Lagrangian Solutions to Systems of Real Principal Type

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Gutachter:	Prof. Dr. Sönke Hansen
	Prof. Dr. Gunther Uhlmann
	Prof. Dr. Hermann Sohr

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Contents

1. Introduction 1
2. Preliminaries 4
2.1. Systems of Pseudodifferential Operators 4
2.2. Lagrangian Distributions 8
2.3. Systems of Real-Principal Type 11
3. Lagrangian Solutions 17
3.1. Statement of the Results 17
3.2. Special Inhomogeneous Equations. 19
3.3. Proof of the Theorems 24
4. Application to the Elastodynamics Equation 26
References 34

1. Introduction

We consider the wave equation

$$\Box v = c^{-2}(x)\,\partial_t^2 v - \Delta v = 0,$$

on $\mathbb{R}_t \times \mathbb{R}_x^3$. This is the prototype of the class of hyperbolic operators, which describe wave-like propagation phenomena.

To find solutions to the wave equation, one can try the classical ansatz of geometrical optics: Consider a function

$$v(t,x) = a(x,\omega)\,e^{i\omega(\phi(x)-t)}, \quad a(x,\omega) = \sum_{k=0}^{\infty}(i\omega)^{-k}a_k(x). \tag{1.1}$$

with amplitude a. Here the principal part a_0 of the amplitude should be unequal to 0.

By inserting (1.1) into the wave equation, an elementary calculation yields the following two conditions: ϕ needs to solve the eikonal equation,

$$c\,|\nabla\phi| = 1,$$

and a_0 needs to solve the transport equation,

$$0 = 2\,(\nabla\phi\cdot\nabla)a_0 + \Delta\phi\,a_0,$$

here ∇ is the formal differential operator $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$; $\nabla\phi$ is the gradient of λ, $\nabla\phi\cdot\nabla$ is the derivation in $\nabla\phi$-direction and so forth. If these two equations are satisfied, one has that $\Box v = O(\omega^{2-2}) = O(\omega^0)$ for $\omega \to \infty$.

The linear transport equation can be solved by reduction to ordinary differential equations along rays, which are the orthogonal trajectories of the wavefronts $\phi =$ constant. By iterating this method with special inhomogeneous equations, one gets an asymptotic solution v such that for all $k \in \mathbb{N}$

$$\Box v = O(\omega^{2-k}) \text{ for } \omega \to \infty. \tag{1.2}$$

In general, the ansatz of geometrical optics does not provide global solutions. In so-called caustics this method breaks down, cf. for example Duistermaat [2, section 5.2].

The ansatz (1.1) translates into special Lagrangian distributions $u \in I^0(X,\Lambda)$: Here $X := \mathbb{R}_t \times \mathbb{R}_x^3$ and u is of the form

$$u(t,x) = (2\pi)^{-3/2}\int e^{i\omega(\varphi(x)-t)}a(x,\omega)\,d\omega.$$

Λ is the Lagrangian submanifold

$$\{(t,x,\tau,\xi) \in T^\star X\backslash 0 \mid t = \varphi(x),\ \tau = -\omega \text{ and } \xi = -\tau\nabla\phi(x)\},$$

where $T^\star X$ is the cotangent bundle of X. The solvability condition (1.2) translates into

$$\Box u \in C^\infty(X, \Omega^{\frac{1}{2}}). \tag{1.3}$$

A solution to the caustics problem was first given by Maslov [8] in 1965. His ideas were included in the development of general Lagrangian distributions, which sense no difficulties with caustics. This is part of the modern theory of linear partial differential equations; the most extensive presentation of this is by Hörmander [6]. In the set-up of this theory, it is natural to look for solutions in the larger class of Lagrangian distributions.

The wave equation belongs to the class of (scalar) real-principal-type operators. The solution theory of these operators has been thoroughly treated by Hörmander and Duistermaat in 1972, see for example their original work [3, section 6] or Hörmander [6, section 26.1]. Their results did not cover systems of equations.

In 1982 Dencker [1] generalized the real-principal-type property to systems of pseudodifferential operators and studied their propagation of singularities. Several important equations from physics classify as such systems of real-principal type, for example the Maxwell-equations of electrodynamics and the Lamé-equations of isotropic elastodynamics.

In this thesis, we investigate Lagrangian solutions to general systems of real-principal type. In particular, we derive the generalized transport equation for these systems, which comprises a quantitative description of the propagation of amplitudes along rays, the bicharacteristic curves. We shall show the necessity and the sufficiency of the transport equation, for Lagrangian solutions. Many of the techniques we use come from Dencker's paper.

This is the structure of the following sections: The preliminaries are given in section 2: In subsections 2.1 and 2.2, we state some facts about systems of pseudodifferential operators and Lagrangian distributions. Subsection 2.3 deals with systems of real-principal type, according to Dencker. We shall show that the system of isotropic elastodynamics is of real-principal type.

The reader who is familiar with the topics of section 2 might directly head to section 3: The statement of the main results is to be found in subsection 3.1. Subsection 3.2 deals with special inhomogeneous equations. The results are required for the proof of the main theorems in subsection 3.3.

Finally, in section 4, we apply the results to the elastodynamics system. We determine the transport equation and we show that it corresponds to a result of Karal and Keller [7], if applied to the geometrical optics ansatz.

I wish to thank my supervisor, Professor Sönke Hansen, for his excellent support and guidance during the work on this thesis.

2. Preliminaries

The purpose of this section is to state definitions, notations and results, to make this treatise more self-contained. It is practical if the reader has some knowledge about the modern theory of linear partial differential operators, namely about distributions, (scalar) pseudodifferential operators, Fourier integral operators and generally the methods of microlocal analysis. The most extensive presentation of this theory is by Hörmander [6].

Manifolds and vector bundles are always meant to be C^∞.

2.1. **Systems of Pseudodifferential Operators.** We assume that the pseudodifferential operators we use are properly supported and polyhomogeneous; the latter means that their full symbol is an asymptotic sum of homogeneous terms.

Hörmander [6][Definition 18.1.32] defines systems of pseudodifferential operators, that act between sections of vector bundles. We restate his definition using frames of the bundles:

Definition 2.1. Let E and F be complex vector bundles over a manifold X. A pseudodifferential operator of order m, from sections of E to sections of F, is a continuous linear map

$$P : C_0^\infty(X, E) \to C^\infty(X, F),$$

that satisfies the following local condition: For every open $Y \subseteq X$, with local frames

$$e_1, \ldots, e_{N_E} : Y \to E|_Y \text{ and } f_1, \ldots, f_{N_F} : Y \to F|_Y,$$

there is an $N_F \times N_E$-matrix of pseudodifferential operators $P_{ij} \in \Psi^m(Y)$, such that for all $u \in C_0^\infty(Y, E)$, $u(x) = \sum_j u_j(x)e_j(x)$

$$(Pu(x))_i = \sum_j (P_{ij}u_j)(x), \quad x \in Y. \tag{2.1}$$

We shall then write $P \in \Psi^m(X; E, F)$.

We call the matrix (P_{ij}) the trivialization of P, according to the chosen frames.

Example 2.1. The special case of trivial vector bundles.

A pseudodifferential operator $P \in \Psi^m(X; \mathbb{C}^N, \mathbb{C}^M)$ corresponds to its trivialization, an $M \times N$-matrix of operators $P_{ij} \in \psi^m(X)$. The image of $u = (u_1, \ldots, u_N) \in C_0^\infty(X, \mathbb{C}^N)$ is given by

$$(Pu)_i = \sum_j P_{ij}u_j.$$

Example 2.2. Operators $P \in \Psi^m(X; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$, which act between the half-density bundle. For a definition of the one-dimensional half-density bundle, cf. Hörmander [6, vol. III, page 92].

Let $Y \subseteq X$ with local coordinates $x = (x_1, \ldots, x_n) : Y \to \mathbb{R}^n$. The corresponding frame of the half-density bundle is usually denoted by

$$|dx|^{\frac{1}{2}} : Y \to \Omega^{\frac{1}{2}}|_Y.$$

Now $P \in \Psi^m(X; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$ if and only if, for every such choice of local coordinates, there exists a $P' \in \Psi^m(Y)$ such that for every $u \in C_0^\infty(Y, \Omega^{\frac{1}{2}})$, $u(x) = u'(x)\,|dx|^{\frac{1}{2}}$,

$$Pu(x) = P'u'(x)\,|dx|^{\frac{1}{2}}.$$

Next, we observe how the definition of the principal symbol carries over to systems of pseudodifferential operators:

Theorem 2.1. *Let $P \in \Psi^m(X; E, F)$; the principal symbol $\sigma^0(P)$ of P is invariantly defined as an element of*

$$S^m(T^\star X, Hom(\hat{E}, \hat{F})).$$

Here $\hat{E}, \hat{F}$ are the vector bundles over $T^\star X$ with fiber at $\gamma \in T^\star X$ equal to the fiber of E, F at $\pi(\gamma)$; $\pi : T^\star X \to X$ is the projection of the cotangent bundle.

Proof. Let $\gamma \in T^\star X$ and $x := \pi(\gamma) \in X$. Let $v \in E_x$, the fiber of E over x; we need to define $p(\gamma)v \in F_x$.

We choose frames

$$e_1, \ldots, e_{N_E} : Y \to E|_Y, \quad f_1, \ldots, f_{N_F} : Y \to F|_Y.$$

in a neighbourhood $Y \subseteq X$ of x and write v in the form $v = \sum_i v_i\, e_i(x)$. Let (P_{ij}) be the trivialization of P, according to these bases and let $p_{ij} \in S(T^\star_Y X)$ be the principal symbol of any $P_{ij} \in \psi^m(Y)$. We define

$$(p(\gamma)v)_i := \sum_j p_{ij}(\gamma)\, v_j. \tag{2.2}$$

The following calculation shows that this definition is invariant under changes of the frames: We choose an $u \in C_0^\infty(Y, E)$ with $v = u(x)$ and

$\phi \in C^\infty(X)$ such that $\gamma = \phi'(x)$, then

$$\begin{aligned}
&\lim_{t\to\infty} t^{-m}\, e^{-it\phi(x)}\, P(e^{it\phi}u)(x) \\
=&\lim_{t\to\infty} t^{-m}\, e^{-it\phi(x)}\, \sum_i (\sum_j P_{ij}(e^{it\phi}u_j))\, f_i \\
=&\sum_i (\sum_j \lim_{t\to\infty} t^{-m}\, e^{-it\phi(x)} P_{ij}(e^{it\phi}u_j))\, f_i \\
=&\sum_i (\sum_j p_{ij}(\gamma)\, v_j) f_i \\
=&\,p(\gamma)v.
\end{aligned} \tag{2.3}$$

Here we used a formula for the principal symbol of the scalar operators P_{ij}, which follows from the so-called Fundamental Asymptotic Expansion Lemma, that is for example treated by Taylor [10, page 184 ff.].

Now the invariance of the principal symbol follows from the invariance of the first term in (2.3). $\square$

We shall write $S^m(T^\star X, \mathrm{Hom}(E,F))$ instead of $S^m(T^\star X, \mathrm{Hom}(\hat{E},\hat{F}))$, for ease of notation.

Let (P_{ij}) be the trivialization of P, according to an arbitrary choice of local frames. Then equation (2.2) means that the trivialization of $\sigma^0(P)$, according to the same choice of local bases, is equal to the matrix (p_{ij}).

Operators $A \in \psi^m(X)$ have an asymptotic expansion of its full symbol $\sigma(A)$ in the form

$$\sigma(A) \sim a_m + a_{m-1} + a_{m-2} + \dots\,,$$

with unique, i-homogeneous symbols $a_i = \sigma_i(A)$. In general, the principal symbol a_m is the only one of these which is invariantly defined as an element of $S^m(T^\star X)$.

Operators $A \in \psi^m(X; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$ have a unique principal symbol $a \in S^m(T^\star X)$, too. In addition one gets an invariant subprincipal symbol $\sigma^s(A) \in S^{m-1}(T^\star X)$ for them, which is, in local coordinates (x,ξ) on $T^\star X$, given by

$$a_{m-1} - \frac{1}{2i}\sum_{j=1}^n \partial_{x_j}\partial_{\xi_j} a,$$

cf. Hörmander [6, Theorem 18.1.33] or Duistermaat [2, Proposition 4.3.1].

In section 3 we shall calculate with the trivializations of systems of pseudodifferential operators. Therefore, we need the following, easy consequence of the calculus of scalar pseudodifferential operators.

Lemma 2.2. *Let $X \subseteq \mathbb{R}^n$ open. Let $A = (A_{ij}) \in \psi^m(X; \mathbb{C}^N, \mathbb{C}^N)$, $B = (B_{ij}) \in \psi^n(X; \mathbb{C}^N, \mathbb{C}^N)$ be pseudodifferential operators with principal symbols $a = (a_{ij})$, $b = (b_{ij})$ and matrices of subprincipal symbols $a^s = (\sigma^s(A_{ij}))$, $b^s = (\sigma^s(B_{ij}))$ respectively. Then*

$$AB \in \psi^{m+n}(X; \mathbb{C}^N, \mathbb{C}^N),$$

its principal symbol is equal to ab and its matrix of subprincipal symbols

$$\sigma^s(AB) = ab^s + a^s b + \frac{1}{2i}\{a, b\}. \tag{2.4}$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket

$$\{a, b\} = \sum_j \partial_{\xi_j} a \, \partial_{x_j} b - \partial_{x_j} a \, \partial_{\xi_j} b.$$

Proof. The operator AB is given by the matrix with the entries

$$(AB)_{ij} = \sum_k A_{ik} B_{kj};$$

every entry is an element of $\psi^{m+n}(X)$.

The expansion-formula for the full symbol of a product of scalar pseudodifferential operators, cf. Hörmander [6, Theorem 18.1.8] or Folland [4, Theorem 8.37], easily carries over to these special systems:

$$\sigma(AB) \sim \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(A)(x, \xi) D_x^\alpha \sigma(B)(x, \xi),$$

here the differentiations are component-wise. As a direct consequence we get that ab is the principal symbol of AB.

An elementary calculation then yields equation (2.4):

$$
\begin{aligned}
&ab^s + a^s b + \frac{1}{2i}\{a,b\} \\
&= ab_{n-1} + a_{m-1}b - \frac{1}{2i}\sum_{j=1}^{n}(a(\partial_{x_j}\partial_{\xi_j}b) + (\partial_{x_j}\partial_{\xi_j}a)b) + \frac{1}{2i}\{a,b\} \\
&= \left(ab_{n-1} + a_{m-1}b + \frac{1}{i}\sum_{j=1}^{n}\partial_{\xi_j}a\,\partial_{x_j}b\right) \\
&\quad - \frac{1}{i}\sum_{j=1}^{n}\partial_{\xi_j}a\,\partial_{x_j}b - \frac{1}{2i}\sum_{j=1}^{n}(a(\partial_{x_j}\partial_{\xi_j}b) + (\partial_{x_j}\partial_{\xi_j}a)b) \\
&\quad + \frac{1}{2i}\sum_{j=1}^{n}(\partial_{\xi_j}a\,\partial_{x_j}b - \partial_{x_j}a\,\partial_{\xi_j}b) \\
&= \sigma_{m+n-1}(ab) \\
&\quad - \frac{1}{2i}\sum_{j=1}^{n}(a(\partial_{x_j}\partial_{\xi_j}b) + (\partial_{x_j}\partial_{\xi_j}a)b + \partial_{x_j}a\,\partial_{\xi_j}b + \partial_{\xi_j}a\,\partial_{x_j}b) \\
&= \sigma_{m+n-1}(ab) - \frac{1}{2i}\sum_{j=1}^{n}\partial_{x_j}\partial_{\xi_j}(ab) \\
&= \sigma^s(AB)
\end{aligned}
$$

□

The formula for the principal symbol of AB above remains valid in the general case of operators between sections of vector bundles.

2.2. **Lagrangian Distributions.**

Lagrangian distributions are invariantly defined by Hörmander [6, Definition 25.1.1]:

Definition 2.2. Let X be a manifold, $\Lambda \subseteq T^\star X\backslash 0$ a closed, conic Lagrangian submanifold and E a complex vector bundle over X. The space $I^m(X,\Lambda;E)$ of Lagrangian distribution sections of E, of order m, is the set of all $u \in \mathfrak{D}'(X,E)$, such that

$$L_1 \dots L_K\, u \in {}^\infty H^{\mathrm{loc}}_{(-m-n/4)}(X,E),$$

for all $K \in \mathbb{N}$ and all properly supported $L_j \in \psi^1(X;E,E)$, with principal symbols $\sigma^0(L_j)$ vanishing on Λ.

These distributions are characterized microlocally, as oscillatory integrals, in [6, Theorem 25.1.5]:

Theorem 2.3. *Let $\phi(x,\theta)$ be a non-degenerate phase function in an open, conic neighbourhood of $(x_0,\theta_0) \in \mathbb{R}^n \times (\mathbb{R}^N \backslash 0)$, such that*

$$(x_0,\theta_0) \in C = \{(x,\theta) \mid \phi'_\theta(x,\theta) = 0\}.$$

We set $\xi_0 := \phi'_x(x_0,\theta_0)$. Let ϕ parametrize the Lagrangian manifold Λ in a neighbourhood U of (x_0,ξ_0):

$$\Lambda \cap U = \{(x,\xi) \mid \xi = \phi'_x(x,\theta) \textit{ for any } (x,\theta) \in C\}.$$

If $a \in S^{m+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$ has support in the interior of a sufficiently small, conic neighbourhood Γ of (x_0,θ_0), then the oscillatory integral

$$u(x) = (2\pi)^{-(n+2N)/4} \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta \tag{2.5}$$

defines a distribution $u \in I^m(\mathbb{R}^n,\Lambda)$.

Conversely, every Lagrangian distribution $u \in I^m(\mathbb{R}^n,\Lambda)$ with $WF(u)$ in a small conic neighborhood of (x_0,θ_0) can, modulo C^∞, be written in the form (2.5).

Here, the case $N = 1$ corresponds to the ansatz of geometrical optics.

The amplitude a, in the microlocal representation (2.5), leads to an invariant definition of principal symbols; cf. Hörmander [6, Theorem 25.1.9] or Duistermaat [2, Definition 4.1.1]:

Theorem 2.4. *There exists an isomorphism*

$$\begin{aligned} I^m(X,&\Lambda;\Omega_X^{\frac{1}{2}} \otimes E)/I^{m-1}(X,\Lambda;\Omega_X^{\frac{1}{2}} \otimes E) \\ &\to S^{m+\frac{n}{4}}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E})/S^{m+\frac{n}{4}-1}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E}), \end{aligned}$$

where $\hat{E}$ is the lifting of the bundle E to Λ. The image under this map is called the principal symbol.

M_Λ is the Maslov bundle on Λ; cf. Hörmander [6, Definition 21.6.5].

The principal symbol of $u \in I^m(X,\Lambda;\Omega^{\frac{1}{2}} \otimes E)$ is given by

$$\Lambda \ni \rho \mapsto < u, e^{-i\psi(\cdot,\rho)}\chi >,$$

Here $\chi \in C_0^\infty(X;\Omega^{\frac{1}{2}} \otimes E^\star)$ and $\psi \in C^\infty(X,\Lambda)$ with $\psi(\pi(\rho),\rho) \equiv 0$ and $\psi'_x(\pi(\rho),\rho) = \rho$, π is the projection of the cotangent bundle. We shall abbreviate

$$S^{m+\frac{n}{4}}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E})$$

to $S^{m+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E)$.

The next two theorems have been derived from the calculus of Fourier integral operators, to the special case of pseudodifferential operators and Lagrangian distributions. Theorem 25.2.3 in Hörmander [6] and

Theorem 4.2.2 in Duistermaat [2] comprise the behavior of Lagrangian distributions, under operation with pseudodifferential operators:

Theorem 2.5. *Let $P \in \psi^m(X; \Omega^{\frac{1}{2}} \otimes E, \Omega^{\frac{1}{2}} \otimes F)$ be a pseudodifferential operator with principal symbol $p \in S^m(T^\star X, Hom(E, F))$ and let $u \in I^\mu(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ be a Lagrangian distribution with principal symbol $\sigma^0(u) = w \in S^{\mu+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E)$. Then*

$$Pu \in I^{m+\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F)$$

and its principal symbol is

$$\sigma^0(Pu) = pw.$$

To be strict, we had to write $\sigma^0(Pu) = \mathrm{id}_M \otimes \mathrm{id}_{\Omega^{\frac{1}{2}}} \otimes p|_\Lambda(w)$ above, but the given short form is common.

Theorem 25.2.4 in Hörmander [6] comprises the next Theorem, which is essential for the derivation of the transport equation in section 3:

Theorem 2.6. *Let $P \in \psi^m(X; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$ be a pseudodifferential operator with principal symbol $p \in S^m(T^\star X)$ and subprincipal symbol $p^s \in S^{m-1}(T^\star X)$. Let $u \in I^\mu(X, \Lambda; \Omega^{\frac{1}{2}})$ be a Lagrangian distribution with principal symbol $w \in S^{\mu+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}})$. If*

$$\Lambda \subseteq Char P = \{\gamma \in T^\star X \backslash 0 \mid p(\gamma) = 0\},$$

then

$$Pu \in I^{m+\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}})$$

and its principal symbol, of this lower order, is

$$\frac{1}{i} \mathfrak{L}_{H_p} w + p^s w. \tag{2.6}$$

Here, $\mathfrak{L}_{H_p}$ is the Lie derivative of half densities, with respect to the vector field H_p on Λ: Let $a \in C^\infty(T^\star X, \Omega^{\frac{1}{2}})$ be a section of the half density bundle. Let $(x, \xi) = ((x', x''), (\xi', \xi'')) : U \to \mathbb{R}^{2n}$ be local coordinates in $T^\star X$ such that $(x', \xi'') : U \cap \Lambda \to \mathbb{R}^n$ are arbitrary local coordinates in Λ. Then $a|_U = a' \, |dx' \, d\xi''|^{\frac{1}{2}}$, with a suitable trivialization $a' \in C^\infty(U)$, and

$$\mathfrak{L}_{H_p}(a' \, |dx' \, d\xi''|^{\frac{1}{2}}) = (H_p a' + \frac{1}{2} \mathrm{div}\,(H_p) a') \, |dx' \, d\xi''|^{\frac{1}{2}}.$$

For the invariance of this definition under changes of coordinates cf. Hörmander [6, vol. IV, page 22].

The notation of the principal symbol in (2.6) is abbreviated; to be precise, one would write $\mathrm{id}_{M_\Lambda} \otimes \mathfrak{L}_{H_p}|_\Lambda(w)$ instead of $\mathfrak{L}_{H_p} w$.

2.3. **Systems of Real-Principal Type.** Let X be an n-dimensional manifold. A scalar pseudodifferential operator Q on X is of real-principal type, if its principal symbol q is real and, in canonical local coordinates (x,ξ) on $T^\star X$, its Hamilton field

$$H_q = \sum_{j=1}^{n} \partial_{\xi_j} q \, \partial_{x_j} - \partial_{x_j} q \, \partial_{\xi_j}$$

is never proportional to the radial vector

$$\sum_{j=1}^{n} \xi_j \, \partial_{\xi_j},$$

on the characteristic set $\{q(x,\xi) = 0, \, \xi \neq 0\}$. The last condition on q implies that $dq \neq 0$, if $q = 0$.

The most important special case of scalar real-principal-type operators are those, whose principal symbol q satisfies $q'_\xi \neq 0$ on $\{q = 0\}$.

Dencker [1] expanded the real-principal-type property to systems $P \in \psi^m(X; \mathbb{C}^N, \mathbb{C}^N)$ and noted that his results immediately carry over to the somewhat more general case of operators $P \in \psi^m(X; E, F)$, with complex vector bundles E, F over X. We formulate things directly in the latter case.

Definition 2.3. Let E and F be complex vector bundles over X. An operator $P \in \psi^m(X; E, F)$ with homogeneous principal symbol

$$p \in S^m(T^\star X, \mathrm{Hom}(E, F)),$$

is of real-principal type at $\gamma \in T^\star X \backslash 0$ if it satisfies the following two conditions:

1. For $l \in \mathbb{N}$ arbitrarily, there exist symbols $\tilde{p} \in S^l(T^\star X, \mathrm{Hom}(F, E))$ and $q \in S^{l+m}(T^\star X)$, q of real-principal type, such that, in a neighbourhood U of γ,
$$\tilde{p} p = qI,$$
where I is pointwise the identity on the fibers.
2. The conic, closed characteristic set
$$\mathrm{Char}\, P = \{\gamma \in T^\star X \backslash 0 \mid \det p(\gamma) = 0\}$$
is, locally in U, equal to
$$\{\gamma \in T^\star X \backslash 0 \mid q(\gamma) = 0\}.$$

We say that P is of real-principal type in $\Omega \subseteq T^\star X \backslash 0$, if it is so at every $\gamma \in \Omega$.

The condition on $\operatorname{Char} P$ implies that it is locally a hypersurface with non-radial Hamilton field.

Since p is homogeneous, $\tilde{p}$ and q can be chosen homogeneous, too. Therefore the set where P is of real-principal type is conical and open in $T^\star X \backslash 0$.

If P is elliptic, i.e. $\det p \neq 0$, it is trivially of real-principal type; take $\tilde{p} = qp^{-1}$. Thus, in general, we only have to check the existence of $\tilde{p}$, q microlocally on $\operatorname{Char} P$.

The condition $\tilde{p}p = qI$ is equivalent to $p\tilde{p} = qI$: This is trivial if $q \neq 0$, since then $\tilde{p} = qp^{-1}$ holds, and follows from continuity in the case $q = 0$, because $dq \neq 0$.

Dencker [1, Proposition 3.2] gives a more geometric characterization of real-principal-type operators, which is independent from the choice of symbols $\tilde{p}$ and q:

Theorem 2.7. *A pseudodifferential operator $P \in \Psi^m(X; E, F)$ is of real-principal type at $\gamma \in Char P$ if and only if the following two conditions are satisfied in a neighborhood of γ:*

1. *$Char P$ is a hypersurface with non-radial Hamilton field; the dimension of $Ker p$ is constant on $Char P$.*
2. *Let π be the quotient bundle-mapping*
$$\pi : F \to F/Im\, p = Coker\, p$$
and $\rho \in N(Char P)$, the normal bundle of $Char P$. Then
$$\pi\, \partial_\rho p|_{Ker p} : Ker\, p \to Coker\, p$$
is a bijection, on $Char P$.

Proof. The proof is rather technical. We first show the necessity of the two conditions:

Let $\tilde{p} \in S^l(T^\star X, \operatorname{Hom}(F, E))$ and $q \in S^{l+m}(T^\star X)$, q of real-principal type, be symbols with
$$\operatorname{Char} P = \{q = 0\}$$
and
$$\tilde{p}p = qI \tag{2.7}$$
in an open neighbourhood U of γ. We can assume that U is sufficiently small, in order to get a chart
$$(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) : T^\star U \to \mathbb{R}^{2n}.$$
with $q(x, \xi) = \xi_n$. We calculate in these local coordinates, on
$$\operatorname{Char} P = \{q(x, \xi) = \xi_n = 0\}.$$

From (2.7) we get that $\operatorname{Im} p \subseteq \operatorname{Ker} \tilde{p}$ and

$$\operatorname{Rank}(p) + \operatorname{Rank}(\tilde{p}) \leq N. \tag{2.8}$$

We differentiate (2.7) in the ξ_n-direction:

$$(\partial_{\xi_n} \tilde{p}) p + \tilde{p}(\partial_{\xi_n} p) = I. \tag{2.9}$$

It follows that

$$\operatorname{Rank}(p) + \operatorname{Rank}(\tilde{p}) \geq N.$$

Together with (2.8), we have that

$$\operatorname{Rank}(p) + \operatorname{Rank}(\tilde{p}) = N \text{ and } \operatorname{Im} p = \operatorname{Ker} \tilde{p}.$$

The rank of a symbol is lower semi-continuous, so

$$\operatorname{Rank} p = N - \operatorname{Rank} \tilde{p}$$

is continuous and integer valued. Thus the dimension of $\operatorname{Ker} p$ is locally constant.

By equation (2.9)

$$\tilde{p}\, \partial_{\xi_n} p|_{\operatorname{Ker} p} = I \text{ and consequently } \tilde{p}\, \partial_{\xi_n} p(\operatorname{Ker} p) = \operatorname{Ker} p;$$

the equality $\operatorname{Ker} \tilde{p} = \operatorname{Im} p$ therefore yields $\partial_{\xi_n} p(\operatorname{Ker} p) \cap \operatorname{Im} p = \{0\}$ and

$$\operatorname{Dim}(\operatorname{Ker} p) = \operatorname{Dim}(\pi\, \partial_{\xi_n} p(\operatorname{Ker} p)) \leq \operatorname{Dim}(\operatorname{Coker} p). \tag{2.10}$$

Together with the fact that

$$\operatorname{Dim}(\operatorname{Coker} p) = \operatorname{Dim}(F/\operatorname{Im} p) = N - \operatorname{Dim}(\operatorname{Im} p) = \operatorname{Dim}(\operatorname{Ker} p),$$

we get equality in (2.10).

To prove the sufficiency of conditions 1 and 2, we can choose a symbol $q \in S^{l+m}(T^\star X)$ of real-principal type, such that, in an open neighbourhood U of γ, we have

$$\operatorname{Char} P = \{q = 0\}.$$

Again, we can assume that U is sufficiently small, in order to get a chart

$$(x, \xi) : U \to \mathbb{R}^{2n},$$

where $\xi = (\xi', \xi_n)$ with $q(x, \xi) = \xi_n$. We calculate in these local coordinates and abbreviate $(x, (\xi', 0))$ to (x, ξ').

Taylor-expansion yields

$$p(x, \xi) = p(x, \xi') + \xi_n\, \partial_{\xi_n} p(x, \xi') + \mathrm{O}(\xi_n^2). \tag{2.11}$$

Condition 2 means that

$$\pi\, \partial_{\xi_n} p|_{\operatorname{Ker} p} : \operatorname{Ker} p \to \operatorname{Coker} P$$

is invertible, on $\operatorname{Char} P = \{(x, \xi')\}$. Therefore we can define

$$\tilde{p}^0(x, \xi') := [(\pi\, \partial_{\xi_n} p|_{\operatorname{Ker} P})^{-1}\, \pi]\, (x, \xi') : F \to \operatorname{Ker} p.$$

Then

$$\tilde{p}^0(x, \xi')\, p(x, \xi') \equiv 0,$$

because $\pi\, p \equiv 0$, and

$$\tilde{p}^0(x, \xi')\, \partial_{\xi_n} p(x, \xi')\, v = v \quad \text{for } v \in \operatorname{Ker} p(x, \xi').$$

The last equation implies that

$$[I - \tilde{p}^0 \partial_{\xi_n} p](x, \xi')\, v = \begin{cases} 0 & \text{if } v \in \operatorname{Ker} p \\ v - \tilde{p}^0(x, \xi')\, \partial_{\xi_n} p(x, \xi')\, v & \text{if } v \notin \operatorname{Ker} p \end{cases}$$

and we can choose a $\tilde{p}^1$, defined in $\operatorname{Char} P$, such that

$$\tilde{p}^1 p = I - \tilde{p}^0 \partial_{\xi_n} p.$$

Equation (2.11) yields

$$\begin{aligned} &[\tilde{p}^0(x, \xi') + \xi_n\, \tilde{p}^1(x, \xi')]\, p(x, \xi) \\ &= \tilde{p}^0(x, \xi') p(x, \xi') + \xi_n E(x, \xi) \\ &= \xi_n E(x, \xi) \\ &= q(x, \xi) E(x, \xi), \end{aligned}$$

with

$$E(x, \xi) = \tilde{p}^0 \partial_{\xi_n} p + \tilde{p}^0 O(\xi_n) + \tilde{p}^1 p + \xi_n \tilde{p}^1 \partial_{\xi_n} p + \tilde{p}^1 O(\xi_n^2)$$

which is equal to $\tilde{p}^0 \partial_{\xi_n} p + \tilde{p}^1 p = I$ in $\operatorname{Char} P$, so E is elliptic in a neighbourhood. Without restriction, E is elliptic in U. Define

$$\tilde{p}(x, \xi) := E^{-1}(x, \xi)\, (\tilde{p}^0(x, \xi') + q(x, \xi)\, \tilde{p}^1(x, \xi')),$$

then $\tilde{p} p = qI$. □

The following, easy consequence shows the connection between kernel and image of $\tilde{p}$ and p.

Corollary 2.8. *Let $P \in \Psi^m(X; E, F)$ be of real-principal type, with homogeneous principal symbol*

$$p \in S^m(T^\star X, Hom(E, F)).$$

Let

$$\tilde{p} \in S^l(T^\star X, Hom(F, E))$$

and

$$q \in S^{l+m}(T^\star X),$$

of real-principal type, be homogeneous symbols with

$$\tilde{p}p = qI \ \text{ and } \ Char\,P = \{q = 0\},$$

in a conical, open set $\Gamma \subseteq T^{\star}X\backslash 0$. *Then*

$$Ker\,\tilde{p} = Im\,p \ \text{ and } \ Ker\,p = Im\,\tilde{p} \quad \text{in } Char\,P \cap \Gamma. \tag{2.12}$$

Proof. Let $\gamma \in \operatorname{Char} P \cap \Gamma$. The equality

$$\operatorname{Ker} \tilde{p}(\gamma) = \operatorname{Im} p(\gamma)$$

was shown in the proof of the necessity in Theorem (2.7). The symmetry

$$\tilde{p}p = qI = p\tilde{p}$$

then yields the second equality. □

We turn towards some examples:

Example 2.3. Let $P \in \Psi^m(X; \mathbb{C}^N, \mathbb{C}^N)$ be an operator, with principal symbol

$$p = \begin{pmatrix} qI_K & 0 \\ 0 & I_{N-K} \end{pmatrix},$$

where $0 \leq K \leq N$ and q is an arbitrary real-principal-type symbol. P is of real-principal type in $T^{\star}X\backslash 0$; take

$$\tilde{p} = \begin{pmatrix} I_K & 0 \\ 0 & qI_{N-K} \end{pmatrix}.$$

Every system of real-principal type can microlocally be transformed to this form, by multiplication with elliptic systems, cf. Dencker [1, page 359].

Dencker [1] also shows that Maxwell's equations correspond to a system of real-principal type. Another important example is the Lamé-equation of isotropic elastodynamics; this is used by Rachele [9] and, in the more general case with residual stress, by Hansen and Uhlmann [5]:

Example 2.4. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, let $0 < \rho \in C^\infty(\overline{\Omega})$; we consider an elastic medium with density ρ in Ω. The linear differential operator

$$L : C^\infty(\mathbb{R}_t \times \Omega, \mathbb{C}^3) \to C^\infty(\mathbb{R}_t \times \Omega, \mathbb{C}^3)$$

of isotropic elastodynamics is given by

$$\begin{aligned} Lv = -\rho\, \partial_t^2 v + (\lambda + \mu)\, \nabla(\nabla \cdot v) + \mu\, \nabla^2 v + (\nabla \cdot v)(\nabla\lambda) \\ + (\nabla\mu) \times (\nabla \times v) + 2\, (\nabla\mu \cdot \nabla)v, \end{aligned}$$

with Lamé parameters $\lambda, \mu \in C^\infty(\overline{\Omega})$, $\lambda > 0$; these parameters represent the elasticity of the medium.

The displacement of the medium is a time-dependent vector field $v(t,\cdot)$ on $\overline{\Omega}$. Small displacements satisfy, in a source-free medium, the homogeneous equation

$$Lv \equiv 0.$$

We want to determine the full symbol of L. Let $a \in \mathbb{C}^3$; an elementary calculation shows that

$$\begin{aligned}
&e^{-i(\tau t+\xi x)} L(e^{i(\tau t+\xi x)} a) \\
&= \rho\tau^2 a - (\lambda+\mu)(a\cdot\xi)\,\xi - \mu\,|\xi|^2 a \\
&\quad + i\,(\xi\cdot a)\nabla\lambda + i\,(\nabla\mu\cdot a)\,\xi + i\,(\nabla\mu\cdot\xi)\,a.
\end{aligned}$$

Therefore the full symbol of L is

$$\sigma(L) = l + l_1$$

with principal symbol

$$l = (\rho\tau^2 - \mu|\xi|^2)I - (\lambda+\mu)\,\xi\otimes\xi$$

and

$$l_1 = i\,(\nabla\mu\cdot\xi)I + i\,\nabla\lambda\otimes\xi + i\,\xi\otimes\nabla\mu.$$

The elastodynamics operator is of real-principal type: We define two scalar real-principal type symbols q_p, q_s by

$$\begin{aligned}
q_s(t,x,\tau,\xi) &:= \rho(x)\,\tau^2 - \mu(x)|\xi|^2 \\
q_p(t,x,\tau,\xi) &:= \rho(x)\,\tau^2 - (\lambda(x)+2\mu(x))|\xi|^2.
\end{aligned}$$

Further let

$$\pi := \frac{\xi\otimes\xi}{|\xi|^2},\ \pi\,a = \frac{\xi\cdot a}{|\xi|^2}\,\xi$$

be the orthogonal projection in ξ-direction, then

$$l = q_s(I-\pi) + q_p\pi.$$

If we take

$$\tilde{l} := q_p(I-\pi) + q_s\pi,$$

we get

$$\tilde{l}l = q_s q_p I.$$

So we can choose $q := q_s q_p$.

3. Lagrangian Solutions

In this section, let X be an n-dimensional manifold and E, F be complex, N-dimensional vector bundles over X. Let

$$P \in \Psi^m(X; \Omega^{\frac{1}{2}} \otimes E, \Omega^{\frac{1}{2}} \otimes F)$$

be a pseudodifferential operator of real-principal type, with principal symbol

$$p \in S^m(T^\star X, \mathrm{Hom}(E, F)).$$

3.1. **Statement of the Results.**

Let $\Lambda \subseteq T^\star X \backslash 0$ be a closed, conic, Lagrangian submanifold. We are looking for Lagrangian distributions

$$u \in I^\mu(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E),$$

with non-zero principal symbol

$$w \in S^{\mu+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E),$$

that solve the homogeneous equation

$$Pu \in C^\infty(X, \Omega^{\frac{1}{2}} \otimes F) \text{ or } Pu \equiv 0 \mod I^{-\infty}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F).$$

We always factor out a half-density bundle here, because this is appropriate for the symbol calculus of Lagrangian distributions, cf. Theorem 2.4.

In particular, we demand that u solves the equation to the highest order, $Pu \equiv 0 \mod I^{m+\mu-1}$, which means that the principal symbol $\sigma^0(Pu) = p\,w$ vanishes, cf. Theorem 2.5. That implies the condition

$$\Lambda \subseteq \mathrm{Char}\, P = \{\gamma \in T^\star X \backslash 0 \mid \det p(\gamma) = 0\} \text{ and } w \in \mathrm{Ker}\, p,$$

which generalizes the eikonal equation in the ansatz of geometrical optics.

Next, we want to declare the generalized transport equation for P. We express it microlocally, by using the real-principal-type property: Let $\gamma \in \Lambda$ arbitrarily. Then there exist homogeneous symbols

$$\tilde{p} = \tilde{p}_\gamma \in S^l(T^\star X, \mathrm{Hom}(F, E))$$

and

$$q = q_\gamma \in S^{l+m}(T^\star X)$$

such that

$$\tilde{p}p = qI,$$

in a conical neighbourhood Γ of γ. By choosing local frames of the involved bundles, over a coordinate neighbourhood of γ, we can trivialize the operator P and the symbols $\tilde{p}$, p, q and w. Let $p^s := \sigma^s(P)$ be the subprincipal-symbol matrix of P, according to such a trivialization. We define a linear partial differential operator

$$T = T_{p,\tilde{p},q} : S^{\mu+\frac{n}{4}}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E) \to S^{l+m+\mu-1+\frac{n}{4}}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)$$

microlocally by

$$Tw = \frac{1}{i}H_q w + \frac{1}{2i}\mathrm{div}(H_q)w + \frac{1}{2i}\{\tilde{p},p\}w + \tilde{p}p^s w. \tag{3.1}$$

Definition 3.1. Let r be the trivialization of a symbol in

$$S^{l+m+\mu-1+\frac{n}{4}}(\Lambda, M\otimes\Omega^{\frac{1}{2}}\otimes E).$$

We say that w satisfies the inhomogeneous, microlocal transport equation in $\gamma\in\Lambda$, with respect to the right side

$$r \in S^{m+\mu-1+\frac{n}{4}}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E),$$

if, for every choice of $\tilde{p}=\tilde{p}_\gamma$ and $q=q_\gamma$ and every choice of trivializations, over a coordinate neighbourhood of γ, there exists a conic neighbourhood $\Gamma_1\subseteq\Gamma$ of γ, such that w solves the linear pde

$$Tw = \tilde{p}\,r \text{ in } \Gamma_1.$$

We say that w satisfies the homogeneous, microlocal transport equation in γ, if it satisfies the inhomogeneous, microlocal transport equation in γ, with respect to the right side $r=0$.

Let $M\subseteq\Lambda$. We say that w satisfies the homogeneous or inhomogeneous, microlocal transport equation in M, if w satisfies the respective equation in every $\gamma\in M$.

Remark 3.1. If $e\in S^{\cdot}(X,\Lambda)$ is elliptic in Γ, then

$$T_{p,e\tilde{p},eq} = e\,T_{p,\tilde{p},q}.$$

That means, the transport equation is invariant under changes of the choice of functions $\tilde{p}$ and q.

Now we are able to state the main results. The first theorem shows that the principal symbol of u must necessarily satisfy the transport equation:

Theorem 3.1. *Let $\Lambda\subseteq \mathit{Char}\,P$ be a closed, conic, Lagrangian submanifold. Let $u\in I^\mu(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)$ be a Lagrangian distribution, with principal symbol $w\in \mathit{Ker}p$, that solves*

$$Pu \equiv 0 \mod I^{-\infty}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes F).$$

Then, w satisfies the homogeneous, microlocal transport equation in Λ.

Conversely, the second theorem shows the sufficiency of the transport equation, if Λ is the bicharacteristic flow-out of a suitable submanifold:

Theorem 3.2. *Let $\Lambda \subseteq \mathit{Char}\, P$ be a closed, conic, Lagrangian submanifold. Let $\Lambda_0 \subseteq \Lambda$ be a conic submanifold of codimension 1, such that any bicharacteristic curve in Λ intersects Λ_0 transversal and exactly once. Let*

$$w \in S^{\mu+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E),$$

be a homogeneous symbol that maps into $\mathit{Ker} p$ and satisfies the homogeneous, microlocal transport equation in Λ. Then there exists a Lagrangian distribution $u \in I^{\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$, with principal symbol w, that solves

$$Pu \equiv 0 \mod I^{-\infty}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F).$$

In particular, one can always find a non-trivial Lagrangian solution $u \in I^{\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ to this equation.

The additional condition on Λ is used to assure the global solvability of the transport equation. For that purpose, one can start with arbitrary, homogeneous values on Λ_0.

The proof of Theorems 3.1 and 3.2 is given in section 3.3. We prepare these proofs with several auxiliary results in section 3.2.

3.2. **Special Inhomogeneous Equations.** Let $\gamma \in \Lambda$ arbitrarily. Let $\tilde{p} = \tilde{p}_\gamma \in S^l(T^\star X, \mathrm{Hom}(F, E))$ and $q = q_\gamma \in S^{l+m}(T^\star X)$ be homogeneous symbols, such that $\tilde{p}p = qI$ in a conical neighbourhood Γ of γ. We choose an operator

$$\tilde{P} = \tilde{P}_\gamma \in \Psi^l(X; \Omega^{\frac{1}{2}} \otimes F, \Omega^{\frac{1}{2}} \otimes E)$$

with principal symbol $\tilde{p}$ and define

$$Q := \tilde{P}P \in \Psi^{l+m}(X; \Omega^{\frac{1}{2}} \otimes E, \Omega^{\frac{1}{2}} \otimes E).$$

Let $D^q \in \psi^{l+m}(X; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$ be a scalar pseudodifferential operator with principal symbol q and vanishing subprincipal symbol; let

$$D^q I \in \Psi^{l+m}(X; \Omega^{\frac{1}{2}} \otimes \mathbb{C}^N, \Omega^{\frac{1}{2}} \otimes \mathbb{C}^N)$$

be the diagonal operator which is defined by operating with D^q on the N half-density components of any

$$v \in C_0^\infty(X, \Omega^{\frac{1}{2}} \otimes \mathbb{C}^N) = (C_0^\infty(X, \Omega^{\frac{1}{2}}))^N.$$

We transform Q into the diagonal operator $D^q I$:

Lemma 3.3. *There exists a pseudodifferential operator*

$$B \in \psi^0(X; \Omega^{\frac{1}{2}} \otimes \mathbb{C}^N, \Omega^{\frac{1}{2}} \otimes E)$$

such that

$$QB \equiv B\,D^q I \quad \bmod \psi^{l+m-2}(X;\Omega^{\frac{1}{2}}\otimes\mathbb{C}^N,\Omega^{\frac{1}{2}}\otimes E), \tag{3.2}$$

and B is elliptic, in $\Gamma_1 \cap Char\,P$, where $\Gamma_1 \subseteq \Gamma$ is a conical neighbourhood of γ.

Proof. We translate (3.2) into an equation for the homogeneous principal symbol

$$b \in S^0(T^\star X, \mathrm{Hom}(\mathbb{C}^N, E))$$

of B. We calculate in Γ. The principal symbols of QB and of BD^qI are both equal to qb; therefore equation (3.2) is equivalent to

$$\sigma^0(QB - B\,D^q I) = 0; \tag{3.3}$$

the left side of this equation is meant to be the principal symbol of $QB - B\,D^qI$, as operator of order $l+m-1$.

We evaluate this equation locally: Let $U \subseteq T^\star X$ be an open coordinate neighbourhood of γ, with local frames of $\Omega^{\frac{1}{2}}$ and E over U. Then we calculate with the corresponding trivializations of the operators und symbols. Lemma 2.2 yields that (3.3) is equivalent to

$$\begin{aligned}
0 &= \sigma^s(QB) - \sigma^s(BD^qI)\\
&= q\,\sigma^s(B) + \sigma^s(Q)\,b + \frac{1}{2i}\{qI, b\} - (b\,\sigma^s(D^qI) + q\,\sigma^s(B) + \frac{1}{2i}\{b, qI\})\\
&= \sigma^s(Q)\,b + \frac{1}{i}\{qI, b\}\\
&= \sigma^s(Q)\,b + \frac{1}{i}H_q b.
\end{aligned}$$

Here we used the fact that $\sigma^s(D^qI) = 0$.

Therefore (3.2) is equivalent to the following linear, first-order pde, in a conical neighbourhood of γ:

$$H_q\,b = \frac{1}{i}\,\sigma^s(Q)\,b. \tag{3.4}$$

We get a 0-homogeneous, elliptic C^∞ solution b to this equation in the intersection of $\mathrm{Char}\,q$ with a conical neighbourhood of γ, by locally solving linear, first-order ordinary differential equations along the bicharacteristic curves. For that purpose, one can start with arbitrary, elliptic 0-homogeneous values on a suitable, conical hypersurface transversal to H_q. □

Transformation (3.2) enables us to deduce the principal symbol of Q applied to a Lagrangian distribution. Later, this provides the transport equation.

Lemma 3.4. *Let $\Lambda \subseteq Char\, P$ be a closed, conic, Lagrangian submanifold. Let $u \in I^\mu(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ be a Lagrangian distribution with principal symbol $w \in Ker p$. Then*

$$Qu \in I^{l+m+\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E).$$

and, after choosing local frames of the bundles over a coordinate neighbourhood of γ, the trivialization of its principal symbol is equal to

$$Tw = \frac{1}{i} H_q w + \frac{1}{2i} div\,(H_q) w + \tilde{p} p^s w + \frac{1}{2i} \{\tilde{p}, p\} w,$$

in the intersection of $Char\, P$ with a conical neighbourhood of γ.

Proof. Let B be the transformation operator of Lemma 3.3, B^{-1} its microlocal parametrix and b, b^{-1} their principal symbols accordingly. Without restriction we can assume that (3.2) is valid in $\mathrm{Char}\, P \cap \Gamma$. Then the principal symbol of Qu is

$$\begin{aligned} \sigma^0(Qu) &= \sigma^0(QB\,(B^{-1}u)) \\ &= \sigma^0(B(D^q I)(B^{-1}u)) \\ &= b\,\sigma^0((D^q I)(B^{-1}u)) \\ &= \frac{1}{i} b\,\mathfrak{L}_{H_q}(b^{-1}w), \end{aligned}$$

For the last equation, notice that we've got the scalar operator D^q acting on the N components of

$$B^{-1}u \in I^\mu(X, \Lambda; \Omega^{\frac{1}{2}} \otimes \mathbb{C}^N) = (I^\mu(X, \Lambda; \Omega^{\frac{1}{2}}))^N.$$

In this situation, we can apply Theorem 2.6 to get a formula for the required principal symbol of one order lower; take into account that D^q has vanishing subprincipal symbol.

We choose local frames of the involved bundles, over a coordinate neighbourhood $U \subseteq T^\star X$ of γ, and calculate with the corresponding trivializations of the symbols in a conical, open neighbourhood $\Gamma_2 \subseteq \Gamma$ of γ:

We eliminate b and b^{-1} from the last term. Observe that

$$0 = H_q(b\, b^{-1}) = H_q b\, b^{-1} + b\, H_q b^{-1}.$$

Together with equation (3.4) we get

$$\frac{1}{i} b\, H_q b^{-1} w = i H_q b\, b^{-1} w = \sigma^s(Q) w = \tilde{p} p^s\, w + \frac{1}{2i} \{\tilde{p}, p\}\, w;$$

the last equation is Lemma 2.2 and $w \in \operatorname{Ker} p$. Therefore

$$\begin{aligned}
\sigma^0(Qu) &= \frac{1}{i} b(H_q(b^{-1}w) + \frac{1}{2}\operatorname{div}(H_q) b^{-1} w) \\
&= \frac{1}{i} b(H_q b^{-1}\, w + b^{-1} H_q w) + \frac{1}{2i}\operatorname{div}(H_q) w) \\
&= \frac{1}{i} H_q w + \frac{1}{2i}\operatorname{div}(H_q) w + \tilde{p} p^s w + \frac{1}{2i}\{\tilde{p}, p\} w.
\end{aligned}$$

□

The transport equation has non-zero solutions in the Lagrangian manifold Λ, if Λ is the bicharacteristic flow-out of a suitable submanifold:

Lemma 3.5. *Let $f \in I^{m+\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F)$. Let $\Lambda \subseteq \operatorname{Char} P$ be a closed, conic, Lagrangian submanifold. Let $\Lambda_0 \subseteq \Lambda$ be a conic submanifold of codimension 1, such that any bicharacteristic curve in Λ intersects Λ_0 transversal and exactly once. Then there exists a non-zero symbol*

$$w \in S^{\mu + \frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E)$$

that maps into $\operatorname{Ker} p$ and solves the inhomogeneous, microlocal transport equation in Λ, with respect to the right side $\sigma^0(f)$.

Proof. We get a μ-homogeneous C^∞ solution w of the transport equation by solving linear, first-order ordinary differential equations along the bicharacteristic curves. For that purpose, we can start with arbitrary μ-homogeneous values in $\operatorname{Ker} p$, on the conic submanifold $\Lambda_0 \subseteq \Lambda$.

To verify that this method yields a solution w which maps into $\operatorname{Ker} p$, let $\gamma = \gamma(t)$ be a bicharacteristic curve of H_q in $\operatorname{Char} P$ and assume that $w \in \operatorname{Ker} p$ at $\gamma_0 = \gamma(t_0) \in \Lambda_0$. We shall show that w maps into $\operatorname{Ker} p$, on all of γ:

$$\begin{aligned}
0 = q\,\sigma^0(f) &= p\tilde{p}\,\sigma^0(f) = p\,Tw \\
&= \frac{1}{i} p H_q w + \frac{1}{2i}\operatorname{div}(H_q)(pw) + q p^s w + \frac{1}{2i}\, p\{\tilde{p}, p\} w \\
&= \frac{1}{i} p H_q w + \frac{1}{2i}\operatorname{div}(H_q)(pw) + \frac{1}{2i}\, p\{\tilde{p}, p\} w.
\end{aligned}$$

Dencker [1, page 366] shows that

$$p\{\tilde{p}, p\} = 2H_q p + \{\tilde{p}, p\} p,$$

which yields

$$0 = \frac{1}{i} H_q(pw) + \frac{1}{2i}\operatorname{div}(H_q)(pw) + \frac{1}{2i}\{\tilde{p}, p\}(pw).$$

This means that pw solves a first-order ordinary differential equation along γ with initial value $pw(\gamma_0) = 0$; by uniqueness we get that $pw \equiv 0$ on γ. $\square$

The next result is an easy consequence of Lemma 3.4. It implies Theorem 3.1 in the special case $f \equiv 0$.

Theorem 3.6. *Let $\Lambda \subseteq \mathit{Char}\, P$ be a closed, conic, Lagrangian submanifold. Let $f \in I^{m+\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F)$ and $u \in I^{\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ be Lagrangian distributions, the latter with principal symbol $w \in \mathit{Ker}\, p$. If u solves*

$$Pu \equiv f \mod I^{m+\mu-2}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F) \tag{3.5}$$

then w satisfies the inhomogeneous, microlocal transport equation in Λ, with respect to the right side $\sigma^0(f)$.

Proof. By applying $\tilde{P}$ to both sides of equation (3.5) we get

$$Qu \equiv \tilde{P}f \mod I^{l+m+\mu-2}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E).$$

Therefore

$$\sigma^0(Qu) = \sigma^0(\tilde{P}f) = \tilde{p}\sigma^0(f)$$

and Lemma 3.4 yields the claim. $\square$

We will iterate the following converse result, in the proof of Theorem 3.2.

Theorem 3.7. *Let $\Lambda \subseteq \mathit{Char}\, P$ be a closed, conic, Lagrangian submanifold. Let $f \in I^{m+\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F)$ be a Lagrangian distribution and*

$$w \in S^{\mu+\frac{n}{4}}(\Lambda, M \otimes \Omega^{\frac{1}{2}} \otimes E)$$

a homogeneous symbol that maps into $\mathit{Ker}\, p$ and satisfies the inhomogeneous, microlocal transport equation in Λ, with respect to the right side $\sigma^0(f)$. Then there exists a Lagrangian distribution

$$u \in I^{\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$$

with principal symbol w that solves

$$Pu \equiv f \mod I^{m+\mu-2}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F). \tag{3.6}$$

Proof. Let $u' \in I^{\mu}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ be a Lagrangian distribution with principal symbol w. We shall show that an $u'' \in I^{\mu-1}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes E)$ exists, such that

$$P(u' + u'') \equiv f \mod I^{m+\mu-2}(X, \Lambda; \Omega^{\frac{1}{2}} \otimes F).$$

From the assumptions on w and Lemma 3.4, we get that microlocally $\sigma^0(Qu') = \tilde{p}\,\sigma^0(f)$, which is equivalent to

$$\begin{aligned} Qu' - \tilde{P}f &\in I^{l+m+\mu-2}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)\\ \Leftrightarrow \tilde{P}(Pu'-f) &\in I^{l+m+\mu-2}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)\\ \Leftrightarrow \sigma^0(Pu'-f) &\in \mathrm{Ker}\ \tilde{p}; \end{aligned}$$

here σ^0 is the principal-symbol mapping for Lagrangian distributions in $I^{m+\mu-1}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes F)$. Corollary 2.8 yields

$$\mathrm{Ker}\ \tilde{p} = \mathrm{Im}\ p$$

microlocally in Λ, which implies that

$$\sigma^0(Pu'-f) \in \mathrm{Im}\ p$$

on all of Λ. So we find an $u'' \in I^{\mu-1}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)$ with

$$\sigma^0(Pu'-f) = p(-\sigma^0(u'')) = \sigma^0(-Pu''),$$

which implies the desired equation. $\square$

The inhomogeneous equation (3.6) is always solvable if Λ is the bicharacteristic flow-out of a suitable submanifold:

Corollary 3.8. *Let $\Lambda \subseteq \mathit{Char}\,P$ be a closed, conic, Lagrangian submanifold. Let $\Lambda_0 \subseteq \Lambda$ be a conic submanifold of codimension 1, such that any bicharacteristic curve in Λ intersects Λ_0 transversal and exactly once. Let $f \in I^{m+\nu-1}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes F)$ be a Lagrangian distribution. Then there exists a Lagrangian distribution*

$$u \in I^{\nu}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes E)$$

that solves

$$Pu \equiv f \mod I^{m+\nu-2}(X,\Lambda;\Omega^{\frac{1}{2}}\otimes F).$$

Proof. This is Theorem 3.7 combined with Lemma 3.5. $\square$

3.3. **Proof of the Theorems.**

Proof of Theorem 3.1. Follows from Theorem 3.6, with $f \equiv 0$. $\square$

We iterate the results of Theorem 3.7 and Corollary 3.8 in the next proof:

Proof of Theorem 3.2. Taking $f \equiv 0$, we get from Theorem 3.7 a Lagrangian distribution $u_1 \in I^{\mu}(X, M \otimes \Omega^{\frac{1}{2}} \otimes E)$ with principal symbol w, that solves

$$Pu_1 \in I^{m+\mu-2}(X, M\otimes\Omega^{\frac{1}{2}}\otimes F).$$

We set $f_1 := -Pu_1$. An application of Corollary 3.8, with $\nu = \mu - 1$, yields an $u_2 \in I^{\mu-1}(X, M \otimes \Omega^{\frac{1}{2}} \otimes E)$ with

$$P(u_1 + u_2) = Pu_2 - f_1 \in I^{m+\mu-3}(X, M \otimes \Omega^{\frac{1}{2}} \otimes F).$$

By iteration we get a sequence of such u_i with

$$P\left(\sum_{i=1}^{k} u_i\right) \in I^{m+\mu-(k+1)}(X, M \otimes \Omega^{\frac{1}{2}} \otimes F).$$

From these u_i, we can compose an $u \in I^{\mu}(X, M \otimes \Omega^{\frac{1}{2}} \otimes E)$, whose full symbol is the asymptotic sum

$$\sigma(u) \sim \sum_{i=1}^{\infty} \sigma(u_i).$$

Then the principal symbol $\sigma^0(u)$ is equal to $\sigma^0(u_1) = w$ and

$$Pu \in I^{-\infty}(X, M \otimes \Omega^{\frac{1}{2}} \otimes F).$$

$\square$

4. Application to the Elastodynamics Equation

We use the notations of example 2.4.

To apply the results of section 3, we interpret the differential operator

$$L : C^\infty(\mathbb{R} \times \Omega, \mathbb{C}^3) \to C^\infty(\mathbb{R} \times \Omega, \mathbb{C}^3)$$

of isotropic elastodynamics,

$$\begin{aligned} Lv = -\rho\,\partial_t^2 v + (\lambda + \mu)\,\nabla(\nabla \cdot v) + \mu\,\nabla^2 v + (\nabla \cdot v)(\nabla\lambda) \\ + (\nabla\mu) \times (\nabla \times v) + 2\,(\nabla\mu \cdot \nabla)v, \end{aligned}$$

as the trivialization of an operator $P \in \psi^2(\mathbb{R} \times \Omega; \Omega^{\frac{1}{2}} \otimes \mathbb{C}^3, \Omega^{\frac{1}{2}} \otimes \mathbb{C}^3)$.

In example 2.4, we have seen that the full symbol of L is

$$\sigma(L) = l + l_1$$

with principal symbol

$$l = (\rho\tau^2 - \mu|\xi|^2)I - (\lambda + \mu)\,\xi \otimes \xi$$

and

$$l_1 = i\,(\nabla\mu \cdot \xi)I + i\,\nabla\lambda \otimes \xi + i\,\xi \otimes \nabla\mu.$$

Recall that L is of real-principal type: If we define $q_s := \rho\,\tau^2 - \mu|\xi|^2$ and $q_p := \rho\,\tau^2 - (\lambda + 2\mu)|\xi|^2$ we get $l = q_s(I - \pi) + q_p\pi$. Then $\tilde{l}l = qI$ holds for $\tilde{l} := q_p(I - \pi) + q_s\pi$ and $q := q_s q_p$.

First, we evaluate the terms of the transport equation in this special case, namely the matrix $l^s = \sigma^s(L)$, of subprincipal symbols, the term $\tilde{l}\,l^s$ and the Poisson-bracket $\{\tilde{l}, l\}$:

Lemma 4.1. *The subprincipal-symbol matrix l^s of L is given by*

$$2i\,l^s = \xi \otimes \nabla(\lambda - \mu) - \nabla(\lambda - \mu) \otimes \xi. \tag{4.1}$$

Proof. From the equation

$$\partial_{x_j}\partial_{\xi_j}(l) = -2(\partial_{x_j}\mu)\xi_j\,I - \partial_{x_j}(\lambda + \mu)\,(e_j \otimes \xi + \xi \otimes e_j),$$

where $e_j = (\delta_{ij})_{1 \le i \le 3}$ is the j-th unit vector of the canonical basis of $\mathbb{R}^3$, we get

$$\sum_j \partial_{x_j}\partial_{\xi_j}(l) = -2(\nabla\mu \cdot \xi)I - (\nabla\lambda + \nabla\mu) \otimes \xi - \xi \otimes (\nabla\lambda + \nabla\mu).$$

Therefore

$$\begin{aligned} l^s &= l^1 - \frac{1}{2i}\sum_j \partial_{x_j}\partial_{\xi_j}(l) \\ &= i(\nabla\mu\cdot\xi)I + i\,\nabla\lambda\otimes\xi + i\,\xi\otimes\nabla\mu \\ &\quad - i(\nabla\mu\cdot\xi)I + \frac{1}{2i}(\nabla\lambda+\nabla\mu)\otimes\xi + \frac{1}{2i}\,\xi\otimes(\nabla\lambda+\nabla\mu) \\ &= \frac{1}{2i}\left(\xi\otimes(\nabla\lambda-\nabla\mu) - (\nabla\lambda-\nabla\mu)\otimes\xi\right). \end{aligned}$$

□

Lemma 4.2. *The subprincipal-symbol term of the transport equation satisfies*

$$\tilde{l}l^s\,\pi = 0 \ \textit{on}\ \mathit{Char}\,q_p$$

and

$$\tilde{l}l^s\,(I-\pi) = 0 \ \textit{on}\ \mathit{Char}\,q_s.$$

Proof. On $\mathrm{Char}\,q_p$

$$\tilde{l} = q_p(I-\pi) + q_s\pi \ = q_s\pi$$

and, with the abbreviation $b := \nabla\lambda - \nabla\mu$, we get from Lemma 4.1 that

$$2i\,\tilde{l}l^s\,\pi = q_s\,\pi(\xi\otimes b - b\otimes\xi)\pi.$$

The matrix $\xi\otimes b - b\otimes\xi$ is skew-symmetric and

$$\pi = \frac{\xi\otimes\xi}{|\xi|^2}$$

is symmetric; therefore $\pi(\xi\otimes b - b\otimes\xi)\pi$ is skew-symmetric, too. The rank of the latter matrix is even and ≤ 1, because the rank of π is equal to 1; therefore it has to be 0. So $\tilde{l}l^s\,\pi = 0$ on $\mathrm{Char}\,q_p$.

Analogous, we get that $\tilde{l} = q_p\,(I-\pi)$ holds on $\mathrm{Char}\,q_s$, and

$$\begin{aligned} 2i\,\tilde{l}l^s\,(I-\pi) &= q_p\,(I-\pi)(\xi\otimes b - b\otimes\xi)(I-\pi) \\ &= -q_p\,(I-\pi)(b\otimes\xi)(I-\pi). \end{aligned}$$

The last equation implies that the rank of the skew-symmetric matrix $(I-\pi)(\xi\otimes b - b\otimes\xi)(I-\pi)$ is ≤ 1; then it has to be 0. In consequence, $\tilde{l}l^s\,(I-\pi) = 0$ on $\mathrm{Char}\,q_s$. □

Lemma 4.3. *The Poisson-bracket term of the transport equation satisfies*

$$\{\tilde{l}, l\}\pi = -2q_s\,H_{q_p}\pi\,\pi + \{q_s, q_p\}\pi \ \textit{on}\ \mathit{Char}\,q_p \tag{4.2}$$

and

$$\{\tilde{l}, l\}(I-\pi) = 2q_p\, H_{q_s}\pi(I-\pi) + \{q_p, q_s\}(I-\pi) \ \text{on } Char\, q_s. \quad (4.3)$$

Proof. We calculate on $\mathrm{Char}\, q_p$:

From $\pi^2 = \pi$ we get

$$\{rI, \pi\} = \{rI, \pi^2\} = \{rI, \pi\}\pi + \pi\{rI, \pi\},$$

which implies the following two equations:

$$\begin{aligned} \pi\{rI, \pi\} &= \{rI, \pi\}(I-\pi), \\ (I-\pi)\{rI, \pi\} &= \{rI, \pi\}\pi. \end{aligned}$$

Then we get

$$\begin{aligned} \{\tilde{l}, l\} &= \{q_p(I-\pi) + q_s\pi, q_s(I-\pi) + q_p\pi\} \\ &= \{q_p(I-\pi), q_s(I-\pi)\} + \{q_p(I-\pi), q_p\pi\} \\ &\quad + \{q_s\pi, q_s(I-\pi)\} + \{q_s\pi, q_p\pi\}. \\ \{q_p(I-\pi), q_s(I-\pi)\}\pi &= q_s(I-\pi)\{q_pI, I-\pi\}\pi \\ &= -q_s(I-\pi)\{q_pI, \pi\}\pi \\ &= -q_s\{q_pI, \pi\}\pi \\ &= -q_sH_{q_p}\pi\,\pi. \\ \{q_p(I-\pi), q_p\pi\}\pi &= 0, \text{ because } q_p = 0. \\ \{q_s\pi, q_s(I-\pi)\}\pi &= q_s\pi\{q_sI, I-\pi\}\pi \\ &= -q_s\pi\{q_sI, \pi\}\pi \\ &= -q_s\{q_sI, \pi\}(I-\pi)\pi \\ &= 0. \\ \{q_s\pi, q_p\pi\}\pi &= \{q_s, q_p\}\pi + q_s\{\pi, q_pI\}\pi \\ &= \{q_s, q_p\}\pi - q_sH_{q_p}\pi\,\pi. \end{aligned}$$

Summed up this gives equation (4.2).

Equation (4.3) follows by symmetry if we exchange π with $I-\pi$ and q_p with q_s. □

Now we are able to derive the transport equation:

Theorem 4.4. *The transport equation of isotropic elastodynamics, for symbols $w \in Ker\, l$ on a Lagrangian manifold $\Lambda \subseteq Char\, q$, is*

$$\pi H_{q_p} w + \frac{1}{2} div(H_{q_p}) w = 0 \ \text{on } \Lambda \cap Char\, q_p, \quad (4.4)$$

and is

$$(I-\pi)H_{q_s}w + \frac{1}{2}div\,(H_{q_s})w = 0 \;\; on \; \Lambda \cap Char\, q_s. \tag{4.5}$$

Or, in a more compact form, it is

$$H_q w + \frac{1}{2}div\,(H_q)w \pm H_q \pi\, w = 0 \;\; on \; \Lambda \cap Char\, q, \;\; for \; q = q_{\substack{s\\p}}. \tag{4.6}$$

Proof. We calculate on $\mathrm{Char}\, q_p$: There $l = q_s(I-\pi)$ and $q_s \neq 0$. The condition $w \in \mathrm{Ker}\, l$ therefore yields $(I-\pi)w = 0$ and $\pi w = w$. Together with Lemma 4.2, we get

$$\tilde{l}\, l^s\, w = \tilde{l}\, l^s\, \pi w = 0.$$

Then

$$(\frac{1}{q_s}\tilde{l})l = q_p I$$

yields the transport equation

$$\begin{aligned} 0 =& H_{q_p}w + \frac{1}{2}\,\mathrm{div}\,(H_{q_p})\,w + \frac{1}{2}\left\{\frac{1}{q_s}\tilde{l}, l\right\} w \\ =& H_{q_p}w + \frac{1}{2}\,\mathrm{div}\,(H_{q_p})\,w + \frac{1}{2q_s}\{\tilde{l}, l\}w - \frac{1}{2q_s}\{q_s, q_p\}w. \end{aligned}$$

The last equation follows from a formula in Dencker [1, page 366]:

$$\{f\,\tilde{l}, l\} = f\{\tilde{l}, l\} + \{f, q\}\, I \text{ on } \mathrm{Char}\, q.$$

Now we apply equation (4.2) an get:

$$0 = H_{q_p}w + \frac{1}{2}\mathrm{div}\,(H_{q_p})w - H_{q_p}\pi\, w.$$

By using the fact that $\pi w = w$ we get the transport equation:

$$0 = \pi H_{q_p}w + \frac{1}{2}\mathrm{div}\,(H_{q_p})w.$$

The transport equation on $\mathrm{Char}\, q_s$ follows analogous. $\square$

Karal and Keller [7] calculated solutions to the isotropic elastodynamics equation on the basis of the classical ansatz of geometrical optics: Consider solutions of the form

$$v(t,x) = a(x,\omega)e^{i\omega(\phi(x)-t)}, \quad a(x,\omega) \sim \sum_{k=0}^{\infty}(i\omega)^{-k}a_k(x). \tag{4.7}$$

with amplitude a.

These special solutions translate into the theory of Lagrangian distributions as the trivializations, with respect to the half-density bundle, of distributions $u \in I^0(\mathbb{R}_t \times \mathbb{R}^3_x, \Lambda, \Omega^{\frac{1}{2}})$,

$$u(t,x) = (2\pi)^{-3/2} \int e^{i\omega(\varphi(x)-t)} a(x,\omega)\, d\omega\, |dx\, d\omega|^{\frac{1}{2}}.$$

Here, the Lagrangian manifold Λ, corresponding to the special phase function

$$\psi(t,x,\omega) = \omega(\varphi(x)-t),$$

is equal to

$$\begin{aligned} &\{(t,x,\tau,\xi) \mid \psi'_\omega(t,x,\omega) = 0 \text{ and } (\tau,\xi) = \psi'_{t,x}(t,x,\omega)\} \\ =&\{(t,x,\tau,\xi) \mid t = \varphi(x),\ \tau = -\omega \text{ and } \xi = -\tau\nabla\phi(x)\} \\ =&\{(\phi(x), x, \tau, -\tau\nabla\phi(x))\}. \end{aligned}$$

We choose coordinates (x,τ) on Λ. The trivialization of the principal symbol of u is equal to the principal part a_0 of the amplitude.

We'll show that our method, if applied to these special Lagrangian distributions, results in the same eikonal- and transport equation as the elementary calculations of Karal and Keller. But in contrast to their result, the generalized transport equation in Theorem 4.4 is in addition correct and meaningful at caustics.

First, we evaluate the eikonal equation:

Theorem 4.5. *The conditions $\Lambda \subseteq \mathrm{Char}\, L$ and $a_0 \in \mathrm{Ker}\, l$ are equivalent to*

$$|\nabla\phi|^2 = \rho/(\lambda+2\mu) \text{ and } a_0 \times \nabla\phi = 0 \qquad \text{on } \mathrm{Char}\, q_p$$

and

$$|\nabla\phi|^2 = \rho/\mu \text{ and } a_0 \cdot \nabla\phi = 0 \qquad \text{on } \mathrm{Char}\, q_s.$$

Proof. The characteristic set $\mathrm{Char}\, L$ is

$$\begin{aligned} &\{(t,x,\tau,\xi) \mid q_p(t,x,\tau,\xi) = 0 \text{ or } q_s(t,x,\tau,\xi) = 0\} \\ =&\left\{(t,x,\tau,\xi) \mid |\xi|^2 = \tau^2 \frac{\rho}{\lambda+2\mu} \text{ or } |\xi|^2 = \tau^2 \frac{\rho}{\mu}\right\}. \end{aligned}$$

Therefore the condition $\Lambda \subseteq \mathrm{Char}\, L$ is equivalent to

$$|\nabla\phi|^2 = \frac{\rho}{\lambda+2\mu} \qquad \text{on } \mathrm{Char}\, q_p$$

and

$$|\nabla\phi|^2 = \frac{\rho}{\mu} \qquad \text{on } \mathrm{Char}\, q_s.$$

Next, we evaluate the condition $a_0 \in \operatorname{Ker} l$.

First we calculate on $\operatorname{Char} q_p$: We saw in the proof of Theorem 4.4 that the condition $a_0 \in \operatorname{Ker} l$ is equivalent to $\pi\, a_0 = a_0$. With $\xi = -\tau\nabla\phi$ on Λ we get that $a_0 \times \nabla\phi = 0$.

On $\operatorname{Char} q_s$, the condition $a_0 \in \operatorname{Ker} l$ is equivalent to $\pi\, a_0 = 0$, which implies $a_0 \cdot \nabla\phi = 0$. □

This corresponds to equations (9) to (12) in Karal and Keller [7].

Theorem 4.6. *Assume that the eikonal equation holds. The transport equation for a_0 on $\operatorname{Char} q_p$, expressed in α_0 such that $a_0 = \alpha_0\nabla\phi$, is equal to*

$$0 = 2(\nabla\phi \cdot \nabla)\, \alpha_0 + \frac{1}{\rho}\, \nabla \cdot (\rho\nabla\phi)\, \alpha_0$$

and on $\operatorname{Char} q_s$ it is

$$0 = 2(\nabla\phi \cdot \nabla) a_0 + \frac{1}{\mu}\, \nabla \cdot (\mu\nabla\phi) a_0 + \frac{\mu}{\rho}\, (a_0 \cdot \nabla(\rho\mu^{-1}))\nabla\phi.$$

Proof. On $\operatorname{Char} q_p$ the transport equation is (4.4); we evaluate the components:

In the coordinates (x, τ) on Λ, the Hamiltonian H_{q_p} is

$$\begin{aligned}
&\sum_j \partial_{\xi_j} q_p \partial_{x_j} - \partial_t q_p \partial_\tau \\
=& -2(\lambda + 2\mu) \sum_j \xi_j\, \partial_{x_j} - 0 \\
=& -2(\lambda + 2\mu)\, (\xi \cdot \nabla) \\
=& \,2\tau(\lambda + 2\mu)\, (\nabla\phi \cdot \nabla).
\end{aligned}$$

The divergence $\operatorname{div} H_{q_p}$ is therefore equal to

$$\nabla \cdot [2\tau\, (\lambda + 2\mu)\, \nabla\phi] = 2\tau\, \nabla \cdot [(\lambda + 2\mu)\, \nabla\phi].$$

By using that

$$|\nabla\phi|^2 = \frac{\rho}{\lambda + 2\mu} \text{ and } \xi = -\tau\nabla\phi$$

on $\Lambda \cap \operatorname{Char} q_p$ we get that the projection

$$\pi = \frac{\lambda + 2\mu}{\rho}\, \nabla\phi \otimes \nabla\phi.$$

We insert this into equation (4.4):

$$\begin{aligned}
0 &= \pi H_{q_p} a_0 + \frac{1}{2} \operatorname{div}(H_{q_p}) a_0 \\
&= 2\tau(\lambda + 2\mu)\, \pi\, [(\nabla\phi \cdot \nabla) a_0] + \tau\, \nabla \cdot [(\lambda + 2\mu)\, \nabla\phi]\, a_0.
\end{aligned}$$

Division by $\tau(\lambda+2\mu)$ yields

$$
\begin{aligned}
0 &= 2\pi\,[(\nabla\phi\cdot\nabla)a_0] + \frac{1}{\lambda+2\mu}\,\nabla\cdot[(\lambda+2\mu)\,\nabla\phi]\,a_0 \\
&= 2\,\frac{\lambda+2\mu}{\rho}(\nabla\phi\cdot[(\nabla\phi\cdot\nabla)a_0])\,\nabla\phi + \frac{1}{\lambda+2\mu}\,\nabla\cdot[(\lambda+2\mu)\,\nabla\phi]\,a_0
\end{aligned}
$$

We next use equation (120) of Karal and Keller [7],

$$
\begin{aligned}
\nabla\phi\cdot[(\nabla\phi\cdot\nabla)a_0] =& \frac{1}{2}\alpha_0\,\nabla\phi\cdot[\nabla(\rho\,(\lambda+2\mu)^{-1}] \\
&+ \frac{\rho}{\lambda+2\mu}\,(\nabla\phi\cdot\nabla)\,\alpha_0,
\end{aligned}
$$

and eliminate $\nabla\phi$ in the transport equation:

$$
\begin{aligned}
0 &= 2(\nabla\phi\cdot\nabla)\,\alpha_0 + \frac{\alpha_0(\lambda+2\mu)}{\rho}\nabla\phi\cdot[\nabla(\rho\,(\lambda+2\mu)^{-1}] \\
&\quad + \frac{1}{\lambda+2\mu}\,\nabla\cdot[(\lambda+2\mu)\,\nabla\phi]\,\alpha_0 \\
&= 2(\nabla\phi\cdot\nabla)\,\alpha_0 + \frac{1}{\rho}\,\nabla\cdot(\rho\nabla\phi)\,\alpha_0.
\end{aligned}
$$

The derivation of the transport equation on $\mathrm{Char}\, q_s$ works likewise: In coordinates (x,τ) on Λ, the Hamiltonian H_{q_s} is

$$
H_{q_s} = 2\tau\mu\,(\nabla\phi\cdot\nabla),
$$

the divergence $\mathrm{div}\, H_{q_s}$ is equal to $2\tau\,\nabla\cdot(\mu\,\nabla\phi)$ and the projection is

$$
\pi = \frac{\mu}{\rho}\,\nabla\phi\otimes\nabla\phi.
$$

On insertion into equation (4.5), we get

$$
\begin{aligned}
0 &= (I-\pi)H_{q_s}a_0 + \frac{1}{2}\mathrm{div}\,(H_{q_s})a_0 \\
&= 2\tau\mu\,(I-\pi)(\nabla\phi\cdot\nabla)a_0 + \tau\,\nabla\cdot(\mu\,\nabla\phi)\,a_0
\end{aligned}
$$

We divide by $\tau\mu$ and get

$$
0 = 2\,(\nabla\phi\cdot\nabla)a_0 - 2\,\pi(\nabla\phi\cdot\nabla a_0) + \frac{1}{\mu}\,\nabla\cdot(\mu\,\nabla\phi)\,a_0.
$$

Now, we use Karal and Keller [7, equation (69)],

$$
\nabla\phi\cdot[(\nabla\phi\cdot\nabla)a_0] = -\frac{1}{2}\,[a_0\cdot\nabla(\rho\mu^{-1})],
$$

to get that

$$
\pi(\nabla\phi\cdot\nabla a_0) = \frac{\mu}{\rho}\,(\nabla\phi\cdot[(\nabla\phi\cdot\nabla)a_0])\,\nabla\phi = -\frac{\mu}{2\rho}\,[a_0\cdot\nabla(\rho\mu^{-1})]\,\nabla\phi.
$$

That yields the transport equation

$$0 = 2\,(\nabla\phi \cdot \nabla)a_0 + \frac{\mu}{\rho}\,(a_0 \cdot \nabla(\rho\mu^{-1}))\,\nabla\phi + \frac{1}{\mu}\,\nabla \cdot (\mu\,\nabla\phi)\,a_0.$$

$\square$

This result corresponds to equations (116) and (72), in Karal and Keller [7].

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