{|n|\le x} \log\frac{x}{|n|} M(\frac{x}{|n|})\Lambda_g(n) + \sum_{|n|\le x} M(\frac{x}{|n|})\log|n|\Lambda_g(n)\\ &+ \log x R_1(x) + \log x R_2(x). \qquad (4.13)\end{aligned}$$

Then, by substituting (4.12) into (4.13) we arrive at

$$\begin{aligned}\log^2 x M(x) = &\sum_{|n|\le x} M(\frac{x}{|n|})(\Lambda_g*\Lambda_g(n) + \log|n|\Lambda_g(n))\\ &+ \sum_{|n|\le x}(R_1+R_2)(\frac{x}{|n|})\Lambda_g(n)\\ &+ \log x(R_1+R_2)(x)\end{aligned}$$

which leads immediately to Lemma 16. □

4.3 Proof of the theorems

Proof of Theorem 5. We apply Lemma 16 and show first that

$$R_1(x) = \mathcal{O}(\frac{x^\delta}{\log x}(\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta} + |A|\log^{(\beta+1)\kappa} x)) \qquad (4.14)$$

and that

$$R_2(x) \ll \frac{x^{\delta-1}}{\log x}\int_1^x |M(\frac{x}{t})|\log t dt + (\eta + \log^{-1} x)|A|x^\delta \log^{(\beta+1)\kappa} x$$
$$+ (\max_{1<u<x} |\delta(u)|\log u + 1)\frac{x^\delta}{\log x}\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}. \quad (4.15)$$

Then we deduce

$$\sum_{|n|\le x} R_1(\frac{x}{|n|})\Lambda_{\tau_\kappa}(n) = \mathcal{O}\left(x^\delta(\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta} + |A|\log^{(\beta+1)\kappa} x)\right),$$

and

$$\sum_{|n|\le x} R_2(\frac{x}{|n|})\Lambda_{\tau_\kappa}(n) = \mathcal{O}\Bigg(x^{\delta-1}\int_1^x |M(\frac{x}{t})|\log t dt + \eta|A|x^\delta\log^{(\beta+1)\kappa+1} x$$
$$+ x^\delta \sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}\Bigg).$$

Now using (4.10) we obtain

$$\begin{aligned}
R_1(x) &= \sum_{|n|\le x}\log\frac{x}{|n|}(f - A\tau_\kappa)(n)\\
&= \sum_{|n|\le x}(f - A\tau_\kappa)(n)\int_{|n|}^x \frac{1}{u}du\\
&= \int_1^x \frac{M(u)}{u}du\\
&= \int_1^x \frac{M_f(u) - AM_{\tau_\kappa}(u)}{u}du\\
&\ll \sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}\int_{1+\epsilon}^x \frac{u^{\delta-1}}{\log u}du + |A|\log^{(\beta+1)\kappa} x\int_{1+\epsilon}^x \frac{u^{\delta-1}}{\log u}du
\end{aligned}$$

for some $\epsilon > 0$. This proves (4.14) since the estimation in (4.8) holds. Since

$$\begin{aligned}\sum_{\substack{|p|^\alpha \leq u \\ \alpha \geq 2}} \alpha(n_2-\lambda)^{\alpha-1}\log|p| \ll& \sum_{|p|\leq\sqrt{u}} \log|p| \sum_{\alpha\leq\frac{\log u}{\log|p|}} \alpha\exp(\alpha\log(n_2-\lambda_0)) \\ \ll& \log u \sum_{|p|\leq n_2^{2\delta}} \exp(\frac{\log(n_2-\lambda_0)}{\log|p|}\log u) \\ &+ u^{\delta/2-\epsilon}\log^2 u \sum_{|p|\leq\sqrt{u}} \log|p| \\ \ll& u^{\delta-\epsilon} \end{aligned} \tag{4.16}$$

for some appropriate $\epsilon > 0$,

$$\begin{aligned}\sum_{|n|\leq u} |\Lambda_f(n)-\Lambda_{\tau_\kappa}(n)| \leq& \eta \sum_{|p|^\alpha\leq u} \alpha(n_2-\lambda)^{\alpha-1}\log|p| \\ =& \eta\sum_{|p|\leq u}\log|p| + \eta\sum_{\substack{|p|^\alpha\leq u \\ \alpha\geq 2}} \alpha(n_2-\lambda)^{\alpha-1}\log|p| \\ \ll& \eta u^\delta \end{aligned} \tag{4.17}$$

holds. This implies

$$\begin{aligned}\sum_{|n|\leq y} \log\frac{x}{|n|}(\Lambda_f-\Lambda_{\tau_\kappa})(n) =& \int_1^y \frac{\sum_{|n|\leq u}(\Lambda_f-\Lambda_{\tau_\kappa})(n)}{u}du \\ \ll& y^\delta. \end{aligned} \tag{4.18}$$

Thus rearranging the terms in the summation,

$$\sum_{|n|\leq x} \log\frac{x}{|n|} f*(\Lambda_f-\Lambda_{\tau_\kappa})(n) \ll x^\delta \sum_{|n|\leq x} \frac{|f(n)|}{|n|^\delta}.$$

Observe that $Lf = \Lambda_f * f$ and that

$$|\Lambda_f-\Lambda_{\tau_\kappa}| \leq \eta\Lambda + c\tilde{\Lambda}$$

where

$$\tilde{\Lambda}(n) = \begin{cases} \alpha(n_2-\lambda_0)^\alpha\log|p| & \text{if } n=p^\alpha, \alpha>1 \\ 0 & \text{otherwise .} \end{cases}$$

This leads to

$$
\begin{aligned}
LR_2(x) = &\sum_{|n|\le x} f * (\Lambda_f * (\Lambda_f - \Lambda_{\tau_\kappa})(n) + L(\Lambda_f - \Lambda_{\tau_\kappa})(n)) + \mathcal{O}(x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta}) \\
\ll &\sum_{|n|\le x} |(f - A\tau_k)(\frac{x}{|n|})| [(\eta\Lambda + \tilde{\Lambda}) * (\Lambda + \tilde{\Lambda}) + L(\eta\Lambda + \tilde{\Lambda})](n) \\
&+ x^\delta \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta} + |A| \sum_{|n|\le x} \tau_k * (|\Lambda_f| * |\Lambda_f - \Lambda_{\tau_\kappa}| + L|\Lambda_f - \Lambda_{\tau_\kappa}|)(n).
\end{aligned} \tag{4.19}
$$

Note that by (4.16)

$$
\sum_{|n|\le x} \tilde{\Lambda} * \tilde{\Lambda}(n) \ll x^\delta \quad \text{and} \quad \sum_{|n|\le x} \Lambda * \tilde{\Lambda}(n) \ll x^\delta.
$$

Thus, by Selberg's formula, Lemma 5 is applicable to the first term on the right hand side of (4.19) and we arrive at

$$
\begin{aligned}
R_2(x) \ll &\frac{x^{\delta-1}}{\log x} \int_1^x |M(\frac{x}{t})| \log t dt + (\eta + \log^{-1} x)|A| x^\delta \log^{(\beta+1)\kappa} x \\
&+ (\max_{1<u<x} |\delta(u)| \log u + 1) \frac{x^\delta}{\log x} \sum_{|n|\le x} \frac{|f(n)|}{|n|^\delta},
\end{aligned}
$$

which proves (4.15). Here in the last step we used the inequality

$$
\sum_{|n|\le x} \tau_k * (|\Lambda_f| * |\Lambda_f - \Lambda_{\tau_\kappa}| + L|\Lambda_f - \Lambda_{\tau_\kappa}|)(n) \ll \sum_{|n|\le x} |\tau_\kappa(n)| \frac{x^\delta}{|n|^\delta} (\eta \log \frac{x}{|n|} + 1),
$$

which is nothing else but

$$
\eta x^\delta \int_1^x \frac{\sum_{|n|\le u} \frac{|\tau_\kappa(n)|}{|n|^\delta}}{u} du + x^\delta \sum_{|n|\le x} \frac{\tau_k(n)}{|n|^\delta} \ll (\eta + \frac{1}{\log x}) x^\delta \log^{(\beta+1)\kappa+1} x.
$$

Concerning $\sum_{|n|\le x} R_1(\frac{x}{|n|}) \Lambda_{\tau_\kappa}(n)$ we have by partial summation that

$$
\begin{aligned}
\sum_{|n|\le y} \log \frac{y}{|n|} \Lambda_{\tau_\kappa}(n) = &\int_1^y \frac{\sum_{|n|\le u} \Lambda_{\tau_\kappa}(n)}{u} du \\
\ll &y^\delta.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\sum_{|n|\le x} R_1(\frac{x}{|n|})\Lambda_{\tau_\kappa}(n) &= \sum_{|n|\le x} \log\frac{x}{|n|}(f-A\tau_\kappa)*\Lambda_{\tau_\kappa}(n)\\ &= \sum_{|n|\le x}(f-A\tau_\kappa)(n)\sum_{|m|\le \frac{x}{|n|}} \log\frac{x}{|nm|}\Lambda_{\tau_\kappa}(m)\\ &\ll x^\delta(\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}+|A|\log^{(\beta+1)\kappa}x).\end{aligned}$$

We have to estimate

$$\sum_{|n|\le x} R_2(\frac{x}{|n|})\Lambda_{\tau_\kappa}(n) = \sum_{|n|\le x} f*(\Lambda_f-\Lambda_{\tau_\kappa})*\Lambda_{\tau_\kappa}(n).$$

Similarly as above,

$$\sum_{|n|\le x}|M(\frac{x}{|n|})|(\Lambda_f-\Lambda_{\tau_\kappa})*\Lambda_{\tau_\kappa}(n)\ll\sum_{|n|\le x}|M(\frac{x}{|n|})|(\Lambda+\tilde{\Lambda})*\Lambda(n).$$

Further

$$\begin{aligned}\sum_{|n|\le x}\tilde{\Lambda}*\Lambda(n) &\ll x^\delta\sum_{\substack{|p|^\alpha\le x\\ \alpha>1}}\frac{\alpha(n_2-\lambda_0)^\alpha}{|p|^\alpha}\\ &\ll x^\delta.\end{aligned}$$

Therefore, Lemma 5 is applicable and we deduce that

$$\begin{aligned}\sum_{|n|\le x} R_2(\frac{x}{|n|})\Lambda_{\tau_\kappa}(n) \ll& x^{\delta-1}\int_1^x |M(\frac{x}{t})|\log t dt\\ &+\eta|A|x^\delta\log^{(\beta+1)\kappa+1}x\\ &+x^\delta\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}(\max_{1<u<x}|\delta(u)|\log u+1),\end{aligned}$$

as asserted. Here in the last step we used that

$$\begin{aligned}\sum_{|n|\le x}\tau_\kappa*(\Lambda_f-\Lambda_{\tau_\kappa})*\Lambda_{\tau_\kappa}(n) &= \sum_{|n|\le x}(\Lambda_f-\Lambda_{\tau_\kappa})*\Lambda_{\tau_\kappa}*\tau_\kappa(n)\\ &\ll\eta\sum_{|n|\le x}(\frac{x}{|n|})^\delta\log|n|\tau_\kappa(n),\end{aligned}$$

and that

$$\sum_{|n|\le x}\frac{\tau_\kappa(n)}{|n|^\delta}\le\zeta(\delta+\frac{1}{\log x})^\kappa.$$

Since

$$\Lambda_{\tau_\kappa}*\Lambda_{\tau_\kappa}(n)+\Lambda_{\tau_\kappa}(n)\log|n|\ll\Lambda*\Lambda(n)+\Lambda(n)\log|n|$$

for each $n\in\mathcal{G}$, therefore, using Lemma 5 again

$$|\sum_{|n|\le x}M\left(\frac{x}{|n|}\right)(\Lambda_{\tau_\kappa}*\Lambda_{\tau_\kappa}(n)+\Lambda_{\tau_\kappa}(n)\log|n|)\,|\ll$$
$$x^{\delta-1}\int_1^x|M\left(\frac{x}{t}\right)|(\log t)dt+x^\delta\sum_{|n|\le x}\frac{|f(n)|}{|n|^\delta}(\max_{1<u<x}|\delta(u)|\log x+1).$$

Observing

$$\int_1^x|M\left(\frac{x}{t}\right)|(\log t)dt\le x\log x\int\limits_1^x\frac{|M\,(u)\,|}{u^2}du,$$

the proof is finished. □

Proof of Theorem 4. We use the estimation

$$\zeta^\kappa(s)=\mathcal{O}(\frac{1}{|s-\delta|^{(\beta+1)\kappa}}),\tag{4.20}$$

which is valid uniformly for all $|\tau|\ll1$, $\delta+1>\sigma>\delta$.

Lemma 17. *Let $x>2$ and let $f:\mathcal{G}\to\mathbb{C}$ be multiplicative. Assume that $\kappa>\frac{1}{2(\beta+1)}$ and $0\le\eta_0<\kappa$, $0<\lambda_0\le n_2$, $c_0>0$. Suppose that $\tilde{f}(p^\alpha)=\kappa$ for all $p\in\mathcal{P}$ and $\alpha\ge1$ such that $|p|^\alpha>x$ and that* (4.9) *and* (4.10) *are satisfied. We have*

$$F'(s)-A(\zeta^\kappa(s))'\ll\frac{|A|}{|s-\delta|^{(\beta+1)\kappa}}\{\eta\log(2+|s-\delta|\log x)\}\frac{1}{\sigma-\delta}\tag{4.21}$$

uniformly for all $\tau\ll1$, $\delta<\sigma\le\delta+1$, $2<x$, as long as $\eta\log(2+|s-\delta|\log x)\ll1$.

Proof. Since for $\sigma > \delta$

$$\zeta^{\kappa}(s) = \exp(\sum_{|n|>1} \frac{\kappa\Lambda(n)}{|n|^s \log|n|}) \quad \text{and } F(s) = \exp(\sum_{|n|>1} \frac{\tilde{f}(n)\Lambda(n)}{|n|^s \log|n|}),$$

we have

$$\begin{aligned} F(s) - A\zeta^{\kappa}(s) =& \zeta^{\kappa}(s)(\exp(\sum_{|n|>1} \frac{\Lambda(n)(\tilde{f}(n)-\kappa)}{|n|^s \log|n|}) - A) \\ \ll& |\zeta^{\kappa}(s)A||\exp(\sum_{|n|>1} \frac{\Lambda(n)(\tilde{f}(n)-\kappa)}{|n|^s \log|n|} - \sum_{|p|\le x} \frac{f(p)-\kappa}{|p|^\delta}) - 1| \\ \ll& |\zeta^{\kappa}(s)A||\exp(\sum_{|p|\le x}(f(p)-\kappa)(\frac{1}{|p|^s} - \frac{1}{|p|^\delta}) + \sum_{|p|>x} \frac{f(p)-\kappa}{|p|^s} \\ &+ \sum_{\substack{p^\alpha \\ \alpha\ge 2}} \frac{\tilde{f}(p^\alpha)-\kappa}{\alpha|p|^{\alpha s}}) - 1|. \end{aligned} \tag{4.22}$$

We compute

$$\sum_{|p|\le x} |\frac{1}{|p|^s} - \frac{1}{|p|^\delta}|$$

for $\delta \le \sigma \le \delta + 1$, $x > 1$. As usual let $a = \exp(\frac{1}{|s-\delta|})$. Then

$$\begin{aligned} \sum_{|p|\le a} |\frac{1}{|p|^s} - \frac{1}{|p|^\delta}| =& \sum_{|p|\le a} \frac{1}{|p|^\delta}|\exp((s-\delta)\log|p|) - 1| \\ \ll& |s-\delta| \sum_{|p|\le a} \frac{\log|p|}{|p|^\delta} \\ \ll& 1. \end{aligned}$$

For the rest we have

$$\begin{aligned} \sum_{a<|p|\le x} |\frac{1}{|p|^s} - \frac{1}{|p|^\delta}| \le& 2 \sum_{a<|p|\le x} \frac{1}{|p|^\delta} \\ \ll& \int_a^x \frac{1}{u^\delta \log u} d\psi(u) \\ \ll& \log\frac{\log x}{\log a} + 1. \end{aligned}$$

Thus,

$$\sum_{|p|\le x} |\frac{1}{|p|^s} - \frac{1}{|p|^\delta}| \ll \log(2+|s-\delta|\log x) \tag{4.23}$$

uniformly for $\delta \le \sigma \le \delta+1$, $1 < x$. Therefore substituting it into the above inequality we have for all s with $\eta\log(2+|s-\delta|\log x) \ll 1$ that

$$\begin{aligned} F(s) - A\zeta^\kappa(s) \ll & |\zeta^\kappa(s)A|\{\eta\log(2+|s-\delta|\log x)+\Sigma_1\} \\ & \times\exp\{\eta\log(2+|s-\delta|\log x)+\Sigma_1\} \\ \ll & |\zeta^\kappa(s)A|\{\eta\log(2+|s-\delta|\log x)+\Sigma_1\}\exp\{\Sigma_1\}, \end{aligned} \tag{4.24}$$

where by the conditions

$$\begin{aligned} (\Sigma_1(\sigma) =)\Sigma_1 = & \sup_\tau |\sum_{|p|>x} \frac{f(p)-\kappa}{|p|^s} + \sum_{\substack{p^\alpha \\ \alpha\ge 2}} \frac{\tilde{f}(p^\alpha)-\kappa}{\alpha|p|^{\alpha s}}| \\ \ll & \eta. \end{aligned}$$

Let Γ be the circular path surrounding s with radius $(\sigma-\delta)/2$. It is easy to check that the conditions for the above inequality are satisfied for the points of Γ, therefore using Cauchy's theorem and (4.20) we obtain

$$\begin{aligned} F'(s) - A(\zeta^\kappa(s))' = & \int_\Gamma \frac{F(z)-A\zeta^\kappa(z)}{(z-s)^2}dz \\ \ll & \frac{|A|\{\eta\log(2+|s-\delta|\log x)\}}{(\sigma-\delta)^2} \int_\Gamma \frac{1}{|z-\delta|^{(\beta+1)\kappa}}dz \\ \ll & \frac{|A|\{\eta\log(2+|s-\delta|\log x)\}}{\sigma-\delta} \frac{1}{|s-\delta|^{(\beta+1)\kappa}} \end{aligned} \tag{4.25}$$

uniformly for $|\tau| \ll 1$, $\delta < \sigma \le \delta+1$, $1 < x$, $\eta\log(2+|s-\delta|\log x) \ll 1$. Here we used that

$$|s-\delta|/2 \le |s-\delta| - |z-s| \le |s-\delta+z-s| = |z-\delta|,$$

and that similarly

$$|z-\delta| \le 3/2|s-\delta|$$

hold for the points of Γ. □

Let

$$F_0(s) = \exp(\sum_p \frac{|f(p)|}{|p|^s}).$$

Lemma 18. *Let $x > 2$ and let $f : \mathcal{G} \to \mathbb{C}$ be multiplicative. Assume that $\kappa > \frac{1}{2(\beta+1)}$ and $0 \le \eta_0 < \kappa$, $0 < \lambda_0 \le n_2$, $c_0 > 0$. Suppose that $\tilde{f}(p^\alpha) = \kappa$ for all $p \in \mathcal{P}$ and $\alpha \ge 1$ such that $|p|^\alpha > x$ and that (4.9) and (4.10) are satisfied. Let $\theta_p = \arg f(p)$ with $-\pi < \arg z \le \pi$ for all complex numbers z. Assume that there are real numbers θ_0, and $1 > \xi > 0$ such that*

$$|e^{i\theta_0} - e^{i\theta_p}| \ge \xi$$

is satisfied. Let $A > 1$ be an arbitrary large number. Then there are positive constants τ_0, K so that the following inequalities are satisfied for $\delta < \sigma \le \delta + 1$:

$$\frac{|F(s)|}{F_0(\sigma)} \le K \exp\left(-\frac{\xi^3(\kappa - \eta_0)}{64\pi} \log \frac{1}{\sigma - \delta}\right) \tag{4.26}$$

if

$$\tau_0 < |\tau| < (\sigma - \delta)^{-A}, \quad \delta < \sigma \le \delta + 1$$

and

$$\frac{|F(s)|}{F_0(\sigma)} \le K \exp\left(-\frac{\xi^3(\kappa - \eta_0)}{32\pi} \log \left(1 + \frac{|\tau|}{\sigma - \delta}\right)\right) \tag{4.27}$$

if $|\tau| \le \tau_0$, $\delta < \sigma \le \delta + 1$.

Proof. One can follow the proof of [11] Lemma 19.6. By the conditions we have

$$\begin{aligned}\frac{|F(s)|}{F_0(\sigma)} &= |\exp(\sum_{\substack{p^\alpha \\ \alpha \ge 2}} \frac{\tilde{f}(p^\alpha)}{\alpha |p|^{s\alpha}})| \exp(\sum_p \frac{(\Re\{e^{i\theta_p}|p|^{-it}\} - 1)|f(p)|}{|p|^\sigma}) \\ &\le \Sigma_2 \exp(\sum_p \frac{(\Re\{e^{i\theta_p}|p|^{-it}\} - 1)|f(p)|}{|p|^\sigma}),\end{aligned}$$

where by the conditions

$$\begin{aligned}(\Sigma_2(\sigma) =)\Sigma_2 = \sup_\tau \exp(|\sum_{\substack{p^\alpha \\ \alpha \ge 2}} \frac{\tilde{f}(p^\alpha)}{\alpha |p|^{s\alpha}}|) \\ \ll 1.\end{aligned}$$

Let $0 < \varrho \le 1$ be a fixed parameter to be determined later. For $a, b \in \mathbb{R}$ we use the notation

$$|a - b| \pmod{2\pi} := \min_{k \in \mathbb{Z}} |a - b + 2k\pi|.$$

Let $\psi(e^{i\theta}) \in C_{2\pi}(\mathbb{R})$ such that it is zero at $\theta_0 \pm \xi/2$, $\varrho\xi^2/8$ at θ_0, and linear on the intervals between these three points, (mod 2π), and zero otherwise. The Fourier series expansion of ψ is

$$\psi(e^{i\theta}) = \sum_{l\in\mathbb{Z}} a_l e^{il\theta} \tag{4.28}$$

where

$$a_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(e^{i\theta})e^{-i\theta l}d\theta.$$

One has $a_0 = \varrho\frac{\xi^3}{32\pi}$, and integrating by parts gives

$$|a_l| \leq \varrho\frac{8}{\pi l^2}$$

for all $l \neq 0$. Since

$$1 - \Re e^{i\theta_p}|p|^{-i\tau} \geq \frac{\xi^2}{8}$$

if $|\theta_0 - \tau \log p| \pmod{2\pi} \leq \xi/2$ and since it is non-negative, it is at least as large as $\psi(|p|^{i\tau})$. Here we used that

$$|1 - \Re e^{i\theta_p}|p|^{-i\tau}| = \frac{|1 - e^{i\theta_p}|p|^{-i\tau}|^2}{2},$$

and that

$$|e^{-i\tau\log|p|} - e^{i\theta_0}| \leq \xi/2.$$

It follows that

$$\sum_p (1 - \Re\{e^{i\theta_p}|p|^{-i\tau}\})|f(p)||p|^{-\sigma} \geq \sum_p (\kappa - \eta_0)\psi(|p|^{i\tau})|p|^{-\sigma}. \tag{4.29}$$

Substituting (4.28) into (4.29) and rearranging the terms we obtain that the sum here on the right side of (4.29) is

$$(\kappa - \eta_0)\sum_{l\in\mathbb{Z}} a_l \sum_p \frac{1}{|p|^{\sigma+il\tau}}.$$

Since using Lemma 15 and the remark after this lemma we have

$$\log\zeta(s) = \sum_p \frac{1}{|p|^s} + \mathcal{O}(1),$$

using that

$$\sum_l |a_l| = |a_0| + \sum_{l\neq 0} |a_l| \leq \frac{\varrho}{32\pi} + 2\sum_{l=1}^{\infty} \frac{1}{\pi l^2}.$$

we obtain that the right side of (4.29) equals

$$\sum_{l\in\mathbb{Z}} (\kappa - \eta_0) a_l \log \zeta(\sigma - il\tau) + \mathcal{O}(1).$$

Let first $|\tau| > \tau_0 > 0$. If $l \neq 0$ then by Lemma 2 we have

$$|\log \zeta(\sigma - il\tau)| \leq \log(2 + |\tau|) + c_1 \log(2 + |l|)$$

with an appropriate $c_1 > 0$ constant. By choosing $\varrho \leq \frac{3\xi^3}{64\pi^2 A}$ we deduce

$$\begin{aligned} |\sum_{l\neq 0} a_l \log\zeta(\sigma - il\tau)| \leq& \varrho\frac{\pi}{3} \log(2 + |\tau|) + c_2 \\ \leq& \frac{\xi^3}{64\pi} \log \frac{1}{\sigma - \delta} + c_2 \end{aligned}$$

where c_2 is a constant. Here we used that

$$\sum_{l=1}^{\infty} \frac{\log(2+l)}{l^2} \leq \infty \quad \text{and that} \quad \sum_{l=1}^{\infty} \frac{1}{l^2} \leq \frac{\pi^2}{6}$$

and that $|\tau| \leq (\sigma - \delta)^{-A}$. Since

$$a_0 \log \zeta(\sigma) \geq a_0 \log \frac{1}{\sigma - \delta} - c_3 \quad \delta < \sigma \leq \delta + 1$$

for some $c_2 > 0$, (4.26) follows. Now suppose that $|\tau| \leq \tau_0$. If $l|\tau| > \tau_0$ then as above

$$\begin{aligned} \log \zeta(\sigma - il\tau) =& \mathcal{O}(\log(2 + |l||\tau|)) \\ =& -(\beta + 1) \log(\sigma - \delta + |\tau|) + \mathcal{O}(\log(2 + |l|)). \end{aligned} \tag{4.30}$$

If $l|\tau| \leq \tau_0$ and τ_0 is small enough, then with an appropriate $c_4 > 0$

$$\zeta(\sigma - il\tau) = \frac{1}{(\sigma - il\tau - \delta)^{\beta+1}} (c_4 + \mathcal{O}(1)),$$

such that (4.30) remains valid. Thus,

$$\begin{aligned}\sum_{l\in\mathbb{Z}} a_l \log\zeta(\sigma - il\tau) =& a_0 \log\zeta(\sigma) - (\beta+1)\log(\sigma-\delta+|\tau|)\sum_{l\neq 0} a_l + \mathcal{O}(1)\\ =& a_0(\beta+1)\log\frac{1}{\sigma-\delta} - (\psi(1)-a_0)(\beta+1)\log(\sigma-\delta+|\tau|)\\ &+ \mathcal{O}(1)\\ \geq& a_0\log(1+\frac{|\tau|}{\sigma-\delta}) - c_5\end{aligned}$$

for some $c_5 > 0$. Since $\psi(1) \geq 0$, this proves (4.27). □

Lemma 19. *Under the conditions of Lemma 18. we have that*

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\exp\left(-\frac{\xi^3(\kappa-\eta_0)}{32\pi(A+2)}\log\left(1+\frac{|\tau|}{\sigma-\delta}\right)\right) \tag{4.31}$$

uniformly for all $|\tau| \leq (\sigma-\delta)^{-A}$.

Proof of Lemma 19. Since

$$\log(1+\frac{|\tau|}{\sigma-\delta}) \leq (A+2)\log(\frac{1}{\sigma-\delta}) + c$$

holds uniformly for all $|\tau| \leq (\sigma-\delta)^{-A}$, substituting this last inequality into (4.26) we obtain that

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\exp\left(-\frac{\xi^3(\kappa-\eta_0)}{32\pi(A+2)}\log\left(1+\frac{|\tau|}{\sigma-\delta}\right)\right)$$

holds for all $\tau_0 \leq |\tau| \leq (\sigma-\delta)^{-A}$. But the same inequality holds by (4.27) for $|\tau| \leq \tau_0$, and we deduce that (4.31) is valid. □

Define $\beta_y = \exp(r)y$, and $v = \exp(r)$ with $2r = \frac{1}{\eta+1/\log\log x}$. Let

$$H^2(1+y) = \int_{-\infty}^{\infty} |\frac{F'(\delta+y+it) - A(\zeta^\kappa(\delta+y+it))'}{\delta+y+it}|^2 dt. \tag{4.32}$$

In the range $1/\log^{-1} x \leq y \leq v\log^{-1} x$ we treat the integral on the right side for $|t| \leq \beta_y$, $\beta_y < |t| \leq T$ and $T < |t|$ separately where $T = y^{-D}$ with an arbitrary large positive constant D. The integral over these three ranges will be denoted by I_{11}, I_{12}, I_{13} respectively. With $s := \delta + u + it$, considering I_{11} we have that

$$\begin{aligned}\eta\log(2+|s-\delta|\log x) \leq & \eta\log(2+y\log x + yv\log x)\\ \ll & \eta\log v^2\\ \ll & 1,\end{aligned}$$

and

$$y \leq \beta_y \leq v^2/\log x \leq 1.$$

Using (4.21) it follows that

$$\begin{aligned}I_{11} \ll & \frac{\eta^2|A|^2}{y^2}\int_{|t|\leq\beta_y}\frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa(\beta+1)}}dt\\ \ll & \frac{\eta^2|A|^2}{y^2}\int_{|t|\leq y}\frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa(\beta+1)}}dt\\ & + \frac{\eta^2|A|^2}{y^2}\int_{y<|t|\leq\beta_y}\frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa(\beta+1)}}dt.\end{aligned}$$

The first term on the most right hand side above does not exceed

$$2\eta^2|A|^2\frac{\log^2(2+2y\log x)}{y^{2\kappa(\beta+1)+1}},$$

while the integral in the second term for a $\kappa > \frac{1}{2(\beta+1)}$ is at most

$$\begin{aligned}2\int_y^\infty\frac{\log^2(2+2t\log x)}{t^{2\kappa(\beta+1)}}dt \ll & \log^{2\kappa(\beta+1)-1}x\int_{y\log x}^\infty\frac{\log^2(2+2u)}{u^{2\kappa(\beta+1)}}du\\ \ll & \log^2(2+2y\log x)y^{-2\kappa(\beta+1)+1}.\end{aligned}$$

Thus

$$I_{11} \ll \eta^2|A|^2\log^2(2+2y\log x)y^{-2\kappa(\beta+1)-1}.$$

Concerning I_{12}, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}I_{12} \ll & \int_{\beta_y\leq|t|\leq T}|\frac{F'(\delta+y+it)}{\delta+y+it}|^2dt + \int_{\beta_y\leq|t|\leq T}|\frac{A(\zeta^\kappa(\delta+y+it))'}{\delta+y+it}|^2dt\\ = & I_{121}+I_{122}.\end{aligned}$$

Keeping in mind that $F(s) \neq 0$ for $\sigma > \delta$ and using the factorization

$$F'(s) = F(s)\frac{F'(s)}{F(s)},$$

which is valid for all $\Re s > \delta$, we obtain

$$I_{121} \ll \sup_{\beta_y \leq |t| \leq T} |F(\delta + y + it)|^2 \int_{-\infty}^{\infty} |\frac{F'(\delta + y + iu)}{F(\delta + y + iu)(\delta + y + iu)}|^2 du.$$

Since using (4.17)

$$\begin{aligned} L(u) :=& \sum_{|n| \leq u} \tilde{f}(n)\Lambda(n) \\ \ll& u^{\delta}, \end{aligned}$$

by an application of Parseval's identity we deduce,

$$\begin{aligned} \int_{-\infty}^{\infty} |\frac{F'(\delta + y + iu)}{F(\delta + y + iu)(\delta + y + iu)}|^2 du =& 2\pi \int_0^{\infty} |L(e^w)|^2 e^{-2(\delta+y)w} dw \\ \ll& y^{-1}. \end{aligned}$$

Furthermore, the conditions of Lemma 18 are fulfilled. Under these circumstances we can choose $\theta_0 = \pi$, and $v \leq \frac{1-2\eta_0}{2-\eta_0}$ in that Lemma, and using Lemma 19 we have that

$$\sup_{\beta_y \leq |t| \leq T} |F(\delta + y + it)|^2 \ll F_0^2(\delta + y) \exp(-2c \log(1 + y^{-1}\beta_y))$$

with some appropriate positive constant $(c_{D,\kappa,\eta_0,\mathcal{G}} =)c$. With $\epsilon = \kappa - \eta_0$

$$\begin{aligned} F_0(\delta + y) \ll& \exp(\sum_{|p| \leq \exp(y^{-1})} \frac{|f(p)| - \epsilon}{|p|^{\delta+y}})y^{-\epsilon} \\ \ll& \exp(\sum_{|p| \leq x} \frac{|f(p)| - \epsilon}{|p|^{\delta+y}})y^{-\epsilon} \\ \ll& \exp(\sum_{|p| \leq x} \frac{|f(p)| - \epsilon}{|p|^{\delta}})y^{-\epsilon} \end{aligned}$$

uniformly for $1/\log x < y < 1$. Here we used that

$$\sum_{|p| > \exp(y^{-1})} \frac{1}{|p|^{\delta+y}} \ll 1,$$

uniformly for $0 < y \leq 1$ which is a direct consequence of (4.23) and the asymptotic estimations

$$\sum_{|p|\leq u} \frac{1}{|p|^\delta} =(\beta+1)\log\log u + \mathcal{O}(1) \qquad (u>2),$$
$$\zeta(\delta+y+it) =\frac{c_1}{(y+it)^{\beta+1}} + \frac{c_2}{(y+it)^{\beta}} + \mathcal{O}(1) \qquad (0<y\leq 1, |t| \ll 1).$$

Using Lemma 2 we have that for some $\tau_0 > 0$

$$\zeta'(\delta+y+it) \ll |y+it|^{-(\beta+2)}$$

uniformly for $0 < y \leq 1, |t| \leq \tau_0$, and

$$\zeta'(\delta+y+it) \ll |t|^2$$

uniformly for $0 < y \leq 1, |t| \geq \tau_0$ and

$$\zeta(\delta+y+it) \ll y^{-(\beta+1)}$$

uniformly for all t. Applying these estimations in this order we obtain with an appropriate $u_1 > 0$

$$\begin{aligned} I_{122} \ll& |A|^2 \int_{\beta_y\leq|t|\leq 1} \frac{1}{|y+it|^{2\kappa(\beta+1)+2}} dt + |A|^2 \int_{1\leq|t|\leq u_1} t^2 dt \\ &+ |A|^2 y^{-2\kappa(\beta+1)} \int_{u_1\leq|t|} |\frac{\zeta^\kappa(\delta+y+it)'}{\zeta^\kappa(\delta+y+it)(\delta+y+it)}|^2 dt \\ \ll& |A|^2(\beta_y^{-2\kappa(\beta+1)-1} + u_1^3 + y^{-2\kappa(\beta+1)-1}u_1^{-1}). \end{aligned}$$

Here we used that a similar argument as in the proof of Lemma 11 shows that

$$\int_{u_1\leq|t|} |\frac{\zeta^\kappa(\delta+y+it)'}{\zeta^\kappa(\delta+y+it)(\delta+y+it)}|^2 dt \ll u_1^{-1}.$$

Choosing $u_1 = y^{-u_2}$ where $u_2 > 0$ is to be determined later we deduce that

$$\begin{aligned} I_{122} \ll& |A|^2(v^{-2\kappa(\beta+1)-1}y^{-2\kappa(\beta+1)-1} + y^{-3u_2} + y^{-2\kappa(\beta+1)-1}u_1^{-1}) \\ =:& E_{122}. \end{aligned}$$

We arrive at

$$I_{12} \ll \exp(2\sum_{|p|\leq x} \frac{|f(p)|-\epsilon}{|p|^\delta}) y^{-2\epsilon-1} \exp(-2c\log(1+y^{-1}\beta_y)) + E_{122}.$$

For I_{13} we use that since

$$|F(\delta+y+it)| \ll y^{-B} \qquad 0<y\le 1, t\in\mathbb{R}$$

for some $0<B$, using Cauchy's theorem

$$|F'(\delta+y+it)| \ll y^{-B-1} \qquad 0<y\le 1, t\in\mathbb{R}.$$

Choosing the constant D large enough we obtain

$$\sup_t |F'(\delta+y+it)|^2 \int_{T\le|t|} \frac{1}{|\delta+y+it|^2}dt \ll y^B.$$

Similarly, we deduce

$$I_{13} \ll (|A|^2+1)y^B.$$

It remains to estimate $H^2(\delta+y)$ for $v/\log x < y \le 1$. In this range we split the integral appearing on the right hand side of (4.32) into two pieces, which we denote by I_{21} and I_{22}. First we estimate the contribution of $|t|\le T$, then that of $T<|t|$ respectively. A similar computation as by I_{12} shows that

$$I_{21} \ll \exp(2\sum_{|p|\le x}\frac{|f(p)|-\epsilon}{|p|^\delta})y^{-2\epsilon-1} + E_{122},$$

and like by I_{13} we obtain that

$$I_{22} \ll (|A|^2+1)y^B.$$

Putting it all together we deduce that $\int_{1/\log x}^1 H(\delta+y)y^{-1/2}dy$ is at most

$$\begin{aligned}\int_{1/\log x}^1 &[\eta|A|\log(2+2y\log x)y^{-\kappa(\beta+1)-1} \\ &+\exp(\sum_{|p|\le x}\frac{|f(p)|-\epsilon}{|p|^\delta})y^{-\epsilon-1}\exp(-c\log(1+y^{-1}\beta_y)) \\ &+|A|v^{-\kappa(\beta+1)-1}y^{-\kappa(\beta+1)-1}+|A|y^{-3u_2/2-1/2} \\ &+|A|y^{-\kappa(\beta+1)-1+u_2/2}+(|A|+1)y^B]dy \\ +\int_{v/\log x}^1 &[\exp(\sum_{|p|\le x}\frac{|f(p)|-\epsilon}{|p|^\delta})y^{-\epsilon-1}+|A|y^{-\kappa(\beta+1)-1} \\ &+|A|y^{-3u_2/2-1/2}+|A|y^{-\kappa(\beta+1)-1+u_2/2}]dy,\end{aligned}$$

which fixing

$$0 < u_2 < (\kappa(\beta+1) - \frac{1}{2})2/3$$

does not exceed

$$\log^{\kappa(\beta+1)} x\{\eta|A| + \exp(\sum_{|p|\le x} \frac{|f(p)|-\kappa}{|p|^\delta})(\exp(-\frac{c}{\eta}) + \log^{-c} x)\}.$$

Here we used that

$$v^{-1} \ll \begin{cases} \exp(-\frac{1}{2\eta}) & \text{if} 1/\log\log x < \eta \\ \log^{-1/2} x & \text{otherwise.} \end{cases}$$

Applying first Theorem 5 then Lemma 7 the proof is finished. □

Chapter 5

The Theorem of Erdős and Wintner

In this chapter we always suppose that $\mathcal{G}$ satisfies the conditions mentioned at the beginning of Chapter 3 with

$$R(x) \ll x^{\delta} \log^{-\eta} x$$

where $0 < \eta_0 < \eta$. Let $f : \mathcal{G} \to \mathbb{R}$ be an arithmetic function. Let

$$F_x(y) := N(x)^{-1} \#\{n \in \mathcal{G}, |n| \leq x : f(n) \leq y\}.$$

For $x > 1$ $F_x(y)$ is a distribution function. We say that f possesses a limit law if there exists a distribution function F such that

$$F_x(y) \to F(y) \quad (x \to \infty)$$

holds for all continuity points of F. In notation

$$F_x \Longrightarrow F \quad (x \to \infty).$$

f is said to be additive if $f(mn) = f(m) + f(n)$ for all $(n, m) = 1$. Erdős and Wintner proved that in the case of $\mathbb{N}$ f has a limit law if and only if the three series

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f^2(p)}{p}, \quad \sum_{|f(p)| > 1} \frac{1}{p}$$

converge (See for example in [10]). Their work was pioneering in the topic initiated by Hardy and Ramanujan in [22] which is known today as probabilistic number theory. Probabilistic properties of primes and arithmetical functions have been intensively investigated by several authors. For further references see [4, 15, 33]. In this chapter we show that the Erdős Wintner Theorem remains valid for quite general arithmetical semigroups. We have

Theorem 6. *An additive arithmetical function $f : \mathcal{G} \to \mathbb{R}$ possesses a limit law if and only if the three series*

$$\sum_{|f(p)|\le 1} \frac{f(p)}{|p|^\delta}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{|p|^\delta}, \quad \sum_{|f(p)|>1} \frac{1}{|p|^\delta} \tag{5.1}$$

converge. The characteristic function $\psi(t)$ of the limit law is given by the convergent product

$$\psi(t) = \prod_p (1 - \frac{1}{|p|^\delta})(1 + \sum_{\alpha\ge 1} \frac{e^{itf(p^\alpha)}}{|p|^{\delta\alpha}}).$$

The limit law is continuous if and only if

$$\sum_{f(p)\ne 0} \frac{1}{|p|^\delta}$$

diverge.

To sketch the proof of the theorem we need some lemmas.

Lemma 20 (Delange). *Let $g : \mathcal{G} \to \mathbb{C}$ be a multiplicative function with values in the unit disc. Then g possesses a non-zero mean value if and only if*

$$\sum_p \frac{1 - g(p)}{|p|^\delta}$$

converges, and

$$\sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)}{|p|^{\alpha\delta}} \ne -1$$

for all primes p with $|p|^\delta \le 2$. The non-zero mean value is given by the convergent product

$$\prod_p (1 - \frac{1}{|p|^\delta})(1 + \sum_{\alpha\ge 1} \frac{g(p^\alpha)}{|p|^{\delta\alpha}})$$

Proof. The lemma is an easy consequence of the generalization of the Theorem of Halász in Chapter 3 and the fact that

$$\sum_{x<|p|\le ax} \frac{1}{|p|^\delta} = o(1) \quad (x \to \infty)$$

holds for all $a > 1$. □

Scatch of the proof of Theorem 6. The characteristic function of F_x equals

$$\psi_x(t) = N(x)^{-1} \sum_{|n|\le x} e^{i\tau f(n)}.$$

Suppose first that F_x has a limit law. By the continuity theorem of Lévy [44] (Theorem 2.4) $\psi_x(t)$ converges uniformly for all bounded values of t to a characteristic function, say ψ. Since ψ is continuous and $\psi(0) = 1$, with an appropriate T we have $|\psi(t)| > 1/2$ for all $|t| < T$. Thus using Lemma 20 we obtain that

$$\sum_p \frac{1 - e^{itf(p)}}{|p|^\delta}$$

converges. Therefore

$$\psi(t) = \prod_p (1 - \frac{1}{|p|^\delta})(1 + \sum_{\alpha\ge 1} \frac{e^{itf(p^\alpha)}}{|p|^{\delta\alpha}}).$$

From this representation we easily obtain that the three series in (5.1) converge. Conversely suppose that the three series converge. The uniform convergence of

$$\sum_p \frac{1 - e^{itf(p)}}{|p|^\delta}$$

for $|t| \le T$ is immediate. Thus using Lemma 20 and then the Theorem of Lévy again, we obtain that the limit law exists. By another theorem of Lévy we have that the limit law is continuous if and only if

$$\sum_{f(p)\ne 0} \frac{1}{|p|^\delta}$$

diverges. □

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