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Doctoral thesis

The Superconductor
as Model for the Hardening
Stages in a Dislocated Crystal

Report presented by José-Luis Marqués-López to the
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*Dedicated to my parents
María Fernanda and José Luis*

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Preface

Facts which at first seem improbable
will, even on scant explanation, drop
the cloak which has hidden them and
stand forth in naked and simply beauty.

Galileo Galilei (1546-1642)

“Dialogues Concerning Two New Sciences”

Although the development to explain the plastic deformation of a crystal began just simultaneously with the 20th century, it was not until 1934 that Taylor, Orowan and Polanyi independently described the linear defects, called dislocations, that were the agents for such a plastic deformation. Since then and especially during the 1950s most of the dislocation properties and mechanisms were explained which are responsible for the changes in the plastic behaviour displayed by the crystal when an external shear stress is imposed on it, changes which form the so-called hardening curve. Today many particular details of the dislocation theory are well proved and checked, but nevertheless a unifying dislocation model able to describe the different hardening stages in a dislocated crystal, without being necessary to include many adjustable material parameters, is still lacking.

On the other hand, the development of the particle physics from the middle of the 1970s on has shown that phase transitions are not only a feature of condensed matter physics but also a general language in many Physics' areas which enables to explain and describe the effective long distance behaviour, and its changes, for very different and complex models. The advantage of phase transitions bases on the fact that they are determined by a very reduced set of parameters and field variables, which allows us to focus our attention only on the main degrees of freedom of the system we want to modelize; if such a model is well constructed, then it will display the expected different behaviours even though many particular details at the very short distance scale are not included. And this is therefore one start point in the direction towards the construction of a dislocation model describing the hardening curve: to treat the different hardening stages along such a curve as different phases, since although dislocation arrangements can become very complex, a correct but simplified model can retain most of the main features of the system to be studied.

The other starting point is the gauge invariance. This concept, firstly introduced by Weyl (in 1918 and definitively in 1929) within the framework of the General Theory of Relativity, has become the fundamental basis of almost all the models developed in

the theoretical physics, both for condensed matter as well as for particle physics. The invariance of the energy (or of the Lagrangian which describes the system) under the transformation in one of the system's field variables by a different amount at every spatial point, invariance ensured because another field variable called the gauge field compensates such a local transformation, has become the standard way to introduce interactions among different fields, especially if any previous, coarser ansatz for such interactions produced diverging results. The combination of the concept of phase transitions in gauge invariant models has led to a general scheme shared by a large family of different models: the so-called spontaneous symmetry breaking. The main characteristic of this breaking is the isolated appearance of certain topological (stable) objects whose energy is maintained finite due to the coupling to the gauge field. The collective action of a large number of these objects can disorder the system in a very effective way leading to a phase transition.

Returning to the dislocation theory, the topological nature of the dislocation lines were early identified, where “topological (stable) object” will mean throughout this whole work the impossibility to remove the object by means of any allowed transformation which leaves the model's energy invariant; only singular transformations which require the “surgery” of cutting the system remove these topological objects. Further, even the analogy between the dislocations and the topological flux lines or vortices in a superconductor — the latter described by a gauge invariant model called the Abrikosov or also the non-relativistic (non-compact) Higgs model — was also established (see N. D. Mermin, *Rev. Mod. Phys.* **51**, 591 (1979)). But nevertheless it has been not taken fully advantage of this result to develop a new dislocation model displaying behaviours analogous to the phases of a superconductor. The reason may have been that, within the framework of the conventional dislocation theory, as soon as the topological character of the dislocation is left, the aspect of this theory has no similarity with the model of superconductivity since the former is based mainly on the theory of the linear elasticity. And also because the direct identification (in function) of, on the one hand, dislocations in a crystal and, on the other hand, vortices in a superconductor is not correct if the superconductor has to be taken as leading model whose different phases correspond to the different plastic stages of the dislocation system: although the vortices in a superconductor are responsible for the phase transitions of the latter, the dislocations able to *glide* and not any dislocation are actually the objects which cause the changes in the plastic behaviour of a dislocated crystal.

And this will be our goal: to construct a new formulation for the dislocation theory which has the same form as the four Maxwell equations of electromagnetism, and then by following the superconductor model, to couple such a formulation in a gauge invariant way to a scalar field: what in the superconductor is the diamagnetism opposing the spreading out inside the system of a magnetic field will become the analogue to the rigidity, displayed by the crystal in the first stages of the plastic deformation, against the spreading out over the crystal of the density and current of gliding dislocations. Within the framework of this analogy to the superconductor, which is the fundamental idea of the work, the conventional linear elasticity theory occurs as a particular case of the more general formulation of the dislocation theory proposed in this work.

It is a bit surprising to obtain that the new model, constructed on so a simple

analogy to the superconductor, results to display a set of different phases whose main characteristics are in a one-to-one correspondence to the different plastic stages of the hardening curve. A bit surprising and, also, the final justification of the model.

The work is structured as follows: the first chapter contains an introduction into the hardening curve and the main properties characterizing its several hardening stages, whose modellization is the task of this work. The second chapter is devoted to the construction of the model along the leading idea introduced above: that of the analogy between the superconductor's diamagnetism and the crystal's rigidity opposing the dislocation glide motion. The third chapter deals with the applications of the model constructed, as well as the discussion of the phase diagrams resulting for several different cases studied.

Since this work relates two different areas of Physics, the plasticity and the particle physics, it has been tried to maintain its contents as self-contained as possible. Hence the reader specialized in the theory of plasticity will find all results about the Abrikosov model, topological vortices, the phase diagram of the Higgs model (compact and non-compact) at zero temperature as well as at very high temperatures, are completely calculated in the Appendices A, B, C and D respectively.

On the contrary, the reader familiar with about particle physics will find in these appendices no new relations which have not yet appeared during the second half of the 1970s, as the first models displaying phenomena similar to the quark confinement were studied in the standard bibliography quoted on the next page (†). This reader will soon notice that the dislocation model constructed in this work is nothing else then a “translation” of the (non-relativistic) periodic scalar electrodynamics. For the latter model, which is also called the compact $U(1)$ Higgs model, the translation rule corresponds with the above mentioned analogy between superconductor's diamagnetism and crystal's rigidity against gliding. Hence the phase which in the compact $U(1)$ Higgs model is characterized by the confinement of electric charges, because the electric field going from a charge to the other has become short range and is restricted to exist only inside certain filaments connecting these electric charges, corresponds in the dislocation formulation to the rapid hardening stage where the current of gliding dislocations — field which in the dislocation equations plays a role similar to that of the electric field in the Maxwell equations — is also confined within a tube-structure delimited by immobile dislocation segments. It may be therefore interesting to see how a model, initially defined for its use in an area like the particle physics which have borrowed many ideas from the condensed matter physics, finds this time the path back to this latter speciality.

As a side-product of the above new formulation of the dislocation theory, a form is proposed to couple the glide motion of the dislocations to the Lagrange formalism for irreversible processes, the latter a theory developed by Anthony and co-workers in the papers K.-H. Anthony, “A New Approach to Thermodynamics of Irreversible Processes by means of Lagrange-Formalism”, in *Disequilibrium and Self-Organisation*, ed. by C. W. Kilmister, 75 (1986); and A. Azirhi, K.-H. Anthony, “Dislocation Dynamics and Plasticity by means of Lagrange Formalism”, in *Continuum Models of Discrete Systems (CMDS8)*, ed. by K. Z. Markov, 490 (1996).

Summarizing, the analogy between the opposing against dislocation gliding shown by a rigid crystal and the diamagnetism of a superconductor, together with the incorporation into such an analogy of a mechanism similar to the confinement of electric charges displayed by certain models with a compact $U(1)$ gauge symmetry, has permitted the development of a simple model which is able to describe all changes between the different plastic behaviours in a crystal as the phase transitions of the modellized theory.

(†) The above mentioned bibliography for phase transitions in gauge invariant systems is formed by the following standard papers: K. G. Wilson, *Phys. Rev. D* **10**, 2445 (1974); L. Dolan, R. Jackiw, *Phys. Rev. D* **9**, 3320 (1974); T. Banks, R. Myerson, J. Kogut, *Nucl. Phys. B* **129**, 493 (1977); A. M. Polyakov, *Nucl. Phys. B* **120**, 429 (1977) and *Phys. Lett. B* **72**, 477 (1978); M. E. Peskin, *Ann. Phys.* **113**, 122 (1978); E. Fradkin, S. H. Shenker, *Phys. Rev. D* **19**, 3682 (1979); B. Svetitsky, L. G. Yaffe, *Nucl. Phys. B* **210**, 423 (1982); and G. 't Hooft, “Extended objects in gauge field theories”, in *Particle and Fields*, edited by D. H. Boal and A. N. Kamal, Plenum Press, New York, 165 (1978).

Compared to the above bibliography, the new results in the appendices are: in Appendix C the application of the “high temperature” power expansion on the correlation function $K(C)_{XY}$ defined for the study of 3-dim. non-compact lattice Higgs model (see relation (C.468) on page 461 and the discussion following there). And in Appendix D the following results:

- The phase diagram of $Z(q)$ Ising model for q large in a 3-dim. lattice which is periodic along one direction, on page 538.
- The Villain approximation for the $Z(q)$ gauge model in a 3-dim. lattice periodic along one direction, from equation (D.191) on page 586 to equation (D.198). And the phase diagram of the $Z(q)$ gauge model for $q = 2$ and for large q in a 3 and 4-dim. lattice periodic along one direction, in the Note on page 594.
- The equivalence between energy increase $\mathcal{V}_{\vec{x}_1, \vec{x}_2}$ created by the introduction of two external opposite charged sources and the correlation function $W(C) = \left\langle \exp \left[i \sum_{\text{loop } C} A_{\mu, \vec{r}} \right] \right\rangle$ in the compact Higgs model, discussed in the Note on page 615.
- The phase diagram of the 3-dim. non-compact Higgs model at high temperatures in the section starting on page 636, together with the calculation for such a model of correlation function $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ with $\mathcal{L}_{\mu, \vec{r}}^{\text{smear}} = -\Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_1} + \Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_2}$ in the Note on page 656.
- The calculation up to one-loop order of the effective potential for the (continuum) Abrikosov model in a 2 and 3-dim. system, in the final result (D.346) on page 723; together with the transition value of the lattice periodicity for the 3-dim. Abrikosov model, calculated in (D.349) on page 727, being represented the form of the effective potential in Fig. D.8 on page 753.
- The complete phase diagrams for the compact lattice Higgs model at large q in a 3-dim. lattice (Fig. D.3 on page 748) as well as in a 4-dim. lattice (Fig. D.5 on page 750).

Notation

$\sim$	Fourier transform
$\vec{r}$	site in a d -dimensional lattice
$\vec{x}$	site in the $(d-1)$ -dimensional sublattice
$\vec{r}^*$	dual lattice site
$\mu, \nu, \lambda, \dots$	Cartesian components of coordinates in physical space
$i, j, k, \dots$	Cartesian components in physical space if the greek indices contain $\mu = 0$ or 4
$a, {}^a, b, {}^b, c, {}^c, \dots$	components in internal symmetry space
τ	lattice direction of taking the continuum limit
ϕ, θ	polar angles in physical space
Φ, Θ	polar angles in internal space
α	Burgers vector components
$\mathbf{B}_i^\alpha, \mathbf{H}_i^\alpha$ and $\mathbf{E}_i^\alpha, \mathbf{D}_i^\alpha$	density and current of able-to-glide dislocations (see Table 2.3 on page 52)
a	lattice spacing
a_τ	lattice spacing in the direction of taking the continuum limit
$A_\mu(\vec{r}), A_{\mu,\vec{r}}$	$U(1)$ gauge field
$\mathcal{A}_C$	area enclosed by loop C
$ \alpha , \beta $	effective parameters for the Higgs potential
b^α	Burgers vector
$\beta_\psi, \beta_e \equiv \frac{1}{e^2}$	coupling strengts in the $Z(N)$ lattice models
c_{sound}	(transversal) sound velocity
$D(\vec{r}^* - \vec{r}), D_{\vec{r}, \vec{r}'}$	propagator or Green's function
D_μ	covariant derivative
Δ_μ	lattice partial derivative
d	lattice dimension
e	coupling constant between the scalar and the gauge field

$\epsilon_{\mu\nu\dots}, \epsilon_{ab\dots}$	completely antisymmetric tensor
η	partition function's scale to which every configuration is compared
F	Helmholtz free energy
$\mathcal{F}$	Helmholtz free energy density
$F_{\mu\nu}(\vec{r}), F_{\mu\nu,\vec{r}}$	gauge invariant curl of the $U(1)$ gauge field
$G_{\mu\nu}^a(\vec{r})$	gauge covariant curl of $SO(3)$ - $SU(2)$ gauge field
G	elastic shear modulus; Gibbs free energy
g_{mg}	magnetic (monopole) charge
$H(C^*), H(\bar{C})$	't-Hooft loop correlation
$I_a^{(r)}$	group generator in the matrix representation (r)
χ	phase change in a gauge transformation $\psi \rightarrow \psi' = e^{-i\chi}\psi$
Λ	d -dimensional square lattice; length dimension of the system's volume
$ \Lambda $	lattice volume (number of lattice sites)
Λ_c	cut-off
λ	penetration length of the gauge field A_μ
$\mathcal{P}_C$	perimeter enclosed by loop C
$\psi(\vec{r}), \psi_{\vec{r}}$	scalar field coupled to a $U(1)$ gauge field
$\Psi^a(\vec{r})$	scalar field coupled to a non-Abelian gauge field
ρ	mass density
$\bar{\rho}$	expectation value for the length of the (effective) scalar field
T	(absolute) temperature
$\hat{T}$	transfer matrix
$(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$	resolved shear stress
$u^\alpha(\vec{r})$	displacement vector
$V_{\text{eff}}(\bar{\rho})$	effective potential
$\mathcal{V}_{\vec{x}_1, \vec{x}_2}$	free energy increase due to two external sources placed at sites $\vec{x}_1$ and $\vec{x}_2$
$W(C)$	Wilson loop correlation

$W_\mu^a(\vec{r})$	$SO(3)$ - $SU(2)$ gauge field
Ω	solid angle
ω	lattice periodicity's length
Ξ	parallax angle
ξ	correlation length for the scalar field
$\mathcal{Z}$	partition function
$\vartheta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$	Heaviside step function, with $\frac{d\vartheta}{dx} = \delta(x)$ the Dirac's delta
$b_{\mu,\vec{r}}, a_{\nu,\vec{r}}, B_{\mu\nu,\vec{r}}, C_{\vec{r}}$	auxiliary lattice fields used in the Villain approximation

Chapter 1

Introduction to dislocation theory

1.1 Concept of dislocation

Because this work deals with plasticity and hardening in a single crystal, let us start by introducing the main ideas of the dislocation theory. At first we shall consider single crystals at low temperatures; the higher temperature range will be studied at the end of the work. Plastic deformation is due to inhomogeneous slip of one plane of atoms over the neighbouring plane through the breaking and new-building of atomic bonds between these two planes. Slip usually occurs on one of the planes of closest packing and in the direction of the line of closest packing on this plane, since perpendicular to a plane of closest packing the interatomic spacing is larger (and thus the binding force weaker) than along such a plane. For face-centered cubic (f.c.c.) crystals (Fig. 1.1), these slip planes are the $\{111\}$ 's¹ and the slip directions are the $\langle 101 \rangle$'s: a slip system gets defined through the specification of a slip plane and a slip direction. A useful representation of the different crystal planes and directions is the stereographic projection Fig.1.2.

Since plastic deformation occurs through the slipping of a particular crystal's plane over another parallel plane under the action of an applied tensile stress σ_T , then the projection of σ_T on this slip plane along the slip direction will be a determining variable, called the resolved shear stress

$$\tau_{\text{resol}} = s_{\alpha}(\sigma_T)_i^{\alpha} n^l, \quad (1.1)$$

where s_{α} is the slip direction and n^l is the normal to the slip plane (Fig. 1.3) with $\alpha, l = 1, 2, 3$; throughout this work we shall use the Einstein's convention of summation

¹As a reminder, a given crystallographic plane is denoted by the the Miller indices, obtained as follows: one finds the lengths c_1, c_2, c_3 along the x, y and z axes of the points where the plane cuts the three axes (one takes $c_i = \infty$ if the plane does not intercept axis i) and calculates $(1/c_1, 1/c_2, 1/c_3)$. Then, one convertes these three last numbers into smallest integers dividing with the largest common factor: the resulting integers, called the Miller indices, are enclosed in parenthesis, (hkl) , and an index with a bar on it denotes a negative value. Because of the crystal's symmetry, there are several planes with the same spacing between atoms, as for example, planes $(111), (11\bar{1}), (\bar{1}11), (\bar{1}\bar{1}1), (1\bar{1}\bar{1})$ in f.c.c. crystals; one denotes these previous equivalent planes by $\{111\}$. For the crystallographic directions, one simply takes the components along the coordinate axes and reduces them to smallest integers, written as $[uvw]$. As for the planes, a crystal shows also equivalent directions: for instance, $\langle 101 \rangle$ represents directions $[101], [110], [011]$.

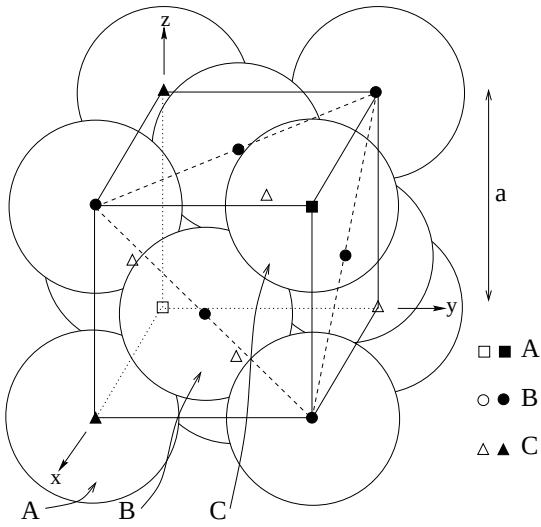


Figure 1.1: The close packed f.c.c. lattice. With the directions along the x, y, z axes defined as $[100], [010], [001]$ respectively, the plane formed for instance by the $\bullet$'s, is plane (111) . Note that in a cubic lattice, direction $[hkl]$ cuts normally plane (hkl) ; a denotes the lattice spacing.

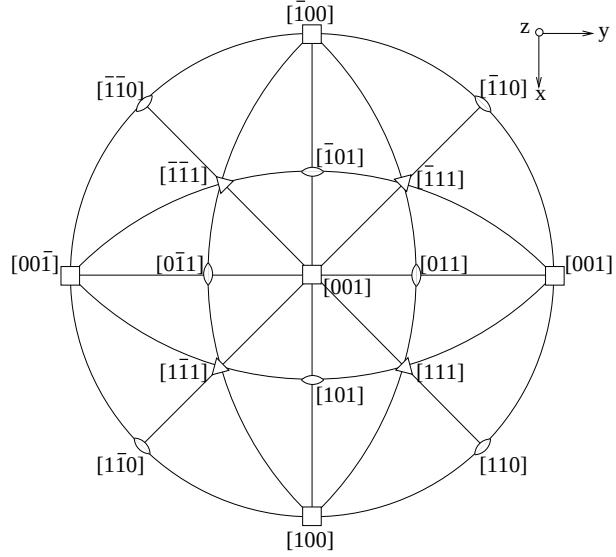


Figure 1.2: The $[001]$ stereographic projection for f.c.c. crystals. This is the projection on the $x-y$ plane (the “picture” taken from a plane of constant positive z , with $\hat{z} = [001]$) of the intersection points of the different crystal’s directions with a sphere. Symbols $(\circ)$, $(\triangle)$ and $(\diamond)$ denote a two-, three- and four-fold symmetry axis.

over repeated indices as long as we are working with the continuous formulation. The Schmid law states that τ_{resol} is the only component of the tensile stress which controls the plastic deformation on a particular slip plane; this law works rather well in most of the cases. Nevertheless, slip on a certain plane can be also influenced by other processes which are controlled by stress’ components different from the resolved shear stress [7] in such plane: these deviations from the Schmid law will be dealt with later in Chapter 2.

When the resolved shear stress in a certain plane P reaches the value τ_0^P , called the *yield point* or the critical resolved shear stress, plastic deformation starts in this plane. For a given orientation of the crystal, the slip system for which its corresponding resolved stress is the largest and thus is the first to be activated for plastic deformation, is called the *primary slip system*. In an f.c.c. crystal, of the four equivalent $\{111\}$ planes, an example of primary system is defined through the $(\bar{1}11)$ plane (normal to the $[\bar{1}11]$ direction, see Fig. 1.2) and the $[110]$ direction, which lies on the $(\bar{1}11)$ plane.

The origin of plastic deformation is the appearance of *mobile* line defects called dislocations. Thus the yield point, as the minimal shear stress necessary to have plastic deformation, is the component of the tensile stress on the primary plane at which these line defects begin to occur in the crystal. However, τ_0^P is not a constant not even for a particular slip plane: it increases with the plastic deformation due to the interactions on the same between the dislocations which are being produced and those already existing. This effect is known under the name of *self-hardening*. Another case where the dislocations can impede reciprocally their motion can be seen from Fig. 1.3: as the crystal

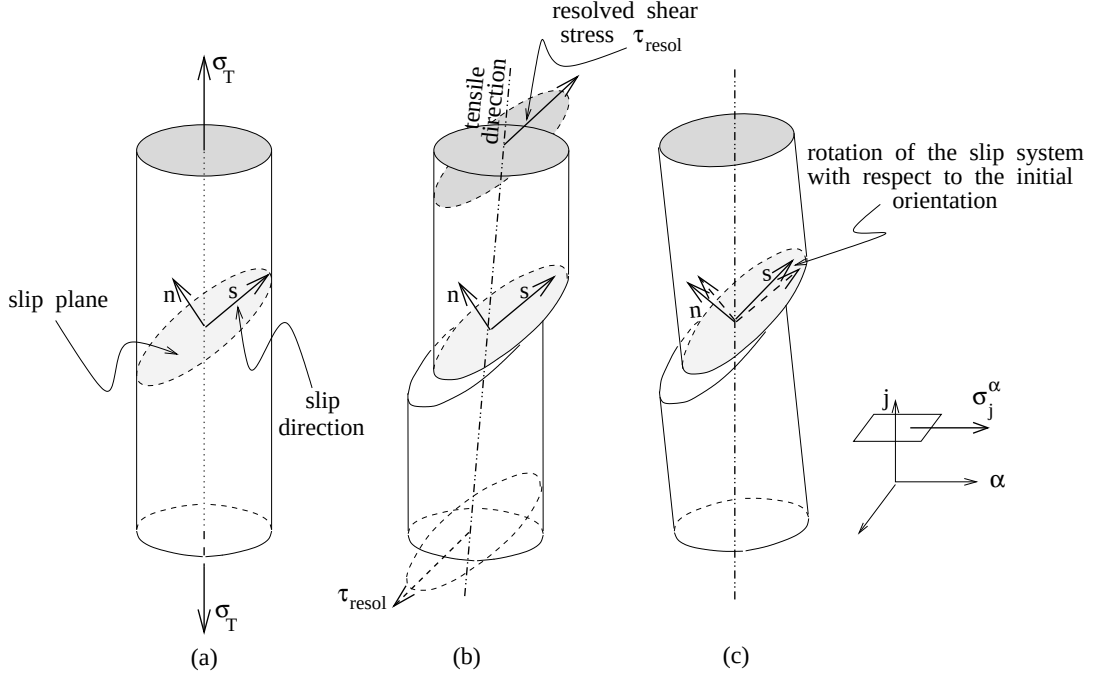


Figure 1.3: Applied tensile stress σ_T and projected component τ_{resol} on the slip plane along the slip direction. If the tensile axis is held fixed, the slip changes the projection of the tensile stress on the first activated slip system. According to the usual index convention, σ_j^α denotes tension along α direction applied to the plane normal to the j direction; both α and j runs from 1 to 3.

slips on the first activated slip plane, the orientation of the tensile axis changes with respect to this slip plane. In most of the machines, the tensile direction is constrained to be held fixed (vertically in Fig. 1.3 (c)), and thus the slip direction rotates toward the tensile direction (and the angle between the normal to the slip plane and the tensile direction increases). This leads to a large projection of the tensile stress also on other slip systems which eventually are activated. These systems, called *secondary slip systems*, start producing mobile dislocations which interact with the dislocations of the primary slip plane, reducing their mobility and the grade of plastic deformation. This is called the *latent hardening* (since the secondary systems were latent, initially not active) and is responsible for the large increase in the minimal stress necessary to maintain the plastic deformation. This hardening, or change in the mechanical strength, is observed in face-centered and body-centered cubic crystals; in hexagonal crystals, on the other hand, the activation of secondary slip planes hardly occurs [23] and therefore no latent hardening is usually observed. Cases like these two ones are generally known as work hardening since an increasing deformation increases also the mechanical strength.

The constants characterizing the elasticity of a homogeneous crystal are G , the elastic shear modulus, and E , the elastic Young's modulus, which describe respectively the crystal's stiffness under shear or under stretching. Both are effective material constants, which in our phenomenological model contain the energy scale of the atomic interactions which are actually taking place (and which are responsible for such elastic resistance to a deformation) but whose details are not relevant in the large distance scale description

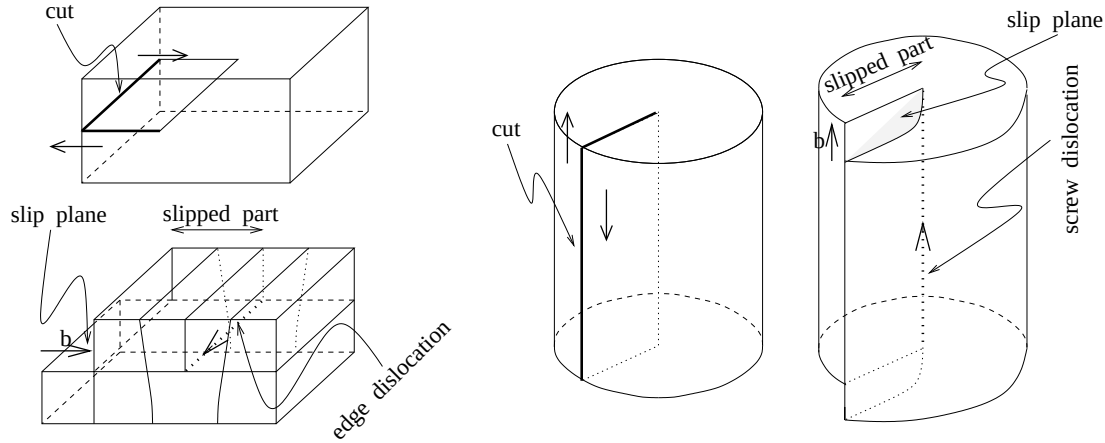


Figure 1.4: Edge and screw dislocation.

which we are carrying out here: the energy scale constructed with G or E and the interatomic spacing a , Ga^3 or Ea^3 , is just of the order of energy of such interatomic interactions. The shear stress necessary to move *simultaneously* all the atoms in the plane a distance of a is of order of G , easily estimated by noting that the work necessary to break all interplane atomic bonds over the plane's surface A (to be formed again after the displacement of length a) is GaA , and thus the force per unit of such surface is G . However, this value is far much larger than the observed yield point stress τ_0 , a signal that the plastic deformation is not due to the simultaneous displacement of the crystal's atoms. Dislocations are the objects responsible for the difference between the theoretical shear stress of order G and τ_0 , since dislocations allow the atoms to slip *consecutively* instead of all together and simultaneously. The dislocation line is thus the boundary which separates the crystal's region, where atoms which have been already displaced an atomic step in the slip direction, from the region where atoms have not yet slipped (Fig. 1.4). Depending on the relative orientation of the displacement direction with

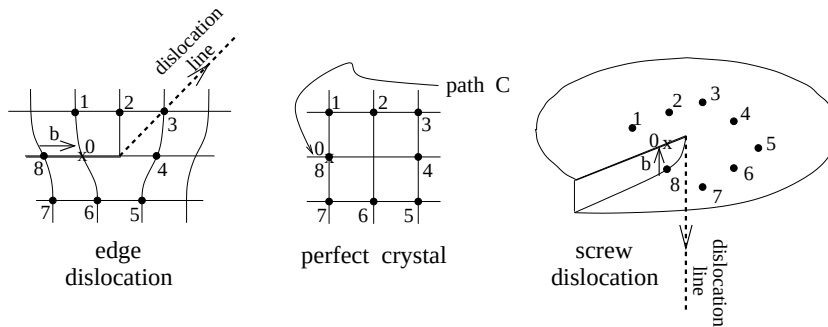


Figure 1.5: An 8-steps path defines, according to the right-hand rule, the direction assigned to the dislocation line. Burgers vector $\mathbf{b}$ points from the begin to the end of the path C . For an edge dislocation, the Burgers vector is orthogonal to the dislocation line and for a screw dislocation, parallel to it.

respect to the line separating the slipped from the still-not-slipped part, dislocations are classified as screw dislocation (if both directions are parallel), edge dislocations (if

these are perpendicular) or of mixed type.

The main characteristic of a dislocation is its Burgers vector, defined as follows: let C be a closed path in a perfect crystal (a crystal without dislocations) as in Fig. 1.5; if we transfer this closed path to a region crossed by a dislocation, the path becomes open and the closing fault is just the difference in translation between the slipped part and the part of the crystal which has not yet been translated. The displacement vector u for the 8-steps path of the figure is given by $u_0 = 0$ and $u_8 = -b$, being b the just mentioned difference, which is called the Burgers vector and is equal to a lattice vector between two neighbour points on the slip plane.

Dislocations are either closed lines or open lines which extend from one crystal side to the other. A dislocation cannot end inside the crystal: since dislocations are the limit between the slipped and the unslipped regions, if a dislocation would end inside the crystal, one could transit from the slipped region into the unslipped around the end of the dislocation without to cross any limit, and unslipped and slipped region would not be well defined. It is important to note that the atomic cells at one side of the half plane containing the dislocation are only translated (but not rotated) with respect to the cells on the other side. This means that in a dislocated crystal there still exists a long distance parallelism, i.e., vertical direction at one side of this half plane containing the dislocation remains vertical at the other side. This parallelism does not exist in crystals with single disclinations and grain boundaries. The energy of a single disclination in a 3 dimensional crystal is nevertheless very high (not so in biological tissues) to occur in a crystal, whilst grain boundaries, for their part, are present in polycrystals but not in a single crystal. Since in this work we are only concerned with single crystals, these grain boundaries will not be studied.

The atoms are slipped when the dislocation *moves* through the crystal. This translational motion of a dislocation is called *glide motion* and the slip plane is also called the glide plane. The glide plane contains the dislocations line and the Burgers vector. For an edge dislocation there exists only one glide plane, whereas for a screw dislocation, whose Burgers vector lies along the dislocation line, any lattice plane containing the dislocation line is already a glide plane. All these possible glide planes for a screw dislocation are called the *cross-slip planes*. An edge dislocation can also move in the direction normal to the glide plane, but this motion can only occur if atoms are extracted (e.g., by diffusion of vacancies). This so-called *climb motion* takes place therefore only at high temperatures.

As already mentioned above, the phenomenological model describing the large distance scale behaviour of a deformed crystal does not take into account all the different interactions occurring at the atomic scale but substitutes them through a limited set of (hopefully enough) effective degrees of freedom and material constants. At such large scales interatomic forces are of very short range: thus, when one considers an infinitesimal volume element dV (still large enough to cover distances larger than the interatomic spacing) in a deformed crystal, the force on dV from the neighbour volume elements only acts through the surface enclosing the volume element; such surface forces are the stress σ_k^α describing to the crystal's deformation. The same argument applies to any finite volume V inside the crystal, $F_{\text{on } V}^\alpha = \int_V f_{\text{on } dV}^\alpha dV = \oint_S \sigma_k^\alpha dS^k$, and by using

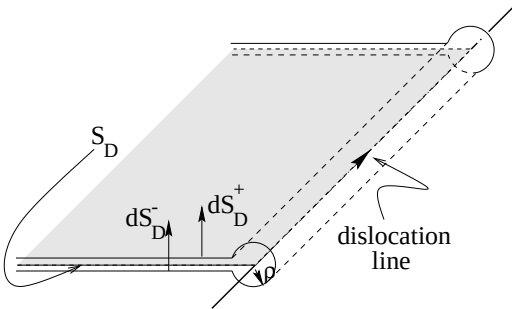
the Gauss theorem, $\oint_S \sigma_k^\alpha dS^k = \int_V \partial_k \sigma_k^\alpha dV$, relation $f^\alpha = \frac{\partial \sigma_k^\alpha}{\partial x^k}$ holds. In the same way, $-\int_V \frac{\partial \sigma_k^\alpha}{\partial x^k} dV$ is the force produced by volume V on its neighbours. If volume forces controlled from the exterior (as for instance the gravity force) acts on V , these push the system out of the equilibrium until the internal forces caused by the appearing deformations compensate this external action, then the relation between the exterior forces and the stress tensor is given by

$$\int_V f_{\text{ext}}^\alpha dV + \oint_S \sigma_k^\alpha dS^k = 0 \quad \Longleftrightarrow \quad f_{\text{ext}}^\alpha = -\partial_k \sigma_k^\alpha. \quad (1.2)$$

Clearly, if the internal response against the external forces is not strong enough to compensate them, the volumen V will adquire an acceleration and the previous result must be extended to include such a contribution.

Let us now study the force per unit length on a dislocation line [65] §28. The elastic stress $(\sigma_{(\text{ex})})_k^\alpha$, acting on the volume V occupied by the dislocation, is created by loads external to the dislocation (the dislocation cannot exert any force on itself since, due to the conservation of impuls, the motion of a system gets changed only by external forces to it). We assume that the crystal is in equilibrium: $\partial_k (\sigma_{(\text{ex})})_k^\alpha = 0$. The work done by this $(\sigma_{(\text{ex})})_k^\alpha$ during an infinitesimal displacement δu^α of the dislocation loop is due to the variation in the inhomogeneous displacement's distribution, the latter measured by gradient $\frac{\partial u^\alpha}{\partial x^k}$, and is given by $\delta W = -\int_V (\sigma_{(\text{ex})})_k^\alpha \delta \left(\frac{\partial u^\alpha}{\partial x^k} \right) dV$. Since the external load $(\sigma_{(\text{ex})})_k^\alpha$ is assumed independent of the dislocation position, we can take the δ -variation outside the volume integral: $\delta W = -\delta \int_V (\sigma_{(\text{ex})})_k^\alpha \partial_k u^\alpha dV \stackrel{\partial_k (\sigma_{(\text{ex})})_k^\alpha = 0}{=} -\delta \int_V \partial_k [(\sigma_{(\text{ex})})_k^\alpha u^\alpha] dV = -\delta \oint_S (\sigma_{(\text{ex})})_k^\alpha u^\alpha dS^k$. When a closed path is drawn around the dislocation line (at a distance large enough to be still working within the frame of an effective model which does not distinguish the atomic scale), displacement vector u^α suffers a jump equal to the Burgers vector of the dislocation. Hence u^α is a function which shows a discontinuity when the surface S_D is crossed, being S_D the surface attached to the dislocation line; in Fig. 1.4, S_D is the surface along which the crystal has been cut. The volume integral in the work which we are calculating can be therefore written as an integral over the closed surface formed by area just above and below of S_D plus the tube (of infinitesimal small radius) around the dislocation line

$$\delta W = -\delta \int_{S_D^+} (\sigma_{(\text{ex})})_k^\alpha u^\alpha dS^k + \delta \int_{S_D^-} (\sigma_{(\text{ex})})_k^\alpha u^\alpha dS^k - \delta \int_{\text{tube}} (\sigma_{(\text{ex})})_k^\alpha u^\alpha dS^k.$$



The integral on the tube tends to zero for $\rho \rightarrow 0$ since u^α goes to infinity more slowly than $1/\rho$ and thus $2\pi\rho u^\alpha \xrightarrow{\rho \rightarrow 0} 0$. On S_D^+ and S_D^- the value of $(\sigma_{(\text{ex})})_k^\alpha$ is the same: the only difference lies in displacement u^α which changes by b^α when going from one surface to the other

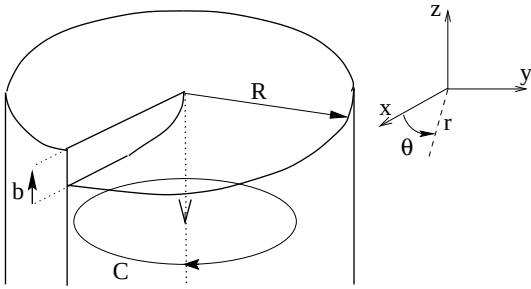
$$\delta W = -b^\alpha \delta \int_{S_D} (\sigma_{(\text{ex})})_k^\alpha dS^k.$$

The force on the dislocation produces a displacement δx^j in each segment dl^i of the dislocation line, incrementing surface S_D by an amount equal to $\delta \int_{S_D} dS^k = \oint \epsilon_{kji} \delta x^j dl^i$, where the $\oint$ is because the dislocation line is a closed path (or an infinity open one). Therefore $\delta \int_{S_D} (\sigma_{(ex)})_k^\alpha dS^k$ is equal to the integral over this surface increment, obtaining

$$\delta W = - \left[\oint_{\text{disloc}} \left(b^\alpha \epsilon_{kji} \hat{t}_i (\sigma_{(ex)})_k^\alpha \right) dl \right] \delta x^j, \quad (1.3)$$

where $\hat{t}$ is the unit vector tangent to the dislocation line ($dl^i = \hat{t}^i dl$). Since by definition force is equal to $-\frac{\delta W}{\delta x^j}$, the force per unit length on the dislocation is given by

$$f_j = \epsilon_{jik} \hat{t}_i (b_\alpha \sigma_k^\alpha), \quad (1.4)$$



and is called the Peach-Köhler force. Let us now consider a straight screw dislocation along the $-\hat{z}$ direction with Burgers vector $\vec{b} = +b\hat{z}$. The displacement u^z along the $\hat{z}$ direction is given by

$$u^z = \frac{b}{2\pi} \theta \quad \text{with} \quad \theta = \arctan\left(\frac{y}{x}\right), \quad (1.5)$$

at a distance far enough from the dislocation line. In the figure, path C is drawn clockwise whilst angle θ is measured anti-clockwise due to the minus in the definition of the Burgers vector, which points from the end of C to its start. Thus, when covering C around the dislocation from $\theta = 2\pi$ to $\theta = 0$, the slip is equal to b . The stresses σ_x^z and σ_y^z created by this dislocation are equal to

$$\sigma_x^z = G u_x^z = G(\partial_x u^z) = -\frac{Gb \sin \theta}{2\pi r} \quad \text{and} \quad \sigma_y^z = G u_y^z = G(\partial_y u^z) = \frac{Gb \cos \theta}{2\pi r}, \quad (1.6)$$

and the corresponding energy E per unit length L , arising from the deformation created by the dislocation, is given by

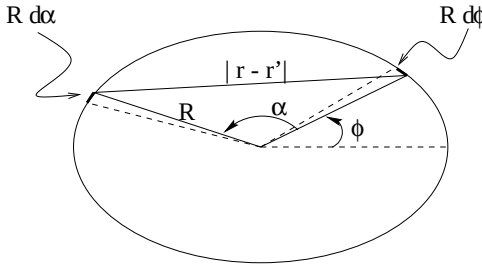
$$\frac{E}{L} = \int_S \frac{1}{2} \sigma_k^\alpha u_k^\alpha dS = G \left(\frac{b}{2\pi} \right)^2 \frac{2\pi}{2} \int_{r=r_0}^{r=\Lambda} \frac{r dr}{r^2} = \frac{Gb^2}{4\pi} \ln\left(\frac{\Lambda}{r_0}\right), \quad (1.7)$$

where Λ the crystal's radius. On the other hand the above integral has been cut-off at very short distances by the introduction of the core radius r_0 , which is of the order of the interatomic distance a and thus the shortest distance we can reach in our effective model for large distances.

In many cases, dislocations are not straight (infinite) lines but form closed loops. This avoids the divergence like the crystal's radius in their energy as one can easily show by noting that the energy of a dislocation loop C is of the order of [61]

$$E_{\text{loop}} \sim \frac{Gb^2}{8\pi(1-\nu)} \oint_C \oint_C \frac{dl^i dl^j}{|\vec{r} - \vec{r}'|}, \quad (1.8)$$

where $|\vec{r} - \vec{r}'|$ is the distance between the dislocation length element dl^i and the element dl'^i . Elastic constant ν is called the Poisson's number² which takes account that along any closed dislocation loop, the Burgers vector does not always point along the dislocation line (as it was the case for the screw dislocation (1.5)) and therefore the stress created by the dislocation is not purely of shear character; in an isotropic material re-



lation $\nu = \frac{E}{2G} - 1$ holds. In the resulting integral for a circular loop of radius R

$$E_{\text{loop}} \sim \frac{Gb^2}{8\pi(1-\nu)} \oint_{\phi=0}^{\phi=2\pi} \int_{\alpha=\frac{a}{R}}^{\alpha=2\pi-\frac{a}{R}} d\alpha d\phi \frac{R^2 \cos \alpha}{2R \sin \frac{\alpha}{2}},$$

the integration over $d\alpha$ is from a/R to $2\pi - a/R$ in order to avoid the (divergent) self-interaction of

any dislocation segment (a is the lattice spacing), self-interaction which already occurs in the atomic range, outside the validity region of our effective model. The angular integral is directly carried out $2\pi R \int_{a/R}^{\pi} d\alpha \frac{\cos \alpha}{\sin \frac{\alpha}{2}} = 4\pi R \left[\ln \tan \frac{\alpha}{4} \right]_{a/R}^{\pi} + 4\pi R \left[\cos \frac{\alpha}{2} \right]_{a/R}^{\pi}$ and one obtains that the energy of a dislocation loop grows like $R \ln R$ for large R ($\ln \tan(a/4R) \approx -\ln 4 - \ln(R/a)$)

$$E_{\text{loop}} \sim \frac{Gb^2}{2(1-\nu)} \left[(-1 + \ln 4)R + R \ln \left(\frac{R}{A} \right) \right] \stackrel{\text{large } R}{\approx} \frac{Gb^2}{2(1-\nu)} R \ln R. \quad (1.9)$$

However, there are cases where dislocations are straight lines and therefore the logarithmic divergence like the crystal's size appears again — the divergence at very short distances remains nevertheless as a footprint of the interactions occurring at a fundamental scale (the lattice spacing) and whose effects the phenomenological model only describes approximatively at large distances. The source of such divergence is the following: $\frac{2\pi}{b} u^z = \arctan\left(\frac{y}{x}\right)$ is the solution of differential equation $\frac{2\pi}{b} \nabla^2 u^z = 0 \pmod{2\pi}$, where $\pmod{b}$ takes account of the jumps corresponding to integer times the Burgers vector when one covers a closed path around the dislocation line; $\nabla^2 = \partial_x^2 + \partial_y^2$ is the Laplace operator in the 2-dimensional plane crossed perpendicularly by the dislocation. The propagator of Green's function $D(\vec{r} - \vec{r}') = D(|\vec{r} - \vec{r}'|)$ of this Laplace operator, i.e., the solution of $\nabla^2 D(|\vec{r} - \vec{r}'|) = -\delta^2(\vec{r} - \vec{r}')$, is equal to³ $-\frac{1}{2\pi} \ln |\vec{r} - \vec{r}'|$. This propagator grows with the distance, which is the origin of the logarithmic divergence of the

²For a crystal in general (which is not an isotropic material), the elastic constants depend not only on whether the deformation is of shear or of tensile type but also on which crystallographic direction this deformation takes place. For the simplified discussion carried out in this introduction the system will be considered isotropic.

³Heaviside function $\vartheta(x - x') = \begin{cases} 1 & x - x' > 0 \\ 0 & x - x' < 0 \end{cases}$ can be written as $\lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{e^{ik(x-x')}}{k - i\epsilon}$.

using the Cauchy's theorem applied to complex integral $\frac{1}{2\pi i} \oint_C d\zeta \frac{e^{i\zeta(x-x')}}{\zeta - i\epsilon}$, with $\zeta = k + i\kappa$ with the closed integration path C going from $-\infty$ to ∞ along the real axis and then closing over a very distant semicircle over the positive (upper) semiplane for $(x - x') > 0$ or over the lower semiplane for $(x - x') < 0$. The calculus of the residuum for the pole placed

energy per length of a straight dislocation. Usually, this divergence is removed [91] by assuming that the high dislocation energy in a rigid crystal makes dislocations distribute themselves all over the crystal in such a form to cancel mutually their respective strain fields. In this case, the range of the dislocation's deformation field is not the whole crystal's radius Λ but a finite distance of the order of the spacing l between dislocations; this changes $D(|\vec{r} - \vec{r}'|) \sim \ln |\vec{r} - \vec{r}'|$ by a short-range propagator. As we shall see in the model developed in the next Chapter, there is a more elegant form to implement this idea of coupling the crystal rigidity to the dislocation's deformation in order to give a finite range to the dislocation strain fields. Provided one assumes the above argument, the energy per unit length for a straight dislocation becomes

$$\frac{E_{\text{straight}}}{L} \approx \frac{Gb^2}{4\pi} \ln\left(\frac{l}{r_0}\right), \quad (1.10)$$

which for typical dislocation densities — dislocation density (number of dislocations per unit area) ρ is related to l by $\rho = 1/l^2$ — with $l \sim 10^4 a - 10^5 a$

$$\frac{E_{\text{straight}}}{L} \approx Gb^2. \quad (1.11)$$

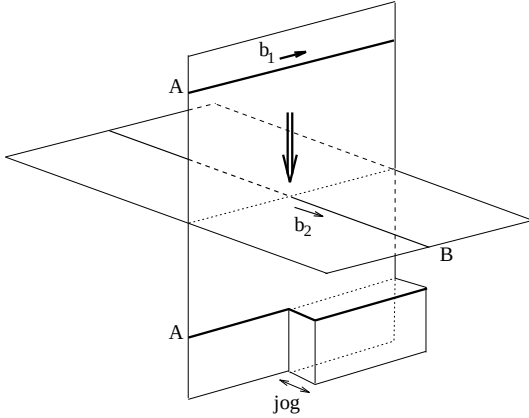


Figure 1.6: Example of the jog formed at screw dislocation 1 after having crossed through screw dislocation 2. Although not represented in the figure, dislocation 2 has suffered an analogous jog.

Returning to the general discussion on dislocations, according to $f_i = \epsilon_{ijk} \hat{t}_j (b_\alpha \sigma_k^\alpha)$ the force per unit length produced by a screw dislocation 1 on another parallel screw dislocation 2 (with $\hat{t}^{(2)} = -\hat{z}$ and $\vec{b}^{(2)} = b^{(2)}\hat{z}$) is given by

$$f_{\text{radial}} = f_x \cos \theta + f_y \sin \theta = \frac{G}{2\pi} \frac{b^{(1)}b^{(2)}}{d}, \quad (1.12)$$

where d is the distance between the two dislocations. For two edge dislocations, the equation is similar by changing G with $G/(1 - \nu)$, because of the tensile components of the stress produced by the dislocations on the surrounding volume. If both Burgers

at $\kappa = \epsilon$ gives then $\frac{1}{2\pi i} \oint_C d\zeta \frac{e^{i\zeta(x-x')}}{\zeta - i\epsilon} = \begin{cases} e^{-\epsilon} & (x - x') > 0 \\ 0 & (x - x') < 0 \end{cases}$. The 2-dimensional Dirac's delta $\delta^2(\vec{r}) \equiv \delta(x)\delta(y) = \frac{1}{2\pi r} \delta(r)$ satisfies therefore $\delta^2(\vec{r}) = \frac{1}{2\pi r} \frac{d}{dr} \vartheta(r)$, from which one obtains immediately the propagator: $\nabla^2 D(r) = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} D(r) \right) = -\frac{1}{2\pi r} \frac{d}{dr} \vartheta(r)$ and hence $\frac{d}{dr} D(r) = -\frac{1}{2\pi r}$.

vectors point in the same direction, $b^{(1)}b^{(2)} > 0$, dislocations repel each other. On the contrary, dislocations with opposite Burgers vectors attract each other: if both are placed on the same plane, they annihilate one another; and if they are on parallel planes, the dislocations form a dipole.

Now let us consider two dislocations whose Burgers vectors point in different directions. When these dislocations cross each other, it can occur that a part of one dislocation is translated a step in the same slip plane (kink) or that a part of one dislocation is translated to another parallel slip plane (jog). In Fig. 1.6 we have the case of two screw dislocations, with Burgers vectors $\vec{b}_1$ and $\vec{b}_2$, crossing each other perpendicularly. After crossing the dislocation 2, a half of the dislocation 1 are translated by an amount equal to b_2 . This half lies on another parallel slip plane and the dislocation 1 has a jog (of course, the dislocation 2 has also a jog in the direction of $\vec{b}_1$). The motion of the jogged part leads to the formation of vacancies and it can only take place at high temperatures. Such a jog obstructs the motion at low temperatures of the dislocation.

1.2 Partial dislocations

The face-centered cubic crystals (elements such as copper, silver or aluminium, or alloys such Ni + Co) are the most studied materials. In this work we shall focus our attention on this type of crystals, since the body-centered cubic crystals have approximately the same characteristics. With respect to the h.c.p. crystals, their properties will be mentioned when they deviate from those of the f.c.c. crystals.

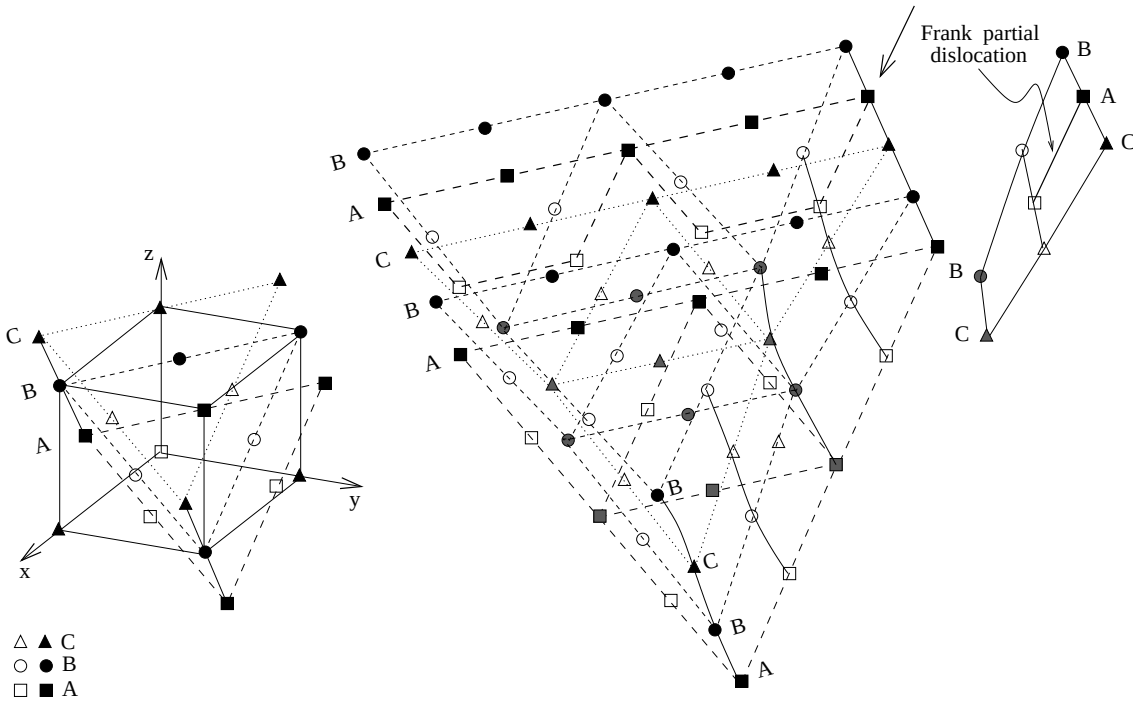


Figure 1.7: Frank partial dislocation after removing a part of the A layer. Below the partial dislocation, the stacking sequence is disturbed: $ABCB$.

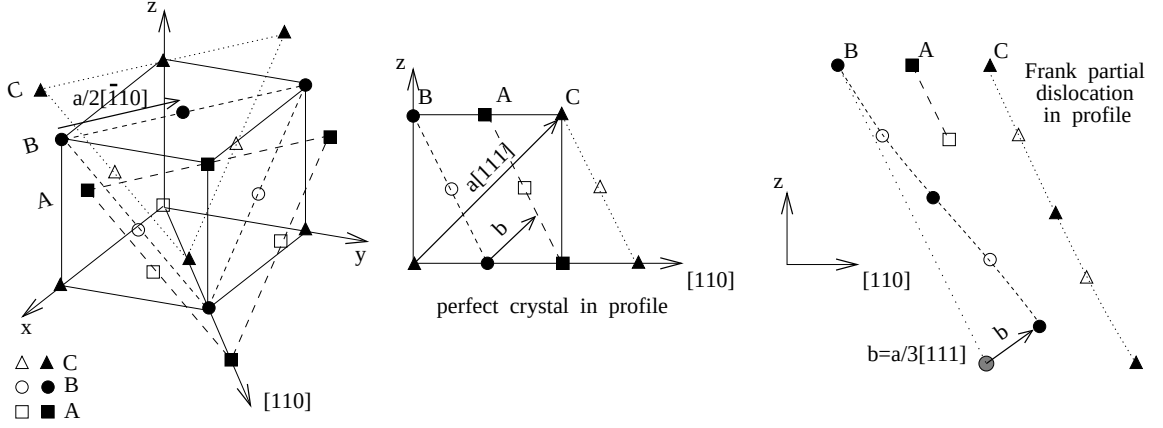


Figure 1.8: Burgers vector of a Frank partial dislocation.

For f.c.c. crystals, the stacking sequence of the atom layers on the $\{111\}$ planes is $ABCABCABC\dots$. The Burgers vector on a certain $\{111\}$ plane is the distance between two nearest lattice points on this plane: for example, on the (111) plane a Burgers vector is $\vec{b}_{\text{total}} = \frac{a}{2}[\bar{1}10]$, where a is the edge length of the cube. On each side of the corresponding dislocation, the atoms have the same stacking sequence $ABCABC\dots$. The stacking sequence can, nevertheless, be damaged and a stacking fault, marked by $|$, is produced: $ABCABC|BCABC\dots$. Let us suppose we remove a half of one A layer (Fig. 1.7): a stacking fault is formed and a dislocation begins where the A layer was removed. Below this dislocation the stacking sequence ($ABC B\dots$) is not equal to the normal sequence above the dislocation ($ABCAB\dots$) and its corresponding Burgers vector $\vec{b}_{\text{partial}} = \frac{a}{3}[111]$ (Fig. 1.8) is smaller than the lattice vector between two neighbour points on the glide plane, i.e., smaller than the Burgers vector of a normal dislocation

$$\vec{b}_{\text{partial}}^2 = \frac{a^2}{3} < \vec{b}_{\text{total}}^2 = \frac{a^2}{2}.$$

Dislocations which do not perturb the stacking sequence are called *total* dislocations and their Burgers vectors are equal to one of the $\langle 101 \rangle$ vectors: for instance, for the (111) plane, all possible Burgers vectors of total dislocations are $[\bar{1}10]$, $[1\bar{1}0]$, $[0\bar{1}1]$, $[01\bar{1}]$, $[10\bar{1}]$, $[\bar{1}01]$. Dislocations created where the stacking sequence is perturbed are called *partial* dislocations and have smaller Burgers vectors than those of total dislocations. The partial dislocation in Fig. 1.7 is called Frank partial dislocation, or sessile partial dislocation, [63] and it cannot glide since its Burgers vector $\vec{b}_{\text{partial}} = \frac{a}{3}[111]$ does not lie on any glide plane $\{111\}$. Partial dislocations whose Burgers vectors are placed on the glide plane and can therefore glide are called Shockley, or glissile, partial dislocations.

At the line where a stacking sequence begins and at the line where it ends there exist partial dislocations. These two dislocations are located on the same plane (the stacking fault plane) and usually repel each other. On the other hand, the stacking fault tends to be as small as possible in order to minimize the disturbed stacking sequence, and thus it takes the two dislocations together. Width d of a stacking fault is given by the equilibrium between the surface tension of the stacking fault γ_s and the repulsive force

between the limiting partial dislocations: $0 = \gamma_s - \frac{D}{2\pi} \frac{b_1 b_2}{d} \Rightarrow d = \frac{D b_1 b_2}{2\pi \gamma_s}$, where D is equal to G for screw dislocations or to $G/(1 - \nu)$ for edge dislocations.

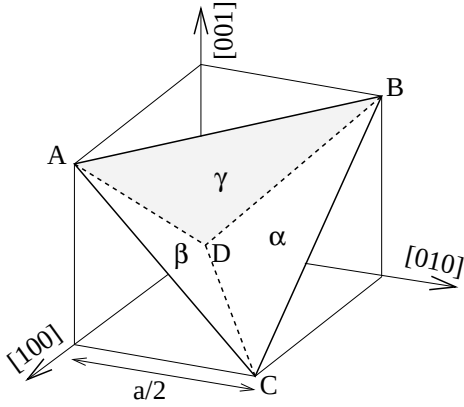


Figure 1.9: Thompson's tetrahedron. The plane (111) is the plane ABC , the cover of the tetrahedron. A, B, C, D denote the vertices of the tetrahedron and greek letters $\alpha, \beta, \gamma, \delta$ identify the possible $\{111\}$ glide planes corresponding to the corners named with latin letters in front of each plane; plane δ is the tetrahedron's cover.

The slip planes in f.c.c. crystals are placed on the faces of a tetrahedron. In order to represent total and partial dislocations, one uses the so-called Thompson's tetrahedron (see Fig. 1.9). The edges of this tetrahedron have a length of $\frac{a}{2}$ and they are all the possible Burgers vectors for total dislocations. For example, on the glide plane (111) the vector $\overrightarrow{AB}$ represents a total edge dislocation and the vector $\overrightarrow{AD}$ a partial Shockley edge dislocation which can glide on the (111) plane. Vectors like $\overrightarrow{\delta\alpha}$ or $\overrightarrow{\delta D}$ represent partial dislocations which cannot glide on (111) because their Burgers vectors are not completely contained in the glide plane. The Frank partial dislocation discussed above is represented in the Thompson's tetrahedron by the vector $2/3\overrightarrow{\delta D}$.

If two (or more) not parallel glide planes, e.g. the primary and any of the secondary planes, are active, dislocations on the primary plane interact with dislocations on the secondary planes leading to the formation of dislocations which cannot glide in either of the interacting planes. The energy of a dislocation with Burgers vector $\vec{b}$ is proportional to $\vec{b}^2$ and since dislocations cannot end inside the crystal, the whole of the Burgers vectors of the reacting dislocations must be transferred to the resulting dislocation: $\vec{b}_{\text{resul}} = \vec{b}_1 + \vec{b}_2$. Thus, two dislocations form a third dislocation if $\vec{b}_1 \cdot \vec{b}_2 < 0$.

Assuming that the primary plane is the (111) (or δ) plane and that the Burgers vector of the gliding dislocations on the primary plane is equal to $\vec{b}_1 = \frac{a}{2}[0\bar{1}1] \equiv \overrightarrow{CA}$, let us consider the following cases [7]:

1. The dislocation on the primary plane interacts with a dislocation on the so-called conjugate plane (11 $\bar{1}$) (plane γ) whose Burgers vector is $\frac{a}{2}[\bar{1}0\bar{1}] \equiv \overrightarrow{AD}$. Since $\vec{b}_1 \cdot \vec{b}_2 < 0$, along the intersection line of the two slip planes both dislocations form a dislocation with Burgers vector equal to $\vec{b}_{\text{LC}} = \frac{a}{2}[0\bar{1}1] + \frac{a}{2}[\bar{1}0\bar{1}] = \frac{a}{2}[\bar{1}\bar{1}0] \equiv \overrightarrow{CD}$.

The vector defining the direction of the formed segment is $\hat{t}_{\text{LC}} = \frac{1}{\sqrt{2}}[\bar{1}\bar{1}0] \parallel \overrightarrow{AB}$, and thus this resulting dislocation is an edge dislocation. Further, it could only glide on a plane which would contain both $\hat{t}_{\text{LC}}$ and $\vec{b}_{\text{LC}}$, for instance plane (001), but such a plane is not one of the $\{111\}$ -planes, where slip takes place in face-centered

cubic crystals. This resulting dislocation is called a *Lomer-Cottrell lock*, cannot glide in any slip plane and obstructs the further glide of the primary dislocation (and of the conjugate dislocation).

2. The dislocation which interacts with the primary dislocation lies on the plane $(1\bar{1}1)$ and has Burgers vector $\frac{a}{2}[10\bar{1}] \equiv \overrightarrow{BC}$. Such a dislocation is called forest dislocation. A dislocation segment with Burgers vector $\frac{a}{2}[1\bar{1}0] \equiv \overrightarrow{BA}$ is formed ($\vec{b}_1 \cdot \vec{b}_2 < 0$) along the line where the primary and the forest planes intersect. Although the resulting Burgers vector lies on the primary plane, the reacting dislocations placed at the beginning and end of this segment become pinned since this segment is energetically more stable. These segments on parallel primary planes form a three-dimensional network [7] which impedes, since the energy of the segments grows like their lengths, the motion of the reacting primary and forest dislocations.

1.3 Plasticity in a crystal under external stress

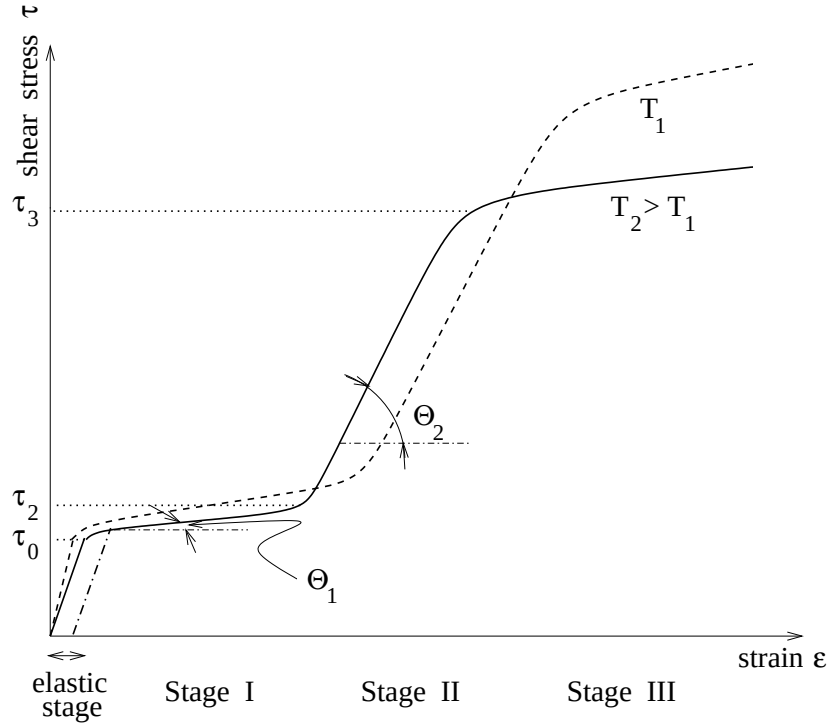


Figure 1.10: Typical hardening curve for (resolved) shear stress versus shear strain in f.c.c. crystals at two different temperatures T_1 and T_2 .

In Fig. 1.10 is represented an example of hardening curve with the resolved shear stress τ_{resol} as a function of the resolved shear strain ε for a face-centered cubic crystal, strain ε being the symmetric part of the deformation tensor $\partial_k u^\alpha$. More properly, the above figure represents only the shear stress, not the resolved shear stress, since the

latter slightly changes during the deformation because of the change in orientation of the slip plane. However, since such a change only implies lesser numerical corrections from the trigonometric factors in the definition of the resolved shear stress, without modifying the characteristics of the hardening curve, we shall consider the hardening curve as the representation of the resolved shear stress. In such curve different regions can be distinguished according to the different slopes, which are called hardening rates. Although Stage I and Stage II are usually not linear regions, at very low temperatures these stages can be considered as approximately linear [13].

1.3.1 Transition from elastic deformation to Stage I

The region, where only elastic deformation occurs, extends usually only until a shear strain ε of the order of 0.01. In this range, if the external stress is removed, the crystal returns to the original size. As the shear stress $\tau \equiv \tau_{\text{resol}}$ is increased above yield point τ_0 , the deformation becomes plastic and after having removed the stress the crystal does not recover its initial shape: the remaining strain is due to the distance covered by the gliding dislocations which have eventually been stopped. Moreover, if after removing one increases again the external stress until reaching the point where the plastic deformation starts, such a stress is bigger than the yield point for an unstrained crystal, i.e., the minimal resolved stress to produce plastic deformation increases with the strain. The interactions among dislocations already produced in the previous stretching with those mobile dislocations which are being created in the second stretching process is the origin of such effect, represented by the dash-dotted line in the elastic range of Fig. 1.10 — if the second stretching begins within the elastic stage but at an already existing strain $\varepsilon \neq 0$ the linear increase characteristic of the elastic behaviour will last until the corresponding minimal shear stress value for the start of plastic behaviour is reached, value which due to the slight slope of Stage I is a bit higher than τ_0 .

1.3.2 Stage I (easy glide)

The dislocations in this stage can glide approximately freely through the crystal. Slip takes place on the set of (parallel) primary planes. The primary dislocations are predominantly edge dislocations and form isolated long bundles (Fig. 1.13), whilst the secondary dislocation density is about two orders of magnitude lower than the primary dislocation density [13]. Thus, only few interactions take place between the two slip systems and the secondary system scarcely hardens the primary one. During the Stage I, long fine slip lines with a length of the order of $1000 \mu m$ (a slip line is the intersection of a slip plane with the external surface of the crystal) are produced and cover uniformly the whole crystal (Fig. 1.12).

The main feature of the Stage I is the production of mobile dislocations. A mechanism to produce dislocations is the Frank-Read source: let be a dislocation line pinned at two points A and B (Fig. 1.13 and Fig. 1.14). Under the applied shear stress the pinned segment bows out. We shall assume that this bowing proceeds in circular segments. Let R be the curvature radius of the bowing segment and $\mathcal{T}$ the line tension, which is of the order of the energy per unit length of a dislocation Gb^2 . The applied

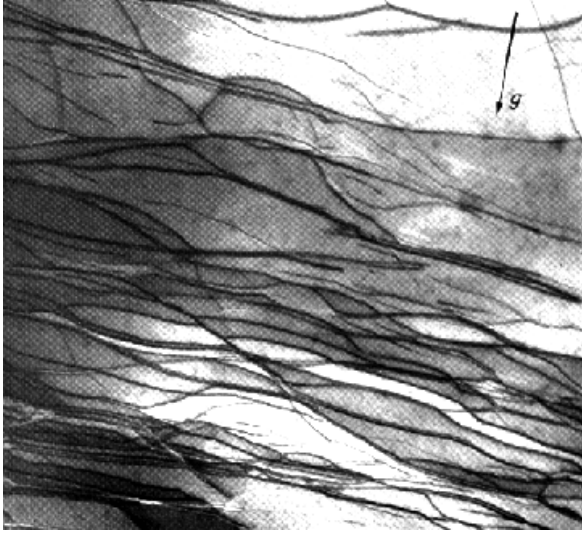


Figure 1.11: Dislocations in Stage I for an f.c.c. Ni+40% Co alloy. Most of the dislocations are perpendicular to the glide direction g , i.e., they are edge dislocations. Glide=0.21. After [67].

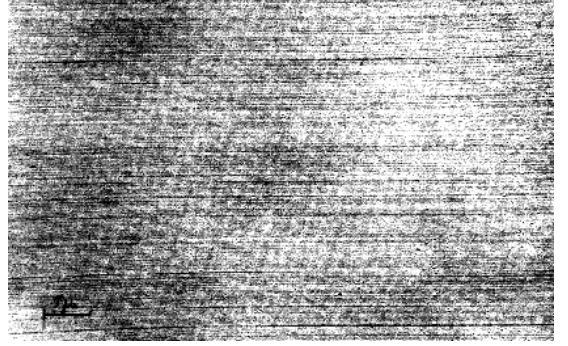


Figure 1.12: Slip lines in Stage I for copper. Glide=0.094. (Mader, S., *Z. Metallk.* **149**, 412 (1957) quoted in [20]).

resolved stress τ is equilibrated by the tension of the dislocation segment

$$\tau b = \frac{\mathcal{T}}{R}.$$

If the shear stress increases, the dislocation segment eventually becomes a semicircle: the bowing is now unstable and the segment begins to close on itself, forming two lobes. The critical stress required for the activation of a Frank-Read source is given by the last equation when the radius of curvature is equal to $\frac{l}{2}$

$$\tau_{\text{crit}} = \frac{2\mathcal{T}}{bl} \sim G \frac{b}{l}, \quad (1.13)$$

where l is the distance between the pinning points A and B . The two lobe segments will touch each other and recombine, since they are in the same glide plane, leaving a closed dislocation loop and the original pinned segment. If this process is repeated, the outer loop expands and other loops are produced at the Frank-Read source. This process stops when the outer loop encounters an obstacle: the inner loops are repealed by the outer one because, according to (1.12), dislocations with the same Burgers vectors repel each other. The inner loops repel the pinned segment which cannot bow out and produce further dislocations.

The characteristics of the hardening curve parameters in Stage I are the following:

- The hardening rate Θ_1 in the Stage I is of the order of $3 \times 10^{-4} G$ [13].
- The extent of Stage I and the hardening rate Θ_1 depend strongly on the orientation. There are soft materials with long extent of Stage I and very low hardening rate,

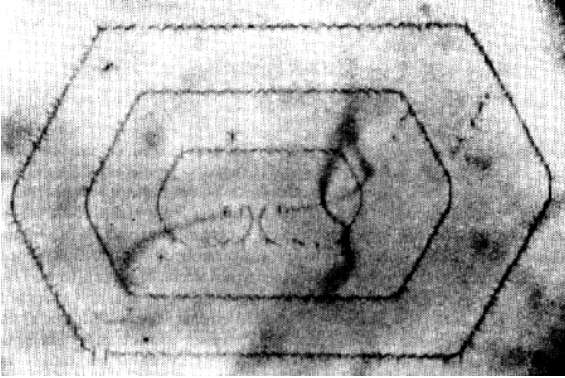


Figure 1.13: A Frank-Read source in silicon decorated with copper.
(W.C. Dash, *J. Appl. Phys.* **27**, 1193 (1956) quoted in [49]).

dislocation pinned at A and B

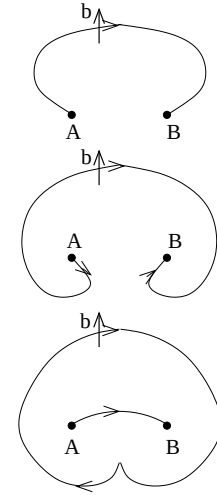
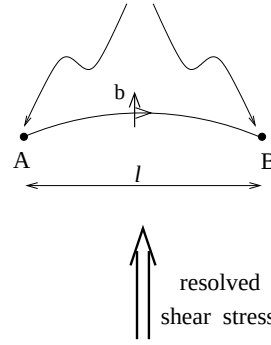
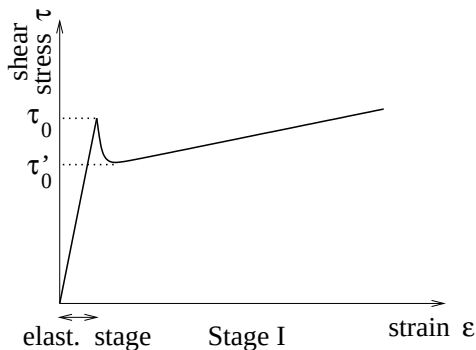


Figure 1.14: The Frank-Read source with a dislocation line pinned at the points A and B separated by a distance l .

and hard materials with very short Stage I and higher hardening rate. At hard orientations, the projection of the tensile external stress on secondary planes are similar to the projection on the primary glide plane, and thus shortly after yielding in the primary plane, the secondary systems are soon activated and interact with the primary, leading to a rapid hardening: the Stage II begins. In the hardest orientation, the Stage I is absent [51].

- The extent of Stage I is increased with decreasing temperature [13]. The hardening rate Θ_1 , on the other hand, can be considered as nearly independent of the temperature [63].

All the above characteristics hold also for body-centered cubic crystals. However, there is a difference, a drop, at the beginning of Stage I in the stress-strain curves of b.c.c. crystals. This is not a feature of this kind of crystals but a consequence of the type of tensile experiment carried out ([84], p. 67). Most of the tensile machines impose a constant deformation velocity: if the dislocation motion for single slip in the



primary plane is strongly obstructed, as it is the case for b.c.c. crystals, the stress supplied by the machine is largely accumulated. Hence when the plastic deformation starts, there is a large production of mobile dislocations which causes the observed drop: the crystal tries to extend more quickly than the stretching produced by the machine stress. After this drop, the hardening curve for body-centered cubic crystals shows the same features as that of the f.c.c. crystals.

1.3.3 Stage II (rapid hardening)

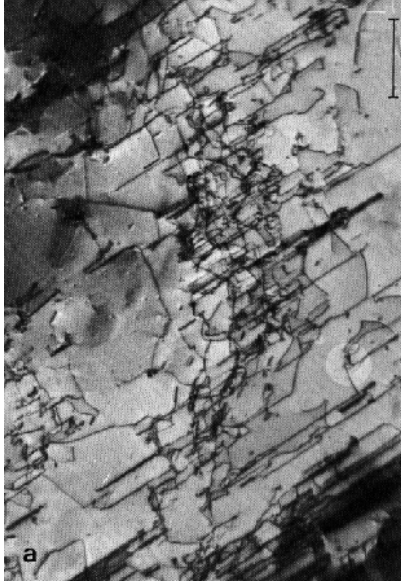


Figure 1.15: Lomer-Cottrell networks at early Stage II in copper at a temperature of 4.2 K. Micrograph of the primary glide plane. After [13].

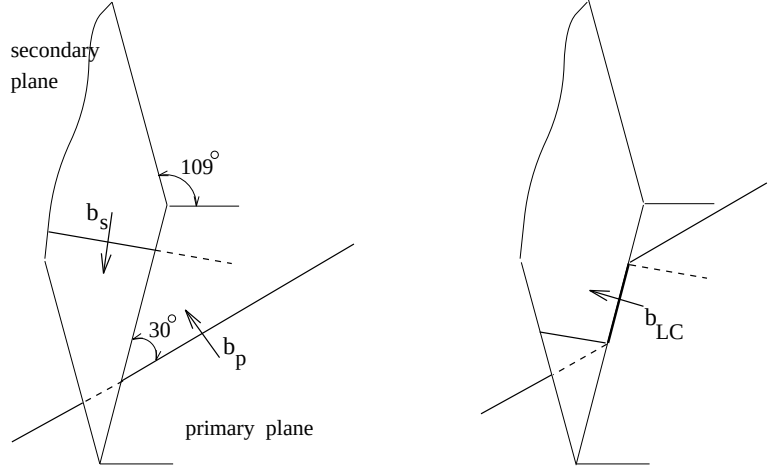
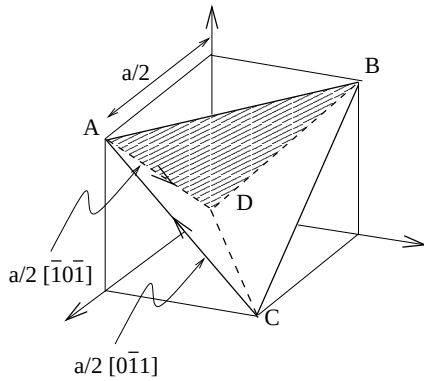
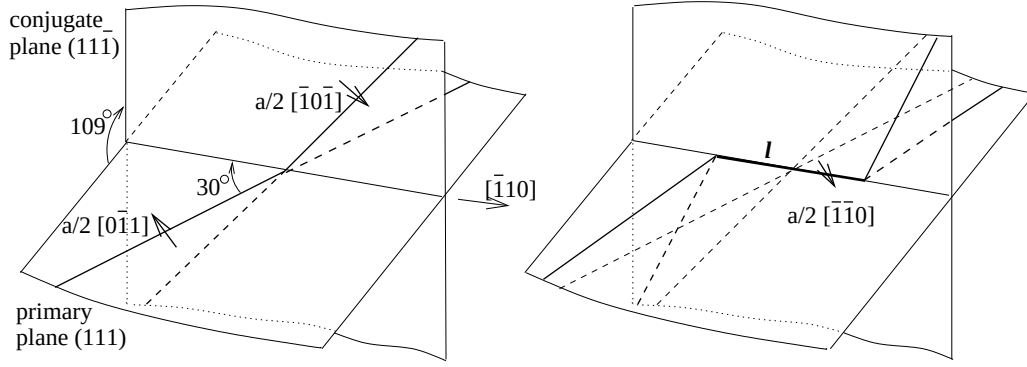


Figure 1.16: Formation of a Lomer-Cottrell segment as in Fig. 1.15 along the intersection line between the primary plane and the secondary one. This Lomer-Cottrell lock cannot glide in any $\{111\}$ slip plane, since none of these planes can contain both the lock and its Burgers vector.

The transition from Stage I to Stage II is characterized by an increase of one order of magnitude in the hardening rate Θ . The slip lines of Stage I become shorter and the density of secondary dislocations reaches the order of magnitude similar of the primary density due to the activity in the secondary systems. The density of obstacles which can impede the free motion of the dislocations on the primary plane are in this stage fairly higher (by one or two orders of magnitude) than the obstacle densities at the begin of the Stage I. In h.c.p. crystals there is only one main glide plane and the activation of secondary glide planes takes hardly place: h.c.p. crystals usually show only the Stage I.



Returning to the f.c.c. and b.c.c. crystals, the dislocation distribution becomes very complex in this stage, appearing patches of straight Lomer-Cottrell segments (Fig. 1.15). The formation of a Lomer-Cottrell lock is described in the following example: let us suppose that the primary slip system is the (111) (the tetrahedron's cover in the figure) with Burgers vector equal to $\vec{b}_p = \frac{a}{2}[0\bar{1}1]$. Moreover, the conjugate slip system (shaded plane in the figure) is the $(11\bar{1})$ plane and the Burgers vector $\vec{b}_s = \frac{a}{2}[\bar{1}0\bar{1}]$.



This two dislocations react with each other ($\vec{b}_p \cdot \vec{b}_s < 0$) along the intersection line between the two planes, forming another dislocation. But the resulting edge dislocation, with Burgers vector $\vec{b}_{LC} = \frac{a}{2}[\bar{1}\bar{1}0]$, cannot glide and thus anchors both the primary and the secondary dislocations which have formed it. Further motion of the primary dislocation on the primary plane would make longer the Lomer-Cottrell segment l increasing its energy (proportional to the length) and thus the system energy. The primary dislocations created at the Frank-Read sources inside the regions surrounded by the Lomer-Cottrell locks can only glide until they reach the Lomer-Cottrell segments [20].



Figure 1.17: Dislocations in Stage II for a f.c.c. Ni+40%Co alloy. The glide direction is denoted by g : the majority of the primary dislocations are edge dislocations. In the upper right corner there are dislocations drawing an angle of 30° with the primary ones. They are the dislocations formed along the intersection line between the primary and the conjugate plane. Glide=0.70. After [67].

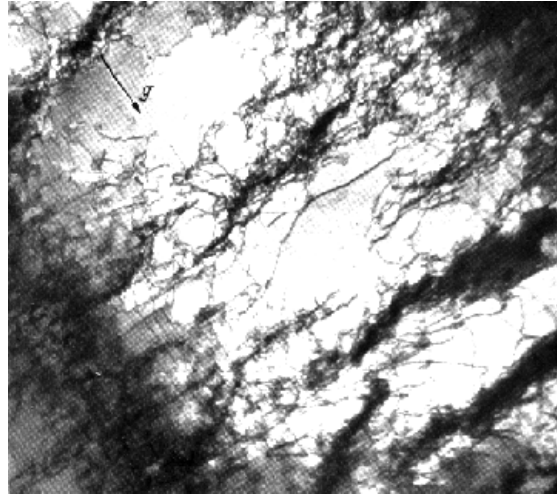


Figure 1.18: Dislocations in late Stage II for a f.c.c. Ni+40%Co alloy. The primary and secondary dislocations (and their products) are grouped together in cords. Between these cords there are still regions of very low dislocation density. Glide=0.83, with g the glide direction. After [67].

As the deformation goes on in Stage II, the mesh length of the Lomer-Cottrell seg-

ments becomes shorter and dense networks are formed approximately parallel to the primary plane [13]. At the end of Stage II (and thus at the begin of Stage III) there are regions parallel to the primary plane containing a low dislocation density because dislocations are collected in cords (Fig. 1.18). A pattern of “discs”, delimited by such cords of Lomer-Cottrell locks and in whose interior dislocations can glide, is formed in the primary plane as represented in Fig. 1.19, “discs” which have a diameter of about $8\mu m$ for copper [67].

Another feature in the dislocation distribution at this stage is the appearance of abrupt changes in the directions of the primary dislocations from a primary plane to another parallel primary plane: these changes are due to the interaction with secondary dislocations which originate a jog in the primary dislocation. Many of these jogs can only move by diffusion of vacancies (see Fig. 1.6) and at low temperatures they act as obstacles for the motion of primary dislocations [67].

The parameters of the hardening curve have the following properties:

- The extent of Stage II depends strongly on the temperature, but the hardening rate Θ_2 is fairly *independent* of temperature, similarly to Stage I. As the temperature increases the extent of Stage II decreases until it disappears and the stress-strain curve is only composed of Stage III [51]. At room temperature there is no well-defined Stage II. Further, the Lomer-Cottrell dislocations play no important role at high temperatures [13].
- The lengths of the slip lines in Stage II decreases with increasing strain, characterized the former by the shortening of the glide distance L_{glide} of screw dislocations according to the relation

$$L_{\text{glide}} = \frac{\Lambda_{\text{hard}}}{\varepsilon - \varepsilon_2}, \quad (1.14)$$

where ε_2 is the shear strain at the start of Stage II and Λ_{hard} a constant of the order of $5 \times 10^{-4} \text{ cm}$ [51] and [50].

1.3.4 Stage III

The distribution of dislocations is characterized by the further development of the cord structure at the end of Stage II, maintaining the “disc”-pattern (Fig. 1.19) delimited by such cords. The Stage III, also called the “parabolic” hardening because of the form in the hardening curve in such stage (its slope decreases gradually with increasing plastic deformation), starts with a sudden decreases in the hardening rate: this signalizes that now certain processes take place which allow the dislocations within the “disc”-structure to overcome the obstacles which are pinning them. Cross-slip is the main part of these processes [67].

Let us consider the partial dislocations A and B with Burgers vectors contained in the primary plane but held by an obstacle such the Lomer-Cottrell locks surrounding the mentioned “discs” — in whose interior dislocations can glide without being impeded. In Fig. 1.21 both partial dislocations arise from the dissociation of a screw dislocation and if



Figure 1.19: “Disc”-structure in the primary plane at the end of Stage II and at the begin of Stage III in copper, where $[101]$ denotes the glide direction. The diameter of the region confined by the dense cords of Lomer-Cottrell locks is about some micrometers. After [67].

they eventually recombine at the pinning points S and S' , building their corresponding screw dislocation again, they can “jump” into the disc placed in an upper primary glide. Once the screw dislocation has reached this new primary plane, it dissociates in the partial dislocations A and B again, which can glide further inside the new disc. The height to be covered in order to reach the new upper primary plane is the height of the piled up cords delimiting the disc in the lower primary plane.

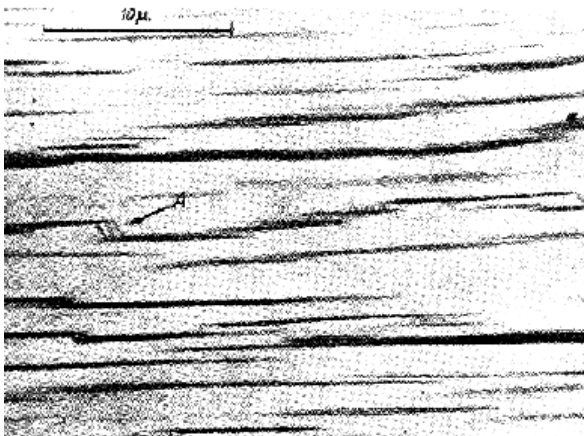


Figure 1.20: Slip bands in Stage III for copper with a cross-slip at A : the step height is about $1\mu m$. (Mader et al., *J. Appl. Phys.* **28**, 53 (1963) quoted in [51]).

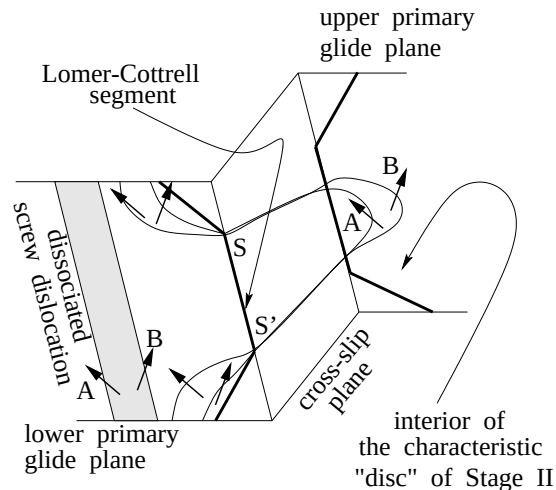


Figure 1.21: Cross-slip of the partial dislocations A and B from the “disc” in a lower primary plane to the “disc” in a higher primary one.

Other process which allows a pinned edge dislocation to move to another glide plane is dislocation climb. Both processes, cross-slip and climb, produce deviations from the Schmid law (1.1) since not only the projection of the external stress on the primary plane but processes at the cross-slip planes or at planes perpendicular to the primary one are also relevant to explain the dislocation distribution in the primary plane. Both processes are strongly dependent on temperature and therefore Stage III does not exist at very low temperatures or it only occurs at very high shear stress [13]. Cross-slip occurs only in f.c.c and b.c.c. crystals because h.c.p. crystals have no slip planes connecting two parallel primary glide planes.

Chapter 2

A gauge principle for the dislocation theory

A compatible or elastic deformed crystal is characterized by the fact that one can relax the crystal to its original size without producing holes inside or without being cut. Compatible deformations are therefore given by a well-defined displacement vector at each point of the crystal. Dislocations, on the other hand, are line defects which separate slipped from unslipped regions in the crystal: a displacement vector which were characterizing a dislocation would suffer a jump across the dislocation line and were therefore not well-defined. Dislocations are not described by displacement vectors, and more important, compatible deformations do not modify the density of dislocations: a compatible deformation only changes the positions of the dislocations but not their number. The changing dislocation density during the different stages shown by a single crystal under stress (stages just produced by the different dislocation configurations) is not due to compatible deformations.

In this Chapter 2, a model will be presented which describes the different stages of a single crystal under stress as different phases of the dislocation system. To achieve such a description, we focus our attention on the density of *gliding* dislocations, because it is not the total dislocation density but the gliding part that responsible for the differences between the elastic and the various plastic stages. Since this density is invariant under compatible deformations, we shall construct a model with an energy invariant under such deformations. The main idea in this work will be to use the Abrikosov model of the Superconductivity (see Appendix A for an introduction to this subject) as model: a superconductor is a system which opposes the penetration of the magnetic field since it destroys the material's rigidity, where this rigidity is defined as the nonzero macroscopic expectation value of a scalar field called order parameter. In the model developed in this work, a rigid crystal is a system which opposes the existence of gliding dislocations since a high gliding dislocation density makes the crystal become soft and destroys its rigidity. With this new model we shall define a partition function and study the different phases characterized by the different behaviours of some correlation functions. As we shall see, these phases show properties similar to those of the different stages of a single crystal under stress. Most of the discussion holds for a system at a very low temperature; the effect of high temperatures will be considered in the next Chapter.

2.1 Definition of dislocation density and current

Let us consider a crystal subjected to a compatible deformation. In order to describe the deformation (Fig. 2.1), the following coordinate systems are used (*Tensor Analysis* by I.S. Sokolnikoff, John Wiley and Sons, New York (1964), §109-110):

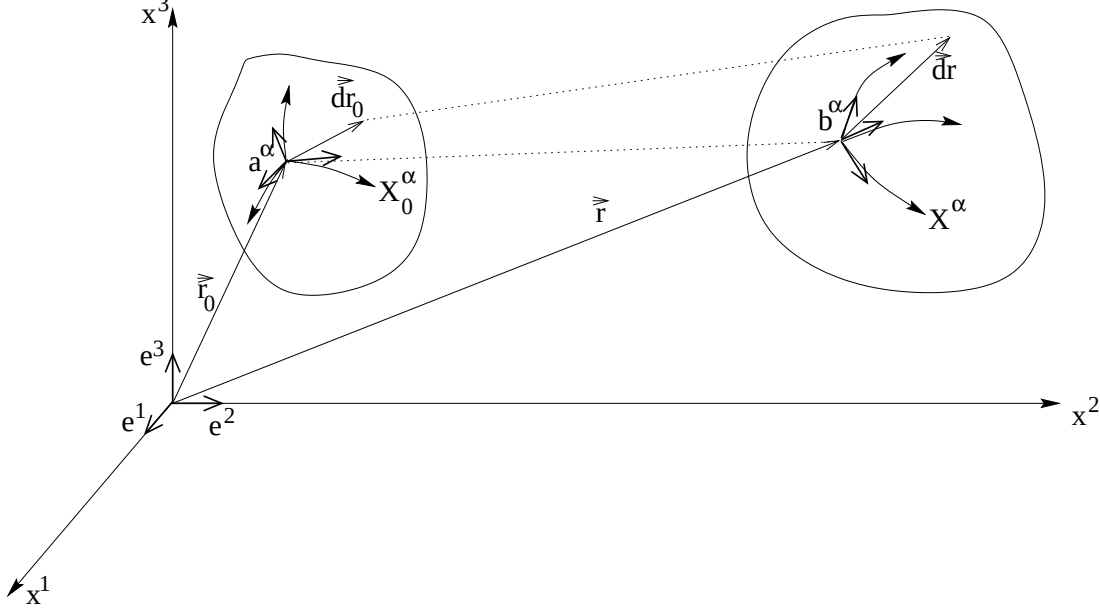


Figure 2.1: External coordinates x^k and substantial (with-material-comoving) coordinates X^α .

- $(x^i)_{i=1,2,3}$ and $\{\vec{e}_i\}_{i=1,2,3}$ are, respectively, the external cartesian coordinates and vectors.
- $(X^\alpha)_{\alpha=1,2,3}$ and $\{\vec{b}_\alpha\}_{\alpha=1,2,3}$ are the (internal) crystallographic coordinates and vectors in the deformed state. They are defined relative to an in-lattice-sitting observer, i.e., $\{\vec{b}_\alpha\}$ are the local vectors of the Bravais lattice. Let $\vec{v}(x^i, t)$ be the velocity of the mass point P , which has coordinates x^i in the external coordinate system

$$v^j(x^i, t) = \lim_{\Delta t \rightarrow 0} \frac{x^j(X^\alpha, t + \Delta t) - x^j(X^\alpha, t)}{\Delta t} = \left. \frac{\partial x^j(X^\alpha, t)}{\partial t} \right|_{X^\alpha = \text{const}}. \quad (2.1)$$

The above derivative is taken for a constant X^α since this last set of coordinates are moving together with the material as the system is deformed. The acceleration vector of point P is then given by

$$a^j(x^i, t) = \lim_{\Delta t \rightarrow 0} \frac{v^j(X^\alpha, t + \Delta t) - v^j(X^\alpha, t)}{\Delta t} = \left. \frac{\partial v^j(X^\alpha, t)}{\partial t} \right|_{X^\alpha = \text{const}}, \quad (2.2)$$

with $v^j(X^\alpha, t) = v^j(x^i(X^\alpha, t), t)$ and thus

$$\begin{aligned} \left. \frac{\partial v^j(X^\alpha, t)}{\partial t} \right|_{X^\alpha = \text{const}} &= \left. \frac{\partial v^j(x^i, t)}{\partial t} \right|_{x^i = \text{const}} + \underbrace{\frac{\partial v^j(x^i, t)}{\partial x^m} \frac{\partial x^m(X^\alpha, t)}{\partial t}}_{v^m(x^i, t)} \Big|_{X^\alpha = \text{const}}, \\ a^j(x^i, t) &= \left. \frac{\partial v^j(x^i, t)}{\partial t} \right|_{x^i} + v^m(x^i, t) \frac{\partial v^j(x^i, t)}{\partial x^m} \stackrel{\text{def}}{=} \frac{d}{dt} v^j(x^i, t), \end{aligned} \quad (2.3)$$

where the differential operator $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$ is called the convective (or material) derivative and can be applied to any function $f = f(x^i, t)$: $\frac{df(x^i, t)}{dt} \equiv \frac{\partial f(X^\alpha, t)}{\partial t} \Big|_{X^\alpha} = \frac{\partial f(x^i, t)}{\partial t} \Big|_{x^i} + v^m \partial_m f(x^i, t)$. Quantity $\frac{\partial f(X^\alpha, t)}{\partial t} \Big|_{X^\alpha}$ is the change of f with the time, measured by an in-lattice-living observer, whereas $\frac{\partial f(x^i, t)}{\partial t} \Big|_{x^i}$ is the temporal change observed in the external labor system. Since coordinates X^α move together with the material during a deformation, they remain constant for the internal observer, $\frac{\partial X^\alpha}{\partial t} \Big|_{X^\alpha = \text{const}} = 0$, or equivalently, their convective derivative vanishes

$$\partial_t X^\alpha = -v^m \partial_m X^\alpha. \quad (2.4)$$

- $(X_0^\alpha)_{\alpha=1,2,3}$ and $\{\vec{\mathbf{a}}_\alpha\}_{\alpha=1,2,3}$ are the crystallographic coordinates and vectors in the initial, undeformed state. The crystallographic coordinates of the undeformed state X_0^α are equal to the crystallographic ones of the deformed state X^α , because these coordinates are dragged along with the material during the deformation.

Let $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{r}}$ be the position vector of the in-lattice-residing observer relative to the external coordinates, for the undeformed and deformed state respectively. The Bravais lattice vectors for the initial and deformed state $\{\vec{\mathbf{a}}_\alpha\}$ and $\{\vec{\mathbf{b}}_\alpha\}$ are given by

$$\vec{\mathbf{a}}_\alpha = \frac{\partial \vec{\mathbf{r}}_0}{\partial X_0^\alpha} = \frac{\partial \vec{\mathbf{r}}_0}{\partial X^\alpha} \quad \text{and} \quad \vec{\mathbf{b}}_\alpha = \frac{\partial \vec{\mathbf{r}}}{\partial X^\alpha}. \quad (2.5)$$

Let us now consider a vector $d\vec{\mathbf{r}}_0$ in the initial state which becomes deformed into $d\vec{\mathbf{r}}$

$$d\vec{\mathbf{r}}_0 = \vec{\mathbf{e}}_j dx_0^j = \vec{\mathbf{a}}_\alpha (\partial_{0k} X^\alpha) dx_0^k \quad \text{and} \quad d\vec{\mathbf{r}} = \vec{\mathbf{e}}_j dx^j = \vec{\mathbf{b}}_\alpha (\partial_k X^\alpha) dx^k.$$

From these relations and using (2.4), one defines vector field $\{\vec{\mathbf{W}}^\alpha = W_k^\alpha \vec{\mathbf{e}}_k, W_0^\alpha\}$ as

$$W_k^\alpha = \partial_k X^\alpha \quad \text{and} \quad W_0^\alpha = v^m W_m^\alpha, \quad (2.6)$$

which, according to $d\vec{r} = \vec{e}_k dx^k = \vec{b}_\alpha W_k^\alpha dx^k$, satisfies $\vec{b}_\alpha \cdot \vec{e}^j \equiv (\vec{b}_\alpha)^j = (W^{-1})_\alpha^j$ and

$$\vec{b}_\beta \cdot \vec{W}^\alpha = \delta_\beta^\alpha. \quad (2.7)$$

Hence, vectors $\vec{W}^\alpha$ are called the reciprocal Bravais lattice vectors [33].

The deformation $d\ell^2$ and the strain (or deformation) tensor $\varepsilon_{\alpha\beta}$ are defined as follows

$$d\ell^2 = d\vec{r}^2 - d\vec{r}_0^2 = (\vec{b}_\alpha \cdot \vec{b}_\beta - \vec{a}_\alpha \cdot \vec{a}_\beta) dX^\alpha dX^\beta \stackrel{\text{def}}{=} 2\varepsilon_{\alpha\beta} dX^\alpha dX^\beta. \quad (2.8)$$

Without loss of generality, we can suppose that the lattice vectors in the initial undeformed state are orthonormal to each other: $(\vec{a}_\alpha)^j = \delta_\alpha^j$ and $(W_0)^\beta_j = \delta_j^\beta$. Let $\vec{u} = u^j \vec{e}_j$ be the displacement of the lattice's cells during a compatible deformation: $d\vec{r} = d\vec{r}_0 + d\vec{u}$. The lattice vectors in the deformed state are given by

$$(\vec{b}_\alpha)^j = (\vec{a}_\alpha)^j + \partial_\alpha u^j \stackrel{\partial_\alpha = (W^{-1})_\alpha^r \partial_r}{=} \delta_\alpha^j + (W^{-1})_\alpha^r \partial_r u^j.$$

In the linear theory, where $\vec{u}$ is small, we obtain the following approximations for the strain tensor ε and for $W_j^{\beta-1}$

$$2\varepsilon_{\alpha\beta} = \vec{b}_\alpha \cdot \vec{b}_\beta - \vec{a}_\alpha \cdot \vec{a}_\beta \stackrel{\text{linear}}{\approx} \partial_\beta u_\alpha + \partial_\alpha u_\beta. \quad (2.9)$$

$$W_j^\beta \approx \delta_j^\beta - W_m^\beta \partial_j u^m \stackrel{\text{linear}}{\approx} \delta_j^\beta - \partial_j u^\beta. \quad (2.10)$$

After this introduction, let us now consider a crystal with dislocations. As we have seen in the first chapter, dislocations are the limiting lines between the slipped and the not-yet-slipped region. At the site of a dislocation, the coordinates $X^\alpha = X^\alpha(x^k)$ are multivalued and two derivatives applied to X^α do not commute

$$\begin{cases} \partial_j \partial_k X^\alpha \neq \partial_k \partial_j X^\alpha & \Longleftrightarrow \partial_j W_k^\alpha \neq \partial_k W_j^\alpha, \\ \partial_t \partial_k X^\alpha \neq \partial_k \partial_t X^\alpha & \Longleftrightarrow \partial_t W_k^\alpha \neq -\partial_k W_0^\alpha. \end{cases}$$

The density α and current or flow density $\mathcal{J}$ of dislocations are defined by

$$\begin{aligned} \alpha_i^\alpha &= \epsilon_{ijk} \partial_j W_k^\alpha, \\ \mathcal{J}_k^\alpha &= -\partial_k W_0^\alpha - \partial_t W_k^\alpha. \end{aligned} \quad (2.11)$$

If there exists long distance parallelism in the crystal, then the (reciprocal) Bravais lattice vectors W_k^α are well-defined and two derivatives applied on them do commute, $\begin{cases} \partial_i \partial_j W_k^\alpha = \partial_j \partial_i W_k^\alpha \\ \partial_i \partial_t W_k^\alpha = \partial_t \partial_i W_k^\alpha \end{cases}$ and $\begin{cases} \partial_i \partial_j W_0^\alpha = \partial_j \partial_i W_0^\alpha \\ \partial_i \partial_t W_0^\alpha = \partial_t \partial_i W_0^\alpha \end{cases}$, property which leads to the following equations

$$\partial_i \alpha_i^\alpha = \epsilon_{ijk} \partial_i \partial_j W_k^\alpha = 0, \quad (2.12)$$

¹From $(\vec{b}_\alpha)^j W_j^\beta = \delta_\alpha^\beta$ and $(\vec{b}_\alpha)^j = \delta_\alpha^j + (W^{-1})_\alpha^r \partial_r u^j$ it follows $W_j^\beta \approx \delta_j^\beta - W_m^\beta \partial_j u^m$.

and

$$\epsilon_{ijk} \partial_j \mathcal{J}_k^\alpha = 0 - \epsilon_{ijk} \partial_j \partial_t W_k^\alpha = -\partial_t \alpha_i^\alpha. \quad (2.13)$$

As an example of the last equation, let us consider the case where during an elastic deformation with constant velocity v_i , dislocations are dragged along with the crystal lattice. Then, dislocation density current $\mathcal{J}_r^\alpha$ is given by

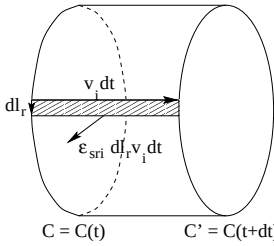
$$\mathcal{J}_r^\alpha = -\epsilon_{ris} v_i \alpha_s^\alpha, \quad (2.14)$$

as one can easily prove: since dislocations are only dragged along, the convective derivative of the dislocation density vanishes and thus $-\partial_t \alpha_j^\alpha = v_k \partial_k \alpha_j^\alpha$

$$\epsilon_{jkr} \partial_k \mathcal{J}_r^\alpha = v_k \partial_k \alpha_j^\alpha - v_j \partial_k \alpha_k^\alpha \stackrel{\partial_k \alpha_k^\alpha = 0}{=} -\partial_t \alpha_j^\alpha,$$

which is general result (2.13) for this particular case; case which also corresponds to the well-known relation in electromagnetism between, on the one hand, the electric field measured in a coordinate frame moving with constant velocity within a region with a nonvanishing magnetic field, and on the other hand, the electric field in the resting coordinate system [66] §49². Also the convective derivative of the dislocation current vanishes: $\partial_t \mathcal{J}_r^\alpha = -\epsilon_{ris} v_i \partial_t \alpha_s^\alpha \stackrel{\partial_t \alpha_j^\alpha = -v_k \partial_k \alpha_j^\alpha}{=} -v_k \partial_k (-\epsilon_{ris} v_i \alpha_s^\alpha).$

²The rate of change with time of the magnetic flux $-\frac{d\Phi}{dt}$ crossing the surface enclosed by a closed path C measured by a coordinate frame K' moving with a constant velocity $\vec{v}$ with respect to another coordinate frame K within a spatial region with a nonvanishing magnetic field $\vec{B}$ is equal to $-\frac{d\Phi}{dt} = \oint_C \vec{E}' \cdot d\vec{l}$, where $\vec{E}'$ is the electric field measured by K' . This $-\frac{d\Phi}{dt}$ is formed by a) the rate of change of the magnetic flux measured by the resting frame K , due to any variation of $\vec{B}$ and equal to $-\frac{\partial\Phi}{\partial t}\Big|_{\vec{v}=\vec{0}} = \oint_C \vec{E} \cdot d\vec{l}$, plus b) the change produced by the motion of the closed path C inside a constant $\vec{B}$. The latter change is given by $\oint_C (\vec{v} \times \vec{B}) \cdot d\vec{l}$ as easily shown: from relation $\oint_C \epsilon_{ris} (v_i dt) B_s \frac{dl_r}{dt} = \int_{S_{\text{lateral}}} \vec{B} \cdot \frac{d\vec{S}_{\text{lateral}}}{dt}$, with $d\vec{S}_{\text{lateral}} = d\vec{l} \times (\vec{v} dt)$ the side surface element covered by C during its motion in the time interval dt , together with the Maxwell equation stating that the magnetic flux crossing any closed surface vanishes identically, leads to the result that $\int_{S_{\text{lateral}}} \vec{B} \cdot d\vec{S}_{\text{lateral}}$ is equal to minus the flux leaving the closed surface of the figure across the area enclosed by $C' = C(t + dt)$ plus the flux entering through $C = C(t)$. Hence equation $\oint_C (\vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{\partial\Phi}{\partial t}\Big|_{\vec{B}=\text{const}}$ holds and therefore $\oint_C \vec{E}' \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\left(\frac{\partial\Phi}{\partial t}\Big|_{\vec{v}=\vec{0}} + \frac{\partial\Phi}{\partial t}\Big|_{\vec{B}=\text{const}}\right) = \oint_C (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$ or equivalently $\vec{E}'\Big|_{\text{measur. by } K'} = \vec{E}\Big|_{\text{measur. by } K} + \vec{v} \times \vec{B}$. If the magnetic field moves



with just the velocity $\vec{v}$ then K' does not observe any variation of the magnetic flux, $\frac{d\Phi}{dt} = 0$, or equivalently $\vec{E}' = \vec{0}$: thus $\vec{E} = -\vec{v} \times \vec{B}$ holds, which is a result analogous to (2.14) since both the dislocation density α as well as the dislocation current $\mathcal{J}$ are defined by the same kind of differential equations as the magnetic and the electric field.

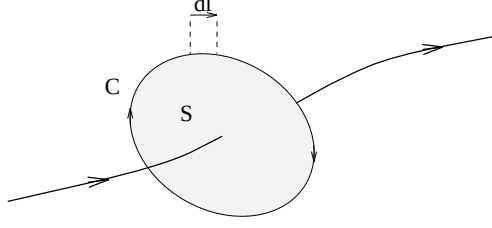


Figure 2.2: Definition of the Burgers vector of a dislocation.

In the non-linear theory, the Burgers vector is defined as flux of the dislocation density (Fig. 2.2)

$$b^\alpha = \int_S \alpha_i^\alpha dS^i = \oint_C W_k^\alpha dl^k. \quad (2.15)$$

This equation turns into equation $b^i = - \oint_C \frac{\partial u^i}{\partial s^k} ds^k$ of the linear theory [65] §27 for small deformations because of relation (2.10), $W_k^\beta \approx \delta_k^\beta - \partial_k u^\beta$. Note that the vector character of b^α (with $\alpha = 1, 2, 3$) is measured with respect to the internal crystallographic system. The usual density ρ_i^α of dislocations along the i direction with Burgers vector in the α direction is expressed as number of dislocations per unit area and it is related to α_i^α through $\alpha_i^\alpha = b^\alpha \rho_i^\alpha$ without summing over index α .

Both equations (2.12) and (2.13) are similar to those equations in the electromagnetism which state the absence of magnetic-monopole densities and currents: $\text{div} \vec{B} = 0$ and $\text{curl} \vec{E} = -\partial_t \vec{B}$. This is, nevertheless, not the usual analogy between the electromagnetism and the dislocation theory found in many books, an analogy which starts with the equilibrium condition $\text{div} \sigma = 0$ (σ is the stress tensor) as the analogue to $\text{div} \vec{B} = 0$ and then defines the Airy's function χ as analogous field to the vector potential in the electromagnetism: $\sigma \sim \text{curl} \chi$. Such an analogy does not work for the dynamical case, where $\text{div} \sigma = \frac{d}{dt}(\rho v)$ with ρ the mass density.

The above analogy between, on the one hand, $\begin{cases} \text{div} \alpha^\alpha = 0 \\ \text{curl} \mathcal{J}^\alpha = -\partial_t \alpha^\alpha \end{cases}$ and the two equations of the electromagnetism $\begin{cases} \text{div} \vec{B} = 0 \\ \text{curl} \vec{E} = -\partial_t \vec{B} \end{cases}$ on the other hand, can be also found in [32].

2.2 Equations of dislocation dynamics *à la* Maxwell

In order to determine any vector field we need the divergence as well as the curl of such a vector. Hence it is necessary to find the equation for $\text{curl} \alpha^\alpha$ and $\text{div} \mathcal{J}^\alpha$, where the differential operators act on the latin (external) indices of both fields, not on the internal index α which denotes the direction of the Burgers vector. In particular, the equations $\begin{cases} \text{div} \alpha^\alpha = 0 \\ \text{curl} \mathcal{J}^\alpha = -\partial_t \alpha^\alpha \end{cases}$ carry the topological (or geometrical) content of the

model we are constructing, i.e., the existence of well-defined Bravais vectors throughout the whole crystal; but the equations which relate how the dislocation get distributed under the action of a stress and how the dislocation contribute to the mass flow in a plastic deformed crystal, are still missing. Therefore, and although not considered in the current works about dislocation theory, the analogy with the Maxwell theory can be completed [68] by finding the dislocation equations analogous to $\text{curl } \vec{H} = \vec{j}_{e1} + \partial_t \vec{D}$ and $\text{div } \vec{D} = \rho_{e1}$.

In order to solve how a stress σ_i^α external to a dislocation distribution modifies such an arrangement, one usually consider the case [65] §30 where all dislocations are parallel to a certain direction (say the z -direction) and their distribution along a direction transversal to the dislocation direction, for instance the x -direction, gets changed by the action of σ_y^α (with $\alpha = x$ if the dislocations under study are of edge type): an integral equation of the kind $\sigma_y^x = -Db \int_{a_1}^{a_2} \frac{\alpha_z^x(x) d\xi}{\xi - x}$ has to be solved, which is the generalization to a continuous dislocation density $\alpha_z^x(x)$ (of edge dislocations parallel to the z -direction and with Burgers vector in the x -direction) of result (1.6), being D an elastic constant equal to $G/(1 - \nu)$ for edge dislocations, and a_1 and a_2 the spatial interval along the x -direction where the dislocations are held. Such an equation is the combination of $\text{div } \alpha^\alpha = 0$ — carrier of the existence of a stable Burgers vector along the whole dislocation line — reduced to this particular 2-dimensional case, plus a material relation which relates the external stress to the strain ε_y^x giving the dislocation density by means of $\alpha_z^x = \partial_x \varepsilon_y^x$: such a relation characteristic of every crystal is called the Hooke's law, expressed by $\sigma_y^x = D\varepsilon_y^x$, which measures the rigidity of the crystal.

Instead of solving the just mentioned integral equation relating the stress on a dislocation group and the dislocation distribution itself, in the model we shall construct in this work, the stress external to the dislocation arrangement as well as the yielded mass flow will be considered as sources for the dislocation density and current, being the former related to these latter fields through differential equations similar to those of the Maxwell theory in which the sources for the magnetic and the electric field enter. Further, such formulation will allow us to obtain a model able to describe all stages during a plastic deformation. And the Hooke's law characterizing the crystal's rigidity, and which in the usual formulation was the link relating stress and dislocation density, will become now a special case (of a more general equation) for the particular cases of elastic and easy glide stages.

Before entering in the discussion of this new model, some remarks are noteworthy. The equations in the form (2.12)-(2.13) or like those in [32] are not useful to describe the hardening curve: they deal with the *total* dislocation density and current, but the great difference between the elastic stage and any plastic stage does reside in the density of dislocations which can *glide* through the crystal and not in the total dislocation density (which in the first steps of stage I is approximately the same as in the elastic stage although both stages display different mechanical properties). It is therefore necessary to focus our attention on the mobile dislocations; and it will result that this standpoint, together with the analogy to how a superconductor works effectively, will enable us to formulate a complete dislocation model to describe the whole hardening curve.

The experimental measurement of the mobile dislocation density in the different

stages of the plastic deformation [77] for single slip³ is represented in Fig. 2.3. Such an

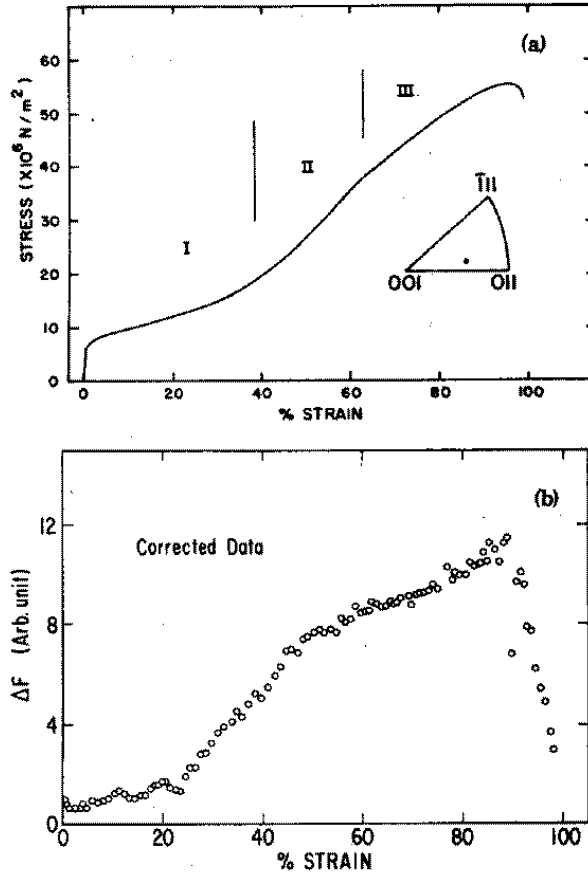
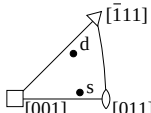


Figure 2.3: Plastic stages in the stress-strain curve (a) with only one slip system activated in a lead-indium type-II superconductor at a temperature of 4.2 K and at a strain rate of 10^{-4} s^{-1} . Representation (b) (in arbitrary units and as a function of the plastic shear strain) of the change in the magnetic flux density accompanying the motion of dislocations which were pinning the flux lines. Diagram (b) represents thus the change of mobile dislocation density $\alpha_{\text{glid}}^{\alpha}$ during the plastic deformation. For the elastic stage (zero plastic strain) there is no change in $\alpha_{\text{glid}}^{\alpha}$. After [77].

³As represented in , if the tensile stress σ_T lies near to a two-fold symmetry axis ()

(point *s*), then only the primary plane ($\bar{1}11$) is activated since the projection of the tensile stress on other planes is nearly vanishing: this case is called the single slip and is the only form to observe the Stage I. On the other hand, if the tensile stress acts near a three-fold axis Δ (point *d*), then the primary slip system as well as the conjugate system — the latter corresponding to the triangle in Fig. 1.2 on page 10 just above and on the other side of axis $[\bar{1}11]$ — are activated nearly with the same tensile projection: as a result, as soon as the plastic deformation starts the sessile dislocations, produced by the reaction between the dislocations in one slip system with the dislocations in the other one, lead directly to Stage II.

experiment bases on the fact that flux-lines in a superconductor of type II are pinned by dislocations and when the latter become mobile because the superconductor is being deformed plastically, a change in the magnetic flux density is observed. That this effect is due to the motion of dislocations gets proved because during the elastic stage no change in magnetic flux is measured in the superconductor.

Since we are interested in describing the hardening curve, and since its behaviour is determined by the mobile dislocations, the gliding part in the dislocation density and current will be separated out, being denoted by $\mathbf{B}_i^\alpha$ and $\mathbf{E}_i^\alpha$ respectively

$$\boldsymbol{\alpha}_{\text{gliding}}^\alpha \equiv \mathbf{B}^\alpha, \quad (2.16)$$

$$\mathcal{J}_{\text{gliding}}^\alpha \equiv \mathbf{E}^\alpha. \quad (2.17)$$

Fields $\mathbf{B}^\alpha$ and $\mathbf{E}^\alpha$ contain the density and current of *all mobile* (or gliding) dislocations in *any* slip system, the primary as well as the other slip systems (which in a face centered cubic crystal corresponds to the $\{111\}$ planes), dislocations which by definition have their Burgers vector contained completely in one of the slip planes. The information about the density and current of all not-able-to-glide or sessile dislocations, $\boldsymbol{\alpha}_{\text{sessile}}^\alpha$ and $\mathcal{J}_{\text{sessile}}^\alpha$, will be carried by other fields to be introduced soon.

The reason for the notation in (2.16)-(2.17), which resembles that of the electromagnetism, will become gradually clearer throughout this Chapter. At first glance, the above separation of the gliding part looks rather arbitrary and deserves a more detailed explanation. In the electromagnetism, one studies how a system reacts under applied external fields and how much the fields which are allowed to exist within the system, $\vec{B}$ and $\vec{E}$, differ from the applied ones, $\mu_0 \vec{H}$ and $\epsilon_0^{-1} \vec{D}$. Returning to the dislocation model, let us firstly assume that only one slip system (the primary one) is activated, so that we need not be worried about other slip systems — and about their accompanying reaction with the primary dislocations to form sessile dislocations. Let us denote by $\mu_0 \mathbf{H}^\alpha$ and $\epsilon_0^{-1} \mathbf{D}^\alpha$, respectively, the density and current of dislocations in the primary system (whose Burgers vector is completely contained on the primary plane), corresponding $\mathbf{B}^\alpha$ and $\mathbf{E}^\alpha$ to the density and current of the dislocations actually allowed by the crystal to glide through the primary slip system (Table 2.1). Dislocations whose motion is not gliding but climbing will be dealt with much later, together with the cross-slip and the temperature effects.

	gliding in the primary slip system	
dislocation density	$\mathbf{B}_i^\alpha = (\boldsymbol{\alpha}_{\text{gliding}})_i^\alpha$	$\mu_0 \mathbf{H}_i^\alpha = (\boldsymbol{\alpha}_{\text{primary}})_i^\alpha$
dislocation current	$\mathbf{E}_i^\alpha = (\mathcal{J}_{\text{gliding}})_i^\alpha$	$\epsilon_0^{-1} \mathbf{D}_i^\alpha = (\mathcal{J}_{\text{primary}})_i^\alpha$

Table 2.1: Definitions of the gliding and primary dislocation density and current with only one slip system activated.

Analogously to the electromagnetism, we impose by external means a dislocation density and current $\mu_0 \mathbf{H}^\alpha$ and $\epsilon_0^{-1} \mathbf{D}^\alpha$ on the primary slip system and then we ask ourselves how many of these dislocations actually glide. Depending on the relation between the dislocation density/current in the primary slip system and the real gliding dislocation density/current, the crystal will show different stages, elastic and plastic (easy glide stage, Stage III, ...). The response of the material to an imposed dislocation density and current on the primary slip system are characterized, as in the electromagnetism, by some material parameters μ_{rel} and ϵ_{rel}

$$\mathbf{B}_i^\alpha = \mu_{\text{rel}}(\mu_0 \mathbf{H}_i^\alpha) \quad \text{and} \quad \mathbf{E}_i^\alpha = \epsilon_{\text{rel}}^{-1}(\epsilon_0^{-1} \mathbf{D}_i^\alpha), \quad (2.18)$$

which are in general tensors. However and for simplicity, we shall assume through this whole work that μ_{rel} and ϵ_{rel} are constants, being its generalization to a tensor straightforward.

If $\mu_{\text{rel}} \approx 1$, or equivalently if $\mathbf{B}_i^\alpha \approx \mu_0 \mathbf{H}_i^\alpha$, nearly all dislocations in the primary slip system — which by definition would be able to glide if they were not pinned since their Burgers vector completely lies on the primary slip — are actually mobile dislocations. This state corresponds to a well developed “parabolic” hardening or Stage III in the hardening curve, or to an equivalent state where the crystal has lost most of its stiffness becoming very soft. On the contrary, in a plastic deformation stage characterized by $\mu_{\text{rel}} \ll 1$ only a small fraction of the would-be gliding dislocations is allowed to move through the crystal, crystal which maintains thus its rigidity. In the language of the electromagnetism, this second case would correspond to a diamagnetic system. And the elastic stage, where the density and current of gliding dislocations vanishes, corresponds to an ideal diamagnetic in electromagnetism: this will be the leading idea through the whole work.

If in the hypothetical case that at the transition from the elastic to the plastic deformation state all dislocations in the primary slip system could glide, $\left\{ \begin{array}{l} \mu_0 \mathbf{H}_i^\alpha = \mathbf{B}_i^\alpha \\ \epsilon_0^{-1} \mathbf{D}_i^\alpha = \mathbf{E}_i^\alpha \end{array} \right.$, then the only material constants occurring, which are μ_0 and ϵ_0 ($\mu_{\text{rel}} = 1 = \epsilon_{\text{rel}}$), would be determined thus only by properties of the crystal previous to the start of the plastic deformation: by the elastic constants of the elastic stage, by the mass density ρ and by the initially *grown-in* dislocation density, where the latter is characterized by the spacing λ_0 between dislocations in the original crystal. In order to avoid an overloaded notation, from hereafter the only elastic constant to be considered will be the elastic shear modulus G : the dependence on the deformation’s direction of the elastic constants (such as the Poisson’s number ν) can be easily included since it is already contained in the tensor character of μ_{rel} and ϵ_{rel} , when such a generalization is considered. All this shows therefore that the material constants μ_0 and ϵ_0 only depend on G , ρ and λ_0 . Lattice spacing a does not play any relevant role since its influence is already “hidden” in the energy scale set by the elastic constants.

Returning to (2.16), from its very definition $\mathbf{B}^\alpha$ does not satisfy to have a vanishing divergence, $\text{div } \mathbf{B}^\alpha \equiv \text{div } \boldsymbol{\alpha}_{\text{gliding}}^\alpha \neq 0$, since a mobile dislocation segment can end (and start) at a pinning point where the dislocation passes from being gliding to being sessile. Formally, this is expressed by equation

$$\text{div } \mathbf{B}^\alpha = (\partial_i \mathbf{B}_i^\alpha) = \varrho_{\text{mg}}^\alpha, \quad (2.19)$$

where the meaning of field $\varrho_{\text{mg}}^\alpha$ is just the mentioned density of points acting as “sinks” for the gliding dislocations. An analogous result holds for the current of mobile dislocations which, as different from (2.13), does not fulfil $\text{curl } \mathbf{E}^\alpha + \partial_t \mathbf{B}^\alpha \equiv \text{curl } \mathcal{J}_{\text{gliding}}^\alpha + \partial_t \boldsymbol{\alpha}_{\text{gliding}}^\alpha$ to vanish, property written formally as

$$\text{curl } \mathbf{E}^\alpha + \partial_t \mathbf{B}^\alpha = (\epsilon_{ijk} \partial_j \mathbf{E}_k^\alpha + \partial_t \mathbf{B}_i^\alpha) = -\mathbf{j}_{\text{mg}}^\alpha. \quad (2.20)$$

Returning for a moment to the language of the electromagnetism, the analogues to fields ϱ_{mg} and $\mathbf{j}_{\text{mg}}$ would correspond to the magnetic monopole density and current respectively, which is the reason for the (formal) label “mg” applied to these field, used to stress the similarity with the Maxwell equations.

Such monopole fields does not occur in the conventional Maxwell theory, at least as a non-singular solution; the only form to introduce monopoles in the usual electromagnetism is by means of a singular line called the Dirac string. In the conventional magnetism, if we have constructed a static field arrangement such that it carries in a stable way a certain amount of magnetic flux, this flux as well as the whole field energy content cannot be changed by means of any transformation of the type $A_k(\vec{r}) \rightarrow A'_k(\vec{r}) = A_k(\vec{r}) - \partial_k \chi(\vec{r})$, where $A_k(\vec{r})$ is the magnetic potential defining the magnetic field and χ is an arbitrary local scalar function. This invariance under the just mentioned transformation is called an Abelian $U(1)$ gauge symmetry. It should be noted that if some transformation of this kind were able to change the magnetic flux of the considered configuration, then at the point where applied would be necessary the existence of a magnetic monopole to produce or to absorb the increase or decrease in the magnetic flux. However, if the $U(1)$ symmetry is not the underlying system’s symmetry but only a subgroup of a larger non-Abelian compact symmetry (as for instance the compact $SO(3)$ group), case called the compact (or the periodic) $U(1)$ electromagnetism, certain field transformations which leaves invariant the energy do exist relating static field configurations with different amount of magnetic flux: in such a case, a monopole must occur to take account of this change in magnetic flux, and thus monopole fields appear in a natural form. A detailed introduction to this subject can be found in the Appendix B: see Note on page 235 and all the discussion on page 239 after equation (B.156).

Translating these comments to the dislocation theory, with the identification of density and current of mobile dislocations as the magnetic and electric field of a compact $U(1)$ electromagnetism, the Burgers vector of this kind of dislocations (Burgers vector which according to the definition (2.15) is nothing else as the flux of the dislocation density) as well as the gliding dislocations themselves can be absorbed by certain fields at the points where these dislocations are stopped, becoming thus immobile dislocations: for instance, through reaction with mobile dislocations gliding on other slip systems to form Lomer-Cottrell locks. Summarizing, the above introduced fields ϱ_{mg} and $\mathbf{j}_{\text{mg}}$ carry the information of the sessile, which cannot glide in any slip plane. Whether the $U(1)$ group, describing the invariance of the dislocation model to be constructed, is or not a embedded into a larger compact group — and therefore whether the model will contain sessile dislocations or not — will be determined by the nature of the crystal: in b.c.c and f.c.c. crystals, sessile dislocations appear from the very beginning, whereas hexagonal packed crystals do not display sessile dislocations since these would be energetically too

unfavourable because of the large deformation they would cause.

The already discussed equations $\begin{cases} \operatorname{div} \boldsymbol{\alpha}^\alpha = 0 \\ \operatorname{curl} \mathcal{J}^\alpha + \partial_t \boldsymbol{\alpha}^\alpha = 0 \end{cases}$ for the total dislocation density (gliding plus non-gliding), or their formulation as $\begin{cases} \operatorname{div} \mathbf{B}^\alpha = \varrho_{\text{mg}}^\alpha \\ \operatorname{curl} \mathbf{E}^\alpha + \partial_t \mathbf{B}^\alpha = -\mathbf{j}_{\text{mg}}^\alpha \end{cases}$ already oriented for the discussion of the hardening curve, only states a property of topological nature, that of the existence of well-defined orientation for the Bravais lattice vectors, or equivalently, the existence of long range parallelism within the crystal. Now it is time to formulate the equations which determine the mechanical part of the dislocation theory: how the dislocations distribute themselves under an imposed stress and flow velocity. Or equivalently, what the sources for dislocation density and current are. The task in the dislocation model is therefore to look for the analogous equations to Maxwell's $\operatorname{curl} \vec{H} = \vec{j}_{\text{el}} + \partial_t \vec{D}$ and $\operatorname{div} \vec{D} = \varrho_{\text{el}}$ [68]. In order to achieve this formulation, we shall firstly consider the simplification that the whole plastic deformation is taking place only in the primary slip system⁴ of primary dislocations; the generalization to the case with several active slip systems will be discussed after considering the former case.

1. static case: $\operatorname{curl} \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha$ with $\mu_0 \mathbf{H}_k^\alpha = (\boldsymbol{\alpha}_{\text{prim}})_k^\alpha$

Let us suppose we have on the primary plane some parallel dislocations, gliding as well as pinned, but *both* eventually able to glide since their Burgers vector is located completely on the slip plane. The density of these primary dislocations is denoted by $\mu_0 \mathbf{H}_k^\alpha = (\boldsymbol{\alpha}_{\text{prim}})_k^\alpha$. Moreover we assume that they are impeded of escaping from a certain large region within the primary plane by certain obstacles: distant from these obstacles and without forces external so this dislocation system, the primary dislocations are uniformly distributed along the direction perpendicular to the dislocation line. Or equivalently, the gradient along the transversal direction — taking the longitudinal direction as that of the (parallel) dislocation lines — for the primary dislocation density vanishes, what mathematically written takes the form $\operatorname{curl}(\mu_0 \mathbf{H}^\alpha) = 0$. In the vicinity of the obstacles, this transversal gradient (or curl) is no longer zero due to the stress created by these obstacles.

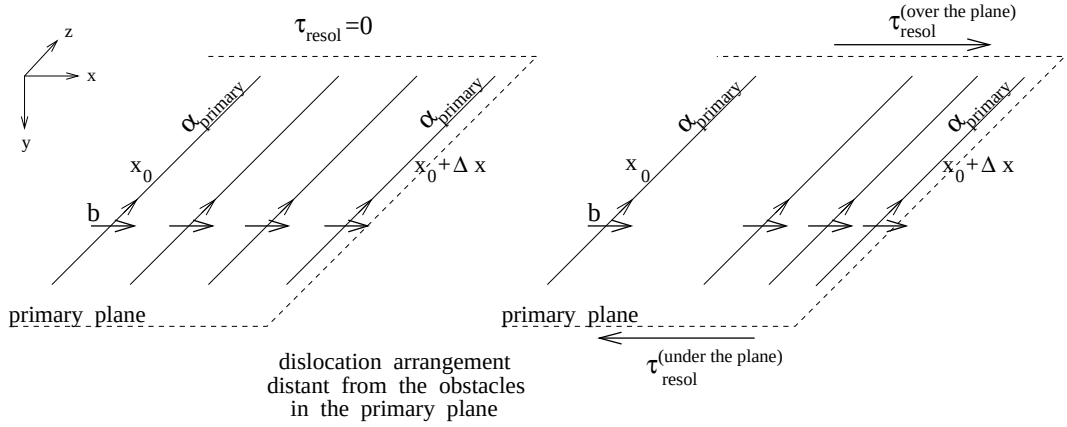
If now, beside the obstacles, an external stress $\boldsymbol{\tau}_{\text{ext}}^\alpha$ acts within the region delimited by the obstacles, the transversal distribution of the primary dislocations, which distant from the obstacles was previously uniform, is no longer so since they tend to group themselves in the direction imposed by the external stress: this re-grouping is proportional to the applied stress, being defined by the proportionality constant as the μ_0 introduced above. It should be noted that since one is considering dislocations with Burgers vector residing completely on the slip plane — Burgers vector given by $\int_{\text{area } S} \mu_0 \mathbf{H}_k^\alpha dS^k$ and whose direction is therefore denoted by $^\alpha$ —

⁴This is achieved by setting up the tensile stress to single slip, in order to have an extensive Stage I, or by working in a well developed Stage III, because in this latter stage, the leading effects producing the plastic deformation occur again in the primary slip systems (since the sessile Lomer-Cottrell locks are overcome by processes activated by the temperature).

and since the regrouping of the dislocations takes place also in this plane, the stress acting on the dislocation system is just the resolved shear stress defined in (1.1). All this stated in an equation gives

$$\epsilon_{ijk} \partial_j (\alpha_{\text{prim}})_k^\alpha \equiv \text{curl}(\mu_0 \mathbf{H}^\alpha) = \mu_0 \boldsymbol{\tau}_{\text{ext}}^\alpha \equiv \mu_0 (\boldsymbol{\tau}_{\text{resol}})_i^\alpha. \quad (2.21)$$

The correspondence of the (latin) indices in the above equation between those of the transversal gradient and that of the resolved stress is the correct by remembering that stress $\boldsymbol{\tau}_i^\alpha$ points along the direction α (which is also the direction of the dislocations' Burgers vector) and is applied to a surface whose normal direction is given by i , as one can easily see in Fig. 2.4. Summarizing, transversal gradients in the dislocation distribution are the result of applied stresses, being the latter the result of an external shear stress applied by a tensile machine as well as that due to (point) fixed obstacles.



$$(\alpha_{\text{prim}})_z^1(x_0) < (\alpha_{\text{prim}})_z^1(x_0 + \Delta x)$$

$$\left(\text{curl}(\mu_0 \mathbf{H}^{\alpha=1}) \right)_y(x_0) = -\partial_x (\alpha_{\text{prim}})_z^1(x_0) < 0$$

Figure 2.4: Distortion of the uniform arrangement (left figure) of primary dislocations, $\partial_x (\alpha_{\text{prim}})_z^1 > 0$, due to an applied stress. According to the convention defining the indices of the stress tensor, the applied stress in the right figure is negative since the y axis points downwards:

$$\boldsymbol{\tau}_{\text{ext}} = -(\boldsymbol{\tau}_{\text{resol}})_y^1 < 0.$$

The form of μ_0 is defined by equation (2.21) and can be estimated in the following way: let us consider two different materials under the same (fixed) stress $\boldsymbol{\tau}_{\text{resol}}^\alpha$. The transversal gradient produced in the dislocation distribution will be more pronounced if the crystal's rigidity is low, i.e., if shear modulus G is small or if the initial grown-in dislocation network is not dense (λ_0 large) and thus this initial network does not strongly impede the dislocation movement. Since in the case under study μ_0 is proportional to this just mentioned gradient, $\mu_0 = \frac{\text{grad}_{\text{trans}} \alpha_{\text{prim}}}{\boldsymbol{\tau}_{\text{resol}} \text{ constant}}$,

then one guesses $\mu_0 \sim \frac{\rho^n}{G^l \lambda_0^m}$ with $l, m \geq 0$ (and ρ the mass density) as the dependence of μ_0 on the material parameters. By adjusting the dimensions

$$\mu_0 \sim \frac{1}{G \lambda_0^2} . \quad (2.22)$$

The more exact form $\mu_0^{-1} = G \lambda_0^2$ will be argued later through the analogy to the superconductor. In addition to μ_0 , determined by the original properties of the crystal before the deformation starts, the material response in the different states of the plastic deformation is carried by μ_{rel} , i.e., by how strong is the “diamagnetism” displayed by the crystal against the existence of a gliding dislocation density — whose role is the same to that of magnetic induction $\vec{B}$ in the Maxwell equations.

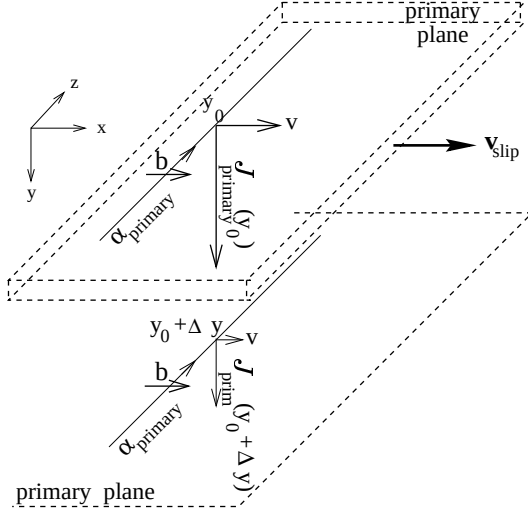
Actually, relation $\text{curl } \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha$ expresses in detailed form the Schmid law: the projection of the external stress on the primary slip system is the responsible for the dislocation distribution in this slip plane. It is important to note that, since the equation we are dealing with here is the analogue to the Ampère law of the magnetostatic $\text{curl } \vec{H} = \vec{j}_{\text{el}}$, the external resolved stress is playing in equation $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha \equiv \text{curl } \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{ext}}^\alpha$ the same role as the electric current $\vec{j}_{\text{el}}$ in the Ampère law.

$$2. \quad \boxed{\text{div } \mathbf{D}^\alpha = -\rho v_{\text{slip}}^\alpha \quad \text{with} \quad \epsilon_0^{-1} \mathbf{D}_k^\alpha = (\mathcal{J}_{\text{prim}})_k^\alpha}$$

Now let us consider two neighbour primary planes (or two layers within the same primary plane) containing each one the same dislocation density: if the dislocations in the upper layer have a larger velocity than in the lower layer, the upper part glides over the lower one. As in the previous case, the dislocations are both gliding as well as pinned primary dislocations, the latter while having their Burgers vector completely contained in the primary plane could otherwise move. According to Fig. 2.5, this situation means that a nonvanishing divergence of the primary dislocation current $\text{div } \mathcal{J}_{\text{prim}}^\alpha \neq 0$ — i.e. a gradient along the (longitudinal) direction of the dislocation current — is proportional to (minus) the slip or glide velocity of the mass elements in the crystal. As in equation (2.21), field $\mathbf{D}_k^\alpha$ or equivalently ϵ_0 is defined through the proportionality constant between, on the one side, the longitudinal gradient of the primary dislocation current, and on the other side, the slip velocity (multiplied by the mass density ρ)

$$\partial_k (\mathcal{J}_{\text{prim}})_k^\alpha \equiv \text{div } (\epsilon_0^{-1} \mathbf{D}^\alpha) = -\epsilon_0^{-1} \rho v_{\text{slip}}^\alpha . \quad (2.23)$$

And just as for the estimation of μ_0 , the dependence of ϵ_0 on the crystal properties before being deformed, i.e. G , the initial dislocation interspacing λ_0 and the mass density, is guessed in a similar way: on two different materials is imposed the same slip velocity (for instance by means of a tensile machine). Large longitudinal changes in the primary dislocation current are due to a low crystal rigidity, i.e.,



$$(\mathcal{J}_{\text{prim}})_y^1(y_0) > (\mathcal{J}_{\text{prim}})_y^1(y_0 + \Delta y)$$

$$\partial_y (\epsilon_0^{-1} \mathbf{D}_y^{\alpha=1})(y_0) < 0$$

Figure 2.5: The dislocation current in the lower primary plane, $(\mathcal{J}_{\text{prim}})_y^1(y_0 + \Delta y)$, is smaller than at y_0 and thus crystal plane at y_0 slips over the plane at $y_0 + \Delta y$ with a velocity $v_{\text{slip}}^1(y_0) > 0$.

it occurs for small G and for non-dense groups of initially grown-in dislocations (large λ_0) such that the latter do not represent a strong hindrance to the dislocation motion: $\text{grad}_{\text{long}} \mathcal{J}_{\text{prim}} \sim \frac{1}{G^{l'} \lambda_0^{m'}}$ with $l', m' \geq 0$. On the other hand, ϵ_0 is inversely proportional to the longitudinal gradient of the primary dislocation current, $\epsilon_0 = \frac{\rho v_{\text{slip}}^\alpha \text{ constant}}{\text{grad}_{\text{long}} \mathcal{J}_{\text{prim}}}$, and therefore estimation $\epsilon_0 \sim G^{l'} \lambda_0^{m'} \rho^{n'}$ holds, which by adjusting dimensions again, leads to relation

$$\epsilon_0 \sim \rho \lambda_0^2. \quad (2.24)$$

Combining this last result with the other estimation (2.22) one obtains

$$\frac{1}{\epsilon_0 \mu_0} \sim \frac{G}{\rho} = c_{\text{sound}}^2, \quad (2.25)$$

which is a result formally equivalent to that in electromagnetism stating the relation between the vacuum's dielectric constant and magnetic permeability and the light velocity. In fact, the (transversal) sound velocity $c_{\text{sound}} = \sqrt{G/\rho}$ is the highest propagation velocity for elastic waves emitted by an oscillating dislocation line and thus equation (2.25) will be taken as an exact relation: once μ_0 is exactly determined (through the analogy of the dislocation model with a superconductor), then $\frac{1}{\epsilon_0 \mu_0} = c_{\text{sound}}^2$ will define constant ϵ_0 . The fact that in the above discussion only the transversal sound velocity appears and not the longitudinal one too is due to the simplification already carried out of restricting the number of elastic constants to only the shear elastic modulus G .

In a similar way as noted in the comparison between $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha \equiv \text{curl } \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{ext}}^\alpha$ and Maxwell's $\text{curl } \vec{H} = \vec{j}_{\text{el}}$, the role played by the mass slip velocity (multiplied by minus the mass density ρ) in the dislocation theory $\epsilon_0 \text{div } \mathcal{J}_{\text{prim}}^\alpha \equiv \text{div } \mathbf{D}^\alpha = -\rho v_{\text{slip}}^\alpha$

is just the same as the electric free charge density ϱ_{el} in the Gauss law $\text{div } \vec{D} = \varrho_{\text{el}}$.

3. dynamical case: $\text{curl } \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha + \partial_t \mathbf{D}^\alpha$ with $\epsilon_0^{-1} \mathbf{D}_k^\alpha = (\mathcal{J}_{\text{prim}})_k^\alpha$

In order to complete the analogy with electromagnetism, it is necessary to relate the primary dislocation density and current in one equation, in the same way as the gliding dislocation density and current are related through the equation $\text{curl } \mathcal{J}_{\text{glid}}^\alpha + \partial_t \boldsymbol{\alpha}_{\text{glid}}^\alpha = -\mathbf{j}_{\text{mg}}$. In electromagnetism, this connecting term is the so-called Maxwell displacement current $+\partial_t \vec{D}$ entering in equation $\text{curl } \vec{H} = \vec{j}_{\text{el}} + \partial_t \vec{D}$. Therefore, one should expect that in the dislocation model we are formulating an analogous term should be included into equation $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha$.

Term $+\partial_t \vec{D}$ was forced in electromagnetism by the conservation of the electric charge, conservation expressed mathematically by relation $\text{div } \vec{j}_{\text{el}} + \partial_t \varrho_{\text{el}} = 0$, since without it one would obtain $\text{div } \vec{j}_{\text{el}} = \text{div curl } \vec{H} = 0$. Moreover, the electric current is also related to the electric field by means of another material relation which is the Ohm's law: $\vec{j}_{\text{el}} = \sigma_{\text{cond}} \vec{E}$, with parameter σ_{cond} (in general a tensor) called

the conductivity. Such a law, together with equations
$$\begin{cases} \text{div } \vec{B} = 0 \\ \text{curl } \vec{E} + \partial_t \vec{B} = 0 \\ \text{curl } \vec{H} = \vec{j}_{\text{el}} + \partial_t \vec{D} \approx \vec{j}_{\text{el}} \end{cases},$$

allows to describe the behaviour of a good conductor characterized by a large conductivity σ_{cond} , where term $+\partial_t \vec{D}$ plays no relevant role since the leading effect is carried by $\sigma_{\text{cond}} \vec{E}$, except for the case of a fast change of $\vec{E}$ with time⁵. For a bad conductor, on the contrary, the addition of $+\partial_t \vec{D}$ is necessary to describe the system.

Returning to the dislocation theory, for a crystal highly deformed plastically as for instance in a well developed Stage III, the gliding dislocation current inside the crystal is high, being its change rather pronounced although the increase in the applied stress is low: in the analogy to the electromagnetism, this corresponds to a bad conductor where $\vec{E}$ (in our case, the primary dislocation current) is weakly coupled to $\vec{j}_{\text{el}}$ (whose role is played in our model by the external stress $\boldsymbol{\tau}_{\text{resol}}^\alpha$). Hence the transversal gradient of the primary dislocation density should receive an important contribution from the primary dislocation current. In a crystal which has largely lost its rigidity — and thus most of the primary dislocations are mobile — a large transversal gradient in the primary dislocation density, with its accompanying large deformation and energy increase, will yield an acceleration of the current of other primary dislocations. This allows us to write as a contribution

to $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha$ the following term

⁵In a more mathematical way, if frequency ω characterizes the time elapsed such that a change in $\vec{E}$ is observable, then term $\partial_t \vec{D}$ contributes to $\text{curl } \vec{H} = \vec{j}_{\text{el}} + \partial_t \vec{D} = \sigma_{\text{cond}} \vec{E} + \epsilon_{\text{rel}} \epsilon_0 \partial_t \vec{E}$ in the same order as $\vec{j}_{\text{el}}$ if $\epsilon_{\text{rel}} \epsilon_0 \omega \geq \sigma_{\text{cond}}$ [66] §56, something which also occurs for not so large ω in the case of a bad conductor (small σ_{cond}).

$$\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha \equiv \text{curl } \mathbf{H}^\alpha = \dots + \underbrace{\epsilon_0 \partial_t \mathcal{J}_{\text{prim}}^\alpha}_{\partial_t \mathbf{D}^\alpha} . \quad (2.26)$$

However this is not the only reason to postulate such a contribution. As in electromagnetism, this additional term is also forced by a conservation law.

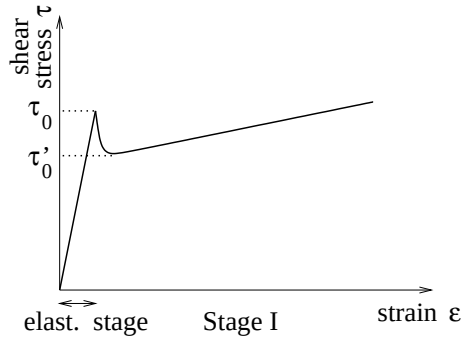
If one were using modified equation $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha + \epsilon_0 \partial_t \mathcal{J}$ together with the already proved equation (2.23), $\epsilon_0 \text{div } \mathcal{J}_{\text{prim}}^\alpha = -\rho v_{\text{slip}}^\alpha$, it would lead to

$$\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha = \partial_t \left(\rho v_{\text{slip}}^\alpha \right) , \quad (2.27)$$

as analogue to Maxwell's conservation law $\text{div } \vec{j}_{\text{el}} = -\partial_t \varrho_{\text{el}}$. At first glance, the fact that a dynamical equation as (2.27) — stating the time-change in the density of mechanical momentum in absence of external volume forces like gravity — arises in the framework of this model as a conservation law similar to that of the electric charge appears to be rather surprising. However, as we shall see on page 68 after having formulated completely the dislocation model, $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha = \partial_t \left(\rho v_{\text{slip}}^\alpha \right)$ will become nothing else as a consequence of an internal invariance of the system's energy under a displacement of the whole crystal's lattice by a global constant vector; and, as it is well-known from the Noether theorem, invariance of the model's energy under rigid translations leads to the conservation of the linear momentum density.

Further, relation (2.27) can be made plausible without resorting to any differential equation of the kind (2.26): the total force per unit volume on the crystal — or also the force acting on the volume containing the dislocations under study, force due to an external load as well as to the crystal elements located outside the volume considered —, acting through its surface because of the phenomenological scales involved (much larger than the interatomic distances where the interaction takes actually place), is given by $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha$ and equal thus to the time-change of the density of linear momentum for the considered crystal's mass elements.

As an example, in most of the tensile experiments the shear stress placed on the upper crystal's surface (which by simplicity we shall assume parallel to the primary plane and pointing in the slip direction, in order to identify directly this stress with the resolved one) is equal to that in the lower one and thus $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha \sim (\boldsymbol{\tau}_{\text{upper}})_k^\alpha - (\boldsymbol{\tau}_{\text{lower}})_k^\alpha = 0$. However, there are some cases where this condition is not satisfied as at the drop displayed in the stress-strain curve by b.c.c. crystals at the beginning of Stage I (see figure, and page 24). At such point, the stress acting on the crystal due to the tensile machine is accumulated because the strong hindrance for dislocation single slip in the primary plane does not permit to relax such a stress. When the plastic deformation finally starts, mobile dislocations are produced



in large amounts and the crystal tries to extend even *more quickly* than the stretching supplied by the machine stress. The result is the observed drop. It is just at this moment when the stress opposed by the upper surface with respect to the lower one is not the same, since one of these surfaces displays a very weakened resistance against deformation. Therefore during this instant of sudden plastic deformation, $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha \neq 0$ holds. And also this drop is accompanied by a large increase in the slip velocity of the mass elements in the crystal. After a short time, nevertheless, the repulsion between the produced dislocations increases and the production of gliding dislocations is stopped: the mass slip velocity is also slowed down and the upper part cannot slip freely with respect to the crystal's part located below the activated primary plane. A further dislocation production must be activated by an external increase in $\boldsymbol{\tau}_{\text{resol}}$: up to this point $\boldsymbol{\tau}_{\text{upper}} = \boldsymbol{\tau}_{\text{lower}}$, and therefore $\text{div } \boldsymbol{\tau}_{\text{ext}} = 0$. And also from this moment on, the slip velocity in the crystal's mass elements remains constant with the time, determined by the tensile machine.

All this discussion justified therefore the generalization of dislocation theory's equation of $\frac{1}{\mu_0} \text{curl } \boldsymbol{\alpha}_{\text{prim}}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha$ into

$$\epsilon_{ijk} \partial_j (\boldsymbol{\alpha}_{\text{prim}})_k^\alpha = \mu_0 (\boldsymbol{\tau}_{\text{resol}})_i^\alpha + \epsilon_0 \mu_0 \partial_t (\boldsymbol{\mathcal{J}}_{\text{prim}})_i^\alpha, \quad (2.28)$$

or in a formulation similar to that of the Maxwell equations

$$\text{curl } \mathbf{H}^\alpha = \boldsymbol{\tau}_{\text{resol}}^\alpha + \partial_t \mathbf{D}^\alpha, \quad (2.29)$$

with the notation $\mu_0 \mathbf{H}_k^\alpha = (\boldsymbol{\alpha}_{\text{prim}})_k^\alpha$ and $\epsilon_0^{-1} \mathbf{D}_k^\alpha = (\boldsymbol{\mathcal{J}}_{\text{prim}})_k^\alpha$. From this result, the identification of $\frac{1}{\sqrt{\epsilon_0 \mu_0}}$ with the (transversal) sound velocity as discussed after (2.25) is justified: calculating the curl of equation's part $\text{curl } \mathbf{H}^\alpha = \dots + \partial_t \mathbf{D}^\alpha$, together with substitution of $\text{curl}(\epsilon_{\text{rel}}^{-1} \mathbf{D}^\alpha) = -\epsilon_0 \mu_0 \partial_t (\mu_{\text{rel}} \mathbf{H}^\alpha) - \dots$, one obtains the equation for the propagation of the elastic waves radiated by a oscillating dislocation configuration.

The whole previous discussion has led us to a model valid to describe effectively the dislocation theory when the motion of dislocations and its corresponding dislocation interactions only occurs in the primary slip system. The resulting equations are summarized in Table 2.2, with the corresponding analogues in the Maxwell theory of electromagnetism.

Let us define fields A_k^α and A_0^α as the vector and the scalar potential for the density and the current of gliding dislocations

$$\mathbf{B}_i^\alpha \stackrel{\text{def}}{=} \epsilon_{ijk} \partial_j A_k^\alpha, \quad (2.30)$$

and

$$\mathbf{E}_k^\alpha \stackrel{\text{def}}{=} -\partial_k A_0^\alpha - \partial_t A_k^\alpha, \quad (2.31)$$

Maxwell theory			Dislocation dynamics		
	magnetic	electric	disloc. density	disloc. current	
external	$\mu_0 H_i$	$\epsilon_0^{-1} D_i$	$\mu_0 \mathbf{H}_i^\alpha = (\boldsymbol{\alpha}_{\text{prim}})_i^\alpha$	$\epsilon_0^{-1} \mathbf{D}_i^\alpha = (\mathcal{J}_{\text{prim}})_i^\alpha$	primary
internal	B_i	E_i	$\mathbf{B}_i^\alpha = (\boldsymbol{\alpha}_{\text{glid}})_i^\alpha$	$\mathbf{E}_i^\alpha = (\mathcal{J}_{\text{glid}})_i^\alpha$	gliding
μ_0, ϵ_0 :			$\mu_0 = \frac{1}{G\lambda_0^2}, \epsilon_0 = \rho\lambda_0^2$:		
$c_{\text{light}}^2 = \frac{1}{\mu_0 \epsilon_0}$			$c_{\text{sound}}^2 = \frac{1}{\mu_0 \epsilon_0} = \frac{G}{\rho}$		
$\vec{j}_{\text{el}}$: electric current			$(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$: external resolved shear stress		
ϱ_{el} : free charge density			$-\rho v_{\text{slip}}^\alpha$: slip velocity of crystal mass elements		
$\begin{aligned} \text{div } \vec{B} &= 0 \\ \text{curl } \vec{E} &= -\partial_t \vec{B} \\ \text{div } (\epsilon_{\text{rel}} \epsilon_0 \vec{E}) &= \varrho_{\text{el}} \\ \text{curl } \left(\frac{\vec{B}}{\mu_{\text{rel}} \mu_0} \right) &= \vec{j}_{\text{el}} + \partial_t (\epsilon \vec{E}) \end{aligned}$			$\begin{aligned} \text{div } \mathbf{B}^\alpha &= \varrho_{\text{mg}}^\alpha \\ \text{curl } \mathbf{E}^\alpha &= -\partial_t \mathbf{B}^\alpha - \mathbf{j}_{\text{mg}}^\alpha \\ \text{div } (\epsilon_{\text{rel}} \epsilon_0 \mathbf{E}^\alpha) &= -\rho v_{\text{slip}}^\alpha \\ \text{curl } \left(\frac{\mathbf{B}^\alpha}{\mu_{\text{rel}} \mu_0} \right) &= \boldsymbol{\tau}_{\text{resol}}^\alpha + \partial_t (\epsilon \mathbf{E}^\alpha) \end{aligned}$		
$\text{div } \vec{j}_{\text{el}} = -\partial_t \varrho_{\text{el}}$			$\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha = \partial_t (\rho v_{\text{slip}}^\alpha)$		

Table 2.2: Analogy between the equations of the Maxwell theory and those of dislocation dynamics for single slip, in the primary slip system. Fields $\varrho_{\text{mg}}^\alpha$ and $\mathbf{j}_{\text{mg}}^\alpha$ takes account of the sessile dislocations, whose Burgers vector is not completely contained in the primary plane. But in the case of only one active slip system, sessile dislocations are not produced and thus $\varrho_{\text{mg}}^\alpha \approx \mathbf{j}_{\text{mg}}^\alpha \approx 0$.

just in the same way as the vector and scalar potentials of the electromagnetic field. It is noteworthy that, since the Burgers vector is defined as the flux of the dislocation density — in our case, the Burgers vector $b^\alpha = \int_S \mathbf{B}_i^\alpha dS^i$ of the dislocations gliding in the primary plane — field A_k^α corresponds (up the gradient of any scalar function) to the reciprocal Bravais vector, which for its part is approximately equal (see (2.10)) to the strain tensor.

In the Maxwell theory, fields $\begin{cases} \vec{B} = \text{curl } \vec{A} \\ \vec{E} = -\text{grad } A_0 - \partial_t \vec{A} \end{cases}$ are invariant under the follow-

ing transformation

$$\begin{cases} \vec{A}(\vec{r}, t) \rightarrow \vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) - \text{grad } \chi(\vec{r}, t), \\ A_0(\vec{r}, t) \rightarrow A'_0(\vec{r}, t) = A_0(\vec{r}, t) + \partial_t \chi(\vec{r}, t), \end{cases} \quad (2.32)$$

with $\chi(\vec{r}, t)$ any arbitrary local function with well-defined derivatives. The above transformation is called a gauge transformation, and $U(1)$ gauge invariance the corresponding invariance of the electromagnetic fields $\vec{B}$ and $\vec{E}$, fields which are the actual variables determining the energy of any electromagnetic field configuration.

In the dislocation model just formulated in Table 2.2 there exists also a similar gauge invariance, with a very intuitive meaning: the addition of the gradient, or the time-derivative, of a $\chi^\alpha(\vec{r}, t)$ (well-defined everywhere through the crystal) to fields A_k^α , A_0^α — the latter determining the dislocation density and current through (2.30)-(2.31) — is equivalent to apply a compatible deformation on the crystal, being $\chi^\alpha(\vec{r}, t)$ the corresponding displacement vector $u^\alpha(\vec{r}, t)$

$$\begin{cases} A_k^\alpha(\vec{r}, t) \rightarrow A_k'^\alpha(\vec{r}, t) = A_k^\alpha(\vec{r}, t) - \partial_k u^\alpha(\vec{r}, t), \\ A_0^\alpha(\vec{r}, t) \rightarrow A_0'^\alpha(\vec{r}, t) = A_0^\alpha(\vec{r}, t) + \partial_t u^\alpha(\vec{r}, t). \end{cases} \quad (2.33)$$

And any compatible deformation leaves invariant the density and current of dislocations, mobile as well as pinned. From hereafter, the meaning of “gauge invariance” when related to the dislocation theory is the following

density and current of gliding and immobile dislocations are invariant under any compatible deformation	(2.34)

As already mentioned, fields ϱ_{mg} and $\mathbf{j}_{\text{mg}}$ are the carrier of the information about sessile dislocations, those which cannot glide in any form in the primary slip system nor in any other slip system because their Burgers vectors do not lie completely included in any slip plane; their corresponding analogues in the Maxwell equations would be the density and current of magnetic monopoles, respectively. In electromagnetism, the form of introducing such monopole fields and simultaneously maintaining relations $\begin{cases} \vec{B} = \text{curl } \vec{A} \\ \vec{E} = -\text{grad } A_0 - \partial_t \vec{A} \end{cases}$ is to allow certain “jumps” or multivalueness of the potentials A_0 , $\vec{A}$ at those points where such monopoles are located, points where equations $\begin{cases} \text{div } \vec{B} = 0 \\ \text{curl } \vec{E} + \partial_t \vec{B} = 0 \end{cases}$ no longer hold. As we shall see in the next sections, the form to deal with such multivalueness is to endow fields A_0 , $\vec{A}$ with an angular character, which is actually an effective way of saying that the above $U(1)$ gauge invariance (2.33) is the residual subgroup left after the partial breakdown of a larger compact gauge symmetry, the latter being the actual underlying symmetry of the system. Similarly in the dislocation model, at the points where a gliding dislocation ends because it has become a reacting part to give a sessile dislocation, fields A_0^α , A_k^α will become multivalued in order to take account of these produced sessile dislocations and of equations

$\begin{cases} \operatorname{div} \mathbf{B}^\alpha \neq 0 \\ \operatorname{curl} \mathbf{E}^\alpha + \partial_t \mathbf{B}^\alpha \neq 0 \end{cases}$. However, as long as only the primary slip system is active — and addition of only primary dislocations cannot produce sessile dislocations, the latter resulting from the reaction of primary dislocations with mobile dislocations produced in slip planes different from the primary one — one need not be worried about these “jumps” in the fields A_0^α , A_k^α since one can safely assume $\varrho_{\text{mg}}^\alpha \approx \mathbf{j}_{\text{mg}}^\alpha \approx 0$; their contribution to the equations in Table 2.2 has been nevertheless maintained in order to obtain an unified picture when the more general case of activation of several slip systems will be considered (see Table 2.3).

Finally, the Lagrangian density describing the dislocation theory in the just formulated model is the analogue to the well-known Lagrangian density of the Maxwell equations

$$\ell = \frac{1}{2} \sum_{i=1}^3 \mathbf{E}_i^\alpha \mathbf{D}_i^\alpha - \frac{1}{2} \sum_{i=1}^3 \mathbf{B}_i^\alpha \mathbf{H}_i^\alpha - \left(-\rho v_{\text{slip}}^\alpha \right) A_0^\alpha + \sum_{k=1}^3 (\boldsymbol{\tau}_{\text{resol}})_k^\alpha A_k^\alpha, \quad (2.35)$$

for every direction $^\alpha$ of the Burgers vector, being necessary to sum over this $^\alpha$ in order to obtain the complete Lagrangian density. If one prefers to absorb μ_0 and ϵ_0 into the fields, as in the formulation change of electromagnetism from the MKS to the CGS unit system (the factor 4π will be left out)

$$\begin{aligned}
 A_k^\alpha|_{\text{CGS}} &= \frac{1}{\sqrt{\mu_0}} A_k^\alpha|_{\text{MKS}} & A_0^\alpha|_{\text{CGS}} &= \sqrt{\epsilon_0} A_0^\alpha|_{\text{MKS}} \\
 \mathbf{B}_k^\alpha|_{\text{CGS}} &= \frac{1}{\sqrt{\mu_0}} \mathbf{B}_k^\alpha|_{\text{MKS}} & \mathbf{E}_k^\alpha|_{\text{CGS}} &= \sqrt{\epsilon_0} \mathbf{E}_k^\alpha|_{\text{MKS}} = \\
 & & &= -\partial_k A_0^\alpha|_{\text{CGS}} - \frac{1}{c_{\text{sound}}} \partial_t A_k^\alpha|_{\text{CGS}} \\
 \mathbf{H}_k^\alpha|_{\text{CGS}} &= \sqrt{\mu_0} \mathbf{H}_k^\alpha|_{\text{MKS}} & \mathbf{D}_k^\alpha|_{\text{CGS}} &= \frac{1}{\sqrt{\epsilon_0}} \mathbf{D}_k^\alpha|_{\text{MKS}} \\
 (\boldsymbol{\tau}_{\text{resol}})_k^\alpha|_{\text{CGS}} &= \frac{1}{\sqrt{\epsilon_0}} (\boldsymbol{\tau}_{\text{resol}})_k^\alpha|_{\text{MKS}} & -\rho v_{\text{slip}}^\alpha|_{\text{CGS}} &= -\frac{1}{\sqrt{\epsilon_0}} \rho v_{\text{slip}}^\alpha|_{\text{MKS}}
 \end{aligned} \quad (2.36)$$

then the Lagrangian density is given by

$$\ell = \sum_\alpha \left\{ \frac{\epsilon_{\text{rel}}}{2} \sum_{i=1}^3 (\mathbf{E}_i^\alpha)^2 - \frac{1}{2\mu_{\text{rel}}} \sum_{i=1}^3 (\mathbf{B}_i^\alpha)^2 - \left(-\rho v_{\text{slip}}^\alpha \right) A_0^\alpha + \frac{1}{c_{\text{sound}}} \sum_{k=1}^3 (\boldsymbol{\tau}_{\text{resol}})_k^\alpha A_k^\alpha \right\}. \quad (2.37)$$

In general the above dislocation fields, which because of gauge invariance (2.34) will be also called gauge invariant fields from here on, are coupled to other fields, yielding such a coupling the source terms for equations in Table 2.2. In this coupled situation, as next one solves the resulting local equations for a certain state — to which many different (microscopic) dislocation configurations may correspond — and calculate the macroscopical response of the system, what actually determines the value of ϵ_{rel} and μ_{rel} . Hence, when one follows this process of coupling the gauge fields to other fields and the

Lagrangian containing purely the gauge fields is only a part of the whole Lagrangian describing the behaviour of the system, (2.37) will be written as

$$\frac{1}{2} \sum_{\alpha, i} \mathbf{E}_i^\alpha \mathbf{E}_i^\alpha - \frac{1}{2} \sum_{\alpha, i} \mathbf{B}_i^\alpha \mathbf{B}_i^\alpha,$$

being ϵ_{rel} , μ_{rel} the result of the different behaviours displayed by the dislocation system. In the 4-dimensional tensor notation, this Lagrangian density takes the form

$$\ell_{\text{pure gauge part}}^\alpha = -\frac{1}{2} \sum_{\mu} \sum_{\nu < \mu} F_{\mu\nu}^\alpha F^{\mu\nu \alpha} = -\frac{1}{4} \sum_{\mu, \nu} F_{\mu\nu}^\alpha F^{\mu\nu \alpha} \quad \mu, \nu = 0, 1, 2, 3, \quad (2.38)$$

with $x^{\mu=0} = c_{\text{sound}} t$ and ⁶

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha,$$

for each direction α of the Burgers vector. Gauge potentials $A_{\mu=1,2,3}^\alpha = A^{\mu=1,2,3 \alpha} = A_k^\alpha$ and $A_{\mu=0}^\alpha = -A^{\mu=0 \alpha} = -A_0^\alpha$ already contain $\frac{1}{\sqrt{\mu_0}}$ and $\sqrt{\epsilon_0}$ respectively, as written in the CGS units system (2.36).

One remark still deserves attention. Up to this point we have considered that the only active slip system was the primary one. And as a consequence there was practically no production of sessile dislocations, whose effect is carried by ρ_{mg}^α and by $\mathbf{j}_{\text{mg}}^\alpha$. However, in order to formulate a general model we need include the case where gliding occurs in several non-parallel slip systems — with the accompanying Lomer-Cottrell locks as

⁶The use of this abbreviated notation requires nevertheless the usual care by lowering and raising the indices of temporal character, since the gauge potential $A^{\mu=0,1,2,3 \alpha} = (A_0^\alpha, A_x^\alpha, A_y^\alpha, A_z^\alpha)$ transforms by definition like the 4-dimensional coordinate vector $x^{\mu=0,1,2,3} = (c_{\text{sound}} t, x, y, z)$, whereas $A_\mu^\alpha = (-A_0^\alpha, A_x^\alpha, A_y^\alpha, A_z^\alpha)$ does it like the coordinate co-vector $x_\mu = (-c_{\text{sound}} t, x, y, z)$. Hence

$$F_{\mu\nu}^\alpha = \begin{pmatrix} 0 & -\mathbf{E}_x^\alpha & -\mathbf{E}_y^\alpha & -\mathbf{E}_z^\alpha \\ \mathbf{E}_x^\alpha & 0 & \mathbf{B}_z^\alpha & -\mathbf{B}_y^\alpha \\ \mathbf{E}_y^\alpha & -\mathbf{B}_z^\alpha & 0 & \mathbf{B}_x^\alpha \\ \mathbf{E}_z^\alpha & \mathbf{B}_y^\alpha & -\mathbf{B}_x^\alpha & 0 \end{pmatrix} \quad \text{and} \quad F^{\mu\nu \alpha} = \begin{pmatrix} 0 & \mathbf{E}_x^\alpha & \mathbf{E}_y^\alpha & \mathbf{E}_z^\alpha \\ -\mathbf{E}_x^\alpha & 0 & \mathbf{B}_z^\alpha & -\mathbf{B}_y^\alpha \\ -\mathbf{E}_y^\alpha & -\mathbf{B}_z^\alpha & 0 & \mathbf{B}_x^\alpha \\ -\mathbf{E}_z^\alpha & \mathbf{B}_y^\alpha & -\mathbf{B}_x^\alpha & 0 \end{pmatrix}. \quad \text{A form to}$$

avoid this difference in the treatment of vector and co-vector indices is to work with an imaginary time $x^{\mu=4} = ic_{\text{sound}} t$ and thus $\begin{cases} x^{\mu=1,2,3,4} = x_{\mu=1,2,3,4} = (x, y, z, ic_{\text{sound}} t) \\ A^{\mu=1,2,3,4 \alpha} = A_{\mu=1,2,3,4}^\alpha = (A_x^\alpha, A_y^\alpha, A_z^\alpha, iA_0^\alpha) \end{cases}$ leads to

$$F_{\mu\nu}^\alpha = F^{\mu\nu \alpha} = \begin{pmatrix} 0 & \mathbf{B}_z^\alpha & -\mathbf{B}_y^\alpha & -i\mathbf{E}_x^\alpha \\ -\mathbf{B}_z^\alpha & 0 & \mathbf{B}_x^\alpha & -i\mathbf{E}_y^\alpha \\ \mathbf{B}_y^\alpha & -\mathbf{B}_x^\alpha & 0 & -i\mathbf{E}_z^\alpha \\ i\mathbf{E}_x^\alpha & i\mathbf{E}_y^\alpha & i\mathbf{E}_z^\alpha & 0 \end{pmatrix}. \quad \text{In any case, } -\frac{1}{2} \sum_{\mu, \nu} F_{\mu\nu}^\alpha F^{\mu\nu \alpha} = \sum_{i=1}^3 [(\mathbf{E}_i^\alpha)^2 - (\mathbf{B}_i^\alpha)^2]$$

$$\text{and } -\sum_{\mu} \partial_\mu F^{\mu\nu \alpha} = \begin{cases} \sum_{j,k=1}^3 \epsilon_{ijk} \partial_j \mathbf{B}_k^\alpha - \frac{1}{c_{\text{sound}}} \partial_t \mathbf{E}_i^\alpha & \nu = i = 1, 2, 3 \\ \sum_{i=1}^3 \partial_i \mathbf{E}_i^\alpha \quad \text{or} \quad i \sum_{i=1}^3 \partial_i \mathbf{E}_i^\alpha & \nu = 0 \text{ or } \nu = 4 \end{cases} = \frac{1}{c_{\text{sound}}} J^{\nu \alpha}, \quad \text{with } J^{\nu \alpha} \equiv$$

$$\equiv \begin{cases} (\boldsymbol{\tau}_{\text{resol}})_i^\alpha & \nu = i = 1, 2, 3 \\ c_{\text{sound}} (-\rho \mathbf{v}_{\text{slip}}^\alpha) \quad \text{or} \quad ic_{\text{sound}} (-\rho \mathbf{v}_{\text{slip}}^\alpha) & \nu = 0 \text{ or } \nu = 4 \end{cases}, \quad \text{so that the equations for the dislocation}$$

model arise as Euler-Lagrange equations of Lagrangian density $-\frac{1}{4} \sum_{\mu, \nu} F_{\mu\nu}^\alpha F^{\mu\nu \alpha} + \frac{1}{c_{\text{sound}}} \sum_{\nu} J^{\nu \alpha} A_\nu^\alpha$.

result of the reaction among gliding dislocations coming from these different slip systems. Hence, those definitions for $\mathbf{H}_i^\alpha$ and $\mathbf{D}_i^\alpha$ in Table 2.1 valid for glide only on the primary plane are now extended to include the projection on the primary plane of the density and current of all able-to-glide dislocations appearing in other slip system, denoting by $\mathbf{B}_i^\alpha$ and $\mathbf{E}_i^\alpha$ the total density and current of the actual gliding dislocations projected on the primary slip system

$$\begin{aligned}\mathbf{B}_i^\alpha &= (\boldsymbol{\alpha}_{\text{prim glid}})_i^\alpha + \hat{b}_{\text{prim}}^\alpha \sum_{\text{second. slip systems}} \hat{b}_{\text{prim}}^{\alpha'} (\boldsymbol{\alpha}_{\text{second glid}})_i^{\alpha'}, \\ \mathbf{E}_i^\alpha &= (\mathcal{J}_{\text{prim glid}})_i^\alpha + \hat{b}_{\text{prim}}^\alpha \sum_{\text{second. slip systems}} \hat{b}_{\text{prim}}^{\alpha'} (\mathcal{J}_{\text{second glid}})_i^{\alpha'},\end{aligned}\tag{2.39}$$

$$\begin{aligned}\mu_0 \mathbf{H}_i^\alpha &= (\boldsymbol{\alpha}_{\text{prim slip system}})_i^\alpha + \hat{b}_{\text{prim}}^\alpha \sum_{\text{second. slip systems}} \hat{b}_{\text{prim}}^{\alpha'} (\boldsymbol{\alpha}_{\text{second slip system}})_i^{\alpha'}, \\ \epsilon_0^{-1} \mathbf{D}_i^\alpha &= (\mathcal{J}_{\text{prim slip system}})_i^\alpha + \hat{b}_{\text{prim}}^\alpha \sum_{\text{second. slip systems}} \hat{b}_{\text{prim}}^{\alpha'} (\mathcal{J}_{\text{second slip system}})_i^{\alpha'},\end{aligned}$$

with $\hat{b}_{\text{prim}}^\alpha$ the unit vector in the Burgers vector of the gliding primary dislocations. One should allways keep in mind that in all the above fields only gliding or would-become gliding dislocations contribute, i.e., those with a Burgers vector completely parallel to some of the slip planes (the primary or the secondary ones): the dislocations which do not fulfil this condition are the sessile dislocations, and their effect is included into $\varrho_{\text{mg}}^\alpha$ and $\mathbf{j}_{\text{mg}}^\alpha$, fields which are non-vanishing now. Of course, for (2.39) the tensor nature of μ_{rel} and ϵ_{rel} (although for simplicity will be not considered during the next discussion to follow) is the real carrier of how different the fraction of gliding dislocations (with respect to the amount of dislocations which could be allowed to glide by the crystal) is depending on which slip system these dislocations are located in.

The resulting equations for fields (2.39), written in Table 2.3, are very similar to those of the single slip case (Table 2.2). And if these gauge fields are coupled to other fields — the latter with own dynamics determined by other terms also included into the Lagrangian — then the Lagrangian density's part containing exclusively the gauge fields is given by

$$\ell_{\text{pure gauge part}} = \frac{1}{2} \sum_{\alpha, i=1}^3 \mathbf{E}_i^\alpha \mathbf{E}_i^\alpha - \frac{1}{2} \sum_{\alpha, i=1}^3 \mathbf{B}_i^\alpha \mathbf{B}_i^\alpha = -\frac{1}{4} \sum_{\alpha=1}^3 \sum_{\mu, \nu} F_{\mu\nu}^\alpha F^{\mu\nu \alpha}, \tag{2.40}$$

with $\mu, \nu = 0, 1, 2, 3$ (or $\mu, \nu = 1, 2, 3, 4$). The gauge potentials in $F_{\mu\nu}^\alpha$ already contain

$$\mu_0 \text{ or } \epsilon_0, \text{ and further, relations } \begin{cases} \mathbf{B}_i^\alpha = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} F_{jk}^\alpha \\ \mathbf{E}_i^\alpha = -F_{0i}^\alpha = F^{0i \alpha} \text{ or } \mathbf{E}_i^\alpha = -i F_{4i}^\alpha = -i F^{4i \alpha} \end{cases} \text{ hold.}$$

After all this preparation, it is time to discuss the leading idea of this work, for which this formulation *à la* Maxwell of the dislocation theory will result to be very useful.

	total gliding, projected on primary slip system	able-to-glide in any slip sys- tem, projected on primary
disl. density	$\mathbf{B}_i^\alpha = (\boldsymbol{\alpha}_{\text{gliding}})_i^\alpha$	$\mu_0 \mathbf{H}_i^\alpha = (\boldsymbol{\alpha}_{\text{any slip system}})_i^\alpha$
disl. current	$\mathbf{E}_i^\alpha = (\mathcal{J}_{\text{gliding}})_i^\alpha$	$\epsilon_0^{-1} \mathbf{D}_i^\alpha = (\mathcal{J}_{\text{any slip system}})_i^\alpha$

$$\begin{aligned} \partial_i \mathbf{B}_i^\alpha &= \varrho_{\text{mg}}^\alpha \\ \epsilon_{ijk} \partial_j \mathbf{E}_k^\alpha &= -\partial_t \mathbf{B}_i^\alpha - (\mathbf{j}_{\text{mg}})_i^\alpha \\ \partial_i (\epsilon \mathbf{E}_i^\alpha) &= -\rho_{\text{slip}}^\alpha \\ \epsilon_{ijk} \partial_j \left(\frac{\mathbf{B}_k^\alpha}{\mu} \right) &= (\boldsymbol{\tau}_{\text{resol}})_i^\alpha + \partial_t (\epsilon \mathbf{E}_i^\alpha) \end{aligned}$$

Table 2.3: Definitions and equations for the density and current for dislocations whose Burgers vector is completely parallel to some of the slip planes, distinguishing the contribution due to the actual gliding dislocations from that arising from pinned dislocations but which otherwise would be able to glide.

2.3 The central idea: the superconductor as model for the plastically deformed crystal [68]

The aim of the previous discussion was not only to obtain a closed set of equations for the dislocation dynamics, but to describe the different plastic states in the hardening curve displayed by a crystal under an applied shear stress. A rigid crystal opposes the existence of a non-vanishing flow of gliding dislocations through it in order to keep its rigidity. In the language of the dislocation theory as the Maxwell formulation of electromagnetism discussed in the previous section (see Tables 2.2 and 2.3), with the role of the magnetic induction and the electric field inside a material played respectively by the density and the current of gliding dislocations, a material deformed plastically but still keeping its mechanical rigidity corresponds to a good diamagnetic. And a crystal in the elastic stage, where there are no gliding dislocations, to the state of perfect diamagnetism.

This perfect diamagnetism — understood as a “magnetic rigidity” against the penetration of a magnetic induction $\vec{B}$ created by an applied electric current — is the main property shown by a superconductor in the Meissner state. The same diamagnetism, this time understood as the mechanical rigidity against the existence of gliding dislocations within a crystal under an applied shear stress, together with the formulation of the dislocation theory in a form completely analogous to that of the Maxwell equations, allows us to set the analogy between, on the hand, the elastic stage for a dislocation system, and on the other hand, the Meissner state for a superconductor.

All superconductors have a critical external electric current, at which the superconductivity — or equivalently, the rigidity against the internal spreading of a magnetic field — is destroyed, what leads to the fact that most of the external magnetic field $\mu_0 \vec{H}$ imposed on the material is permitted within the system, being this latter normal conducting state denoted as the Coulomb phase. And this phase can be also found in the stress-strain curve of a crystal: it corresponds to the well developed Stage III (or “parabolic hardening”), where most of the dislocations in the slip systems are not longer impeded to glide in such slip systems, and as a result, the crystal loses its rigidity. It should be remembered that in the formulation *à la* Maxwell of the dislocation theory, the role of the external magnetic field is played by the density of all dislocations which are potentially able to glide in any slip system.

However there exist superconductors called of type II which are able to maintain, at least partially, its superconductivity or magnetic rigidity for a range of applied external magnetic field — or of applied external electric current which produces this magnetic field trying to penetrate inside the superconductor — larger than that in the superconductors where the superconductivity exists either completely over the whole system or it does not exist at all; this last group of superconductors are called of type I. In the superconductors of type II, for an external magnetic field under a certain value H_{c1} the system is in the pure superconducting (or Meissner) state. By increasing the external magnetic field (or its source, the electric current) beyond this H_{c1} , the magnetic field is allowed inside *but* only confined within certain tubes called vortices or flux lines, vortices which due to Maxwell equation $\text{div } \vec{B} = 0$ go from side to side of the superconductor or form closed rings — at least, as long as the superconductor does not contain magnetic monopoles which could act as sinks for these vortices.

In the model which describes effectively this type of superconductors, the stability of the flux lines is ensured by the coupling of the gauge potential $\vec{A}$ (defining the magnetic field by means of $\vec{B} = \text{curl } \vec{A}$) to a complex scalar field ψ , called the order parameter and whose non-zero length $|\psi|^2$ just characterizes the magnetic rigidity. The coupling between these two fields is made in a gauge invariant way, i.e., the energy (or the Lagrangian) describing the system remains unchanged under a local (dependent on $\vec{r}$) change of the order parameter’s phase: the model thus constructed, containing this gauge invariance as well as solutions characterized by a non-zero $|\psi|^2$, is called the Abrikosov model and is considered in detail in Appendix A. It is important to note that throughout this work we will directly consider this model as an *effective* description of a superconductor, being its deduction from the fundamental microscopic scale, the Quantum Field Theory, of no interest here. As a consequence of the stability of the vortex lines, the magnetic flux carried by them can only change in integer units of a certain amount Φ_0 determined by the strength of the coupling between gauge potential $\vec{A}$ and order parameter ψ . The state in superconductors of type II characterized by the existence as isolated objects of the just discussed vortices is called the Shubnikov state: its destruction in the transition towards the normal conducting state is produced when the vortices condense, permitting thus that the magnetic field — which up to this moment was confined inside the vortices — spread over the whole material.

Returning to the dislocation system, and keeping in mind the analogy between gliding dislocation density and magnetic field allowed into a material, the Shubnikov phase

corresponds to the Stage I or easy glide stage: the start of Stage I is due to the appearance of gliding dislocations in small number, being these gliding dislocations, also stable line defects, the analogues to the vortices in a superconductor. Flux Φ_0 carried by the superconductor's vortex corresponds to the Burgers vector of the gliding dislocation. And the external electric current to produce the magnetic field at the transition point from the pure (Meissner) superconducting to the Shubnikov phase becomes the analogue to the yield stress for which the elastic state ceases and the easy glide stage begins.

Although the discussion up to this point has been carried out in qualitative terms, it will gradually result that the similarities in the behaviours displayed by these two systems, a superconductor and a crystal with dislocations, are rather pronounced: both are characterized by a certain rigidity opposing an external applied influence, something which will be described by a scalar field with non-vanishing length. And both show gauge invariance, which corresponds, for the case of the dislocation model, to the invariance (2.34) of the plastic deformation state (as well as of the dislocation density) under any compatible deformation. As a guide for the following discussion, the analogy between, on the one hand, the two superconducting and the one conducting phases described by the Abrikosov model of superconductivity, and on the other hand, the elastic and the two plastic stages — characterized the latter by a large mobility of gliding dislocations — is summarized in Table 2.4.

Abrikosov model		Dislocation model
perfect diamagnetism	$\longleftrightarrow$	rigidity $\equiv$ no dislocation mobility
gauge invariance	$\longleftrightarrow$	invariance under compatible deformations
vortex	$\longleftrightarrow$	gliding dislocation
magnetic flux Φ_0 carried by the vortex	$\longleftrightarrow$	Burgers vector of the gliding dislocation
Meissner state	$\longleftrightarrow$	elastic stage
Shubnikov state	$\longleftrightarrow$	easy glide stage (Stage I)
Normal or Coulomb conducting state	$\longleftrightarrow$	well developed “parabolic hardening” (Stage III)

Table 2.4: Analogies between the states displayed by the Abrikosov model of superconductivity and the stages shown by a crystal containing dislocations.

In a superconductor, of type II as well as of type I, the relation which takes effectively account of its behaviour is the London equation,

$$\vec{j}_{el} = -\frac{1}{\mu_0 \lambda^2} \vec{A} \quad (\text{MKS}) \text{ units}, \quad \vec{j}_{el} = -\frac{c}{\lambda^2} \vec{A} \quad (\text{CGS}) \text{ units}, \quad (2.41)$$

coupling directly the electric current $\vec{j}_{\text{el}}$, source of the external magnetic field applied on the superconductor, and the gauge potential $\vec{A}$ determining the magnetic induction allowed inside the superconductor. Penetration length λ is the characteristic length describing the range where the magnetic induction can spread inside the superconducting material. The London equation is a term which also arises in the Abrikosov model; let us study in more detail its correspondence within the framework of the dislocation model. In the formulation of Table 2.2 for the dislocation dynamics, the role which in the Maxwell equations corresponds to the electric current, is now played by the external applied resolved stress; on the other hand, the gauge potential A_k^α giving $(\alpha_{\text{glid}})_i^\alpha \equiv \mathbf{B}_i^\alpha = \epsilon_{ijk} \partial_j A_k^\alpha$ corresponds, up to a compatible deformation⁷, to the reciprocal Bravais lattice vector (in the active slip system), vector which in the linear approximation is equal to minus the deformation or strain tensor (see relation (2.10)). Hence, the London equation becomes in the dislocation model an equation relating the applied stress to the deformation due to the gliding dislocations produced by this applied stress.

Closing this argumentation, by “translating” superconductor’s $\vec{j}_{\text{el}} = -\frac{1}{\mu_0 \lambda^2} \vec{A}$ into the dislocation language and by considering the estimation for μ_0 obtained in the dislocation model, $\frac{1}{\mu_0} \sim G \lambda_0^2$, we see that the London equation carries just the same information as a (generalized) Hooke’s law in the dislocation theory, provided the penetration length λ in the superconductor is given here by the spacing λ_0 between the initial grown-in dislocations. This is actually not a surprise, since the Hooke law only states the *rigidity* of the crystal, and similarly, the London equation characterized the *resistance* displayed by a superconductor against the spreading through the material of a magnetic induction $\vec{B}$ — the latter, as often mentioned, analogue in the dislocation model to the density of mobile gliding dislocations. Hence, the conventional form discussed on page 35 of solving the dislocation distribution created by an applied stress, solution which bases on the Hooke’s law, is also contained as a particular case in the dislocation model being formulated now. The new model, nevertheless, is more general since it can be applied on states of plastic deformation where the linear relation of the Hooke’s law works no longer, as for instance in the Stage III or “parabolic hardening” (as its name points out).

As a byproduct of the analogy between Stage I and Shubnikov state, one obtains in a immediate way the removing of the divergence in the energy per unit length of an isolated dislocation straight line in a crystal: as seen in the discussion previous to equation (1.10), the usual solution for the dislocation energy diverges logarithmically like the crystal’s size, $\ln \Lambda$, and in order to remove this infinity one is forced to substitute by hand that crystal’s size with distance λ_0 between dislocations. In the dislocation model based on the analogy to the superconductor, nevertheless, the energy per unit length of a straight dislocation as analogue of a vortex in the superconductor is already finite as a consequence of the London equation. According to result (A.30), the energy per length

⁷Since the dislocation density is invariant under the addition to A_k^α of the gradient of any local displacement vector, i.e., is invariant under any compatible deformation. This is the gauge principle discussed in (2.34).

(written in MKS units) for one straight vortex in the Shubnikov state is equal to

$$\frac{E}{L} = \epsilon_{\text{vortex}} = \frac{1}{\mu_0 \lambda^2} \frac{\Phi_0^2}{4\pi} \ln\left(\frac{\lambda}{\xi}\right), \quad (2.42)$$

where ξ characterizes the radius of the vortex' core. Translating this equation into the

dislocation model we are formulating, $\begin{cases} \lambda & \rightarrow \lambda_0 \\ \mu_0^{-1} & \rightarrow G\lambda_0^2 \\ \Phi_0 & \rightarrow b \end{cases}$, one obtains the announced fi-

nite energy (1.10). In fact, the conventional argument in dislocation theory of substituting “by hand” Λ with λ_0 , based on the assumption that dislocations arrange themselves in order to cancel out their corresponding strain fields outside a certain range λ_0 , is only a form of introducing *ad hoc* a finite range for the deformation field, short range which nevertheless arises in a natural way in the just introduced analogy with the superconductor. Correlation length ξ in the superconductor corresponds in the dislocation model to the length scale ⁸ of the dislocation's core radius r_0 .

By advancing results to be discussed in detail later, the analogy to the superconductor already gives the correct behaviour of Stage I with increasing temperature. In a superconductor, the destruction of the “magnetic rigidity” or diamagnetism is favoured by increasing the temperature; and reached a certain value of temperature, the superconducting phase is forbidden at all since it has been stopped to be energetically favourable. Also in a crystal with dislocations a similar loss of rigidity appears at high temperatures, loss equivalent to the pass toward a well developed “parabolic hardening” (Stage III) — plastic stage characterized by the fact that most of the able-to-glide dislocations in the slip systems are actually allowed (by means of cross-slip and other mechanisms activated by the temperature) to become gliding dislocations — stage which, nevertheless, does not occur at low temperatures. In this latter Stage III the Hooke's law is no longer valid, just as one should expect from the analogy to a superconductor which has become a normal conductor and where the London equation no longer determines the behaviour of the material.

It should be noted that in the previous discussion the rapid hardening stage (or Stage II) does not correspond to any phase of the usual superconductor. This was already to be expected since the properties shown by such stage depend on sessile dislocations, which in the framework of the dislocation theory formulated in this work correspond to magnetic monopole fields, something not occurring in the conventional superconductor. As we shall see later, if the gauge field in a superconductor is allowed to have angular character such that a certain multivalueness (associated to the magnetic monopoles) appears, then a new phase is displayed by the superconductor with a behaviour similar to that of Stage II. In this way, the initially formal classification in able-to-glide dislocations (described by $\mathbf{H}_i^\alpha, \mathbf{B}_i^\alpha$) and in sessile dislocations (described by ϱ_{mg}) results at the end to be well justified since it yields the correct description of the crystal

⁸Superconductor's scale ξ is actually controlled by interactions occurring at a microscopic quantum scale: it depends on the energy gap characteristic of a superconductor. However, now we are dealing with effective models, without taking care of which are the actual interactions occurring at a scale smaller than the shortest distance we are able to observe, and thus such a scale is set simply by the limit of validity of our formulated model.

deformed plastically. The superconductor for which the gauge potential has angular character will be called the compact superconductor, one of whose simplest forms is the “ $SO(3)$ ”-superconductor, dealt with on page 233.

2.4 Formulating the model in four steps

After having seen the similarities between the phases of a superconductor and the different stages of plastic deformation, it is time to construct for the dislocation theory a model based in the Abrikosov model of superconductivity. The main ingredient in the latter is the invariance under local (gauge) changes in the phase of the scalar field which is the carrier of a large non-vanishing rigidity or diamagnetism. This is therefore the first step to follow in the construction of our model: to define by means of a scalar field a model to describe the elastic rigidity or “diamagnetism” against the existence within the crystal of mobile gliding dislocations. As next, the coupling (in a gauge covariant way) of such scalar field to the gauge field defining the able-to-glide dislocations will yield one model with the main characteristics of the Abrikosov model.

2.4.1 Step 1: the gauge invariant energy for dislocation configurations

For the following, except when indicating the contrary, we shall work in the CGS system of units (2.36). For each direction α of the crystal lattice let us define a scalar field $\psi^\alpha = |\psi^\alpha|e^{i\varphi^\alpha}$. Such scalar field can be represented in an internal 2-dimensional space as an “arrow” of length $|\psi^\alpha|$, whose orientation with respect to a certain internal reference direction is just given by φ^α : thus at each point of the (external) physical crystal lattice and for each lattice’s direction, one of these 2-dim. internal planes, where the scalar field “lives”, is defined.

Now we set the scalar field’s phase at point $\vec{r}$ proportional to the displacement $u^\alpha(\vec{r})$ measured with respect to the undeformed state of the crystal

$$\psi^\alpha = |\psi^\alpha|e^{i\varphi^\alpha}, \quad (2.43)$$

$$\text{with phase } \varphi^\alpha = \frac{2\pi}{a} u^\alpha \quad \alpha = 1, 2, 3,$$

being a the interatomic distance or spacing of the crystal lattice. For the following, let us assume that the scalar field’s length $|\psi^\alpha|$ is a nonzero constant everywhere throughout the crystal; how to achieve it will be discussed soon in (2.45).

The elastic deformation state of the crystal lattice results thus easily to visualize: the undeformed crystal is represented as the configuration of parallel scalar fields where, at each physical point of the crystal, ψ^α in its internal 2-dim. space is pointing along the reference direction. A translation of each crystal cell by the same amount is represented through the rotation of every ψ^α by the *same* angle; and if the displacement which each cell center suffers during this rigid translation is an integer multiple of the interatomic distance a , the ψ^α assigned to the cell center draws also such a multiple of complete

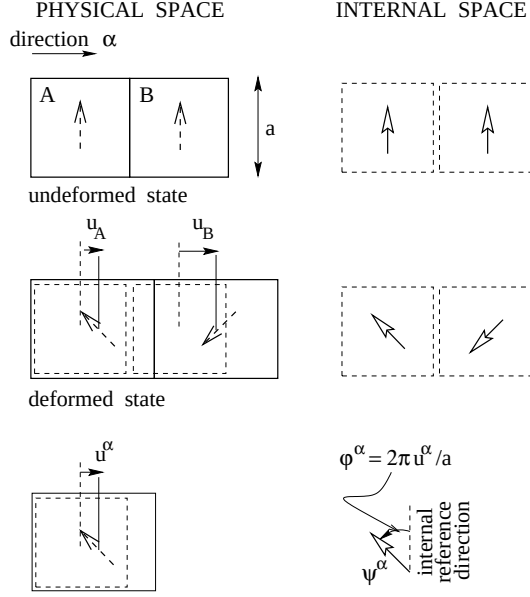


Figure 2.6: Example of a deformation of two neighbour atomic cells along direction α , whereas at the initial undeformed state the scalar field for both cells was pointing (within its internal space) in the same reference direction.

internal rotations, reflecting thus the periodic nature of the crystal lattice. In the same way if, by looking at two neighbour lattice cells along a certain direction α , the scalar field changes its internal orientation when going from one cell to the other, then the crystal lattice is deformed at that place; as already mentioned, the change in orientation of ψ^α occurs in the internal space where the scalar field is represented as an arrow, although for simplicity, in Fig. 2.6 this arrow has been represented directly on the physical crystal plane. The scalar field is defined at every point of a continuum which modellizes the system of the crystal; the fact that in the previous examples we have focussed our attention on the center of the crystal cells was only due to make more easily understandable the model, but we are working with a continuous model.

Any deformation state of a crystal will be represented thus by some configuration of the scalar field ψ^α , assigning to each point of the *perfect*, undeformed crystal an internal orientation of the scalar field. Or equivalently, by placing this configuration of ψ^α into a continuum with the same form as the original undeformed crystal. After this assignation, one forgets the real crystal lattice and focus the attention on the arrangement of the scalar fields, knowing that deformations are signalized when, by passing from a crystal point to another one in the external space, there exists a gradient (a change) in the orientation of ψ^α . The energy contained in such a deformed state can be modellized as being proportional to the just mentioned gradient

$$\frac{1}{2} \sum_{\alpha, \mu=1}^3 \int d^3 \vec{r} (\partial_\mu \psi^\alpha) (\partial_\mu \psi^\alpha)^* \quad |\psi^\alpha| \equiv \text{const} \quad \frac{1}{2} \sum_{\alpha, \mu=1}^3 |\psi^\alpha|^2 \int d^3 \vec{r} (\partial_\mu \varphi^\alpha)^2 . \quad (2.44)$$

To ensure $|\psi^\alpha| = \text{const.}$ in the energy density $\frac{1}{2} \sum_{\alpha, \mu=1}^3 (\partial_\mu \psi^\alpha) (\partial_\mu \psi^\alpha)^*$, a potential of the Landau-Ginzburg form

$$V(|\psi^\alpha|) = -|\alpha| |\psi^\alpha|^2 + \frac{|\beta|}{2} |\psi^\alpha|^4 , \quad (2.45)$$

has to be included, with $|\alpha|, |\beta|$ constant parameters. Since

$$\epsilon_{\text{energy}} = \sum_{\alpha=1}^3 \left[\frac{1}{2} \sum_{\mu} (\partial_{\mu} |\psi^{\alpha}|)^2 + V(|\psi^{\alpha}|) + \frac{1}{2} |\psi^{\alpha}|^2 \sum_{\mu} (\partial_{\mu} \varphi^{\alpha})^2 \right], \quad (2.46)$$

and since $\frac{1}{2} |\psi^{\alpha}|^2 \sum_{\mu} (\partial_{\mu} \varphi^{\alpha})^2$ is positive, minimizing the energy requires minimizing the above underlined terms, something achieved by a constant nonzero $|\psi^{\alpha}|$ — any gradient in $|\psi^{\alpha}|$ yields a positive contribution to the energy — solution of $\frac{d}{d|\psi^{\alpha}|} V(|\psi^{\alpha}|) = 0$:

$$|\psi^{\alpha}| = \sqrt{\frac{|\alpha|}{|\beta|}}. \quad (2.47)$$

A minimum of energy (2.46) corresponds to a scalar field whose length is given by the previous result for any physical point of the crystal, and whose phase is also a constant $\forall \vec{r}$: thus there are (infinite) many different lowest energy configurations, each characterized by the value of the constant scalar field's phase. Note that although energy density (2.46) is invariant under the subtraction of a constant value χ_0 to the phase φ^{α} , each of the just discussed lowest energy configurations does not have such a symmetry: change $\varphi^{\alpha} \rightarrow \varphi^{\alpha} - \chi_0$ takes one minimum into another different one. The fact that a configuration of lowest energy is not left invariant by a set of symmetry operations under which the energy itself does remain invariant is called the spontaneous symmetry breaking, where the word “spontaneous” means that the election of one particular minimum (in our case, the election of the particular constant φ^{α} defining the lowest energy configuration) has to be imposed by external means, as for instance, through certain boundary conditions.

Returning to energy (2.44), and taking into account the relation of the scalar field's phase to the displacement of crystal point during a deformation, $\varphi^{\alpha} = \frac{2\pi}{a} u^{\alpha}$, one sees that such an energy is just the usual elastic energy $\frac{G}{2} (\vec{\nabla} u^{\alpha})^2$ with the identification of the constant scalar field's length as the shear modulus: $|\psi^{\alpha}|^2 \leftrightarrow \frac{Ga^2}{4\pi^2}$. As already mentioned, in the general case there are several different elastic constants depending on the direction α for the deformation, and this would require a different potential $V(|\psi^{\alpha}|)$ for each of these directions; but for the sake of simplicity we shall further consider only one single elastic constant, G , and therefore the index α for the scalar field's length will be not written hereafter.

Of course the election of the energy (2.44) was not arbitrary if one keeps in mind the central scope of our task: to describe the hardening curve by constructing a dislocation theory based on the Abrikosov model of superconductivity. In the latter model, a potential of the type (2.45) produces a large $|\psi|^2$ responsible for the ideal diamagnetism of a superconductor⁹. And according to the qualitative discussion summarized in Ta-

⁹In fact, the ideal diamagnetism or Meissner effect in a superconductor is not only the result of a large constant $|\psi|^2$ but also of the coupling, by means of the covariant derivative, of the scalar field to the gauge field $\vec{A}$ which determines the allowed magnetic induction $\vec{B} = \text{curl } \vec{A}$ inside the superconductor.

ble 2.4, this diamagnetism plays the role of the rigidity against the action of gliding dislocations in a crystal, rigidity characterized by the elastic shear modulus G , as we have just obtained above

$$\frac{\hbar^2 |\psi|^2}{m} \Big|_{\text{Abrikosov model}} \longleftrightarrow |\psi|^2 \Big|_{\text{dislocation model}} = \frac{Ga^2}{4\pi^2}. \quad (2.48)$$

This was the reason of having chosen a potential to produce the symmetry breaking, i.e., to force a constant scalar field's length carrier of the system's rigidity, procedure which is one of the two main components of the Abrikosov model.

The other central component of the Abrikosov model is the gauge invariance. According to the definition of the scalar field (2.43), a compatible deformation defined by a displacement vector $u^\alpha(\vec{r})$ is expressed as a local transformation of the scalar field's phase

$$\varphi^\alpha \rightarrow \varphi'^\alpha = \varphi^\alpha + u^\alpha(\vec{r}), \quad (2.49)$$

change under which the energy (2.44), $\frac{1}{2} \sum_{\alpha, \mu=1}^3 |\psi^\alpha|^2 \int d^3 \vec{r} (\partial_\mu \varphi^\alpha)^2$, does not remain invariant. However, compatible deformations do not produce the changes of plastic behaviour observed along the hardening curve, being the description of such curve the aim of this work.

The analogy to the Abrikosov model has allowed us to identificate the different plastic stages as different phases displayed by the system. The phase transitions connecting these different stages are signalized by changes in the behaviour of certain correlation functions calculated by means of the model's energy. If this energy changes under transformations like the compatible ones (and although the crystal deformed compatibly does have an energy higher than in the undeformed state), the model constructed with this energy is not valid to describe the phase transitions actually occuring, since otherwise some compatible deformation could exist which would cause a phase transition. The model's energy has to be extended in order to remove the variance of the energy under such transformations, or more properly, by identifiating the entire set of these compatible transformations with the identity transformation for the extended energy. Although a part dependent on compatible deformations can still be present in the energy, this part will *not* enter in the energy's term responsible for the phase transitions.

The solution to this question completes the analogy to the Abrikosov model. Since we have already formulated the dislocation dynamics by means of a set of equations which remain invariant under compatible deformations and which have the form of the Maxwell equations of the electromagnetism, the form to implement also for the scalar part the invariance of the model's energy under any transformation of the type (2.49) is to couple through the covariant derivative (also called to couple minimally)

$$D_\mu \psi^\alpha = \partial_\mu \psi^\alpha - ie \psi^\alpha A_\mu^\alpha, \quad (2.50)$$

the scalar field ψ^α to the gauge field A_μ^α which determines the density of the able-to-glide dislocations. And simultaneously, to include into the system's energy the contribution

arising from pure gauge field part $\frac{1}{2} \sum_{\mu=1}^3 \mathbf{B}_\mu^\alpha \mathbf{B}_\mu^\alpha$. It should be noted that, in the same way as in the Abrikosov model, the pure gauge field part only contains the static term $\frac{1}{2} \sum_{\mu=1}^3 \mathbf{B}_\mu^\alpha \mathbf{B}_\mu^\alpha$, since in the static case the current of gliding dislocations $\mathbf{E}_\mu^\alpha$ (role played by the electric field in a superconductor) is completely decoupled of the density of dislocations which can become gliding. To consider only the static case is because we are not interested in *how* a phase transition starts and evolves along time until reaching the equilibrium condition in the new phase, but in which are the *initial* and the *final configurations* characterizing the two phases between which the phase transition has already taken place. In other words, we are only interested in those configurations describing the old phase and in those corresponding to the new phase, with both phases already considered in equilibrium: the transition point is thus defined as the point where the energy of equilibrium configurations characterizing the new phase have become energetically so favourable as the configurations which were describing (in equilibrium) the old phase. The actual details during the phase transition itself about how the new phase begins to form, details which would require the dynamics of all fields and thus the contribution arising from $\mathbf{E}_\mu^\alpha$, will be not studied.

Hence one obtains the following energy density

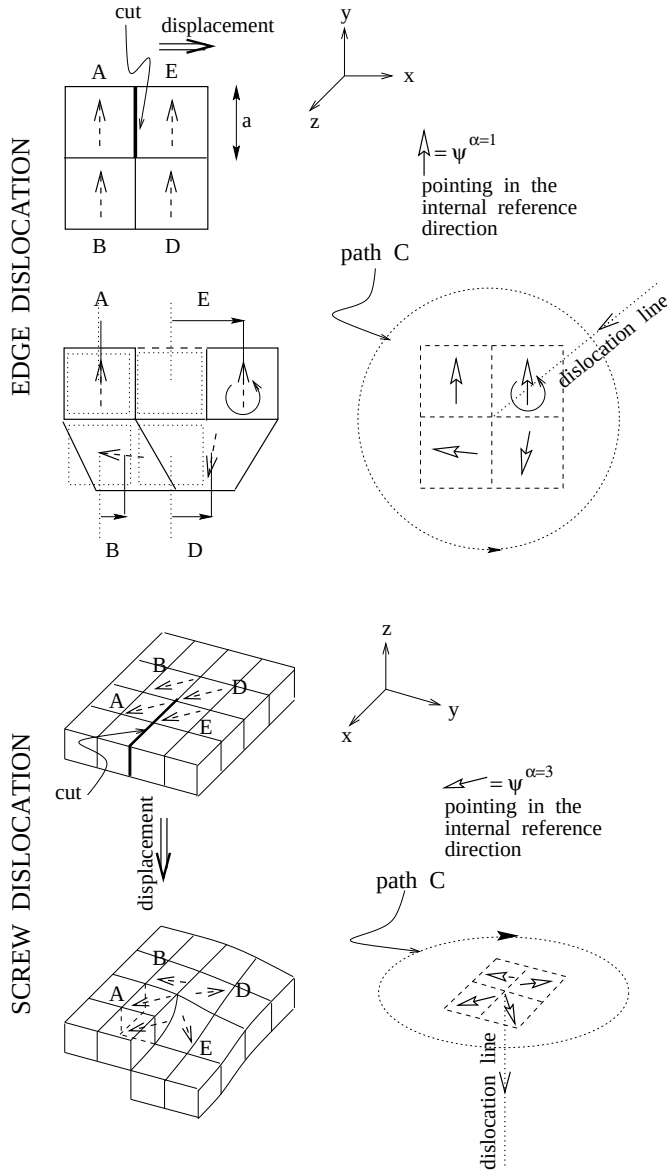
$$\begin{aligned} \epsilon_{\text{energy}} &= \sum_{\alpha, \mu=1}^3 \left[\frac{1}{2} \mathbf{B}_\mu^\alpha \mathbf{B}_\mu^\alpha + \frac{1}{2} (\partial_\mu \psi^\alpha - i e \psi^\alpha A_\mu^\alpha) (\partial_\mu \psi^\alpha - i e \psi^\alpha A_\mu^\alpha)^* + V(|\psi^\alpha|) \right] \Big|_{|\psi^\alpha| \equiv \text{const}} \\ &= \sum_{\alpha=1}^3 \left[\frac{1}{4} \sum_{\mu, \nu=1}^3 F_{\mu\nu}^\alpha F_{\mu\nu}^\alpha + \frac{|\psi|^2}{2} (\partial_\mu \varphi^\alpha - e A_\mu^\alpha)^2 \right], \end{aligned} \quad (2.51)$$

with $F_{\mu\nu}^\alpha = F^{\mu\nu \alpha} = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha$ and $\mu, \nu = 1, 2, 3$, which is invariant under any compatible deformation of local well-defined displacement $\chi^\alpha(\vec{r})$

$$\begin{cases} \varphi^\alpha & \rightarrow \varphi'^\alpha = \varphi^\alpha - \chi^\alpha, \\ A_\mu^\alpha & \rightarrow A'^\alpha_\mu = A_\mu^\alpha - \frac{1}{e} \partial_\mu \chi^\alpha, \end{cases} \quad (2.52)$$

as required in order to be able to construct a model which could describe the different behaviours of the hardening curve. This local transformation will be also called a $U(1)$ gauge transformation. And further the scalar field ψ^α , borrowing the notation from the Abrikosov model, will be also denoted as the order parameter. Although not working with a full 4-dimensional model, the greek notation for the spatial indices will be maintained in order to reserve the latin ones for the description of the configurations appearing at high temperatures.

The actual object of study in this work, the dislocations, can also be visualized by means of scalar field $\psi^\alpha = |\psi| e^{i\varphi^\alpha}$ (with $|\psi|$ constant). Let us suppose we have produced a dislocation in a perfect crystal by cutting along a lattice plane, then displacing a part by one (or several) atomic distance in the direction α with respect to the other side,



and finally connecting and gluing the two parts. As for any deformation, the configuration of the scalar field corresponding to the dislocation results from assigning to every point of the deformed physical crystal one point in an undeformed plane-continuum, where arrow ψ^α is placed and oriented by an angle proportional to the displacement suffered by each crystal point in physical space; actually these are three planes, one for each ψ^α (or for each crystal direction α). In the figures can be seen the representation of an edge dislocation along the z direction (and with Burgers vector in the x direction) as well as a screw dislocation along the $-z$ direction. Characteristic of any dislocation is the fact that by covering a closed path C encircling the dislocation in the physical space at any distance from the dislocation, the scalar field ψ^α draws a complete rotation in its internal plane. Index $\alpha = 1, 2, 3$ denotes therefore the direc-

tion of the Burgers vector. It is thus how the angular nature of the scalar field's phase really becomes apparent: assuming the scalar field's length is nonzero for points well distant from a dislocation, a configuration of the scalar such as those just seen in the figures above — where the internal orientation of the scalar field corresponds to the angle in external physical space defining a closed path C around the dislocation line and thus ψ^α completes an internal rotation after having covered C , yielding a nonvanishing gradient $\partial_\mu \varphi^\alpha$ maintained even at distances $|\vec{r}| \rightarrow \infty$ from the dislocation —, produces a diverging contribution to $\int d^3 \vec{r} (\partial_\mu \varphi^\alpha)^2$. This infinite energy (the same divergence as discussed in (1.7)) cannot be relaxed by the crystal, at least without repeating in the contrary sequence the cutting process followed to produce the dislocation. The divergence is nevertheless avoided through the coupling to the gauge field by imposing

that the covariant derivative vanishes rapidly enough by approaching, in the physical space, distances very far from the dislocation

$$\left(\partial_\mu \psi^\alpha - ie\psi^\alpha A_\mu^\alpha\right)\Big|_{r \rightarrow \infty} = \underbrace{|\psi|}_{\neq 0} \Big|_{r \rightarrow \infty} \left(\partial_\mu \varphi^\alpha - eA_\mu^\alpha\right)\Big|_{r \rightarrow \infty} = 0, \quad (2.53)$$

or equivalently

$$\oint_{\text{circumference } C \text{ at } r \rightarrow \infty} dl^\mu A_\mu^\alpha = \frac{1}{e} \int_{\varphi^\alpha=0}^{\varphi^\alpha=2\pi\hat{u}^\alpha} dl^\mu \partial_\mu \varphi^\alpha = \frac{2\pi}{e} \hat{u}^\alpha, \quad (2.54)$$

with $\hat{u}^\alpha$ a unit vector in the direction where the cut part has been displaced during the process of producing the dislocation: $\hat{u}^\alpha$ is the unit vector in the direction of the Burgers vector. Result (2.54) states simply the topological nature of the dislocation line and its Burgers vector: it is, as expected, the analogue to the quantization in units of $\Phi_0 = 2\pi \frac{\hbar c}{e}$ of the magnetic flux carried by a vortex in the Shubnikov phase of a superconductor. A complete introduction into the topological stability of defects such as the vortex in a superconductor can be found in Appendix B.

The above discussion on topologically stable objects described by configurations of the scalar field, whose phase is proportional to a polar angle in physical space describing a contour around such objects, is not only applicable to the dislocations which can be able to glide but to any type of dislocations. But it should be stressed once more that alone the topological stability, understood as the impossibility¹⁰ of the system (without being cut) to relax the energy contained in such objects, cannot take account of the different stages during the plastic deformation. It is by focussing the attention on the able-to-glide dislocations what allows us to set an analogy with the behaviour of a type II superconductor: it is the appearance of vortices in the latter what signalizes the transition to the Shubnikov phase, and it is the condensation of these vortices what destroys the almost ideal diamagnetism of a superconductor; similarly, in a crystal it is the appearance of gliding dislocations (not simply of dislocations of any type, which almost any crystal contains in large number even in the elastic stage) what defines the transition to the easy glide stage, and what destroys the crystal's rigidity is just the spreading throughout the crystal of a density of gliding dislocations which has become equal to all dislocations which could be able to glide.

Once the analogy to the Abrikosov model has been set up completely, it only remains to identify the value of constant e appearing in the covariant derivative $\partial_\mu \psi^\alpha - ie\psi^\alpha A_\mu^\alpha$, characterizing the strength of the coupling between the scalar and the gauge field. Return for a moment to (2.51), one sees that the term responsible for the limited spreading of the gliding dislocation density within a rigid crystal (as analogue to the short range of the magnetic induction $\vec{B}$ within a superconductor) is given in the energy density by

¹⁰Impossibility which reflects again the invariance of the dislocation density under compatible deformations, being the latter just the crystal deformations which cannot contain the “surgery” of cutting the crystal.

the underlined term

$$\epsilon_{\text{energy}} = \sum_{\alpha, \mu=1}^3 \left[\frac{1}{2} \mathbf{B}_\mu^\alpha \mathbf{B}_\mu^\alpha + \dots + \underline{\frac{1}{2} |\psi|^2 e^2 A_\mu^\alpha A_\mu^\alpha} \right]. \quad (2.55)$$

In the language of superconductivity, it is $\frac{1}{2} |\psi|^2 e^2 A_\mu^\alpha A_\mu^\alpha$ the term which yields ¹¹ the London equation for the external source of the magnetic field $\vec{B}$; or for the external source of the field analogue to $\vec{B}$ (the gliding dislocation density) in the language of the dislocation model. From relation (2.41) therefore, $\frac{1}{\sqrt{|\psi|^2 e^2}}$ in the elastic and in the easy glide stage, plastic stages similar in behaviour to the two superconducting phases (Meissner and Shubnikov), corresponds to the initial grown-in dislocation spacing λ_0 . Combining this result with the relation between the scalar field's length and the rigidity modulus G obtained in (2.48), it follows

$$e = \frac{2\pi}{\sqrt{Ga^2}} \frac{1}{\lambda_0}. \quad (2.56)$$

And the Burgers vector carried by a gliding dislocation becomes equal to the crystal lattice spacing a as it should

$$b^\alpha = \oint_{\text{contour } C \text{ arbitrarily distant}} dl^\mu A_\mu^\alpha = \sqrt{\mu_0} \frac{2\pi}{e} \hat{u}^\alpha \stackrel{\mu_0=1/G\lambda_0^2}{=} a \hat{u}^\alpha, \quad (2.57)$$

where the result is written in the system of units MKS, without having included μ_0 into the gauge field.

Note. Gauge invariance and the Noether theorem.

Let us consider a general scalar field ψ^a , where a denotes here some *internal* index; this internal index a is not related to index α used to denote in the previous discussion the direction of the Burgers vector, or equivalently, of the displacement imposed to one part of the crystal's lattice after the cutting procedure explained above. For instance, a complex scalar $\psi = |\psi| e^{i\varphi}$ can be written as a 2-dimensional vector $\begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \end{pmatrix} \equiv \psi^a = |\psi| \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$, what defines an internal 2-dim. symmetry plane where the scalar field is oriented as an arrow of length $|\psi|$, determining φ this orientation. A gauge transformation corresponds to a change — which can be local, dependent on the coordinates $\vec{r}$ of the physical space — of the scalar field's phase without any modification of the scalar field's length $|\psi|$, or equivalently, it corresponds to a change in orientation of the scalar field in its internal symmetry space. A more general scalar field corresponds to an arrow in an internal symmetry space of dimensions higher than 2, as for instance $\begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \\ \psi^{a=3} \end{pmatrix} \equiv \psi^a = |\psi| \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}$: in such case, as different from the simplest

¹¹This is easily seen by noting that, for the case of a $U(1)$ gauge group, the sources of the gauge field are completely given in the Euler-Lagrange equations by the term $\frac{\partial \ell}{\partial A_\mu^\alpha}$; and in the static case we are considering here, Lagrangian density ℓ is equal to minus the energy density. For (2.55), such source is $\frac{\partial \ell}{\partial A_\mu^\alpha} = -|\psi|^2 e^2 A_\mu^\alpha$.

case $\psi^a = |\psi| \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$, two consecutive gauge transformations do not commute for the same reason as rotations in ordinary 3-dim. space (or in a space of higher dimensionality) do not commute. Hence the gauge group describing the latter local internal rotations (or gauge transformations) is a non-Abelian group, whereas the group for the 2-dim. internal space is Abelian (and called the $U(1)$ gauge group).

Allways summing over repeated indices, a general infinitesimal gauge transformation is written as

$$\psi^a \rightarrow \psi'^a = \psi^a + \delta\psi^a = \psi^a - i\chi^{(c)} I_{ab}^{(c)} \psi^b, \quad (2.58)$$

with $I_{ab}^{(c)}$ the generator of the internal rotations, and where $^{(c)}$ indicates the number of different internal axes for such rotations. For the case of the Abelian $U(1)$ gauge symmetry, where rotations takes place only on one plane, there is only one axis and index $^{(c)}$ is not necessary; moreover, $I_{ab} = -i\epsilon_{ab}$ (with ϵ_{ab} the completey antisymmetric tensor in 2 dimensions) since matrix $I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the generator of

any planar rotation $e^{-i\chi I} = \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix}$, and equivalently, of any $U(1)$ gauge transformation.

If the model we are studying contains some derivative of the scalar field, i.e., the system energy depends on gradients in physical space created by a non-uniform internal orientation of ψ , and furthermore if one requires the model to remain invariant under any change (even local) of this scalar field's orientation, then ψ has to be coupled to a another field $A_\mu^{(c)}$, called the gauge field, in order to ensure such invariance. Such coupling occurs through the covariant derivative

$$D_\mu \psi^a = \partial_\mu \psi^a - ie I_{ab}^{(c)} \psi^b A_\mu^{(c)}, \quad (2.59)$$

which is required to transform like the scalar field itself, so that the contraction in the internal index a of the scalar derivative with the scalar field or with another covariant derivative results to be invariant under any gauge transformation (or internal rotation). If the internal space for the scalar field has different independent axes, denoted by $^{(c)}$, then for each axis it is necessary the coupling to a different gauge field $A_\mu^{(c)}$ (except for the $U(1)$ case, where there is only one axis). For the following, the internal symmetry space where the scalar field is represented as an arrow will be called isospace; and the rotations taking place inside it, isorotations. A field of the form $\psi^a = |\psi| \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}$ transforms in isospace like a vector, but in the external physical space it remains a scalar field.

The transformation law for gauge field $A_\mu^{(c)}$ under an infinitesimal gauge transformation is obtained from the following relation

$$\begin{cases} \psi^a \rightarrow \psi'^a = \psi^a - i\chi^{(c)} I_{ab}^{(c)} \psi^b, \\ D_\mu \psi^a \rightarrow \left(\partial_\mu \psi'^a - ie I_{ab}^{(e)} \psi'^b A_\mu^{(e)} \right) = D_\mu \psi^a - ie \chi^{(c)} I_{ab}^{(c)} \left(\partial_\mu \psi^b - ie I_{bd}^{(e)} \psi^d A_\mu^{(e)} \right). \end{cases} \quad (2.60)$$

By operating

$$\begin{aligned} \partial_\mu \psi^a &\rightarrow \partial_\mu \psi'^a = \partial_\mu \psi^a - i \left(\partial_\mu \chi^{(c)} \right) I_{ab}^{(c)} \psi^b - i\chi^{(c)} I_{ab}^{(c)} \partial_\mu \psi^b, \\ -ie I_{ab}^{(e)} \psi^b A_\mu^{(e)} &\rightarrow -ie I_{ab}^{(e)} \psi'^b A_\mu^{(e)} = -ie I_{ab}^{(e)} \psi^b A_\mu^{(e)} - e\chi^{(c)} I_{ab}^{(e)} I_{bd}^{(c)} \psi^d A_\mu^{(e)} = \\ &= -ie I_{ab}^{(e)} \psi^b A_\mu^{(e)} - \underline{e\chi^{(c)} I_{ab}^{(e)} I_{bd}^{(c)} \psi^d A_\mu^{(e)}}, \end{aligned}$$

where the substitution by $A_\mu^{(e)}$ of $A_\mu'^{(e)}$ in the last equality (yielding the underlined term) holds to leading order in infinitesimal $\chi^{(c)}$. Comparing the right hand term with the result of transforming the covariant derivative like the scalar field, $D_\mu \psi^a - ie \chi^{(c)} I_{ab}^{(c)} \left(\partial_\mu \psi^b - ie I_{bd}^{(e)} \psi^d A_\mu^{(e)} \right)$, one obtains

$$-ie I_{ab}^{(e)} \psi^b A_\mu'^{(e)} = -ie I_{ab}^{(e)} \psi^b A_\mu^{(e)} + i \left(\partial_\mu \chi^{(c)} \right) I_{ab}^{(c)} \psi^b - e\chi^{(c)} \left[I_{ab}^{(c)}, I_{bd}^{(e)} \right] \psi^d A_\mu^{(e)},$$

where the commutator of the two transformation generators does not vanish for a non-Abelian group and is given by $\left[I^{(c)}, I^{(e)} \right]_{ad} = C_{(f)}^{(ce)} I_{ad}^{(f)}$, with $C_{(f)}^{(ce)}$ the structure constants of the Lie algebra to which the gauge group belongs; more about Lie algebras can be found in Appendix B after equation (B.57) on page 213. For the scope of this work we shall restrict ourselves to simple compact Lie algebras, for which the structure constants satisfy relation $C_{(f)}^{(ce)} = C_{(cef)}$, being the latter totally antisymmetric under the change of two of its indices (see equation (B.76)): as an example of this case is the Lie algebra to which the group of 3-dimensional rotations belong, defined by the structure constants $C_{cef} = \epsilon_{cef}$. Applying this consideration, one obtains finally the transformation law for the gauge field under an infinitesimal (local) gauge transformations

$$A_\mu^{(c)} \rightarrow A'_\mu^{(c)} = A_\mu^{(c)} - \frac{1}{e} \partial_\mu \chi^{(c)} + C_{(cde)} \chi^{(d)} A_\mu^{(e)}. \quad (2.61)$$

For the Abelian $U(1)$ gauge group, on the contrary, the structure constants are identically zero and one obtains the well-known transformation $A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \chi$ for the gauge potential of the electromagnetism.

The (local) Lagrangian density ℓ — if we are studying the dynamical evolution in time ($\mu = 0, 1, 2, 3$) of field arrangements — or the energy density ϵ_{energy} — if we are considering the energy of several equilibrium ($\mu = 1, 2, 3$) configurations describing different states in order to characterize a phase transition between them — is a function of the covariant derivative $\partial_\mu \psi^a - ie I_{ab}^{(c)} \psi^b A_\mu^{(c)}$ and this is the only form of coupling between scalar field ψ^a and gauge field $A_\mu^{(c)}$ in order to ensure the gauge invariance of the Lagrangian/energy. Hence relation

$$\frac{\partial}{\partial(\partial_\mu \psi^a)} \left[\ell \left(\psi^a, \partial_\mu \psi^a, A_\mu^{(c)}, \partial_\nu A_\mu^{(c)} \right) \right]_{\epsilon_{\text{energy}} \left(\psi^a, \partial_\mu \psi^a, A_\mu^{(c)}, \partial_\nu A_\mu^{(c)} \right)} = -ie I_{ab}^{(c)} \psi^b \frac{\partial}{\partial A_\mu^{(c)}} \left[\ell \left(\psi^a, \partial_\mu \psi^a, A_\mu^{(c)}, \partial_\nu A_\mu^{(c)} \right) \right]_{\epsilon_{\text{energy}} \left(\psi^a, \partial_\mu \psi^a, A_\mu^{(c)}, \partial_\nu A_\mu^{(c)} \right)} \Big|_{\text{source}} \quad (2.62)$$

is satisfied. In the previous result, the label “source” refers to the contributions arising from the Lagrangian/energy terms where the scalar is coupled to the gauge field. It should be noted that for non-Abelian gauge fields, also the pure gauge field part (that equal to the square of the gauge field’s curl) contributes to $\frac{\partial \ell}{\partial A_\mu^{(c)}}$: such contribution does not enter in the $\frac{\partial \ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}}$ of equation (2.62)¹².

For non-Abelian fields holds thus

$$\frac{\partial \ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}} = \frac{\partial \ell}{\partial A_\mu^{(c)}} - e C_{(cde)} A_\nu^{(d)} \left(\partial_\mu A_\nu^{(e)} - \partial_\nu A_\mu^{(e)} + e C_{(efg)} A_\mu^{(f)} A_\nu^{(g)} \right), \quad (2.63)$$

whereas for the Abelian case it reduces to $\frac{\partial \ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}} = \frac{\partial \ell}{\partial A_\mu^{(c)}}$ since the structure constants in such case are equal to zero.

¹²In the non-Abelian case (when index (c) plays a determinat role and the structure constants do not vanish) one must be more careful with the identification of the sources for the gauge field, which do not correspond to the whole $\frac{\partial \ell}{\partial A_\mu^{(c)}}$.

The non-commutativity in this case makes that the correct curl for the gauge field is not simply $\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)}$ nor $-\frac{1}{4} \left(\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} \right)^2$ is the Lagrangian density which should remain invariant under gauge transformation (2.61)

$$\begin{aligned} \delta \left[-\frac{1}{4} \left(\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} \right) \left(\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} \right) \right] &= - \underbrace{\left(\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} \right) C_{(cde)} \left(\partial_\mu \chi^{(d)} \right) A_\nu^{(e)}}_{\neq 0} - \\ &\quad - \frac{1}{2} \chi^{(d)} \underbrace{C_{(cde)} \left(\partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} \right) \left(\partial_\mu A_\nu^{(e)} - \partial_\nu A_\mu^{(e)} \right)}_{= 0, \text{ due to antisymmetry of } C_{(cde)}} \neq 0. \end{aligned}$$

Now let us apply all these results to the case of a internal rotation of the scalar field by the *same* amount at all points of the physical space

$$\psi^a \rightarrow \psi'^a = \psi^a - i\chi^{(c)} I_{ab}^{(c)} \psi^b, \quad (2.64)$$

with $\chi^{(c)}$ an infinitesimal *constant* parameter. Since the Lagrangian (or the energy) density is invariant under such a transformation (it is only a particular case of the more general local gauge transformation), relation $0 = \delta\ell = \frac{\partial\ell}{\partial\psi^a} \delta\psi^a + \frac{\partial\ell}{\partial(\partial_\mu\psi^a)} \delta(\partial_\mu\psi^a)$ holds, where $\delta(\partial_\mu\psi^a) = \partial_\mu(\delta\psi^a)$ since χ is constant. By applying the Euler-Lagrange equations, $\partial_\mu \left[\frac{\partial\ell}{\partial(\partial_\mu\psi^a)} \right] = \frac{\partial\ell}{\partial\psi^a}$, it follows the conservation law

$$0 = \delta\ell = \partial_\mu \left[\frac{\partial\ell}{\partial(\partial_\mu\psi^a)} \delta\psi^a \right] \quad \chi^{(c)} \stackrel{\text{constant}}{=} \frac{1}{e} \chi^{(c)} \partial_\mu \left[-ie I_{ab}^{(c)} \psi^b \frac{\partial\ell}{\partial(\partial_\mu\psi^a)} \right] = 0. \quad (2.65)$$

The same discussion can be carried out for the energy density, with the only difference that the resulting conservation law is only a static one: it states the vanishing divergence of a vector field (in external physical space) equal to $-ie I_{ab}^{(c)} \psi^b \frac{\partial\epsilon_{\text{energy}}}{\partial(\partial_\mu\psi^a)}$ (with $\mu = 1, 2, 3$).

Nevertheless, and according to (2.62), the conserved $J^{(c)\mu} = -ie I_{ab}^{(c)} \psi^b \frac{\partial\ell}{\partial(\partial_\mu\psi^a)}$ of equation (2.65) is also equal to $J^{(c)\mu} = \frac{\partial\ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}}$ due to the coupling between scalar and gauge field through the covariant derivative. And it is just $\frac{\partial\ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}}$ the term containing, in the Euler-Lagrange equations for $A_\mu^{(c)}$, the sources for this gauge field. Since these sources are the electric current and charge density — or the fields which play the analogous role in the equations, as the applied stress and the slip velocity for mass flow in the dislocation model — the invariance of the model under a constant transformation of the kind $\psi^a \rightarrow \psi'^a = \psi^a - i\chi^{(c)} I_{ab}^{(c)} \psi^b$ (with $\chi^{(c)}$ constant) leads to the result that the electric charge is conserved: $\partial_\mu J^{(c)\mu} = 0$. This is a particular case of a more general result known as the Noether theorem, which states that to every invariance of the Lagrangian/energy corresponds a conservation law.

This theorem is also the reason for equation (2.27): a global addition of the same constant to the phase of the scalar field coupled to the gauge field of the dislocation model is nothing else as the translation of the whole lattice by the same displacement vector. The invariance of the energy under such a transformation (the formulated energy actually satisfies a much stronger condition which is the invariance under a compatible deformation) — transformation which, both local (compatible deformation) as global (rigid translation), is an Abelian $U(1)$ transformation — leads to the conservation

As discussed in Appendix B at equation (B.81) on page 220, the correct curl tensor is given by

$$G_{\mu\nu}^{(c)} = \partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} + eC_{(cde)} A_\mu^{(d)} A_\nu^{(e)},$$

which transforms (result (B.82)) as $G_{\mu\nu}^{(c)} \rightarrow G_{\mu\nu}^{(c)} + C_{(cde)} \chi^{(d)} G_{\mu\nu}^{(e)}$ and therefore $-\frac{1}{4} G_{\mu\nu}^{(c)} G_{\mu\nu}^{(c)}$ does remain invariant under a gauge transformation

$$\delta \left[-\frac{1}{4} G_{\mu\nu}^{(c)} G_{\mu\nu}^{(c)} \right] = -\frac{1}{2} \chi^{(d)} C_{(cde)} G_{\mu\nu}^{(c)} G_{\mu\nu}^{(e)} = 0,$$

as a result of the antisymmetry of $C_{(cde)}$. Hence, even for free fields not coupled to any other source — whose complete Lagrangian density is then $-\frac{1}{4} G_{\mu\nu}^{(c)} G_{\mu\nu}^{(c)}$ — Euler-Lagrange term $\frac{\partial\ell}{\partial A_\mu^{(c)}}$ becomes non-zero and equal to $+eC_{(cde)} A_\nu^{(d)} G_{\nu\mu}^{(e)}$ due to the non-linear term $+eC_{(cde)} A_\mu^{(d)} A_\nu^{(e)}$ within $G_{\mu\nu}^{(c)}$. For non-Abelian gauge fields, therefore, the actual source is given by $\frac{\partial\ell}{\partial A_\mu^{(c)}} \Big|_{\text{source}} = \frac{\partial\ell}{\partial A_\mu^{(c)}} - eC_{(cde)} A_\nu^{(d)} G_{\nu\mu}^{(e)}$.

of the sources for the gauge field defining the density and current of the able-to-glide dislocations, sources which as mentioned above corresponds to the resolved stress and the mass flow velocity

$$\operatorname{div} \boldsymbol{\tau}_{\text{resol}}^{\alpha} - \partial_t (-\rho v_{\text{slip}}^{\alpha}) = 0. \quad (2.66)$$

As mentioned in the discussion following (2.27), this is not a surprising result since invariance under a rigid translation leads to the conservation of linear momentum, which is the actual content of equation $\operatorname{div} \boldsymbol{\tau}_{\text{resol}}^{\alpha} = \partial_t (\rho v_{\text{slip}}^{\alpha})$.

Relation (2.56), $e = \frac{1}{\sqrt{Ga^2}} \frac{1}{\lambda_0}$, has been obtained for the elastic and the easy glide stage, analogues respectively to the Meissner and Shubnikov phases of a conventional superconductor, phases described in the latter model by a weak coupling $\frac{e}{\hbar c}$. These two stages are characterized by the fact that the dislocation spacing remains nearly constant. During the rapid hardening stage, behaviour which does not correspond to any phase of the conventional superconductor, the distance between dislocations deviates considerably from λ_0 , this an effect of the produced sessile dislocations. Therefore, the coupling constant between the scalar field and the gauge field — gauge field which now, by means of certain multivalueness, has to take account of the sessile dislocations, the latter working in the dislocation equations as magnetic monopole fields would do in the Maxwell equations — also changes. In general, for any plastic stage (and for the elastic stage), the coupling constant will be defined as

$$e = \frac{1}{\sqrt{Ga^2}} \frac{1}{\lambda}, \quad (2.67)$$

with λ the actual dislocation spacing for the plastic behaviour under study. Advancing results to be discussed later, if the elastic stage and the easy glide stage are described by phases occurring at weak coupling, the rapid hardening stage, on the contrary, will correspond to a phase characteristic for large coupling e .

Of course, in the most general formulation, the spacing λ would depend on the crystal direction, just in the same form as the elastic constant is really a tensor. This generalization would require a coupling e^{α} specific for every direction α , although in the following such a detail will be ignored in order not to make obscure the construction and main features of the model proposed in this work.

This whole section has been devoted to put flesh into the schematic analogy proposed in Table 2.4 on page 54 between the states displayed by the Abrikosov model — Meissner state, characterized by the perfect diamagnetism, Shubnikov state, where the still existing high rigidity against the spreading of $\vec{B}$ allows its penetration only confined in vortices, and Coulomb state, where the condensation of vortices leads finally to the destruction of the diamagnetism — and the elastic and the plastic stages where sessile dislocations do not determine the plastic behaviour. The parameters of the dislocation model, constructed on the base of this analogy, are given by the following “translation

rules”

Abrikosov model param.

Dislocation model param.

$$\begin{aligned}
\text{scalar field's length } \frac{\hbar^2 |\psi|^2}{m} &\longleftrightarrow |\psi|^2 = \frac{Ga^2}{4\pi^2} \quad \text{or} \quad \sim G \\
\text{coupling constant } \frac{e}{\hbar c} &\longleftrightarrow e = \frac{2\pi}{\sqrt{Ga^2}} \frac{1}{\lambda} \quad \text{or} \quad \sim \frac{1}{\lambda} \\
\text{correlation length } \xi &\longleftrightarrow \text{radius of the dislocation core,} \\
&\quad \text{on the order of lattice spacing } a
\end{aligned} \tag{2.68}$$

Now it is time to extend the conventional superconductor model by including magnetic monopoles. As we shall see, this extension will allow us to describe correctly the rapid hardening stage (or Stage II).

2.4.2 Step 2: putting the model on a discrete lattice and introducing some multivalueness in the gauge field A_μ^α

Although the divergence occurring in the energy in the range of large distances has been removed by means of the coupling to the gauge field (see (2.55)), it still remains another source of divergences: the range of very small distances. This problem has been already found in (1.7), as one had to introduce a cut-off r_0 (in the form of the core radius) in order to make finite the integral for the crystal containing one dislocation. Or more properly, in order to indicate the lowest limit up to which the continuum formulation of the elasticity theory is still valid.

This problem is the price to be paid for working with an effective continuum model which ignores, because not observable, the fields and the interactions between them which actually occur at a “hidden” fundamental scale, substituting these unknown fundamental fields by a limited number of effective degrees of freedom. The necessity of the mentioned cut-off is nothing else as the “footprint” of this hidden scale.

Any effective model, formulated to describe the behaviour of the studied system at a certain scale much larger than the unknown fundamental one, will depend on a limited set of effective parameters which in general will become different according to the range of distances for which the model is supposed to work: the parameters modelling the system will vary depending on how fine is the picture we are taking of the system (a finer picture shows interactions and fluctuations which were not observable at the coarser pictures). And equivalently, the coarser pictures are obtained from the finer ones by integrating (or summing) the details which cannot be distinguished in the final coarser picture: in this process, especially if the difference in magnification power between both pictures is large — and if the interactions taking place do not become enhanced by this change of scale —, many interaction types of the finer scale become irrelevant (non-observable) in the coarser picture. This latter feature enables us to construct an effective model at coarser scales containing a very restricted number of parameters, which (if one has not reached a too coarse scale) hopefully contains the main characteristics of the

model under study. An example is just the continuum dislocation theory, where the fundamental interatomic interactions are substituted at a much coarser scale (where such interactions cannot be distinguished) by simple effective parameters, the elastic constants, multiplying gradients of a scalar field. In the Abrikosov formulation of the superconductivity, also an effective model derived from a more fundamental quantum mechanical model called the BCS theory, the fundamental quantum scale for the actual interaction leaves its footprint in the scalar field's correlation length ξ , introduced in the Abrikosov model “by hand” through effective parameters in the potential's part of the energy.

In order to be able to carry out calculations with the model, especially when fluctuations in the scalar as well as in the gauge field are considered to determine the existence of a phase transition and its character, one need have a recipe to deal with the divergences at very short distances plaguing all such calculations. The first step is to introduce a distance cut-off $1/\Lambda_c$ (with Λ_c large), which delimits the lowest distance up to which the formulated model is still valid. As next step, one reduces the distance cut-off and arranges the already existing model's effective parameters such that the model remains finite and able to describe the new finer scale — parameters which, as expected, become thus dependent on the scale we can still distinguish —, in the hope that the thus arranged model will still remain finite even in the limit of sending the distance cut-off towards zero. In general, this limit may well fail since by going to shorter scales, new interactions appear which cannot be described by the effective terms already contained in the initially proposed model — consequence of having formulated an effective model at a scale too coarse to observe all the main features of the system. However, if we were clever enough modelling the model's energy, the description of the finer pictures can still be achieved by the *same* effective model used for the coarser pictures, modifying only the model's effective parameters: a model which fulfils this property is called renormalizable. As a result of this latter “clever” case, a model formulated by a rather coarse picture of the system still works by going to finer, shorter distances, i.e., by letting the distance cut-off go to zero. But even in such a clever case, the hidden, non-reachable fundamental scale has already left its footprint at least through the necessity of introducing an arbitrary scale at which the value of every of the scale-dependent effective parameters has to be taken as reference. This is the so-called renormalization group approach; an introduction into this subject, giving the criterium for choosing the “clever” (renormalized) models, is described in the Note of Appendix B on page 291.

One example of this strategy is to put the proposed model on a discrete lattice: the lattice spacing itself provides thus the short distance cut-off, $a \sim 1/\Lambda_c$, which is of the order of the fundamental scale. The continuum model, where the lattice discreteness is washed out, will be obtained by looking for a phase transition point of critical kind for the lattice model, working then at such a point [88]¹³. For the following (as well as in

¹³Let us suppose we are studying a certain state of the system. Now we watch it at a distance large enough such that we can only distinguish distances equal to k -times the initial lattice spacing a ; the same process can be achieved if instead of observing the system under a powerful magnifying-glass we watch it under a k -times lesser magnification power. This procedure is equivalent to the following steps: first, to sum over all possible configurations occuring at scales between the lattice spacing a and a distance ka ; and second, to re-arrange the new lattice spacing a' — in the resulting model,

Appendix C and D) we shall use this procedure. To work with the discrete formulation allows us to take over all the tools of Statistical Physics developed for spin systems, making more comfortable the calculations. Further, the introduction of multivalueness in the gauge field in order to include the effects of monopoles is especially easy in this formulation. But in the framework of our discussion, the continuum limit for the lattice model will not be considered. In our case, the reason for keeping the lattice discrete without looking for a continuous model is the very existence of the crystal lattice, which actually defines a fundamental graininess: here the lattice is not an intermediate step to introduce a distance cut-off in the way towards a continuum model but a real feature. Of course, one can nevertheless ask for a continuum model which reproduces the characteristics of the lattice model: this will be dealt with in the part of Appendix D devoted to the effective potential. But in such a case, the renormalization approach will be required to define a sensible continuum model for long distances which still works at short distances.

In a d -dimensional discrete lattice of lattice spacing a (which as already noted, need not coincide exactly with the lattice spacing of the real crystal) one defines:

- The vertices or *sites* of this lattice, denoted by a vector $\vec{r}$ formed by d integers. Any scalar field, like ψ^α or its phase φ^α , reside at these sites. The scalar field of nonzero length (or order parameter) coupled in our model to the gauge field is

neighbour degrees of freedom are separated a distance of k -times the initial lattice spacing — to have the initial length a ; it can also be necessary to re-arrange the field degrees of freedom themselves in order to maintain their self-strength without disappearing during the defocussing. It is to be noted that the state is just the same, only the picture is fine or coarse focussed. Let us assume that the interaction types in the initially modellized energy suffice also to describe completely the interactions in the resulting picture's model, whose parameters will have changed in general their strength: under this assumption, and repeating the defocussing process, we shall obtain a series of pictures of the same state described by the same energy form but described by the evolution of the effective parameters until reaching a macroscopical scale where only long distance effects occur. And the latter is the desired result: to formulate a model based on large distance effects (which we can observe) but which also works at much smaller scales without being necessary to modify the form of the interaction terms of the model. It is important to stress that all this process is completely equivalent to, instead of moving ourselves away from the system (such that we cannot distinguish so many details as before), getting the lattice spacing finer and finer, i.e., is equivalent to going towards the continuum.

The above introduced assumption is a rather strong one since, in general, by moving away new interaction types may have appeared or, on the contrary, relevant interaction at the finer scale are no longer distinguishable at the coarser scale. The form to ensure this behaviour is that the effective model's parameters are adjusted so that the model displays a phase transition of critical type: the defining characteristic of such a phase transition is the scale invariance, i.e., the system looks very similarly in the very long distance range as well as a very fine picture. In other words, the long distance behaviour is not sensitive to many of the short-distance details (especially to how the distance cut-off was introduced) and as a consequence, there exist many effective models, differing in the short-distance details, which display the same behaviour at long distances. Hence the apparent relevance about the new interaction terms, which could appear in the intermediate steps towards the very coarse (long-distance) picture, is completely washed out if the model's parameters correspond to a critical phase transition point. And as a result, a meaningful formulation is obtained in the limit $a \rightarrow 0$ — or equivalently, in the very long distance, very coarse picture — at such a critical point for the lattice formulation.

written as

$$\psi_{\vec{r}}^\alpha = |\psi| e^{i\varphi_{\vec{r}}^\alpha}. \quad (2.69)$$

- The link or *bond* $(\vec{r}, \mu)$ is the segment connecting site $\vec{r}$ with site $\vec{r} + \hat{\mu}a$, where $\hat{\mu}$ denotes an unit vector along the μ direction ($\mu = 1, 2, \dots, d$). The gauge vector field A_μ^α (which defines the density of dislocation which could glide) is placed on these bonds, $A_{\mu, \vec{r}}^\alpha$, and since $(\vec{r}, \mu)$ and $(\vec{r} + \hat{\mu}a, -\mu)$ denotes the same bond, the vector field fulfils the following convention

$$A_{-\mu, \vec{r} + \hat{\mu}a}^\alpha = -A_{\mu, \vec{r}}^\alpha. \quad (2.70)$$

- The *plaquette* $(\vec{r}, \mu, \nu)$ is the unit closed quadrat formed by bonds $(\vec{r}, \mu)$, $(\vec{r} + \hat{\mu}a, \nu)$, $(\vec{r} + \hat{\mu}a + \hat{\nu}a, -\mu)$ and $(\vec{r} + \hat{\nu}a, -\nu)$. Curl field $F_{\mu\nu, \vec{r}}^\alpha$ is located at these plaquettes and defined as the sum of gauge field $A_{\mu, \vec{r}}^\alpha$ along one unit plaquette beginning and ending at site $\vec{r}$: $F_{\mu\nu, \vec{r}}^\alpha$ is the discrete formulation of the density of able-to-glide dislocations (or in the Maxwell formulation, of the magnetic field) crossing the

plaquette:

$$F_{\mu\nu, \vec{r}}^\alpha = A_{\mu, \vec{r}}^\alpha + A_{\nu, \vec{r} + \hat{\mu}a}^\alpha + \underline{A_{-\mu, \vec{r} + \hat{\mu}a + \hat{\nu}a}^\alpha} + \underline{A_{-\nu, \vec{r} + \hat{\nu}a}^\alpha} =$$

$$= \left(A_{\nu, \vec{r} + \hat{\mu}a}^\alpha - \underline{A_{\nu, \vec{r}}^\alpha} \right) - \left(\underline{A_{\mu, \vec{r} + \hat{\nu}a}^\alpha} - A_{\mu, \vec{r}}^\alpha \right) \quad (2.71)$$

In order to simplify a little the notation, let us introduce the discrete partial derivative Δ_μ

$$\Delta_\mu f_{\vec{r}} \stackrel{\text{def}}{=} f_{\vec{r} + \hat{\mu}a} - f_{\vec{r}}. \quad (2.72)$$

Hence equation (2.71) takes the more manageable form of

$$F_{\mu\nu, \vec{r}}^\alpha = \Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha, \quad (2.73)$$

which is the discrete form of $F_{\mu\nu}^\alpha(\vec{r}) = \partial_\mu A_\nu^\alpha(\vec{r}) - \partial_\nu A_\mu^\alpha(\vec{r})$ in the continuum formulation. The usual (continuum) derivative ∂_μ is related to the discrete one through $\Delta_\mu f_{\vec{r}} = a \partial_\mu f(\vec{r}) + \mathcal{O}(a^2)$, or equivalently $\partial_\mu = \lim_{a \rightarrow 0} a^{-1} \Delta_\mu$.

Operating with this discrete partial derivative is very similar as with the usual continuum one. For instance, an equation like $\sum_{\mu=1}^3 \Delta_\mu b_{\mu, \vec{r}} = 0$ (the discrete formulation of the vanishing divergence of a vector field) has the solution

$$b_{\mu, \vec{r}} = \sum_{\nu, \lambda=1}^3 \epsilon_{\mu\nu\lambda} \Delta_\nu a_{\lambda, \vec{r}} = \sum_{\nu, \lambda=1}^3 \epsilon_{\mu\nu\lambda} \left(a_{\lambda, \vec{r} + \hat{\nu}a} - a_{\lambda, \vec{r}} \right),$$

as one can easily prove

$$\begin{aligned}
\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r}} &= \sum_{\mu} \left(b_{\mu, \vec{r} + \hat{\mu}} - b_{\mu, \vec{r}} \right) = \sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} \left(a_{\lambda, \vec{r} + \hat{\mu} + \hat{\nu}} - a_{\lambda, \vec{r} + \hat{\mu}} - a_{\lambda, \vec{r} + \hat{\nu}} + a_{\lambda, \vec{r}} \right) = \\
&= \underbrace{\left[\sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} a_{\lambda, \vec{r} + \hat{\mu} + \hat{\nu}} \right]}_0 - \left[\sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} a_{\lambda, \vec{r} + \hat{\mu}} + \underbrace{\sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} a_{\lambda, \vec{r} + \hat{\nu}}}_{-\sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} a_{\lambda, \vec{r} + \hat{\mu}}} \right] + \underbrace{\left[\sum_{\mu, \nu, \lambda} \epsilon_{\mu \nu \lambda} a_{\lambda, \vec{r}} \right]}_0.
\end{aligned}$$

Another example is the partial integration in the discrete formulation: assuming that the fields vanish at the infinitely distant boundary of the system, the following relation holds

$$\sum_{\vec{r}} f_{\vec{r}} \Delta_{\mu} g_{\vec{r}} = \sum_{\vec{r}} f_{\vec{r}} g_{\vec{r} + \hat{\mu}} - \sum_{\vec{r}} f_{\vec{r}} g_{\vec{r}} = \sum_{\vec{r}} f_{\vec{r} - \hat{\mu}} g_{\vec{r}} - \sum_{\vec{r}} f_{\vec{r}} g_{\vec{r}} = - \sum_{\vec{r}} \left(\Delta_{\mu} f_{\vec{r} - \hat{\mu}} \right) g_{\vec{r}},$$

which corresponds to $\int_{-\infty}^{\infty} d^d \vec{r} f(\vec{r}) \partial_{\mu} g(\vec{r}) = - \int_{-\infty}^{\infty} d^d \vec{r} \left(\partial_{\mu} f(\vec{r}) \right) g(\vec{r})$ in the continuum formulation.

The $U(1)$ gauge transformation (2.52) in the discrete formulation is written as follows

$$\begin{cases} \varphi_{\vec{r}}^{\alpha} \rightarrow \varphi'_{\vec{r}}{}^{\alpha} = \varphi_{\vec{r}}^{\alpha} - \chi_{\vec{r}}^{\alpha}, \\ A_{\mu, \vec{r}}^{\alpha} \rightarrow A'_{\mu, \vec{r}}{}^{\alpha} = A_{\mu, \vec{r}}^{\alpha} - \frac{1}{ae} \left(\chi_{\vec{r} + \hat{\mu}}^{\alpha} - \chi_{\vec{r}}^{\alpha} \right) = A_{\mu, \vec{r}}^{\alpha} - \frac{1}{ae} \Delta_{\mu} \chi_{\vec{r}}^{\alpha}, \end{cases} \quad (2.74)$$

with $\chi_{\vec{r}}^{\alpha}$ an arbitrary local function. In order to avoid the ae which occurs in the above transformation, in the lattice formulation the gauge field is usually defined including this factor

$$A_{\mu, \vec{r}}^{\alpha} \equiv ae A_{\mu, \vec{r}}^{\alpha}. \quad (2.75)$$

For the following, a will be taken as 1; it will be only written down as indication to keep the correct dimensions.

With respect to energy's term containing the covariant derivative which couples the scalar and the gauge field, the naive translation of the continuum formulation would yield

$$\begin{aligned}
\frac{|\psi|^2}{2} \int d^d \vec{r} \sum_{\alpha, \mu} \left(\partial_{\mu} \varphi^{\alpha}(\vec{r}) - e A_{\mu}^{\alpha}(\vec{r}) \right)^2 &\rightarrow \frac{a^d |\psi|^2}{2} \sum_{\vec{r}} \sum_{\alpha, \mu} \left(\frac{\varphi_{\vec{r} + \hat{\mu}}^{\alpha} - \varphi_{\vec{r}}^{\alpha}}{a} - \frac{ae}{a} A_{\mu, \vec{r}}^{\alpha} \right)^2 = \\
&\stackrel{(2.75)}{=} \frac{a^{d-2} |\psi|^2}{2} \sum_{\vec{r}} \sum_{\alpha, \mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - A_{\mu, \vec{r}}^{\alpha} \right)^2.
\end{aligned}$$

but, nevertheless, this energy term cannot reproduce the main feature of the Abrikosov model: the existence of isolated vortices as topological stable objects; or in the dislocation model, of isolated stable gliding (and would-become gliding) dislocations.

Let us see this last statement in more detail by considering only the energy's part depending exclusively on the scalar field's phase. In the continuum formulation this term is $\frac{|\psi|^2}{2} \int d^d \vec{r} \sum_{\alpha, \mu} (\partial_\mu \varphi^\alpha)^2$ and the configurations which minimize this energy part are therefore solutions of $\sum_\mu \partial_\mu \partial_\mu \varphi^\alpha = 0$. Beside the trivial solution $\varphi^\alpha = \text{const}$, corresponding to the absolute minima of the energy, there are other smooth configurations of higher energy but which are *stable* due to the topology of the gauge group which leaves invariant the whole energy — in our case, such a symmetry is defined by the $U(1)$ group of local planar internal rotations of the scalar field, with the stability meaning the impossibility to deform the configuration (without cutting the crystal) into the trivial configuration $\varphi^\alpha = \text{const}$ —; examples of these stable arrangements, the dislocations, have been already seen on page 62, when the orientation of the scalar field in its internal 2-dim. plane corresponds to the azimuthal angle in physical space. Or in other words, when the scalar field draws a complete internal rotation when also a circumference in external space is covered. Such solutions are thus due to the angular character of the scalar field's phase, something which is mathematically written by stating that φ^α is solution of

$$\sum_\mu \partial_\mu \partial_\mu \varphi^\alpha = 0 \pmod{2\pi}. \quad (2.76)$$

The above “mod 2π ” just takes account of the jumps of value $2\pi k$ (k an integer) suffered by φ^α when in external physical one ends at the start point of a closed path. It should be noted once more that the configurations of φ^α just discussed are smooth. Hence, to translate this property into a discrete lattice requires a further implementation than the simply $\frac{a^{d-2}|\psi|^2}{2} \sum_{\vec{r}} \sum_{\mu, \alpha} (\Delta_\mu \varphi_{\vec{r}}^\alpha)^2$. Such an extension is provided by

$$a^{d-2}|\psi|^2 \sum_{\vec{r}} \sum_{\mu, \alpha} \left[1 - \cos(\Delta_\mu \varphi_{\vec{r}}^\alpha) \right], \quad (2.77)$$

which, beside yielding the correct continuum formulation in the limit $a \rightarrow 0$ (using $1 - \cos ax \stackrel{a \rightarrow 0}{\approx} a^2 x^2 / 2$)

$$\frac{1}{a^2} \left[1 - \cos(\underbrace{\Delta_\mu \varphi_{\vec{r}}^\alpha}_{\stackrel{a \rightarrow 0}{\approx} a \partial_\mu \varphi^\alpha(\vec{r})}) \right] \stackrel{a \rightarrow 0}{\approx} \frac{1}{2} \left(\partial_\mu \varphi^\alpha(\vec{r}) \right)^2,$$

contains the required periodicity to allow the “jumps” in φ^α characteristic of the topological objects (named dislocations or vortices). Further, any statistical calculation carried out by means of discrete formulation (2.77) leads to the results expected for the continuum formulation of the model with a scalar field's phase containing multivaluenesses just at physical sites around a topological stable object. The proof of this property can be found in the Note of Appendix B on page 344.

Summarizing, the energy's term we are looking for in the lattice formulation for the covariant derivative is given by

$$a^{d-2}|\psi|^2 \sum_{\vec{r}} \sum_{\mu, \alpha} \left[1 - \cos(\Delta_\mu \varphi_{\vec{r}}^\alpha - A_{\mu, \vec{r}}^\alpha) \right], \quad (2.78)$$

term which, due to the constraint of finiteness in the energy at large distances around any dislocation — or equivalently, of a non-diverging energy for a configuration in which the scalar field's internal orientation is proportional to the external azimuthal angle —, yields the topological quantization of the Burgers vector in the discrete formulation

$$\sum_{\vec{r} \in \text{distant closed path in phys. space}} \Delta_\mu \varphi_{\vec{r}}^\alpha = 2\pi \implies \sum_{\vec{r} \in \text{distant closed path around disloc.}} A_{\mu, \vec{r}}^\alpha = 2\pi, \quad (2.79)$$

in the same way as (2.54) for the continuum formulation (remembering that coupling constant e is usually included into the gauge field in the lattice formulation).

Now let us continue with the lattice formulation of the energy part containing the curl of the gauge field, also called the pure gauge field part: for the (Abelian) $U(1)$ case under study now, this part is equal to $\frac{1}{4} \int d^d \vec{r} \sum_{\alpha, \mu, \nu} F_{\mu\nu}^\alpha(\vec{r}) F_{\mu\nu}^\alpha(\vec{r})$. The Abelian $U(1)$ gauge field describes the conventional electromagnetism (or its analogue, the dislocation model for single slip), where there are no magnetic monopoles (or the fields corresponding to the sessile dislocations in the dislocation model). In other words, the integral over a closed surface of the magnetic field is zero. For this case, the lattice corresponding energy term is just the naive discrete version

$$\begin{aligned} \frac{1}{2} \int d^d \vec{r} \sum_{\alpha} \sum_{\mu < \nu} \left(F_{\mu\nu}^\alpha(\vec{r}) \right)^2 &\rightarrow \frac{a^d}{2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left(\frac{A_{\nu, \vec{r} + \vec{\mu}}^\alpha - A_{\nu, \vec{r}}^\alpha}{a} - \frac{ae}{ae} \frac{\Delta_\nu A_{\mu, \vec{r}}^\alpha}{a} \right)^2 = \\ &\stackrel{(2.75)}{=} \frac{a^{d-4}}{2e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha \right)^2, \end{aligned} \quad (2.80)$$

However, although the conventional electromagnetism was useful to set the analogy with the dislocation model when only the primary slip system was active (see Table 2.2 on page 47), we still need the extension of such model for the case of several non-parallel activated slip system with the resulting sessile dislocations (extension expressed in Table 2.3 on page 52). In this last case, $\mathbf{B}_i^\alpha$ contains first the dislocations allowed by the crystal to glide throughout the primary plane, as well as the projection onto such plane of the dislocation density which is gliding in the secondary systems. Hence, one expects that the Burgers vector corresponding to a certain configuration of $\mathbf{B}_i^\alpha$ will become reducible to a smaller amount of gliding Burgers vector simply because some primary gliding dislocations have react with some secondary ones — the latter also contributing to $\mathbf{B}_i^\alpha$ as long as they are able to glide — to form sessile dislocations. This unstability of the flux carried by the total gliding dislocations, flux which corresponds by definition to their Burgers vector, is something which did not occur in the case where only one slip system was active. And this unstability is equivalent to the appearance of sessile dislocations, fields playing in the formulated dislocation equations *à la* Maxwell a role similar to that of the magnetic monopoles in the electromagnetism.

In order to understand how magnetic monopoles arise, let us return for a while to the leading model throughout this work: the superconductor. The conventional superconductor model, that which opposes the spreading in its interior of the conventional magnetic field, has no magnetic monopoles and as a result vortices carrying any integer

number of the fundamental unit of flux Φ_0 can exist. This last statement arises from a general relation regarding the topology of the (Abelian) $U(1)$ gauge group (more exactly, its first homotopy group), result proved in Appendix B. But one can see it in a more mechanical way: let us suppose that, from the combination of two vortices, the resulting flux (some multiple of Φ_0) would be unstable in this model. Then, due to the conservation of magnetic flux, a monopole should occur at the point where the two vortices have react, absorbing the flux of these two vortices. Since there exist no monopole in usual superconductor, any multiple of Φ_0 is stable. Another form to understand this result, and now returning to the dislocation model, is that in the framework of the Abelian $U(1)$ gauge symmetry no gauge transformation (read no compatible deformation in the dislocation model) is able to change the number of flux units (read Burgers vector for dislocations) carried by a certain field configuration.

There is nevertheless a whole family of superconductor models where monopoles arise in a natural way. Characteristic of all these models is the fact that the $U(1)$ gauge symmetry is not the real symmetry displayed by the system but only a residual one: the $U(1)$ is a subgroup of a larger non-Abelian compact gauge group, the latter being the actual underlying symmetry of the system. The larger group means that, for the scalar field coupled to the gauge field, the dimensions in the internal space where this scalar field rotates are higher than the two dimensions for the $U(1)$ planar rotations: the non-Abelian character of the gauge transformations is just the result of the non-commutativity of rotations in space of dimensions higher than 2; see for instance the Note on page 64. Since the non-Abelian compact group is the actual symmetry of the system, certain gauge transformations (or equivalently, certain internal rotations) can relate two vortex configurations of different flux content, deforming one into the other, without changing the energy of the system which, by definition of the now non-Abelian compact gauge symmetry, remains invariant under such gauge transformations. And thus a magnetic monopole appears in order to absorb the difference in magnetic flux. One of the simplest models displaying this behaviour is the “ $SO(3)$ ”-superconductor [9], where the only topologically stable vortices ¹⁴ are those with one unit of Φ_0 : any vortex

¹⁴ Although this kind of superconductor model is studied in Appendix B’s section on page 233, let us repeat here the origin of the topological instability of configurations that were stable as the model’s symmetry was $U(1)$ alone. In fact, the discussion about the stability of line defects such as vortices or dislocation lines (once the direction of the Burgers vector is specified) runs as follows: at a point crossed by the line defect and on the plane normal to the latter, one covers a closed path C surrounding that point at a distance far from it. Since at long distances of the defect the scalar field’s length has to tend towards the constant value which minimizes the energy’s part $\frac{1}{2} \sum_{\mu} (\partial_{\mu} |\psi|)^2 - |\alpha| |\psi|^2 + \frac{|\beta|}{2} |\psi|^4$,

the value taken on by the scalar field (actually, only the orientation since the length is already fixed) at any point of path C is actually a mapping from every point on the circumference C , placed in the physical external space, onto elements of model’s symmetry group. A deformation or local modification of this mapping corresponds to a gauge transformation belonging to the symmetry group. Further, the mapping characteristic of a field configuration without any line defect is just that given by the correspondence which sends the whole C onto only one element of the symmetry group; or more accurately, given by any configuration obtained from the latter by any gauge transformation (read deformation) belonging to the symmetry group.

In the case of being $U(1)$ such a symmetry, any element of this group is completely determined by only one parameter $\chi \in [0, 2\pi)$, $U(1) = \{e^{-i\chi} : \chi \in [0, 2\pi)\}$, and one says that $U(1)$ has the topology of

of even number in Φ_0 can be deformed to a configuration of zero magnetic flux, and the same occurs to any vortex of odd number in Φ_0 and configurations carrying only one unit of Φ_0 .

The consequence of this feature is that, when we focus our attention only on the gauge subgroup $U(1)$ deriving from this model, its corresponding gauge field shows “jumps”, shows angular character: just as a single-valued scalar field’s phase $\varphi(\vec{r})$

a circumference S^1 — the circumference on which χ takes on its value —; the mapping is thus between the external azimuthal angle ϕ defining C , and the χ characterizing S^1 , or equivalently, between circumference C and circumference S^1 . For the mapping of one circumference onto another one there are numerable many different forms which are non-deformable into each other; in particular, these mappings cannot be deformed to the trivial mapping where every point in C is taken into the same χ . The pictorial way of visualize these non-deformable mappings are by considering the different forms of wrapping a rubber band (representing C) around a ring (representing the topology of $U(1)$): see Fig. B.1 on page 191. A trivial map, characterizing thus a configuration without stable topological content, corresponds to a partial wrapping which can be shrunk to a point without leaving the circumference S^1 (or without cutting any of the two circumferences). Summarizing, if the system’s symmetry is described by the Abelian $U(1)$ gauge group, the numerable different line defects, which exist and are topologically stable, are classified by the number of times C is wrapped around circumference S^1 : this number of times is called the winding number and just denotes the units of flux Φ_0 , or of the elemental Burgers vector in the dislocation model, carried by the line defect around which the closed path C has been covered.

A case completely different occurs if the system’s symmetry is described by a compact (non-Abelian) group which contains many more transformations than $U(1)$, as for instance the group $SO(3)$ of rotations in a 3-dim. space. This last group is determined by four parameters: three real numbers to fix the rotation axis — parameters which were not necessary for the $U(1)$ rotations, which occur on only one fixed plane — and given by the unit vector $\hat{n}_{a=1,2,3}$, and then the rotation angle χ . In the same manner as for $U(1)$ (whose topology has been shown to correspond to a circumference S^1) let us calculate the geometric figure topologically equivalent to group $SO(3)$. As a previous step, let us obtain such a result for a group closely related to $SO(3)$, the group $SU(2)$ of the 2×2 unitary matrices with determinant unity: $e^{-i/2 \chi \hat{n}_a \sigma_a} = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i \hat{n}_a \sigma_a \sin\left(\frac{\chi}{2}\right) = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}$, where $\sigma_{a=1,2,3}$ are the Pauli matrices, $\chi \in [0, 4\pi)$, and (a_0, a_1, a_2, a_3) are real numbers given by $\begin{cases} a_0 = \cos(\chi/2) \\ a_k = -\hat{n}_k \sin(\chi/2) \end{cases}$.

The topology of $SU(2)$ is immediately calculated: since $a_0^2 + \sum_{k=1}^3 a_k^2 = 1$, $SU(2)$ has the same topology as a 3-dimensional spherical surface (living in a 4-dim. space), which is denoted by S^3 . The correspondence between group $SU(2)$ and $SO(3)$ is also easily determined: both parameters (a_0, a_k) as well as $(-a_0, -a_k)$ correspond to the same $SO(3)$ matrix since χ and $\chi + 2\pi$ yields the same rotation matrix; therefore the correspondence between both groups is 2-to-1 — two different elements of $SU(2)$, i.e., two different points on S^3 corresponds to only one element of $SO(3)$ — and as a result, the topology of $SO(3)$ is the 3-dimensional spherical surface *but* with diametrical points identified, surface being denoted by $\mathcal{S}^3$. Now let us return to the mapping from the physical circumference C , around the defect whose stability we are studying, onto the symmetry space $SO(3)$ of the proposed model and whose topology is $\mathcal{S}^3$: the mapping can again be graphically represented as the action of placing the rubber band but this time on the spherical surface $\mathcal{S}^3$. As a result, many of the mappings (actually half of them) are deformable to a point simply by sliding the rubber band over the surface: see Fig. B.2 on page 192. Only those mappings for which the rubber band ends at the $\mathcal{S}^3$ ’s point antipodal to the start point — and since both points are identified, the rubber band remains uncut — cannot be shrunk to a point [87]: see Fig. B.6 on page 228. This is the reason for the fact that there is only one kind of topological stable line defects: it is that determined by the latter discussed type of mappings.

would fulfil $\oint_{\text{closed path}} dl^\mu \partial_\mu \varphi(\vec{r}) = 0$ and the non-fulfilment of such a relation is the indication of the angular character of $\varphi(\vec{r})$ and the corresponding “jumps” by 2π after covering a closed path (as for instance when this closed path surrounds in physical space one vortex), in the same way a single valued gauge field gets determined by $\oint_{\text{closed surface}} dS^{\mu\nu} \partial_\mu A_\nu(\vec{r}) = 0$ whereas the multivalueness characteristic of an angular character yields $\oint_{\text{closed surface}} dS^{\mu\nu} \partial_\mu A_\nu = \frac{1}{2} \oint_{\text{closed surface}} dS^{\mu\nu} F_{\mu\nu} \neq 0$, equation corresponding to a monopole.

Back to the dislocation model, and with the experience gained in the discussion from equation (2.76) — which, applied to the scalar field’s phase, states the angular character in the continuum formulation — to result (2.77), where the lattice formulation was given, the pure gauge field part for a $U(1)$ model with objects similar to the magnetic monopoles takes the following form

$$\frac{a^{d-4}}{e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left[1 - \cos(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha) \right], \quad (2.81)$$

extension of (2.80) for the case of a $U(1)$ gauge field which can display certain “jumps” due to its angular nature. This latter type of gauge symmetry will be called from here on the compact $U(1)$ gauge symmetry. On the other hand, the $U(1)$ gauge field described by (2.80) will be denoted as the non-compact $U(1)$ gauge symmetry

$$\text{Energy} \Big|_{\text{pure g. field}} = \begin{cases} \frac{1}{a^{4-d} e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left[1 - \cos(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha) \right] & \text{compact } U(1), \\ \frac{1}{2a^{4-d} e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} (\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha)^2 & \text{non-compact } U(1). \end{cases} \quad (2.82)$$

The use of one or the other $U(1)$ symmetry will be determined by the type of crystal we are studying and whether in such a crystal slip systems other than the primary one can be activated with the accompanying sessile dislocations in the positive case. A compact $U(1)$ gauge symmetry, when applied to the dislocation model based on the Abrikosov model, will contain the effect of the fields which are characterizing the non-vanishing divergence for the density of all dislocations which are either mobile by glide or which were otherwise able to glide. It will result, when we shall study the phase diagram of such model, that the new phase introduced by this angular character of the gauge field displays the behaviour one should expect from the rapid hardening stage.

The reader may now think that we are moving in a circle: in equation (2.19) we have separated “by hand” the sessile dislocations, with $\varrho_{\text{mg}}^\alpha$ carrying their degrees of freedom, and now we are claiming that this separation reproduces the plastic behaviour of such a kind of dislocations. However this conclusion is not correct: the in principle formal separation in able-to-glide and sessile dislocations is not responsible — since it only enters in the definition of $F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha$ and in the fact whether the gauge field can be multivalued at some points or not — for the rapid hardening behaviour. It is the other

field equation determining the spatial distribution of able-to-glide dislocations, that $\epsilon_{ijk}\partial_j\mathbf{B}_k^\alpha = \mu(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$ analogue to the Ampère law, which describes (or should describe if the formulated model is well constructed) all different plastic stages. In other words, it must be the *complete* analogy with the superconductor, and not the formal separation into glissile and sessile dislocations, which should modellize the crystal containing dislocations and which should eventually justify the correctness of the whole proposed theory.

We have arrived finally at the energy density which, by following the analogy with the superconductor and then by extending it to include some multivalueness in the gauge field for the case of glide activation in several non-parallel slip systems, will define our dislocation model. Only a small connecting step remains. The covariant derivative term, coupling the scalar to the gauge field, written in the lattice formulation (2.78), $a^{d-2}|\psi|^2 \sum_{\vec{r}} \sum_{\mu,\alpha} \left[1 - \cos\left(\Delta_\mu\varphi_{\vec{r}}^\alpha - A_{\mu,\vec{r}}^\alpha\right)\right]$, was obtained for the non-compact $U(1)$ gauge field. In the compact case, there is further slight modification to be included, that given in the following equation

$$a^{d-2}|\psi|^2 \sum_{\vec{r},\alpha} \sum_{\mu} \left[1 - \cos\left(\Delta_\mu\varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha\right)\right], \quad (2.83)$$

which changes too the form of the gauge transformation, $\begin{cases} \varphi_{\vec{r}}^\alpha \rightarrow \varphi_{\vec{r}}^{\prime\alpha} = \varphi_{\vec{r}}^\alpha - q^\alpha \chi_{\vec{r}}^\alpha \\ A_{\mu,\vec{r}}^\alpha \rightarrow A_{\mu,\vec{r}}^{\prime\alpha} = A_{\mu,\vec{r}}^\alpha - \Delta_\mu \chi_{\vec{r}}^\alpha \end{cases}$, under which the term remains invariant. In order to explain the physical meaning of the just introduced constant factor q^α , let us return again to the superconductor (where index α does not appear) but this time with a compact $U(1)$ gauge field, i.e., for a $U(1)$ gauge symmetry which is only a residual one, being the actual symmetry that described by a compact non-Abelian group. In a more mathematical sense, factor q labels the representation of the compact non-Abelian group chosen for the scalar field — being this latter scalar field other than the scalar field whose phase is coupled to the $U(1)$ gauge field A_μ^α — in order to break partially such a compact non-Abelian symmetry down to the $U(1)$ gauge symmetry. In a more practical sense, factor q denotes the number of units of magnetic flux which are absorbed by a magnetic monopole: see equation (B.242) on page 289 and the comments there.

As an example, let us consider the easiest of all non-Abelian Lie algebras, that defined by the structure constants ϵ_{abc} . Such a Lie algebra is shared by the two compact groups $SU(2)$ and $SO(3)$. Factor $q = 1$ means that a scalar field in the fundamental representation (the representation of lowest rank for the $SU(2)$ group) has been used to break $SU(2)$ down to $U(1)$; on the other hand, factor $q = 2$ indicates that the obtention of $U(1)$ as the residual symmetry has been achieved by a scalar field in the adjoint representation (the representation of lowest rank for the $SO(3)$ group) which broke this $SO(3)$ symmetry down to $U(1)$.

But as we have seen in the long footnote on page 76, the superconductor with an underlying $SO(3)$ symmetry has only one quantity of stable vorticity — and the superconductor with $SU(2)$ as actual symmetry has no stable vortices at all because of an argument analogous to that discussed in that footnote: the topology of $SU(2)$ is a 3-dim. spherical surface without any identification of its points (as it was the

case for $SO(3)$) and thus *any* map from the physical circumference C surrounding the vortex, onto $SU(2)$ can be shrunk to a constant trivial configuration (just as it is not possible to arrange a stable configuration of a rubber band on a spherical surface) —; and therefore, the vorticity resulting from the sum of *two* vortices in the “ $SO(3)$ ”-superconductor ($q = 2$) is unstable and thus gets absorbed by one magnetic monopole of charge equal to *two* units of vortex’ flux. And also in the similar way, the sole vorticity of one vortex in the case of a “ $SU(2)$ ”-superconductor ($q = 1$) is already topologically unstable, being absorbed by the corresponding monopole (endowed thus by only *one* unit of vortex’ flux). Higher value of q corresponds to partial breaking of $SU(2)$ or $SO(3)$ down to $U(1)$ carried out by a scalar field in the group representation of the corresponding rank; all these questions are considered in the Appendix B’s section on page 251.

The main function of factor q is therefore to denote if the compact $U(1)$ model, resulting from the discussed partial breakdown of compact gauge symmetry, has ($q \neq 1$) or not ($q = 1$) topologically stable vortices. And in the cases where a certain amount of vorticity results to be unstable, factor q determines how large is the magnetic charge which absorbs such vortices. Translating these results to our dislocation model, factor q^α indicates which kind of Burgers vector (as analogue to the vorticity in the superconductor) will correspond to glissile stable dislocations — i.e., to dislocations which can maintain their mobile character —, and which amount of Burgers vector characterizes dislocations which will react to produce sessile dislocations (as analogue to the magnetic monopoles). Depending on the value of q^α — or equivalently on how the $U(1)$ gauge symmetry, corresponding to the invariance under compatible deformations, has been extracted from the non-Abelian compact underlying symmetry — one will obtain different types for the smallest able-to-glide Burgers vectors, characteristic for the different types of crystal arrangement (f.c.c. or b.c.c.). These types are discussed in the section on page 251.

As it should be, factor q^α will result to play no role in the model with a non-compact $U(1)$ gauge symmetry. This case, which corresponds to the a $U(1)$ group as actual underlying symmetry, describes a dislocation system where only a sole slip system (the primary one) is active. And hence sessile dislocations do not appear and “jumps” in the gauge field cannot occur.

Summarizing, the dislocation model for a crystal in which different non-parallel slip systems can be activated (as the f.c.c. and b.c.c. crystals) will be modellized by the following lattice energy

$$\begin{aligned} \text{Energy}|_{\text{compact Higgs model}} &= \frac{1}{a^{4-d}e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left[1 - \cos(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha) \right] + \\ &+ a^{d-2} |\psi|^2 \sum_{\vec{r}, \alpha} \sum_{\mu} \left[1 - \cos(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu, \vec{r}}^\alpha) \right], \end{aligned} \quad (2.84)$$

which, borrowing the nomenclature of Particle Physics, will be also called the compact Higgs model. And for a crystal which is deformed plastically due to the activation of only one slip system, the lattice formulation will be called the non-compact Higgs model:

its energy is similar to the above equation but instead of the periodic function in the part depending exclusively on the $U(1)$ gauge field, one only keeps the quadratic term as follows

$$\begin{aligned} \text{Energy}_{\text{non-compact Higgs model}} &= \frac{1}{2a^{4-d}e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha \right)^2 + \\ &+ a^{d-2} |\psi|^2 \sum_{\vec{r}, \alpha} \sum_{\mu} \left[1 - \cos \left(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu, \vec{r}}^\alpha \right) \right], \end{aligned} \quad (2.85)$$

which will yield a linear part in the gauge field's equations, removing thus the non-linear terms responsible for the underlying non-Abelian compact character of the model and its accompanying multivalueness in the gauge field. For the non-compact model, factor q^α can be maintained although it will not determine any property of the model.

2.4.3 Step 3: scale η and partition function $\mathcal{Z}$ for equilibrium phase transitions

As mentioned several times along this work, our aim is to describe the different hardening behaviours of a crystal containing dislocations, description achieved by setting analogies with the phases displayed by the superconductor, either with a conventional $U(1)$ gauge symmetry or with a compact $U(1)$ one. The study of the phase transitions requires the definition and calculation of the partition function $\mathcal{Z}$ by starting from the energy E_{conf} modelling each of the different (static) equilibrium configurations which the system under study can assume. This just mentioned energy has been already obtained in (2.84) and (2.85), and it is our task now to discuss how phase transitions can be studied within this framework.

Phase transitions are characterized by the change in the behaviour at large distances of correlation functions, the latter being calculated by means of the partition function

$$\mathcal{Z} = \sum_{\text{all configurations}} \exp \left[-\frac{E_{\text{conf}}}{\eta} \right]. \quad (2.86)$$

Closely related to the partition function is the (Helmholtz) free energy F of the system, i.e., the average energy characterizing the macroscopic states displayed by the system at large distances. This free energy is proportional to minus the logarithm of the partition function. Hence those configurations of the model's fields which yield the leading contribution to the partition function determine the macroscopic physical situation, the latter defined therefore as the state which minimizes the free energy, state corresponding to all these dominating microscopic configurations — where “microscopic” means in this context the scale where the degrees of freedom occurring in the modelized energy (2.84) and (2.85) live, whilst “macroscopic” refers to the long distance behaviour determined collectively by many field variables. As an example of how this works, let us return to the superconductor model again: here the microscopic configurations, in the type II superconducting as well as in the normal conducting state, correspond to the flux lines and to any field arrangement without topological content (the latter, configurations

which can be transformed by means of an allowed gauge transformation into a trivial configuration where the scalar field's phase is constant and the gauge field is zero). Nevertheless the macroscopic situation differs greatly from one state to the other, due to the fact that in the normal conducting (or Coulomb) phase the vortex lines have condensed together so that the magnetic field confined in their core can spread out throughout the whole material. In this latter phase, dense arrangement of vortices are the microscopic field configurations which dominate into the partition function, whereas in the type II superconductor (or Shubnikov phase), such configurations were strongly suppressed.

In definition (2.86) a certain scale η has been introduced, which sets up the energy scale to which any equilibrium configuration assumed by the model's degrees of freedom is compared in order to determine the contribution to the partition function by each configuration. Depending on how strong are the coupling parameters — modelling the strength in the interactions among the different degrees of freedom themselves and with external sources, in our case (2.84) and (2.85), $|\psi|^2$ and e^2 — when compared to this scale η , certain configurations contribute more strongly than others to the partition function. Phase transitions occur when these relations of the coupling parameters to η start to largely enable configurations yielding a new minimum in the free energy $F \sim -\ln \mathcal{Z}$, i.e., to enable new field configurations (previously forbidden energetically and which contributed very weakly) which eventually dominate in the partition function $\mathcal{Z}$ and in the correlation functions derived from it. And it is this new minimum in the free energy which is also responsible for changes in the correlation functions' behaviour; or equivalently, for the nonanalytic form in the free energy's derivatives, signal and definition of a phase transition. A toy model to explain the relation between nonanalyticity in the free energy and change in the correlation function as characteristics of a phase transition can be found in equation (C.7) on page 309 and in the discussion which follows there.

Since the phase transition ¹⁵ from the Shubnikov to the Coulomb phase — produced by the condensation of topological vortices — has been used as the leading idea to modellize in an analogous way the loss of rigidity characteristic for the plastic behaviour, most of the phase transitions occurring in the model under construction here will be also signalized by condensation of certain topologically stable defects. This is actually not so

¹⁵It should be noted that the description of such a transition in the framework of the continuous Abrikosov model, as an effective model for the large distance behaviour of a superconductor, gets described in a different form as that resulting from the lattice Higgs model. In the Abrikosov model is the whole scalar field, as different from the lattice formulation — where only its phase is the actual variable, being its length (considered as a model's coupling strength) independent on the point of physical space —, what is taken as field variable, and the transition towards the Coulomb (normal conducting) state is described by the fact that such a scalar field tends towards zero since the cores of the flux lines — the only region in a superconductor where the material can exist in the normal conducting state characterized by a vanishing scalar field — are touching each other. In the lattice model, on the contrary, the length of the scalar field always remains nonzero and it is the correlation function of distantly separated scalar fields which changes from being nonzero to tend towards zero at large distances by gradually approaching the phase transition point. The treatment of this transition by means of the Abrikosov model is considered after equation (A.48) on page 187, whereas its study within the framework of the lattice model is carried out in the discussion previous to equation (D.252) on page 656, through the calculation of a special correlation function which displays the same behaviour at large distances as the effective scalar field of the continuous Abrikosov model.

surprising for a dislocation model describing changes in hardening characterized by the massive arrangement of certain types of dislocations. And it is at such condensations where the role of scale η becomes relevant: although the topological defects have a large energy, if the model's coupling parameters lie in a range of values such that, when compared to η , are strong enough to produce a large number of such configurations — i.e., to enable a large amount of fluctuations in the field variables yielding configurations with high content in topologically stable defects —, their contributions to the partition function can become important and even dominant; something what actually occurs in the condensation of many topological defects with its accompanying interchange in their mutual positions producing thus the high number of similar configurations.

As a remark, up to this point we have focussed our attention only on two phases displayed by the conventional superconductor: *a*) the phase where vortices (or their correspondence in the dislocation model, the able-to-glide dislocations) exist as isolated objects; and *b*) the phase where such topological defects condense allowing the field, which was previously restricted inside the vortices, to spread out over the whole system. However, if the $U(1)$ gauge field in the superconductor model is of *compact* or angular character, and if the factor q introduced after (2.83) is different from 1, there exist configurations which can connect field arrangements corresponding to different amounts of vorticity (or of different content in Burgers vector). When such “connecting configurations” begin to yield the leading contribution to the partition function, also a phase transition takes place, phase which has no analogue in the conventional (non-compact) superconductor and is characterized by the strongest decay with the distance of certain correlation functions of the gauge field to be studied later ¹⁶. Moreover, this new phase

¹⁶Although this point is discussed in detail on Appendix C's page 442, the main argument can be easily abbreviated here. As it is well known, a system of neighbouring coupled classical spins — classical in the sense that such spins are represented by a scalar field taking on two (or more) discrete values “up” and “down” without deriving it from any representation of the Poincaré group — where the energy is invariant under a simultaneous global inversion in the orientation of all spins, two different phases are displayed: one ordered phase for a large coupling strength between the neighbour spins, phase where spins try to align themselves parallel to each other and where the correlation of two distantly separated spins does not vanish at large distances; and a second, disordered, phase occurring at a weak coupling strength where the same correlation function decays exponentially with the distance. The transition towards this latter phase is produced when field configurations called *domain walls*, which can connect different spatial regions where the spin orientation is reversed, are not energetically suppressed and appear in large amounts. The higher the number of the orientations available for the spin field, the more favoured are domains walls and the easier to transit towards the disordered phase.

For the gauge field, on the other hand, the situation is quite similar as long as the gauge field $A_{\mu,\vec{r}}$ has *angular* or compact character and thus $e^{iA_{\mu,\vec{r}}}$ can be considered as a classical spin field. As one can see from equation (2.84), $\frac{1}{a^{4-d}e^2} \sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})] + \dots = \frac{1}{a^{4-d}e^2} \sum_{\vec{r}} \sum_{\mu < \nu} \left[1 - \frac{1}{2} (e^{iA_{\mu,\vec{r}}} e^{iA_{\nu,\vec{r}+\vec{\mu}}} e^{iA_{-\mu,\vec{r}+\vec{\mu}+\vec{\nu}}} e^{iA_{-\nu,\vec{r}+\vec{\nu}}} + \text{c.c.}) \right] + \dots$, the coupling strength between neighbour $e^{iA_{\mu,\vec{r}}}$'s is equal to $1/e^2$ in this case; the $\dots$ in the previous equation denotes the term where the gauge field is coupled to the scalar field (characterizing the rigidity of the system), term which includes the factor q . In the gauge invariant case there is a further complication which does not appear in the usual spin system discussed above: the infinite repetition of the *same* contribution to the partition function due to the gauge invariance in the energy describing the system. During the calculation of the partition function or of any (gauge invariant) correlation function, this infinite number of

will result to describe the main characteristics of the rapid hardening stage, something to be expected since the above connecting configurations have the same origin (and their number of different classes also coincides) as the objects whose function is analogue to that of magnetic monopoles in an electromagnetic context, and the latter just correspond in the dislocation model to fields carrying the information about sessile dislocations, which are responsible for the Stage II (or rapid hardening stage) when they condense into dense networks.

For the dislocation theory the scale η necessary for the partition function which will modelize the equilibrium phase transitions cannot be the temperature scale $k_B T$ (with k_B the Boltzmann's constant): the dislocation energy is, for the usual temperature range, much larger than $k_B T$ and thus dislocations cannot be created or destroyed by temperature fluctuations. This is the same case as in a superconductor, where any vortex line lies at a energy scale much higher than the temperature scale — what leads to the permanence during a very long time of the electric current localized around the vortex line, property which in the dislocation model corresponds to the localization around any able-to-glide dislocation of a shear stress —: vortices in a superconductor are created by the external magnetic field, or by the external electric current source of such a magnetic field; and similarly, the source for the dislocations is an external applied stress. How to define scale η in our dislocation model?

The Abrikosov model, as an effective description of superconductivity, arises from the formulation at the quantum scale of a model called the BCS theory: through the calculation of a partition function in which the different configurations to be summed over are those due to the fluctuations in the electron's matter field. The scale η used in such a partition function corresponds to the Planck's scale $\hbar c$. In the deduction of the Abrikosov model, the gauge field coupled to the matter field is assumed to be controlled from the exterior, and its fluctuations are not taken into account in the partition function leading from the quantum scale towards the (effective) Abrikosov model. In fact, when such gauge field's fluctuations become relevant since the transition to the normal conducting state is being approached, their effect is indirectly considered by solving the field equations for a nearly vanishing effective scalar field describing the

times has to be factorized out, something which is achieved by removing explicitly the gauge invariance, i.e., by choosing among all equivalent configurations — configurations taken into each other through a gauge transformation — of a representative and then summing over the different representatives to evaluate the partition function. Hence, *after* having chosen the representative for the gauge equivalent configurations and working sole with them, a very high rigidity (controlled by the scalar field's length $|\psi|^2$) fixes the number of different available orientations for “spin” $e^{iA_{\mu,\mathbf{r}}}$ which just coincides with factor q — without forgetting that factor q in the compact Higgs model also indicates the number of different stable vorticities (and also the number of vortices attached to one magnetic monopole) and that $q = 1$ is equivalent to the non-existence on any topologically stable vortex. And for a large coupling strength $1/e^2$ (small e^2), in a way similar to the scalar spin system, most of $e^{iA_{\mu,\mathbf{r}}}$'s tend to be aligned along one of the q available orientations, having as a consequence that the (gauge invariant) correlation function for the gauge field shows a weak decay with the distance. On the contrary, a weaker coupling strength $1/e^2$ (strong e^2) permits larger fluctuations of the $e^{iA_{\mu,\mathbf{r}}}$'s, and thus field configurations which, in the manner of domain walls, can connect different “spin”-oriented parts of the system, leading thus to a disordered phase characterized by a stronger decay of the correlation functions. The lower the rigidity, the more favoured this disorder since then the fluctuations in the “spin” corresponding to the compact gauge field are still stronger.

paired electrons. Another more detailed form to describe this latter transition, without leaving the framework of the Abrikosov model, is by constructing again a partition function but this time taking as configuration energy that given by the Abrikosov model, using again as scale the Planck's constant $\hbar c$: summing over the fluctuations in the gauge field leads to a resulting free energy for the paired electrons' effective scalar field (or order parameter) ψ where the minimum of the free energy gets eventually displaced towards a solution of zero $|\psi|$; this last approach is the so-called effective potential for the Abrikosov model, dealt with in detail in the Appendix D's section on page 700.

But the scale η for the dislocation model cannot be taken over from the Abrikosov model by setting it equal to $\hbar c$ — although the quantum scale is already “hidden” behind the energy scale determined jointly by the elastic constant and by the interatomic spacing —; the value to which the model's coupling parameters are to be compared to in order to know whether a certain configuration largely contributes or not to the partition function cannot be of quantum origin since quantum fluctuations are not able to modify such configurations — the activation of the dislocation motion through quantum tunneling plays no relevant role in the plastic behaviour. However, something like a fundamental uncertainty similar to that corresponding to the Planck's scale can also be found in a dislocation system, by following an original idea by Kondo quoted by Kröner [64]: the measurement of dislocation positions carries always an uncertainty due to the Burgers vector, which is of the order of one lattice spacing. Such an uncertainty can be considered as similar to the “zero oscillation” of Quantum Mechanics, allowing us to close the analogy to the Abrikosov model by defining a scale η playing a role similar to the Planck's scale. Let us consider a dislocation segment “zero oscillating” due to a position uncertainty x of the order of the Burgers vector: its energy is given by the energy per length $\frac{G}{4\pi}x^2 \ln\left(\frac{\Lambda}{r_0}\right)$ (where Λ is the length dimension of the plane crossed perpendicularly by the dislocation, and r_0 is the dislocation's core radius) multiplied by the length of the segment “oscillating”. In the partition function, the contribution produced by the above dislocation gets multiplied by a factor due to the different positions assumed by the dislocation because of the position uncertainty. Hence, the leading contribution to free energy $F \sim -\ln \mathcal{Z}$, i.e. the contribution dominating in partition function $\mathcal{Z} = \sum_{\text{configurations}} \exp\left[-\frac{E_{\text{conf}}}{\eta}\right]$, results to be

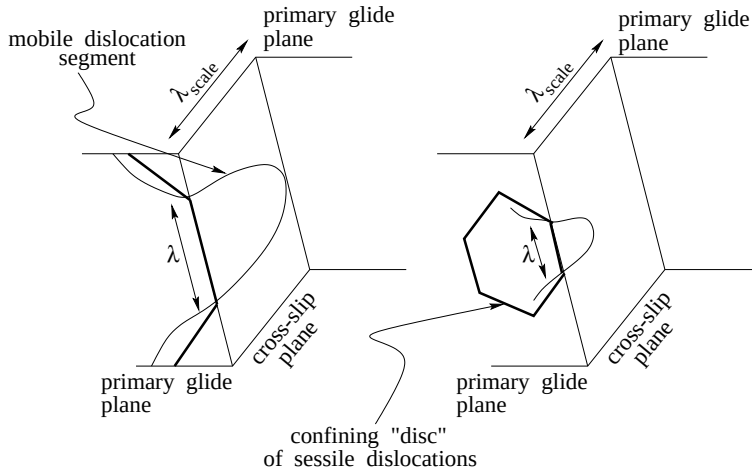
$$F \sim \underbrace{\frac{1}{\eta} \frac{G}{4\pi} x^2 \lambda_{\text{smallest between sessile p.}} \ln\left(\frac{\Lambda}{r_0}\right)}_{E_{\text{conf}}/\eta \text{ due to the “zero oscillating” disl.}} - \underbrace{\ln\left(\frac{x}{r_0}\right)}_{\text{position uncertainty}}, \quad (2.87)$$

for the smallest possible “zero oscillating” dislocation segment between the two points at which the segment becomes sessile, immobile. The value of x which minimizes F , $\frac{\partial F}{\partial x} = 0$, corresponds to the lattice spacing (the smallest possible spacing in the crystal) and this defines the scale η

$$\eta \stackrel{\text{def}}{=} \frac{Ga^2}{4\pi^2} \lambda_{\text{scale}}, \quad (2.88)$$

by means of the introduction of a length λ_{scale} which is of the order of the just discussed shortest dislocation segment of uncertain, unable-to-exactly-specify position. One can argue that the η defined in (2.88) is not, because of the elastic constant G , a constant for all crystals, but what actually enters into the partition function is $\frac{E_{\text{conf}}}{\eta}$ where its dependence on G disappears due to the quotient.

The physical meaning of $\lambda_{\text{scale}} \sim \lambda_{\text{smallest between sessile p.}}$ can be obtained by the formal similarity between, on the one hand, what the minimal configuration is for a mobile dislocation segment to escape from a confining barrier of sessile dislocations, and on the other hand, how a particle escapes from a potential barrier due to the uncertainty in its position and energy. As represented in the following figure, the escape of a mobile segment of length λ trapped by a “disc” of immobile dislocations can only be achieved if its length is of the order of the shortest step-height to be jumped — by means of cross-slip — to reach the next glide plane. Further, and according to the definition of



scale η , the contribution of the dislocation arrangement in the figure to the partition function is $e^{-\lambda/\lambda_{\text{scale}}}$, which will play a non-vanishing role only if the mobile segment has a length λ of the order of λ_{scale} , what allows us to identify the length scale defining η with the shortest cross-slip step-height, height which for most of the crystals has a constant value about $1 \mu\text{m}$.

How length λ_{scale} is related to characteristic length Λ_{hard} of equation (1.14) will be discussed in the next Chapter in equation (3.11). Firstly, in order to close the analogy relations between our model and the Abrikosov model, let us consider in the next Note the equivalent to scale $\hbar c$ in the framework of the dislocation model under study.

Note. The analogue to the Abrikosov model's $\hbar c$.

As the analogy between the Abrikosov model and the dislocation model was for the first time discussed, the two main coupling parameters in the first model, $\frac{\hbar^2 |\psi|^2}{m}$ and $\frac{e}{\hbar c}$, were identified as parallel to the two main parameters in the dislocation model, G and $\frac{1}{\lambda}$, which characterize the rigidity and the interaction range between dislocations. In the last identification, it was not set separately the correspondence of e and $\hbar c$ with any parameters in the dislocation model, task to be carried out in order to obtain the corresponding parameter for the superconductor's dimensionless coupling parameter $\frac{e^2}{\hbar c} = \left(\frac{e}{\hbar c}\right)^2 (\hbar c)$. Thus based on the above result for scale η , which has the dimension of energy, let us obtain the “energy×length” scale which would correspond, within the framework of the dislocation model, to the scale $\hbar c$ of the Abrikosov model. Since this formal equivalence for $\hbar c$ is related to the

energy scale η , one should expect a dependence of the following type

$$\hbar c \longleftrightarrow \frac{Ga^4}{4\pi^2} \left(\frac{\lambda_{\text{scale}}}{a} \right)^m, \quad (2.89)$$

with $m \geq 1$ (to agree with definition (2.88)), as well as a result

$$\frac{e^2}{\hbar c} \longleftrightarrow \left(\frac{\lambda_{\text{scale}}}{\lambda} \right)^k \left(\frac{\lambda_{\text{scale}}}{a} \right)^l, \quad (2.90)$$

for the Abrikosov's dimensionless coupling constant, remembering that λ is the spacing between dislocations characterizing thus the strength of the coupling between the latter and the scalar field which carries the information about the crystal rigidity. Because of the correspondence between the parameters of the superconductor and the dislocation model stated in relations (2.68), two of the exponents m, k, l for the proposed dependences (2.89)-(2.90) become determined

$$\frac{4\pi^2}{Ga^2} \frac{1}{\lambda^2} \xleftrightarrow{\text{eq. (2.68)}} \left(\frac{e}{\hbar c} \right)^2 = \frac{e^2}{\hbar c} \frac{1}{\hbar c} \longleftrightarrow \left(\frac{\lambda_{\text{scale}}}{\lambda} \right)^k \left(\frac{\lambda_{\text{scale}}}{a} \right)^l \frac{4\pi^2}{Ga^4} \left(\frac{a}{\lambda_{\text{scale}}} \right)^m, \quad (2.91)$$

obtaining $k = 2$ and $l - m = -2$.

In order to fix m , we shall consider a case similar to that discussed previous to equation (2.87). Among the quantum uncertainty relations studied by Heisenberg [48], that interesting for our discussion now states the impossibility of a simultaneous vanishing in the uncertainties of the magnetic and electric field measured inside a certain small volume δV of length dimension δl (with $\delta V = (\delta l)^3$):

$$\Delta H_z \Delta E_y \sim \frac{\hbar c}{(\delta l)^4}.$$

In the crystal, the uncertainty or imprecision in the dislocation position caused by the dislocation's Burgers vector $b \sim a$ has as consequence that the density B_z of mobile dislocations crossing perpendicularly the small surface $(\delta l)^2$ becomes uncertain by the amount $\Delta B_z \sim \frac{b}{(\delta l)^2}$ and therefore

$\Delta H_z = \frac{1}{\mu_0} \Delta B_z \sim \frac{b}{(\delta l)^2} G \lambda_0^2$. And in a similar way for the uncertainty in the gliding dislocation current perpendicular to such dislocations, $\Delta E_y \sim \Delta B_z \times \frac{v_{\text{glide}}}{c}$, obtaining thus

$$\Delta H_z \Delta E_y \sim \frac{1}{(\delta l)^4} G b^2 \lambda_0^2 \frac{v_{\text{glide}}}{c_{\text{sound}}}.$$

This last relation is what corresponds for the case of matter fields coupled to the gauge field (as for instance in the superconductor) to the Heisenberg uncertainty relation $\Delta H_z \Delta E_y \sim \frac{1}{(\delta l)^4} \hbar c$, what

allows us to set the correspondence $\hbar c \longleftrightarrow G b^2 \lambda_0^2 \frac{v_{\text{glide}}}{c_{\text{sound}}}$. For the typical values [91] of initially grown-in dislocation spacing and of glide velocity, $\left\{ \begin{array}{l} \lambda_0 \sim 10^5 a \\ \frac{v_{\text{glide}}}{c_{\text{sound}}} \sim 10^{-4} \end{array} \right.$, one obtains that $\sqrt{\lambda_0^2 \frac{v_{\text{glide}}}{c_{\text{sound}}}}$ has the typical size of the step-height to be overcome in order to escape from the "discs" of sessile dislocation which impedes the glide motion in the primary plane. Or equivalently

$$\sqrt{\lambda_0^2 \frac{v_{\text{glide}}}{c_{\text{sound}}}} \sim \lambda_{\text{scale}},$$

what leads finally to the formal correspondence between the quantum mechanical scale $\hbar c$ of the Abrikosov model and the value of such scale within the dislocation model

$$\hbar c \longleftrightarrow \frac{Ga^2}{4\pi^2} \lambda_{\text{scale}}^2, \quad (2.92)$$

which is the case $m = 2$ in (2.89) and therefore $l = 0$ in (2.91). This result completes relations (2.68), as summarized in Table 2.5. It is to be noticed that the number of independent parameters in both models coincides: coupling strengths $\frac{\hbar^2|\psi|^2}{m}$ and $\frac{e}{\hbar c}$, scale $\hbar c$ and effective correlation length ξ in the Abrikosov model, and on the other hand, the strengths characterizing the interactions of the dislocations among them and with the crystal rigidity, G and $\frac{1}{\lambda}$, the scale λ_{scale} defining the energy scale for the partition function, and finally the size of the dislocation core.

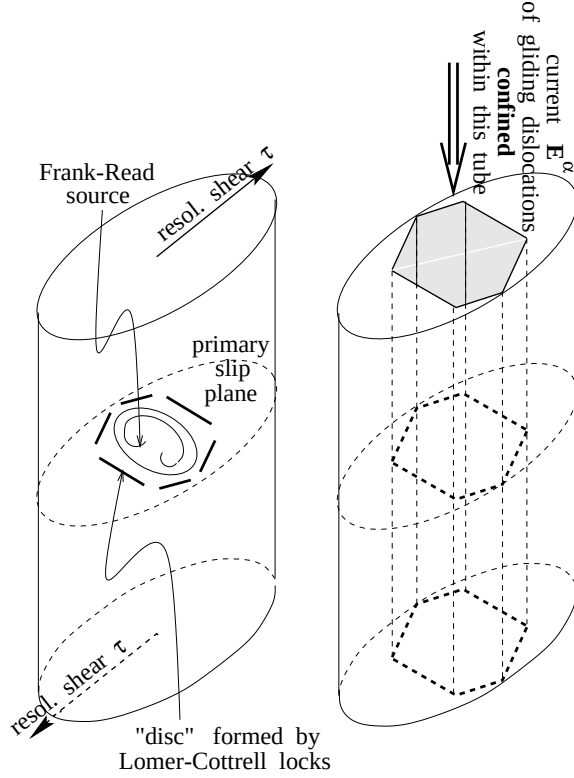
Abrikosov model parameters	Dislocation model parameters
scalar field's length $\frac{\hbar^2 \psi ^2}{m}$	$\longleftrightarrow \frac{Ga^2}{4\pi^2}$
coupling constant $\frac{e}{\hbar c}$	$\longleftrightarrow \frac{2\pi}{\sqrt{Ga^2}} \frac{1}{\lambda}$
dimensionless coupling $\frac{e^2}{\hbar c}$	$\longleftrightarrow \left(\frac{\lambda_{\text{scale}}}{\lambda}\right)^2$
scale $\hbar c$	$\longleftrightarrow \frac{Ga^2}{4\pi^2} \lambda_{\text{scale}}^2 = \eta \lambda_{\text{scale}}$
correlation length ξ	$\longleftrightarrow$ radius r_0 of the dislocation core

Table 2.5: Correspondence among the parameters of the Abrikosov model and those of the dislocation model.

Closing this discussion on the scale for the partition function and advancing the next point to be considered, by looking at the modellized energy for the compact Higgs model (2.84), $\frac{1}{a^{4-d}e^2} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left[1 - \cos(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha)\right] + \dots$ (or at the energy for the non-compact model described by $(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha)^2$ in the pure gauge field part), one notices that large fluctuations in the gauge field only occurs at large ηe^2 . And these large gradients in the field configurations, necessary condition for topologically stable objects, become relevant in the partition function in the case of condensation of the latter defects. As mentioned above on page 83, exclusively for the model with a $U(1)$ gauge symmetry of angular or compact character, the condensation of certain topological “connecting configurations” will lead to a transition towards a phase characterized by a restricted dislocation mobility confined within certain spatial “tubes”, being the latter the main characteristic of Stage II. On the other hand, and according to Table 2.5, a strong e^2 corresponds to a very close dislocation spacing λ , what also coincides with the tangled dislocation arrangement leading to the Stage II.

2.4.4 Step 4: incorporating the effect of the temperature

Now it is time to deal with the last component to be included into the model: the temperature. Its effect does not enter in the equations through η , scale which is usually much larger than $k_B T$. As we have seen in the introduction (Chapter 1 on page 27), high temperatures only play a relevant role in the breaking of the rapid hardening stage (Stage II) through the activation of cross-slip. Therefore, let us consider firstly this Stage II. In such a stage, barriers built by sessile dislocations impede the glide motion



of the mobile dislocations created on the primary planes at the Frank-Read sources. In order to keep up with the strain imposed by external forces, the Frank-Read sources produce more and more dislocations (the largest dislocation production takes place in the Stage II, as it can be seen in Fig 2.3 on page 36) which eventually build very complex and tangled networks of immobile dislocations. On the primary slip planes, only inside the “discs” delimited by the sessile dislocations, as the Lomer-Cottrell locks, the produced dislocations can glide. And for a single crystal at very low temperature, such discs are piled in “tubes”, like that represented by the hollow hexagonal tube in the figure, along the whole crystal. Hence the

current of dislocations gliding through the primary planes gets confined inside such “tubes” — such current, which in the dislocation model formulated previously plays a role analogous to the electric field in the Maxwell equations, points along the tube as one can easily check from equation $\text{curl } \alpha_{\text{glid}} = \dots + (\epsilon_0 \mu_0) (\epsilon_{\text{rel}} \mu_{\text{rel}}) \partial_t \mathcal{J}_{\text{glid}}$. Summarizing, for the rapid hardening stage we obtain a picture strongly reminding to that of the usual superconductor where the magnetic field can only exist within the vortices, but with the difference that the field confined now is the analogue to the electric, not to the magnetic, field ¹⁷.

¹⁷The reader knowledgeable about particle physics may have already noticed that the case we are dealing here is nothing else as the analogue, applied to the dislocation model, of the dual superconductor. The dual superconductor displays a behaviour similar to the usual superconductor but whose characteristics, instead of being applied to the magnetic field and its sources, apply to the electric field. Such a particular superconducting state occurs as a phase in a system modellized by a scalar field coupled to a *compact* $U(1)$ gauge field, i.e. in a system whose energy is defined by (2.84), and in the range of strong coupling ηe^2 . Since in this phase the electric field cannot spread out over the system, the configuration formed by two opposite electric charges increases its energy when the electric charges are separated: the energy increases proportionally to the length of the tube where the electric field (going from one electric charge to the other) is confined. This is called the *confinement of electric charges*. In

However, by increasing the temperature T a dislocation segment will eventually become able to escape from the enclosing ring of Lomer-Cottrell locks and to jump to another slip plane located above the initial primary glide plane. But as long as the temperature is low enough, the Frank-Read sources in the new slip plane are also enclosed by Lomer-Cottrell locks, without the possibility of escaping from these locks: therefore the just discussed tube-structure gets repeated in the new reached glide system. As a result, the “tube” initially stretching along the whole crystals’ height becomes now cut into two pieces. And by further increasing, one obtains a *periodically repeated pattern* of the mobile dislocations. It should be noted that for a single and homogeneous crystal this repetition can be considered as actually periodic, although in a real crystal local inhomogeneties as well as the dependence on the direction of the elastic constants cause this repetition to become non-periodic.

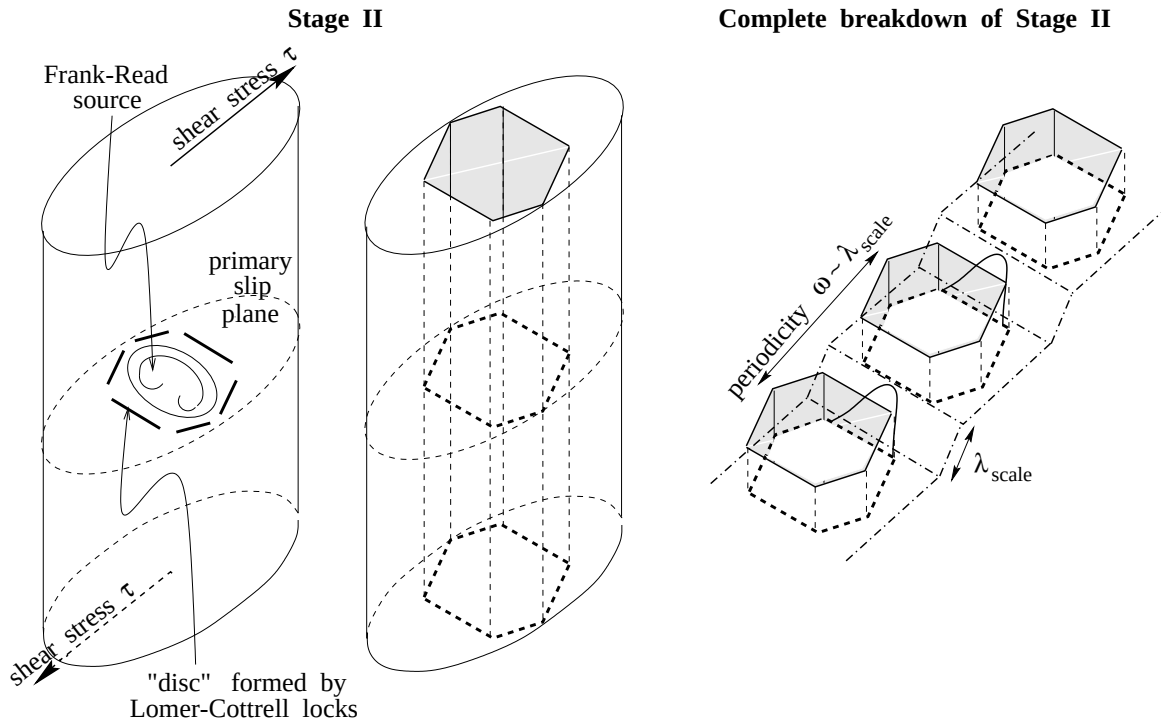


Figure 2.7: Left, dislocation arrangement in the rapid hardening stage with the characteristic confinement of the current of gliding dislocations within the tube delimited by “discs” of sessile dislocations. Right, complete breaking of such configuration resulting in periodically repeated pieces due to the thermal activation of cross-slip. Length ω denotes the periodicity.

Denoting by ω the periodicity lattice’s length of the above repetition, one arrives thus at the ansatz of how ω is related to temperature T : when the temperature reaches an energy scale of the order of η , T is then able to activate the cross-slip throughout the whole crystal, leading to the complete breakdown of Stage II as represented in the

the dislocation model, on the other, such a behaviour will correspond to the energy increase when the crystal is stretched along the direction of confinement of the gliding dislocation current — stretching imposed by the slip velocity externally applied on the crystal.

right picture of Fig. 2.7. But in such case, periodicity ω is of the order of λ_{scale} , the latter identified as the characteristic shortest step-height to be overcome by cross-slip. Hence the quotient of lengths $\frac{\omega}{\lambda_{\text{scale}}}$ is assumed inversely proportional to the quotient of the energy corresponding to scale length λ_{scale} and that of the thermal energy $k_B T$

$$\frac{\omega}{\lambda_{\text{scale}}} = \frac{\eta}{k_B T}. \quad (2.93)$$

The form to include the temperature effect in the modellized energy runs as follows: along the direction (say z) where the periodicity will occur due to the activation of cross-slip, one substitutes the infinite range in the integral (for the continuum formulation) or in the sum (for the lattice formulation) which appeared as cross-slip was not energetically possible with the finite range from $z = 0$ to $z = \omega \sim \frac{1}{T}$; and simultaneously, one makes periodic along such a coordinate the field variables entering in the energy density $\epsilon[A_k^\alpha, \varphi^\alpha]$, $\begin{cases} A_k^\alpha(z = \omega) = A_k^\alpha(z = 0) \\ \varphi^\alpha(z = \omega) = \varphi^\alpha(z = 0) \end{cases}$, in order to take account of the periodic repetition in the confining “disc”-structure caused by the cutting of the initial “tube” which ran in one piece along the whole crystal. Of course, this periodicity is irrelevant for $T = 0$ ($\omega = \infty$) where the cross-slip for the mobile dislocations — to jump into another slip system producing thus the tube to be cut into fragments — cannot occur. Hence, the energy which appears in the partition function at $T \neq 0$ is given by

$$\text{Energy} = \oint_{z=0}^{z=\omega} dz \int dxdy \epsilon[A_k^\alpha, \varphi^\alpha], \quad (2.94)$$

$\left| A_k^\alpha(z=\omega) = A_k^\alpha(0), \varphi^\alpha(z=\omega) = \varphi^\alpha(0) \right.$

where, for simplicity, the integral notation has been used instead of the more involved lattice discrete summation of models (2.84) and (2.85). However it is worthy to notice that for the complex arrangement of dislocations we are studying now, the lattice spacing a plays no longer any role along the direction of the periodic repetition: the relevant length now is λ_{scale} , which is much larger than a . And therefore, even in the lattice formulation, the continuum formulation along such a periodic direction is fully justified.

The whole Appendix C is devoted to the obtention of the phase diagrams for these modellized lattice models at $T = 0$, i.e., for a lattice non-periodic in any direction. In Appendix D it is studied how the periodicity along one of the lattice directions, direction whose coordinate can be furtherly considered as continuous, modifies the results of Appendix C, obtaining thus the effect of high temperature on the formulated dislocation model.

2.5 Gauge invariant correlation functions

After all the preparatory discussion to construct a consistent dislocation model to describe the different hardening stages, we have reached the point of applying this model and seeing how the different hardening behaviours arise as equilibrium phases of the model. As argued previously to equation (2.50), and as it was the leading idea throughout the whole work, the occurrence of phase transitions must be exclusively due

to a modellized energy which remains invariant under $U(1)$ gauge transformations, corresponding the latter to compatible deformations $\begin{cases} \varphi_{\vec{r}}^\alpha \rightarrow \varphi_{\vec{r}}^\alpha - q^\alpha u_{\vec{r}}^\alpha \\ A_{\mu,\vec{r}}^\alpha \rightarrow A_{\mu,\vec{r}}^\alpha - \Delta_\mu u_{\vec{r}}^\alpha \end{cases}$ of non-singular displacement vector u^α (with $\mu = 1, 2, 3$). In fact, the only logical form to proceed while studying phase transitions not caused by compatible deformations (read gauge transformations) is, in the partition function, to take account of only the gauge inequivalent configurations. In other words, after classifying all possible field configurations by identifying together in a class all (infinite) configurations mutually related by means of any gauge transformation, to work in the partition function with only one representative for each class. Example of these gauge inequivalent configurations are field arrangements of different content in topological defects but which cannot be deformed into one other by means of gauge transformations. Mathematically this form of operating runs as follows: after separating all configurations as follows

$$\text{all configurations} = \begin{cases} \text{all gauge inequivalent (representative) configurations} \\ + \\ \text{all compatible deformations (= gauge transformations)} \\ \text{about each of the above gauge inequivalent configurations} \end{cases}$$

the contributions of the second part (that of all possible gauge transformations applied to every representative configuration) are factorized out in the partition function defined by the gauge invariant energy

$$\begin{aligned} \mathcal{Z} &= \sum_{\text{all configurations}} \exp \left[-\frac{E_{\text{conf}}}{\eta} \right] = \left(\sum_{\text{representative gauge inequivalent conf.}} \exp \left[-\frac{E_{\text{conf}}}{\eta} \right] \right) \times \\ &\times \underbrace{\left(\sum_{\text{gauge transf. about representatives}} 1 \right)}_{\text{constant } \mathcal{N}}. \end{aligned} \quad (2.95)$$

Factor $\mathcal{N}$, counting the (infinite) number of times that the same contribution arising from a representative configuration appears in the partition function, and in order to obtain finite results, such a factor has to be removed during any calculation carried out by means of the partition function. Moreover, $\mathcal{N}$ only yields a constant additive term in the (Helmholtz) free energy $\mathcal{F} = -\eta \ln \mathcal{Z}$.

Phase transitions are signalized by the change in the behaviour of the correlation functions calculated with the partition function (2.95). Such correlation functions are defined as the expectation value

$$\langle (\cdot) \rangle = \frac{1}{\mathcal{Z}} \sum_{\text{all configurations}} (\cdot) \exp \left[-\frac{E_{\text{conf}}}{\eta} \right], \quad (2.96)$$

of a certain function of the field variables, being such function itself gauge invariant or not. As a consequence of the Elitzur theorem ([36] and [59]), nevertheless, only gauge invariant correlation functions have sense when calculated by means of a partition function constructed from a gauge invariant energy: the calculation, by means of such a

gauge invariant partition, of any correlation function non-invariant under gauge transformations vanishes identically. Let us see how this statement arises: if a function (more properly functional) f depending on the scalar and gauge fields placed at certain lattice points is not gauge invariant, then the sum of the f -values produced by the infinite many gauge transformations about *every* gauge inequivalent configuration is zero

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \sum_{\text{gauge inequiv. conf.}} \left(\exp \left[-\frac{E_{\text{conf}}}{\eta} \right] \underbrace{\left(\sum_{\text{all gauge transf. about each gauge ineq. conf.}} f \right)}_{=0} \right), \quad (2.97)$$

since function f does change its value under each one of these gauge transformations but factor $\exp \left[-\frac{E_{\text{conf.}}}{\eta} \right]$ remains the same. Only if f is gauge invariant, factor $\mathcal{N}$ can be factorized out also in the numerator of (2.97), in the same way as in equation (2.95), since f is kept unchanged under all the gauge transformations about each representative configuration. On the contrary, such a vanishing of the correlation function as in (2.97) would not hold if also E_{conf} were gauge dependent, since in such a case, every different value of f would get multiplied by a different factor $\exp \left[-\frac{E_{\text{conf.}}}{\eta} \right]$. A more detailed treatment of this question can be found in the Appendix B's Note on page 210.

It is worth noting that the schematically discussed Elitzur theorem does not rule out a nonzero value for a gauge dependent correlation function if the gauge invariance in the energy functional defining the partition function is removed explicitly, by hand, through the choice on the model's fields of a boundary condition strong enough to eliminate any gauge transformation which would still leave invariant the energy. This choice is called the gauge fixing and corresponds to work, also in the calculation of expectation values for correlation functions, exclusively with the representatives of each class of gauge related configurations. Hence, instead of the vanishing (2.97), for a gauge dependent function f one would calculate

$$\langle f \rangle = \frac{\sum_{\text{gauge inequiv. representative conf.}} f \exp \left[-\frac{E_{\text{conf}}}{\eta} \right]}{\sum_{\text{gauge inequiv. representative conf.}} \exp \left[-\frac{E_{\text{conf}}}{\eta} \right]}, \quad (2.98)$$

which is not necessarily zero. Summarizing, with a model invariant under local or gauge transformations, only the expectation value of gauge invariant expressions can be nonzero.

The just mentioned theorem rules out as correlation function to distinguish the phase transitions occurring in the system the usual $\langle e^{i\varphi_{\vec{r}_1}^\alpha} e^{-i\varphi_{\vec{r}_2}^\alpha} \rangle$ for two distant scalar fields: this correlation function changes under any compatible deformation (or gauge transformation) $\left\{ \begin{array}{l} \varphi_{\vec{r}}^\alpha \rightarrow \varphi_{\vec{r}}^{\alpha \text{ gauged}} = \varphi_{\vec{r}}^\alpha - q^\alpha \chi_{\vec{r}}^\alpha \\ A_{\mu, \vec{r}}^\alpha \rightarrow A_{\mu, \vec{r}}^{\alpha \text{ gauged}} = A_{\mu, \vec{r}}^\alpha - (\chi_{\vec{r}+\hat{\mu}}^\alpha - \chi_{\vec{r}}^\alpha) \end{array} \right.$. Following Wilson [98], one of the gauge invariant correlation functions able to distinguish the different phases, is given by

$$W(C) = \left\langle \exp \left[ip^\alpha \sum_{\text{loop } C} A_{\mu, \vec{r}}^\alpha \right] \right\rangle, \quad (2.99)$$

where C is a large, oriented, lattice closed path. Another form to write this correlation function is obtained by introducing a unit lattice vector $J_{\mu,\vec{r}}$ tangent to loop C

$$J_{\mu,\vec{r}} = \begin{cases} +1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{coincides with the direction of } C \\ -1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{is the opposite to the direction of } C \\ 0 & \text{on all other bonds and for all other directions} \end{cases} \quad (2.100)$$

with $\mu = 1, 2, 3$. This vector field satisfies the following relation

$$\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} = \sum_{\mu} (J_{\mu,\vec{r}} - J_{\mu,\vec{r}-\hat{\mu}}) = 0, \quad (2.101)$$

and therefore

$$W(C) = \left\langle \exp \left[i \frac{p^{\alpha}}{q^{\alpha}} q^{\alpha} \sum_{\vec{r}, \mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}^{\alpha} \right] \right\rangle. \quad (2.102)$$

This correlation function $W(C)$, also called the Wilson loop correlation, is invariant under gauge transformation $A_{\mu,\vec{r}}^{\alpha} \rightarrow A_{\mu,\vec{r}}^{\alpha \text{ gauged}} = A_{\mu,\vec{r}}^{\alpha} - (\chi_{\vec{r}+\hat{\mu}}^{\alpha} - \chi_{\vec{r}}^{\alpha})$ as it can be easily proved by using (2.101)

$$\begin{aligned} \sum_{\text{closed path } C} A_{\mu,\vec{r}}^{\alpha \text{ gauged}} &= \sum_{\vec{r}, \mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}^{\alpha} - \underbrace{\sum_{\vec{r}, \mu} J_{\mu,\vec{r}} (\chi_{\vec{r}+\hat{\mu}}^{\alpha} - \chi_{\vec{r}}^{\alpha})}_{=0} = \sum_{\text{closed path } C} A_{\mu,\vec{r}}^{\alpha}. \quad (2.103) \\ &\quad - \sum_{\vec{r}} \left(\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} \right) \chi_{\vec{r}}^{\alpha} \stackrel{\text{eq. (2.101)}}{=} 0 \end{aligned}$$

Note that $\sum_{\text{loop } C}$ is the discrete version of contour integral $\oint_{\text{loop } C}$ and thus $\sum_{\text{loop } C} A_{\mu,\vec{r}}^{\alpha}$ measures the total vorticity or Burgers vector crossing the area enclosed by path C . Integer factor p^{α} will be set equal to 1 for the following and its function will be considered later, after having explained the meaning of $W(C)$.

2.5.1 Meaning of $W(C)$

In order to explain the physical function of correlation function $W(C)$, let us firstly focus our attention on the 3-dimensional model with a non-compact gauge field, model corresponding to the conventional superconductor: $W(C)$ is given by

$$\frac{\int \prod_{\vec{r}, \alpha} \frac{d\varphi_{\vec{r}}^{\alpha}}{2\pi} \prod_{\vec{r}, \mu, \alpha} \frac{dA_{\mu,\vec{r}}^{\alpha}}{2\pi} \left\{ \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \alpha, \mu} \left(1 - \cos \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu,\vec{r}}^{\alpha} \right) \right) \right] \exp \left[i \sum_{\vec{r}, \mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}^{\alpha} \right] \right\}}{\int_{-\pi}^{\pi} \prod_{\vec{r}} \prod_{\alpha} \frac{d\varphi_{\vec{r}}^{\alpha}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}} \prod_{\mu, \alpha} \frac{dA_{\mu,\vec{r}}^{\alpha}}{2\pi} \left\{ \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \alpha} \sum_{\mu=1}^3 \left(1 - \cos \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu,\vec{r}}^{\alpha} \right) \right) \right] \right\} \times}$$

$$\frac{\times \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\lambda,\nu} A_{\lambda,\vec{r}} \left(-\delta_{\lambda\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} A_{\nu,\vec{r}-\hat{\rho}} + \Delta_{\lambda} \Delta_{\nu} A_{\nu,\vec{r}-\hat{\lambda}} \right) \right]}{\times \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\lambda,\nu} A_{\lambda,\vec{r}} \left(-\delta_{\lambda\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} A_{\nu,\vec{r}-\hat{\rho}} + \Delta_{\lambda} \Delta_{\nu} A_{\nu,\vec{r}-\hat{\lambda}} \right) \right]}, \quad (2.104)$$

where after integrating the angular variable $d\varphi_{\vec{r}}^{\alpha}$ gives

$$W(C) = \frac{\int \prod_{\vec{r},\mu,\alpha} dA_{\mu,\vec{r}}^{\alpha} \exp \left[i \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}^{\alpha} \right] \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\alpha,\mu} A_{\mu,\vec{r}}^{\alpha} (\text{differ.op.}) A_{\mu,\vec{r}}^{\alpha} \right]}{\int_{-\infty}^{\infty} \prod_{\vec{r}} \prod_{\mu,\alpha} dA_{\mu,\vec{r}}^{\alpha} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\mu} A_{\mu,\vec{r}}^{\alpha} (\text{differential operator}) A_{\mu,\vec{r}}^{\alpha} \right]}. \quad (2.105)$$

In the above “differential operator” the effect of the scalar field’s phase $\varphi_{\vec{r}}^{\alpha}$ is already contained, which for a certain range of $|\psi|^2$ will make the gauge field short range. The Gaussian integration over $dA_{\mu,\vec{r}}^{\alpha}$ in (2.105) yields finally

$$W(C) = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} \right], \quad (2.106)$$

where $D_{\vec{r},\vec{r}'}$ is the propagator (or Green’s function) of the just mentioned differential operator, propagator defined through relation $(\text{differential operator})_{\text{acting on } \vec{r}} D_{\vec{r},\vec{r}'} = \delta_{\vec{r},\vec{r}'}$. The difficulty during the integration regarding the overcounting of contributions due to the gauge invariance is of no interest here: this is solved by means of the Faddeev-Popov method, studied in the Appendix C’s Note on page 366, which is nothing else as to divide the partition function by the (infinite) factor $\mathcal{N}$ discussed in (2.95), factor arising from the overcounting of the same contribution an infinite number of times — each time due to a new configuration obtained from a previous one through a gauge transformation which nevertheless leaves invariant the configuration energy and produces thus the same contribution.

Returning to (2.106), $\sum_{\vec{r},\vec{r}'} \sum_{\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'}$ will show a stronger or a weaker dependence on the dimension of loop C depending on the strength of propagator $D_{\vec{r},\vec{r}'}$, which for its part depends on the value of the coupling parameters compared to scale η :

- For a short range propagator, like type $D_{\vec{r},\vec{r}'} = \frac{1}{4\pi} \frac{e^{-|\vec{r}-\vec{r}'|/\lambda_0}}{|\vec{r}-\vec{r}'|}$ in a 3-dimensional space, the $J_{\mu,\vec{r}}$ at a certain lattice site can only interact with the neighbours located within range λ_0 , and therefore the leading dependence results to be equal to $\sum_{\vec{r},\vec{r}'} \sum_{\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} \sim \sum_{\vec{r},\mu} J_{\mu,\vec{r}}^2 = \mathcal{P}_C$, with $\mathcal{P}_C$ the perimeter of C . It is noteworthy that such a dependence holds in the non-compact model for any form of the short range propagator. The latter is, in a superconductor, the cause of the Meissner effect: in other words, it is equivalent to the high rigidity of the system against the spreading out of a magnetic field inside the material. This rigidity, expressed by the London equation, together with the angular character of the scalar field’s

phase $\varphi_{\vec{r}}^\alpha$ — whose coupling to the gauge field has been the responsible for the short range in the latter — results in the existence of the vortices characteristic of the Shubnikov state. Since in the dislocation model this state corresponds to the easy glide stage, where also the density of gliding dislocations (analogue in this model to the magnetic field) cannot spread out over the crystal in order to maintain the latter its mechanical rigidity — remaining such a density field confined within the isolated able-to-glide dislocations — $W(C) \sim e^{-\mathcal{P}_C}$ will characterize the easy glide stage or Stage I.

Note. Behaviour of $W(C)$ in the elastic stage.

A special case is the elastic stage, where there are no mobile (gliding) dislocations, in the same manner as the superconducting Meissner state, to which the elastic stage has been set as analogue, shows no vortices. The Meissner state shares with the Shubnikov state the feature which produces $W(C) \sim e^{-\text{Perimeter}}$, namely, the short range of the gauge field, mark of the material's rigidity. One could argue, nevertheless, that since in this state there is no able-to-glide dislocation, then the sum of all Burgers vectors belonging to the mobile dislocations crossing the interior of C would become identically zero: $\sum_{\text{loop } C} A_{\mu, \vec{r}}^\alpha = 0$, and therefore $W(C) = 1$ for the elastic stage. But this conclusion is not correct. In the lattice model as long as $e^2 \neq 0$ the gauge field is always coupled to the scalar field's phase which is an angular variable, what together with the short range in the gauge field will *always* yield vortices (for a superconductor) or able-to-glide dislocations (for our dislocation model) as microscopic degrees of freedom. Remaining in the context of superconductivity (the same argument holds for the dislocation in a similar way), it depends on how many of such stable vortices are allowed by the coupling parameters of the model, ranging from being very dilute to the condensation of all them, that the macroscopic realization is characterized respectively by no vortex or by a magnetic field spreading out over the material — state corresponding to the condensation of the lattice model's vortices, vortices which still remain as microscopic variables although they cannot be distinguished macroscopically as such —; and it is this macroscopic situation what the Abrikosov model as effective model actually describes. Hence one should be a bit careful by distinguishing these two slight different descriptions, for although the Abrikosov model has been the basis for the analogy used along this work, now we are working in the framework of the lattice model and only the behaviour of the phases displayed by the latter model — but not the microscopic field variables which arrange themselves in many different configurations contributing to the partition function to yield the macroscopic phase — can be made correspond to the description given by the Abrikosov model.

- For the long range 3-dimensional Coulomb propagator $D_{\vec{r}, \vec{r}'} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}$, correlation function $W(C)$ shows a decay $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \sim \mathcal{P}_C \ln \mathcal{P}_C$ (see result (C.221) on page 375), stronger than that in the Shubnikov state just considered. That now the gauge field has become long range is the main characteristic of the normal conducting (or Coulomb) phase in the Abrikosov model, or of a well developed “parabolic hardening” (Stage III) in the dislocation model: this is the signal of the loss of rigidity accompanying the fact that, depending on the model, either the magnetic field or the field density of able-to-glide dislocations is no longer

restricted into certain spatial regions due to the condensation of the topological line defects which previously contained confined such field. Simultaneously, the London equation in the Abrikosov model does not determine the behaviour of the Abrikosov model in the same way as the Hooke's law holds no longer in Stage III where, as its name indicates, stress $\sim \sqrt{\text{strain}}$.

It is important to stress that the above cases have been considered in the situation that no periodical repetition of dislocation patterns as in Fig. 2.7 (page 90) takes place; or equivalently, that we are studying a single crystal at very low temperatures.

These two cases, whose correspondence to some hardening stage has been already established, allow us to obtain an interpretation for $W(C)$ which can be later tested on the case of the compact gauge field. A pure electromagnetic interpretation of the exponent of $W(C)$ for the non-compact case, $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}$, would give that it corresponds

to the energy due to the interaction (carried by propagator $D_{\vec{r}, \vec{r}'}$) of the electric current element at point $\vec{r}$ with the current element at $\vec{r}'$ and summed over all elements of closed path C . The strength of such interaction depends on the arrangement (or correlation) and number of topological line defects *crossing* — the direction of the electric current corresponds to the direction of transversal changes in the distribution of magnetic field contained into the vortices — the spatial region enclosed by C , being such number given by $\sum_{\text{loop } C} A_{\mu, \vec{r}}^{\alpha}$. Hence configurations of these defects, where element $e_{\text{unit plaq.}}^i A_{\mu, \vec{r}}^{\alpha}$ at

one of the unit plaquettes covering the interior of C is weakly correlated with another plaquette element, will produce a decay in $W(C) = \left\langle \prod_{\text{all plaq. inside } C} e_{\text{unit plaq.}}^i A_{\mu, \vec{r}}^{\alpha} \right\rangle$ with the dimensions of C more pronounced than in a more strongly correlated state. Such a weak correlation occurs for instance in the case where there are many of these topological line defects are trapped inside C in a *stable form*, since this accumulation actually points out towards a highly disordered state.

Back again to the dislocation formulation, the above electric current $J_{\mu, \vec{r}}$ flowing along C represents in the equations *à la* Maxwell of our model the shear stress τ_i^{α} — whose direction i runs along loop C , determining the shear direction α the Burgers vector direction (see Fig. 2.4) —, which is the source for the density of gliding dislocations (and of dislocations which could otherwise glide for their Burgers vector is completely contained in a glide plane). Energy $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}$ depends on the arrangement

of the dislocations crossing the area encircled by C and therefore $W(C)$ characterizes the response of the dislocated crystal to the action of such an external stress on the dislocations inside C . By following an argument similar to that in [52], depending on how the system reacts by holding them trapped within C in a stable form — accumulation what signalizes disorder and a very weak correlation — or, on the contrary, by permitting them to escape or to be screened, $W(C)$ will display a stronger or a weaker decay respectively. It is here where the role of integer factor p^{α} becomes relevant: it determines if the Burgers vector of the able-to-glide dislocations which external stress $J_{\mu, \vec{r}}$ would produce is of the same kind as the Burgers vector which the model itself contains, the latter being determined by index q^{α} in the covariant derivative term

$a^{d-2}|\psi|^2 \sum_{\vec{r}, \alpha} \sum_{\mu} \left[1 - \cos \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu, \vec{r}}^{\alpha} \right) \right]$ coupling the gauge field to the scalar field.

Let us consider this last point in more detail. Correlation function $W(C)$ follows as a special case of making the usual scalar-scalar correlation function $\langle e^{i\varphi_{\vec{r}_1}^{\alpha}} e^{-i\varphi_{\vec{r}_2}^{\alpha}} \rangle$ a gauge invariant correlation function $\left\langle e^{i\varphi_{\vec{r}_1}^{\alpha}} \exp \left[i q^{\alpha} \sum_{C_{1 \rightarrow 2}} A_{\mu, \vec{r}}^{\alpha} \right] e^{-i\varphi_{\vec{r}_2}^{\alpha}} \right\rangle$: the change in $\varphi_{\vec{r}_1}^{\alpha}$ gets compensated by a part of the transformation in gauge field $A_{\mu_1, \vec{r}_1}^{\alpha}$ placed on the lattice bond connecting the site $\vec{r}_1$ with the next site along path $C_{1 \rightarrow 2}$; the other part of the change in gauge field $A_{\mu_1, \vec{r}_1}^{\alpha}$ is compensated by a part of the transformation of the gauge field on the next bond, proceeding in the same until reaching the end site $\vec{r}_2$

$$\begin{aligned}
\varphi_{\vec{r}_1}^{\alpha} &\rightarrow \varphi_{\vec{r}_1}^{\alpha} - \underline{q^{\alpha} \chi_{\vec{r}_1}^{\alpha}} \\
q^{\alpha} A_{\mu_a, \vec{r}_1}^{\alpha} &\rightarrow q^{\alpha} A_{\mu_a, \vec{r}_1}^{\alpha} - \underline{q^{\alpha} \chi_{\vec{r}_1 + \hat{\mu}_a}^{\alpha}} + \underline{q^{\alpha} \chi_{\vec{r}_1}^{\alpha}} \\
q^{\alpha} A_{\mu_b, \vec{r}_1 + \hat{\mu}_a}^{\alpha} &\rightarrow q^{\alpha} A_{\mu_b, \vec{r}_1 + \hat{\mu}_a}^{\alpha} - \underline{q^{\alpha} \chi_{\vec{r}_1 + \hat{\mu}_a + \hat{\mu}_b}^{\alpha}} + \underline{q^{\alpha} \chi_{\vec{r}_1 + \hat{\mu}_a}^{\alpha}} \\
\dots &\dots \\
q^{\alpha} A_{\mu_{\text{end of path } C_{1 \rightarrow 2}}, \vec{r}_2 - \hat{\mu}_{\text{end}}}^{\alpha} &\rightarrow q^{\alpha} A_{\mu_{\text{end}}, \vec{r}_2 - \hat{\mu}_{\text{end}}}^{\alpha} - \underline{q^{\alpha} \chi_{\vec{r}_2}^{\alpha}} + \underline{q^{\alpha} \chi_{\vec{r}_2 - \hat{\mu}_{\text{end}}}^{\alpha}} \\
-\varphi_{\vec{r}_2}^{\alpha} &\rightarrow -\varphi_{\vec{r}_2}^{\alpha} + \underline{q^{\alpha} \chi_{\vec{r}_2}^{\alpha}}
\end{aligned}$$

where it is clear that the integer factor q^{α} multiplying $\sum_{C_{1 \rightarrow 2}} A_{\mu, \vec{r}}^{\alpha}$ gets fixed by the coupling to the scalar field's phase in order to ensure the gauge invariance of the correlation function. Correlation function $W(C)$ is thus equal to $\left\langle e^{i\varphi_{\vec{r}_1}^{\alpha}} \exp \left[i q^{\alpha} \sum_{C_{1 \rightarrow 2}} A_{\mu, \vec{r}}^{\alpha} \right] e^{-i\varphi_{\vec{r}_2}^{\alpha}} \right\rangle$ for the case that start site $\vec{r}_1$ coincides with end site $\vec{r}_2$ (and thus $C_{1 \rightarrow 2}$ becomes a closed loop C). But since in this latter case the scalar field does not appear in the correlation function, a different factor p^{α} can be introduced multiplying the gauge field $A_{\mu, \vec{r}}^{\alpha}$. This allows us to consider separately case $p^{\alpha} = q^{\alpha}$ — where the dislocation's Burgers vector inserted by the external stress $J_{\mu, \vec{r}}$ is of the same type as, and thus can *always* be screened by, the Burgers vector produced dynamically by the system when the scalar field's phase (simultaneously to the covering of a closed path in physical space) draws a complete number of rotations — and case $p^{\alpha} \neq q^{\alpha}$. Advancing results to be discussed later, for the compact model (the only model where q^{α} plays a real role) it will result that the behaviour changes significantly depending on one case or the other. The election of $p^{\alpha} = 1$ is due to the fact that for $q^{\alpha} = p^{\alpha} = 1$ the Burgers vector or vorticity of the line defects is the largest possible [41]: this derives from energy term $a^{d-2}|\psi|^2 \sum_{\vec{r}, \alpha} \sum_{\mu} \left[1 - \cos \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu, \vec{r}}^{\alpha} \right) \right]$ where for the just mentioned configurations,

with one (or more) complete rotation of the scalar field's phase $\varphi_{\vec{r}}^{\alpha}$ after walking along C , $\sum_{\text{loop } C} A_{\mu, \vec{r}}^{\alpha} = \frac{2\pi}{q^{\alpha}}$ must hold in order to keep finite the energy ¹⁸.

¹⁸ There exist an interesting feature: in a dislocation system with $q^{\alpha} = 2$, able therefore to produce topological stable line defects of vorticity (or Burgers vector) equal to one half of those created by the external source (the shear stress in the dislocation model) $J_{\mu, \vec{r}}$ occurring in

Summarizing, correlation function $W(C)$ measures the response of the dislocation system under the effect of an external action which would create dislocations of the largest Burgers vector possible: for $q^\alpha \neq 1$, i.e., when the system itself produces able-to-glide dislocations of smaller Burgers vector, $W(C)$ measures if these latter remain trapped inside C or if they can escape from the region delimited by C . In other words, in such a case $W(C)$ represents the probability of escaping from the area spanned by C for the able-to-glide dislocations corresponding to the system's dynamics.

This interpretation is consistent with the two previous identified cases: in the easy glide state characterized by $W(C) \sim e^{-\text{Perimeter}_C}$, only a slip system (the primary one) has been activated and there is no reaction with other not-parallel glide systems which could lead to formation of sessile dislocations. Since the gliding dislocations produced at the Frank-Read sources after the yield point are only weakly impeded and do not exert a large load on these sources, there is no strong production of gliding dislocations whose number remains low (see Fig. 2.3 on page 36). Thus for a low number of isolated gliding dislocations, the probability of being retained inside C is just proportional to the limiting contour C itself which could link them. On the other hand, in the advanced parabolic hardening where $W(C)$ displays a stronger decay, $W(C) \sim e^{-\mathcal{P}_C \ln \mathcal{P}_C}$, the number of gliding dislocations is much higher and they have condensed such that the gliding dislocation density spreads out unimpeded, and thus the escape probability is also lower than in the previous state where isolated gliding dislocations had a much more ordered glide motion. In fact, a correlation function sensitive to the dislocation arrangement crossing the area delimited by C is expected to display a stronger decay in the state where mobile dislocations condense — marking the loss of rigidity which restricted the extension of the density of gliding dislocations — and thus where dislocations are disordered, mutually weaker correlated.

Now let us study the Higgs model but with a *compact* gauge field: the calculation leading to (2.105) and to (2.106) does not work in this case due to the non-linear terms introduced by the periodic function in the pure gauge field part. But the interpretation of $W(C)$ must remain. As a result of the angular character of the gauge field exclusively, for $T = 0$ (or equivalently, for a non-periodic lattice as discussed in Step 4) most of the ηe^2 -

$$W(C) = \left\langle \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}}^\alpha \right] \right\rangle \quad (p^\alpha = 1), \text{ these latter externally introduced line defects nevertheless cannot be screened (as one could naively expect) by } \textit{two} \text{ of the topological line defects produced dynamically by the system. The reason is, as explained in the long discussion after (2.83), that for the system with } q^\alpha = 2 \text{ the combination of two stable line defects is no longer topologically stable and, using the language of the dislocation formulation, they will eventually become attached to a sessile segment. And the same holds for other } q^\alpha \neq 1. \text{ This is the actual reason of choosing } p^\alpha = 1: \text{ to avoid that the line defects which } J_{\mu, \vec{r}} \text{ would introduce could not become instantaneously screened by being surrounded by stable line defects which the system would produce dynamically due to the coupling to the (angular) scalar field's phase, in whose case one would not observe any effect of these introduced defects. Of course, it remains the situation where the system's dynamics is determined also by a } q^\alpha = p^\alpha = 1, \text{ but in such a case the system cannot produce any topologically stable line defect from the very beginning, and thus the failing in the screening is also satisfied here. All these comments, which just as the role of } q^\alpha \text{ only work for the compact model, are nothing else a the consequence of such a compact nature in the gauge field; or more properly, of the representation chosen to break partially down to } U(1) \text{ the non-Abelian group which described initially the actual symmetry of the system.}$$

less cannot be screened (as one could naively expect) by *two* of the topological line defects produced dynamically by the system. The reason is, as explained in the long discussion after (2.83), that for the system with $q^\alpha = 2$ the combination of two stable line defects is no longer topologically stable and, using the language of the dislocation formulation, they will eventually become attached to a sessile segment. And the same holds for other $q^\alpha \neq 1$. This is the actual reason of choosing $p^\alpha = 1$: to avoid that the line defects which $J_{\mu, \vec{r}}$ would introduce could not become instantaneously screened by being surrounded by stable line defects which the system would produce dynamically due to the coupling to the (angular) scalar field's phase, in whose case one would not observe any effect of these introduced defects. Of course, it remains the situation where the system's dynamics is determined also by a $q^\alpha = p^\alpha = 1$, but in such a case the system cannot produce any topologically stable line defect from the very beginning, and thus the failing in the screening is also satisfied here. All these comments, which just as the role of q^α only work for the compact model, are nothing else a the consequence of such a compact nature in the gauge field; or more properly, of the representation chosen to break partially down to $U(1)$ the non-Abelian group which described initially the actual symmetry of the system.

range where the phase characterized by $W(C) \sim e^{-\mathcal{P}_C \ln \mathcal{P}_C}$ appeared, is occupied in the case of a large $q^\alpha \neq 1$ by a new phase described by a still stronger decay $W(C) \sim e^{-\mathcal{A}_C}$ ¹⁹. This is actually the behaviour we should expect in the rapid hardening stage (Stage II) where gliding dislocations (and their currents) are *confined* inside the “discs” delimited by Lomer-Cottrell segments. And since the number of (temporary) mobile dislocations in this stage is high (see again Fig. 2.3 on page 36), the number of gliding dislocations

¹⁹ This strongest area-decay in $W(C)$ is the result of the so-called “high temperature” power expansion method. When evaluating $W(C)$ in the compact gauge model, one expands term $e^{-\frac{1}{ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha)]}$ and term $e^{-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\alpha,\mu} [1 - \cos(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha)]}$ in powers of $\frac{1}{2ae^2\eta} \left(e^{-i \underbrace{(A_{\mu,\vec{r}}^\alpha + A_{\nu,\vec{r}+\hat{\mu}}^\alpha + A_{-\mu,\vec{r}+\hat{\mu}+\hat{\nu}}^\alpha + A_{-\nu,\vec{r}+\hat{\nu}}^\alpha)}_{\text{unit plaquette}}} + \text{c.c.} \right)$ and of $\frac{a|\psi|^2}{2\eta} \left(e^{i \underbrace{(\varphi_{\vec{r}+\hat{\mu}}^\alpha - \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha)}_{\text{bond}}} + \text{c.c.} \right)$

respectively. In the compact gauge model, relations $\int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}^\alpha}{2\pi} e^{iA_{\mu,\vec{r}}^\alpha} = 0$ and

$\int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}^\alpha}{2\pi} e^{iA_{\mu,\vec{r}}^\alpha} e^{-iA_{\mu,\vec{r}}^\alpha} = 1$ — the compact interval where the $U(1)$ gauge field takes on its values is due to its angular nature — make that each $e^{iA_{\mu,\vec{r}}^\alpha}$ placed on each bond along the closed path C of $W(C) = \prod_{\text{bond } (\vec{r},\mu) \in \text{loop } C} e^{iA_{\mu,\vec{r}}^\alpha}$ must be just covered by one $e^{-iA_{\mu,\vec{r}}^\alpha}$ arising from the above mentioned

expansion. Since the expansion of $e^{-\frac{1}{ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha)]}$ is in plaquette-powers, the form to cover once all bonds along C is by tiling the interior of C with unit plaquettes — plus other forms coming from “bumps” of unit-plaquettes in such interior, but whose contribution is much weaker in the range of small $1/\eta e^2$ since they are proportional to a higher power of $\frac{1}{2ae^2\eta}$ — and hence the leading

contribution produced by this tiling is equal to $\left(\frac{1}{2ae^2\eta} \right)^{\mathcal{A}_C} = e^{-\mathcal{A}_C \ln(2ae^2\eta)}$. On the other hand,

the same covering of all C 's bonds by means of the bond-powers $\frac{a|\psi|^2}{2\eta} e^{i(\varphi_{\vec{r}+\hat{\mu}}^\alpha - \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha)}$ arising from

the expansion of $e^{-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\alpha,\mu} [1 - \cos(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha)]}$, something which would produce as contribution to

$W(C)$ the term $\left(\frac{a|\psi|^2}{2\eta} \right)^{\mathcal{P}_C} = e^{-\mathcal{P}_C \ln(2\eta/a|\psi|^2)}$, cannot be achieved in the case of $q^\alpha \neq 1$ because such

bond-powers contain $e^{-iq^\alpha A_{\mu,\vec{r}}^\alpha}$ and not simply the $e^{-iA_{\mu,\vec{r}}^\alpha}$ required — although the resting superfluous $(q^\alpha - 1)$ times the same bond has been covered by $\frac{a|\psi|^2}{2\eta} e^{i(\varphi_{\vec{r}+\hat{\mu}}^\alpha - \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha)}$ would be compensated

by $(q^\alpha - 1)$ times the $e^{iA_{\mu,\vec{r}}^\alpha}$ from the plaquette-expansion of $e^{-\frac{1}{ae^2\eta} \sum_{\vec{r},\alpha} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha)]}$, it would result that now the other plaquette's bonds attached to the bonds of path C had been left uncovered, being the sole form to cover them and to yield a non-zero contribution that of tiling again the interior of C , with its corresponding area decay contribution. Summarizing, for $q^\alpha \neq 1$ and in the range of small $1/\eta e^2$, the behaviour of $W(C)$ is given by $W(C) = e^{-\mathcal{A}_C \ln(2ae^2\eta)}$. Let us stress once more that the just obtained result does *not depend* on the form of any propagator $D_{\vec{r},\vec{r}'}$ but only on the compact character of the $U(1)$ gauge field $A_{\mu,\vec{r}}$.

Further, since the compact model at $|\psi|^2 = 0$ shows no phase transition in 3 dimensions (see the proof due to Polyakov and discussed in Appendix C from equation (C.297) on page 401 to equation (C.314)), the strongest decay in $W(C)$ extends also to the range of large $1/\eta e^2$ as long as $|\psi|^2/\eta$ remains small.

All the above discussion could be generalized for the case that integer factor p^α in (2.102) is different from 1: $W(C) = \prod_{\text{bond } (\vec{r},\mu) \in \text{loop } C} e^{ip^\alpha A_{\mu,\vec{r}}^\alpha}$; the conclusions are analogous by changing the cases $q^\alpha \neq 1$ and $q^\alpha = 1$ with, respectively, $q^\alpha \neq p^\alpha$ and $q^\alpha = p^\alpha$.

retained inside C and thus the energy due to such a configuration becomes proportional to the area enclosed by C . If this identification between $W(C) \sim e^{-\mathcal{A}_C}$ and the rapid hardening is correct, the gauge field must result to be short range because of the spatial confinement, not in the same manner as in Stage I — where the short range was a consequence of the high rigidity of the system, described by the length of the scalar field coupled to the gauge field — but this time caused by the compact character of the gauge field.

But this short range is what actually occurs for the phase with $W(C) \sim e^{-\mathcal{A}_C}$, and its origin is also the angular nature of the gauge field in the compact Higgs model. Let us consider firstly the result about the short range: the range of a field is determined by the connected correlation function of two of such fields separated a large distance $|\vec{r}_2 - \vec{r}_1|$, being of finite range if the connected correlation function shows an exponential decay with the distance, or of infinite range if the dependence with the distance is weaker (a power-law). In the case of the gauge field, such a connected correlation must be gauge invariant (since otherwise it would become zero identically) and is given by [41]

$$\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) = \langle e^{iF_{\mu\nu, \vec{r}_1}} e^{-iF_{\mu\nu, \vec{r}_2}} \rangle - \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle \langle e^{iF_{\mu\nu, \vec{r}_2}} \rangle, \quad (2.107)$$

whose gauge invariance is ensured by the fact that $F_{\mu\nu, \vec{r}}$ is gauge invariant (and not only covariant) for a $U(1)$ gauge field. The calculation of such correlation function is carried out in Appendix C after equation (C.476) on page 466, obtaining the following parallel behaviours between $W(C)$ and $\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|)$

$$W(C) \sim e^{-\mathcal{P}_C} \longleftrightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \frac{a\eta e^2 \sqrt{a^2 e^2 |\psi|^2}}{|\vec{r}_2 - \vec{r}_1|^2} \exp\left[-\sqrt{a^2 e^2 |\psi|^2} |\vec{r}_2 - \vec{r}_1|\right],$$

$$W(C) \sim e^{-\mathcal{P}_C \ln \mathcal{P}_C} \longleftrightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \frac{a\eta e^2}{|\vec{r}_2 - \vec{r}_1|^3}, \quad (2.108)$$

$$W(C) \sim e^{-\mathcal{A}_C} \longleftrightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \underline{\exp[-4 \ln(2a\eta e^2) |\vec{r}_2 - \vec{r}_1|]}.$$

As already identified, Stage I and $W(C) \sim e^{-\mathcal{P}_C}$ are characterized by a short range gauge field, and advanced Stage III with $W(C) \sim e^{-\mathcal{P}_C \ln \mathcal{P}_C}$ by a long range gauge field. And as expected, also the Stage II and $W(C) \sim e^{-\mathcal{A}_C}$ correspond to a gauge field which cannot spread out over the system (as the underlined exponential points out).

Moreover, this allows us to gather together ideas about the compactness of the $U(1)$ gauge field which lie scattered in the footnotes on pages 83, 76 and on page 100. As discussed in equation (2.95), the only form which actually makes sense of calculating the (gauge invariant) partition function is that by working not directly with all possible configurations but with only one representative for each class formed by the configurations related to one other by allowed gauge transformations. It is within this framework where the origin for the strongest decay $W(C) \sim e^{-\mathcal{A}_C}$ becomes clear (in 4 steps):

1. For a gauge field of angular nature (and where “jumps” by a multiple of 2π are allowed) it is possible to define a gauge spin $e^{iA_{\mu, \vec{r}}}$; if the model’s parameters are

chosen such that the dominating configurations in the partition function are *represented* by the field arrangement where all these gauge spins are aligned — where, stressing it once more, this configuration is the representative for all configurations gained from it through the application of all possible non-singular gauge transformations — one speaks of “breakdown” of the gauge symmetry, just in the same manner as when working with a system of usual (classical) spins described by a energy of coupled nearest neighbours invariant under the simultaneous inversions of all spins, a magnetized state defined by the parallel alignment of all spins along one of their possible orientations is also denoted as that of broken symmetry — since this state does not display a symmetry, that of the global inversion of all spins, which the energy describing the system does have. Such an aligned state is also called the ordered state.

2. If, on the contrary, the coupling parameters of the model permit strong fluctuations of the spins such that field arrangements, which connect a spatial region where the spins are aligned along one orientation with another region where the alignment orientation is different, are not suppressed, the system transits towards a state characterized by the absence of alignment, either for the usual spins (where one speaks of a non-magnetized state) or for the representative configurations of gauge field spins. This case is called the restoring of the broken symmetry or the disordered state, and as a consequence, the correlation functions defined for such spins display a decay with the distance stronger than in the case of symmetry breaking. Such a restoring occurs when the coupling strength for the spin-spin interaction (what in the case of the gauge field is equal to $1/e^2$) compared to scale η is small.
3. For a compact gauge field the number of the just discussed possible orientations for the gauge spin $e^{iA_{\mu,\vec{r}}}$ gets determined by the integer q , and $q = 1$ means that only one orientation is available and that the symmetry remains always broken without reaching the correlation function the strongest decay characteristic of the disordered state; of course, for a non-compact gauge field all this discussion has no sense since the absence of jumps (or of the angular character) in $A_{\mu,\vec{r}}$ impedes the definition of the gauge spins.
4. On the other hand, it is the “breakdown” of the model’s gauge symmetry — where the quotation marks are to signalize that this holds as long as working with one representative for each class of gauge related configurations — what is equivalent to the existence of isolated vortices (or of isolated able-to-glide dislocations in the dislocation model), as studied in detail in the last part of Appendix B (see the discussion previous to result (B.217)). And this property is that characteristic of the Shubnikov state/Stage I, corresponding to a behaviour $W(C) \sim e^{-\mathcal{P}C}$. Hence the “restoring” of the gauge symmetry (only possible if $q^\alpha \neq 1$ and only possible for a compact gauge field) will lead to a stronger decay — and this *independently* of the form of the propagator (since the latter behaviour is exclusively a consequence of the compact nature of the gauge field) and for any lattice dimensionality — i.e., this stronger decay can only correspond to $W(C) \sim e^{-\mathcal{A}C}$ (what actually results

when the correlation function is calculated in the range of small $1/\eta e^2$, as explained in the footnote on page 100).

The main element in all this argument has been the angular character of the $U(1)$ gauge field, character also responsible for the existence of sessile dislocation segments (acting as magnetic monopoles would do in the electromagnetism), which are the actual agents of Stage II. This closes the circle on the identification of rapid hardening and $W(C) \sim e^{-\mathcal{A}_C}$: behaviour of correlation function $W(C)$ in the state disordered with respect to the gauge field — state without *isolated* able-to-glide dislocations which were characteristic of the symmetry breakdown — exclusive for a compact gauge field with $q^\alpha \neq 1$ — being this the factor determining in the symmetry breakdown phase the existence stable gliding dislocations as well as of sessile segments to which the reaction of number equal to q^α of able-to-glide dislocations becomes attached —; together with the fact that in Stage II the many mobile dislocations become characteristically trapped, being the energy of the resulting configurations proportional to the area where dislocations are being retained and thus $W(C)$, as escape probability from the region delimited by C , must behave in this plastic stage as $W(C) \sim e^{-\mathcal{A}_C}$. Moreover, the symmetry restoring characterized by this latter decay in $W(C)$ occurs for weak $\frac{1}{\eta e^2}$, or equivalently and by considering relation $\frac{1}{\eta e^2} \sim \frac{\lambda_{\text{scale}}}{\lambda}$ (see Table 2.5), this phase occurs for a small spacing λ between dislocations, property also characteristic for the dense dislocation networks appearing in the rapid hardening stage (Stage II).

It only remains to discuss which hardening behaviour corresponds to the a bit special compact case of $q^\alpha = 1$. As already explained, in such a case all able-to-glide dislocations are not topologically stable: any long dislocation line will always break into different pieces, each one finally “absorbed” by the start points of the sessile segments which work as magnetic monopoles would do in an electromagnetic context. These latter start/end points of sessile segments are the only topologically stable (point) objects of the model which, as we already know from the footnote on page 76, corresponds to a $U(1)$ symmetry remaining from the partial breakdown of a symmetry group of topology equivalent to a geometric figure on which it is impossible any stable wrapping of a rubber band. Since for $q^\alpha = 1$ the gauge $U(1)$ symmetry remains (when referred to the representative configurations) broken from the very beginning, the strongest decay in $W(C)$ is ruled out — except for the special limiting case at $|\psi|^2 = 0$ where fluctuations in the gauge field spins $e^{iA_{\mu,\vec{r}}}$ are strong enough to produce disorder —; further, if there are no stable able-to-glide dislocations, then they cannot be retained confined inside a spatial region. In agreement with this qualitative argument, the power expansion sketched in the footnote on page 100 yields a perimeter decay for $W(C)$ since for $q^\alpha = 1$ it does become possible to cover all bonds along C simply and completely with bond-powers instead of the plaquette-powers tiling the area inside C . And the same power expansion gives also $\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim e^{-|\vec{r}_2 - \vec{r}_1|}$ — what it is also to be expected since the just mentioned symmetry breaking is always accompanied by a short range gauge field.

However, physically in a dislocated single crystal at very low temperatures, this case of $q^\alpha = 1$ does not take place for the absence of long topological stable line defects would describe a crystal without stable mobile dislocations, without a real Stage I nor a real Stage II (since there is no confinement of mobile dislocations): only with stable sessile dislocations. And this would imply that the system has no well defined slip system, not even only one, the primary one.

Note. The complementary correlation function $H(|\vec{r}_b - \vec{r}_a|)$ in the 3-dimensional Higgs model.

Another further argument justifying the characterization of Stage II by means of $W(C) \sim e^{-\mathcal{A}_C}$ runs as follows: when the angular nature of the $U(1)$ gauge field is manifested the consequence is $W(C) \sim e^{-\mathcal{A}_C}$ as well as the existence of points where sessile segments ends or starts, points acting therefore as sinks or sources for the able-to-glide dislocation lines. Let us therefore calculate the free energy increase ΔE in the system when two of such sink/sources are introduced by hand, one acting as sink and the other acting as source. In the context of the superconductor, this would correspond to the free energy increase due to the introduction by hand of two opposite charged magnetic monopoles. From the relation between the free energy and the partition function, $F = -\eta \ln \mathcal{Z}$, the just mentioned free energy increase is obtained by calculating the quotient to the system's partition function of the partition function where following substitution

$$F_{\mu\nu,\vec{r}}^\alpha = \Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha \rightarrow F_{\mu\nu,\vec{r}}^\alpha + (F_{\mu\nu,\vec{r}}^\alpha)_{\text{monop.}} = \Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha - \frac{2\pi}{q^\alpha} \sum_\lambda \epsilon_{\mu\nu\lambda} L_{\lambda,\vec{r}-\hat{\lambda}}, \quad (2.109)$$

has been carried out ($^\alpha$ fixed)

$$e^{-\Delta E/\eta} \equiv H(|\vec{r}_b - \vec{r}_a|) = \frac{\mathcal{Z} \left[\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha - 2\pi/q^\alpha \sum_\lambda \epsilon_{\mu\nu\lambda} L_{\lambda,\vec{r}-\hat{\lambda}} \right]}{\mathcal{Z} \left[\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha \right]}, \quad (2.110)$$

with the numerator equal to

$$\sum_{\text{all configur.}} \exp \left[-\frac{1}{a^{4-d}\eta e^2} \sum_{\vec{r},\alpha} \sum_{\mu < \nu} \left[1 - \cos \left(\Delta_\mu A_{\nu,\vec{r}}^\alpha - \Delta_\nu A_{\mu,\vec{r}}^\alpha - \frac{2\pi}{q^\alpha} \sum_\lambda \epsilon_{\mu\nu\lambda} L_{\lambda,\vec{r}-\hat{\lambda}} \right) + \dots \right] \right].$$

Integer vector field $L_{\lambda,\vec{r}}$ is defined, in a similar way as (2.100), as the unit tangent vector to some path connecting the two opposite charged sinks — points where a gliding dislocation become sessile — placed at sites $\vec{r}_a$ and $\vec{r}_b$: field $L_{\lambda,\vec{r}}$ satisfies therefore equation

$$\sum_\lambda \Delta_\lambda L_{\lambda,\vec{r}-\hat{\lambda}} = \delta_{\vec{r},\vec{r}_a} - \delta_{\vec{r},\vec{r}_b}. \quad (2.111)$$

The added term in substitution (2.109) yields

$$\frac{1}{2} \sum_{\rho,\mu,\nu} \epsilon_{\rho\mu\nu} \Delta_\rho (F_{\mu\nu,\vec{r}}^\alpha)_{\text{monop. pair}} = -\frac{2\pi}{q^\alpha} (\delta_{\vec{r},\vec{r}_a} - \delta_{\vec{r},\vec{r}_b}), \quad (2.112)$$

or expressed in the electromagnetic language, $(F_{\mu\nu,\vec{r}}^\alpha)_{\text{monop. pair}}$ corresponds to the magnetic field created by two opposite charged monopoles. Due to the term coupling the scalar to the gauge field, $a^{d-2}|\psi|^2 \sum_{\vec{r},\alpha} \sum_\mu \left[1 - \cos \left(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu,\vec{r}}^\alpha \right) \right]$, a complete rotation of the scalar field's phase $\varphi_{\vec{r}}^\alpha$ after having covered a closed path in physical space around a topological line defect leads to the fact that the vorticity carried by such defect is equal to $\frac{2\pi}{q^\alpha}$, which is also the vorticity absorbed by the sinks (or

start points of sessile segments): this is the reason for the factor $\frac{2\pi}{q^\alpha}$ appearing in (2.109). Further, it becomes clear that for $q^\alpha = 1$ and since $L_{\lambda, \vec{r}}$ is an integer field, following relation holds

$$H(|\vec{r}_b - \vec{r}_a|) \Big|_{q^\alpha=1} = 1, \quad (2.113)$$

which is a result of the periodic cosine function in the pure gauge field part of the compact model.

This correlation function $H(|\vec{r}_b - \vec{r}_a|)$ is studied in Appendix C (see equation (C.474) for the non-compact Higgs model) and in Appendix D (see scheme on page 610 for a model whose limit is the compact Higgs model) resulting that the behaviours of $W(C)$ and of $H(|\vec{r}_b - \vec{r}_a|)$, in the compact as well as in the non-compact Higgs model, are complementary: the strongest decay with the distance in $H(|\vec{r}_b - \vec{r}_a|)$ accompanies to the weakest decay in $W(C)$ like the perimeter of C , and vice versa, the area decay in $W(C)$ corresponds to no-decay with the distance in $H(|\vec{r}_b - \vec{r}_a|)$. This is actually not so surprising: $W(C)$ displays its strongest decay in the compact model for the phase of symmetry restoring, or equivalently, for the phase of disorder in the (representative) gauge field spin; and since the correlation between two of the fields which just cause such a disorder, the sessile segments and their starting points, is measured by $H(|\vec{r}_b - \vec{r}_a|)$ one should already expect a very weak decay (actually none) in the latter for such a disordered phase. And on the contrary, a strong decay with the distance in the “disorder correlation function” $H(|\vec{r}_b - \vec{r}_a|)$, indicating a complete absence of disorder, points out towards a weak decay in the “order correlation function” $W(C)$, indicating order in the spin gauge field

$$\begin{aligned} W(C) \sim e^{-\mathcal{A}C} &\implies H(|\vec{r}_b - \vec{r}_a|) \sim \text{constant} \\ H(|\vec{r}_b - \vec{r}_a|) \sim e^{-|\vec{r}_b - \vec{r}_a|} &\implies W(C) \sim e^{-\mathcal{P}C}. \end{aligned} \quad (2.114)$$

Of course, the strongest decay in *both*, mutually complementary, correlation functions $H(|\vec{r}_b - \vec{r}_a|)$ and $W(C)$ cannot occur: this would be a signal of complete absence of disorder and simultaneously, a signal of complete absence of order, what is contradictory. On the contrary, a weak decay in both correlation functions is not ruled out, since this would correspond to an intermediate phase partially ordered; it is only the strongest decay in one correlation function what implies the weak decay in the complementary one.

On the other hand, result (2.114) confirms the identification $\begin{cases} W(C) \sim e^{-\mathcal{P}C} \longleftrightarrow \text{Stage I} \\ W(C) \sim e^{-\mathcal{A}C} \longleftrightarrow \text{Stage II} \end{cases}$ for very low temperatures, since in the phase where isolated able-to-glide dislocations exist (Stage I), the two start/end points of sessile segments are joined by a mobile gliding segment, whose energy increases proportional to the length of the mobile segment, i.e., proportional to the separation between the two points: $H(|\vec{r}_b - \vec{r}_a|) \sim e^{-\Delta E} \sim e^{-|\vec{r}_b - \vec{r}_a|}$. And in the phase where such sinks for gliding dislocation lines are dense, strongly present and uncorrelated, the energy of two opposite charged sinks separated by a long distance actually remains constant due to the just mentioned uncorrelation; but this dislocation arrangement is characteristic for Stage II.

And finally, the phase corresponding to $W(C) \sim e^{-\mathcal{P}C \ln \mathcal{P}C}$ and determined by a gliding dislocation density which spread out over the whole lattice, being the interaction carried by the long range Coulomb propagator $D_{\vec{r}, \vec{r}'} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}$, will be also characterized by a free energy increase ΔE where the interaction among the points acting as sinks for the mobile dislocation lines is also defined by this same Coulomb propagator: $\Delta E \sim \frac{1}{|\vec{r}_b - \vec{r}_a|}$ and $H(|\vec{r}_b - \vec{r}_a|) = e^{-\Delta E/\eta} \sim e^{-1/|\vec{r}_b - \vec{r}_a|}$.

Correlation function $H(|\vec{r}_b - \vec{r}_a|)$ is a modification, for a case of a 3-dimensional space, of the so-called 't-Hooft loop correlation.

All this concludes the construction of a dislocation model able to describe the different plastic behaviours in a crystal containing mobile dislocations. This construction has been carried out by following the leading idea of the analogy between the perfect

Phase	range	correlation functions
① : Stage I	$\frac{1}{\eta e^2}$ large, $\frac{ \psi ^2}{\eta}$ large	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C} \\ \Gamma_{\text{gauge}} & \sim \frac{e^{-\sqrt{a^2 e^2 \psi ^2} \vec{r}_2 - \vec{r}_1 }}{ \vec{r}_2 - \vec{r}_1 ^2} \\ H(\vec{r}_a, \vec{r}_b) & \sim e^{- \vec{r}_b - \vec{r}_a } \end{cases}$
③ : Stage II (compact, $q^\alpha \neq 1$)	$\frac{1}{\eta e^2}$ small	$\begin{cases} W(C) & \sim e^{-\ln(2a\eta e^2) \mathcal{A}_C} \\ \Gamma_{\text{gauge}} & \sim e^{-4 \ln(2a\eta e^2) \vec{r}_2 - \vec{r}_1 } \\ H(\vec{r}_a, \vec{r}_b) & \sim \text{const} \end{cases}$
④ : loss of rigidity advanced Stage III (non – compact)	$\frac{1}{\eta e^2}$ small	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C \ln \mathcal{P}_C} \\ \Gamma_{\text{gauge}} & \sim \frac{1}{ \vec{r}_2 - \vec{r}_1 ^3} \\ H(\vec{r}_a, \vec{r}_b) & \sim e^{-1/ \vec{r}_b - \vec{r}_a } \end{cases}$
absence of primary slip (compact, $q^\alpha = 1$)	$\frac{1}{\eta e^2}$ small $\frac{1}{\eta e^2}$ large, $\frac{ \psi ^2}{\eta}$ large	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C} \\ \Gamma_{\text{gauge}} & \sim e^{- \vec{r}_2 - \vec{r}_1 } \\ H(\vec{r}_a, \vec{r}_b) & = 1 \end{cases}$

Table 2.6: Behaviour of the correlation functions for the 3-dimensional lattice dislocation model (analogue to the 3-dim. lattice Higgs model) at very low temperatures, showing the correspondence to the different hardening stages.

diamagnetism in a superconductor and the rigidity against the existence of gliding dislocations in a crystal. Analogy implemented later by the possibility of a compact $U(1)$ gauge field, which leads to a new phase corresponding to the rapid hardening stage (or Stage II) and characterized by the strong suppression for the escape probability of the mobile dislocations from a certain region where the former remain confined, effect due to such a compact gauge field, which is also the origin for the sessile segments within the framework of the formulated dislocation model. Table 2.6 will be very useful to identificate the different model's hardening stages by considering the behaviours displayed by the three different correlation function studied above. Such behaviours have been obtained always assuming that Stage II is not destroyed, maintaining the latter its characteristic confinement (for a perfect single crystal) of the current of mobile dislocations within a tube stretching along near the whole height of the crystal under external stress: in other words, assuming no lattice periodicity along such height, or assuming a low temperature T .

Chapter 3

Applications of the dislocation model: the different plastic behaviours in the hardening curve

The final scope of this work is to describe the hardening curve as a phase diagram of the lattice dislocation model developed in the previous Chapter, obtaining the phases characterized by the behaviours in the correlation functions of Table 2.6 (page 106). However, before starting in this direction, there are two particular questions to be dealt with, one for the easy glide stage and one for the rapid hardening stage, which strengthen on a more quantitative basis the analogy leading to the dislocation model — especially the similarity between the rapid hardening (Stage II) and the confinement of the electric field displayed by the compact Higgs model.

3.1 Stage I in a crystal and Shubnikov state in a superconductor

After having set the analogy between, on the one hand, the Shubnikov state of the Abrikosov model and, on the other hand, the easy glide stage of a dislocated crystal, let us translate the main equations of the first model (explained in Appendix A) into the language of the dislocation model.

For the reasons argued in Step 2 on page 69 the lattice Higgs model is the appropriate framework for the treatment of phase transitions, although nevertheless the transition analogous to that in a superconductor from the Meissner to the Shubnikov state becomes clearer in the continuum effective formulation, formulation which for a superconductor is the Abrikosov model and which will be considered now.

Characteristic for the superconductivity is the external magnetic field H_{c1} at which a type II superconductor transits from the Meissner to Shubnikov phase, and the external magnetic field H_c for the direct destruction of the Meissner state towards the normal conducting (or Coulomb) state occurring in a superconductor of type I. These two magnetic fields are given by, respectively, (A.31) (page 180 in Appendix A) and (A.5) (page

$$H_{c1} = \frac{\epsilon_{1 \text{ vortex}}}{\Phi_0} \approx H_c \frac{\xi}{\lambda}, \quad (3.1)$$

being $\epsilon_{1 \text{ vortex}}$ the energy per unit length of one vortex and Φ_0 the magnetic flux carried by such vortex (also called its vorticity); ξ is the length of the order of the vortex core's radius, and λ the penetration length of the short range magnetic field inside the superconductor. The magnetic field for which the Shubnikov phase is destroyed and the system transits to the Coulomb phase is given by (A.49) (page 187)

$$H_{c2} = H_c \frac{\lambda}{\xi}. \quad (3.2)$$

For the further translation of these results into the dislocation model, it is more useful to speak about the electric current which, applied to the superconductor, produces magnetic fields H_{c1} and H_c . These currents are easily estimated by considering that the length range where the magnetic field can spread out within the superconductor is, for the pure Meissner state (which will eventually transit directly to the normal conducting state) as well as for the Shubnikov state, equal to λ

$$|\vec{j}_{el}| = |\text{curl } \vec{H}| \sim \frac{|\vec{H}|}{\lambda}. \quad (3.3)$$

A different case occurs near the transition point from the Shubnikov to the Coulomb phase where, due to the fact that vortices are very near to each other, the spreading of the magnetic field is no longer controlled by penetration length λ_0 but it is much longer. Hence the electric current corresponding to H_{c1} and to H_c are given by

$$\begin{aligned} |\vec{j}_{el}|_{\text{for } H_{c1}} &\sim \frac{4\pi \epsilon_{1 \text{ vortex}}}{\Phi_0 \lambda}, \\ |\vec{j}_{el}|_{\text{for } H_c} &\sim |\vec{j}_{el}|_{\text{for } H_{c1}} \frac{\lambda}{\xi}. \end{aligned} \quad (3.4)$$

The application of the above results to the dislocation model defined in the previous chapter — with the identification of the electric current as the resolved shear stress, and of the magnetic field as the density of able-to-glide dislocations — produces that the external stress τ_0 applied on the crystal at the yield point, i.e., at the transition from the elastic stage (analogue to the Meissner state) to the easy glide stage (analogue to the Shubnikov state) is given by $|\vec{j}_{el}|_{\text{for } H_{c1}}$

$$\left\{ \begin{array}{ll} \lambda = \text{range of magnetic induct.} \equiv \text{spacing among initially grown-in disloc.} = \lambda_0, \\ \epsilon_1 = \text{vortex energy/length} \quad \equiv \text{gliding dislocation energy/length} \approx \frac{Gb^2}{4\pi}, \\ \Phi_0 = \text{flux carried by vortex} \quad \equiv \text{Burgers vector} = b, \end{array} \right.$$

$$\tau_0 \sim G \frac{b}{\lambda_0}, \quad (3.5)$$

which is the correct result of the critical stress at the yield point for the appearance of the first isolated gliding dislocations at the Frank-Read sources (see equation (1.13)), just as the transition to the Shubnikov phase from the Meissner state is characterized by the appearance of isolated vortices.

Of course, the above results for the superconductor applies when this is of type II, or equivalently, when the penetration length of the magnetic field inside the superconductor is much larger than the correlation length of the scalar field, correlation length which lies on the order of the radius of the vortex core. For the dislocation model such a condition is automatically satisfied since the spacing λ_0 (analogue to λ in a superconductor) among the dislocations initially contained in the crystal is larger than the lattice spacing a , the latter also the typical size of the dislocation core. Continuing with the second result in (3.4), the direct loss of rigidity corresponding to the direct transition from the elastic stage to the well developed parabolic stage (or Stage III) — analogue to the transition in a type I superconductor from the Meissner state to the Coulomb state, where the long range magnetic field becomes able to spread over the whole material — would imply that all dislocation whose Burgers vector lies completely contained in the primary plane would become mobile *simultaneously*, leading to the spreading out over the whole crystal of the gliding dislocation density (which is the characteristic of the developed Stage III) without having passed by a intermediate state where only some isolated mobile dislocations are allowed to glide. The resolved shear stress responsible for this particular transition will be given by the same equation as the electric current $|\vec{j}_{el}|_{\text{for } H_c}$ necessary in a type I superconductor for the destruction of the superconductivity

$$\tau_{th} \sim \tau_0 \frac{\lambda_0}{b} \sim G. \quad (3.6)$$

Not very surprising, the above shear stress lies in the order of the theoretical shear stress necessary to produce the glide of simultaneously the whole slip plane. The fact that $\tau_{th} \sim G$ is not the real critical shear stress at the yield point is therefore related to the just mentioned fact that the formulated dislocation model is analogue to a superconductor of type II: a crystal whose describing model would correspond to a superconductor of type I would require a λ_0 of the order of the lattice spacing, a dislocation density much higher than that which any crystal can bear.

Further one can exploit other results ¹ of the Abrikosov model of superconductivity and apply them to the description of the Stage I. As an example, one can obtain the

¹The electric current (and thus the analogous resolved shear stress in the dislocation model) corresponding to critical magnetic field H_{c2} for the transition from the Shubnikov to the Coulomb phase cannot be estimated as in the above (3.4). The reason is that the condensation of vortices responsible for such a transition causes the magnetic field to spread out over near the whole system just previous to the transition — since the vortex cores, where such magnetic field is confined, are very near to one other — and thus $|\vec{j}_{el}|_{\text{for } H_{c2}} \sim \frac{H_{c2}}{\Lambda} \stackrel{\text{eq.(3.2)}}{=} \frac{1}{\Lambda} \frac{H_c}{\lambda} \frac{\lambda^2}{\xi}$, being Λ the length dimension of the system. Or equivalently, $|\vec{j}_{el}|_{\text{for } H_{c2}} \sim \frac{1}{\Lambda} |\vec{j}_{el}|_{\text{for } H_c} \frac{\lambda^2}{\xi}$ which, in the analogy between the superconductot and the

evolution of the hardening rate Θ_1 during the first moments of the easy glide stage. This is obtained as follows: since at the start of Stage I the production of dislocations is weak and therefore the spacing among dislocations is still given by λ_0 , the relation on the one hand between the resolved shear stress $\boldsymbol{\tau}$ and the density $\mu_0 \mathbf{H}$ of able-to-glide dislocations on the primary slip plane (see Table 2.1), and on the other hand, between the plastic strain $\varepsilon_{\text{flow}}$ due to the dislocation glide and the density $\mathbf{B}$ of dislocations which are actually gliding, is given by

$$\begin{cases} \Delta \boldsymbol{\tau} \sim \frac{\Delta \mathbf{H}}{\lambda_0} = \frac{\Delta(\mu_0 \mathbf{H})}{\mu_0 \lambda_0} = \frac{\Delta \boldsymbol{\alpha}_{\text{primary}}}{\mu_0 \lambda_0}, \\ \Delta \varepsilon_{\text{flow}} \sim \frac{\Delta \mathbf{B}}{\lambda_0} = \frac{\Delta \boldsymbol{\alpha}_{\text{gliding}}}{\lambda_0}, \end{cases}$$

and thus the hardening rate Θ_1 becomes equal to

$$\Theta_1 = \frac{d\boldsymbol{\tau}}{d\varepsilon_{\text{flow}}} \sim \underbrace{\left(\frac{1}{\mu_0 \lambda_0^2} \right)}_G \frac{1}{\left(\frac{d\boldsymbol{\alpha}_{\text{glid}}}{d\boldsymbol{\alpha}_{\text{prim}}} \right)}. \quad (3.7)$$

In other words, the difference between Θ_1 and the elastic shear modulus G is determined by the effective response of the system on how many of the dislocation which could glide on the primary slip system (since their Burgers vector are completely contained in such slip system) are actually allowed to glide: this response is given by $\left(\frac{d\boldsymbol{\alpha}_{\text{glid}}}{d\boldsymbol{\alpha}_{\text{prim}}} \right)^{-1}$, which is the analogue to the inverse of the (differential) diamagnetic constant $\frac{dB}{d(\mu_0 H)}$ (or to $\frac{dH}{dB}$ in CGS units) in a superconductor. In order to translate Appendix A's relation (A.45) on page 186 (already in CGS units) into the corresponding dislocation model's result one should note that $\frac{\mu_0 (H_{\text{ext}} - H_{c1})}{\Phi_0 / \lambda^2}$ is the analogue to

$$\left(\underbrace{\frac{H}{\lambda}}_{j_{\text{el}} \equiv \boldsymbol{\tau}_{\text{resol}}} - \underbrace{\frac{H_{c1}}{\lambda}}_{\boldsymbol{\tau}_0} \right) \underbrace{\frac{\mu_0}{\Phi_0 / \lambda}}_{\frac{1}{G\lambda_0^2} \frac{1}{b/\lambda_0} = \frac{1}{\boldsymbol{\tau}_0}} = \frac{\boldsymbol{\tau}_{\text{resol}} - \boldsymbol{\tau}_0}{\boldsymbol{\tau}_0},$$

by using result (3.5). The dependence of fraction $\frac{\Theta_1}{G}$ is represented in Fig. 3.1, where as expected for the hardening stage with a very low number of mobile dislocations (see page 22), the hardening rate Θ_1 is 3-4 magnitude orders weaker than G . Note that since we are working in the initial state of Stage I, we need not consider the activation of secondary slip systems, which its corresponding reaction with the primary dislocations to produce sessile dislocation segments,

modellized dislocation theory, corresponds to the external resolved stress becomes $\boldsymbol{\tau} \sim \frac{G \lambda_0^2}{\Lambda b}$.

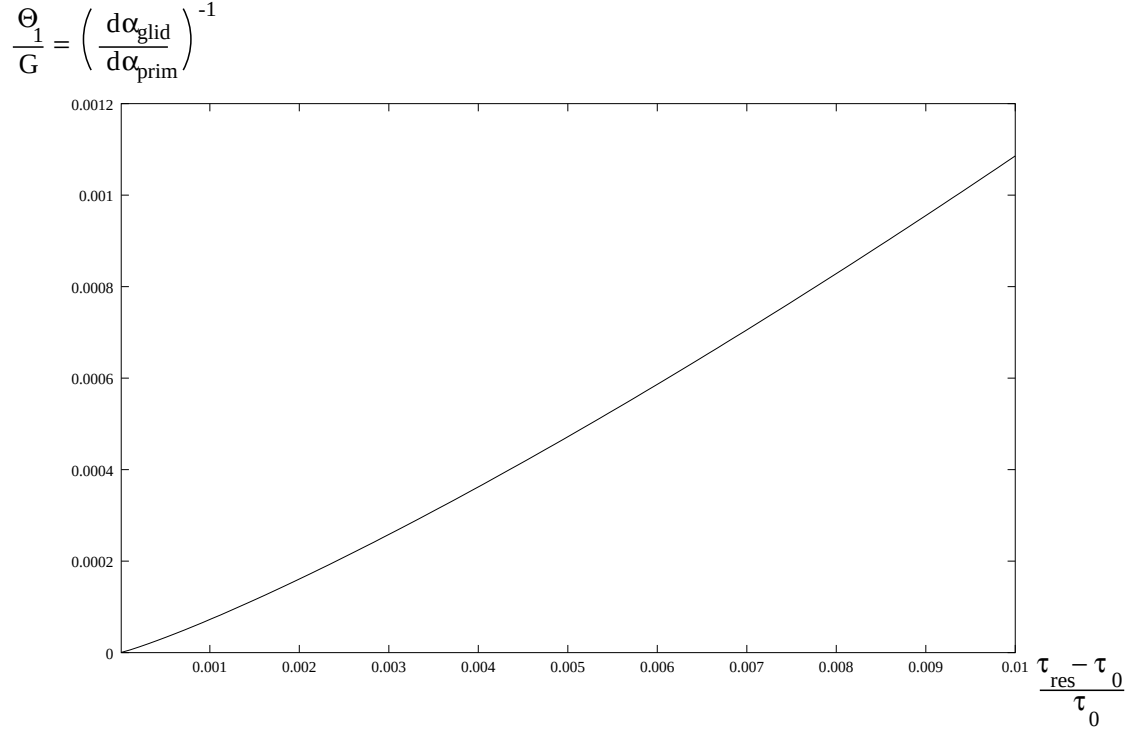


Figure 3.1: Fraction Θ_1/G in dependence on the applied resolved shear stress for the start of the easy glide stage.

Another interesting system is that formed, in a highly inhomogeneous crystal, by a narrow layer of material crossed by gliding dislocations, which still maintains its rigidity and is immersed in a bulk system which already shows nearly no rigidity. Such an arrangement, which corresponds to the superconducting 2-dimensional film in a 3-dimensional physical space studied in the Note on page 182, allows us to consider a system which for the same initial grown-in dislocation spacing λ_0 as the bulk crystal shows nevertheless properties characteristic of a lattice spacing $\lambda_{\text{eff}} = \frac{\lambda_0^2}{\bar{d}} \gg \lambda_0$, being $\bar{d} \ll \lambda_0$ the thickness of the layer where gliding dislocations are not still permitted to spread out. The fact that the interaction between the topological line defects decreases at long distances like $\frac{1}{|\vec{r}' - \vec{r}|}$ in the case under study now, different from the 3-dimensional homogeneous crystal where such an interaction increases logarithmically with the distance, allows for an easier condensation of the gliding dislocations which leads to the loss of the remaining rigidity of the layer.

3.2 Stage II and the confinement of the current of gliding dislocations

As it has been argued in the previous Chapter, the rapid hardening stage or Stage II is characterized by the area-decay in the correlation function $W(C)$, this a consequence of the suppressed escape probability of the gliding dislocations crossing the area enclosed

by C . As it can be seen in the first diagram in Fig. 3.2, the mobile dislocation rings created at the primary plane's Frank-Read sources (with Burgers vector b pointing in the shear direction) are *retained* inside the discs located on the primary planes and formed by sessile Lomer-Cottrell locks, and therefore the number of gliding dislocations crossing (without being able to escape) the interior of the closed path C defining $W(C)$ is proportional to its area. Further under the action of the external shear stress, the

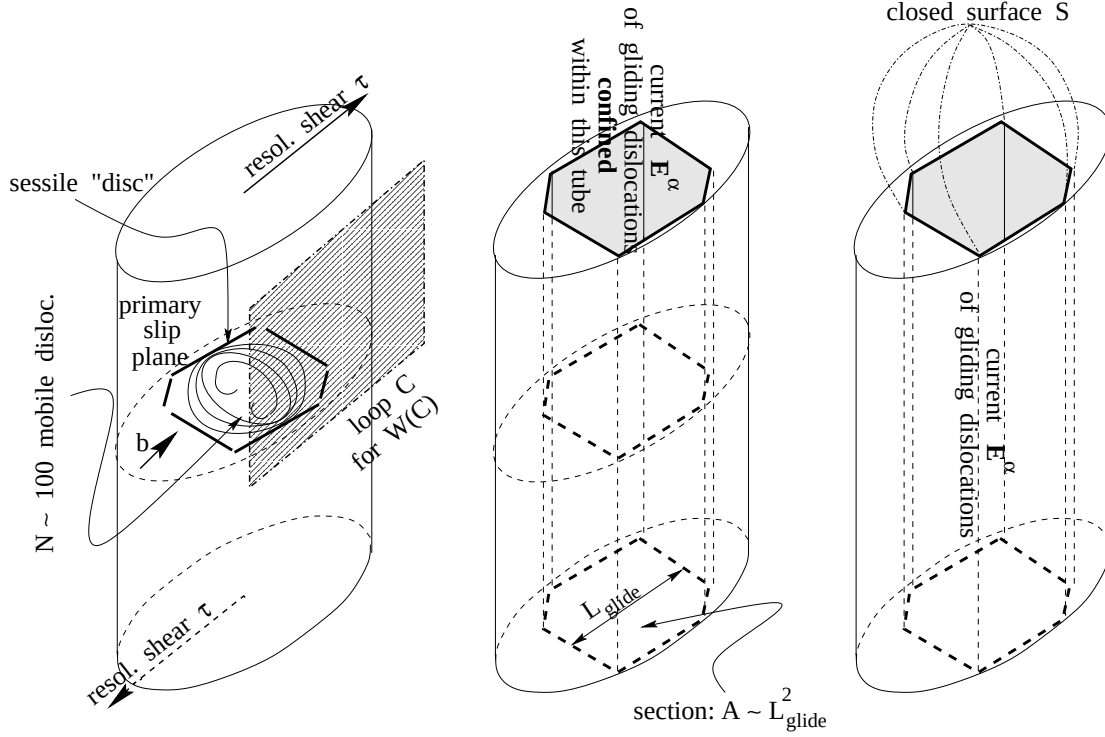


Figure 3.2: In the Stage II the gliding dislocations crossing the area enclosed by loop C of correlation function $W(C)$ cannot escape from the disc-structure of sessile segments. The number of the retained mobile dislocations is of the order of $N \sim 100$. Current $\mathbf{E}_i^\alpha$ of gliding dislocations is confined within a tube formed by these discs of section $A \sim L_{\text{glide}}^2$, with $\mathbf{E} \sim \frac{N}{A}$ obtained from the Gauss theorem applied on the closed surface S around the end of such tube.

dislocation rings glide outwards until reaching the sessile segments,,: the direction of $\text{curl } \alpha_{\text{glid}}$, due to the inhomogeneous radial distribution of such mobile dislocation rings inside the discs, is perpendicular to the primary plane. And thus, according to the proved dislocation equation (see Table 2.3 on page 52) $\epsilon_{ijk} \partial_j (\alpha_{\text{glid}})_k^\alpha = \dots + \epsilon \mu \partial_t (\mathcal{J}_{\text{glid}})_i^\alpha$, the direction of the current of gliding dislocations results to be confined within tubes stretching along the crystal (at least for very low temperatures), being the section of these tubes the disc of sessile segments (see the middle diagram in Fig. 3.2).

It is the application of the model formulated in the previous Chapter on the just discussed dislocation arrangement what, together with a) the experimental result on the number $N \sim 100$ [20] of mobile dislocations confined inside the disc-structure and with b) the ansatz for the λ_{scale} defining the scale η (analogue to the Planck scale $\hbar c$),

allows us to obtain a result consistent with the equation (1.14) which describes the rapid hardening stage, estimating also correctly the order of the constant Λ_{hard} appearing in such an equation. The argument is nothing else as the translation into the dislocation language of the dual superconductor, which is a system where instead of being confined the magnetic field inside tubes, like it was the case in the ordered phase of the usual superconductor characterized by the existence of isolated vortices, it is now the electric field that is constrained into certain finite spatial regions, occurring such a behaviour only for a $U(1)$ gauge field of compact nature and only in the phase characterized by the disorder in the (representative) spins defined for such a gauge field (see the four steps on page 101).

Such argument runs as follows: since the gliding dislocation current $\mathbf{E}_i^\alpha$ only exists constrained inside the tube (for simplicity, only one tube-system is considered and the current is assumed the same for all points inside), let us integrate this current over a closed surface S surrounding, at the external surface of the crystal, one of the tube's ends. As it is clear from the right diagram in Fig. 3.2, such an integration is equal to $\mathbf{E}^\alpha \times A$, where the Burgers vector's direction $^\alpha$ is fixed by the shear direction and A is the section of the disc of sessile segments, section which is of the order of L_{glide}^2 with L_{glide} the glide distance available for the mobile dislocations. According to the dislocation formulation obtained in Table 2.3 (see page 52), the analogue to the electromagnetic Gauss theorem states that $\mathbf{E} \times A$ is equal to the analogue to the electric charge located in the crystal's external surface enclosed by the integration surface S . Let us continue the explanation using the more common notation of the superconductivity, whose correspondence to the notation of the dislocation model is found in Table 2.5 on page 88. Hence

$$\mathbf{E} \times A = Ne \quad (\text{in CGS units}), \quad \mathbf{E} \times A = \frac{Ne}{\epsilon_0} \quad (\text{in MKS units}), \quad (3.8)$$

where e is the model's parameter determining the strength of coupling to the $U(1)$ gauge field describing the gliding dislocation density and current; and N is the number of dislocations which can slip, since it is the slip velocity imposed by the stress machine what plays the role of the electric charge in the field equations of the dislocation model. Continuing with the CGS units, where material constant ϵ_0 is included into the fields — the material response, characterized by ϵ_{rel} , is actually determined by the behaviour of the model —, the energy accumulated by the dislocation arrangement of the above Figure under study is

$$\text{Energy} = \frac{1}{2} \int \mathbf{E}^2 d^3\vec{r} = \frac{1}{2} \left(\frac{Ne}{A} \right)^2 A |\vec{x}_2 - \vec{x}_1|,$$

with $|\vec{x}_2 - \vec{x}_1|$ the height of the tube where $\mathbf{E}$ is confined, tube which covers the whole height of the single crystal. By writing the energy as function of parameters with direct translation into the dislocation model, one obtains

$$\text{Energy} = 2\pi^2 \left(\frac{N\hbar c}{\Phi_0} \right)^2 \frac{|\vec{x}_2 - \vec{x}_1|}{L_{\text{glide}}^2}. \quad (3.9)$$

Note that this energy is proportional to the tube's height $|\vec{x}_2 - \vec{x}_1|$ and inversely proportional to the tube's section L_{glide}^2 . Since the field variables with a function in the

dislocation equations similar to that of the electric charge are placed below and above of the tube-structure under study here, and since the energy grows like distance $|\vec{x}_2 - \vec{x}_1|$ separating these two places, in the electromagnetic language one would speak in this case of confinement of the electric charges.

Now let us show that the just obtained result, especially its dependence on $\frac{|\vec{x}_2 - \vec{x}_1|}{L_{\text{glide}}^2}$,

is consistent with the rapid hardening's equation (1.14), $L_{\text{glide}} = \frac{\Lambda_{\text{hard}}}{\varepsilon - \varepsilon_2}$, stating how the glide distance available decreases inversely proportional to the shear strain part $(\varepsilon - \varepsilon_2)$ of the Stage II. The crystal energy satisfying such relation for the shear strain is, according to the relation between the shear stress and the Stage II's shear strain part through the hardening rate Θ_2 , equal to

$$\text{Energy} = \frac{1}{2} \int \Theta_2 (\varepsilon - \varepsilon_2)^2 d^3\vec{r} = \frac{\Theta_2}{2} \left(\frac{\Lambda_{\text{hard}}}{L_{\text{glide}}} \right)^2 \int d^3\vec{r},$$

where the volume integral, beside yielding the $|\vec{x}_2 - \vec{x}_1|$ corresponding to the crystal's height, gives $\bar{\lambda}^2$; length $\bar{\lambda}$ is the spacing between two of the confining tubes such that $\frac{\Theta_2}{2} \left(\frac{\Lambda_{\text{hard}}}{L_{\text{glide}}} \right)^2 \bar{\lambda}^2$ is the energy pro length of a one-tube configuration. Such spacing is experimentally measured to be about some micrometers, also the order of the λ_{scale} defined in (2.88) as the shortest length necessary to overcome a confining barrier by means cross-slip, escape achieved through the “zero-oscillation” jumps which the position uncertainty (due to the lattice Burgers vector) causes. Hence

$$\text{Energy} = \frac{\Theta_2}{2} (\Lambda_{\text{hard}} \bar{\lambda})^2 \frac{|\vec{x}_2 - \vec{x}_1|}{L_{\text{glide}}^2}, \quad (3.10)$$

energy which, obtained from experimental results alone, displays the same dependence on the size of the confining tube and of the crystal's height as energy (3.9). Further, the order of Λ_{hard} can also be obtained comparing both results

$$\sqrt{\Theta_2} \Lambda_{\text{hard}} \bar{\lambda} \sim N \frac{\hbar c}{\Phi_0},$$

where by taking into account that result (3.9) has been written in CGS units and that $\frac{\hbar c}{\Phi_0}|_{\text{CGS}} = \sqrt{\mu_0} \frac{\hbar c}{\Phi_0}|_{\text{MKS}}$ corresponds in the dislocation model to $\frac{Ga^2\lambda_{\text{scale}}^2}{4\pi^2} \frac{1}{b} \frac{1}{\sqrt{G\lambda_0^2}} \sim \sim \sqrt{G} \frac{a}{\lambda_0} \lambda_{\text{scale}}^2$, one obtains

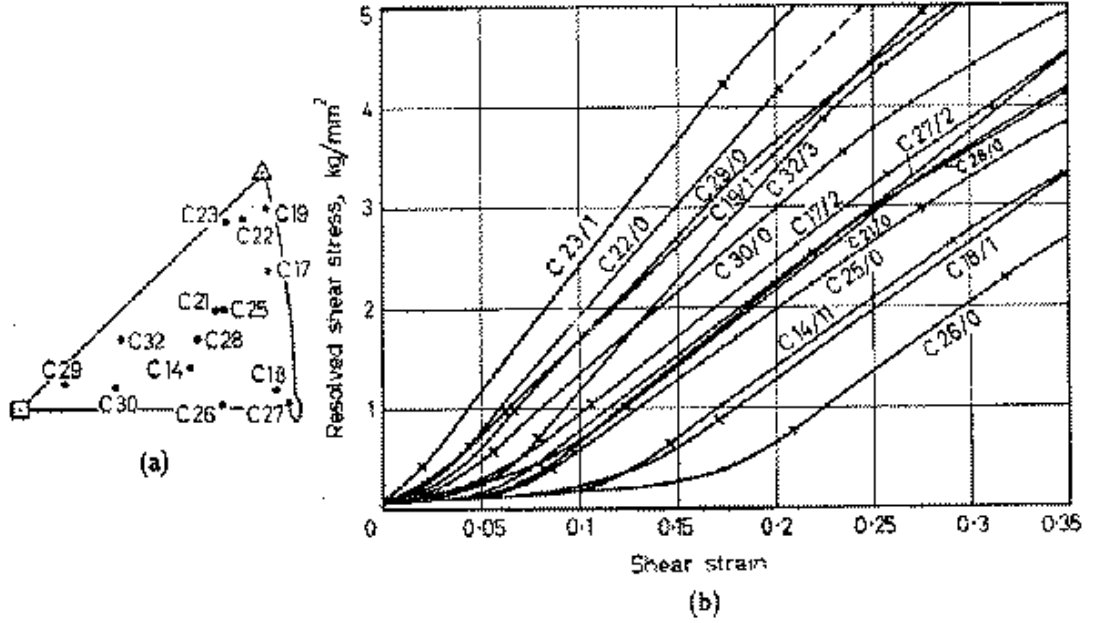
$$\begin{cases} \sqrt{\Theta_2} \sim \sqrt{G} \times 10^{-1}, & N \sim 10^2, \\ \bar{\lambda} \sim \lambda_{\text{scale}}, & \frac{a}{\lambda_0} \sim 10^{-3} \quad (\text{for the Stage II directly}), \end{cases}$$

$$\Lambda_{\text{hard}} \sim \lambda_{\text{scale}} \sim 10^{-6} m, \quad (3.11)$$

which is the correct order of magnitude for Λ_{hard} .

3.3 Phase diagram for the 3-dimensional lattice dislocation model at $T = 0$

After having presented the analogy between the superconductor's diamagnetism and the rigidity against the spreading of gliding dislocations in the dislocation model, it is now time to consider the phase diagram for the constructed lattice model based on such a correspondence: the analogue to the lattice Higgs model, with both a compact as well as a non-compact $U(1)$ gauge field.



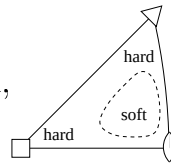
Rates of hardening Θ (kg/mm^2):

	Stage I	Stage II		Stage I	Stage II
C14	0.77	13.5	C25	1.22	13.5
C17	2.71	15.1	C26	0.76	13.8
C18	1.01	13.7	C27	3.41	15.7
C19	7.0	18.5	C28	1.10	14.7
C21	0.99	15.0	C29	4.1	20.4
C22	5.1	23.0	C30	2.32	17.1
C23	11.7	24.9	C32	1.73	21.8

Figure 3.3: Influence of the orientation in the hardening curve for copper crystals (J. Diehl, *Z. Metallk.* **47**, 331 (1956) quoted in [23] and in [51]).

In the case of the compact model, it has still to be established how integer q^α is related to the initial orientation of the applied stress and how this leads to the production of few or of many different mobile dislocation types, being the latter possibility that leading to a more strongly favoured reaction between them to build sessile dislocations. The

stereographic triangle for the primary system,



(see also Fig. 1.2 on page

10), shows [51] the orientations for the application of the external stress and how the hardening rate for both the easy glide as well as the rapid hardening stage depends on this orientation. Not only the hardening rate, but also the extent of these two plastic stages gets modified very strongly by the orientation, as one can see in Fig. 3.3. This dependence is a consequence of how many non-parallel slip systems beside the primary one have been actually activated by the applied stress, since the reaction among the mobile dislocations created in each of these slip systems produces the sessile dislocations determining the end of Stage I and begin of Stage II.

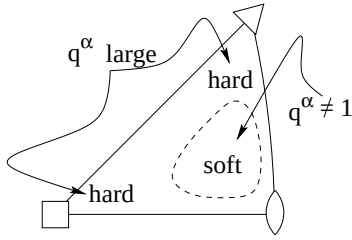
Within the framework of the (lattice) dislocation model derived from the lattice $U(1)$ Higgs model — through the analogy between the superconductor’s diamagnetism and the crystal’s rigidity which tends to oppose the spreading out over the system of gliding dislocations — if the gauge $U(1)$ symmetry describing the invariance under compatible deformations is of compact nature, q^α labels the number of different types of stable able-to-glide dislocations; actually, in a compact $U(1)$ gauge model, q^α labels the representation chosen by the scalar field coupled to the gauge field of a compact non-Abelian symmetry, to break partially down this group to the $U(1)$ of the Higgs model under study, $U(1)$ symmetry which by means of the coupling of its gauge field to another scalar field will be broken down completely, except for a set of discrete symmetry operations, number of operations which just depend on the scalar field’s representation to reduce the non-Abelian group to $U(1)$ and thus, number of operations which depend of q^α , with $q^\alpha = 1$ denoting that the only discrete symmetry remaining after all this process is the identity alone (see the discussion on page 290).

Only as a reminder, the just mentioned expression “breakdown” in the context of a gauge symmetry has only sense if it is used after having removed by hand the gauge invariance itself, or in other words, if it is applied only on the set formed by choosing one representation from each class of field configurations related to one other by allowed gauge transformations. Moreover, for the application to a single crystal the whole process must end in a complete breakdown of the gauge symmetry (up to the above discrete remaining transformations), without leaving unbroken any continuous subgroup (see result (B.217) on page 271); on the contrary closed sessile dislocations would exist without needing to be attached to segments of able-to-glide dislocations — one should remember that the start/end points of the sessile segments act in the dislocation formulation as the magnetic monopoles do in the Maxwell equations —, and this for its part would correspond to a crystal for which mobile dislocations can be produced only very hardly, what for a single crystal (as long as it if has no defects of those called disclinations) does not usually occur since there is always a well defined primary slip system at least.

In the Appendix B’s section which begins on page 251 are studied in detail all the different simplest cases about how to break down completely the non-Abelian group, or equivalently, the different patterns of Burgers vector resulting from the corresponding complete breaking of the compact $U(1)$ gauge symmetry arising as a partial reduction of the symmetry described by such a non-Abelian group, especially in the sections on page 274 and on page 297. Also the the effect of entanglement of mobile dislocations, or in other words, the feature that two reacting gliding dislocations become connected by a segment of an able-to-glide dislocation (what reduces considerably the the glide motion of the reacting dislocations) is considered in section on page 271. Entanglement occurs

in crystal with several non-parallel slip systems strongly active and is the responsible for the forest dislocations : as it was to be expected this property is only shown in the compact model, not in the non-compact model describing a dislocated crystal where exclusively one slip system can be activated.

Summarizing: a crystal where more than one non-parallel slip system can become active, leading to the production of sessile dislocations which are completely unable to glide, has to be described by a model with a compact $U(1)$ symmetry². Depending on the orientation of the applied stress, different patterns of Burgers vector sizes (or vorticities) will be obtained, just as studied in the cases of Appendix B. However for the mechanical use through the following, it suffices to know that a large q^α characterizes the fact that for the direction $^\alpha$ there exists a large set of different Burgers vector components (along such a direction $^\alpha$) available for the *able-to-glide* dislocations. And



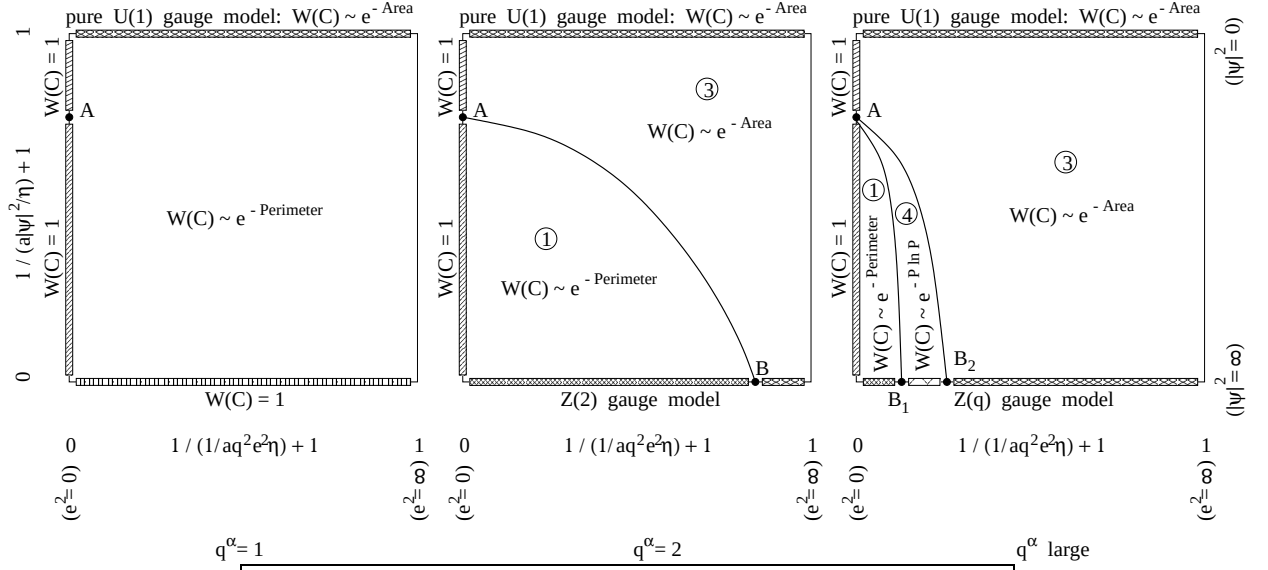
thus, a large q^α signalizes the strong activation of several non-parallel slip systems. One should expect therefore that the hardening behaviours associated to the region labeled as “soft” in the stereographic triangle will be described by a small $q^\alpha \neq 1$, whereas those corresponding to the “hard” orientations, by a large q^α — as it is already known, case $q^\alpha = 1$ corresponds to have no stable able-to-glide

dislocation. This expected behaviour is what actually follows from the phase diagram for the compact model, as we shall see now.

²At the risk of boring the reader, let us repeat again the main idea leading to how the compact character of the $U(1)$ gauge field is responsible for the behaviour associated, in the dislocation model, to the role of the sessile dislocations. At small ηe^2 , only slight oscillations about a certain orientation are energetically permitted for the gauge field spin $e^{iA_{\mu,\pi}}$ — again, when speaking about one sole orientation for all $e^{iA_{\mu,\pi}}$'s, it refers to the configuration *representing* all those field configurations obtained from this nearly complete alignment by means of any allowed gauge transformation —, without “noticing” this gauge field its angular character: the oscillation only covers a short path of the whole rotation which $e^{iA_{\mu,\pi}}$ can draw, path which can be approximated by a straight tangent segment without the curvature characteristic of the compact nature. This corresponds to have active only one slip system (the primary one) and therefore whether the gauge field is compact or not plays no role at the behaviour of the Stage I (or of the Shubnikov phase in the superconductor). On the contrary, at large ηe^2 the oscillations of $e^{iA_{\mu,\pi}}$ — now energetically not suppressed in the partition function — are large enough to draw a complete rotation sometimes, something which corresponds to the jumps by a multiple of 2π in the compact gauge field. And these jumps, which for their part describe in the dislocation model the sessile segments formed by the reaction between different non-parallel active slip systems, begin to disorder the alignment which occurred at weak ηe^2 . Such a disordering will eventually dominate in the partition function, yielding the macroscopical behaviour characterized by the area-decay in the correlation function $W(C)$. This disordering occurs more easily depending *a*) on how weak is the system's rigidity characterized by $|\psi|^2$, and *b*) on how many are the possible orientations for the representative configurations at $|\psi|^2 = \infty$ — the closer these orientations, the easier to achieve that a strong oscillation of the $e^{iA_{\mu,\pi}}$'s can connect different orientations yielding the disorder looked for —, being such number of orientations controlled by factor q^α ; for $q^\alpha = 1$, on the other hand, there is only one of such (representative) orientations and the disordering takes place only when $|\psi|^2 = 0$ is reached. For a non-compact model, nevertheless, spin $e^{iA_{\mu,\pi}}$ cannot be defined since $A_{\mu,\pi}$ is not a angular variable — or equivalently, the spin gauge field $e^{iA_{\mu,\pi}}$ would always remain covering a straight segment, not a circumference — and thus the conclusion obtained for the compact model cannot occur in the non-compact model.

Let us therefore study the lattice dislocation model described by a compact $U(1)$ gauge symmetry: the translation into the dislocation formulation of the 3-dimensional compact Higgs model. The parameters on which the modelled interaction depends are $|\psi|^2$, carrying the information about the crystal rigidity, and e^2 which measures the strength of the coupling to such a rigidity by the gauge field describing the able-to-glide dislocations. As already mentioned in the Note on page 96, one should be aware that the order parameter of the Abrikosov model, which when nonzero characterizes the superconducting state (or Stage I in the analogous dislocation language) and when zero the loss of the strong superconductivity's diamagnetism (or advanced Stage III in our dislocation model), does *not* correspond directly to the parameter $|\psi|^2$ of the lattice model, which is always nonzero although the macroscopic phase already displays the infinite spreading of the gauge field characteristic of such a loss of magnetic rigidity (or of the analogous mechanical one against dislocation glide in a crystal). The order parameter measuring the effective rigidity in the continuum Abrikosov model is actually defined through the expectation value of correlation functions depending on the lattice model's degrees of freedom: one of such correlation functions in the lattice model which displays the behaviour to be expected for the order parameter is $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ with $\mathcal{L}_{\mu, \vec{r}}^{\text{smear}} = -\frac{\Delta}{(\vec{r})} \mu D_{\vec{r}, \vec{r}_1} + \frac{\Delta}{(\vec{r})} \mu D_{\vec{r}, \vec{r}_2}$, which is studied in Appendix D after the equation (D.244) on page 653 (see also the Remark on page 663 of the same Appendix). One should keep in mind that every microscopic configuration in the lattice Higgs model ("translated" into the dislocation context) gets always determined by its content in stable able-to-glide dislocations (and in other topological stable objects); and this independent of the fact whether the macroscopic realization, to which this configuration and many others corresponds within the partition function, is also characterized by the isolated existence of such gliding dislocations — and in this case one speaks of the Stage I — or not because some kind of more complex arrangement of the gliding dislocations have occurred — as for instance the condensation of the gliding dislocations to produce the unhidden spreading out of the gliding dislocation density — [10]. For the following, nevertheless, the different macroscopic states or phases will be identified without referring to such an order parameter of the Abrikosov model but through the changes in behaviour of the correlation functions studied in the previous Chapter and summarized in Table 2.6 on page 106.

The phase diagram at very low temperatures for the model under study now, the compact Higgs model in a 3-dimensional lattice, is determined in the Appendix C (see Fig. C.7 on page 450 and Fig. C.6 on page 449) together with the character of the transition between the phases occurring. Here only the final result will be given, which can be seen in Fig. 3.4. In order to keep the same names as in the theory of phase transitions, limit $|\psi|^2 = 0$ will be called the (compact) pure $U(1)$ gauge model — model described only by the gauge field since the interaction with the scalar field has disappeared —, limit $e^2 = 0$ the XY model — only described by the scalar field — and limit $|\psi|^2 = \infty$ will be named the pure $Z(q)$ gauge model. This latter model, where the gauge field is constrained to take only a discrete set of values, $A_{\mu, \vec{r}}^\alpha = \frac{2\pi}{q^\alpha} n_{\mu, \vec{r}}$ with $n_{\mu, \vec{r}} = 0, 1, \dots, (q^\alpha - 1)$



Phase	correlation functions
① : Stage I	$\begin{cases} W(C) \sim e^{-P_C} \\ H(\vec{r}_a, \vec{r}_b) \sim e^{- \vec{r}_b - \vec{r}_a } \end{cases}$
③ : Stage II ($q^\alpha \neq 1$)	$\begin{cases} W(C) \sim e^{-\ln(2a\eta e^2)A_C} \\ H(\vec{r}_a, \vec{r}_b) \sim \text{const} \end{cases}$
④ : loss of rigidity advanced Stage III	$\begin{cases} W(C) \sim e^{-P_C \ln P_C} \\ H(\vec{r}_a, \vec{r}_b) \sim e^{-1/ \vec{r}_b - \vec{r}_a } \end{cases}$

Figure 3.4: Expected phase diagram for the 3-dimensional compact Higgs model at zero temperature for $q^\alpha = 1$, $q^\alpha = 2$ and for large q^α . Axis $|\psi|^2 = 0$ corresponds to the (compact) pure $U(1)$ gauge model, axis $e^2 = 0$ to the XY model and axis $|\psi|^2 = \infty$ is the pure $Z(q)$ gauge model. Point A is the critical value (or second-order transition point) of the 3-dimensional XY model: $(a|\psi|^2/\eta)_{\text{crit}} = 0.322$; point B denotes the critical value of the 3-dimensional $Z(2)$ gauge model: $(ae^2\eta)_{\text{crit}} = 1.314$. Both points B_1 and B_2 tend towards zero with increasing q^α . All lines separating the different phases correspond to second-order phase transitions.

³, determines together with limit $|\psi|^2 = 0$ the form of the phase diagram almost completely. It is to be noted that for large q^α there exists a small region of phase ④ as a consequence of the self-duality of the pure $Z(q)$ gauge model for q^α large enough, as it

³The fact of this discreteness in the gauge field for $|\psi|^2 = \infty$ derives from the following argument: the evaluation of the partition function requires to remove explicitly the gauge invariance by choosing one configuration among all those configurations gauge equivalent to the former. One of such choices is the so-called unitary gauge which is obtained by absorbing scalar field's phase $\varphi_{\vec{r}}^\alpha$ into the gauge field, becoming thus the partition function equal to

$$\mathcal{Z} = \int_{-\pi}^{\pi} \prod_{\vec{r}, \alpha, \mu} \frac{dA_{\mu, \vec{r}}^\alpha}{2\pi} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha)] - \frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \alpha} \sum_{\mu} [1 - \cos(q^\alpha A_{\mu, \vec{r}}^\alpha)] \right].$$

Limit $|\psi|^2 = \infty$ forces that from compact gauge field's range $[0, 2\pi)$, only the values fulfilling $q^\alpha A_{\mu, \vec{r}}^\alpha = \text{integer} \times 2\pi$ are allowed. These are also the values, which substituted into the “spin” $e^{iA_{\mu, \vec{r}}^\alpha}$, define the q^α different representatives for the gauge equivalent configurations at $|\psi|^2 = \infty$.

is briefly discussed in Appendix D together with the phase diagram on page 609.

The absence of a phase transition for $q^\alpha = 1$ is discussed in the Appendix C's Note on page 451, by following [41], and its relation to the absence of any well defined slip system has been discussed in Chapter 2 previous to the Note on page 104. The character of all phase transitions in the phase diagram of the 3-dimensional compact model is continuous or second-order.

For large q^α most of the phase diagram in the compact model is occupied by the phase with the area-decay in $W(C)$, what characterizes the rapid hardening stage (or Stage II) when many able-to-glide dislocations are retained within a region whose area they are densely crossing. On the contrary, for small $q^\alpha \neq 1$, a large part of the phase diagram correspond to the easy glide stage (or Stage I) with its characteristic perimeter-decay in $W(C)$ which is the consequence from the restricted spreading out of the density of gliding dislocations; the rest of the phase diagram still corresponds to the Stage II. Comparing these two phase diagrams with Fig. 3.3, together with the argument about q^α , one obtains that the phase diagram just reflects the properties of the hardening curve to be expected, respectively, for the “hard” — when near the whole hardening curve is formed by Stage II alone — and for to the “soft” region — when there exists a long well defined Stage I too — of the stereographic triangle in single crystals at very low temperatures and where different slip systems can become active. Hence at $T \approx 0$ Stage I and Stage II dominate in crystals like the face centered and body centered cubic crystals, which is the feature experimentally observed. It must be stressed one final time again that it has been *not* the formal separation between mobile and sessile dislocations, being described through a compact $U(1)$ gauge symmetry those systems which have both types of dislocations, what has allowed us to formulate a model whose phase diagram corresponds to the different hardening behaviours: the latter result has actually been a consequence of the analogy established with the superconductor as model for a rigid system restricting the existence through the whole crystal of gliding dislocations.

If a single crystal at $T \approx 0$ with several non-parallel active slip systems is described by a model analogous to the lattice Higgs model with a compact $U(1)$ gauge symmetry, a different case occurs if only the primary slip system is active, as it is the case in many of the hexagonal packed crystals. The model which describes its behaviour is the 3-dimensional non-compact Higgs model, where the gauge symmetry now is the conventional non-compact $U(1)$ since the production of sessile dislocations is strongly suppressed: there is no further slip system, beside the primary one, which could produce mobile dislocations which by reacting with the primary ones could create sessile segments. The phase diagram of this model at very low temperatures, as well as the character of the phase transitions, is also studied in Appendix C (see Fig. C.8 on page 465), resulting in the diagram of Fig. 3.5. Limit $|\psi|^2 = \infty$ is called the Peskin model. And limit $|\psi|^2 = 0$ is called the non-compact pure $U(1)$ gauge model where, as different from the compact model, correlation function $W(C)$ only shows the decay like $\mathcal{P}_c \ln \mathcal{P}_c$, and not the stronger area-decay of the compact model, which indicates the loss of rigidity in the system. Since this weaker decay holds for $|\psi|^2/\eta = 0$ and $e^2\eta = \infty$, and these are the values in the partition function which according to (2.85)

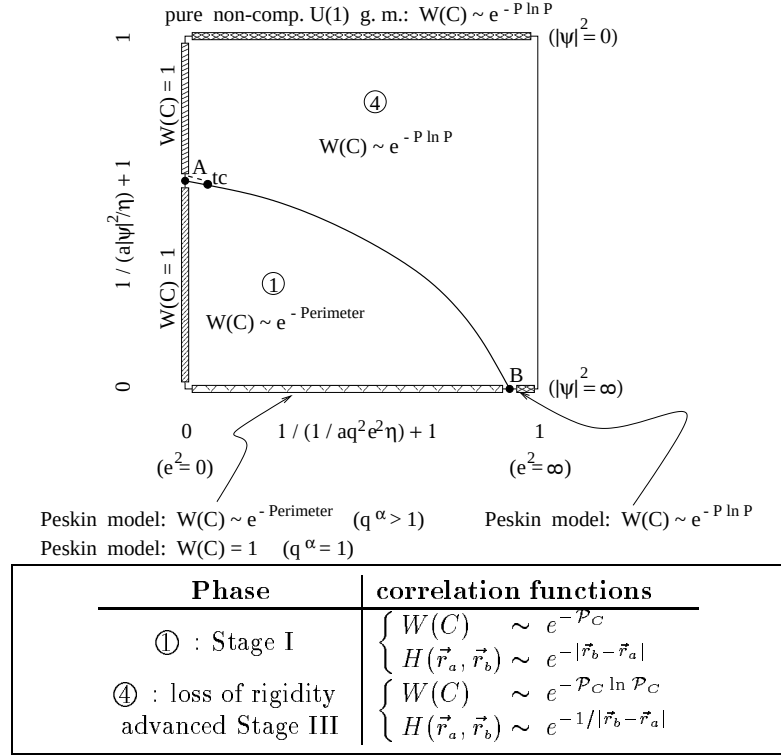


Figure 3.5: Expected phase diagram for the 3-dimensional non-compact Higgs model at zero temperature for any q^α . Axis $|\psi|^2 = 0$ corresponds to the (non-compact) pure $U(1)$ gauge model, axis $e^2 = 0$ to the XY model and axis $|\psi|^2 = \infty$ is the Peskin model. Point A is the critical value of the 3-dimensional XY model: $(a|\psi|^2/\eta)_{\text{crit}} = 0.322$; point B denotes the critical value of the 3-dimensional Peskin model: $(aq^2e^2\eta)_{\text{crit}} = 12.71$. The short solid-dashed part of line AB near to A denotes first-order transitions; the rest of AB is expected to be of second-order kind. Point tc denotes the tricritical point separating the segment of first-order transitions from the line of transitions of second-order type.

$$\frac{E}{\eta} = \frac{1}{2ae^2\eta} \sum_{\vec{r}, \alpha} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^\alpha - \Delta_\nu A_{\mu, \vec{r}}^\alpha \right)^2 + \frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \alpha} \sum_{\mu} \left[1 - \cos \left(\Delta_\mu \varphi_{\vec{r}}^\alpha - q^\alpha A_{\mu, \vec{r}}^\alpha \right) \right]$$
 allow for the largest fluctuations in the field variables and thus lead to the strongest decay in $W(C)$ inside the whole phase diagram, then $W(C)$ is impeded to reach the decay like the area of C . And this for its part means that this system never displays the rapid hardening stage, associated as we know to such an area-decay. Further, factor q^α results to play no role in the phase diagram, something to be already expected since any possible Burgers vector for able-to-glide dislocations keeps its gliding character in a stable form, without becoming attached to sessile dislocations certain combinations of Burgers vectors (as it was the case in the compact model), sessile dislocations which cannot be produced in a system with only the primary slip system activated.

All these features in the phase diagram coincide again with the results following from the experimental measurements for crystals where glide motion in other slip systems non-

parallel to the primary one can occur only very hardly, and where only the easy glide stage and the stage characterized by the loss of rigidity — and the associated long range spreading of the gliding dislocation density — are observed. Most of the line between these two phases is formed by second-order transitions points, except for a very short segment in the range of very small e^2 , where the phase transitions are discontinuous or of first-order; the point where this segment of first-order phase transition ends, and where the line of second-order phase transition begins, is a tricritical point.

It is interesting to note that the line of phase transitions separating phase ① or Stage I — analogue to the (superconducting) Shubnikov phase of the Abrikosov model — from the well developed Stage III or phase ④ characterized by the rigidity loss — corresponding to the normal conducting phase in the Abrikosov model — is drawn such that the hardening stage where, macroscopically, mobile dislocations are still isolated (easy glide or Stage I) becomes gradually restricted, with increasing coupling ηe^2 , to the range of a larger rigidity parameter $|\psi|^2/\eta$. In fact, and although experimentally it is the strength of the external applied stress what is modified, one can also look at the system as under the action of a fixed external stress and then, by modifying the coupling between the gliding dislocation (gauge) field and the scalar field describing the rigidity, to change the effect of such an external source on the dislocated crystal. This equivalence in effect can be at best seen by returning for a while to the picture of the superconductor where the Shubnikov phase is constrained, with increasing external magnetic field H_{ext} , to a range of higher $|\psi|^2$ -values until a certain external field is reached where superconductivity is energetically unfavourable for any $|\psi|^2$. If instead of increasing, $H_{\text{ext}} > H_{c1}$ is held fixed and the coupling of the gauge field to the scalar field ψ is increased, the number of vortices increases — since they can only carry in their cores integer portions $\Phi_0 = 2\pi \frac{\hbar c}{e}$ of such an external magnetic field, portion which decreases with increasing e — and also the range $\lambda \sim \frac{1}{e}$ of the magnetic field inside the superconductor decreases in such a process, allowing the vortices to come near to each other: this increasing of the coupling constant e makes therefore the system tend towards the condensation of the vortices and thus towards the destruction of the superconducting Shubnikov state, which is just the same effect as that obtained by increasing H_{ext} .

Note. The phase transition in the 3-dim. non-compact model at $T = 0$ described through the continuum model.

The discussion of the phase diagram for the non-compact model — i.e., the model where the $U(1)$ gauge symmetry describing the invariance under compatible deformations is analogous to the conventional electromagnetism — has been carried out up to this point by means of the lattice model, since the latter has the advantage of already having a built-in cut-off in the form of the lattice spacing. This cut-off necessary to avoid the very short distance divergences arising from the existence of a hidden scale which cannot be reached nor described with the model formulated. But if we are interested in going to the continuum limit in the lattice model, as well as in formulating a continuum model from the very beginning, we have to solve this problem. Under the condition that all the main interactions have been correctly identified and included into the model constructed, the divergences in the continuum model can be removed by means of the renormalization (see Step 2 on page 69), i.e., by making the model's parameters scale dependent — plus the introduction of a reference scale — but without being necessary to introduce, each time we go towards shorter distances, new interaction terms which were

not already contained in the energy initially modellized at coarser distances. This continuum approach is that dealt with in the last part of Appendix D under the name of the effective potential (see section on page 700), when into the effective continuum model (in our case, the analogue to the Abrikosov model) one implements the corrections due to fluctuations occurring at distances shorter than the long distance scale at which the continuum model was formulated.

The characterization of the phases within such a continuum formulation is achieved by means of distinguishing between the vanishing or non-vanishing constant expectation value of the scalar field (called the order parameter) coupled to the (non-compact) gauge field; of course, such an expectation value can be calculated only after removing explicitly the gauge invariance since otherwise the expectation value is identically zero. The resulting description corresponds to an effective potential for such an expectation value $\bar{\rho}$ after having integrated out the short-distance interactions between the scalar field and the gauge field; and the macroscopic state is given by calculating the value of $\bar{\rho}$ corresponding to the minimum of this potential. The transition from Stage I to the advanced Stage III with its associated loss in the rigidity against glide — or in the Abrikosov model, from the state characterized by the existence of isolated vortices to the state with a long range magnetic field — is signaled by the fact that such a minimum transits from being located at a non-vanishing constant $\bar{\rho}$ to a zero value. At a zero temperature and for the range of small ηe^2 , the phase transition occurs when by decreasing the parameter $\eta|\beta|$ of the effective potential, relation (D.347) on page 723 is satisfied in first approximation

$$(\eta e^2)^3 \geq \eta|\beta|_{\text{ren}} (16\pi^2|\alpha|_{\text{ren}}) . \quad (3.12)$$

Both $|\alpha|$ and $|\beta|$ are the initial parameters of the usual potential energy $-|\alpha|\bar{\rho}^2 + \frac{|\beta|}{2}\bar{\rho}^4$ as in the Abrikosov model formulated at large distances, and the label “ren” denotes that we are working with the above mentioned scale dependent parameters necessary to absorb the finite contributions of the interactions which become observable as we go to distance scales shorter than the initial one. According to (3.12), if we place ourselves in a range of ηe^2 and $\eta|\beta|_{\text{ren}}$ for which Stage I is the macroscopic state displayed by the system, and then we increase slightly ηe^2 , the value of $\frac{1}{\eta|\beta|_{\text{ren}}}$ necessary to be reached such that the transition towards the hardening stage of very weak rigidity takes place is smaller than its previous value as ηe^2 was not yet increased. As expected, this is the behaviour shown by the line separating the two plastic states in the phase diagram of the non-compact lattice model: with increasing ηe^2 , the highest value of $\frac{|\psi|^2}{\eta}$ — which in the continuum model is related to the value of the nonzero order parameter $\bar{\rho} = \langle \phi_{\text{scal}} \rangle \sim \frac{1}{|\beta|_{\text{ren}}}$ as the carrier of the crystal rigidity — for which Stage I can still exist decreases.

From the figure representing the effective potential for the 3-dimensional model on page 724, the phase transition in the range of small ηe^2 is first-order since the metastable minimum at the nonzero $\bar{\rho}$ is separated for $(\eta e^2)^3 \geq \eta|\beta|_{\text{ren}} (16\pi^2|\alpha|_{\text{ren}})$ from the actual absolute minimum at $\bar{\rho} = 0$ by a barrier. And this phase transition character coincides again with that across the separating line in the phase diagram of the lattice non-compact model for small ηe^2 .

The above presentation of the phase diagrams for the compact as well as for the non-compact model has shown that the analogy “superconductivity” – “rigidity against gliding” established at the beginning as a qualitative idea has eventually led to a model able to describe all the hardening behaviours displayed by a single crystal at zero temperature. Before proving that this coincidence between the phase diagram of the modellized theory and the experimental measurements holds also at high temperatures, let us consider briefly the particular case of a 2-dimensional crystal layer containing (point) dislocations described by the analogue of the Higgs model in a 2-dimensional lattice.

The phase diagram of this model at $T = 0$, both for a compact as well as for a non-compact $U(1)$ symmetry, is studied in Appendix C (see Fig. C.11 on page 487) and

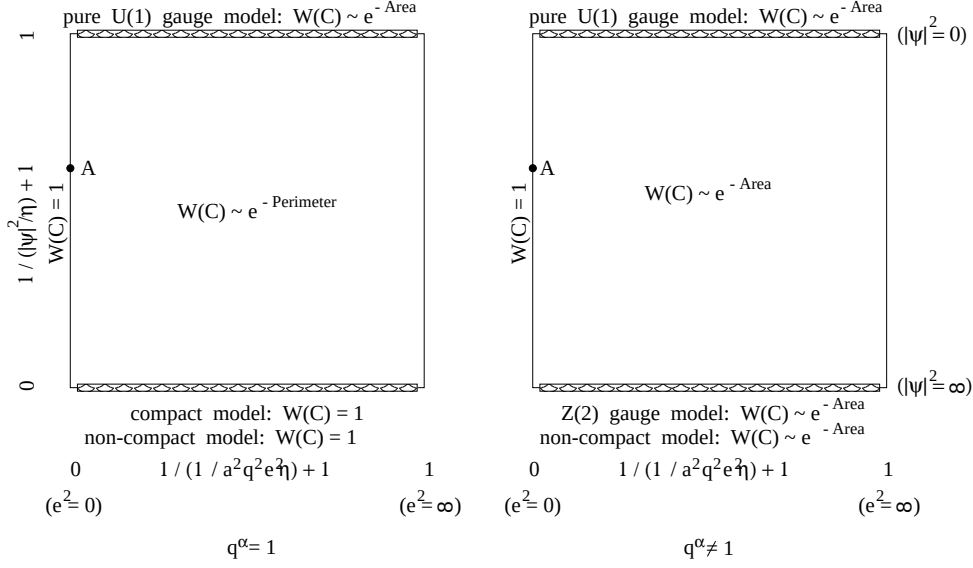


Figure 3.6: Expected phase diagram for the 2-dimensional compact and non-compact Higgs model: left, for $q^\alpha = 1$, and right, for $q^\alpha \neq 1$. Point A denotes the Kosterlitz-Thouless transition of the 2-dimensional XY model, $(|\psi|^2/\eta)_{\text{crit}} = 0.637$, which remains isolated. The area-decay of the only phase displayed has nothing to do with sessile dislocations but with the very dimensionality of the lattice.

is represented in Fig. 3.6. The main characteristic is the fact that in the phase diagram there is no difference between the compact and the non-compact model, something to be already expected since in a 2-dimensional crystal there is only one possible slip system (the system itself) which cannot interact with another not-parallel slip system to form sessile dislocations, being the latter characteristic for the compact model in higher dimensions.

Correlation function $W(C)$ shows only one behaviour: $W(C) \sim e^{-A_C}$ for any $q^\alpha \neq 1$. This is a consequence of the fact that the energy of every point dislocation in the gauge invariant model is a finite quantity, without any dependence on the size of the crystal where they are placed; and thus, they can be produced in large numbers which cross densely the area encircled by C . Any pinning obstacle for such point dislocations make them be retained in a permanent form since in the 2-dimensional case there is only one slip system, without the possibility of escaping by using other slip systems (and cross-slip). It is important to stress that the area-decay in $W(C)$ is not due to sessile dislocations — which do not exit in this case — nor to the compact character of the gauge field — the non-compact model also shows the same behaviour for $q^\alpha \neq 1$ — but only to the dimensionality of the system ⁴. The perimeter decay in $W(C)$ for the case

⁴At the limit $|\psi|^2 = 0$ and e^2 large, where $W(C)$ shows its strongest decay since it is in this region of the phase diagram where field configurations of high energy and large fluctuations are at weakest suppressed, the area-decay in $W(C)$ follows alone from the dimensionality of the system under study, with the same argument being satisfied also for $|\psi|^2 = 0$ and small e^2 : see equation (C.320) on page 406 for the compact model, and result (C.321) on the same page for the non-compact model. The area-decay and the absence of a phase transition hold also for the opposite limit $|\psi|^2 = \infty$ (see (C.347)

$q^\alpha = 1$ is only a manifestation of the system rigidity (elastic stage) in the situation that no able-to-glide dislocations exist (see the argument on page 103).

The same absence of any phase transition in the non-compact model follows also from the continuum formulation of the effective potential, as discussed in Appendix D after equation (D.346) on page 723.

It is noteworthy that the existence of only one phase for the 2-dimensional model is a direct consequence of the gauge invariance. The scalar model alone (the XY model, limit $e^2 = 0$ of the Higgs model), which is only invariant under a simultaneous global transformation in the scalar field's phase by the same amount, does have one phase transition: point A in Fig. 3.6. The XY model has also topological (point) dislocations which nevertheless, because of the energy's logarithmic divergence like the size of the system for each isolated defect — in such a model there is no coupling to the gauge field which can compensate a too large gradient in the scalar field —, are forced to occur building pairs of opposite oriented point dislocations. The transition of the model takes place when every pair breaks and its two dislocations can get separated. However, this model now introduced is not suitable for our purpose because it has been the coupling of the gauge field to the scalar field describing the crystal's rigidity what has allowed us to explain the different response of the system against the activation of gliding within the system. And further, such a response must remain invariant under compatible deformations since the latter does not change the density and current of gliding dislocations, what has a consequence that the energy modelling the behaviour of interest must be invariant under such deformations. However, in the pure scalar (XY) model the energy does change under any compatible deformation. In fact the XY model has been used [62] to modelize the melting of a 2-dimensional crystal; but in the cases under study in this work, nevertheless, we are interested in the changes of the plastic behaviour in a crystal, without going with the deformation so far that the gliding dislocations are not influenced by the crystal's rigidity because they decouple completely from the latter, what would signalize that the crystalline structure has got destroyed (what, on the other hand, is the meaning of melting).

3.4 Phase diagram for the 3-dimensional lattice dislocation model at high temperatures

Now it is time to see that the formulated model also reproduces the behaviour one should expect in a dislocated crystal at high temperatures: the gradual disappearing of Stage I and Stage II in a crystal where different non-parallel slip systems can become active — situation described in the dislocation model by the coupling of the scalar field to a compact $U(1)$ gauge field of $q^\alpha \neq 1$ —, or of Stage I alone if only the primary slip system has been activated — system described by the non-compact model — with increasing temperature T .

As already discussed in Step 4 on page 89, the effect of a high temperature on the arrangement of the gliding dislocations in a single crystal is to enable “jumps”

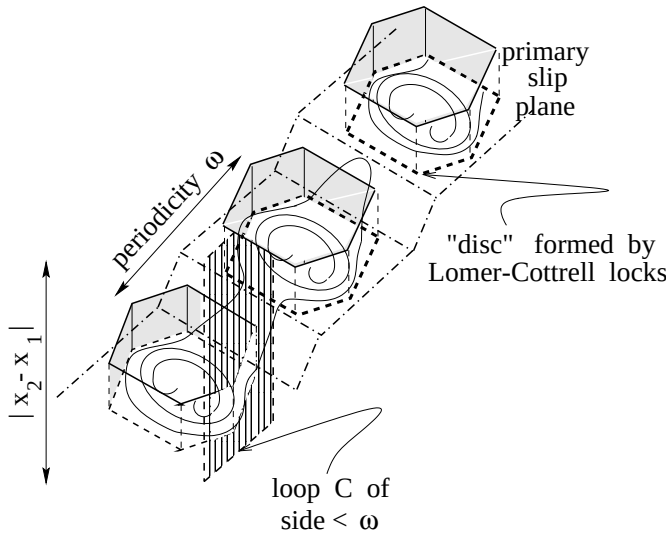
on page 416 for the compact model, and (C.517) on page 488 for the non-compact model), being again the reason the dimensionality. The whole phase diagram is therefore occupied by only one phase.

between different parallel primary planes, what leads to the fragmentation and periodical repetition of the same dislocation pattern, pattern which had been characteristic of the situation as no jump among primary slip systems was energetically possible and thus the current of gliding dislocations was confined within a spatial region (see Fig. 2.7 on page 90). This periodicity in the gliding dislocation arrangements must be implemented also in the partition function describing the hardening stages at high temperatures: by studying the effect when the lattice becomes periodic along the direction where the repetition takes place (or in other words, where the just discussed jumps activated by T occur), being lattice periodicity ω given by relation (2.93)

$$\omega = \frac{\eta \lambda_{\text{scale}}}{k_B T} \stackrel{\text{eq. (2.88)}}{=} \frac{Ga^2}{4\pi^2} \frac{\lambda_{\text{scale}}^2}{k_B T}. \quad (3.13)$$

Further, along the periodic direction the lattice spacing stops being the relevant length since any length must be compared now to scale λ_{scale} , which is the shortest height to be jumped, and such a scale is much larger than the spacing a ; thus the lattice discreteness along the periodic direction, which for the following will be named the τ -direction, becomes irrelevant and one can take the continuum limit along such a direction. Most of the Appendix D is devoted to study the effect of such periodicity on the behaviour of the different models, and only the final phase diagrams will be mentioned in this section.

The behaviour of the correlation functions used to identificate the different hardening stages at $T = 0$, Table 2.6 on page 106, is modified in the periodic situation corresponding to a high temperature. Let us assume that one of the sides of rectangular path C is placed along the τ -direction, just as in the left picture of Fig. 3.2, but this time covering



its whole periodicity ω . The reason for choosing in C its side length along the τ -direction equal to ω is to avoid that the mobile dislocations, jumping from a primary plane to another one, can slip away of crossing the interior of C , as it is the case in the figure. On the contrary, if one of the side of C covers ω and thus C gets pierced by one sessile disc as well as by the periodic repetition of the latter: the loop defining $W(C)$ becomes repeated together with the lattice and thus the able-to-glide dislocations, even those

which escape by jumping or cross-slipping, cross the interior of C in one or in the other primary plane. Path C crosses thus one sessile disc as well as the periodic repetition of the latter occurring, due to cross-slip, at a higher primary slip plane: by calling $|\vec{x}_2 - \vec{x}_1|$ the length of C perpendicular to the periodic direction (and parallel to the direction of the tube where the current of gliding dislocations remains confined), C encircles

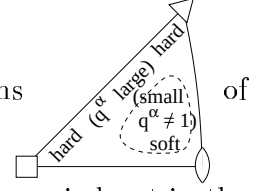
a rectangle of dimension $|\vec{x}_2 - \vec{x}_1| \times \omega$ (plus its repetitions along the direction where the dislocation patterns are repeated). In the compact model, the behaviour of $W(C)$ which signalizes that a large number of gliding dislocations are being retained (Stage II) and is a direct consequence of the compact character of the gauge field as well as of the fact $q^\alpha \neq 1$, corresponds at very low temperatures to the area-decay. And in the lattice periodic along one direction (or equivalently, at high temperatures), such a behaviour becomes characterized by $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ ⁵: it should be expected that this retention of mobile dislocations — $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ as a escape probability points out to a energy for the retained dislocations which grows like the side length perpendicular to the direction where the periodic repetition takes place, dependence what actually occurs in (3.9) — gradually disappears with increasing temperature promoting the jumping away from the confining discs of sessile dislocations, the retention being only maintained for a very tangled and dense dislocation arrangement, i.e., for a small λ or equivalently for a large ϵ^2 (what actually occurs in the phase diagram of the compact model). It should be noted that we are speaking of the area-decay actually due to the compact gauge field, not as in the 2-dimensional model where the area-decay was only a consequence of the dimensionality; of course, the periodical repetition of the dislocation arrangements caused by the temperature cannot take place in the 2-dimensional lattice since there is no remaining direction for the jumps between parallel slip systems.

The complete discussion about the behaviour of $W(C)$ and other correlation functions at high temperatures (or small periodicity ω) corresponding to the dependence with the distance at $T = 0$ ($\omega = \infty$) is the subject of section on page 611 in Appendix D, especially in the Note on page 630: it bases on the result that for the phases where the behaviour of $W(C)$ is not determined by the compact character of the gauge field but by the propagator, equation (D.139) $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ holds relating the propagator in the d -dim. lattice of periodicity ω only along one direction with the propagator in the $(d - 1)$ -dim. sublattice orthogonal to the periodic direction.

⁵The feature that leads to $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ is the equivalence found by Polyakov [82] and by Susskind [90] between, on the one hand, the compact Higgs model at large $\epsilon^2 \eta$ and small $|\psi|^2/\eta$ (the parameters at which the area-decay is more strongly favoured) for a d -dimensional lattice periodic along one direction and, on the other hand, the pure scalar XY model in a $(d - 1)$ -dimensional non-periodic lattice — described by energy $|\psi|_{XY}^2 \sum_{\vec{x} \in (d-1)\text{-dim. lat.}} \sum_{k=1}^{d-1} [1 - \cos(\theta_{\vec{x}+\hat{k}} - \theta_{\vec{x}})]$ — at small coupling strength $|\psi|_{XY}^2$ and under the action of an external magnetic field's term $h \cos(q^\alpha \theta_{\vec{x}})$. This last term $h \cos(q^\alpha \theta_{\vec{x}})$, which leaves a discrete symmetry $\theta_{\vec{x}} \rightarrow \theta_{\vec{x}} + \frac{2\pi}{q^\alpha}$ for $q^\alpha \neq 1$ (the range of $\theta_{\vec{x}}$ is $[0, 2\pi)$) and allows thus a behaviour similar to that of the model of classical spins with q^α different orientations and whose energy is determined by the interaction between nearest neighbours — but *without* any coupling to an external magnetic field —, has again to do with the different representatives for each one of the q^α different classes of gauge equivalent configurations (see the argument leading to result (D.225) on page 617). Or in other words, a $q^\alpha \neq 1$ allows for the restoring of a broken gauge symmetry, broken symmetry as always understood as the choice among the set of representative configurations — and after having fixed a representative for each class formed by configurations related to one other by means of gauge transformations — of one particular representative; an unbroken symmetry, which is the consequence of field arrangement connecting different representatives, leads to the exponential decay of $W(C)$ like the distance measured in the $(d - 1)$ -dimensional (non-periodic) sublattice orthogonal to the direction where the lattice has become periodic.

Finally, the phase diagrams of the 3-dimensional dislocation model with a compact $U(1)$ gauge symmetry (the compact Higgs model) for $q^\alpha = 2$ and for q^α large are represented, respectively, in Fig. 3.8 and in Fig. 3.9 (pages 136 and 137), and correspond also

respectively to the stereographic triangle's soft and hard orientations



the external applied stress. The deduction of these phase diagrams is carried out in the section of Appendix D which begins on page 611. It is important to note that in those diagrams, although continued until limit $T = \infty$ (something which would correspond to a senseless too rapid repetition of the dislocation patterns), the tendency with increasing temperature is only meaningful up to temperatures $k_B T > \eta$ which correspond to the complete breakdown of the rapid hardening stage. In fact, the repetition of periodicity smaller than λ_{scale} would indicate that the temperature is able to liberate the retained gliding dislocations without activating cross-slip (whose characteristic height is carried by λ_{scale}), simply by creating thermally the mobile dislocations; and thus would eventually lead to the melting of the crystalline structure.

Compared to the phase diagram of the compact model at very low temperatures, where the phases where the easy glide stage ① and the rapid hardening stage ③, for small $q^\alpha \neq 1$ a new state occurs at high temperatures: the phase denoted by ② and characterized by a weak (actually none) decay like the distance at *both* $W(C)$ and $H(\vec{r}_a, \vec{r}_b)$. For $W(C)$ such a behaviour at high temperatures corresponds to the perimeter-decay at $T = 0$, which was characteristic of the more ordered glide motion of a small number of isolated gliding dislocations. However, it cannot describe the easy glide stage now since this is also associated with an exponential decay like the distance in the complementary correlation function $H(\vec{r}_a, \vec{r}_b)$ — one should keep in mind that, as mentioned in result (2.114), only the strongest decay in one of the two complementary correlation leads to any conclusion —; since $H(\vec{r}_a, \vec{r}_b)$ is defined as the exponential of the energy increase due to the introduction of a sink and a source for the able-to-glide dislocation lines, and this energy in phase ② does not grow like the distance separating the sink from the source, then the phase is also characterized by a condensation of uncorrelated sessile segments such that their mutually interaction screens the dependence with the distance. This indicates a tangled configuration of sessile dislocations, which nevertheless does not impede, as the behaviour of $W(C)$ points out, the existence of isolated mobile dislocations escaping from the former (if the latter were retained by the sessile segments, then $W(C)$ would have to display the characteristic exponential decay). This last feature allows us to identificate phase ② as the breakdown of Stage II but where a rest of rigidity — which still forms a certain restriction for the extending of the gliding dislocation density throughout the system, as the existence of isolated gliding dislocations points out — still remains; and this is the behaviour one should expect in a crystal with fewer non-parallel slip systems to become strongly active. Such a phase a phase ② will therefore be denoted as a soft loss of rigidity (or weak Stage III). Further, that phase ② at small $q^\alpha \neq 1$ becomes substituted

by phase ④ at large q^α is to be expected since the more numerous set of non-parallel slip systems which can be activated in a pronounced form and thus become available for large q^α allows in an easier way the liberation of retained mobile dislocations, and thus of the spreading out of their density over the crystal. And such a loss of the rigidity against this spreading is associated with the well developed Stage III or phase ④. Either phase ② or phase ④, both describing the parabolic hardening stage in different types of crystals, tends to dominate in the phase diagram with increasing temperature.

The character of the phase transitions between the different phases displayed by the compact model at high temperatures are most of them second-order, i.e., of continuous type, just as at $T = 0$. Only in the case of a strong activation of many different slip systems (large q^α), the breakdown of the rapid hardening stage to transit towards the well developed Stage III is of Kosterlitz-Thouless type. Such a kind of phase transition is, like the second-order type, a continuous phase transitions without “jumps” at the transition point from one phase to the other (jumps which are nevertheless characteristic for a first-order phase transition due to the existence of a energy barrier separating the phases). And also just as the second-order phase transition, a Kosterlitz-Thouless type displays critical behaviour at the transition point, what means that the aspect of the system is at such transition point the same independently of the smallest scale one can distinguish in it. The difference between both types of critical phase transitions lies in the more severe divergence of the correlation length — note that according to Table 2.6 on page 106, phase ③ has a finite correlation length in connected correlation function Γ_{gauge} , whereas in phase ④ the correlation length is infinite since the fall out of Γ_{gauge} is weaker than an exponential decay — by approaching a Kosterlitz-Thouless transition point.

Regarding the effect of a high temperature on the non-compact Higgs model, which describes the crystal where only the primary slip system can become active, the study of its phase diagram is presented in the section on page 636 of Appendix D and the final result can be seen in Fig. 3.10 (page 138). As it occurred at very low temperatures, the only phases being displayed by the system are the easy glide stage ① and the well advanced parabolic hardening stage ④. With increasing temperature, the latter phase tends to occupy nearly the whole phase diagram. The phase transitions between both phases, which at $T = 0$ were of second-order type for almost the whole range of $e^2\eta$, become gradually of first-order character with the tricritical point t_c going towards the transition point B of the Peskin model at high temperatures.

Analogous results are also obtained if instead of the lattice non-compact model one goes to the continuum limit by using the effective potential formulation applied to the Abrikosov model (described by a non-compact $U(1)$ gauge field). Working in the range of ηe^2 small one calculates in first perturbative order the corrections which have to be added to $-|\alpha|\bar{\rho}^2 + \frac{|\beta|}{2}\bar{\rho}^4$; corrections which are due to the fluctuations in the fields of the non-compact models which become gradually observable by making finer the distance scale and leaving thus the coarse scale at which potential $-|\alpha|\bar{\rho}^2 + \frac{|\beta|}{2}\bar{\rho}^4$ determined the behaviour of the expectation value of the scalar field (by looking for the value of $\bar{\rho}^2$ at the minimum of the potential). Depending on the relation between ηe^2

and the renormalized $\eta|\beta|_{\text{ren}}$, the model will have only one minimum or two different minima, one of them deeper placed than the other; in the latter case, the higher local minimum is separated by a barrier from the absolute minimum, and if such a barrier is high enough, the macroscopic state can correspond to the local minimum (called then metastable $\bar{\rho} = \langle \phi_{\text{scal}}^{(m)} \rangle$) for a long time before transiting as a first-order transition to the real absolute minimum. The complete calculation lies in Appendix D starting at page 700.

If the system becomes periodic along one of its direction as a consequence of dislocation configuration's repetitions activated by a high temperature, it will result that for a certain temperature the position of the absolute minimum of the effective potential transits from being located at $\bar{\rho}^2 \neq 0$ for the easy glide stage ① to the position at $\bar{\rho}^2 = 0$ for the well advanced parabolic hardening stage ④. Such a transition temperature is given by results (D.348) and (D.349) on page 727

$$\begin{aligned} \eta|\beta|_{\text{ren}} \gg \eta e^2 \quad (\text{large } |\alpha|_{\text{ren}}) \quad k_B T_{\text{crit}} &\approx \frac{-4\pi\eta\lambda_{\text{scale}} |\alpha|_{\text{ren}}/|\beta|_{\text{ren}}}{\eta \left(6 \ln \left| 2|\alpha|_{\text{ren}} \left(\frac{\eta\lambda_{\text{scale}}}{k_B T} \right)^2 \right| + 2 \lim_{p \rightarrow 0} \ln p \right)} \left[1 - \frac{\eta e^2}{\eta|\beta|_{\text{ren}}} \right], \\ \begin{cases} \eta|\beta|_{\text{ren}} \ll \eta e^2 \\ k < \frac{3}{2} \end{cases} \quad k_B T_{\text{trans}} &\approx \eta\lambda_{\text{scale}} \left(\frac{\pi}{2\zeta(3)} \frac{|\beta|_{\text{ren}} \langle \phi_{\text{scal}}^{(m)} \rangle^4}{\eta} \left[1 - \frac{2}{3}k \right] \right)^{1/3}, \end{aligned} \quad (3.14)$$

$$\begin{cases} \eta|\beta|_{\text{ren}} \ll (\eta e^2)^3 \\ k \geq \frac{3}{2} \end{cases} \quad k_B T_{\text{trans}} = 0,$$

with $\zeta(3) = 1.202 \dots$ and $k = \frac{1}{4\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}}$. The character of the phase transition is of critical type for $\eta|\beta|_{\text{ren}} \gg \eta e^2$, and of first-order type for $\eta|\beta|_{\text{ren}} \ll \eta e^2$. The form of the effective potential, together with the evolution in the position of the absolute minimum, can be seen in Fig. 3.10 on page 138, where for $1 < k < \frac{3}{2}$ and at low temperatures the well developed Stage III ④ already corresponds to a local metastable minimum; minimum which becomes absolute for $k > \frac{3}{2}$.

Summarizing the effect of a high temperature T , both in the compact as well as in the non-compact model, with increasing temperature the hardening curve becomes dominated by the parabolic hardening or Stage III, which is the feature observed experimentally for any type of dislocated crystal.

3.5 A speculative short excursus: modelling disordered single crystals and the introduction of a fourth spatial dimension

Up to this point, and throughout this whole work, the attention has been focussed exclusively on a perfect single crystal. In such a system the position of the primary slip system is well determined and thus the periodicity occurring at high temperatures takes place only along one direction, that of the glide on the primary plane. On the other hand, if the single crystal contains grain boundaries formed by a piled amount of dislocations such that the orientation at one side of the boundary is not parallel to the orientation at the other side, then the periodic repetition of dislocation patterns can occur in many different directions since one must take into account even the jumps made by the gliding dislocations between crystals regions of different orientations.

A possible approach to describe such a situation is the following: the initial idea behind the formulation of λ_{scale} in (2.88) was the smallest segment of a gliding, mobile dislocation which, because of the “oscillating” state permitted (or more precisely, unable to be determined exactly) by the fundamental position uncertainty which the Burgers vector causes, was able to jump in order to escape from a barrier which impedes the motion of the dislocation segment; in other words, the idea behind was the quantum tunnel effect. Hence let us assume that the field variables of our dislocation model depend also on an additional spatial coordinate x_o (which should not be confused with the $x_{\mu=0}$ denoting the time coordinate) which carries the possibility of jumping over an energy barrier in any direction, even to primary slip planes located in crystals of different orientation across a grain boundary. Such a dependence must contain: *a*) parameter e giving the strength of the coupling between the $U(1)$ gauge field describing the able-to-glide dislocations and the scalar field carrier of the crystal rigidity, since $e = 0$ would imply the complete decoupling of the dislocation field from any reaction which could display the crystal to oppose the glide motion and therefore this would have as result that the model’s degrees of freedom would show no dependence on x_o since any jump could occur without being impeded at all; and *b*) length scale λ_{scale} which determines the shortest length any gliding segment of a dislocation has to overcome to jump over the barrier: a too large λ_{scale} would suppress the jumps to other primary slip systems, removing thus any dependence on x_o . Therefore, the dependence on x_o for the model describing a system formed by non-parallel oriented single crystals must have the following form

$$e^{-x_o e / \lambda_{\text{scale}}}, \quad (3.15)$$

and since in the dislocation model $e = \frac{2\pi}{\sqrt{Ga^2}} \frac{1}{\lambda} \sim \frac{1}{\lambda}$, and by adjusting dimensions, a plausible ansatz for the dependence on x_o becomes

$$e^{-x_o a / (\lambda \lambda_{\text{scale}})}. \quad (3.16)$$

And the energy defining any *equilibrium* configuration of the model — it should be noticed that the time coordinate plays no role in all these considerations — also becomes

extended to $\int dx dy dz \epsilon[A_\mu^\alpha, \varphi^\alpha] \rightarrow \int dx_o \int dx dy dz \epsilon[A_\mu^\alpha, \varphi^\alpha, (A^*)_\mu^\alpha, (\varphi^*)^\alpha]$, where fields $(A^*)_\mu^\alpha, (\varphi^*)^\alpha$ describe (in the same form as $A_\mu^\alpha, \varphi^\alpha$ do for the density and current of the able-to-glide dislocations) the density and current of the dislocations mobile through jumps between primary slip planes located at different heights and in different oriented crystal's regions. Such a model corresponds thus to the Higgs model describing the proposed dislocation formulation but set in a 4-dimensional lattice. The effect of the temperature, which makes energetically more favourable the above discussed jumps will be implemented in the model by making the lattice periodic along the direction of x_o ⁶, with periodicity ω defined as in the previous section

$$\omega = \frac{Ga^2 \lambda_{\text{scale}}^2}{4\pi^2 k_B T}.$$

It is important to note that any coordinate transformation defining a compatible deformation, deformation which within the framework of the dislocation model corresponds to a gauge transformation, takes the following form

$$x_o \rightarrow x_o + \chi(x^{\mu=1,2,3,4}), \quad (3.17)$$

⁶The form of the extended dislocation equations containing the density and current deriving from the fields $(A^*)_\mu^\alpha, (\varphi^*)^\alpha$ can be easily obtained: it is *formally* the same as that resulting in the usual dislocation formulation of Table 2.3 (page 52) but written compactly in a 4-notation with the introduction of an imaginary (or Euclidean) time $x^{\mu=4} = ic_{\text{sound}} t$ as a coordinate of spatial dimension, just as discussed in the footnote on page 50. It must be stressed that this is only a formal equivalence, since when studying equilibrium phase transitions, time variations and the current of gliding dislocations play no role. In fact, the density of dislocations mobile by “jumps” across a barrier (density deriving from $(A^*)_\mu^\alpha$ by the calculation of its curl) assumes — in the equilibrium configurations entering in the partition function which has to describe the overcoming of a barrier by jumps of height λ_{scale} in any direction within an ensemble of non-parallel oriented single crystals — the effect which the current of gliding dislocations had in the time-dynamical case as source for the deviations from the Schmid law: such a law is described by equation (2.21), $\epsilon_{ijk} \partial_j (\alpha_{\text{prim}})_k^\alpha = \mu_o (\tau_{\text{resol}})_i^\alpha$, and any addition arising from the new introduced field $(A^*)_\mu^\alpha$ in the spatially 4-dimensional model will suppose a deviation, which on the other hand is to be expected since cross-slip among different primary slip planes is just the origin for such a deviation.

This formal interplay in the dislocation equations between the added fourth spatial dimension and the role of the current of mobile dislocations (which, it should be remembered, plays the same role as the electric field does in the Maxwell equations of the electromagnetism) is further supported by the discussion in the footnote on page 544. Hence when studying in a d -dim. system the effect of the periodicity along one of the lattice directions and its relation with high temperatures, the resulting partition function for the model becomes completely *similar* to that of a $(d-1)$ -dim. quantum system (and as such, containing the time as a coordinate, although written as an imaginary time) where the gauge field component, corresponding in the classical energy to the coordinate which becomes periodic, plays the role through the functional derivative after such a gauge field component, of the electric field in the formally equivalent $(d-1)$ -dim. quantum system. In our case, the coordinate which will become periodic is x_o and thus the gauge field component corresponding to such a coordinate, $(A^*)_\mu^\alpha$, will become associated to the “electric field” in the form to read the resulting partition function with the words of the quantum language. This is actually not so surprising since, according to ansatz (3.15), $e^{-x_o e/\lambda_{\text{scale}}}$, the coordinate x_o whose periodicity will become inversely proportional to the temperature which activates the overcoming of barriers, is related to: *a*) coupling constant e is, which in the analogy to the superconductor (basis of the whole theory developed in this work) corresponds to the *electric* charge; and further *b*) to the length scale λ_{scale} which determines in the dislocation model the effect of a fundamental uncertainty position (as in the quantum theory).

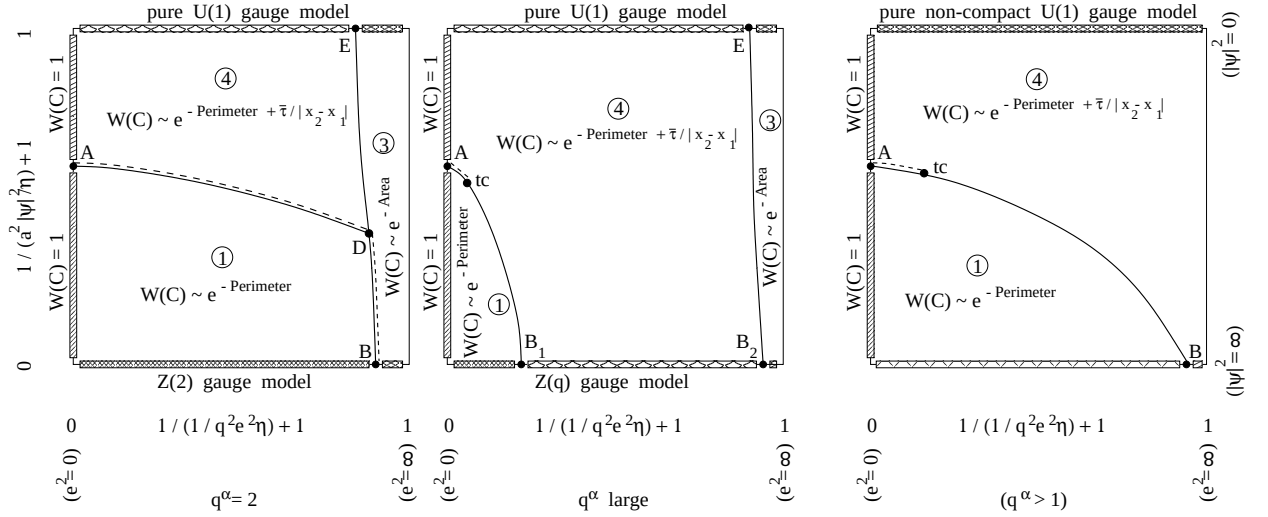


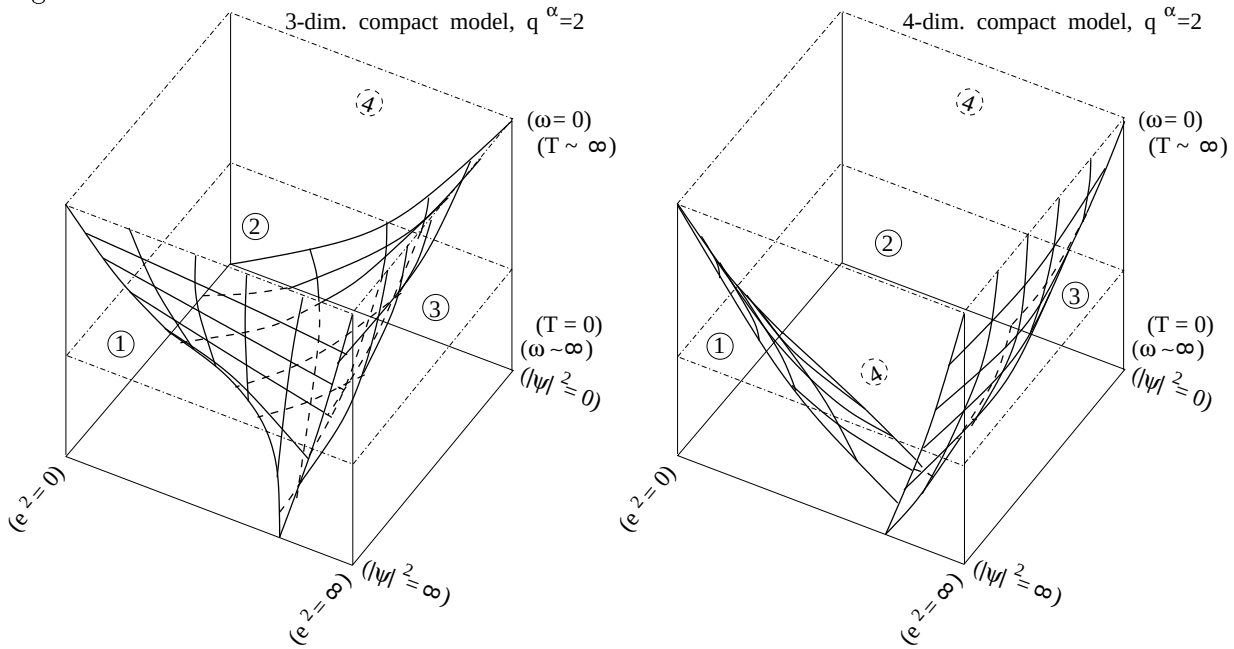
Figure 3.7: Left and middle, expected phase diagram for the 4-dimensional compact Higgs model with $q^\alpha = 2$ and q^α large, respectively; right, phase diagram for the 4-dimensional non-compact Higgs model with $q^\alpha > 1$. Point A is the critical value of the XY model: $(a^2|\psi|^2/\eta)_{\text{crit}} = 0.526$. Left point B is the first-order transition point of the pure $Z(2)$ gauge model: $(e^2\eta)_{\text{crit}} = 2.268$; right point B is the critical value of the Peskin model: $(q^2e^2\eta)_{\text{crit}} = 25.11$; point E denotes the second-order transition point of the pure compact $U(1)$ gauge model: $(e^2\eta)_{\text{crit}} = 1.572$. In the middle diagram, B_1 decreases like $1/q^2$ and B_2 tends to the critical value of the pure compact $U(1)$ model when $q^\alpha \rightarrow \infty$. Solid-dashed lines are lines of first-order phase-transitions and solid lines of second-order type. Point D or tc denotes a tricritical point separating the segment of first-order transitions from the line of second-order transitions.

but where $\chi(x^\mu)$ does *not* depend on x_o . This is due to the role of the introduced 4th spatial coordinate x_o , as related to the probability of jumping between primary slip systems, as well as due to the invariance of the dislocation model (which describes the changes of hardening as phase transitions) under compatible deformations. For instance, if transformation (3.17) would contain a duplication of x_o this would imply a strong suppression of the jumps across the barriers; but since such barriers are formed by grain boundaries or by other arrangements of dislocations, and the density (and current) of dislocations remain unchanged under any compatible deformation, then the result stated in (3.17) is fully justified.

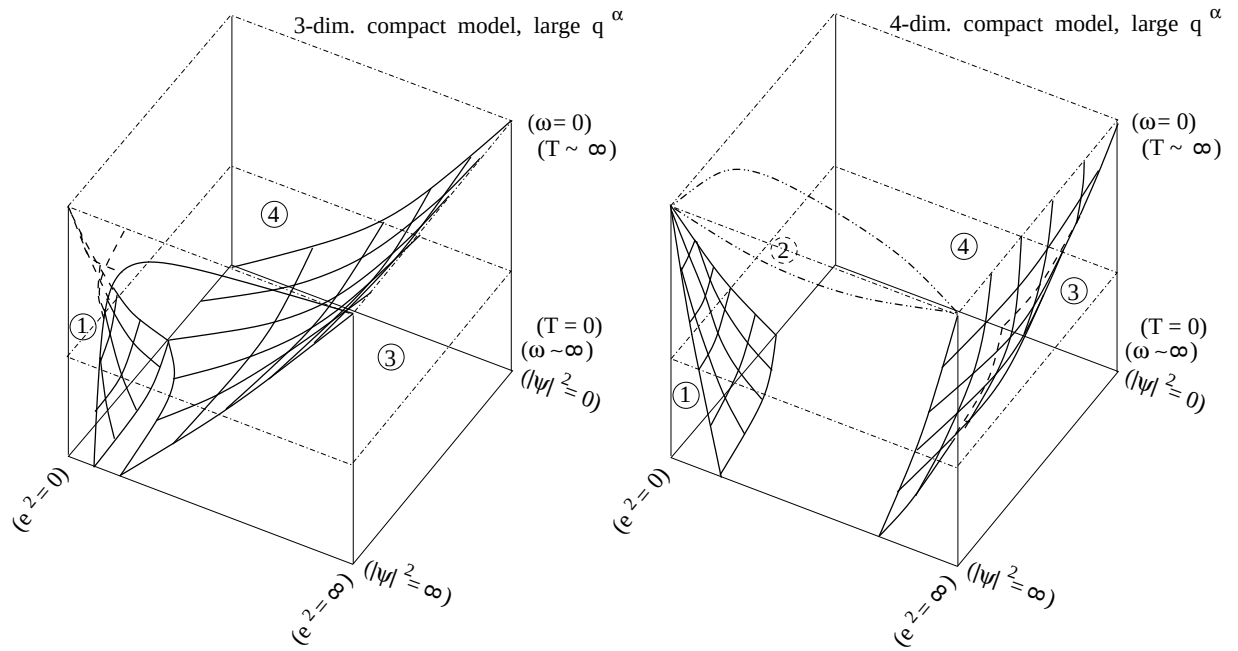
The diagram for the *equilibrium* phase transitions in the Higgs model, compact as well as non-compact, placed in 4-dimensional spatial lattice and at very low temperatures is studied in the Note on page 472 of Appendix C, resulting in the phase diagram Fig. C.10 (page 481) for the compact model and Fig. C.9 (page 477) for the non-compact model, phase diagrams which can be also seen in Fig. 3.7. The non-compact Higgs model, i.e. the model describing a crystal where mainly the primary slip system alone can be activated, shows a phase diagram very similar to that of the 3-dimensional case (Fig. 3.5, page 121) formed exclusively by the expected easy glide stage ① and the well advanced

parabolic hardening stage ④. However, the main difference with the 3-dimensional case occurs for the compact model, which corresponds to the modellized description of a crystal containing several non-parallel slip systems which can become strongly active: when compared to the 3-dimensional compact Higgs model, one notices that even at $T = 0$ the rapid hardening stage ③ has been relegated to a small region of the phase diagram, with its previous place occupied now by the phase corresponding to the well developed Stage III ④. The behaviour of $W(C)$ in this latter state is characterized, for a rectangular loop C of dimensions $\bar{\tau} \times |\vec{x}_2 - \vec{x}_1|$, by the decay $W(C) \sim e^{-\text{Perimeter}_C + \bar{\tau}/|\vec{x}_2 - \vec{x}_1|}$ since the long range — long range since the main characteristic of this phase is the spreading out over the system of the gliding dislocation density — Coulomb propagator in four spatial dimensions, $D_{\vec{r}, \vec{r}'} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|^2}$, decreases faster than in the 3-dimensional case.

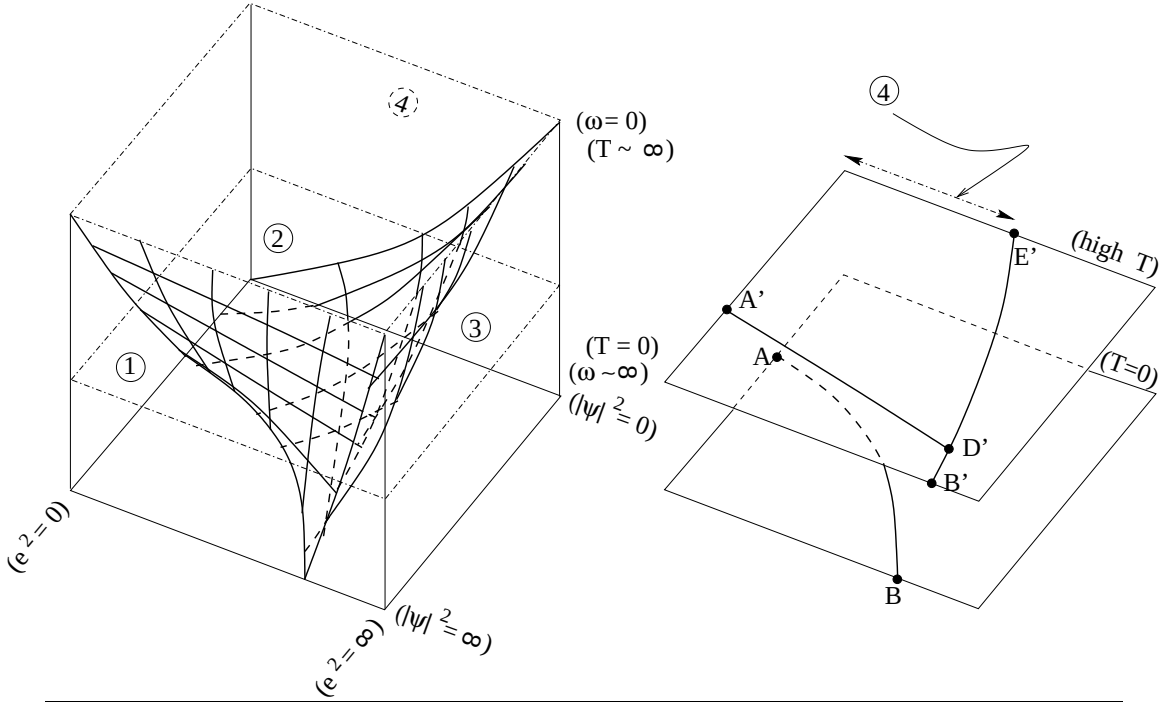
The effect of high temperatures in the 4-dimensional model is studied, starting at the section on page 611 of Appendix D, through the effect of the lattice becoming periodic along one of its direction, direction which in our case corresponds with the just introduced fourth spatial dimension x_o . As one can see in the phase diagrams of the 4-dimensional model at any temperature, for the compact $U(1)$ symmetry in Fig. D.4 on page 749 and Fig. D.5 on page 750 and for the non-compact ⁷ gauge field in Fig. D.7 on page 752, the breaking of Stage I and especially of Stage II — the latter a tendency already observed in the phase diagrams at $T = 0$ — is much more effective than in the 3-dimensional case. For the compact model the two cases are compared in the next figure:



⁷The same conclusion as in the non-compact lattice formulation follows also from the continuum effective potential for the Abrikosov model, for which the temperature at the transition point from Stage I to a well advanced Stage III is calculated in equations (D.348) and (D.349) on page 727. The evolution of the effective potential is very similar to that in the 3-dimensional effective potential: see Fig. 3.11 (page 139), as well as Fig. D.8 on page 753 of Appendix D together with the discussion following result (D.347) on page 723 of the same Appendix. The only difference when compared to the 3-dimensional model is the stronger dependence with the temperature in the 4-dimensional case.

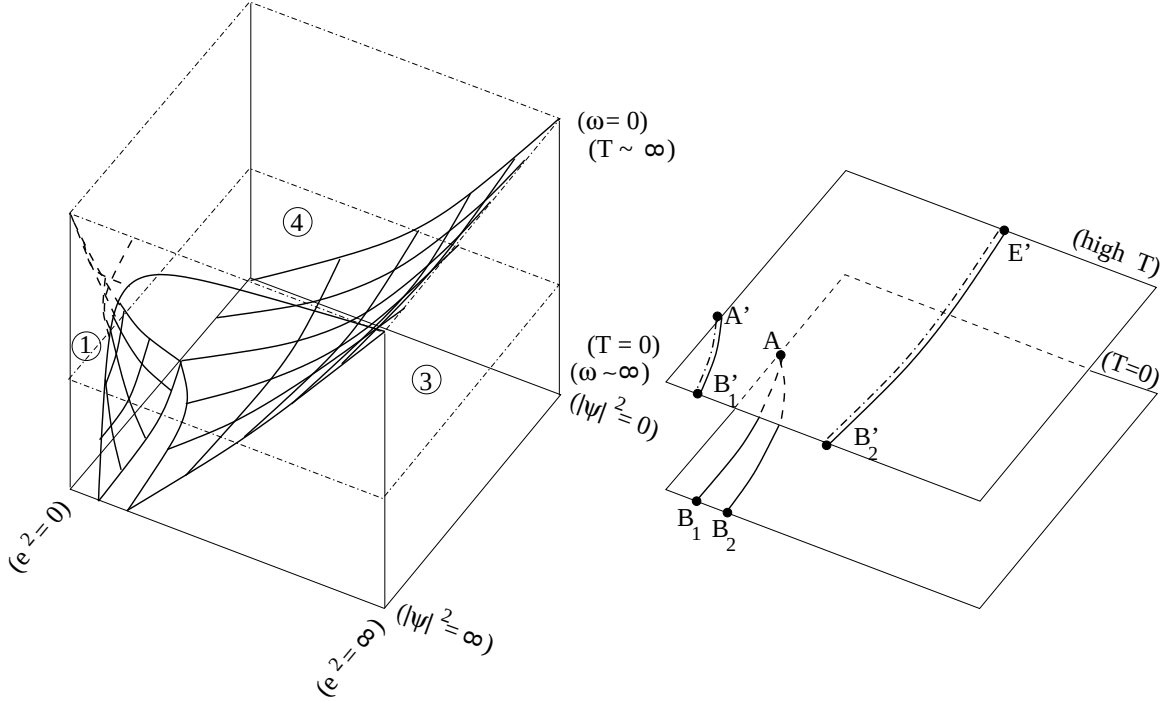


where, especially for the crystal with a lower number of different secondary slip systems to be activated (described by a small q^α), the easy glide stage as well as the rapid hardening stage occupy in the phase diagram of the 3-dimensional model a far larger region than in the 4-dimensional case. But this is just the feature measured experimentally for polycrystals former by single crystals oriented in different directions, for which Stage I and Stage II are nearly absent, being the hardening curve formed almost exclusively by the parabolic hardening stage ([84], p. 82).



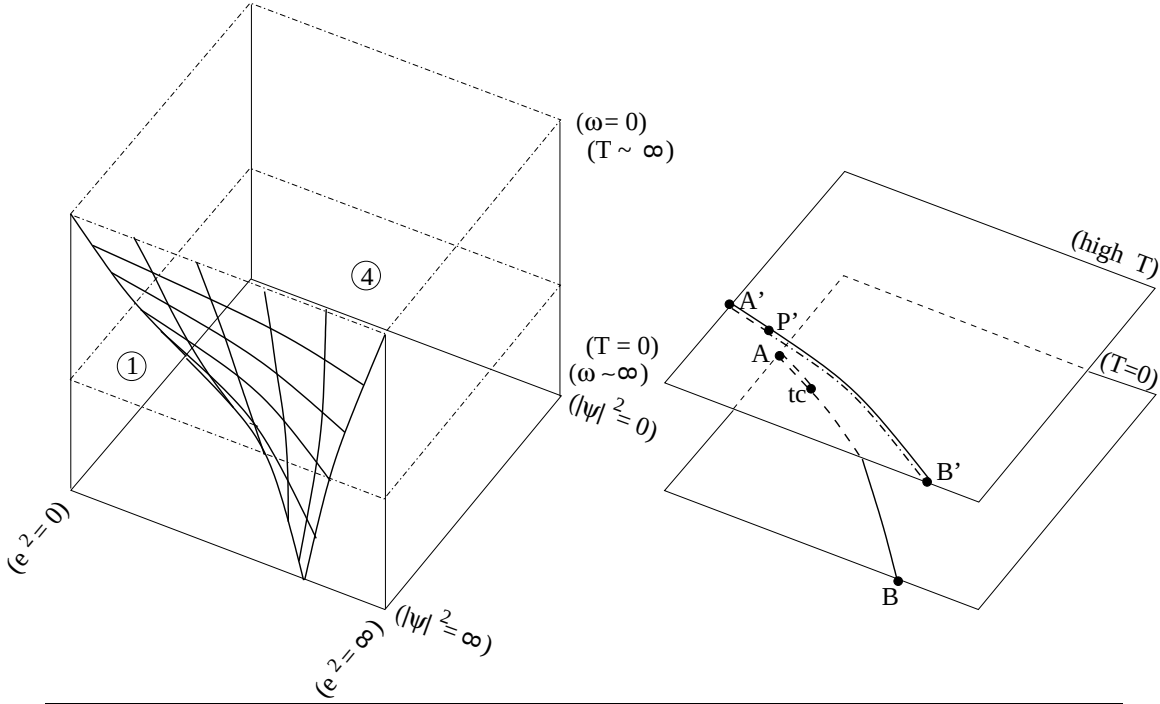
Phase	high T	$T = 0$
① : Stage I	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a, \vec{x}_b) \sim e^{- \vec{x}_b - \vec{x}_a } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(\vec{r}_a, \vec{r}_b) \sim e^{- \vec{r}_b - \vec{r}_a } \end{cases}$
② : soft loss of rigidity weak Stage III	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a, \vec{x}_b) \sim \text{const.} \end{cases}$	
③ : Stage II	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a, \vec{x}_b) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(\vec{r}_a, \vec{r}_b) \sim \text{const.} \end{cases}$
④ : loss of rigidity advanced Stage III	$\begin{cases} W(C) \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a, \vec{x}_b) \sim e^{-\ln \vec{x}_b - \vec{x}_a } \end{cases}$	

Figure 3.8: Expected phase diagram for the 3-dimensional compact Higgs model with $q^\alpha = 2$ showing the dependence on the temperature through the periodic lattice's length from $\omega = \infty$ (no periodicity, $T = 0$) to a very small ω (high T). Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2}{\pi} \frac{ak_B T}{\eta \lambda_{\text{scale}}}$, point B' by $ae^2 \eta = 2.268 \frac{ak_B T}{\eta \lambda_{\text{scale}}}$ and point E' by $ae^2 \eta = \frac{\pi}{2} \frac{ak_B T}{\eta \lambda_{\text{scale}}}$. Points A and B at $T = 0$ are equal to those in Fig. 3.4, page 119. $\frac{a|\psi|^2}{\eta} = 0.322$ and $ae^2 \eta = 1.314$ respectively. All lines are lines of second-order phase transitions.



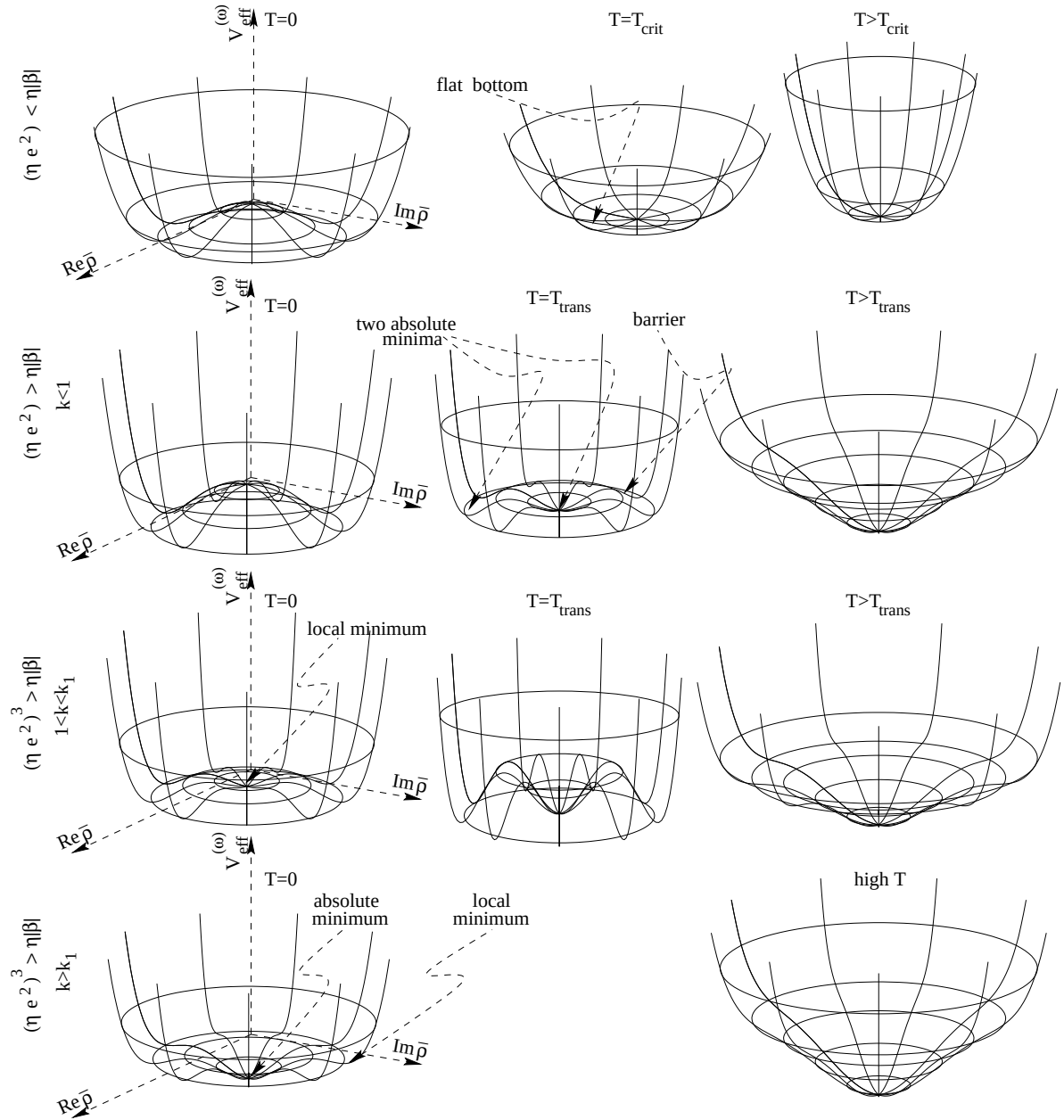
Phase	high T	$T = 0$
① : Stage I	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a, \vec{x}_b) \sim e^{- \vec{x}_b - \vec{x}_a } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}C} \\ H(\vec{r}_a, \vec{r}_b) \sim e^{- \vec{r}_b - \vec{r}_a } \end{cases}$
③ : Stage II	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a, \vec{x}_b) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}C} \\ H(\vec{r}_a, \vec{r}_b) \sim \text{const.} \end{cases}$
④ : loss of rigidity advanced Stage III	$\begin{cases} W(C) \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a, \vec{x}_b) \sim e^{-\ln \vec{x}_b - \vec{x}_a } \end{cases}$	$\begin{cases} W(C) \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a, \vec{r}_b) \sim e^{1/ \vec{r}_b - \vec{r}_a } \end{cases}$

Figure 3.9: Expected phase diagram for the 3-dimensional compact Higgs model with large q^α showing the dependence on the temperature through the periodic lattice's length from $\omega = \infty$ (no periodicity, $T = 0$) to a very small ω (high T). Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2ak_B T}{\pi\eta\lambda_{\text{scale}}}$ and point E' by $ae^2\eta = \frac{\pi ak_B T}{2\eta\lambda_{\text{scale}}}$. Point B'_2 tends toward the value of E' in the limit $q^\alpha \rightarrow \infty$. Both point B'_1 and point B_1 decreases with increasing q^α like $1/q^2$; transition point B_2 tends to $e^2 = 0$ too, since the 3-dimensional $U(1)$ gauge model displays no phase transition points at $T = 0$. Point A is equal to $\frac{a|\psi|^2}{\eta} = 0.322$. Both solid lines at $T = 0$ are expected to be formed by second-order transition points, whilst the dashed-solid lines (high T) are formed by transition points of Kosterlitz-Thouless type.



Phase	high T	$T = 0$
① : Stage I	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a, \vec{x}_b) \sim e^{- \vec{x}_b - \vec{x}_a } \end{cases}$	$\begin{cases} W(C) \sim e^{-p_c} \\ H(\vec{r}_a, \vec{r}_b) \sim e^{- \vec{r}_b - \vec{r}_a } \end{cases}$
④ : loss of rigidity advanced Stage III	$\begin{cases} W(C) \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a, \vec{x}_b) \sim e^{-\ln \vec{x}_b - \vec{x}_a } \end{cases}$	$\begin{cases} W(C) \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a, \vec{r}_b) \sim e^{1/ \vec{r}_b - \vec{r}_a } \end{cases}$

Figure 3.10: Expected phase diagram for the 3-dimensional non-compact Higgs model with any q^α showing the dependence on the temperature through the periodic lattice's length from $\omega = \infty$ (no periodicity, $T = 0$) to a very small ω (high T). Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2}{\pi} \frac{ak_B T}{\eta \lambda_{\text{scale}}}$ and point B' by $ae^2 \eta = \frac{8\pi}{q^2} \frac{ak_B T}{\eta \lambda_{\text{scale}}}$. Points A and B at $T = 0$ are equal to those in Fig. 3.5 on page 121: $\frac{a|\psi|^2}{\eta} = 0.322$ and $ae^2 \eta = \frac{12.71}{q^2}$ respectively. Dashed-solid line $P'B'$ at small ω is formed by Kosterlitz-Thouless transition points; short segment $A'P'$ is expected to be of first-order type. At $T = 0$, except for the short segment represented by a dashed-dashed line which starts at A and finishes at the tricritical point tc , segment where the phase transitions are of first-order kind, the rest of line AB is of second-order type.



	$T = 0$	$T = T_{\text{crit}}$ (eq.(3.14))	$T > T_{\text{crit}}$
$\eta \beta \gg \eta e^2$	①: Stage I	critical point	④: advanced Stage III
$\eta \beta \ll \eta e^2$	①: Stage I	$T = T_{\text{trans}}$ (eq.(3.14))	$T > T_{\text{trans}}$
$k < 1$		1st order transition	④: advanced Stage III
$\eta \beta \ll (\eta e^2)^3$	①, (④ metast.)	1st order transition	④: advanced Stage III
$1 < k < k_1$			
$\eta \beta \ll (\eta e^2)^3$	④: advanced Stage III	no phase transition	④: advanced Stage III
$k > k_1$			

Figure 3.11: Expected phases of the effective potential in first approximation for the 3-dim. Abrikosov model. Stage I ① gets characterized by symmetry breaking, by a nonvanishing expectation value of the effective potential's scalar field. Well developed Stage III ④ denotes the state of unimpeded spreading out of the density of gliding dislocations, with its associated loss of rigidity, and corresponds to a zero expectation value $\bar{\rho}^2 = 0$ as lowest minimum of the effective potential. Value k_1 is defined by $k = \frac{1}{4\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} = \frac{3}{2}$.

Chapter 4

Conclusions and further development

4.1 Conclusions

As a summary, let us present the main ideas of this work.

- A new formulation of the equations for the dislocation dynamics has been obtained, taking as model the four Maxwell equations of the electromagnetism. For such a formulation it was determining the analogy between, on the one hand, the diamagnetism displayed by a material and, on the other hand, the effective rigidity of a crystal with respect to which fraction of the dislocations are allowed to glide through the system. Hence, the density and current of dislocations which could glide are set as analogue respectively to $\vec{H}$ and $\vec{D}$, whereas the density and current of the dislocations which actually gliding play the role of $\vec{B}$ and $\vec{E}$. The set of the resulting equations, beside of characterizing how dislocations arrange themselves under the action of an external shear stress and of an externally imposed slip velocity — these latter field variables playing in the equations the same role as the sources of the electromagnetic field in the Maxwell equations, i.e., the role of the electric current and of the electric charge density respectively —, are characterized by the $U(1)$ gauge invariance corresponding to the property that the dislocation density and current remain invariable under any compatible deformation.

The dislocation equations thus obtained can be found in Table 2.2 (page 47) for the activation of one single slip system (the primary) and in Table 2.3 (page 52) for the more general case of gliding in different non-parallel slip systems, with the accompanying reaction between them which yields the formation of sessile dislocations. For the formulation in the manner of the Maxwell equations, these just mentioned sessile dislocations enter in the equations as the density and current of magnetic monopoles would do.

- The aim of the work has been to construct an effective model not only to describe the dislocation dynamics but to reproduce the different plastic behaviours shown by the hardening curve of any *single* crystal containing dislocations. The dislocation fields therefore have to be coupled to another field which describes the rigidity

of the crystal and, by studying the phases which result from such a model, to determine whether this rigidity is strong enough to restrict in an effective way the number of mobile dislocations or whether, on the contrary, the gliding dislocations can spread out over the whole crystal, characterizing thus the loss of rigidity seen in the hardening curve for an advanced state of plastic deformation.

By taking advantage of the ($U(1)$ gauge) invariance under compatible deformations in the formulated equations, together with the analogy diamagnetism-rigidity, the fundamental step towards the modelling of the different hardening behaviours has been to set the similarity between the Meissner phase of a superconductor and the elastic stage in the dislocated crystal, and between the Shubnikov phase and the easy glide stage. The Meissner state is characterized by the ordered state of perfect diamagnetism which does not allow any penetration of the external magnetic field inside the system, similarly to the elastic stage where there is no density of gliding dislocations — being the role of such a density in the equations that of the magnetic field $\vec{B}$ in the Maxwell equations —; and the Shubnikov state gets defined by the appearance of isolated flux lines carrying retained in their interior integer portions of the externally imposed magnetic field, preserving thus the system most of its rigidity against the spreading out over the superconductor of such a magnetic field. And also similarly, the easy glide system is characterized by a rigidity which only permits a very low gliding dislocation density retained inside a low number of mobile dislocations, being the transition from the elastic to the easy glide stage just determined by the occurrence of the first gliding dislocation, in an analogous way as the first vortex signalizes the transition in a superconductor from the Meissner to the Shubnikov phase.

Within the framework of the above analogy, whose qualitative correspondences are summarized in Table 2.4 (page 54), the equation defining the crystal's rigidity, the Hooke's law, becomes completely similar to the London equation, being the latter the effective carrier within the modelled energy of the superconductor's perfect diamagnetism.

- For the partition function which will characterize all hardening behaviours, the next step was to define an effective energy scale η by considering the fact that the size of the Burgers vector introduces a fundamental uncertainty in any position measurement within a dislocated crystal. In the same way as for the quantum uncertainty relations, the just mentioned incapacity to determine exactly the dislocation position within the crystal leads to an uncertainty in the energy of a gliding dislocation segment, a kind of “zero oscillation” energy, which marks the scale for the smallest energy necessary for a gliding segment to overcome a barrier. More exactly, this energy scale gets defined λ_{scale} which is of the order of the shortest path (approximately $1\ \mu\text{m}$ for most crystals) to be covered by cross-slip in order to “jump” over a hindrance. To such energy scale any dislocation configuration has to be compared in order to know, for instance, if the dislocation arrangement under study is able or not to produce a dominant contribution to the partition which would yield a transition towards a phase of high dislocation mobility.

This λ_{scale} , similar to the role of the quantum scale $\hbar c$ in the (effective) Abrikosov model of superconductivity, completes the analogy with the latter model not only in the displayed phases but also in the number of effective parameters necessary for its description, as stated in Table 2.5 on page 88.

- The dislocation theory constructed by using the conventional superconductor as the leading model is able to describe a crystal where only one slip system can be activated. It contains three phases: the elastic stage without any gliding dislocations, the easy glide stage with some isolated mobile dislocations, and the stage where nearly all primary dislocations are allowed to glide through the whole crystal and thus the latter loses its rigidity; in the Abrikosov model describing effectively the superconductor, these three phases are the analogues to the Meissner, the Shubnikov and to the Coulomb state, with the latter characterized by the condensation of the isolated vortices of the Shubnikov phase what leads to the spreading out over the material of the magnetic field. However such a model cannot take account of the hardening stage in a dislocated crystal where different non-parallel slip systems are active, with its associated sessile dislocation segments produced as result of the reaction among dislocations arising from these different slip systems. The implementation of this last feature runs through the substitution in the conventional superconductor model of the usual $U(1)$ gauge field by a gauge field of angular or compact character: it is just the multivalueness in such an angular variable what causes the non-vanishing differential divergence of the density of able-to-glide dislocations; or equivalently, that the gliding dislocation lines end/start at certain points where such lines have become sessile in character.

The different phases displayed by the partition function calculated by means of the 3-dimensional model thus constructed, with a $U(1)$ non-compact symmetry as well as with a compact one, have been identified by the different behaviours shown at large distance by certain gauge invariant correlation functions. Among these latter special attention has been paid to the correlation function $W(C)$, interpreted as the probability for the gliding dislocations crossing the interior of C of escaping from this enclosing loop C . Such a correlation function behaves as it should do for the three above phases in the system with only the primary slip system active. In the case of a *compact* gauge field, and for the range of strong ηe^2 , another further, different phase occurs characterized by a decay in $W(C)$ like the area inside C : such a behaviour — which did not occur for any of the phases of the non-compact model — is the expected one for a phase characterized by mobile dislocations which have become trapped inside certain delimited spatial regions, being able to glide only within such regions (see Fig. 3.2 on page 112). And, on the other hand, this is also the dislocation arrangement occurring in the rapid hardening stage with its characteristic tube formed by the piling up, at very low temperatures, of the disc-structure of sessile Lomer-Cottrell locks which confines inside it the current of gliding dislocations.

This closes the justification for the choice of the superconductivity as model for the description of the hardening curve: the identification, on the one hand, of the vortices in a superconductor with the able-to-glide dislocations, and on the other

hand, the superconductivity diamagnetism with the rigidity against dislocation motion, leads to a dislocation model which displays just the same number of phases as a dislocated crystal, coinciding the characteristics of each phase with those of each plastic stage. Even results as the resolved stress at the yield point for the start of the plastic deformation (see equation (3.5)) or as the shortening of the glide distance during the rapid hardening stage (see discussion leading to (3.9) and further to (3.10)) follows directly from the formulated model.

Hence it has been obtained the phase diagram corresponding to the case of activation of several non-parallel slip systems, in Fig. 3.4 on page 119, as well as the phase diagram for only one slip system activated, in Fig. 3.5 on page 121, displaying just the behaviours to be expected for the actual hardening stages corresponding experimentally to these two cases. Also the 2-dimensional case for the formulated model has been studied, which only shows one phase and therefore only one plastic stage, a correct result for a system with only one slip system which cannot be hardened by any other non-parallel system or which cannot cross-slip to another parallel slip system.

- And finally, the effect of high temperatures on the phase diagrams has been studied. This effect has been implemented in the model by considering the periodic repetition of the dislocation arrangement displayed especially by the single crystal during the breaking of the rapid hardening stage. And also this time the corrected experimental results are reflected in the phase diagram: not only the rapid hardening but also the easy glide stage occupy a gradually smaller place in the phase diagram with increasing temperature, tending the system towards one unique phase which occurs for nearly any value of the model's coupling strengths and which is characterized by the spreading out of the glide motion over the crystal. This phase is that corresponding to the well developed parabolic hardening.

These are all the results which follow from the partition function for this new modelized dislocation theory

$$\begin{aligned} \mathcal{Z} = & \int_{-\pi}^{\pi} \prod_{\vec{r}, \alpha} \frac{d\varphi_{\vec{r}}^{\alpha}}{2\pi} \int_{-\pi}^{\pi} \prod_{\vec{r}, \alpha, \mu} \frac{dA_{\mu, \vec{r}}^{\alpha}}{2\pi} \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\alpha=1}^3 \sum_{\mu, \nu > \mu=1}^3 \left[1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}}^{\alpha} - \Delta_{\nu} A_{\mu, \vec{r}}^{\alpha}) \right] \right] \times \\ & \times \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \alpha, \mu} \left[1 - \cos(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu, \vec{r}}^{\alpha}) \right] \right]. \end{aligned} \quad (4.1)$$

Or with an infinite integration range for the gauge field $A_{\mu, \vec{r}}^{\alpha}$ and with the underlined term substituted by $\frac{1}{2} (\Delta_{\mu} A_{\nu, \vec{r}}^{\alpha} - \Delta_{\nu} A_{\mu, \vec{r}}^{\alpha})^2$ for the case with a non-compact $U(1)$ gauge symmetry. The above partition function can be also re-written by using the Villain approximation as follows

$$\mathcal{Z} = \int_{-\infty}^{\infty} \prod_{\vec{r}, \alpha, \mu} \frac{d\varphi_{\vec{r}}^{\alpha}}{2\pi} \frac{dA_{\mu, \vec{r}}^{\alpha}}{2\pi} \prod_{\vec{r}, \alpha} \prod_{\mu < \nu} \sum_{g_{\mu\nu, \vec{r}}^{\alpha} = -\infty}^{+\infty} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \alpha, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}}^{\alpha} - \Delta_{\nu} A_{\mu, \vec{r}}^{\alpha} - 2\pi g_{\mu\nu, \vec{r}}^{\alpha})^2 \right] \times$$

$$\times \prod_{\vec{r}, \alpha} \prod_{\mu} \sum_{m_{\mu, \vec{r}}^{\alpha} = -\infty}^{+\infty} \exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \alpha, \mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\mu, \vec{r}}^{\alpha} - 2\pi m_{\mu, \vec{r}}^{\alpha} \right)^2 \right], \quad (4.2)$$

where the integer fields $g_{\mu\nu, \vec{r}}^{\alpha}$ and $m_{\mu, \vec{r}}^{\alpha}$ carry the periodicity of the two cosine functions in (4.1): these measures, respectively, the jumps by 2π in the gauge field $A_{\mu, \vec{r}}^{\alpha}$ and in the scalar field's phase $\varphi_{\vec{r}}^{\alpha}$ — for the non-compact model $g_{\mu\nu, \vec{r}}^{\alpha} = 0$ holds.

It is curious to notice that all the above results — correct at least qualitatively — have been obtained from such a partition function in spite of the main simplification carried through the whole work about assuming a homogeneous system described by only one elastic constant. The generalization is straightforward by introducing a symmetric tensor $G_{\lambda\rho, \vec{r}}^{\alpha\beta}$, in general dependent of the lattice site, to replace

$$\left\{ \begin{aligned} & \sum_{\lambda, \alpha} \left[\sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{\mu} A_{\nu, \vec{r}}^{\alpha} - \Delta_{\nu} A_{\mu, \vec{r}}^{\alpha} - 2\pi g_{\mu\nu, \vec{r}}^{\alpha} \right) \right] \left[\sum_{\mu' < \nu'} \epsilon_{\lambda\mu'\nu'} \left(\Delta_{\mu'} A_{\nu', \vec{r}}^{\alpha} - \Delta_{\nu'} A_{\mu', \vec{r}}^{\alpha} - \dots \right) \right], \\ & \sum_{\lambda, \alpha} \left[\Delta_{\lambda} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\lambda, \vec{r}}^{\alpha} - 2\pi m_{\lambda, \vec{r}}^{\alpha} \right] \left[\Delta_{\lambda} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\lambda, \vec{r}}^{\alpha} - 2\pi m_{\lambda, \vec{r}}^{\alpha} \right], \end{aligned} \right.$$

with

$$\left\{ \begin{aligned} & \sum_{\alpha, \beta} \sum_{\lambda, \rho} G_{\lambda\rho, \vec{r}}^{\alpha\beta} \left[\sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{\mu} A_{\nu, \vec{r}}^{\alpha} - \Delta_{\nu} A_{\mu, \vec{r}}^{\alpha} - 2\pi g_{\mu\nu, \vec{r}}^{\alpha} \right) \right] \left[\sum_{\mu' < \nu'} \epsilon_{\rho\mu'\nu'} \left(\Delta_{\mu'} A_{\nu', \vec{r}}^{\beta} - \dots \right) \right], \\ & \sum_{\alpha, \beta} \sum_{\lambda, \rho} G_{\lambda\rho, \vec{r}}^{\alpha\beta} \left[\Delta_{\lambda} \varphi_{\vec{r}}^{\alpha} - q^{\alpha} A_{\lambda, \vec{r}}^{\alpha} - 2\pi m_{\lambda, \vec{r}}^{\alpha} \right] \left[\Delta_{\rho} \varphi_{\vec{r}}^{\beta} - q^{\beta} A_{\rho, \vec{r}}^{\beta} - 2\pi m_{\rho, \vec{r}}^{\beta} \right]. \end{aligned} \right. \quad (4.3)$$

This tensor $G_{\lambda\rho, \vec{r}}^{\alpha\beta}$, symmetric both in the indices of the Burgers vector direction as well as in the coordinate indices, does not affect nevertheless to the internal (2-dimensional) symmetry space where the $U(1)$ gauge transformation takes place. For this generalization therefore the $U(1)$ gauge symmetry — one of the main components of the model, as expression of the invariance of the density of dislocations under any compatible deformation — remains unchanged. By means of $G_{\lambda\rho, \vec{r}}^{\alpha\beta}$ one can also include the effect, for instance, of local impurities or of the dependence on the direction and on the lattice site of the different elastic constants.

A more complete integration of the local geometry in the dislocation model, taking account of non-parallel orientations of the single crystals building the system under study or even the effect of disclinations, will result by coupling the constructed energy defining the partition function (4.1) to the curvature tensor R describing such a local geometry. An especially elegant form of achieving such a coupling is by means of the Kaluza-Klein formulation, which follows from the introduction of a further spatial coordinate which does not enter in the gauge transformations and whose metric's component carries the $U(1)$ gauge field. The complete lagrangian density of the model results thus to take a very simple form, being proportional to the curvature tensor defined in the space extended by this new coordinate. Following the original idea of Klein [73], it is in such a

context where the 4th additional spatial coordinate x_o , introduced in the short excursus at the end of the last chapter, arises in a natural way as “bridge” to implement the effect of the barriers, which the mentioned single crystals non-parallel orientated represent, on the glide motion which describes the dislocation theory formulated as a gauge model.

All the phase transitions considered along this work have been equilibrium phase transitions, i.e., all field configurations entering in the partition function corresponded to macroscopic equilibrium states. Hence another interesting point to be investigated is how the correlation function behave when, during a phase transition, the system starts to leave the old phase to evolve towards the new equilibrium state. This is a question which belongs to subject of the so-called critical dynamics, mainly developed by Hohenberg and Halperin [99], chapter 29.

4.2 Ansatz for the connection to the Anthony’s model of Irreversible Thermodynamics

Before ending definitely, there still remains a point to be discussed briefly: that of the coupling of the here modellized description of the dislocated crystal to the energy dissipation associated to the glide motion of the dislocations. This will be done by incorporating the dislocation model constructed in this work into the Lagrange-formalism of Irreversible Thermodynamics developed by Anthony.

This approach has been led by the idea of unifying within the framework of the Lagrange-formalism also the dissipative processes. The initial steps in this new direction had as result a Lagrangian density for the heat conduction obtained through the duplication of the field variables on which such a Lagrangian density ℓ_{th} depends. Let us introduce the main argument for this field duplication [3]. Even for the stationary heat conduction there exists a non-zero energy current $\vec{J}_{\text{energy}}$. But if the Lagrangian density describing the system would depend exclusively on the temperature T and on its derivatives, $\ell = \ell(T, \partial_t T, \partial_i T)$, the energy current — resulting from the Noether theorem for the particular case of the system’s invariance under a shift in time — would become equal to $\vec{J}_{\text{energy}} = \frac{\partial \ell}{\partial(\vec{\nabla} T)} \partial_t T$ which for the stationary case, $\partial_t T = 0$, vanishes nevertheless. The solution in order to find a Lagrangian density which describes such a process is to consider not directly T as the unique degree of freedom but a complex scalar field $\bar{\psi}_{\text{th}}$ whose length is defined as equal to $\sqrt{T}$

$$T(\vec{r}, t) = \bar{\psi}_{\text{th}}^*(\vec{r}, t) \bar{\psi}_{\text{th}}(\vec{r}, t), \quad (4.4)$$

where the phase of such scalar field, $\bar{\psi}_{\text{th}} = \sqrt{T} e^{i\bar{\phi}_{\text{th}}}$, is related to the velocity of propagation of temperature gradients throughout the system. The duplication in the number of variables in the Lagrangian density, $\ell = \ell(\bar{\psi}_{\text{th}}, \bar{\psi}_{\text{th}}^*, \partial_t \bar{\psi}_{\text{th}}, \partial_t \bar{\psi}_{\text{th}}^*, \partial_i \bar{\psi}_{\text{th}}, \partial_i \bar{\psi}_{\text{th}}^*)$ or equivalently $\ell = \ell(T, \bar{\phi}_{\text{th}}, \partial_t T, \partial_t \bar{\phi}_{\text{th}}, \partial_i T, \partial_i \bar{\phi}_{\text{th}})$, allows us to obtain from the Noether theorem a non-vanishing energy current $\vec{J}_{\text{energy}} = \frac{\partial \ell}{\partial(\vec{\nabla} T)} \partial_t T + \frac{\partial \ell}{\partial(\vec{\nabla} \bar{\phi}_{\text{th}})} \partial_t \bar{\phi}_{\text{th}}$ even in the case of no time-variation in the temperature.

The *effective* Lagrangian density formulated for the heat conduction reads thus [3]

$$\ell_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}}, \quad (4.5)$$

summing over the repeated indices, with $i = 1, 2, 3$. In the above equation $\bar{c}_{\text{th}}$ denotes the specific heat, $\bar{\lambda}_{\text{th}}$ the heat diffusion constant, and $1/\bar{\omega}_{\text{th}}$ the constant which sets the time scale for the smearing out in time of a temperature inhomogeneity — this latter is an effective parameter corresponding to a hidden scale where the interactions actually leading to such smearing out take place. One of the Euler-Lagrange equations following from (4.5) is the Fourier's law for the heat diffusion

$$\begin{aligned} \frac{\partial \ell_{\text{th}}}{\partial(\partial_t \bar{\phi}_{\text{th}})} &= -\frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T, & \frac{\partial \ell_{\text{th}}}{\partial(\partial_i \bar{\phi}_{\text{th}})} &= -\frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T, & \frac{\partial \ell_{\text{th}}}{\partial \bar{\phi}_{\text{th}}} &= 0, \\ -\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T &= 0. \end{aligned} \quad (4.6)$$

This equation, as well as the above Lagrangian density ℓ_{th} , is an effective modellization at large distances by means of a low number of parameters and field variables of much more complex interactions occuring at a very short scale which remains unreachable. And it will require to be extended if more complicated phenomena are to be described. For instance, $\ell_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}}$ does not contain dissipation yet, something which would correspond to an Euler-Lagrange equation of the type

$$-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = \text{rate of energy dissipation}. \quad (4.7)$$

Let us now focus our attention on the gliding dislocations since their glide motion is what causes the energy dissipation due to the breaking and consecutive building of interatomic bonds during the glide motion; note that, on the contrary, the dragging along of dislocations together with the crystal lattice during an elastic (or compatible) deformation such breaking and building of bonds does not occur and thus dislocation dragging is reversible. For the dislocation dynamics the Lagrangian density has been already obtained as a result of the formulation in the same manner as the Maxwell equations (see Tables 2.2 and 2.3): it is given by equation (2.37)

$$\ell_{\text{gauge}} = \frac{1}{2} \mathbf{E}_i^\alpha \mathbf{D}_i^\alpha - \frac{1}{2} \mathbf{B}_i^\alpha \mathbf{H}_i^\alpha - \left(-\rho v_{\text{slip}}^\alpha \right) A_0^\alpha + \frac{1}{c_{\text{sound}}} (\boldsymbol{\tau}_{\text{resol}})_k^\alpha A_k^\alpha. \quad (4.8)$$

And now the gauge field $A_\mu^\alpha(\vec{r}, t)$ describing the dislocations has to be coupled to the scalar field $\bar{\psi}_{\text{th}}(\vec{r}, t)$. For the following, and since gliding is the source of dissipation, we shall restrict ourselves to the plastic stage characterized by a high dislocation mobility: the well developed parabolic hardening stage or Stage III. In fact, this is the opposite point to that studied for the equilibrium phase transitions of the previous Chapters, where the number of mobile dislocations and their static arrangements were the subject to pay attention to. In the case under study now, nevertheless, we are in the hardening stage where the production of gliding dislocations is nearly saturated (see Fig. 2.3 on page 36) and relations $\begin{cases} \sqrt{\mu_0} \mathbf{B}_i^\alpha \equiv (\boldsymbol{\alpha}_{\text{glid}})_i^\alpha \approx (\boldsymbol{\alpha}_{\text{slip on prim}})_i^\alpha \equiv \sqrt{\mu_0} \mathbf{H}_i^\alpha \\ \sqrt{\epsilon_0} \mathbf{E}_i^\alpha \equiv (\boldsymbol{\mathcal{J}}_{\text{glid}})_i^\alpha \approx (\boldsymbol{\mathcal{J}}_{\text{slip on prim}})_i^\alpha \equiv \sqrt{\epsilon_0} \mathbf{D}_i^\alpha \end{cases}$ have been

reached, what indicates that the material is no longer able to restrict the gliding dislocation density into spatial delimited regions: in the language of the superconductor model, the system has lost its diamagnetism — and similarly for the Stage II, the system has lost its capacity of restricting the gliding dislocation current into the tubes delimited by sessile segments. Further, we shall assume that the temperature is high enough so that this Stage III is approximately the only phase displayed by the system. Hence, the question about sessile dislocations will be avoided and the dislocation equations describing the state correspond to that of only one slip system active, as in Table 2.2 on page 47. And also the coupling of the gauge field $A_\mu^\alpha(\vec{r}, t)$ to the scalar field, which carries the information about the crystal rigidity and which was responsible for the limited spreading out of the gliding dislocation density, plays no role since in this stage: (4.8) describes completely the behaviour of the dislocation system.

As argued, and returning to (4.7), the dissipation rate must be proportional to the current of gliding dislocations $\mathbf{E}_i^\alpha$. The correct ansatz can be directly taken from the electromagnetism where $\frac{\vec{H} \cdot \vec{B}}{2} + \frac{\vec{D} \cdot \vec{E}}{2}$ is the energy density and according to result $\vec{j}_{\text{el}} \cdot \vec{E} = -\text{div}(c\vec{E} \times \vec{H}) - \partial_t \left(\frac{\vec{H} \cdot \vec{B}}{2} + \frac{\vec{D} \cdot \vec{E}}{2} \right)$ the dissipation rate is given by $\vec{j}_{\text{el}} \cdot \vec{E}$, which is the energy dissipated from the system by the Joule effect¹. And in the dislocation model, therefore, this is given by

$$\text{rate of energy dissipated from the dislocation system} = (\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha, \quad (4.9)$$

since in the dislocation formulation *à la* Maxwell, the external applied stress $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$ plays the same role as the electric current $(j_{\text{el}})_i$, as well as the current of gliding dislocations does as the electric field. The equation for the temperature field becomes

$$-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = -(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha, \quad (4.10)$$

where the change in sign for $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ is due to the fact that the energy loss in the dislocation system is absorbed, and thus $\bar{c}_{\text{th}} \partial_t T = \dots + (\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$, by the temperature field. This ensures the conservation of the whole energy arising from the temperature and from the dislocation fields

$$\partial_t \left(\bar{c}_{\text{th}} T + \left(\frac{\mathbf{H}_i^\alpha \mathbf{B}_i^\alpha}{2} + \frac{\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha}{2} \right) \right) + \text{div} \left(-\bar{\lambda}_{\text{th}} \vec{\nabla} T + c_{\text{sound}} \epsilon_{jik} \mathbf{E}_i^\alpha \mathbf{H}_k^\alpha \right) = 0. \quad (4.11)$$

¹The same deduction can be carried out for the dislocation model where, always summing over repeated indices, one starts with $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ and by using the dynamical equations first $\epsilon_{ijk} \partial_j \mathbf{H}_k^\alpha = \frac{1}{c_{\text{sound}}} (\boldsymbol{\tau}_{\text{res}})_i^\alpha + \frac{1}{c_{\text{sound}}} \partial_t \mathbf{D}_i^\alpha$ and second $\epsilon_{kji} \partial_j \mathbf{E}_i^\alpha = -\frac{1}{c_{\text{sound}}} \partial_t \mathbf{B}_k^\alpha$ (the contributions of the sessile segments are irrelevant in Stage III at high temperatures), one obtains $(\boldsymbol{\tau}_{\text{res}})_i^\alpha \mathbf{E}_i^\alpha = c_{\text{sound}} (\epsilon_{ijk} \partial_j \mathbf{H}_k^\alpha) - \frac{1}{2} \partial_t (\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha) = -\partial_j \left(c_{\text{sound}} \epsilon_{jik} \mathbf{E}_i^\alpha \mathbf{H}_k^\alpha \right) + c_{\text{sound}} \mathbf{H}_k^\alpha \underbrace{\epsilon_{kji} \partial_j \mathbf{E}_i^\alpha}_{-\frac{1}{c_{\text{sound}}} \partial_t \mathbf{B}_k^\alpha} - \frac{1}{2} \partial_t (\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha) = -\partial_j \left(c_{\text{sound}} \epsilon_{jik} \mathbf{E}_i^\alpha \mathbf{H}_k^\alpha \right) - \partial_t \left(\frac{\mathbf{H}_i^\alpha \mathbf{B}_i^\alpha}{2} + \frac{\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha}{2} \right).$

In order to yield $-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = -(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$, one must be a bit careful in the manner how (4.9) is implemented into the Lagrangian density ℓ_{th} . The sensitive point is the invariance of $\ell_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}}$ under the global transformation in the phase of $\bar{\psi}_{\text{th}}$: $\bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}_0$, with $\bar{\chi}_0$ constant. One cannot incorporate the source term $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ for the inhomogeneous equation $-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = \text{“source”}$ through the (double underlined) term

$$\ell'_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}} + \frac{1}{\bar{\omega}_{\text{th}}} \bar{\phi}_{\text{th}} [-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha], \quad (4.12)$$

without removing the just mentioned global invariance. This statement deserves two precissions:

1. A partial solution in order to keep the global invariance is the introduction of a so-called transfer variable $\bar{\Phi}$ [5] in the form

$$\ell'_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}} + \frac{1}{\bar{\omega}_{\text{th}}} (\bar{\phi}_{\text{th}} - \bar{\Phi}) [-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha],$$

and then to require that $\bar{\Phi}$ has to transform in the same way as $\bar{\phi}_{\text{th}}$ under any global transformation, $\begin{cases} \bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}_0 \\ \bar{\Phi} \rightarrow \bar{\Phi} + \bar{\chi}_0 \end{cases}$, what ensures thus the global invariance of the Lagrangian density. This procedure is only meaningful if we are interested in studying exclusively the dynamics of $\bar{\psi}_{\text{th}}$ (and its phase) without investigating which are the actual degrees of freedom whose effect is represented by field $\bar{\Phi}$, field which for its part will have its own dynamics to be determined. However, in the case under study now the real origin of the dissipation has been already identified (the current of gliding dislocations) and the just presented procedure is not useful since we are looking for a non-mediate coupling between $\bar{\phi}_{\text{th}}$ and the gauge field A_μ^α defining the current and density of the able-to-glide dislocations.

2. The global invariance is still maintained if the source term can be written as $\partial_t \mathcal{M} + \partial_i \mathcal{N}_i$ since in such a case the Lagrangian density will become equal to

$$\begin{aligned} \ell'_{\text{th}} &= \dots + \frac{1}{\bar{\omega}_{\text{th}}} \bar{\phi}_{\text{th}} (\partial_t \mathcal{M} + \partial_i \mathcal{N}_i) = \dots - \frac{1}{\bar{\omega}_{\text{th}}} \mathcal{M} \partial_t \bar{\phi}_{\text{th}} - \frac{1}{\bar{\omega}_{\text{th}}} \mathcal{N}_i \partial_i \bar{\phi}_{\text{th}} = \\ &= -\bar{c}_{\text{th}} T - \frac{1}{\bar{\omega}_{\text{th}}} (\bar{c}_{\text{th}} T + \mathcal{M}) \partial_t \bar{\phi}_{\text{th}} - \frac{1}{\bar{\omega}_{\text{th}}} (-\bar{\lambda}_{\text{th}} \partial_i T + \mathcal{N}_i) \partial_i \bar{\phi}_{\text{th}}. \end{aligned} \quad (4.13)$$

Therefore, the correct statement is that the invariance under a global transformation in the phase $\bar{\phi}_{\text{th}}$ is removed if this $\bar{\phi}_{\text{th}}$ is coupled directly to a source term which is not equal to the 4-divergence of a certain 4-vector $(\mathcal{M}, \mathcal{N}_x, \mathcal{N}_y, \mathcal{N}_z)$: “source term” $\neq \partial_t \mathcal{M} + \partial_i \mathcal{N}_i$. But even this latter case, as one can see in the final result of (4.13), does not represent a real coupling of the phase of field $\bar{\psi}_{\text{th}}$ to the degrees of freedom responsible for the source term. An example of this

case is just the naive ansatz which would arise from using relation $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha = \partial_t \left(\frac{\mathbf{H}_i^\alpha \mathbf{B}_i^\alpha}{2} + \frac{\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha}{2} \right) + \partial_j \left(c_{\text{sound}} \epsilon_{jik} \mathbf{E}_i^\alpha \mathbf{H}_k^\alpha \right)$ and which would result in

$$\begin{aligned} \ell'_{\text{th}} = & -\bar{c}_{\text{th}} T - \frac{1}{\bar{\omega}_{\text{th}}} \left(\bar{c}_{\text{th}} T + \left(\frac{\mathbf{H}_i^\alpha \mathbf{B}_i^\alpha}{2} + \frac{\mathbf{D}_i^\alpha \mathbf{E}_i^\alpha}{2} \right) \right) \partial_t \bar{\phi}_{\text{th}} - \\ & - \frac{1}{\bar{\omega}_{\text{th}}} \left(-\bar{\lambda}_{\text{th}} \partial_i T + c_{\text{sound}} \epsilon_{jik} \mathbf{E}_i^\alpha \mathbf{H}_k^\alpha \right) \partial_i \bar{\phi}_{\text{th}}, \end{aligned}$$

since this yields nothing else as the trivial conservation (4.11) of the sum of all energy parts but without modifying the field equation for $\bar{\phi}_{\text{th}}$, which according to $\partial_t \left(\frac{\partial \ell'_{\text{th}}}{\partial(\partial_t T)} \right) + \partial_i \left(\frac{\partial \ell'_{\text{th}}}{\partial(\partial_i T)} \right) = \frac{\partial \ell'_{\text{th}}}{\partial T}$ remains the same as when $\bar{\phi}_{\text{th}}$ was not coupled to $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ — this last property is due to the fact that ℓ_{th} is linear in the terms containing both T as well as $\bar{\phi}_{\text{th}}$.

In fact the only forms to couple $\bar{\phi}_{\text{th}}$ to the dissipation's source term are already fixed, if the global invariance is to be maintained, by this invariance itself. The invariance of Lagrangian density $\ell_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}}$ under the global transformation $\bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}_0$ is the actual responsible for equation $-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = 0$ as it can be easily shown: from the discussion about the Noether theorem accompanying result (2.65) (see page 67), with the global invariance is associated the conservation

of a density and current given by
$$\begin{cases} J_0 &= \frac{\partial \ell}{\partial(\partial_t \bar{\phi}_{\text{th}})} \delta \bar{\phi}_{\text{th}} \\ J_{i=1,2,3} &= \frac{\partial \ell}{\partial(\partial_i \bar{\phi}_{\text{th}})} \delta \bar{\phi}_{\text{th}} \end{cases} \text{ respectively. But the con-}$$

servation law $\partial_t J_0 + \partial_i J_i = 0$ derived from the Noether theorem is equivalent to the Euler-Lagrange equation (4.6), by noticing that $\delta \bar{\phi}_{\text{th}} = \bar{\chi}_0 = \text{const.}$ This is a result to be expected since this global invariance of the Lagrangian density is actually the content of equation $\frac{\partial \ell_{\text{th}}}{\partial \bar{\phi}_{\text{th}}} = 0$ in (4.6): ℓ_{th} does not depend explicitly on phase $\bar{\phi}_{\text{th}}$ — or

more exactly, since the ℓ_{th} 's terms are linear in the derivatives of $\bar{\phi}_{\text{th}}$, any Lagrangian's term for which ℓ_{th} would depend explicitly on $\bar{\phi}_{\text{th}}$ can only be of the form $\bar{\phi}_{\text{th}} \partial_t ()$ or of the form $\bar{\phi}_{\text{th}} \partial_i ()$, and thus to be re-written as $-() \partial_t \bar{\phi}_{\text{th}}$ and as $-() \partial_i \bar{\phi}_{\text{th}}$ — and therefore $\partial_t J_0 + \partial_i J_i = 0$ corresponds to $\partial_t \left(\frac{\partial \ell_{\text{th}}}{\partial(\partial_t \bar{\phi}_{\text{th}})} \right) + \partial_i \left(\frac{\partial \ell_{\text{th}}}{\partial(\partial_i \bar{\phi}_{\text{th}})} \right) = \frac{\partial \ell_{\text{th}}}{\partial \bar{\phi}_{\text{th}}} = 0$.

Hence, if we want to keep the global invariance, the implementation towards an inhomogeneous equation $-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T \neq 0$ can only be accomplished by some term *non-linear* in the *derivatives* of phase $\bar{\phi}_{\text{th}}$, such that $\frac{\partial \ell'_{\text{th}}}{\partial \bar{\phi}_{\text{th}}} = 0$ still holds for the

extended $\ell'_{\text{th}} = \ell_{\text{th}} + \ell_{\text{coupling}}$ but $\frac{\partial \ell'_{\text{th}}}{\partial(\partial_t \bar{\phi}_{\text{th}})}$ or $\frac{\partial \ell'_{\text{th}}}{\partial(\partial_i \bar{\phi}_{\text{th}})}$ (or both) receives an additional contribution to yield the inhomogeneous equation we are searching for

$$\underbrace{\partial_t \left(\frac{\partial \ell_{\text{th}}}{\partial(\partial_t \bar{\phi}_{\text{th}})} \right) + \partial_i \left(\frac{\partial \ell_{\text{th}}}{\partial(\partial_i \bar{\phi}_{\text{th}})} \right)}_{\frac{1}{\bar{\omega}_{\text{th}}} (-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T)} = \underbrace{\frac{\partial \overbrace{\ell'_{\text{th}} + \ell_{\text{coupling}}}}{\partial \bar{\phi}_{\text{th}}}}_{=0} - \underbrace{\partial_t \left(\frac{\partial \ell_{\text{coupling}}}{\partial(\partial_t \bar{\phi}_{\text{th}})} \right) + \partial_i \left(\frac{\partial \ell_{\text{coupling}}}{\partial(\partial_i \bar{\phi}_{\text{th}})} \right)}_{\neq 0}. \quad (4.14)$$

The conclusion that the source for the inhomogeneous heat conduction equation has to be introduced by means of such non-linear terms is also proved, by other methods, in [4]. Although also in result (4.14) the source term has the form $\partial_t \mathcal{M} + \partial_i \mathcal{N}_i$ as that discussed in (4.13), these terms $\mathcal{M}$ and $\mathcal{N}_i$ are now not coupled directly to $\bar{\phi}_{\text{th}}$ but to its derivatives, what ensures the global invariance of the extended $\ell'_{\text{th}} = \ell_{\text{th}} + \ell_{\text{coupling}}$.

Why are we so much interested in preserving the invariance of the Lagrangian density under any transformation $\bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}_0$ with $\bar{\chi}_0$ constant? First of all because, physically, the removing of a global invariance can only be achieved by imposing, through *external* means, boundary conditions on the system². A well known system where a similar breaking occurs is the classical spin model where its energy is determined by the relative orientation between two nearest neighbour spins. A simultaneous change in the orientation by the same amount in all spins leaves unchanged the energy. Only a boundary condition or the application of an external magnetic field breaks such global invariance by choosing a preferred orientation for all spins; and the effect of the external magnetic field on the spins is described by a term where this magnetic field is directly coupled to the spin's orientation angle. Returning to our general considerations, the removing of the global invariance from the Lagrangian density caused by the direct coupling of $\bar{\phi}_{\text{th}}$ to the source term for the inhomogeneous equation $-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = \text{“source”}$ could be only justified therefore if such a source term were controlled by external means. But, on the contrary, if such “source” is originated by the interaction with other field variables whose dynamics is completely determined by the Lagrangian and not by the exterior, such a symmetry cannot be broken — again the fact that interactions with other field variables can enable the dynamical breaking of a global invariance (just as in equation (D.306) on page 693) has to be distinguished from the removing of the global invariance from the Lagrangian (or from the energy) since in the former case the invariance has only disappeared from the macroscopic realization of lowest energy but not from the energy describing the system —; the above argument is the idea behind equation (8.13) in [12] p. 104, ansatz which breaks the invariance under $\bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}_0$ because $\bar{\phi}_{\text{th}}$ (and not its derivatives) is directly coupled to the dislocation variables (in the same way as discussed in (4.12)), coupling which can therefore only hold if these field variables are controlled externally.

Actually the source term $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ contains an externally determined field, stress $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$, and thus one could expect that the global invariance under discussion now

²One should be very careful and not confuse two different concepts: that of the *removing* of a global invariance and that of the *breaking* of a global invariance (spontaneous as well as dynamical). The latter occurs in a system described by an energy (or by a Lagrangian) which is invariant under the simultaneous addition to all field's phases of the same amount but whose state of lowest energy does not display such a symmetry; the global invariance nevertheless remains as symmetry in the Lagrangian, something not occurring in the case under study now where such Lagrangian's invariance is destroyed.

would get lost. However, and in order to complete the analogy with the electromagnetism's Joule effect — analogy from which $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha$ has followed —, $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha$ can be directly related to $\mathbf{E}_i^\alpha$ through a material equation of the kind $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha = \bar{\kappa} \mathbf{E}_i^\alpha$, with $\bar{\kappa}$ measuring the effectiveness of the external stress to produce a current of gliding dislocations, parameter which plays the same role as the electric conductivity in the Maxwell theory. In this latter case, source term $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha = -\bar{\kappa} (\mathbf{E}_i^\alpha)^2$ depends on degrees of freedom, gauge field A_μ^α , which are freely variable and whose dynamics becomes controlled by the Lagrangian³. And thus the Lagrangian's symmetry represented by the global invariance in $\bar{\phi}_{\text{th}}$ cannot be broken by the coupling to other Lagrangian's field variables.

There is a further reason to maintain the global symmetry: this allows us to introduce the interaction of $\bar{\psi}_{\text{th}}$ to other fields by raising such a global invariance to a local one, through replacing the derivatives of $\bar{\phi}_{\text{th}}$ by covariant derivatives. The phase of $\bar{\psi}_{\text{th}}$ carries the information of the propagation of any temperature inhomogeneous distribution throughout the system. A *local* transformation in such a phase, $\bar{\phi}_{\text{th}} \rightarrow \bar{\phi}_{\text{th}} + \bar{\chi}(\vec{r}, t)$, corresponds to the choice at each system's point of a different propagation rate. But the accumulation of the temperature at certain sites which such a choice would produce has to be compensated by the coupling to some gauge field which “absorbs” these local changes. And it is this feature what we were looking for in order to couple $\bar{\phi}_{\text{th}}$ to the field describing the gliding dislocations: the local stresses, created by the different rapidity in the propagation of T which the transformation in $\bar{\phi}_{\text{th}}$ by a local scalar function $\chi(\vec{r}, t)$ would yield at certain points, are compensated by gliding dislocation's gauge field $A_\mu^\alpha \equiv (A_0^\alpha, A_{i=1,2,3}^\alpha)$ whose motion is made easier or impeded at such points.

Summarizing: the implementation in ℓ_{th} of the dissipation due to the glide motion of the dislocations is achieved by means of two consecutive steps:

1. To introduce a term where the field variables origin of the dissipation are coupled to non-linear terms of the derivatives of $\bar{\phi}_{\text{th}}$ — but not to $\bar{\phi}_{\text{th}}$ directly, in order to maintain the global invariance.
2. And then to generalize the global invariance to a local symmetry by substituting with the covariant derivatives the partial derivatives initially contained in the Lagrangian (in CGS units) $\begin{cases} \partial_t \bar{\phi}_{\text{th}} \rightarrow \partial_t \bar{\phi}_{\text{th}} - g^\alpha c_{\text{sound}} A_0^\alpha \\ \partial_i \bar{\phi}_{\text{th}} \rightarrow \partial_i \bar{\phi}_{\text{th}} - g^\alpha A_i^\alpha \end{cases}$, where g^α is a constant (actually depending on the direction of the Burgers vector of dislocations) to be determined which characterizes the strength of the coupling between $\bar{\phi}_{\text{th}}$ and the able-to-glide dislocations.

And these two step have to be completed by including into the complete system's Lagrangian the pure gauge field part $\frac{1}{2} \mathbf{E}_i^\alpha \mathbf{D}_i^\alpha - \frac{1}{2} \mathbf{B}_i^\alpha \mathbf{H}_i^\alpha$ which determines the dynamics

³Another possible relation between the stress and the gauge degrees of freedom just as the $-(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \sim A_i^\alpha$ responsible for the high material rigidity and the corresponding restricted spreading out of the gliding dislocation density (characteristic of the elastic stage and of Stage I) is of no interest for the hardening stage we are studying now and which is characterized by the opposite feature of very low rigidity and high dislocation mobility throughout the whole crystal.

of the gauge field coupled to the phase of the temperature field.

One arrives thus at the following ansatz for the incorporation of dissipation into the initial Lagrangian $\ell_{\text{th}} = -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \partial_t \bar{\phi}_{\text{th}} + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \partial_i \bar{\phi}_{\text{th}}$

$$\begin{aligned} \ell = & -\bar{c}_{\text{th}} T - \frac{\bar{c}_{\text{th}}}{\bar{\omega}_{\text{th}}} T \left(\partial_t \bar{\phi}_{\text{th}} - \underline{\underline{g^\alpha c_{\text{sound}} A_0^\alpha}} \right) + \frac{\bar{\lambda}_{\text{th}}}{\bar{\omega}_{\text{th}}} \partial_i T \left(\partial_i \bar{\phi}_{\text{th}} - \underline{\underline{g^\alpha A_i^\alpha}} \right) + \\ & + \frac{1}{2} \mathbf{E}_i^\alpha \mathbf{D}_i^\alpha - \frac{1}{2} \mathbf{B}_i^\alpha \mathbf{H}_i^\alpha + \frac{1}{\underline{\underline{\bar{\omega}_{\text{th}} g c_{\text{sound}}}}} \left(\partial_t \bar{\phi}_{\text{th}} - g^\alpha c_{\text{sound}} A_0^\alpha \right) \left(\partial_i \bar{\phi}_{\text{th}} - g^\alpha A_i^\alpha \right), \end{aligned} \quad (4.15)$$

where the last double underlined term is the above argued non-linear term in the derivatives of $\bar{\phi}_{\text{th}}$. The field equation for T deriving from (4.15) is the following

$$\begin{aligned} -\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = & \bar{\omega}_{\text{th}} \overbrace{\frac{\partial \ell}{\partial \bar{\phi}_{\text{th}}}}^{\equiv 0} - \partial_t \left(\frac{(\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \left(\partial_i \bar{\phi}_{\text{th}} - g^\alpha A_i^\alpha \right) \right) - \\ & - \partial_i \left(\frac{(\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \left(\partial_t \bar{\phi}_{\text{th}} - g^\alpha c_{\text{sound}} A_0^\alpha \right) \right), \end{aligned} \quad (4.16)$$

or equivalently

$$\begin{aligned} -\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T = & -(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \overbrace{\left(-\frac{1}{c_{\text{sound}}} \partial_t A_i^\alpha - \partial_i A_0^\alpha \right)}^{\equiv \mathbf{E}_i^\alpha} - \\ & - 2 \frac{(\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \partial_i \partial_t \bar{\phi}_{\text{th}} - \frac{\partial_t (\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \left(\partial_i \bar{\phi}_{\text{th}} - g^\alpha A_i^\alpha \right) - \\ & - \frac{\partial_i (\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \left(\partial_t \bar{\phi}_{\text{th}} - g^\alpha c_{\text{sound}} A_0^\alpha \right). \end{aligned} \quad (4.17)$$

Although the above ansatz does not yield alone the equation we are looking for, the superfluous terms in the two last lines of the previous result can be shown to yield an almost vanishing contribution in the case of a crystal which under a tensile machine is plastically deformed into the well developed parabolic hardening stage (Stage III) at high temperatures. Hence $\partial_t (\boldsymbol{\tau}_{\text{resol}})_i^\alpha$ for a stage described by $\boldsymbol{\tau}^\alpha \approx \Theta' \sqrt{\text{strain } \varepsilon^\alpha}$ with Θ' small, strain $\varepsilon \sim 1$ (due to the advanced state of deformation) and at a constant rate $\partial_t \varepsilon$ imposed by the machine which is typically of the order of $10^{-4} \text{ seconds}^{-1}$, can be neglected. On the other hand, $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha$ is related to the change in time of the slip velocity of the crystal mass elements, and this for its part is due to a pronounced production of gliding dislocations in the primary plane (see argument following equation (2.27) on page 45); but in the well developed Stage III at high temperatures there is nearly no production of gliding dislocations as one can check from the final part of Fig. 2.3 on page 36 and therefore $\text{div } \boldsymbol{\tau}_{\text{resol}}^\alpha \approx 0$. And finally, in [3] it is shown that solution $\bar{\phi}_{\text{th}} = -\bar{\omega}_{\text{th}} t$ is consistent with the Euler-Lagrange equations for the phase of field $\bar{\psi}_{\text{th}}$ — solution which

also holds in the generalized case where the model is invariant under any local change in $\bar{\phi}_{\text{th}}$ simply by fixing the appropriate relation for the gauge field among all possible gauge choices with describe the same macroscopic state —, what leads to $\partial_i \partial_t \bar{\phi}_{\text{th}} = 0$ and to

$$-\bar{c}_{\text{th}} \partial_t T + \bar{\lambda}_{\text{th}} \partial_i \partial_i T \approx -(\boldsymbol{\tau}_{\text{resol}})_i^\alpha \mathbf{E}_i^\alpha, \quad (4.18)$$

for the advanced deformation state within the Stage III. Moreover, for such a plastic stage characterized by a high dislocation mobility (since the rigidity which opposed such mobility has already disappeared), inhomogeneities in the temperature can be rapidly smeared out over the whole crystal through the glide motion of the temperature: since phase $\bar{\phi}_{\text{th}}$ determines how these inhomogeneities propagate, then for an extended smearing out second derivative $\partial_i \partial_t \bar{\phi}_{\text{th}}$ is small and can be neglected at a first approximation.

But (4.18) is not the only result deriving from Lagrangian density (4.15). Also the source for the dislocation gauge field, defined by $\epsilon_{ijk} \partial_j \mathbf{H}_k^\alpha - \frac{1}{c_{\text{sound}}} \partial_t \mathbf{D}_i^\alpha = \text{source}$, contains now contributions from the temperature field and its phase

$$\frac{\partial \ell}{\partial A_i^\alpha} = g^\alpha \frac{(-\bar{\lambda}_{\text{th}} \partial_i T)}{\bar{\omega}_{\text{th}}} - (\boldsymbol{\tau}_{\text{resol}})_i^\alpha \frac{1}{\bar{\omega}_{\text{th}}} \left(\frac{1}{c_{\text{sound}}} \partial_t \bar{\phi}_{\text{th}} - g^\alpha A_0^\alpha \right). \quad (4.19)$$

First term $g^\alpha \frac{(-\bar{\lambda}_{\text{th}} \partial_i T)}{\bar{\omega}_{\text{th}}}$ states that a gradient in the temperature contributes to the variation in time of the dislocation current $\mathbf{D}_i^\alpha$ in the primary slip system, which for the advanced Stage III is approximately equal to the current of gliding dislocations; the direction i of such a current coincides with the direction of the temperature gradient. The second term in the right hand side of relation (4.19) is just the contribution to transversal gradients in the dislocation distribution in the primary slip system and to the change of the dislocation current created by the dissipation. It is in this latter term where the analogue in the dislocation model to the Ohm law, $(\boldsymbol{\tau}_{\text{resol}})_i^\alpha = \bar{\kappa} \mathbf{E}_i^\alpha$, can be substituted in order to study how the applied stress affects on the dislocation system through its relation to the current of gliding dislocations, and from it, how strong the effect on the dynamics of the temperature field becomes.

The plausibility of the ansatz in Lagrangian density (4.15) can be further supported by the following consideration. Constant $1/\bar{\omega}_{\text{th}}$ is the characteristic time constant for the evolution of any non-homogenous temperature distribution. Hence limit $\lim_{\bar{\omega}_{\text{th}} \rightarrow \infty}$ corresponds to a state characterized by an (instantaneously) homogeneous distribution of the temperature throughout the whole system. And in such a case one should expect the decoupling of the dislocation degrees of freedom from the temperature field,

what actually occurs by taking into account $\lim_{\bar{\omega}_{\text{th}} \rightarrow \infty} \frac{(-\bar{\lambda}_{\text{th}} \partial_i T)}{\bar{\omega}_{\text{th}}} = 0$ together with the above discussed solution for $\bar{\phi}_{\text{th}}$ which leads to relation $\lim_{\bar{\omega}_{\text{th}} \rightarrow \infty} \frac{1}{\bar{\omega}_{\text{th}}} \left(\frac{1}{c_{\text{sound}}} \partial_t \bar{\phi}_{\text{th}} - g^\alpha A_0^\alpha \right) = \frac{1}{c_{\text{sound}}} \lim_{\bar{\omega}_{\text{th}} \rightarrow \infty} \frac{1}{\bar{\omega}_{\text{th}}} \partial_t \bar{\phi}_{\text{th}} \Big|_{\bar{\phi}_{\text{th}} \sim -\bar{\omega}_{\text{th}} t} + 0 = -\frac{1}{c_{\text{sound}}}$ and thus to $\lim_{\bar{\omega}_{\text{th}} \rightarrow \infty} \frac{\partial \ell}{\partial A_i^\alpha} = \frac{1}{c_{\text{sound}}} (\boldsymbol{\tau}_{\text{resol}})_i^\alpha$, resulting in the relation (2.29) obtained as the dissipation had no influence on the dislocation system. Concluding this section, the just presented argument as well as re-

sult (4.18) justify the choice of ansatz $\frac{1}{\bar{\omega}_{\text{th}}} \frac{(\boldsymbol{\tau}_{\text{resol}})_i^\alpha}{g c_{\text{sound}}} \left(\partial_t \bar{\phi}_{\text{th}} - g^\alpha c_{\text{sound}} A_0^\alpha \right) \left(\partial_i \bar{\phi}_{\text{th}} - g^\alpha A_i^\alpha \right)$ for the description of the dissipation occuring in the plastic stage characterized by an non-restricted spreading out of the gliding dislocations over the crystal ⁴.

⁴Of course, all the above discussion can be repeated word for word if field $\bar{\psi}_{\text{th}}$ is coupled to the gauge field of the electromagnetism.

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 Numbers in *italic* refer to footnotes.

Appendix A

The Abrikosov model for Superconductivity

A.1 The Meissner phase

Works [39] and [72] will be followed and we shall operate in the CGS system. Einstein convention of summation over repeated indices is used throughout the following discussion.

The superconductor is described by an effective Lagrangian density ℓ of Landau-Ginzburg kind for an order parameter ψ , which is a complex scalar field coupled to a gauge field A_j ($j = 1, 2, 3$)¹

$$\ell = -\frac{1}{4}F_{jk}F^{jk} - \frac{\hbar^2}{2m}\left|\left(\partial_j - \frac{iqe}{\hbar c}A_j\right)\psi\right|^2 + |\alpha||\psi|^2 - \frac{|\beta|}{2}|\psi|^4, \quad (\text{A.1})$$

where $F_{jk} = \partial_j A_k - \partial_k A_j$ is the field strength or magnetic field. The coupling constant e is the electron's charge or fundamental charge (the smallest of all charges) and q is number of fundamental charges carried by the order parameter: for a superconductor where ψ characterizes the density of Cooper pairs, $q = 2$. It is to be stressed that (A.1) is an *effective* model, in the sense that neither ψ nor $\vec{A}$ have to be the real microscopic variables which describes each one of the possible microscopic configurations which are compatible with a certain macroscopic state.

Since we shall only consider time independent states, (Helmholtz) free energy density $\mathcal{F}$ is equal, up to an additive constant corresponding to the free energy density of the normal conductor, to minus the Lagrangian density ℓ : $\mathcal{F}_{\text{super, Hext}} = \mathcal{F}_{\text{normal, Hext}=0} - \ell$. The

¹Actually, working in the CGS system would demand a Lagrangian for the electromagnetic fields equal to $-\frac{1}{16\pi}F_{jk}F^{jk}$ which yields $\text{curl } \vec{H} = \frac{4\pi}{c}\vec{j}$ as equation. But in order not to get too involved with factor 4π when one changed from the MKS to the CGS system, we shall consider the Lagrangian as $-\frac{1}{4}F_{jk}F^{jk}$, $j, k = 1, 2, 3$, which gives equation $\text{curl } \vec{H} = \frac{1}{c}\vec{j}$. The differences with the results in [39] are due to this change.

Euler-Lagrange's equations are

$$-\frac{\hbar^2}{2m} \left(\partial_k - \frac{iqe}{\hbar c} A_k \right)^2 \psi - |\alpha| \psi + |\beta| |\psi|^2 \psi = 0,$$

and

$$(\text{curl } B)_k = -\frac{iqe\hbar}{2mc} \left[\psi^* \left(\partial_k - \frac{iqe}{\hbar c} A_k \right) \psi - \text{c.c.} \right] = \frac{1}{c} (\vec{j})_k,$$

(A.2)

where c.c. means complex conjugated. $\vec{B} = \text{curl } \vec{A}$ is the local magnetic induction and the (macroscopic) magnetic induction is defined by $B = \frac{1}{\text{volume}} \int \mathcal{B} dv$.

Let us study the macroscopic states of lowest free energy. Assuming an infinite system, in the pure superconducting state (also called the Meissner phase) magnetic induction inside the system, F_{jk} , as well as gauge field $\vec{A}$ vanishes everywhere. Further, since any spatial change in the order parameter produces an increase in the free energy through the term quadratic in $\vec{\nabla}\psi$, the state of minimal free energy gets characterized by an homogeneous $|\psi| = \text{const.}$, being the order parameter's phase also a constant everywhere. As long as the Meissner phase is maintained, the free energy of the superconducting state is equal to $\mathcal{F}_s = \mathcal{F}_{n, H_{\text{ext}}=0} - |\alpha| |\psi|^2 + \frac{|\beta|}{2} |\psi|^4$. Therefore, the constant value of $|\psi|$ which minimizes $\mathcal{F}_s$ in the Meissner state is ²

$$|\psi| = \sqrt{\frac{|\alpha|}{|\beta|}},$$

(A.3)

value for which the free energy density becomes equal to

$$\mathcal{F}_{\text{Meissner}} = \mathcal{F}_{n, H=0} - \frac{|\alpha|^2}{2|\beta|} < \mathcal{F}_{n, H=0},$$

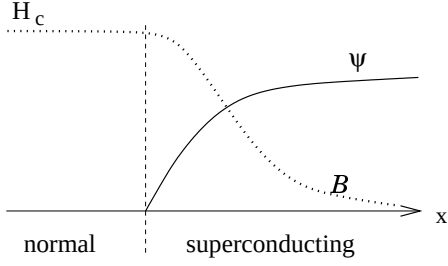
(A.4)

lower than that of the normal conducting phase. Let us increase the external magnetic field on the superconductor until, reached a critical magnetic field H_c , the pure superconducting state is destroyed and the system transits to the normal conducting phase, where the magnetic penetrates through the whole system. The free energy increase produced by the loss of the Meissner state is equal to $\frac{1}{2} H_c^2$ (since the magnetic permeability of a normal conductor is 1) and thus, the considered transition occurs when such an increase compensates the decrease $-\frac{|\alpha|^2}{2|\beta|}$ due to the pure superconducting state

$$H_c = \frac{|\alpha|}{\sqrt{|\beta|}}.$$

(A.5)

²Currently in Particle Physics the Lagrangian describing the model is written as $\ell = -\frac{1}{4} F_{jk} F^{jk} - \frac{\hbar^2}{m} \left| \left(\partial_j - \frac{iqe}{\hbar c} A_j \right) \hat{\psi} \right|^2 + |\hat{\alpha}| |\hat{\psi}|^2 - \frac{|\hat{\beta}|}{2} |\hat{\psi}|^4$, without denominator 2 before the energy's part of the scalar field. The correspondence with the here used fields and parameters are: $\hat{\psi} = \frac{1}{\sqrt{2}} \psi$, $\hat{\alpha} = 2\alpha$ and $\hat{\beta} = 4\beta$. This causes that correlation length ξ for the scalar field, when written as function of the $\hat{\psi}$ -parameters, takes as slight different form from that in equation (A.17).



Now let us assume that the system under study does not fill the whole space, and consider the surface separating a normal conducting region and a region in the Meissner state at $H_{\text{ext}} = H_c$: let us study for which value of $|\beta|$ the pure superconducting state is still more stable than the normal conducting one.

In order to deal with the stability question of a system under the action of a constant *external* magnetic field one has to use Gibbs free energy G instead of Helmholtz free energy F : $G = \int \mathcal{G} dv = F - \int \mathcal{B} H dv = \int (\mathcal{F} - \mathcal{B} H) dv$. For the previous considerations above, as the Meissner phase spread over the whole space, it was not relevant to distinguish between F or G because we were considering the case where the magnetic induction inside was zero or the case as superconductivity was destroyed and thus $\mathcal{B} = H_{\text{ext}}$.

Let us consider the difference Π in the Gibbs free energy between the conducting and the superconducting region of the above figure at $H = H_c$, which is given by

$$\Pi = L \int (\mathcal{G}_{s, H_c} - \mathcal{G}_{n, H_c}) dx dy \quad \text{with} \quad \mathcal{G}_{n, H_c} = \left(\mathcal{F}_{n, H=0} + \frac{1}{2} H_c^2 \right) - H_c^2, \quad (\text{A.6})$$

where L is the length of the material (normal to the picture); for the normal region $B = H_c$. Before continuing with the expression of Π , one needs the solution of the equation for the order parameter ψ

$$-\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 \psi - |\alpha| \psi + |\beta| |\psi|^2 \psi = 0. \quad (\text{A.7})$$

Let z be the direction of the external magnetic field: we shall work on the surface (x, y) , to which the external magnetic is normal and therefore we need not to consider the dependence on the coordinate z of the order parameter ψ and of the magnetic field inside the superconductor, which is given by $\mathcal{B} = \partial_x A_y - \partial_y A_x$. The equation (A.7) can be rewritten by means of the definitions

$$\partial_{\pm} \stackrel{\text{def}}{=} \left(\partial_x - \frac{iqe}{\hbar c} A_x \right) \pm i \left(\partial_y - \frac{iqe}{\hbar c} A_y \right), \quad \partial_{\mp} \partial_{\pm} = \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 \pm \frac{qe}{\hbar c} \mathcal{B}. \quad (\text{A.8})$$

as $-\frac{\hbar^2}{2m} \partial_- \partial_+ \psi + \left(\frac{qe\hbar}{2mc} \mathcal{B} - |\alpha| + |\beta| |\psi|^2 \right) \psi = 0$. In order to solve it, let us assume that for a certain value $|\beta| = \beta_0$, relation $\partial_+ \psi = 0$ holds $\forall r$. Provided this, the two equations relating ψ and $\vec{\mathcal{B}} = \text{curl } \vec{A}$ become ³

$$\text{curl}(\mathcal{B} \hat{z}) = -\beta_0 \frac{2mc}{qe\hbar} \text{curl}(|\psi|^2 \hat{z}), \quad (\text{A.9})$$

³The first relation arises from taking the curl of order parameter's equation $\frac{qe\hbar}{2mc} \mathcal{B} - |\alpha| + \beta_0 |\psi|^2 = 0$ after substituting assumption $\partial_+ \psi = 0$. The second relation is direct result of field equation for the gauge field $\text{curl}(\mathcal{B} \hat{z}) = -\frac{iqe\hbar}{2mc} \left\{ \hat{x} \left[\psi^* \left(-i\partial_y - \frac{qe}{\hbar c} A_y \right) \psi - \text{c.c.} \right] + \hat{y} \left[\psi^* \left(i\partial_x + \frac{qe}{\hbar c} A_x \right) \psi - \text{c.c.} \right] \right\} = \partial_+ \psi = 0 - \frac{qe\hbar}{2mc} [\hat{x} \partial_y |\psi|^2 - \hat{y} \partial_x |\psi|^2]$.

and

$$\text{curl}(\mathcal{B}\hat{z}) = -\frac{qe\hbar}{2mc} \text{curl}(|\psi|^2\hat{z}), \quad (\text{A.10})$$

which combined together gives

$$\beta_0 = \left(\frac{qe\hbar}{2mc}\right)^2, \quad (\text{A.11})$$

for which $\partial_+\psi = 0$ is a general solution for the whole range of r . The meaning of β_0 becomes now clear by rewriting Π as follows: ⁴

$$\Pi = \frac{L}{\sqrt{2}|\beta|} \int dxdy \left[\frac{1}{\sqrt{2}}(\mathcal{B} - H_c) - \sqrt{\frac{|\beta|}{2}}|\psi|^2 \right] \left[\sqrt{|\beta|}\mathcal{B} - |\alpha| + |\beta||\psi|^2 \right], \quad (\text{A.12})$$

where the second factor vanishes for $|\beta| = \beta_0$ since $\frac{qe\hbar}{2mc}\mathcal{B} - |\alpha| + \beta_0|\psi|^2 = 0$ holds. This value β_0 for which $\Pi = 0$ is called critical value β_{crit} .

According to numerical simulations (see for example [83]), $\Pi < 0$ for $|\beta| > \beta_{\text{crit}}$ and thus the Meissner superconducting state is more stable than the normal conducting state at $H = H_c$. We have already seen that for $H_{\text{ext}} > H_c$ the state with *complete* magnetic field's exclusion has a higher energy than the normal conducting state. However and still for $H_{\text{ext}} > H_c$, if $|\beta| > \beta_{\text{crit}}$ it is more favourable for the system to allow the external magnetic field to penetrate only *inside certain regions*, so that the superconducting state is not completely destroyed. Such regions crossed by the external magnetic field inside a superconductor are called flux lines or vortices, and the state characterized by their presence, the mixed or Shubnikov state. Now we shall firstly consider the Shubnikov state with only one flux line, which corresponds to the minimal pattern for the mixed state.

A.2 The Shubnikov phase

A.2.1 Energy of one flux line in the mixed or Shubnikov state

Let us continue working on the surface described by the coordinates (r, ϕ) , plane which is perpendicularly crossed by the flux line or vortex. Beginning with the following ansatz

$$\vec{A}(\vec{r}) = -\hat{\phi}rf(r) \quad \text{and} \quad \psi(\vec{r}) = Q(r)e^{in\phi}, \quad (\text{A.13})$$

⁴From the definition of free energy difference Π/L at the separating surface between a normal conductor and a superconductor, $\int dxdy \left(\mathcal{F}_{s, H_c} - \mathcal{B}H_c - \mathcal{F}_{n, H=0} + \frac{H_c^2}{2} \right) = \int dxdy \left[\frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 - |\alpha||\psi|^2 + \frac{|\beta|}{2}|\psi|^4 + \frac{(\mathcal{B} - H_c)^2}{2} \right]$, one obtains through partial integration $\Pi = \int dxdy \psi^* \left[-\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 - |\alpha| + |\beta||\psi|^2 \right] \psi + \int dxdy \left[\frac{1}{\sqrt{2}}(\mathcal{B} - H_c) - \sqrt{\frac{|\beta|}{2}}|\psi|^2 \right] \left[\frac{1}{\sqrt{2}}(\mathcal{B} - H_c) + \sqrt{\frac{|\beta|}{2}}|\psi|^2 \right]$, where the first integral vanishes because of field equation for ψ . By means of the result for critical magnetic field (A.5), $H_c = \frac{|\alpha|}{\sqrt{|\beta|}}$, the second integral can be rewritten as in (A.12).

the equation for Q and f are given by ⁵

$$-\frac{\hbar^2}{2m} \left[\frac{d^2 Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} - \left(\frac{n}{r} + \frac{qe}{\hbar c} r f \right)^2 Q \right] - |\alpha|Q + |\beta|Q^3 = 0, \quad (\text{A.14})$$

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r^2 f) \right] - Q^2 \frac{qe\hbar}{mc} \left(\frac{n}{r} + \frac{qe}{\hbar c} r f \right) = 0, \quad (\text{A.15})$$

whose solutions for $r \rightarrow \infty$ are

$$r^2 f(r) \xrightarrow{r \rightarrow \infty} -n \frac{\hbar c}{qe} \left[1 - \frac{r}{\lambda} K_1 \left(\frac{r}{\lambda} \right) \right] \quad \text{and} \quad Q(r) \xrightarrow{r \rightarrow \infty} \sqrt{\frac{|\alpha|}{|\beta|}} + C e^{-\frac{r}{\xi}}, \quad (\text{A.16})$$

where characteristic lengths λ and ξ are yet to be specified, and $K_1(x)$ is a Henkel function (which for $r \rightarrow \infty$ decays like $e^{-\frac{r}{\lambda}}$). Factors like $r^{-n} e^{-r}$ or e^{-nr} $n \geq 1$ are for $r \rightarrow \infty$ negligibly compared to e^{-r} . The equations for $Q(r)$ and $f(r)$ take the following form ⁶

$$C e^{-\frac{r}{\xi}} \left[-\frac{\hbar^2}{2m} \frac{1}{\xi^2} - |\alpha| + 3|\beta| \frac{|\alpha|}{|\beta|} \right] = 0 \quad \Rightarrow \quad \xi^2 = \frac{\hbar^2}{4m|\alpha|}, \quad (\text{A.17})$$

$$\left(\frac{n\hbar c}{qe} \right) K_1 \left(\frac{r}{\lambda} \right) \left[\frac{1}{\lambda^3} - \frac{|\alpha|}{|\beta|} \frac{q^2 e^2}{mc^2} \frac{1}{\lambda} \right] = 0 \quad \Rightarrow \quad \lambda^2 = \frac{mc^2}{q^2 e^2} \frac{|\beta|}{|\alpha|}, \quad (\text{A.18})$$

where we are implicitly assuming $\xi \ll \lambda$ in order to neglect term $e^{-\frac{r}{\xi}}$ when compared to $e^{-\frac{r}{\lambda}}$ in (A.18). This is not a too strong restriction since this will be the case we shall consider later through the whole Appendix.

Remark. Another (more usual) form to define the correlation or coherence length ξ is as the characteristic length which describes the scalar field's phase for a truncated Lagrangian (or energy density) only quadratic in the scalar field, i.e., for the linear equation $-\left(\frac{\hbar^2}{2m} \vec{\nabla}^2 - |\alpha|\right) \psi = 0$ obtained from (A.2) after neglecting term $|\beta||\psi|^2$ and

⁵By using the following relations: $\left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}\right)^2 = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\phi^2 + \frac{2iqe}{\hbar c} f \partial_\phi - \frac{q^2 e^2}{\hbar^2 c^2} r^2 f^2$, $\left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}\right) \psi = e^{in\phi} \left[\hat{r} \frac{dQ}{dr} + i\hat{\phi} \left(\frac{n}{r} + \frac{qe}{\hbar c} r f \right) Q \right]$, $\vec{j}_s = -\frac{iqe\hbar}{2m} \left[\psi^* \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi - \psi \left(\vec{\nabla} + \frac{iqe}{\hbar c} \vec{A} \right) \psi^* \right] = \hat{\phi} \frac{qe\hbar}{m} \left(\frac{n}{r} + \frac{qe}{\hbar c} r f \right) Q^2$, $\mathcal{B} \equiv \mathcal{B}_z = -\frac{1}{r} \frac{d}{dr} (r^2 f)$ and finally $\text{curl curl } \vec{A} = \hat{\phi} \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r^2 f) \right]$. It

should be noted that, assumed equations (A.14)-(A.15) hold, $\begin{cases} \vec{A}(\vec{r}) = -\hat{\phi} r f(r) \\ \psi(\vec{r}) = Q(r) e^{in\phi} \end{cases}$ are not the only possible solutions of the field equations: due to the invariance of the model's free energy (and Lagrangian)

under local transformation $\begin{cases} \psi(\vec{r}) \rightarrow \psi'(\vec{r}) = e^{-iq\chi(\vec{r})} \psi \\ \vec{A}(\vec{r}) \rightarrow \vec{A}'(\vec{r}) = \vec{A} - \frac{\hbar c}{e} \vec{\nabla} \chi \end{cases}$, called a gauge transformation, any

ansatz derived from (A.13) through such a transformation satisfies also the equations for the scalar and the gauge fields. This point will be considered in more detail in the following Appendices.

⁶Taking into account relations $\frac{d}{dx} [x^n K_n(x)] = -x^n K_{n-1}(x)$ and $K_{-n}(x) = K_n(x)$ for $n = 0, 1, 2, \dots$ [89], p. 139.

the coupling to the gauge field. Taking $\psi \sim e^{ix/\xi'}$, coherence length ξ' results to be equal to $\xi' = \sqrt{\frac{\hbar^2}{2m|\alpha|}}$, which differs from (A.17) only by a factor $\sqrt{2}$. However, we shall keep the definition $\xi = \sqrt{\frac{\hbar^2}{4m|\alpha|}}$ since the latter stresses the decay of order parameter ψ .

Note. The non-relativistic Higgs mechanism.

We could obtain the same behaviour of f and Q for very large r in the following way: let rf and Q at $r \rightarrow \infty$ be defined as

$$Q(r) = Q_0 + q(r) \quad \text{with} \quad Q_0 = \sqrt{\frac{|\alpha|}{|\beta|}} \quad \text{and} \quad \lim_{r \rightarrow \infty} q(r) = 0,$$

$$rf(r) = (rf)_0 + a(r) \quad \text{with} \quad (rf)_0 = -\frac{\hbar c n}{qe r} \quad \text{and} \quad \lim_{r \rightarrow \infty} a(r) = 0.$$

Functions $q(r)$ and $a(r)$ are to be considered as perturbations of Q_0 and $(rf)_0$ when r is large but not equal to infinity. When we substitute this in the equations (A.14) and (A.15) and we neglect terms which become very small for $r \rightarrow \infty$, such as $\frac{1}{r} \frac{d}{dr}$ and q^2 , we obtain

$$\frac{d^2 q}{dr^2} - \frac{2m}{\hbar^2} [-|\alpha|Q_0 - |\alpha|q + |\beta|Q_0^3 + 3|\beta|Q_0^2 q] \stackrel{Q_0^2 = \frac{|\alpha|}{|\beta|}}{=} \frac{d^2 q}{dr^2} - 2|\alpha|\frac{2m}{\hbar^2} q = 0, \quad (\text{A.19})$$

$$\frac{d^2 a}{dr^2} - \frac{q^2 e^2}{mc^2} \frac{|\alpha|}{|\beta|} a = 0, \quad (\text{A.20})$$

whose solutions are easy to get from:

$$q(r) \sim e^{-\sqrt{\frac{4m|\alpha|}{\hbar^2}} r} \quad \text{and} \quad a(r) \sim e^{-\sqrt{\frac{q^2 e^2 |\alpha|}{mc^2 |\beta|}} r}. \quad (\text{A.21})$$

This is nothing else as non-relativistic (we have restrict ourselves to space-like coordinates) version of the so-called Higgs mechanism of Particle Physics.

Let us define the following parameter

$$\kappa = \frac{\lambda}{\xi} = \frac{2mc}{qe\hbar} \sqrt{|\beta|}. \quad (\text{A.22})$$

whose value corresponding to β_{crit} is

$$\beta_{\text{crit}} = \left(\frac{qe\hbar}{2mc} \right)^2 \Rightarrow \kappa_{\text{crit}} = 1. \quad (\text{A.23})$$

If κ is defined as λ/ξ' , according to the definition of $\xi' = \sqrt{2}\xi$ in the above Remark, then $\kappa_{\text{crit}} = 1/\sqrt{2}$. Further, the magnetic flux crossing through the vortex is given

$$\oint_{\text{circumference at } r \rightarrow \infty} \vec{A} \cdot d\vec{l} = \int_{\phi=0}^{\phi=2\pi} d\phi r A_\phi \Big|_{r \rightarrow \infty} = n 2\pi \left(\frac{\hbar c}{qe} \right) \equiv n \Phi_0. \quad (\text{A.24})$$

Note that for $r \rightarrow \infty$, ψ and $\vec{A}$ fulfill $\partial_+ \psi = 0$, although this does not mean that $|\beta| = \beta_{\text{crit}}$ since here $\partial_+ \psi = 0$ only holds asymptotically, i.e. for $r \rightarrow \infty$, and not $\forall r$. For range $r \rightarrow \infty$, the magnetic induction and its source the supercurrent are given by

$$\vec{\mathcal{B}}(\vec{r}) = -\hat{z} \frac{1}{r} \frac{d}{dr} (r^2 f) = \hat{z} n \frac{\Phi_0}{2\pi\lambda^2} K_0 \left(\frac{r}{\lambda} \right), \quad (\text{A.25})$$

and

$$\vec{j}_s = \hat{\phi} Q^2 \frac{qe\hbar}{m} \left(\frac{n}{r} + \frac{qe}{\hbar c} r f \right) = \hat{\phi} n \Phi_0 \frac{c}{2\pi\lambda^3} K_1 \left(\frac{r}{\lambda} \right), \quad (\text{A.26})$$

while for $\xi \ll r \ll \lambda$ these take the form ⁷

$$\vec{\mathcal{B}} \stackrel{\frac{r}{\lambda} \approx 0}{\approx} -\hat{z} n \frac{\Phi_0}{2\pi\lambda^2} \ln \left(\frac{r}{\lambda} \right) \quad \text{and} \quad \vec{j}_s \stackrel{\frac{r}{\lambda} \approx 0}{\approx} \hat{\phi} n \Phi_0 \frac{c}{2\pi\lambda^2} \frac{1}{r}. \quad (\text{A.27})$$

Now we shall consider the energy ϵ_1 per unit length of one vortex, defined as the free energy increase over the pure superconducting state's energy ⁸

$$\epsilon_1 = \int (\mathcal{F}_{\text{1 vortex}} - \mathcal{F}_{\text{s without vort.}}) dx dy = \int \left(\frac{1}{2} (\text{curl } \vec{A})^2 + \frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 \right) dx dy.$$

In the above integral we separate a core and a long distance part, using $\psi = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\phi}$

and $\vec{A} = \hat{\phi} \frac{\hbar c}{qer} \left[1 - \frac{r}{\lambda} K_1 \left(\frac{r}{\lambda} \right) \right]$ for the latter ⁹

$$\epsilon_1 = \frac{\Phi_0^2}{4\pi\lambda^2} \int_{\frac{a}{\lambda}}^{\infty} \left[x [K_0(x)]^2 + x [K_1(x)]^2 \right] dx + \epsilon_{\text{core}}. \quad (\text{A.28})$$

Core radius a has to be determined, but from the just written equation we are already assuming that outside the core, part $Ce^{-r/\xi}$ of scalar field ψ is already damped, i.e., a is of the order of correlation length ξ : this assumption will be confirmed later. Integration by parts in ϵ_1 , by using relation $\frac{d}{dx} [x^n K_n(x)] = -x^n K_{n-1}(x)$, gives $\int_{\frac{a}{\lambda}}^{\infty} x [K_0(x)]^2 dx =$

⁷ $K_0(x) \stackrel{x \approx 0}{\approx} -\left(\ln \frac{x}{2} + \gamma\right) \approx -\ln x$, $K_1(x) \stackrel{x \approx 0}{\approx} \frac{1}{x}$ with $\gamma = 0.5772 \dots$ the Euler's constant [89], p.138.

⁸ In fact, we can also consider ϵ_1 as the energy increase over the normal conducting state, since according to (A.4) the Meissner state's energy differs from the normal conductor's energy only in a constant.

⁹ For the long distance range and from equation (A.25), the energy's part coming from the pure gauge field's term is $\int \frac{1}{2} (\text{curl } \vec{A})^2 dx dy = \frac{1}{2} \left(\frac{\Phi_0}{2\pi\lambda^2} \right)^2 \int \left[K_0 \left(\frac{r}{\lambda} \right) \right]^2 \lambda^2 2\pi \frac{r}{\lambda} \frac{dr}{\lambda} = \frac{\Phi_0^2}{4\pi\lambda^2} \int [K_0(x')]^2 dx'$, with $x' = \frac{r}{\lambda}$. On the other hand, following holds: $\left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi = \hat{\phi} \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\phi} \frac{1}{\lambda} K_1 \left(\frac{r}{\lambda} \right)$ since $\vec{\nabla} \phi = \frac{\hat{\phi}}{r}$ with $\phi = \arctan \left(\frac{y}{x} \right)$. And therefore $\int \frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 dx dy = \frac{\Phi_0^2}{4\pi\lambda^2} \int \left[K_1 \left(\frac{r}{\lambda} \right) \right]^2 \frac{r}{\lambda} \frac{dr}{\lambda}$ using definition (A.18) $\frac{1}{\lambda^2} = \frac{q^2 e^2 |\alpha|}{mc^2 |\beta|}$ and flux (A.24) $\Phi_0 = 2\pi \frac{\hbar c}{qe}$.

$= - \left[x K_0(x) K_1(x) \right]_{\frac{a}{\lambda}}^{\infty} - \int_{\frac{a}{\lambda}}^{\infty} x [K_1(x)]^2 dx$; since both $K_0(x)$, $K_1(x)$ fall off at $x \rightarrow \infty$ faster than x , and from $\begin{cases} K_0(x) \approx -\ln x + \ln 2 - \gamma \\ x K_1(x) \approx 1 \end{cases}$ valid for $x \approx 0$, one obtains

$$\epsilon_1 = \frac{\Phi_0^2}{4\pi\lambda^2} \left[\ln \left(\frac{\lambda}{a} \right) + \ln 2 - \gamma \right] + \epsilon_{\text{core}}. \quad (\text{A.29})$$

An estimation of ϵ_{core} is formed by a contribution $\frac{1}{2}(H_{\text{ext}})^2 \pi a^2$, which arises from the magnetic field spreading through the normal conducting vortex' core, plus contribution $\frac{|\alpha|^2}{2|\beta|} \pi a^2$ due to the energetical cost of the absence of superconducting state inside the core. The external magnetic field is approximately equal to vortex flux Φ_0 divided by area $\pi\lambda^2$ where the magnetic field can penetrate. And thus

$$\epsilon_{\text{core}} \approx \pi a^2 \left[\frac{H_{\text{ext}}^2}{2} + \frac{|\alpha|^2}{2|\beta|} \right] = a^2 \frac{\Phi_0^2}{4\pi\lambda^2} \left[\frac{2}{\lambda^2} + \frac{1}{8\xi^2} \right].$$

For $\lambda \gg \xi$, we can neglect the term proportional to $1/\lambda^2$ and the core radius a must be arranged in order to minimize ϵ_{total} , being the result $a = 2\xi$. The total energy per unit length of a flux line is therefore ¹⁰

$$\epsilon_{1,\text{total}} \stackrel{\lambda \gg \xi}{\approx} \frac{\Phi_0^2}{4\pi\lambda^2} \left[\ln \left(\frac{\lambda}{\xi} \right) - 0.577 + \frac{1}{2} \right] \approx \frac{\Phi_0^2}{4\pi\lambda^2} \ln \left(\frac{\lambda}{\xi} \right). \quad (\text{A.30})$$

In equation (A.5) we obtained the minimal external magnetic field at which it can spread over the whole material. The next step is to obtain the external magnetic field H_{c1} at which the magnetic field can penetrate inside the superconductor but only through one vortex, i.e., the magnetic field at which the pure superconducting state transits to the Shubnikov state maintaining wide regions where the spreading of magnetic field is forbidden. The Gibbs energy for the pure superconducting state is given by $G_s = F_s$, since B vanishes in the Meissner state. In the mixed state, free energy G is given by $G = F - H \int \mathcal{B} dv = F - (\text{volume}) \times BH$ (the external magnetic field is constant), and thus the Gibbs free energy per unit length for the state with only one vortex is $\frac{G_{1\text{ vortex}}}{L} = \left(\frac{F_s}{L} + \epsilon_1 \right) - (\text{area}) \times B_{1\text{ vortex}} H$, with $B = \frac{1}{\text{area}} \int \mathcal{B} dx dy = \frac{\Phi_0}{\text{area}}$ the macroscopic magnetic induction. Therefore, H_{c1} is the magnetic field at which the formation of a single vortex just becomes energetically favourable, i.e., at which $G_{1\text{ vortex}} = G_s$,

$$H_{c1} = \frac{\epsilon_1}{\Phi_0} \frac{\frac{|\alpha|}{\sqrt{|\beta|}} = \frac{\Phi_0}{4\pi} \frac{1}{\lambda\xi}}{\frac{\Phi_0}{4\pi} \frac{1}{\lambda\xi}} H_c \frac{\xi}{\lambda} \ln \left(\frac{\lambda}{\xi} \right). \quad (\text{A.31})$$

When the external magnetic field surpasses $H_{c1} = H_c \frac{\xi}{\lambda} \ln \left(\frac{\lambda}{\xi} \right)$, and as long as the would-be magnetic flux created by the external magnetic field corresponds to an integer number of Φ_0 , the number of vortices begins to increase. The increase in free energy F

¹⁰Another different argument based in the scaling properties of the transition from normal conductor to a superconductor can be found in [24], p. 528.

per unit length for a set of flux lines distributed over the whole material is given by $\int (\mathcal{F}_{s, \text{with vort.}} - \mathcal{F}_{s, \text{without vort.}}) dxdy = \int \left(\frac{1}{2} (\text{curl } \vec{A})^2 + \frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 \right) dxdy$, with gauge field $\vec{A} = \sum_a \vec{A}_a$ (summing over the contributions of all vortices). Distant enough

from all vortices, the order parameter is given approximately by $\psi = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\varphi}$ and

thus $\left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi = \psi \left(\vec{\nabla} \varphi - \frac{qe}{\hbar c} \vec{A} \right)$, satisfying ψ 's phase φ the following relation:

$\partial_j \varphi = - \sum_k \epsilon_{jk} \sum_a \frac{(\vec{r} - \vec{r}_a)_k}{|\vec{r} - \vec{r}_a|^2}$. And further, the gauge potential becomes at such distances

equal to $\vec{A} = \frac{\hbar c}{qe} \sum_a \hat{\phi}_a \left[\frac{1}{|\vec{r} - \vec{r}_a|} - \frac{1}{\lambda} K_1 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) \right]$, being $\hat{\phi}_a = -\hat{x} \frac{y - y_a}{|\vec{r} - \vec{r}_a|} + \hat{y} \frac{x - x_a}{|\vec{r} - \vec{r}_a|}$ the unit vector in the direction of the azimuthal angle around the a -vortex. Repeating the same calculations and approximations as after (A.28) ¹¹, one obtains

$$\frac{F}{L} = N \frac{\Phi_0^2}{4\pi\lambda^2} \ln \left(\frac{\lambda}{\xi} \right) + N \frac{\Phi_0^2}{4\pi\lambda^2} \sum_{b \neq 0} K_0 \left(\frac{|\vec{r}_b|}{\lambda} \right),$$

$$^{11} \sum_a \sum_b K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) K_0 \left(\frac{|\vec{r} - \vec{r}_b|}{\lambda} \right) = \sum_a \left[K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) \right]^2 + \sum_a \sum_{b \neq a} K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) K_0 \left(\frac{|\vec{r} - \vec{r}_b|}{\lambda} \right)$$

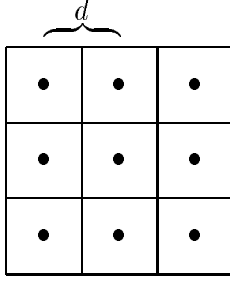
and integral $\int \sum_a \sum_{b \neq a} K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) K_0 \left(\frac{|\vec{r} - \vec{r}_b|}{\lambda} \right) \frac{1}{\lambda^2} dxdy$ is solved by means of an already used result:

$$\int_{\text{start long range}}^{\infty} \left[K_0 \left(\frac{r}{\lambda} \right) \right]^2 \frac{1}{\lambda^2} dxdy \approx 2\pi K_0 \Big|_{\text{start long range}} - \int_{\text{start long range}}^{\infty} \left[K_1 \left(\frac{r}{\lambda} \right) \right]^2 \frac{1}{\lambda^2} dxdy.$$

Such a relation, when applied to integral $\int K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) K_0 \left(\frac{|\vec{r} - \vec{r}_b|}{\lambda} \right) \frac{1}{\lambda^2} dxdy$ yields $2\pi K_0 \left(\frac{|\vec{r}_a - \vec{r}_b|}{\lambda} \right) - \int K_1 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) K_1 \left(\frac{|\vec{r} - \vec{r}_b|}{\lambda} \right) \frac{1}{\lambda^2} dxdy$, and therefore

$$\begin{aligned} \int \left(\frac{1}{2} (\text{curl } \vec{A})^2 + \frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 \right) dxdy &= \frac{\Phi_0^2}{8\pi^2\lambda^2} 2\pi \sum_a \sum_{b \neq a} K_0 \left(\frac{|\vec{r}_a - \vec{r}_b|}{\lambda} \right) + \\ &+ \underbrace{\sum_a \frac{\Phi_0^2}{8\pi^2\lambda^2} \int \left[\left[K_0 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) \right]^2 + \left[K_1 \left(\frac{|\vec{r} - \vec{r}_a|}{\lambda} \right) \right]^2 \right] \frac{dxdy}{\lambda^2}}_{\epsilon_1 \stackrel{\text{eq. (A.30)}}{\approx} \frac{\Phi_0^2}{4\pi\lambda^2} \ln \left(\frac{\lambda}{\xi} \right)}. \end{aligned}$$

The approximation that ξ is much smaller than any distance $|\vec{r}_a - \vec{r}_b|$ between two vortex lines, i.e., $\frac{\xi}{\lambda} - \frac{|\vec{r}_a - \vec{r}_b|}{\lambda} \approx -\frac{|\vec{r}_a - \vec{r}_b|}{\lambda}$ has been also used; such an approximation holds as long as vortices are distant to each other and, as we shall see in the following, this physically corresponds to the case of H_{ext} far from the magnetic field which produces the transition toward the normal conducting state.



where vortex coordinates has been chosen in such a way that $\vec{r}_b$ are the positions of all vortices referred to the position of the vortex placed at the origin $\vec{r}_0 = \vec{0}$. Number of flux lines N is given by the material section divided by area d^2 of the cell containing one vortex. We obtain thus the free energy F per unit length of a set of flux lines

$$\frac{F}{L} = (\text{area}) \times \left[\frac{1}{d^2} \epsilon_1 + \frac{1}{d^2} \frac{\Phi_0^2}{4\pi\lambda^2} \sum_{b \neq 0} K_0 \left(\frac{r_b}{\lambda} \right) \right]. \quad (\text{A.32})$$

Note. The superconducting film [71].

During the whole previous discussion we have assumed that vortex lines completely cover the third dimension (the $\hat{z}$ -direction) normal to the 2-dimensional plane where the degrees of freedom for the discussed model have been located. Of course, the same 2-dimensional model works also if we were studying a 2-dimensional superconductor which “lives” in a 2-dimensional space. But a different case appears if we are considering a 2-dimensional superconducting film embedded into the (usual) 3-dimensional space: the field lines of the external magnetic field above and below the superconductor spread over the whole space and only when they cross the film, they are constrained inside the flux-lines. This spreading of the magnetic field, which was confined inside the vortices, as soon as the film’s boundaries are left causes that the (3-dimensional) energy of the vortex deviates from the above result (A.30).

Before we start considering this last case, let us write the energy per unit length of a superconductor with one vortex, $\epsilon_1 = \int d^2\vec{r} \left(\frac{1}{2} (\text{curl } \vec{A})^2 + \frac{\hbar^2}{2m} \left| \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi \right|^2 \right)$, in a more compact form. For r large compared to core radius $a \sim \xi$, this energy is, up to the finite contribution arising from the core’s region, given by $\int d^2\vec{r} \left(\frac{1}{2} (\text{curl } \vec{A})^2 + \frac{\hbar^2}{2m} \frac{|\alpha|}{|\beta|} \left(n \vec{\nabla} \phi - \frac{qe}{\hbar c} \vec{A} \right)^2 \right)$, obtained after substituting $\psi(\vec{r}) = \sqrt{\frac{|\alpha|}{|\beta|}} e^{in\phi}$, valid for $r \gg \xi$; (r, ϕ) are the 2-dim. polar coordinates on the plane crossed perpendicularly by the vortex. Taking into account that the supercurrent in such range of distances is equal to $\vec{j}_s = -\frac{iqe\hbar}{2m} \left[\psi^* \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right) \psi - \text{c.c.} \right] = \frac{qe\hbar}{m} \frac{|\alpha|}{|\beta|} \left(n \vec{\nabla} \phi - \frac{qe}{\hbar c} \vec{A} \right)$, one obtains approximately

$$\epsilon_1 \approx \frac{1}{2} \int_{\xi}^{\infty} d^2\vec{r} \left[(\text{curl } \vec{A})^2 + \lambda^2 \vec{j}_s^2 \right], \quad (\text{A.33})$$

where the approximation holds as long as $\lambda \gg \xi$. Since both $\text{curl } \vec{A}$ as well as $\vec{j}_s$ decay exponentially at large distances as $e^{-r/\lambda}$, the energy of the superconductor with one vortex would not depend on the superconductor’s size if we would consider a large but *finite* size over the 2-dimensional plane normal to the vortex; of course, this only holds as long as this size is large compared to λ , the length characterizing the decay of $\vec{\nabla} \times \vec{A}$ and $\vec{j}_s$.

Now let us consider a superconducting film (its plane still described by the coordinates (r, ϕ)) embedded into a 3-dim. space, being $\bar{d}$ the film’s thickness along the $\hat{z}$ -direction which is the direction of the vortex tube placed at the coordinate origin and which carries n units of flux Φ_0 ; thickness $\bar{d}$ will be assumed to be much smaller than penetration length λ , such that we can speak about a 2-dim. film and not about a 3-dim. bulk system. For the following discussion, subindices “2d” and “3d” will denote that the field resides, respectively, on the 2-dim. plane described by $\vec{r} \equiv (r, \phi)$ or in the whole 3-dim,

space described by $(\vec{r}, z)$. Let us suppose that the supercurrent is concentrated on the superconducting plane $z = 0$: the full 3-dimensional current, source of the magnetic field is then equal to

$$\vec{j}_{s,3d}(\vec{r}, z) = \underbrace{\vec{j}_{s,2d}(\vec{r})}_{\delta(z)\vec{d}} = \frac{\Phi_0 c}{2\pi\lambda^2} \left(\hat{\phi} \frac{n}{r} - \frac{qe}{\hbar c} \vec{A}_{2d}(\vec{r}) \right) \delta(z)\vec{d}, \quad (\text{A.34})$$

$$\frac{qe\hbar}{m} \frac{|\alpha|}{|\beta|} \left(n\vec{\nabla}\phi - \frac{qe}{\hbar c} \vec{A}_{2d} \right)$$

through the definition of the Dirac delta and of $\vec{j}_{s,2d} = \frac{1}{d} \int_0^d dz \vec{j}_{s,3d}$. In the case under study, the A_{2d} 's dependence on r does not correspond to $\frac{n\Phi_0}{2\pi} \left[\frac{1}{r} - \frac{1}{\lambda} K_1 \left(\frac{r}{\lambda} \right) \right]$ (as it was the case for the superconductor whose vortex covered the whole $\hat{z}$ -direction) due to the film's boundaries along this $\hat{z}$ -direction: now $\vec{A}_{2d}$ has to be calculated from the field equations for the magnetic field. The magnetic induction created by current (A.34), $\vec{B} = \text{curl} \vec{A}_{3d}$, spreads over the whole 3-dimensional space, when it leaves the superconducting film, and is defined through equation $\text{curl} \vec{B} = \frac{1}{c} \vec{j}_{s,3d}$. As mentioned for ansatz (A.13), one can exploit the invariance of the energy under a so-called gauge transformation

$$\begin{cases} \psi(\vec{r}) \rightarrow \psi'(\vec{r}) = e^{-q\chi(\vec{r})} \psi \\ \vec{A}(\vec{r}) \rightarrow \vec{A}'(\vec{r}) = \vec{A} - \frac{\hbar c}{e} \vec{\nabla}\chi \end{cases} \quad \text{in order to choose, among the infinite many different but equiv-}$$

alent gauge fields $\vec{A}$ (equivalent in the sense that each one yields the same energy), that $\vec{A}$ which is more comfortable to operate with: this is called to fix (or to choose) the gauge. In our case, this gauge field corresponds to a field in the direction of azimuthal unit vector $\hat{\phi}$: $\begin{cases} \vec{A}_{3d}(\vec{r}, z) = \hat{\phi} A_{3d}(\vec{r}, z) \\ \vec{A}_{2d}(\vec{r}) = \hat{\phi} A_{2d}(\vec{r}) \end{cases}$ ¹².

Working under this gauge choice, the resulting equation for the gauge fields is

$$-\nabla^2 A_{3d}(\vec{r}, z) = \frac{n\Phi_0 d}{2\pi\lambda^2} \frac{1}{r} \delta(z) - \frac{\vec{d}}{\lambda^2} \delta(z) A_{2d}(\vec{r}),$$

with Laplace operator $\nabla^2 = \nabla_{\perp}^2 + \frac{\partial^2}{\partial z^2}$, denoting by $\perp$ the two coordinates on the superconducting film. In order to solve the previous differential equation, one formulates the equation for the Fourier

transforms defined through $\begin{cases} A_{3d}(\vec{r}, z) = \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{i\vec{k}_{\perp} \cdot \vec{r}} e^{ik_z z} \tilde{A}_{3d}(\vec{k}_{\perp}, k_z) \\ A_{2d}(\vec{r}) = \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} e^{i\vec{k}_{\perp} \cdot \vec{r}} \tilde{A}_{2d}(\vec{k}_{\perp}) \end{cases}$

$$\tilde{A}_{3d}(\vec{k}_{\perp}, k_z) = -\frac{in\Phi_0 d}{\lambda^2} \frac{1}{k_{\perp}(\vec{k}_{\perp}^2 + k_z^2)} - \frac{d}{\lambda^2} \frac{1}{\vec{k}_{\perp}^2 + k_z^2} \tilde{A}_{2d}(\vec{k}_{\perp}), \quad (\text{A.35})$$

with $k_{\perp} = \sqrt{\vec{k}_{\perp}^2}$ and after having used definition $\delta(z) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{ik_z z}$ and relation ¹³

$$\int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}} = \frac{i}{2\pi r}. \quad (\text{A.36})$$

¹²This gauge fixing is denoted as the Coulomb gauge.

¹³As one can easily check, $D(\vec{r}) = \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}^2}$ is propagator (or Green's function) of the 2-dim. Laplace operator: $\nabla^2 \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}^2} = -\delta^2(\vec{r})$. Further, it is well-known that such a propagator is equal to $-\frac{1}{2\pi} \ln r$ (see, for instance, the footnote on page 339) and since this propagator only depends on $r = |\vec{r}|$, one obtains that $\vec{\nabla} D(\vec{r}) = \hat{r} \frac{d}{dr} \left(-\frac{1}{2\pi} \ln r \right)$, with $\hat{r}$ a unit vector in the (2-dim.) radial direction. On the other hand we have $\vec{\nabla} D(\vec{r}) = i \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{\vec{k}_{\perp} e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}^2}$ which for symmetry gives $i \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{\vec{k}_{\perp} e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}^2} = \hat{r} i \int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{k_{\perp} e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}^2}$ and therefore $\int \frac{d^2 \vec{k}_{\perp}}{(2\pi)^2} \frac{e^{i\vec{k}_{\perp} \cdot \vec{r}}}{k_{\perp}} = -i \frac{d}{dr} \left(-\frac{1}{2\pi} \ln r \right)$.

Since the full 3-dimensional gauge field at $z = 0$ (on the superconducting plane) coincides with $A_{2d}(\vec{r})$, or equivalently, $\int \frac{d^2 \vec{k}_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{r}} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \tilde{A}_{3d}(\vec{k}_\perp, k_z) = A_{3d}(\vec{r}, z=0) = A_{2d}(\vec{r}) = \int \frac{d^2 \vec{k}_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{r}} \tilde{A}_{2d}(\vec{k}_\perp)$, equation (A.35) gives after integration over dk_z

$$\begin{aligned}\tilde{A}_{2d}(\vec{k}_\perp) &= -\frac{in\Phi_0}{2\lambda^2} \frac{\bar{d}}{k_\perp^2} - \frac{\bar{d}}{\lambda^2} \frac{1}{2k_\perp} \tilde{A}_{2d}(\vec{k}_\perp), \\ A_{2d}(\vec{k}_\perp) &= -\frac{in\Phi_0}{k_\perp} \frac{1}{1 + 2\frac{\lambda^2}{\bar{d}}k_\perp}.\end{aligned}\tag{A.37}$$

For the sake of completeness, by substituting this last result into equation (A.35) one gets the Fourier transform of full 3-dim. gauge field $A_{3d}(\vec{k}_\perp, k_z) = -\frac{2in\Phi_0}{k_\perp^2 + k_z^2} \frac{1}{1 + 2\frac{\lambda^2}{\bar{d}}k_\perp}$.

From (A.37) it follows

$$A_{2d}(\vec{r}) = -\frac{in\Phi_0}{\lambda_{\text{eff}}} \int \frac{d^2 \vec{q}_\perp}{(2\pi)^2} \frac{e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}}{q_\perp (1 + 2q_\perp)},\tag{A.38}$$

with

$$\lambda_{\text{eff}} = \frac{\lambda^2}{\bar{d}} \gg \lambda,\tag{A.39}$$

and $\vec{q}_\perp \equiv \lambda_{\text{eff}} \vec{k}_\perp$ dimensionless. For $r \gg \lambda_{\text{eff}}$, the relevant contributions to $A_{2d}(\vec{r})$ comes from the range of small $\vec{q}_\perp$ since otherwise exponential $e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}$ oscillates too fast to yield a nonvanishing contribution to the integral. Thus one gets $A_{2d}(\vec{r}) \Big|_{\text{large } |\vec{r}|} \approx -\frac{in\Phi_0}{\lambda_{\text{eff}}} \int \frac{d^2 \vec{q}_\perp}{(2\pi)^2} \frac{e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}}{q_\perp}$ and from result (A.36)

$$\vec{A}_{2d}(\vec{r}) \Big|_{\text{large } |\vec{r}|} \approx \hat{\phi} \frac{\Phi_0}{2\pi} \frac{n}{r},\tag{A.40}$$

just as in the gauge field's solution (A.16) for the ansatz (A.13) used in 3-dim. model when the vortices were infinite straight lines along the $\hat{z}$ -direction. According to equation (A.33), we also need supercurrent $\vec{j}_{s,2d}$ in order to calculate the energy of superconductor with one vortex. The Fourier transform of such a current is easily deduced from (A.34): $\vec{j}_{s,2d}(\vec{r}) = \hat{\phi} j_{s,2d}(\vec{r})$ plus result (A.36) gives

$$\begin{aligned}\tilde{j}_{s,2d}(\vec{k}_\perp) &= -\frac{in\Phi_0 c}{\lambda^2} \frac{1}{k_\perp} - \frac{c}{\lambda^2} \tilde{A}_{2d}(\vec{k}_\perp) = -\frac{2in\Phi_0 c}{\bar{d}} \frac{1}{1 + 2\frac{\lambda^2}{\bar{d}}k_\perp}, \\ \vec{j}_{s,2d}(\vec{r}) &= -\hat{\phi} \frac{2in\Phi_0 c}{\lambda_{\text{eff}}^2 \bar{d}} \int \frac{d^2 \vec{q}_\perp}{(2\pi)^2} \frac{e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}}{1 + 2q_\perp}.\end{aligned}\tag{A.41}$$

According to [71], for the range of $r \gg \lambda_{\text{eff}}$ integral (A.41) is approximately given by ¹⁴

$$\vec{j}_{s,2d}(\vec{r}) \Big|_{\text{large } |\vec{r}|} \approx -\hat{\phi} \frac{n\Phi_0 c}{\pi \lambda_{\text{eff}}^2 \bar{d}} \frac{1}{r^2}.\tag{A.42}$$

¹⁴A rapid form to “see” this result is by comparing Fourier transforms (A.38), $A_{2d}(\vec{r}) = -\frac{in\Phi_0}{\lambda_{\text{eff}}} \int \frac{d^2 \vec{q}_\perp}{(2\pi)^2} \frac{e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}}{q_\perp (1 + 2q_\perp)}$, and (A.41), $j_{s,2d}(\vec{r}) = -\frac{2in\Phi_0 c}{\lambda_{\text{eff}}^2 \bar{d}} \int \frac{d^2 \vec{q}_\perp}{(2\pi)^2} \frac{e^{i\vec{q}_\perp \cdot \vec{r}/\lambda_{\text{eff}}}}{1 + 2q_\perp}$: up to factor $\frac{2c}{\lambda_{\text{eff}} \bar{d}}$, $\vec{j}_{s,2d}(\vec{r})$ has to be derivative $\frac{d}{dr}$ of $\vec{A}_{2d}(\vec{r})$, gauge field which for large r is approximately equal to $\hat{\phi} \frac{\Phi_0}{2\pi} \frac{n}{r}$.

It is important to note that in this discussion, the relevant length is no longer λ (as it was the case in the 3-dim. model) but $\lambda_{\text{eff}} \gg \lambda$ since film's thickness $\tilde{d}$ has been assumed to be much smaller than λ .

Let us compare at *large distances* supercurrent (A.42), $j_{s,2d} \sim \frac{1}{r^2}$ which flows on the superconducting film, to supercurrent (A.26) $j_s = n \Phi_0 \frac{c}{2\pi\lambda^3} K_1\left(\frac{r}{\lambda}\right) \sim e^{-r/\lambda}$ which flows on any plane intersecting the infinite long vortex of a 3-dim. superconductor (or which flows on the 2-dim. superconductor “living” in a 2-dim. space): although both currents fall off to zero for a system spreading to infinity in the 2-dimensional plane, it is clear nevertheless that for a large but finite system the energy will depend on system's size Λ^2 for the case of the superconducting system, $\epsilon_1 \sim \int_{\text{finite plane}} d^2\vec{r} j_{s,2d}^2 \sim \frac{1}{\Lambda^2}$, whereas for the superconductor covering the whole $\hat{z}$ -direction this energy will be independent of the size over the 2-dim. plane due to the exponential decay of both $\vec{j}_s$ and $\vec{B}$. This difference becomes especially relevant when one calculates the energy of the superconductor with a set of vortices. Apart from the contribution arising from each isolated flux line, there is another contribution due to the interaction among vortices, being this latter the integral over the 2-dim. plane (where flux lines are placed) of the gauge field corresponding to one vortex multiplied by the supercurrent another different vortex. In the case of an infinite system along the third dimension, such an interactive part is, in first approximation, only restricted to near vortices because of $j_s \sim e^{-r/\lambda}$, being given this near interaction's energy by a logarithm of the distance separating the vortices, as one can see by considering free energy (A.32) and relation $K_0(x) \stackrel{x \gg 0}{\sim} \ln x$ ¹⁵. On the other hand, for the superconducting film there are not only interactions between near vortices but *also* between vortices separated by *large distances*: such

an interaction is of the type $\frac{1}{r}$ as it follows from
$$\begin{cases} \vec{A}_{2d}|_{\text{large } r} \sim \frac{1}{r} \\ \vec{j}_{s,2d}(\vec{r})|_{\text{large } r} \sim \frac{1}{r^2} \end{cases} \text{ and from interaction's energy}$$

$$\int d^2\vec{r} \vec{A}_{2d} \cdot \vec{j}_{s,2d} \sim \frac{1}{r}$$
. It is just this long range potential the responsible for the differences between the bulk 3-dim. superconductor with straight infinite vortices (or the 2-dim. superconductor in a 2-dim. space) and the superconducting 2-dim. film embedded in a 3-dim. space.

Let us return to the 3-dimensional superconductor in the mixed state, with all vortices (of infinite length) along the direction perpendicular to the plane where we are working. We shall now consider three different ranges for the magnetic field applied on the system:

1. for an external magnetic field $H_{\text{ext}} - H_{c1} \ll H_{c1}$,
2. at $H_{c1} \ll H_{\text{ext}} \ll H_{c2}$,
3. and for an external magnetic field $H_{\text{ext}} \approx H_{c2}$,

where H_{c2} is the external magnetic field at which even the mixed state has a higher energy than the normal conducting state and thus the system transits to the state of a normal conductor.

¹⁵ An easy estimation leading to this logarithm runs as follows: since
$$\begin{cases} \vec{A}(\vec{r}) \sim \frac{1}{r} \\ \vec{j}_s(\vec{r}) \sim K_1\left(\frac{r}{\lambda}\right) \end{cases}, \text{ then the interaction's energy}$$
 is of the order of $\int d^2\vec{r} \vec{A} \cdot \vec{j}_s \sim K_0\left(\frac{r}{\lambda}\right)$ just as in the steps leading to equation (A.29): by using $\frac{d}{dx}(K_0(x)) = -K_1(x)$.

A.2.2 Range $H_{\text{ext}} - H_{c1} \ll H_{c1}$

In this region the distance d between adjacent flux lines is assumed to be much larger than the magnetic field's penetration depth λ , such that in first approximation there exists no interaction between flux lines which are not nearest neighbours. According to equation (A.32), the free energy per unit volume is given by

$$\frac{F_{\text{mixed}}}{\text{volume}} = \frac{1}{d^2} \epsilon_1 + \frac{1}{d^2} \frac{\Phi_0^2}{4\pi\lambda^2} \sum_{b \neq 0} K_0 \left(\frac{r_b}{\lambda} \right) \approx \frac{1}{d^2} \epsilon_1 + z \frac{1}{d^2} \frac{\Phi_0^2}{4\pi\lambda^2} K_0 \left(\frac{d}{\lambda} \right),$$

where z is the number of nearest neighbours to a given vortex (e.g., $z = 4$ for a square distribution of flux lines). The Gibbs free energy G per unit volume in this state is $\frac{G_{\text{mixed}}}{\text{volume}} = \frac{F_{\text{mixed}}}{\text{volume}} - B H_{\text{ext}}$, with $B = \frac{\Phi_0}{d^2}$ the macroscopic magnetic induction. The equilibrium state is characterized by $\frac{\partial G_{\text{mixed}}}{\partial d} = 0$. Since $d \gg \lambda$, the asymptotic form of the Henkel functions $K_0 \left(\frac{d}{\lambda} \right) \sim \sqrt{\frac{\pi\lambda}{2d}} e^{-\frac{d}{\lambda}}$ and $\frac{d}{\lambda} K_1 \left(\frac{d}{\lambda} \right) \sim \frac{d}{\lambda} \sqrt{\frac{\pi\lambda}{2d}} e^{-\frac{d}{\lambda}} \gg K_0 \left(\frac{d}{\lambda} \right)$ can be used, obtaining thus

$$H_{\text{ext}} - H_{c1} \approx z \frac{\Phi_0}{8\pi\lambda^2} \sqrt{\frac{\pi d}{2\lambda}} e^{-\frac{d}{\lambda}}, \quad (\text{A.43})$$

after having substituted (A.31) $H_{c1} = \frac{\epsilon_1}{\Phi_0}$. And approximating recursively

$$\ln \left[\frac{z \Phi_0}{(H_{\text{ext}} - H_{c1}) 8\pi\lambda^2} \right] = \frac{d}{\lambda} - \frac{1}{2} \ln \left(\frac{\pi d}{2\lambda} \right) \stackrel{d \gg \lambda}{\approx} \frac{d}{\lambda}, \quad (\text{A.44})$$

the macroscopic magnetic induction B becomes

$$B = \frac{\Phi_0}{d^2} \approx \frac{\Phi_0}{\lambda^2} \frac{1}{\left(\ln \frac{z \Phi_0}{(H_{\text{ext}} - H_{c1}) 8\pi\lambda^2} \right)^2}. \quad (\text{A.45})$$

A.2.3 Range $H_{c1} \ll H_{\text{ext}} \ll H_{c2}$

In this case the number of flux lines is large and the distance d between flux lines is much smaller than penetration length λ (but $d \gg \xi$ still holds) and we can replace the sum over the positions of the vortex lines by integral $\sum_{j \neq 0} \approx \frac{1}{d^2} \int_d^\infty d^2 r$

$$\frac{F_{\text{mixed}}}{\text{volume}} = \frac{1}{d^2} \epsilon_1 + \frac{1}{d^4} \frac{\Phi_0^2}{4\pi\lambda^2} \int_d^\infty \lambda^2 2\pi \frac{r}{\lambda} \frac{dr}{\lambda} K_0 \left(\frac{r}{\lambda} \right) = \frac{1}{d^2} \epsilon_1 + \frac{1}{d^4} \frac{\Phi_0^2}{2} \frac{d}{\lambda} K_1 \left(\frac{d}{\lambda} \right).$$

From the equilibrium condition $\frac{\partial G_{\text{mixed}}}{\partial d} = 0$, one obtains

$$H_{\text{ext}} - H_{c1} = \frac{\Phi_0}{4d^2} \frac{d}{\lambda} \left[4K_1 \left(\frac{d}{\lambda} \right) + \frac{d}{\lambda} K_0 \left(\frac{d}{\lambda} \right) \right]. \quad (\text{A.46})$$

Using the asymptotic behaviour of the Henkel's functions for $x \approx 0$ ¹⁶ and since now d/λ is very small, we obtain for the macroscopic magnetic induction the following equation

$$H_{\text{ext}} - H_{c1} \stackrel{B=\frac{\Phi_0}{d^2}}{\approx} B + \frac{\Phi_0}{8\lambda^2} \left[\ln \left(\frac{\Phi_0}{4B\lambda^2} \right) + 2\gamma - 2 \right]. \quad (\text{A.47})$$

A.2.4 Range $H_{\text{ext}} \approx H_{c2}$

Approaching the point where also the Shubnikov state is destroyed and the systems transits towards a normal conductor, $H_{\text{ext}} \approx H_{c2}$, the now very small order parameter ψ fulfils no longer $|\psi| = \sqrt{\frac{|\alpha|}{|\beta|}}$. Therefore, we cannot use the previous field equations which held when a large part of the material was in the Meissner state, i.e., when the distance between flux lines was much greater than the radius of the flux line ($d \gg \xi$). Let us start at a magnetic field very near to $H_{\text{ext}} = H_{c2}$, at which the order parameter is ψ_1 . The contribution of the supercurrent due to the magnetic induction is assumed negligibly and thus $\mathcal{B} = |\text{curl } \vec{A}| \approx H_{c2}$. Magnetic field H_{c2} will be obtained from the equation which satisfies the small value ψ_1 of the scalar field

$$-\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right)^2 \psi_1 - |\alpha| \psi_1 = 0, \quad (\text{A.48})$$

where term $|\beta||\psi_1|^2\psi_1$ has been neglected when compared to terms of order ψ_1 . Taking $\hat{z}$ as the direction of local magnetic induction $\vec{\mathcal{B}} = \text{curl } \vec{A}$, its corresponding vector potential is $\vec{A} = \hat{y} x \mathcal{B}$. By means of the ansatz $\psi_1 = u(x)e^{ik_y y}$, equation (A.48) becomes

$$u'' + u \left[- \left(\frac{qe\mathcal{B}}{\hbar c} \right)^2 \left(x - k_y \frac{\hbar c}{qe\mathcal{B}} \right)^2 + \frac{2m|\alpha|}{\hbar^2} \right] = 0.$$

With definitions $\zeta \stackrel{\text{def}}{=} \sqrt{\frac{qe\mathcal{B}}{\hbar c}} \left(x - k_y \frac{\hbar c}{qe\mathcal{B}} \right)$ and $u \stackrel{\text{def}}{=} v(\zeta)e^{-\frac{\zeta^2}{2}}$, the above equation takes the form of an Hermite differential equation $v'' - 2\zeta v' + \left[\left(\frac{\hbar c}{qe\mathcal{B}} \right) \frac{2m|\alpha|}{\hbar^2} - 1 \right] v = 0$, which can be solved if $\left(\frac{\hbar c}{qe\mathcal{B}} \right) \frac{2m|\alpha|}{\hbar^2} - 1 = 2n$, $n = 0, 1, 2, \dots$ is satisfied, or equivalently, if $\mathcal{B} = \frac{1}{2n+1} \frac{2mc|\alpha|}{qe\hbar}$. The solution for $n = 0$ corresponds to the largest local magnetic induction which the system can still hold up and therefore it corresponds to critical magnetic field H_{c2} we are looking for

$$H_{c2} = \frac{2mc}{qe\hbar} |\alpha| = \frac{\lambda}{\xi} H_c = \frac{\Phi_0}{4\pi\xi^2}. \quad (\text{A.49})$$

$$^{16}K_0(x) \approx - \left(\ln \frac{x}{2} + \gamma \right) \left(1 + \frac{x^2}{4} \right) + \frac{x^2}{4}, \quad xK_1(x) + \frac{1}{4}x^2K_0(x) \frac{dK_0}{dx} \approx -K_1 \frac{x^2}{4} \left(\ln \frac{x}{2} + \gamma - 1 \right) + 1.$$

Since $H_{c2} \approx \frac{\Phi_0}{d^2}$, it becomes clear that at $H_{\text{ext}} \approx H_{c2}$ the distance d between flux lines is of the order of core radius ξ , i.e., the flux lines are very close to each other: vortices condense. If one prefers to use the definition for the scalar field's correlation length like that in the Remark on page 177, $\xi' = \sqrt{2}\xi$, then critical field H_{c2} is given by $\frac{\Phi_0}{2\pi\xi'^2} = \sqrt{2} \left(\frac{\lambda}{\xi'} \right) H_c$.

Further, at $\mathcal{B} \approx H_{c2}$ and by using $\partial_{\mp}\partial_{\pm} = \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 \pm \frac{qe}{\hbar c} \mathcal{B}$, the equation for scalar field ψ_1 , $\left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right)^2 \psi_1 = - \left(\frac{qeH_{c2}}{\hbar c} \right) \psi_1$, is equivalent to $\partial_+ \psi_1 \Big|_{\vec{A}=\vec{A}_{c2}} = 0$. One obtains again equation $\partial_+ \psi = 0$, just as in the case where the equation for β_0 (A.11) was solved, since we are working again at the limit to the normal conducting state; the difference lies in that now we are studying the *mixed* or Shubnikov state at $|\beta| > \beta_{\text{crit}}$ whereas in the other case the *Meissner* state at $|\beta| = \beta_{\text{crit}}$ was considered.

In order to obtain the dependence of the magnetic induction $\mathcal{B}$ on the external applied magnetic field for the range of $H_{\text{ext}} \approx H_{c2}$, we shall proceed along the following self-consistent approximation: one starts at the just discussed ψ_1 with $\mathcal{B} = H_{c2}$ (neglecting any contribution of the supercurrent to the magnetic induction), and after decreasing slightly the external magnetic field, one looks for the solution for ψ by taking into account in the local magnetic induction the contribution due to the initial ψ_1 , i.e., by solving the equation for ψ for a magnetic induction equal to $\mathcal{B} = H_{\text{ext}} + h_s(\psi_1)$, being $h_s(\psi_1)$ the magnetic field created by the supercurrent $\vec{j}_s(\psi_1)$ corresponding to ψ_1 . From

$$\vec{j}_s(\psi_1) = -\frac{iqe\hbar}{2m} \left[\psi_1^* \left(\vec{\nabla}_{(x,y)} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right) \psi - \text{c.c.} \right] \Big|_{\vec{A}=\vec{A}_{c2}}^{\partial_+ \psi_1=0} - \frac{qe\hbar}{2m} \vec{\nabla} \times (|\psi_1|^2 \hat{z}),$$

and according to $\vec{\nabla} \times \vec{h}_s(\psi_1) = \frac{1}{c} \vec{j}_s(\psi_1)$, the contribution $\vec{h}_s(\psi_1)$ to the magnetic field we are looking for is $\vec{h}_s(\psi_1) = -\frac{qe\hbar}{2mc} |\psi_1|^2 \hat{z}$. After having decreased the external magnetic

$$\text{field, and considering } \begin{cases} \psi \text{ solution of } -\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 \psi - |\alpha|\psi + |\beta||\psi|^2\psi = 0 \\ \psi_1 \text{ solution of } -\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right)^2 \psi_1 - |\alpha|\psi_1 = 0 \end{cases}, \text{ the}$$

vanishing integral

$$\begin{aligned} & \left\langle \psi \left| -\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 - |\alpha| + |\beta||\psi|^2 \right| \psi \right\rangle \stackrel{\text{def}}{=} \\ & = \frac{1}{\text{area}} \int \psi^* \underbrace{\left[-\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 - |\alpha| + |\beta||\psi|^2 \right] \psi}_{=0} d^2\vec{r} = 0, \end{aligned} \tag{A.50}$$

will be expanded in powers of $\mathcal{B} - H_{c2} = (H_{\text{ext}} - H_{c2}) - \frac{qe\hbar}{2mc} |\psi_1|^2$ in order to obtain the dependence of ψ_1 on H_{ext} ; the above integral is carried out on the plane normal to the

direction of the magnetic field.

The expansion of (A.50) for $\begin{cases} \vec{A} \approx \vec{A}_{c2} + (\mathcal{B} - H_{c2}) \frac{\partial \vec{A}}{\partial \mathcal{B}} \Big|_{\mathcal{B}=H_{c2}} \\ \psi \approx \psi_1 + (\mathcal{B} - H_{c2}) \chi \end{cases}$ becomes for the leading (first) order in $(\mathcal{B} - H_{c2})$ equal to

$$\begin{aligned} 0 = & \left\langle \psi \left| -\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A} \right)^2 - |\alpha| \right| \psi \right\rangle + |\beta| \langle |\psi|^4 \rangle \approx |\beta| \langle |\psi|^4 \rangle + \\ & + \underbrace{\left\langle \psi_1 \left| -\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right)^2 - |\alpha| \right| \psi_1 \right\rangle}_{=0} + \\ & + \left[\left\langle \chi \left| (\mathcal{B} - H_{c2}) \left[-\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{iqe}{\hbar c} \vec{A}_{c2} \right)^2 - |\alpha| \right] \right| \psi_1 \right\rangle + \text{c.c.} \right] - \end{aligned} \quad (\text{A.51})$$

$$\begin{aligned} & - \frac{\hbar^2}{2m} \left\langle \psi_1 \left| (\mathcal{B} - H_{c2}) \left[\frac{\partial}{\partial \mathcal{B}} \left(\vec{\nabla}^2 - \frac{2iqe}{\hbar c} x \mathcal{B} \partial_y - \frac{q^2 e^2}{\hbar^2 c^2} x^2 \mathcal{B}^2 + \frac{2m|\alpha|}{\hbar^2} \right) \right] \right|_{\mathcal{B}=H_{c2}} \right| \psi_1 \right\rangle = \\ & = \left(-\frac{2iqe}{\hbar c} x \partial_y - \frac{2q^2 e^2}{\hbar^2 c^2} x^2 H_{c2} \right) \psi_1 = \frac{2qe}{\hbar c} x \partial_x \psi_1 \\ & = |\beta| \langle |\psi|^4 \rangle - 2 \left(\frac{qe\hbar}{2mc} \right) (H_{\text{ext}} - H_{c2}) \langle \psi_1 | x \partial_x | \psi_1 \rangle + 2 \left(\frac{qe\hbar}{2mc} \right)^2 \langle \psi_1 | |\psi_1|^2 x \partial_x | \psi_1 \rangle, \end{aligned}$$

where in the last term previous to the final result, $\frac{2qe}{\hbar c} x \partial_x \psi_1 \Big|_{\vec{A}=\vec{A}_{c2}} = 0 = \frac{2qe}{\hbar c} x \partial_x \psi_1 + i \frac{2qe}{\hbar c} x \partial_y \psi_1 + \frac{2q^2 e^2}{\hbar^2 c^2} x (A_{c2})_y$ with $\vec{A}_{c2} = \hat{y} x H_{c2}$ was used. Equalities $\langle \psi_1 | x \partial_x | \psi_1 \rangle = \frac{1}{\text{area}} \frac{1}{2} \int x \partial_x (|\psi_1|^2) d^2 \vec{r} \stackrel{\text{integr. by parts}}{=} 0 - \frac{1}{2} \langle |\psi_1|^2 \rangle$ leads from (A.51) to the following result

$$\begin{aligned} 0 = & \underbrace{\left(|\beta| - \left(\frac{qe\hbar}{2mc} \right)^2 \right) \langle |\psi_1|^4 \rangle + (H_{\text{ext}} - H_{c2}) \left(\frac{qe\hbar}{2mc} \right) \langle |\psi_1|^2 \rangle}_{\left(\frac{qe\hbar}{2mc} \right)^2 (\kappa^2 - 1)} \end{aligned} \quad (\text{A.52})$$

which is the searched dependence of ψ_1 on the external applied magnetic field near to the transition point towards the normal conducting state. By writting $\langle |\psi_1|^2 \rangle = \frac{\langle |\psi_1|^2 \rangle^2}{\langle |\psi_1|^2 \rangle} =$

$= -\frac{\langle |\psi_1|^2 \rangle^2}{\langle |\psi_1|^4 \rangle} \frac{(H_{\text{ext}} - H_{c2})}{\left(\frac{qe\hbar}{2mc}\right)(\kappa^2 - 1)}$ and by taking at first approximation $\frac{\langle |\psi_1|^2 \rangle^2}{\langle |\psi_1|^4 \rangle} \approx 1$, the macroscopic magnetic induction $B = \frac{1}{\text{area}} \int \mathcal{B} d^2\vec{r} = \frac{1}{\text{area}} \int (H_{\text{ext}} + h_s) d^2\vec{r}$ becomes then given by the following equation

$$B \stackrel{h_s = -\frac{qe\hbar}{2mc} |\psi_1|^2}{=} H_{\text{ext}} - \left(\frac{qe\hbar}{2mc} \right) \langle |\psi_1|^2 \rangle \approx H_{\text{ext}} + \frac{(H_{\text{ext}} - H_{c2})}{(\kappa^2 - 1)}. \quad (\text{A.53})$$

Or if the notation in the Remark on page 177, is used $\kappa' = \frac{\lambda}{\xi'} = \frac{\lambda}{\sqrt{2}\xi} = \frac{\kappa}{\sqrt{2}}$, the previous relation is written as $B = H_{\text{ext}} + \frac{(H_{\text{ext}} - H_{c2})}{(2\kappa'^2 - 1)}$.

Appendix B

Informal introduction to homotopy theory. Line and point defects.

B.1 The first homotopy group π_1

A good and detailed introduction into this subject can be found in the excellent papers by G. 't Hooft [9], N.D. Mermin [69] and R. Shankar [87]. All these papers will be followed in this Appendix. Let us start with a closed rubber band, parametrized by a variable ϕ with $0 \leq \phi \leq 2\pi$, and a manifold Y , for example, a ring. When we wrap the closed rubber band on the manifold Y , each point ϕ of the band is in contact with a point on the manifold, and this defines the map

$$f(\phi) : \text{rubber band} \longrightarrow \text{manifold } Y,$$

which is called a loop. Hereafter the closed rubber band will be denoted by S^1 . If the manifold Y is a ring, the rubber band can be wrapped around the ring as in Fig. B.1.

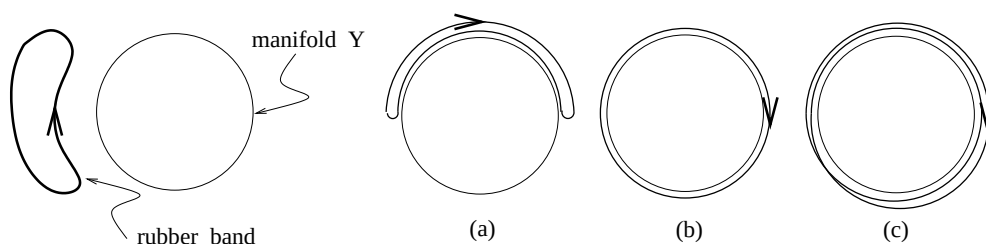


Figure B.1: Different forms to wrap a rubber band around a ring.

Because the configuration in (a) can be deformed to a point, it is called a topologically trivial configuration. The other configurations are stable and topologically nontrivial. The number of times which the band winds around the ring is called the winding number. For the case where the manifold Y is a ring S^1 , a configuration with a certain winding number cannot be deformed into another configuration with different winding number without cutting the rubber band or without leaving the manifold Y . For the case where the manifold Y is a 2-dimensional sphere's surface (or of higher dimensionality), the rubber band can always be deformed into a point: the band can slide on the surface, being contracted to a point (Fig. B.2). In this case, all configurations are trivial.

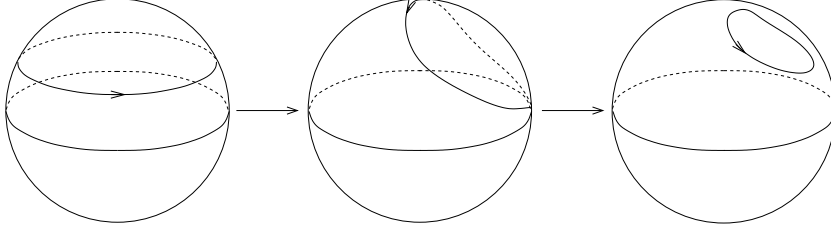


Figure B.2: All loops on a spherical surface can be shrunk to a point.

Let now consider loops based at the manifold point y_0 : $f(\phi = 0) = f(\phi = 2\pi) = y_0$. Given two maps f and g , we define the product map $f \circ g$

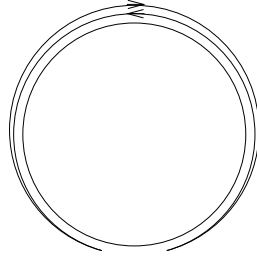
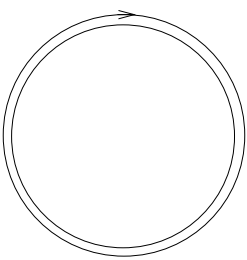
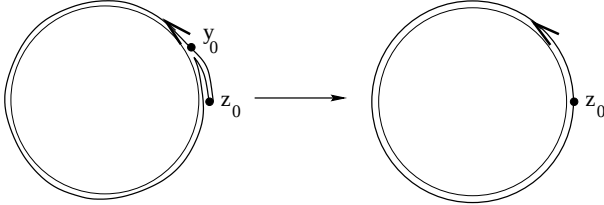
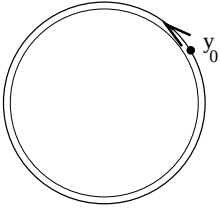
$$(f \circ g)(\phi) = \begin{cases} f(2\phi) & 0 \leq \phi \leq \pi \\ g(2\phi - 2\pi) & \pi \leq \phi \leq 2\pi, \end{cases}, \quad (\text{B.1})$$

and the inverse map

$$f^{-1}(\phi) = f(2\pi - \phi). \quad (\text{B.2})$$

As an example, if f is the loop (b) in Fig. B.1, the loop (c) is given by $f \circ f$ and the inverse loop of (b) has the rubber band wrapped in the contrary direction. The identity or constant map $\tilde{e}(\phi)$ is defined by the loop

$$\tilde{e}(\phi) = y_0 \quad 0 \leq \phi \leq 2\pi, \quad (\text{B.3})$$



and corresponds to the loop (a) in Fig. B.1 shrunk to a point. The choice of point y_0 to classify the different loops is irrelevant: to study loops based at another point z_0 , we only need “travel” to y_0 by means of a connecting path, study the loop based at y_0 and return to z_0 (as in the figure). If we have a topologically nontrivial loop — i.e., which cannot be contracted to a point — based at y_0 , the

loop is also nontrivial when based at any other point of the manifold Y . Now, let $f(\phi)$ be the loop of the adjacent figure: loop $f \circ f^{-1}$ is not exactly $\tilde{e}(\phi) = y_0$, but it can

be deformed to $\tilde{e}(\phi)$. Thus, in order to study the different topologically stable loops based at y_0 it is more useful to group all loops based at y_0 which can be deformed into each other. Such loops are called homotopic and form a class with respect to the equivalence relation $\sim$ of “being homotopic (or deformable) to”; the mathematical definition of a

deformation which takes loop $f_1(\phi)$ into $f_2(\phi)$ is a parametric function $h(t, \phi)$ (with $0 \leq t \leq 1$) satisfying $\begin{cases} h(t, \phi) = f_1(\phi) \\ h(t, \phi) = f_2(\phi) \end{cases}$, with $h(t, \phi = 0) = y_0$.

This grouping into equivalence classes allows us to define the product $*$ of two classes $[f]$ and $[g]$ as

$$[f] * [g] = [f \circ g],$$

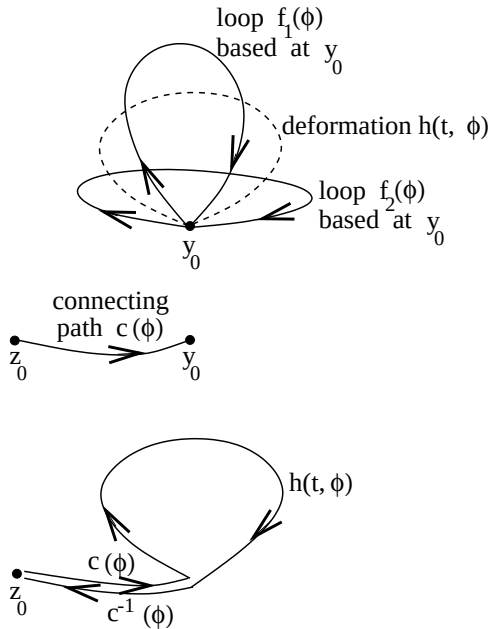
product which satisfies the associative law, $([f] * [g]) * [h] = [f] * ([g] * [h])$, since loop $(f \circ g) \circ h$ is just loop $f \circ (g \circ h)$ but with a different parametrization of ϕ (see definition (B.1)). The identity or trivial class $[\tilde{e}]$ is formed by all those loops deformable to a point and therefore, the product map of a loop belonging to class $[\tilde{e}]$ times a loop of class $[f]$ can be deformed to a loop of class $[f]$: $[\tilde{e}] * [f] = [f] * [\tilde{e}] = [f]$. Further, to state that $f^{-1}(\phi)$ is the inverse map of $f(\phi)$ is equivalent to say that $(f \circ f^{-1})(\phi)$ is deformable to a single point, $f \circ f^{-1} \sim \tilde{e}$: the inverse $[f]^{-1}$ of class $[f]$ is just class $[f^{-1}]$ since $[f] * ([f]^{-1}) = [f \circ f^{-1}] = [\tilde{e}]$. Thus, the classes of loops based at point y_0 have the structure of a group¹ called the based first (or fundamental) homotopy group $\pi_1(Y, y_0)$: this group contains the set of all different topologically stable forms to wrap a rubber band S^1 on Y beginning and ending at y_0 , where “different” means not deformable (=not homotopic) into one other.

As already noted, the base point y_0 is not relevant in the above classification of topologically stable loops: among the classes of loops based at two different points y_0 and z_0 one can always construct a correspondence which is one-to-one and onto. For simplicity, let us represent a loop only by means of the points in Y which are being vi-

sited as one covers in an external space the rubber band S^1 . Hence, let $f_1(\phi)$ and $f_2(\phi)$ of the figure represent two loops based at y_0 which belong to the same class $[f] \subset \pi_1(y_0, Y)$ and are thus deformable into each other: an intermediate stage of the deformation $h(t, \phi)$ is also represented as dashed line. And let us denote by $c(\phi)$ the connecting path between point z_0 and y_0 ; since $c(\phi = 0) = z_0 \neq c(\phi = 2\pi) = y_0$, $c(\phi)$ is not a loop. Loops $c \circ f_1 \circ c^{-1}$ and $c \circ f_2 \circ c^{-1}$ are based at z_0 and belong to the same class of $\pi_1(z_0, Y)$: the deformation which takes the first z_0 -based loop into the second one is given by

$$h'(t, \phi) = (c \circ h(t) \circ c^{-1})(\phi),$$

as represented in the last figure. The latter construction allows us to associate with the connecting path $c(\phi)$ a correspondence c among each y_0 -based



¹Only as a reminder, a group is a set with an combination operation $*$ defined in it fulfilling the associative law, the existence of the identity element (in this case, the class of loops deformable to a point), and the existence of the inverse for each element of the group.

class $[f]$ and the z_0 -based class $[c \circ f \circ c^{-1}]$

$$\begin{aligned} c : \pi_1(Y, y_0) &\longrightarrow \pi_1(Y, z_0) \\ \text{class } [f] &\rightarrow \text{class } c([f]) \equiv [c \circ f \circ c^{-1}] \end{aligned} \quad (\text{B.4})$$

where, from the very definition of equivalence class as subset of loops related to each other by means of a deformation, the just discussed correspondence does not depend on the representative f_1 or f_2 of the y_0 -based class $[f]$ we have chosen for above construction. Correspondence (B.4) is one-to-one: if two images through this correspondence coincides, $c([f]) = c([g])$, then $[f] = [g]$ follows from $c^{-1}(c([f]) = c([g]))$ and from result $c^{-1}(c([f])) = c^{-1}([c \circ f \circ c^{-1}]) = [c^{-1} \circ (c \circ f \circ c^{-1}) \circ c] = [f]$. This one-to-one correspondence is also onto, and thus isomorphism, since all z_0 -based loops can be obtained, by the “connection” through path $c(\phi)$, from the y_0 -based loops. Further, this isomorphism between the different classes of loops based at two different points keeps the combination law of classes: if $f_1 \circ g_1 \in [f] * [g]$ is a loop representative of the y_0 -based product class $[f] * [g] = [f \circ g]$, then the z_0 -based loop $c \circ (f_1 \circ g_1) \circ c^{-1}$ satisfies

$$c \circ (f_1 \circ g_1) \circ c^{-1} = (c \circ f_1 \circ c^{-1}) \circ (c \circ g_1 \circ c^{-1}) \in \underbrace{[c \circ f \circ c^{-1}] * [c \circ g \circ c^{-1}]}_{c([f]) * c([g])},$$

or equivalently

$$c \left(\underbrace{[f] * [g]}_{y_0 - \text{based}} \right) = c([f]) * \underbrace{c([g])}_{z_0 - \text{based}}. \quad (\text{B.5})$$

Isomorphism (B.4) between based group $\pi_1(Y, y_0)$ and based group $\pi_1(Y, z_0)$ allows us to “reconstruct”, for an arbitrary base point, its number of different classes of topologically stable loops once we are given the number of classes of loops based at another point. Moreover, this allows us also to define an abstract group $\pi_1(Y)$, being each one of the based groups $\pi_1(Y, y_0)$, $\pi_1(Y, z_0)$... particular realizations of this abstract group $\pi_1(Y)$ and related to each other through the one-to-one correspondence $[f] \rightarrow c([f])$.

From the considered cases on different forms to wrap a rubber band around Y , with Y a ring (Fig. B.1) and with Y a sphere (Fig. B.2), we obtain the following results for the (abstract) first homotopy group

$$\pi_1(S^1) = \mathbf{Z}, \quad (\text{B.6})$$

$$\pi_1(S^n) = 0 \quad \text{for } n > 1. \quad (\text{B.7})$$

We can generalize this definition and consider the topologically different forms to wrap a 2-dimensional spherical surface S^2 (for example, an orange’s skin) on Y : this set is called $\pi_2(Y)$, and so on.

Before starting with the use of the first homotopy group as tool to classify topological line defects, let us consider one interesting point which will play an important role much later in this Appendix (see page 271): the consequence of being the first homotopy group $\pi_1(Y)$ Abelian (=commutative) or not [69]. The correspondence (B.4) among classes of

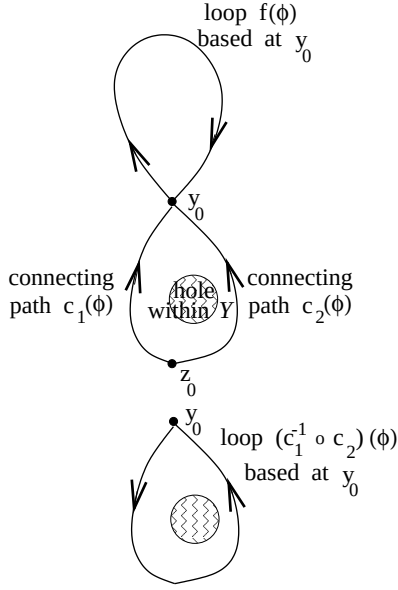
loops based at two different points may well depend on the path $c(\phi)$ which connects these two base points. In general, for two different connecting paths $c_1(\phi)$ and $c_2(\phi)$ between points z_0 and y_0 and for a class of loops $[f]$ in $\pi_1(Y, y_0)$ it may occur that

$$\underbrace{c_1([f])}_{[c_1 \circ f \circ c_1^{-1}]} \neq \underbrace{c_2([f])}_{[c_2 \circ f \circ c_2^{-1}]} \quad , \quad (\text{B.8})$$

and the z_0 -based loops $c_1 \circ f \circ c_1^{-1}$ and $c_2 \circ f \circ c_2^{-1}$ are not deformable to one other

$$\underbrace{c_1 \circ f \circ c_1^{-1}}_{z_0 - \text{based}} \not\sim \underbrace{c_2 \circ f \circ c_2^{-1}}_{z_0 - \text{based}} \iff \underbrace{f}_{y_0 - \text{based}} \not\sim \underbrace{(c_1^{-1} \circ c_2)}_{y_0 - \text{based}} \circ f \circ \underbrace{(c_2 \circ c_1^{-1})}_{y_0 - \text{based}} \quad , \quad (\text{B.9})$$

$$= (c_1^{-1} \circ c_2)^{-1}$$



where $\not\sim$ means “not deformable (or not homotopic) to”. An example where this occurs (see figure) is the case where manifold Y has a hole and thus loop $c_1 \circ f \circ c_1^{-1}$, drawn from z_0 avoiding the hole from the left, cannot be taken to the from-the-right-drawn loop $c_2 \circ f \circ c_2^{-1}$ without crossing the hole and thus leaving Y . Loop $g = (c_1^{-1} \circ c_2)(\phi)$, which surrounds the hole and cannot be shrunk to a point, is based at y_0 . Hence relation (B.9), $f \not\sim g \circ f \circ g^{-1}$ or equivalently $f \circ g \not\sim g \circ f$, means that the product of two different loops based at y_0 — and thus the product of two classes of $\pi_1(Y, y_0)$ — is not commutative: as a consequence, and due to the isomorphism among based first homotopy groups, this non-commutativity extends to all these and the (abstract) first homotopy group $\pi_1(Y)$ is non-Abelian.

The converse, i.e., a non-Abelian $\pi_1(Y)$ implies the existence of different connecting paths (not deformable into each other) between base points, and thus the existence of different isomorphisms between based first homotopy groups, holds also true². Therefore, one cannot directly assign to each loop class belonging to based $\pi_1(Y, y_0)$ an element of group $\pi_1(Y)$: although we had assumed that the correspondence between each element of the first homotopy group $\pi_1(Y)$ and each class of loops based at y_0 would be unique, to which z_0 -based class should correspond the $\pi_1(Y)$ ’s element related to a certain y_0 -based class if there exist different correspondences between $\pi_1(Y, y_0)$ and $\pi_1(Y, z_0)$? However, according to result (B.9), the two non-homotopic z_0 -based loops $c_1 \circ f \circ c_1^{-1}$ and $c_2 \circ f \circ c_2^{-1}$ — images of the same y_0 -based loop f through the two

²Since both $\pi_1(Y, y_0)$ and $\pi_1(Y, z_0)$ are non-commutative (and the same holds for any other based first homotopy group), there exist two y_0 -based loops g and f which satisfy $f \circ g \not\sim g \circ f$ or $f \not\sim g \circ f \circ g^{-1}$. Assuming that (closed) loop g goes through point z_0 , g is written as the product $c_1^{-1} \circ c_2$, where $c_1(\phi)$ and $c_2(\phi)$ are two (open) paths connecting z_0 and y_0 ; if loop g does not cover point z_0 , one can always slightly deform the loop in such a way that from an arbitrary point of g an “arm” is detached from the loop, reaches z_0 and then comes back to g along the same “arm”. Thus, $f \not\sim g \circ f \circ g^{-1}$ and $g = c_1^{-1} \circ c_2$ combine to $c_1 \circ f \circ c_1^{-1} \not\sim c_2 \circ f \circ c_2^{-1}$, or equivalently, the image of the y_0 -based loop class $[f]$ through the two isomorphisms $[f] \rightarrow c_1([f])$ and $[f] \rightarrow c_2([f])$ is not the same.

different isomorphisms $[f] \rightarrow c_1([f]) = [c_1 \circ f \circ c_1^{-1}]$ and $[f] \rightarrow c_2([f])$ — are related by the z_0 -based loop $k = c_1 \circ c_2^{-1}$ (to be distinguished from the loop $c_1^{-1} \circ c_2$ based at y_0)

$$c_1 \circ f \circ c_1^{-1} \sim k \circ (c_2 \circ f \circ c_2^{-1}) \circ k^{-1}. \quad (\text{B.10})$$

Or equivalently, the two isomorphic images of based group $\pi_1(Y, y_0)$ differ only by the following one-to-one correspondence among z_0 -based loop classes

$$\underbrace{[\alpha]}_{z_0 - \text{based}} \longrightarrow \underbrace{[k \circ \alpha \circ k^{-1}]}_{z_0 - \text{based}} = [k] * [\alpha] * [k]^{-1}, \quad (\text{B.11})$$

which is an inner isomorphism (also called an automorphism) of $\pi_1(Y, z_0)$ onto the same $\pi_1(Y, z_0)$. And conversely, given any certain automorphism in $\pi_1(Y, z_0)$ defined by the loop class $[k]$, by taking a representative k of such homotopy class and descomposing it into $k = c_1 \circ c_2^{-1}$, we obtain two different isomorphism from $\pi_1(Y, z_0)$ to $\pi_1(Y, y_0)$ (different, of course, if class $[k]$ is not trivial or formed by loops deformable to a point) characterized by the two paths $c_1(\phi)$ and $c_2(\phi)$.

If, in the non-Abelian case, instead of considering the group $\pi_1(Y, \text{base point})$ of based homotopy classes — where for any based loop class there can be different images when transporting to another base point — one considers *conjugacy* classes³, it is then possible to define an *unique* 1-to-1 correspondence between the conjugacy classes of the based first homotopy groups: because from a first homotopy group based at a certain point, any set of different images (B.8) at another transported base point are related to each other by some automorphism (B.11) defined in the set of loops based at the transported point⁴. And this unique 1-to-1 correspondence among the conjugacy classes of the based $\pi_1(Y, \text{base point})$'s at any base point can be also extended to an unique 1-to-1 correspondence to the conjugacy classes of the (abstract) first homotopy groups $\pi_1(Y)$, providing us with a tool to classify loops independently of the non-commutative character of the first homotopy group.

In an informal way of speaking and returning to the figure of the previous page, to deal directly with the conjugacy classes allows us to combine the loop under study $f(\phi)$ with the operation of surrounding of all possible holes — described by the loops $g(\phi)$ which do not commute with $f(\phi)$ — which the manifold Y could have and, by putting all these combinations into one conjugacy class, to “forget” the existence of the holes and

³Let G a group and a one of its elements: the conjugacy class of a is defined as the set of all elements $b \circ a \circ b^{-1}$ with any $b \in G$. The advantage of dealing with conjugacy classes is that we need not to be concerned with the order in the product of loops: if $a_1 \circ a_2$ belongs to a certain conjugacy class, the same holds for $a_2 \circ a_1 = a_2 \circ (a_1 \circ a_2) \circ a_2^{-1}$. Of course, if the elements of G commute among them, $b \circ a = a \circ b \quad \forall b$, then the conjugacy class of a reduces to a itself.

⁴And automorphism (B.11) does not mix conjugacy classes, i.e., does not take members of a certain conjugacy class into members of a different conjugacy class. Let α be a certain z_0 -based loop of homotopy class $[\alpha]$: for a certain (fixed) homotopy class $[k]$ of z_0 -loops, the image through automorphism $[\alpha] \rightarrow [k \circ \alpha \circ k]$ for the conjugacy class $\{g \circ \alpha \circ g^{-1} : \text{any loop } g \in \pi_1(Y, z_0) \text{ based at } z_0\}$ belongs also to the same conjugacy class

$$\underbrace{g \circ \alpha \circ g^{-1}}_{\in \text{conj. class}} \xrightarrow{\text{autom. (B.11)}} k \circ (g \circ \alpha \circ g^{-1}) \circ k^{-1} = \underbrace{(k \circ g) \circ \alpha \circ (k \circ g)^{-1}}_{\substack{z_0 - \text{based} \\ \in \text{same conj. class}}}.$$

to concentrate ourselves in the classification of the loop $f(\phi)$ itself. Of course, if $\pi_1(Y)$ is Abelian, all this discussion is unnecessary since the conjugacy class of any element is the element itself: there is only one unique isomorphism between the homotopy loop classes based at two different points, independently of the connecting path $c(\phi)$ chosen between these two points since any other path $c'(\phi)$ can be deformed to $c(\phi)$.

B.2 Applications of group π_1 : line defects in a 3-dimensional system (I)

For a crystal, let us define a complex scalar field ψ at the center of each atomic cell

$$\psi = e^{i\varphi}, \quad (\text{B.12})$$

where phase φ can be also viewed as the angle, in a internal 2-dimensional space, between a certain reference direction and the orientation taken by the “arrow” ψ . Now this phase φ is set proportional to the displacement u suffered by each crystal cell’s center in external physical space

$$\varphi = \frac{2\pi}{a}u, \quad (\text{B.13})$$

where a is the lattice spacing. When the displacement of the cell is equal to a , the arrow has drawn a complete rotation. The fact that the arrow is pointing in different internal directions at two neighbour external points means that the crystal is deformed in that region: one lattice cell has been displaced a different distance than the other. On the contrary, the undeformed state is represented by a configuration where all arrows point in the same internal direction. And a rigid translation of the crystal as a whole is represented by an internal rotation of the arrows by the *same* amount χ_0

$$\varphi \rightarrow \varphi - \chi_0 \quad \text{or} \quad u \rightarrow u - \frac{a}{2\pi}\chi_0 \quad \text{with} \quad \chi_0 = \text{constant}. \quad (\text{B.14})$$

Let us now produce a dislocation in a perfect crystal by cutting along a lattice plane, then displacing a cut side by one atomic distance with respect to the other side, and finally gluing the two parts. The dislocation will be represented by assigning an arrow ψ to each lattice cell of the original perfect crystal, arrow which suffers one *complete rotation* in its internal plane when a closed path is covered in physical space around the dislocation (Fig. B.3). Each complete rotation of the arrow indicates one elementary Burgers vector. This model is of course oversimplified, since we are dealing with a deformation produced only along one direction; further, we are supposing that dislocations are straight lines, reducing thus the problem to a 2-dimensional model for the external physical plane, perpendicular to which the dislocation lines are placed. The generalization is immediate and with the simplified model one is able to reproduce the relevant physical characteristics without being involved with many indices ⁵.

⁵As already discussed in the Note on page 182, an actual 2-dimensional system — its field variables only depends on two coordinates — embedded into a 3-dimensional space deviates in its behaviour from that displayed by the simplified model presented here when is coupled to a gauge field.

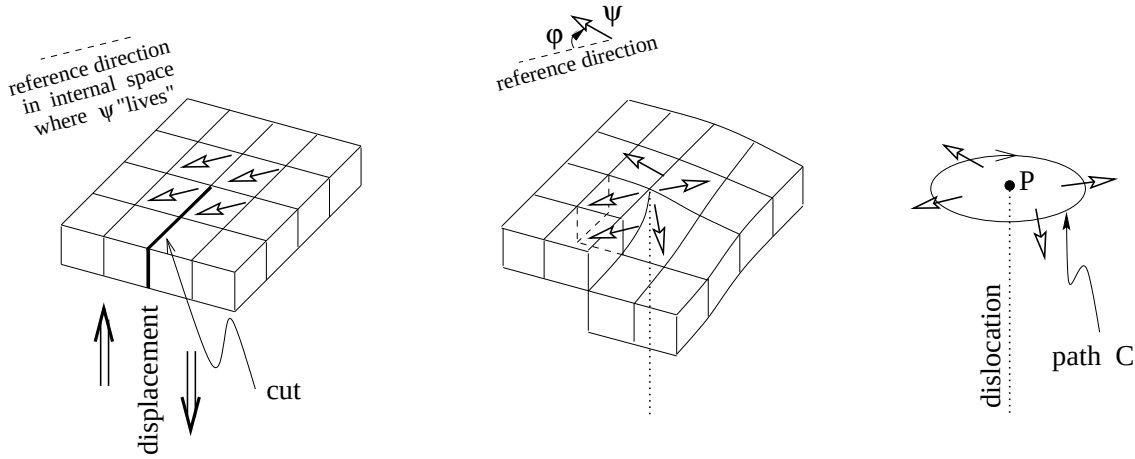


Figure B.3: Screw dislocation represented by means of complex scalar field ψ , whose orientation in an internal plane indicates the displacement in the physical space of each lattice cell. Covering circumference $C = S^1$, around and arbitrarily distant from the dislocation, ψ draws a complete internal rotation.

A local elastic deformation at a certain point P , such as the shear of the crystal, changes the orientation of ψ at this point and at its neighbours, but following a closed path around P , the arrow ψ will not draw a complete rotation in its internal space. If such a lattice is allowed to relax, the final state will be that with all ψ pointing in the same direction. These deformations which can be removed by relaxing without cutting are called compatible deformations. This is not the case for a dislocation, where the complete rotation drawn by the arrow cannot be removed without cutting again the crystal. At point P therefore any deformation, compatible as well as due to dislocations, can be classified by the number of entire internal rotations which are drawn by scalar field ψ when a distant closed path C around P is covered in the physical external space: i.e., by considering all possible maps from C to the set of ψ values. It is important to stress that the changes in orientation of scalar field ψ , although represented in the external physical space in Fig B.3, actually occurs in an internal space.

Now let us suppose that our system is 2-dimensional (it contains only fields depending only on two coordinates), is placed also in a 2-dim. space and is described by the following Lagrangian density ℓ

$$\ell = -\frac{1}{2} (\partial_\mu \psi^*) (\partial^\mu \psi) - V(\psi) \quad \text{with} \quad \mu = 1, 2, \quad (\text{B.15})$$

where, as usual, the Einstein's convention of summation over repeated indices is understood; $|\psi|$ is the length of field ψ . Since we shall only consider stationary systems, the Lagrangian (up to a constant) is also minus the energy density. The potential $V(\psi)$ contains no derivative terms and is constructed so that it is invariant under changes in the phase of ψ : $\psi \rightarrow \psi' = e^{-i\chi} \psi$ with χ a real number or a real function. This can be achieved by a potential of the form $V(\psi) = V(|\psi|)$. With such a potential, the Lagrangian density (B.15) is invariant under the *global* phase transformation

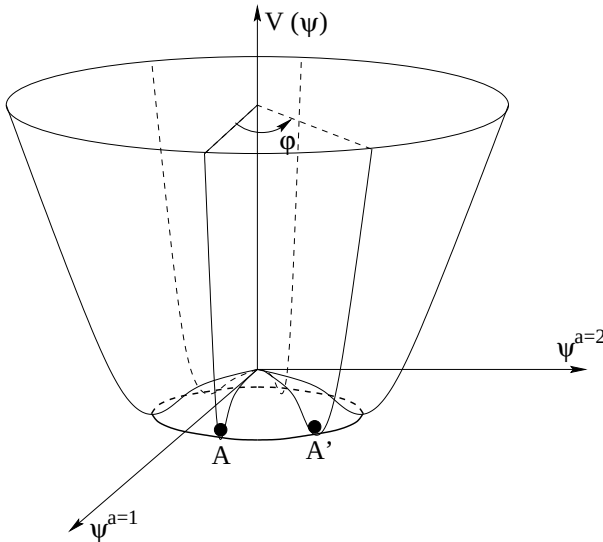
$$\psi \rightarrow \psi' = e^{-i\chi_0} \psi, \quad (\text{B.16})$$

with χ_0 a constant phase. Let us denote by G the set of all these global transformations, which form a group called $U(1)$. Now we assume that a *constant nonzero* ψ_0 minimizes the potential V . Such a state is a lowest energy state, since $|\partial_\mu \psi|^2$ vanishes for it. Due to the invariance of the Lagrangian density under the group G of global transformations (B.16), all states constructed from ψ_0 by a constant phase transformation $g \in G$ are also minima of $V(\psi)$ and have the same energy as ψ_0 . Let us choose one of these lowest energy states: ψ_0^A . Although all $g\psi_0^A$ have the same energy as ψ_0^A , all of them may be not equal to ψ_0^A . Let $H \subset G$ be the subgroup of transformations h_j , under which ψ_0^A is invariant. Therefore, $g_i h_j \psi_0^A = g_i \psi_0^A \forall h_j \in H$ and all $\{g_i h_1, g_i h_2, \dots\}$ produces the same effect on ψ_0^A as g_i . If we now identify all these $\{g_i h_j\}$ with g_i , one obtains the quotient or coset space G/H . Hence, the elements of G/H represent the different possible lowest energy states. The transformations contained in G/H connect one lowest energy state ψ_0^A with another different (lowest energy) state $\psi_0^{A'}$.

The case, where the lowest energy states remain invariant under a smaller set of transformations than the Lagrangian density (the lowest energy states have a lower symmetry than that of the Lagrangian), is called the symmetry breakdown. To make this point more concrete, let us consider an example of potential which shows this property: the well-known “mexican hat” potential as in the Landau-Ginzburg model of second order phase transitions

$$V(\psi) = -|\alpha||\psi|^2 + \frac{|\beta|}{2}|\psi|^4, \quad (\text{B.17})$$

represented in the following figure, with $\psi^{a=1} = \text{Re } \psi$ and $\psi^{a=2} = \text{Im } \psi$. Note that the two components ψ^a are referred to the internal symmetry space and have nothing to do with the two dimensions of the ordinary space. It is in the internal space where the phase transformations $\psi \rightarrow \psi e^{-i\chi}$ takes place. Angles or phases in the internal symmetry space will be denoted by φ , while angles in physical space will be by ϕ .



The potential has a minimum for all ψ_0 located at the bottom of the “hat”

$$\psi_0 = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\varphi} \quad 0 \leq \varphi < 2\pi. \quad (\text{B.18})$$

In this example, the different ψ_0 's are easy to classify by the angle φ for each point at the bottom of the “hat”. Starting from a reference state A , all different lowest energy states are obtained from by a phase transformation $\psi \rightarrow \psi' = e^{-i\chi_0} \psi$ with $\chi_0 = -\varphi$. Topologically, this set of different lowest energy states has the structure of a circumference: $G/H = S^1$.

The same result can be obtained in a more formal way as follows. If transformations $\psi \rightarrow \psi' = \psi e^{-i\chi_0}$ are restricted to $0 \leq \chi_0 < 2\pi$ and thus $G = S^1$, then the identity

(represented by $\chi_0 = 0$) is the only transformation belonging to H . Therefore, $G/H = G = S^1$ as expected ⁶.

Before continuing with the discussion, let us connect these ideas with the dislocation model introduced above. A good approximation for the energy density ε of a deformed crystal is $\varepsilon = \frac{K}{2} (\vec{\nabla} u)^2$, where K is an elastic constant and u the displacement vector of the atomic cell ⁷. The same energy is obtained from the model described by the arrow ψ (B.12) and the Lagrangian density (B.15) with a potential equal to that in (B.17)

$$-\ell = \frac{|\psi|^2}{2} (\vec{\nabla} \varphi)^2 = \frac{|\psi|^2}{2} \frac{4\pi^2}{a^2} (\vec{\nabla} u)^2, \quad (\text{B.19})$$

where the parameters $|\alpha|$ and $|\beta|$ of the potential $V(|\psi|)$ are chosen to give

$$|\psi|^2 = \frac{|\alpha|}{|\beta|} = \frac{K a^2}{4\pi^2}. \quad (\text{B.20})$$

As already mentioned in (B.14), the energy invariance under the global $U(1)$ symmetry states the translational invariance of the elastic energy. A ground state is where each arrow points in the same direction and the set of different ground states is described by the different directions of the arrow.

The above treatment of G/H would be unnecessary if we were interested only in the ground state, which is represented by one point of the space G/H : the angle φ along which all ψ fields are directed in the internal symmetry space. This angle φ is the *same* for all points $\vec{r}$ of the material. Let us now consider a richer variety of configurations, which for small $|\vec{r}|$ can differ from the ground state but which approach one of the lowest energy states ψ_0 for $|\vec{r}| \rightarrow \infty$. One can nevertheless approach spatial infinity through several directions ϕ and ϕ' , and in each of these directions, ψ can take different values, for example ψ_0 and ψ'_0 . In the above dislocation model, the arrow can point in different internal directions at different points of the external physical path C . In order to classify these configurations, we have to consider all maps f from the spatial infinity into the set of different lowest energy states G/H

$$f : \text{spatial infinity} \longrightarrow (G/H).$$

The application of this method will be the classification of line defects in a 3-dimensional material. Let us suppose we have a system described by a Lagrangian containing certain ψ fields, and we want to know if there exist topologically stable line defects. Let be C a closed path distant enough from the line defect and with the line defect crossing the area enclosed by C . This path C represents the spatial infinity in the previous 2-dimensional case. We now study the different classes of ψ configurations along C : since $C = S^1$, the system will have stable line defects if $\pi_1(G/H)$ contains more

⁶If χ_0 is not restricted, G is the whole real line $T(1)$: $\chi_0 = \varphi + 2\pi n$ with $0 \leq \varphi < 2\pi$ and n and integer. Only transformations with $\chi_0 = 2\pi n$ let invariant ψ_0^A and the result for G/H is again the same: $G/H = \{\varphi : 0 \leq \varphi < 2\pi\} = S^1$.

⁷In fact, this energy density is only valid if in the crystal there is only one kind of screw dislocations. For the general case, K is a tensor and we have to consider displacements in the three spatial directions. This generalization is straightforward and does not yield new relevant properties.

classes than the identity $\tilde{e}$ class. In the above case described by a complex scalar ψ field and the Lagrangian (B.15) (or equivalently, the dislocation model of Fig. B.3), we have obtained $G/H = S^1$. According to equation (B.6), the system contains topologically stable line defects

$$\pi_1(S^1) = \mathbf{Z} = 0, \pm 1, \pm 2, \dots, \quad (\text{B.21})$$

describing respectively the absence of a dislocation, the dislocation with Burgers vector equal to an interatomic distance in the two opposite orientations, the dislocation with two lattice distances as Burgers vector, ... Relation (B.21) can be also expressed by a map f between the angle ϕ describing C in the physical (real) space, and the angle φ characterizing each lowest energy state:

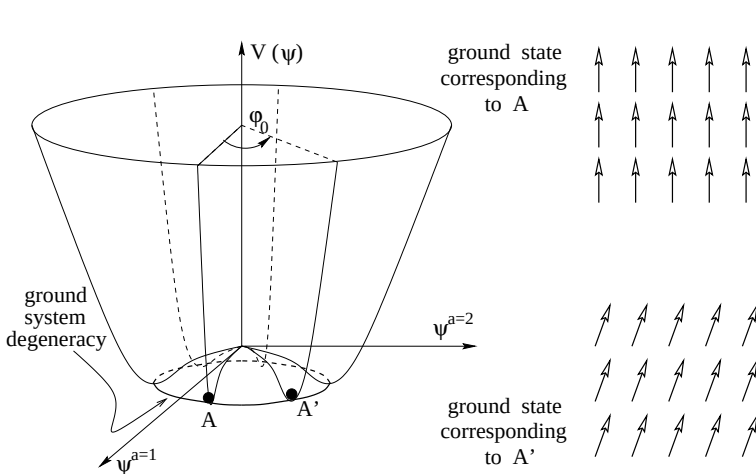
- The trivial class is represented by $\varphi = f(\phi) = \text{constant}$.
- The configuration with winding number equal to $\pm n$, by $\varphi = f(\phi) = \pm n \phi$.

Note. About the spontaneous symmetry breakdown (I).

The system under study appears under many different configurations and the information which we can state about it is of a statistical nature: we have to calculate expectation values of the fields involved in the model description. As seen in Chapter 2, an energy scale η to which the energy of every field configuration is compared is introduced, and then a summation is made over all possible configurations to obtain the partition function and the expectation values. Although this calculation will be dealt with in detail in the next Appendix, let us introduce some of the lines of the discussion now. For the moment, we are interested in the ground state of the system: this is obtained by setting the scale η to very small values, such that only those configurations with minimal energy create the leading contributions to the partition function.

Let us consider in detail what “symmetry breaking” means by looking at Lagrangian density (B.15), invariant under global transformations. The lowest energy configurations, when the system energy contains a potential like the “mexican hat”, are represented in the next figure. Different ground states appear depending on which point of the bottom of the potential is chosen. Perturbations around these lowest energy states are described by the following energy

$$E_{\text{global inv.}} = \frac{|\alpha|}{2|\beta|} \int d^d \vec{r} \left(\vec{\nabla} \varphi \right)^2, \quad (\text{B.22})$$



with d the dimensionality of the ordinary space where field ψ is located. These perturbations arise when the orientation in the internal symmetry space of two neighbouring ψ fields is not the same and therefore $\nabla \varphi \neq 0$. The two neighbouring fields sit at $\vec{r}$ and at $\vec{r} + d\vec{r}$. We can obtain a more manageable representation if we formulate the model in a infinite discrete lattice of spacing a , with fields $\varphi_{\vec{r}}$ located at the vertices of each lattice cell,

$$E_{\text{global inv.}} \approx \frac{a^d |\alpha|}{2|\beta|} \sum_{\vec{r}} \sum_{\mu=1,2,\dots,d} \left(\frac{\varphi_{\vec{r}+a\hat{\mu}} - \varphi_{\vec{r}}}{a} \right)^2 = \frac{a^{d-2} |\alpha|}{2|\beta|} \sum_{\vec{r}, \mu} (\varphi_{\vec{r}+a\hat{\mu}} - \varphi_{\vec{r}})^2, \quad (\text{B.23})$$

where $\mu = 1, 2$ and $\hat{\mu}$ is the unit vector along the μ -direction. After all calculations have been carried out, we take the limit $a \rightarrow 0$. More exactly, we can take as lattice energy any form which is invariant under global transformations and that at $a \rightarrow 0$ tends to the quadratic term $(\vec{\nabla}\varphi)^2$, for instance

$$E_{\text{global inv.}} = \frac{a^{d-2}|\alpha|}{|\beta|} \sum_{\vec{r}, \mu} [1 - \cos(\varphi_{\vec{r}+a\hat{\mu}} - \varphi_{\vec{r}})] , \quad (\text{B.24})$$

which at the limit $a \rightarrow 0$ yields the correct continuum form

$$(\varphi_{\vec{r}+a\hat{\mu}} - \varphi_{\vec{r}}) \rightarrow a\partial_{\mu}\varphi \quad \text{and} \quad 1 - \cos(\varphi_{\vec{r}+a\hat{\mu}} - \varphi_{\vec{r}}) \rightarrow a^2 \frac{1}{2} (\partial_{\mu}\varphi)^2 .$$

Such a generalization allows us to consider not only trivial topological configurations around a ground state but also configurations ψ whose $|\psi|$ tends at $r \rightarrow \infty$ to the values at the bottom of the potential but which at $r \approx 0$ vanishes, i.e., configurations with nonvanishing winding number. There is no difference in the partition behaviour at low η between using energy (B.23) or energy (B.24) as long as we don't forget the angular character of the phase φ in (B.23) (see the comment previous to equation (C.132) in the next Appendix). For the dislocation model seen above, the discrete lattice is drawn through the cell centers, where field ψ is located.

With the energy $E_{\text{global inv.}}$, we construct the partition function

$$\mathcal{Z} = \int \mathcal{D}\varphi e^{-\frac{E_{\text{global inv.}}}{\eta}} \rightarrow \prod_{\vec{r}} \left(\int d\varphi_{\vec{r}} \right) \exp \left[-\frac{E_{\text{global inv.}}[\varphi]}{\eta} \right] . \quad (\text{B.25})$$

In Condensed Matter Physics, the case we are studying is represented by a magnet at low temperature (or more exactly, at large $J/k_B T$, with J the coupling strength): the energy describing the magnet is invariant when subjected to a spin rotation of the same amount at every site. In this case the phase φ in the internal symmetry space corresponds to the orientation of the spins in the real space. There are as many ground states as there are different spin orientations when all spins are oriented in the same direction. If we now take the average over all these ground states (which is really done when the partition function is calculated) the average magnetization is exactly zero. This holds, of course, if we have not imposed any boundary conditions on the material and there is no external magnetic field. Now let us apply an external magnetic field h to this material: h tends to align all spins in the direction of the magnetic field. In this state, we calculate the average magnetization and then take the limit $h \rightarrow 0$ to study if the system remains magnetized. If this is the case, one says that the system has acquired a spontaneous magnetization and that the symmetry has been spontaneously broken down.

Similarly, into the energy (B.23), we introduce the term $-h \sum_{\vec{r}} \cos \varphi_{\vec{r}}$, where $h > 0$ is a field controlled from outside the system. The finality of this external field is to select one of the lowest energy state from the many which are into the bottom of the potential. This new term is not invariant under the global transformation (B.16) and has its minimum for $\varphi = 0$, or equivalently, the main contribution to the partition comes from the lowest energy state A ⁸. The expectation value

$$\langle \psi \rangle_h = \sqrt{\frac{|\alpha|}{|\beta|}} \langle e^{i\varphi} \rangle_h = \sqrt{\frac{|\alpha|}{|\beta|}} \frac{1}{\mathcal{Z}} \sum_{\text{all configurations}} e^{i\varphi_{\vec{r}}} \exp \left[-\frac{1}{\eta} \left(E_{\text{global inv.}}[\varphi] - h \sum_{\vec{r}} \cos \varphi_{\vec{r}} \right) \right] , \quad (\text{B.26})$$

is the analogue to the macroscopic magnetization (density). The “all configurations” of the previous equations are the lowest energy configuration plus all not large perturbations around them (remember we are studying the system at low η). We now ask whether $\langle e^{i\varphi} \rangle$ vanishes or not at $h \rightarrow 0$. The answer can also be obtained if the phase correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ between two distant points $\vec{r}_1$ and $\vec{r}_2$ vanishes or not in the absence of the external field h . If the correlation function does not decay at large distances, a nonzero magnetization exists and $\langle e^{i\varphi} \rangle \neq 0$. This calculation will be not carried out here, since it will be studied in the next Appendix. We only state the final result here, which can be read off from Tables C.2 and C.3 for $a|r_2 - r_1| \rightarrow \infty$

⁸We could select another ground state, for instance A' , by means of the term $-h \sum_{\vec{r}} \cos(\varphi_{\vec{r}} - \varphi_0)$.

- For $d = 2$, $\langle e^{i\varphi} \rangle$, and therefore $\langle \psi \rangle$, vanishes.
- For $d \geq 3$, $\langle e^{i\varphi} \rangle$ and $\langle \psi \rangle$ are nonzero at $\eta \rightarrow 0$.

The origin of the $\langle \psi \rangle$ vanishing for the two dimensional case is the Green's function of the Laplace operator in two dimensions, which grows with the distance and thus can disorder the system over wide regions. The nonvanishing of $\langle \psi \rangle$, as analogue of the magnetization in a magnet, is the characterization of spontaneous symmetry breaking. Note that in the case $d \geq 3$ and at low η , where $\langle \psi \rangle \neq 0$, the expectation value fulfils the same equation as the microscopic variable ψ

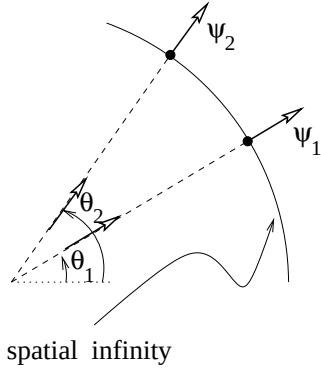
$$E_{\text{effective}} = \frac{1}{2} \langle \psi \rangle^2 \int d^d \vec{r} \left(\vec{\nabla} \bar{\varphi} \right)^2, \quad (\text{B.27})$$

with $\bar{\varphi}$ the phase for field $\bar{\psi} = \langle \psi \rangle e^{i\bar{\varphi}}$. Here “microscopic” variable has to be understood as the variable describing a configuration, configuration which together with many others produces the expectation value. The expectation value $\langle \psi \rangle$, if nonvanishing, is also called order parameter.

We already know how to find out whether a given configuration is or not topologically stable. But what about its energy? Let us return to the model (B.15) and consider the configuration where field ψ points in the radial spatial direction ($\varphi = \phi$) at the points on the distant closed path C

$$\lim_{|\vec{r}| \rightarrow \infty, \text{ along } \phi} |\psi| e^{i\phi} = |\psi| e^{i\phi} \quad \text{with} \quad \lim_{|\vec{r}| \rightarrow \infty} |\psi| = \sqrt{\frac{|\alpha|}{|\beta|}}.$$

This configuration is topologically stable. The question now is whether such a configuration has a finite energy. Starting at two different points of the spatial infinity and approaching $r \rightarrow 0$, continuity requires that ψ has a zero at $r = 0$.



Otherwise nothing prevents that at $r = 0$, ψ has different values depending on the direction we follow. Note that the topological stability causes that ψ already takes, at a close distance from $r = 0$ along a certain direction ϕ_i , a value very near to its corresponding value $|\psi| e^{i\phi_i}$ at the spatial infinity along this direction. However, this zero in ψ costs energy because of the gradient $\partial_\mu \psi$ in the Lagrangian

$$(\nabla \psi)_\phi = \frac{1}{r} \partial_\phi (|\psi| e^{i\phi}) = i |\psi| e^{i\phi} \frac{1}{r},$$

$$\int d^d \vec{r} (\nabla \psi)^2 \sim \int |\psi|^2 \frac{1}{r^2} d^d \vec{r}, \quad (\text{B.28})$$

which diverges at $r \rightarrow \infty$ for $d \geq 2$. We have already met this divergence in Chapter 1 as we calculated the energy per length of a straight dislocation (1.7). In order to avoid that this gradient contributes to the energy with an nonzero amount in the region $r \rightarrow \infty$, we couple ψ to a gauge field A_μ and look for the A_μ arrangement such that the covariant derivative $D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi$ vanishes rapidly enough for $r \rightarrow \infty$

$$(\partial_\mu - ie A_\mu) \psi \Big|_{r \rightarrow \infty} = 0. \quad (\text{B.29})$$

The parameter e is the coupling constant. The condition (B.29) leads to the quantization of line integral of A_μ around a distant closed path C . In a superconductor⁹, this means the quantization of the magnetic flux Φ_0 (A.24)

$$\Phi_0 = \oint_{r \rightarrow \infty} A_\mu dx^\mu = \frac{1}{e} \oint_{r \rightarrow \infty} \partial_\mu \phi dx^\mu = \frac{2\pi}{e}. \quad (\text{B.30})$$

For a configuration of winding number n , $\psi|_{r \rightarrow \infty} = \sqrt{\frac{|\alpha|}{|\beta|}} e^{in\phi}$, the magnetic flux is equal to $n2\pi/e$. Up to this moment, the gauge field A_μ is only an external field. In order to allow A_μ to have its own dynamics, we must include into the Lagrangian density the term $F_{\mu\nu}^2$, obtaining thus

$$\ell = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu \psi^* + ie A_\mu \psi^*) (\partial_\mu \psi - ie A_\mu \psi) + |\alpha| |\psi|^2 - \frac{|\beta|}{2} |\psi|^4. \quad (\text{B.31})$$

Under a *local* phase transformation $\psi \rightarrow \psi' = e^{-i\chi} \psi$ with $\chi = \chi(x)$ a scalar, the gauge field transforms as $A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \chi$ such that the covariant derivative transforms in the same form as scalar field ψ

$$(D_\mu \psi)' \equiv \partial_\mu \psi' - ie A'_\mu \psi' = e^{-i\chi} (D_\mu \psi).$$

The Lagrangian is thus invariant under the following local transformation

$$\begin{cases} \psi & \rightarrow \psi' = e^{-i\chi(\vec{r})} \psi, \\ A_\mu & \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \chi, \end{cases} \quad (\text{B.32})$$

called an $U(1)$ gauge or local transformation.

The coupling to the gauge field produces no increase in the energy at spatial infinity as a result of the vanishing covariant derivative. The zero contribution at $r \rightarrow \infty$ arising from $F_{\mu\nu}^2$ is explained by the following equation

$$D_\mu D_\nu = \left(\partial_\mu \partial_\nu - ie A_\nu \partial_\mu - ie A_\mu \partial_\nu - e^2 A_\mu A_\nu \right) - ie \partial_\mu A_\nu, \\ [D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = -ie F_{\mu\nu}. \quad (\text{B.33})$$

At the spatial infinity, $\lim_{r \rightarrow \infty} \psi \neq 0$ but $\lim_{r \rightarrow \infty} D_\mu \psi = 0$, and $F_{\mu\nu}$ vanishes because of $\left([D_\mu, D_\nu] \psi \right)|_{r \rightarrow \infty} = -ie F_{\mu\nu} \psi|_{r \rightarrow \infty}$. Hence, the states with constant $|\psi|$ (although varying phase from point to point) are at $r \rightarrow \infty$ lowest energy states.

The question is now whether these new allowed local transformations can connect two configurations belonging to different homotopy classes. If this were the case, the energetical stability would be ensured by the coupling to a gauge field but at the expense of breaking the topological stability and thus a perturbation could change the winding number and deform a nontrivial topological configuration into a trivial one. However,

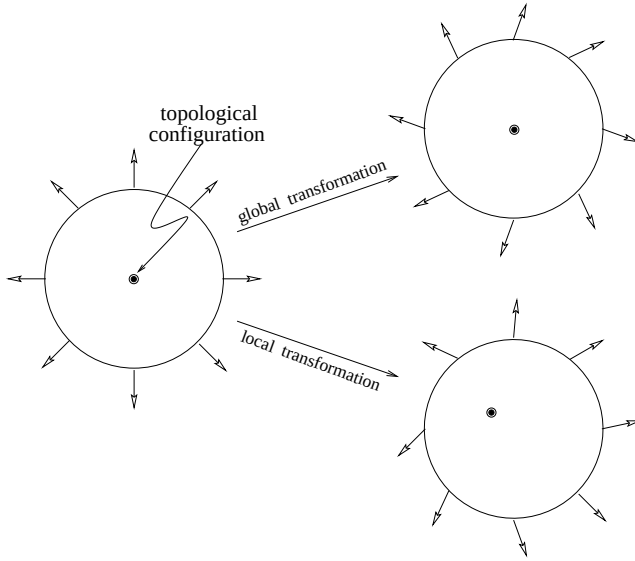
⁹In order to compare this result with that of a superconductor, we have to substitute the coupling constant e by $q e / \hbar c$, where q is the charge of the ψ field measured in units of the electron charge e .

this is not the case: let us suppose we start with a configuration which on C are described by $\psi = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i2\phi}$ (with $\phi = \arctan y/x$) and we are looking for a gauge transformation which produces $\psi' = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\phi}$. The transformation is just $\chi = \phi$, but when substituted into A_μ produces a singularity, since $\vec{\nabla}\phi$ is not well-defined at $\rho = \sqrt{x^2 + y^2} = 0$: $\vec{\nabla}\phi = \frac{1}{\rho}\hat{\phi}$.

That transformation $\chi = \phi$ is not an allowed gauge transformation for the discussed model can be also seen as follows: gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \delta A_\mu$ leaves invariant tensor $F_{\mu\nu}$, i.e., $\delta F_{\mu\nu} = \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)$ is exactly zero $\forall \mu, \nu$. However for $\delta A_\mu = -\frac{1}{e}\partial_\mu\phi$, tensor δF_{12} is not zero but contains a singularity line at $x, y = 0$. Using relations $\partial_1\phi = -\partial_2 \ln \rho$ and $\partial_2\phi = \partial_1 \ln \rho$, one obtains

$$\delta F_{12} = \frac{1}{e} (-\partial_1^2 - \partial_2^2) \ln \rho = \frac{2\pi}{e} \delta(x)\delta(y), \quad (\text{B.34})$$

since $1/2\pi \ln \rho$ is the Green's function of 2-dimensional Laplace operator $(\partial_x^2 + \partial_y^2)$. This singularity line carries the flux $2\pi/e$ removed by the singular gauge transformation. All this discussion leads us to the result that local transformations can only relate ψ configurations which are inside the *same* homotopy class.



In the case of global transformations already discussed, the homotopy class of winding number n is formed by the configuration which at spatial infinity is given by $\psi = |\psi|_0 e^{in\phi}$ (with ϕ the polar angle in real space), plus all other configurations obtained from this by changing phase φ by the same amount at all points. The generalization to local transformations increases the number of configurations contained in each homotopy class but the number of classes remains the same. In this case, each homotopy

class contains all local deformations of $\psi = |\psi|_0 e^{in\phi}$ without changing the winding number. The classification according to the winding number made for global transformations holds also for local gauge transformations, as long as we are considering the above model. In the next sections, nevertheless, we shall see that if the above $U(1)$ gauge symmetry (B.32) is not the original but only the residual symmetry arising from the breakdown of a compact larger symmetry, then there can exist *allowed non-singular* gauge transformations which do connect configurations of different winding number. In that case, the winding number does not characterize the homotopy classes, since one homotopy class

contains configurations with several different magnetic fluxes. Only configurations which belong to different homotopy classes cannot be connected by a gauge transformation.

In the dislocation model, local transformation (B.32) represents a displacement field different at each atomic cell: a compatible deformation. If we postulate a Lagrangian which includes covariant derivatives as dislocation energy, it becomes invariant under compatible deformations. Let us consider a configuration of winding number (or Burgers vector) 1 as in the above figure: it represents a dislocation located at $r = 0$. A global transformation only represents a translation of the lattice as a whole and therefore the lattice remains at the lattice center. For a local transformation ¹⁰ with an invariant energy under this kind of transformations, the energy increase due to the inhomogeneous gradients in ψ field are compensated by the re-locating of the dislocation: in the figure above, by bringing the dislocation nearer to the upper left corner. Hence, a partition function constructed from an invariant energy describes a model which is invariant under dislocation relocations. This is useful if we are trying to describe the different hardening states in a crystal, which are due to the changing dislocation density. Note that according to the previous discussion, a compatible deformation is not able to change the winding number of a configuration and therefore cannot change the dislocation density. The energy due exclusively to compatible deformations does contribute to the crystal's energy but does not take account of the changes in the dislocation density which cause the hardening. With a gauge invariant energy, we are focusing our attention only on the origin of the hardening. Moreover, the vortex energy in a noninvariant model under gauge transformations contains divergences which are canceled by the coupling to a gauge field.

Now let us return to the 2-dimensional model described by the Lagrangian density (B.31), but instead of working with a complex scalar field, we shall formulate the model in the two real components $\psi^{a=1,2}$: $\psi^{a=1} \equiv \text{Re } \psi$, $\psi^{a=2} \equiv \text{Im } \psi$ with $|\psi|^2 = \psi^a \psi^a$. These components define a two-dimensional internal symmetry space, where field ψ is represented by an arrow which points in the direction given by the scalar's phase φ ¹¹. According to this notation, the Lagrangian density becomes

$$\ell = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\partial_\mu\psi^a + eA_\mu\epsilon_{ab}\psi^b\right)\left(\partial_\mu\psi^a + eA_\mu\epsilon_{ac}\psi^c\right) + |\alpha||\psi|^2 - \frac{|\beta|}{2}|\psi|^4, \quad (\text{B.35})$$

with $\epsilon_{12} = -\epsilon_{21} = 1$ and zero otherwise. In this notation, the $U(1)$ transformation $\psi \rightarrow \psi' = e^{-i\chi}\psi$ is represented by means of the following $SO(2)$ matrix ¹²

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \rightarrow \begin{pmatrix} (\psi')^1 \\ (\psi')^2 \end{pmatrix} = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}. \quad (\text{B.36})$$

¹⁰A transformation, which could change the Burgers vector of the line defect, would require the process cutting-displacement-gluing already described and does not correspond to a compatible deformation. This is the equivalent in the dislocation model to the statement that a local transformation cannot relate configurations from different homotopy classes.

¹¹The utility of this notation is to stress for topological configurations as in the next equation (B.37) the connection between the phase φ in the internal symmetry space and the polar angle ϕ of ordinary space.

¹² $SO(2)$ is the group of rotations in two dimensions and is formed by 2×2 orthogonal real matrices with determinant 1.

A gauge transformation is represented by a rotation in the internal symmetry space.

With the above Lagrangian density, we study the topological configuration of winding number 1

$$\psi|_{r \rightarrow \infty} = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\phi} \quad \text{or} \quad \psi^a|_{r \rightarrow \infty} = \frac{r^a}{r} \sqrt{\frac{|\alpha|}{|\beta|}}, \quad (\text{B.37})$$

with ϕ being the polar angle in the ordinary space, $\phi = \arctan(y/x)$, and $r^a = (x, y)$. Taking into account the form of ψ^a at $r \rightarrow \infty$, the following ansatz is proposed

$$\psi^a = \frac{r^a}{r} Q(r) \quad \Rightarrow \quad \lim_{r \rightarrow \infty} Q = \sqrt{\frac{|\alpha|}{|\beta|}}, \quad (\text{B.38})$$

which describes ψ^a at any $\vec{r}$. For $r \rightarrow 0$, the boundary condition is that the scalar field has a zero, as we have already discussed: $\lim_{r \rightarrow 0} Q = 0$. For gauge field A_μ , the following form

$$A_\mu = \epsilon_{\mu b} r^b f(r), \quad (\text{B.39})$$

will be shown that it is consistent with the condition that the energy of this topological configuration, named a vortex for the following, remains finite at $r \rightarrow \infty$. Since the energy density is minus the Lagrangian density, the vortex energy is given by

$$E_{\text{vortex}} = \int d^2 \vec{r} \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \psi^a + e A_\mu \epsilon_{ab} \psi^b)^2 - |\alpha| |\psi|^2 + \frac{|\beta|}{2} |\psi|^4 - \varepsilon_{\text{ground}} \right), \quad (\text{B.40})$$

where we have subtracted energy $\varepsilon_{\text{ground}} = -\frac{|\alpha|^2}{2|\beta|}$, due to the constant configuration

$\psi = \sqrt{\frac{|\alpha|}{|\beta|}}$, in order to obtain the net energy increase produced by the topological object.

Substituting into E_{vortex} the partial results $F_{\mu\nu} = -2\epsilon_{\mu\nu} f + \frac{r^b}{r} \frac{df}{dr} (\epsilon_{\nu b} r_\mu - \epsilon_{\mu b} r_\nu)$ with

$$F_{\mu\nu}^2 = \frac{4}{r} \frac{d}{dr} (r^2 f^2) + 2r^2 \left(\frac{df}{dr} \right)^2, \text{ and } D_\mu \psi^a = Q \left(\frac{\delta_{a\mu}}{r} - \frac{r_a r_\mu}{r^3} + e \epsilon_{\mu c} \epsilon_{ab} \frac{r_c r_b}{r} f \right) + \frac{r_a r_\mu}{r^2} \frac{dQ}{dr}$$

with $(D_\mu \psi^a)^2 = Q^2 \left(\frac{1}{r} + e r f \right)^2 + \left(\frac{dQ}{dr} \right)^2$, we obtain

$$\begin{aligned} E_{\text{vortex}} = \int_0^\infty 2\pi r dr & \left[\frac{r^2}{2} \left(\frac{df}{dr} \right)^2 + \frac{1}{2} \left(\frac{dQ}{dr} \right)^2 + \frac{Q^2}{2} \left(\frac{1}{r} + e r f \right)^2 + \frac{|\beta|}{2} \left(Q^2 - \frac{|\alpha|}{|\beta|} \right)^2 \right] + \\ & + 2\pi \left[r^2 f^2 \right]_0^\infty. \end{aligned} \quad (\text{B.41})$$

The boundary conditions for $f(r)$ and $Q(r)$ at $r \rightarrow \infty$ can be directly read off from the underlined terms

$$f(r)|_{r \rightarrow \infty} = -\frac{1}{er^2} \quad \text{and} \quad Q(r)|_{r \rightarrow \infty} = \sqrt{\frac{|\alpha|}{|\beta|}}, \quad (\text{B.42})$$

where the boundary condition for $Q(r)$ was already expected from (B.38). With the behaviour of $f(r)$ at large r , and provided $f(r)$ remains finite at $r = 0$, the term outside the integral in (B.41) vanishes. Note that the term $\frac{1}{r} + erf$, which imposes the boundary condition on $f(r)$, comes from the covariant derivative. As expected, the flux Φ_0 is given by ¹³

$$\Phi_0 = \oint_{\text{circumference at } r \rightarrow \infty} A_\mu dl^\mu = \int_{\phi=0}^{\phi=2\pi} d\phi \frac{r^2}{er^2} = \frac{2\pi}{e}. \quad (\text{B.43})$$

Fields $\psi^a = \frac{r^a}{r}Q(r)$ and $A_\mu = \epsilon_{\mu b}r^b f(r)$ do not form the unique ansatz for a configuration of winding number 1. In fact, any allowed gauge transformation ¹⁴ applied to it yields also a winding number 1 configuration whose energy is the same as the untransformed configuration, since the Lagrangian is gauge invariant. Therefore, we can choose the most manageable ansatz, which fulfils the condition of vanishing covariant derivative at spatial infinity, to calculate the energy of such a configuration.

Only topological arguments are not sufficient to obtain a final result for the vortex energy; the Euler-Lagrange equations for the fields must be solved and their solutions substituted into the energy (B.41). According to (B.35), these equations are given by

$$\begin{cases} \partial_\mu F_{\mu\nu} &= -e\epsilon_{ca}\psi^c D_\mu \psi^a, \\ D_\mu D_\mu \psi^a &= \frac{\psi^a}{|\psi|} \frac{dV(|\psi|)}{d|\psi|}, \end{cases} \quad (\text{B.44})$$

which expressed in function of $f(r)$ and $Q(r)$ take the form ¹⁵

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (r^2 f) \right) - eQ^2 \left(\frac{1}{r} + erf \right) = 0, \quad (\text{B.45})$$

$$\frac{d^2 Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} - Q \left(\frac{1}{r} + erf \right)^2 + 2|\alpha|Q - 2|\beta|Q^3 = 0. \quad (\text{B.46})$$

Note that since the function between brackets inside the integral (B.41) is minus the Lagrangian density, the above Euler-Lagrange equations can be also obtained from this

¹³With vector dl^μ tangent to the circumference defined by $dl^\mu = -\epsilon_{\mu\nu}r^\nu$.

¹⁴“Allowed” means gauge transformations from those not presenting singularities (i.e., whose derivatives are finite overall).

¹⁵For the first Euler-Lagrange equation:

$$\partial_\mu F_{\mu\nu} = \epsilon_{\nu a} \frac{r^a}{r} \left[\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (r^2 f) \right) \right]; \quad -e\epsilon_{ca}\psi^c D_\mu \psi^a = \epsilon_{\nu a} \frac{r^a}{r} eQ^2 \left(\frac{1}{r} + erf \right),$$

and for the second one:

$$\partial_\mu D_\mu \psi^a = \frac{r^a}{r} \left(\frac{d^2 Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} - \frac{Q}{r} \left(\frac{1}{r} + erf \right) \right); \quad eA_\mu \epsilon_{ab} D_\mu \psi^b = -\frac{r^a}{r} erf Q \left(\frac{1}{r} + erf \right).$$

bracketed expression by using $\frac{1}{r} \frac{d}{dr} \left(r \frac{\partial[]}{\partial \left(\frac{df, Q}{dr} \right)} \right) = \frac{\partial[]}{\partial f, Q}$ ¹⁶. These equations can be solved only by means of numerical methods, but one can nevertheless obtain the analytical behaviour of A_μ and ψ^a at large r by extending the solutions (B.42)

$$\begin{aligned} rf(r) &= (rf)_0 + a(r) \quad \text{with} \quad (rf)_0 = -\frac{1}{er} \quad \text{and} \quad \lim_{r \rightarrow \infty} a(r) = 0, \\ Q(r) &= Q_0 + q(r) \quad \text{with} \quad Q_0 = \sqrt{\frac{|\alpha|}{|\beta|}} \quad \text{and} \quad \lim_{r \rightarrow \infty} q(r) = 0. \end{aligned} \tag{B.47}$$

After neglecting terms, like $r^{-2}a(r)$ or $q(r)a(r)$, which at large r are much smaller than $a(r)$ and $q(r)$, the solutions are

$$\begin{aligned} a(r) &\sim e^{-\frac{r}{\lambda}} \quad \text{with} \quad \lambda^2 = \frac{|\beta|}{e^2|\alpha|} \\ q(r) &\sim e^{-\frac{r}{\xi}} \quad \text{with} \quad \xi^2 = \frac{1}{4|\alpha|} \end{aligned} \tag{B.48}$$

Thus, the variables characterizing any field configuration are the vortices located at points r_m^a and described by

$$\left\{ A_\mu \Big|_{r \rightarrow \infty} = -\epsilon_{\mu b} \frac{(r - r_m)^b}{e(r^c r^c - r_m^c r_m^c)}, \psi^a \Big|_{r \rightarrow \infty} = \frac{(r - r_m)^a}{\sqrt{r^c r^c - r_m^c r_m^c}} \sqrt{\frac{|\alpha|}{|\beta|}} \right\},$$

plus the fields with no topological content

$$\left\{ A_\mu = \epsilon_{\mu b} \frac{r^b}{r} a(r), \psi^a = \frac{r^a}{r} q(r) \right\}, \quad \lim_{r \rightarrow 0} a(r) = \text{finite} \quad \text{and} \quad \lim_{r \rightarrow 0} q(r) = 0.$$

Having arrived at this point, the reader may be thinking that we are repeating the same discussion as in Appendix A for the Landau-Ginzburg theory of superconductivity (the Abrikosov model). This is (partially) right: the model presented in Appendix A is an *effective* theory for the order parameter ψ which in the current discussion is denoted by $\bar{\psi} = \langle \psi \rangle e^{i\bar{\varphi}}$, with $\langle \psi \rangle$ the expectation value of the microscopic variable ψ^a ¹⁷. “Microscopic” variables are to be understood here as the variables defining every field configuration; we carry out a sum over these to obtain the partition function and the correlations. The model description with the expectation value (or order parameter) and the description with the microscopic variables coincide only in the case where $\langle \psi \rangle$

¹⁶Since f and Q are functions of r only, $\partial_{\mu=1,2}(f, Q) = \frac{r_\mu}{r} \frac{d(f, Q)}{dr}$, one obtains $\partial_\mu \left(\frac{\partial[]}{\partial (\partial_\mu f, Q)} \right) = \partial_\mu \left(\frac{r_\mu}{r} \frac{\partial[]}{\partial \left(\frac{df, Q}{dr} \right)} \right) = \frac{1}{r} \frac{\partial[]}{\partial \left(\frac{df, Q}{dr} \right)} + \frac{d}{dr} \left(\frac{\partial[]}{\partial \left(\frac{df, Q}{dr} \right)} \right)$

¹⁷The reader should excuse this confusing notation, which is unfortunately very common. This confusion is further complicated by the fact that $\langle \psi \rangle$ is the expectation value of microscopic variable ψ only *after* having imposed on the system a constraint strong enough to remove the gauge invariance.

does not vanish [11]. Hence, for the superconducting phase in the Abrikosov model, the current discussion is a repetition of Appendix A: equation (B.48) is similar to equations (A.18) and (A.17) in the Abrikosov model¹⁸. In the normal conducting phase, the microscopic variables which define a configuration and over which we are summing to obtain the expectation value, are still $\{A_\mu = \epsilon_{\mu b} \frac{r^b}{r} a(r), \psi^a = \frac{r^a}{r} q(r)\}$ plus the vortices. However, the corresponding macroscopic state obtained from the expectation value is completely different to the microscopic states: $|\langle\psi\rangle| = 0$ (although $|\psi^a| \neq 0$). This is described in the Abrikosov model by adjusting the parameters of the *effective* potential such that its minimum is for the value of the order parameter $\bar{\psi} = 0$; the microscopical variables ψ nevertheless remain distinct from zero. Moreover, in the normal conducting phase there are no flux lines in a normal conductor, since the microscopic vortices condense to form the usual force lines of the magnetic field. In fact, the very existence of isolated flux lines as a *macroscopic* state (more than $|\langle\psi\rangle| \neq 0$) is the main characteristic of superconductivity.

Note. About the spontaneous symmetry breakdown (II).

And still another question: can a local gauge symmetry be broken spontaneously? Or stated in a more exact form: if we switch on an external magnetic field h by adding the term $-h \int d^d \vec{r} \cos \varphi$ to the system's energy, which is invariant under local transformations of the type (B.32), is the expectation value $\langle\psi\rangle$ at $h \rightarrow 0$ nonzero? i.e., does the magnetic field select one of the states at the bottom of the potential? The answer is no, and the reason is the local invariance itself ([36] and [59]).

In order to represent the system state in a clearer form, we return to the discrete lattice formulation. Let us start by considering the expectation value of $e^{i\varphi_{\vec{p}}}$, where $\varphi_{\vec{p}}$ is the phase of field ψ (again written as complex valued field) at a certain site $\vec{p}$

$$\langle e^{i\varphi_{\vec{p}}} \rangle = \lim_{h \rightarrow 0} \langle e^{i\varphi_{\vec{p}}} \rangle_h ,$$

with

$$\langle e^{i\varphi_{\vec{p}}} \rangle_h = \lim_{|\Lambda| \rightarrow \infty} \frac{1}{\mathcal{Z}} \sum_{\text{all configurations}} e^{i\varphi_{\vec{p}}} \exp \left[-\frac{1}{\eta} \left(E_{\text{local inv.}}[\varphi, A_\mu] - h \sum_{\vec{r}} \cos \varphi_{\vec{r}} \right) \right] . \quad (\text{B.49})$$

The following discussion holds for any lattice dimension and for any value of the scale η , since we can include as many terms as necessary in the energy to describe the model at any scale η . The only condition is that all these terms are powers of the covariant derivative $\partial_\mu \varphi - eA_\mu$, or in the discrete lattice formulation, of the discrete covariant derivative $\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - eA_{\mu,\vec{r}}$ with $A_{\mu,\vec{r}}$ the gauge field located between site $\vec{r}$ and site $\vec{r} + \hat{\mu}$. Now we carry out a local transformation to the new variables φ' , equal to the original except for ψ at $\vec{p}$ which now is pointing in the opposite direction

$$\begin{cases} \varphi_{\vec{p}} \rightarrow \varphi'_{\vec{p}} = \varphi_{\vec{p}} - \pi \\ \varphi_{\vec{r}} \rightarrow \varphi'_{\vec{r}} = \varphi_{\vec{r}} & \forall \vec{r} \neq \vec{p} \end{cases} . \quad (\text{B.50})$$

This change gets compensated by the gauge transformation of A_μ

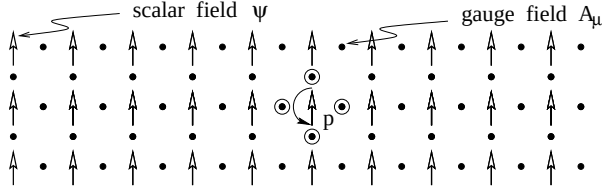
$$A_{\mu,\vec{r}} \rightarrow A'_{\mu,\vec{r}} = A_{\mu,\vec{r}} - \frac{1}{e} (\chi_{\vec{r}+\hat{\mu}} + \chi_{\vec{r}})$$

¹⁸In order to compare these results with those obtained in the Abrikosov model of Appendix A, one has to consider the change $e \rightarrow eq/\hbar c$ and the fact that in the present discussion the constant $\hbar^2/m$ of the Abrikosov model has been set equal to 1.

with $\chi_{\vec{r}}$ given by

$$\chi_{\vec{r}} = \begin{cases} \pi & \vec{r} = \vec{p} \\ 0 & \forall \vec{r} \neq \vec{p} \end{cases}. \quad (\text{B.51})$$

In the figure (a 2-dimensional lattice for the sake of simplicity), gauge fields are represented by points and those gauge fields which compensate the local transformation, by encircled points. Since $E_{\text{local inv.}}$ in (B.49) is invariant under this transformation, the only change in the expectation value arises from the term containing the external magnetic field



$$h \sum_{\vec{r}} \cos \varphi_{\vec{r}} = h \sum_{\vec{r}} \cos \varphi'_{\vec{r}} - h \sum_{\vec{r}} \delta(\cos \varphi_{\vec{r}}) = h \sum_{\vec{r}} \cos \varphi'_{\vec{r}} - h \underbrace{\sum_{\vec{r}} [\cos \varphi'_{\vec{r}} - \cos \varphi_{\vec{r}}]}_{-2 \cos \varphi_{\vec{p}} \times 2d}, \quad (\text{B.52})$$

where the factor $2d$ arises from the number of neighbouring ψ fields surrounding the reversed $\psi_{\vec{p}}$. The expectation value (B.49) can be also written

$$\lim_{|\Lambda| \rightarrow \infty} \frac{1}{\mathcal{Z}} \sum_{\text{configurations}} \left(-e^{i\varphi'_{\vec{p}}} \right) \exp \left[-\frac{1}{\eta} \left(E_{\text{local inv.}}[\varphi', A'_\mu] - h \sum_{\vec{r}} \cos \varphi'_{\vec{r}} \right) \right] \exp \left[-\frac{h}{\eta} \sum_{\vec{r}} \delta(\cos \varphi_{\vec{r}}) \right],$$

and therefore

$$\langle e^{i\varphi_{\vec{p}}} \rangle_h = \left\langle \left(-e^{i\varphi'_{\vec{p}}} \right) \exp \left[-\frac{h}{\eta} \sum_{\vec{r}} \delta(\cos \varphi_{\vec{r}}) \right] \right\rangle_h. \quad (\text{B.53})$$

From the last equation, we produce the following bound

$$\left| \langle e^{i\varphi_{\vec{p}}} \rangle_h - \langle (-e^{i\varphi_{\vec{p}}}) \rangle_h \right| = \left| \left\langle \left(-e^{i\varphi'_{\vec{p}}} \right) \underbrace{\left[e^{-\frac{h}{\eta} \sum_{\vec{r}} \delta(\cos \varphi_{\vec{r}})} - 1 \right]}_{\leq e^{\frac{4dh}{\eta}} - 1} \right\rangle_h \right|.$$

Since the bound only depends from *one* site alone, it can be take outside the thermodynamic limit $\lim_{|\Lambda| \rightarrow \infty}$ included in the expectation value $\langle \rangle_h$

$$\left| \langle e^{i\varphi_{\vec{p}}} \rangle_h - \langle (-e^{i\varphi_{\vec{p}}}) \rangle_h \right| \leq \left[e^{\frac{4dh}{\eta}} - 1 \right] \langle (-e^{i\varphi_{\vec{p}}}) \rangle_h, \quad (\text{B.54})$$

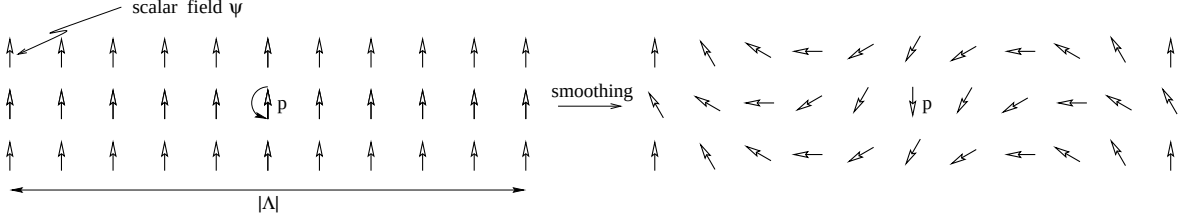
and taking now the limit $\lim_{h \rightarrow 0}$, we obtain

$$\langle e^{i\varphi_{\vec{p}}} \rangle = \langle (-e^{i\varphi_{\vec{p}}}) \rangle = 0. \quad (\text{B.55})$$

Let us stress the main feature leading to this result. Due to the invariance of $E_{\text{local inv.}}$ under a local transformation, we can make field ψ at a certain site $\vec{p}$ point in the opposite direction, producing a change (due to the h -term) which involves only a *finite* number of sites: for $\langle e^{i\varphi_{\vec{p}}} \rangle$ only one site, for the correlation function $\langle e^{i(\varphi_{\vec{p}_1} - \varphi_{\vec{p}_2})} \rangle$, two sites. Therefore the energetic increase due to the h -term in the exponent can be taken outside the thermodynamic limit, and limit $h \rightarrow 0$ removes this increase.

On the other hand, for a model invariant only under global transformations, the reversing of only one ψ field produces a large increase in $E_{\text{global inv.}}$. The system tries to smooth out through the whole lattice the large gradient in the orientation accumulated around $\vec{p}$ in order to reduce the energy. But

this smoothing gives rise to a lot of contributions in the h -term and the above procedure cannot be applied, since the bound involves almost all of the lattice sites: it cannot be brought outside the thermodynamic limit. Therefore, one can obtain a spontaneous symmetry breakdown, $\langle e^{i\varphi_{\vec{p}}} \rangle \neq 0 \forall \vec{p}$, for a model invariant only under global transformations.



For a local invariant model, it is clear that if the expectation value of a gauge invariant function Γ had been calculated, the above discussion would not hold since local transformations do not change Γ . Therefore, with a model invariant under local transformations, only the expectation value of gauge invariant expressions can be nonzero. This result is known as Elitzur theorem.

However, the Elitzur theorem does not rule out the existence of a nonvanishing $\langle e^{i\varphi} \rangle$ if we remove *explicitly* the local invariance by imposing a condition on the fields on which every configuration depends. This condition, which is called a gauge fixing, is defined by means of a boundary condition. An example of a common gauge fixing is the so-called Coulomb gauge $\text{div } \vec{A} = 0$, in which the gauge field is constrained to point in the $\hat{\phi}$ direction¹⁹. Let us suppose that after removing the local invariance, the energy invariance under global transformations is still allowed. In this case, spontaneous symmetry breakdown $\langle \psi \rangle \neq 0$ may occur, depending on the strength of the coupling with the boundary expressed in the gauge fixing [43]. This is the case of the Abrikosov model for superconductivity: the gauge fixing is the Coulomb gauge and the order parameter $\langle \psi \rangle$ may be nonvanishing, this occurs in the superconducting phase.

Before going further, let us finish the discussion about the vortex and obtain a close expression for its energy equal to that in the Abrikosov model. We define the dimensionless coordinate $\bar{r} = r/\lambda$, with λ the range of the gauge field without topological content (B.48), and the following dimensionless functions

$$\bar{Q}(\bar{r}) = \sqrt{\frac{|\beta|}{|\alpha|}} Q \quad \text{and} \quad \bar{f}(\bar{r}) = e\lambda^2 f,$$

and thus the vortex energy becomes

$$\begin{aligned} E_{\text{vortex}} &= \frac{\pi}{e^2 \lambda^2} \int_0^\infty \bar{r} d\bar{r} \left[\bar{r}^2 \left(\frac{d\bar{f}}{d\bar{r}} \right)^2 + \left(\frac{d\bar{Q}}{d\bar{r}} \right)^2 + \bar{Q}^2 \left(\frac{1}{\bar{r}} + \bar{r}\bar{f} \right)^2 + \frac{1}{4} \kappa^2 (\bar{Q}^2 - 1)^2 \right] = \\ &= \frac{\Phi_0^2}{4\pi\lambda^2} C(\kappa) \quad \text{with} \quad \kappa = \lambda/\xi. \end{aligned} \quad (\text{B.56})$$

¹⁹Considering the antisymmetry in the indices of $\epsilon_{\mu b}$, one can easily show that $A_\mu = \epsilon_{\mu b} r^b f(r)$ is divergenceless:

$$\partial_\mu A_\mu = \epsilon_{\mu b} \delta_{\mu b} f + \epsilon_{\mu b} r^\mu r^b \frac{1}{r} \frac{df}{dr} = 0.$$

In addition to ensure the finiteness of the vortex energy, the above ansatz $\psi^a = \frac{r^a}{r} Q(r)$ and $A_\mu = \epsilon_{\mu b} r^b f(r)$ fulfils the Coulomb gauge.

Summarising, in a 3-dimensional system described by a Lagrangian invariant under a gauge symmetry G , topological line defects are possible if $\pi_1(G/H)$ is not trivial, whilst the energy nondivergence at $r \rightarrow \infty$ is ensured by the covariant derivative. In a two dimensional system, these line defects become points. The exact form of the Lagrangian is not relevant (except if we are interested in obtaining the exact energy of the topological line defect). We only need that the coupling between scalar field ψ and gauge field A_μ always appears in the form of covariant derivatives, and a potential V which allows for a subgroup $H \subset G$ such to provide a nontrivial $\pi_1(G/H)$. In fact, in the model discussed in this section, V has been chosen in order to break G completely ($H = 1$): $G/H = G = U(1)$.

Similarly, if we want to study point defects in a 3-dimensional crystal, we must consider the different ψ configurations on a distant sphere surrounding the point defect: we have to consider all mappings from a sphere, S^2 , into the quotient space G/H . The point defect will be topologically stable if $\pi_2(G/H)$ contains more classes than the identity (and all configurations which can be deformed into the identity).

Up to this moment, we have only studied the easy case of breaking $G = U(1)$ completely for which $\pi_1(G/H) = \pi_1(U(1))$ can be directly obtained. However, we shall find another interesant case where G is a bit more complicated, but whose topological configurations are very different from those discussed in this section. This will be presented in the next section, after a short introduction to continuous (or Lie) symmetry groups.

B.3 A little excursus: Lie groups and Lie algebras

The next step in our discussion is to consider a Lagrangian density similar to that of the superconductor, but where the gauge symmetry is a non-Abelian one. In this model the existence of topological line defects will be studied. However, the discussion can be followed better if some ideas about Lie groups are firstly introduced. This section is intended only as an introduction, following the introductions in the books [53] and [80], and some results will not be proved.

In the systems in which we are interested, transformations act on a scalar field ψ conserving its length $|\psi|$. Moreover, the change produced by the transformation on ψ can be infinitesimal. Therefore we restrict ourselves to continuous or Lie groups, where there are transformations arbitrarily close to the identity and of the form ²⁰

$$U_{\text{infinitesimal}} = \mathbf{1} - i\chi^a I^a. \quad (\text{B.57})$$

Any transformation U can be constructed by successive application of these infinitesimal transformations (unless the contrary is stated, summation on repeated indices is to be understood). χ^a , with $a = 1, 2, \dots, N$, are real parameters (constants or functions), and I^a are Hermitian operators (matrices) and are called *generators*. Any group

²⁰A mathematician defines a Lie group in a more general form, but for the applications which we shall need later, this definition suffices.

transformation U is therefore an unitary operator (matrix). The generators close under commutations: the commutator of two generators is proportional to one of them

$$[I_a, I_b] = I_a I_b - I_b I_a = i C_{ab}^c I_c, \quad (\text{B.58})$$

with the real numbers C_{ab}^c being called structure constants of the Lie group. With these definitions, any group transformation is of the form

$$U = e^{-i\chi^a I_a}. \quad (\text{B.59})$$

That these elements form a group arises from the Baker-Hausdorff-Campbell formula: let I_a and I_b be two of the generators

$$e^{-iI_a} e^{-iI_b} = e^{-iI_a - iI_b - \frac{1}{2}[I_a, I_b] - \frac{i}{12}[[I_a, I_b], I_a] + \frac{i}{12}[[I_a, I_b], I_b] + \dots}, \quad (\text{B.60})$$

where $\dots$ contains relations involving more than 2 commutators, like $[[[I_a, I_b], I_a], I_a]$ and so on. According to (B.58), the exponent in the Baker-Hausdorff-Campbell equation contains only a *sum* of generators and therefore $e^{-iI_a} e^{-iI_b}$ is of the form $e^{-i\sum_a \chi^a I_a}$, belonging thus to the group. The factors $\frac{1}{2}, \frac{1}{12}, \dots$ appearing in the Baker-Hausdorff-Campbell formula can be directly obtained in the following form: let us introduce a parameter α multiplying the generators

$$e^{-i\alpha I_a} e^{-i\alpha I_b} = \left(1 - i\alpha I_a - \frac{\alpha^2}{2} I_a^2 + \frac{i}{3!} I_a^3 + \dots\right) \left(1 - i\alpha I_b - \frac{\alpha^2}{2} I_b^2 + \frac{i}{3!} I_b^3 + \dots\right) =$$

$$= 1 - i\alpha (I_a + I_b) - \alpha^2 \left(\frac{1}{2} I_a^2 + I_a I_b + \frac{1}{2} I_b^2\right) + i\alpha^3 \left(\frac{1}{3!} I_a^3 + \frac{1}{2} I_a^2 I_b + \frac{1}{2} I_a I_b^2 + \frac{1}{3!} I_b^3\right) + \dots,$$

and compare to the expansion of $e^{-i\alpha(I_a + I_b) - \alpha^2 k_2 [I_a, I_b] - i\alpha^3 k_3 [[I_a, I_b], I_a] - i\alpha^3 k'_3 [[I_a, I_b], I_b] + \dots}$ power by power of α to obtain the parameters $k_2, k_3, k'_3, \dots$

$$1 - i\alpha(I_a + I_b) - \alpha^2 \left(k_2 [I_a, I_b] + \frac{1}{2} I_a^2 + \frac{1}{2} I_b^2 + \frac{1}{2} I_a I_b + \frac{1}{2} I_b I_a\right) +$$

$$+ i\alpha^3 \left(k_3 [[I_a, I_b], I_a] + k'_3 [[I_a, I_b], I_b] + \frac{k_2}{2} I_a [I_a, I_b] + \frac{k_2}{2} [I_a, I_b] I_a + \frac{k_2}{2} I_b [I_a, I_b] + \right.$$

$$\left. + \frac{k_2}{2} [I_a, I_b] I_b + \frac{1}{3!} (I_a^3 + I_a^2 I_b + I_a I_b I_a + I_b I_a^2 + I_a I_b^2 + I_b I_a I_b + I_b^2 I_a + I_b^3)\right) + \dots,$$

which gives $k_2 = 1/2$ and $k_3 = -k'_3 = -1/12$.

Returning to main relation of a Lie group (B.58), if all C_{ab}^c are zero, the Lie group is called Abelian. The easiest example of an Abelian Lie group is the $U(1)$ group

$$\psi \rightarrow e^{-i\chi} \psi, \quad (\text{B.61})$$

or equivalently, the group of rotations in two dimensions $SO(2)$

$$\begin{pmatrix} \text{Re } \psi' \\ \text{Im } \psi' \end{pmatrix} = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} \text{Re } \psi \\ \text{Im } \psi \end{pmatrix}. \quad (\text{B.62})$$

This group has only one generator, the matrix $I = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ for $SO(2)$ ²¹, and the group is Abelian since the generator trivially commutes with itself. If the number of generators N of the Abelian group is larger than 1, each generator commutes with the others. Therefore, the Lie group is the direct product of N $U(1)$ groups, since due to the definition of direct product, all group transformations of each group commutes with all those of another group.

If, on the other hand, some of the structure constants do not vanish, the group is said non-Abelian. The C_{ab}^c 's are not unique for a given (non-Abelian) Lie group, since if we construct linear combinations of the generators I^a , $I'_a = \Lambda_a^d I_d$, and calculate the commutators for this transformed generators, the structure constants change into

$$\begin{aligned} iC_{ab}^{\prime c} I'_c &= [I'_a, I'_b] = \Lambda_a^d \Lambda_b^e [I_d, I_e] = i\Lambda_a^d \Lambda_b^e C_{de}^f (\Lambda^{-1})_f^c I'_c, \\ C_{ab}^{\prime c} &= \Lambda_a^d \Lambda_b^e C_{de}^f (\Lambda^{-1})_f^c. \end{aligned} \quad (\text{B.63})$$

Two properties of C_{ab}^c , however, are invariant and arise from equation (B.58)

$$C_{ba}^c = -C_{ab}^c, \quad (\text{B.64})$$

and from the so-called Jacobi identity $[[I_a, I_b], I_c] + [[I_b, I_c], I_a] + [[I_c, I_a], I_b] = 0$ (which can be proved by direct calculation)

$$C_{ab}^e C_{ec}^d + C_{bc}^e C_{ea}^d + C_{ca}^e C_{eb}^d = 0. \quad (\text{B.65})$$

Given a set of structure constants which fulfil (B.64) and (B.65), we have defined a Lie algebra, which is the vector space spanned by the generators I^a together with the commutation relations (B.57). More than one Lie group can exist sharing the same Lie algebra (sharing the same structure constants).

Now some definitions. When the range of the parameters χ^a in $U = e^{-i\chi^a I_a}$ is bounded, we call the Lie algebra *compact*. If a linear transformation such as (B.63) allows us to divide the set of generators $\{I'_a\}$ into two subsets, such that $C_{ab}^{\prime c} = 0$ if a belongs to one subset and b to the other, then the Lie algebra is broken down into two mutually commuting subalgebras and the original Lie group is factorized into two independent Lie groups. Note that each Lie subgroup is by definition a continuous subgroup; there may be discrete subgroups which commute with the rest of the generators, but they are not Lie groups and the following definitions do not apply to them.

Let us consider further the case where a Lie group has been factorized into two mutually commuting Lie subgroups, which we call G_1 and G_2 . In the case where one of

²¹One can easily prove that any $SO(2)$ matrix can be written as $U = e^{-i\chi I}$: $I = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ has the eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$ whose respective eigenvalues are 1, -1, and therefore

$$e^{-i\chi I} = \frac{e^{-i\chi}}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{e^{i\chi}}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix}.$$

the subgroups (for example G_1) only contains one generator, then $G_1 = U(1)$ and the factorization of the original Lie group contains an Abelian $U(1)$ Lie group. The immediate generalization follows when subgroup G_1 contains more than one generator and each of these generators commutes with all the others: G_1 is an Abelian group and is formed as the direct product of $U(1)$ Lie groups. If the Lie algebra does not contain any $U(1)$ factor, it is called a *semi-simple* Lie algebra; if the algebra cannot be factorized in any way into two mutually commuting sets, it is then called a *simple* Lie algebra. The direct product of two *simple* Lie groups $A \otimes B$ is a semi-simple Lie group, since the group transformations belonging to A commute with those belonging to B , and because A and B , being simple groups, are not Abelian. In fact, every semi-simple Lie group can be written as a direct product of simple Lie groups.

From here on, we shall focus our attention on Lie groups which admit a representation of all group transformations by means of finite-dimensional matrices. The group generators are represented therefore by (hermitian) finite-dimensional matrices, for which the commutation relations $[I_a, I_b] = iC_{ab}^c I_c$ hold. This restriction in the matrix representation ($U = e^{-i\chi^a I_a}$ with χ^a real and I_a finite-dimensional hermitian matrices) makes the Lie algebras we shall consider be compact. As an example of a Lie algebra, which does not permit representation by means of finite-dimensional unitary matrices, let us consider the well-known case of the group of Lorentz transformations. The finite-dimensional matrix representations of this group²² are not generated by hermitian matrices, and the unitary representations of the group (i.e. those representations generated by hermitian matrices) are infinite-dimensional matrices. The Lorentz group is not compact since its parametrization is not bounded: $0 \leq v/c \leq 1$ and therefore $0 \leq \chi < \infty$. One can show that in general, a non-compact Lie algebra permits no finite-dimensional representation of the kind $U = e^{-i\chi^a I_a}$ with χ^a real and I_a hermitian [44].

The faithful²³ representation formed by the lowest dimensional matrices which fulfil

²²For instance, a Lorentz boost for the case of a reference system moving along the x -direction, $\begin{pmatrix} ct' \\ x \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & -\frac{v/c}{\sqrt{1-v^2/c^2}} \\ -\frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$, can be rewritten in a form resembling a rotation matrix

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \chi & -\sinh \chi & 0 & 0 \\ -\sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{with} \quad \chi = \operatorname{arctanh} \left(\frac{v}{c} \right).$$

An infinitesimal Lorentz transformation ($v/c \approx 0$) is of the form $L = \mathbf{1} - i\chi \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

therefore, the generators are not hermitian.

²³If the correspondence between group transformations and representation matrices is an isomorphism, the representation is called “faithful”. An isomorphism means that each group transformation corresponds to a matrix in the representation considered, and each matrix of the representation corresponds to a group transformation. Moreover, if g_1 and g_2 are two arbitrary group transformations and $r(g_1)$ and $r(g_2)$ their respective matrix representations, the representation of the transformation $g_1 g_2$,

the commutation relations is called *fundamental representation*. For an arbitrary representation, one can find a basis (a set of generators) in which all representation matrices are simultaneously block-diagonal. In this basis, one can write the representation as a direct sum of *irreducible* representations. Another irreducible matrix representation, which always exists, is that completely determined by the structure constants of the Lie algebra,

$$(I_a^{(\text{adj.})})_b^c = -iC_{ab}^c. \quad (\text{B.66})$$

The dimensionality of the matrices in this representation is N (the range for the indices $a, b, c, \dots$), equal to the number of generators; this representation is called the *adjoint representation*. Such a representation is constituted by hermitian matrices because of (B.63) and fulfils the commutation relations due to the Jacobi identity

$$\begin{aligned} [I_a^{(\text{adj.})}, I_b^{(\text{adj.})}]_r^s &= -C_{ar}^m C_{bm}^s + C_{br}^m C_{am}^s = C_{ar}^m C_{mb}^s + C_{rb}^m C_{am}^s \stackrel{\text{Jacobi}}{=} -C_{ba}^m C_{mr}^s, \\ [I_a^{(\text{adj.})}, I_b^{(\text{adj.})}]_r^s &= iC_{ab}^m (I_a)_r^s. \end{aligned} \quad (\text{B.67})$$

For a *simple* Lie algebra, one can define the following metric (also called Killing metric or Killing form)

$$g_{ab} = \text{Tr} (I_a^{(\text{adj.})} I_b^{(\text{adj.})}) = -C_{an}^m C_{bm}^n. \quad (\text{B.68})$$

The condition on the Lie algebra to be simple has been required in order to ensure that the structure constants do not vanish due to the existence of an invariant subset. Let us show that the tensor (B.68) is really a metric. Firstly, g_{ab} is a symmetric tensor: $g_{ba} = -C_{bn}^m C_{am}^n \stackrel{n \leftrightarrow m}{=} -C_{an}^m C_{bm}^n = g_{ab}$. Since the generators are hermitian, for each generator there exists a unitary matrix U such that $U I_a^{(\text{adj.})} U^\dagger = I_a'^{(\text{adj.})}$ is in diagonal form, with all eigenvalues real numbers. Therefore, g_{aa} (without summing over a) is always positive since²⁴

$$\begin{aligned} g_{aa} &= \text{Tr} (I_a^{(\text{adj.})} I_a^{(\text{adj.})}) = \text{Tr} (U I_a'^{(\text{adj.})} U^\dagger U I_a'^{(\text{adj.})} U^\dagger) = \text{Tr} (U^\dagger U I_a'^{(\text{adj.})} I_a'^{(\text{adj.})}) = \\ &= \text{Tr} \begin{pmatrix} + & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & + \end{pmatrix} > 0. \end{aligned} \quad (\text{B.69})$$

This property together with the symmetry $g_{ab} = g_{ba}$ mean that the tensor g_{ab} has the properties of a metric. Defining new structure constants

$$C_{abc} = g_{nc} C_{ab}^n, \quad (\text{B.70})$$

$r(g_1 g_2)$ is equal to the matrix product of the representations of g_1 and g_2 : $r(g_1 g_2) = r(g_1) r(g_2)$.

²⁴From the general property of the matrix trace $\text{Tr} (AB) = \text{Tr} (BA)$ with A, B two arbitrary matrices, it follows $\text{Tr} (\underline{ABC}) = \text{Tr} (C \underline{AB})$ and $\text{Tr} (\underline{ABC}) = \text{Tr} (\underline{BCA})$. And in general

$$\text{Tr} (ABC \dots PQR) = \text{Tr} (RABC \dots PQ) .$$

we shall now show that, by means of the general properties (B.63) and (B.65) of C_{ab}^n , the tensor C_{abc} is completely antisymmetric in its indices

$$C_{abc} = -C_{ns}^r C_{cr}^s C_{ab}^n = -C_{cr}^s \underbrace{(C_{ab}^n C_{ns}^r)}_{= -C_{sa}^n C_{nb}^r C_{rc}^s + C_{sb}^n C_{na}^r C_{rc}^s} = -C_{sa}^n C_{nb}^r C_{rc}^s + C_{sb}^n C_{na}^r C_{rc}^s, \\ -C_{sa}^n C_{nb}^r - C_{bs}^n C_{na}^r$$

and therefore by interchanging the names of the repeated indices

$$C_{cab} = -\underbrace{C_{sc}^n C_{na}^r C_{rb}^s}_{n \rightarrow s \rightarrow r \rightarrow n} + \underbrace{C_{sa}^n C_{nc}^r C_{rb}^s}_{r \rightarrow s \rightarrow n \rightarrow r} = C_{abc}. \quad (\text{B.71})$$

From $C_{bac} = g_{nc} C_{ba}^n = -g_{nc} C_{ab}^n = -C_{abc}$ and from this last equation, it follows also the antisymmetry in the second and third indices of C_{abc} : $-C_{acb} = C_{cab} = C_{abc}$. Further, under a linear transformation of the generators, the structure constants change according to (B.63) and the metric will change as follows

$$g'_{ab} = -C'^e_{ad} C'^d_{be} = -\Lambda_a^m \Lambda_d^n C_{mn}^p \left(\Lambda^{-1} \right)_p^e \Lambda_b^r \Lambda_e^s C_{rs}^t \left(\Lambda^{-1} \right)_t^d = -\Lambda_a^m \Lambda_b^r C_{mn}^p C_{rp}^n, \\ g'_{ab} = \Lambda_a^m \Lambda_b^r g_{mr}. \quad (\text{B.72})$$

Thus, by means of a linear transformation, g_{ab} can be diagonalized; and by re-scaling of the generators, all eigenvalues can be made unity

$$g_{ab} = \delta_{ab} \quad \text{and} \quad C_{abc} = g_{nc} C_{ab}^n = C_{ab}^c. \quad (\text{B.73})$$

Hence, we shall write C_{abc} instead of C_{ab}^c for the following.

The possibility of bringing $g_{ab} = \text{Tr} (I_a I_b)$ into a diagonal form of all positive eigenvalues has been deduced for the adjoint representation, but for a simple (and compact) Lie group, holds also for any matrix representation. In order to prove it, we shall show that in the case of a *simple* Lie group, for any matrix representation (r) of the group transformations one can find a set of generators $\{I_a^{(r)}\}$ such that the following holds

$$\text{Tr} \left(I_a^{(r)} I_b^{(r)} \right) = K^{(r)} \delta_{ab}, \quad (\text{B.74})$$

where $K^{(r)}$ is a constant depending only on the matrix representation (r) but not on the index a . Due to the property of the trace, $\text{Tr} \left(I_b^{(r)} I_a^{(r)} \right) = \text{Tr} \left(I_a^{(r)} I_b^{(r)} \right)$, and thus one can find a basis of generators so that $\text{Tr} \left(I_a^{(r)} I_b^{(r)} \right)$ is diagonal

$$\text{Tr} \left(I_a^{(r)} I_b^{(r)} \right) = \begin{cases} 0 & \text{if } a \neq b \\ K_a^{(r)} & \text{if } a = b \end{cases}.$$

Now we need to show that the constant $K_a^{(r)}$ does not depend on a . Let us define

$$d_{abc} = \text{Tr} \left(\left[I_a^{(r)}, I_b^{(r)} \right] I_c^{(r)} \right) = \text{Tr} \left(I_a^{(r)} I_b^{(r)} I_c^{(r)} \right) - \text{Tr} \left(I_b^{(r)} I_a^{(r)} I_c^{(r)} \right).$$

This is completely antisymmetric in a, b, c

$$d_{bac} = \text{Tr} \left(\left[I_b^{(r)}, I_a^{(r)} \right] I_c^{(r)} \right) = -\text{Tr} \left(\left[I_a^{(r)}, I_b^{(r)} \right] I_c^{(r)} \right) = -d_{abc},$$

and according to the relation $\text{Tr}(ABC) = \text{Tr}(CAB)$

$$d_{cab} = \text{Tr} \left(I_c^{(r)} I_a^{(r)} I_b^{(r)} \right) - \text{Tr} \left(I_a^{(r)} I_c^{(r)} I_b^{(r)} \right) = \text{Tr} \left(I_a^{(r)} I_b^{(r)} I_c^{(r)} \right) - \text{Tr} \left(I_b^{(r)} I_a^{(r)} I_c^{(r)} \right) = d_{abc},$$

from which it follows the antisymmetry in the two last indices $-d_{acb} = d_{cab} = d_{abc}$. The constants d_{abc} are related to the structure constants by means of

$$d_{abc} = \text{Tr} \left(iC_{abm} I_m^{(r)} I_c^{(r)} \right) = iC_{abm} \delta_{mc} K_m^{(r)} = iC_{abc} K_c^{(r)} \quad \text{without summing on } c.$$

Interchanging of the indices b and c gives $d_{acb} = iC_{acb} K_b^{(r)}$, again without summing on b . But since $d_{acb} = -d_{abc}$ and $C_{acb} = -C_{abc}$, we obtain $K_c^{(r)} = K_b^{(r)}$ whenever $C_{abc} \neq 0$ (whenever $[I_c^{(r)}, I_b^{(r)}] \neq 0$). Since the group is simple, *all* group generators can be obtained starting from two arbitrary generators by applying successively commutators. From the above discussion for $d_{abc} = \text{Tr} \left([I_a^{(r)}, I_b^{(r)}] I_c^{(r)} \right)$ follows therefore that, for a given representation, all $K_c^{(r)}$'s are equal. The constant $K^{(r)}$ is related to the Casimir operator $C^{(r)} = \sum_a I_a^{(r)} I_a^{(r)}$ ²⁵ by the following relation

$$\sum_a \text{Tr} \left(I_a^{(r)} I_a^{(r)} \right) = K^{(r)} N \quad \implies \quad K^{(r)} = \frac{C^{(r)} d^{(r)}}{N}, \quad (\text{B.75})$$

where N is the number of generators and the representation (r) is formed by $d^{(r)} \times d^{(r)}$ matrices. The result (B.74) and the commutation relations $[I_a, I_b] = iC_{abc} I_c$ allows us to obtain the following general form for the structure constants

$$C_{abc} = -\frac{i}{K^{(r)}} \text{Tr} \left([I_a^{(r)}, I_b^{(r)}] I_c^{(r)} \right). \quad (\text{B.76})$$

From this equation, the complete antisymmetry in the indices of the structure constants for a simple compact Lie algebra is directly evident.

Closing this introduction, a Lie algebra whose structure constants $C_{ab}^c = C_{abc}$ are totally antisymmetric and fulfil the Jacobi identity, and for which relation (B.74) holds, is a simple compact Lie algebra.

B.4 Applications of group π_1 : line defects in a 3-dimensional system (II)

We shall focus our attention on simple compact non-Abelian Lie algebras, defined by the structure constants C_{abc} . With such a Lie algebra we shall construct a model with a scalar field Ψ coupled to a gauge field by means of the covariant derivative

$$D_\mu \Psi = \partial^\mu \Psi^{(r)} - ie W_\mu^a I_a^{(r)} \Psi^{(r)}, \quad (\text{B.77})$$

²⁵The Casimir operator $C^{(r)} = \sum_a I_a^{(r)} I_a^{(r)}$ commutes with all group generators (represented by $d^{(r)} \times d^{(r)}$ matrices) due to the complete antisymmetry of C_{abc}

$$[C^{(r)}, I_b^{(r)}] = \sum_a I_a^{(r)} [I_a^{(r)}, I_b^{(r)}] + \sum_a [I_a^{(r)}, I_b^{(r)}] I_a^{(r)} = iC_{abc} I_a^{(r)} I_c^{(r)} + \underbrace{iC_{abc} I_c^{(r)} I_a^{(r)}}_{c \leftrightarrow a} = 0.$$

Therefore $C^{(r)}$ is a constant times the $d^{(r)} \times d^{(r)}$ unit matrix and depends only on the matrix representation: $I_a^{(r)} I_a^{(r)} = C^{(r)} \mathbf{1}$ (sum on a), with $\mathbf{1}$ the $d^{(r)} \times d^{(r)}$ unit matrix.

where W_μ^a is the gauge field, e the coupling constant, $I_a^{(r)}$'s are the generators in the matrix representation (r) and $\Psi^{(r)}$ is the corresponding matrix representation of the scalar field. Under an infinitesimal local (gauge) transformation (B.57)

$$\delta\Psi = -i\chi^a(x)I_a\Psi, \quad (\text{B.78})$$

gauge field W_μ^a must transform in such a way so that the covariant derivative transforms as the scalar field

$$\delta(D_\mu\Psi^{(r)}) = -i\chi^a I_a^{(r)} D_\mu\Psi^{(r)}. \quad (\text{B.79})$$

By substituting (B.78) into this last equation, we obtain

$$\begin{aligned} \delta(D_\mu\Psi^{(r)}) &= \partial_\mu(\delta\Psi^{(r)}) - ie(\delta W_\mu^b)I_b^{(r)}\Psi^{(r)} - ieW_\mu^b I_b^{(r)}\delta\Psi^{(r)} = \\ &= -i(\partial_\mu\chi^a)I_a^{(r)}\Psi^{(r)} - i\chi^a I_a^{(r)}\partial_\mu\Psi^{(r)} - ie(\delta W_\mu^b)I_b^{(r)}\Psi^{(r)} - eW_\mu^b\chi^a I_b^{(r)}I_a^{(r)}\Psi, \\ &-i(\partial_\mu\chi^b)I_b^{(r)}\Psi^{(r)} - ie(\delta W_\mu^b)I_b^{(r)}\Psi^{(r)} - eW_\mu^b\chi^a I_b^{(r)}I_a^{(r)}\Psi = -eW_\mu^b\chi^a I_b^{(r)}I_a^{(r)}\Psi, \end{aligned}$$

which, according to $[I_a^{(r)}, I_b^{(r)}] = iC_{abc}I_c^{(r)}$, leads to ²⁶

$$\delta W_\mu^b = -\frac{1}{e}\partial_\mu\chi^b + C_{bac}\chi^a W_\mu^c. \quad (\text{B.80})$$

Similarly to the Abelian $U(1)$ case, one has to include a term, quadratic in the space derivatives of the gauge field, to describe the spatial variation of the gauge field. Such a term must be invariant under a gauge transformation (B.77). Term $F_{\mu\nu}^a F_{\mu\nu}^a$ with $F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a$ as in the Abelian case, does not work since it is not invariant under a gauge transformation

$$\delta(F_{\mu\nu}^a F_{\mu\nu}^a) = 2F_{\mu\nu}^a (\partial_\mu\delta W_\nu^a - \partial_\nu\delta W_\mu^a) = 4F_{\mu\nu}^a C_{abc} ((\partial_\mu\chi^b)W_\nu^c + \chi^b\partial_\mu W_\nu^c) \neq 0.$$

The correct tensor is given by

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + eC_{abc}W_\mu^b W_\nu^c. \quad (\text{B.81})$$

Considering the complete antisymmetry of the structure constants and Jacobi identity (B.65) $C_{acm}C_{med} + C_{cem}C_{mad} + C_{eam}C_{mcd} = 0$

$$C_{abc}C_{bde}\chi^d W_\mu^e W_\nu^c = +C_{acm}C_{med}\chi^d W_\mu^e W_\nu^c = C_{adm}\chi^d (C_{mec}W_\mu^e W_\nu^c) - C_{aem}C_{mdc}\chi^d W_\mu^e W_\nu^c,$$

²⁶For a finite transformation $\Psi^{(r)} \rightarrow \Psi'^{(r)} = U\Psi^{(r)} = e^{-i\chi^a I_a^{(r)}}\Psi^{(r)}$, the transformed gauge field is $(D_\mu\Psi^{(r)})' = (\partial_\mu U)\Psi^{(r)} + U\partial_\mu\Psi^{(r)} - ieW_\mu'^a I_a^{(r)}U\Psi^{(r)} = UD_\mu\Psi^{(r)} = U(\partial_\mu\Psi^{(r)} - ieW_\mu^a I_a^{(r)}\Psi^{(r)})$ and thus

$$[(\partial_\mu U) - ieW_\mu'^a I_a^{(r)}U = -ieW_\mu^a U I_a^{(r)}] U^{-1} \implies W_\mu'^a I_a^{(r)} = W_\mu^a U I_a^{(r)} U^{-1} - \frac{i}{e}(\partial_\mu U)U^{-1}.$$

Multiplying by $I_b^{(r)}$, taking the trace and noting the relation $\text{Tr}(I_a^{(r)}I_b^{(r)}) = K^{(r)}\delta_{ab}$, one obtains

$$W_\mu'^b = \frac{1}{K^{(r)}}W_\mu^a \text{Tr}(U I_a^{(r)} U^{-1} I_b^{(r)}) - \frac{i}{eK^{(r)}}\text{Tr}((\partial_\mu U)U^{-1} I_b^{(r)}).$$

the tensor $G_{\mu\nu}^a$ transforms under an infinitesimal local change according to

$$\begin{aligned} \delta G_{\mu\nu}^a = & C_{ade} \left[(\partial_\mu \chi^d) W_\nu^e - (\partial_\nu \chi^d) W_\mu^e \right] + C_{ade} \chi^d (\partial_\mu W_\nu^e - \partial_\nu W_\mu^e) - C_{abc} \partial_\mu \chi^b W_\nu^c - \\ & - C_{abc} W_\mu^b \partial_\nu \chi^c + e C_{abc} C_{bde} \chi^d W_\mu^e W_\nu^c + e \underbrace{C_{abc} C_{cde}}_{b \rightarrow e \rightarrow c \rightarrow m : C_{aem} C_{mdc}} \chi^d W_\mu^b W_\nu^e, \end{aligned}$$

obtaining

$$\delta G_{\mu\nu}^a = C_{adm} \chi^d (\partial_\mu W_\nu^m - \partial_\nu W_\mu^m + e C_{mec} W_\mu^e W_\nu^c) = C_{adm} \chi^d G_{\mu\nu}^m. \quad (\text{B.82})$$

This equation makes the term $G_{\mu\nu}^a G_{\mu\nu}^a$ be invariant under a gauge transformation

$$\delta (G_{\mu\nu}^a G_{\mu\nu}^a) = 2 \chi^d C_{ade} G_{\mu\nu}^a G_{\mu\nu}^e = 0, \quad (\text{B.83})$$

due to the antisymmetry of the structure constants. Note that both W_μ^a and $G_{\mu\nu}^a$ do not depend on the matrix representation (r) chosen for the generators and for the scalar field. A Lagrangian density containing the following terms

$$-\frac{1}{4} (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e C_{abc} W_\mu^b W_\nu^c)^2 - \frac{1}{2} \left| \partial^\mu \Psi^{(r)} - i e W_\mu^a I_a^{(r)} \Psi^{(r)} \right|^2, \quad (\text{B.84})$$

is therefore invariant under the gauge transformation

$$\begin{cases} \Psi^{(r)} \rightarrow \Psi'^{(r)} = U \Psi^{(r)} = e^{-i \chi^a I_a^{(r)}} \Psi^{(r)} \\ W_\mu^b \rightarrow W'^b_\mu = \frac{1}{K^{(r)}} W_\mu^a \text{Tr} \left(U I_a^{(r)} U^{-1} I_b^{(r)} \right) - \frac{i}{e K^{(r)}} \text{Tr} \left((\partial_\mu U) U^{-1} I_b^{(r)} \right) \end{cases} \quad (\text{B.85})$$

From here on, the index (r) will be not written down more, unless it is necessary for the discussion.

Now we shall study a model described by the above Lagrangian density (B.84) for a simple compact Lie algebra, whose structure constants are the following C_{abc} , with $a, b, c = 1, 2, 3$

$$C_{abc} = \epsilon_{abc} = \begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1 \\ 0 \text{ otherwise} \end{cases}. \quad (\text{B.86})$$

The first two representations of lowest dimensionality are the fundamental and the adjoint representation.

1. The fundamental representation: the generators are given by the 2×2 matrices $I_a = \frac{1}{2} \sigma_a$, with $\{\sigma\}$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B.87})$$

These generators are hermitian matrices and fulfil the commutation relations $[I_a, I_b] = i\epsilon_{abc}I_c$, as one can easily show by using the following relation for the Pauli matrices

$$\sigma_a \sigma_b = \delta_{ab} \mathbf{1} + i\epsilon_{abc} \sigma_c, \quad (\text{B.88})$$

with $\mathbf{1}$ the 2×2 unit matrix. Scalar field Ψ is given by

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_0 \end{pmatrix}, \quad (\text{B.89})$$

where ψ_+ and ψ_0 are complex functions. By considering the relation (B.88) and the antisymmetry of ϵ_{abc} (underlined in the next relation)

$$\begin{aligned} (\chi^a \sigma_a)^2 &= \chi^a \chi^b \sigma_a \sigma_b = \chi^2 \mathbf{1} + i\chi^a \chi^b \epsilon_{abc} \sigma_c = \chi^2 \mathbf{1}, \\ (\chi^a \sigma_a)^3 &= \chi^a \chi^b \chi^c \delta_{ab} \sigma_c + i\chi^a \chi^b \chi^c \epsilon_{abm} \sigma_m \sigma_c = \chi^2 \chi^a \sigma_a, \end{aligned}$$

one obtains in general

$$(\chi^a \sigma_a)^{2n} = (\chi)^{2n} \mathbf{1} \quad \text{and} \quad (\chi^a \sigma_a)^{2n+1} = (\chi)^{2n+1} \hat{n}^a \sigma_a, \quad (\text{B.90})$$

where $\chi = \sqrt{\chi^a \chi^a}$ and $\hat{n}^a = \chi^a / \chi$. Thus, any group transformation in the fundamental representation $U = e^{-i/2 \chi^a \sigma_a}$ takes the following form

$$\begin{aligned} U &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(-i\chi^a \frac{\sigma_a}{2} \right)^m = \mathbf{1} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \left(\frac{\chi}{2} \right)^{2n} - i\hat{n}^a \sigma_a \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\chi}{2} \right)^{2n+1}, \\ U &= e^{-i\chi/2 \hat{n}^a \sigma_a} = \mathbf{1} \cos \left(\frac{\chi}{2} \right) - i\hat{n} \cdot \vec{\sigma} \sin \left(\frac{\chi}{2} \right) = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix}, \quad (\text{B.91}) \end{aligned}$$

with

$$\begin{aligned} a_0 &= \cos \left(\frac{\chi}{2} \right), \quad a_k = -\hat{n}_k \sin \left(\frac{\chi}{2} \right) \quad 0 \leq \chi < 4\pi, \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1. \end{aligned} \quad (\text{B.92})$$

According to the relation $\sigma_a \sigma_b = \delta_{ab} \mathbf{1} + i\epsilon_{abc} \sigma_c$, the value of the Casimir operator $C = \sum_a \text{Tr} (I_a I_a)$ for the fundamental representation is

$$C^{(\text{fund.})} = \sum_a \text{Tr} \left(\frac{\sigma_a}{2} \frac{\sigma_a}{2} \right) = \frac{3}{4} \mathbf{1}. \quad (\text{B.93})$$

In general, the representation (r) yields the infinitesimal transformation

$$\Psi \rightarrow \Psi' = (1 - i\chi^a I_a^{(r)}) \Psi.$$

The complex conjugate of this transformation

$$\Psi^* \rightarrow \Psi'^* = (1 + i\chi^a (I_a^{(r)})^*) \Psi^*,$$

defines another representation, called the conjugate of the representation (r) , whose generators are denoted by $\{I_a^{(\bar{r})}\}$

$$\Psi^* \rightarrow \Psi'^* = (1 - i\chi^a I_a^{(\bar{r})}) \Psi^* \quad \text{with} \quad I_a^{(\bar{r})} = -(I_a^{(r)})^*. \quad (\text{B.94})$$

Since the generators are hermitian matrices, representation (r) and its associated conjugate representation $(\bar{r})$ are equivalent, if there is an unitary transformation $\mathcal{U}$ such that $I_a^{(\bar{r})} = \mathcal{U} I_a^{(r)} \mathcal{U}^\dagger$. In the case of the fundamental representation of the Lie algebra defined by the structure constants $C_{abc} = \epsilon_{abc}$, the conjugate representation is equivalent to the fundamental representation, with $\mathcal{U} = a \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $|a|^2 = 1$. For other simple compact Lie algebras, the conjugate fundamental representation is usually non-equivalent to the fundamental representation.

2. The adjoint representation: the generators are the 3×3 matrices $(I_a)_{bc} = -i\epsilon_{abc}$, which are hermitian and fulfil the commutation relations as have been proved for any simple compact Lie algebra in (B.67). The scalar field in this representation is given by

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}, \quad (\text{B.95})$$

with $\Psi^{a=1,2,3}$ real functions. Since the tensor ϵ_{abc} fulfils the following relation $\epsilon_{arm}\epsilon_{bms} = \delta_{as}\delta_{rb} - \delta_{ab}\delta_{rs}$, we obtain

$$(\hat{n}^a I_a^{(\text{adj.})})_{rs}^2 = (\hat{n}^a \hat{n}^b I_a^{(\text{adj.})} I_b^{(\text{adj.})})_{rs} = \delta_{rs} - \hat{n}_r \hat{n}_s,$$

$$(\hat{n}^a \hat{n}^b \hat{n}^c I_a^{(\text{adj.})} I_b^{(\text{adj.})} I_c^{(\text{adj.})})_{rs} = -i\hat{n}_c \epsilon_{crs},$$

with $\hat{n}$ the unit vector $\hat{n}^a = \chi^a / \chi$. In general

$$(\hat{n}^a I_a^{(\text{adj.})})^{2n} = \mathbf{1} - \hat{n} \otimes \hat{n} \quad \text{and} \quad (\hat{n}^a I_a^{(\text{adj.})})^{2n+1} = \hat{n}^a I_a^{(\text{adj.})}, \quad (\text{B.96})$$

where $\mathbf{1}$ is the 3×3 unit matrix and $\hat{n} \otimes \hat{n}$ is the matrix defined by $(\hat{n} \otimes \hat{n})_{rs} = \hat{n}_r \hat{n}_s$. Hence, a general group transformation in the adjoint representation $U = e^{-i\chi^a I_a^{(\text{adj.})}}$ is given by

$$U = \hat{n} \otimes \hat{n} + (\mathbf{1} - \hat{n} \otimes \hat{n}) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \chi^{2n} - i \hat{n}^a I_a^{(\text{adj.})} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \chi^{2n+1},$$

$$U = e^{-i\chi^a I_a^{(\text{adj.})}} = \hat{n} \otimes \hat{n} + (\mathbf{1} - \hat{n} \otimes \hat{n}) \cos \chi - i \hat{n}^a I_a^{(\text{adj.})} \sin \chi. \quad (\text{B.97})$$

The Casimir operator in the adjoint representation is given by

$$(C^{(\text{adj.})})_{rs} = -\epsilon_{arc} \epsilon_{acs} = 2\delta_{rs} \quad \implies \quad C^{(\text{adj.})} = 2 \mathbf{1}. \quad (\text{B.98})$$

It is clear from the definition of adjoint representation $(I_a)_{bc} = -i\epsilon_{abc}$ that its conjugate representation $\overline{I_a^{(\text{adj.})}} = -(I_a^{(\text{adj.})})^*$ is the adjoint representation itself.

Both representations, the fundamental and the adjoint, will play an important role in the next discussion. There is an alternative form of writing the adjoint representation: if X_i , with $i = 1, 2$, transforms according to the fundamental representation

$$\delta X_i = -i\chi^a \frac{(\sigma_a)_{ij}}{2} X_j.$$

then Y_a , with $a = 1, 2, 3$, defined by

$$Y_a = (X^* I_a^{(\text{fund.})} X) = \frac{1}{2} X_i^* (\sigma_a)_{ij} X_j, \quad (\text{B.99})$$

is real

$$Y_a^* = \frac{1}{2} X_i (\sigma_a)_{ij}^* X_j^* \stackrel{\sigma_a^+ = \sigma_a}{=} \frac{1}{2} X_i (\sigma_a)_{ji} X_j^* = \frac{1}{2} X_j^* (\sigma_a)_{ji} X_i = Y_a,$$

and transforms as the adjoint representation

$$\begin{aligned} \delta Y_a &= \frac{1}{2} \left(\frac{i}{2} \chi^b \underbrace{(\sigma_b)_{ir}^*}_{(\sigma_b)_{ri}} X_r^* \right) (\sigma_a)_{ij} X_j + \frac{1}{2} X_i^* (\sigma_a)_{ij} \left(-\frac{i}{2} \chi^b (\sigma_b)_{jr} X_r \right) = \\ &= i\chi^b X_i^* \left[\frac{\sigma_b}{2}, \frac{\sigma_a}{2} \right]_{ij} X_j = -i\chi^b (-i\epsilon_{bac}) \left(\frac{1}{2} X_i^* (\sigma_c)_{ij} X_j \right), \\ \delta Y_a &= -i\chi^b (I_b^{(\text{adj.})})_{ac} Y_c. \end{aligned} \quad (\text{B.100})$$

One can also associate a field Y_a , which transforms as the adjoint representation, to a 2×2 complex traceless tensor T given by

$$T = Y_a I_a^{(\text{fund.})} = Y_a \frac{\sigma_a}{2}, \quad (\text{B.101})$$

such that a group transformation in the fundamental representation $U^{(\text{fund.})} = e^{-i/2 \chi^a \sigma_a}$ acting on this tensor corresponds to a group transformation in the adjoint representation $U^{(\text{adj.})} = e^{-i\chi^a I_a^{(\text{adj.})}}$ on Y_a . This statement can be easily shown: according to (B.91), T changes as follows under a transformation $U^{(\text{fund.})}$

$$\begin{aligned} T &\rightarrow T' = U^{(\text{fund.})} T U^{(\text{fund.})+} = \\ &= \left(\mathbf{1} \cos \left(\frac{\chi}{2} \right) - i \hat{n} \cdot \vec{\sigma} \sin \left(\frac{\chi}{2} \right) \right) Y_b \frac{\sigma_b}{2} \left(\mathbf{1} \cos \left(\frac{\chi}{2} \right) + i \hat{n} \cdot \vec{\sigma} \sin \left(\frac{\chi}{2} \right) \right). \end{aligned} \quad (\text{B.102})$$

By noting the relation $\sigma_a \sigma_b = \delta_{ab} \mathbf{1} + i \epsilon_{abc} \sigma_c$, equation (B.102) leads to

$$\begin{aligned} T' &= \cos^2 \left(\frac{\chi}{2} \right) Y_b \frac{\sigma_b}{2} - i 2 \sin \left(\frac{\chi}{2} \right) \cos \left(\frac{\chi}{2} \right) \hat{n}_a Y_b \left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2} \right] + \\ &\quad + \frac{1}{2} \sin^2 \left(\frac{\chi}{2} \right) Y_b \hat{n}_a \hat{n}_c \underbrace{\sigma_a \sigma_b \sigma_c}_{i \epsilon_{abc} \mathbf{1} - \delta_{ac} \sigma_b + \delta_{ab} \sigma_c + \delta_{bc} \sigma_a} = \frac{i}{2} \sin^2 \left(\frac{\chi}{2} \right) Y_b \hat{n}_a \hat{n}_c \epsilon_{abc} \mathbf{1} + \\ &\quad + Y_b \left[\left(\cos^2 \left(\frac{\chi}{2} \right) - \sin^2 \left(\frac{\chi}{2} \right) \right) \frac{\sigma_b}{2} + (1 - \cos \chi) \hat{n}_b \hat{n}_c \frac{\sigma_c}{2} + \sin \chi \hat{n}_a \epsilon_{abc} \frac{\sigma_c}{2} \right], \\ T' &= \left(\hat{n}_a \hat{n}_b + (\delta_{ab} - \hat{n}_a \hat{n}_b) \cos \chi - i \hat{n}_c (I_c^{(\text{adj.})})_{ab} \sin \chi \right) Y_b \frac{\sigma_a}{2}. \end{aligned} \quad (\text{B.103})$$

Thus, transformation $T \rightarrow T' = U^{(\text{fund.})} T U^{(\text{fund.})+}$ yields the transformation in Y_a according to the adjoint representation (B.97). Note that $U^{(\text{fund.})}(\chi + 2\pi)$ produces the *same* transformation on Y_a as $U^{(\text{fund.})}(\chi)$ although $U^{(\text{fund.})}(\chi + 2\pi) \neq U^{(\text{fund.})}(\chi)$

$$\begin{aligned} U^{(\text{fund.})}(\chi + 2\pi) &= \mathbf{1} \cos \left(\frac{\chi + 2\pi}{2} \right) - i \hat{n} \cdot \vec{\sigma} \sin \left(\frac{\chi + 2\pi}{2} \right) = \\ &= -\mathbf{1} \cos \left(\frac{\chi}{2} \right) + i \hat{n} \cdot \vec{\sigma} \sin \left(\frac{\chi}{2} \right) = -U^{(\text{fund.})}(\chi). \end{aligned} \quad (\text{B.104})$$

Let us consider in detail the simple compact Lie algebra defined by the structure constants $C_{abc} = \epsilon_{abc}$. This Lie algebra is shared by two different Lie groups: $SO(3)$ and $SU(2)$. $SO(3)$ is the group of proper rotations (without reflections) in three dimensions. Its lowest representation of lowest dimensionality are 3×3 real orthogonal matrices, with determinant one. This corresponds to the adjoint representation ²⁷ $(U^{(\text{adj.})})_{bc} = \hat{n}_b \hat{n}_c (1 - \cos \chi) + \delta_{bc} \cos \chi - \hat{n}_a \epsilon_{abc} \sin \chi$, as one can easily show: firstly, the adjoint representation is formed by orthogonal matrices

$$\begin{aligned} &\left(U^{(\text{adj.})} \right)_{bc} \left(U^{(\text{adj.})T} \right)_{cd} = (U^{(\text{adj.})})_{bc} (U^{(\text{adj.})})_{dc} = \\ &= \left(\hat{n}_b \hat{n}_c (1 - \cos \chi) + \delta_{bc} \cos \chi - \hat{n}_a \epsilon_{abc} \sin \chi \right) \left(\hat{n}_d \hat{n}_c (1 - \cos \chi) + \delta_{dc} \cos \chi - \hat{n}_e \epsilon_{edc} \sin \chi \right) = \end{aligned}$$

²⁷This correspondence is better seen if we consider an infinitesimal rotation by an angle $\vec{\theta}$ of a vector $\vec{r}$ in three dimensions: $\delta \vec{r} = \vec{\theta} \times \vec{r}$ or $\delta r_i = \epsilon_{ijk} \theta_j r_k$. This is just the transformation law for a field Y_a transforming according to the adjoint representation: $\delta Y_a = -i \chi^b (I_b^{(\text{adj.})})_{ac} Y_c = -\chi^b \epsilon_{bac} Y_c = \epsilon_{abc} \chi^b Y_c$.

$$= \hat{n}_b \hat{n}_d (1 - \cos \chi)^2 + 2 \hat{n}_b \hat{n}_d \cos \chi (1 - \cos \chi) - \hat{n}_b \hat{n}_d \sin^2 \chi + \delta_{bd} \cos^2 \chi + \delta_{bd} \sin^2 \chi,$$

$$\left(U^{(\text{adj.})} \right)_{bc} \left(U^{(\text{adj.})T} \right)_{cd} = \delta_{bd}. \quad (\text{B.105})$$

And the determinant of the any transformation in the adjoint representation is one ²⁸

$$\det \left(U^{(\text{adj.})} \right) = \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} \left(U^{(\text{adj.})} \right)_{ia} \left(U^{(\text{adj.})} \right)_{jb} \left(U^{(\text{adj.})} \right)_{kc} =$$

$$= \frac{2}{3!} \left(\hat{n}_k \hat{n}_c (1 - \cos \chi) + \delta_{kc} \cos \chi - \hat{n}_a \epsilon_{akc} \sin \chi \right)^2,$$

$$\det \left(U^{(\text{adj.})} \right) = 1. \quad (\text{B.106})$$

On the other hand, the representation of lowest dimensionality for $SU(2)$ is the group of 2×2 unitary matrices with determinant unity. This representation corresponds to the fundamental representation $U^{(\text{fund.})} = \mathbf{1} \cos(\chi/2) - i \hat{n}_a \sigma_a \sin(\chi/2)$, which are formed by unitary matrices

$$U^{(\text{fund.})} U^{(\text{fund.})+} = \left(\mathbf{1} \cos \left(\frac{\chi}{2} \right) - i \hat{n}_a \sigma_a \sin \left(\frac{\chi}{2} \right) \right) \left(\mathbf{1} \cos \left(\frac{\chi}{2} \right) + i \hat{n}_b \sigma_b \sin \left(\frac{\chi}{2} \right) \right) =$$

$$= \cos^2 \left(\frac{\chi}{2} \right) \mathbf{1} + \hat{n}_a \hat{n}_b \underbrace{\sigma_a \sigma_b}_{\delta_{ab} \mathbf{1} + i \epsilon_{abc} \sigma_c} \sin^2 \left(\frac{\chi}{2} \right) = \mathbf{1}, \quad (\text{B.107})$$

whose determinant is one

$$\det \begin{pmatrix} \cos \left(\frac{\chi}{2} \right) - i \hat{n}_3 \sin \left(\frac{\chi}{2} \right) & -i \hat{n}_1 \sin \left(\frac{\chi}{2} \right) - \hat{n}_2 \sin \left(\frac{\chi}{2} \right) \\ -i \hat{n}_1 \sin \left(\frac{\chi}{2} \right) + \hat{n}_2 \sin \left(\frac{\chi}{2} \right) & \cos \left(\frac{\chi}{2} \right) + i \hat{n}_3 \sin \left(\frac{\chi}{2} \right) \end{pmatrix} = 1. \quad (\text{B.108})$$

The difference between $SU(2)$ and $SO(3)$ can be better understood by noting that from a general $SU(2)$ transformation $U = e^{-i\chi \hat{n} \cdot \vec{\sigma}/2}$, one obtains the following homomorphism ²⁹ between the elements of $SU(2)$ and those of the group of rotations in three dimensions $SO(3)$

$$f: U(\hat{n}, \chi) \in SU(2) \longrightarrow O(\hat{n}, \chi) \in SO(3). \quad (\text{B.109})$$

For any rotation matrix $O \in SO(3)$, $O(\hat{n}, \chi) = O(\hat{n}, \chi + 2\pi)$ holds. On the other hand, $SU(2)$ matrices do not fulfil such a relation. According to

$$e^{-i/2 \chi \hat{n}_a \sigma_a} = \mathbf{1} \cos \left(\frac{\chi}{2} \right) - i \hat{n}_a \sigma_a \sin \left(\frac{\chi}{2} \right) = a_0 \mathbf{1} + i a_k \sigma_k \quad \text{with} \quad \begin{cases} a_0 = \cos(\chi/2) \\ a_k = -\hat{n}_k \sin(\chi/2) \end{cases}$$

²⁸Using $\epsilon_{ajk} \epsilon_{abc} = \delta_{jb} \delta_{kc} - \delta_{jc} \delta_{kb}$, one obtains $\epsilon_{ijk} \epsilon_{abc} \left(\hat{n}_i \hat{n}_a (1 - \cos \chi) + \delta_{ia} \cos \chi - \hat{n}_d \epsilon_{dia} \sin \chi \right) \times$
 $\times \left(\hat{n}_j \hat{n}_b (1 - \cos \chi) + \delta_{jb} \cos \chi - \hat{n}_e \epsilon_{ejb} \sin \chi \right) = 2 (\delta_{kc} - \hat{n}_k \hat{n}_c) \cos \chi (1 - \cos \chi) - 2 \hat{n}_a \epsilon_{akc} \sin \chi (1 - \cos \chi) +$
 $+ 2 \delta_{kc} \cos^2 \chi + 2 \hat{n}_e \epsilon_{eck} \sin \chi \cos \chi + 2 \hat{n}_k \hat{n}_c \sin^2 \chi = 2 \left(\hat{n}_k \hat{n}_c (1 - \cos \chi) + \delta_{kc} \cos \chi - \hat{n}_a \epsilon_{akc} \sin \chi \right).$

²⁹A homomorphism is a correspondence between the elements of two groups G and R , such that if g_1 and g_2 are two arbitrary group transformations of G , and $r(g_1)$ and $r(g_2)$ are their respective images in the group R , it holds $r(g_1 g_2) = r(g_1) r(g_2)$. This correspondence need not be 1-to-1 (the image of different elements of G can be the same element in R) or onto (some elements of R can be left unassociated with any element of G). If a homomorphism is 1-to-1 and onto, is called an isomorphism.

if rotation $(\hat{n}, \chi)$ corresponds to parameters (a_0, a_1, a_2, a_3) , then rotation $(\hat{n}, \chi + 2\pi)$ corresponds to $(-a_0, -a_{k=1,2,3})$ and thus $U(\hat{n}, \chi + 2\pi) = -U(\hat{n}, \chi) \neq U(\hat{n}, \chi)$. Only rotation $(\hat{n}, \chi + 4\pi)$ is the same $SU(2)$ rotation as $(\hat{n}, \chi)$. The correspondence between $SU(2)$ and $SO(3)$ is therefore 2-to-1: $U(\hat{n}, \chi)$ and $U(\hat{n}, \chi + 2\pi) \neq U(\hat{n}, \chi)$ correspond to the same rotation matrix $O(\hat{n}, \chi)$.

This homomorphism has been already constructed in equation (B.101), where a $U(\hat{n}, \chi) \in SU(2)$ transformation on tensor $T = 1/2 Y_a \sigma_a$ induces a $O(\hat{n}, \chi) \in SO(3)$ transformation on Y_a

$$\begin{aligned} T &\rightarrow T' = U(\hat{n}, \chi) T U(\hat{n}, \chi)^\dagger \\ Y &\rightarrow Y' = O(\hat{n}, \chi) Y \end{aligned} .$$

Further,

$$U(\hat{n}, \chi + 2\pi) T U(\hat{n}, \chi + 2\pi) = U(\hat{n}, \chi) T U(\hat{n}, \chi) \iff O(\hat{n}, \chi + 2\pi) Y = O(\hat{n}, \chi) Y ,$$

although $U(\hat{n}, \chi + 2\pi) \neq U(\hat{n}, \chi)$.

The difference between $SU(2)$ and $SO(3)$ can be “visualized” by the following “experiment” due to Dirac [53]: let us hang a pencil from our table with two rubber bands, and anchor it to the floor with two rubber bands (Fig. B.4). We rotate the pencil as in the figure: the twist produced by a rotation of $\chi = 2\pi$ cannot be unravelled by deforming the rubber bands without cutting them. Only the twist by $\chi = 4\pi$ can be unravelled

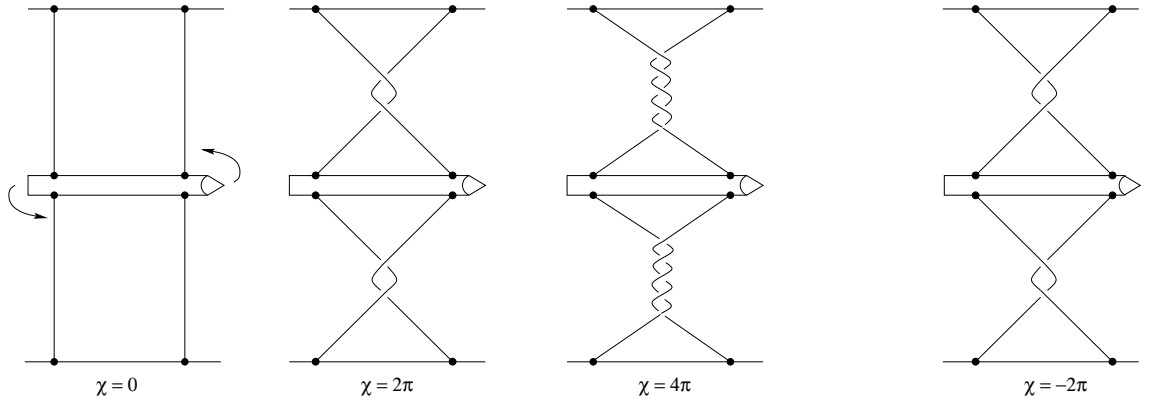


Figure B.4: Twist produced by a rotation through angle $\chi = 2\pi$, $\chi = 4\pi$ and $\chi = -2\pi$ of a pencil hung and anchored by two rubber bands.

continuously by means of the steps represented in Fig. B.5; “backpassing” means bringing the left twisted bands at the rear of the pencil behind the twisted bands on the right (similarly, for the left twisted bands in the front part of the pencil, we bring them over the bands on the right). These same steps applied on $\chi = 2\pi$ twist gives $\chi = -2\pi$ twist. If we have twisted the pencil in the contrary direction as that in Fig. B.4, the steps in Fig. B.5 have to be carried on in the inverse sequence.

Hence, only rotation $\chi = 4\pi$ is equivalent or can be deformed to the non-rotation $\chi = 0$. Group $SU(2)$ distinguishes between rotation $\chi = 0$ and $\chi = 2\pi$ ³⁰; group $SO(3)$,

³⁰The pencil fixed by the rubber bands is here the “representation” of the isospinor, i.e., the scalar field in the fundamental representation, which is only invariant under a $\chi = 4\pi$ rotation.

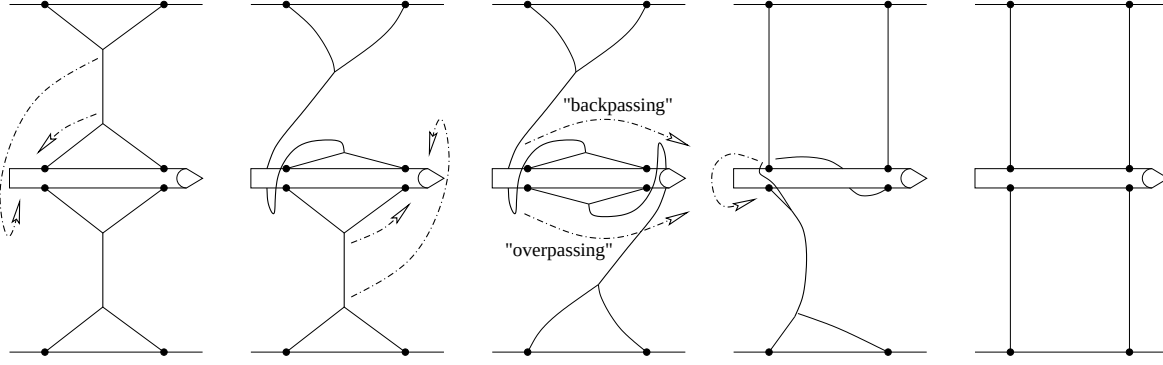


Figure B.5: Unravelling the $\chi = 4\pi$ twist by deforming the rubber bands.

nevertheless, identifies both rotations by ignoring the rubber bands. $SO(3)$ is therefore written as the coset space $SU(2)/Z_2$, where Z_2 is the additive group of the integers modulo 2. Since any $U \in SU(2)$ matrix is of the form $U = a_0 \mathbf{1} + i a_k \sigma_k$ with $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$, $SU(2)$ is parametrized by S^3 (which should be imagined as a 3-dimensional “sphere’s surface” embedded in a 4-dimensional space). On the other hand, $SO(3)$ is parametrized by the same surface *with the identification of the diametrically opposite points* (a_0, a_k) and $(-a_0, -a_k)$. This sphere with identified opposite points will be denoted by $\mathcal{S}^3$.

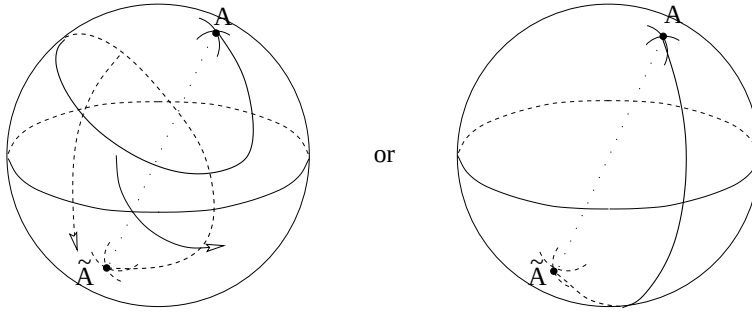


Figure B.6: Topologically stable loop on a sphere with the opposite points identified ($\tilde{A} \equiv A$).

As already seen, wrapping a rubber band S^1 around a sphere (denoted by S^2) has no topologically stable configurations, since a loop lying on a sphere can always be deformed to a point without abandoning the sphere’s surface. The same holds for higher dimensional spheres such as S^3 . Since $SU(2)$ has the same topology as S^3

$$\pi_1(SU(2)) = \pi_1(S^3) = 0. \quad (\text{B.110})$$

However, wrapping a rubber band around sphere $\mathcal{S}^3$, where opposite points are identified, does have one stable configuration. The path of Fig. B.6 beginning at A and ending at $\tilde{A}$ is a closed loop because $\tilde{A}$ is the same point as A ; but it cannot be deformed to a point since when A is moved, $\tilde{A}$ moves also, remaining opposite to A . Other loops lying on the surface, beginning at A without reaching the opposite point $\tilde{A}$ can be deformed to a point (see Fig. B.2). Hence, we obtain

$$\pi_1(SO(3)) = \pi_1(\mathcal{S}^3) = Z_2 \equiv 0, 1. \quad (\text{B.111})$$

According to this equation, if we combine two stable loops, the resulting loop can be deformed to a point. This can be shown by means of the following steps [85]: each stable loop begins at P (or Q), goes to A , “exits” at $\tilde{A}$ and return to P (or Q). The combination of both loops begins at P , goes to A , exits at $\tilde{A}$, covers the loop through Q until reaching A , exits again at $\tilde{A}$ and finishes at P (Fig. B.7). Let us deform this loop to the following loop

$$P \rightarrow A \equiv \tilde{A} \rightarrow Q \rightarrow B \equiv \tilde{B} \rightarrow P.$$

Now by approaching B to $\tilde{A}$ and $\tilde{B}$ to A , sliding on the sphere’s surface, we deform the the path $\tilde{A} \rightarrow Q \rightarrow B$ to point $\tilde{A}$ and similarly, the path $\tilde{B} \rightarrow P \rightarrow A$ to point A . At the end of this process, we only have points A and $\tilde{A}$, which on $\mathbb{S}^3$ are the same point. Of course, these results do not depend on the chosen sphere’s point A .

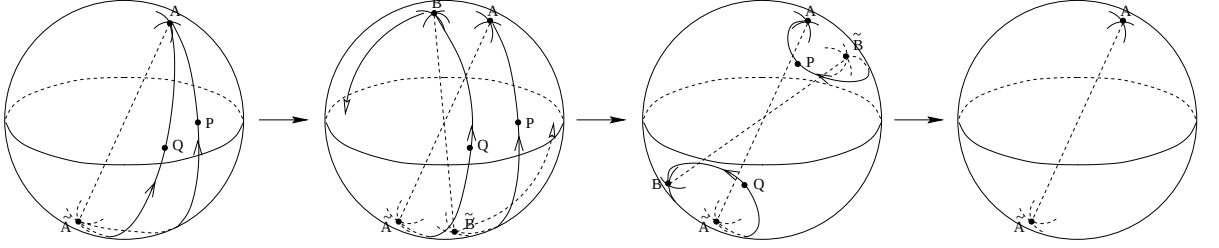
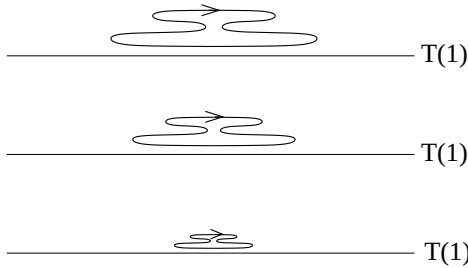


Figure B.7: The resulting loop of the reaction between two $SO(3)$ stable loops can be deformed to a point.

A Lie group G is called *simply-connected* if $\pi_1(G) = 0$ (if any loop can be shrunk to a point). And it is called *multiply-connected* otherwise. According to equations (B.110) and (B.111), group $SU(2)$ is simply-connected and $SO(3)$ is doubly-connected. One can show that, among all Lie groups sharing the same structure constants, only one is simply-connected: this group is called the *covering group*. In the case of the structure constants given by ϵ_{abc} , shared only by the Lie groups $SU(2)$ and $SO(3)$, $SU(2)$ is the covering group. Another example of Lie group, which we already know, is the Abelian group $U(1)$. The transformations of this group are of the form $U = e^{-i\chi}$ with $0 \leq \chi < 2\pi$ if acting on a complex scalar field, or equivalently, $U = e^{-i\chi I}$ with $0 \leq \chi < 2\pi$ and $I = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ if scalar field ψ is represented as $\begin{pmatrix} \text{Im } \psi \\ \text{Re } \psi \end{pmatrix}$. Due to $0 \leq \chi < 2\pi$, $U(1)$ is a compact group and has the topology of a circumference S^1 : since $\pi_1(S^1) = \mathbf{Z}$, $U(1)$ is multiply-connected. Its covering group is the real line $T(1)$

$$T(1) = \{\chi : -\infty < \chi < \infty\}. \quad (\text{B.112})$$



$T(1)$ is simply-connected, since any loop can be shrunk to a point (in the figure, the loop has been depicted over the real line, but it has to be understood as lying on it). But as different from the compact Lie group $SU(2)$, $T(1)$ is not a compact group, since it is parametrized by a real number with an infinite range.

It is finally time, we return to the study of a system described by the following Lagrangian density

$$\ell = -\frac{1}{4} \left(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e \epsilon_{abc} W_\mu^b W_\nu^c \right)^2 - \frac{1}{2} \left| \partial_\mu \Psi - i e W_\mu^a I_a \Psi \right|^2 - V(\Psi), \quad (\text{B.113})$$

invariant under any local (gauge) infinitesimal non-Abelian transformation of the kind

$$\begin{cases} \delta \Psi &= -i \chi^a I_a \Psi, \\ \delta W_\mu^a &= -\frac{1}{e} \partial_\mu \chi^a + \epsilon_{abc} \chi^b W_\mu^c, \end{cases} \quad (\text{B.114})$$

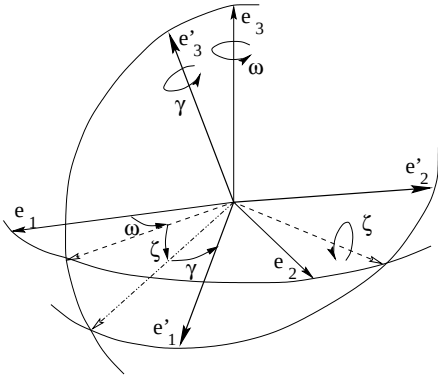
with $[I_a, I_b] = i \epsilon_{abc} I_c$. The potential energy $V(\Psi)$ must be invariant under gauge transformations and contains only nonderivative terms. The indices of the internal symmetry space $a, b, c, \dots$ should be not confused with those of the ordinary space $\mu, \nu, \lambda, \dots$. Field Ψ , in the fundamental, in the adjoint or in any other representation, is a *scalar* field in the external space. Borrowing the usual notation in Particle Physics, we shall denote the internal degrees of freedom with the prefix “iso”: thus, the internal space of symmetry will be called isospace, scalar field Ψ in the fundamental representation will be called isospinor, and Ψ in the adjoint representation, isovector Ψ^a .

Note. Isospace rotations of isovector Ψ^a .

Any $SO(3)$ gauge transformation acting on isovector Ψ^a produces a change in its orientation in isospace. A general rotation in three dimensions is parametrized by the three Euler angles (ω, ζ, γ) : angles $0 \leq \omega < 2\pi$ and $0 \leq \zeta < \pi$ take the axis system to the new rotation plane, and angle $0 \leq \gamma < 2\pi$ rotates the system about the new third axis e'_3 . In general, all these angles change from point to point $\vec{r}$ of the ordinary (external) space. Let us write a general rotation on a isovector Ψ^a by means of the more manageable tensor notation (B.101): $T = \frac{1}{2} \Psi^a \sigma_a \rightarrow T' = \frac{1}{2} \Psi'^a \sigma_a = \Omega(\omega, \zeta, \gamma) T \Omega^\dagger(\omega, \zeta, \gamma)$, with

$$\Omega(\chi) = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i \hat{n}_a \sigma_a \sin\left(\frac{\chi}{2}\right) = \begin{pmatrix} \cos(\chi/2) - i \hat{n}_3 \sin(\chi/2) & (-i \hat{n}_1 - \hat{n}_2) \sin(\chi/2) \\ (-i \hat{n}_1 + \hat{n}_2) \sin(\chi/2) & \cos(\chi/2) + i \hat{n}_3 \sin(\chi/2) \end{pmatrix}, \quad (\text{B.115})$$

with χ the angle and the unit vector $\hat{n}$ the rotation axis.



- First rotation Ω_1 with angle $\chi = \omega$ and rotation axis $\hat{n} = (0, 0, 1)$ (components referred to the fixed axis system $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$)

$$\Omega_1 = \begin{pmatrix} e^{-i\omega/2} & 0 \\ 0 & e^{i\omega/2} \end{pmatrix}. \quad (\text{B.116})$$

- Second rotation Ω_2 with angle $\chi = \zeta$ and rotation axis $\hat{n} = (-\sin \omega, \cos \omega, 0)$

$$\Omega_2 = \begin{pmatrix} \cos(\zeta/2) & -e^{-i\omega} \sin(\zeta/2) \\ e^{i\omega} \sin(\zeta/2) & \cos(\zeta/2) \end{pmatrix}. \quad (\text{B.117})$$

- Third rotation Ω_3 with angle $\chi = \gamma$ and rotation axis $\hat{n} = (\sin \zeta \cos \omega, \sin \zeta \sin \omega, \cos \zeta)$

$$\Omega_3 = \begin{pmatrix} \cos(\gamma/2) - i \cos \zeta \sin(\gamma/2) & -i e^{-i\omega} \sin \zeta \sin(\gamma/2) \\ -i e^{i\omega} \sin \zeta \sin(\gamma/2) & \cos(\gamma/2) + i \cos \zeta \sin(\gamma/2) \end{pmatrix}. \quad (\text{B.118})$$

After operating, the general rotation matrix $\Omega(\omega, \zeta, \gamma) = \Omega_3(\gamma)\Omega_2(\zeta)\Omega_1(\omega)$ takes the following form

$$\begin{pmatrix} \cos(\gamma/2) - i \cos \zeta \sin(\gamma/2) & -ie^{-i\omega} \sin \zeta \sin(\gamma/2) \\ -ie^{i\omega} \sin \zeta \sin(\gamma/2) & \cos(\gamma/2) + i \cos \zeta \sin(\gamma/2) \end{pmatrix} \begin{pmatrix} e^{-i\omega/2} \cos(\zeta/2) & -e^{-i\omega/2} \sin(\zeta/2) \\ e^{i\omega/2} \sin(\zeta/2) & e^{i\omega/2} \cos(\zeta/2) \end{pmatrix} = \\ = \begin{pmatrix} e^{-i\omega/2} e^{-i\gamma/2} \cos(\zeta/2) & -e^{-i\omega/2} e^{i\gamma/2} \sin(\zeta/2) \\ e^{i\omega/2} e^{-i\gamma/2} \sin(\zeta/2) & e^{i\omega/2} e^{i\gamma/2} \cos(\zeta/2) \end{pmatrix} = \Omega(\omega, \zeta, \gamma). \quad (\text{B.119})$$

If we prefer to work with the scalar field written as isovector $\Psi^a = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}$ instead of the tensor notation $T = \frac{1}{2} \begin{pmatrix} \Psi^3 & \Psi^1 - i\Psi^2 \\ \Psi^1 + i\Psi^2 & -\Psi^3 \end{pmatrix}$, the general gauge rotation is given by the following 3×3 orthogonal real matrix with determinant one ³¹

$$\begin{pmatrix} \cos^2(\zeta/2) \cos(\omega + \gamma) - \sin^2(\zeta/2) \cos(\omega - \gamma) & -\cos^2(\zeta/2) \sin(\omega + \gamma) - \sin^2(\zeta/2) \sin(\omega - \gamma) & \sin \zeta \cos \omega \\ \cos^2(\zeta/2) \sin(\omega + \gamma) - \sin^2(\zeta/2) \sin(\omega - \gamma) & \cos^2(\zeta/2) \cos(\omega + \gamma) + \sin^2(\zeta/2) \cos(\omega - \gamma) & \sin \zeta \sin \omega \\ -\sin \zeta \cos \gamma & \sin \zeta \sin \gamma & \cos \zeta \end{pmatrix} = \\ = \begin{pmatrix} \cos \zeta \cos \omega \cos \gamma - \sin \omega \sin \gamma & -\cos \zeta \cos \omega \sin \gamma - \sin \omega \cos \gamma & \sin \zeta \cos \omega \\ \cos \zeta \sin \omega \cos \gamma + \cos \omega \sin \gamma & -\cos \zeta \sin \omega \sin \gamma + \cos \omega \cos \gamma & \sin \zeta \sin \omega \\ -\sin \zeta \cos \gamma & \sin \zeta \sin \gamma & \cos \zeta \end{pmatrix}. \quad (\text{B.120})$$

As for isovector Ψ^a , for gauge field W_μ^a we define the traceless 2×2 tensor $\mathcal{W}_\mu = \frac{1}{2} W_\mu^a \sigma_a$. Looking at the transformation law for the gauge field (B.114), $W_\mu'^a I_a^{(r)} = W_\mu^a U I_a^{(r)} U^\dagger - \frac{i}{e} (\partial_\mu U) U^\dagger$, tensor $\mathcal{W}_\mu$ transforms as follows under isorotation $\Omega(\omega, \zeta, \gamma)$

$$\mathcal{W}_\mu \rightarrow \mathcal{W}_\mu' = \Omega(\omega, \zeta, \gamma) \mathcal{W}_\mu \Omega^\dagger(\omega, \zeta, \gamma) - \frac{i}{e} \left(\partial_\mu \Omega(\omega, \zeta, \gamma) \right) \Omega^\dagger(\omega, \zeta, \gamma). \quad (\text{B.121})$$

The first contribution $\Omega(\omega, \zeta, \gamma) \frac{1}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \Omega^\dagger(\omega, \zeta, \gamma)$ is due to the non-Abelian character of the gauge group (i.e. Ω does not commute with the Pauli matrices) and is equal to

$$\begin{pmatrix} e^{-i\omega/2} e^{-i\gamma/2} \cos(\zeta/2) & -e^{-i\omega/2} e^{i\gamma/2} \sin(\zeta/2) \\ e^{i\omega/2} e^{-i\gamma/2} \sin(\zeta/2) & e^{i\omega/2} e^{i\gamma/2} \cos(\zeta/2) \end{pmatrix} \mathcal{W}_\mu \begin{pmatrix} e^{i\omega/2} e^{i\gamma/2} \cos(\zeta/2) & e^{-i\omega/2} e^{i\gamma/2} \sin(\zeta/2) \\ -e^{i\omega/2} e^{-i\gamma/2} \sin(\zeta/2) & e^{-i\omega/2} e^{-i\gamma/2} \cos(\zeta/2) \end{pmatrix} = \\ = \frac{1}{2} \begin{pmatrix} W_\mu^1 (-\sin \zeta \cos \gamma) + \\ + W_\mu^2 (\sin \zeta \sin \gamma) + \\ + W_\mu^3 (\cos \zeta) & W_\mu^1 (e^{-i\omega} e^{-i\gamma} \cos^2(\zeta/2) - e^{-i\omega} e^{i\gamma} \sin^2(\zeta/2)) + \\ + W_\mu^2 (-ie^{-i\omega} e^{-i\gamma} \cos^2(\zeta/2) - ie^{-i\omega} e^{i\gamma} \sin^2(\zeta/2)) + \\ + W_\mu^3 (e^{-i\omega} \sin \zeta) \\ W_\mu^1 (e^{i\omega} e^{i\gamma} \cos^2(\zeta/2) - e^{i\omega} e^{-i\gamma} \sin^2(\zeta/2)) + \\ + W_\mu^2 (ie^{i\omega} e^{i\gamma} \cos^2(\zeta/2) + ie^{i\omega} e^{-i\gamma} \sin^2(\zeta/2)) + \\ + W_\mu^3 (e^{i\omega} \sin \zeta) & W_\mu^1 (\sin \zeta \cos \gamma) + \\ + W_\mu^2 (-\sin \zeta \sin \gamma) + \\ + W_\mu^3 (-\cos \zeta) \end{pmatrix}. \quad (\text{B.122})$$

In addition to this non-Abelian contribution, which as we shall see corresponds to a rotation of the vector field within the internal space, the term $-\frac{i}{e} \left(\partial_\mu \Omega(\omega, \zeta, \gamma) \right) \Omega^\dagger(\omega, \zeta, \gamma)$ is given by

$$-\frac{i}{e} \left(\partial_\mu \Omega \right) \Omega^\dagger = -\frac{1}{2e} \begin{pmatrix} \partial_\mu \omega + \cos \zeta \partial_\mu \gamma & (\sin \zeta \cos \omega \partial_\mu \gamma - \sin \omega \partial_\mu \zeta) - \\ -i (\sin \zeta \sin \omega \partial_\mu \gamma + \cos \omega \partial_\mu \zeta) & \\ (\sin \zeta \cos \omega \partial_\mu \gamma - \sin \omega \partial_\mu \zeta) + & -\partial_\mu \omega - \cos \zeta \partial_\mu \gamma \\ +i (\sin \zeta \sin \omega \partial_\mu \gamma + \cos \omega \partial_\mu \zeta) & \end{pmatrix}. \quad (\text{B.123})$$

³¹This matrix can be directly obtained by comparing $\Omega T \Omega^\dagger$ with $\frac{1}{2} \begin{pmatrix} \Psi'^3 & \Psi'^1 - i\Psi'^2 \\ \Psi'^1 + i\Psi'^2 & -\Psi'^3 \end{pmatrix}$. After carrying out the matrix product, one obtains $\Omega T \Omega^\dagger = \frac{1}{2} \begin{pmatrix} \alpha & \beta^* \\ \beta & -\alpha \end{pmatrix}$ with $\alpha = -\sin \zeta \cos \gamma \Psi^1 + \sin \zeta \sin \gamma \Psi^2 + \cos \zeta \Psi^3$ and $\beta = \cos^2(\zeta/2) e^{i(\omega+\gamma)} (\Psi^1 + i\Psi^2) - \sin^2(\zeta/2) e^{i(\omega-\gamma)} (\Psi^1 - i\Psi^2) + \sin \zeta e^{i\omega} \Psi^3$.

Comparing results (B.122) and (B.123) with the form of tensor $\mathcal{W}_\mu = \frac{1}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}$, we obtain the transformation of gauge field W_μ^a

$$\left\{ \begin{array}{l} W_\mu^1 \rightarrow W'^1_\mu = W_\mu^1 (\cos \zeta \cos \omega \cos \gamma - \sin \omega \sin \gamma) + W_\mu^2 (-\cos \zeta \cos \omega \sin \gamma - \sin \omega \cos \gamma) + \\ \quad + W_\mu^3 \sin \zeta \cos \omega - \frac{1}{e} (\sin \zeta \cos \omega \partial_\mu \gamma - \sin \omega \partial_\mu \zeta), \\ W_\mu^2 \rightarrow W'^2_\mu = W_\mu^1 (\cos \zeta \sin \omega \cos \gamma + \cos \omega \sin \gamma) + W_\mu^2 (-\cos \zeta \sin \omega \sin \gamma + \cos \omega \cos \gamma) + \\ \quad + W_\mu^3 \sin \zeta \sin \omega - \frac{1}{e} (\sin \zeta \sin \omega \partial_\mu \gamma + \cos \omega \partial_\mu \zeta), \\ W_\mu^3 \rightarrow W'^3_\mu = -W_\mu^1 \sin \zeta \cos \gamma + W_\mu^2 \sin \zeta \sin \gamma + W_\mu^3 \cos \zeta - \frac{1}{e} (\partial_\mu \omega + \cos \zeta \partial_\mu \gamma). \end{array} \right. \quad (\text{B.124})$$

Note that, as already advanced after result (B.122), the part of the transformed gauge field which does not contain the gradients of the angles (the part not proportional to $1/e$) is actually that resulting from a rotation in the isospace of the isovector character (index a) of the gauge field, as one can easily see by considering the rotation matrix (B.120) for an isovector.

The parameters of potential V in Lagrangian density (B.113) are chosen in such a way that its minimum is reached for a constant nonzero scalar field. Of course, there are more than one lowest energy state because of the invariance under any local transformation g of kind (B.114): if Ψ_0 is a lowest energy state, any $g\Psi_0$ is also a lowest energy state. Let G be the group of local transformations, which leaves invariant the Lagrangian, and H the subgroup $H \subset G$ containing all transformations under which each potential's minimum remains invariant. We want to know if the system contains topologically stable line defects. Or stated in a more mathematically form, if G/H contains more configurations than those deformable to a point, if $\pi_1(G/H)$ contains more elements than 0.

Let us follow the same strategy as in the case of the Abelian $U(1)$ gauge symmetry: in that case, by means of the potential $V(\psi)$ we produced a nonzero ψ_0 which breaks the $U(1)$ symmetry completely. Complete breakdown means that ψ_0 remains invariant only under the identity: $H = 1$ and $G/H = G = U(1)$. Since $U(1)$ has the topology of the circumference S^1 and $\pi_1(G/H) = \pi_1(S^1) = \mathbf{Z}$, the system contains stable line defects. In the case we are studying now, we break the symmetry G completely with a scalar field Ψ or a set of scalar fields $\Psi_1, \Psi_2, \dots$, such that $G/H = G$. But for a Lagrangian invariant under all transformations of groups with structure constants ϵ_{abc} , G can be $SU(2)$ or $SO(3)$, depending on the representation chosen for scalar field Ψ . For the fundamental representation, G is $SU(2)$ and therefore there are no line defects since $\pi_1(G/H) = \pi_1(SU(2)) = 0$. On the other hand, for the adjoint representation the Lagrangian's symmetry is $SO(3)$, which fulfils $\pi_1(G/H) = \pi_1(SO(3)) = 0, 1$, and there is (only) one set of topological line defects. Moreover, the reaction of two of these line defects is not stable.

B.5 Scalar field coupled to an $SO(3)$ gauge field: the “ $SO(3)$ ”-superconductor

Let us study Lagrangian density (B.113) in the adjoint representation

$$\ell = -\frac{1}{4} \left(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \epsilon \epsilon_{abc} W_\mu^b W_\nu^c \right)^2 - \frac{1}{2} \left(\partial_\mu \Psi^a + \epsilon \epsilon_{abc} W_\mu^b \Psi^c \right)^2 - V(\Psi^a), \quad (\text{B.125})$$

which has thus $SO(3)$ as symmetry group, with the potential given by

$$V(\Psi^a) = -|\mathcal{A}| \Psi^a \Psi^a + \frac{\mathcal{B}}{2} (\Psi^a \Psi^a)^2 \quad \text{with } \mathcal{B} > 0. \quad (\text{B.126})$$

Any isovector Ψ^a of length $\sqrt{|\mathcal{A}|/\mathcal{B}}$ is a lowest energy state. Such a isovector can point in any direction in the 3-dimensional isospace. Let Θ and Φ denote the polar angles in the isospace, then the lowest energy states are given by³²

$$\Psi_0^a \equiv \vec{\Psi}_0 = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix} \quad \text{with } 0 \leq \Theta < \pi \quad \text{and} \quad 0 \leq \Phi < 2\pi, \quad (\text{B.127})$$

which, as in (B.101), can be also represented by the 2×2 traceless tensor $T_0 = \frac{1}{2} \Psi_0^a \sigma_a$

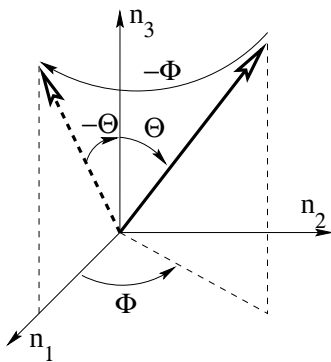
$$T_0 = \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix}. \quad (\text{B.128})$$

The invariant subgroup H is $U_{\vec{\Psi}_0}(1)$, the rotations in the isospace about $\vec{\Psi}_0$. Considering $\Omega(\gamma) = \mathbf{1} \cos(\gamma/2) - i \hat{n}_a \sigma_a \sin(\gamma/2)$, these rotations about scalar field $\vec{\Psi}_0$, which is pointing in iso-direction $\hat{n} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)$, are represented by

$$\Omega_{U(1)}(\gamma) = \begin{pmatrix} \cos(\gamma/2) - i \cos \Theta \sin(\gamma/2) & -i e^{-i\Phi} \sin \Theta \sin(\gamma/2) \\ -i e^{i\Phi} \sin \Theta \sin(\gamma/2) & \cos(\gamma/2) + i \cos \Theta \sin(\gamma/2) \end{pmatrix}. \quad (\text{B.129})$$

The set of different lowest energy states, $SO(3)/U_{\vec{\Psi}_0}(1)$, corresponds to all possible directions taken by the isovector in the 3-dimensional isospace and is thus parametrized by the (usual) 2-dimensional sphere's surface

$$SO(3)/U_{\vec{\Psi}_0}(1) = S^2. \quad (\text{B.130})$$



Let us choose Ψ^a pointing, for instance, in the $\hat{z}$ direction in isospace

$$\Psi_{0,z}^a = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\text{B.131})$$

or equivalently

$$T_{0,z} = \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

³²Similar to the Abelian $U(1)$ case, where ψ_0 was given by $\sqrt{|\alpha|/|\beta|} e^{i\varphi} \equiv \sqrt{|\alpha|/|\beta|} \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$ with $0 \leq \varphi < 2\pi$ the angle in the internal (2-dimensional) symmetry space.

This can be obtained from $\vec{\Psi}_0 = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}$ by means of the following gauge transformation: a rotation in isospace around the axis $\hat{n}_3$ through an angle $\chi = -\Phi$, followed by a rotation about $\hat{n}_2$ with $\chi = -\Theta$

$$O(\Phi, \Theta) = e^{-i(-\Theta)I_2^{(\text{adj.})}} e^{-i(-\Phi)I_3^{(\text{adj.})}} = \begin{pmatrix} \cos \Theta \cos \Phi & \cos \Theta \sin \Phi & -\sin \Theta \\ -\sin \Phi & \cos \Phi & 0 \\ \sin \Theta \cos \Phi & \sin \Theta \sin \Phi & \cos \Theta \end{pmatrix}. \quad (\text{B.132})$$

Or if the notation with tensor $T_0 = \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix}$ is preferred ³³

$$\begin{aligned} \Omega(\Phi, \Theta) &= \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} = \\ &= \begin{pmatrix} e^{i\Phi/2} \cos(\Theta/2) & e^{-i\Phi/2} \sin(\Theta/2) \\ -e^{i\Phi/2} \sin(\Theta/2) & e^{-i\Phi/2} \cos(\Theta/2) \end{pmatrix}. \end{aligned} \quad (\text{B.133})$$

After this gauge rotation, the model described by the above Lagrangian contains a long-range $U(1)$ gauge field given by $W_\mu^{a=3}$. In addition to this long-range gauge field, W_μ^1 and W_μ^2 are short-range. A short-range gauge field is defined taking as model the London equation of superconductivity: magnetic potential $\vec{A}$ and field $\vec{B}$ in Maxwell theory become short-range if the electric current $\vec{j}$, which creates the magnetic field, is given by $\vec{j}_{\text{London}} = -\frac{1}{\lambda^2} \vec{A}$. The length λ is called penetration length or range. After substituting London equation into the Ampere law, one obtains

$$\text{curl } \vec{B} = \text{grad} (\text{div } \vec{A}) - \Delta \vec{A} = -\frac{1}{\lambda^2} \vec{A} \quad \text{or} \quad \partial_\mu \partial_\mu A_\nu - \frac{1}{\lambda^2} A_\nu - (\partial_\nu \partial_\mu A_\mu) = 0,$$

with $\mu, \nu = 1, 2, 3$. The Euler-Lagrange equation for gauge field W_μ^a is equal to ³⁴ $\partial_\mu G_{\mu\nu}^a + e\epsilon_{abc} W_\mu^b G_{\mu\nu}^c = e\epsilon_{abc} \Psi^b (\partial_\nu \Psi^c + e\epsilon_{cde} W_\nu^d \Psi^e)$ with $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e\epsilon_{abc} W_\mu^b W_\nu^c$. The minimum in energy is obtained for a Ψ_0^a , which minimizes the potential (in our case, $\Psi_0^a = \delta_{a3} \sqrt{|\mathcal{A}|/\mathcal{B}}$), and all gauge fields equal to zero. Starting from this configuration,

$$\begin{aligned} &^{33}\text{Simply by calculating } \Omega(\Phi, \Theta) \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix} \Omega^+(\Phi, \Theta): \\ &\begin{pmatrix} e^{i\Phi/2} \cos(\Theta/2) & e^{-i\Phi/2} \sin(\Theta/2) \\ -e^{i\Phi/2} \sin(\Theta/2) & e^{-i\Phi/2} \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix} \begin{pmatrix} e^{-i\Phi/2} \cos(\Theta/2) & -e^{-i\Phi/2} \sin(\Theta/2) \\ e^{i\Phi/2} \sin(\Theta/2) & e^{i\Phi/2} \cos(\Theta/2) \end{pmatrix} = \\ &= \begin{pmatrix} e^{i\Phi/2} \cos(\Theta/2) & e^{-i\Phi/2} \sin(\Theta/2) \\ e^{i\Phi/2} \sin(\Theta/2) & -e^{-i\Phi/2} \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{-i\Phi/2} \cos(\Theta/2) & -e^{-i\Phi/2} \sin(\Theta/2) \\ e^{i\Phi/2} \sin(\Theta/2) & e^{i\Phi/2} \cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

³⁴From Lagrangian density (B.125), it follows

$$\frac{\partial \ell}{\partial (\partial_\mu W_\nu^a)} = -G_{\mu\nu}^a; \quad \frac{\partial \ell}{\partial W_\nu^a} = -G_{\nu\mu}^b e\epsilon_{bac} W_\mu^c - (D_\nu \Psi^c) e\epsilon_{cab} \Psi^b = e\epsilon_{abc} W_\mu^b G_{\mu\nu}^c - e\epsilon_{abc} \Psi^b D_\nu \Psi^c.$$

solutions whose energies are near to the minimum, are obtained by solving the Euler-Lagrange equations for $\Psi^a = \delta_{a3}\sqrt{|\mathcal{A}|/\mathcal{B}} + \eta^a$ and $W_\mu^a \neq 0$ with η^a and W_μ^a small. Neglecting powers in η^a or W_μ^a higher than the first, we obtain for the gauge field

$$\partial_\mu \partial_\mu W_\nu^a - (\partial_\nu \partial_\mu W_\mu^a) = e^2 \frac{|\mathcal{A}|}{\mathcal{B}} (W_\nu^a - \delta_{a3} W_\nu^3). \quad (\text{B.134})$$

Therefore, W_μ^3 is a infinite long-range field, whereas $W_\mu^{a=1,2}$ have a finite range equal to $\xi_w = \sqrt{\mathcal{B}/e^2|\mathcal{A}|}$.

Under a gauge transformation of those still available (i.e. rotations about $\hat{z}$ in isospace with $\chi^b = \chi^{b=3} = \gamma$), gauge field W_μ^3 transforms as one could expect for an (Abelian) $U(1)$ gauge field

$$\delta A_\mu = \delta W_\mu^3 = -\frac{1}{e} \partial_\mu \gamma + \epsilon_{33d} \gamma W_\mu^d = -\frac{1}{e} \partial_\mu \gamma \quad \text{with} \quad 0 \leq \gamma < 2\pi. \quad (\text{B.135})$$

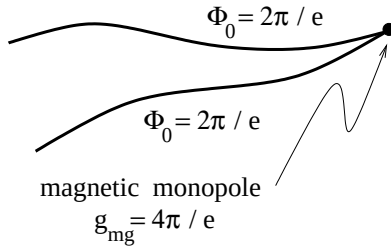
And now, let us couple this $U(1)$ gauge field $A_\mu \equiv W_\mu^3$ to a scalar field ψ as in an ordinary superconductor (or as in our model describing dislocations)

$$\ell' = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} |\partial_\mu \psi - ie A_\mu \psi|^2 - V'(\psi). \quad (\text{B.136})$$

To couple only the long-range field to ψ is legitime as long as we are studying the model at distaces (much) larger than the range ξ_w of the short-range fields. The coupling constant e between W_μ^a and Ψ^a in (B.125), and between A_μ and ψ in (B.136), is the same. If the coupling between A_μ and ψ were different, e.g. e' , then equations (B.135) and (B.136) would imply that the gauge transformations allowed for ψ would have to be $\psi \rightarrow \psi' = e^{-i\chi} \psi$ with $\chi = \frac{e}{e'} \gamma$ and therefore $0 \leq \chi < e/e' 2\pi$. However, the fact of $\chi < e/e' 2\pi \neq 2\pi$ means that this is not the phase of a $U(1)$ transformation.

If in Lagrangian density (B.136), ψ is chosen such that the residual gauge symmetry $U(1)$ is broken down³⁵, we expect topologically stable line defects, as in an ordinary

superconductor. However, since the underlying symmetry for this superconductor is $SO(3)$ and not $U(1)$, two stable vortices carrying one unit of magnetic flux annihilate each other at the point where they meet. Magnetic flux is conserved and therefore, a magnetic monopole g_{mg} is formed at the point where the two vortices annihilate, absorbing the magnetic flux of both vortices.



Note. Magnetic monopoles [11].

The magnetic charge g_{mg} or magnetic monopole is defined by the equation

$$\text{div } \vec{B} = \varrho_{\text{mg}} \quad \text{and} \quad \oint_S \vec{B} \cdot d\vec{S} = g_{\text{mg}}. \quad (\text{B.137})$$

Let us return to the model with a scalar field coupled to a $U(1)$ gauge field

$$\ell = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} |\partial_\mu \psi - ie A_\mu \psi|^2 - V(|\psi|). \quad (\text{B.138})$$

From the definition of gauge invariant tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, magnetic field $B_\lambda = \frac{1}{2}\epsilon_{\lambda\nu\mu}F_{\mu\nu}$ is divergenceless. A monopole in this model cannot exist as an isolated object and must be visualized as the end point of an infinitely thin solenoid carrying a magnetic flux Φ_0

$$\oint_C A_\mu dl^\mu = \Phi_0, \quad (\text{B.139})$$

where C is any closed path around the coil. Let us suppose we isolate the solenoid's end, i.e., the magnetic monopole. The magnetic flux crossing a closed surface S surrounding it, is equal to

$$g_{\text{mg}} = \oint_S \vec{B} \cdot d\vec{S} = \Phi_0, \quad (\text{B.140})$$

and thus flux Φ_0 “injected” by the solenoid is equal to the magnetic monopole sitting at the solenoid's end.

We have introduced the solenoid as an “artifact” to describe a magnetic monopole: we assume that the effect on the coil's surroundings is created only by the end point. Let us study a point close to this solenoid, where A_μ is very large (since C and dl^μ are very small). If we have constructed a good “artifact”, we can remove this large A_μ by means of a gauge transformation

$$\begin{cases} \psi & \rightarrow \psi' = e^{-i\Lambda}\psi, \\ A_\mu & \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\Lambda = 0, \end{cases} \quad (\text{B.141})$$

such that the effect by the coil arises only from the monopole at its end. Field Λ , which defines the above transformation, is multi-valued

$$\oint_C A'_\mu dl^\mu = 0 = \oint_C A_\mu dl^\mu - \frac{1}{e} \oint_C \partial_\mu \Lambda dl^\mu = \Phi_0 - \frac{1}{e} [\Lambda_{\text{end of } C} - \Lambda_{\text{begin of } C}],$$

and therefore

$$[\Lambda_{\text{end of } C} - \Lambda_{\text{begin of } C}] = e\Phi_0. \quad (\text{B.142})$$

However, we require that scalar field ψ remains single-valued under such a transformation, i.e., scalar field ψ does not “see” the coil

$$\psi' = e^{-i\Lambda_{\text{end of } C}}\psi = e^{-i\Lambda_{\text{begin of } C}}\psi,$$

which produces

$$\Lambda_{\text{end of } C} - \Lambda_{\text{begin of } C} = n2\pi, \quad (\text{B.143})$$

with n an integer. Therefore, the (singular) gauge transformation connects single-valued configurations if the magnetic flux fulfils

$$\Phi_0 = n \frac{2\pi}{e}. \quad (\text{B.144})$$

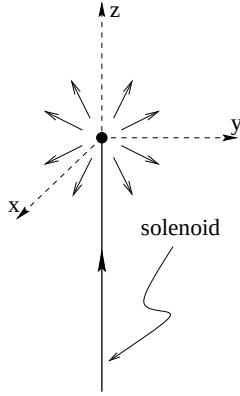
Monopole charge $g_{\text{mg}} = \Phi_0$ and coupling constant e are related to each other by

$$g_{\text{mg}}e = n2\pi. \quad (\text{B.145})$$

This relation was firstly formulated by Dirac and it is named after him. The thin solenoid, which we have used as a (necessary) trick to introduce monopoles in the $U(1)$ model, is called the Dirac string.

Note that g_{mg} is equal to the flux of a vortex in a superconductor (or with the notation of Appendix A, $\Phi_0 = 2\pi\hbar c/qe$). Hence, where a vortex ends inside a superconductor, a monopole sits. However, the origin of result (B.144) is different for the superconductor and for the Dirac string: in the latter the gauge potential is very large near to the coil and the flux quantization arises from the single-valueness

of scalar field ψ . On the contrary, A_μ is finite at the vortex center in the superconductor and the flux quantization is obtained from the energy finiteness at large distances which constrains the vanishing of the covariant derivative. The Dirac string is only an artifact whereas a vortex is a real object.



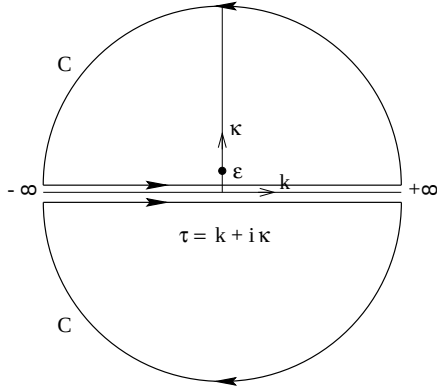
If we want to obtain an explicit result, let us consider a thin straight solenoid located in the $-z$ -axis (in ordinary space), carrying a magnetic field oriented in $\hat{z}$ -direction with flux Φ_0 . In this case, the magnetic field is given by

$$\vec{B} = \frac{g_{\text{mg}}}{4\pi} \frac{\hat{r}}{r^2} + \hat{z} \Phi_0 \delta(x) \delta(y) \vartheta(-z), \quad (\text{B.146})$$

where $\hat{r}$ and $\hat{z}$ are the unit vectors in the radial direction and in the z -direction respectively, $\delta(x)$ the Dirac delta distribution and $\vartheta(x)$ the Heaviside distribution (also called step function). The term $\frac{g_{\text{mg}}}{4\pi} \frac{\hat{r}}{r^2}$ is the magnetic field due to a monopole (similar to the electric field created by a point charge), and $\hat{z} \Phi_0 \delta(x) \delta(y) \vartheta(-z)$ is the field produced by the solenoid. The function ϑ is defined by

$$\vartheta(x) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k - i\epsilon} dk = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}. \quad (\text{B.147})$$

The last equality can be easily proved by means of the Cauchy theorem applied to the following integral along the closed path C of the following figure



$$\frac{1}{2\pi i} \oint_C \frac{e^{i\tau x}}{\tau - i\epsilon} d\tau, \quad \tau = k + i\kappa, \quad i\tau x = ikx - \kappa x$$

where the limit $\epsilon \rightarrow 0^+$ must be taken. The closed path C is chosen with the upper circumference ($\kappa > 0$) for $x > 0$, or with the lower circumference ($\kappa < 0$) for $x < 0$, since the integrand vanishes on the semicircumference for each respective case. Therefore, it only remains the integral along the x axis. The integral along the closed path C does not vanish if it surrounds the pole $\kappa = \epsilon$

$$\vartheta(x) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k - i\epsilon} dk = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \oint_C \frac{e^{i\tau x}}{\tau - i\epsilon} d\tau = \begin{cases} e^{-\epsilon} = 1 & x > 0 \\ 0 & x < 0 \end{cases}. \quad (\text{B.148})$$

Further, the step function fulfils the following important relation

$$\frac{d}{dx} \vartheta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk = \delta(x). \quad (\text{B.149})$$

Applying this relation to (B.146) and noting that $\frac{\hat{r}}{r^2} = -\vec{\nabla} \left(\frac{1}{r} \right)$, one obtains

$$\text{div } \vec{B} = -g_{\text{mg}} \nabla^2 \left(\frac{1}{4\pi r} \right) - \Phi_0 \delta(x) \delta(y) \underbrace{\left(\frac{d}{d(-z)} \vartheta(-z) \right)}_{\delta(-z) = \delta(z)} = (g_{\text{mg}} - \Phi_0) \delta^3(\vec{r}) = 0. \quad (\text{B.150})$$

In the last equation, Green's function $D(r) = 1/4\pi r$ of the 3-dimensional Laplace operator ($\nabla^2 D(r) = -\delta^3(\vec{r})$) have been used

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} D(r) = -\delta^3(\vec{r}) = -\frac{\delta(r)}{4\pi r^2} = -\frac{1}{4\pi r^2} \frac{d}{dr} r \quad \Rightarrow \quad \frac{d}{dr} D(r) = -\frac{1}{4\pi r^2}. \quad (\text{B.151})$$

The above magnetic field is divergenceless and therefore can be obtained as a curl of the following gauge potential A_μ [16]

$$A_\mu = \frac{g_{\text{mg}}}{4\pi}(1 - \cos \theta)\partial_\mu \phi, \quad (\text{B.152})$$

with $\mu = 1, 2, 3$ and $\{\theta, \phi\}$ the polar angles in the 3-dimensional space: $\theta = \arctan(\sqrt{x^2 + y^2}/z)$ and $\phi = \arctan(y/x)$. Gauge potential $A_\mu^{\text{sing}} = \partial_\mu \phi$ or $\vec{A}^{\text{sing}} = \hat{\phi} \frac{1}{\rho}$, with $\rho = \sqrt{x^2 + y^2}$ and $\hat{\phi}$ the unit vector in the ϕ -direction, produces a magnetic field confined to the z -axis

$$\vec{A}^{\text{sing}} = \hat{\phi} \frac{1}{\rho} = \hat{\phi} \frac{d}{d\rho} \ln \rho \quad \implies \quad \vec{B}^{\text{sing}} = \hat{z} \frac{1}{\rho} \frac{d}{d\rho} (\rho A_\phi^{\text{sing}}) = \hat{z} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \ln \rho \right) \right].$$

The term between brackets [] is the 2-dimensional Laplace operator applied to $\ln \rho$. However, $-\frac{1}{2\pi} \ln \rho$ is just the Green's function of the Laplace operator in two dimensions³⁶ and therefore

$$\vec{B}^{\text{sing}} = \hat{z} 2\pi \delta(x) \delta(y), \quad (\text{B.153})$$

a result which we already have met in equation (B.34). Substituting this result into (B.152), we obtain

$$B_\lambda = \frac{g_{\text{mg}}}{4\pi} \epsilon_{\lambda\nu\mu} \partial_\nu \left((1 - \cos \theta) \partial_\mu \phi \right) = \frac{g_{\text{mg}}}{4\pi} \epsilon_{\lambda\nu\mu} \sin \theta (\partial_\nu \theta) (\partial_\mu \phi) + \frac{g_{\text{mg}}}{4\pi} \delta_{\lambda 3} (1 - \cos \theta) 2\pi \delta(x) \delta(y),$$

$$\vec{B} = \frac{g_{\text{mg}}}{4\pi} \frac{\hat{r}}{r^2} + \frac{g_{\text{mg}}}{2} \hat{z} (1 - \cos \theta) \delta(x) \delta(y). \quad (\text{B.154})$$

The term $(1 - \cos \theta) \delta(x) \delta(y)$ vanishes for $(x = 0, y = 0, z > 0)$ or equivalently, for $\theta = 0$. Hence, magnetic field $\vec{B}$ takes the same form as in (B.146)

$$\vec{B} = \frac{g_{\text{mg}}}{4\pi} \frac{\hat{r}}{r^2} + \hat{z} g_{\text{mg}} \delta(x) \delta(y) \vartheta(-z). \quad (\text{B.155})$$

Summarizing, the Dirac string was only a necessary artifact to introduce monopoles in the usual $U(1)$ Magnetism. However, if this $U(1)$ symmetry is a subgroup of $SO(3)$, isolated magnetic monopoles exist *without* introducing Dirac strings. This occurs because of the non-linear interactions due to the non-Abelian character (the nonvanishing structure constants) of the underlying gauge symmetry $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \epsilon \epsilon_{abc} W_\mu^b W_\nu^c$.

Let us continue with the $SO(3)$ gauge field coupled to an isovector Ψ^a to break down the gauge symmetry to $U(1)$ and later coupled to another scalar field ψ to break the residual symmetry in order to obtain vortices. At the point where two vortices react, a vortex of double flux is formed. Following the arguments by 't Hooft [8] and Tze & Ezawa [94], such a vortex can be seen as a configuration with isovector pointing in a fixed direction in isospace (for instance, the $\hat{z}$ direction) and gauge field W_μ^a equal to zero plus a Dirac string in the vortex' core (which carries the double flux). If we can eliminate the Dirac string by means of a gauge rotation, then the double flux vortex is not stable and after removing it, only the monopole located at the string's end remains. In a model whose original symmetry is $U(1)$, this elimination is not possible. However if

³⁶Proved in a similar way as in (B.151)

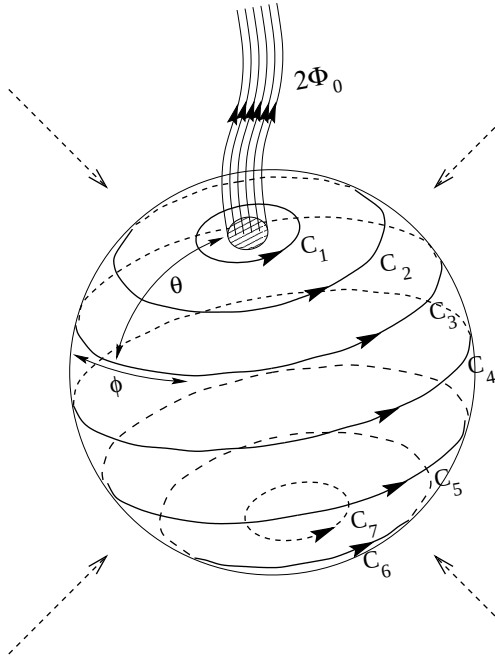
$$\nabla^2 D(\rho) = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} D(\rho) \right) = -\delta(x) \delta(y) = -\frac{\delta(\rho)}{2\pi\rho} = -\frac{1}{2\pi\rho} \frac{d}{d\rho} \vartheta(\rho) \quad \implies \quad \frac{d}{d\rho} D(\rho) = -\frac{1}{2\pi\rho}.$$

$U(1)$ is embedded in $SO(3)$, we can construct the required gauge rotation to remove the Dirac string of double flux. This is another way of understanding that vortices carrying two units of flux are not topologically stable. Hence, monopoles in the $SO(3)$ model should carry two units of flux: $g_{\text{mg}}e = 4\pi$. In the following discussion, we shall see how such a gauge transformation can be obtained.

In the model with $SO(3)$ broken down to $U(1)$ by scalar field Ψ^a pointing in the $\hat{z}$ direction of isospace, let us consider a flux $2\Phi_0 = 4\pi/e$ carried by a Dirac string which emerges from a region surrounded by the sphere of the following figure. The sphere is large enough such that the short-range gauge fields are vanishing small on its surface. Therefore, the Dirac string along the $+z$ -axis is described by

$$W_\mu^{a=1,2} = 0 \quad \text{and} \quad W_\mu^3 = \frac{1}{e} (1 + \cos \theta) \partial_\mu \phi. \quad (\text{B.156})$$

Along path C_1 , drawn around the point where the flux crosses the sphere, a gauge field A_μ must exist with $\oint_{C_1} A_\mu dl^\mu = 2\Phi_0$. As already discussed in the ordinary superconductor (with $U(1)$ the system's original symmetry), we can create such a gauge field A_μ by means of a singular gauge transformation: $A_\mu = -e^{-1} \partial_\mu \Lambda$ with Λ multivalued to produce $\oint_{C_1} A_\mu dl^\mu \neq 0$. As a result of this transformation, when we “travel” along path C_1 , the scalar field changes its orientation in the internal symmetry space. Although Λ can make a jump at a certain point (for instance, at the beginning and so the end of C_1), we require that the scalar field remains single-valued and thus the jumps in Λ must be multiples of 2π . Each 2π corresponds to a complete rotation of the scalar field. In our case, a flux of $4\pi/e$ means that the rotation in the internal symmetry space is 4π (without considering the sign) after covering path C_1 . To be exact, the rotation is -4π since the singular gauge transformation is given by $\Lambda = -2\phi$. However, this transformation is no longer singular if $U(1)$ is a subgroup of $SO(3)$.



In a system with an underlying Abelian gauge symmetry, if we eliminate the Dirac string by means of gauge transformation $\Lambda = 2\phi$, we create a Dirac string which enters the sphere through the “south pole”. In a model where $U(1)$ is embedded in an $SO(3)$ symmetry, we can construct a gauge transformation which rotates the scalar field through 4π at the “north pole”; but when moving on the sphere ($C_1 \rightarrow C_2 \rightarrow \dots \rightarrow C_7$), the 4π rotation is shrunk to a zero rotation ([9] and [6]). In this case, only the magnetic monopole (represented by the dashed arrows pointing inwards) is left inside the sphere.

The rotation which we are looking for must produce a 4π rotation at the “north pole”

$$\Omega(\phi, \theta = 0) = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \quad (\text{B.157})$$

If Θ and Φ are the polar angles determining the direction in isospace, when we follow the path C_1 from $\phi = 0$ to $\phi = 2\pi$, the above matrix makes the scalar field rotate from Φ to $\Phi + 2 \times 2\pi$. This can be seen by looking at equation (B.116) or by direct calculation³⁷. On the other hand, at the south pole ($\theta = \pi$) there must be no magnetic flux coming out. Therefore, as we “travel” along C_7 from $\phi = 0$ to $\phi = 2\pi$, scalar field does not change its orientation in isospace

$$\Omega(\phi, \theta = \pi) = \text{independent of } \phi. \quad (\text{B.158})$$

Note that θ and ϕ are angles in the real space, not in isospace.

Starting from a state with isovector Ψ^a pointing in the $\hat{z}$ isodirection and with a Dirac string, we can obtain a state with an isolated (negative) monopole $g_{\text{mg}} = -4\pi/e$ at the origin of ordinary space, by applying a gauge transformation equal (in tensor notation) to the following unitary matrix³⁸

$$\Omega(\phi, \theta) = \begin{pmatrix} e^{-i\phi} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & e^{i\phi} \cos(\theta/2) \end{pmatrix}, \quad (\text{B.159})$$

or in isovector notation Ψ^a , the orthogonal matrix $O(\phi, \theta)$ given by

$$\begin{pmatrix} \cos \theta \cos^2 \phi - \sin^2 \phi & -\cos \theta \cos \phi \sin \phi - \cos \phi \sin \phi & \sin \theta \cos \phi \\ \cos \theta \cos \phi \sin \phi + \cos \phi \sin \phi & -\cos \theta \sin^2 \phi + \cos^2 \phi & \sin \theta \sin \phi \\ -\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix}. \quad (\text{B.160})$$

This transformation connects or deforms double flux state (B.157) to no flux state (B.158) without singularities. The only point where it is not defined is the monopole's site. In the ordinary superconductor, where the gauge symmetry $U(1)$ was not a subgroup of a larger compact group, vortices of double winding number could not be transformed into zero winding number by means of a gauge transformation. Here, as different from this case, there are non-singular gauge transformations which do connect states whose winding numbers differ in 2 units. From here on, the $U(1)$ superconductor model, where symmetry $U(1)$ is embedded in a compact group, will be called the compact $U(1)$ superconductor. In order to avoid a too long notation, the compact $U(1)$ model arising from the breaking of the $SO(3)$ symmetry will be abbreviated as “ $SO(3)$ ”-superconductor.

³⁷This rotation acting on a general isovector (in tensor notation) $T = \frac{1}{2}\sqrt{\frac{|A|}{B}} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix}$ gives

$$\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i(\Phi+2\phi)} \\ \sin \Theta e^{i(\Phi+2\phi)} & -\cos \Theta \end{pmatrix}.$$

³⁸The form of transformation matrix (B.159) can easily be found: the 4π internal space rotation created by $\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$ must be multiplied by $\cos(\theta/2)$ in order that it vanish at the south pole: $\Omega = \begin{pmatrix} e^{-i\phi} \cos(\theta/2) & a \\ b & e^{i\phi} \cos(\theta/2) \end{pmatrix}$, where a and b are complex functions of θ alone since Ω at the south pole is independent of ϕ . Ω is an unitary matrix with determinant one, and thus, a and b must fulfil the following conditions: $|a|^2 = |b|^2 = \sin^2(\theta/2)$ and $a + b^* = 0$.

Transformation (B.159) is a special case of general rotation (B.118), with the angles in isospace (ω, ζ, γ) equal to (ϕ, θ, ϕ) respectively. Hence, after this gauge rotation, isovector Ψ^a does not longer point in the $\hat{z}$ direction in isospace, but in the radial direction of external space, i.e., $\Theta = \theta$ and $\Phi = \phi$ ³⁹

$$\Psi_0^a = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \Psi_{\text{monop.}}^a = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \frac{1}{|\vec{r}|} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (\text{B.161})$$

As in the case of a vortex, the fact that depending on which direction we approach the sphere's surface, Ψ^a takes a different orientation in isospace, requires that Ψ^a has a zero at $r = 0$. This zero was “hidden” in the description of the monopole by means of a Ψ^a pointing everywhere in the same isodirection, at the cost of introducing a line of singularities (the Dirac string) in the gauge potential.

On the other hand and according to (B.124), gauge field W_μ^a becomes

$$\left\{ \begin{array}{ll} W_\mu^1 = 0 & \rightarrow W_\mu'^1 = \frac{1}{e} (\cos \theta \sin \theta \cos \phi \partial_\mu \phi + \sin \phi \partial_\mu \theta), \\ W_\mu^2 = 0 & \rightarrow W_\mu'^2 = \frac{1}{e} (\cos \theta \sin \theta \sin \phi \partial_\mu \phi - \cos \phi \partial_\mu \theta), \\ W_\mu^3 = \frac{1}{e} (1 + \cos \theta) \partial_\mu \phi & \rightarrow W_\mu'^3 = -\frac{1}{e} \sin^2 \theta \partial_\mu \phi. \end{array} \right. \quad (\text{B.162})$$

The line of singularities is removed by the diagonal elements (those contributing to $W_\mu'^3$) of $-\frac{i}{e} (\partial_\mu \Omega) \Omega^+$, which contain the Dirac string but with opposite sign.

Of course, (B.159) is not the only gauge transformation which produces the monopole configuration above discussed. Starting from $\Psi_0^a = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ plus the Dirac string,

any matrix obtained from $\begin{pmatrix} e^{-i\phi} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & e^{i\phi} \cos(\theta/2) \end{pmatrix}$ by means of a gauge transformation “creates” the monopole configuration eliminating the string. This process is called to gauge away the Dirac string. Here the gauge transformation belongs to the set of non-singular transformations, i.e., without line of singularities in their derivatives. Since the Lagrangian is invariant under non-singular gauge transformations, all these configurations obtained from gauge equivalent transformations have the same energy. Transformation (B.159) has been chosen because it produces a manageable form for scalar isovector Ψ^a ⁴⁰.

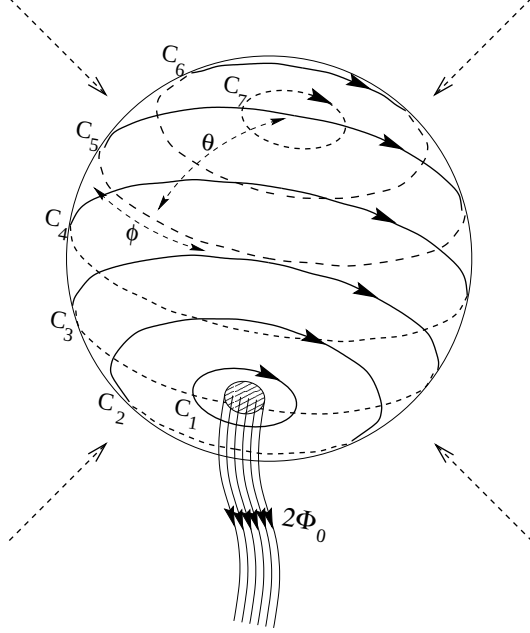
³⁹As one can easily read off from equation (B.160) or by direct calculation on $T_{0,z} = \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and comparing the result with tensor $T = \frac{1}{2} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i\Phi} \\ \sin \Theta e^{i\Phi} & -\cos \Theta \end{pmatrix}$

$$\begin{pmatrix} e^{-i\phi} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & e^{i\phi} \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}.$$

⁴⁰In the same way as in an ordinary superconductor: from all equivalent ψ configurations describing a vortex of winding number 1, we choose that whose orientation in internal space corresponds to the radial direction in the ordinary space.

The negative monopole can be also obtained from a Dirac string along the $-z$ -axis as in the figure

$$W_\mu^{a=1,2} = 0 \quad \text{and} \quad W_\mu^3 = -\frac{1}{e}(1 - \cos \theta) \partial_\mu \phi. \quad (\text{B.163})$$



Such a string can be gauge away by means of the following transformation

$$\Omega(\phi, \theta) = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Under this gauge rotation, scalar field Ψ^a and gauge field W_μ^a take the same form as in equations (B.161) and (B.162). We can convince ourselves of this result by considering the general relations (B.118) and (B.124) with $(\omega, \zeta, \gamma) = (\phi, \theta, -\phi)$.

At the risk of boring the reader, let us stress once more that these results hold for distances larger than the range of the short-range fields. Monopole gauge field (B.162) can be also written

$$W_\mu^1 = \frac{1}{e} \begin{pmatrix} 0 \\ \frac{\cos \theta}{r} \\ -\frac{\sin \theta \sin \phi}{r} \end{pmatrix}, \quad W_\mu^2 = \frac{1}{e} \begin{pmatrix} -\frac{\cos \theta}{r} \\ 0 \\ \frac{\sin \theta \cos \phi}{r} \end{pmatrix}, \quad W_\mu^3 = \frac{1}{e} \begin{pmatrix} \frac{\sin \theta \sin \phi}{r} \\ -\frac{\sin \theta \cos \phi}{r} \\ 0 \end{pmatrix},$$

as one can easily calculate ⁴¹. A more concise form of writing this configuration is

$$W_\mu^a = -\frac{1}{er} \epsilon_{\mu ab} \frac{r^b}{r} \quad r^b = (x, y, z). \quad (\text{B.164})$$

Closing this section, it is worth noting that magnetic monopoles are possible if the covering group of the Lie groups, which share the same Lie algebra, is compact [8]. If not, any group transformation of the kind $U = e^{-i\chi^a I_a}$ with χ^a real and I_a hermitian, can only be represented by infinite-dimensional matrices. In such a case, the Jacobi identity can fail. However, this identity was just the main property necessary to show the complete antisymmetry of the structure constants. And without this antisymmetry, the previous discussion and its results no longer hold. Hence, if the original symmetry of our model is described by Lie group $U(1)$, the model cannot have magnetic monopoles since the covering group of $U(1)$ is the real line, which is not compact. No gauge transformation can change the winding number of a configuration. If, on the contrary, $U(1)$

⁴¹Considering $\partial_\mu \phi = \left(-\frac{\sin \phi}{r \sin \theta}, \frac{\cos \phi}{r \sin \theta}, 0 \right)$ and $\partial_\mu \theta = \left(\frac{\cos \theta \cos \phi}{r}, \frac{\cos \theta \sin \phi}{r}, -\frac{\sin \theta}{r} \right)$.

is the subgroup of compact Lie group $SO(3)$, there are allowed gauge transformations which can connect configurations of different winding number by “moving” through the compact covering group. This moving is represented in the $SO(3)$ case by the shrinking of the rubber band on the sphere’s surface S^3 as in Fig. B.7: its result, jumps in A_μ to set double flux back to zero flux.

On the other hand, topologically stable line defects are only possible if the underlying group G describing the gauge transformations is multiply-connected. Thus, after breaking completely the symmetry ($G/H = G$), we obtain $\pi_1(G/H) = \pi_1(G) \neq 0$. Lie group $U(1)$ has an infinite number of different stable line defects (although no kind of monopoles), $SO(3)$ has only one kind of line defects, and single-connected $SU(2)$ (i.e., the fundamental representation of the Lie algebra defined by $C_{abc} = \epsilon_{abc}$) has no stable line defect.

All this discussion was for a superconductor. Let us consider the dislocation model presented in Chapter 2, where gliding dislocations are described as vortices whilst non-able-to-glide dislocations (for example, Lomer-Cottrell locks) play the same role in the equations as magnetic monopoles in Maxwell’s theory. The above results mean that, if interactions between gliding dislocations are described by means of a Lagrangian invariant under gauge $SO(3)$ transformations, a non-gliding dislocation will be formed at the point where two dislocations meet.

B.6 The ’t Hooft–Polyakov monopole

After the qualitative discussion in the previous section, we shall calculate now the form and energy of the magnetic monopole in the model described by a unique isovector Ψ^a coupled to a $SO(3)$ gauge field W_μ^a

$$\ell = -\frac{1}{4} (G_{\mu\nu}^a)^2 - \frac{1}{2} (D_\mu \Psi^a)^2 - V(\Psi^a), \quad V(\Psi^a) = -|\mathcal{A}| \Psi^a \Psi^a + \frac{\mathcal{B}}{2} (\Psi^a \Psi^a)^2, \quad (\text{B.165})$$

with

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e\epsilon_{abc} W_\mu^b W_\nu^c \quad \text{and} \quad D_\mu \Psi^a = \partial_\mu \Psi^a + e\epsilon_{abc} W_\mu^b \Psi^c. \quad (\text{B.166})$$

We should remember that latin indices $a, b, c, \dots = 1, 2, 3$ denote components in isospace, whilst greek indices $\mu, \nu, \dots = 1, 2, 3$ denote the coordinates in the ordinary space.

Before we begin, let us obtain some useful relations based exclusively on the complete antisymmetry of the structure constants ϵ_{abc} . From its definition, $G_{\mu\nu}^a$ fulfils the following equation⁴²

$$\epsilon_{\lambda\mu\nu} D_\lambda G_{\mu\nu}^a = 0. \quad (\text{B.167})$$

⁴²Proved by using $\epsilon_{abc}\epsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$ (underlined indices denote where the ϵ_{abc} antisymmetry has been used)

$$\epsilon_{\lambda\mu\nu}\epsilon_{abc}W_\lambda^b(\partial_\mu W_\nu^c - \partial_\nu W_\mu^c + e\epsilon_{cde}W_\mu^dW_\nu^e) = 2e \underbrace{\epsilon_{\lambda\mu\nu}\epsilon_{abc}W_\lambda^b\partial_\mu W_\nu^c}_{-\epsilon_{\lambda\mu\nu}\epsilon_{abc}W_\nu^c\partial_\lambda W_\mu^b} + e^2\epsilon_{\lambda\mu\nu}W_\mu^aW_\lambda^bW_\nu^b + 0.$$

Moreover, $G_{\mu\nu}^a$ and the covariant derivative of any isovector ξ^a are related by ⁴³

$$D_\mu D_\nu \xi^a - D_\nu D_\mu \xi^a = e \epsilon_{abc} G_{\mu\nu}^b \xi^c. \quad (\text{B.168})$$

The derivative of isoscalar $\xi^a \zeta^a$ (sum on a), formed by the product of an iso(co)vector and an isovector, is related to the covariant derivatives of ξ^a and ζ^a as follows

$$\partial_\mu (\xi^a \zeta^a) = (\partial_\mu \xi^a) \zeta^a + (\epsilon_{abc} W_\mu^b \xi^c \zeta^a - \underbrace{\epsilon_{abc} W_\mu^b \xi^c \zeta^a}_{-\epsilon_{abc} \xi^a W_\mu^b \zeta^c}) + \xi^a \partial_\mu \zeta^a = (D_\mu \xi^a) \zeta^a + \xi^a D_\mu \zeta^a. \quad (\text{B.169})$$

From this property, the following relation for an arbitrary unit isovector $\hat{\Psi}^a = \frac{\Psi^a}{|\Psi|}$ is obtained

$$\hat{\Psi}^a \partial_\mu \hat{\Psi}^a = 0 = \hat{\Psi}^a D_\mu \hat{\Psi}^a. \quad (\text{B.170})$$

And finally, using again the antisymmetry of the ϵ_{abc} ,

$$\epsilon_{abc} \hat{\Psi}^a \partial_\mu \hat{\Psi}^b = \frac{1}{|\Psi|} \epsilon_{abc} \hat{\Psi}^a \partial_\mu \Psi^b. \quad (\text{B.171})$$

As we have seen, potential $V(\Psi^a)$ in (B.165) has as minima nonzero constant Ψ_0^a 's. This minima are lowest energy states. Any Ψ_0^a has $U_{\Psi_0}(1)$ as symmetry, i.e., all rotations about Ψ_0^a leave it invariant. Since Ψ_0^a can be oriented in any direction in the 3-dimensional isospace, the set of different lowest energy states has the topology of a sphere's surface S^2 : $G/H = SO(3)/U_{\Psi_0}(1) = S^2$. It is this topology which ensures the existence of monopoles.

In a 3-dimensional material, let us consider a point defect (such as a magnetic monopole). We surround it with a sphere large enough such that on its surface, scalar field Ψ^a is approximately given by one of the $V(\Psi^a)$ minima. However, isovector Ψ^a must not assume the same minimum value at all sphere's points: it can occur that when approaching the surface along different directions, Ψ^a takes different orientations in isospace. Some of these configuration can be topologically stable and therefore can create a zero in Ψ^a . The increase in the energy due to this zero is maintained within a finite value by the coupling to the gauge field, such that covariant derivative $D_\mu \Psi^a$ vanishes on the sphere's surface (and thus so $G_{\mu\nu}^a$ because of equation (B.168)). Therefore, on the surface we have lowest energy states and if we ask ourselves about the defect's topological stability, we must study the maps from the sphere's surface to the set of different lowest energy states $G/H = S^2$, i.e., we have to consider $\pi_2(G/H)$.

Analogously to the infinite numerable forms of wrapping a rubber band around a ring, $\pi_1(S^1) = \mathbf{Z}$, the forms of wrapping a ball with an elastic sphere's surface are classified by a winding number too: $\pi_2(S^2) = \mathbf{Z}$. Hence, $\pi_2(G/H) = \mathbf{Z}$ and our model has stable point defects. In the following, we shall consider the configuration belonging to the homotopy class of winding number 1 which, in a colloquial way of speaking,

⁴³Shown by means of $\epsilon_{abc}\epsilon_{bde}W_\mu^dW_\nu^e\xi^c = W_\mu^bW_\nu^a\xi^b - W_\mu^aW_\nu^b\xi^b$ and by considering

$$D_\mu D_\nu \xi^a = \partial_\mu \partial_\nu \xi^a + e \epsilon_{abc} (W_\mu^b \partial_\nu \xi^c + W_\nu^b \partial_\mu \xi^c) + e \epsilon_{abc} (\partial_\mu W_\nu^b) \xi^c + e^2 W_\mu^b W_\nu^a \xi^b - e^2 W_\mu^b W_\nu^b \xi^a.$$

corresponds to the map between the orange's skin and the orange. To construct it is easy: if Θ and Φ are the polar angles in isospace and determines the isovector orientation, the map between the points on the sphere and G/H is given by $\Theta = \theta$ and $\Phi = \phi$, with θ and ϕ the polar angles in the external space. At large distances, this configuration is just (B.161). Any local perturbation in the radial arrangement produced by a (non-singular) gauge transformation is equivalent to the displacement of the monopole from the coordinates origin to another point. Such a gauge transformation, nevertheless, does not change the winding number or the configuration's energy. We can therefore choose the configuration which we find more manageable.

In the previous section we have discussed that if we choose Ψ^a pointing in to $\hat{z}$ direction of isospace, we obtain an $U(1)$ "electromagnetic" (i.e., infinite-range) gauge field $A_\mu \equiv W_\mu^3$. However, working with this choice is not comfortable since it requires a Dirac string to describe the monopole. On the contrary, working with the isovector pointing in the radial direction of external space does not make clear which are the components of the $U(1)$ field. We need therefore a relation for the magnetic field $F_{\mu\nu}$, which is invariant under rotations in isospace. Such a relation has been given by 't Hooft [8]

$$F_{\mu\nu} = \frac{\Psi^a}{|\Psi|} G_{\mu\nu}^a - \frac{1}{e|\Psi|^3} \epsilon_{abc} \Psi^a (D_\mu \Psi^b) (D_\nu \Psi^c) , \quad (\text{B.172})$$

which using (B.171) can be written as

$$F_{\mu\nu} = \hat{\Psi}^a G_{\mu\nu}^a - \frac{1}{e} \epsilon_{abc} \hat{\Psi}^a (D_\mu \hat{\Psi}^b) (D_\nu \hat{\Psi}^c) , \quad (\text{B.173})$$

where $\hat{\Psi}^a = \Psi^a/|\Psi|$. Field $F_{\mu\nu}$ is a scalar in the internal space of symmetry, since it is invariant under any gauge rotation in isospace. Since Ψ^a and $G_{\mu\nu}^a$ transforms according to the adjoint transformation under a gauge rotation, $\delta\Psi^a = \epsilon_{abc}\chi^b\Psi^c$ and $\delta G_{\mu\nu}^a = \epsilon_{abc}\chi^b G_{\mu\nu}^c$ ⁴⁴, the isoscalar $\hat{\Psi}^a G_{\mu\nu}^a$ is invariant under a gauge rotation (the length $|\Psi|$ is invariant from the very definition of rotation). The second term in $F_{\mu\nu}$ is also invariant due to (B.170) (used in the underlined terms)

$$\begin{aligned} \delta F_{\mu\nu} = & -\frac{1}{e} \underbrace{\epsilon_{abd}\epsilon_{ade}\chi^d \hat{\Psi}^e (D_\mu \hat{\Psi}^b) (D_\nu \hat{\Psi}^c)}_{\chi^b (D_\mu \hat{\Psi}^b) \hat{\Psi}^c (D_\nu \hat{\Psi}^c) - \hat{\Psi}^b (D_\mu \hat{\Psi}^b) \chi^c D_\nu \hat{\Psi}^c} - \underline{0} = 0 . \end{aligned} \quad (\text{B.174})$$

⁴⁴The transformation of gauge field W_μ^a , $\delta W_\mu^a = -\frac{1}{e}\partial_\mu\chi^a + \epsilon_{ade}\chi^d W_\mu^e$, makes $G_{\mu\nu}^a$ transform as follows

$$\begin{aligned} \delta G_{\mu\nu}^a = & \epsilon_{ade} [(\partial_\mu\chi^d) W_\nu^e - (\partial_\nu\chi^d) W_\mu^e] + \epsilon_{ade}\chi^d (\partial_\mu W_\nu^e - \partial_\nu W_\mu^e) - \epsilon_{abc}\partial_\mu\chi^b W_\nu^c - \epsilon_{abc}W_\mu^b\partial_\nu\chi^c + \\ & + e \underbrace{\epsilon_{abc}\epsilon_{bde}\chi^d W_\mu^e W_\nu^c}_{W_\mu^a\chi_b W_\nu^b - \chi^a W_\mu^b W_\nu^b} + e \underbrace{\epsilon_{abc}W_\mu^b\epsilon_{cde}\chi^d W_\nu^e}_{\chi^a W_\mu^b W_\nu^b - W_\nu^a\chi_b W_\mu^b} = \epsilon_{ade}\chi^d (\partial_\mu W_\nu^e - \partial_\nu W_\mu^e) + e\epsilon_{ebc}W_\mu^b W_\nu^c . \end{aligned}$$

Moreover, with the choice $\hat{\Psi}^a = \delta_{a3}$, $F_{\mu\nu}$ becomes $\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3$ as expected

$$F_{\mu\nu} = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3 + e\epsilon_{3bc}W_\mu^b W_\nu^c - \frac{1}{e}\epsilon_{3bc} \underbrace{\left(D_\mu \hat{\Psi}^b\right)}_{e\epsilon_{bd3}W_\mu^d} \underbrace{\left(D_\nu \hat{\Psi}^c\right)}_{e\epsilon_{ce3}W_\nu^e} = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3.$$

B.6.1 Maxwell equations for $F_{\mu\nu}$

In this section, we shall see that at those points where $|\Psi| \neq 0$, $F_{\mu\nu}$ fulfils the usual equations of the magnetic field. However at the point where $|\Psi| = 0$, $F_{\mu\nu}$ has a singularity. This will we proved for the stable configuration where Ψ^a is pointing in the radial direction of ordinary space at large distances. Similarly to the case of $U(1)$ vortices, the topological stability makes that Ψ^a has also the radial orientation at points near to the origin. If Ψ^a were not zero at $r = 0$, starting at the spatial we could approach the origin along different directions and obtain different Ψ^a orientations at $r = 0$. The coordinates origin is just the place where the monopole defined by this configuration sits. The same stated in one equation

$$\frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = g_{\text{mg}} \delta^3(\vec{r}), \quad (\text{B.175})$$

where g_{mg} is the monopole magnetic charge. Let us begin with this equation: by using (B.169), (B.167) and (B.168), one obtains

$$\begin{aligned} \frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda F_{\mu\nu} &= \frac{1}{2}\epsilon_{\lambda\mu\nu} \left(D_\lambda \hat{\Psi}^a\right) G_{\mu\nu}^a + \frac{1}{2}\hat{\Psi}^a \underbrace{\epsilon_{\lambda\mu\nu} D_\lambda G_{\mu\nu}^a}_{0 \text{ (B.167)}} - \\ &- \frac{1}{2e}\epsilon_{\lambda\mu\nu}\epsilon_{abc} \left(D_\lambda \hat{\Psi}^a\right) \left(D_\mu \hat{\Psi}^b\right) \left(D_\nu \hat{\Psi}^c\right) - \frac{2}{2e}\epsilon_{abc} \hat{\Psi}^a \underbrace{\left(\epsilon_{\lambda\mu\nu} D_\lambda D_\mu \hat{\Psi}^b\right)}_{\frac{e}{2}\epsilon_{\lambda\mu\nu}\epsilon_{bde}G_{\lambda\mu}^d \hat{\Psi}^e \text{ (B.168)}} D_\nu \hat{\Psi}^c, \end{aligned}$$

which after operating and considering (B.170) becomes

$$\frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = -\frac{1}{2e}\epsilon_{\lambda\mu\nu}\epsilon_{abc} \left(D_\lambda \hat{\Psi}^a\right) \left(D_\mu \hat{\Psi}^b\right) \left(D_\nu \hat{\Psi}^c\right).$$

This equation can be further simplified, by using again $\hat{\Psi}^a D_\mu \hat{\Psi}^a = 0 = \hat{\Psi}^a \partial_\mu \hat{\Psi}^a$ in the underlined parts

$$\begin{aligned} &-\frac{1}{e}\epsilon_{\lambda\mu\nu}\epsilon_{abc} \left(\partial_\lambda \hat{\Psi}^a\right) \left(D_\mu \hat{\Psi}^b\right) \left(D_\nu \hat{\Psi}^c\right) - \epsilon_{\lambda\mu\nu} \underbrace{\epsilon_{abc}\epsilon_{ade}W_\lambda^d \hat{\Psi}^e \left(D_\mu \hat{\Psi}^b\right) \left(D_\nu \hat{\Psi}^c\right)}_{W_\lambda^b \left(D_\mu \hat{\Psi}^b\right) \underline{\hat{\Psi}^c (D_\nu \hat{\Psi}^c)} - W_\lambda^b \left(D_\nu \hat{\Psi}^b\right) \underline{\hat{\Psi}^c (D_\mu \hat{\Psi}^c)}} = \\ &= -\frac{1}{e}\epsilon_{\lambda\mu\nu}\epsilon_{abc} \left(\partial_\lambda \hat{\Psi}^a\right) \left(\partial_\mu \hat{\Psi}^b\right) \left(D_\nu \hat{\Psi}^c\right) - \epsilon_{\lambda\mu\nu} \underbrace{\left(\partial_\lambda \hat{\Psi}^a\right) \epsilon_{abc}\epsilon_{bde}W_\mu^d \hat{\Psi}^e \left(D_\nu \hat{\Psi}^c\right)}_{\underline{(\partial_\lambda \hat{\Psi}^a) \hat{\Psi}^a W_\mu^c (D_\nu \hat{\Psi}^c)} - (\partial_\lambda \hat{\Psi}^a) W_\mu^a \underline{\hat{\Psi}^c (D_\mu \hat{\Psi}^c)}} = \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{e}\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right) - \epsilon_{\lambda\mu\nu}\underbrace{\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\epsilon_{abc}\epsilon_{cde}W_\nu^d\hat{\Psi}^e}_{\left(\partial_\lambda\hat{\Psi}^a\right)W_\nu^a\left(\partial_\mu\hat{\Psi}^b\right)\hat{\Psi}^b - \left(\partial_\lambda\hat{\Psi}^a\right)\hat{\Psi}^a\left(\partial_\mu\hat{\Psi}^b\right)W_\nu^b}, \\
&\frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = -\frac{1}{2e}\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right). \tag{B.176}
\end{aligned}$$

In the following Note it will shown that $\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right)$ is proportional to the Dirac delta $\delta^3(\vec{r})$ for isovector Ψ^a in the radial direction (or equivalently, if the isovector has a zero).

Note. Calculation of $\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right)$.

Due to the antisymmetry of $\epsilon_{\lambda\mu\nu}$, the expression we want to calculate can be written as follows

$$\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right) = \epsilon_{\lambda\mu\nu}\partial_\lambda\left[\epsilon_{abc}\hat{\Psi}^a\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right)\right]. \tag{B.177}$$

For the winding number 1 configuration, i.e., that where isovector in the isospace is pointing in the radial direction of the real space, the term inside the brackets $[\]$ takes the following form

$$\hat{\Psi}^a = \frac{r^a}{r} \implies \epsilon_{abc}\frac{r_a}{r}\left(\frac{\delta_{b\mu}}{r} - \frac{r_b r_\mu}{r^3}\right)\left(\frac{\delta_{c\nu}}{r} - \frac{r_c r_\nu}{r^3}\right) = \epsilon_{a\mu\nu}\frac{r_a}{r^2}, \tag{B.178}$$

where $r^a = (x, y, z)$ are the coordinates in the ordinary space. Taking into account that $\partial_a\left(\frac{1}{r}\right) = -\frac{r_a}{r^2}$ and that $1/4\pi r$ is the Green's function of differential operator $\partial_\lambda\partial_\lambda$ (see (B.151)), the above expression becomes

$$\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right) = -\underbrace{\epsilon_{\lambda\mu\nu}\epsilon_{a\mu\nu}}_{2\delta_{\lambda a}}\partial_\lambda\partial_a\left(\frac{1}{r}\right) = 8\pi\delta^3(\vec{r}). \tag{B.179}$$

According to (B.176), $\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(\partial_\lambda\hat{\Psi}^a\right)\left(\partial_\mu\hat{\Psi}^b\right)\left(\partial_\nu\hat{\Psi}^c\right) = \epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(D_\lambda\hat{\Psi}^a\right)\left(D_\mu\hat{\Psi}^b\right)\left(D_\nu\hat{\Psi}^c\right)$, expression which is invariant under gauge rotations due to the antisymmetry of $\epsilon_{\lambda\mu\nu}$

$$\begin{aligned}
&\delta\left[\epsilon_{\lambda\mu\nu}\epsilon_{abc}\left(D_\lambda\hat{\Psi}^a\right)\left(D_\mu\hat{\Psi}^b\right)\left(D_\nu\hat{\Psi}^c\right)\right] = 3\epsilon_{\lambda\mu\nu}\underbrace{\epsilon_{abc}\epsilon_{ade}}_{\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}}\chi^d\left(D_\lambda\hat{\Psi}^e\right)\left(D_\mu\hat{\Psi}^b\right)\left(D_\nu\hat{\Psi}^c\right) = \\
&= 3\epsilon_{\underline{\lambda}\underline{\mu}\underline{\nu}}\chi^b\left(D_{\underline{\lambda}}\hat{\Psi}^c\right)\left(D_{\underline{\mu}}\hat{\Psi}^b\right)\left(D_{\underline{\nu}}\hat{\Psi}^c\right) - \underline{0} = 0. \tag{B.180}
\end{aligned}$$

Therefore, result (B.179) holds for any local deformation of the radial configuration.

Considering the result of the previous Note, we obtain

$$\frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = -\frac{4\pi}{e}\delta^3(\vec{r}), \tag{B.181}$$

which defines a negative monopole of $|g_{\text{mg}}| = \frac{4\pi}{|e|}$, the charge of the configuration studied in the general discussion of the last section. Note that up to this moment we have

not used the equations describing the system, which one obtains from the Lagrangian density. The above result holds because of the topology alone.

Now we shall prove that for the radial $\Psi^a = \frac{r^a}{r}Q(r)$ configuration (with $Q(r)$ a certain function), $F_{\mu\nu}$ fulfils the other Maxwell equation $\partial_\mu F_{\mu\nu} = 0$. For the studied configuration, let us choose gauge field W_μ^a as follows

$$W_\mu^a = \epsilon_{\mu ab} r^b f(r), \quad (\text{B.182})$$

required to render the covariant derivative at large distances zero

$$D_\mu \Psi^a \Big|_{r \rightarrow \infty} = \left[Q(1 + er^2 f) \left(\frac{\delta_{a\mu}}{r} - \frac{r_a r_\mu}{r^3} \right) + \frac{dQ}{dr} \frac{r_a r_\mu}{r^2} \right] \Big|_{r \rightarrow \infty} = 0. \quad (\text{B.183})$$

The length of the scalar field $Q(r)$ must tend to the constant value $\sqrt{|\mathcal{A}|/\mathcal{B}}$ at large distances, and therefore its derivative vanishes in the above relation. Function $f(r)$ must be equal to $-\frac{1}{er^2}$ at large distances in order to make zero the covariant derivative. With this result, ansatz (B.182) is just gauge field (B.164).

The vanishing of $\partial_\mu F_{\mu\nu}$ is nevertheless independent of the form of $Q(r)$ or $f(r)$. Moreover, configuration $\left\{ \Psi^a = \frac{r^a}{r}Q(r), W_\mu^a = \epsilon_{\mu ab} r^b f(r) \right\}$ is not the unique valid ansatz for the monopole: any gauge transformation applied to it (which produces a local deformation of the radial arrangement) yields the same form of $F_{\mu\nu}$ or of the energy, since both are gauge invariant. We shall use this ansatz because to work with it is easier (note the similarity with that in (B.38) and (B.39)). Direct substitution of the ansatz into $F_{\mu\nu}$ produces ⁴⁵

$$F_{\mu\nu} = -\epsilon_{\mu\nu a} \frac{1}{e} \frac{r^a}{r^3}, \quad (\text{B.184})$$

which is independent of $Q(r)$ and $f(r)$. From this result it is clear that the magnetic field, defined as $B_\lambda = \frac{1}{2}\epsilon_{\lambda\mu\nu} F_{\mu\nu}$, fulfils the Maxwell equations in absence of currents

$$\epsilon_{\nu\rho\lambda} \partial_\rho B_\lambda = -\partial_\mu F_{\mu\nu} = 0. \quad (\text{B.185})$$

Also the previous result (B.181) can be directly obtained from (B.184), by noting that the magnetic field has radial direction

$$\partial_\lambda B_\lambda = -\partial_\lambda \left(\frac{r_\lambda}{er^3} \right) = \frac{4\pi}{e} \nabla^2 \left(\frac{1}{4\pi r} \right) = -\frac{4\pi}{e} \delta^3(\vec{r}). \quad (\text{B.186})$$

This $F_{\mu\nu}$ was the infinite-range $U(1)$ gauge field of the previous section, which was later coupled to scalar field ψ to obtain the superconductor whose vortices decay into a monopole when two of them react.

⁴⁵Considering $G_{\mu\nu}^a = 2\epsilon_{\mu\nu a} f + \frac{r^b}{r} \frac{df}{dr} (\epsilon_{\nu ab} r_\mu - \epsilon_{\mu ab} r_\nu) + e\epsilon_{\mu\nu c} r^a r^c f^2$, $\hat{\Psi}^a G_{\mu\nu}^a = \frac{r^a}{r} G_{\mu\nu}^a = 2\epsilon_{\mu\nu a} \frac{r^a}{r} f + e\epsilon_{\mu\nu a} r^a r f^2$ and $\frac{1}{|\Psi|^3} \epsilon_{abc} \Psi^a D_\mu \Psi^b D_\nu \Psi^c = \frac{1}{Q^2} \frac{\epsilon_{abc} r^a}{r} \left[Q(1 + er^2 f) \left(\frac{\delta_{b\mu}}{r} - \frac{r_b r_\mu}{r^3} \right) + \frac{dQ}{dr} \frac{r_b r_\mu}{r^2} \right] D_\nu \Psi^c = \frac{1}{Q} (1 + er^2 f) \frac{\epsilon_{a\mu c} r^a}{r^2} \left[Q(1 + er^2 f) \left(\frac{\delta_{c\nu}}{r} - \frac{r_c r_\nu}{r^3} \right) + \frac{dQ}{dr} \frac{r_c r_\nu}{r^2} \right] = (1 + er^2 f)^2 \epsilon_{a\mu\nu} \frac{r^a}{r^3}.$

B.6.2 The energy of the 't Hooft–Polyakov monopole

Although the equations for $F_{\mu\nu}$ are independent of the specific form of the Lagrangian density (we only needed a potential which ensures constant nonzero minima and that the coupling to the gauge field appears in the covariant derivative). To obtain the monopole energy, we have to solve the Euler-Lagrange equations corresponding to Lagrangian density (B.165) ⁴⁶

$$D_\mu G_{\mu\nu}^a = e\epsilon_{abc}\Psi^b D_\nu \Psi^c, \quad (\text{B.187})$$

$$D_\mu D_\mu \Psi^a = \frac{\Psi^a}{|\Psi|} \frac{dV(|\Psi|)}{d|\Psi|}. \quad (\text{B.188})$$

The energy of the spherical symmetric monopole is given by

$$E = \int d^3\vec{r} (\varepsilon_{\text{monopole}} - \varepsilon_{\text{ground}}) \quad \text{with} \quad \varepsilon_{\text{ground}} = -\frac{|\mathcal{A}|^2}{2\mathcal{B}}, \quad (\text{B.189})$$

where energy density ε is equal to $-\ell$ since we are studying a static case. In the previous equation, we have subtracted energy $\varepsilon_{\text{ground}} = -|\mathcal{A}|^2/2\mathcal{B}$ corresponding to the state where the isovector is pointing in the same direction at any point and the gauge field is zero. Thus, one obtains the net energy increase produced by the topological object.

The above energy will be studied for the configuration of winding number 1, which we shall describe with the ansatz where the isovector is pointing in the radial direction of ordinary space and the gauge field is given by (B.182)

$$\begin{cases} \Psi^a = \frac{r^a}{r} Q(r) & \text{with} \quad \lim_{r \rightarrow \infty} Q = \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}}, \\ W_\mu^a = \epsilon_{\mu ab} r^b f(r) & \text{with} \quad \lim_{r \rightarrow \infty} f = -\frac{1}{er^2}. \end{cases} \quad (\text{B.190})$$

The Euler-Lagrange equations become thus ⁴⁷

$$r \frac{d^2 f}{dr^2} + 4 \frac{df}{dr} - 3erf^2 - e^2 r^3 f^3 - eQ^2 \left(\frac{1}{r} + erf \right) = 0, \quad (\text{B.191})$$

⁴⁶For gauge field W_ν^a :

$$\frac{\partial \ell}{\partial (\partial_\mu W_\nu^a)} = -G_{\mu\nu}^a; \quad \frac{\partial \ell}{\partial W_\nu^a} = -G_{\nu\mu}^b e\epsilon_{bac} W_\mu^c - (D_\nu \Psi^c) e\epsilon_{cab} \Psi^b = e\epsilon_{abc} W_\mu^b G_{\mu\nu}^c - e\epsilon_{abc} \Psi^b D_\nu \Psi^c,$$

and for scalar field Ψ^a :

$$\frac{\partial \ell}{\partial (\partial_\mu \Psi^a)} = -D_\mu \Psi^a; \quad \frac{\partial \ell}{\partial \Psi^a} = -D_\mu \Psi^c e\epsilon_{cba} W_\mu^b - \frac{dV}{d|\Psi|} \frac{\Psi^a}{|\Psi|}.$$

⁴⁷For the first Euler-Lagrange equation:

$$\partial_\mu G_{\mu\nu}^a = \epsilon_{\nu ab} \frac{r^b}{r} \left(r \frac{d^2 f}{dr^2} + 4 \frac{df}{dr} - erf^2 \right); \quad e\epsilon_{ade} W_\mu^d G_{\mu\nu}^e = \epsilon_{\nu ab} \frac{r^b}{r} (-2erf^2 - e^2 r^3 f^3),$$

and for the second equation:

$$\partial_\mu D_\mu \Psi^a = \frac{r^a}{r} \left(\frac{d^2 Q}{dr^2} + \frac{2}{r} \frac{dQ}{dr} - 2Q \left(\frac{1}{r^2} + ef \right) \right); \quad e\epsilon_{abc} W_\mu^b D_\mu \Psi^c = \frac{r^a}{r} 2Q(ef + e^2 r^2 f^2).$$

$$\frac{d^2 Q}{dr^2} + \frac{2}{r} \frac{dQ}{dr} - 2Q (1 + er^2 f)^2 + 2|\mathcal{A}|Q - 2\mathcal{B}Q^3 = 0. \quad (\text{B.192})$$

The monopole energy (B.189), for its part, takes the following form ⁴⁸

$$E = 4\pi \int_0^\infty r^2 dr \left[\frac{2}{r^2} \frac{d}{dr} (r^3 f^2) + r^2 \left(\frac{df}{dr} \right)^2 + \frac{1}{2} e^2 r^4 f^4 + 2er^2 f^3 + \right. \\ \left. + Q^2 \left(\frac{1}{r} + erf \right)^2 + \frac{1}{2} \left(\frac{dQ}{dr} \right)^2 + \frac{\mathcal{B}}{2} \left(Q^2 - \frac{|\mathcal{A}|}{\mathcal{B}} \right)^2 \right]. \quad (\text{B.193})$$

Note that we can also obtain equations (B.191) and (B.192) from the expression (equal to $-\ell$) inside the brackets of (B.193) by using $\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{\partial[\cdot]}{\partial \left(\frac{df, Q}{dr} \right)} \right) = \frac{\partial[\cdot]}{\partial f, Q}$ ⁴⁹. In order to calculate E we need solve the Euler-Lagrange equations, which can be solved only by means of numerical methods. However, we can obtain the behaviour of f and Q at large r by extending solutions (B.190) as follows

$$f \sim -\frac{1}{er^2} \left[1 - \frac{r}{\xi_w} C_1 e^{-\frac{r}{\xi_w}} \right] \quad \text{and} \quad Q \sim \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} \left[1 + C_2 e^{-\frac{r}{\xi_\Psi}} \right], \quad (\text{B.194})$$

where ξ_w and ξ_Ψ have to be calculated by substitution into equations (B.191) and (B.192). Note that this ansatz is very similar to that in the Abrikosov model (A.16). Neglecting factors like $r^{-n}e^{-r}$ or e^{-nr} $n = 1, 2, \dots$ when compared to e^{-r} , we obtain

$$C_1 e^{-\frac{r}{\xi_w}} \left[\frac{1}{e\xi_w^3} - \frac{e}{r} \frac{r}{\xi_w} \frac{|\mathcal{A}|}{\mathcal{B}} \right] = 0 \quad \Rightarrow \quad \xi_w^2 = \frac{\mathcal{B}}{e^2 |\mathcal{A}|}, \quad (\text{B.195})$$

$$C_2 e^{-\frac{r}{\xi_\Psi}} \left[\frac{1}{\xi_\Psi^2} + 2|\mathcal{A}| - 6\mathcal{B} \frac{|\mathcal{A}|}{\mathcal{B}} \right] = 0 \quad \Rightarrow \quad \xi_\Psi^2 = \frac{1}{4|\mathcal{A}|}.$$

Decay lengths ξ_w and ξ_Ψ have the same form as those of equations (B.48). A closer expression for the monopole energy is obtained by defining the dimensionless coordinate $\bar{r} = r/\xi_w$ and the following dimensionless functions

$$\bar{Q}(\bar{r}) = \sqrt{\frac{\mathcal{B}}{|\mathcal{A}|}} Q \quad \text{and} \quad \bar{f}(\bar{r}) = e\xi_w^2 f.$$

Hence, the monopole energy becomes

$$E = \frac{4\pi}{e^2 \xi_w^2} C(\kappa), \quad (\text{B.196})$$

⁴⁸ $(G_{\mu\nu}^a)^2 = \frac{8}{r^2} \frac{d}{dr} (r^3 f^2) + 4r^2 \left(\frac{df}{dr} \right)^2 + 2e^2 r^4 f^4 + 8er^2 f^3$ and $(D_\mu \Psi^a)^2 = 2Q^2 \left(\frac{1}{r} + erf \right)^2 + \left(\frac{dQ}{dr} \right)^2$.

⁴⁹Since f and Q are functions of r only, $\partial_\mu f, Q = \frac{r_\mu}{r} \frac{df, Q}{dr}$, one obtains then $\partial_\mu \left(\frac{\partial[\cdot]}{\partial (\partial_\mu f, Q)} \right) = \partial_\mu \left(\frac{r_\mu}{r} \frac{\partial[\cdot]}{\partial \left(\frac{df, Q}{dr} \right)} \right) = \frac{2}{r} \frac{\partial[\cdot]}{\partial \left(\frac{df, Q}{dr} \right)} + \frac{d}{dr} \left(\frac{\partial[\cdot]}{\partial \left(\frac{df, Q}{dr} \right)} \right)$.

where $\kappa = \xi_w/\xi_\psi$, and $C(\kappa)$ is given by $\int_0^\infty \bar{r}^2 d\bar{r} \left[\frac{2}{\bar{r}^2} \frac{d}{d\bar{r}} (\bar{r}^3 \bar{f}^2) + \bar{r}^2 \left(\frac{d\bar{f}}{d\bar{r}} \right)^2 + \frac{1}{2} \bar{r}^4 \bar{f}^4 + \right.$
 $\left. + 2\bar{r}^2 \bar{f}^3 + \bar{Q}^2 \left(\frac{1}{\bar{r}} + \bar{r} \bar{f} \right)^2 + \frac{1}{2} \left(\frac{d\bar{Q}}{d\bar{r}} \right)^2 + \frac{1}{8} (\bar{Q}^2 - 1)^2 \right]$ and is a slowly varying function of κ with $C(0) = 1$ ([8] and [81]).

B.7 The most important part of the Appendix: monopole and vortex configurations with an isotensor $\Psi^{ab\dots}$ of second (and higher) rank

Let us summarize the results of previous sections before starting with the general results regarding monopoles and vortices in a model where the $U(1)$ gauge symmetry is the residual part of an underlying non-Abelian compact group describing the symmetry of the model.

We have dealt (and shall further deal) with models described by an energy functional (or by a Lagrangian) which is invariant under the transformations of a group G . If the model is invariant when, in general, the transformation applied at the site $\vec{r}$ is different from that applied at another site $\vec{r}'$, one speaks of a gauge invariant model. Further, the potential part of the energy (that not containing any gradient of the involved fields) is formed by self-interactions of the scalar field which allows for minimum energy states characterized by a nonvanishing constant scalar field. Once one of these states, denoted by ψ_0 (or by Ψ_0 if the symmetry under study is non-Abelian), has been found, the whole family $\{g\psi_0 : g \in G\}$ contains all possible minimum energy states. This does not mean, nevertheless, that all $g\psi_0$ are different: there may exist transformations $h \in H \subset G$ which leave invariant a nonzero constant lowest energy state (for a zero lowest energy state, $H = G$ trivially), and therefore the set of *different* lowest energy states corresponds to the quotient space G/H , set where the whole subgroup H is identified with the identity transformation [87]. The topological content of the model is determined by this quotient space G/H , also called the orbit space, and according to it we have found the following cases.

- We have started with a scalar model invariant under *global* transformations, i.e., invariant under the application of the same transformation at all sites $\vec{r}$ of the real space; these transformations are given by the fundamental representation of the the Abelian $U(1) \equiv SO(2)$ group. The potential part ensured the existence of a nonvanishing constant $|\psi|_0$ as the potential's minimum, and thus, as the lowest energy solution since the gradient part $|\partial_\mu \psi|^2$ in the energy is zero for a constant scalar field. There are, nevertheless, infinite many lowest energy solutions, each of them described by a constant phase attached to such a nonzero $|\psi|_0$: $|\psi|_0 e^{i\varphi}$ with $0 \leq \varphi < 2\pi$. Or equivalently, if the scalar field is written as a real vector within an internal 2-dimensional space, $\begin{pmatrix} \psi^{a=1} = \text{Re } \psi \\ \psi^{a=2} = \text{Im } \psi \end{pmatrix} = |\psi|_0 \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$, each configuration of lowest energy is characterized by the orientation φ inside this internal

2-dimensional (iso)space. In this last formulation, a global $U(1)$ transformation $\psi \rightarrow \psi' = e^{-i\chi}\psi$ (with $\chi = \text{constant}$) is written as a rigid rotation matrix in 2 dimensions (since $U(1) \equiv SO(2)$)

$$\begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \end{pmatrix} \rightarrow \begin{pmatrix} \psi'^{a=1} \\ \psi'^{a=2} \end{pmatrix} = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \end{pmatrix} \quad \text{with } \chi = \text{constant}.$$

For the case under study, with a constant nonzero scalar field pointing along a fixed φ_0 direction in the internal 2-dim. isospace, the orbit space G/H is trivially calculated: $H = \mathbf{1}$ and thus $G/H = G = U(1) \equiv S^1$, denoting S^1 a circumference-line whose points are described by the phase angle φ . Any case like this, when the remaining symmetry transformations for the lowest energy states are described by $H = \mathbf{1}$, is called the complete breakdown of the model's symmetry.

Beside the solutions of minimal energy, $\psi(\vec{r}) = |\psi|_0 e^{i\varphi}$ with φ and $|\psi|_0$ constant $\forall \vec{r}$, there exist other solutions of finite energy, all of them characterized by the fact that the scalar field has to tend for $|\vec{r}| \rightarrow \infty$ towards one of the lowest energy states in order to have a finite contribution of the potential. For the global invariant model (the pure scalar model), nevertheless, the scalar field has to tend for $|\vec{r}| \rightarrow \infty$ to the *same* lowest energy state in order not to have an infinite large contribution to the energy arising from $|\partial_\mu \psi|^2$. In other words, in a finite energy configuration for the pure scalar model, the scalar field's orientation φ within the internal 2-dimensional isospace must be the same when approaching the spatial infinity, independently of the direction of approach. A different case occurs for the gauge invariant model.

If the Abelian $U(1)$ symmetry becomes *local* through coupling of the scalar field to a gauge field A_μ , both G as well as H includes now all non-singular gauge transformations, i.e., transformations of the type $\begin{cases} \psi \rightarrow \psi' = e^{-i\chi(\vec{r})}\psi \\ A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \chi \end{cases}$ with $\chi(\vec{r})$ any local function such that its gradient $\partial_\mu \chi$ is *well-defined everywhere*. Such a gauge transformation applied to the fields at external site $\vec{r}$ can be written as a local rotation in the internal 2-dimensional isospace, being in general the isorotation angle at another external site $\vec{r}'$ different from that at $\vec{r}$

$$\begin{cases} \begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \end{pmatrix} \rightarrow \begin{pmatrix} \psi'^{a=1} \\ \psi'^{a=2} \end{pmatrix} = \begin{pmatrix} \cos \chi(\vec{r}) & \sin \chi(\vec{r}) \\ -\sin \chi(\vec{r}) & \cos \chi(\vec{r}) \end{pmatrix} \begin{pmatrix} \psi^{a=1} \\ \psi^{a=2} \end{pmatrix}, \\ A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \chi(\vec{r}). \end{cases} \quad (\text{B.197})$$

The addition of the gauge invariance in the model's symmetry, nevertheless, does not change the orbit space G/H , which remains $G/H = S^1$ as in the pure scalar model: the introduction of the gauge freedom in both G and H has increased (by an infinite amount) the number of configurations which describes the same state. But these latter configurations, although different, are related to one other by means of the just discussed gauge transformations and therefore belong to

the same equivalence class (the equivalence between two configurations is defined as “to be transformed into each other through (B.197)”) and do not modify the quotient space G/H .

In the gauge invariant model, H contains more symmetry operations than the identity alone: once a configuration describing a lowest energy state — as the configuration with a) scalar field $\psi(\vec{r})$ of constant nonzero length (to minimize the potential) pointing in the same internal direction for any $\vec{r}$ of the physical external space, and with b) gauge field $A_\mu(\vec{r}) = 0 \quad \forall \vec{r}$ — or describing a finite energy state with vorticity — as the configuration $\begin{cases} \psi(\vec{r}) = Q(r)e^{i\varphi} \\ \vec{A} = -\hat{\phi} r f(r) \end{cases}$ with the scalar field’s internal orientation φ equal to the external space’s azimuthal angle ϕ — has been found, any local transformation (B.197) applied on this found configuration yields another configuration which, due to the gauge invariance of the model, describes the *same* state. However, if from each class of gauge equivalent configurations one would choose a representative, then for the set of non-equivalent representatives one would obtain again $H = \mathbf{1}$. This is the reason for, in a gauge invariant model, one usually still speaks of complete symmetry breaking if H is formed by the identity transformation alone after having removed the invariance under local transformations by choosing (or fixing) a representative for each class of gauge related configurations. This should be kept in mind since this lax way of speaking is very common and will be often used hereafter.

For the discussion of finite energy configurations, let us consider the sites in the external space placed along a circumference infinitely distant with center at $\vec{r} = \vec{0}$, circumference denoted by S_∞^1 , and let us study a configuration where the scalar field at those sites is pointing in different isodirections: for instance, the orientation of the scalar field in the internal symmetry space coincide with the direction under which we are reaching the infinitely distant circumference (of course, at $\vec{r} \rightarrow \infty$ it only has sense to speak about isodirections for ψ_0 if the scalar field is nonzero). Such a configuration has a higher energy than that where ψ_0 is pointing everywhere in the same direction of the internal space, but even if the different orientations of ψ_0 are still maintained for sites infinitely distant, the resulting energy is *finite* due to the coupling of the scalar field to the gauge field.

This solves the energy finiteness, but what about the stability of such a configuration?: can such a configuration be deformed by means of some of the above discussed gauge transformations into another configuration where the scalar field would be pointing in the same internal direction at all sites of the infinitely distant circumference. As we already know for the case of the Abelian symmetry $U(1)$, this is not possible since these two configurations belong to different homotopy classes. One homotopy class is formed by all those different configurations related to each other by means of any non-singular gauge transformation: if a certain configuration cannot be brought into another configuration then each configuration belongs to a different homotopy class. As long as the gauge symmetry of the system is described by the Abelian $U(1)$ group — whose covering group, the real line $T(1)$, is not compact — the set of different homotopy classes is given by

$\pi_1(G/H) = \pi_1(S^1) = \mathbf{Z}$, where $\pi_1(S^1)$ denotes the different forms to wrap a rubber band around a circumference S^1 : each complete rotation the scalar field draws in its internal (iso)space characterizes completely a homotopy class, a topologically stable configuration. The number of these complete rotations in isospace is also called the winding number. The words “topologically stable” applied to a configuration of a certain homotopy class means that such configuration cannot be deformed into a configuration belonging to another class by means of any allowed gauge transformation.

The study of all configurations assumed by the nonzero scalar field along a circumference of very large radius (placed in the d -dim. external space) is useful to obtain the different topologically stable point defects (for a $(d = 2)$ -dim. space) and line defects (for a $(d = 3)$ -dim. space) located at the center of the circumference. The finite energy of a configuration, where for instance the scalar field’s isodirection winds once when S^1_∞ is covered — described by a phase φ equal to the polar angle ϕ in the real space —, forces the gauge field A_μ through the vanishing of the covariant derivative at $|\vec{r}| \rightarrow \infty$, $(\partial_\mu - ieA_\mu)\psi|_{r \rightarrow \infty} = 0$, to satisfy

$$\partial_\mu \psi|_{r \rightarrow \infty} = i \frac{\hat{\phi}}{r} \psi|_{r \rightarrow \infty} \implies \oint_{\text{circumference at } r \rightarrow \infty} A_\mu dl^\mu = \int_{\phi=0}^{\phi=2\pi} \frac{1}{er} r d\phi = \frac{2\pi}{e}. \quad (\text{B.198})$$

Because of this property, which is invariant under an allowed gauge transformation since $\oint \partial_\mu \chi dl^\mu = 0$, these topological defects are also called vortices, which form a infinitely long line in a 3-dimensional space.

- The next step was to consider the easiest of the non-Abelian simple compact Lie algebras, that characterized by the structure constants $C_{abc} = \epsilon_{abc}$. This algebra is shared by two non-Abelian groups: $SU(2)$ and $SO(3)$, being the first one simply-connected — $\pi_1(SU(2)) = 0$ — and thus the covering group of the second, which is multiply-connected — $\pi_1(SO(3)) \neq 0$. Choosing the $SO(3)$ group as the gauge symmetry for the system, its lowest dimensionality representation is the adjoint representation of the Lie algebra; this representation is generated by the 3×3 matrices $(I_a^{(\text{adj.})})_{bc} = -i\epsilon_{abc}$ (the generators of the rotations in 3 dimensions) which act on a vector called isovector to stress that these rotations are taking place in the internal symmetry space, not in the external space. The higher representations of $SO(3)$ are constructed by means of direct products of the adjoint representation: these higher representations act on tensors of second and higher rank.

A real isovector $\Psi^{a=1,2,3} \equiv \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}$, transforming thus as the adjoint representation

$$(I_a^{(\text{adj.})})_{bc} = -i\epsilon_{abc}$$

$$\begin{aligned} \Psi^a \rightarrow \Psi'^a &= (O_{\hat{n}}(\chi))_{ab} \Psi^b, \\ (O_{\hat{n}}(\chi))_{ab} &= \left(e^{-i\chi \hat{n}_c I_c^{(\text{adj.})}} \right)_{ab} = \hat{n}_a \hat{n}_b (1 - \cos \chi) + \delta_{ab} \cos \chi - \epsilon_{abc} \hat{n}_c \sin \chi, \end{aligned}$$

with χ and $\hat{n}$ the angle and axis of the isorotation respectively (and summing over repeated indices), can be also written as the complex traceless 2×2 tensor $T \equiv \frac{1}{2} \Psi^a \sigma_a = \frac{1}{2} \begin{pmatrix} \Psi^3 & \Psi^1 - i\Psi^2 \\ \Psi^1 + i\Psi^2 & -\Psi^3 \end{pmatrix}$: the rotation of the isovector in the internal (iso)space written in tensor form looks as

$$T \rightarrow T' = \Omega_{\hat{n}}(\chi) T \Omega_{\hat{n}}^+(\chi),$$

$$\Omega_{\hat{n}}(\chi) = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i \hat{n}_a \sigma_a \sin\left(\frac{\chi}{2}\right).$$

Since the energy and the Lagrangian are invariant under such a rotation, the covariant derivative of the scalar field has to transform just as the scalar field and this leads to the transformation law for the gauge field W_μ^a (the notation A_μ for the gauge field is reserved for the Abelian case)⁵⁰

$$\begin{cases} \Psi^a \rightarrow \Psi'^a = \left(O_{\hat{n}}(\chi)\right)_{ab} \Psi^b, \\ W_\mu^a \rightarrow W_\mu'^a = \left(O_{\hat{n}}(\chi)\right)_{ab} W_\mu^b - \frac{1}{e} \hat{n}_a \partial_\mu \chi. \end{cases} \quad (\text{B.199})$$

As also mentioned in equation (B.124), it should be noted that in the transformed gauge field, beside the usual gradient of the phase change $e^{-1} \hat{n}_a \partial_\mu \chi$ characteristic of a gauge transformation, there is also a contribution arising from the isorotation of the gauge field in the internal space: W_μ^a is not only a vector (index μ) in the real space but also a vector (index a) in the isospace and transforms therefore according to the adjoint representation of the group $SO(3)$. This internal structure in the gauge field was absent in the Abelian $U(1)$ model since its corresponding real gauge field A_μ is a scalar in the 2-dimensional internal isospace.

⁵⁰The isovector scalar field transforms as $\Psi^{(\text{vec.})} = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix} \rightarrow \Psi'^{(\text{vec.})} = \left(e^{-i\chi \hat{n}_c I_c^{(\text{adj.})}}\right) \Psi^{(\text{vec.})}$

and this is also the transformation law for the covariant derivative $D_\mu \Psi^{(\text{vec.})} = (\partial_\mu \Psi^{(\text{vec.})} - ie W_\mu^a I_a^{(\text{adj.})} \Psi^{(\text{vec.})}) \rightarrow (D_\mu \Psi^{(\text{vec.})})' = (\partial_\mu \Psi'^{(\text{vec.})} - ie W_\mu'^a I_a^{(\text{adj.})} \Psi'^{(\text{vec.})}) = O_{\hat{n}}(\chi) D_\mu \Psi^{(\text{vec.})}$ with $O_{\hat{n}}(\chi) = \left(e^{-i\chi \hat{n}_c I_c^{(\text{adj.})}}\right)$. The gauge field transforms $W_\mu'^a I_a^{(\text{adj.})} = W_\mu^a O_{\hat{n}} I_a^{(\text{adj.})} O_{\hat{n}}^{-1} - \frac{i}{e} (\partial_\mu O_{\hat{n}}) O_{\hat{n}}^{-1}$. Multiplying this last result by $I_b^{(\text{adj.})}$ and calculating the trace, plus relation $\text{Tr}(I_a^{(\text{adj.})} I_b^{(\text{adj.})}) = (-i)^2 \epsilon_{acd} \epsilon_{bdc} = 2\delta_{ab}$, leads to $W_\mu'^b = \frac{1}{2} W_\mu^a \text{Tr}(O_{\hat{n}} I_a^{(\text{adj.})} O_{\hat{n}}^{-1} I_b^{(\text{adj.})}) - \frac{i}{2e} \text{Tr}((\partial_\mu O_{\hat{n}}) O_{\hat{n}}^{-1} I_b^{(\text{adj.})})$, which operating yields: $\frac{1}{2} \text{Tr}(O_{\hat{n}} I_a^{(\text{adj.})} O_{\hat{n}}^T I_b^{(\text{adj.})}) = -\frac{1}{2} [\hat{n}_c \hat{n}_d (1 - \cos \chi) + \delta_{cd} \cos \chi - \epsilon_{cde} \hat{n}_e \sin \chi] \epsilon_{adf} \times [\hat{n}_f \hat{n}_g (1 - \cos \chi) + \delta_{fg} \cos \chi + \epsilon_{fgh} \hat{n}_h \sin \chi] \epsilon_{bgc} = \hat{n}_b \hat{n}_a (1 - \cos \chi) + \delta_{ba} \cos \chi + \epsilon_{bca} \hat{n}_c \sin \chi$ or equivalently $\frac{1}{2} \text{Tr}(O_{\hat{n}} I_a^{(\text{adj.})} O_{\hat{n}}^T I_b^{(\text{adj.})}) = (O_{\hat{n}}(\chi))_{ba}$, as well as $-\frac{i}{2e} \text{Tr}((\partial_\mu O_{\hat{n}}) O_{\hat{n}}^T I_b^{(\text{adj.})}) = -\frac{1}{2e} (\partial_\mu \chi) \times [\hat{n}_c \hat{n}_d \sin \chi - \delta_{cd} \sin \chi - \epsilon_{cde} \hat{n}_e \cos \chi] [\hat{n}_d \hat{n}_f (1 - \cos \chi) + \delta_{df} \cos \chi + \epsilon_{dfg} \hat{n}_g \sin \chi] \epsilon_{bfc} = -\frac{1}{e} \hat{n}_b \partial_\mu \chi$.

For an infinitesimal transformation, it holds $(O_{\hat{n}}(\chi))_{ba} = \left(e^{-i\chi \hat{n}_c I_c^{(\text{adj.})}}\right)_{ba} \approx \delta_{ba} + \epsilon_{bca} \chi \hat{n}_c$ and thus

$$\begin{cases} \delta \Psi^{(\text{vec.})} = \Psi'^b - \Psi^b = \epsilon_{bca} \chi \hat{n}_c \Psi^a = -i\chi \hat{n}_c I_c \Psi^{(\text{vec.})}, \\ \delta W_\mu^b = W_\mu'^b - W_\mu^b = \epsilon_{bca} \chi \hat{n}_c W_\mu^a - \frac{1}{e} \hat{n}_b \partial_\mu \chi, \end{cases} \quad \text{just as in equation (B.114), with } \chi_c = \hat{n}_c \chi.$$

Now let us repeat the same discussion as for the Abelian $U(1)$ gauge symmetry. If the potential containing the self-interactions of the scalar field Ψ^a ⁵¹ forces lowest energy states Ψ_0^a 's characterized by a nonvanishing constant value $|\Psi_0|$, we may ask for the topological stability of configurations where the scalar field draws a complete rotation in its internal space as we cover a circumference S_∞^1 at $|\vec{r}| \rightarrow \infty$. The answer is to calculate $\pi_1(G/H)$. The orbit space G/H formed by all different lowest energy states is in this case $G/H = SO(3)/U_{\vec{\Psi}_0}(1)$, since all rotations around the direction where isovector $\Psi_0^{a=1,2,3}$ is pointing (rotations denoted by the group $U_{\vec{\Psi}_0}(1)$) leave invariant this isovector. The topological structure of $SO(3)/U_{\Psi_0}(1)$ is that of the surface of a sphere living in a 3-dimensional space, i.e., it corresponds to all possible orientations of a nonvanishing isovector in its 3-dimensional internal isospace: $G/H = SO(3)/U_{\Psi_0}(1) = S^2$. Since $\pi_1(S^2) = 0$ (a rubber band lying on an orange can always be deformed to a point), there is no topologically stable line defects in a 3-dimensional space if the gauge invariant model describing them has $SO(3)$ as symmetry group; the same holds for the 2-dimensional space and its point defects.

However, for the 3-dimensional $SO(3)$ gauge invariant model with one isovector Ψ^a , there exists another kind of topologically stable configurations: point defects. The existence of such defects in a 3-dimensional space is ensured by the fact $\pi_2(G/H) = \pi_2(S^2) \neq 0$, i.e., by the existence of maps from a distant spherical surface S_∞^2 enclosing the point defect onto the orbit space $G/H = S^2$, maps which are not deformable by means of gauge transformations to a mapping where every point on S_∞^2 is related to only one point (=one configuration) of the orbit space $S^2 = G/H$. As an example of a topologically stable configuration is just that where, at each site on the spatially distant sphere (site given by the radial direction of approaching infinity), the isovector of constant length is also pointing in the internal space along the (external) radial direction, $\Psi^a(\vec{r})|_{|\vec{r}| \rightarrow \infty} = \frac{r^a}{r} \times \text{constant}$: the points of orbit space $G/H = S^2$ cover point to point the distant 2-dim. sphere S_∞^2 , and such a mapping cannot be shrunk to a point. With respect to the finite energy of such a configuration, this is ensured by the form of the gauge field which makes vanish the covariant derivative at the sites on S_∞^2

$$D_\mu \Psi^a|_{r \rightarrow \infty} = (\partial_\mu \Psi^a + e \epsilon_{abc} W_\mu^b \Psi^c)|_{r \rightarrow \infty} = 0 \quad \text{with} \quad \partial_\mu \Psi^a|_{r \rightarrow \infty} = \text{const.} \times \left(\frac{\delta_{a\mu}}{r} - \frac{r_a r_\mu}{r^3} \right).$$

Such a gauge field is given by $W_\mu^a(\vec{r})|_{|\vec{r}| \rightarrow \infty} = -\frac{1}{er} \epsilon_{\mu ab} \frac{r^b}{r}$, with $r^b = (x, y, z)$. And the field (see (B.181)) defined as $F_{\mu\nu} = \hat{\Psi}^a (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e \epsilon_{abc} W_\mu^b W_\nu^c) - \frac{\epsilon_{abc}}{e} \hat{\Psi}^a (D_\mu \hat{\Psi}^b) (D_\nu \hat{\Psi}^c)$, with $\hat{\Psi}^a = \frac{\Psi^a}{|\Psi|}$, fulfils the equation of the magnetic field created by a point magnetic monopole, $\begin{cases} \partial_\mu F_{\mu\nu} = 0 \\ \frac{1}{2} \epsilon_{\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = -\frac{4\pi}{e} \delta^3(\vec{r}) \end{cases}$, when rela-

⁵¹In the internal symmetry space, Ψ^a transforms as a vector; in the external space, it is a scalar.

direction, $\Psi^a|_{|\vec{r}| \rightarrow \infty} = \delta_{a3} \sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}}$, being $\sqrt{\frac{|\mathcal{A}|}{\mathcal{B}}} = |\Psi_0|$ a constant value determined by the potential. The resulting equation for the gauge field, $\partial_\mu G_{\mu\nu}^a + e\epsilon_{abc}W_\mu^b G_{\mu\nu}^c = e\epsilon_{abc}\Psi^b(\partial_\nu\Psi^c + e\epsilon_{cde}W_\nu^d\Psi^e)$ with $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + e\epsilon_{abc}W_\mu^b W_\nu^c$, takes at $|\vec{r}| \rightarrow \infty$ the following form

$$\left(\partial_\mu\partial_\mu W_\nu^a - \partial_\nu\partial_\mu W_\mu^a\right)|_{r \rightarrow \infty} = \underbrace{\frac{e^2|\mathcal{A}|}{\mathcal{B}}(1 - \delta_{a3})W_\nu^a|_{r \rightarrow \infty}}_{\text{non-linear terms}} - e\epsilon_{abc}\left[\partial_\mu(W_\mu^b W_\nu^c) + W_\mu^b G_{\mu\nu}^c\right]|_{r \rightarrow \infty}.$$

The underlined term makes $W_\mu^{a=1}$ and $W_\mu^{a=2}$ display an exponential decay and thus, at $r \rightarrow \infty$ both are negligible small. On the other hand, $W_\mu^{a=3}$ is a long-range field and it transforms as the usual Abelian gauge field A_μ under a local isorotation which leaves invariant the chosen direction of $\vec{\Psi}_0$, i.e., under an isorotation around the internal $\hat{z}$ -direction which is also the direction along which $W_\mu^{a=3}$ is pointing and thus the only change in this component of the gauge field is $-\frac{1}{e}\hat{n}_3\partial_\mu\chi$. This is the reason why the choice of isovector $\vec{\Psi}_0$ pointing along the internal $\hat{z}$ -direction is also called to work in the ‘‘Abelian gauge’’⁵³. Now let us substitute into the gauge field the singular solution $W_\mu^{a=3}|_{r \rightarrow \infty} = -\frac{1}{e}(1 - \cos\theta)\partial_\mu\phi$ which describes, within the framework of the usual Abelian $U(1)$ model, a monopole located at $\vec{r} = \vec{0}$ *plus* a line of singularities called the Dirac string; θ and ϕ are the spherical angles in the external space. Of course, such a solution fulfils the equation — for the short-range components $W_\mu^{a=1,2}$ the equation is trivially satisfied at large distances —

$$\left(\partial_\mu\partial_\mu W_\nu^a - \partial_\nu\partial_\mu W_\mu^a\right)|_{r \rightarrow \infty} = \frac{e^2|\mathcal{A}|}{\mathcal{B}}(1 - \delta_{a3})W_\nu^a|_{r \rightarrow \infty} - e\epsilon_{abc}\left[\partial_\mu(W_\mu^b W_\nu^c) + W_\mu^b G_{\mu\nu}^c\right]|_{r \rightarrow \infty},$$

⁵³And this is also the reason why in the curl tensor $F_{\mu\nu} = \frac{\Psi^a}{|\Psi|}G_{\mu\nu}^a + \bar{\alpha}D_\mu\left(\frac{\Psi^a}{|\Psi|}\right)D_\nu\left(\frac{\Psi^a}{|\Psi|}\right) + \bar{\beta}\epsilon_{abc}\frac{\Psi^a}{|\Psi|}D_\mu\left(\frac{\Psi^b}{|\Psi|}\right)D_\nu\left(\frac{\Psi^c}{|\Psi|}\right)$, $\bar{\alpha} = 0$ and $\bar{\beta} = -1/e$ must hold. Since $F_{\mu\nu}$ is invariant under isorotations, we can choose at *every* site $\vec{r}$ of the external space an isorotation such that the resulting Ψ^a is pointing in the $\hat{z}$ -direction of the isospace, $\frac{\Psi^a}{|\Psi|} = \delta_{a3}$ and thus $D_\mu\left(\frac{\Psi^b}{|\Psi|}\right) = e\epsilon_{bd3}W_\mu^d$. In such a case

$$\begin{aligned} \frac{\Psi^a}{|\Psi|}G_{\mu\nu}^a &= \partial_\mu W_\nu^{a=3} - \partial_\nu W_\mu^{a=3} + e\epsilon_{3bc}W_\mu^b W_\nu^c = \\ &= (\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3) + e(W_\mu^1 W_\nu^2 - W_\mu^2 W_\nu^1) \\ + \bar{\alpha}D_\mu\left(\frac{\Psi^a}{|\Psi|}\right)D_\nu\left(\frac{\Psi^a}{|\Psi|}\right) &= +\bar{\alpha}e^2(W_\mu^1 W_\nu^1 + W_\mu^2 W_\nu^2) \\ + \bar{\beta}\epsilon_{abc}\frac{\Psi^a}{|\Psi|}D_\mu\left(\frac{\Psi^b}{|\Psi|}\right)D_\nu\left(\frac{\Psi^c}{|\Psi|}\right) &= +\bar{\beta}e^2\epsilon_{3bc}\epsilon_{bd3}W_\mu^d\epsilon_{ce3}W_\nu^e = +\bar{\beta}e^2(W_\mu^1 W_\nu^2 - W_\mu^2 W_\nu^1) \\ \hline F_{\mu\nu} &= \frac{(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3) + \bar{\alpha}e^2(W_\mu^1 W_\nu^1 + W_\mu^2 W_\nu^2) + (e + \bar{\beta}e^2)(W_\mu^1 W_\nu^2 - W_\mu^2 W_\nu^1)}{1} \end{aligned}$$

By setting $\bar{\alpha} = 0$ and $\bar{\beta} = -1/e$, tensor $F_{\mu\nu} = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3$ takes the well-known form of the usual magnetic field, already knowing that $W_\mu^{a=3}$ is a long-range gauge field which transforms as the gauge potential A_μ of the Abelian model under the gauge transformation which are still available: the isorotations around the $\hat{z}$ -isodirection.

which for $W_\mu^{a=3}$ reduces to $(\partial_\mu \partial_\mu W_\nu^{a=3} - \partial_\nu \partial_\mu W_\mu^{a=3})|_{r \rightarrow \infty} = 0$, because this is just the role of the singular line (=Dirac string) stretching along the $-z$ -direction of the external space; note that $\hat{z}$ is also the direction in the internal isospace where the scalar field is pointing. **Second step:** by means of a gauge isorotation which makes Ψ^a be pointing along the external radial direction, the singular Dirac string is removed and the resulting gauge field at large distances takes the expected form $W_\mu^a(\vec{r})|_{|\vec{r}| \rightarrow \infty} = -\frac{1}{er} \epsilon_{\mu ab} \frac{r^b}{r}$. This last gauge transformation is also called to work in the “spherical gauge”, and it is given by $e^{-i\phi I_3^{(\text{adj.})}} e^{-i\theta I_2^{(\text{adj.})}} e^{-i(-\phi) I_3^{(\text{adj.})}} =$
 $= \begin{pmatrix} \cos \theta \cos^2 \phi + \sin^2 \phi & \cos \theta \cos \phi \sin \phi - \cos \phi \sin \phi & \sin \theta \cos \phi \\ \cos \theta \cos \phi \sin \phi - \cos \phi \sin \phi & \cos \theta \sin^2 \phi + \cos^2 \phi & \sin \theta \sin \phi \\ -\sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \end{pmatrix}$. If in the first step one chooses the $+z$ -direction of the external space as the place for the singularity line, which corresponds to solution $W_\mu^{a=3}|_{r \rightarrow \infty} = \frac{1}{e}(1 + \cos \theta) \partial_\mu \phi$, then the gauge transformation in the second step is given by $e^{-i\phi I_3^{(\text{adj.})}} e^{-i\theta I_2^{(\text{adj.})}} e^{-i\phi I_3^{(\text{adj.})}} =$
 $= \begin{pmatrix} \cos \theta \cos^2 \phi - \sin^2 \phi & -\cos \theta \cos \phi \sin \phi - \cos \phi \sin \phi & \sin \theta \cos \phi \\ \cos \theta \cos \phi \sin \phi + \cos \phi \sin \phi & -\cos \theta \sin^2 \phi + \cos^2 \phi & \sin \theta \sin \phi \\ -\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix}$.

It must be noted that although the behaviour of the topological point defect has been studied in detail only in the region of large distances, region where the configuration energy might have become infinite if there were no gauge field, the radial arrangement of the isovector $\Psi^a \sim \frac{r^a}{r}$ and $W_\mu^a \sim \epsilon_{\mu ab} r^b$ can be implemented at any $\vec{r}$ (achieved by the corresponding local isorotation) by introducing a multiplicative function in each field: $\Psi^a(\vec{r}) = \frac{r^a}{r} Q(r)$ and $W_\mu^a(\vec{r}) = \epsilon_{\mu ab} r^b f(r)$ with $\begin{cases} \lim_{r \rightarrow \infty} Q(r) = \sqrt{|\mathcal{A}|/|\mathcal{B}|} = \text{const} \\ \lim_{r \rightarrow \infty} f(r) = -1/er^2 \end{cases}$. Since this implementation at small distances is the result of local isorotations, tensor $F_{\mu\nu}$ remains unchanged and satisfying

$$F_{\mu\nu} = -\frac{1}{e} \epsilon_{\mu\nu a} \frac{1}{r^2} \frac{r^a}{r} \implies \oint_{\text{distant closed surface}} F_{\mu\nu} dS^{\mu\nu} = -\frac{4\pi}{e}. \quad (\text{B.200})$$

- In the third last model which was studied is that given in equation (B.136): the so-called “ $SO(3)$ ”-superconductor. A scalar isovector Ψ^a , characterized by its nonzero length and two angles (Θ, Φ) defining the isovector’s direction in the 3-dimensional isospace, broke the $SO(3)$ symmetry down to the residual $U(1)$ symmetry of isorotations around the internal $\hat{z}$ -isodirection where the constant isovector is pointing; and the further introducing of an $U(1)$ scalar field ψ , characterized by its length and angle φ in the 2-dimensional internal symmetry space, broke completely this residual symmetry. Their corresponding potentials $V(\Psi^a)$ and $V'(\psi)$ were chosen in such a way that their minima were obtained for a constant nonzero length of the respective scalar fields. As a result of the complete breakdown of the $SO(3)$ symmetry, the orbit space is given by $G/H = G = SO(3)$,

and due to the topology of $SO(3)$ stable vortices appear again: this latter group is topologically equivalent to $\mathbb{S}^3$ (the 3-dimensional surface of a sphere “living” in a 4 dimensions, with the antipodal points identified) and according to (B.111), $\pi_1(G/H) = \pi_1(\mathbb{S}^3) = Z_2 \equiv 0, 1$, which ensures thus the existence of one type of stable line defects in the 3-dimensional (physical) real space. The following implication is noteworthy ⁵⁴ for the $SO(3)$ gauge group

$$\boxed{\text{complete “breakdown” of gauge symmetry} \implies \text{existence of vortices}} \quad (\text{B.201})$$

Vortices as stable field configurations was something which does not occurred if there was only a partial breaking of $SO(3)$ down to $U(1)$ through the presence of only one scalar isovector, since then $G/H = SO(3)/U(1) \equiv S^2$ and $\pi_1(S^2) = 0$. Another form of understand the appearance of vortices in the above model is by noting that in the model with only one scalar isovector (model with $G/H = S^2$), the magnetic field created by the monopole spread everywhere, in all directions: it is the further coupling of such a magnetic field to another scalar field which causes the gauge field to become short-range, only existing confined within certain regions, the vortices.

We shall now formulate the model with the two different scalar fields in a bit more general way, in preparation for more complicated schemes of breakdown of the symmetry $SO(3)$, always keeping in mind that a complete breakdown of the gauge group $SO(3)$ means appearance of stable line defects (=vortices) in a 3-dimensional (external) space. Following [72], let us start the discussion by means of the use of two scalar isovectors Ψ^a and $\tilde{\Psi}^a$, provided they are not pointing in the same (or in the opposite) direction in the 3-dimensional isospace. Constant nonvanishing Ψ_0^a and $\tilde{\Psi}_0^a$ are ensured by means of the following potential

$$\begin{aligned} V(\Psi^a, \tilde{\Psi}^a) = & -|\mathcal{A}|\Psi^a\Psi^a + \frac{|\mathcal{B}|}{2}(\Psi^a\Psi^a)^2 - |\tilde{\mathcal{A}}|\tilde{\Psi}^a\tilde{\Psi}^a + \frac{|\tilde{\mathcal{B}}|}{2}(\tilde{\Psi}^a\tilde{\Psi}^a)^2 - \\ & -|\mathcal{C}|\Psi^a\tilde{\Psi}^a + \frac{|\mathcal{D}|}{2}(\Psi^a\tilde{\Psi}^a)^2. \end{aligned} \quad (\text{B.202})$$

For $\begin{cases} -2\mathcal{A}|\Psi| + 2|\mathcal{B}||\Psi|^3 - \mathcal{C}|\tilde{\Psi}|\cos\Theta_0 + |\mathcal{D}||\Psi||\tilde{\Psi}|^2\cos\Theta_0 = 0 \\ -2\tilde{\mathcal{A}}|\tilde{\Psi}| + 2|\tilde{\mathcal{B}}||\tilde{\Psi}|^3 - \mathcal{C}|\Psi|\cos\Theta_0 + |\mathcal{D}||\Psi|^2|\tilde{\Psi}|\cos\Theta_0 = 0 \end{cases}$ the minimum occurs, where Θ_0 is the fixed angle in isospace between Ψ^a and $\tilde{\Psi}^a$, and the solu-

⁵⁴This “breakdown” of a gauge symmetry means that, once the orientations for the scalar fields have been chosen in their respective isospaces for every site $\vec{r}$ of the external physical space, plus their corresponding gauge field arranged in order to minimize and make finite the configuration energy, there is no symmetry operation left except for local deformations (=gauge transformations) of such a configuration which, due to the gauge invariance of the model, does not modify its energy. Or equivalently, if one chooses a representative of all these configurations with the same energy, then the only symmetry operation which leaves invariant this representative is the identity. If one can find at least two representatives which can not be taken into one other by means of any gauge transformation, then there exist at least two different classes of configurations, each class formed by field arrangements related among them through gauge transformations but being a configuration of one class not gauge transformable into a configuration of the other class(es). This case corresponds to a homotopy group $\pi_n(G/H) \neq 0$, with $n = 1$ if we are studying line defects in the 3-dimensional physical external space or with $n = 2$ if the subject under study are point defects in the same physical space.

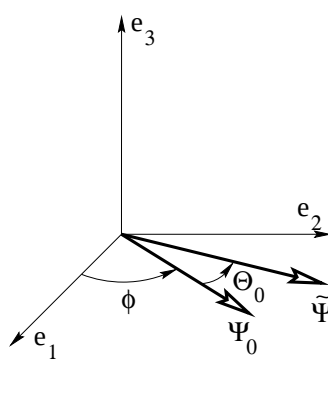
tions are $|\Psi_0|^2 = \frac{|\mathcal{A}|}{|\mathcal{B}|}$, $|\tilde{\Psi}_0|^2 = \frac{|\tilde{\mathcal{A}}|}{|\tilde{\mathcal{B}}|}$ and $\cos^2 \Theta_0 = \frac{\mathcal{C}^2}{|\mathcal{D}|^2} \frac{\mathcal{A}\tilde{\mathcal{A}}}{|\mathcal{B}||\tilde{\mathcal{B}}|} < 1$. The condition on $\cos^2 \Theta_0$ ensures that Ψ_0^a and $\tilde{\Psi}_0^a$ are not parallel in isospace: the first isovector Ψ^a breaks $SO(3)$ down to rotations about Ψ_0^a , rotations described by the group $U_{\Psi_0}(1)$. But since $\tilde{\Psi}_0^a$ is not parallel to Ψ_0^a , $U_{\Psi_0}(1)$ rotations do not leave invariant $\tilde{\Psi}_0^a$ and this residual symmetry $U_{\Psi_0}(1)$ is broken down too by the second isovector. The Lagrangian density describing this model is given by

$$\ell = -\frac{1}{4}(G_{\mu\nu}^a)^2 - \frac{1}{2}(\partial_\mu \Psi^a + e\epsilon_{abc}W_\mu^b\Psi^c)^2 - \frac{1}{2}(\partial_\mu \tilde{\Psi}^a + e\epsilon_{abc}W_\mu^b\tilde{\Psi}^c)^2 - V, \quad (\text{B.203})$$

with $V = V(\Psi^a, \tilde{\Psi}^a)$ equal to (B.202). Let us study the vortex configurations by considering the model at large distances $|\vec{r}|$ normal to the vortex, required that the scalar isovectors tend to one of the minima of the potential when we approach this spatial infinity S_∞^1 . Both covariant derivatives must vanish at these large distances to avoid divergences caused when the nonzero scalar isovectors take *different* orientations in isospace depending on which direction one approaches in the physical space the spatial infinity:

$$(\partial_\nu \Psi^c + e\epsilon_{cde}W_\nu^d\Psi^e)\big|_{r \rightarrow \infty} = (\partial_\nu \tilde{\Psi}^c + e\epsilon_{cde}W_\nu^d\tilde{\Psi}^e)\big|_{r \rightarrow \infty} = 0.$$

Multiplying these equations respectively by $\epsilon_{abc}\Psi_0^b$ the first and by $\epsilon_{abc}\tilde{\Psi}_0^b$ the second one, and summing them together, one obtains the following relation



$$\begin{aligned}
0 &= \epsilon_{abc}\Psi_0^b D_\nu \Psi^c \big|_{r \rightarrow \infty} + \epsilon_{abc}\tilde{\Psi}_0^b D_\nu \tilde{\Psi}^c \big|_{r \rightarrow \infty} = \\
&= \epsilon_{abc} \left(\Psi_0^b \partial_\nu \Psi_0^c + \tilde{\Psi}_0^b \partial_\nu \tilde{\Psi}_0^c \right) - e \Psi_0^a \tilde{\Psi}_0^a \cdot \vec{W}_\nu \big|_{r \rightarrow \infty} - \\
&\quad - e \tilde{\Psi}_0^a \tilde{\Psi}_0^a \cdot \vec{W}_\nu \big|_{r \rightarrow \infty} + e W_\nu^a \big|_{r \rightarrow \infty} (|\Psi_0|^2 + |\tilde{\Psi}_0|^2). \quad (\text{B.204})
\end{aligned}$$

From the many directions which the two isovectors can take in isospace as minima of $V(\Psi^a, \tilde{\Psi}^a)$, let us choose those where both isovectors are lying in the $\hat{e}_1\hat{e}_2$ -plane

$$\begin{aligned}
\Psi_0^a &= \sqrt{\frac{|\mathcal{A}|}{|\mathcal{B}|}} \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}, \\
\tilde{\Psi}_0^a &= \sqrt{\frac{|\tilde{\mathcal{A}}|}{|\tilde{\mathcal{B}}|}} \begin{pmatrix} \cos(\phi + \Theta_0) \\ \sin(\phi + \Theta_0) \\ 0 \end{pmatrix},
\end{aligned} \quad (\text{B.205})$$

with $\Theta_0 \neq 0, \pi$, and ϕ as usual the azimuthal angle around the z -direction of the

external space. Applying these isovectors to (B.204)

$$\begin{aligned}
a = 1 : \quad 0 &= W_\mu^{a=1} \Big|_{r \rightarrow \infty} \left(|\Psi_0|^2 \sin^2 \phi + |\tilde{\Psi}_0|^2 \sin^2(\phi + \Theta_0) \right) - \\
&\quad - \frac{1}{2} W_\mu^{a=2} \Big|_{r \rightarrow \infty} \left(|\Psi_0|^2 \sin 2\phi + |\tilde{\Psi}_0|^2 \sin(2\phi + 2\Theta_0) \right), \\
a = 2 : \quad 0 &= W_\mu^{a=2} \Big|_{r \rightarrow \infty} \left(|\Psi_0|^2 \cos^2 \phi + |\tilde{\Psi}_0|^2 \cos^2(\phi + \Theta_0) \right) - \\
&\quad - \frac{1}{2} W_\mu^{a=1} \Big|_{r \rightarrow \infty} \left(|\Psi_0|^2 \sin 2\phi + |\tilde{\Psi}_0|^2 \sin(2\phi + 2\Theta_0) \right), \\
a = 3 : \quad 0 &= |\Psi_0|^2 \partial_\nu \phi + |\tilde{\Psi}_0|^2 \partial_\nu(\phi + \Theta_0) + e W_\nu^{a=3} \Big|_{r \rightarrow \infty} \left(|\Psi_0|^2 + |\tilde{\Psi}_0|^2 \right),
\end{aligned}$$

it results that at large distances $W_\nu^{a=1} = W_\nu^{a=2} = 0$, and $W_\nu^{a=3}$ has the form of an Abelian $U(1)$ gauge field describing a vortex parallel to the z -direction

$$W_\nu^{a=1} \Big|_{r \rightarrow \infty} = W_\nu^{a=2} \Big|_{r \rightarrow \infty} = 0 \quad \text{and} \quad W_\nu^{a=3} \Big|_{r \rightarrow \infty} = -\frac{1}{e} \partial_\nu \phi. \quad (\text{B.206})$$

Therefore, when the pair of isovectors draw a complete rotation about the $\hat{e}_3$ -direction of isospace, in the ordinary external space this isorotation yields a vortex pointing in the external z -direction, whose flux is equal to

$$\Phi_0 = \oint_{C \text{ at } r \rightarrow \infty} W_\mu^{a=3} dl^\mu = -\frac{2\pi}{e}, \quad (\text{B.207})$$

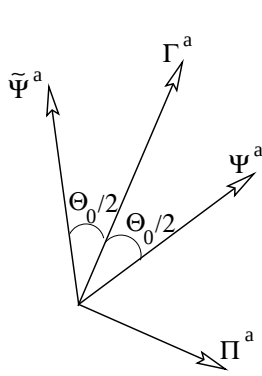
always remembering that, within a model with an underlying $SO(3)$ gauge symmetry, a double isorotation which would have a flux of $4\pi \times \frac{1}{e}$ can be deformed to a zero rotation and thus $|\Phi_0| = \frac{2\pi}{e}$ is the only possible flux topologically stable.

Of course, the direction along which the vortex is placed depends on the isodirections chosen (depends on the choice of gauge) for the two scalar isovectors, but the flux amount carried by the vortex is gauge invariant.

Monopoles, as isolated point defects in the 3-dimensional physical space, do not exist in this model: their existence is determined by the second homotopy group $\pi_2(G/H)$, which in the case of complete breakdown of the $G = SO(3)$ gauge symmetry is equal to $\pi_2(SO(3)/H) = \pi_2(SO(3)) = 0$, due to the Cartan theorem which states that any compact Lie group G satisfies $\pi_2(G) = 0$ [69]. Monopoles only occur at the sites where two stable vortices react forming a double flux object, flux which is not topologically stable and gets thus absorbed by the monopole appearing there. We obtain thus the picture of two vortices running approximately parallel and in that point where they cross, a pair of monopoles of opposite charged flux $\mp \frac{4\pi}{e}$ are formed: one of these monopoles absorbs the magnetic flux carried by the two vortices reacting there, and the other opposite charged monopole acts as a source for the two vortices which continue again without touching each other.

Note. Flux lines (or vortices) attached to a monopole [21].

The above discussion leading to result (B.207) may appear obscure, so let us see in more detail why the above model, characterized by the complete breakdown of the $SO(3)$ gauge symmetry, cannot contain isolated magnetic monopoles: they only exist if attached to flux lines. Without fixing yet the self-interactions of the scalar fields contained into the potential $V(\Psi^a, \tilde{\Psi}^a)$, although assuming (for the sake of simplicity) that the lengths of both scalar fields at large distances are the same, the equation for the (non-Abelian) gauge field becomes equal to $(\partial_\mu \partial_\mu W_\nu^a - \partial_\nu \partial_\mu W_\mu^a) = -e\epsilon_{abc} [\partial_\mu (W_\mu^b W_\nu^c + W_\mu^c W_\nu^b)] + e\epsilon_{abc} (\Psi^b \partial_\nu \Psi^c + \tilde{\Psi}^b \partial_\nu \tilde{\Psi}^c) + e^2 (|\Psi|^2 + |\tilde{\Psi}|^2) W_\nu^a - e^2 [\Psi^a (\Psi^c W_\nu^c) + \tilde{\Psi}^a (\tilde{\Psi}^c W_\nu^c)]$. In order to keep the same scheme with the two steps as in the isolated monopole configuration (see discussion previous to (B.200)), let us introduce the following linear combinations of the scalar isovectors, $\Gamma^a = \frac{\Psi^a + \tilde{\Psi}^a}{\sqrt{2}}$ and



$\Pi^a = \frac{\Psi^a - \tilde{\Psi}^a}{\sqrt{2}}$, as represented in the figure: the equation for the gauge field keeps the same form as when expressed in terms of $\Psi^a, \tilde{\Psi}^a$

$$\begin{aligned} (\partial_\mu \partial_\mu W_\nu^a - \partial_\nu \partial_\mu W_\mu^a) = & -e\epsilon_{abc} [\partial_\mu (W_\mu^b W_\nu^c + W_\mu^c W_\nu^b)] + \\ & + e\epsilon_{abc} (\Gamma^b \partial_\nu \Gamma^c + \Pi^b \partial_\nu \Pi^c) + \\ & + e^2 (|\Gamma|^2 + |\Pi|^2) W_\nu^a - e^2 [\Gamma^a (\Gamma^c W_\nu^c) + \Pi^a (\Pi^c W_\nu^c)]. \end{aligned} \quad (\text{B.208})$$

Now let us study the solutions of the above equations at large distances, where the length of both scalar isovectors becomes a constant, by following the same steps as in the discussion of an isolated monopole.

First step: by choosing the “Abelian gauge” $\Gamma^a|_{r \rightarrow \infty} = |\Gamma|_{r \rightarrow \infty} \delta_{a3}$, with $|\Gamma|_{r \rightarrow \infty}$ constant — isovector Π^a , orthogonal to Γ^a , has no component for $a = 3$ in this choice of configuration — and by assuming that at such long distances the gauge field’s components $W_\mu^{a=1,2}$ are already damped⁵⁵ (i.e., are of shorter range than $W_\mu^{a=3}$), the equation for $W_\mu^{a=3}$ becomes

$$(\partial_\mu \partial_\mu W_\nu^{a=3} - \partial_\nu \partial_\mu W_\mu^{a=3})|_{r \rightarrow \infty} = e\epsilon_{3bc} (\Pi^b \partial_\nu \Pi^c)|_{r \rightarrow \infty} + \underline{e^2 |\Pi|^2 W_\nu^{a=3}}|_{r \rightarrow \infty}, \quad (\text{B.209})$$

indicating the underlined term — as different from the case where the $SO(3)$ gauge model was coupled

to only one scalar isovector — that $W_\mu^{a=3}$ is also short-range. By writing isovector $\Pi^a = |\Pi| \begin{pmatrix} \cos \Phi \\ \sin \Phi \\ 0 \end{pmatrix} =$

$= |\Psi| \sqrt{2(1 - \cos \Theta_0)} \begin{pmatrix} \cos \Phi \\ \sin \Phi \\ 0 \end{pmatrix}$ (with Φ the internal orientation of Π^a on the plane normal to Γ^a) as

a $U(1)$ scalar field $\psi^{a=1,2} = |\Psi| \begin{pmatrix} \cos \Phi \\ \sin \Phi \end{pmatrix}$, equation (B.209) together with the equation for the just defined scalar field⁵⁶

$$D_\mu D_\mu \psi^b = \frac{\psi^b}{|\psi|} \frac{dV}{d|\psi|} \quad \text{with} \quad D_\mu \psi^b = \partial_\mu \psi^b + e\epsilon_{bc} W_\mu^{a=3} \psi^c, \quad (\text{B.210})$$

⁵⁵From (B.208) the linear terms determining the behaviour of $W_\mu^{a=1,2}$ at large distances are, for instance for $a = 1$, $(\partial_\mu \partial_\mu W_\nu^{a=1} - \partial_\nu \partial_\mu W_\mu^{a=1}) = e^2 W_\mu^{a=1} (|\Gamma|^2 + |\Pi|^2 - (\Pi^{a=1})^2)$, and thus condition $|\Gamma| \gg |\Pi|$ suffices (see (B.209)) to enforce that the range of $W_\mu^{a=1,2}$ is shorter than that of $W_\mu^{a=3}$. Relation $|\Gamma| \gg |\Pi|$ is equivalent to $\Theta_0 \ll 1$.

⁵⁶From (B.188) $D_\mu D_\mu \Psi^b = \frac{\Psi^b}{|\Psi|} \frac{dV}{d|\Psi|} = 2\Psi^b \frac{dV}{d|\Psi|^2}$ and from $|\Psi|^2 = |\tilde{\Psi}|^2 = \frac{|\Pi|^2}{2(1 - \cos \Theta_0)}$ it follows $D_\mu D_\mu \Pi^b = \frac{D_\mu D_\mu \Psi^b - D_\mu D_\mu \tilde{\Psi}^b}{\sqrt{2}} = 4(1 - \cos \Theta_0) \Pi^b \frac{dV}{d|\Pi|^2}$ and thus (B.210).

are just the equations (B.44) for the usual superconductor coupled to a $U(1)$ gauge field $A_\mu \equiv W_\mu^{a=3}$. It is important to note that if both original scalar fields $\Psi^a, \tilde{\Psi}^a$ were parallel to each other, something which would correspond to the $SO(3)$ gauge field coupled to only one isovector, then $\Pi^a = 0$ and therefore $W_\mu^{a=3}$ would become a infinite-range field, just as in the isolated monopole model.

As in the gauge model where the $SO(3)$ symmetry was partially broken down to $U(1)$ through the coupling to one scalar isovector, the singular solution corresponding to a Dirac string in the “Abelian gauge” is the solution of the homogeneous equation $(\partial_\mu \partial_\mu W_\nu^{a=3} - \partial_\nu \partial_\mu W_\mu^{a=3}) \Big|_{r \rightarrow \infty} = 0$ for a monopole placed at the coordinate origin. In case under study now, the general solution with a nonvanishing magnetic flux crossing any closed surface around the coordinate origin is given by this Dirac string *plus* a particular non-singular solution of the short-range field equation (B.209). The magnetic field $F_{\mu\nu} = \frac{\Gamma^a}{|\Gamma|} G_{\mu\nu}^a - \frac{1}{e} \epsilon_{abc} \frac{\Gamma^a}{|\Gamma|} D_\mu \left(\frac{\Gamma^b}{|\Gamma|} \right) D_\nu \left(\frac{\Gamma^c}{|\Gamma|} \right)$ — field which is *invariant*, and not simply covariant, under any $SO(3)$ internal isorotation and thus *independent* of the choice for the gauge field W_μ^a — reduces itself, as expected, to $\partial_\mu W_\nu^{a=3} - \partial_\nu W_\mu^{a=3}$ in the “Abelian gauge” and therefore $F_{\mu\nu}$ is short-range, according to (B.209).

Second step: the change from the “Abelian” to the “spherical gauge” for scalar isovector Γ^a — choice of, among all gauge equivalent configurations, that where isovector Γ^a points inside the internal space in the radial direction of the physical external space $\frac{\Gamma^a}{|\Gamma|} \sim \frac{r^a}{r}$ — removes the Dirac string leaving the (point) monopole. Such a removing runs as follows: $F_{\mu\nu} = \frac{\Gamma^a}{|\Gamma|} G_{\mu\nu}^a - \frac{1}{e} \epsilon_{abc} \frac{\Gamma^a}{|\Gamma|} D_\mu \left(\frac{\Gamma^b}{|\Gamma|} \right) D_\nu \left(\frac{\Gamma^c}{|\Gamma|} \right)$ can also be written [6] as ⁵⁷

$$F_{\mu\nu} = \partial_\mu \left(\frac{\Gamma^a}{|\Gamma|} W_\nu^a \right) - \partial_\nu \left(\frac{\Gamma^a}{|\Gamma|} W_\mu^a \right) - \frac{1}{e} \epsilon_{abc} \frac{\Gamma^a}{|\Gamma|} \partial_\mu \left(\frac{\Gamma^b}{|\Gamma|} \right) \partial_\nu \left(\frac{\Gamma^c}{|\Gamma|} \right), \quad (\text{B.211})$$

and it is just the last term in $F_{\mu\nu}$ which for the “spherical gauge” carries the magnetic field which had been created by the singular Dirac string in the “Abelian gauge”. This results from relation (B.178)

$$\left(\epsilon_{abc} \frac{\Gamma^a}{|\Gamma|} \partial_\mu \left(\frac{\Gamma^b}{|\Gamma|} \right) \partial_\nu \left(\frac{\Gamma^c}{|\Gamma|} \right) \right) \Big|_{\text{spher. gauge}} = -\epsilon_{a\mu\nu} \partial_a \left(\frac{1}{r} \right), \quad (\text{B.212})$$

$$\frac{1}{2} \epsilon_{\lambda\mu\nu} \partial_\lambda \left[\left(\epsilon_{abc} \frac{\Gamma^a}{|\Gamma|} \partial_\mu \left(\frac{\Gamma^b}{|\Gamma|} \right) \partial_\nu \left(\frac{\Gamma^c}{|\Gamma|} \right) \right) \Big|_{\text{spher. gauge}} \right] = 4\pi \delta^3(\vec{r}),$$

and hence in the “spherical gauge”, $\frac{\Gamma^a}{|\Gamma|} W_\nu^a$ contains only the non-singular part of the solution for $W_\mu^{a=3}$ in the “Abelian gauge” ⁵⁸.

Further, since the magnetic field $F_{\mu\nu}$ cannot spread everywhere because of its short-range, and on the other hand the magnetic flux across any closed surface must be nonzero, then the magnetic

⁵⁷Since for any Ψ^a relation (B.170) holds, $\frac{\Psi^c}{|\Psi|} D_\mu \left(\frac{\Psi^c}{|\Psi|} \right) = 0$, and further $\epsilon_{abc} \frac{\Psi^a}{|\Psi|} D_\mu \left(\frac{\Psi^b}{|\Psi|} \right) = \epsilon_{abc} \frac{\Psi^a}{|\Psi|} \partial_\mu \left(\frac{\Psi^b}{|\Psi|} \right) - e \frac{\Psi^c}{|\Psi|} \left(\frac{\Psi^e}{|\Psi|} W_\mu^e \right) + e W_\mu^c$, then one obtains relation $\epsilon_{abc} \frac{\Psi^a}{|\Psi|} D_\mu \left(\frac{\Psi^b}{|\Psi|} \right) D_\nu \left(\frac{\Psi^c}{|\Psi|} \right) = \epsilon_{abc} \frac{\Psi^a}{|\Psi|} \partial_\mu \left(\frac{\Psi^b}{|\Psi|} \right) \partial_\nu \left(\frac{\Psi^c}{|\Psi|} \right) - e W_\nu^c \partial_\mu \left(\frac{\Psi^c}{|\Psi|} \right) + e W_\mu^c \partial_\nu \left(\frac{\Psi^c}{|\Psi|} \right) + e^2 \epsilon_{abc} \frac{\Psi^a}{|\Psi|} W_\mu^b W_\nu^c$. Hence part $\frac{\Psi^a}{|\Psi|} G_{\mu\nu}^a$ becomes $\partial_\mu \left(\frac{\Psi^a}{|\Psi|} W_\nu^a \right) - \partial_\nu \left(\frac{\Psi^a}{|\Psi|} W_\mu^a \right) + \frac{1}{e} \epsilon_{abc} \frac{\Psi^a}{|\Psi|} D_\mu \left(\frac{\Psi^b}{|\Psi|} \right) D_\nu \left(\frac{\Psi^c}{|\Psi|} \right) - \frac{1}{e} \epsilon_{abc} \frac{\Psi^a}{|\Psi|} \partial_\mu \left(\frac{\Psi^b}{|\Psi|} \right) \partial_\nu \left(\frac{\Psi^c}{|\Psi|} \right)$.

⁵⁸Although $\Gamma^a W_\mu^a$ is a scalar product of two isovectors and thus it should remain invariant under any isorotation, it does change under any isorotation because of the external index μ of the gauge field and its corresponding gradient of the isorotation matrix characteristic of any gauge transformation. The easiest form to obtain the transformation equation is by working with the $SU(2)$ tensor notation for the scalar field, $T \equiv \frac{1}{2} \Gamma^a \sigma_a = \frac{1}{2} \begin{pmatrix} \Gamma^{a=3} & \Gamma^{a=1} - i\Gamma^{a=2} \\ \Gamma^{a=1} + i\Gamma^{a=2} & -\Gamma^{a=3} \end{pmatrix}$, and for the gauge field, $W_\mu = \frac{1}{2} W_\mu^a \sigma_a$. The transformation of tensor W_μ under an isorotation of a scalar isovector

field of the monopole has to be confined inside a region which can maintain stably this magnetic flux. And such a region are the flux lines or vortices. As a result, therefore, as long as the magnetic field is exponentially damped, and this occurs for $|\Pi| \neq 0$ or equivalently for $\Theta_0 \neq 0$, any magnetic monopole has to be attached to a certain number of vortices which carry inside them the monopole's magnetic field.

Other forms of breaking the compact $SO(3)$ gauge symmetry are obtained through the coupling of the gauge field to a scalar (scalar in the external space) isotensor (tensor in the 3-dimensional internal space) of rank 2 or higher: $\Psi^{ab\dots}$, with $a, b, \dots = 1, 2, 3$. If the energy's potential part is chosen with the appropriate self-interactions of the scalar isotensor, such a breaking of the compact gauge symmetry will cause a spectrum of vortices in an external 3-dimensional space richer than that obtained for the scalar isovector alone.

B.7.1 The fundamental theorem on π_1 and on π_2

But before embarking on such a discussion, let us firstly consider a more general form of calculating $\pi_1(G/H)$. As mentioned at the begin of this Appendix (see comments after equation (B.3)), the classes of loops or maps $g(\phi) \in G$ from S_∞^1 — where S_∞^1 is the spatial infinity parametrized by azimuthal angle $0 \leq \phi < 2\pi$ and which surrounds the vortex whose topological stability we want to study — onto G/H can be calculated based at any “point” y_0 of G/H , where here “point” y_0 refers to a configuration assumed by the model's fields. For the following calculation it is more useful to consider loops based at the whole subgroup H , the set of transformations which leave invariant the lowest energy states: in the coset space G/H , the subgroup H is identified with only

$$T \rightarrow T' = \frac{1}{2} \Psi'^a \sigma_a = \Omega T \Omega^+, \text{ with } \Omega(\omega, \zeta, \gamma) = \begin{pmatrix} e^{-i\omega/2} e^{-i\gamma/2} \cos(\zeta/2) & -e^{-i\omega/2} e^{i\gamma/2} \sin(\zeta/2) \\ e^{i\omega/2} e^{-i\gamma/2} \sin(\zeta/2) & e^{i\omega/2} e^{i\gamma/2} \cos(\zeta/2) \end{pmatrix}, \text{ when this scalar}$$

field is coupled to the gauge field through a covariant derivative is given by (B.121): $\mathcal{W}_\mu \rightarrow \mathcal{W}'_\mu = \Omega \mathcal{W}_\mu \Omega^+ - \frac{i}{e} (\partial_\mu \Omega) \Omega^+$.

On the other hand, relation $\Gamma^a W_\mu^a = 2\text{Tr}(T \mathcal{W}_\mu)$ gives as result

$$\Gamma'^a W_\mu'^a = \underbrace{2\text{Tr}(\Omega T \Omega^+ \Omega \mathcal{W}_\mu \Omega^+)}_{2\text{Tr}(\Omega^+ \Omega T \mathcal{W}_\mu)} - \frac{2i}{e} \text{Tr}(\Omega T \Omega^+ (\partial_\mu \Omega) \Omega^+) = \Gamma^a W_\mu^a - \frac{2i}{e} \text{Tr}(T \Omega^+ \partial_\mu \Omega),$$

$$\text{with } -\frac{2i}{e} \Omega^+ \partial_\mu \Omega = -\frac{1}{e} \begin{pmatrix} \partial_\mu \omega + \cos \zeta \partial_\mu \gamma & -(\sin \zeta \cos \omega \partial_\mu \gamma - \sin \omega \partial_\mu \zeta) - i(\sin \zeta \sin \omega \partial_\mu \gamma + \cos \omega \partial_\mu \zeta) \\ -(\sin \zeta \cos \omega \partial_\mu \gamma - \sin \omega \partial_\mu \zeta) + i(\sin \zeta \sin \omega \partial_\mu \gamma + \cos \omega \partial_\mu \zeta) & -\partial_\mu \omega - \cos \zeta \partial_\mu \gamma \end{pmatrix}. \text{ If for the “Abelian}$$

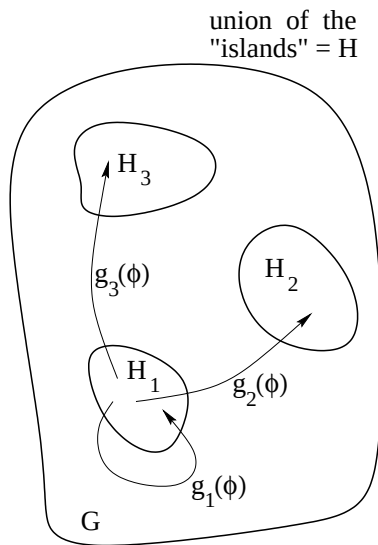
gauge” $\Gamma^a W_\mu^a = |\Gamma| W_\mu^{a=3}$ with $W_\mu^{a=3}$ containing a Dirac string of the form $\frac{1}{e}(1 + \cos \theta) \partial_\mu \phi$ (θ, ϕ are the polar angles in external space) then the “spherical gauge” is obtained by means of rotation matrix $\Omega(\omega = \phi, \zeta = \theta, \gamma = \phi)$ and hence

$$\frac{\Gamma'^a}{|\Gamma|} W_\mu'^a \equiv \left(\frac{\Gamma^a}{|\Gamma|} W_\mu^a \right) \Big|_{\text{spher. gauge}} = \underbrace{\left(\frac{\Gamma^a}{|\Gamma|} W_\mu^a \right) \Big|_{\text{Abel. gauge}}}_{W_\mu^{a=3} \Big|_{\text{Abel. gauge}}} - \frac{1}{e} (1 + \cos \theta) \partial_\mu \phi,$$

proving that the “spherical gauge” removes the Dirac string from the $\frac{\Gamma^a}{|\Gamma|} W_\mu^a$ calculated at the “Abelian gauge”.

one transformation (=one “point” y_0) of the energy’s symmetry group G .

Let us assume Lie group G is simply-connected (the reason for such an assumption will be explained later), $\pi_1(G) = 0$, or equivalently, one can deform to the identity any closed map covering configurations contained in G : for instance, a map which makes each point along a closed path in the external space correspond a (in general) different field configuration, can be brought by means of gauge transformations to a correspondence where each external point is mapped to the same configuration; the just mentioned configurations have not to be necessarily one of those corresponding to nonzero scalar field (those which belong to the more restricted orbit space G/H). The subgroup H nevertheless can be multiply-connected, i.e., formed by several separated “islands”, being each piece connected (its configurations are related to each other by means of transformations located within the “island” itself) or formed by one sole transformation. One of these islands must contain the identity transformation, since the latter always belongs



to the subgroup H . As one covers S^1_∞ in the external space from $\phi = 0$ to $\phi = 2\pi$, loop $g(\phi)$ follows a path within G starting at one of the islands and ending at the same or at another of the pieces forming H : by definition, a loop begins and finishes at the same base “point”, only that here the base “point” is the whole subgroup H , since as already explained we are interested in the loops mapping on G/H . In the figure there are three of these loops, acting as representatives for all those loops in G which connects configurations belonging to H ; in fact, there are many more loops as the three represented, for instance, a loop which starts at any of the island points and, without leaving this island (i.e., this loop’s part belongs entirely to H), connects with the starting point of one of the three loops.

It is clear from the previous figure that depending whether the number of separated “islands” constituting H is larger than one — there is at least one island formed by the identity transformation — or not, we shall have topologically stable loops, $\pi_1(G/H) \neq 0$, or not. Note that loops from one island of H to another different island of H (and thus a closed path since in G/H the whole H is identified to a point) cannot be shrunk to a point. And note also that G has to be simply-connected, this means, without holes or folded regions glued together: otherwise, even if H were formed by only one piece, had G for instance a hole which would be placed inside H , then any two arbitrary paths within H could not be deformed into each other since the deformation process would have to go across the hole, abandoning therefore H . All this is nothing else as a sketched proof of the fundamental theorem on $\pi_1(G/H)$, whose detailed proof can be found in [69]. If H is only formed by one island, this has to contain the identity (H always contains the identity) and any transformation of H can be deformed, by passing through other transformations also contained in H , to the identity. Hence, a more mathematical form of counting the separated islands in H is to count the classes in quotient space (and group) H/H_0 , where H_0 is the subgroup of H containing all transformations which can be taken

into the identity without leaving H . If there are at least two separated islands in H then H/H_0 is formed by two classes: one where all transformations deformable to the identity fall into, and the other class which contain all transformations $h \in H$ belonging to the island which is not H_0 . On the contrary, if $H = H_0$ because there is only one island, then $H/H_0 = 1$ and thus only one class, the trivial one, is present. For convenience, H/H_0 is sometimes written as $\pi_0(H)$, in order to keep a notation similar to the fundamental theorem on the second homotopy group (see (B.215)). The fundamental theorem on the first homotopy group $\pi_1(G/H)$, stated mathematically, takes the following form

$$\pi_1(G/H) = H/H_0 \equiv \pi_0(H), \quad (\text{B.213})$$

always remembering that in order to apply this result, G must simply-connected.

Now to the practical part. We have several times seen that $\pi_1(U(1)) = \mathbf{Z}$, i.e., there are infinite many numerable different non-deformable forms of wrapping a rubber band around a circumference. Let us obtain this result by applying the just proved fundamental theorem on $\pi_1(G/H)$, with $G/H = U(1)$. Lie group $U(1)$, however, has the topology of a ring S^1 and is therefore multiply-connected: it has a hole at the center. Its covering group (its enlargement to a simply-connected group) is the real line $T(1)$, as discussed in (B.112). In the easiest of the gauge models considered, the (usual) superconductor with a non-compact gauge field of Abelian $G = U(1)$ symmetry, the set of nonzero minima for the model's potential is described by $G/H = U(1)$, by all possible orientations of the nonvanishing scalar model within its 2-dimensional internal isospace, being $H = 1$ because the only transformation which leaves invariant the isodirection for a 2-dimensional internal space is the identity itself (plus all those pure gauge transformations obtained from it, as seen in the comments after equation (B.197)). When $G = U(1)$ is enlarged to its corresponding covering group, the real line $\bar{G} = T(1) = \{\chi : -\infty < \chi < \infty\}$, then also H has to be enlarged to $\bar{H}$ such that the orbit space $\bar{G}/\bar{H}$ still remains topologically equivalent to $G/H = S^1$

$$\bar{H} = \{\chi_0 + 2\pi n : n = 0, \pm 1, \pm 2, \dots\}, \quad (\text{B.214})$$

with χ_0 an arbitrary isodirection: the scalar field's orientation in isospace is now given by a real number χ , and the numerable set of complete rotations $\bar{H}$ acting on an arbitrary orientation χ_0 leaves the scalar field pointing in the same isodirection. But now $\bar{G} = T(1)$ is simply-connected and one can use the fundamental theorem applied to $\bar{G}$ and $\bar{H}$, whose result is trivial since $\bar{H}$ is formed by a numerable set of separated point-islands characterized by the integer n of (B.214). Thus we recover that the number of separated connected islands of $\bar{H}$ is $\mathbf{Z}$, and $\pi_1(G/H) = \pi_1(U(1)) = \pi_1(T(1)/\bar{H}) = \mathbf{Z}$.

Let us continue with the calculation of $\pi_1(SO(3))$, since this is the first homotopy group characterizing the line defects in the (compact) “ $SO(3)$ ”-superconductor, where the non-Abelian gauge symmetry is completely broken — “broken” for a gauge symmetry has to be understood as applied not to the set of all possible configurations but only to the set of representatives of each class of gauge equivalent configurations. Lie group $G = SO(3)$ is a multiply-connected group, and therefore $\pi_1(SO(3)) \neq 0$, since it has the topology of $\mathcal{S}^3$, the 3-dimensional spherical surface (in a 4-dimensional space) with antipodal points identified: this identification makes impossible to shrink a closed path lying on the sphere and which ends at the antipode of the start point

(point identified with the start point, as seen in Figure B.6 on page 228). Let us obtain this result about the multiple connectivity for $SO(3)$ applying the fundamental theorem proved in the previous page. The covering group of $G = SO(3)$ is Lie group $\tilde{G} = SU(2)$, being the latter topologically equivalent to 3-dimensional sphere's surface S^3 and thus simply-connected, since any closed path wrapping a part of a sphere's surface can always glide on the surface until being deformed to a point. As already discussed after equation (B.109), there exists a 2-to-1 correspondence between, on the one hand two $SU(2)$ -transformations (fundamental representation of the Lie algebra)

$$\begin{cases} U_{\hat{n}}(\chi) = e^{-i\chi\hat{n}_a\sigma_a/2} \\ U_{\hat{n}}(\chi + 2\pi) \neq U_{\hat{n}}(\chi) \end{cases}, \text{ and on the other hand one } SO(3)\text{-transformation (adjoint rep-}$$

resentation) $O_{\hat{n}}(\chi) = e^{-i\chi\hat{n}_a I_a^{(\text{adj.})}} = \hat{n} \otimes \hat{n} + (\mathbf{1} - \hat{n} \otimes \hat{n}) \cos \chi - i\hat{n}_a I_a^{(\text{adj.})} \sin \chi$ which satisfies $O_{\hat{n}}(\chi + 2\pi) = O_{\hat{n}}(\chi)$ ⁵⁹. Hence, Lie group $SO(3)$ can be written as the quotient group $SU(2)/Z_2$ where Z_2 is the additive group module 2 formed by the two elements $\{0, 1\}$, or equivalently, the multiplicative group formed by the two elements $\{1, -1\}$ ⁶⁰. Now the calculation of $\pi_1(SO(3)) = \pi_1(\tilde{G}/\bar{H})$, with $\tilde{G} = SU(2)$ simply-connected and $\bar{H} = Z_2$, is straightforward by applying the general theorem on simply-connected $\tilde{G}$: $\pi_1(SO(3)) = \pi_1(SU(2)/Z_2) = Z_2$ since two ⁶¹ are the isolated (point-)pieces of $\bar{H} = Z_2$.

And what about point defects in the 3-dim. external space? Let us surround, in the physical space, a point defect with a spherical surface S_∞^2 distant enough such that the

⁵⁹Since for any $SU(2)$ matrix $e^{-i/2\chi\hat{n}_a\sigma_a} = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i\hat{n}_a\sigma_a \sin\left(\frac{\chi}{2}\right) = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}$

with (a_0, a_1, a_2, a_3) real numbers given by $\begin{cases} a_0 = \cos(\chi/2) \\ a_k = -\hat{n}_k \sin(\chi/2) \end{cases}$, $a_0^2 + \sum_{k=1}^3 a_k^2 = 1$ holds, then $SU(2)$ has

the same topology as a 3-dimensional spherical surface S^3 . And the fact that (a_0, a_k) and $(-a_0, -a_k)$ (or equivalently χ and $\chi + 2\pi$) correspond to the same $SO(3)$ matrix, is responsible for the topological equivalence of group $SO(3)$ to a the 3-dimensional spherical surface with diametrical points identified.

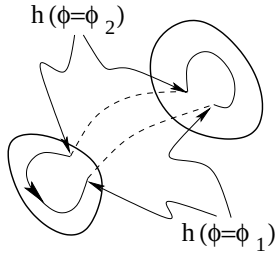
⁶⁰It should be noted that the very properties defining a group — associative law, existence of identity and of inverse element — does not contain the definition of $-g \in G$ as the multiplication of the *real number* -1 times group element $g \in G$. The definition of $-g$ as group element is the following [69]: let $\tilde{i}$ denote a group element which, commuting with every $g \in G$, is different of the identity $\tilde{e}$ and fulfils $\tilde{i} \circ \tilde{i} = \tilde{e}$, being $\circ$ the combination operation defined in the group G ; then $-g$ is only an abbreviated form of writing $\tilde{i} \circ g$. Since $\tilde{i}$ commutes with all group elements and $\tilde{i} \circ \tilde{i} = \tilde{e}$, following holds: $(\tilde{i} \circ g) \circ (\tilde{i} \circ f) = g \circ \tilde{i} \circ \tilde{i} \circ f = g \circ f$ and $\tilde{i} \circ (\tilde{i} \circ g) = g$, which correspond in the (naive) abbreviated notation to $(-g) \circ (-f) = g \circ f$ and $-(-g) = g$. Hence, element -1 of Z_2 (element which could be represented, for instance, by a diagonal matrix of -1's) differs from identity $\tilde{e} \equiv 1$ and corresponds to $\tilde{i}$.

⁶¹On the contrary, when the $SO(3)$ symmetry is not completely broken and isorotations around the isodirection “chosen” by the nonzero scalar isovector still remain, the calculation of $\pi_1(G/H)$ with $G/H = SO(3)/U_{\Psi_0}(1)$ yields the expected result of the absence of topologically stable line defects (in a 3 dimensional physical space). The above mentioned 2-to-1 correspondence between group $SU(2)$, the covering group of $SO(3)$, and $SO(3)$ itself allows us to find the extension $\bar{H}$ in the covering group for $H = U_{\Psi_0}(1)$ such that orbit space $\tilde{G}/\bar{H}$ is equal to G/H : $\bar{H}$ is the set of $SU(2)$ matrices describing rotations about angle χ around a fixed isodirection along which the scalar isovector is pointing, $\bar{H} = \left\{ e^{-i\chi\hat{n}_a\sigma_a/2} : 0 \leq \chi < 2 \times 2\pi \text{ and } \hat{n} \parallel \Psi_0 \text{ fixed} \right\}$. Since χ has one continuous range, there is no matrix in $\bar{H}$ which would form an isolated element without being connected to the other elements by varying χ . There are no different isolated “islands” in $\bar{H}$ and thus $\pi_1(G/H) = \pi_1(SU(2)/\bar{H}) = 0$ by applying the fundamental theorem on the first homotopy group; one should remember that if the only class present is the trivial one, formed by the transformations deformable to the identity, one usually writes $\pi_1(G/H) = 0$ instead of $\pi_1(G/H) = 1$ to stress that there is no *non-trivial* class.

scalar field, at the points on S_∞^2 , has already tended to the same nonzero length value which minimizes the potential: one studies if configurations where the scalar field points in different isodirections in the internal isospace, by approaching S_∞^2 along different directions in the physical space, are topologically stable or on the contrary, can be taken by means of gauge transformations into a configuration where the isodirection of the scalar field is the same for all points on S_∞^2 . This is obtained by calculating the number of different classes of group $\pi_2(G/H)$, the classes of mapping a spherical surface onto the set G/H of different configurations for a nonzero scalar field, and by looking if there is more than the trivial class or not. For a simply-connected group G ($\pi_1(G) = 0$) which also satisfies $\pi_2(G) = 0$ ⁶², the fundamental theorem on $\pi_2(G/H)$ states

$$\pi_2(G/H) = \pi_1(H), \quad (\text{B.215})$$

in a similar way to the fundamental theorem on the first homotopy group (B.213), although this time the proof will be not given: it can be found in [69] and [87]. If H is only formed of discrete elements, as for instance $H = Z_2$, then $\pi_1(H) = 0$. This result can



be seen in the following figure: let us consider H' being formed by two isolated connected pieces and a loop in H' such that when a circle is covered in the external space (by varying ϕ from 0 to 2π), the loop covers the path represented in the figure by jumping from an isolated piece to the other. If now these two islands are shrunk to a point in order to obtain $H = Z_2$, then each part of the loop $h(\phi)$ is also deformed to a point and thus $\pi_1(Z_2) = 0$. Another form of obtaining this result runs as follows: as we already know, $SO(3) = SU(2)/Z_2$ and therefore $Z_2 = SU(2)/SO(3)$, i.e., for a fixed axis $\hat{n}$ in the internal isospace, the $SU(2)$ -isrotations $U_{\hat{n}}(0 \leq \chi < 2\pi)$ already covers the whole $SO(3)$ group and are therefore identified to one element in $SU(2)/SO(3)$, and the same holds for the $SU(2)$ -isrotations $U_{\hat{n}}(2\pi \leq \chi < 4\pi)$ which are identified to the other element in the quotient group since these transformation are different from $U_{\hat{n}}(0 \leq \chi < 2\pi)$. Since $SU(2)$ is simply-connected, one can apply the fundamental theorem (B.213) on $\pi_1(SU(2)/SO(3))$ which yields 0 because all $SO(3)$ -transformations, $O_{\hat{n}}(\chi) = e^{-i\chi\hat{n}_a I_a^{(\text{adj.})}}$, can be deformed to the identity transformation simply by varying χ . Hence, $\pi_1(H = Z_2) = 0$; a similar result holds for other H 's containing exclusively discrete operations.

The application of this partial result and the fundamental theorem on $\pi_2(G/H)$

⁶²As already mentioned after equation (B.207), a theorem due to Cartan shows that $\pi_2(G) = 0$ holds for any compact Lie group: any mapping of the usual spherical surface S^2 into G can be taken into the identity by mean of transformations all belonging to G . The proof of such a theorem is not necessary here since the only compact group to which it will apply on is $SU(2)$, which satisfies the theorem: compact Lie group $SU(2)$ topologically equivalent to S^3 (the surface of a 4-dim. sphere) and thus any form of covering with a (usual) 2-dim. spherical surface S^2 a part of the 3-dim. spherical surface S^3 can be shrunk to a point, in the same way as there were no stable form of extending rubber band on a 2-dim. spherical surface. This last case, represented in Fig. B.2 on page 192, can be actually seen as a section through a plane keeping constant one of the four coordinates of the space where S^3 is living. Hence $\pi_2(SU(2)) = \pi_2(S^3) = 0$. For the case of the compact group $T(1)$, alone for dimensional reasons ($T(1)$ is the 1-dim. real line and S^2 is already a surface), $\pi_2(T(1)) = 0$ holds immediately.

allows us to obtain directly

$$\pi_2(SO(3)) = \pi_2(SU(2)/Z_2) = \pi_1(Z_2) = 0. \quad (\text{B.216})$$

Or equivalently, in a system where the gauge symmetry $SO(3)$ is completely broken (with the meaning for “broken” as discussed in (B.201)), just as in the “ $SO(3)$ ”-superconductor or as in the model of equation (B.205), there exist no monopole as isolated point defect⁶³. This, nevertheless, does not mean that monopoles cannot occur as sources or as ending points attached to two reacting vortices: in the above discussion on $\pi_2(G/H)$, one has begun by considering that on all points on a distant spherical surface surrounding the monopole the scalar field is nonzero. But this cannot be the case if this spherical surface is crossed by the pair of stable vortices which emanate from the monopole: inside the vortex the scalar field is zero, property required by the topological stability of the vortex itself in order to avoid that the scalar field would be pointing in different isodirections at the center of the vortex when one were approaching it along different directions from the spatial infinity of the physical external space. Therefore, the vanishing $\pi_2(SO(3)) = 0$ for the “ $SO(3)$ ”-superconductor does not exclude monopoles connected to vortices: in fact, these attached monopoles are the footprint of the compact $SO(3)$ group, as different from the usual superconductor with an original $U(1)$ gauge symmetry not resulting from the breakdown of a larger non-Abelian group.

Returning to the line defects of vortices in the external 3-dimensional space, the introduction of the two nonzero scalar isovectors as lowest energy states in order to break completely the gauge $G = SO(3)$ symmetry opens a new path to obtain new types of topological stable line defects in a 3-dimensional physical space. If the arrangement of these two scalar isovectors (or of a higher number of them) in the internal isospace is so to allow some *discrete* symmetry transformations, then H (and so $\bar{H}$) are increased by these isolated operations, and thus the first homotopy group contains classes of stable configurations non-deformable into each other by means of a gauge transformation [15] — stable configurations which, without these discrete symmetries, would not be possible because $\bar{H}$ was simply connected and thus consisting in only one island. If moreover the original $\bar{H}$ satisfied $\pi_1(\bar{H}) \neq 0$, then also stable isolated monopoles occur in the model, having the addition of the just discussed discrete elements no effect on $\pi_1(\bar{H})$. It should be stressed that if we are only interested in having vortices without isolated monopoles, except for these discrete transformations (which do not correspond to any gauge transformation, by definition, a continuous transformation), the arrangement of the isovectors in the internal isospace must remove any continuous subgroup as the remaining symmetry H in order to break completely the gauge symmetry, since $\pi_1(\bar{H})$ is zero when formed exclusively by discrete operations. For the compact gauge group

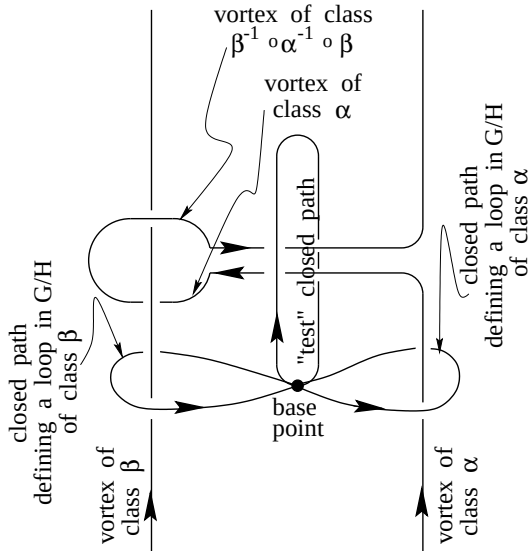
⁶³On the other hand, applying theorem (B.215) to the $SO(3)$ gauge invariant model with only one scalar isovector which breaks the symmetry down to $U_{\Psi_0}(1)$, yields again that such a model does have magnetic monopoles: starting at $\pi_2(SO(3)/U_{\Psi_0}(1)) = \pi_2(SU(2)/\bar{H})$, with $\bar{H} = \left\{ e^{-i\chi \hat{n}_a \sigma_a / 2} : 0 \leq \chi < 4\pi \text{ and } \hat{n} \parallel \Psi_0 \text{ fixed} \right\}$, and since $SU(2)$ fulfils the requisites of the fundamental theorem on π_2 , one obtains $\pi_2(SO(3)/U_{\Psi_0}(1)) = \pi_1(\bar{H})$. But this $\bar{H}$, the planar isorotations around an axis given by the direction of the scalar field in isospace, is topologically equivalent to S^1 and therefore $\pi_1(\bar{H}) = \mathbf{Z} = \pi_2(SO(3)/U_{\Psi_0}(1)) \neq 0$, as expected.

$SO(3)$, this completes result (B.201) into the following equivalence

complete “breakdown” of gauge sym. $\iff$	existence of vortices absence of isolated monopol.	(B.217)

B.7.2 Entanglement of line defects in 3-dim. real space [69]

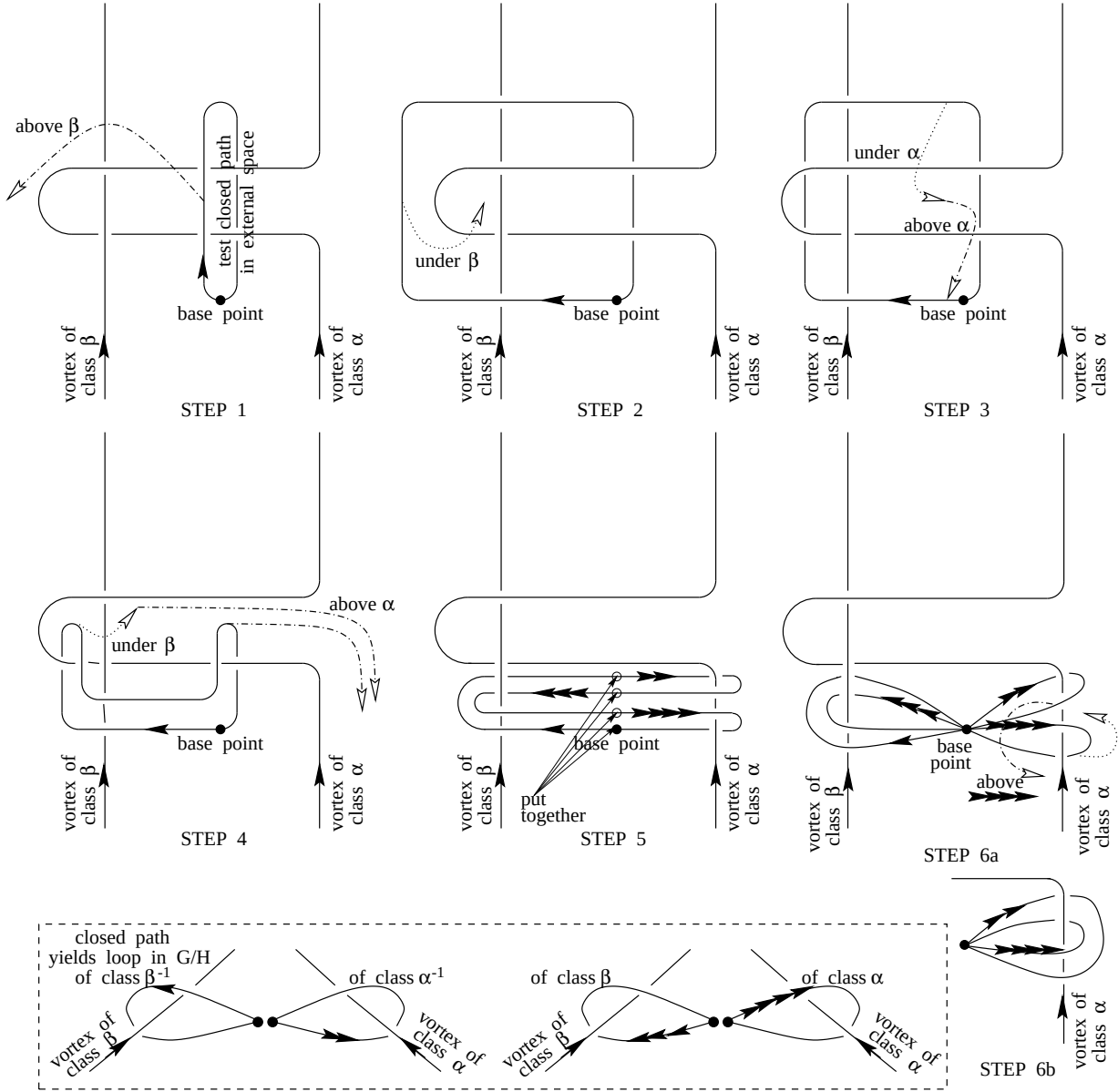
Still more interesting for the dislocation model we intend to define in this work is the $SO(3)$ gauge invariant model where, after a complete symmetry breakdown, only some discrete symmetries remain in $\bar{H}$ — thus $\pi_1(G/H) = \pi_1(SO(3)/H) = \pi_1(SU(2)/\bar{H}) = \bar{H}/H_0 = \bar{H}$ — and these are non-commutative operations. This implies a non-Abelian $\pi_1(G/H)$, and as in the comments following equation (B.11), in this case the general discussion on the fundamental (non-based) homotopy group requires to work with the conjugacy classes for each element of the discrete subgroup $\bar{H}$ in order to define an unique correspondence among first homotopy groups based at different points. The use of conjugacy classes allows this correspondence since these “delete” the information along which path, from a certain base point, has been drawn the contour which encircles a certain region whose interior is crossed by a topological line defect. Another form of working, without the loss of information caused by the use of conjugacy classes, is to consider all possible classes of topological line defects based at exclusively only one reference point, since there is an isomorphism between the abstract first homotopy $\pi_1(G/H)$ and any based $\pi_1(G/H, \text{base point})$. As already shown in (B.9), in a system with a non-Abelian first homotopy group, it can occur that two based loops α and β belonging to different homotopy classes fulfil $\alpha \circ \beta \not\sim \beta \circ \alpha$, or equivalently, that based loop $\beta^{-1} \circ \alpha^{-1} \circ \beta \circ \alpha$ is not deformable to the identity operation. Let us consider the following figure, where



two oriented line defects of the vortex classes just mentioned are represented in the real physical space. From the reference base point, when one draws a closed path (distant enough) around each vortex, in the internal space G/H one is also covering a configuration where the scalar field is changing its isodirection (and the corresponding gauge field arranges itself to make finite the configuration energy). According to the right-hand rule, each of the two horizontal closed contours of the figure defines in G/H a loop of class β and α respectively. Now vortex of loop class α moves across vortex β , as in the middle of the figure: the homotopy class of the first vortex after this passing need

not be equal to class α . Let us explain this assertion. Based at the reference base point, one covers a test closed path which encircles, firstly the in-going vortex previous to surround vortex β , and then the out-going (“piercing” the test contour in the opposite direction) vortex after the interaction with β : one could naively expect that the vorticity measured by the test closed path would correspond to $\alpha \circ \alpha^{-1}$, i.e., zero vorticity.

Nevertheless, this is in general not true: without touching both vortices, the test contour in physical space can be deformed through the steps represented in the following figure:



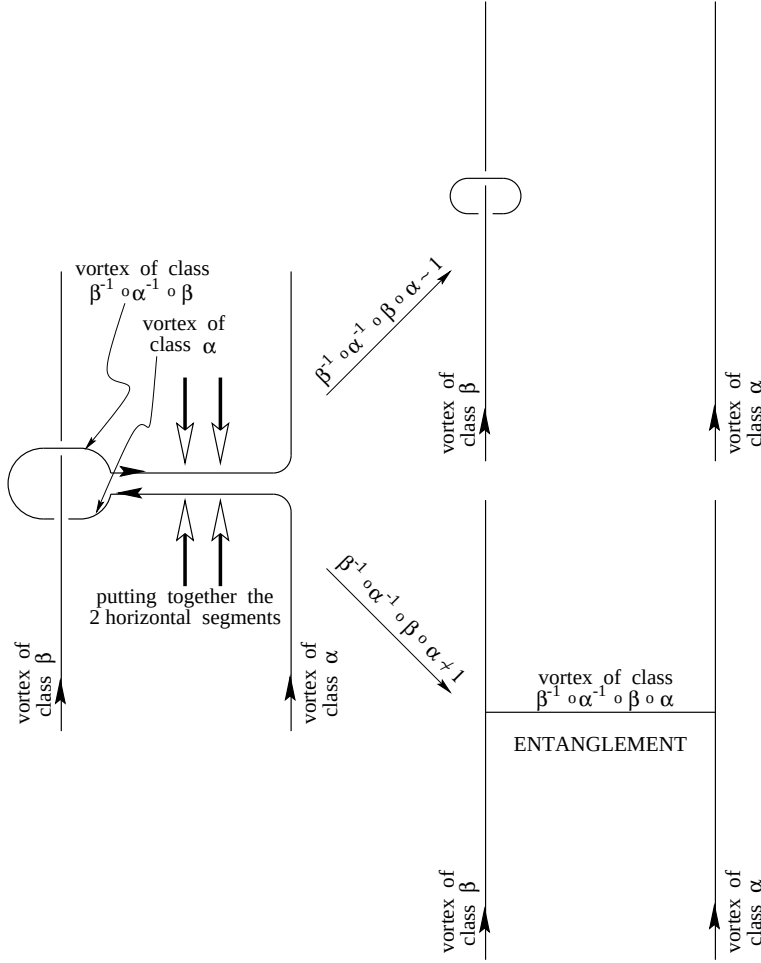
Therefore this test path in external space defines a loop, in the internal space of different configurations for the scalar field, of class $(\beta^{-1} \circ \alpha^{-1} \circ \beta) \circ \alpha$. Or equivalently the outgoing vortex, which as ingoing vortex was of class α , has “mutated” its loop class into $\beta^{-1} \circ \alpha^{-1} \circ \beta$, and if we are dealing with a system characterized by a non-Abelian first homotopy group and thus $\alpha^{-1} \circ \beta \not\sim \beta \circ \alpha^{-1}$ — the product $\alpha^{-1} \circ \beta$ does not belong to the same homotopy class as $\beta \circ \alpha^{-1}$ and are not deformable into each other —, then the outgoing vortex does actually belong to a loop class different from that of the ingoing vortex. Hence, if would put together the two horizontal segments of the vortex which before moving across vortex β was of class α , then the product of the reaction between

these two segments would *not* be topologically unstable

$$\underbrace{\alpha}_{\text{ingoing vortex}} \circ \underbrace{\alpha^{-1}}_{\text{outgoing v. (opposite dir.)}} \sim 1,$$

but

$$\underbrace{\alpha}_{\text{ingoing vortex}} \circ \underbrace{\beta^{-1} \circ \alpha^{-1} \circ \beta}_{\text{outgoing vortex}} \not\sim 1 \quad (\text{B.218})$$



it would belong to a topologically stable class. This has important physical consequences: in the case of an Abelian $\pi_1(G/H)$, i.e., when the two horizontal segments annihilate each other because the vortex α has not mutated its loop class after having passed across, one can move away vortex α from vortex β and it only remains a small ring around the passed vortex as footprint. On the contrary, if $\pi_1(G/H)$ is non-Abelian this across-moving leaves a stable vortex which connects vortices α and β , impeding the further displacement of each vortex. This behaviour is called the entanglement of vortices.

In all those models considered in this section, the usual Abelian superconductor with one scalar field as well as the compact “ $SO(3)$ ”-superconductor when two scalar isovectors were used, the first homotopy group was always commutative and thus entanglement did not occur. Focussing our attention in models with a $SO(3)$ gauge symmetry but this time working with scalar isotensors of second and higher ranks, it depends on the parameters of the potential and on the direction chosen by the scalar field in the isospace that the resulting first homotopy group can be non-Abelian. These scalar isotensors (tensor field in the internal isospace, scalar field in the physical external space) result from the direct product of the isovector representation of the $SO(3)$ group (the adjoint

representation of the Lie algebra); and they, together with the discrete symmetry subgroups in $\pi_1(SO(3)/H) = \pi_1(SU(2)/\bar{H})$ and their corresponding classes of vortices, will be the subject of the last part of this Appendix.

B.7.3 The scalar irreducible isotensor Ψ^{ab} of second rank

The lowest rank representation of the $SO(3)$ group, other than the trivial 0th rank isoscalar representation, is the first rank or (iso)vector representation, i.e., the adjoint representation of the simple compact Lie algebra characterized by the structure constants $C_{abc} = \epsilon_{abc}$. Representations of higher integer ranks for the $SO(3)$ group occur as results from the direct product of isovector representations. Let Ψ^a and $\tilde{\Psi}^a$ be two isovectors, transforming thus as the adjoint representation: the 3×3 matrix constructed from the direct product of both isovectors

$$T^{ab} = \Psi^a \tilde{\Psi}^b,$$

transforms as follows under an isorotation about angle χ around internal axis $\hat{n}$ (that for a external site $\vec{r}$, since at another site the internal isorotation must not be the same in a gauge invariant model)

$$T^{ab} \rightarrow T'^{ab} = \underbrace{(O_{\hat{n}}(\chi))_{ac} \Psi^c (O_{\hat{n}}(\chi))_{bd}}_{(O_{\hat{n}}(\chi))_{ac} \Psi^c \tilde{\Psi}^d (O_{\hat{n}}^{-1}(\chi))_{db}} \tilde{\Psi}^d = \left(O_{\hat{n}}(\chi) T O_{\hat{n}}^{-1}(\chi) \right)_{ab}. \quad (\text{B.219})$$

This matrix of 9 independent components, when diagonalized, can be descomposed into the three following sub(iso)spaces which are invariant, not mixed among them, under any $SO(3)$ isorotation:

1. A subspace of dimension 1 corresponding to the *trace* of T^{ab} , i.e., the scalar product $\Psi^a \tilde{\Psi}^a$, clearly invariant under any $SO(3)$ transformation by the very definition of the adjoint representation, which is formed by orthogonal real matrices.
2. A subspace of dimension 3 corresponding to $T^c = \epsilon_{abc} T^{bc} = \epsilon_{abc} \Psi^b \tilde{\Psi}^c$, i.e., the vector product of the two isovectors, which is also invariant under any $SO(3)$ isorotation since it only depends on the isovector's lengths and on the angle between the two isovectors (all of them kept invariant in an isorotation). Note that only the *antisymmetric* components $\frac{1}{2} (T^{ab} - T^{ba})$ contribute to $\epsilon_{abc} T^{bc}$.
3. A subspace of dimension $9-1-3=5$ corresponding to the following *traceless* (the trace has been removed since it belongs to the firstly mentioned invariant subspace) and *symmetric* (the antisymmetric components belong to the second invariant subspace) tensor

$$\Psi^{ab} = \frac{1}{2} (\Psi^a \tilde{\Psi}^b + \tilde{\Psi}^a \Psi^b) - \frac{1}{3} \delta^{ab} \overbrace{(\Psi^{a=1} \tilde{\Psi}^{a=1} + \Psi^{a=2} \tilde{\Psi}^{a=2} + \Psi^{a=3} \tilde{\Psi}^{a=3})}^{T^{cc}}, \quad (\text{B.220})$$

which is invariant since the trace, and the antisymmetric components, of $\Psi^a \tilde{\Psi}^b$ have been removed because they form, each for their part, own invariant subspaces. It

is not only invariant but also *irreducible*, i.e., it cannot be further decomposed into invariant subspaces of smaller dimension. Any $SO(3)$ transformations acting on this tensor (B.220) belongs also to the representation of rank 2 of such a Lie group. According to (B.219), and taking into account that the scalar product of two isovectors is invariant under any internal rotation of both vectors, isotensor Ψ^{ab} transforms as follows under a $SO(3)$ isorotation

$$\Psi^{ab} \rightarrow \Psi'^{ab} = (O_{\hat{n}}(\chi))_{af} (O_{\hat{n}}(\chi))_{bg} \Psi^{fg}. \quad (\text{B.221})$$

All the above results are usually abbreviated by writing $1 \otimes 1 = 0 \oplus 1 \oplus 2$, where each number represents the rank of each irreducible representation of the $SO(3)$ group. The same discussion can be carried out with the direct product of three adjoint representations $1 \otimes 1 \otimes 1 = 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus \underline{\underline{3}}$ in order to find the tensor of third rank, depending on isovectors $\{\Psi^a, \tilde{\Psi}^b, \check{\Psi}^c\}$, which transforms as the third rank irreducible representation of group $SO(3)$. For instance, the 1-dim. invariant subspace corresponds to $\epsilon_{abc} \Psi^a \tilde{\Psi}^b \check{\Psi}^c$; and the 7-dim. invariant irreducible subspace, we are interested in, is formed by tensors of the form

$$\begin{aligned} \Psi^{abc} = & \frac{1}{6} \left(\Psi^a \tilde{\Psi}^b \check{\Psi}^c + \Psi^a \check{\Psi}^b \tilde{\Psi}^c + \tilde{\Psi}^a \Psi^b \check{\Psi}^c + \tilde{\Psi}^a \check{\Psi}^b \Psi^c + \check{\Psi}^a \Psi^b \tilde{\Psi}^c + \check{\Psi}^a \tilde{\Psi}^b \Psi^c \right) - \\ & - \frac{1}{15} \left[\delta^{ab} \left((\Psi \cdot \tilde{\Psi}) \check{\Psi}^c + (\Psi \cdot \check{\Psi}) \tilde{\Psi}^c + (\tilde{\Psi} \cdot \check{\Psi}) \Psi^c \right) + \right. \\ & + \delta^{bc} \left((\Psi \cdot \tilde{\Psi}) \check{\Psi}^a + (\Psi \cdot \check{\Psi}) \tilde{\Psi}^a + (\tilde{\Psi} \cdot \check{\Psi}) \Psi^a \right) + \\ & \left. + \delta^{ca} \left((\Psi \cdot \tilde{\Psi}) \check{\Psi}^b + (\Psi \cdot \check{\Psi}) \tilde{\Psi}^b + (\tilde{\Psi} \cdot \check{\Psi}) \Psi^b \right) \right], \end{aligned} \quad (\text{B.222})$$

where, for example $(\Psi \cdot \tilde{\Psi})$ means $\Psi^d \tilde{\Psi}^d$. This tensor is symmetric in any two of its indices, $\Psi^{abc} = \Psi^{bac} = \Psi^{acb} = \Psi^{cba}$, and also the trace calculated for two of such indices is always zero: thus of the $3 \times 3 \times 3$ components of this cubic matrix, only $10 - 3 = 7$ are independent (10 from the symmetric condition and 3 from the three possible traces one can calculate). The number of independent components for this tensor as well as for the 2nd rank tensor (B.220) becomes clearer if one notices that these are the number of different spherical harmonics for angular momentum $l = 2$ and $l = 3$, respectively: as it is well known, the spherical harmonics Y_l^m form a basis for the irreducible representation of rank l for the $SO(3)$ group. An example of this statement will be discussed in the next Note.

Note. The scalar isotensor describing a topologically stable point defect in a 3-dim. physical space.

Let us consider in the 3-dim. (external) physical space a spherical surface S_∞^2 surrounding a point defect at a distance large enough such that the scalar field of the model takes on S_∞^2 one of the constant nonzero values which minimizes the potential. As it was done for the sole isovector in the discussion of the magnetic monopole, we will now study the case where at each external site of $\vec{x} \in S_\infty^2$ the isodirection of the scalar isovectors entering in the isotensor field, isodirection chosen in the 3-dim. (internal) isospace, corresponds to the direction in the external space of approaching such a site $\vec{x}$. This

represents thus a non-deformable-to-the-identity map from S_∞^2 into the internal space's configuration ⁶⁴ which is also topologically equivalent to S^2 — or to $\mathcal{S}^2$ depending whether the scalar isotensor is formed by the product of an even number of scalar isovectors and thus the inversion in the isodirection of half of these isovectors is a symmetry too —; of course, this topologically stable map is possible since the gauge group $G = SO(3)$ and the potential of the model yields a G/H with the topology of S^2 . By equating the internal isodirections with the directions in the external physical space, the second rank scalar isotensor (B.220) describing the point defect is given by

$$\begin{aligned}\Psi^{ab} &= \sqrt{(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi})} \hat{\Psi}^{ab} \\ \text{with } \tilde{\Psi}^{ab} &= \frac{r_a r_b}{r^2} - \frac{1}{3} \delta_{ab},\end{aligned}\tag{B.223}$$

where $r_a = (x, y, z)$ refers to coordinates in external space. Although this result has been only studied for sites distant from the point defect, thanks to the gauge invariance of the model's energy, equation (B.223) defines a representative configuration (representative of all configurations gauge equivalent to it) which describes the point defect for *any* distance; it must be only required that $\sqrt{(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi})}$, as a function of the distance, tends to a constant nonzero value when the spatial infinity is reached, nonzero value determined by the energy's potential. In the same way as for the isotensor of second rank, in a model whose scalar field has tensor character of third rank (B.222) in isospace, a representative configuration for all those field arrangements describing a stable point defect is

$$\begin{aligned}\Psi^{abc} &= \sqrt{(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) (\check{\Psi} \cdot \check{\Psi})} \hat{\Psi}^{abc} \\ \text{with } \hat{\Psi}^{abc} &= \frac{r_a r_b r_c}{r^3} - \frac{1}{5} \left(\frac{\delta_{ab} r_c}{r} + \frac{\delta_{ac} r_b}{r} + \frac{\delta_{bc} r_a}{r} \right),\end{aligned}\tag{B.224}$$

together with its corresponding gauge field W_μ^a which ensures the vanishing of the covariant derivative at all sites on S_∞^2 . And in general for the n th-rank isotensor, $\hat{\Psi}^{abc\dots} = \frac{r_a r_b r_c \dots}{r^n} - \frac{\text{traces}}{r^n}$, where each of the “traces” are multiplied by the corresponding numerical factor.

Due to the identification of the internal isodirections and the direction in physical space for a topological stable point defect, all the just discussed scalar isotensors $\hat{\Psi}^{ab}, \hat{\Psi}^{abc}$ are formed by the product of trigonometric relations of exterior angles. It results that each of the 5 components of the symmetric and traceless isotensor (B.223), or of the 7 components of the third rank isotensor (B.224), can be completely expanded in a linear combination of, respectively, the 5 spherical harmonics Y_2^m of angular momentum 2, or of the 7 spherical harmonics Y_3^m of angular momentum 3:

$$1. \text{ Functions } \begin{cases} Y_2^{+2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{i2\phi} & Y_2^{+1} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} & Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \\ Y_2^{-2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{-i2\phi} & Y_2^{-1} = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\phi} \end{cases} \text{ are}$$

enough to write any quadratic function of the cartesian coordinates of type $\hat{\Psi}^{ab} = \frac{r_a r_b}{r^2} - \frac{1}{3} \delta_{ab}$ with $r_{a=1,2,3} \equiv x, y, z$. Let us begin with by looking for the term which has to be added to x^2/r^2 in the form $x^2/r^2 - ()$ such that the complete expression can be written as function of spherical harmonics of angular momentum $l = 2$ exclusively. To obtain $\frac{x^2}{r^2} = \sin^2 \theta \cos^2 \phi$ we have the following chain of equalities, where the underlined terms is expressed in the next step as function of the corresponding spherical harmonics: $\sin^2 \theta \cos^2 \phi = \frac{1}{2} \sin^2 \theta \cos 2\phi + \frac{1}{2} \sin^2 \theta = \sqrt{\frac{2\pi}{15}} (Y_2^{+2} + Y_2^{-2}) - \frac{1}{2} \cos^2 \theta + \frac{1}{2} = \sqrt{\frac{2\pi}{15}} (Y_2^{+2} + Y_2^{-2}) - \frac{1}{3} \sqrt{\frac{4\pi}{5}} Y_2^0 - \frac{1}{6} + \frac{1}{2}$, and thus the linear combination de-

⁶⁴More properly, into the class of all configurations obtained from this first representative by means of any allowed gauge transformation.

pending only on $Y_{i=2}^m$'s is equal to $\sqrt{\frac{2\pi}{15}}(Y_2^{+2} + Y_2^{-2}) - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 = \frac{x^2}{r^2} - \frac{1}{3}$, as expected from (B.223). The same argument for the other combinations $\frac{r_a r_b}{r^2}$ leads to

$$\begin{cases} \frac{xy}{r^2} = \sqrt{\frac{2\pi}{15}}(-iY_2^{+2} + iY_2^{-2}) & \frac{x^2}{r^2} - \frac{1}{3} = \sqrt{\frac{2\pi}{15}}(Y_2^{+2} + Y_2^{-2}) - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 \\ \frac{xz}{r^2} = \sqrt{\frac{2\pi}{15}}(-Y_2^{+1} + Y_2^{-1}) & \frac{y^2}{r^2} - \frac{1}{3} = -\sqrt{\frac{2\pi}{15}}(Y_2^{+2} + Y_2^{-2}) - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 \\ \frac{yz}{r^2} = \sqrt{\frac{2\pi}{15}}(iY_2^{+1} + iY_2^{-1}) & \frac{z^2}{r^2} - \frac{1}{3} = \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 \end{cases}.$$

2. Similarly, third rank isotensor (B.224) can be completely written as function of spherical harmon-

$$\text{ics} \begin{cases} Y_3^{+3} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{i3\phi} & Y_3^{+2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta \cos\theta e^{i2\phi} & Y_3^0 = \sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \\ Y_3^{-3} = \frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{-i3\phi} & Y_3^{-2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta \cos\theta e^{-i2\phi} \\ Y_3^{+1} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta - 1)e^{i\phi} & Y_3^{-1} = \frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta - 1)e^{-i\phi} \end{cases}.$$

For instance: $\frac{x^2 y}{r^3} = \sin^3\theta \cos^2\phi \sin\phi = \frac{1}{4}\frac{\sin^3\theta \sin 3\phi}{\sin\phi} + \frac{\sin^3\theta \sin\phi}{4} = \sqrt{\frac{\pi}{35}}(iY_3^{+3} + iY_3^{-3}) - \frac{1}{4}\frac{\sin\theta \cos^2\theta \sin\phi}{\sin\phi} + \frac{1}{4}\frac{y}{r} = \sqrt{\frac{\pi}{35}}(iY_3^{+3} + iY_3^{-3}) - \frac{1}{5}\sqrt{\frac{\pi}{21}}(iY_3^{+1} + iY_3^{-1}) - \frac{1}{20}\frac{y}{r} + \frac{1}{4}\frac{y}{r}$, and therefore $\frac{x^2 y}{r^3} - \frac{1}{5}\frac{y}{r} = \sqrt{\frac{\pi}{35}}(iY_3^{+3} + iY_3^{-3}) - \frac{1}{5}\sqrt{\frac{\pi}{21}}(iY_3^{+1} + iY_3^{-1})$.

And for the sake of completeness, let us state the corresponding result for the isovector (tensor of first rank): $\frac{x}{r} = \sqrt{\frac{2\pi}{3}}(-Y_1^{+1} + Y_1^{-1})$, $\frac{y}{r} = \sqrt{\frac{2\pi}{3}}(iY_1^{+1} + iY_1^{-1})$ and $\frac{z}{r} = \sqrt{\frac{4\pi}{3}}Y_1^0$.

The just proved result holds in general [87] for the $2n+1$ components of the symmetric traceless isotensor $\hat{\Psi}^{abc\dots} = \frac{r_a r_b r_c \dots}{r^n} - \frac{\text{traces}}{r^n}$ and the $2n+1$ spherical harmonics $Y_I^{m_I}$ with $I=n$. Since the application of any rotation to a spherical harmonic of a certain angular momentum produces a linear combination of other spherical harmonics but all of the same angular momentum as the original one, then any isorotation on isotensor $\Psi^{ab} = \sqrt{(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi})}\left(\frac{r_a r_b}{r^2} - \frac{1}{3}\delta_{ab}\right)$, or on isotensor $\Psi^{abc} = \sqrt{(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi})}\left(\frac{r_a r_b r_c}{r^3} - \frac{1}{5}\left(\frac{\delta_{ab} r_c}{r} + \frac{\delta_{ac} r_b}{r} + \frac{\delta_{bc} r_a}{r}\right)\right)$, will leave it invariant.

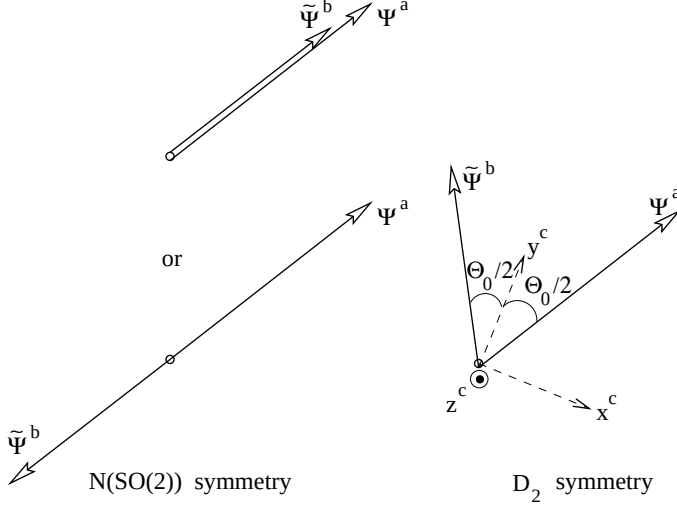
Now let us return our attention to second rank isotensor $\Psi^{ab} = \frac{1}{2}(\Psi^a \tilde{\Psi}^b + \tilde{\Psi}^a \Psi^b) - \frac{1}{3}\delta^{ab}(\Psi \cdot \tilde{\Psi})$ to discuss all possible stable defects (not only the point ones). Depending on the directions chosen in isospace by the two (nonzero) isovectors which define Ψ^{ab} , the symmetries which will form H and leave thus invariant a nonzero Ψ^{ab} are [76]:

- $H = N(SO(2))$ if both isovectors Ψ^a and $\tilde{\Psi}^b$ are parallel or antiparallel in isospace. The normalizer $N(SO(2))$ of group $SO(2)$ is formed by all isorotations around one of the two nonzero isovectors, i.e. the group $SO(2)$ itself, plus any isorotation in the 3-dim. internal isospace by angle π around any axis orthogonal to both isovectors, i.e. the discrete symmetry of sending $\tilde{\Psi}^a$ to $-\tilde{\Psi}^a$ which clearly leaves invariant symmetric tensor Ψ^{ab}

$$N(SO(2)) = \left\{ e^{-i\chi \hat{n}^c I_c^{(\text{adj.})}}, e^{-i\pi \hat{\xi}_1^c I_c^{(\text{adj.})}} e^{-i\chi \hat{n}^c I_c^{(\text{adj.})}} : 0 \leq \chi < 2\pi, \hat{n}_\perp^a \parallel \Psi^a, \hat{\xi}_\perp^a \perp \Psi^a \right\}, \quad (\text{B.225})$$

being $(I_c^{(\text{adj.})})_{ab} = -i\epsilon_{cab}$ the generators of the rotations on vectors in the 3-dimensional internal isospace. In the next figure it becomes evident that it is sufficient to consider only one axis $\xi_\perp^c$ perpendicular to the line along which Ψ^a and $\tilde{\Psi}^b$ are placed: any other perpendicular axis will have the same effect as $\xi_\perp^c$.

- $H = D_2$ if isovectors Ψ^a and $\tilde{\Psi}^b$ are not colinear in isospace, forming an angle Θ_0 .



Then the isorotations by angle π around the isospace axes $\{x^c, y^c, z^c\}$ of the figure (axis y^c bisects angle Θ_0 and axis z^c is normal to this page) are the only symmetries, being also discrete operations, of H which is called the dihedral group of order 4. For instance, the π -isorotations around x^c corresponds to transformation

$$\begin{cases} \Psi^a \rightarrow -\sqrt{\frac{(\Psi \cdot \Psi)}{(\tilde{\Psi} \cdot \tilde{\Psi})}} \tilde{\Psi}^a \\ \tilde{\Psi}^a \rightarrow -\sqrt{\frac{(\tilde{\Psi} \cdot \tilde{\Psi})}{(\Psi \cdot \Psi)}} \Psi^a \end{cases}$$

$$D_2 = \left\{ \mathbf{1}, e^{-i\pi \hat{x}^c I_c^{(\text{adj.})}}, e^{-i\pi \hat{y}^c I_c^{(\text{adj.})}}, e^{-i\pi \hat{z}^c I_c^{(\text{adj.})}} \right\}. \quad (\text{B.226})$$

It should be noted that the above discrete symmetry of a rotation by π around an axis normal to the plane where both isovectors lie is also a symmetry in the case of colinearity of Ψ^a and $\tilde{\Psi}^a$: it is just the responsible for being the symmetry of the isotensor in that case $N(SO(2))$ and not simply $SO(2)$.

In order to determine the different classes of topological stable line defects and of point defects in the 3-dimensional usual space, and since the gauge symmetry of the model is $G = SO(3)$ and multiply-connected therefore, the first and second homotopy group is calculated by the respective fundamental theorem (B.213) or (B.215) but not applied on G/H but on $\bar{G}/\bar{H}$, with $\bar{G} = SU(2)$ the (simply-connected) covering group of $SO(3)$. The corresponding extension $\bar{H}$ for $H = N(SO(2))$, such that $SO(3)/N(SO(2)) = SU(2)/\bar{H}$, is given by

$$\bar{H} = \left\{ e^{-i\chi \hat{n}^c I_c^{(\text{fund.})}}, e^{-i\pi \hat{\xi}_\perp^c I_c^{(\text{fund.})}}, e^{-i\chi \hat{n}^c I_c^{(\text{fund.})}} : 0 \leq \chi < 2 \times 2\pi, \begin{matrix} \hat{n}^a \parallel \Psi^a \\ \hat{\xi}_\perp^a \perp \Psi^a \end{matrix} \right\}, \quad (\text{B.227})$$

with $I_c^{(\text{fund.})} = \frac{1}{2}\sigma_c$, the Pauli matrices multiplied by $1/2$, the generators of the lowest rank representation of group $SU(2)$. In the 2-to-1 correspondence between $SU(2)$ and $SO(3)$,

$$\left. \begin{aligned} U_{\hat{n}}(0 \leq \chi < 2\pi) &= e^{-i\chi\hat{n}_c\sigma_c/2} \\ U'_{\hat{n}}(0 \leq \chi < 2\pi) &= e^{-i(\chi+2\pi)\hat{n}_c\sigma_c/2} \neq e^{-i\chi\hat{n}_c\sigma_c/2} \end{aligned} \right\} \rightarrow O_{\hat{n}}(0 \leq \chi < 2\pi) = e^{-i\chi\hat{n}^c I_c^{(\text{adj.})}}, \text{ one}$$

need not to take into account for $\bar{H}$ elements of the kind $e^{-i(\pi+2\pi)\hat{\xi}_\perp^c I_c^{(\text{fund.})}} e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}}$ since due to the range of angle $0 \leq \chi < 4\pi$ for the $SU(2)$ transformations, such an element are already included in $e^{-i\pi\hat{\xi}_\perp^c I_c^{(\text{fund.})}} e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}}$ since $e^{-i2\pi\hat{\xi}_\perp^c I_c^{(\text{fund.})}} = -\mathbf{1}$ (for any $\hat{\xi}_\perp^c$) and $e^{-i(\chi+2\pi)\hat{n}^c I_c^{(\text{fund.})}} = (-\mathbf{1}) e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}}$.

On the other hand, the extension of $H = D_2 = \left\{ \mathbf{1}, e^{-i\pi\hat{x}^c I_c^{(\text{adj.})}}, e^{-i\pi\hat{y}^c I_c^{(\text{adj.})}}, e^{-i\pi\hat{z}^c I_c^{(\text{adj.})}} \right\}$ as we pass from $G = SO(3)$ to $\bar{G} = SU(2)$ is equal to

$$\begin{aligned} Q &= \left\{ \begin{array}{cccc} \mathbf{1}, & e^{-i\pi\hat{x}^c I_c^{(\text{fund.})}}, & e^{-i\pi\hat{y}^c I_c^{(\text{fund.})}}, & e^{-i\pi\hat{z}^c I_c^{(\text{fund.})}}, \\ -\mathbf{1}, & e^{-i(\pi+2\pi)\hat{x}^c I_c^{(\text{fund.})}}, & e^{-i(\pi+2\pi)\hat{y}^c I_c^{(\text{fund.})}}, & e^{-i(\pi+2\pi)\hat{z}^c I_c^{(\text{fund.})}}, \end{array} \right\} = \\ &= \left\{ \mathbf{1}, -\mathbf{1}, -i\sigma_x, +i\sigma_x, -i\sigma_y, +i\sigma_y, -i\sigma_z, +i\sigma_z \right\}, \end{aligned} \quad (\text{B.228})$$

according to relation $e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}} = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i\hat{n}^c \sigma_c \sin\left(\frac{\chi}{2}\right)$. This discrete group is called the quaternion group. As mentioned in the footnote on page 268, and although in the fundamental representation of the Lie algebra under discussion here group element $\tilde{i} \equiv -\mathbf{1}$ corresponds to identity element $\tilde{e} \equiv \mathbf{1}$ multiplied by real number -1, both elements are different: a group by definition has not the algebraic structure (and it is just this structure which determines the topological structure) of a vector space — the Lie algebra, on the other hand, does have this structure what allows us to construct the different Lie groups sharing the same commutation relations.

Through the fundamental theorem on $\pi_{1,2}(SU(2)/\bar{H})$, the following two cases result.

$$\mathbf{H} = \mathbf{N}(SO(2)) \quad \text{For } \bar{H} = \left\{ e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}}, e^{-i\pi\hat{\xi}_\perp^c I_c^{(\text{fund.})}} e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}} : 0 \leq \chi < 4\pi, \begin{array}{l} \hat{n}^a \parallel \Psi^a \\ \hat{\xi}_\perp^a \perp \Psi^a \end{array} \right\},$$

or equivalently for $H = N(SO(2))$, following holds

$$\begin{aligned} \pi_1(SO(3)/N(SO(2))) &= \pi_1(SU(2)/\bar{H}) = Z_2, \\ \pi_2(SO(3)/N(SO(2))) &= \pi_2(SU(2)/\bar{H}) = \pi_1(\bar{H}) = \mathbf{Z}. \end{aligned} \quad (\text{B.229})$$

Subgroup $\bar{H}$ is formed by two isolated pieces, that containing $SU(2)$ -rotations $e^{-i\chi\hat{n}^c I_c^{(\text{fund.})}}$ around the fixed axis given by the direction of the nonzero colinear isovectors, and the other one “island” containing the same transformations but multiplied by rotation $-i\sigma_{\text{axis}} \hat{\xi}_\perp^c$, transformations which are not related to those of the first island since the axis of this latter rotation is perpendicular to the direction of $\tilde{\Psi}^a$. The calculation of $\pi_1(\bar{H})$ necessary for the second homotopy group is easily carried out by noting that $\bar{H}$ is isomorphic to the rotations around one fixed axis, i.e., $\bar{H}$ is topologically equivalent to S^1 — more exactly, each “island” of $\bar{H}$ is already equivalent to S^1 — and thus $\pi_1(\bar{H}) = \mathbf{Z}$.

Without using the fundamental theorem on $\pi_{1,2}$, a more direct form [87] of obtaining that $\pi_{1,2}(SO(3)/N(SO(2)))$ contains at least one non-trivial

class is to notice that the set of all possible different configurations for a nonzero scalar field, $G/H = SO(3)/N(SO(2))$, corresponds in this case to the set of all possible isodirections which can be chosen by the (nonzero) scalar isovectors of the isotensor — it is the quotient $SO(3)/SO(2)$, i.e. the result of identifying all transformations which leaves unchanged an isovector with the identity, what allows us to set the correspondence between all possible transformations producing different configurations, on the one hand, and the isodirections of the isovector characterizing each configuration, on the other hand —; such $SO(3)/N(SO(2))$ is topologically equivalent to $\mathbb{S}^2$, the usual 2-dim. spherical surface with diametrical points identified. If for only one nonzero isovector, its possible isodirections in the internal 3-dimensional isospace cover completely S^2 , in the case of the scalar isotensor $\Psi^{ab} = \frac{1}{2} (\Psi^a \tilde{\Psi}^b + \tilde{\Psi}^a \Psi^b) - \frac{1}{3} \delta^{ab} (\Psi \cdot \tilde{\Psi})$ formed by two isovectors, the set of possible isodirections of its isovectors cover $\mathbb{S}^2$ since $\tilde{\Psi}^a$ as well as the opposite pointing $-\tilde{\Psi}^a$ generates the same Ψ^{ab} . If, when a closed contour is covered in external physical space around a line defect from azimuthal angle $\phi = 0$ to $\phi = 2\pi$, the loop drawn on internal $G/H = \mathbb{S}^2$ ends at the antipodal point of the start point (and thus, through the identification with the latter, in internal space this path corresponds to a closed loop), then such a loop belongs to a configuration describing a topologically stable line defect (also described by any other configuration gained from the latter through a gauge transformation). Hence $\pi_1(SO(3)/N(SO(2))) = \pi_1(\mathbb{S}^2) = Z_2$ and there is only one kind of stable, non-deformable, line defects; the line defects which can be removed by a gauge transformation are described by loops in the space of isodirections which starts and ends at the same point, loop which therefore can be deformed to a point by gliding and contracting it on the spherical surface. For the case of point defects, and since $G/H = SO(3)/N(SO(2)) = \mathbb{S}^2$, at least the following map from the distant spherical surface S_∞^2 (surrounding the point defect in physical space) onto G/H is topologically stable: that which makes correspond the external direction of tending towards S_∞^2 to $\pm$ the isodirection of the isovectors in Ψ^{ab} . Therefore, $\pi_2(SO(3)/N(SO(2))) \neq 0$.

According to (B.229), and as different from the “ $SO(3)$ ”-superconductor with two isovectors (not grouped into an isotensor) pointing in different isodirections which only contains one class of vortices and no isolated monopole — or as different from the gauge model with only one scalar isovector which contains isolated monopoles but no vortices — the model under discussion here with a scalar isotensor of second rank coupled to a gauge field contains both topological objects: one class of vortices (plus monopoles attached to the point where 2 of such vortices react) *and* isolated monopoles. In order to obtain the form of the gauge field, which together with the scalar isotensor form one configuration describing each one of the just mentioned topological objects, the covariant derivative for the scalar second rank isotensor is needed

$$D_\mu \Psi^{ab} = \partial_\mu \Psi^{ab} - e \epsilon_{adc} \Psi^{db} W_\mu^c - e \Psi^{ad} \epsilon_{bdc} W_\mu^c. \quad (\text{B.230})$$

This expression, apart from being directly shown to transform covariantly⁶⁵, is easily constructed by noting that the covariant derivative of isovector Ψ^a or $\tilde{\Psi}^b$, $D_\mu \tilde{\Psi}^a = \partial_\mu \tilde{\Psi}^a - e \epsilon_{adc} \tilde{\Psi}^d W_\mu^c$, transforms by definition as the isovector itself. Hence, for the 2nd rank isotensor $\Psi^{ab} = \frac{1}{2} (\Psi^a \tilde{\Psi}^b + \tilde{\Psi}^a \Psi^b) - \frac{1}{3} \delta^{ab} (\Psi \cdot \tilde{\Psi})$ its covariant derivative must correspond to

$$\begin{aligned} & \frac{1}{2} \left[\underbrace{(D_\mu \Psi^a) \tilde{\Psi}^b + \Psi^a (D_\mu \tilde{\Psi}^b)}_{\partial_\mu (\Psi^a \tilde{\Psi}^b) - e (\epsilon_{adc} \Psi^d \tilde{\Psi}^b + \Psi^a \tilde{\Psi}^d \epsilon_{bdc}) W_\mu^c} + (D_\mu \tilde{\Psi}^a) \Psi^b + \tilde{\Psi}^a (D_\mu \Psi^b) \right] - \\ & - \frac{\delta^{ab}}{3} [(D_\mu \Psi^d) \tilde{\Psi}^d + \Psi^d (D_\mu \tilde{\Psi}^d)] = \partial_\mu \Psi^{ab} - e \epsilon_{adc} \Psi^{db} W_\mu^c - e \Psi^{ad} \epsilon_{bdc} W_\mu^c, \quad (\text{B.231}) \end{aligned}$$

in order to transform as the isotensor Ψ^{ab} . At large distances of a topological stable object, both point as well as line defect, gauge field W_μ^c has to be arranged so that the covariant derivative of the scalar field vanishes, since otherwise the different isodirections chosen by the scalar field at all those large distances would produce an infinite contribution to the energy. From result (B.231), the vanishing of the covariant derivative applied on each of two isovectors Ψ^a and $\tilde{\Psi}^b$ separately implies $D_\mu \Psi^{ab} \Big|_{r \rightarrow \infty} = 0$.

Continuing with the particular case of $H = N(SO(2))$ where both isovectors contained in the isotensor are parallel (or antiparallel), for the case of one isolated monopole, the scalar field has already been obtained in (B.223) simply by substituting the isovector field describing the monopole, $\tilde{\Psi}^a \sim \frac{r^a}{r}$, into isotensor Ψ^{ab}

$$\Psi^{ab}(\vec{r}) \Big|_{|\vec{r}| \rightarrow \infty} = \text{constant} \times \left(\frac{r^a r^b}{r^2} - \frac{1}{3} \delta^{ab} \right), \quad (\text{B.232})$$

⁶⁵Since $(O_{\hat{n}}(\chi))_{af} = \hat{n}_a \hat{n}_f (1 - \cos \chi) + \delta_{af} \cos \chi - \epsilon_{afc} \hat{n}_c \sin \chi$, by starting at the relations $\partial_\mu (O_{\hat{n}}(\chi))_{af} = \partial_\mu \chi [(\hat{n}_a \hat{n}_f - \delta_{af}) \sin \chi - \epsilon_{afc} \hat{n}_c \cos \chi]$ and $\epsilon_{cad} (O_{\hat{n}}(\chi))_{df} \hat{n}_c = -(\hat{n}_a \hat{n}_f - \delta_{af}) \sin \chi + \epsilon_{caf} \hat{n}_c \cos \chi$ one arrives at the result that (B.230) transforms as the scalar isotensor Ψ^{ab} , $\Psi^{ab} \rightarrow \Psi'^{ab} = (O_{\hat{n}}(\chi))_{af} (O_{\hat{n}}(\chi))_{bg} \Psi^{fg}$, under a gauge transformation:

$$\begin{aligned} D_\mu \Psi^{ab} \rightarrow (D_\mu \Psi^{ab})' &= \frac{(\partial_\mu (O_{\hat{n}}(\chi))_{ah}) (O_{\hat{n}}(\chi))_{bg} \Psi^{hg} + (O_{\hat{n}}(\chi))_{af} (\partial_\mu (O_{\hat{n}}(\chi))_{bh}) \Psi^{fh} +}{+ (O_{\hat{n}}(\chi))_{af} (O_{\hat{n}}(\chi))_{bg} \partial_\mu \Psi^{fg} -} \\ &\quad - e \epsilon_{adc} (O_{\hat{n}}(\chi))_{dh} (O_{\hat{n}}(\chi))_{bg} \Psi^{hg} \left[(O_{\hat{n}}(\chi))_{ce} W_\mu^e - \frac{1}{e} \hat{n}^c \partial_\mu \chi \right] + \\ &\quad - e (O_{\hat{n}}(\chi))_{af} (O_{\hat{n}}(\chi))_{dh} \Psi^{fh} \epsilon_{bdc} \left[(O_{\hat{n}}(\chi))_{ce} W_\mu^e - \frac{1}{e} \hat{n}^c \partial_\mu \chi \right] = \\ &\quad \underline{\epsilon_{bdc} O_{dh} O_{ce} = O_{bg} \epsilon_{ghe}} (O_{\hat{n}}(\chi))_{af} (O_{\hat{n}}(\chi))_{bg} \left[\partial_\mu \Psi^{fg} - e \epsilon_{fhe} \Psi^{hg} W_\mu^e - e \Psi^{fh} \epsilon_{ghe} W_\mu^e \right], \end{aligned}$$

where the underlined terms cancel respectively against one other due to the relation proved at the begin of this footnote. It is curious to note that the covariant derivative can be also written as $D_\mu \Psi^{ab} = \partial_\mu \Psi^{ab} - i e [I_c^{(\text{adj.})} W_\mu^c, \Psi \cdot]_{ab}$, with $(I_c^{(\text{adj.})})_{ad} = -i \epsilon_{cad}$ the infinitesimal generators of the isovector representation of the Lie algebra of structure constants ϵ_{abc} : $[I_c^{(\text{adj.})} W_\mu^c, \Psi \cdot]_{ab} \equiv (I_c^{(\text{adj.})} W_\mu^c)_{ad} \Psi^{db} - \Psi^{ad} (I_c^{(\text{adj.})} W_\mu^c)_{db} = -i \epsilon_{adc} \Psi^{db} W_\mu^c - i \Psi^{ad} \epsilon_{bdc} W_\mu^c$.

where the constant value is determined by the potential, and r^a corresponds to the cartesian coordinates in the external space. And the gauge field necessary to avoid a infinitely large covariant derivative is therefore the same as that in the gauge model with only one isovector

$$W_\mu^c(\vec{r}) \Big|_{|\vec{r}| \rightarrow \infty} = -\frac{1}{er} \epsilon_{\mu cb} \frac{r^c}{r} . \quad (\text{B.233})$$

Any other configuration resulting from the application of any (allowed) gauge transformation on (B.232)-(B.233) also describes the same monopole. And this freedom in determining the representative configuration allows us to choose, as solution describing the monopole at any distance $|\vec{r}|$ of the point defect, equations (B.232)-(B.233) multiplied, respectively, by a scalar function depending on $r = |\vec{r}|$

$$\begin{cases} \Psi^{ab} = Q(r) \left(\frac{r_a r_b}{r^2} - \frac{1}{3} \delta_{ab} \right) & \text{with } \lim_{r \rightarrow \infty} Q = \text{constant} , \\ W_\mu^c = f(r) \epsilon_{\mu cb} r^b & \text{with } \lim_{r \rightarrow \infty} f = -\frac{1}{er^2} . \end{cases} \quad (\text{B.234})$$

Functions $Q(r)$ and $f(r)$ have to be solved by substituting them into the Euler-Lagrange equations of the model.

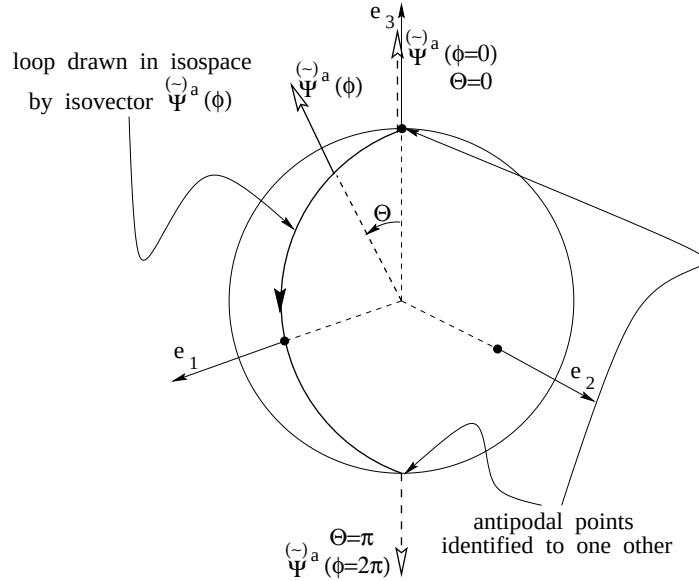


Figure B.8: One topologically stable map (or loop) from the contour S_∞^1 in physical space defined by azimuthal angle $0 \leq \phi \leq 2\pi$ onto isodirection Θ of the two isovectors entering in isotensor Ψ^{ab} for $H = N(SO(2))$: $\tilde{\Psi}^a(\phi) = \text{length} \times (\sin(\phi/2), 0, \cos(\phi/2))$.

The solution for the only class of stable vortices characterized by the unique non-trivial class of $\pi_1(SO(3)/N(SO(2))) = Z_2$ is obtained as follows: since $G/H = SO(3)/N(SO(2)) = \mathbb{S}^2$ and thus a loop in G/H can be also identified

by the “walk” on $\mathcal{S}^2$ of the isovector tip, any stable loop results from a such “walk” where the isovector ends at the antipodal point of the start point after following a closed path S_∞^1 in physical space (parametrized by $0 \leq \phi \leq 2\pi$). One of such loops, closed loops in internal space since diametrical points on surface $\mathcal{S}^2$ are identified to each other, is represented in Fig. B.8 on the plane spanned by the unit isovectors $\hat{e}_1$ and $\hat{e}_3$: each point of the loop, determined by angle Θ , indicates the isodirection taken by the two isovectors

$$\frac{\Psi^{a=1,2,3}}{\sqrt{(\Psi \cdot \Psi)}} \Big|_{|\vec{\rho}| \rightarrow \infty} = \pm \frac{\tilde{\Psi}^{a=1,2,3}}{\sqrt{(\tilde{\Psi} \cdot \tilde{\Psi})}} \Big|_{|\vec{\rho}| \rightarrow \infty} = \begin{pmatrix} \sin \Theta \\ 0 \\ \cos \Theta \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\phi}{2}\right) \\ 0 \\ \cos\left(\frac{\phi}{2}\right) \end{pmatrix}, \quad (\text{B.235})$$

$$\Psi^{ab} \Big|_{|\vec{\rho}| \rightarrow \infty} = \underbrace{\frac{\sqrt{(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi})}}{2} \Big|_{|\vec{\rho}| \rightarrow \infty}}_{\text{constant}} \begin{pmatrix} 1 - \cos \phi & 0 & \sin \phi \\ 0 & 0 & 0 \\ \sin \phi & 0 & 1 + \cos \phi \end{pmatrix},$$

where $\vec{\rho}$ is the radial distance in physical space measured on the plane containing the closed contour, i.e., the distance normal to the line defect whose stability we are studying and which we will assume to be stretching along the z -direction of the external space; this notation for the distance has been used to distinguish it from the distance measured from a point defect (no distinction will be made between these two distances if there is no confusion). The length of the scalar isovectors and isotensor for $|\vec{\rho}| \rightarrow \infty$ must tend to a constant value which will be calculated later in (B.252). The gauge field which compensates too large gradients in the isodirection of the isovectors at $|\vec{\rho}| \rightarrow \infty$ is given by the condition $D_\mu \tilde{\Psi}^a \Big|_{|\vec{\rho}| \rightarrow \infty} = 0$

$$D_\phi \tilde{\Psi}^a \Big|_{|\vec{\rho}| \rightarrow \infty} = \left(\frac{1}{\rho} \frac{\partial \tilde{\Psi}^a}{\partial \phi} - e \epsilon_{abc} \tilde{\Psi}^b W_\phi^c \right) \Big|_{|\vec{\rho}| \rightarrow \infty} = 0,$$

$$-\frac{1}{2\rho} \begin{pmatrix} -\cos\left(\frac{\phi}{2}\right) \\ 0 \\ \sin\left(\frac{\phi}{2}\right) \end{pmatrix} = e \begin{pmatrix} -\cos\left(\frac{\phi}{2}\right) W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} \\ \cos\left(\frac{\phi}{2}\right) W_\phi^{c=1} \Big|_{|\vec{\rho}| \rightarrow \infty} - \sin\left(\frac{\phi}{2}\right) W_\phi^{c=3} \Big|_{|\vec{\rho}| \rightarrow \infty} \\ \sin\left(\frac{\phi}{2}\right) W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} \end{pmatrix},$$

and

$$D_\rho \tilde{\Psi}^a \Big|_{|\vec{\rho}| \rightarrow \infty} = \left(\underbrace{\frac{\partial \tilde{\Psi}^a}{\partial \rho}}_{=0, \text{ at } \rho \rightarrow \infty} - e \epsilon_{abc} \tilde{\Psi}^b W_\rho^c \right) \Big|_{|\vec{\rho}| \rightarrow \infty} = 0,$$

whose one possible solution is

$$\begin{cases} W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} = -\frac{1}{2e\rho}, & W_\phi^{c=1} \Big|_{|\vec{\rho}| \rightarrow \infty} = 0 = W_\phi^{c=3} \Big|_{|\vec{\rho}| \rightarrow \infty}, \\ W_\rho^c \Big|_{|\vec{\rho}| \rightarrow \infty} = 0. \end{cases} \quad (\text{B.236})$$

Therefore, the vortex carries a flux equal to

$$\Phi_0 = \oint_{C \text{ at } \rho \rightarrow \infty} W_\phi^{c=2} \rho d\phi = -\frac{\pi}{e}, \quad (\text{B.237})$$

For the behaviour of equations (B.235) and (B.236) at smaller distances, one has to multiply the above large distance solutions by arbitrary functions of ρ to be solved from the Euler-Lagrange equations. Of course, the above solution (B.235)-(B.236) is not the only one configuration describing the line defect, since any other configuration related to this one by means of a gauge transformation (an isorotation by an angle different at each site of physical space) also describes the vortex: the only condition required is that, when in physical space one “walks” along S_∞^1 , the two isovector’s isodirection follows in the internal space the same path on $\mathbb{S}^2$ ending at the diametrical point of the start. And such a condition cannot be removed by any (allowed) gauge transformation.

Note that $\pi_1(G/H)$ is Abelian for $H = N(SO(2))$. A different case will occur for the next scheme of symmetry breaking due to the fact that the directions of two isovectors are not parallel to one other.

$H = D_2$ For $H = D_2$, and thus $\bar{H} = Q$, one obtains

$$\begin{aligned} \pi_1(SO(3)/D_2) &= \pi_1(SU(2)/Q) = Q, \\ \pi_2(SO(3)/D_2) &= \pi_2(SU(2)/Q) = \pi_1(Q) = 0. \end{aligned} \quad (\text{B.238})$$

The first homotopy group is quickly calculated since the quaternion group is exclusively formed by discrete operations, i.e., by isolated elements or “islands”. For the second homotopy group, using again the absence of continuous transformations in Q , $\pi_1(\bar{H}) = 0$ analogously as shown for the particular case of equation (B.216).

Hence, the gauge model for a 2nd rank scalar isotensor formed by two non-parallel scalar isovectors has no isolated monopoles but only vortices, just as the model described in (B.206) ⁶⁶: in both cases the appearance of vor-

⁶⁶In that model (B.206), where the two isovectors form a nonzero angle Θ_0 without being combined into an isotensor, a rigid complete rotation of both isovectors cannot be deformed to a point simply

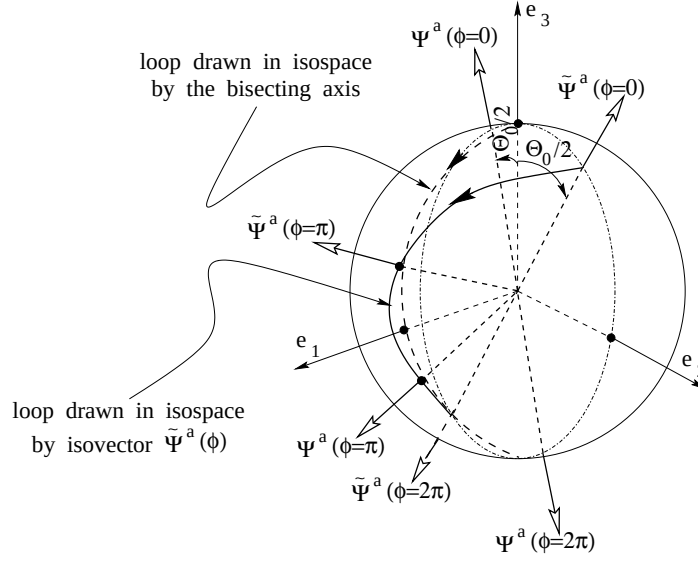
tices alone ⁶⁷ is consequence of the complete, up to some discrete residual symmetries, “breakdown” (B.217) of the gauge symmetry described by the multiply-connected $SO(3)$ group. The combination of two isovectors into an isotensor enables, nevertheless, a number of different stable vortex classes higher than that when both are left uncombined: this is due to the discrete symmetries caused by the different arrangement of the isovectors within the isotensor. If in the model (B.206), once a certain projection of $SO(3) \equiv \mathcal{S}^3$ onto the space for two non-colinear isovectors was chosen, there was only one vortex of flux $\frac{2\pi}{e}$, now in the model with a scalar second rank irreducible isotensor there will occur different vortices all carrying a flux of $\frac{1}{2} \frac{2\pi}{e}$.

How to find out the trajectory in isospace followed by the two isovectors which corresponds to a stable vortex for $H = D_2$? Although particular cases like $G/H = SO(3)/SO(2)$ (for only one isovector) and $G/H = SO(3)/N(SO(2))$ (for two colinear isovectors combined in a symmetric isotensor) allowed us to identify (with slight modifications) quotient space G/H with the possible orientations taken on by the isovector(s) in the internal space, in general both spaces are different. G/H comprises all the different transformations which, leaving invariant the energy, produces from a certain scalar field’s configuration of lowest energy all the other scalar field’s arrangements with the same lowest energy. Only if H corresponds to the operations leaving unchanged directly the isovector (plus some additional discrete symmetries) — the symmetries of a nonzero scalar field, described by subgroup H , have not to coincide with the symmetries of the isovector(s) forming the tensor representation of such scalar field — then G/H can be make correspond to the space S^2 of isovector’s orientations (the tip of the isovector covers S^2), where the additional discrete symmetries causes some points on S^2 to be identified together as in the case of $H = N(SO(2))$. In the case under study now, $G/H = SO(3)/D_2$, this correspondence is not possible since H is much more reduced than $N(SO(2))$. For stable vortices contained in any of the classes $\pi_1(SO(3)/D_2)$, nevertheless, one can still guess the form of the trajectory drawn by the corresponding isovectors.

As discussed in the sketched proof of the fundamental theorem on π_1 (see figure on page 266) a topologically stable configuration corresponds to a path in G/H which starts at one “island” of H (more properly, of $\bar{H}$ if G is not simply-connected, as it is the case for $G = SO(3)$) and ends at another different island; this is a closed path in G/H since the whole H is identified as only one point in the quotient space. The existence of vortices for $G/H = SO(3)/N(SO(2))$, different from $G/H = SO(3)/SO(2)$ which has no stable vortex, is just due to the discrete symmetry of inversion of both isovec-

because, although such a deformation could be carried out for the loop of one of the isovectors, $\Theta_o \neq 0$ impedes the simultaneous deformation of the loop drawn by the other isovector.

⁶⁷Point defects (monopoles) require a continuous subgroup H of remaining symmetries, since otherwise $\pi_1(\bar{H}) = 0$ for the subgroup H formed exclusively by discrete operations.



$$\Psi^a = \text{length} \times \begin{pmatrix} \frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) \sin \phi + \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) \\ -\sin\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \\ \frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) (\cos \phi - 1) + \cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \end{pmatrix}$$

Figure B.9: One topologically stable map (or loop) from the distant contour S^1_∞ in physical space defined by azimuthal angle $0 \leq \phi \leq 2\pi$ onto the directions of the two isovectors which are combined into Ψ^{ab} , isovectors which for $H = D_2$ form always an angle $\Theta_0 \neq 0$. Isovector $\tilde{\Psi}^a$ is given, up to its length, by an equation similar to that of Ψ^a but with the change $\Theta_0/2 \rightarrow -\Theta_0/2$.

tors which produces thus the two isolated “islands” forming $H = N(SO(2))$, and which are responsible for non-deformable loops on the spherical surface topologically equivalent to G/H by identifying diametrically placed points on that. Such discrete symmetry also occurs for the case $G/H = SO(3)/D_2$ as mentioned after (B.226): it corresponds to the π -rotation around the axis normal to the plane of the two isovectors. The particular transformation in $H = D_2$ of a rotation by π around the axis y^c — axis which bisects the angle $\Theta_0 \neq 0$ between the two isovectors and which becomes the isodirection of the colinear isovectors in $H = N(SO(2))$ — can be viewed as a special case of a $G/H = SO(3)/N(SO(2))$ where from all $SO(2)$ -rotations around such axis y^c only the rotation by π is allowed. Therefore, from Fig. B.8 one can already determine a possible trajectory of the isovectors which corresponds to a class of stable vortices: at the same time as we cover a closed path S^1_∞ encircling one of such vortices in external physical space, bisecting axis y^c draws a half circumference in isospace and the two isovectors rotate simultaneously π around such an axis, as represented in Fig. B.9, such that at the end each isovector points towards the isospace antipodal point of its start point (which are identified with the former).

In fact, this is not the unique loop topologically stable. Let us consider in more detail loop-maps between the physical space's S_∞^1 and the quotient space $G/H = SO(3)/D_2 = SU(2)/Q$: since $SU(2)$ is topologically equivalent to the 3-dim. spherical surface S^3 embedded into a 4-dim. space, the eight elements of quaternion group $\bar{H} = Q$ — represented as 8 particular points on S^3 and identified together in quotient space $SU(2)/Q = S^3/Q$ — cause G/H to have the topology of S^3 with antipodal points in the 4 orthogonal isodirections (of the 4-dim. space) identified⁶⁸. To each of the 7 different identifications in couples of such different points on S^3 corresponds as copy or realization in the internal space where the isovectors (not the scalar isotensor, whose internal space is S^3/Q) are moving a spherical surface $\mathbb{S}^2$ with diametrical points equated as the same; and as a non-deformable mapping one like in Fig. B.9, or as well, any mapping resulting from this by means of any gauge transformation. Apart from the trivial, to-the-identity-deformable mapping (where the isovectors end pointing towards the same isospace's point as at the start), there is 7 further loop-classes non-deformable into each other, each one corresponding to the 7 different projections of S^3/Q onto one $\mathbb{S}^2$, being the latter the isodirections available to the tip of the isovectors.

If H becomes again $H = N(SO(2))$, i.e. if both isovectors become colinear, 2 of the non-trivial different vortices (those corresponding to $\pm i\sigma_y \in Q$ since the bisecting axis coincides now with the directions of both isovectors) go into the trivial class, whereas the 4 other non-deformable vortices (those corresponding to $\pm i\sigma_x \in Q$ and $\pm i\sigma_z \in Q$, axes which produce the same symmetry when both isovector are (anti)parallel) coincide with the non-trivial class already existing for $G/H = SO(3)/N(SO(2))$.

$$\text{Starting at } \Psi^a|_{|\vec{\rho}| \rightarrow \infty} = \text{length} \times \begin{pmatrix} 0 \\ -\sin\left(\frac{\Theta_0}{2}\right) \\ \cos\left(\frac{\Theta_0}{2}\right) \end{pmatrix}, \quad \tilde{\Psi}^a|_{|\vec{\rho}| \rightarrow \infty} = \text{length} \times \begin{pmatrix} 0 \\ \sin\left(\frac{\Theta_0}{2}\right) \\ \cos\left(\frac{\Theta_0}{2}\right) \end{pmatrix},$$

the two isovectors represented in Fig. B.9 are obtained through the following isorotations: by angle $\zeta = \phi/2$ around axis $\hat{e}_2$ to locate the bisecting axis, followed by a rotation of angle $\gamma = \phi/2$ around this bisecting axis

$$\begin{pmatrix} \sin(\phi/2) \\ 0 \\ \cos(\phi/2) \end{pmatrix}. \text{ The resulting rotation matrix is read off relation (B.120) for}$$

⁶⁸For $\bar{H} = Q$, this identification among the eight S^3 's points diametrically placed in the 4 directions is easily seen by looking at the correspondence between: a) any $SU(2)$ matrix $e^{-i\chi/2 \hat{n}_a \sigma_a} = \mathbf{1} \cos\left(\frac{\chi}{2}\right) - i\hat{n}_a \sigma_a \sin\left(\frac{\chi}{2}\right) = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}$, and b) the four real coordinates $a_0 = \cos(\chi/2)$, $a_k = -\hat{n}_k \sin(\chi/2)$ of a point on S^3 since $a_0^2 + a_k^2 = 1$: the 8 elements of the quaternion group $Q = \{\pm 1, \mp i\sigma_x, \mp i\sigma_y, \mp i\sigma_z\}$ correspond thus to the following 8 points on S^3 whose coordinates are $(a_0, a_1, a_2, a_3) = \{(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)\}$ or to any other set of coordinates resulting from any global isorotation applied on the former set. In quotient space $SU(2)/Q \equiv S^3/Q$, these 8 points are mutually identified.

$$\omega = 0, \zeta = \phi/2, \gamma = \phi/2$$

$$O = \begin{pmatrix} \frac{1}{2}(\cos \phi + 1) & -\frac{1}{2} \sin \phi & \sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) & 0 \\ -\frac{1}{2} \sin \phi & -\frac{1}{2}(\cos \phi - 1) & \cos\left(\frac{\phi}{2}\right) \end{pmatrix}, \quad (\text{B.239})$$

where as usual ϕ denotes the azimuthal angle in external physical space. Thus

$$\begin{aligned} \Psi^a \Big|_{|\vec{\rho}| \rightarrow \infty} &= \text{length} \times \begin{pmatrix} \frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) \sin \phi + \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) \\ -\sin\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \\ \frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) (\cos \phi - 1) + \cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \end{pmatrix}, \\ \tilde{\Psi}^a \Big|_{|\vec{\rho}| \rightarrow \infty} &= \tilde{\text{length}} \times \begin{pmatrix} -\frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) \sin \phi + \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \\ -\frac{1}{2} \sin\left(\frac{\Theta_0}{2}\right) (\cos \phi - 1) + \cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \end{pmatrix}. \end{aligned} \quad (\text{B.240})$$

Its corresponding gauge field follows from the condition of vanishing of the covariant derivative applied on each isovector

$$\begin{aligned} 0 &= \left(\frac{D_\phi \Psi^a + D_\phi \tilde{\Psi}^a}{2} \right) \Big|_{|\vec{\rho}| \rightarrow \infty} \implies \frac{1}{2\rho} \partial_\phi (\Psi^a + \tilde{\Psi}^a) \Big|_{|\vec{\rho}| \rightarrow \infty} = e \epsilon_{abc} (\Psi^b + \tilde{\Psi}^b) \Big|_{|\vec{\rho}| \rightarrow \infty} W_\phi^c \Big|_{|\vec{\rho}| \rightarrow \infty} \\ &= e \begin{pmatrix} -\cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) \\ 0 \\ \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) \end{pmatrix} = e \begin{pmatrix} -\cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} \\ \cos\left(\frac{\Theta_0}{2}\right) \cos\left(\frac{\phi}{2}\right) W_\phi^{c=1} \Big|_{|\vec{\rho}| \rightarrow \infty} - \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) W_\phi^{c=3} \Big|_{|\vec{\rho}| \rightarrow \infty} \\ \cos\left(\frac{\Theta_0}{2}\right) \sin\left(\frac{\phi}{2}\right) W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} \end{pmatrix}, \end{aligned}$$

with the same solution as (B.236)

$$\begin{cases} W_\phi^{c=2} \Big|_{|\vec{\rho}| \rightarrow \infty} = -\frac{1}{2e\rho}, & W_\phi^{c=1} \Big|_{|\vec{\rho}| \rightarrow \infty} = 0 = W_\phi^{c=3} \Big|_{|\vec{\rho}| \rightarrow \infty}, \\ W_\rho^c \Big|_{|\vec{\rho}| \rightarrow \infty} = 0. \end{cases} \quad (\text{B.241})$$

As usual, this is only one representative configuration of the infinite many gauge-related configurations which describe the same vortex. Clearly, the flux carried by such a vortex is half of that obtained in model (B.206) where two non-parallel separated isovectors were coupled to a gauge field. In the

model where the two isovectors are combined into a isotensor, the additional discrete symmetry — introducing thus a further type of stable vortex to share the flux content of the model with the two separated isovectors — of interchanging the two isovectors within the isotensor is the responsible for the identification of certain isovector's directions: responsible thus for the fact that the stable configuration's isovectors only cover a half circumference and the flux is therefore $\frac{1}{2} \frac{2\pi}{e}$.

This division by an integer of the vortex' flux — that flux arising from the model with the $SO(3)$ gauge symmetry completely broken by two non-parallel isovectors — when one works with a scalar field in increasing higher rank representations of the compact gauge group is a general characteristic resulting from the just mentioned appearance of a larger number of discrete symmetries and the corresponding larger $\pi_1(G/H)$; another example will be briefly studied later for the 3rd rank irreducible representation of $SO(3)$. Although only discussed for $G = SO(3)$, this also holds for group $G = SU(2)$: scalar field in the $SU(2)$ lowest rank representation, the isospinor or fundamental representation of the Lie algebra (with structure constants $C_{abc} = \epsilon_{abc}$), does not allow any vortex when the symmetry is completely broken down since group $SU(2) \equiv S^3$ is simply-connected; on the contrary, a higher rank representation of $SU(2)$ allows for discrete symmetries which can make $G/H = SU(2)/H$ become multiply-connected (and thus $\pi_1(SU(2)/H) \neq 0$) as a consequence of the mutual identification of several points on the 3-dim. spherical surface S^3 . For the following Appendices we will focus our attention only on models whose gauge symmetry is described by the Abelian $U(1)$ group, where such a group can be either the original symmetry of the system (this case will be denoted as the non-compact $U(1)$ gauge model) or the residual symmetry resulting from the partial breakdown of the actual underlying system's symmetry described by a non-Abelian compact group such as $SO(3)$ or $SU(2)$ (case called the compact $U(1)$ gauge model). Hence, especially for the compact $U(1)$ model where the above patterns of division of the flux can occur, one implements the covariant derivative by introducing an integer q in the coupling between the $U(1)$ gauge field A_μ ⁶⁹ and the scalar field ψ , whose nonzero length eventually breaks completely the $U(1)$ gauge symmetry (original as well as residual coming from a larger non-Abelian compact group)

$$\text{covariant derivative in } U(1) \text{ model: } (\partial_\mu - iqeA_\mu) \psi. \quad (\text{B.242})$$

A nonzero scalar field which far around a line defect takes the value of $\psi = |\psi|_0 e^{i\phi}$, with ϕ the azimuthal angle in external space and $|\psi|_0$ constant, will force the $U(1)$ gauge field at those distant sites encircling the vortex to satisfy the following equation: $2\pi - qe \oint_{C \text{ at } \rho \rightarrow \infty} A_\mu dl^\mu = 0$. As it should be,

⁶⁹ $U(1)$ gauge field A_μ equated to the $W_\mu^{c=2}$ of (B.241) if one chooses that solution as representative among all configurations describing the topological line defect.

q will result to play no relevant role in the behaviour of the non-compact $U(1)$ gauge model, since then there is no underlying non-Abelian compact group which could be responsible for such different cases as the “ $SU(2)$ ”-superconductor — where the $U(1)$ symmetry results from the partial breaking of $SU(2)$ through a scalar field in the fundamental representation —, which displays no vortex, or as the “ $SO(3)$ ”-superconductor — where the $U(1)$ symmetry is the residuum from the partial breakdown of $SO(3)$ through a scalar field in the isovector representation —, which shows only one class of stable vortices.

It should be noted that the just introduced compact $U(1)$ gauge model, where through a previous step a non-Abelian compact group is already broken partially down to $U(1)$ to be broken the latter completely by means of the coupling to ψ , does not correspond exactly to the model studied above where the non-Abelian compact group $SO(3)$ is broken completely down to $H = D_2$ *at once*. The mentioned previous step for the compact $U(1)$ gauge model runs along the same process as that seen in the Note on page 263, where the coupling to *one* of the orthogonal isovectors — those resulting from the linear combination of the two non-colinear isovectors which are responsible, in the isotensor model, for the complete breakdown of the $SO(3)$ — reduced $SO(3)$ down to $U(1)$: it was then the coupling to the *other* isovector which yielded the field equations analogous to the that describing the $U(1)$ superconductor and thus characteristic for the breaking of the $U(1)$ (residual) gauge symmetry.

Another important question for the case $H = D_2$ is the fact that the group of the different loop-classes, $\pi_1(SO(3)/D_2)$, is non-Abelian and therefore the product of some of its classes does not commute. For example, let α and β be vortices of the classes represented, respectively, by quaternion group’s elements $-i\sigma_x$ and $-i\sigma_y$ — and thus α^{-1} and β^{-1} corresponds to Q -elements $+i\sigma_x$ and $+i\sigma_y$ —; then following holds

$$\beta^{-1} \circ \alpha^{-1} \circ \beta \circ \alpha \in \text{class } -\mathbf{1} \not\sim \text{trivial class } \mathbf{1}, \quad (\text{B.243})$$

and therefore entanglement will occur when one vortex belonging to one of the above classes encircles a vortex of the other class, just as discussed after equation (B.218). This is nothing else as the consequence of the usual non-commutativity of rotations in (iso)space when applied to the (dimensionful) object formed by the two points on spherical surface S^3 separated an angular distance Θ_0 and which indicate, measured from the center of such a sphere, the isodirections of the two non-parallel isovectors. Only for the case of $\Theta_0 = 0, \pi$ the isotensor is represented by only one (dimensionless) point on that spherical surface (which moreover is then of a lower dimension) and then the order in the isorotations is not relevant and the first homotopy group becomes Abelian.

After having presented in general the two possible patterns of “breaking” the gauge symmetry $SO(3)$ for a model with a scalar isotensor of second rank, let us finish the dis-

cussion by considering the relations among the parameters of the potential (containing the self-interactions of this isotensor) which yields each one of these two patterns. Further, a specific form for the potential is also necessary in order to calculate the constant nonzero value to which the length $\sqrt{\Psi^{ab}\Psi^{ab}}$ of the isotensor (and of its corresponding isovectors) tends by approaching the spatial infinity.

The generalization of potential $V(\Psi^a) = -|\mathcal{A}|(\Psi^a)^2 + \frac{|\mathcal{B}|}{2}(\Psi^a)^4$, used in the $SO(3)$ gauge model with one scalar isovector field, for the case of a scalar field in representation of $SO(3)$ of higher rank can involve different combinations of up to the fourth power of the scalar field. For instance, although for the isovector or adjoint representation it was not possible to include a term with a third power of Ψ^a (term $\Psi^a\Psi^a\Psi^b$ would not be a scalar function), for the isotensor of second rank such a term can be constructed: $\Psi^{ab}\Psi^{bc}\Psi^{ca}$, but not $\Psi^{ab}\Psi^{ba}\Psi^{cc}$ since Ψ^{ab} is traceless. For the isotensor of third rank Ψ^{abc} , the third power in the potential $V(\Psi)$ cannot again be defined due to the impossibility to couple completely one index of one isotensor with the index of another isotensor, coupling in pairs forced by the vanishing of the isotensor's trace for two any of the scalar's internal indices. This impossibility appears always for isotensors in representations of $SO(3)$ of odd rank and thus the third power of the scalar field will only occur in the potential for representations of even rank.

The fact that in the potential breaking the symmetry we shall not consider powers of the scalar field Ψ higher than the fourth one is required in order the whole model to remain finite as the effect of fluctuations is taken into account in the calculation of correlation functions. Before going into the details of the breaking patterns of the potential for the isotensor of second rank, let us gain in the next Note a bit more insight on the last affirmation about the restriction up to the fourth scalar field's power in the potential.

Note. The general form of potential $V(\Psi)$: informal comments.

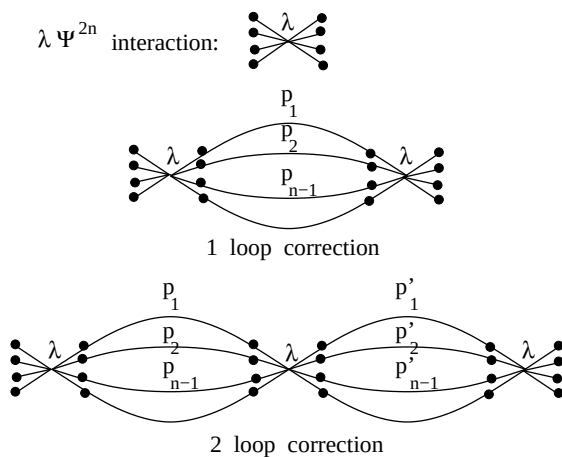
When one writes down the energy (or the Lagrangian) for the system under study, one is usually observing it in a distance range too less detailed to guess really which kind of interactions are actually taking place: one is actually modelling the behaviour of such a system at the distance scale one observes. If one looks at the system under a magnifying glass, new interactions and new fields may become observable which were not distinguishable in the less detailed picture of the system. And also the already existing model's parameters of the first picture may change their strength because at distances smaller than those firstly observed screening/strengthening effects take place: interactions which at a coarse scale looked as immediate may actually become interactions mediated by many intermediate degrees of freedom (when looking at a more detailed picture of the system) and whose effect is to modify the strength of the modelled interaction's parameters.

Hence the model originally written down is actually an effective model only, which depends on fields which may have nothing to do with the actual degrees of freedom of the system, and which is valid for a certain interval of distances much larger than a fundamental scale (hidden to our eyes) where the actual interactions, responsible for the system's behaviour and usually much more different than those modelled, occur. If we are not lucky with our proposed model because new kinds of interactions, different from the modelled ones, appear when one increases the magnifying power under which we observe the system, then our original model is valid only for the distance scale initially observed (or for scales less finer than that, where the initial details go further lost); this does not mean, nevertheless, that the latter model is a bad or false one but that it is a too (over)simplified model. On the contrary, if we have carefully constructed the model, the validity of the model can comprise scales more accurate than the scale at which we have formulated the model: the kind of interactions remain the same

through all these scales varying only, when the magnifying power is modified, the coupling parameters characterizing the strength of such interactions, as we shall now see.

Let us make this discussion a bit more concrete: we suppose that at a certain coarse, not detailed, distance scale we are studying a self-interaction in the scalar field which at that scale gets described by $\lambda(\Psi)^{2n}$, being λ the corresponding coupling parameter. Beside this self-interaction, the energy part due to an inhomogeneous distribution of the scalar field is described by the usual gradient square $|\partial_\mu \Psi|^2$ (which can also contain the coupling to a gauge field). If at the initial scale the value of λ is λ_{coarse} (also called the bare coupling parameter), what is its value λ_{fine} at a much finer scale, *assuming* that at the shorter distances our effective model is still valid and the interaction remains of the form $(\Psi)^{2n}$ without introducing new interaction types? Coupling parameter λ is actually determined by the form how $2n$ scalar fields interact with each other, or stated more mathematically, this coupling is defined through the strength of a correlation function among the $2n$ scalar fields. To calculate any correlation function requires a partition function by summing over all possible configurations: the configuration which minimizes the energy yields the main contribution but there are also other possible higher energy configurations which can produce a relevant contribution. These configurations of higher energy, also called fluctuations about the configuration of lowest energy, actually arise from effects occurring at distance scales shorter than that of the initial coarse picture — the latter only allowed us to distinguish the roughly main features without the detail of the fluctuations —, always bearing in mind that such fluctuations in the partition function arise exclusively from the combination of the interaction terms already present in the original model's energy. In order to define a configuration contribution to the partition it is necessary an energy reference η to which the energy of each configuration is compared to. Since most of the interactions types involve powers of the fields higher than the quadratic one, the resulting partition function cannot be calculated directly (only for quadratic interactions the partition function reduces itself to Gaussian integrals): a perturbative expansion is carried out in powers of perturbative parameter $\eta\lambda$, supposed small, such that each perturbative term can be computed because it contains only a Gaussian integral of the energy's quadratic part. The details of this expansion are not relevant here and can be found in the Note on page 730.

The same perturbative expansion will be applied to the calculation of the correlation functions which gives the form about how the coupling parameter evolves from λ_{coarse} to λ_{fine} when the smallest distance still observable changes. Instead of speaking about short distances it is more comfortable to work with the Fourier transforms: a small distance $\vec{r}$ corresponds to a large Fourier momentum $\vec{p}$ (since otherwise the exponential $e^{i\vec{p}\cdot\vec{r}}$ in the Fourier integral would oscillate too rapidly to produce a contribution). Regarding the just mentioned perturbative expansion, we only need the statement that the effective varying coupling λ_{fine} can be graphically represented as the sum of diagrams like those of the following figure.



Interaction $\lambda(\Psi)^{2n}$ among the $2n$ scalar fields at a certain “point” V is represented by $2n$ legs attached to a vertex of strength λ and corresponds just to the only interaction which one can distinguish at the coarse scale: it corresponds thus to λ_{coarse} . At finer distance scales where one can distinguish the fluctuations, the interaction among the $2n$ scalar fields at V contains also intermediate ways due to the self-interactions of the scalar field mediated by such fluctuations. Some of these intermediate ways are graphically represented by the so-called 1-loop and 2-loop corrections of the figure, where according to the rules explained in (D.276), one has to integrate over all independent Fourier momenta in the internal lines of each diagram; the momentum in the last internal line is not independent but given, for the calculation of the coupling parameter at finer scales, by constrain $\vec{0} = \vec{p}_1 + \dots + \vec{p}_{n-1} + \vec{p}_n$. Each internal

line represents the kernel of the Gaussian integral in the perturbative expansion, i.e., it is the propagator, or (minus) the inverse, of the energy part quadratic in its fields. The form of this propagator is given by $\frac{1}{\vec{p}^2 + \dots}$, since the energy terms quadratic exclusively in the scalar fields is formed by the part from the gradient in the scalar field, $|\partial_\mu \Psi|^2$, as well as other possible quadratic interactions which are not relevant in the following discussion about the behaviour at small scales ($|\vec{p}| \rightarrow \infty$).

Hence the coupling parameter at the finer scale, where all the above diagram-interactions have become distinguishable, is given by

$$\begin{aligned} \lambda_{\text{fine}} = & \lambda_{\text{coarse}} + C_1 \eta \lambda_{\text{coarse}}^2 \int \frac{d^d \vec{p}_1 d^d \vec{p}_2 \dots d^d \vec{p}_{n-1}}{(\vec{p}_1^2 + \dots)(\vec{p}_2^2 + \dots) \dots (\vec{p}_{n-1}^2 + \dots)(\vec{p}_n^2 + \dots)} + \\ & + C_2 \eta^2 \lambda_{\text{coarse}}^3 \int \frac{d^d \vec{p}_1 \dots d^d \vec{p}_{n-1}}{(\vec{p}_1^2 + \dots) \dots (\vec{p}_n^2 + \dots)} \frac{d^d \vec{p}'_1 \dots d^d \vec{p}'_{n-1}}{(\vec{p}'_1^2 + \dots) \dots (\vec{p}'_n^2 + \dots)} + \dots, \end{aligned} \quad (\text{B.244})$$

where $C_1, C_2, \dots$ are specific numerical factors and d is the dimensionality of the external physical space. Note the different powers in η : they indicate the number of intermediate λ_{coarse} -vertices contributing to λ_{fine} , and thus the order of how many fluctuations are being considered. In the range of very short distances, i.e., of large Fourier momentum $\vec{p}_i$, the integrals in (B.244) generally diverge. Such divergences are “avoided” by the change of $\int_0^\infty dp$ into $\int_0^{\Lambda_c} dp$, with Λ_c a large but finite cut-off, change equivalent to modify the model for the range of distances shorter than the cut-off scale $1/\Lambda_c$. The necessity of introducing the cut-off in the relation between λ_{fine} and λ_{coarse} (i.e. its divergence at very short distances) signals that these coupling parameters are only an effective description of unknown interactions occurring at a “hidden” very short-distance fundamental scale which remains far away from being available to our magnifying power [80]. Such a fundamental scale, like for instance the lattice spacing in an effective model describing a system which actually lives in a lattice (or any other fundamental discreteness in the model), sets the shortest distance until which our model can still describe well the system (under the fulfilling of a condition which will be discussed now): for smaller distances, interactions appear which may be well different from those contained in the original model.

However, we are not interested in such very short distances, with all their interactions which cannot be observed by us, but in constructing a model accurate enough and valid for the observable large distance scale. We need know which fields of all the short scale degrees of freedom remain in the long range scale as effective fields for our model. This simplification is carried out at the price of “dragging” the cut-off Λ_c all throughout the calculation. We may ask ourselves, whether the coarse coupling λ_{coarse} (measured at a very less detailed picture) and the cut-off can be traded off, through a *redefinition* of the original effective model, by a model independent of Λ_c and thus finite at $\Lambda_c \rightarrow \infty$. The answer is yes if a certain condition is satisfied: by looking at (B.244)

$$\lambda_{\text{fine}} = \lambda_{\text{coarse}} + C_1 \eta \lambda_{\text{coarse}}^2 \left[\frac{\Lambda_c^{d(n-1)}}{\Lambda_c^{2n}} - \text{constant} \right] + \dots,$$

one sees that λ_{fine} will remain finite at $\Lambda_c \rightarrow \infty$ if

$$\text{interaction } \lambda(\Psi)^{2n} : \quad n(d-2) - d \leq 0. \quad (\text{B.245})$$

Although this condition is not strong enough to ensure that all the finer pictures for any possible correlation function taken of the system will be described by finite coupling parameters, it suffices nevertheless to state that the *number* of such divergent cases will be *limited* since the fulfilment of (B.245) already ensures that integrals in more complicated diagrams will not behave at $\Lambda_c \rightarrow \infty$ worse than λ_{finite} (B.244) ⁷⁰. Therefore, if the divergences in λ_{fine} at $\Lambda_c \rightarrow \infty$ can be removed by substituting the initial effective model by another one containing the same kind of interactions but different coupling

⁷⁰In fact, criterion (B.245) does not exclude the less severe logarithmic divergence $\lim_{\Lambda_c \rightarrow \infty} \int_0^{\Lambda_c} \frac{dp}{p}$. Nor it does if some of the internal lines in a loop correction have attached to them another internal line so that there is an integral of

parameters, the resulting model is finite at any order in the perturbative expansion, and as such, a valid model for any distance scale independently of how far or near to the fundamental scale we are observing the system. One of such substitution procedures is the following one: assuming (B.245) holds true, we can separate in the limited number of divergent integrals the diverging part, simply by cutting arbitrarily the integral as $\lim_{\Lambda_c \rightarrow \infty} \int_0^{\Lambda_c} dp = \int_0^\mu dp + \lim_{\Lambda_c \rightarrow \infty} \int_\mu^{\Lambda_c} dp$ and then by modifying the original effective model at the coarse scale through the introduction of a limited number of terms such that these cancel the diverging part $\lim_{\Lambda_c \rightarrow \infty} \int_\mu^{\Lambda_c} dp$. An example of this is presented after (D.292).

It is very important to stress that this process can be successful only if at each progressive step of making finer the distance scale one can distinguish — i.e., of considering a wider set of more complicated diagrams by going towards perturbative expansion's terms of higher powers in $\eta\lambda$ —, no additional diverging diagram appears of a kind not present at those diagrams of a less finer scale. Only in such a case the correcting terms already introduced in the effective model at coarser distances can also take care of the divergences of more complicated diagrams. Models whose number of different diverging diagrams is finite, or equivalently which satisfy (B.245), are called renormalizable models and are the only ones for which limit $\Lambda_c \rightarrow \infty$ is well defined without knowing exactly which kinds of interactions are actually occurring at very short distances: these are the models useful to describe in a phenomenological way the system under study with only the reduced knowledge of the interactions occurring at large scales. Re-arrangement of the initial interaction terms found at a coarse scale — redefinition which implies the introduction of an arbitrary scale μ (it can be proved that the form of the final finite results are independent of how this introduction is carried out), being μ a footprint of the existence of fundamental hidden scale — is already enough for such models to take account of all divergences at $\Lambda_c \rightarrow \infty$ appearing at any order in the perturbative expansion of any correlation function. As a result, the strength of the coupling parameters depends on the scale, or mathematically expressed, λ_{fine} is a function of the arbitrary μ : $\lambda_{\text{fine}}(\text{certain scale } \mu) = \lambda_{\text{coarse}} f(\mu)$, where $f(\mu = 0) = 1$ corresponds to λ_{coarse} . The choice of another scale μ' , to separate the diverging part in the finite number of divergent diagram's integrals, leads to another strength of the resulting $\lambda_{\text{fine}}(\mu')$ but always the form of the interaction terms in the original modelled energy remains the same. Because of the scale-depending relation, $\lambda_{\text{fine}}(\mu)$ is also called a running coupling parameter.

Applying all this discussion, condition (B.245) rules out as an effective interaction at large distances a scalar self-interaction of the type $\lambda(\Psi)^{2n}$ with $n > 3$ for $d = 3$, or a potential's term $\lambda(\Psi)^{2n}$ with $n > 2$ for $d = 4$. This is the reason because we are always dealing with a potential whose scalar self-interaction terms only contain up to the fourth power of the scalar field.

As a departing comment [88]: the construction of an effective model for the long range scale requires that the characteristic length ξ , which gives how fast the correlation (in the effective model) between two distant fields decays, must satisfy $\xi \gg 1/\Lambda_c$ being $1/\Lambda_c$ on the order of the fundamental unknown

closed-in-itself momenta as $\int_0^{\Lambda_c} \frac{d^d \vec{p}}{p^2}$, divergent for $d - 2 \geq 0$ at $\Lambda_c \rightarrow \infty$.

Now let us consider a certain arbitrary correlation function represented by a diagram like those of the previous page but this time with N external legs and V vertices of interaction $\lambda(\Psi)^{2n}$. Of the total number of $2nV$ legs corresponding to the V vertices, N remain free and the rest, $2nV - N$, attached to an internal line: the number of propagators are thus $\frac{2nV - N}{2}$ and the number of independent momenta to integrate over is $\frac{2nV - N}{2} - (V - 1)$ (since to each vertex, except for the first one, corresponds the condition that the sum of all momenta arriving at the vertex is zero). Thus, the diagram representing this arbitrary correlation function behaves at large Λ_c as $(\Lambda_c)^{d \times \left(\frac{2nV - N}{2} - (V - 1) \right) - 2 \times \frac{2nV - N}{2}} = (\Lambda_c)^\delta$, with $\delta = V \left(n(d - 2) - d \right) - N \left(\frac{d - 2}{2} \right) + d$. The strongest dependence with Λ_c when this cut-off tends to infinity occurs for $N = 0$ and therefore if the diagram has to remain finite for $\Lambda_c \rightarrow \infty$ then condition $V \left(n(d - 2) - d \right) + d \leq 0$ must be satisfied, with $V > 1$ ($V = 1$ correspond to the vertex alone, without any loop correction): this holds for any number of vertices if $n(d - 2) - d \leq 0$, which is condition (B.245). Therefore this latter criterion already takes account of possible divergences in those diagrams belonging to any other correlation function, always with the remark that logarithmic divergences or the closed loops discussed at the beginning of this footnote are not included in this criterion: but if we achieve to remove in some way these two kinds of special divergences, then all results will be finite once (B.245) holds.

scale. This condition can be fulfilled by avoiding any interaction term in the effective field such that the latter could become short-range (and thus disappear at large distances). But it can also occur that the model does contain some of these interactions which make the effective field short-range, as for instance $-|\mathcal{A}|(\Psi)^2$. In this case, what is to be done such that the resulting correlation length ξ is still much larger than the fundamental graining $1/\Lambda_c$? The solution, if the model shows a phase transition of critical type, is to work at a critical point: by definition, at the critical point the system becomes independent of Λ_c , “forgets” any characteristic look at a very short fundamental distance by displaying the same picture in the range of large as well as of short distances, something mathematically expressed through a diverging correlation length ξ . Hence the existence of an effective field still relevant at the long distance scale is immediately fulfilled as long as our effective model works near enough to a critical point. Still more important: independently of how many and complicated the actual interactions in the system at the fundamental scale could be, by reaching the critical point all the different effective models for such a system starts to depend on a very reduced set of fields and parameters, begins to display the same behaviour in their correlation functions. Such belonging to the same critical behaviour by a wide set of different models allows us to construct easily the effective model at large distances, knowing that only very few degrees of freedom are relevant near to a critical point, and that any fundamental scale plays no role at such a phase transition point.

If we are not interested in the effective field becoming of infinite range exactly, but its range being of the order of a certain physical finite length l_{phys} , then the model must be tuned to work near to the critical point. In such a case the similarity in the look at different scales gets lost and the use of a renormalization point μ is again required in order to indicate if the scale we are observing is that of l_{phys} or not.

After having seen the reason for taking only terms in the potential up to the fourth power in the scalar field, let us return to the potential for the case of the 2nd rank isotensor. The only possible combinations are thus: $\Psi^{ab}\Psi^{ba}$, $\Psi^{ab}\Psi^{bc}\Psi^{ca}$, $(\Psi^{ab}\Psi^{ba})^2$ and $\Psi^{ab}\Psi^{bc}\Psi^{cd}\Psi^{da}$. But the latter is not actually independent⁷¹ of the three other combinations due to the following relation

$$\Psi^{ab}\Psi^{bc}\Psi^{cd}\Psi^{da} = \frac{1}{2} (\Psi^{ab}\Psi^{ab})^2. \quad (\text{B.246})$$

The most general potential for the energy modelling the system is

$$V(\Psi^{ab}) = -|\mathcal{A}| (\Psi^{ab}\Psi^{ab}) + \frac{2\mathcal{D}}{3} (\Psi^{ab}\Psi^{bc}\Psi^{ca}) + \frac{|\mathcal{B}|}{2} (\Psi^{ab}\Psi^{ab})^2, \quad (\text{B.247})$$

or written in function of the two isovectors Ψ^a and $\tilde{\Psi}^a$ forming the isotensor

$$\begin{aligned} V = & -|\mathcal{A}| \left(\frac{1}{2} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{6} (\Psi \cdot \tilde{\Psi})^2 \right) + \\ & + \frac{2\mathcal{D}}{3} (\Psi \cdot \tilde{\Psi}) \left(\frac{1}{4} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) - \frac{1}{36} (\Psi \cdot \tilde{\Psi})^2 \right) + \\ & + \frac{|\mathcal{B}|}{2} \left(\frac{1}{2} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{6} (\Psi \cdot \tilde{\Psi})^2 \right)^2, \end{aligned} \quad (\text{B.248})$$

⁷¹From relation $\Psi^{ab}\Psi^{bc} = \frac{1}{4}\Psi^a\Psi^c(\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{4}\tilde{\Psi}^a\tilde{\Psi}^c(\Psi \cdot \Psi) + \frac{1}{9}\delta^{ac}(\Psi \cdot \tilde{\Psi})^2 - \frac{1}{12}\Psi^a\tilde{\Psi}^c(\Psi \cdot \tilde{\Psi}) - \frac{1}{12}\tilde{\Psi}^a\Psi^c(\Psi \cdot \tilde{\Psi})$ one obtains: $\Psi^{ab}\Psi^{ab} = \frac{1}{2}(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{6}(\Psi \cdot \tilde{\Psi})^2$, $\Psi^{ab}\Psi^{bc}\Psi^{ca} = \frac{1}{4}(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi})(\Psi \cdot \tilde{\Psi}) - \frac{1}{36}(\Psi \cdot \tilde{\Psi})^3$ and $\Psi^{ab}\Psi^{bc}\Psi^{cd}\Psi^{da} = \frac{1}{8}(\Psi \cdot \Psi)^2(\tilde{\Psi} \cdot \tilde{\Psi})^2 + \frac{1}{12}(\Psi \cdot \Psi)(\tilde{\Psi} \cdot \tilde{\Psi})(\Psi \cdot \tilde{\Psi})^2 + \frac{1}{72}(\Psi \cdot \tilde{\Psi})^4 = \frac{1}{2}(\Psi^{ab}\Psi^{ab})^2$.

with $(\Psi \cdot \tilde{\Psi}) \equiv \Psi^a \tilde{\Psi}^a$. Let us suppose $\mathcal{D} \neq 0$: the nonvanishing minima of this potential are determined by equation

$$\frac{\partial V}{\partial (\tilde{\Psi} \cdot \tilde{\Psi})} = \frac{(\Psi \cdot \Psi)}{2} \left[-|\mathcal{A}| + \frac{\mathcal{D}}{3} (\Psi \cdot \tilde{\Psi}) + \right. \\ \left. + |\mathcal{B}| \left(\frac{1}{2} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{6} (\Psi \cdot \tilde{\Psi})^2 \right) \right] = 0, \quad (\text{B.249})$$

for $\frac{\partial V}{\partial (\Psi \cdot \Psi)} = 0$ an analogous result with the interchange $(\Psi \cdot \Psi) \leftrightarrow (\tilde{\Psi} \cdot \tilde{\Psi})$, and

$$\frac{\partial V}{\partial (\Psi \cdot \tilde{\Psi})} = \frac{(\Psi \cdot \tilde{\Psi})}{3} \left[-|\mathcal{A}| + \left(\frac{\mathcal{D}}{2} \frac{(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi})}{(\Psi \cdot \tilde{\Psi})} - \frac{\mathcal{D}}{6} (\Psi \cdot \tilde{\Psi}) \right) + \right. \\ \left. + |\mathcal{B}| \left(\frac{1}{2} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) + \frac{1}{6} (\Psi \cdot \tilde{\Psi})^2 \right) \right] = 0, \quad (\text{B.250})$$

where in the last equation term $(\Psi \cdot \tilde{\Psi})$ has been factorized out under the assumption that it does not vanish; otherwise, according to (B.248), $(\Psi \cdot \tilde{\Psi}) = 0$ has the same effect as $\mathcal{D} = 0$, case which will be considered later in (B.253). For scalar isovectors of nonzero length, as well as not orthogonal to each other in isospace, the minimum condition is solved by equating the underlined terms in (B.249)-(B.250)

$$(\Psi \cdot \tilde{\Psi})^2 = (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}), \quad (\text{B.251})$$

and thus, substituting into (B.250), one obtains

$$(\Psi \cdot \tilde{\Psi}) = \frac{-\mathcal{D} \pm \sqrt{\mathcal{D}^2 + 24|\mathcal{A}||\mathcal{B}|}}{4|\mathcal{B}|}. \quad (\text{B.252})$$

Solution $(\Psi \cdot \tilde{\Psi})^2 = (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi})$ implies that both isovectors are parallel or antiparallel to one other: if they are parallel, $(\Psi \cdot \tilde{\Psi}) > 0$, then the solution for the lengths of

each isovector is $(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) = \frac{-\mathcal{D} + \sqrt{\mathcal{D}^2 + 24|\mathcal{A}||\mathcal{B}|}}{4|\mathcal{B}|}$, independently of the sign of $\mathcal{D}$;

and if the two isovectors are antiparallel, then $(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) = \frac{-\mathcal{D} - \sqrt{\mathcal{D}^2 + 24|\mathcal{A}||\mathcal{B}|}}{4|\mathcal{B}|}$,

also for any sign of $\mathcal{D}$. For $\mathcal{D} \neq 0$, these are the constant nonzero values taken on by the scalar isotensor when approaching the spatial infinity around a topological object. Both cases have $H = N(SO(2))$ as remaining symmetry subgroup for a nonzero Ψ^{ab} , and thus they are characterized by the existence of only one kind of stable vortices as well as numerable different types of isolated monopoles, as discussed in (B.229).

The other symmetry breaking pattern for the 2nd rank isotensor, that characterized by $H = D_2$ and studied after (B.238), occurs when equations (B.249)-(B.250) do not imply result (B.251), i.e., for $\mathcal{D} = 0$. In this case (B.249) (or (B.250)) gives the result

$$(\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) = \frac{2|\mathcal{A}|}{|\mathcal{B}|} \frac{1}{1 + \frac{1}{3} \cos \Theta_0}, \quad (\text{B.253})$$

where Θ_0 is the angle formed by the two isovectors Ψ^a and $\tilde{\Psi}^a$ in isospace. Such arrangement of the two isovectors causes the non-existence of any isolated monopole: only 7 different types of stable vortices exist with the property of some of them not being able to go across one other. Returning to the comments of the previous Note on page 291, although this breaking pattern can take place at the distance scale for which the potential without term $\frac{2\mathcal{D}}{3} (\Psi^{ab} \Psi^{bc} \Psi^{ca})$ has been modellized, this does not mean however that such potential's term can be maintained zero when the effect of fluctuations (in more detailed pictures at finer scales) are taken into account: it may occur that modifications of the form $\tilde{\mathcal{D}} (\Psi^{ab} \Psi^{bc} \Psi^{ca})$ are necessary to be introduced in the effective model in order to cancel divergences arising from complicated fluctuation's diagrams — note that the third power of the scalar isotensor satisfies condition (B.245) and thus its inclusion will not make larger the number of divergent diagrams, diagrams representing the effect of fluctuations corresponding a higher order in the perturbative expansion in powers of $\eta|\mathcal{B}|$ —; or equivalently, this correction in the form of a term which was not present in the original model at a coarse scale is required in order to render the effective model finite when sending the cut-off Λ_c to infinity and to ensure therefore that the effective model constructed at large scales *describes* also the behaviour of the system at any scale. In such a case, the introduction of $\tilde{\mathcal{D}} (\Psi^{ab} \Psi^{bc} \Psi^{ca})$ will drive again the symmetry breaking pattern towards $H = N(SO(2))$ [15], changing thus the form of the nonvanishing minima to (B.251). It depends on how weak fluctuations really are ($\eta|\mathcal{B}| \rightarrow 0$) whether such modifications in the effective model are required at a finer or, on the contrary, already at a larger distance range. An example where in a potential, with a vanishing minimum at the original coarse scale where it was formulated, the fluctuation corrections change the symmetry breakdown's result, by inducing a nonvanishing minimum, is given in the effective model discussed in (D.350).

B.7.4 The scalar irreducible isotensor Ψ^{abc} of third rank

To conclude this section let us consider briefly a nonzero scalar third rank (irreducible) isotensor, formed by three isovectors Ψ^a , $\tilde{\Psi}^a$ and $\check{\Psi}^a$ combined into a traceless and symmetric (in any two of its three internal indices) tensor $\Psi^{abc} = \frac{1}{6} (\Psi^a \tilde{\Psi}^b \check{\Psi}^c + 5 \text{ perm.}) - \frac{1}{15} [\delta^{ab} ((\Psi \cdot \tilde{\Psi}) \check{\Psi}^c + (\Psi \cdot \check{\Psi}) \tilde{\Psi}^c + (\tilde{\Psi} \cdot \check{\Psi}) \Psi^c) + 2 \text{ perm.}]$. The general powers of the scalar field, up to the fourth grade, are $\Psi^{abc} \Psi^{abc}$

$$\begin{aligned} \Psi^{abc} \Psi^{abc} = & \frac{1}{6} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \tilde{\Psi}) (\check{\Psi} \cdot \check{\Psi}) + \left[\frac{1}{10} (\Psi \cdot \Psi) (\tilde{\Psi} \cdot \check{\Psi})^2 + \frac{1}{10} (\tilde{\Psi} \cdot \check{\Psi}) (\Psi \cdot \check{\Psi})^2 + \right. \\ & \left. + \frac{1}{10} (\check{\Psi} \cdot \check{\Psi}) (\Psi \cdot \tilde{\Psi})^2 \right] - \frac{1}{15} (\Psi \cdot \tilde{\Psi}) (\tilde{\Psi} \cdot \check{\Psi}) (\check{\Psi} \cdot \Psi), \end{aligned} \quad (\text{B.254})$$

$(\Psi^{abc}\Psi^{abc})^2$ and $\Psi^{apb}\Psi^{pbc}\Psi^{cqd}\Psi^{qda}$; the other possible combination of the indices into a fourth power, $\Psi^{pab}\Psi^{pbc}\Psi^{qcd}\Psi^{qda}$, is actually a function of the previous combinations according to the following result [76]⁷²

$$\Psi^{pab}\Psi^{pcd}\Psi^{qbc}\Psi^{qda} = -\Psi^{apb}\Psi^{pbc}\Psi^{cqd}\Psi^{qda} + \frac{1}{2}(\Psi^{pab}\Psi^{pab})^2. \quad (\text{B.255})$$

A third power of the scalar field vanishes due to the traceless-property of the isotensor.

Hence the most general potential, containing self-interactions of the scalar field up to a number of four, is given by

$$V(\Psi^{abc}) = -|\mathcal{A}|(\Psi^{abc}\Psi^{abc}) + \mathcal{E}(\Psi^{apb}\Psi^{pbc}\Psi^{cqd}\Psi^{qda}) + \frac{|\mathcal{B}|}{2}(\Psi^{abc}\Psi^{abc})^2, \quad (\text{B.256})$$

and depending on the value of $\mathcal{E}$, the minimum will correspond to one or other of the possible arrangements of the isovectors within the isotensor. Such possible symmetries of Ψ^{abc} are described by the following cases.

- $H = U(1) \equiv SO(2)$ if the three isovectors are colinear, since then any rotation around the axis parallel to the three isovectors leaves all three invariant. Note that in this case $H = SO(2)$ and not $H = N(SO(2))$ since the change in isodirection of the three isovectors to be pointing in the opposite direction, $\Psi^a \rightarrow -\Psi^a$ (simultaneously to the same change for the other two isovectors), is not a symmetry of isotensor Ψ^{abc} since the latter is the product of an odd number of isovectors. This subgroup H differs from that shown by the 2nd rank isotensor when its two isovectors were (anti)parallel: there this inversion in the isovectors was a symmetry of the isotensor, something characteristic for any isotensor of even rank. And this difference is just the responsible for $\pi_1(G/H) = \pi_1(SO(3)/SO(2)) = 0$, for the non-existence of stable vortices in the case under study: any internal transformations in $H = SO(2)$ is related to any other by combining transformations all belonging

⁷²By defining the 2nd rank isotensor $L^{ab} = \Psi^{abm}\zeta^m$ and then applying (B.246), $2L^{ab}L^{bc}L^{cd}L^{da} = (L^{ab}L^{ab})(L^{cd}L^{cd})$, one obtains

$$(2\Psi^{mab}\Psi^{nbc}\Psi^{pcd}\Psi^{qda} - \Psi^{mab}\Psi^{nab}\Psi^{pcd}\Psi^{qcd})\zeta^m\zeta^n\zeta^p\zeta^q = 0,$$

without forgetting to sum over repeated indices. Since $\zeta^m\zeta^n\zeta^p\zeta^q$ is symmetric under the change of any two of its indices, the symmetric part of the previous underlined term can be directly equated to zero, which after contracting m with p and n with q gives:

$$\begin{aligned} (m=p, n=q, p, q) & \quad 2\Psi^{pab}\Psi^{qbc}\Psi^{pcd}\Psi^{qda} - \Psi^{pab}\Psi^{qab}\Psi^{pcd}\Psi^{qcd} + \\ (m=p, n=q, q, p) & \quad + \underline{2\Psi^{pab}\Psi^{qbc}\Psi^{qcd}\Psi^{pda}} - \underline{\Psi^{pab}\Psi^{qab}\Psi^{qcd}\Psi^{pcd}} + \\ (m=p, p, n=q, q) & \quad + \underline{2\Psi^{pab}\Psi^{pbc}\Psi^{qcd}\Psi^{qda}} - \underline{\Psi^{pab}\Psi^{pab}\Psi^{qcd}\Psi^{qcd}} + \\ (m=p, p, q, n=q) & \quad + \underline{2\Psi^{pab}\Psi^{pbc}\Psi^{qcd}\Psi^{qda}} - \underline{\Psi^{pab}\Psi^{pab}\Psi^{qcd}\Psi^{qcd}} + \\ (m=p, q, n=q, p) & \quad + \underline{2\Psi^{pab}\Psi^{qbc}\Psi^{qcd}\Psi^{pda}} - \underline{\Psi^{pab}\Psi^{qab}\Psi^{qcd}\Psi^{pcd}} + \\ (m=p, q, p, n=q) & \quad + \underline{2\Psi^{pab}\Psi^{qbc}\Psi^{pcd}\Psi^{qda}} - \underline{\Psi^{pab}\Psi^{qab}\Psi^{pcd}\Psi^{qcd}} + \\ 18 \text{ further permutat.} & \quad + \dots \dots \dots = \\ & \quad = 4 \left[4\Psi^{pab}\Psi^{pcd}\Psi^{qbc}\Psi^{qda} + 4\Psi^{pab}\Psi^{pbc}\Psi^{qcd}\Psi^{qda} - 2(\Psi^{abc}\Psi^{abc})^2 \right] = 0 \end{aligned}$$

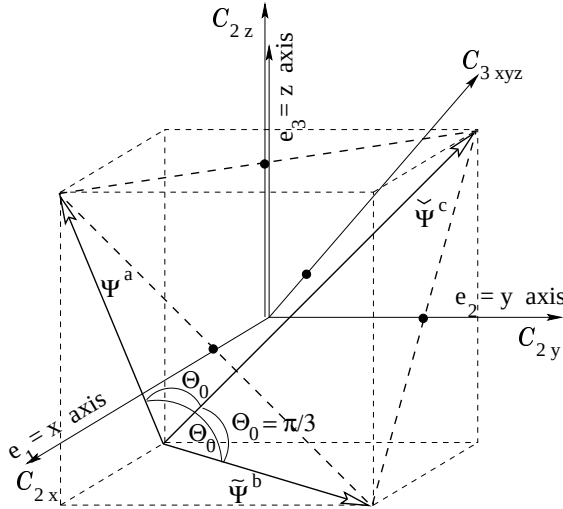
where the previous double underlined terms cancel against the simply underlined ones, by interchanging in the latter the internal indices $\begin{cases} p \leftrightarrow b \\ d \leftrightarrow q \end{cases}$ and taking into account the full symmetry of isotensor Ψ^{abc} .

to H (i.e., by varying the angle which defines the transformation) and therefore $SO(2)$ is formed by only one “island” — different from $N(SO(2))$, where the rotations by π around an axis perpendicular to the axis of the $SO(2)$ rotations made H be constituted by two isolated islands. Regarding monopoles in a 3-dim. physical space, they exist for the same reason as in (B.229): $SO(3)/SO(2)$ is topological equivalent to a 2-dim. spherical surface with antipodal points identified.

- If the three isovectors are not colinear and form angles among them, then the $SO(3)$ gauge symmetry is, up to some discrete transformations, “completely broken”: for a given non-parallel arrangement of the three isovectors — as representative at physical spatial infinity of the many gauge equivalent configurations obtained from this arrangement by any gauge transformation — any isorotation around one of the isovectors affects the other two, not leaving them invariant. Therefore, and because of the multiply-connectivity of $G = SO(3)$, one can expect topologically stable vortices depending on the number of the discrete symmetries, discrete symmetries due to the product of the three isovectors to form the isotensor. Monopoles can only exist attached to the point where two vortices have been combined and the result of such combination is not topologically stable: isolated monopoles, nevertheless, require a partially unbroken continuous symmetry subgroup H .

Depending on the arrangements of non-colinear isovector Ψ^a , $\tilde{\Psi}^a$ and $\check{\Psi}^a$ in isospace, the discrete symmetries still allowed in isotensor Ψ^{abc} are [76]:

- $H = T$, the tetrahedral group, if the three isovectors have the same length



tetrahedral T symmetry

and the angle between two any of them is $\Theta_0 = \frac{\pi}{3}$. This subgroup is formed by: a) the identity transformation; b) rotations by π around the three 2-fold symmetry axes, going from the center to the middle point of each side and represented in the figure as the orthogonal Cartesian axes; and c) rotations by $2\pi/3$ and by $4\pi/3$ around the four 3-fold symmetry axes drawn from the center to the corner of the tetrahedron (one of such axes is represented in the above figure crossing one of the tetrahedron's covers). However

rotation by $4\pi/3$ around each one of the 3-fold symmetry axes is the product of two rotations by $2\pi/3$ around the *same* axis and thus both rotations belong to the same “island” or class within H , classes denoted by

$$\{1, C_{2x}, C_{2y}, C_{2z}, C_{3xyz}, C_{3-x-y-z}, C_{3x-y-z}, C_{3-x-y-z}\}, \quad (\text{B.257})$$

where, for instance, C_{2x} represents the (only one) rotation by π around the $\hat{e}_1$ -axis, and C_{3xyz} is the “island” formed by the two rotations by $\{2\pi/3, 4\pi/3\}$ around the $\frac{\hat{e}_1+\hat{e}_2+\hat{e}_3}{\sqrt{3}}$ -axis.

It is worth noting that the isolated islands in H , and therefore the different types of vortices, are classified through the classes (of discrete elements) of H (more properly by the classes in $\bar{H}$ as we shall see soon) and not directly through the separated discrete elements themselves. Why? If at $\phi = 0$ in physical space the scalar field’s configuration corresponds to the element $2\pi/3$ of a certain class of H , and at the end of external S_∞^1 ($\phi = 2\pi$) the configuration is that corresponding to $4\pi/3$, where both angles refer to the same 3-fold symmetry axis and belongs thus to the same class in (B.257), then we can transform the start’s $2\pi/3$ rotation to a $4\pi/3$ one simply by applying on the initial configuration a further $2\pi/3$ rotation, which leaves invariant this configuration since the latter already belonged initially to such class of H ; but a loop in internal G/H which begins and ends at the same point of H can be shrunk to a point in such internal space, is deformable to a trivial configuration where the scalar field has the same value and orientation at each site of external spatial infinity. Hence the identifications of points on the spherical surface topologically equivalent to G/H must take care of such double occupied “islands” of H . As we have seen after relations (B.238), each discrete symmetry of only one element causes, when G/H is projected onto the internal space of directions of the isovectors, the identification of antipodal points on the sphere’s surface where the isovector’s arrow is moving; now, where the discrete “island” contains two transformations, the projection onto the 3-dim. isovector space will cause the isovector’s tip to cover a (usual) spherical surface where the three points located on every largest circumference and separated a symmetric angle of $2\pi/3$ are identified to one other.

The paths drawn the isovectors in their internal space, when one walks along a closed contour from $\phi = 0$ to $\phi = 2\pi$ in physical space surrounding a vortex of a class single-occupied ($C_{2x,y,z}$), are similar to that represented in Fig. B.9, where the identification of (only) antipodal points produces a flux of half the value of a complete rotation: $\Phi_0 = \frac{1}{2} \frac{2\pi}{e}$. If, on the other hand, the vortex belongs to double-occupied “islands” $C_{3xyz,-x-yz,\dots}$, then the trajectory followed by the isovectors is analogous to Fig. B.9 but now the bisecting axis only covers, as the same time as we cover completely S_∞^1 in physical space, a *third* of circumference — due to identification on the sphere of isovector’s orientations of points separated an angular distance of $1/3$ of largest circumference —; simultaneously, the three isovectors have rotated $2\pi/3$ around this bisecting axis when the external contour S_∞^1 is completed. The resulting isorotation matrix is given by (B.239) but with the substitution $\phi/2 \rightarrow \phi/3$, and thus the flux ⁷³ carried by such vortices is $\Phi_0 = \frac{1}{3} \frac{2\pi}{e}$.

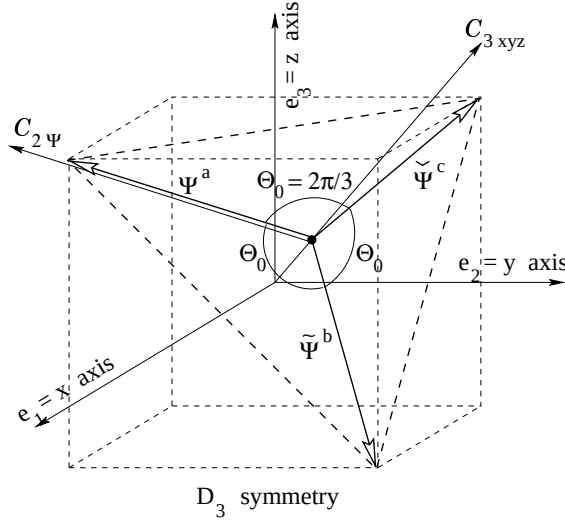
⁷³ When evaluated along the external S_∞^1 , the vanishing $\left(\frac{1}{\rho} \frac{\partial \Psi^a}{\partial \phi} - e \epsilon_{abc} \Psi^b W_\phi^c \right) \Big|_{|\vec{r}| \rightarrow \infty} = 0$ yields

More exactly, for the determination of the loop classes we need actually subgroup $\bar{H}$, the extension of the above H within the covering group $\bar{G} = SU(2)$ such that $SO(3)/H = SU(2)/\bar{H}$, with the corresponding duplication of elements in $\bar{H}$ with respect to those in H (due to the 2-to-1 correspondence between $SU(2)$ and $SO(3)$). The isolated islands or classes of $\bar{H}$ are

$$\left\{ \begin{array}{l} \mathbf{1}, \quad e^{-i\pi \hat{x}^c I_c^{(\text{fund.})}}, \quad e^{-i\pi \hat{y}^c I_c^{(\text{fund.})}}, \quad e^{-i\pi \hat{z}^c I_c^{(\text{fund.})}}, \\ -\mathbf{1}, \quad e^{-i(\pi+2\pi) \hat{x}^c I_c^{(\text{fund.})}}, \quad e^{-i(\pi+2\pi) \hat{y}^c I_c^{(\text{fund.})}}, \quad e^{-i(\pi+2\pi) \hat{z}^c I_c^{(\text{fund.})}}, \\ e^{-i\frac{2\pi}{3} \frac{(\hat{x}^c + \hat{y}^c + \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \quad e^{-i\frac{2\pi}{3} \frac{(-\hat{x}^c - \hat{y}^c + \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \\ e^{-i(\frac{2\pi}{3} + 2\pi) \frac{(\hat{x}^c + \hat{y}^c + \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \quad e^{-i(\frac{2\pi}{3} + 2\pi) \frac{(-\hat{x}^c - \hat{y}^c + \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \\ e^{-i\frac{2\pi}{3} \frac{(\hat{x}^c - \hat{y}^c - \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \quad e^{-i\frac{2\pi}{3} \frac{(-\hat{x}^c + \hat{y}^c - \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \\ e^{-i(\frac{2\pi}{3} + 2\pi) \frac{(\hat{x}^c - \hat{y}^c - \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}}, \quad e^{-i(\frac{2\pi}{3} + 2\pi) \frac{(-\hat{x}^c + \hat{y}^c - \hat{z}^c)}{\sqrt{3}} I_c^{(\text{fund.})}} \end{array} \right\}, \quad (\text{B.258})$$

what signals the existence of 15 different types of stable vortices (plus the trivial one), arising from the identification in $SU(2)/\bar{H}$ of 16 symmetrically distributed points on the 3-dim. spherical surface S^3 , topologically equivalent to $SU(2)$. Further, the non-commutativity of some of the $\bar{H}$'s elements leads to entanglement of their respective vortices, as explained in (B.218).

- A case similar to the previous one occurs if the tetrahedron is “opened” to be projected on a plane to which axis C_{3xyz} is normal: the three isovectors (of the same length) lies on such a plane, forming an angle of $\Theta_0 = \frac{2\pi}{3}$. Therefore, the discrete symmetries which leave invariant Ψ^{abc} are, other than the identity, rotation by $2\pi/3$ and by $4\pi/3$ around the axis normal to this plane, and rotations by π around each of the three isovectors. And the classes in H , subgroup denoted as the trihedral D_3 group, are given by



$$\{\mathbf{1}, C_{3xyz}, C_{2\Psi}, C_{2\tilde{\Psi}}, C_{2\tilde{\tilde{\Psi}}}\}. \quad (\text{B.259})$$

$-\frac{i}{N} \hat{n}_c (T_c^{(\text{adj.})})_{ab} \Psi^b d\phi \equiv \Psi^a(\phi + d\phi) - \Psi^a(\phi) = ie (I_c^{(\text{adj.})} W_\phi^c)_{ab} \Psi^b d\phi$, where $I_c^{(\text{adj.})}$ are the generators of the adjoint representation of the Lie algebra (to which group $SO(3)$ belongs). On the other hand, $\hat{n}_c T_c^{(\text{adj.})}$ are the linear combination of such generators which define the partial (infinitesimal) isorotation of the isovector when one walks along S_∞^1 : factor N (in the case above, $N = 3$) carries the circumference's fraction which has to be covered on S^2 to reach a point identified with the starting position, being such identification the result of $(N - 1)$ -occupation in the isolated islands forming $\bar{H}$.

Hence the flux is given by $(I_c^{(\text{adj.})})_{ab} \left(\oint_{C \text{ at } \rho \rightarrow \infty} W_\phi^c \rho d\phi \right) = -\frac{1}{N} \frac{2\pi}{e} \hat{n}_c (T_c^{(\text{adj.})})_{ab}$.

With the experience already gained with the previous case, the discussion from equation (B.257) to (B.258) can be repeated again here: one obtains that also in this case there are 9 classes of (nontrivial) stable vortices, 1×2 (the $\times 2$ because of the duplication of classes within the covering group) of these classes are vortices of flux $\Phi_0 = \frac{1}{3} \frac{2\pi}{e}$, the other non-deformable configurations carry flux $\Phi_0 = \frac{1}{2} \frac{2\pi}{e}$. Moreover, some of these vortices can also become entangled.

- For a reduced symmetrical arrangements of the three isovectors, with different lengths and angles, the only discrete symmetry remaining is the interchange of the three isovectors among them, ensured by the invariance under the interchange of indices in the (symmetric) 3rd rank isotensor.

The value of the isovector's lengths for each pattern of symmetry breaking would be calculated only for the two cases, $H = D_3$ and $H = T$, with the largest content in vortices; $H = SO(2)$ has no stable line defects. From the general form of potential (B.256), $V(\Psi^{abc}) = -|\mathcal{A}|(\Psi^{abc}\Psi^{abc}) + \mathcal{E}(\Psi^{apb}\Psi^{pbc}\Psi^{cqd}\Psi^{qda}) + \frac{|\mathcal{B}|}{2}(\Psi^{abc}\Psi^{abc})^2$, for the case where all isovectors have the same length, $(\Psi \cdot \Psi) = (\tilde{\Psi} \cdot \tilde{\Psi}) = (\check{\Psi} \cdot \check{\Psi})$, and the same angle Θ_0 , $(\Psi \cdot \tilde{\Psi}) = (\tilde{\Psi} \cdot \check{\Psi}) = (\check{\Psi} \cdot \Psi) \equiv ()$, the potential becomes ⁷⁴ $V = -|\mathcal{A}|(\Psi^{abc}\Psi^{abc}) + \left(\frac{|\mathcal{B}|}{2} + \frac{c}{2}\mathcal{E}\right)(\Psi^{abc}\Psi^{abc})^2$, with $c = 1$ for $H = D_3$ or $c = 2/3$ for $H = T$. The constant nonzero value of the isotensor length which minimizes the potential is therefore

$$\sqrt{\Psi^{abc}\Psi^{abc}} = (\Psi \cdot \Psi)^{3/2} = \begin{cases} \sqrt{\frac{|\mathcal{A}|}{|\mathcal{B}| + \mathcal{E}}} & \text{with } \mathcal{E} > -|\mathcal{B}| \quad H = D_3, \\ \sqrt{\frac{3|\mathcal{A}|}{3|\mathcal{B}| + 2\mathcal{E}}} & \text{with } \mathcal{E} > -\frac{3}{2}|\mathcal{B}| \quad H = T. \end{cases} \quad (\text{B.260})$$

⁷⁴For the case where the three isovectors have the same length, abbreviated as $||$, and form the same angle between two any of them, $(\Psi \cdot \tilde{\Psi}) = (\tilde{\Psi} \cdot \check{\Psi}) = (\check{\Psi} \cdot \Psi) \equiv ()$, following holds

$$\begin{aligned} \Psi^{apb}\Psi^{pbc} = & \frac{||^2}{18} \left[\left(\Psi^a \tilde{\Psi}^c + \tilde{\Psi}^a \Psi^c \right) \left(\frac{()}{||} + \frac{()^2}{||^2} \right) + 2 \text{ perm.} + \left(\Psi^a \Psi^c + \tilde{\Psi}^a \tilde{\Psi}^c + \check{\Psi}^a \check{\Psi}^c \right) \left(1 + \frac{()^2}{||^2} \right) \right] - \\ & - \frac{||^2}{45} \left[\left(\Psi^a (\Psi^c + \tilde{\Psi}^c + \check{\Psi}^c) + (\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a) \Psi^c \right) \frac{()^2}{||^2} + 2 \text{ perm.} \right] - \\ & - \frac{2||^2}{45} \left[\left(\Psi^a \tilde{\Psi}^c + \tilde{\Psi}^a \Psi^c \right) \left(\frac{()}{||} + 2 \frac{()^2}{||^2} \right) + 2 \text{ perm.} \right] + \\ & + \frac{3||^2}{(15)^2} \left[2 \left(\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a \right) \left(\Psi^c + \tilde{\Psi}^c + \check{\Psi}^c \right) \frac{()^2}{||^2} + 3\delta^{ac} || \left(\frac{()^2}{||^2} + 2 \frac{()^3}{||^3} \right) \right], \end{aligned}$$

where “2 perm.” means change of Ψ by $\tilde{\Psi}$ and by $\check{\Psi}$ (and the corresponding change in $\tilde{\Psi}$ and in $\check{\Psi}$). Hence by completing terms in $(\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a)(\Psi^c + \tilde{\Psi}^c + \check{\Psi}^c)$, one obtains

As different from the case of a 2nd rank irreducible isotensor, where a nonvanishing additional $\mathcal{D}$ -term (other than the pure quadratic and quartic powers of the scalar field) containing a third power of the isotensor selected as symmetry breaking pattern the parallel (or antiparallel) arrangements of the two isovectors, in our case where such third power is forbidden by the traceless condition on the isotensor, the additional term proportional to $\mathcal{E}$ forces the symmetry breaking just on the patterns of complete breakdown of the $SO(3)$ gauge group not on the pattern with partial breakdown characterized by $H = SO(2)$.

Let us finish with some general considerations [11]. Up to this point we have studied the different forms of breaking a gauge symmetry group in order to obtain topological objects. The concept of symmetry breaking must be understood in the framework of having firstly eliminated the freedom to choose any possible configuration describing a certain macroscopical state: only after this choice of a particular representative among all those gauge-equivalent ones, it makes sense to speak about breaking of a gauge symmetry. The models considered up to now where effective models, with a limited number of degrees of freedom which allowed us to describe the main features as stable line and point defects of the system. The use of a gauge field and a scalar field (in different tensor representations) does not mean that these are actually the unique fields actually occur-

$$\frac{\Psi^{apb}\Psi^{pbc}}{||^2} = \left[\left(\Psi^a \Psi^c + \tilde{\Psi}^a \tilde{\Psi}^c + \check{\Psi}^a \check{\Psi}^c \right) \left(\frac{1}{18} - \frac{\cos \Theta_0}{18} + \frac{2 \cos \Theta_0}{45} + \frac{4 \cos^2 \Theta_0}{45} \right) \right] - \\ + \left[\left(\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a \right) \left(\Psi^c + \tilde{\Psi}^c + \check{\Psi}^c \right) \left(\frac{\cos \Theta_0}{90} - \frac{23 \cos^2 \Theta_0}{450} \right) \right] + \frac{|| \delta^{ac}}{25} (\cos^2 \Theta_0 + 2 \cos^3 \Theta_0) .$$

For $H = D_3$, or equivalently, for $(\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a) = 0$ and $\cos \Theta_0 = -1/2$, one obtains $\frac{\Psi^{apb}\Psi^{pbc}}{||^2} =$
 $= \frac{1}{12} (\Psi^a \Psi^c + \tilde{\Psi}^a \tilde{\Psi}^c + \check{\Psi}^a \check{\Psi}^c)$ and $(\Psi^{apb}\Psi^{pbc})^2 = \frac{||^6}{(12)^2} (3 + 6 \cos^2 \Theta_0) = \frac{1}{32} ||^6$. For $H = T$ (or
 $\cos \Theta_0 = 1/2$): $\frac{\Psi^{apb}\Psi^{pbc}}{||^2} = \frac{13}{180} (\Psi^a \Psi^c + \tilde{\Psi}^a \tilde{\Psi}^c + \check{\Psi}^a \check{\Psi}^c) - \frac{13}{1800} (\Psi^a + \tilde{\Psi}^a + \check{\Psi}^a) (\Psi^c + \tilde{\Psi}^c + \check{\Psi}^c) +$
 $+ \frac{|| \delta^{ac}}{50}$. The calculation of $(\Psi^{apb}\Psi^{pbc})^2$ for this last case is more easily carried out by writing the
three isovectors in function of the 3 orthonormal Cartesian axes, $\Psi^a = \sqrt{||} \frac{x^a + z^a}{\sqrt{2}}$, $\tilde{\Psi}^a = \sqrt{||} \frac{x^a + y^a}{\sqrt{2}}$
and $\check{\Psi}^a = \sqrt{||} \frac{y^a + z^a}{\sqrt{2}}$, and then noticing relations $x^a x^c + y^a y^c + z^a z^c = \delta^{ac}$ (due to the orthonormal-
ity condition) and $(\Psi^a \Psi^c + \tilde{\Psi}^a \tilde{\Psi}^c + \check{\Psi}^a \check{\Psi}^c) = \frac{||}{2} (\delta^{ac} + (x^a + y^a + z^a)(x^c + y^c + z^c))$: $\Psi^{apb}\Psi^{pbc} =$
 $= \frac{||^3}{180} \left(\frac{101}{10} \delta^{ac} + \frac{39}{10} (x^a + y^a + z^a)(x^c + y^c + z^c) \right)$ and $(\Psi^{apb}\Psi^{pbc})^2 = \frac{||^6}{(180)^2} \left(\frac{30603 + 23634 + 4563}{100} \right) =$
 $= \frac{588}{(180)^2} ||^6$. Since $\Psi^{abc}\Psi^{abc} = ||^3 \left(\frac{1}{6} + \frac{3 \cos^2 \Theta_0}{10} - \frac{\cos^3 \Theta_0}{15} \right) = \begin{cases} \frac{1}{4} ||^3 & H = D_3 \\ \frac{7}{30} ||^3 & H = T \end{cases}$, then $(\Psi^{apb}\Psi^{pbc})^2 \equiv$
 $\equiv \Psi^{apb}\Psi^{pbc}\Psi^{cqd}\Psi^{qda} = \begin{cases} \frac{1}{2} (\Psi^{abc}\Psi^{abc})^2 & H = D_3 \\ \frac{1}{3} (\Psi^{abc}\Psi^{abc})^2 & H = T \end{cases}$.

ing at the fundamental scale. And, on the contrary, the fields describing macroscopically the system need not to coincide with the field variables used in the models considered in this Appendix: as in an already mentioned example, the usual superconductor model gets defined through the existence of stable vortices, although the characterization of its two different phases (superconducting and normal conducting) by means of these vortices is completely different, since in the normal conducting phase these vortices, although remaining the describing variables of the model, condense to form the usual everywhere-spreading lines of the magnetic field and thus the macroscopical degrees of freedom look rather different from the field variables used to define the model. It is the appearance of transitions between the phases displayed by the model what can make less transparent the effect of the separated model's field variables by taking account of them collectively — the other reason for a non-transparent identification of the actual degrees of freedom is gauge invariance with its infinite large amount of equivalent configurations describing the same macroscopical state.

Just the study of such phase transitions is the subject of the two next long Appendices. Although this can be carried out directly on the effective models already discussed, the infinities which occur at soon as we start to observe the system at scales finer than that where we have constructed our model — what, as we already know, points to the existence of a hidden, non-accessible fundamental scale — make more comfortable to work with a model already containing a built-in shortest distance scale in the form of the spacing in a lattice. Further, we shall only consider models invariant under a gauge $U(1)$ symmetry, where such group can describe the actual symmetry of the system or can be a residual symmetry of a compact non-Abelian group which corresponds to the real system's symmetry. In the last case, called the compact $U(1)$ gauge model, the effect of irreducible representations for the compact non-Abelian group of rank higher than the adjoint representation — with the resulting different patterns for the complete breakdown of such compact non-Abelian symmetry and the accompanying characteristic division of the magnetic flux — will be taken into account *effectively*, in the (lattice) compact $U(1)$ gauge model, through an integer parameter q , just as mentioned in the discussion after (B.242) in page 290. The main part of the next Appendices will be dedicated to the study of the different phases occurring in the lattice model thus defined, both for the compact as well as for the non-compact $U(1)$ gauge symmetry (corresponding the latter to the case of $U(1)$ as the original underlying symmetry of the system). The coupling between the scalar and the gauge field as well as the constant length of the scalar field in the lattice model are parameters whose strength is given at the lattice spacing scale: by looking at coarser scales, the effect of all those field configurations whose details get lost by the coarsening process may screen/reinforce this parameter's strength. Of course, if we are interested in formulating an effective continuous model independent of the shortest scale contained in the model, we have to work with the value of the lattice model's parameters at a critical phase transition point — or very near to it, if the effective continuous model must contain some kind of interaction which makes some fields not of infinite range completely — since for such kind of phase transitions the model “forgets” its lattice spacing.

The different phases displayed in the lattice model by varying its parameters — variation which can be produced simply due to the integration of the effects of fine-scale

lattice configurations because we are progressively taking coarser pictures of the system described by the lattice model — can also be reproduced in the effective models of the type discussed up to now in this Appendix. The last part of Appendix D will deal with such a connection between the lattice non-compact $U(1)$ model and the continuous effective model, where in the latter the different phases occur due (again) to the small-distance fluctuations whose effect is to modify the parameters of the continuous model. The parallels between the lattice and the continuous formulations also apply to the compact $U(1)$ gauge model: this is done according to the “bridge” between both formulations offered by a certain correlation function (D.244), which already takes care of the behaviour in the phase characteristic only for the lattice compact model (see remark on page 663).

Appendix C

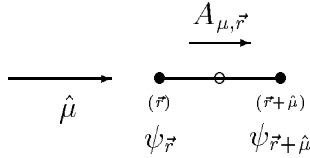
Correlation functions and phase diagrams at temperature $T = 0$

In this rather long appendix we shall present the calculation of the correlation function $W(C)$ (2.102) and study the phase diagram of the model presented in Chapter 2. In order to introduce techniques we need for the calculation of $W(C)$ and present results which allow us to describe the different phases of our dislocation model, we shall firstly start by discussing some models which are easier than the model in Chapter 2, . As introduction and first step we shall study the 2-dimensional Ising model, proving by means of the Peierls' argument the existence of spontaneous magnetization at large coupling strength (where large means when compared to a reference scale). Introducing a very useful technique, called the “high temperature” expansion method, the two-point correlation function at small coupling strength will be calculated, resulting in an exponential decay characteristic of many models. At a second step a general and very illustrative, even though not very rigorous, method based on the Peierls contours will be introduced, which allows us to obtain the previous results for the correlation function in a more intuitive way. We shall use this tool several times during the discussion of the different models presented in this appendix.

As next step we shall study the model of a scalar field ψ represented as an “arrow” in a internal 2-dimensional symmetry space. The arrow's orientation is defined by scalar field's phase φ with a compact range $0 \leq \varphi < 2\pi$. This model is the so-called planar or XY model of Statistical Mechanics. In the dislocation model, it describes the deformation state of a crystal. In this model, the scalar field is yet not coupled to the gauge field describing the gliding dislocations. The correlation function between two of such scalar fields will be calculated for large and small $|\psi|^2$ (parameter analogous to the coupling strength in the Ising model), firstly in a 2-dimensional lattice and later in a 3-dimensional one. The correlation's behaviour for small $|\psi|^2$ will be calculated by means of the “high temperature” power expansion (obtaining an exponential decay as in the Ising model) and the result will be re-obtained using the illustrative method of the Peierls contours already introduced for the Ising model; in the range of large $|\psi|^2$, the so-called Villain approximation will be used. Finally we shall consider the model with the scalar field of the Gaussian model coupled to a gauge field, the complete model (2.84) we have presented in Chapter 2. With this model (which in Statistical Mechanics

is called the lattice Higgs model) we shall calculate the correlation function $W(C)$ for weak and strong coupling e^2 and apply the method based on Peierls contours in order to characterize the different phases of the system. A good introduction into this subject can be found in the reports by Kogut [59] and by Brower [22], and in the papers by Banks et al. [18], by Einhorn and Savit [35] and by Fradkin and Shenker [41].

Let us now introduce some of the notation we shall use. The calculations will be carried out on a square lattice Λ of dimension d and lattice spacing a , which for the sake of clarity we shall take equal to 1. Each lattice site is labeled by $\vec{r}$ and the number of lattice sites by $|\Lambda|$. A general scalar field, denoted by σ or by ψ , is located at the lattice sites and is written as $\sigma_{\vec{r}}$ or $\psi_{\vec{r}}$; if the scalar field has only a discrete set of values, we speak of a spin field¹. The values taken on by the scalar field can be represented as different orientations inside an internal symmetry space. We also define the link or bond $b = \langle \vec{x}\vec{y} \rangle$ as the segment connecting two nearest neighbour sites. A link will be also labeled by $(\vec{r}, \mu)$, $\mu = 1, 2, \dots, d$, where d is the lattice dimensionality and μ is the direction from site $\vec{x} \equiv \vec{r}$ to neighbour site $\vec{y} \equiv \vec{r} + \hat{\mu}$, with $\hat{\mu}$ being a unit vector in the μ direction.



A general vector field A_μ is placed on bonds: thus $A_{\mu, \vec{r}}$ denotes the vector field pointing in the μ direction and located on bond $(\vec{r}, \mu)$. Since $(\vec{r} + \hat{\mu}, -\mu)$ labels the same bond as $(\vec{r}, \mu)$, vector field $A_{-\mu, \vec{r} + \hat{\mu}}$, which points in the $-\mu$ direction and sits on bond $(\vec{r} + \hat{\mu}, -\mu)$, is defined as

$$A_{-\mu, \vec{r} + \hat{\mu}} \equiv -A_{\mu, \vec{r}}. \quad (\text{C.1})$$

All model energies we shall deal with, contain interactions only between nearest neighbouring scalar fields, i.e., are local interactions. For instance, the spin energy we shall consider in the Ising model is given by

$$E = J \sum_{\vec{x}, \vec{y} \text{ nearest neighbours}} [1 - \sigma_{\vec{x}} \sigma_{\vec{y}}] = J \sum_{\vec{r}} \sum_{\mu} [1 - \sigma_{\vec{r}} \sigma_{\vec{r} + \hat{\mu}}], \quad (\text{C.2})$$

where J a positive constant measuring the strength of the spin interaction and it is therefore called the coupling strength. With this energy one constructs the partition function and calculates expectation values $\langle \cdot \rangle_\Lambda$ as follows

$$\mathcal{Z}_\Lambda = \sum_{\text{all configurations in } \Lambda} \exp \left[-\frac{E_{\text{conf}}}{\eta} \right], \quad (\text{C.3})$$

and

$$\langle \cdot \rangle_\Lambda = \frac{1}{\mathcal{Z}_\Lambda} \sum_{\text{all configurations in } \Lambda} \cdot \exp \left[-\frac{E_{\text{conf}}}{\eta} \right]. \quad (\text{C.4})$$

Each configuration energy is compared to an energy scale η : for example, in the current formulation of the Ising model this scale is $k_B T$, with T the absolute temperature and k_B Boltzmann's constant. The free energy F is defined as

$$F_\Lambda = -\eta \ln \mathcal{Z}_\Lambda, \quad (\text{C.5})$$

¹Spins in this context are to be understood as classical fields with discrete values, “up” and “down”, not as quantum fields.

and the free energy density $\mathcal{F}$ (the lattice spacing a has been set equal to 1) as follows

$$\mathcal{F}_\Lambda = -\frac{\eta}{|\Lambda|} \ln \mathcal{Z}_\Lambda . \quad (\text{C.6})$$

Note the partition function, the expected value and the free energy have been calculated in a finite lattice Λ . However, the application to our model is for infinitely extended systems and we must take limit $|\Lambda| \rightarrow \infty$, called the thermodynamic limit, and show that this limit really exists. Moreover, the very existence of phase transitions depends on $|\Lambda|$ becoming infinite. The existence of the thermodynamic limit for the model we have constructed in Chapter 2 will be briefly discussed near to the end of this Appendix (see page 453).

Before we begin with the Ising model, let us consider in more detail the above statement about the non existence of phase transitions in a finite lattice. Phase transitions are defined as singularities in the free energy or in its derivatives. The partition function $\mathcal{Z}_\Lambda$ depends on several parameters, like the scale η , the coupling strength J or like fields controlled from outside (an external magnetic field). All contributions to $\mathcal{Z}_\Lambda$ are positive as long as these parameters are all real; therefore the free energy density $\mathcal{F}_\Lambda = -\frac{\eta}{|\Lambda|} \ln \mathcal{Z}_\Lambda$ has no singularity. A singularity can be produced by a zero in $\mathcal{Z}_\Lambda$ but this can only be achieved by a complex value of some of the parameters on which the partition function depends. Let us make clear this point with a “toy model” [78]: we consider a spin system with only two states, all spins “up” and all spins “down”. Both states have the same energy E_0 per spin. The total magnetization $\bar{M}$ is $\pm|\Lambda|$. Now we couple this system to an external magnetic field h and calculate the free energy density as function of h

$$\begin{aligned} \mathcal{Z}_\Lambda &= e^{-\frac{E_0}{\eta}|\Lambda| + \frac{h}{\eta}|\Lambda|} + e^{-\frac{E_0}{\eta}|\Lambda| - \frac{h}{\eta}|\Lambda|} = 2e^{-\frac{E_0}{\eta}|\Lambda|} \cosh\left(\frac{h|\Lambda|}{\eta}\right) , \\ \mathcal{F}_\Lambda &= E_0 - \frac{\eta}{|\Lambda|} \ln \left[2 \cosh\left(\frac{h|\Lambda|}{\eta}\right) \right] . \end{aligned} \quad (\text{C.7})$$

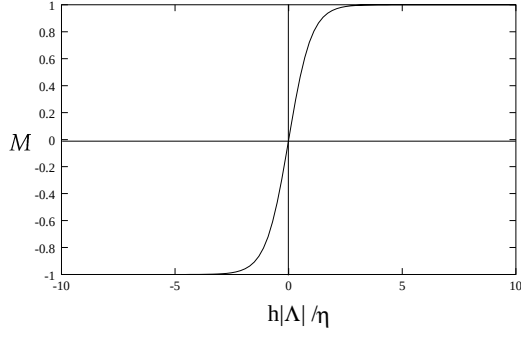
As we can see from this last result, the zeros in the free energy density are due to the following imaginary values of h

$$h = i \frac{(2n+1)\pi\eta}{2|\Lambda|} . \quad (\text{C.8})$$

However, in the infinite-lattice limit, these complex zeros reach the real h axis at $h = 0$ and the free energy density becomes nonanalytic². The consequence is that the magnetization density $\mathcal{M}$ (the total magnetization divided by the lattice “volume” $|\Lambda|$)

$$\mathcal{M}_\Lambda = -\frac{\partial \mathcal{F}_\Lambda}{\partial h} = \tanh\left(\frac{h|\Lambda|}{\eta}\right) ,$$

²This is an special case of a more general result called the Yang-Lee theorem.



can have two different values at $h = 0$: let us fix h at a small positive value and then take the limit $|\Lambda| \rightarrow \infty$. The magnetization tends toward 1. If we fix h at a small but negative value, the magnetization is then -1 . Therefore the magnetization at the infinite-lattice limit becomes the following function

$$\mathcal{M}_{|\Lambda| \rightarrow \infty} = \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases} . \quad (\text{C.9})$$

C.1 The 2-dimensional Ising model

As introduction to the calculus of correlation functions, let us begin with the 2-dimensional Ising model: spin variables $\sigma_{\vec{r}} = \pm 1$ at each site $\vec{r}$ of a 2-dimensional square lattice Λ with an energy describing the system equal to

$$E = J \sum_{\vec{r}} \sum_{\mu=1,2} [1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}] \quad \text{with} \quad J > 0 . \quad (\text{C.10})$$

and a partition function given by

$$\mathcal{Z}_{\Lambda} = \sum_{\text{all configurations in } \Lambda} \exp \left[-\frac{J}{\eta} \sum_{\vec{r}, \mu} [1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}] \right] . \quad (\text{C.11})$$

Let us take a given spin configuration: if we invert all spins (those spins “up” become “down” and viceversa) the configuration energy does not change. Therefore the expected value $\langle \sigma \rangle_{\Lambda}$ vanishes since it is an odd function under the change $\sigma \rightarrow -\sigma$ while the energy is an even function. Let us now switch on an external magnetic field h

$$E = J \sum_{\vec{r}} \sum_{\mu} [1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}] - h \sum_{\vec{r}} \sigma_{\vec{r}} . \quad (\text{C.12})$$

We are interested in whether the system has or not a magnetization. As we have just discussed, if we go to the infinite-lattice limit with a magnetic field exactly equal to zero, the magnetization will be vanish. What happens for a small but nonvanishing h ? Let us consider the free energy density in the infinite-lattice limit

$$\mathcal{F}(h) = - \lim_{|\Lambda| \rightarrow \infty} \frac{\eta}{|\Lambda|} \ln \left(\sum_{\text{all config.}} \exp \left[-\frac{J}{\eta} \sum_{\vec{r}, \mu} [1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}] + \frac{h}{\eta} \sum_{\vec{r}} \sigma_{\vec{r}} \right] \right) , \quad (\text{C.13})$$

and the magnetization density $\mathcal{M}_{|\Lambda| \rightarrow \infty}$

$$\mathcal{M}_{|\Lambda| \rightarrow \infty}(h) = -\frac{\partial \mathcal{F}}{\partial h} . \quad (\text{C.14})$$

According to the invariance under $\sigma \rightarrow -\sigma$ in energy term (C.10), $\mathcal{F}(-h) = \mathcal{F}(h)$ holds since we can change all spins in the term $h \sum_{\vec{r}} \sigma_{\vec{r}}$ to point in the other direction and the

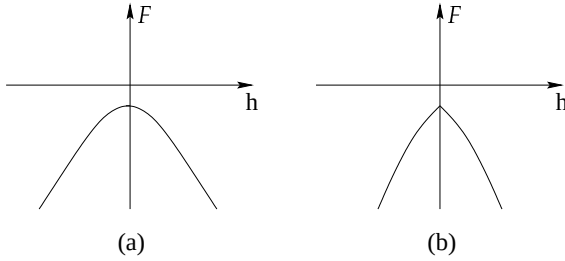
energy part due to spin interactions, $\sum_{\vec{r},\mu} [1 - \sigma_{\vec{r}}\sigma_{\vec{r}+\hat{\mu}}]$, does not change. Analogously, the magnetization fulfils $\mathcal{M}_{|\Lambda|\rightarrow\infty}(-h) = -\mathcal{M}_{|\Lambda|\rightarrow\infty}(h)$. To obtain the form of $\mathcal{F}$ as function of h , we shall now show that the free energy density always lies under its tangent (its second derivative is negative)

$$\begin{aligned}\mathcal{Z}_\Lambda &= \sum_{\text{all config.}} \exp \left[-\frac{J}{\eta} \sum_{\vec{r},\mu} [1 - \sigma_{\vec{r}}\sigma_{\vec{r}+\hat{\mu}}] + \frac{h}{\eta} S \right] \quad \text{with} \quad S = \sum_{\vec{r}} \sigma_{\vec{r}}, \\ \frac{\partial \mathcal{Z}_\Lambda}{\partial h} &= \frac{1}{\eta} \sum_{\text{all config.}} S \exp \left[-\frac{J}{\eta} \sum_{\vec{r},\mu} [1 - \sigma_{\vec{r}}\sigma_{\vec{r}+\hat{\mu}}] + \frac{h}{\eta} S \right] = \frac{1}{\eta} \mathcal{Z}_\Lambda \langle S \rangle_{\Lambda,h}, \\ \frac{\partial \langle S \rangle_{\Lambda,h}}{\partial h} &= -\frac{1}{\eta} \langle S \rangle_{\Lambda,h}^2 + \frac{1}{\eta} \frac{1}{\mathcal{Z}_\Lambda} \sum_{\text{all config.}} S S \exp \left[-\frac{J}{\eta} \sum_{\vec{r},\mu} [1 - \sigma_{\vec{r}}\sigma_{\vec{r}+\hat{\mu}}] + \frac{h}{\eta} S \right].\end{aligned}$$

From these results, it follows immediately

$$\frac{\partial \mathcal{F}_\Lambda}{\partial h} = -\frac{1}{|\Lambda|} \langle S \rangle_{\Lambda,h}, \quad \frac{\partial^2 \mathcal{F}_\Lambda}{\partial h^2} = -\frac{1}{|\Lambda|\eta} \underbrace{\left\langle \left(S - \langle S \rangle_{\Lambda,h} \right)^2 \right\rangle_{\Lambda,h}}_{\text{positive}} \leq 0. \quad (\text{C.15})$$

The above properties hold also in the thermodynamic limit. As discussed for the toy model in (C.7)-(C.8), the singularities of free energy density $\mathcal{F} = \lim_{|\Lambda|\rightarrow\infty} \mathcal{F}_\Lambda$ appears only at $h = 0$. Thus, we can have the following two possibilities:



either $\mathcal{F}$ is differentiable at $h = 0$ and therefore $\mathcal{M}_{|\Lambda|\rightarrow\infty}$ is continuous at this point (case (a)). In this case, the magnetization vanishes because of $\mathcal{M}_{|\Lambda|\rightarrow\infty}(-h) = -\mathcal{M}_{|\Lambda|\rightarrow\infty}(h)$. Or magnetization $\mathcal{M}_{|\Lambda|\rightarrow\infty}$ is discontinuous at $h = 0$ (case (b))

$$\mathcal{M}_{|\Lambda|\rightarrow\infty}^+ = \lim_{h \rightarrow 0^+} \mathcal{M}_{|\Lambda|\rightarrow\infty}(h) = m_{\text{sp}} \neq \mathcal{M}_{|\Lambda|\rightarrow\infty}^- = \lim_{h \rightarrow 0^-} \mathcal{M}_{|\Lambda|\rightarrow\infty}(h) = -m_{\text{sp}}. \quad (\text{C.16})$$

In this last case, an infinitesimal magnetic field already produces a nonvanishing magnetization $m_{\text{sp}} = \langle \sigma \rangle_{\Lambda \rightarrow \infty}$. This is a case of spontaneous symmetry breakdown: although the energy is symmetric under the transformation $\sigma \rightarrow -\sigma$, the system's state is it not. This takes place when there is more than one lowest energy state.

Although the exact results will be proved in the next sections, let us introduce now the main characteristics of the Ising model. We suppose that our system shows a behaviour described by the case (b). As we shall see, by varying J/η toward smaller values, the peak in (b) get smoothed and eventually the derivative of $\mathcal{F}$ at $h = 0$ becomes continuous and the spontaneous magnetization vanishes. The value of J/η at which this occurs, is called the critical point.

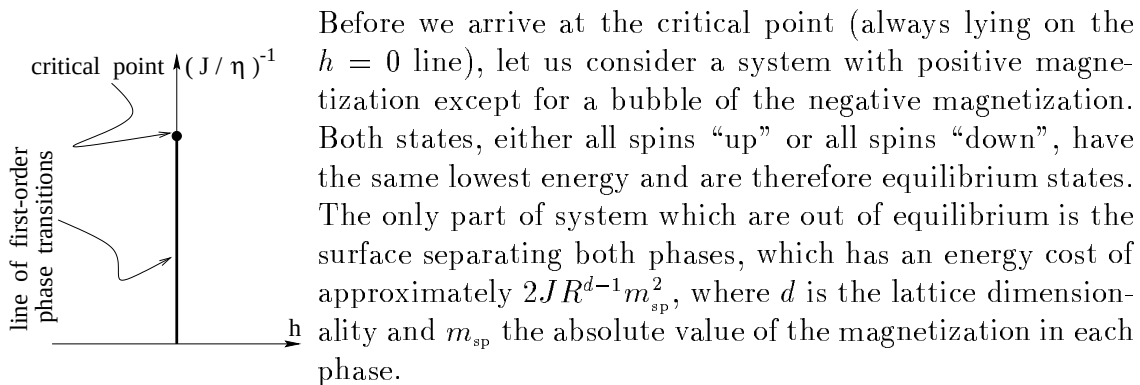
Let us consider the system at a J/η higher than the critical point and with an infinitesimal positive external magnetic field: the state has a positive spontaneous magnetization. The same effect can be reached by imposing the boundary condition of spins pointing in the “up” direction at the boundary of the finite lattice. Configuration with all spins “up”, and configuration with all spins “down”, have the lowest possible energy; although both configurations are equally probable, by imposing the boundary conditions we slightly increase the energy of one of the two configurations. When taking the infinite-lattice limit, this energy difference goes to infinity with the boundary surface and only one of the configurations dominates in the partition function (for the case of an infinitesimal h , the energy difference goes to infinity with the lattice volume). Note that the barrier between the two most likely configurations grows infinitely only in the thermodynamic limit [78].

Now, we decrease very slowly the external magnetic field to a infinitesimal negative value. After a time (which can be very long), a large fluctuation produces a bubble where the spins are in the “down” orientation. This bubble grows very rapidly and a transition to the phase where all spins are “down” takes place: the fluctuation makes the system “discover” that its real energy minimum lies at a negative spontaneous magnetization and thus the system “springs” to the state with this magnetization. Such a phase transition, where the magnetization is discontinuous at the transition point, is called a discontinuous or first order phase transition.

Let us introduce the correlation length ξ as a measure of the size of the largest possible coherent fluctuations in the system. Fluctuations of size larger than ξ have a exponentially small probability [78]. Placed on the line of first-order transitions ($h = 0$), we shall now study the change in the form of the transtion between the phase of positive magnetization and that of negative when we approach the critical point. At this point

$$\lim_{h \rightarrow 0^+} \mathcal{M}_{|\Lambda| \rightarrow \infty}(h) = 0 = \lim_{h \rightarrow 0^-} \mathcal{M}_{|\Lambda| \rightarrow \infty}(h),$$

and there is no difference between the spontaneous magnetization of the two phases. As we shall see, this makes the correlation length go to infinity.



This surface contributes to the partition function with a factor $e^{-\frac{J}{\eta}2R^{d-1}m_{sp}^2}$ which is still not too small if the exponent is of the order of -1 : this defines the radius (correlation

length) of the largest bubble which is still possible in the system

$$\frac{J}{\eta} 2R^{d-1} m_{\text{sp}}^2 \sim 1 \quad \implies \quad R_{\text{larges}} \sim \text{constant} \times \left(\frac{1}{m_{\text{sp}}} \right)^{\frac{2}{d-1}}, \quad (\text{C.17})$$

which diverges at $m_{\text{sp}} \rightarrow 0$. At the critical point, the change from phase with positive magnetization to the other phase is done continuously. This is called a second order phase transition. Although the magnetization does not show any discontinuity, some of its derivatives are discontinuous: these derivatives are the correlation functions, from which we shall study the two-point correlation in the following sections.

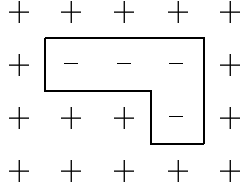
For values $(J/\eta)^{-1}$ above the critical point (small J/η), there is no phase transition since neither the partition function (or the free energy density) nor its derivatives exhibit any nonanalyticity. After the introduction of the correlation function of two spins separated a distance $|\vec{r}_2 - \vec{r}_1|$, we shall see that the phase diagram for the Ising model looks as follows: at large J/η , there is a *nonzero* magnetization and the two-point correlation tends toward a constant when the distance between the two spins goes to infinity. This phase is called the ordered phase; since the generalized scale η plays the same role as $k_B T$, the phase at large J/η can be also called the “low temperature” phase. At the critical point (or second-order phase-transition point), the correlation function decays like a power of $|\vec{r}_2 - \vec{r}_1|$. And for values J/η smaller than the corresponding to the critical point, the correlation shows an exponential decay with distance $|\vec{r}_2 - \vec{r}_1|$ and there is no magnetization. This last phase is called the disordered phase (or the “high temperature” phase).

C.1.1 Peierls’ proof of the nonvanishing spontaneous magnetization at large $\beta = J/\eta$

For the following, coupling strength J divided by scale η will be abbreviated by β . After the previous introductory discussion, we shall now see that the 2-dimensional Ising model exhibits a spontaneous magnetization at large β . The proof is due to Peierls [46] and follows in two steps: firstly and for a finite lattice, one obtains an strictly positive lower bound for the magnetization density $\mathcal{M}_\Lambda$ at $h = 0$ with the special boundary condition of all spins at the lattice’s boundary pointing upward. The obtained lower bound results to be independent of the lattice size. Then one argues that the spontaneous magnetization m_{sp} at $h \rightarrow 0^+$ and for the infinite lattice is not smaller than this bound.

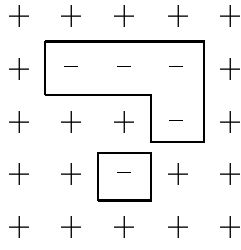
Let us consider a finite square lattice with the boundary condition that all spins outside our square lattice point upward. We have used this condition in order to ensure a strictly positive bound. If we had considered free boundary conditions (all spins outside the finite lattice we are considering are zero), this would have lead to a zero magnetization since both configurations all “up” and all “down” would have the same probability. Further, one can prove that the free energy density is independent of the boundary condition in the limit of a infinite lattice ³.

³Although we shall not present here the rigorous proof, one can approximately argue that the effect of the boundary produces a multiplicative contribution to the partition of the order $e^{-\beta S_{\text{boundary}}}$, where



For a given spin configuration, a Peierls' contour is defined as the set of bonds $b = \langle \vec{x}\vec{y} \rangle$ such that the spins have opposite signs to each side of this contour

$$\sigma_{\vec{x}}\sigma_{\vec{y}} = -1.$$



Peierls' contours are closed, and a contour like that in the figure (left) is to be considered as two non intersecting contours. Let N_+ denote the number of spins with value $+1$ and N_- those with value -1 , with the total number of spins equal to the number of lattice sites $N = N_+ + N_- = |\Lambda|$. The magnetization density is defined as the expectation value for the spin to have the value $+1$ like at the boundary

$$\mathcal{M}_\Lambda = \left\langle \frac{N_+}{N} - \frac{N_-}{N} \right\rangle_\Lambda = 1 - 2 \left\langle \frac{N_-}{N} \right\rangle_\Lambda. \quad (\text{C.18})$$

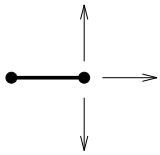
the expectation value is calculated according to partition function (C.11).

The main idea to obtain the bound is that energy $J \sum_{\vec{r}, \mu} [1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}]$ due to the neighbour spin interactions, plus the condition of all spins “up” at the boundary, tries to align all the spins inside the boundary at large $\beta = J/\eta$. Large Peierls' contours, which strongly disorder the system, are therefore suppressed.

Let us firstly numerate all possible contours of length L . Now we consider a given spin configuration and define the parameter $\chi_L^j = 0, 1$ as follows: if the contour of length L and number j appears in our configuration $\chi_L^j = 1$, and $\chi_L^j = 0$ otherwise. The largest area enclosed by a closed contours in two dimensions is that of a square. Therefore, a contour of length L includes at most $\left(\frac{L}{4}\right)^2$ spins $\sigma = -1$, and for our given configuration, the number N_- of spins $\sigma = -1$ is bounded by

$$N_- \leq \sum_{L=4,6,8,\dots} \left(\frac{L}{4}\right)^2 \sum_{\text{all Peierls' contours of length } L} \chi_L^j. \quad (\text{C.19})$$

In order to estimate $\langle N_- \rangle_\Lambda$ we need an upper bound for the number of all Peierls' contours of length L and for $\langle \chi_L^j \rangle_\Lambda$.



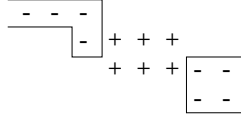
Let us consider firstly a path of L bonds which never turns directly on itself: the first bond can be put in one of the $N = |\Lambda|$ lattice sites. Since the lattice is two dimensional, there are 3 possible directions for the next bond. The number of all possible paths which begin anywhere without turning on themselves is equal to $N3^{L-1}$. However, Peierls' contours are only closed contours, obtaining thus the following upper bound for the number of Peierls' contours

$$\text{number of Peierls' contours} \leq N3^{L-1}. \quad (\text{C.20})$$

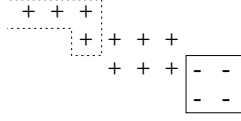
S_{boundary} is the finite lattice's surface. Thus, the free energy density contains the term $J \frac{S_{\text{boundary}}}{|\Lambda|}$ with $|\Lambda|$ the lattice's volume: at the thermodynamic limit $|\Lambda| \rightarrow \infty$, this contribution tends toward zero.

Now let us obtain the estimation for $\langle \chi_L^j \rangle_\Lambda$

$$\langle \chi_L^j \rangle_\Lambda = \frac{1}{\mathcal{Z}_\Lambda} \sum_{\text{all configurations in } \Lambda} \chi_L^j e^{-E_{\text{conf}}/\eta} = \frac{1}{\mathcal{Z}_\Lambda} \sum_{\substack{\text{config. containing} \\ \text{the } j\text{-th contour}}} e^{-E_{\text{conf}}/\eta}. \quad (\text{C.21})$$



configuration C



configuration C^*

In $\mathcal{Z}_\Lambda = \sum_{\text{all configurations}} e^{-E_{\text{conf}}/\eta}$ we separate those C^* configurations which are obtained from the C configurations in the $\langle \chi_L^j \rangle_\Lambda$'s numerator by reversing all spins inside the j -th contour of length L (but only inside this contour)

$$\mathcal{Z}_\Lambda = \sum_{\text{configurations } C^*} e^{-E_{\text{conf}}/\eta} + \dots$$

This means to remove the j -th contour. A C^* configuration has an energy which is lower than that of its corresponding C configuration by the following amount

$$E_{\text{conf. } C^*} = E_{\text{conf. } C} - 2JL, \quad (\text{C.22})$$

and thus

$$\langle \chi_L^j \rangle_\Lambda = \frac{\sum_{\text{configurations } C} e^{-E_C/\eta}}{\sum_{\text{configurations } C^*} e^{-E_{C^*}/\eta} + \dots} \leq \frac{\sum_{\text{config. } C^*} e^{-\frac{E_{C^*} + 2JL}{\eta}}}{\sum_{\text{config. } C^*} e^{-\frac{E_{C^*}}{\eta}}} = e^{-2\beta L}. \quad (\text{C.23})$$

Substituting results (C.20) and (C.23) into (C.19), we obtain

$$\begin{aligned} \left\langle \frac{N_-}{N} \right\rangle_\Lambda &\leq \frac{1}{N} \sum_{L=4,6,8,\dots} \left(\frac{L}{4} \right)^2 \sum_{\substack{\text{all Peierls} \\ \text{contours of} \\ \text{length } L}} \langle \chi_L^j \rangle_\Lambda \leq \frac{N}{N} \sum_{L=4,6,8,\dots} 3^{L-1} \left(\frac{L}{4} \right)^2 e^{-2\beta L}, \\ \left\langle \frac{N_-}{N} \right\rangle_\Lambda &\leq \frac{1}{48} \sum_{L=4,6,8,\dots} L^2 \left(3e^{-2\beta} \right)^L. \end{aligned} \quad (\text{C.24})$$

Note that this estimation does not depend on the number $N = |\Lambda|$ of lattice sites. At $\beta \rightarrow \infty$, $\left(3e^{-2\beta} \right)^L$ can be made arbitrary small, for instance such that the magnetization density is strictly positive

$$\mathcal{M}_\Lambda = 1 - 2 \left\langle \frac{N_-}{N} \right\rangle_\Lambda \geq 1 - \frac{1}{24} \sum_{L=4,6,8,\dots} L^2 \left(3e^{-2\beta} \right)^L \geq B > 0. \quad (\text{C.25})$$

We can now proceed with the second step of the Peierls' argument. According to the general form of $\mathcal{F}_\Lambda$ as function of h (C.15), we obtain for $h \geq 0$

$$\mathcal{F}_\Lambda(h) = \mathcal{F}_\Lambda(0) - \mathcal{M}_\Lambda h \leq \mathcal{F}_\Lambda(0) - Bh, \quad (\text{C.26})$$

with B the obtained bound which is independent of $|\Lambda|$. Taking now the thermodynamic limit $|\Lambda| \rightarrow \infty$ and by noting $m_{\text{sp}} = -\frac{\partial \mathcal{F}(h)}{\partial h}$, we obtain

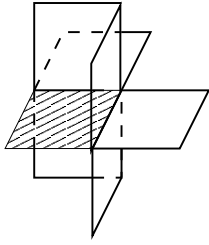
$$\mathcal{F}(h) = \lim_{|\Lambda| \rightarrow \infty} \mathcal{F}_\Lambda(h) = \text{constant} - m_{\text{sp}} h \leq \lim_{|\Lambda| \rightarrow \infty} \mathcal{F}_\Lambda(0) - Bh,$$

and therefore

$$m_{\text{sp}} \geq B > 0. \quad (\text{C.27})$$

The Peierls' argument works also for a 3-dimensional lattice (and for lattices of higher dimensionality) [45]. For a lattice in three dimensions, the Peierls' contours are closed polyhedra of area A and equation (C.19) must be changed into

$$N_- \leq \sum_{A=6,10,\dots} \left(\sqrt{\frac{A}{6}} \right)^3 \sum_{\text{all Peierls' contours of area } A} \chi_A^j, \quad (\text{C.28})$$



since the largest volume enclosed by a closed polyhedra is that of a cube. For the number of (closed) Peierls' contours the upper bound is $N3^{A-1}$, similar to (C.20), since to the edge of a given surface (shaded in the figure) a unit square can be attached in three different ways. Finally, the energy of a configuration C^* is different from that of configuration C by an amount of $E_{\text{conf. } C^*} = E_{\text{conf. } C} - 2JA$ and therefore

$$\langle \chi_A^j \rangle_\Lambda \leq e^{-2\beta A}. \quad (\text{C.29})$$

Substitution into (C.28) produces

$$\left\langle \frac{N_-}{N} \right\rangle_\Lambda \leq \frac{1}{18\sqrt{6}} \sum_{L=6,10,\dots} A^{3/2} (3e^{-2\beta})^A. \quad (\text{C.30})$$

which, as in the 2-dimensional case, can be made so small as we like.

C.1.2 The two-point correlation function $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$

After having seen the existence of an ordered phase (nonvanishing magnetization) at large $\beta = J/\eta$, and thus the existence of first-order phase transitions when we change from an (infinitesimal) positive external magnetic field to a negative h , let us place ourselves on the line $h = 0$ and study at which value of β^{-1} this line of first-order transitions finishes. We shall now consider the correlation function of two spins separated by a large distance $|\vec{r}_2 - \vec{r}_1|$

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda = \frac{1}{\mathcal{Z}_\Lambda} \sum_{\{\sigma_{\vec{r}} = \pm 1\}} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \exp \left[-\beta \sum_{\vec{r}} \sum_{\mu=1,2} (1 - \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}) \right], \quad (\text{C.31})$$

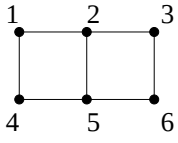
where the following notation is used

$$\sum_{\{\sigma_{\vec{r}}=\pm 1\}} \equiv \underbrace{\sum_{\sigma_{\vec{1}}=\pm 1} \sum_{\sigma_{\vec{2}}=\pm 1} \cdots \sum_{\sigma_{\vec{N}}=\pm 1}}_{\text{all the } N = |\Lambda| \text{ sites of the lattice}} = \prod_{\vec{r}} \sum_{\sigma_{\vec{r}}=\pm 1} .$$

This two-point correlation function (which for the following will be simply called correlation function) will be studied for small and large β . Before the behaviour at $\beta \rightarrow 0$ is calculated, it is useful to rewrite (C.31) in the following form by considering the relation $1 \pm \tanh \beta = e^{\pm \beta} / \cosh \beta$

$$\begin{aligned} \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda} &= \frac{1}{\mathcal{Z}_{\Lambda}} \sum_{\{\sigma_{\vec{r}}=\pm 1\}} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \prod_{\vec{r}} \prod_{\mu=1,2} \left[(1 + \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}} \tanh \beta) (e^{-\beta} \cosh \beta) \right] = \\ &= \frac{e^{-2N\beta} (\cosh \beta)^{2N} \sum_{\sigma_{\vec{1}}=\pm 1} \sum_{\sigma_{\vec{2}}=\pm 1} \cdots \sum_{\sigma_{\vec{N}}=\pm 1} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \prod_{\vec{r}, \mu} [1 + \tanh \beta \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}]}{e^{-2N\beta} (\cosh \beta)^{2N} \sum_{\sigma_{\vec{1}}=\pm 1} \sum_{\sigma_{\vec{2}}=\pm 1} \cdots \sum_{\sigma_{\vec{N}}=\pm 1} \prod_{\vec{r}, \mu} [1 + \tanh \beta \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}]} , \end{aligned} \quad (\text{C.32})$$

with $N = |\Lambda|$ the number of lattice sites. By means of the above equation, it is easy to calculate the largest contributions to the correlation for $\beta \rightarrow 0$ ($\tanh \beta \approx \beta$); in order to introduce the technique to compute this kind of expressions, let us firstly consider a simplified lattice formed by six sites



$$\sum_{\{\sigma_{\vec{r}}=\pm 1\}} \prod_{\vec{r}, \mu} (1 + \beta \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}}) = \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \cdots \sum_{\sigma_6=\pm 1} \{ (1 + \beta \sigma_1 \sigma_2) \times \\ \times (1 + \beta \sigma_2 \sigma_3) (1 + \beta \sigma_4 \sigma_5) (1 + \beta \sigma_5 \sigma_6) (1 + \beta \sigma_1 \sigma_4) (1 + \beta \sigma_2 \sigma_5) (1 + \beta \sigma_3 \sigma_6) \} .$$

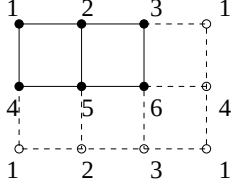
But most of the products in this last equality vanish when summing over $\{\sigma_{\vec{r}} = \pm 1\}$

$$\sum_{\sigma_{\vec{r}}=\pm 1} \sigma_{\vec{r}} = 0 \quad \text{and} \quad \sum_{\sigma_{\vec{r}}=\pm 1} \sigma_{\vec{r}}^2 = \sum_{\sigma_{\vec{r}}=\pm 1} 1 = 2. \quad (\text{C.33})$$

Only those products containing only even powers of $\sigma_{\vec{r}}$ contribute (underlined in the following expression)

$$\begin{aligned} \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \sum_{\sigma_6=\pm 1} \{ & 1 + \beta [\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \cdots] + \\ & + \beta^2 [\sigma_1 \sigma_2^2 \sigma_3 + \cdots] + \\ & + \beta^3 [\sigma_1 \sigma_2^2 \sigma_3^2 \sigma_6 + \cdots] + \\ & + \beta^4 [\sigma_1^2 \sigma_2^2 \sigma_5^2 \sigma_4^2 + \sigma_2^2 \sigma_3^2 \sigma_6^2 \sigma_5^2 + \cdots] + \\ & + \beta^5 [\sigma_1 \sigma_2^3 \sigma_3^2 \sigma_6^2 \sigma_5^2 + \cdots] + \\ & + \beta^6 [\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_6^2 \sigma_5^2 \sigma_4^2 + \cdots] + \\ & + \beta^7 \sigma_1^2 \sigma_2^3 \sigma_3^2 \sigma_6^2 \sigma_5^3 \sigma_4^2 \} = \\ & = 2^6 [1 + \beta \cdot 0 + \beta^2 \cdot 0 + \beta^3 \cdot 0 + \beta^4 (\underline{1} + \underline{1} + 0) + \beta^5 \cdot 0 + \beta^6 (\underline{1} + 0) + \beta^7 \cdot 0] . \end{aligned}$$

Note that only closed paths of bonds produce a nonvanishing contribution. This fact is due to the form of the spin energy, $J[1 - \sigma_{\vec{r}+\hat{\mu}}\sigma_{\vec{r}}]$, which makes that a closed path of bonds contains only even powers of the spin variables. In our example, only the contributions by the square with four bonds and by the rectangle with six bonds do not vanish. For $\beta \rightarrow 0$ the contribution of the longer paths is much smaller than the contribution coming from short paths.



To avoid boundary effects, we can use periodic conditions on the border like in the figure. The lattice has the geometry of a torus. In this case we can place the squared path of four bonds on each of the 6 lattice sites and the rectangular path in 2×6 different forms (6 from the lattice sites and 2 from the two possible rectangle's orientations in two dimensions: vertical and horizontal)

$$\mathcal{Z}_\Lambda = (e^{-\beta} \cosh \beta)^{12} 2^6 [1 + \beta^4 \cdot 6 + \beta^6 \cdot 12] .$$

For a large lattice with $N = |\Lambda|$ sites, a square path of four bonds can be placed at every of the N lattices sites, the smallest rectangular path at $2N$ sites and so on

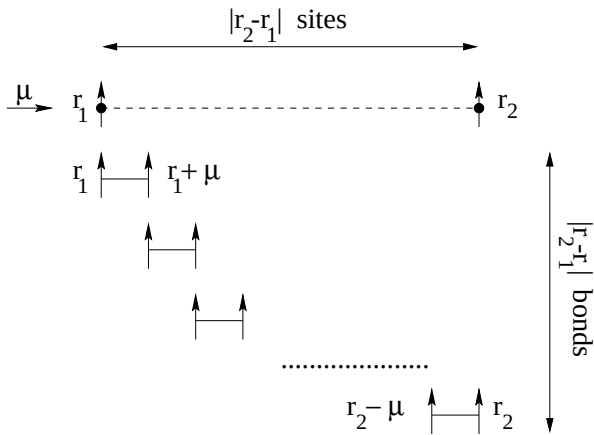
$$\mathcal{Z}_\Lambda = (e^{-\beta} \cosh \beta)^{2N} 2^N [1 + \beta^4 \cdot N + \beta^6 \cdot 2N + \dots] , \quad (\text{C.34})$$

where $\dots$ are contributions containing higher powers of β . With the same arguments, we can calculate the leading contribution for $\beta \rightarrow 0$ to

$$\sum_{\{\sigma_{\vec{r}}=\pm 1\}} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \prod_{\vec{r}, \mu} (1 + \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}} \tanh \beta) ,$$

contribution which comes from a direct path of bonds connecting the points $\vec{r}_1$ and $\vec{r}_2$. In this way all spin variables are squared and when summing over all spins $\sum_{\{\sigma_{\vec{r}}=\pm 1\}}$ it does not vanish

$$\sum_{\{\sigma_{\vec{r}}=\pm 1\}} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \prod_{\vec{r}, \mu} (1 + \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{\mu}} \beta) = 0 + \sum_{\{\sigma_{\vec{r}}=\pm 1\}} \sigma_{\vec{r}_1}^2 \sigma_{\vec{r}_1+\hat{\mu}}^2 \cdots \sigma_{\vec{r}_2}^2 + \dots = 2^N \beta^{|\vec{r}_2-\vec{r}_1|} + \dots ,$$



where $\dots$ contains higher powers in β , arising from walks from $\vec{r}_1$ to $\vec{r}_2$ but not along the shortest path, which is the straight line connecting $\vec{r}_1$ and $\vec{r}_2$. For a better understanding, in the figure the bonds of the shortest path are displayed above each other, although they should be represented aligned along the path between $\vec{r}_1$ and $\vec{r}_2$. The correlation function for two spins separated by a distance $|\vec{r}_2 - \vec{r}_1|$ in the range of low β is therefore equal to

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda} \stackrel{\beta \rightarrow 0}{\approx} \frac{(e^{-\beta} \cosh \beta)^{2N} 2^N \beta^{|\vec{r}_2 - \vec{r}_1|}}{(e^{-\beta} \cosh \beta)^{2N} 2^N [1 + N\beta^4 + 2N\beta^6]} \approx \beta^{|\vec{r}_2 - \vec{r}_1|},$$

or equivalently

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda}^{\text{low } \beta} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(1/\beta)}. \quad (\text{C.35})$$

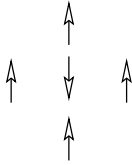
The method used to obtain this result is called the “high temperature” power expansion, since it holds for small $\beta = J/\eta$ and η plays the role of $k_B T$. The above calculation can be repeated for any lattice dimension and thus equation (C.35) holds for any dimension.

Now let us consider the case of large β and return to the definition of the two-point correlation function

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda} = \frac{\sum_{\{\sigma_{\vec{r}} = \pm 1\}} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \exp \left[-\beta \sum_{\vec{r}, \mu} (1 - \sigma_{\vec{r}} \sigma_{\vec{r} + \hat{\mu}}) \right]}{\sum_{\{\sigma_{\vec{r}} = \pm 1\}} \exp \left[-\beta \sum_{\vec{r}, \mu} (1 - \sigma_{\vec{r}} \sigma_{\vec{r} + \hat{\mu}}) \right]}. \quad (\text{C.36})$$

The exponent in the partition function forces most of the spins to be oriented in the same direction ($\sigma_{\vec{r}} \sigma_{\vec{r} + \hat{\mu}} = 1$) at very large β . Let us suppose that this direction is spin “up”. For large β , we can expand both numerator and denominator of the correlation in powers of $e^{-\beta}$. Starting with the partition function (the correlation’s denominator), we see that the largest contribution is 1 and arises from the configuration where all spins are pointing upward.

The configuration with only a spin “down” produces the next leading contribution: this reversed spins can be placed at any of the $N = |\Lambda|$ lattice sites and its contribution is $e^{-4 \times 2\beta}$ since four bonds have opposite directed spins at their extremes. Such a bond, where the spin at one bond’s extreme is pointing in the opposite direction to the spin at the other bond’s extreme, will be called “broken” bond. The partition function becomes thus



$$\mathcal{Z}_{\Lambda} = 1 + |\Lambda| e^{-8\beta} + \dots, \quad (\text{C.37})$$

where $\dots$ contains configurations with more than one flipped spin and therefore with higher powers of $e^{-\beta}$. The same argumentation can be carried out for the correlation’s numerator: for the configuration with all spins “up” the value of $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2}$ is 1. When only one spin is flipped, one can place it at $\vec{r}_1$ or $\vec{r}_2$, producing thus $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2} = -1$, or at any of the other $|\Lambda| - 2$ sites ($\sigma_{\vec{r}_1} \sigma_{\vec{r}_2} = 1$). The correlation function in the limit large β takes the following form

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda} = \frac{1 + (|\Lambda| - 2) e^{-8\beta} - 2e^{-\beta} + \dots}{1 + |\Lambda| e^{-8\beta} + \dots} \stackrel{\beta \rightarrow \infty}{\approx} \frac{\exp [(|\Lambda| - 4) e^{-8\beta}]}{\exp [|\Lambda| e^{-8\beta}]},$$

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda} \approx \exp [-4e^{-8\beta}]. \quad (\text{C.38})$$

By comparing this result with equation (C.35), one sees that when distance $|\vec{r}_2 - \vec{r}_1|$ becomes infinite, the correlation function tends toward zero at small β but remains nonzero at large β

$$\begin{aligned} \lim_{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda}^{\text{low } \beta} &= 0, \\ \lim_{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda}^{\text{high } \beta} &= \text{constant} \neq 0. \end{aligned} \quad (\text{C.39})$$

Repeating the same steps leading to (C.38) this time for the expectation value $\langle \sigma_{\vec{r}_1} \rangle_{\Lambda}$, one obtains

$$\langle \sigma_{\vec{r}_1} \rangle_{\Lambda}^{\text{high } \beta} \approx \exp \left[-2e^{-8\beta} \right], \quad (\text{C.40})$$

which is nonzero. We obtain again the the Peierls' result of a nonvanishing magnetization, although the Peierls' argument is a more powerful tool, since it holds in the thermodynamic limit whilst we are working here on a finite lattice. In fact, by means of a slight modification of the Peierls' argument, one can prove that the two-point correlation function does not vanish at large β for an infinite lattice; this result be presented at the end of the next section (see equation (C.63)).

One should note that although the two-point correlation function does not vanish as $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$, this does not imply the existence of long-range order in the system. One says there is long-range order in the system if the *connected* correlation function

$$\Gamma_{\vec{r}_1 \vec{r}_2} = \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle - \langle \sigma_{\vec{r}_1} \rangle \langle \sigma_{\vec{r}_2} \rangle, \quad (\text{C.41})$$

does not vanish in the limit $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$ [45]. Long-range order means that at large distances one cannot consider the spin at $\vec{r}_1$ as approximately independent of the spin at $\vec{r}_2$: $\lim_{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle \neq \langle \sigma_{\vec{r}_1} \rangle \langle \sigma_{\vec{r}_2} \rangle$.

By looking at equations (C.38) and (C.40), we see that even at large β there is no long-range order for the Ising model. An exact calculation (without the approximations done to obtain (C.38)) gives that the connected correlation function decays like the distance for large values of β . On the other hand, at small β (above the critical point) the magnetization is zero ⁴ and therefore, the connected correlation function becomes the two-point correlation. According to (C.35), $\Gamma_{\vec{r}_1 \vec{r}_2}$ shows also an exponential decay with the distance in this range of β .

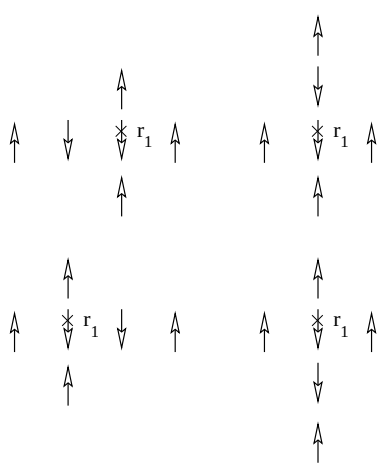
Note. Evaluation of $(\Gamma_{\vec{r}_1 \vec{r}_2})_{\Lambda}$ at large β .

In this note we shall show that the connected correlation function $(\Gamma_{\vec{r}_1 \vec{r}_2})_{\Lambda}$ decays with distance $|\vec{r}_2 - \vec{r}_1|$ not only at low but also at high β ; the subindex Λ denotes that we are working with a finite lattice. The behaviour of $\Gamma_{\vec{r}_1 \vec{r}_2}$ in the thermodynamic limit is the same as for a finite lattice since both $\langle \sigma_{\vec{r}_1} \rangle_{\Lambda}$ as well as $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda}$ keep their behaviour in the limit of an infinite lattice (for magnetization $\langle \sigma_{\vec{r}_1} \rangle$ this has been already proved by means of the Peierl's argument, and for correlation $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ it will be shown in the next section that it does not vanish at large β in a infinite lattice).

⁴This can be easily seen by calculating $\langle \sigma_{\vec{r}_1} \rangle$ by means of the "high temperature" expansion. One cannot construct any configuration which starting at $\vec{r}_1$ only contains quadratic powers of every of its spins. Thus we can either construct an open path beginning at $\vec{r}_1$ but whose ending spin is only included once, or a closed path starting and ending at $\vec{r}_1$ but then $\sigma_{\vec{r}_1}$ is included three times.

In the above partition function (C.37), we go a step further and consider the contribution due to two flipped spins. If these are adjacent to each other, we can locate them in $\frac{1}{2}|\Lambda|4$ different forms, each of them contributing with $e^{-6 \times 2\beta}$ since there are six “broken” bonds (the bonds which surround the two nearest neighbouring flipped spins). The factor 1/2 in the number of different forms has the function of avoiding the overcounting of configurations which differs from each other only by interchanging the positions of the two flipped spins. On the other hand, if the reversed spins lie apart from each other, the contribution to the partition function is $e^{-(4 \times 2\beta + 4 \times 2\beta)}$ a number of times given by $\frac{1}{2}|\Lambda|(|\Lambda| - 5)$. However, these are not the only configurations whose energy lies $16J$ above the configuration with all spins “up”: for instance, a configuration with three reversed spins put along a straight segment or a configuration with four reversed spins building a square produce the same contribution. We break our expansion at power $e^{-12\beta}$, and so we obtain

$$Z_\Lambda \approx 1 + |\Lambda|e^{-8\beta} + \frac{1}{2}4|\Lambda|e^{-12\beta}. \quad (\text{C.42})$$



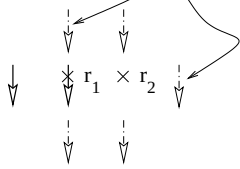
The expectation value $\langle \sigma_{\vec{r}_1} \rangle_\Lambda$ is calculated in a similar way, taking into account whether the reversed spins are placed at $\vec{r}_1$ (and thus producing a negative contribution) or not. The two adjacent spins can be placed in 4 different forms in which one of the reversed spins is at $\vec{r}_1$ (see figure); the rest of the $4|\Lambda| - 4$ forms produce positive contributions

$$\begin{aligned} \langle \sigma_{\vec{r}_1} \rangle_\Lambda &\approx \frac{1 + (|\Lambda| - 2)e^{-8\beta} + \frac{1}{2}(4|\Lambda| - 8)e^{-12\beta}}{1 + |\Lambda|e^{-8\beta} + \frac{1}{2}4|\Lambda|e^{-12\beta}} \approx \\ &\approx \left\{ \begin{array}{cc} 1 & + (|\Lambda| - 2)e^{-8\beta} + \frac{1}{2}(4|\Lambda| - 8)e^{-12\beta} \\ -|\Lambda|e^{-8\beta} & - \frac{1}{2}4|\Lambda|e^{-12\beta} \end{array} \right\}, \end{aligned}$$

$$\langle \sigma_{\vec{r}_1} \rangle_\Lambda \approx 1 - 2e^{-8\beta} - 4e^{-12\beta}. \quad (\text{C.43})$$

Now let us evaluate $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ when $|\vec{r}_2 - \vec{r}_1| = 1$ (one lattice spacing); again we break the expansion at power $e^{-12\beta}$. The case of two nearest reversed spins is a bit different from the calculation of $\langle \sigma_{\vec{r}_1} \rangle_\Lambda$.

possible positions
for a negative contribution



Since $|\vec{r}_2 - \vec{r}_1| = 1$, if both spins are located at $\vec{r}_1$ and $\vec{r}_2$ we have a positive value of $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2}$: only the 6 positions of the figure produce a negative contribution, which is equal to

$$\frac{1}{2}(4|\Lambda| - 2 \times 6)e^{-12\beta}. \quad (\text{C.44})$$

The other contributions in the numerator are 1 for all spins “up” and $(|\Lambda| - 4)e^{-8\beta}$ for one reversed spin. Thus the two-point correlation becomes

$$\text{but not } \begin{array}{c} \downarrow \quad \downarrow \\ \times \quad \times \\ \downarrow \quad \downarrow \end{array} \quad \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda \approx 1 - 4e^{-8\beta} - 6e^{-12\beta}. \quad (\text{C.45})$$

and the connected correlation (up to order $e^{-12\beta}$) is given by

$$(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda \approx [1 - 4e^{-8\beta} - 6e^{-12\beta}] - [1 - 2e^{-8\beta} - 4e^{-12\beta}]^2 = 2e^{-12\beta}, \quad (\text{C.46})$$

where we have used $\langle \sigma_{\vec{r}_1} \rangle_\Lambda = \langle \sigma_{\vec{r}_2} \rangle_\Lambda$ due to the translation invariance of the partition function (coupling strength J does not depend on the lattice position and the effects of the boundary are neglected).

$\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_1 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_2 \quad \text{and} \quad \times r_1 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_2 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$

However, if the distance between $\vec{r}_1$ and $\vec{r}_2$ were larger than 1, the two adjacent reversed spins would produce a contribution in $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ equal to $\frac{1}{2} (4|\Lambda| - 16) e^{-12\beta}$, which is the quadratic power of the contribution in $\langle \sigma_{\vec{r}_1} \rangle_\Lambda$. Hence, the coefficient of power $e^{-12\beta}$ in $(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda$ vanishes if $|\vec{r}_2 - \vec{r}_1| > 1$

$$(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda \approx [1 - 4e^{-8\beta} - 8e^{-12\beta}] - [1 - 2e^{-8\beta} - 4e^{-12\beta}]^2 = 0 + \dots,$$

and we must go to a configuration which a higher number of reversed spins.

$\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_1 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_2 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$

From the previous discussion, it is clear that the first configuration of reversed spins which produces a contribution in the connected correlation is that formed by a straight chain of adjacent flipped spins with a length $|\vec{r}_2 - \vec{r}_1|$. The reason is that to obtain a negative contribution in $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$'s numerator, the chain must avoid to stretch from $\vec{r}_1$ to $\vec{r}_2$. And these not available positions make the contribution to $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ be different from the quadratic power of the contribution to $\langle \sigma_{\vec{r}_1} \rangle_\Lambda$ produced by this chain.

but not $\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_1 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \times r_2 \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$

All the other spin configurations which cannot cover the complete distance between $\vec{r}_1$ and $\vec{r}_2$ produce contributions in $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ which are equal to the quadratic power of the corresponding contribution in $\langle \sigma_{\vec{r}_1} \rangle_\Lambda$.

Since a chain of $|\vec{r}_2 - \vec{r}_1|$ of reversed spins has an energy $4(|\vec{r}_2 - \vec{r}_1| + 1)J$ above the energy of the configuration with all spins aligned, the leading contribution to the connected correlation of two spins separated a distance $|\vec{r}_2 - \vec{r}_1|$ is (except for a combinatorial factor) equal to

$$(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda \sim e^{-4(|\vec{r}_2 - \vec{r}_1| + 1)\beta}. \quad (\text{C.47})$$

Therefore, the connected correlation shows an exponential decay with the distance even at large β .

The result of the previous Note, together with the behaviour of the two-point correlation at small β (C.35) allows us to estimate correlation length ξ . As we have seen, ξ is the length-dimension of the largest possible coherent fluctuations in the system, i.e, the extension of the long-range order. Since the connected correlation “always”⁵ shows an exponential decay

$$(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda^{\text{large } \beta} \sim e^{-|\vec{r}_2 - \vec{r}_1| 4\beta}, \quad (\text{C.48})$$

$$(\Gamma_{\vec{r}_1 \vec{r}_2})_\Lambda^{\text{small } \beta} = \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda^{\text{small } \beta} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(1/\beta)},$$

ξ is the inverse of the coefficient multiplying $|\vec{r}_2 - \vec{r}_1|$

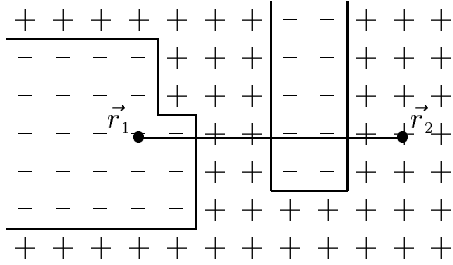
$$\xi = \begin{cases} \frac{1}{4\beta} & \text{at large } \beta \\ -\frac{1}{\ln \beta} & \text{at small } \beta \end{cases} \quad (\text{C.49})$$

Hence, correlation length ξ is small in both ranges.

⁵As we shall soon discuss, this “always” means at any β except at the critical point.

C.2 An illustrative method to estimate correlation functions

Following [22], we shall now introduce an illustrative technique to obtain the exponential decay of equation (C.35) for small β values in a very clear way, which allows us to visualize the origin of this behaviour. As we have seen, large Peierls contours disorder very effectively the spin arrangement but they are suppressed at large β (C.24). Now let us suppose that there are many large Peierls contours inside the system, all of the same length $\omega(\beta)$. The argument we are introducing is not very rigorous because of the fact we are supposing that all contours are large and have the same length ω ; however, the method works since small contours are not able to disorder efficiently the system and thus, they are not relevant to cause the exponential decay in the correlation function. Contour length ω depends on $\beta = J/\eta$, but the exact dependence is of no interest for the argument.



Let us suppose that there are no interactions between contours: this is the case if there are many contours and they are distributed randomly such that their respective interactions cancel each other. The energy increase due to k contours is therefore the k -times product of the one contour energy. To avoid overcounting because the k contours cannot be distinguished, we divide by $k!$. Within this approximation, correlation $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ is given by

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda \approx \frac{1}{\mathcal{Z}'_\Lambda} \sum_{\text{k contours}} \frac{1}{k!} \Sigma_k \zeta^k. \quad (\text{C.50})$$

In these equations we are summing over all possible arrangements of the Peierls contours. Σ_k is the sum over the values of $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2}$ for all possible distributions of k large Peierls contours

$$\Sigma_k = \sum_{\substack{\text{all distributions} \\ \text{of } k \text{ contours}}} (\sigma_{\vec{r}_1} \sigma_{\vec{r}_2})_{\text{each distribution}}. \quad (\text{C.51})$$

In ζ is the energy increase due to have distorted the order with a contour, analogously to the factor $\exp(-2\beta L)$ in equation (C.22) as $\langle \chi_L^j \rangle_\Lambda$ was estimated for the Ising model. The partition function $\mathcal{Z}'_\Lambda$ is equal to $\sum_{\text{k contours}} \frac{1}{k!} |\Lambda|^k \zeta^k$, with $|\Lambda|$ the number of lattice sites.

Let us now consider the number of possible manners for one large contour to surround $\vec{r}_1$ or $\vec{r}_2$ but not both, since only when the line connecting $\vec{r}_1$ and $\vec{r}_2$ is crossed an odd number of times the value of $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2}$ is negative, differing thus the contribution (negative) by such a contour configuration to the numerator of (C.50) from the contribution (positive) by the same configuration to the denominator. For a large contour of perimeter length $\omega(\beta)$, this number of possible sites such that the contour surrounds

only one of the ending points, is proportional to the distance $|\vec{r}_2 - \vec{r}_1|$ multiplied by $\omega(\beta)/4$ (Fig. C.1). The probability for one large contour to surround one of the ending points is given by

$$\gamma \approx \frac{|\vec{r}_2 - \vec{r}_1| \omega(\beta)}{4|\Lambda|a^2}, \quad (\text{C.52})$$

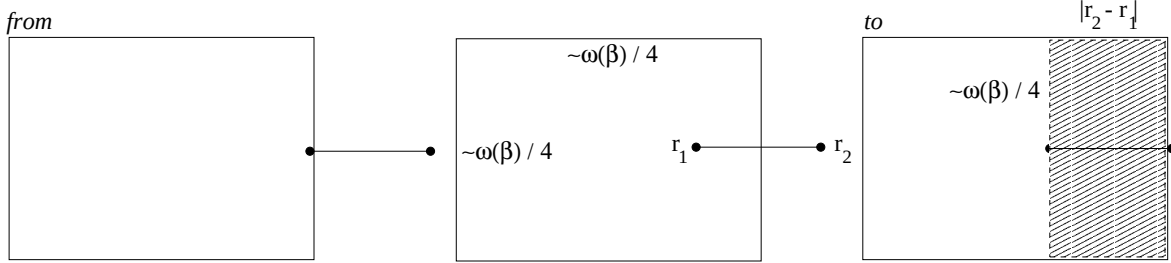
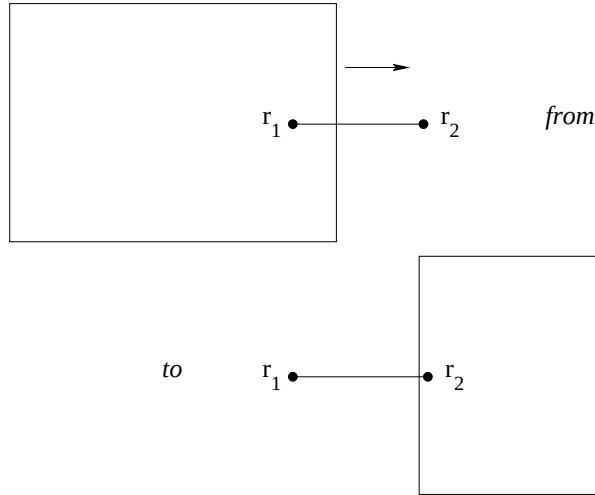


Figure C.1: In a 2-dimensional Ising model, a large contour which surrounds only one of the ending points $\vec{r}_{1,2}$ can be placed at $|\vec{r}_2 - \vec{r}_1| \omega(\beta)/4$ positions.

where $|\Lambda|$ is the total number of lattice sites and a is the space between two contiguous sites, being $|\Lambda|a^2$ the volumen of our 2-dimensional system. This a is needed if we are working with a dimensionful $|\vec{r}_2 - \vec{r}_1|$, otherwise we can take $a = 1$ and denote with $|\vec{r}_2 - \vec{r}_1|$ the number of sites between $\vec{r}_1$ and $\vec{r}_2$. Let us consider, for instance, the contribution of one large contour to the correlation function: the number of places which



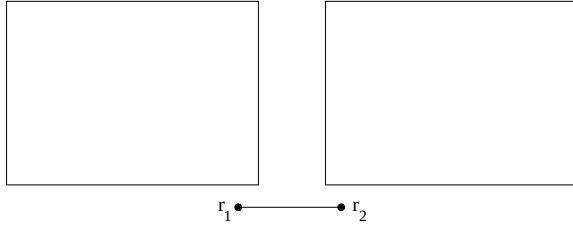
a contour can take avoiding both ending points is equal to $|\Lambda| - |\vec{r}_2 - \vec{r}_1| \omega/4$ and all these distributions produces $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2} = 1$. For the other $|\vec{r}_2 - \vec{r}_1| \omega/4$ sites, the contribution is -1 : the large contour can cover the whole distance $|\vec{r}_2 - \vec{r}_1|$ and simultaneously produce $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2} = -1$, if we take into account that, when already surrounding one end point and when the contour is approaching the second one, the only form to surround only one point is to change its orientation like in the figure.

We obtain thus

$$\Sigma_1 = (|\Lambda| - \frac{|\vec{r}_2 - \vec{r}_1| \omega}{4}) \cdot 1 + \frac{|\vec{r}_2 - \vec{r}_1| \omega}{4} \cdot (-1) = |\Lambda|(1 - 2\gamma).$$

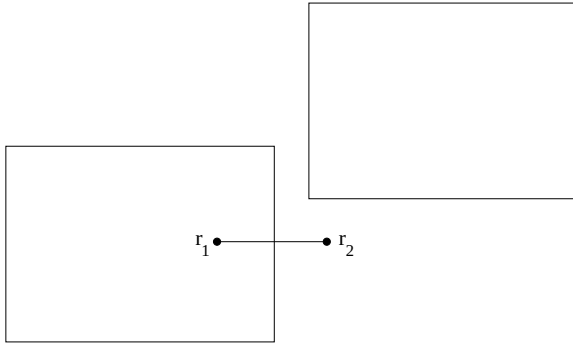
For two large contours to place, there are three different possibilities:

- No contour surrounds any of the end points $\vec{r}_1$ or $\vec{r}_2$. The value of $\sigma_{\vec{r}_1}\sigma_{\vec{r}_2}$ is 1. If the first contour sits at one of the $|\Lambda| - |\vec{r}_2 - \vec{r}_1|\omega/4$ possible sites the second contour has as its service $|\Lambda| - |\vec{r}_2 - \vec{r}_1|\omega/4$ sites, including the same site of the first contour and cancelling thus the effect of the first contour.
- Only one contour crosses the line connecting $\vec{r}_1$ and $\vec{r}_2$. The value of $\sigma_{\vec{r}_1}\sigma_{\vec{r}_2}$ in this case is -1 . If the first contour surrounds one of the ending point ($|\vec{r}_2 - \vec{r}_1|\omega/4$ possible sites) the other can sit on $|\Lambda| - |\vec{r}_2 - \vec{r}_1|\omega/4$ sites. The same if it is the second contour which is surrounding the ending point.
- Both contours surround one of the ending points, being $\sigma_{\vec{r}_1}\sigma_{\vec{r}_2}$ equal to $+1$. In the case where both contours are around the same point, their respective effects are cancelled. The number of sites available to each contour is $|\vec{r}_2 - \vec{r}_1|\omega/4$.



$$\text{probability} = (1 - \gamma)(1 - \gamma),$$

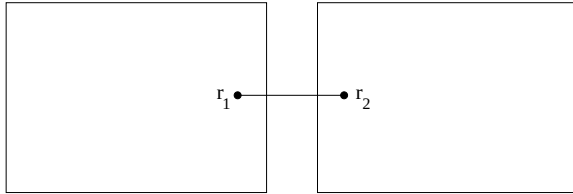
$$\sigma_{\vec{r}_1}\sigma_{\vec{r}_2} = +1.$$



$$\text{probability} = \gamma(1 - \gamma),$$

$$\sigma_{\vec{r}_1}\sigma_{\vec{r}_2} = -1,$$

in two different forms: the intersecting contour can be one of the two contours.

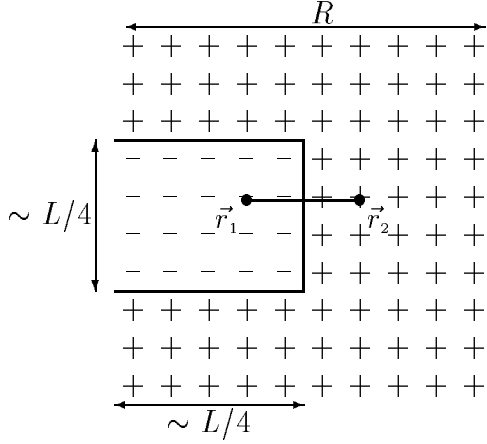


$$\text{probability} = \gamma\gamma,$$

$$\sigma_{\vec{r}_1}\sigma_{\vec{r}_2} = (-1)^2 = +1.$$

Hence for two contours, the contribution to $\langle \sigma_{\vec{r}_1}\sigma_{\vec{r}_2} \rangle_\Lambda$ is the following

$$\Sigma_2 = |\Lambda|^2(1 - 2\gamma)^2,$$



The value $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2}$ becomes negative if a Peierls contour surrounds $\vec{r}_1$ or $\vec{r}_2$ (but not both). Defining the parameter $\bar{\chi}_L^j$ as equal to 1 if in a given spin configuration, the j -th contour of length L appears and surrounds only one of the points $(\vec{r}_1, \vec{r}_2)$, and $\bar{\chi}_L^j = 0$ otherwise, we obtain

$$\langle (\sigma\sigma)_- \rangle_\Lambda = \left\langle \sum_L \sum_{j \text{ contours of length } L} \bar{\chi}_L^j \right\rangle_\Lambda. \quad (\text{C.57})$$

The expectation value $\langle \bar{\chi}_L^j \rangle_\Lambda$ is calculated as in equation (C.23), giving the same result

$$\langle \bar{\chi}_L^j \rangle_\Lambda \leq e^{-2\beta L}. \quad (\text{C.58})$$

For the estimation of the number of closed contours which surrounds one of the points, we must distinguish between large and small contours. As already known, of all $R^2 3^{L-1}$ contours only the fraction $\frac{|\vec{r}_2 - \vec{r}_1| L}{4R^2}$ of them can be placed surrounding only one of the points $\vec{r}_1$ or $\vec{r}_2$ if $\frac{L}{4} > |\vec{r}_2 - \vec{r}_1|$; for $\frac{L}{4} < |\vec{r}_2 - \vec{r}_1|$, the fraction of surrounding contours is approximately equal to $\frac{L^2}{16R^2}$. Hence for any L , the following bound holds

$$\text{number of (closed) contours surrounding either } \vec{r}_1 \text{ or } \vec{r}_2 \leq \frac{|\vec{r}_2 - \vec{r}_1| L}{4R^2}. \quad (\text{C.59})$$

The contribution to (C.57) due only to the largest contours $\frac{L}{4} \sim \frac{R}{2}$ is given by

$$\langle (\sigma\sigma)_- \rangle_\Lambda^{\text{large contours}} \leq \frac{|\vec{r}_2 - \vec{r}_1| L}{4R^2} 3^{L-1} e^{-2\beta L} \sim \frac{1}{3} \frac{|\vec{r}_2 - \vec{r}_1|}{2R} (3e^{-2\beta})^R, \quad (\text{C.60})$$

which at large enough β , $3e^{-2\beta} < 1$, vanishes in the thermodynamic limit $R \rightarrow \infty$. This is the relation expressing the absence of large Peierls contours at $\beta \rightarrow \infty$.

Substituting results (C.58) and (C.59) into (C.57), one obtains for any contour size

$$\langle (\sigma\sigma)_- \rangle_\Lambda \leq \frac{1}{3} \frac{|\vec{r}_2 - \vec{r}_1|}{R} \sum_L \frac{L}{4R} (3e^{-2\beta})^L, \quad (\text{C.61})$$

and therefore

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda \geq 1 - \frac{2}{3} \frac{|\vec{r}_2 - \vec{r}_1|}{R} \sum_L \frac{L}{4R} (3e^{-2\beta})^L, \quad (\text{C.62})$$

where the second summand can be made arbitrarily small for a large enough β and for a infinite large $R \rightarrow \infty$, independently whether $|\vec{r}_2 - \vec{r}_1|$ is small or large

$$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda \rightarrow \infty}^{\text{large } \beta} > 0. \quad (\text{C.63})$$

Thus, at high β the correlation does not vanish and there is a spontaneous magnetization. These two facts are not unrelated, but on the contrary. Since the connected correlation function shows an exponential decay with the distance, at large distances correlation $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ tends toward m_{sp}^2 : at large J/η the two-point correlation does not vanish. Similarly, the vanishing of the correlation function at small J/η is an indication of the zero magnetization at this β -range. Note that the magnetization, which is an expectation value, can vanish although $|\sigma| = 1$ always. The critical point, at which the transition from the phase with nonzero magnetization to that with a vanishing magnetization occurs, will be calculated later (see (C.366)).

Now we consider briefly the second-order phase-transition point (or critical point). At this point, correlation function $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_\Lambda$ (and connected correlation $\Gamma_{\vec{r}_1 \vec{r}_2}$, since at the critical point $\Gamma_{\vec{r}_1 \vec{r}_2} = \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ because the magnetization vanishes) shows a *power decay* with distance $|\vec{r}_2 - \vec{r}_1|$. Let us see how this result is obtained: according to the Widom or scaling hypothesis, correlation length ξ is the only characteristic length near the transition point, and since at a critical point ξ diverges (see (C.17)) the system has thus no characteristic length and is scale invariant [54]. Scale invariance means that if we observe a portion of the system under a microscope such that the portion's picture is so large as the picture of the whole system without magnification, then there is no difference between both pictures. Since $\xi = \infty$ at the critical point, there are fluctuations of all possible sizes and the images obtained if we magnify or if we observe far apart the system are similar. The same stated mathematically: the correlation function has to be an homogeneous function of the distance. Let us suppose we take a look of the system from a distance such that a $s \times s$ cell has the same size as a 1×1 cell observed from a very close distance: under this “distant looking” procedure, the coordinate transforms as $x \rightarrow x' = x/s$ and hence correlation functions such as $\Gamma(r)$ (with $r = |\vec{r}_2 - \vec{r}_1|$) or $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle(r)$ transform *at the critical point* as follows

$$\Gamma\left(\frac{r}{s}\right) = s^p \Gamma(r), \quad (\text{C.64})$$

where, due to the scale invariance, Γ is the *same* function for the distant as well as for the close picture; p is the dimension of Γ and is a positive number since the correlation increases with decreasing separation. However, relation (C.64) cannot be fulfilled by an exponential function but by a power function

$$\Gamma_{\vec{r}_1 \vec{r}_2}^{\text{crit}} = \langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle^{\text{crit}} \sim \frac{1}{|\vec{r}_2 - \vec{r}_1|^p}. \quad (\text{C.65})$$

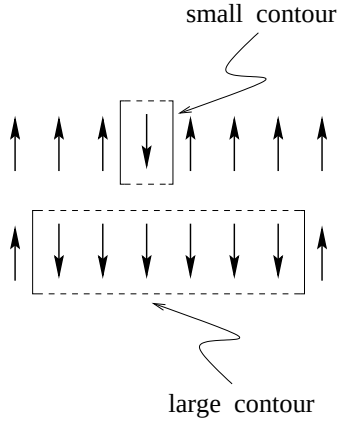
It is worth noting that the above argument only applies to the second order phase transition. A first order phase transition has a finite correlation length even at the transition point and therefore there is no long-range order and the connected correlation function decays exponentially also at the transition point.

Closing this section, let us summarize the phase structure of the 2-dimensional Ising model at $h = 0$. A phase at high β with nonvanishing $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ for large distances; we shall call it the ordered phase. A phase at low β where $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ decays exponentially

with the distance, and which we shall call the disordered phase. Separating both phases, a critical point where $\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ decays like a power of the distance (see Table C.1).

β	$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle$ for $ \vec{r}_2 - \vec{r}_1 \rightarrow \infty$	Peierls contours
high β	$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda \rightarrow \infty} > 0$	small contours
low β	$\langle \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \rangle_{\Lambda \rightarrow \infty} \sim e^{- \vec{r}_2 - \vec{r}_1 \ln(1/\beta)} \rightarrow 0$	large contours

Table C.1: Ising spin correlation for a 2-dimensional lattice.



The above diagram holds true for lattice dimension $d \geq 2$. For the 1-dimensional case, nevertheless, no spontaneous magnetization exists since to produce large Peierls contours costs little energy: at $d = 2$ a large contour as well as a small one costs only the two non parallel spins at both contour borders. Therefore for a nonvanishing β , large contours are possible and there exists no permanent order in the system. As we shall see in the next models, the existence of different phases depends not only on the model parameters like coupling strength J or energy scale η but also on the lattice dimension.

From here on and in order to avoid a complicated notation, the subindex Λ will be not written down, unless the thermodynamic limit is discussed. However, the reader should bear in mind that all the calculations we are carrying out are for a finite lattice.

C.3 The 2-dimensional XY model

The XY or planar model (see [59]) is similar to the Ising model but now the variable $\psi_{\vec{r}}$ on each site $\vec{r}$ has not only two possible values ± 1 but a (compact) continuous range defined by

$$\psi_{\vec{r}} = |\psi| e^{i\varphi_{\vec{r}}} \quad \text{with} \quad -\pi \leq \varphi_{\vec{r}} < \pi \quad \text{and} \quad |\psi| = \text{constant}. \quad (\text{C.66})$$

We can “visualize” this scalar field as an arrow in a 2-dimensional internal symmetry space (different from the ordinary coordinate space), where $\varphi_{\vec{r}}$ denotes the orientation of $\psi_{\vec{r}}$ in this space. Stated in a equation, we represent $\psi_{\vec{r}}$ as $\psi_{\vec{r}} = |\psi| S_{\vec{r}}^{\text{scalar}}$, where $S_{\vec{r}}^{\text{scalar}}$ is an “arrow” pointing in this internal space $S_{\vec{r}}^{\text{scalar}} = e^{i\varphi_{\vec{r}}}$; or equivalently, the arrow can be given by the following vector of two components

$$S_{\vec{r}}^{\text{scalar}} = \begin{pmatrix} \cos \varphi_{\vec{r}} \\ \sin \varphi_{\vec{r}} \end{pmatrix}. \quad (\text{C.67})$$

The energy due to the interaction between nearest neighbours is given by

$$E = |\psi|^2 \sum_{\vec{r}} \sum_{\mu=1,2} \left[1 - S_{\vec{r}}^{\text{scalar}} \cdot S_{\vec{r}+\hat{\mu}}^{\text{scalar}} \right] = |\psi|^2 \sum_{\vec{r},\mu} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})] , \quad (\text{C.68})$$

which can also be written as $\sum_{\vec{r}} \sum_{\mu} \left[|\psi|^2 - \frac{1}{2}(\psi_{\vec{r}+\hat{\mu}} \psi_{\vec{r}}^* + \psi_{\vec{r}+\hat{\mu}}^* \psi_{\vec{r}}) \right]$. The scalar field model presented in Appendix B Fig. B.3, or in Chapter 2 (2.84) when not coupled to the gauge field, is an example of XY model. The partition function describing this system is the following

$$\mathcal{Z} = \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left\{ -\frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu=1,2} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})] \right\} , \quad (\text{C.69})$$

where instead of temperature $k_B T$ as escale to which the energy of a configuration is compared, we have written a general escale η . Now the correlation between two scalar fields separated by a distance $|\vec{r}_2 - \vec{r}_1|$ will be studied

$$\begin{aligned} \langle \psi_{\vec{r}_1} \psi_{\vec{r}_2}^* \rangle &= |\psi|^2 \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \\ &= |\psi|^2 \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \exp \left\{ -\frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu=1,2} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})] \right\} , \end{aligned} \quad (\text{C.70})$$

where it is to be integrated over each $d\varphi_{\vec{r}}$.

As in the Ising model the behaviour of the correlation function for small and large $|\psi|^2/\eta$ will be studied. Let us start with the small $|\psi|^2/\eta$ range by using the “high temperature” power expansion method. “High temperature” has been put between quotation marks because although we are not dealing with the temperature as scale, the method is used when $|\psi|^2/\eta$ (equivalent to $J/k_B T$ in the current formulation of the Ising model) is small. Term $e^{-\frac{|\psi|^2}{\eta}[1-\cos()]}$ is expanded in powers of $\cos()$ and the leading contribution at small $|\psi|^2/\eta$ is obtained by breaking this expansion after the first term

$$e^{-\frac{|\psi|^2}{\eta}[1-\cos()]} \approx 1 - \frac{|\psi|^2}{\eta} [1 - \cos()] .$$

Thus, the correlation becomes

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &\approx \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \times \\ &\times \prod_{\vec{r}} \prod_{\mu=1,2} \left[\left(1 - \frac{|\psi|^2}{\eta} \right) + \frac{|\psi|^2}{2\eta} \left(e^{i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})} + e^{-i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})} \right) \right] . \end{aligned} \quad (\text{C.71})$$

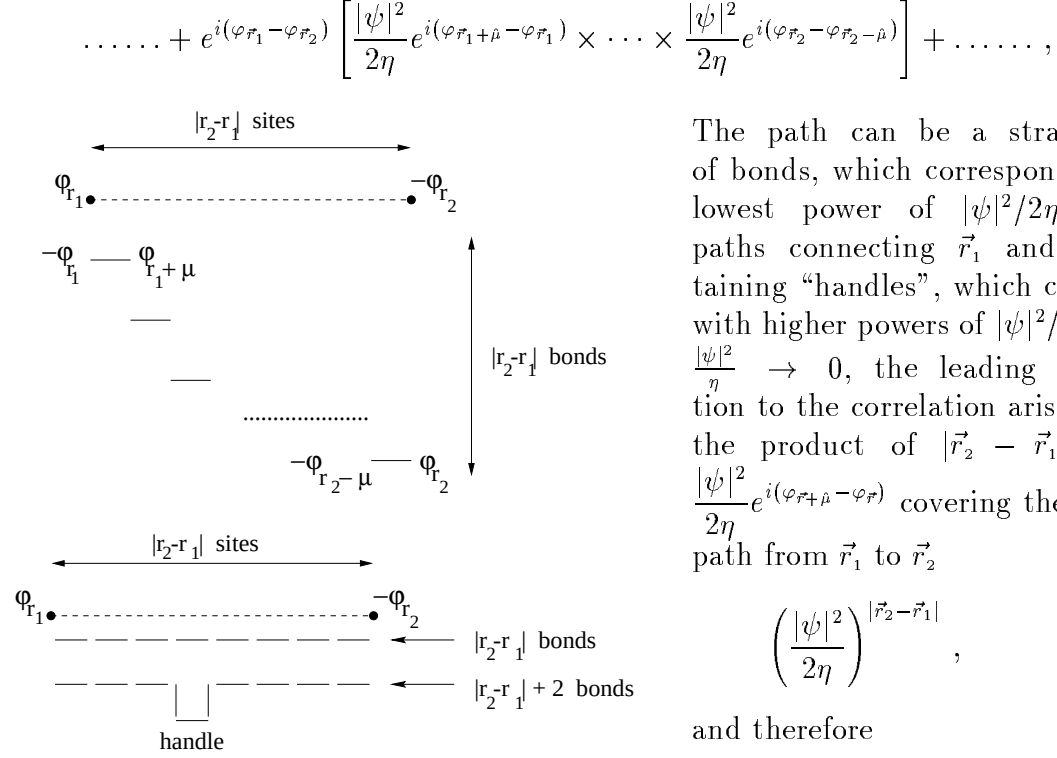
Due to the following relations, similar to (C.33) in the Ising model,

$$\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i\varphi_{\vec{r}}} = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i\varphi_{\vec{r}}} e^{-i\varphi_{\vec{r}}} = 1 , \quad (\text{C.72})$$

the only products contained in

$$e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \prod_{\vec{r}} \prod_{\mu=1,2} \left\{ \left(1 - \frac{|\psi|^2}{\eta} \right) + \frac{|\psi|^2}{2\eta} \left[e^{i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})} + e^{-i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})} \right] \right\},$$

which do not vanish when integrated are those in which the bonds cover continuously a path between $\vec{r}_1$ and $\vec{r}_2$



The path can be a straight line of bonds, which corresponds to the lowest power of $|\psi|^2/2\eta$, or by paths connecting $\vec{r}_1$ and $\vec{r}_2$ containing “handles”, which contribute with higher powers of $|\psi|^2/2\eta$. Since $\frac{|\psi|^2}{\eta} \rightarrow 0$, the leading contribution to the correlation arises from the product of $|\vec{r}_2 - \vec{r}_1|$ factors $\frac{|\psi|^2}{2\eta} e^{i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})}$ covering the straight path from $\vec{r}_1$ to $\vec{r}_2$

$$\left(\frac{|\psi|^2}{2\eta} \right)^{|\vec{r}_2 - \vec{r}_1|}, \quad (\text{C.73})$$

and therefore

$$\left\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \right\rangle_{\text{small } \frac{|\psi|^2}{\eta}} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/|\psi|^2)}. \quad (\text{C.74})$$

The correlation thus obtained by the power expansion method shows an exponential decay as in the Ising model (C.53) for βJ large. Before continuing with the correlation function at small $|\psi|^2/\eta$, we shall re-express the XY model in a form which highlights the role of topological defects.

C.3.1 The Villain transformation in the XY model

For the next discussion the papers [79] and [55] will be followed. Because of the periodicity in the energy (C.68), $\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}}$ and $\varphi_{\vec{r}+\hat{\mu}} + 2\pi - \varphi_{\vec{r}}$ gives the same energy. In our model of Chapter 2 and Appendix B, where the phase $\varphi_{\vec{r}}$ represents the deformation of the atomic cell, this complete rotation of the scalar field means displacement of the center of the atomic cell by a lattice space. The energy periodicity allows the system to have topological defects, as we shall see. For $|\psi|^2/\eta \rightarrow \infty$ the energy in the partition

function peaks those configurations with $\cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}}) \approx 1$. The same effect can be reached by substituting the periodic cosine by the following function

$$e^{-\frac{|\psi|^2}{\eta}[1-\cos(\varphi_{\vec{r}+\hat{\mu}}-\varphi_{\vec{r}})]} \approx \sum_{m_{\mu,\vec{r}}=-\infty}^{m_{\mu,\vec{r}}=+\infty} \exp \left[-\frac{|\psi|^2}{2\eta} (\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right]. \quad (\text{C.75})$$

This approximation is due to Villain and, besides to render the integrals easy to calculate, works rather well even for low $|\psi|^2/\eta$; it does not change the number and characteristics (such as the critical exponents) of the system's phases [59], since these phases are caused by the topological defects and the approximation preserves the energy periodicity, origin of these topological defects

$$\mathcal{Z} = \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \prod_{\vec{r}} \prod_{\mu=1,2} \sum_{m_{\mu,\vec{r}}=-\infty}^{+\infty} \exp \left[-\frac{|\psi|^2}{2\eta} (\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right]. \quad (\text{C.76})$$

The integrand remains constant under the following transformation

$$\begin{cases} \varphi_{\vec{r}} \rightarrow \varphi_{\vec{r}} + 2\pi N_{\vec{r}}, & \varphi_{\vec{r}+\hat{\mu}} \rightarrow \varphi_{\vec{r}+\hat{\mu}} + 2\pi N_{\vec{r}+\hat{\mu}}, \\ m_{\mu,\vec{r}} \rightarrow m_{\mu,\vec{r}} + (N_{\vec{r}+\hat{\mu}} - N_{\vec{r}}), \end{cases} \quad (\text{C.77})$$

with $N_{\vec{r}}$ an integer field. Due to this invariance, since all the $m_{\mu,\vec{r}}$'s are included in the sum of Villain's approximation, we can extend the integration range in $\varphi_{\vec{r}}$ from $[-\pi, +\pi]$ to every interval $[k\pi, (k+1)\pi]$, with k an integer, covering thus the whole interval $(-\infty, +\infty)$. The extension to $(-\infty, +\infty)$ in the φ 's range has been made because the integrals which result are all Gaussian integrals, which are easy to calculate. However, one should be careful with this extension: as φ ranged from $-\pi$ to $+\pi$, an arbitrary orientation difference between two neighbour sites of $\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi$, contributes only once to the partition

$$\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi \quad \text{with} \quad m_{\mu,\vec{r}} = 0, m_{\mu,\vec{r}} = \pm 1, m_{\mu,\vec{r}} = \pm 2, m_{\mu,\vec{r}} = \pm 3, \dots$$

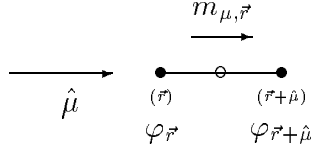
Now with the integration range extended to $(-\infty, +\infty)$, the same contribution is reached an infinite number of times ([79] and [34]):

$$\begin{array}{llllll} \varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi & \text{with} & m_{\mu,\vec{r}} = 0, & m_{\mu,\vec{r}} = 1, & m_{\mu,\vec{r}} = -1, & m_{\mu,\vec{r}} = 2, & \dots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \\ \varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi + 2\pi & \text{with} & m_{\mu,\vec{r}} = 1 & m_{\mu,\vec{r}} = 2 & m_{\mu,\vec{r}} = 0 & m_{\mu,\vec{r}} = 3 & \dots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \\ \varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi - 2\pi & \text{with} & m_{\mu,\vec{r}} = -1 & m_{\mu,\vec{r}} = 0 & m_{\mu,\vec{r}} = 2 & m_{\mu,\vec{r}} = 1 & \dots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \\ \varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} = \xi + 4\pi & \text{with} & m_{\mu,\vec{r}} = 2 & m_{\mu,\vec{r}} = 3 & m_{\mu,\vec{r}} = 1 & m_{\mu,\vec{r}} = 4 & \dots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \dots \end{array}$$

To avoid such a infinite repetition, we have to impose a constraint in the field $m_{\mu,\vec{r}}$ in such a way that each contribution due to every $m_{\mu,\vec{r}}$ only appears once. Such a constraint can be

$$\sum_{\mu} (m_{\mu,\vec{r}+\hat{\mu}} - m_{\mu,\vec{r}}) = 0. \quad (\text{C.78})$$

For the following, the sum over $m_{\mu,\vec{r}}$ should be intended as subjected to a constraint which eliminates this redundancy.



Integer field $m_{\mu,\vec{r}}$ carry the number of complete rotations (in the internal symmetry space) drawn by $\varphi_{\vec{r}}$ in relation to $\varphi_{\vec{r} + \hat{\mu}}$. This integer vector field is defined on lattice bonds (or links) and according to our convention, field $m_{-\mu,\vec{r} + \hat{\mu}}$ which points in the $-\mu$ direction and sits on bond $(\vec{r} + \hat{\mu}, \vec{r})$ is related to $m_{\mu,\vec{r}}$ which points in the contrary direction and sits on the same bond by

$$m_{-\mu,\vec{r} + \hat{\mu}} \equiv -m_{\mu,\vec{r}} ,$$

A good approximation for the partition function is therefore given by

$$\mathcal{Z} = \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \prod_{\vec{r}} \prod_{\mu=1,2} \sum_{m_{\mu,\vec{r}}=-\infty}^{+\infty} \exp \left[-\frac{|\psi|^2}{2\eta} (\varphi_{\vec{r} + \hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right] , \quad (\text{C.79})$$

and for the correlation function

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \prod_{\vec{r}} \prod_{\mu=1,2} \sum_{m_{\mu,\vec{r}}} e^{-\frac{|\psi|^2}{2\eta} (\varphi_{\vec{r} + \hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2} . \quad (\text{C.80})$$

We can write $\prod_{\vec{r},\mu} \sum_{m_{\mu,\vec{r}}}$ in the following form $\sum_{\{m_{\mu,\vec{r}}\}} \prod_{\vec{r},\mu}$ where $\sum_{\{m_{\mu,\vec{r}}\}}$ denotes the summation over all values of $m_{\mu,\vec{r}}$ in all sites of the lattice

$$\sum_{\{m_{\mu,\vec{r}}=-\infty, \dots, +\infty\}} = \underbrace{\sum_{m_{\mu,1}=-\infty, \dots, +\infty} \sum_{m_{\mu,2}=-\infty, \dots, +\infty} \dots \sum_{m_{\mu,N}=-\infty, \dots, +\infty}}_{\text{all the } N = |\Lambda| \text{ sites and all the } \mu \text{ directions}} \quad (\text{C.81})$$

Before we start with the calculation of (C.80), let us transformate it as follows

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r},\mu} \left(\int_{-\infty}^{\infty} db_{\mu,\vec{r}} \sqrt{\frac{\eta}{2\pi|\psi|^2}} \right) \times \\ &\times e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}} (\varphi_{\vec{r} + \hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}}) \right] , \quad (\text{C.82}) \end{aligned}$$

with

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \left(\frac{\eta}{2\pi|\psi|^2} \right)^{\frac{1}{2}} db_{\mu,\vec{r}} \sum_{\{m_{\mu,\vec{r}}\}} \times \\ &\times \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}} (\varphi_{\vec{r} + \hat{\mu}} - \varphi_{\vec{r}} - 2\pi m_{\mu,\vec{r}}) \right] . \quad (\text{C.83}) \end{aligned}$$

This last form leads to (C.80) by a Gaussian integration over all $db_{\mu,\vec{r}}$ (see equations (C.87) and (C.89) of the following Note). In the next discussion of this partition function, we shall integrate all the fields $d\varphi_{\vec{r}}$ and $db_{\mu,\vec{r}}$ out in order to stress the role played

by the topological defects, described by $m_{\mu,\vec{r}}$, in the different system phases.

Note. Gaussian integrals.

Let us begin with a Gaussian integral for one variable

$$Z_1 = \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}, \quad (\text{C.84})$$

$$I_1 = \frac{1}{Z_1} \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = e^{\frac{\beta^2}{4\alpha}} \frac{1}{Z_1} \int_{-\infty}^{\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2} = e^{\frac{\beta^2}{4\alpha}}, \quad (\text{C.85})$$

and

$$\int_{-\infty}^{\infty} dx x e^{-\alpha x^2} = 0, \quad C_1 = \frac{1}{Z_1} \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = 2\alpha^{-1}. \quad (\text{C.86})$$

For the case of N variables

$$Z_N = \prod_{k=1}^{k=N} \int_{-\infty}^{\infty} dx_k e^{-\alpha \sum_{i,j} x_i \tilde{M}_{ij} x_j},$$

with $\tilde{M}_{ij}$ a symmetrical matrix, the solution is similar: let $\mathcal{A}_{ri}$ be the orthogonal matrix which diagonalizes $\tilde{M}_{ij}$, $\sum_{i,j} \mathcal{A}_{ri} \tilde{M}_{ij} \mathcal{A}_{js}^{-1} = \delta_{rs} \lambda_s$ (no summation over s), and with λ_s $s = 1, 2, \dots, N$ the eigenvalues. By transforming the integration variables to $y_r = \sum_i \mathcal{A}_{ri} x_i$, the integral becomes only the product of N Gaussian integrals of one variable (C.84)

$$Z_N = \prod_{k=1}^{k=N} \int_{-\infty}^{\infty} 1 dy_k e^{-\alpha \sum_i y_i^2 \lambda_i} = \left(\sqrt{\frac{\pi}{\alpha}} \right)^N \frac{1}{\sqrt{\prod_{i=1}^{i=N} \lambda_i}} = \left(\sqrt{\frac{\pi}{\alpha}} \right)^N \frac{1}{\sqrt{\det \tilde{M}}}. \quad (\text{C.87})$$

The next integral we have to calculate is

$$I_N = \frac{1}{Z_N} \prod_{k=1}^{k=N} \int_{-\infty}^{\infty} dx^k e^{-\alpha \sum_{i,j} (x_i \tilde{M}_{ij} x_j) + \beta_{i_1} x_{i_1} + \beta_{i_2} x_{i_2}},$$

which transforming to the y_k variables and using $\beta_{i_1} x_{i_1} = \sum_k \beta_{i_1} \mathcal{A}_{i_1 k}^{-1} y_k = \sum_k y_k \mathcal{A}_{k i_1} \beta_{i_1}$, takes the form of N independent integrals of the kind (C.85)

$$I_N = \exp \left[\frac{1}{4\alpha} \sum_k \frac{1}{\lambda_k} (\mathcal{A}_{k i_1} \beta_{i_1} + \mathcal{A}_{k i_2} \beta_{i_2})^2 \right].$$

The inverse matrix of $\tilde{M}_{ij}$ is $\tilde{M}_{ij}^{-1} = \sum_k \mathcal{A}_{ik}^{-1} \frac{1}{\lambda_k} \mathcal{A}_{kj} = \sum_k \frac{1}{\lambda_k} \mathcal{A}_{ki} \mathcal{A}_{kj}$, which leads us to the result

$$\sum_k \frac{1}{\lambda_k} \mathcal{A}_{k i_1} \mathcal{A}_{k i_2} = \tilde{M}_{i_1 i_2}^{-1}, \quad (\text{C.88})$$

and thus

$$I_N = \exp \frac{1}{4\alpha} \begin{pmatrix} \beta_{i_1} & \beta_{i_2} \end{pmatrix} \begin{pmatrix} \tilde{M}_{i_1 i_1}^{-1} & \tilde{M}_{i_1 i_2}^{-1} \\ \tilde{M}_{i_2 i_1}^{-1} & \tilde{M}_{i_2 i_2}^{-1} \end{pmatrix} \begin{pmatrix} \beta_{i_1} \\ \beta_{i_2} \end{pmatrix}. \quad (\text{C.89})$$

Closing this note, let us calculate the following integral

$$C_N = \frac{1}{Z_N} \prod_{k=1}^{k=N} \int_{-\infty}^{\infty} dx_k x_{i_1} x_{i_2} e^{-\alpha \sum_{i,j} x_i \tilde{M}_{ij} x_j}.$$

As in the previous integral, by transforming to the y_k variables and by considering (C.86), we obtain

$$C_N = 2\alpha^{-1} \sum_k \sum_m \mathcal{A}_{i_1 k}^{-1} \mathcal{A}_{i_2 m}^{-1} \delta_{km} \frac{1}{\lambda_m},$$

which together with (C.88) produces

$$C_N = \frac{1}{Z_N} \prod_{k=1}^{k=N} \int_{-\infty}^{\infty} dx_k x_{i_1} x_{i_2} e^{-\alpha \sum_{i,j} x_i \tilde{M}_{ij} x_j} = 2\alpha^{-1} \tilde{M}_{i_1 i_2}^{-1}. \quad (\text{C.90})$$

Now we can integrate over all $d\varphi_{\vec{r}}$ in equation (C.82), which contains terms such as

$$\int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i\varphi_{\vec{r}} \sum_{\mu} (b_{\mu, \vec{r}-\hat{\mu}} - b_{\mu, \vec{r}})}, \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}_1}}{2\pi} e^{i\varphi_{\vec{r}_1} \sum_{\mu} (b_{\mu, \vec{r}_1-\hat{\mu}} - b_{\mu, \vec{r}_1})}.$$

Their integration produces a product of Dirac's deltas

$$\prod_{\vec{r}} \delta \left(\left[\sum_{\mu} b_{\mu, \vec{r}} - b_{\mu, \vec{r}-\hat{\mu}} \right] - \delta_{\vec{r}, \vec{r}_1} + \delta_{\vec{r}, \vec{r}_2} \right),$$

and this means that field $b_{\mu, \vec{r}}$ is constrained to fulfil the following relation

$$\sum_{\mu} [b_{\mu, \vec{r}} - b_{\mu, \vec{r}-\hat{\mu}}] = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}, \quad (\text{C.91})$$

with $\delta_{\vec{r}, \vec{r}_1}$ and $\delta_{\vec{r}, \vec{r}_2}$ Kronecker's deltas. This constraint is the discrete form of differential equation

$$\text{div } \vec{b}(\vec{r}) = \partial_{\mu} b_{\mu}(\vec{r}) = \delta(\vec{r} - \vec{r}_1) - \delta(\vec{r} - \vec{r}_2).$$

From here on, in the continuum differential equations the summation over repeated indices should be intended although not explicitly written down. In order to write the equations in a form which resembles the usual differential calculus, we introduce the discrete derivative Δ_{μ} , defined as follows

$$\Delta_{\mu} f_{\vec{r}} = f_{\vec{r}+\hat{\mu}} - f_{\vec{r}}. \quad (\text{C.92})$$

Equation (C.91) reads now

$$\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r}-\hat{\mu}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}. \quad (\text{C.93})$$

To incorporate this constraint (C.93) into equation (C.82), the field $b_{\mu, \vec{r}}$ will be written in the following form

$$b_{\mu, \vec{r}} = b_{\mu, \vec{r}}^0 + L_{\mu, \vec{r}}, \quad (\text{C.94})$$

where the field $L_{\mu, \vec{r}}$ is an integer vector field directed along a path C from $\vec{r}_1$ to $\vec{r}_2$

$$L_{\mu, \vec{r}} = \begin{cases} +1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{coincides with the direction of } C, \\ -1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{is the opposite to the direction of } C \\ 0 & \text{on all other bonds and for all other directions,} \end{cases} \quad (\text{C.95})$$

$$\sum_{\mu} \Delta_{\mu} L_{\mu, \vec{r}-\hat{\mu}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2} . \quad (\text{C.96})$$

$L_{\mu, \vec{r}}$ is the discrete formulation of a particular solution of the differential equation $\text{div } \vec{L} = \delta(\vec{r} - \vec{r}_1) - \delta(\vec{r} - \vec{r}_2)$, for example, the force line of the electric field beginning at a positive charge at point $\vec{r}_1$ and ending at a negative charge at $\vec{r}_2$; the integer field $L_{\mu, \vec{r}}$ is the tangent unit vector of any path C connecting $\vec{r}_1$ and $\vec{r}_2$. Path C can be chosen freely because as we shall see later, its contribution of the correlation depend only on $\vec{r}_1$ and $\vec{r}_2$ and not on the path self. Field $b_{\mu, \vec{r}}^0$ in (C.94) is the general solution of $\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r}-\hat{\mu}} = 0$

$$b_{\mu, \vec{r}}^0 = \sum_{\nu} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} ,$$

which is the discrete form of the curl in two dimensions and the solution of a vanishing discrete divergence, as it can be easily seen

$$\begin{aligned} b_{1, \vec{r}}^0 &= +a_{\vec{r}} - a_{\vec{r}-\hat{2}} & -b_{1, \vec{r}-\hat{1}}^0 &= -a_{\vec{r}-\hat{1}} + a_{\vec{r}-\hat{1}-\hat{2}} \\ b_{2, \vec{r}}^0 &= -a_{\vec{r}} + a_{\vec{r}-\hat{1}} & -b_{2, \vec{r}-\hat{2}}^0 &= +a_{\vec{r}-\hat{2}} - a_{\vec{r}-\hat{2}-\hat{1}} \end{aligned}$$

$$\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r}-\hat{\mu}}^0 = (b_{1, \vec{r}}^0 - b_{1, \vec{r}-\hat{1}}^0) + (b_{2, \vec{r}}^0 - b_{1, \vec{r}-\hat{2}}^0) = 0 .$$

The same calculations but with $L_{\mu, \vec{r}} = 0$ are to be made for partition function $\mathcal{Z}$ in the denominator of the correlation function. Now we can change the integration over $db_{\mu, \vec{r}}$, which is a constrained field due to (C.93), into the integration over the fields $da_{\vec{r}}$, which are not subjected to any condition

$$\left(\sqrt{\frac{\eta}{2\pi|\psi|^2}} \right)^{2|\Lambda|} \prod_{\vec{r}, \mu=1,2} db_{\vec{r}, \mu} = \left(\sqrt{\frac{\eta}{2\pi|\psi|^2}} \right)^{2|\Lambda|} \prod_{\vec{r}} da_{\vec{r}} = \prod_{\vec{r}} \left(\frac{\eta}{2\pi|\psi|^2} da_{\vec{r}} \right) ,$$

and

$$\sum_{\mu} b_{\mu, \vec{r}}^2 = \sum_{\nu} (\Delta_{\nu} a_{\vec{r}-\hat{\nu}})^2 + 2 \sum_{\mu, \nu} L_{\mu, \vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} + \sum_{\mu} L_{\mu, \vec{r}}^2 , \quad (\text{C.97})$$

The correlation takes now the form

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \int_{-\infty}^{\infty} \left(\frac{\eta}{2\pi|\psi|^2} \right) da_{\vec{r}} \sum_{\{m_{\mu, \vec{r}}\}} \times \\ &\times \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 - i2\pi \sum_{\vec{r}, \mu, \nu} m_{\mu, \vec{r}} (\epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} + L_{\mu, \vec{r}}) \right] , \end{aligned} \quad (\text{C.98})$$

with

$$\mathcal{Z} = \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{\eta da_{\vec{r}}}{2\pi|\psi|^2} \sum_{\{m_{\mu, \vec{r}}\}} \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}, \nu} (\Delta_{\nu} a_{\vec{r}-\hat{\nu}})^2 - i2\pi \sum_{\vec{r}, \mu, \nu} m_{\mu, \vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} \right] . \quad (\text{C.99})$$

Equation (C.98) can be simplified due to the fact that the term $i2\pi \sum_{\mu} L_{\mu,\vec{r}} m_{\mu,\vec{r}}$ does not contribute to the integral, since both $L_{\mu,\vec{r}}$ and $m_{\mu,\vec{r}}$ are integers. The terms in the sum $\sum_{\vec{r},\mu,\nu} L_{\mu,\vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}}$ can be regrouped in the following form

$$\begin{aligned} \sum_{\vec{r},\mu,\nu} L_{\mu,\vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} &= \sum_{\mu,\nu} \left[\cdots + L_{\mu,\vec{r}} \epsilon_{\mu\nu} (\underline{a}_{\vec{r}} - a_{\vec{r}-\hat{\nu}}) + L_{\mu,\vec{r}+\hat{\nu}} \epsilon_{\mu\nu} (a_{\vec{r}+\hat{\nu}} - \underline{a}_{\vec{r}}) + \cdots \right] = \\ &= - \sum_{\vec{r},\mu,\nu} a_{\vec{r}} \epsilon_{\mu\nu} (L_{\mu,\vec{r}+\hat{\nu}} - L_{\mu,\vec{r}}) = - \sum_{\vec{r},\mu,\nu} a_{\vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} L_{\mu,\vec{r}}, \end{aligned} \quad (\text{C.100})$$

which is the discrete version of the integration by parts

$$\int_{-\infty}^{\infty} L_{\mu} \epsilon_{\mu\nu} \partial_{\nu} a = - \int_{-\infty}^{\infty} a \epsilon_{\mu\nu} \partial_{\nu} L_{\mu}.$$

With the same argument, it can be proved: $\sum_{\vec{r},\mu,\nu} m_{\mu,\vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} = - \sum_{\vec{r},\mu,\nu} a_{\vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} m_{\mu,\vec{r}}$ and $\sum_{\vec{r},\nu} (\Delta_{\nu} a_{\vec{r}-\hat{\nu}})^2 = - \sum_{\vec{r},\nu} a_{\vec{r}} \Delta_{\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}}$

$$\sum_{\vec{r},\nu} (a_{\vec{r}} - a_{\vec{r}-\hat{\nu}})^2 = \sum_{\vec{r},\nu} a_{\vec{r}}^2 + \underbrace{\sum_{\vec{r},\nu} a_{\vec{r}-\hat{\nu}}^2}_{\sum_{\vec{r},\nu} a_{\vec{r}}^2} - \sum_{\vec{r},\nu} a_{\vec{r}} a_{\vec{r}-\hat{\nu}} - \underbrace{\sum_{\vec{r},\nu} a_{\vec{r}} a_{\vec{r}-\hat{\nu}}}_{\sum_{\vec{r},\nu} a_{\vec{r}+\hat{\nu}} a_{\vec{r}}} = - \sum_{\vec{r}} a_{\vec{r}} (\Delta_{\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}}). \quad (\text{C.101})$$

Since the term $L_{\mu,\vec{r}}^2$ in (C.97) does not depend on $a_{\vec{r}}$, it can be taken out of the integral in the numerator, obtaining thus

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \exp \left[- \frac{\eta}{2|\psi|^2} \sum_{\vec{r},\mu} L_{\mu,\vec{r}}^2 \right] \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \int_{-\infty}^{\infty} \left(\frac{\eta}{2\pi|\psi|^2} \right) da_{\vec{r}} \times \\ &\times \sum_{\{M_{\vec{r}}\}} \exp \left[- \frac{\eta}{2|\psi|^2} \sum_{\vec{r},\nu} (-a_{\vec{r}} \Delta_{\nu} \Delta_{\nu} a_{\vec{r}}) - \frac{\eta}{|\psi|^2} \sum_{\vec{r},\mu,\nu} a_{\vec{r}} \epsilon_{\nu\mu} \Delta_{\nu} L_{\mu,\vec{r}} - i2\pi \sum_{\vec{r}} a_{\vec{r}} M_{\vec{r}} \right]. \end{aligned} \quad (\text{C.102})$$

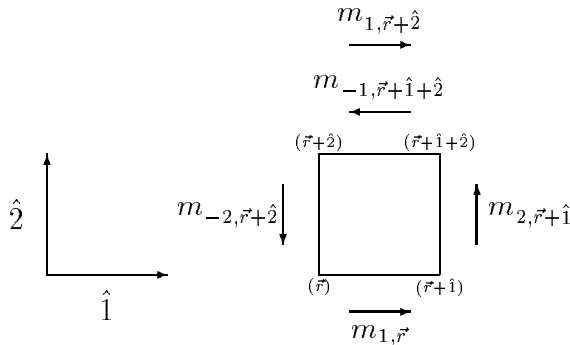
Integer field $M_{\vec{r}}$, ranging from $-\infty$ to $+\infty$, is given by the discrete 2-dimensional curl of $m_{\mu,\vec{r}}$

$$M_{\vec{r}} = \sum_{\nu,\mu} \epsilon_{\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}} = m_{2,\vec{r}+\hat{1}} - m_{2,\vec{r}} - m_{1,\vec{r}+\hat{2}} + m_{1,\vec{r}}, \quad (\text{C.103})$$

where $m_{\mu,\vec{r}}$ was the number of complete rotations made by the phase $\varphi_{\vec{r}}$ when we walk from $\vec{r}$ to $\vec{r} + \hat{\mu}$. According to the definition $m_{-\mu,\vec{r}+\hat{\mu}} \equiv -m_{\mu,\vec{r}}$

$$m_{-1,\vec{r}+\hat{1}+\hat{2}} = -m_{1,\vec{r}+\hat{2}},$$

$$m_{-2,\vec{r}+\hat{2}} = -m_{2,\vec{r}},$$



the sum of the $m_{\mu,\vec{r}}$ along a closed path beginning at $\vec{r}$ and formed by one step in each direction, is just $M_{\vec{r}}$

$$\begin{aligned} m_{1,\vec{r}} + m_{2,\vec{r}+\hat{1}} + m_{-1,\vec{r}+\hat{1}+\hat{2}} + m_{-2,\vec{r}+\hat{2}} &= \\ &= m_{2,\vec{r}+\hat{1}} - m_{2,\vec{r}} - m_{1,\vec{r}+\hat{2}} + m_{1,\vec{r}} = M_{\vec{r}}. \end{aligned}$$

The meaning of $M_{\vec{r}}$ becomes thus clear: it is the number of complete rotations drawn by the scalar field in its internal symmetry space when a unit closed path located at $\vec{r}$ (in the ordinary space) is covered; this is what in Appendix B, Fig. B.3, was defined as the Burgers vector of the dislocation at crystal site $\vec{r}$. Since we are dealing with a 2-dimensional lattice, dislocations are represented by the point where the dislocation line crosses perpendicularly the lattice plane, and therefore $M_{\vec{r}}$ is only a scalar field. Note that correlation function (C.102) depends on the vorticity $M_{\vec{r}}$ and not longer on $m_{\mu,\vec{r}}$. From the definition of the vorticity, there is an infinite number of combination of integers $m_{\mu,\vec{r}}$ which would produce the same $M_{\vec{r}}$ (and the same contribution to the correlation) if the sum over $m_{\mu,\vec{r}}$ had not been constrained by a equation such as (C.78). Without this constraint would not be possible to substitute the sum over $m_{\mu,\vec{r}}$ by the sum over $M_{\vec{r}}$.

After all these manipulations, we can carry out the $a_{\vec{r}}$ Gaussian $a_{\vec{r}}$ integrals in the correlation. In our case the matrix $\tilde{M}$ of equation (C.89) is (minus) the discrete version of the Laplace's operator

$$-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} f_{\vec{r}-\hat{\nu}} = -\sum_{\nu} (f_{\vec{r}+\hat{\nu}} + f_{\vec{r}-\hat{\nu}} - 2f_{\vec{r}}) \equiv -\Delta \Delta f_{\vec{r}}, \quad (\text{C.104})$$

and the inverse matrix is the lattice propagator (or Green's function) $D_{\vec{r},\vec{r}'}$

$$-\Delta_{(\vec{r})} \Delta_{(\vec{r})} D_{\vec{r},\vec{r}'} \equiv -\sum_{\nu} \Delta_{\nu} \Delta_{\nu} D_{\vec{r}-\hat{\nu},\vec{r}'} = \delta_{\vec{r},\vec{r}'}. \quad (\text{C.105})$$

In order to avoid boundary effects, we can use periodic boundary conditions and then the propagator (a real function) is given by

$$D_{\vec{r},\vec{r}'} = \sum_{\vec{k}} e^{i\vec{k} \cdot (\vec{r}-\vec{r}')} \tilde{D}(k). \quad (\text{C.106})$$

Let $\sqrt{|\Lambda|}$ be the lattice length, or $\sqrt{|\Lambda|}a$ with a the lattice spacing, if we prefer to work with dimensionful variables for the lattice sites $\vec{r}$. The sum over $\vec{k}$ in (C.106) ranges from $|\vec{k}| = 2\pi/\sqrt{|\Lambda|}$ to $|\vec{k}| = \pi$ (with length dimensions, from $2\pi/\sqrt{|\Lambda|}a$ to π/a , as in a crystal). If we now take the limit of a infinite lattice, $|\Lambda|a^2 \rightarrow \infty$, the periodic conditions are not longer necessary; moreover, we can approximate the sum by an integral

$$D_{\vec{r},\vec{r}'} \approx \left(\int_{-\frac{\pi}{a}}^{-\frac{2\pi}{\sqrt{|\Lambda|}a}} + \int_{\frac{2\pi}{\sqrt{|\Lambda|}a}}^{\frac{\pi}{a}} \right) \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{i\vec{k} \cdot (\vec{r}-\vec{r}')} \tilde{D}(k) \approx \int_{-\pi/a}^{\pi/a} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{i\vec{k} \cdot (\vec{r}-\vec{r}')} \tilde{D}(k).$$

The solution of $\tilde{D}(k)$ is thus directly obtained

$$-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} D_{\vec{r}-\hat{\nu},\vec{r}'} = -\int_{-\pi/a}^{\pi/a} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{i\vec{k} \cdot (\vec{r}-\vec{r}')} \tilde{D}(k) (2 \cos(ak_x) + 2 \cos(ak_y) - 4) = \delta_{\vec{r},\vec{r}'},$$

$$D_{\vec{r},\vec{r}'} = \int_{-\pi/a}^{\pi/a} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{e^{i\vec{k} \cdot (\vec{r}-\vec{r}')}}{(4 - 2 \cos(ak_x) - 2 \cos(ak_y))/a^2}. \quad (\text{C.107})$$

Let us consider in detail this propagator: for the case $\vec{r} = \vec{r}'$ we obtain

$$D_{\vec{r},\vec{r}} \equiv D(0) = 4 \int_0^{\pi/a} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{a^2}{4 - 2 \cos(ak_x) - 2 \cos(ak_y)},$$

which is logarithmically divergent in the range $k_x, k_y \rightarrow 0$, i.e. for those contributions due to the infinite lattice size $\sqrt{|\Lambda|}a$,

$$\frac{dk_x dk_y}{k_x^2 + k_y^2} = \frac{2\pi k dk}{k^2} = 2\pi d(\ln k) \Rightarrow D(0) = \dots - 2\pi \lim_{k=\frac{2\pi}{\sqrt{|\Lambda|}} \rightarrow 0} \ln k \sim \ln \sqrt{|\Lambda|}, \quad (\text{C.108})$$

where the finite part of $D(0)$ is contained in $\dots$. This divergence is the origin of the divergence in the dislocation energy discussed in Chapter 1 (1.7). For $|\vec{r} - \vec{r}'|$ large enough, the discreteness of the lattice is not relevant and the finite part of the propagator $D'_{\vec{r},\vec{r}'} = D_{\vec{r},\vec{r}'} - D(0)$ can be well approximated [59] by the continuum 2-dimensional propagator⁶

$$D'_{\vec{r},\vec{r}'} = \int_{-\pi/a}^{\pi/a} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} - 1}{((4 - 2 \cos(ak_x) - 2 \cos(ak_y)) / a^2)} \approx -\frac{1}{2\pi} \ln \frac{|\vec{r} - \vec{r}'|}{a} - \frac{1}{4}, \quad (\text{C.109})$$

where a is the lattice space; we can also work further with a dimensionless $\vec{r}$ and set a equal to 1. This Green's function for $|\vec{r} - \vec{r}'| \gg 1$ works quite good even for much smaller distances [59], and for the following it will be taken as propagator for distances larger than one lattice space. Note that $D'(0)$ is equal to 0.

Carrying out the Gaussian integrals

$$\begin{aligned} & \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} \frac{\eta}{2\pi |\psi|^2} da_{\vec{r}} \right) \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}} a_{\vec{r}} (-\Delta \Delta) a_{\vec{r}} - i \sum_{\vec{r}} a_{\vec{r}} ()_{\vec{r}} \right] \\ & \longrightarrow \left(\frac{\eta}{2\pi |\psi|^2} \right)^{|\Lambda|} \left(\sqrt{\frac{2\pi |\psi|^2}{\eta}} \right)^{|\Lambda|} \frac{1}{\sqrt{\det(-\Delta \Delta)}} \exp \left[-\frac{|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}'} ()_{\vec{r}} D_{\vec{r}, \vec{r}'} ()_{\vec{r}'} \right], \end{aligned}$$

⁶An easy deduction of the logarithm in the 2-dim. continuum propagator $D(\vec{r} - \vec{r}') = D(|\vec{r} - \vec{r}'|)$ is achieved by considering the following representation of Dirac's delta $\delta(x - x') = \frac{d}{dx} \vartheta(x - x')$ as the derivative of Heaviside function $\vartheta(x - x') = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{e^{ik(x-x')}}{k - i\epsilon}$. By means of Cauchy's theorem applied to $\frac{1}{2\pi i} \oint_C d\zeta \frac{e^{i\zeta(x-x')}}{\zeta - i\epsilon}$ with the integration path C over the positive (upper) semiplane for $(x - x') > 0$ or over the lower semiplane for $(x - x') < 0$, it is shown that $\vartheta(x - x')$ is equal to 1, if $(x - x') > 0$, or to 0 if $(x - x') < 0$. The Dirac's delta in two dimensions is $\delta(x)\delta(y) = \frac{1}{2\pi r} \delta(r)$, where the factor $2\pi r$ comes from $dx dy = 2\pi r dr$. By applying the discrete 2-dimensional Laplace's operator to $D(r)$, we obtain

$$\sum_{\mu=1,2} \partial_{\mu}^2 D(r) = \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} D(r) = -\frac{1}{2\pi r} \delta(r) = -\frac{1}{2\pi r} \frac{d}{dr} \vartheta(r) \Rightarrow \frac{d}{dr} D(r) = -\frac{1}{2\pi r}.$$

in (C.102), correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ takes the form

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \frac{1}{\mathcal{Z}} \exp \left[-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 \right] \mathcal{Z}_0 \sum_{\{M_{\vec{r}}\}} \times \\ &\times \exp \left[-\frac{|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}'} \left(2\pi M_{\vec{r}} - i \frac{\eta}{|\psi|^2} \sum_{\mu, \nu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^\nu L_{\mu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(2\pi M_{\vec{r}'} - i \frac{\eta}{|\psi|^2} \sum_{\mu', \nu'} \epsilon_{\nu'\mu'} \Delta_{(\vec{r}')}^{\nu'} L_{\mu', \vec{r}'} \right) \right] \end{aligned} \quad (\text{C.110})$$

where according to (C.87) $\mathcal{Z}_0$ is equal to $\left(\sqrt{\eta/2\pi|\psi|^2} \right)^{|\Lambda|} \left(\sqrt{\det(-\Delta\Delta)} \right)^{-1}$, with $|\Lambda|$ the total number of lattice sites and $\det(-\Delta\Delta)$ the determinant of (minus) the discrete Laplace's operator. Such $\mathcal{Z}_0$ can be also written as

$$\mathcal{Z}_0 = \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left[-\frac{|\psi|^2}{2\eta} \sum_{\vec{r}} \varphi_{\vec{r}} \left(-\sum_{\mu} \Delta_{\mu} \Delta_{\mu} \varphi_{\vec{r}-\hat{\mu}} \right) \right]. \quad (\text{C.111})$$

Comparing this equation with (C.79), we see that $\mathcal{Z}_0$ contains the effects of gradients $\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}}$ which do not exceed 2π and thus have all the $m_{\mu, \vec{r}}$ equal to zero. In the language of the model of Chapter 2, such gradients represents compatible deformations. Determinant $\det(-\Delta\Delta)$ can be expressed in a more precise way by working with the Fourier components $\tilde{\varphi}(\vec{k})$

$$\begin{aligned} \mathcal{Z}_0 &= \prod_{\vec{k}} \left(\int_{-\infty}^{\infty} d\tilde{\varphi}(\vec{k}) \right) \exp \left[-\frac{2\pi^2|\psi|^2}{\eta} \sum_{\vec{k}} \tilde{\varphi}(\vec{k})^2 \sum_{\mu=1,2} \left(-e^{ik_{\mu}} - e^{-ik_{\mu}} + 2 \right) \right] = \\ &= \left(\sqrt{\frac{\eta}{2\pi|\psi|^2}} \right)^{|\Lambda|} \left(\sqrt{\prod_{\vec{k}} (4 - 2 \sum_{\mu} \cos k_{\mu})} \right)^{-1} = \left(\sqrt{\frac{\eta}{2\pi|\psi|^2}} \right)^{|\Lambda|} \left(\sqrt{\det(-\Delta\Delta)} \right)^{-1}. \end{aligned} \quad (\text{C.112})$$

Further, it is important that partition function $\mathcal{Z}_0$ describes only one phase: if we calculate $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ by means only of $\mathcal{Z}_0$, we obtain using (C.89) and (C.109)

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_0 &= \frac{1}{\mathcal{Z}_0} \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \right) e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \exp \left[-\frac{|\psi|^2}{2\eta} \sum_{\vec{r}} \varphi_{\vec{r}} \left(-\sum_{\mu} \Delta_{\mu} \Delta_{\mu} \varphi_{\vec{r}-\hat{\mu}} \right) \right] = \\ &= \exp \left[\frac{\eta}{2|\psi|^2} (-D_{\vec{r}_1, \vec{r}_1} - D_{\vec{r}_2, \vec{r}_2} + 2D_{\vec{r}_1, \vec{r}_2}) \right] = e^{\frac{\eta}{|\psi|^2} D'_{\vec{r}_1, \vec{r}_2}} = e^{-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1|}, \end{aligned} \quad (\text{C.113})$$

where we have neglected the $1/4$ in $D'_{\vec{r}_1, \vec{r}_2}$ compared to $\ln |\vec{r}_2 - \vec{r}_1|$ for large distances. This result differs from the decay obtained in (C.74), what could be already expected since that result holds for small $|\psi|^2/\eta$, whereas the Villain approximation without any $m_{\mu, \vec{r}}$ holds only for very large $|\psi|^2/\eta$, as we shall see when we discuss the different correlation behaviours.

Note that the Villain approximation separates the contributions arising from compatible deformations from those produced by dislocations, which are included in the exponential factor of the following equation

$$\mathcal{Z} = \mathcal{Z}_0 \sum_{\{M_{\vec{r}} = -\infty, \dots, +\infty\}} \exp \left[-\frac{2\pi^2|\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} M_{\vec{r}} D_{\vec{r}, \vec{r}'} M_{\vec{r}'} \right]. \quad (\text{C.114})$$

The sum in the exponent can be separated in two parts (note that $D'_{\vec{r},\vec{r}} = 0$)

$$\sum_{\vec{r},\vec{r}'} M_{\vec{r}} D'_{\vec{r},\vec{r}'} M_{\vec{r}'} = D(0) \left(\sum_{\vec{r}} M_{\vec{r}} \right) \left(\sum_{\vec{r}'} M_{\vec{r}'} \right) + \sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} D'_{\vec{r},\vec{r}'} M_{\vec{r}'} . \quad (\text{C.115})$$

The first part contains, among other terms, the self-energy $2\pi^2|\psi|^2 M_{\vec{r}}^2 D(0)$ of each dislocations, which according to (C.108) diverges logarithmically with the lattice size, analogously to (1.7). Due to this divergence, the partition function by means of (C.115) singles out those configurations with zero vorticity

$$\sum_{\vec{r}} M_{\vec{r}} = 0 . \quad (\text{C.116})$$

After all the transformations up to this point, we have obtained a $\mathcal{Z}$ which is the partition function of a dislocation ensemble with an energy given by the interaction of the dislocation at $\vec{r}$ with the dislocation at $\vec{r}'$, interaction carried by the propagator $D'_{\vec{r},\vec{r}'}$, and subjected to the constraint that the total Burgers vector must be zero. Moreover, since $D'_{\vec{r},\vec{r}'}$ is negative, dislocations of opposite vorticity or Burgers vector attract each other (we have already obtained this result in the discussion following equation (1.12) in Chapter 1).

Noting that $\sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} M_{\vec{r}'} = -\sum_{\vec{r}} M_{\vec{r}}^2$, due to the zero vorticity condition $(\sum_{\vec{r}} M_{\vec{r}})^2 = 0$, and using (C.109), the partition function takes the following form

$$\mathcal{Z} \approx \mathcal{Z}_0 \sum_{\{M_{\vec{r}}\}}' \exp \left[-\frac{\pi^2|\psi|^2}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 + \frac{\pi|\psi|^2}{\eta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right] , \quad (\text{C.117})$$

where $\sum_{\{M_{\vec{r}}\}}'$ means sum over dislocation configurations with zero total Burgers vector.

In order to obtain a final result for correlation function (C.110), in its numerator we have to calculate:

$$\bullet \sum_{\vec{r},\mu,\nu} \epsilon_{\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} \right) D'_{\vec{r},\vec{r}'} , \quad (\text{C.118})$$

$$\bullet \sum_{\vec{r},\mu} L_{\mu,\vec{r}}^2 - \sum_{\vec{r},\mu,\nu} \sum_{\vec{r}',\mu',\nu'} \epsilon_{\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} \right) D'_{\vec{r},\vec{r}'} \epsilon_{\nu'\mu'} \Delta_{(\vec{r}')}^{\nu'} L_{\mu',\vec{r}'} . \quad (\text{C.119})$$

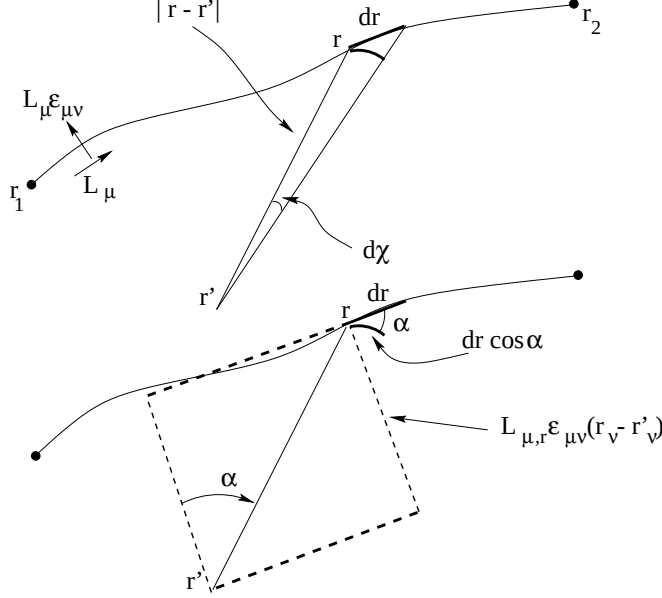
Let us begin with the first sum (C.118): this term occurs in the numerator as part of

$$\sum_{\vec{r},\vec{r}',\mu,\nu} \epsilon_{\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} \right) D'_{\vec{r},\vec{r}'} M_{\vec{r}'} = \sum_{\vec{r},\vec{r}' \neq \vec{r},\mu,\nu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} D'_{\vec{r},\vec{r}'} M_{\vec{r}'} + D(0) \left(\sum_{\vec{r},\mu,\nu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} \right) \left(\sum_{\vec{r}'} M_{\vec{r}'} \right) ,$$

where the second term vanishes due to the zero vorticity condition (C.116). If the distances $|\vec{r}_1 - \vec{r}'|$ and $|\vec{r}_2 - \vec{r}'|$ are large compared to the lattice spacing, we can approximate the discrete sum by an integral and the propagator $D'_{\vec{r},\vec{r}'}$ by (C.109)

$$\sum_{\vec{r},\mu,\nu} \epsilon_{\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu,\vec{r}} \right) D'_{\vec{r},\vec{r}'} = - \sum_{\vec{r},\mu,\nu} L_{\mu,\vec{r}} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} D'_{\vec{r}-\hat{\nu},\vec{r}'} \approx -\frac{1}{2\pi} \int_C dr L_{\mu} \epsilon_{\mu\nu} \partial_{\nu} \ln |\vec{r} - \vec{r}'| , \quad (\text{C.120})$$

where the integral is along the path C , connecting $\vec{r}_1$ and $\vec{r}_2$, defined by the field $L_{\mu,\vec{r}}$ of equation (C.96). Let us now consider the angle $\Xi_{\vec{r}'}$ under which the path C is seen from the point $\vec{r}'$: this parallax $\Xi_{\vec{r}'}$ is the sum of all parallaxes $\delta\chi_{\vec{r}}$ of each small segment of the path C between $\vec{r}$ and $\vec{r} + L_{\mu,\vec{r}}$



$$\delta\chi_{\vec{r}} = \frac{\cos \alpha_{\vec{r}}}{|\vec{r} - \vec{r}'|},$$

or in the continuum formulation

$$d\chi = \frac{\cos \alpha}{|\vec{r} - \vec{r}'|} dr.$$

Angle α for the point $\vec{r}'$ is given by

$$\cos \alpha_{\vec{r}} = \frac{\sum_{\mu,\nu} L_{\mu,\vec{r}} \epsilon_{\mu\nu} (r_\nu - r'_\nu)}{|\vec{r} - \vec{r}'|},$$

since, being $L_{\mu,\vec{r}}$ the direction of the path C at point $\vec{r}$, the direction perpendicular to it is equal to $\sum_{\mu} L_{\mu,\vec{r}} \epsilon_{\mu\nu}$, the projection of $|\vec{r} - \vec{r}'|$ on this direction becomes $\sum_{\mu,\nu} L_{\mu,\vec{r}} \epsilon_{\mu\nu} (r_\nu - r'_\nu)$.

Taking into account that for the 2-dimensional propagator the following relation holds

$$\partial_\nu \left(-\frac{1}{2\pi} \ln |\vec{r} - \vec{r}'| \right) = -\frac{1}{2\pi} \frac{r_\nu - r'_\nu}{|\vec{r} - \vec{r}'|^2},$$

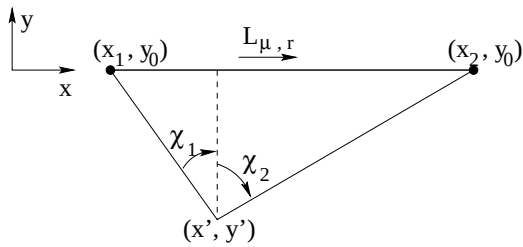
and using (C.120), the total parallax $\Xi_{\vec{r}'}$ becomes

$$\Xi_{\vec{r}'} = \sum_{\vec{r},\mu,\nu} \frac{L_{\mu,\vec{r}} \epsilon_{\mu\nu} (r_\nu - r'_\nu)}{|\vec{r} - \vec{r}'|^2} = -2\pi \sum_{\vec{r},\mu,\nu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^\nu L_{\mu,\vec{r}} D'_{\vec{r},\vec{r}'} . \quad (\text{C.121})$$

As an example of equation (C.121), let us suppose $L_{\mu,\vec{r}} = 1$ along an horizontal ($\mu = 1$) straight path between (x_1, y_0) and (x_2, y_0) : in this case (C.118) is equal to

$$-\frac{1}{2\pi} \int_{x=x_1}^{x=x_2} dx \frac{y_0 - y}{(x - x')^2 + (y_0 - y')^2} = -\frac{1}{2\pi} \left[\arctan \left(\frac{x - x'}{y_0 - y'} \right) \right]_{x=x_1}^{x=x_2} = -\frac{1}{2\pi} (\chi_2 + \chi_1),$$

which leads again to (C.121)



$$\sum_{\vec{r},\mu,\nu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^\nu L_{\mu,\vec{r}} D'_{\vec{r},\vec{r}'} = -\frac{1}{2\pi} \Xi_{\vec{r}'} .$$

From the site $\vec{r}'$, the path from $\vec{r}_1$ to $\vec{r}_2$ is observed under the angle

$$\Xi_{\vec{r}'} = \left[\arctan \left(\frac{x - x'}{y_0 - y'} \right) \right]_{x=x_1}^{x=x_2} . \quad (\text{C.122})$$

The parallax angle is positive if it is clockwise when measured from $\vec{r}_1$ to $\vec{r}_2$. Starting just below the path and going far away from it ($y' \rightarrow -\infty$), the angle Ξ changes from π to 0. On the other hand, if we approach the path from above and go from infinitely distant, $y' \rightarrow \infty$, up to just above the path, Ξ changes from 0 to $-\pi$ (since $y' > y_0$ and we count the angle clockwise).

For the second sum (C.119), we approximate again with the continuum notation. Let us start rewriting its second term by means of (C.100)

$$\begin{aligned} \sum_{\vec{r}, \mu, \nu} \sum_{\vec{r}', \mu', \nu'} \epsilon_{\nu\mu} \left(\frac{\Delta}{(\vec{r})} \right) L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} \epsilon_{\nu'\mu'} \frac{\Delta}{(\vec{r}')} L_{\mu', \vec{r}'} &= \sum_{\vec{r}, \vec{r}'} \sum_{\mu, \nu} L_{\mu, \vec{r}} \epsilon_{\mu\nu} \sum_{\mu', \nu'} L_{\mu', \vec{r}'} \epsilon_{\mu'\nu'} \frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r}')} D_{\vec{r}-\hat{\nu}, \vec{r}'-\hat{\nu}'} \approx \\ &\approx \int d^2 \vec{r} \int d^2 \vec{r}' L_{\mu}(\vec{r}) \epsilon_{\mu\nu} L_{\mu'}(\vec{r}') \epsilon_{\mu'\nu'} \frac{\partial}{\partial_{(\vec{r})}} \frac{\partial}{\partial_{(\vec{r}')}} D(|\vec{r}-\vec{r}'|) = \int \int d\vec{r} d\vec{r}' \frac{\partial}{\partial_{\perp}} \frac{\partial}{\partial_{\perp}} D, \end{aligned} \quad (\text{C.123})$$

with summation over repeated indices. The formulation in the continuum notation is only in order not to get too embarrassed with the index $\vec{r}-\hat{\nu}, \vec{r}'-\hat{\nu}'$ in the propagator, but at the end we shall come back to the discrete notation. Derivative $\frac{\partial}{\partial_{(\vec{r})}} = \sum_{\mu, \nu} L_{\mu}(\vec{r}) \epsilon_{\mu\nu} \partial_{\nu}$ is the derivative perpendicular to the direction of the path C at point $\vec{r}$. For the sake of clarity, in the following $\frac{\partial}{\partial_{(\vec{r})}}$ will be abbreviated by ∂' . Similarly, we define the derivative parallel to the path at $\vec{r}$ as $\partial_{\parallel} = \sum_{\mu} L_{\mu}(\vec{r}) \partial_{\mu}$. Since we are working in a 2-dimensional space, at each point $\vec{r}$ there is only one direction perpendicular to $L_{\mu}(\vec{r})$ (and only one parallel). Let us consider different possibilities for the orientations of L_{μ} at $\vec{r}$ and $\vec{r}'$:

- (i) Both $L_{\mu}(\vec{r})$ and $L_{\mu'}(\vec{r}')$ point in the same $\mu = \mu' = 1$ direction: $\partial_{\perp} = \partial_2$, $\partial_{\parallel} = \partial_1$ and $\partial'_{\perp} = \partial'_2$, $\partial'_{\parallel} = \partial'_1$. Since the Green's function only depends on $|\vec{r}-\vec{r}'|$, it holds

$$\partial_{\mu} \partial'_{\mu'} D(|\vec{r}-\vec{r}'|) = -\partial_{\mu} \partial_{\mu'} D(|\vec{r}-\vec{r}'|),$$

and therefore $\partial_{\perp} \partial'_{\perp} D = -\partial_2 \partial_2 D$ and $\partial_{\parallel} \partial'_{\parallel} D = -\partial_1 \partial_1 D$.

- (ii) $L_{\mu}(\vec{r})$ points in the $\mu = 1$ direction and $L_{\mu'}(\vec{r}')$ in the $\mu' = 2$ direction. In this case, $\partial_{\perp} \partial'_{\perp} D = \partial_2 \partial_1 D$ and $\partial_{\parallel} \partial'_{\parallel} D = -\partial_1 \partial_2 D$.

Summarizing these results,

$$\partial_{\perp} \partial'_{\perp} D + \partial_{\parallel} \partial'_{\parallel} D = - \sum_{\mu, \mu'} L_{\mu}(\vec{r}) L_{\mu'}(\vec{r}') \delta_{\mu\mu'} \underbrace{\left[\sum_{\rho} \partial_{\rho} \partial_{\rho} D \right]}_{-\delta(\vec{r}-\vec{r}')}, \quad (\text{C.124})$$

where on the last term it was used the equation defining the Green's function (C.105). Returning to the formulation with discrete derivatives and using the definition of parallel derivative $\partial_{\parallel} = \sum_{\mu} L_{\mu}(\vec{r}) \partial_{\mu}$, we obtain from (C.123) the following equation

$$\begin{aligned} \sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 - \sum_{\vec{r}, \mu, \nu} \sum_{\vec{r}', \mu', \nu'} \epsilon_{\nu\mu} \left(\frac{\Delta}{(\vec{r})} \right) L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} \epsilon_{\nu'\mu'} \frac{\Delta}{(\vec{r}')} L_{\mu', \vec{r}'} &= \\ = \sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 - \sum_{\vec{r}, \vec{r}'} \sum_{\mu, \mu'} L_{\mu, \vec{r}} L_{\mu', \vec{r}'} \delta_{\mu\mu'} \delta_{\vec{r}, \vec{r}'} + &\underbrace{\sum_{\vec{r}, \vec{r}'} \sum_{\mu, \mu'} L_{\mu, \vec{r}} L_{\mu', \vec{r}'} \frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r}')} D_{\vec{r}-\hat{\mu}, \vec{r}'-\hat{\mu}'}}_{\text{C.125}} \\ + \sum_{\vec{r}, \vec{r}'} \left(\sum_{\mu} \frac{\Delta}{(\vec{r})} L_{\mu, \vec{r}} \right) \left(\sum_{\mu'} \frac{\Delta}{(\vec{r}')} L_{\mu', \vec{r}'} \right) D_{\vec{r}, \vec{r}'} & \end{aligned}$$

where the last equality was achieved by regrouping the terms like in (C.100). Taking into account the definition of the $L_{\mu,\vec{r}}$ (C.96), the sum (C.125) becomes

$$\begin{aligned} & \sum_{\vec{r},\mu} L_{\mu,\vec{r}}^2 - \sum_{\vec{r},\mu,\nu} \sum_{\vec{r}',\mu',\nu'} \epsilon_{\nu\mu} \Delta_{\vec{r}}^{\nu} L_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} \epsilon_{\nu'\mu'} \Delta_{\vec{r}'}^{\nu'} L_{\mu',\vec{r}'} = \\ & = \sum_{\vec{r},\vec{r}'} (\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}) D_{\vec{r},\vec{r}'} (\delta_{\vec{r}',\vec{r}_1} - \delta_{\vec{r}',\vec{r}_2}) = -2[D_{\vec{r}_1,\vec{r}_2} - D(0)] \approx \frac{1}{\pi} \ln |\vec{r}_1 - \vec{r}_2|, \end{aligned} \quad (\text{C.126})$$

for large distances $|\vec{r}_1 - \vec{r}_2|$. Finally we are in the position of writing in a compact form correlation function (C.110) for the whole range of $|\psi|^2/\eta$, better approximated for large $|\psi|^2/\eta$ than for small values of $|\psi|^2/\eta$

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \exp \left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right) \times \\ & \times \frac{\mathcal{Z}_0 \sum'_{\{M_{\vec{r}}\}} \exp \left[-i \sum_{\vec{r}} M_{\vec{r}} \Xi_{\vec{r}} - \frac{\pi^2 |\psi|^2}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 + \frac{\pi |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right]}{\mathcal{Z}_0 \sum'_{\{M_{\vec{r}}\}} \exp \left[-\frac{\pi^2 |\psi|^2}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 + \frac{\pi |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right]}. \end{aligned} \quad (\text{C.127})$$

Remember that $\Xi_{\vec{r}}$ is the angle under which the dislocation sitting at $\vec{r}$ “sees” the path $L_{\mu,\vec{r}}$ connecting $\vec{r}_1$ and $\vec{r}_2$. This angle will play a very important role in the next discussion. The first factor in the correlation is just the result obtained for a partition function $\mathcal{Z}_0$ without dislocations (C.113). Equation (C.127) can be also written as

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \exp \left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right) \left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M, \quad (\text{C.128})$$

where the expectation value $\langle \rangle_M$ is calculated by means of the following partition function

$$\mathcal{Z}_M = \sum'_{\{M_{\vec{r}}\}} \exp \left[-\frac{\pi^2 |\psi|^2}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 + \frac{\pi |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right]. \quad (\text{C.129})$$

Note. About the topological fields (I) and the Villain approximation.

There is another form [81] (see also [86]) to obtain the previous partition function (C.129), which is more elegant although perhaps not so intuitive as the above transformation. Let us come back to the original partition function

$$\mathcal{Z} = \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left\{ -\frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu} [1 - \cos \Delta_{\mu} \varphi_{\vec{r}}] \right\}. \quad (\text{C.130})$$

For large $|\psi|^2/\eta$, we can approximate the cosine in the energy by its quadratic term and allow the φ integration range from $-\infty$ to ∞ and thus express the energy in the continuum formulation

$$E[\varphi] = \frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu} [1 - \cos \Delta_{\mu} \varphi_{\vec{r}}] \approx \frac{|\psi|^2}{2\eta} \int d^2 \vec{r} \left(\vec{\nabla} \varphi \right)^2. \quad (\text{C.131})$$

But φ is really an angular variable, and therefore jumps of $2\pi k$ (with k an integer) at some points are allowed: this discontinuity still corresponds to a smooth configuration of φ at most of the lattice sites. According to (C.131), the main contribution to the partition arises from the minimum of $E[\varphi]$, which is obtained by solving

$$\nabla^2 \varphi = 0 \pmod{2\pi}, \quad (\text{C.132})$$

where $\pmod{2\pi}$ takes account of $2\pi k$ jumps. Solution φ_{cl} of this equation is

$$\varphi_{\text{cl}}(x, y) = \sum_{\vec{r}_m} M_{\vec{r}_m} \text{Im} \ln(z - z_m), \quad (\text{C.133})$$

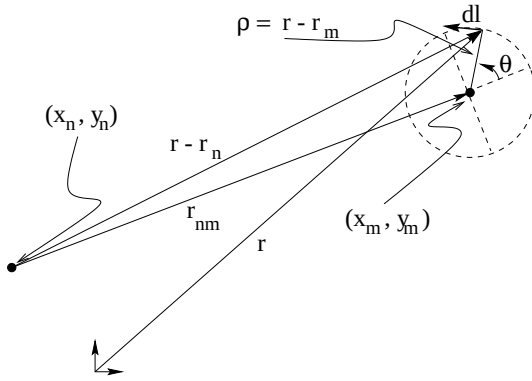
where $\{M_{\vec{r}_m}\}$ is a set of integers, $z = x + iy$ and $\{z_m\}$ are arbitrary sites. This solution can be easily checked by noting

$$\partial_\mu \varphi_{\text{cl}} = -\epsilon_{\mu\nu} \sum_{\vec{r}_m} M_{\vec{r}_m} \frac{(\vec{r} - \vec{r}_m)_\nu}{|\vec{r} - \vec{r}_m|^2}, \quad (\text{C.134})$$

with $\mu, \nu = 1, 2$, and $\epsilon_{12} = -\epsilon_{21} = 1$ and zero otherwise. Due to the antisymmetry of $\epsilon_{\mu\nu}$, it follows that $\nabla^2 \varphi_{\text{cl}} = 0$. However, the integral of $\vec{\nabla} \varphi_{\text{cl}}$ along a path, which encloses only one of the sites $\{\vec{r}_m = (x_m, y_m)\}$

$$\oint_{\text{around } z_m} \partial_\mu \varphi_{\text{cl}} dl^\mu = M_m \int_{\theta=0}^{\theta=2\pi} \frac{\epsilon_{\mu\nu} (\vec{r} - \vec{r}_m)_\mu dl^\nu}{|\vec{r} - \vec{r}_m|^2} + \sum_{\vec{r}_n \neq \vec{r}_m} M_{\vec{r}_n} \int_{\theta=0}^{\theta=2\pi} \frac{\epsilon_{\mu\nu} (\vec{r} - \vec{r}_n)_\mu dl^\nu}{|\vec{r} - \vec{r}_n|^2}, \quad (\text{C.135})$$

does not vanish, as one can easily show using the following partial results



$$\epsilon_{\mu\nu} (\vec{r} - \vec{r}_n)_\mu dl^\nu = (\rho r_{nm} \cos \theta + \rho^2) d\theta,$$

$$|\vec{r} - \vec{r}_n|^2 = \rho^2 + r_{nm}^2 + 2\rho r_{nm} \cos \theta,$$

and since the integral contour only surrounds $M_{\vec{r}_m}$, or equivalently, $\rho < r_{nm} \forall n \neq m$

$$\int_{\theta=0}^{\theta=2\pi} \frac{(\rho r_{nm} \cos \theta + \rho^2) d\theta}{\rho^2 + r_{nm}^2 + 2\rho r_{nm} \cos \theta} = \begin{cases} 2\pi & r_{nm} = 0 \\ 0 & r_{nm} \neq 0 \end{cases}.$$

Substituting these results into the integral (C.135), we obtain

$$\oint_{\text{around } z_m} \partial_\mu \varphi_{\text{cl}} dl^\mu = 2\pi M_m. \quad (\text{C.136})$$

Solution φ_{cl} corresponds to a set of vortices or dislocations located at sites $\vec{r}_m$ with a vorticity given by $M_{\vec{r}_m}$. At large $|\psi|^2/\eta$, the most probable φ configurations are given by $\varphi = \varphi_{\text{cl}} + \gamma$, where γ is only a small perturbation about φ_{cl} , without topological content. Substituting into $E[\varphi]$, it gives

$$E[\varphi] \approx E[\varphi_{\text{cl}}] + \int d^2\vec{r} \underbrace{\frac{\delta E[\varphi]}{\delta \varphi(\vec{r})}}_0 \gamma(\vec{r}) + \frac{1}{2} \int d^2\vec{r} d^2\vec{r}' \underbrace{\frac{\delta^2 E[\varphi]}{\delta \varphi(\vec{r}) \delta \varphi(\vec{r}')}}_{-\frac{|\psi|^2}{\eta} \nabla^2 \delta^2(\vec{r} - \vec{r}')} \gamma(\vec{r}) \gamma(\vec{r}'),$$

and therefore partition function (C.130) is approximated as follows

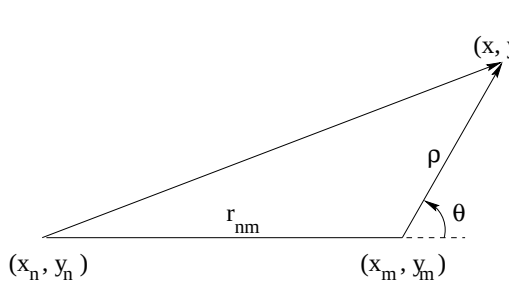
$$\mathcal{Z} \approx e^{-E[\varphi_{\text{cl}}]} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\gamma_{\vec{r}}}{2\pi} \exp \left[-\frac{|\psi|^2}{2\eta} \sum_{\vec{r}} \gamma_{\vec{r}} \left(-\sum_{\rho} \Delta_{\rho} \Delta_{\rho} \right) \gamma_{\vec{r}} \right], \quad E[\varphi_{\text{cl}}] = \frac{|\psi|^2}{2\eta} \int d^2\vec{r} \left(\vec{\nabla} \varphi_{\text{cl}} \right)^2, \quad (\text{C.137})$$

where the integral over $d\gamma_{\vec{r}}$ (with the integrand written again in the discrete formulation) is just partition $\mathcal{Z}_0$ due to compatible deformations (C.111). We need only to calculate $E[\varphi_{cl}]$

$$E[\varphi_{cl}] = \frac{|\psi|^2}{2\eta} \sum_{\vec{r}_m} \sum_{\vec{r}_n} M_{\vec{r}_m} M_{\vec{r}_n} \int d^2\vec{r} \frac{(x-x_m)(x-x_n) + (y-y_m)(y-y_n)}{[(x-x_m)^2 + (y-y_m)^2][(x-x_n)^2 + (y-y_n)^2]}.$$

In this integral, we firstly fix $\vec{r}_m$ and sum over all $\vec{r}_n$, choosing for each $\vec{r}_n$ the coordinates of the figure

$$\sum_{\vec{r}_n} M_{\vec{r}_n} \int d^2\vec{r} \frac{(x-x_m)(x-x_n) + (y-y_m)(y-y_n)}{[(x-x_m)^2 + (y-y_m)^2][(x-x_n)^2 + (y-y_n)^2]} = M_{\vec{r}_m} \int_c^{\sqrt{|\Lambda|}} d\rho \frac{2\pi}{\rho} +$$

$$+ \sum_{\vec{r}_n \neq \vec{r}_m} M_{\vec{r}_n} \int_0^{\sqrt{|\Lambda|}} d\rho \int_0^{2\pi} d\theta \frac{\rho + r_{nm} \cos \theta}{\rho^2 + r_{nm}^2 + 2\rho r_{nm} \cos \theta},$$


where $\sqrt{|\Lambda|}$ is the length of the whole lattice. In the first integral, we have had to introduce a cut-off c to avoid the divergence of the ρ integral at very short distances, where the continuum approximation (C.131) is not valid. The angular integral of the second summand is zero for $\rho < r_{nm}$ and $2\pi/\rho$ for $\rho > r_{nm}$. Hence, $E[\varphi_{cl}]$ takes the following form

$$E[\varphi_{cl}] = \pi \frac{|\psi|^2}{\eta} \sum_{\vec{r}_m, \vec{r}_n} M_{\vec{r}_m} M_{\vec{r}_n} [\ln |\Lambda| - \ln r_{nm}], \quad (\text{C.138})$$

where r_{nm} must be substituted by the cut-off c . We recover again the logarithmical divergence with the lattice size (C.115)

$$\sum_{\vec{r}_m} \sum_{\vec{r}_n} M_{\vec{r}_m} M_{\vec{r}_n} \ln |\Lambda| = \left(\sum_{\vec{r}_m} M_{\vec{r}_m} \right)^2 \ln |\Lambda|$$

which makes the vortices form dipoles

$$\sum_{\vec{r}_m} M_{\vec{r}_m} = 0. \quad (\text{C.139})$$

Finally, we obtain

$$E[\varphi_{cl}] = -\pi \frac{|\psi|^2}{\eta} \sum_{\vec{r}_m} M_{\vec{r}_m}^2 \ln c - \pi \frac{|\psi|^2}{\eta} \sum_{\vec{r}_m, \vec{r}_n \neq \vec{r}_m} M_{\vec{r}_m} M_{\vec{r}_n} \ln r_{nm}, \quad (\text{C.140})$$

which together with (C.137) is the already known partition in the Villain approximation. The advantage of this formulation is that the meaning of $M_{\vec{r}}$ field becomes clear from the very beginning (C.136) and it is not obscured by the transformations of the Villain formulation.

C.3.2 Different behaviours of correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$

In equation (C.74) we have already seen the exponential decay with the distance $|\vec{r}_2 - \vec{r}_1|$ in the correlation function for small $|\psi|^2/\eta$. Now, let us consider the correlation for large values of $|\psi|^2/\eta$. Due to the zero vorticity condition (C.116), there are so many positive dislocation points as negative ones. Two dislocation points of opposite Burgers vectors are bound by the logarithmic potential given by the propagator. In three dimensions this corresponds to a dislocation ring, but in a 2-dimensional space we only “see” the two points where the dislocation ring crosses the plane. At large $|\psi|^2/\eta$, the term $\pi^2 |\psi|^2 / 2\eta \sum_{\vec{r}} M_{\vec{r}}^2$ in partition function (C.129) suppresses many dislocations, and due to the term $\pi |\psi|^2 / \eta \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'}$, the few existing dislocations have to build small pairs (or dipoles), where the dislocation points are near to each other. For the correlation function at large $|\psi|^2/\eta$, we can take as first approximation that there is no dislocation at all $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } \frac{|\psi|^2}{\eta}} \approx \exp \left(-\frac{\eta}{2\pi |\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right)$. We obtain thus that $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ decays toward zero for $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$ both at small *and* at large $|\psi|^2/\eta$

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } \frac{|\psi|^2}{\eta}} = \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)^{\frac{\eta}{2\pi |\psi|^2}}, \quad (\text{C.141})$$

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{small } \frac{|\psi|^2}{\eta}} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/|\psi|^2)}. \quad (\text{C.142})$$

A second-order phase-transition point (as in the 2-dimensional Ising model), is characterized by the fact that correlation function behaves as $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \xrightarrow{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \text{constant} \neq 0$ in the phase at $|\psi|^2/\eta > (|\psi|^2/\eta)_{\text{critical}}$, whilst the correlation only decays like a power of the distance at exactly the critical point $|\psi|^2/\eta = (|\psi|^2/\eta)_{\text{critical}}$ (see (C.65)). In the present case, nevertheless, the correlation function shows the power decay for the complete range $|\psi|^2/\eta > (|\psi|^2/\eta)_{\text{critical}}$, not only at a point. Such a phase change, which as the second-order one is a continuous phase transition, is called the Kosterlitz-Thouless phase transition [62]⁷. Although the transition point, which separates the two phases characterized by the different correlation’s behaviours (C.141) and (C.142), is not a second-order phase-transition point, we shall also call it a critical point for the following, since the system shows scale invariance (the system looks the same at all scales) at this point.

The mechanism of the Kosterlitz-Thouless transition can be explained by considering (C.128). The energy periodicity in the XY model allows for dislocations which can disorder $\varphi_{\vec{r}}$ the directions of the scalar field $\psi_{\vec{r}}$, specially if these dislocations are isolated. When a dislocation crosses the line connecting $\vec{r}_1$ with $\vec{r}_2$, phase difference $\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2}$ changes by π and therefore $\psi_{\vec{r}_1} \psi_{\vec{r}_2}^*$ becomes the negative value $-|\psi|^2$, or equivalently $S_{\vec{r}_1}^{\text{scalar}} \cdot S_{\vec{r}_2}^{\text{scalar}} = -1$ (Fig. C.2). This fact is completely analogous to the crossing by a Peierls contour of the line between $\sigma_{\vec{r}_1}$ and $\sigma_{\vec{r}_2}$ in the Ising model; this crossing produces $\sigma_{\vec{r}_1} \sigma_{\vec{r}_2} = -1$.

⁷The rigorous proof of the existence of the Kosterlitz-Thouless transition can be found in the paper by J. Fröhlich and T. Spencer, *Comm. Math. Phys.* **81**, 527 (1981). For our discussion, nevertheless, it will suffice to make plausible the form in which this transition takes place.

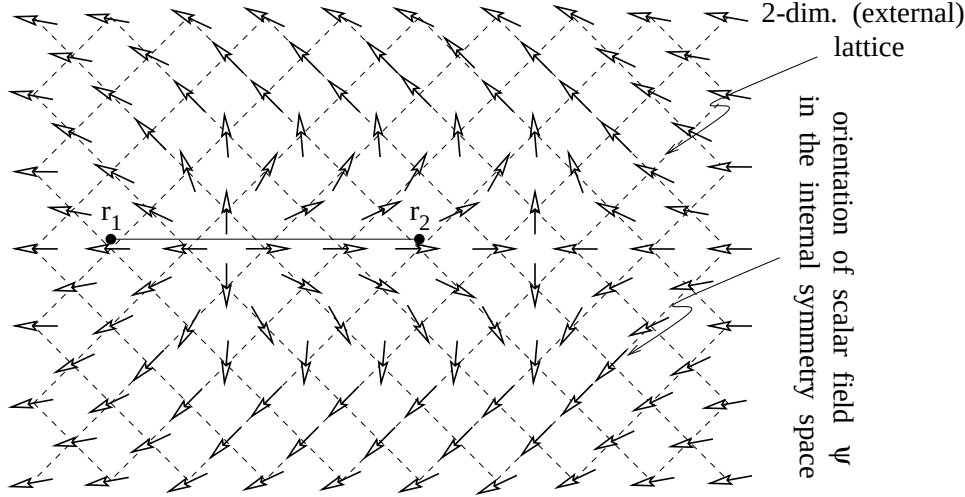
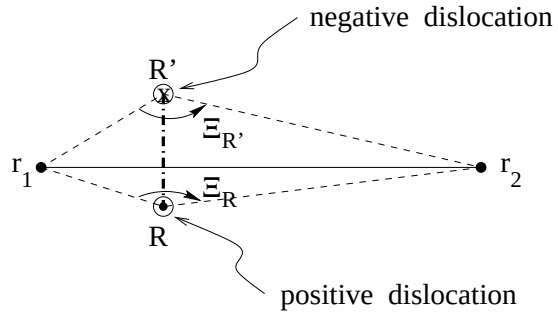


Figure C.2: Dislocation pair and its representation by means of “arrow” $S_{\vec{r}}^{\text{scalar}}$, with scalar field $\psi_{\vec{r}} = |\psi| S_{\vec{r}}^{\text{scalar}}$.

The negative contribution in the correlation’s numerator differs from the contributions in the partition function (in the denominator), which are always positive. If these negative contributions are very numerous, they make the correlation change its behaviour (as seen in the Ising model with the effect due to large Peierls contours), but this can only occur in the case of many large dislocation pairs, which are possible only at small $|\psi|^2/\eta$. Small dislocation pairs, on the other hand, produce few negative contributions to (C.128) as we can see from the following examples, where for the sake of clarity we have assumed that the path defined by $L_{\mu, \vec{r}}$ is a straight horizontal line.

1. Pair $R - R'$ is across the path connecting $\vec{r}_1$ and $\vec{r}_2$. The short distance ρ between the two dislocations forming the pair leads to the fact that the parallax at point $\vec{R}$ (where for example the positive dislocation sits) is approximately π ; at the point of the negative dislocation $\vec{R}'$, the parallax is $-\pi$. The dislocation configuration containing this pair at this position, $M_{\vec{R}} = 1$ and $M_{\vec{R}'} = -1$, produces the following contribution to the correlation’s numerator (C.128)



$$(\cdots) \exp(i\Xi_{\vec{R}} - i\Xi_{\vec{R}'}) \exp \left[-\frac{\pi^2 |\psi|^2}{2\eta} \times 2 - \frac{\pi |\psi|^2}{\eta} \times 2 \ln |\vec{R} - \vec{R}'| \right],$$

where $(\cdots)$ contains the contributions by other dislocations at other sites. We see that such a contribution is approximately the same as the contribution in the

denominator arising from the partition function

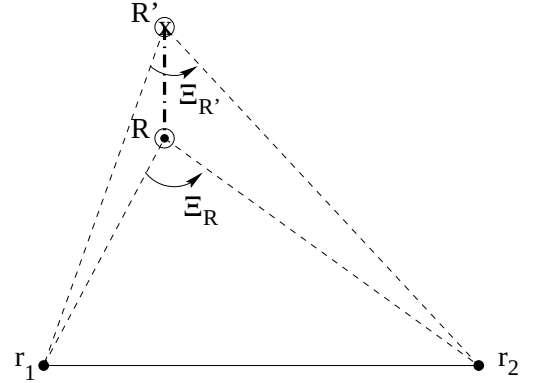
$$\frac{\dots\dots + (\dots)e^{\overbrace{(i\Xi_{\vec{R}} - i\Xi_{\vec{R}'})}^{\approx i\pi - i(-\pi)}} \exp\left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln|\vec{R} - \vec{R}'|\right] + \dots\dots}{\dots\dots + (\dots)\exp\left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln|\vec{R} - \vec{R}'|\right] + \dots\dots}, \quad (\text{C.143})$$

and therefore it causes no change in the correlation's behaviour.

2. Pair $R - R'$ is distant from the path connecting both ending points. In this case both parallaxes are small and approximately the same

$$e^{i\Xi_{\vec{R}}M_{\vec{R}} + i\Xi_{\vec{R}'}M_{\vec{R}'}} \approx e^{1 \cdot \Xi + (-1) \cdot (\Xi)} = 1,$$

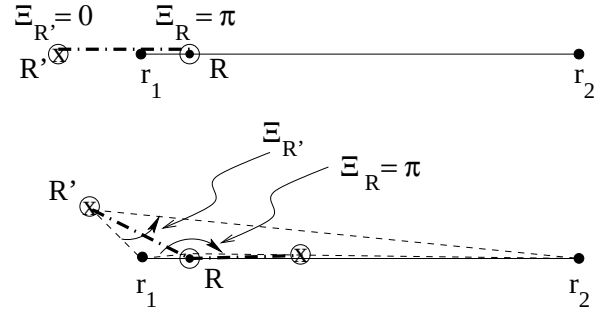
producing the same contribution in both numerator and denominator. The same holds for other orientations of the pair as long as it is distant.



3. A more interesting case is when the pair is aligned with the path and one of the dislocation rests on the path (with $\Xi_{\vec{R}} = \pi$) and the other outside the path. This pair produces a change in the contribution to the correlation's numerator since

$$e^{i\Xi_{\vec{R}}M_{\vec{R}} + i\Xi_{\vec{R}'}M_{\vec{R}'}} = -1,$$

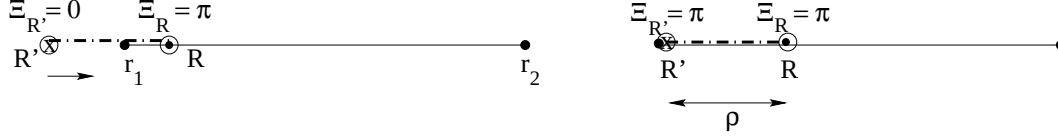
whereas the same dislocation configuration contributes with 1 to the partition function in the denominator



$$\frac{\dots\dots + (\dots)(-1) \exp\left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln|\vec{R} - \vec{R}'|\right] + \dots\dots}{\dots\dots + (\dots)\exp\left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln|\vec{R} - \vec{R}'|\right] + \dots\dots}. \quad (\text{C.144})$$

If the pair is rotated about $\vec{R}$, the value of $\Xi_{\vec{R}'}$ changes from 0 to $-\pi$ (when it rests on the path like the other dislocation $M_{\vec{R}}$), and further from π just below the path to 0 again. Therefore $\Xi_{\vec{R}}M_{\vec{R}} + \Xi_{\vec{R}'}M_{\vec{R}'}$ changes from π to 2π above the path, and continuing with the rotation, from 0 to π . The contribution of all these

possible orientations to the numerator is zero, whereas the one to the denominator is nonvanishing. This effect, however, can only be maintained for the ρ positions of the dislocation resting on the path, since at the moment when the dislocation which was outside the path comes into the path, both $\Xi_{\vec{R}}$ and $\Xi_{\vec{R}'}$ are equal to $|\pi|$. Due to the large value $|\psi|^2/\eta$, we are dealing with small pairs and the distance ρ between the dislocations which form the pair is much smaller than $|\vec{r}_2 - \vec{r}_1|$: the discussed contribution in the numerator has no dependence on $|\vec{r}_2 - \vec{r}_1|$.



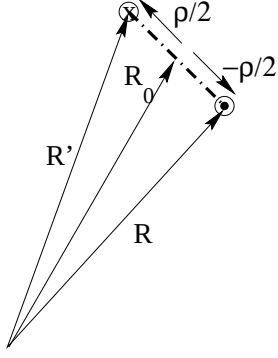
Thus these few 0 contributions are not able to spoil the $\exp(-\eta/2\pi|\psi|^2 \ln |\vec{r}_2 - \vec{r}_1|)$ behaviour from the correlation function. Note that if the pair had been much larger, so that we could move the pair through the whole path without resting both dislocations on the path ($\rho > |\vec{r}_2 - \vec{r}_1|$), we would have obtained $|\vec{r}_2 - \vec{r}_1|$ contributions of -1 in correlation's numerator (C.128): these $|\vec{r}_2 - \vec{r}_1|$ contributions cover from the situation where the first dislocation is resting in one of the ending points up to the moment where the first dislocation goes out of the path and the second has not yet come into ⁸. This large amount of contributions to the numerator, different from the terms occurring in the denominator, causes the change in the behaviour of the correlation function.

Although small dislocation dipoles do not change the correlation's behaviour from $\exp(-\ln |\vec{r}_2 - \vec{r}_1|)$ to another form of decay, they have an effect on the factor multiplying the logarithm in the exponential. Therefore, before we study large dislocation pairs, let us firstly consider the effect of small pairs formed by $M_{\vec{R}}$ and $M_{\vec{R}'} = -M_{\vec{R}} = \pm 1$. We are not considering dislocations with Burgers vector larger than 1 since these, for large $|\psi|^2/\eta$, are strongly suppressed by the term $\pi^2|\psi|^2/2\eta \sum_{\vec{r}} M_{\vec{r}}^2$ in the partition function (such a dislocation has an energy four times higher). Distance ρ between the two dislocations will be assumed to be much smaller than the distances from each dislocation to the ending points $\vec{r}_1$ and $\vec{r}_2$. To make clearer the calculation we shall use the following notation:

$$\vec{R}' = \vec{R}_0 + \frac{1}{2}\vec{\rho}, \quad \vec{R} = \vec{R}_0 - \frac{1}{2}\vec{\rho},$$

⁸At the moment where both dislocations are resting on path, we have only positive contributions to the numerator. Even if we had rotate the pair around one of the dislocations, the parallax of the other dislocation would not become considerably different from $-\pi$, due to the smallness of ρ ; its contribution to the numerator would differ thus scarcely from the contribution in the denominator

$$\frac{\dots\dots + (\dots)(+1) \exp \left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln |\vec{R} - \vec{R}'| \right] + \dots\dots}{\dots\dots + (\dots) \exp \left[-\frac{\pi^2|\psi|^2}{\eta} - \frac{2\pi|\psi|^2}{\eta} \ln |\vec{R} - \vec{R}'| \right] + \dots\dots}.$$



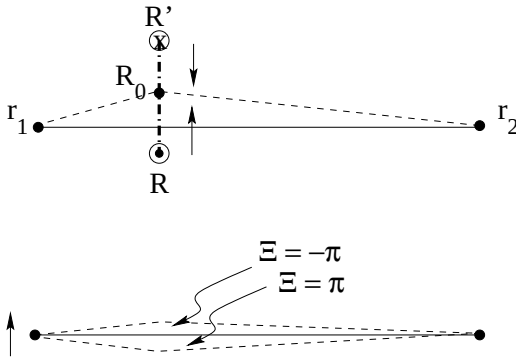
where $\vec{R}_0$ is the position of the pair's center. The sum in the partition function over all dislocation positions $\vec{R}$ and $\vec{R}'$ can be written as the sum over all positions $\vec{R}_0$ followed by the sum over $\vec{\rho}$. Partition function for these small dislocation pairs becomes

$$\mathcal{Z}_{\text{small pairs}} = 1 + \sum_{\vec{R}_0} \sum_{\vec{\rho}} \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right]. \quad (\text{C.145})$$

For the numerator of equation (C.128) we obtain

$$\begin{aligned} \sum_{\{M_{\vec{r}}\}}' \exp \left[i \sum_{\vec{r}} M_{\vec{r}} \Xi_{\vec{r}} - \frac{\pi^2 |\psi|^2}{\eta} \sum_{\vec{r}} M_{\vec{r}}^2 + \frac{\pi |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right] = \\ = 1 + \sum_{\vec{R}_0} \frac{1}{2} \sum_{\vec{\rho}} 2 \cos(\Xi_{\vec{R}} - \Xi_{\vec{R}'}) \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right]. \end{aligned} \quad (\text{C.146})$$

The cosine term arises from $\exp[i \sum_{\vec{r}} M_{\vec{r}} \Xi_{\vec{r}}]$ when summed over all possible positions which can be taken by the two pair's dislocations, once sitting the positive dislocation at $\vec{R}$ (and the negative at $\vec{R}'$) and once with $\vec{R}'$ for the positive dislocation (and thus $\vec{\rho}$ pointing in the opposite direction). Parallax difference $(\Xi_{\vec{R}} - \Xi_{\vec{R}'})$ has different values depending on the example we have considered before: it is close to 2π if the pair crosses the path between $\vec{r}_1$ and $\vec{r}_2$, approximately zero if the pair is far away from the path or close to $\pm\pi$ for the case one of the dislocations is resting on the path. For the first case we can use the approximation $\cos(\Xi_{\vec{R}} - \Xi_{\vec{R}'}) \approx 1 - \frac{1}{2}(\Xi_{\vec{R}} - \Xi_{\vec{R}'} - 2\pi)^2$; similar approximations have to be used for the other cases. Let us further consider this case where the pair lies across the path:



in order to express the previous relation in terms of $\Xi_{\vec{R}_0}$ we have to be careful, since as we go from $\vec{R}$, below the path, to the point $\vec{R}_0$, $\Xi_{\vec{R}}$ makes a jump of 2π (Ξ is close to $-\pi$ just above the path, and equal to π just below). In this case $\Xi_{\vec{R}}$ in relation to $\Xi_{\vec{R}_0}$ is given by

$$\Xi_{\vec{R}} \approx (2\pi + \Xi_{\vec{R}_0}) - \frac{1}{2} \sum_{\mu} \rho_{\mu(\vec{R}_0)}^{\Delta_{\mu}} \Xi_{\vec{R}_0}, \quad (\text{C.147})$$

whereas $\Xi_{\vec{R}'}$ is related to $\Xi_{\vec{R}_0}$ by means of $\Xi_{\vec{R}'} = \Xi_{\vec{R}_0} + 1/2 \sum_{\mu} \rho_{\mu(\vec{R}_0)}^{\Delta_{\mu}} \Xi_{\vec{R}_0}$.

Thus $\cos(\Xi_{\vec{R}} - \Xi_{\vec{R}'})$ in this case is equal to

$$\cos(\Xi_{\vec{R}} - \Xi_{\vec{R}'}) \approx 1 - \frac{1}{2}(\Xi_{\vec{R}} - \Xi_{\vec{R}'} - 2\pi)^2 \approx 1 - \frac{1}{2} \sum_{\mu, \nu} \rho_\mu \rho_\nu \Delta_{(\vec{R}_0)}^\mu \Xi_{\vec{R}_0} \Delta_{(\vec{R}_0)}^\nu \Xi_{\vec{R}_0} \quad (\text{C.148})$$

The same argument has to be applied for the case when one of the dislocations is resting on the path. For the case where both dislocations are distant from the path (and therefore $\Xi_{\vec{R}}$ and $\Xi_{\vec{R}'}$ are either both positive or both negative) it is straightforward to show (C.148). With this result we can return to (C.146)

$$1 + \sum_{\vec{R}_0} \sum_{\vec{\rho}} \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right] - \frac{1}{2} \sum_{\vec{R}_0} \sum_{\vec{\rho}} \sum_{\mu, \nu} \rho_\mu \rho_\nu \Delta_{(\vec{R}_0)}^\mu \Xi_{\vec{R}_0} \Delta_{(\vec{R}_0)}^\nu \Xi_{\vec{R}_0} \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right]. \quad (\text{C.149})$$

The first two terms are the same as partition function (C.145), and thus correlation function (C.128) takes the following form

$$\left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M = 1 - \frac{\frac{1}{2} \sum_{\vec{R}_0} \sum_{\vec{\rho}} \sum_{\mu, \nu} \rho_\mu \rho_\nu \Delta_{(\vec{R}_0)}^\mu \Xi_{\vec{R}_0} \Delta_{(\vec{R}_0)}^\nu \Xi_{\vec{R}_0} \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right]}{1 + \sum_{\vec{R}_0} \sum_{\vec{\rho}} \exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right]}.$$

Since $|\psi|^2/\eta$ is large, the fraction can be approximated by the numerator alone. By using the following relation⁹

$$\sum_{\vec{\rho}} \rho_\mu \rho_\nu = \frac{1}{2} \delta_{\mu\nu} \vec{\rho}^2,$$

and by noting that, due to the smallness of the distance d between the pair's dislocations we can approximate $1 - \rho^2(\dots)$ by $\exp[-\rho^2(\dots)]$, we obtain

$$\left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M \approx \exp \left[-\frac{1}{4} \sum_{\vec{\rho}} \vec{\rho}^2 \sum_{\vec{R}_0, \mu} \left(\Delta_{(\vec{R}_0)}^\mu \Xi_{\vec{R}_0} \right)^2 e^{-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}|} \right]. \quad (\text{C.150})$$

In order to obtain a final result from (C.150) we have to calculate $\sum_{\vec{R}_0} \sum_{\mu} \left(\Delta_{(\vec{R}_0)}^\mu \Xi_{\vec{R}_0} \right)^2$.

Since the distances we are considering $|\vec{r}_{1,2} - \vec{R}_0|$ are much larger than the lattice space, we can approximate the discrete sums and differences by integrals and derivatives. Taking into account (C.122)

$$\partial_{X_0} \Xi(\vec{R}_0) = -2\pi \partial_{Y_0} \left[D'(|\vec{r}_1 - \vec{R}_0|) - D'(|\vec{r}_2 - \vec{R}_0|) \right],$$

⁹Shown by returning to the continuum formulation: $\sum_{\vec{\rho}} \rho_x \rho_y \rightarrow \int_a^\infty \left(\frac{\rho}{a}\right)^3 \frac{d\rho}{a} \int_0^{2\pi} \cos \phi \sin \phi d\phi = 0$

and $\sum_{\vec{\rho}} \rho_x^2 = \sum_{\vec{\rho}} \rho_y^2 \rightarrow \int_a^\infty \left(\frac{\rho}{a}\right)^3 \frac{d\rho}{a} = \frac{1}{2} \frac{1}{a^2} \int d^2 \vec{\rho} \vec{\rho}^2 \rightarrow \frac{1}{2} \sum_{\vec{\rho}} \vec{\rho}^2$, where the lattice spacing a has been included since $\vec{r}$ and $\vec{\rho}$ are vectors of natural numbers in the discrete formulation, and thus dimensionless, but have dimensions in the continuum formulation.

$$\partial_{Y_0} \Xi(\vec{R}_0) = +2\pi \partial_{X_0} \left[D'(|\vec{r}_1 - \vec{R}_0|) - D'(|\vec{r}_2 - \vec{R}_0|) \right],$$

we obtain

$$\begin{aligned} 4\pi^2 \int d^2 \vec{R}_0 \partial_{X_\mu} \left[D'(|\vec{r}_1 - \vec{R}_0|) - D'(|\vec{r}_2 - \vec{R}_0|) \right] \partial_{X_\mu} \left[D'(|\vec{r}_1 - \vec{R}_0|) - D'(|\vec{r}_2 - \vec{R}_0|) \right] = \\ = -8\pi^2 D'(|\vec{r}_1 - \vec{r}_2|) = 4\pi \ln(|\vec{r}_2 - \vec{r}_1|). \end{aligned}$$

In the last steps of the previous equality we have integrated by parts, used the main property of a Green's function ($\partial_\mu \partial_\mu D' = -\delta(r)$) and applied the formula (C.109) for the two dimensional propagator (with $D'(0) = 0$). Substituting into (C.150) the following equation for the correlation function is obtained

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \exp \left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right) \left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M = \\ = \exp \left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right) \exp \left[-\pi e^{-\frac{\pi^2|\psi|^2}{\eta}} \sum_{\vec{\rho}} \left(\frac{1}{|\vec{\rho}|} \right)^{\frac{2\pi|\psi|^2}{\eta} - 2} \ln |\vec{r}_2 - \vec{r}_1| \right]. \quad (C.151) \end{aligned}$$

As it was discussed, small dislocation pairs are not able to change the correlation behaviour, which remains $\sim \exp(-\ln |\vec{r}_2 - \vec{r}_1|)$. The effect of small pairs for large $|\psi|^2/\eta$ is only to replace the factor multiplying the logarithm by an effective value

$$\frac{\eta}{|\psi|^2} \longrightarrow \left(\frac{\eta}{|\psi|^2} \right)_{\text{eff}} = \frac{\eta}{|\psi|^2} + 2\pi^2 \exp \left(-\frac{\pi^2|\psi|^2}{\eta} \right) \sum_{\vec{\rho}} \left(\frac{1}{|\vec{\rho}|} \right)^{\frac{2\pi|\psi|^2}{\eta} - 2}, \quad (C.152)$$

pushing $|\psi|^2/\eta$ to a bit smaller value and thus to a state with a bit larger dislocation pairs. If we consider now these (bit) larger dislocations and sum them up, the result will be again a decay like in equation (C.151) but at a still lower value of $|\psi|^2/\eta$. The sum in (C.152) can be approximated in a continuum form as an integral (after restoring the length dimension for ρ)

$$\left(\frac{\eta}{|\psi|^2} \right)_{\text{eff}} = \frac{\eta}{|\psi|^2} + 4\pi^3 \exp \left(-\frac{\pi^2|\psi|^2}{\eta} \right) \int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi|\psi|^2}{\eta}}. \quad (C.153)$$

Our approximation leading to this result, which was based on the absence of large pairs, breaks down when the power of ρ is stronger than ρ^{-1} ([62] and [55])

$$3 - \frac{2\pi|\psi|^2}{\eta} \geq -1 \quad \implies \quad \left(\frac{|\psi|^2}{\eta} \right)_{\text{critical}} = \frac{2}{\pi} = 0.637. \quad (C.154)$$

since sum (C.152) (or integral (C.153)) diverges for large pairs. At this moment, when the approximation does not longer holds because large pairs begin to be dominant, we cannot state longer that the decay in the correlation function is given by (C.151) (and as we shall now see, there is a change in the decay of the correlation function for large dislocations). A more detailed discussion will be done on page 567. Returning

for a moment to our model introduced in Chapter 2 where $|\psi|^2$ is proportional to the crystal's rigidity, equation (C.152) tell us that the effect of small dislocations is to make the crystal a bit softer.

Finally, let us consider the effect of large pairs in the correlation function. A high number of large dislocation pairs is only possible for small $|\psi|^2/\eta$ since in this range the energy term $\pi|\psi|^2/\eta \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'}$ allows for large separations of the two dislocations forming the pair. Under large separations, $|\vec{\rho}| \gg |\vec{r}_2 - \vec{r}_1|$ is to be understood. Correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$, related to $\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \rangle_M$ by equation (C.128), was already calculated for small $|\psi|^2/\eta$ by means of the power expansion method in (C.74), and it shows an exponential decay with the distance $|\vec{r}_2 - \vec{r}_1|$

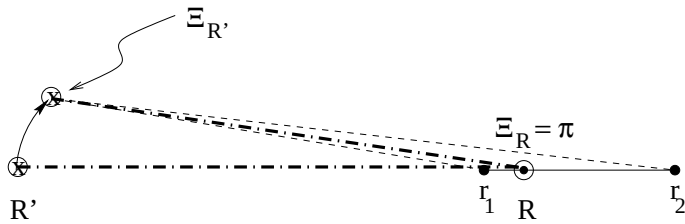
$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{small } \frac{|\psi|^2}{\eta}} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/|\psi|^2)} = \exp \left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1| \right) \left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M.$$

Therefore we shall expect that, as distinct from the phase with large $|\psi|^2/\eta$, the contributions of large pairs makes the decay of $\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \rangle_M$ go like the distance instead of like $\ln |\vec{r}_2 - \vec{r}_1|$. Since we are studying the correlation for large $|\vec{r}_2 - \vec{r}_1|$, the exponential decay with the distance is stronger and causes the leading behaviour of correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$.

From the three different cases discussed at the beginning of this section, in which the contributions of dislocation pairs were studied depending on their position in relation to $\vec{r}_1$ and $\vec{r}_2$, only that pair position with one of the dislocations resting on the path (or very near to it) is relevant; the other dislocation has a parallax $\Xi_{\vec{R}'} \approx 0$ since it is distant. For such pairs

$$\Xi_{\vec{R}} - \Xi_{\vec{R}'} = \pm \pi, \quad (\text{C.155})$$

and its contribution to the numerator of $\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \rangle_M$ is negative whereas its contribution to the partition function (in the denominator) is always positive as in (C.144).



Further, if the pair is rotated, the negative contribution is maintained along the whole rotation since the outer dislocation is too distant and its $\Xi_{\vec{R}'}$ is always very small. Due to $|\vec{\rho}| \gg |\vec{r}_2 - \vec{r}_1|$, this dislocation $M_{\vec{R}'}$ can never rest on the path simultaneously with the other dislocation $M_{\vec{R}}$, what would produce $\Xi_{\vec{R}'} = \pi$, destroying thus the negative contribution.

As we can see from this example, a large pair with a dislocation on the path is acting as if this intersecting dislocation were isolated. The total number of sites where a large pair can be placed producing $\Xi_{\vec{R}} - \Xi_{\vec{R}'} = \pm \pi$ is approximately equal to $|\vec{r}_2 - \vec{r}_1|$,

which corresponds to all the sites available for the dislocation resting on the path. In the correlation's numerator we obtain

$$\underbrace{1}_{\text{no pair}} + \sum_{\text{large } \vec{\rho}} \underbrace{\exp \left[-\frac{\pi^2 |\psi|^2}{\eta} - \frac{2\pi |\psi|^2}{\eta} \ln |\vec{\rho}| \right] \left[(|\Lambda| - 2\pi |\vec{r}_2 - \vec{r}_1|) \cdot 1 + 2\pi |\vec{r}_2 - \vec{r}_1| \cdot (-1) \right]}_{\text{one large pair}} + \dots,$$

where ... contains contributions with more than one large pair. This last equation requires a comment: we have only considered large pairs since, as already seen, small pairs only modifies the value of $|\psi|^2$ but have no influence on the correlation's behaviour. Therefore the effect of small pairs can be included by the multiplication of $\exp(-\pi^2 |\psi|^2 / \eta - 2\pi |\psi|^2 / \eta \ln |\vec{\rho}|)$ by some factor near to 1, and it will not be dealt with in the following. The problem is to evaluate the contributions of more than one pair due to the many interactions between pairs. Following [18], let us start at a value of $|\psi|^2 / \eta$ where there are few large pairs. Approaching the transition point (C.154), the number of large pairs increases strongly and they become randomly distributed, so that we can suppose that the interactions between the dislocations are considerably cancelled by this random arrangement. In such a case, we can neglect the interaction term between point $\vec{R}$ and $\vec{R}'$, and thus both partition and correlation become easy to evaluate, since they only contain products of a sum referred to the same point. Note that this approximation is equivalent to consider dislocations as isolated objects and not longer bound into dislocation pairs. Using (C.81), the partition takes the following form

$$\mathcal{Z}_M \approx \sum_{\{M_{\vec{R}}\}}' \exp \left[\sum_{\vec{R}} \left(i \Xi_{\vec{R}} M_{\vec{R}} - \frac{\pi^2 |\psi|^2}{2\eta} M_{\vec{R}}^2 \right) \right] = \prod_{\vec{R}} \sum_{M=-\infty}^{+\infty}' \exp \left[i \Xi_{\vec{R}} M - \frac{\pi^2 |\psi|^2}{2\eta} M^2 \right].$$

For the correlation function we have two cases: if $\vec{R}$ belongs (or lies near) to the path, $\Xi_{\vec{R}} = \pi$; the contribution, arising from each of the possible $|\vec{r}_2 - \vec{r}_1|$ sites where the intersecting isolated dislocation can be placed, is the following

$$\sum_M' \exp \left[i \Xi_{\vec{R}} M - \frac{\pi^2 |\psi|^2}{2\eta} M^2 \right] = 1 + 2(-1) \exp \left[-\frac{\pi^2 |\psi|^2}{2\eta} \right] + 2(+1) \exp \left[-\frac{2\pi^2 |\psi|^2}{\eta} \right] + \dots$$

On the contrary, if $\vec{R}$ lies far away outside the path connecting $\vec{r}_1$ and $\vec{r}_2$, parallax angle $\Xi_{\vec{R}}$ is very small; contributions from a point far from the path are all positive. Calling y to $\exp(-\frac{\pi^2 |\psi|^2}{2\eta})$, which for small $|\psi|^2 / \eta$ is very near to 1, the correlation function takes thus the form

$$\begin{aligned} \left\langle \prod_{\vec{r}} \exp(i M_{\vec{r}} \Xi_{\vec{r}}) \right\rangle_M &= \frac{\left(2 \sum_{M=0}^{\infty} (-1)^M y^{M^2} - 1 \right)^{|\vec{r}_2 - \vec{r}_1|} \left(2 \sum_{M=0}^{\infty} y^{M^2} - 1 \right)^{|\Lambda| - |\vec{r}_2 - \vec{r}_1|}}{\left(2 \sum_{M=0}^{\infty} y^{M^2} - 1 \right)^{|\vec{r}_2 - \vec{r}_1|} \left(2 \sum_{M=0}^{\infty} y^{M^2} - 1 \right)^{|\Lambda| - |\vec{r}_2 - \vec{r}_1|}} = \\ &= \exp \left[-\kappa |\vec{r}_2 - \vec{r}_1| \right]. \end{aligned} \quad (\text{C.156})$$

Constant κ is given by

$$\kappa = -\ln \frac{2 \sum_{M=0}^{\infty} y^{(2M)^2} - 2 \sum_{M=0}^{\infty} y^{(2M+1)^2} - 1}{2 \sum_{M=0}^{\infty} y^{(2M)^2} + 2 \sum_{M=0}^{\infty} y^{(2M+1)^2} - 1}. \quad (\text{C.157})$$

To obtain a closer form for κ , let us separate in the above sum the even powers in E from those odd, contained in O

$$\kappa = -\ln \frac{2 \left(1 - \frac{O}{E}\right) - \frac{1}{E}}{2 \left(1 + \frac{O}{E}\right) - \frac{1}{E}} \approx -\ln \frac{1 - \frac{O}{E}}{1 + \frac{O}{E}} \quad \text{with} \quad \begin{aligned} O &= y + y^9 + y^{25} + \dots \\ E &= 1 + y^4 + y^{16} + \dots \end{aligned} \quad (\text{C.158})$$

where the term $1/E$ can be neglected for $y \rightarrow 1$ (small $|\psi|^2/\eta$). Now let us demonstrate that O/E is finite and equal to y for $y \rightarrow 1^-$ ¹⁰: let be h_k the sum up to $M = k$ in both numerator and denominator

$$\frac{O}{E} = y \frac{1 + y^8 + y^{24} + \dots}{1 + y^4 + y^{16} + \dots}, \quad h_k = \frac{\sum_{M=0}^k y^{(2M+1)^2-1}}{\sum_{M=0}^k y^{(2M)^2}} = \frac{1 + y^8 + y^{24} + \dots}{1 + y^4 + y^{16} + \dots}.$$

In order to prove the finiteness of O/E , it is sufficient to show that h_k is a decreasing function of k (including the next power makes the quotient smaller). We shall use the following result: if $\frac{a}{b} < \frac{c}{d}$ with a, b, c, d positive, the fraction $\frac{a+c}{b+d}$ lies between them

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Since $y^8/y^4 < 1/1$ for y between 0 and 1, we obtain

$$\frac{y^8}{y^4} < \frac{1 + y^8}{1 + y^4} < \frac{1}{1},$$

and analogously, since $y^{24}/y^{16} < y^8/y^4$ for $0 < y < 1$

$$\frac{y^{24}}{y^{16}} < \frac{1 + y^8 + y^{24}}{1 + y^4 + y^{16}} < \frac{1 + y^8}{1 + y^4} < \frac{1}{1}.$$

We arrive thus at the result $y^{4k} < \mathbf{h}_k < \mathbf{h}_{k-1} < 1$ and taking the limit $y = e^{-\frac{\pi^2 |\psi|^2}{2\eta}} \rightarrow 1^-$ (small $|\psi|^2/\eta$) we obtain

$$\lim_{y \rightarrow 1^-} \frac{O}{E} = y,$$

¹⁰I am very grateful to Prof. Dr. Juan Luis Varona for the proof of this step.

and substituting into (C.158)

$$\kappa = -\ln \left[\frac{1}{2}(1-y) \right] \stackrel{|\psi|^2/\eta \rightarrow 0}{\approx} \ln 2 - \ln \left(\frac{\pi^2 |\psi|^2}{2\eta} \right) \approx \ln \left(\frac{2\eta}{|\psi|^2} \right). \quad (\text{C.159})$$

For the correlation the following form is thus obtained

$$\left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)^{\frac{\eta}{2\pi|\psi|^2}} \exp \left[-|\vec{r}_2 - \vec{r}_1| \ln \left(\frac{2\eta}{|\psi|^2} \right) \right] \approx \exp \left[-|\vec{r}_2 - \vec{r}_1| \ln \left(\frac{2\eta}{|\psi|^2} \right) \right], \quad (\text{C.160})$$

which is the result we have expected, an exponential decay with the distance as in equation (C.74).

As in the Ising model, the exponential decay will be reobtained by means of the estimation method developed for the Ising model. We have already discussed that for small $|\psi|^2/\eta$ a large pair can be considered as two “almost isolated” dislocations, since the distance between them is large. The effect of large Peierls contours in the Ising model is produced in this model by isolated dislocations (Fig. C.3); small pairs, on the contrary, are not able to disorder efficiently and thus to change the form of the correlation behaviour. A random distribution of many large pairs makes that the interaction between them is cancelled to a great extent, and therefore we need not consider the interactions between large pairs. The correlation function is thus given by

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \approx \frac{1}{\mathcal{Z}'} \sum_{\text{k isolated dislocations}} \frac{1}{k!} \Sigma_k \zeta^k, \quad (\text{C.161})$$

where the contribution of k dislocations is the k -times product of the contribution of one isolated dislocation, due to the fact that there is approximately no interaction between dislocations. As in equation (C.50) for the Ising model the contribution of k dislocations must be divided by $k!$ to avoid overcounting when we consider the same k dislocations in other order. Σ_k is the sum over the values of $e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})}$ for all possible positions of k isolated dislocations

$$\Sigma_k = \sum_{\substack{\text{all distributions} \\ \text{of } k \text{ isolated dislo-} \\ \text{cations}}} \left(e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \right)_{\text{each distribution}}, \quad (\text{C.162})$$

as in equation (C.51) for the Ising model. The energy cost of one isolated dislocation (and a small correcting factor due to small dislocation pairs) is contained in ζ and it is approximately given by $\exp(-\pi^2 |\psi|^2/\eta)$. Partition function $\mathcal{Z}'$ is equal to $\sum_{\text{k isolated dislocations}} \frac{1}{k!} \zeta^k$.

Each time an isolated dislocation lies on the path (a large pair has one of its dislocations resting on the path and the other one far outside the path), the phase difference between $\vec{r}_1$ and $\vec{r}_2$ acquires a contribution of $\pm\pi$ depending on the dislocation sign

$$\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2} = \pm\pi. \quad (\text{C.163})$$

Note that this π in the phase difference has the same origin as the discussed contribution of π in $\Xi_{\vec{R}} - \Xi_{\vec{R}'}$ (see for example equation (C.143)): for a large dislocation pair with $\vec{R}$

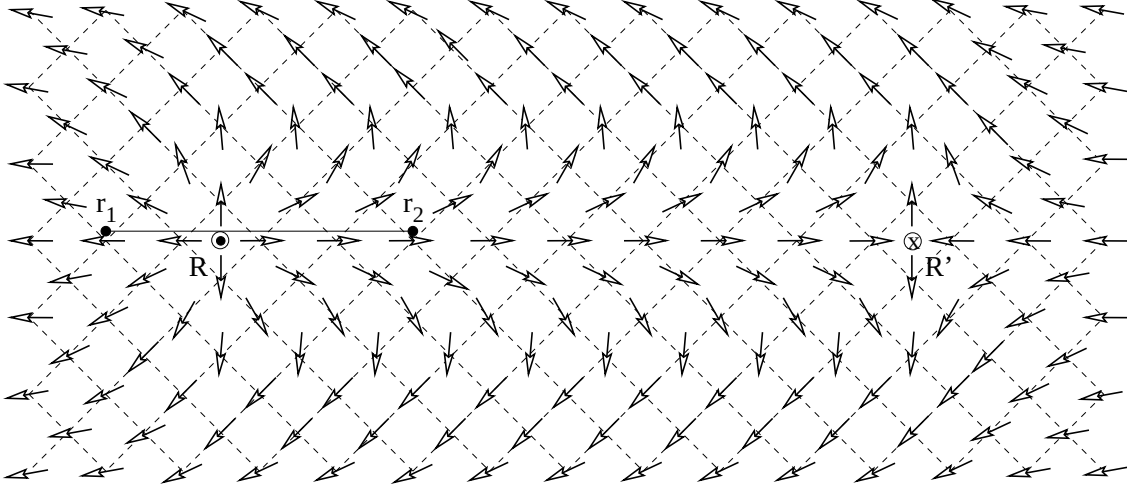


Figure C.3: Large dislocation pair formed by nearly isolated dislocations.

belonging to the path we have $\Xi_{\vec{R}} = \pi$ since from this point the path between $\vec{r}_1$ and $\vec{r}_2$ is seen under an angle of π , and $\Xi_{\vec{R}'} = 0$ since the other dislocation is very distant.

If we consider now the different contributions to the correlation by two isolated dislocations (or by two large pairs), we obtain

- No dislocation lies on the path. The number of sites available for each dislocation is $|\Lambda| - |\vec{r}_2 - \vec{r}_1|$ and the phase difference between the end points $\vec{r}_1$ and $\vec{r}_2$ is zero

$$\text{sites} = (|\Lambda| - |\vec{r}_2 - \vec{r}_1|)(|\Lambda| - |\vec{r}_2 - \vec{r}_1|) \quad \text{and} \quad \varphi_{\vec{r}_1} = \varphi_{\vec{r}_2} + 0.$$

- One isolated dislocation is placed on the path (which can occur in two different forms depending which dislocation is outside).

$$\text{sites} = |\vec{r}_2 - \vec{r}_1|(|\Lambda| - |\vec{r}_2 - \vec{r}_1|) \quad \text{and} \quad \varphi_{\vec{r}_1} = \varphi_{\vec{r}_2} + \pi.$$

- Both isolated dislocations (each one belonging to a different pair) rest on the path.

$$\text{sites} = |\vec{r}_2 - \vec{r}_1||\vec{r}_2 - \vec{r}_1| \quad \text{and} \quad \varphi_{\vec{r}_1} = \varphi_{\vec{r}_2} + \pi + \pi.$$

For two dislocations the contribution to $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ is the following

$$\Sigma_2 = (|\Lambda| - |\vec{r}_2 - \vec{r}_1|)^2 e^{i0} + 2|\vec{r}_2 - \vec{r}_1|(|\Lambda| - |\vec{r}_2 - \vec{r}_1|) e^{i\pi} + |\vec{r}_2 - \vec{r}_1|^2 e^{i2\pi} = (|\Lambda| - 2|\vec{r}_2 - \vec{r}_1|)^2.$$

Repeating the same argument for k dislocations, as in the Ising model, and substituting into (C.161), we obtain

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \approx e^{-2|\vec{r}_2 - \vec{r}_1|\zeta}. \quad (\text{C.164})$$

If all pairs had been short and there were no isolated dislocation, the number of sites such that only one of the pair's dislocations lies on the path would be approximately equal to the pair's length ρ . Substituting this result into the correlation, we would obtain

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \approx e^{-2\rho\zeta}, \quad (\text{C.165})$$

which has no dependence with the distance $|\vec{r}_2 - \vec{r}_1|$ between the scalar fields we are studying. Note that the estimation method based on the effect of large disordering contours cannot reproduce the weak power decay with the distance (C.141); it only distinguishes between a constant correlation (C.165) for small pairs (large $|\psi|^2/\eta$) and an exponential decay (C.164) for large pairs where their dislocations act as isolated dislocations (low $|\psi|^2/\eta$). The finer and more rigorous method based on the study of equation (C.127) produces a weak power decay influenced weakly by small pairs, which is surpassed by a much stronger exponential decay (C.160) for large dislocation pairs. As in the Ising model, let us summarize in the next Table C.2 the results about the correlation function.

$ \psi ^2/\eta$	$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$	Dislocation pairs
low $ \psi ^2/\eta$	$ \vec{r}_2 - \vec{r}_1 ^{-\eta/2\pi \psi ^2}$	small pairs
high $ \psi ^2/\eta$	$e^{- \vec{r}_2 - \vec{r}_1 \ln(2\eta/ \psi ^2)}$	large pairs, acting as isolated dislocations

Table C.2: Correlation in the 2-dimensional XY model.

C.4 The 3-dimensional XY model

The model introduced in Chapter 2, a scalar field ψ with constant $|\psi|$ coupled to a gauge field, is defined in a 3-dimensional lattice, but we have firstly studied in the previous section the 2-dimensional XY model because some of calculations (specially the contribution of small dislocation pairs and the critical point (C.154)) were easier to be carried out in two dimensions. Once we have introduced the necessary techniques to study this model, let us now repeat step by step the same calculations for the 3-dimensional XY model. It should be remembered that the three dimensional space is the coordinate space, since the internal symmetry one remains two dimensional, defined by the scalar field's orientation $\varphi_{\vec{r}}$. We advance the main differences with the previous model: dislocations build closed lines (in the 2-dimensional model we have only seen the points where this closed line crossed the lattice plane), and since the Green's function of the Laplace's operator, $1/|\vec{r} - \vec{r}'|$, is now weaker than the logarithm of the distance, the system has an ordered state for large $|\psi|^2/\eta$, with a nonvanishing correlation at $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$.

The model is described by the same energy as in equation (C.68), with $\mu = 1, 2, 3$

$$E = a|\psi|^2 \sum_{\vec{r}} \sum_{\mu=1,2,3} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})] . \quad (\text{C.166})$$

Lattice spacing a has been introduced to maintain the same notation as in equation (2.84) of Chapter 2, although for the following calculations it will be set equal to 1.

Partition function $\mathcal{Z}$ and correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ are calculated by means of the same formulas as in the previous section

$$\mathcal{Z} = \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left\{ -\frac{a|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu=1,2,3} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})] \right\}, \quad (\text{C.167})$$

and

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} e^{-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu} [1 - \cos(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}})]}. \quad (\text{C.168})$$

For small values of $a|\psi|^2/\eta$ the same power expansion as for the 2-dimensional model is carried out (equation (C.71) and (C.73)), obtaining thus the same exponential decay with the distance

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{small } \frac{a|\psi|^2}{\eta}} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/a|\psi|^2)}, \quad (\text{C.169})$$

which is characteristic for all XY models in any dimension at small $|\psi|^2/\eta$.

For the range large $a|\psi|^2/\eta$, the Villain approximation is again used, as in (C.82)

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r}, \mu=1,2,3} \left(\int_{-\infty}^{\infty} db_{\mu, \vec{r}} \sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right) \times \\ &\times e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \sum_{\{m_{\mu, \vec{r}}\}} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - 2\pi m_{\mu, \vec{r}}) \right], \end{aligned} \quad (\text{C.170})$$

with partition function $\mathcal{Z}$ given by

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \prod_{\vec{r}, \mu} \int_{-\infty}^{\infty} \left(\frac{\eta}{2\pi a|\psi|^2} \right)^{\frac{1}{2}} db_{\mu, \vec{r}} \sum_{\{m_{\mu, \vec{r}}\}} \times \\ &\times \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - 2\pi m_{\mu, \vec{r}}) \right]. \end{aligned} \quad (\text{C.171})$$

Note that we have to impose a condition over the $m_{\mu, \vec{r}}$ sum in order to avoid the overcounting of the same configuration, as was discussed for the 2-dimensional XY model in (C.78). Such a constraint over $m_{\mu, \vec{r}}$, for example

$$\sum_{\mu} \Delta_{\mu} m_{\mu, \vec{r}} = 0, \quad (\text{C.172})$$

must be included in the next manipulations, although it will not be written down. Now we shall repeat the same steps as in the 2-dimensional model to obtain the correlation in a closer form with all continuous fields integrated out.

1. Integration over all $d\varphi_{\vec{r}}$ produces the following constraint on the fields $b_{\mu,\vec{r}}$ in the correlation's numerator and in the denominator (partition function) respectively

$$\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}, \quad \text{and} \quad \sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} = 0. \quad (\text{C.173})$$

General solution $b_{\mu,\vec{r}}^0$ of the vanishing discrete divergence is the discrete curl in three dimensions of a vector field $a_{\lambda,\vec{r}}$

$$b_{\mu,\vec{r}}^0 = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}, \quad (\text{C.174})$$

and a particular solution of equation $\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}$ is the same $L_{\mu,\vec{r}}$ as in equation (C.120), a vector field only nonzero on bonds belonging to a path C between $\vec{r}_1$ and $\vec{r}_2$. Therefore the general solution of $b_{\mu,\vec{r}}$ for the denominator is $b_{\mu,\vec{r}}^0$ and for the numerator

$$b_{\mu,\vec{r}} = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + L_{\mu,\vec{r}}, \quad (\text{C.175})$$

and substituting into the numerator term $b_{\mu,\vec{r}}^2$, we obtain

$$\begin{aligned} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 = & \underbrace{\sum_{\vec{r},\nu,\lambda} (\Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}})^2}_{\sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(- \sum_{\nu} \Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} \right)} - \underbrace{\sum_{\vec{r},\nu,\lambda} (\Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}) (\Delta_{\lambda} a_{\nu,\vec{r}-\hat{\lambda}})}_{- \sum_{\vec{r},\nu,\lambda} a_{\lambda,\vec{r}} \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}}} + \\ & + 2 \underbrace{\sum_{\vec{r},\mu,\nu,\lambda} L_{\mu,\vec{r}} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}}_{+ \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_{\nu} L_{\mu,\vec{r}}} + \sum_{\mu} L_{\mu,\vec{r}}^2, \end{aligned} \quad (\text{C.176})$$

where we have rearranged the terms in the sums (equivalent to the integration by parts in the continuum case).

2. Since both $L_{\mu,\vec{r}}$ and $m_{\mu,\vec{r}}$ are integers, we have $\exp \left[-i2\pi \sum_{\vec{r},\mu} L_{\mu,\vec{r}} m_{\mu,\vec{r}} \right] = 1$, and the correlation function takes the form

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = & \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\mu} L_{\mu,\vec{r}}^2 \right] \frac{1}{\mathcal{Z}} \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{3|\Lambda|} \prod_{\vec{r},\lambda} \int_{-\infty}^{\infty} da_{\lambda,\vec{r}} \times \\ & \times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(- \sum_{\nu} \Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + \sum_{\nu} \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} \right) \right] \times \end{aligned}$$

$$\times \exp \left[-\frac{\eta}{a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(\sum_{\mu,\nu} \epsilon_{\lambda\nu\mu} \Delta_\nu L_{\mu,\vec{r}} \right) - i2\pi \underbrace{\sum_{\vec{r},\mu,\nu,\lambda} m_{\mu,\vec{r}} \epsilon_{\mu\nu\lambda} \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}}}_{+ \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_\nu m_{\mu,\vec{r}}} \right]. \quad (\text{C.177})$$

Partition function $\mathcal{Z}$ is equal to

$$\begin{aligned} \mathcal{Z} = & \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{3|\Lambda|} \prod_{\vec{r},\lambda} \int_{-\infty}^{\infty} da_{\lambda,\vec{r}} \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_\nu m_{\mu,\vec{r}} \right] \times \\ & \times \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(-\sum_{\nu} \Delta_\nu \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}} + \sum_{\nu} \Delta_\lambda \Delta_\nu a_{\nu,\vec{r}-\hat{\lambda}} \right) \right]. \quad (\text{C.178}) \end{aligned}$$

We have in equation (C.177) and (C.178) a difficulty: let us firstly look at in detail the term in the integrand which only depends on $a_{\lambda,\vec{r}}$

$$\Sigma = \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(-\sum_{\nu} \Delta_\nu \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}} + \Delta_\lambda \sum_{\nu} \Delta_\nu a_{\nu,\vec{r}-\hat{\lambda}} \right),$$

which expressed in relation to the Fourier transforms, denoted by the tilde $\tilde{a}_\lambda$, has the following form

$$\tilde{\Sigma} = \sum_{\lambda,\nu} \tilde{a}_\lambda \left(\sum_{\mu=1,2,3} (-e^{ik_\mu} - e^{-ik_\mu} + 2) \delta_{\lambda\nu} + (e^{ik_\lambda} - 1)(e^{ik_\nu} - 1)e^{-ik_\lambda} \right) \tilde{a}_\nu. \quad (\text{C.179})$$

Let us now define the transversal and the longitudinal component of $a_{\lambda,\vec{r}}$

$$a_{\lambda,\vec{r}} = a_{\lambda,\vec{r}}^T + a_{\lambda,\vec{r}}^L = \sum_{\rho} P_{\lambda\rho}^T a_{\rho,\vec{r}} + \sum_{\rho} P_{\lambda\rho}^L a_{\rho,\vec{r}},$$

where $P_{\lambda\rho}^T$ and $P_{\lambda\rho}^L$ acting on the Fourier components are given by

$$P_{\lambda\rho}^T = \delta_{\lambda\rho} + \frac{e^{-ik_\lambda}(e^{ik_\lambda} - 1)(e^{ik_\rho} - 1)}{6 - 2 \sum_{\mu} \cos k_\mu}, \quad P_{\lambda\rho}^L = -\frac{e^{-ik_\lambda}(e^{ik_\lambda} - 1)(e^{ik_\rho} - 1)}{6 - 2 \sum_{\mu} \cos k_\mu}. \quad (\text{C.180})$$

The notation “transversal” and “longitudinal” become clear by noting

$$\sum_{\lambda} (e^{ik_\lambda} - 1) P_{\lambda\rho}^T = 0, \quad \sum_{\lambda} (e^{ik_\lambda} - 1) P_{\lambda\rho}^L = (e^{ik_\rho} - 1), \quad (\text{C.181})$$

$$\sum_{\nu,\lambda} \epsilon_{\sigma\nu\lambda} (e^{ik_\nu} - 1) P_{\lambda\rho}^T = \sum_{\nu} \epsilon_{\sigma\nu\rho} (e^{ik_\nu} - 1), \quad \sum_{\nu,\lambda} \epsilon_{\sigma\nu\lambda} (e^{ik_\nu} - 1) P_{\lambda\rho}^L = 0. \quad (\text{C.182})$$

Since $(e^{ik_\lambda} - 1)$ is the Fourier transform of the discrete derivative Δ_λ , equation (C.181) states that the discrete divergence of $a_{\lambda,\vec{r}}^T$ vanishes, $\sum_\lambda \Delta_\lambda a_{\lambda,\vec{r}}^T = 0$ (and $\sum_\lambda \Delta_\lambda a_{\lambda,\vec{r}}^L = \sum_\lambda \Delta_\lambda a_{\lambda,\vec{r}}$), whereas the second equation means $\sum_{\nu,\lambda} \epsilon_{\sigma\nu\rho} \Delta_\nu a_{\rho,\vec{r}}^L = 0$. Using the following relation

$$\sum_\nu P_{\lambda\nu}^L P_{\nu\rho}^L = \frac{e^{-ik_\lambda}(e^{ik_\lambda} - 1)(e^{ik_\rho} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \frac{\sum_\nu (e^{ik_\nu} + e^{-ik_\nu} - 2)}{6 - 2 \sum_\mu \cos k_\mu} = P_{\lambda\rho}^L,$$

it is easy to show that $\sum_\nu P_{\lambda\nu}^T P_{\nu\rho}^L = 0$ and $\sum_\nu P_{\lambda\nu}^T P_{\nu\rho}^T = P_{\lambda\rho}^T$ hold. Returning to equation (C.179) and taking into account these properties, we obtain

$$\sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(-\sum_\nu \Delta_\nu \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}} + \Delta_\lambda \sum_\nu \Delta_\nu a_{\nu,\vec{r}-\hat{\lambda}} \right) = \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^T \left(-\sum_\nu \Delta_\nu \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}}^T \right), \quad (\text{C.183})$$

with no dependence on the longitudinal component. Moreover, if we look at (C.177) and (C.178), in the terms which include only one $a_{\lambda,\vec{r}}$, this field is coupled to the curl of $L_{\mu,\vec{r}}$ or to the curl of $m_{\mu,\vec{r}}$. If we regroup terms as in the last step of equation (C.177), $m_{\mu,\vec{r}}$ or $L_{\mu,\vec{r}}$ are coupled to the curl, $\sum_{\nu,\lambda} \epsilon_{\sigma\nu\lambda} \Delta_\nu a_{\lambda,\vec{r}}$, which according to (C.182) is equal to the curl of $a_{\lambda,\vec{r}}^T$. Therefore, both the term (C.183) and the terms where $a_{\lambda,\vec{r}}$ is coupled to $L_{\mu,\vec{r}}$ and $m_{\mu,\vec{r}}$, depend only on the transversal component $a_{\lambda,\vec{r}}^T$. The integral over $da_{\lambda,\vec{r}}$ in the partition and the correlation can be splitted into an integral over the two independent transversal components, and an integral over the only independent longitudinal component¹¹. Since the integrand only depends on $a_{\lambda,\vec{r}}^T$, the integration over $da_{\lambda,\vec{r}}^L$ can be factorized out obtaining thus for the partition function

$$\begin{aligned} \mathcal{Z} &= \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \left[\prod_{\vec{r},\lambda} \left(\int_{-\infty}^{\infty} da_{\lambda,\vec{r}}^L \right) \prod_{\vec{r}} \delta \left(\sum_{\nu,\lambda} \epsilon_{\sigma\nu\lambda} \Delta_\nu a_{\lambda,\vec{r}}^L \right) \right] \times \\ &\times \prod_{\vec{r},\lambda} \left(\int_{-\infty}^{\infty} da_{\lambda,\vec{r}}^T \right) \prod_{\vec{r}} \delta \left(\sum_\lambda \Delta_\lambda a_{\lambda,\vec{r}}^T \right) \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^T \left(-\sum_\nu \Delta_\nu \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}}^T \right) - i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}}^T \epsilon_{\lambda\nu\mu} \Delta_\nu m_{\mu,\vec{r}} \right], \quad (\text{C.184}) \end{aligned}$$

and similar, plus the terms including $L_{\mu,\vec{r}}$, for the correlation function. Let us return for a moment to the 2-dimensional case: there we changed the integration

¹¹With $\prod_{\lambda,\vec{r}} da_{\lambda,\vec{r}} = \mathcal{J} \prod_{\lambda,\vec{r}} da_{\lambda,\vec{r}}^L \prod_{\lambda,\vec{r}} da_{\lambda,\vec{r}}^T$ ($\mathcal{J}$ is the Jacobian, equal to 1 due to $P_{\lambda\nu}^T P_{\nu\rho}^L = 0$) we have not duplicated the variables, since both $a_{\lambda,\vec{r}}^T$ and $a_{\lambda,\vec{r}}^L$ are subjected to constraints: one for the transversal field, $\sum_\lambda \Delta_\lambda a_{\lambda,\vec{r}}^T$, and two for the longitudinal component, $\Delta_1 a_{2,\vec{r}}^L - \Delta_2 a_{1,\vec{r}}^L = 0$ and $\Delta_2 a_{3,\vec{r}}^L - \Delta_3 a_{2,\vec{r}}^L$ (the third equation coming from $\sum_{\nu,\lambda} \epsilon_{\sigma\nu\lambda} \Delta_\nu a_{\lambda,\vec{r}}^L$ can be deduced from the two previous constraints for $a_{\lambda,\vec{r}}^L$). At every site there is only two transversal directions and one longitudinal.

over a constrained field $b_{\mu,\vec{r}}$ by the integration over an unconstrained one $a_{\vec{r}}$, knowing that because of $b_{\mu,\vec{r}} - L_{\mu,\vec{r}} = \sum_{\nu} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}}$ (with $L_{\mu,\vec{r}}$ given) the field $a_{\vec{r}}$ fixes $b_{\mu,\vec{r}}$ without ambiguity. In our case there is a freedom in $a_{\lambda,\vec{r}}$ since the addition of a discrete gradient $\Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}$ to $a_{\vec{r}}$ has no influence on $b_{\mu,\vec{r}}$

$$\sum_{\nu\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}-\hat{\nu}} = \sum_{\nu\lambda} \epsilon_{\mu\nu\lambda} \underbrace{(\chi_{\vec{r}} - \chi_{\vec{r}-\hat{\lambda}} - \chi_{\vec{r}-\hat{\nu}} + \chi_{\vec{r}-\hat{\nu}-\hat{\lambda}})}_{\text{symmetrical under } \nu \leftrightarrow \lambda} = \chi_{\vec{r}} \sum_{\nu\lambda} \epsilon_{\mu\nu\lambda} = 0,$$

$$\begin{aligned} a_{\lambda,\vec{r}} &\longrightarrow a'_{\lambda,\vec{r}} = a_{\lambda,\vec{r}} - \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}, \\ b_{\mu,\vec{r}} - L_{\mu,\vec{r}} &\longrightarrow b'_{\mu,\vec{r}} - L_{\mu,\vec{r}} = b_{\mu,\vec{r}} - L_{\mu,\vec{r}} \longrightarrow b'_{\mu,\vec{r}} = b_{\mu,\vec{r}}. \end{aligned} \quad (\text{C.185})$$

The problem is now easy to identify. In the original integral (C.171), we had one contribution from each configuration of $b_{\mu,\vec{r}}$; but the change to the $a_{\lambda,\vec{r}}$ variables means, that we now have an infinite number of $a_{\lambda,\vec{r}}$ configurations, all of them corresponding to *only one* $b_{\mu,\vec{r}}$ contribution. The addition of $\Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}$ has an effect on only the longitudinal field $a_{\lambda,\vec{r}}^L$, as we can see by considering the definition of $P_{\lambda\rho}^T$ (C.180)

$$\sum_{\rho} P_{\lambda\rho}^T e^{-ik_{\rho}} (e^{ik_{\rho}} - 1) = e^{-ik_{\lambda}} (e^{ik_{\lambda}} - 1) \left[1 + \frac{\sum_{\rho} e^{-ik_{\rho}} (e^{ik_{\rho}} - 1)^2}{6 - 2 \sum_{\mu} \cos k_{\mu}} \right] = 0.$$

Moreover, since $b_{\mu,\vec{r}}$ is obtained as a discrete curl of $a_{\lambda,\vec{r}}$, only the transversal components $a_{\lambda,\vec{r}}^T$ fix $b_{\mu,\vec{r}}$. The integrand in (C.184) depends only on $a_{\lambda,\vec{r}}^T$, and therefore the change of $a_{\lambda,\vec{r}} \rightarrow a_{\lambda,\vec{r}} - \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}$ ¹² has an effect only on the integration over $da_{\lambda,\vec{r}}^L$ which we have factorized out in (C.184). It is this integral over the longitudinal components which is problematic: for each $a_{\lambda,\vec{r}}^T$ value (or each $b_{\mu,\vec{r}}$), we have an infinite number of possible changes of the (C.185) type contributing to $\prod_{\vec{r},\lambda} \left(\int_{-\infty}^{\infty} da_{\lambda,\vec{r}}^L \right)$. In order to obtain just one contribution for each $b_{\mu,\vec{r}}$ configuration, we have to carry out the integral over the transversal fields alone, removing the integral over the longitudinal fields¹³. Before we present a general method to solve this problem, let us calculate the partition function (C.184) separating the part which depends only on the longitudinal fields.

To evaluate the partition, we have to incorporate into the partition the constraint of the vanishing divergence of the transversal components and the zero curl of the longitudinal fields. To achieve this implementation, we shall work with the Fourier

¹²In the literature on these topics, such a change is called gauge transformation; this use has been here avoided in order to reserve the name “gauge” for the compatible deformations in the dislocation theory developed in Chapter 2.

¹³One could argue that, since we are calculating correlation functions and this “infinite number” appears in both numerator and denominator, it is not important to deal with this question. But there are cases, as in the estimation of the critical point of the 3-dimensional XY model (C.229), where we are interested in the partition function alone and we then have to solve this problem then.

components, since the notation is not so complicated. From the three transversal components only two are independent, for exmaple a_x^T and a_y^T ; and from the three longitudinal components, only one (a_z^L). We can change the integration variables from a_x, a_y, a_z to a_x^T, a_y^T, a_z^L according to (C.180)

$$\begin{aligned}\tilde{a}_x^T &= \left(1 + \frac{e^{-ik_x}(e^{ik_x} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu}\right) \tilde{a}_x + \frac{(1 - e^{-ik_x})(e^{ik_y} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_y + \frac{(1 - e^{-ik_x})(e^{ik_z} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_z, \\ \tilde{a}_y^T &= \frac{(1 - e^{-ik_y})(e^{ik_x} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_x + \left(1 + \frac{e^{-ik_y}(e^{ik_y} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu}\right) \tilde{a}_y + \frac{(1 - e^{-ik_y})(e^{ik_z} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_z, \\ \tilde{a}_z^L &= -\frac{(1 - e^{-ik_z})(e^{ik_x} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_x - \frac{(1 - e^{-ik_z})(e^{ik_y} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_y - \frac{e^{-ik_z}(e^{ik_z} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu} \tilde{a}_z,\end{aligned}$$

and therefore

$$\begin{aligned}d\tilde{a}_x^T d\tilde{a}_y^T d\tilde{a}_z^L &= \frac{d\tilde{a}_x^T d\tilde{a}_y^T d\tilde{a}_z^L}{\begin{vmatrix} 1 + \frac{e^{-ik_x}(e^{ik_x} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu} & \frac{e^{-ik_x}(e^{ik_x} - 1)(e^{ik_y} - 1)}{6 - 2 \sum_\mu \cos k_\mu} & \frac{e^{-ik_x}(e^{ik_x} - 1)(e^{ik_z} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \\ \frac{e^{-ik_y}(e^{ik_y} - 1)(e^{ik_x} - 1)}{6 - 2 \sum_\mu \cos k_\mu} & 1 + \frac{e^{-ik_y}(e^{ik_y} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu} & \frac{e^{-ik_y}(e^{ik_y} - 1)(e^{ik_z} - 1)}{6 - 2 \sum_\mu \cos k_\mu} \\ -\frac{e^{-ik_z}(e^{ik_z} - 1)(e^{ik_x} - 1)}{6 - 2 \sum_\mu \cos k_\mu} & -\frac{e^{-ik_z}(e^{ik_z} - 1)(e^{ik_y} - 1)}{6 - 2 \sum_\mu \cos k_\mu} & -\frac{e^{-ik_z}(e^{ik_z} - 1)^2}{6 - 2 \sum_\mu \cos k_\mu} \end{vmatrix}} = \\ &= d\tilde{a}_x^T d\tilde{a}_y^T d\tilde{a}_z^L \frac{6 - 2 \sum_\mu \cos k_\mu}{-e^{-ik_z}(e^{ik_z} - 1)^2},\end{aligned}\tag{C.186}$$

together with the constraints (in Fourier components)

$$(e^{ik_x} - 1)\tilde{a}_y^L - (e^{ik_y} - 1)\tilde{a}_x^L = 0, \quad (e^{ik_y} - 1)\tilde{a}_z^L - (e^{ik_z} - 1)\tilde{a}_y^L = 0,$$

$$(e^{ik_x} - 1)\tilde{a}_x^T + (e^{ik_y} - 1)\tilde{a}_y^T + (e^{ik_z} - 1)\tilde{a}_z^T = 0.$$

As already noted, under a gauge transformation $a_{\lambda, \vec{r}} \rightarrow a_{\lambda, \vec{r}} - \Delta_\lambda \chi_{\vec{r} - \hat{\lambda}}$, or equivalently $\tilde{a}_\lambda \rightarrow \tilde{a}_\lambda - (e^{ik_\lambda} - 1)e^{-ik_\lambda} \tilde{\chi}$, the longitudinal component transforms as $\tilde{a}_z^L \rightarrow \tilde{a}_z^L - e^{-ik_z}(e^{ik_z} - 1)\tilde{\chi}$ whereas the transversal components remain unchanged. Therefore, in order to isolated the contribution of the longitudinal component, we have to separate the factor $-e^{-ik_z}(e^{ik_z} - 1)$ out from the result (C.186)

$$d\tilde{a}_x^T d\tilde{a}_y^T d\tilde{a}_z^L = \left(\frac{d\tilde{a}_z^L}{-e^{-ik_z}(e^{ik_z} - 1)} \right) \left(d\tilde{a}_x^T d\tilde{a}_y^T \frac{6 - 2 \sum_\mu \cos k_\mu}{(e^{ik_z} - 1)} \right).\tag{C.187}$$

For the 3-dimensional case, it holds $\prod_{\vec{k}} (6 - 2 \sum_\mu \cos k_\mu) = \det(-\Delta \Delta)$ (as in equation (C.112) for two dimensions)

$$\prod_{\vec{k}} d\tilde{a}_x^T d\tilde{a}_y^T d\tilde{a}_z^L = \prod_{\vec{r}} da_{x, \vec{r}} da_{y, \vec{r}} da_{z, \vec{r}} = \prod_{\vec{r}} \frac{da_{z, \vec{r}}^L}{\det(-\Delta_z)} \prod_{\vec{r}} da_{x, \vec{r}}^T da_{y, \vec{r}}^T \frac{\det(-\Delta \Delta)}{\det(\Delta_z)}.$$

The finite part of the partition function, due to the transversal components, becomes thus

$$\begin{aligned} \mathcal{Z}_{\text{finite}} &= \frac{\det(-\Delta\Delta)}{\det(\Delta_z)} \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{3|\Lambda|} \times \\ &\times \prod_{\vec{r}} \int_{-\infty}^{\infty} da_{x,\vec{r}}^T da_{y,\vec{r}}^T da_{z,\vec{r}}^T \prod_{\vec{r}} \delta \left(a_z^T + \frac{\Delta_x}{\Delta_z} a_x^T + \frac{\Delta_y}{\Delta_z} a_y^T \right) \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^T \left(-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}^T \right) - i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}}^T \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}} \right]. \end{aligned} \quad (\text{C.188})$$

The Dirac's delta is the transversality condition fixing the third (dependent) component $a_{z,\vec{r}}^T$: it reads $\tilde{a}_z^T + (e^{ik_z} - 1)^{-1}(e^{ik_x} - 1)\tilde{a}_x^T + (e^{ik_z} - 1)^{-1}(e^{ik_y} - 1)\tilde{a}_y^T = 0$ in Fourier components.

In the previous case it was rather immediate to isolate the field component which has no influence on the integrand. In a more general case, this procedure can be very complicated and a more direct method is necessary. The problem stated in general: since the integrand is constant under $a_{\lambda,\vec{r}} \rightarrow a_{\lambda,\vec{r}} - \Delta_{\lambda}\chi_{\vec{r}-\hat{\lambda}}$, all these transformations produce the same contribution. Integrating over all possible $a_{\lambda,\vec{r}}$ would mean integrating over each different contribution an infinite number of times: we have to choose one $\bar{a}_{\lambda,\vec{r}}$ as representative of all these transformations and not integrate over the others which produce the same integrand value (in the case discussed above, $\bar{a}_{\lambda,\vec{r}}$ was the transversal component). In the language of fiber bundles, we have to choose one $\bar{a}_{\lambda,\vec{r}}$ on every orbit, represented by $a_{\lambda,\vec{r}} - \Delta_{\lambda}\chi_{\vec{r}-\hat{\lambda}}$ with $\chi_{\vec{r}}$ arbitrary [38]. There is another way to solve the problem: we can integrate over all $a_{\lambda,\vec{r}}$, both transversal and longitudinal, without worrying about the respective contributions of each; in order to obtain a finite result, we then include in the integral a factor which compensates for the infinite number of (gauge) transformations in $a_{\lambda,\vec{r}}$ that produce the same integrand value. To achieve this, we shall use the method developed by Faddeev and Popov.

Note. The Faddeev-Popov recipe [25].

Firstly, a result we shall use several times during the following discussion

$$\delta(x - c) = \delta(y - f(c)) \frac{df}{dx}, \quad (\text{C.189})$$

which is proved by noting that $g(c) = \int \delta(x - c)g(x)dx = \int \delta(y - f(c))g(f^{-1}(y))dy$, with $y = f(x)$, and $dy = \frac{df}{dx}dx$. The generalization for n variables with $y_i = f_i(x_1, \dots, x_n) \forall i = 1, \dots, n$, is straightforward

$$\int \prod_i dx_i \delta(x_1 - c_1) \cdots \delta(x_n - c_n) g(x) = \int \prod_i dy_i \det \left(\frac{\partial f_i}{\partial x_j} \right) \prod_k \delta(y_k - f(x)) g(f^{-1}(y)),$$

$$\prod_k \delta(x_k - c_k) = \det \left(\frac{\partial f_i}{\partial x_j} \right) \prod_k \delta(y_k - f(x)). \quad (\text{C.190})$$

Let us consider a space with $n + m$ variables denoted by $a = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n, \chi_1, \chi_2, \dots, \chi_m)$ in which we calculate the partition function by means of an energy E which is independent of the last m variables $\chi_1, \dots, \chi_m$; the energy E only depends on the first n variables $\bar{a}_1, \dots, \bar{a}_n$. The $\bar{a}$ variables represent the transversal field $a_{\lambda, \vec{r}}^T$ and the χ the longitudinal field in the case discussed above. Since the energy does not depend on χ , if we integrated over all χ variables, we would obtain an infinite result because of the infinite number of χ 's which produce the same energy. In order to obtain a finite partition function we have to integrate only over the variables on which the energy depends

$$\mathcal{Z} = \int d\bar{a}_1 \cdots d\bar{a}_n e^{-E}, \quad (\text{C.191})$$

which can also be written as follows

$$\mathcal{Z} = \int d\bar{a}_1 \cdots d\bar{a}_n \int d\chi_1 \cdots d\chi_m e^{-E} \prod_j \delta(\chi_j) = \int da_1 \cdots da_{n+m} e^{-E} \prod_j \delta(\chi_j). \quad (\text{C.192})$$

In this last equation the integration, over all a variables, is carried out along the (m dimensional) surface $(\chi_1, \dots, \chi_m) = (0, \dots, 0)$. But since the resulting partition does not depend on the χ variables, we could integrate along any surface defined by $\chi_j = f_j(\bar{a}_1, \dots, \bar{a}_n)$, $\forall j = 1, \dots, m$, obtaining thus

$$\mathcal{Z} = \int da_1 \cdots da_{n+m} e^{-E} \prod_j \delta(\chi_j - f_j(\bar{a})). \quad (\text{C.193})$$

The surface along which we are integrating the χ variables can be given also in implicit form

$$F_j(\bar{a}_1, \dots, \bar{a}_n, \chi_1, \dots, \chi_m) = 0 \quad \forall j = 1, \dots, m,$$

and, using (C.190), the partition function becomes

$$\mathcal{Z} = \int da_1 \cdots da_{n+m} e^{-E} \det \left(\frac{\partial F_i}{\partial \chi_j} \right) \prod_j \delta(F_j(a)). \quad (\text{C.194})$$

Summarising: in order to obtain a finite partition, we can integrate only over the variables on which the energy depends (as in (C.191)); or we can as well integrate over all variables (for example, because isolating those variables which have an influence on E is difficult) including $\det \left(\frac{\partial F_i}{\partial \chi_j} \right) \prod_j \delta(F_j)$, where $F_j(a)$ is a set of conditions we want to impose on the integration variables (for example, because this choice makes easier the calculation), but which has no effect on the partition function.

Let us now return to our partition and correlation function (C.184). In that case was rather easy to isolate the field which caused the problems. But we can also obtain a finite partition using the result (C.194): let us choose as gauge the following one

$$F_{\vec{r}} = \sum_{\lambda} \Delta_{\lambda} a_{\lambda, \vec{r}} - f_{\vec{r}}, \quad (\text{C.195})$$

with $f_{\vec{r}}$ an arbitrary function at each site $\vec{r}$. Under a transformation $a_{\lambda,\vec{r}} \rightarrow a_{\lambda,\vec{r}} - \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}$, $F_{\vec{r}}$ changes as follows

$$F_{\vec{r}} = \sum_{\lambda} \Delta_{\lambda} a_{\lambda,\vec{r}} - f_{\vec{r}} \rightarrow F'_{\vec{r}} = F_{\vec{r}} - \sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}},$$

and therefore

$$\det \left(\frac{\partial F_{\vec{r}}}{\partial \chi_{\vec{r}'}} \right) = \det(-\Delta \Delta), \quad (\text{C.196})$$

which does not depend on $a_{\lambda,\vec{r}}$ and can be brought out of the integral. Note that the obtained finite partition

$$\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \det(-\Delta \Delta) \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} a_{\lambda,\vec{r}} \right) \exp \left[-\frac{\eta}{2a |\psi|^2} E[a_{\lambda,\vec{r}}] \right],$$

is equal to (C.188), since from the relation (C.190)

$$\prod_{\vec{r}} \delta (\Delta_z a_{z,\vec{r}} + \Delta_x a_{x,\vec{r}} + \Delta_y a_{y,\vec{r}}) = \prod_{\vec{r}} \delta \left(a_{z,\vec{r}} + \frac{\Delta_x}{\Delta_z} a_{x,\vec{r}} + \frac{\Delta_y}{\Delta_z} a_{y,\vec{r}} \right) \frac{1}{\det(\Delta_z)}.$$

At the end of all this process we shall calculate Gaussian integrals and it would be very useful if the factor containing the Dirac's deltas could be rewritten in the form of an exponential. This can be achieved by means of the following trick: since the final result does not depend on the choice of $F_{\vec{r}}$ we have made, the function $f_{\vec{r}}$ included in (C.195) has no effect on the partition. If we now integrate the partition over $\prod_{\vec{r}} df_{\vec{r}}$ with an arbitrary functional $G[f_{\vec{r}}]$

$$\begin{aligned} & \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \det(-\Delta \Delta) \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \exp \left[-\frac{\eta}{2a |\psi|^2} E[a_{\lambda,\vec{r}}] \right] \times \\ & \times \prod_{\vec{r}} \int df_{\vec{r}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} a_{\lambda,\vec{r}} - f_{\vec{r}} \right) G[f_{\vec{r}}] \end{aligned} \quad (\text{C.197})$$

and we divide the result by $\mathcal{N} = \prod_{\vec{r}} \int df_{\vec{r}} G[f_{\vec{r}}]$, the partition remains the same. In order to have a Gaussian integral we choose the following functional

$$G[f_{\vec{r}}] = \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}} f_{\vec{r}}^2 \right], \quad \text{with} \quad \mathcal{N} = \left(\sqrt{\frac{2\pi a |\psi|^2}{\eta}} \right)^{|\Lambda|}, \quad (\text{C.198})$$

obtaining thus

$$\begin{aligned} \mathcal{Z} &= \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{(3+1)|\Lambda|} \det(-\Delta \Delta) \prod_{\vec{r}} \int df_{\vec{r}} \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \sum_{\{m_{\mu,\vec{r}}\}} \times \\ & \times \exp \left[-\frac{\eta}{2a |\psi|^2} \left(E[a_{\lambda,\vec{r}}, m_{\mu,\vec{r}}] + \sum_{\vec{r}} f_{\vec{r}}^2 \right) \right] \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} a_{\lambda,\vec{r}} - f_{\vec{r}} \right). \end{aligned} \quad (\text{C.199})$$

Integration over $df_{\vec{r}}$ produces finally a partition function where we need not to be worried about which fields have to be factorized out or about infinities

$$\begin{aligned} \mathcal{Z} = & \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{4|\Lambda|} \det(-\Delta\Delta) \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \sum_{\{m_{\mu},\vec{r}\}} \exp \left[-i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}} \right] \times \\ & \times \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda,\nu} a_{\lambda,\vec{r}} \left(-\Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} \right) \right] \times \\ & \times \exp \left[-\frac{\eta}{2a|\psi|^2} \underbrace{\left(\sum_{\vec{r},\lambda,\nu} (\Delta_{\lambda} a_{\lambda,\vec{r}}) (\Delta_{\nu} a_{\nu,\vec{r}}) \right)}_{-\sum_{\vec{r},\lambda,\nu} a_{\lambda,\vec{r}} \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}}} \right]. \end{aligned} \quad (\text{C.200})$$

The added term coming from the series of Dirac's delta is cancelled out by the second term inside in the exponent, remaining thus only $\sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} (-\Delta\Delta) a_{\lambda,\vec{r}}$. Instead of the $G[f_{\vec{r}}]$ of equation (C.198) we could as well choose a generalization of it, containing a parameter α (with no influence on the partition, since this does not depend on $G[f_{\vec{r}}]$)

$$G[f_{\vec{r}}] = \exp \left[-\frac{1}{\alpha} \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} f_{\vec{r}}^2 \right], \quad \text{with} \quad \mathcal{N} = \left(\sqrt{\frac{\alpha 2\pi a|\psi|^2}{\eta}} \right)^{|\Lambda|}, \quad (\text{C.201})$$

and then the partition function becomes

$$\begin{aligned} \mathcal{Z} = & \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{4|\Lambda|} \frac{\det(-\Delta\Delta)}{(\sqrt{\alpha})^{|\Lambda|}} \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \sum_{\{m_{\mu},\vec{r}\}} \exp \left[-i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}} \right] \times \\ & \times \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda,\nu} a_{\lambda,\vec{r}} \left(-\Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} - \frac{1}{\alpha} \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} \right) \right]. \end{aligned} \quad (\text{C.202})$$

The case $\alpha = 1$ is called the Feynman gauge, the choice we have used in the previous (C.200). The case $\alpha = 0$ is called the Landau gauge; note that in $G[\Delta_{\lambda} a_{\lambda,\vec{r}}] = \exp \left[-\frac{1}{\alpha} \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} (\Delta_{\lambda} a_{\lambda,\vec{r}})^2 \right]$, limit $\alpha \rightarrow 0$ forces $\Delta_{\lambda} a_{\lambda,\vec{r}} = 0$, since on the contrary, the exponential would vanish.

We shall continue the discussion of the 3-dimensional XY model with partition function (C.200)

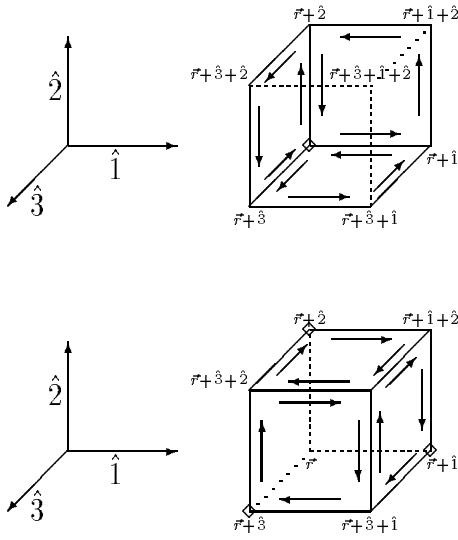
$$\begin{aligned} \mathcal{Z} = & \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{4|\Lambda|} \det(-\Delta\Delta) \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \sum_{\{m_{\mu},\vec{r}\}} \times \\ & \times \exp \left[-i2\pi \sum_{\vec{r},\mu,\nu,\lambda} a_{\lambda,\vec{r}} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}} - \frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda,\nu} a_{\lambda,\vec{r}} \left(-\Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} \right) \right]. \end{aligned} \quad (\text{C.203})$$

3. Looking at the last equation, the dislocation field at site $\vec{r}$ is given, as in the 2-dimensional case, by the curl of the $m_{\mu,\vec{r}}$ fields

$$M_{\lambda,\vec{r}} = \sum_{\nu,\mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}}. \quad (\text{C.204})$$

Note that without a constraint on the sum over $m_{\mu,\vec{r}}$ as (C.172), we would obtain the same $M_{\lambda,\vec{r}}$ arising from an infinite number of different $m_{\mu,\vec{r}}$ configurations. Field $M_{\lambda,\vec{r}}$ has vector character and fulfils the following relation

$$\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}} = 0, \quad (\text{C.205})$$



Remembering the definition $m_{-\mu,\vec{r}+\hat{\mu}} = m_{\mu,\vec{r}}$, this equation can be proved from the figure (left). Fields $m_{\mu,\vec{r}}$ are represented by arrows. For the upper figure

$$M_{3,\vec{r}} = m_{1,\vec{r}} + m_{2,\vec{r}+\hat{1}} - m_{1,\vec{r}+\hat{2}} - m_{2,\vec{r}}$$

$$M_{1,\vec{r}} = m_{2,\vec{r}} + m_{3,\vec{r}+\hat{2}} - m_{2,\vec{r}+\hat{3}} - m_{3,\vec{r}}$$

$$M_{2,\vec{r}} = m_{3,\vec{r}} + m_{1,\vec{r}+\hat{3}} - m_{3,\vec{r}+\hat{1}} - m_{1,\vec{r}}$$

and for the lower figure

$$-M_{3,\vec{r}+\hat{3}} = m_{2,\vec{r}+\hat{3}} + m_{1,\vec{r}+\hat{3}+\hat{2}} - m_{2,\vec{r}+\hat{3}+\hat{1}} - m_{1,\vec{r}+\hat{3}}$$

$$-M_{1,\vec{r}+\hat{1}} = m_{3,\vec{r}+\hat{1}} + m_{2,\vec{r}+\hat{3}+\hat{1}} - m_{3,\vec{r}+\hat{1}+\hat{2}} - m_{2,\vec{r}+\hat{1}}$$

$$-M_{3,\vec{r}+\hat{2}} = m_{1,\vec{r}+\hat{2}} + m_{3,\vec{r}+\hat{1}+\hat{2}} - m_{1,\vec{r}+\hat{3}+\hat{2}} - m_{3,\vec{r}+\hat{2}}$$

In equation (C.205) are contained the sum of the four $m_{\mu,\vec{r}}$ on each side or plaquette of the unit cube. The sum vanishes since each $m_{\mu,\vec{r}}$ appears twice, once in the contrary direction to the other time. Physically, the equation states that the vorticity contained in the three back plaquettes ($M_{3,\vec{r}}, M_{1,\vec{r}}, M_{2,\vec{r}}$) is conserved and get distributed among the front plaquettes. In the continuum case it would mean that vector $\vec{M}(\vec{r})$ has vanishing divergence $\text{div } \vec{M}(\vec{r}) = \vec{0}$. In the 2-dimensional XY model, $M_{\vec{r}}$ was only constrained by the diverging propagator part $D(0)$ to build pairs with dislocations of opposite vorticity. In three dimensions there is no diverging $D(0)$ (as we shall see now) but $M_{\lambda,\vec{r}}$ has to fulfil equation (C.205), which states that $M_{\lambda,\vec{r}}$ build closed loops. The positive and negative dislocation points bound in a pair for the 2-dimensional case can be viewed as the crossing points of the dislocation loop $M_{\lambda,\vec{r}}$ with the 2-dimensional lattice.

4. The Green's function of the discrete Laplace's operator is given by ($a = 1$)

$$D_{\vec{r},\vec{r}'} = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} \frac{e^{i\vec{k} \cdot (\vec{r}-\vec{r}')}}{6 - 2\cos k_x - 2\cos k_y - 2\cos k_z}. \quad (\text{C.206})$$

As distinct from the 2-dimensional propagator, $D(0)$ is non-divergent for $\vec{k} \approx \vec{0}$

$$\frac{1}{\pi^3} dk_x dk_y dk_z \frac{1}{k_x^2 + k_y^2 + k_z^2} = \frac{4}{\pi} \frac{k^2 dk}{k^2} = \frac{4}{\pi} dk \quad \Rightarrow \quad D(0) = \dots + \lim_{k \rightarrow 0} \frac{4}{\pi} k,$$

and it is approximately equal to 0.253 [18]. Since $D(0)$ is finite, we need not to subtract it from the propagator for distances larger than the lattice spacing $D'_{\vec{r}, \vec{r}'}$ (as was necessary for the 2-dimensional XY model), which is given by the continuum 3-dimensional Coulomb propagator¹⁴ $1/|\vec{r} - \vec{r}'|$

$$D_{\vec{r}, \vec{r}'} = D(0)\delta_{\vec{r}, \vec{r}'} + D'_{\vec{r}, \vec{r}'} \quad \text{with} \quad \begin{cases} D(0) \approx 0.253 \\ D'_{\vec{r}, \vec{r}'} = 0 \\ D'_{\vec{r}, \vec{r}'} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}, \quad |\vec{r} - \vec{r}'| \neq 0 \end{cases} \quad (\text{C.207})$$

By integrating over $da_{\lambda, \vec{r}}$ the equation (C.203) and taking into account the constraint on $M_{\lambda, \vec{r}}$, the partition function takes the following form (similar to (C.117))

$$\begin{aligned} \mathcal{Z} = \mathcal{Z}_0 \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \times \\ \times \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 - \frac{\pi a |\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\lambda} \frac{M_{\lambda, \vec{r}} M_{\lambda, \vec{r}'}}{|\vec{r} - \vec{r}'|} \right], \end{aligned} \quad (\text{C.208})$$

with $\mathcal{Z}_0$ equal to

$$\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{4|\Lambda|} \det(-\Delta\Delta) \left(\sqrt{\frac{2\pi a |\psi|^2}{\eta}} \right)^{3|\Lambda|} \left(\frac{1}{\sqrt{\det(-\Delta\Delta)}} \right)^3.$$

$\mathcal{Z}_0$ corresponds to the partition function of compatible deformations

$$\mathcal{Z}_0 = \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left[-\frac{a |\psi|^2}{2\eta} \sum_{\vec{r}} \varphi_{\vec{r}} \left(-\sum_{\mu=1,2,3} \Delta_{\mu} \Delta_{\mu} \varphi_{\vec{r}-\hat{\mu}} \right) \right], \quad (\text{C.209})$$

and is equal to $(\sqrt{\eta/2\pi a |\psi|^2})^{|\Lambda|} (\sqrt{-\Delta\Delta})^{-1}$. The sum with the product of Dirac's deltas $\sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right)$ in equation (C.208) contains the constraint of

¹⁴The deduction of this power behaviour for the Green's function is again made using $\frac{d}{dx} \vartheta(x-x') = \delta(x-x')$ and $\delta(x)\delta(y)\delta(z) = 1/4\pi r^2 \delta(r)$

$$\sum_{\mu=1,2,3} \partial_{\mu}^2 D(r) = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} D(r) = -\frac{1}{4\pi r^2} \delta(r) = -\frac{1}{4\pi r^2} \frac{d}{dr} \vartheta(r) \quad \Rightarrow \quad \frac{d}{dr} D(r) = -\frac{1}{4\pi r^2}.$$

closed dislocation loops (instead of sum $\sum'_{\{M_{\vec{r}}\}}$ over dislocation dipoles in the 2-dimensional model).

Carrying out the $da_{\lambda,\vec{r}}$ integrations in the correlation's numerator of (C.203), we obtain

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \frac{1}{\mathcal{Z}} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\mu} L_{\mu,\vec{r}}^2 \right] \mathcal{Z}_0 \sum_{\{M_{\lambda,\vec{r}}\}} \sum_{\{M_{\lambda,\vec{r}'}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}}, 0 \right) \times (C.210)$$

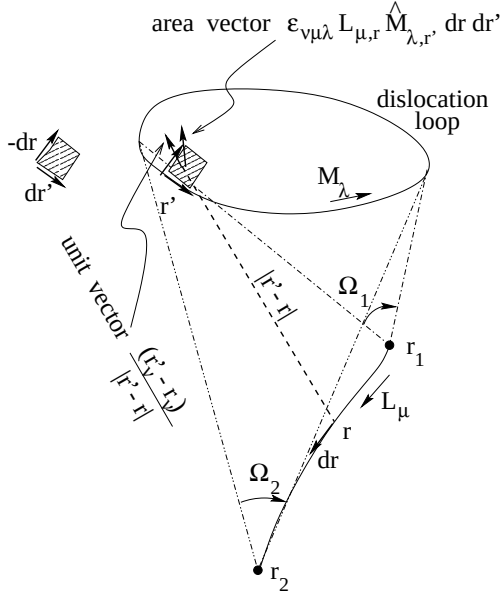
$$\exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r},\vec{r}',\lambda} \left(2\pi M_{\lambda,\vec{r}} - \frac{i\eta}{a|\psi|^2} \sum_{\mu,\nu} \epsilon_{\lambda\nu\mu} \Delta_{\vec{r}}^{\nu} L_{\mu,\vec{r}} \right) D_{\vec{r},\vec{r}'} \left(2\pi M_{\lambda,\vec{r}'} - \frac{i\eta}{a|\psi|^2} \sum_{\mu',\nu'} \epsilon_{\lambda\nu'\mu'} \Delta_{\vec{r}'}^{\nu'} L_{\mu',\vec{r}'} \right) \right]$$

5. For distances $|\vec{r}' - \vec{r}|$ larger than the lattice spacing, we can approximate $D_{\vec{r},\vec{r}'}$ by $1/4\pi|\vec{r}' - \vec{r}|$, and thus sum $\sum_{\vec{r},\vec{r}'} \sum_{\lambda,\mu,\nu} \epsilon_{\lambda\nu\mu} (\Delta_{\vec{r}}^{\nu} L_{\mu,\vec{r}}) D_{\vec{r},\vec{r}'} M_{\lambda,\vec{r}'}$ can be rewritten as follows

$$\sum_{\vec{r},\vec{r}' \neq \vec{r}} \sum_{\lambda,\mu,\nu} \epsilon_{\lambda\nu\mu} (\Delta_{\vec{r}}^{\nu} L_{\mu,\vec{r}}) D'_{\vec{r},\vec{r}'} M_{\lambda,\vec{r}'} = - \sum_{\vec{r},\vec{r}' \neq \vec{r},\lambda,\nu,\mu} M_{\lambda,\vec{r}'} \epsilon_{\lambda\nu\mu} L_{\mu,\vec{r}} \Delta_{\vec{r}}^{\nu} D'_{\vec{r}-\hat{\nu},\vec{r}'} . \quad (C.211)$$

For a closed dislocation loop, a concise expression for (C.211) can be easily calculated. Approximating differences and sums by derivatives and integrals respectively (something which can be made for distances larger than the lattice spacing), one

obtains $\partial_{\nu} \left(\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\frac{1}{4\pi} \frac{r_{\nu} - r'_{\nu}}{|\vec{r} - \vec{r}'|^3}$ and thus sum (C.211) for a closed loop can be written as $-|M| \frac{1}{4\pi} \int_C dr \oint_{\text{disloc. loop}} dr' \epsilon_{\nu\mu\lambda} \frac{r'_{\nu} - r_{\nu}}{|\vec{r}' - \vec{r}|^3} L_{\mu}(\vec{r}) \hat{M}_{\lambda}(\vec{r}')$ with



C the path from $\vec{r}_1$ to $\vec{r}_2$ and $\hat{M}_{\lambda}$ an unit vector along the dislocation loop. From the figure we can see that the infinitesimal solid angle $\delta\Omega_{\vec{r}}$ is given by

$$\delta\Omega_{\vec{r}} = \sum_{\nu,\mu,\lambda} \frac{(r'_{\nu} - r_{\nu})}{|\vec{r}' - \vec{r}|^3} \epsilon_{\nu\mu\lambda} L_{\mu,\vec{r}} \hat{M}_{\lambda,\vec{r}'} .$$

and the integral along the closed dislocation loop

$$\oint_{\text{disloc. loop}} dr' \epsilon_{\nu\mu\lambda} \frac{r'_{\nu} - r_{\nu}}{|\vec{r}' - \vec{r}|^3} L_{\mu}(\vec{r}) \hat{M}_{\lambda}(\vec{r}') ,$$

is the difference in the loop's parallax solid angle between watching from $\vec{r}$ and watching from $\vec{r} + d\vec{r}$, with both $\vec{r}$ and $\vec{r} + d\vec{r}$ on the path C

$$\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2} = \int_C dr \oint_{\text{disloc. loop}} dr' \epsilon_{\nu\mu\lambda} \frac{r'_{\nu} - r_{\nu}}{|\vec{r}' - \vec{r}|^3} L_{\mu}(\vec{r}) \hat{M}_{\lambda}(\vec{r}') .$$

For a closed loop of vorticity (or Burgers vector) $|M|$, the sum (C.211) is equal to

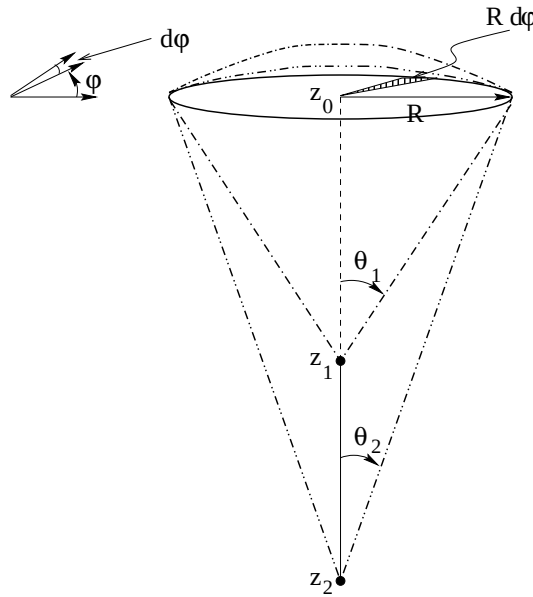
$$\sum_{\vec{r}, \vec{r}'} \sum_{\nu, \mu, \lambda} M_{\lambda, \vec{r}'} \epsilon_{\nu \mu \lambda} \Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} = -|M| \frac{1}{4\pi} (\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2}) , \quad (\text{C.212})$$

and in general

$$\sum_{\vec{r}, \vec{r}'} \sum_{\nu, \mu, \lambda} M_{\lambda, \vec{r}'} \epsilon_{\nu \mu \lambda} \Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} = -\frac{1}{4\pi} \sum_{\vec{r}', \lambda} M_{\lambda, \vec{r}'} \left(\delta\Omega_{\lambda, \vec{r}'}^{(\vec{r}_1)} - \delta\Omega_{\lambda, \vec{r}'}^{(\vec{r}_2)} \right) . \quad (\text{C.213})$$

In the last equation, $\delta\Omega_{\lambda, \vec{r}'}^{(\vec{r}_1)}$ is the solid parallax, observed from $\vec{r}_1$, of the loop sector stretched by the loop segment between $\vec{r}'$ and $\vec{r}' + \hat{M}_{\lambda, \vec{r}'}$. As an example of this result, let us consider a vertical path C along the $\mu = 3$ direction and a segment $Rd\varphi$ of circular loop of vorticity M placed on the plane z_0 . The parallax solid angle of the whole loop seen from $\vec{r}_1$ is equal to

$$\Omega_{\vec{r}_1} = \int_0^{\theta_1} d\theta \, 2\pi \sin \theta = 2\pi(1 - \cos \theta_1) ,$$



For the loop sector stretched by the segment $Rd\varphi$, the solid parallax is

$$\delta\Omega_{z_0}^{(\vec{r}_1)} = d\varphi \int_0^{\theta_1} d\theta \sin \theta = d\varphi(1 - \cos \theta_1) ,$$

and $\sum_{\lambda} \hat{M}_{\lambda, \vec{r}'} \sum_{\vec{r}, \nu, \mu} \epsilon_{\nu \mu \lambda} \Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}$ takes the form

$$\begin{aligned} & + Rd\varphi \frac{1}{4\pi} \int_{z_1}^{z_2} dz \frac{R}{[R^2 + (z_0 - z)^2]^{\frac{3}{2}}} = \\ & = -Rd\varphi \frac{1}{4\pi} \left[\frac{(z_0 - z)}{\sqrt{R^2 + (z_0 - z)^2}} \right]_{z=z_1}^{z=z_2} = \\ & = -\frac{d\varphi}{4\pi} (\cos \theta_2 - \cos \theta_1) = -\frac{\delta\Omega_{z_0}^{(\vec{r}_1)} - \delta\Omega_{z_0}^{(\vec{r}_2)}}{4\pi} . \end{aligned}$$

6. The other sum in the correlation (C.210) which must be evaluated is

$$\begin{aligned} & \sum_{\vec{r}, \vec{r}', \lambda} \sum_{\mu, \mu', \nu, \nu'} \epsilon_{\lambda \nu \mu} \epsilon_{\lambda \nu' \mu'} (\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}}) (\Delta_{(\vec{r}')}^{\nu'} L_{\mu', \vec{r}'}) D_{\vec{r}, \vec{r}'} = \\ & = \underbrace{\sum_{\vec{r}, \vec{r}'} \sum_{\mu, \nu} (\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}}) (\Delta_{(\vec{r}')}^{\nu} L_{\mu, \vec{r}'}) D_{\vec{r}, \vec{r}'}}_{\text{}} - \sum_{\vec{r}, \vec{r}'} \sum_{\mu, \nu} (\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}}) (\Delta_{(\vec{r}')}^{\mu} L_{\nu, \vec{r}'}) D_{\vec{r}, \vec{r}'} . \quad (\text{C.214}) \\ & + \sum_{\vec{r}, \vec{r}'} \sum_{\mu} L_{\mu, \vec{r}} L_{\mu, \vec{r}'} \sum_{\nu} \Delta_{(\vec{r})}^{\nu} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}-\hat{\nu}, \vec{r}'-\hat{\nu}} \end{aligned}$$

The propagator depends only on the distance $|\vec{r} - \vec{r}'|$ and therefore $D_{\vec{r}-\hat{\nu}, \vec{r}'-\hat{\nu}} = D_{\vec{r}, \vec{r}'}$. Moreover, due to this dependence the propagator fulfils the following property

$$\Delta_{(\vec{r}')\mu} D_{\vec{r}, \vec{r}'} = D_{\vec{r}, \vec{r}'+\hat{\mu}} - D_{\vec{r}, \vec{r}'} = D_{\vec{r}-\hat{\mu}, \vec{r}'} - D_{\vec{r}, \vec{r}'} = -\Delta_{(\vec{r})\mu} D_{\vec{r}-\hat{\mu}, \vec{r}'} . \quad (\text{C.215})$$

Using this property and the propagator definition, $-\sum_{\nu} \Delta_{(\vec{r})\nu} \Delta_{(\vec{r}')\nu} D_{\vec{r}-\hat{\nu}, \vec{r}'} = \delta_{\vec{r}, \vec{r}'}$, the first term in the previous sum is equal to $\sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2$ and is compensated by the term outside the integral in correlation function (C.210). By regrouping the terms inside the second sum in (C.214), it is obtained

$$\begin{aligned} \sum_{\substack{\vec{r}, \vec{r}' \\ \mu, \nu}} (\Delta_{(\vec{r})\nu} L_{\mu, \vec{r}}) (\Delta_{(\vec{r}')\mu} L_{\nu, \vec{r}'}) D_{\vec{r}, \vec{r}'} &= \sum_{\substack{\vec{r}, \vec{r}' \\ \mu, \nu}} \left[\begin{array}{ll} +L_{\mu, \vec{r}+\hat{\nu}} L_{\nu, \vec{r}'+\hat{\mu}} D_{\vec{r}, \vec{r}'} & \rightarrow +L_{\mu, \vec{r}-\hat{\mu}} L_{\nu, \vec{r}'+\hat{\nu}} D_{\vec{r}, \vec{r}'} \\ -L_{\mu, \vec{r}} L_{\nu, \vec{r}'+\hat{\mu}} D_{\vec{r}, \vec{r}'} & \rightarrow -L_{\mu, \vec{r}-\hat{\mu}} L_{\nu, \vec{r}'} D_{\vec{r}, \vec{r}'} \\ -L_{\mu, \vec{r}+\hat{\nu}} L_{\nu, \vec{r}'} D_{\vec{r}, \vec{r}'} & \rightarrow -L_{\mu, \vec{r}} L_{\nu, \vec{r}'+\hat{\nu}} D_{\vec{r}, \vec{r}'} \\ +L_{\mu, \vec{r}} L_{\nu, \vec{r}'} D_{\vec{r}, \vec{r}'} & \end{array} \right] = \\ &= \sum_{\vec{r}, \vec{r}'} D_{\vec{r}, \vec{r}'} \left(\sum_{\mu} \Delta_{(\vec{r})\mu} L_{\mu, \vec{r}-\hat{\mu}} \right) \left(\sum_{\nu} \Delta_{(\vec{r}')\nu} L_{\nu, \vec{r}'+\hat{\nu}} \right) , \end{aligned} \quad (\text{C.216})$$

where it was used that the propagator depends only on the absolute value of the distance

$$\sum_{\vec{r}, \vec{r}'} L_{\mu, \vec{r}+\hat{\nu}} D_{\vec{r}, \vec{r}'} L_{\nu, \vec{r}'+\hat{\mu}} = \sum_{\vec{r}, \vec{r}'} L_{\mu, \vec{r}} \underbrace{D_{\vec{r}-\hat{\nu}, \vec{r}'+\hat{\mu}}}_{D_{\vec{r}+\hat{\mu}, \vec{r}'+\hat{\nu}}} L_{\nu, \vec{r}'} = \sum_{\vec{r}, \vec{r}'} L_{\mu, \vec{r}-\hat{\mu}} D_{\vec{r}, \vec{r}'} L_{\nu, \vec{r}'+\hat{\nu}} .$$

Substituting these results into (C.214) and considering the equation which $L_{\mu, \vec{r}}$ fulfils, $\sum_{\mu} \Delta_{\mu} L_{\mu, \vec{r}-\hat{\mu}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}$, we obtain

$$\begin{aligned} \sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 - \sum_{\vec{r}, \vec{r}', \lambda} \sum_{\mu, \mu', \nu, \nu'} \epsilon_{\lambda \nu \mu} \epsilon_{\lambda \nu' \mu'} (\Delta_{(\vec{r})\nu} L_{\mu, \vec{r}}) (\Delta_{(\vec{r}')\nu'} L_{\mu', \vec{r}'}) D_{\vec{r}, \vec{r}'} &= \\ &= \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) = -\frac{1}{2\pi} \frac{1}{|\vec{r}_1 - \vec{r}_2|} + 2D(0) . \end{aligned} \quad (\text{C.217})$$

Constant factor $\exp[-\eta/a|\psi|^2 D(0)]$ has no dependence on the distance and plays no role in the different behaviours presented by the correlation function. Therefore it will be absorbed into the correlation and not further written in the following.

7. Finally, the correlation function takes the form

$$\begin{aligned} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle &= \exp \left[\frac{\eta}{4\pi a |\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|} \right] \frac{\mathcal{Z}_0}{\mathcal{Z}} \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \times \\ &\exp \left[-\frac{i}{2} \sum_{\vec{r}', \lambda} M_{\lambda, \vec{r}'} \left(\delta \Omega_{\lambda, \vec{r}'}^{(\vec{r}_1)} - \delta \Omega_{\lambda, \vec{r}'}^{(\vec{r}_2)} \right) - \frac{2\pi^2 a |\psi|^2}{\eta} D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 - \frac{\pi a |\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}, \lambda} \frac{M_{\lambda, \vec{r}'} M_{\lambda, \vec{r}}}{|\vec{r} - \vec{r}'|} \right] \end{aligned} \quad (\text{C.218})$$

with $\mathcal{Z}$ given by equation (C.208) (or, up to the factor $\mathcal{Z}_0$, by equation (C.220)). This result is very similar to that of the 2-dimensional XY model (C.127), where instead of the dislocations points building pairs (or dipoles), here they are closed loops. The other difference is the form of the propagator, which makes the first factor change from $\exp[-(\eta/2\pi|\psi|^2)\ln|\vec{r}_2 - \vec{r}_1|]$ to $\exp[(\eta/4\pi a|\psi|^2)\mathbf{1}/|\vec{r}_2 - \vec{r}_1|]$.

The correlation can be also written as follows

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \exp\left(\frac{\eta}{4\pi a|\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|}\right) \left\langle \prod_{\vec{r}', \lambda} \exp\left[-\frac{i}{2} M_{\lambda, \vec{r}'} \left(\delta\Omega_{\lambda, \vec{r}'}^{(\vec{r}_1)} - \delta\Omega_{\lambda, \vec{r}'}^{(\vec{r}_2)}\right)\right] \right\rangle_M \quad (\text{C.219})$$

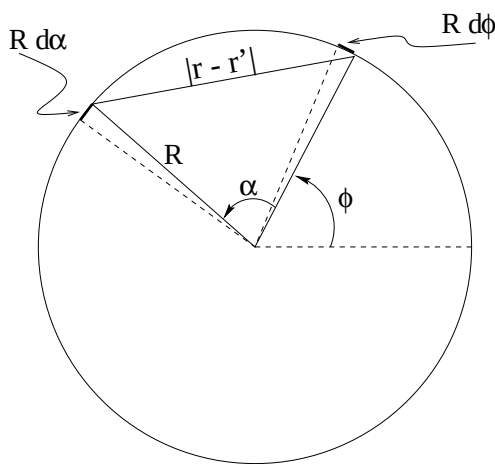
with the expectation value $\langle \rangle_M$ calculated using the following partition function

$$\begin{aligned} \mathcal{Z}_M &= \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0\right) \times \\ &\times \exp\left[-\frac{2\pi^2 a|\psi|^2}{\eta} D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 - \frac{\pi a|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\lambda} M_{\lambda, \vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} M_{\lambda, \vec{r}'}\right]. \end{aligned} \quad (\text{C.220})$$

C.4.1 Phases in the 3-dimensional XY model

The discussion about the correlation function for the 3-dimensional case runs parallel to that of the 2-dimensional model. The energy of a dislocation pair in the 2-dimensional case (C.129) was proportional to $|\psi|^2/\eta \ln d/a$, with d the distance between the two dislocations and a the lattice spacing. Due to the logarithm, large pairs were strongly suppressed at large values of $|\psi|^2/\eta$. In our case, one can read off equation (C.220) that the energy of a dislocation loop of radius R (for the sake of clarity, we shall consider a circular loop defined by $M_{\lambda, \vec{r}} = |M|R\hat{\phi}$) is given by

$$\frac{2\pi^2 a|\psi|^2}{\eta} |M|^2 D(0) 2\pi R + \frac{\pi a|\psi|^2}{2\eta} |M|^2 \oint_{\phi=0}^{\phi=2\pi} \int_{\alpha=\frac{a}{R}}^{\alpha=2\pi-\frac{a}{R}} d\alpha d\phi \frac{R^2 \cos \alpha}{2R \sin \frac{\alpha}{2}},$$



where the second integration over $d\alpha$ is from a/R to $2\pi - a/R$ and not $[0, 2\pi)$ because the sum in the partition function is $\sum_{\vec{r}, \vec{r}' \neq \vec{r}}$. Carrying out the second integral

$$\begin{aligned} 2\pi R \int_{\frac{a}{R}}^{\pi} d\alpha \frac{\cos \alpha}{\sin \frac{\alpha}{2}} &= 4\pi R \left[\ln \tan \frac{\alpha}{4} \right]_{\frac{a}{R}}^{\pi} + \\ &+ 4\pi R \left[\cos \frac{\alpha}{2} \right]_{\frac{a}{R}}^{\pi}, \end{aligned}$$

we obtain that the energy of a dislocation loop grows like $R \ln R$ for large R

$$\frac{2\pi^2 a|\psi|^2}{\eta} |M|^2 \left[(2\pi D(0) - 1 + \ln 4)R + R \ln \frac{R}{a} \right]. \quad (\text{C.221})$$

At large $a|\psi|^2/\eta$, large dislocation loops are strongly suppressed in the partition and in the correlation due to the high energy they cost: in this range as first approximation we can suppose all $M_{\lambda,\vec{r}} = 0$ and thus $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } \frac{a|\psi|^2}{\eta}} \approx \exp\left(\frac{\eta}{4\pi a|\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|}\right)$. Note that in three dimensions the XY model does have an ordered state, since for large values of $a|\psi|^2/\eta$ the correlation function does not vanish for large distances. At small $a|\psi|^2/\eta$, it decays exponentially (C.169)

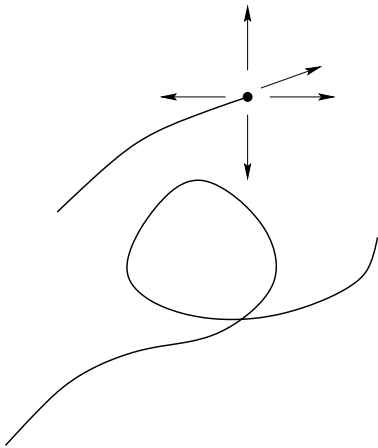
$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } \frac{a|\psi|^2}{\eta}} \approx \exp\left(\frac{\eta}{4\pi a|\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|}\right) \xrightarrow{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} 1, \quad (\text{C.222})$$

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{small } \frac{a|\psi|^2}{\eta}} = e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/a|\psi|^2)} \xrightarrow{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} 0. \quad (\text{C.223})$$

The phase structure is similar to that in the 2-dimensional Ising model and there is a critical point separating the ordered from the disordered phase (just at the critical point, the correlation decays like a power of $|\vec{r}_2 - \vec{r}_1|$). Note that the factor $e^{-\frac{\eta}{a|\psi|^2} D(0)}$ we have neglected after equation (C.217) does not change in anything this result: for large $a|\psi|^2/\eta$ the factor is approximately equal to 1, and for the other range, it is only multiplying the vanishing expectation value.

In the 3-dimensional case, it is more difficult to evaluate the critical $a|\psi|^2/\eta$ value at which the transition from one behaviour to the other one takes place, due to the many possible forms of the dislocation loop and orientations of the loop in relation to the path between $\vec{r}_1$ and $\vec{r}_2$. However we can estimate this critical value by means of the following argument, taken from the paper by Banks, Myerson and Kogut [18]: as we approach the transition point, the size and the distribution of the loops become random and we can suppose that the interactions between dislocations, and between the elements of the same dislocation loop, are cancelled out against each other. In this case, the interaction term $\sum_{\vec{r}, \vec{r}' \neq \vec{r}, \lambda} M_{\lambda, \vec{r}} M_{\lambda, \vec{r}'} 1/|\vec{r} - \vec{r}'|$ can be neglected. We suppose further that at the beginning of the transition, only $|M_{\lambda, \vec{r}}| = 1$ occurs and the contribution of a loop of length L to the partition function is then

$$N(L) \exp\left[-\frac{2\pi^2 a|\psi|^2}{\eta} D(0)L\right], \quad (\text{C.224})$$



where $N(L)$ is the number of closed loops of length L . Let us start at any arbitrary point: the number of walks with L steps which end anywhere and never turn directly on themselves is equal to 5^L , since in three dimensions at each new step there are 5 possible directions available. Although these walks cannot immediately turn back in the opposite direction of the previous step, they can firstly turn aside and then come back.

Not all of these 5^L walks are closed loops and some of them can re-cross a site which was already covered (as in the figure). In this case we would have two loops instead

of one, neither of them with length L . We can roughly estimate the number of closed walks in the following way: the average effective distance covered by these walks is not L (straight walks) but $\sqrt{L}$, meaning that many of the walks partially turn back and some of them are closed ¹⁵. Therefore we have to correct 5^L by a fraction $1/L^{\frac{3}{2}}$ which is the ratio between the volume of the sphere filled by those walks which can turn back, and that filled by straight walks. For large L , this correction is unimportant compared with 5^L and the number of closed loops of length L is roughly given by

$$N(L) \sim f(L)\mu^L \quad \text{with} \quad \mu \lesssim 5, \quad (\text{C.225})$$

where $f(L)$ is a function which varies with L much more slowly than μ^L . In order not to take into account the self-crossing walks, 5 must be replaced by $\mu \lesssim 5$. Substituting into (C.221), we obtain

$$f(L)\mu^L \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 \right] = f(L) \exp \left[L \left(\ln \mu - \frac{2\pi^2 a |\psi|^2}{\eta} D(0) \right) \right],$$

with $D(0) = 0.253$. The contribution of very large dislocations to the partition and to the correlation is nonvanishing if the exponent of the last equation is positive

$$\frac{a |\psi|^2}{\eta} \leq \frac{\ln \mu}{2\pi^2 D(0)} \Rightarrow \left(\frac{a |\psi|^2}{\eta} \right)_{\text{critical}} = \frac{\ln \mu}{2\pi^2 D(0)} \approx \frac{\ln 5}{2\pi^2 \times 0.253} = 0.322. \quad (\text{C.226})$$

From Monte Carlo simulations, the estimated value of the critical value is given by $(a |\psi|^2 / \eta)_{\text{crit}} = 0.33 \pm 0.005$ [30]. Specific to the 3-dimensional XY model there is another argument to estimate the critical point [18], which gives a similar result.

Note. Critical point in the 3-dimensional XY model.

Let us return to partition function (C.171) after the integration over $d\varphi_{\vec{r}}$, which has produced the constraint $\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r} - \hat{\mu}} = 0$

$$\mathcal{Z} = \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \prod_{\vec{r}, \mu} \int_{-\infty}^{\infty} db_{\mu, \vec{r}} \sum_{\{m_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r} - \hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 - i2\pi \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} m_{\mu, \vec{r}} \right].$$

Using the Poisson sum formula ([89], p.109) $\sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} dx F(x) e^{-i2\pi m x} = \sum_{l=-\infty}^{+\infty} F(l)$ we can rewrite the above partition function in the following form

$$\mathcal{Z} = \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \sum_{\{l_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r} - \hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 \right]. \quad (\text{C.227})$$

¹⁵One may remember the problem about the typical distance covered by a person who has the same probability of taking a step forward as a step back. After $n+1$ steps, the covered distance x_{n+1} is related to the distance after n steps by $x_{n+1} = x_n \pm 1$ (the length of the step is taken as 1). The average value of the square of this equation is $\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + 1$ because of $\langle x_n \rangle = 0$ (the person has the same probability for going in both directions). Since $x_1 = \pm 1$ and $\langle x_1^2 \rangle = 1$, we obtain $\langle x_n^2 \rangle = n$ and therefore the effective distance covered on average is $\bar{x} = \sqrt{n}$.

Now let us consider our final partition function (C.220) for the case of random distribution of randomly sized loops, where the interaction terms for $\vec{r}' \neq \vec{r}$ are supposed to be canceled

$$\mathcal{Z} \approx \mathcal{Z}_0 \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 \right]. \quad (\text{C.228})$$

The cancel of interactions between different points is equivalent to the substitution of the Laplace's operator $-\sum_{\mu} \Delta_{\mu} \Delta_{\mu}$ by $D(0)^{-1}$, since the origin of the interaction comes from the operator Δ_{μ} relating two different points. Therefore in this approximation $\mathcal{Z}_0$ is given by $(\sqrt{\eta/2\pi a |\psi|^2})^{|\Lambda|} (\sqrt{D(0)})^{|\Lambda|}$. Defining

$$\tilde{\mathcal{Z}}[\beta] = \sum_{\{l_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}}, 0 \right) \exp \left[-\pi \beta \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 \right],$$

and comparing now (C.227) and (C.228), we obtain

$$\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{2|\Lambda|} \tilde{\mathcal{Z}} \left[\frac{\eta}{2\pi a |\psi|^2} \right] = \left(\sqrt{D(0)} \right)^{|\Lambda|} \tilde{\mathcal{Z}} \left[\frac{2\pi a |\psi|^2 D(0)}{\eta} \right],$$

or equivalently

$$\frac{\tilde{\mathcal{Z}} \left[\frac{\eta}{2\pi a |\psi|^2} \right]}{\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{|\Lambda|}} = \frac{\tilde{\mathcal{Z}} \left[\left(\frac{\eta}{2\pi a |\psi|^2} \right)^* \right]}{\left(\sqrt{\left(\frac{\eta}{2\pi a |\psi|^2} \right)^*} \right)^{|\Lambda|}}, \quad \text{with} \quad \left(\frac{\eta}{2\pi a |\psi|^2} \right)^* = \left(\frac{\eta}{2\pi a |\psi|^2} \right)^{-1} D(0). \quad (\text{C.229})$$

This relation states that the behaviour of our dislocation system (C.228) for high $2\pi a |\psi|^2/\eta$ (but only slightly higher than the critical value, otherwise the approximation of neglecting interactions between different points could not be made) is described by the behaviour at low $(2\pi a |\psi|^2/\eta)^*$ of (C.227) (where the integers $l_{\mu, \vec{r}}$ have directly no physical meaning); and viceversa. Decreasing $2\pi a |\psi|^2/\eta$ in the dislocation system means increasing $(2\pi a |\psi|^2/\eta)^*$ in the other system, and at the critical point both values $(2\pi a |\psi|^2/\eta)$ and $(2\pi a |\psi|^2/\eta)^*$ coincide

$$\left(\frac{\eta}{2\pi a |\psi|^2} \right) = \left(\frac{\eta}{2\pi a |\psi|^2} \right)^{-1} D(0) \implies \left(\frac{a |\psi|^2}{\eta} \right)_{\text{critical}} = \frac{1}{2\pi \sqrt{D(0)}} = 0.316, \quad (\text{C.230})$$

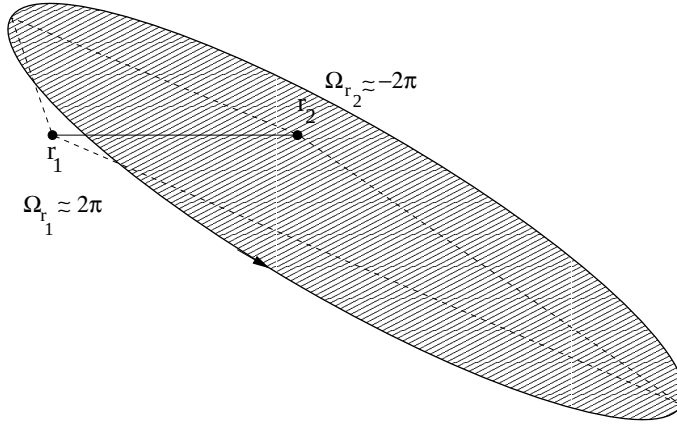
very near to the estimation in (C.226).

Returning to the discussion about the phases presented by the 3-dimensional XY model, and by considering that the loop energy grows like $R \ln R$, we expect the same phases as in two dimensions, where the energy grew like $\ln R$. At large $a |\psi|^2/\eta$, the term $a |\psi|^2/\eta R \ln R$ only allows for small loops and there is no relevant effect due to dislocations; for small $a |\psi|^2/\eta$ values, there exist a great amount of large loops, which disorder the system. We shall now consider the effect of large loops on the factor $\exp \left[-\frac{i}{2} \sum_{\vec{r}', \lambda} M_{\lambda, \vec{r}'} \left(\delta \Omega_{\lambda, \vec{r}'}^{(\vec{r}_1)} - \delta \Omega_{\lambda, \vec{r}'}^{(\vec{r}_2)} \right) \right]$, which can be written as sum over the sum over all closed loops of the difference in the solid parallax seen from $\vec{r}_1$ and from $\vec{r}_2$

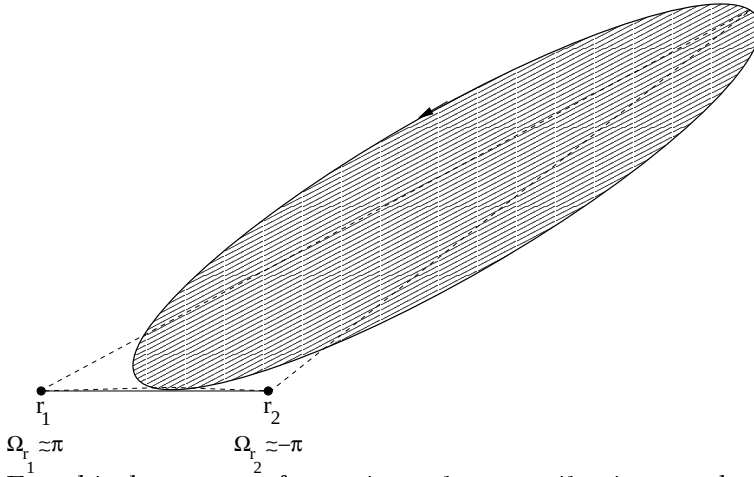
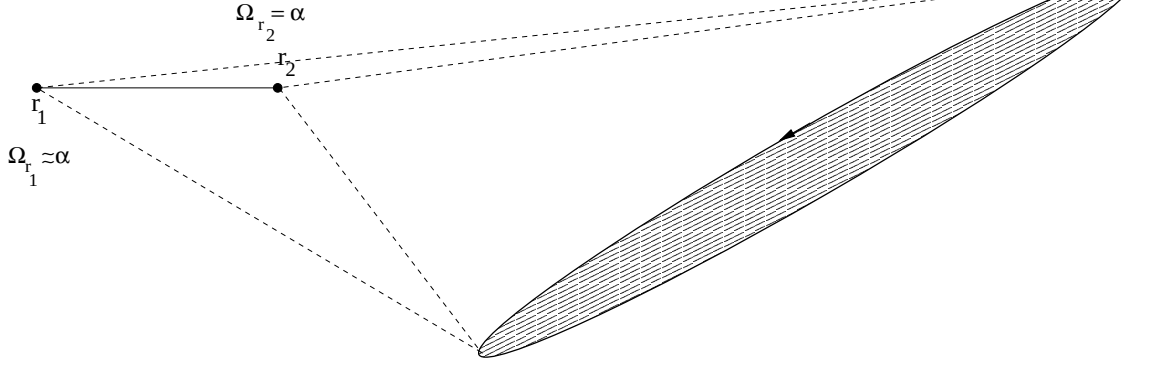
$$\exp \left[-\frac{i}{2} \sum_{\text{loops}} |M|^{\text{loop}} \left(\Omega_{\vec{r}_1}^{\text{loop}} - \Omega_{\vec{r}_2}^{\text{loop}} \right) \right].$$

At small $a |\psi|^2/\eta$ and supposing the self-cancel of interaction between loops and between the elements of the same loops, we have the following contributions, depending

on whether the loop crosses or not the path between $\vec{r}_1$ and $\vec{r}_2$.



If the loop does not cross the path, the value of $\exp\left[-\frac{i}{2}(\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2})\right]$ for this loop is very near to 1, as we can see from the figures. If the large loop lies between $\vec{r}_1$ and $\vec{r}_2$ but without crossing the path, $\Omega_{\vec{r}_1}$ and $\Omega_{\vec{r}_2}$ are approximately equal to 2π since the loop is much larger than the distance $|\vec{r}_2 - \vec{r}_1|$. If the loop is distant from the path, $\Omega_{\vec{r}_1} \approx \Omega_{\vec{r}_2}$ and the difference is zero.



In the case where a large loop crosses the path, the difference $\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2}$ becomes $\pm 2\pi$. If the loop lies as in the figure, $\Omega_{\vec{r}_1} \approx -\Omega_{\vec{r}_2} = \pi$ since the loop is much larger than the path distance. In the case where the loop is on the same plane as the path, one of the $\Omega_{\vec{r}}$ becomes 2π whereas the other one is zero, $\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2} = \pm 2\pi$.

For this last case of crossing, the contribution made by a loop of vorticity M to the numerator is equal to $(-1)^M$. Moreover, these contributions (negative for odd M) can be maintained for the whole distance $|\vec{r}_2 - \vec{r}_1|$. Let us consider the dislocation bond which rests on the path: if it has a vorticity of M , it can belong to one dislocation of the same vorticity or there can be several loops, whose total vorticity is M , which just go across this point. Always in the approximation of no interaction between different points (C.227) and supposing that all loops have approximately the

same size R , in the case of only one loop of vorticity M , the contribution $(-1)^M$ is multiplied by the factor $\exp\left[-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)M^2 2\pi R\right]$, and in the case of several loops, by $\sum_{i=1}^{i=k} \exp\left[-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)m_i^2 2\pi R\right]$ with $\sum_{i=1}^{i=k} m_i = M$

$$(-1)^M \left[e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)M^2 2\pi R} + \sum_{i=1}^{i=k} e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)m_i^2 2\pi R} \right] = (-1)^M e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)M^2 2\pi R},$$

with $2 < \mu < 3$. Since the crossing bond can be placed on every of the $|\vec{r}_2 - \vec{r}_1|$ path sites, the correlation becomes

$$\begin{aligned} & \left\langle \exp \left[-\frac{i}{2} \sum_{\text{loops}} |M|^{\text{loop}} \left(\Omega_{\vec{r}_1}^{\text{loop}} - \Omega_{\vec{r}_2}^{\text{loop}} \right) \right] \right\rangle_M = \\ & = \frac{\left[\sum_{M=-\infty}^{+\infty} (-1)^M \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} \right]^{|\vec{r}_2 - \vec{r}_1|} \left[\sum_{M=-\infty}^{+\infty} \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} \right]^{|\Lambda| - |\vec{r}_2 - \vec{r}_1|}}{\left[\sum_{M=-\infty}^{+\infty} \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} \right]^{|\vec{r}_2 - \vec{r}_1|} \left[\sum_{M=-\infty}^{+\infty} \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} \right]^{|\Lambda| - |\vec{r}_2 - \vec{r}_1|}}, \end{aligned} \quad (\text{C.231})$$

which shows the expected exponential decay, $\exp[-\kappa|\vec{r}_2 - \vec{r}_1|]$, with κ equal to

$$\kappa = -\ln \frac{\left[2 \sum_{M=0}^{\infty} (-1)^M \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} - 1 \right]^{|\vec{r}_2 - \vec{r}_1|}}{\left[2 \sum_{M=0}^{\infty} \left(e^{-\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R} \right)^{M^\mu} - 1 \right]^{|\vec{r}_2 - \vec{r}_1|}}. \quad (\text{C.232})$$

The correlation length, defined as the distance where the fields are still correlated, is equal to κ^{-1} . Now the demonstration runs completely parallel to that from equation (C.158) to (C.159), obtaining thus at small $a|\psi|^2/\eta$

$$\kappa = \ln 2 - \ln \left(\frac{2\pi^2 a|\psi|^2}{\eta}D(0)2\pi R \right) \approx \ln \frac{\eta}{a|\psi|^2} - \ln R. \quad (\text{C.233})$$

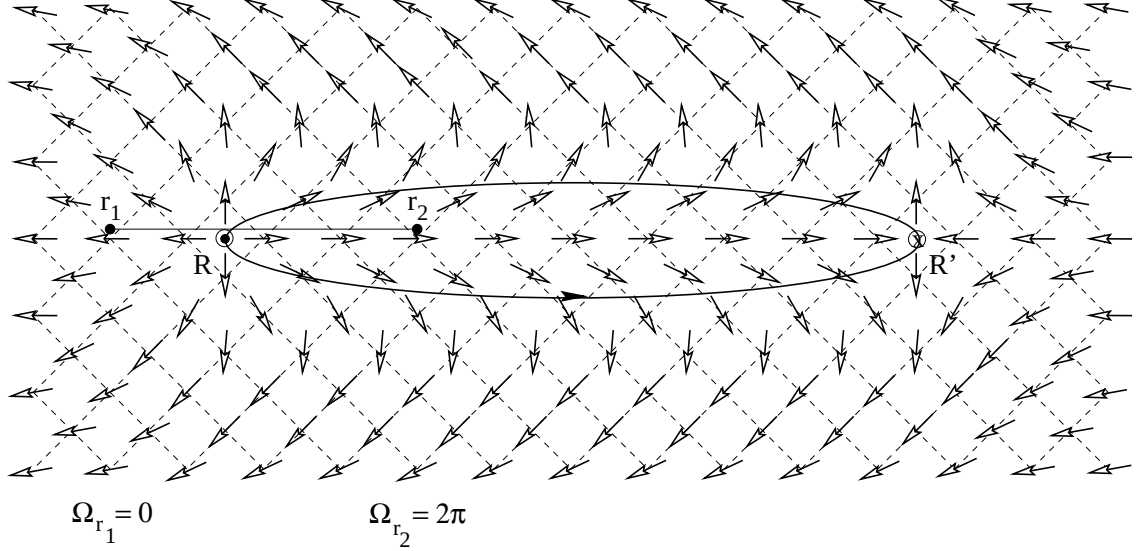
In order to prove that this result is the same as in equation (C.169) we have to estimate R . Noting that the larger the loop size R , the weaker the correlation and the smaller the correlation length κ^{-1} , R and κ are approximately related by $R \sim \kappa$; looking at (C.233), R is estimated by

$$R \sim \ln \frac{\eta}{a|\psi|^2}, \quad (\text{C.234})$$

and therefore for small $a|\psi|^2/\eta$ values $\ln \frac{\eta}{a|\psi|^2} - \ln R \approx \ln \frac{\eta}{a|\psi|^2}$, similar to the decay calculated by means of the “high temperature” power expansion in equation (C.169).

To show that small loops cannot change the behaviour of the correlation function is in this model rather more complicated than in two dimensions. However we can use the estimation method based on disordering contours, Peierls’ contours in the Ising model and in this model large dislocation loops, to estimate that the contributions of

small loops are approximately a constant. To produce a negative contribution to the correlation it is necessary that the loop crosses the path and surrounds only one of the ending points, producing thus $\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2} = \pm\pi$. A π phase difference between the scalar fields at $\vec{r}_1$ and $\vec{r}_2$ is equivalent to a difference in the solid angle parallax of $\Omega_{\vec{r}_1} - \Omega_{\vec{r}_2} = 2\pi$.



One large loop (only possible at small $a|\psi|^2/\eta$) which crosses the path and surround only one of the points, can be placed in the following number of positions

$$R|\vec{r}_2 - \vec{r}_1|, \quad (\text{C.235})$$

with R the radius (size) of the loop, since for every path site, the crossing loop can have its center aligned with the path or up to a distance approximately equal to R displaced to the left or to the right (always crossing the path). The value of $\exp[i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})]$ is equal to -1 for all these positions and $+1$ for the remaining $|\Lambda| - R|\vec{r}_2 - \vec{r}_1|$. The contribution by one large loop to the correlation function is therefore

$$\Phi = (|\Lambda| - R|\vec{r}_2 - \vec{r}_1|) \cdot (+1) + R|\vec{r}_2 - \vec{r}_1| \cdot (-1) = |\Lambda| - 2R|\vec{r}_2 - \vec{r}_1|. \quad (\text{C.236})$$

In the approximation of no interaction between loops, the contribution by k loops is the product of k factors like (C.236). As in equation (C.50) for the 2-dimensional model, the correlation is estimated by

$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \approx \frac{1}{\mathcal{Z}'} \sum_{k \text{ large loops}} \frac{1}{k!} \Phi^k \zeta^k = e^{-2|\vec{r}_2 - \vec{r}_1|R\zeta}. \quad (\text{C.237})$$

Note that this argument is only a coarse form of the above discussion for large loops, which has led to equation (C.231): here $\zeta \sim \exp[-2\pi^2 a|\psi|^2/\eta D(0)2\pi R]$, which together with the result (C.234), $R \sim \ln \frac{\eta}{a|\psi|^2}$ produces the exact result.

Then, if the loop is small, the number of places where it can be placed surrounding only one of the ending points and crossing simultaneously the path, is not longer proportional to the distance $|\vec{r}_2 - \vec{r}_1|$ but only to the area πR^2 enclosed by the loop. Substituting this result into (C.237), we obtain

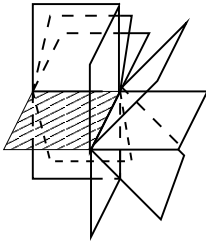
$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \approx \frac{1}{\mathcal{Z}'} \sum_{k \text{ large loops}} \frac{1}{k!} \Phi^k \zeta^k = e^{-2\pi R^2 \zeta}, \quad (\text{C.238})$$

which has no dependence on $|\vec{r}_2 - \vec{r}_1|$ (in ζ is only included the energy cost of dislocation loops). Therefore small loops are not able to change the form of the correlation function, and at large values of $a|\psi|^2/\eta$ where only small loops are possible, there is no change in the correlation behaviour (C.222) which was calculated without taking into account the effect of dislocations $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } \frac{|\psi|^2}{\eta}} = \exp\left(-\eta/4\pi a|\psi|^2|\vec{r}_2 - \vec{r}_1|\right)$. These results are listed in Table C.3.

$a \psi ^2/\eta$	$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$	Dislocation loops
high $a \psi ^2/\eta$	$e^{\frac{\eta}{4\pi a \psi ^2} \frac{1}{ \vec{r}_2 - \vec{r}_1 }}$	small loops
low $a \psi ^2/\eta$	$e^{- \vec{r}_2 - \vec{r}_1 \ln(2\eta/a \psi ^2)}$	large loops

Table C.3: Correlation in 3-dimensional XY model.

As in the 2-dimensional Ising model, the 3-dimensional XY model shows a phase where correlation $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ does not vanish at large distances. The same holds for the XY model in a 4-dimensional lattice; its propagator at large distances ¹⁶ is $1/|\vec{r} - \vec{r}'|^2$ and $D(0)$ is approximately equal to 0.155 [60]. All the four dimensions are of spatial nature. The topological defects now are closed surfaces: the dislocation loops of the 3-dimensional model can be viewed as the section's profile of these surfaces, whose energy grows like the surface area S for large S . The phase structure is therefore the same as in three dimensions: at high $a^2|\psi|^2/\eta$, large topological defects are strongly suppressed and there is an ordered phase since the Coulomb propagator, contained in the correlation exponent, decreases with the distance. At low $a^2|\psi|^2/\eta$, the “high temperature” expansion gives again the characteristic exponential decay with distance $|\vec{r}_2 - \vec{r}_1|$.



The transition point between these two phases can be estimated again by the same argument as in equation (C.226): the number of closed random surfaces is roughly estimated as 5^S , where 5 is in four dimensions the number of available directions to attach a unit area element to a given surface without turning back directly on itself (in the figure, the fourth dimension has been represented in an oblique direction). Neglecting mutual interactions for randomly distributed surfaces, we obtain the following contribution in the partition function

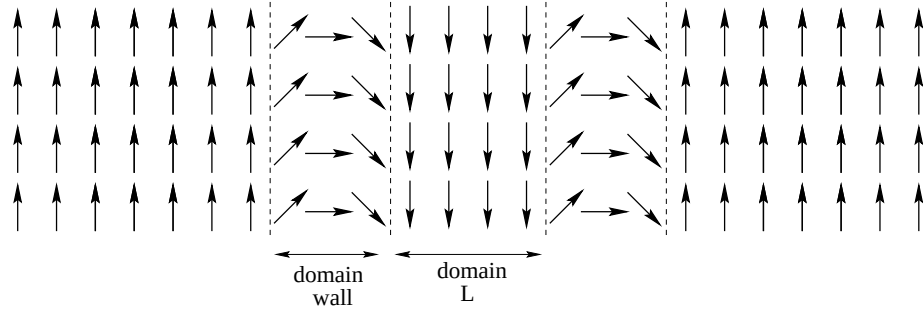
$$\exp \left[S \left(\ln 5 - \frac{2\pi^2 a^2 |\psi|^2}{\eta} D(0) \right) \right],$$

¹⁶By noting that $\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4) = \frac{1}{2\pi^2} \frac{1}{r^3} \delta(r)$ and $\frac{d}{dx} \vartheta(x-x') = \delta(x-x')$, the propagator's form is easily deduced: $\sum_{\mu=1,2,3,4} \partial_\mu^2 D(r) = \frac{1}{r^3} \frac{d}{dr} r^3 \frac{d}{dr} D(r) = -\frac{1}{2\pi^2 r^3} \frac{d}{dr} \vartheta(r)$ leads to $\frac{d}{dr} D(r) = -\frac{1}{2\pi^2 r^3}$.

which for large surfaces gives a nonvanishing contribution for $a^2|\psi|^2/\eta$ values smaller than

$$\left(\frac{a^2|\psi|^2}{\eta}\right)_{\text{critical}} \approx \frac{\ln 5}{2\pi^2 D(0)} = 0.526. \quad (\text{C.239})$$

In general, the XY model for a lattice dimension greater than 2 presents two phases: an ordered state for large $a^{d-2}|\psi|^2/\eta$ characterized by a nonvanishing correlation function at large distances $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \xrightarrow{|\vec{r}_2 - \vec{r}_1|} \text{constant} \neq 0$; and at low $a^{d-2}|\psi|^2/\eta$, an exponential decay with the distance. The reason for the absence of an ordered phase in the 2-dimensional case lies in the Coulomb propagator, which is not bounded for large distances in two dimensions (note that at large $a^{d-2}|\psi|^2/\eta$, the correlation can be approximated by term containing only the Coulomb propagator in the exponent). Another form to see this fact is to consider if an ordered state is stable against the formation of domain walls [54]. A domain wall is the transition zone between two regions where scalar field ψ is pointing in different directions. We shall consider domains which disorder in the most effective way: those in which ψ has the opposite direction to the rest of the lattice.



The energy cost due to the formation of a domain arises from the gradient in orientation of ψ , $\Delta_\mu \varphi_{\vec{r}}$, in the domain wall. Let L be the linear dimension of the domain: if L is not large, the transition to the other orientation must be fast and therefore the wall thickness is small but the gradient $\Delta_\mu \varphi_{\vec{r}}$ is large; on the contrary, if the domain is very wide, the transition can be made smoother (a smaller gradient $\Delta_\mu \varphi_{\vec{r}}$) but then the wall thickness is larger. This means that the thickness is of the order of L whereas the gradient $\Delta_\mu \varphi_{\vec{r}}$ is of the order of $1/L$, which leads to an energy for the domain wall given by

$$E_{\text{wall}} \sim \sum_{\vec{r}, \mu} (\Delta_\mu \varphi_{\vec{r}})^2 \sim \frac{\text{domain area } L^{d-1} \times \text{domain thickness } L}{L^2} = L^{d-2}, \quad (\text{C.240})$$

where d is the lattice dimensionality. Therefore, for $d \leq 2$ the energy cost for a large domain, which strongly disorders the system, is nonmacroscopic. This argument is similar to the fact that the Ising model has no magnetization for $d = 1$, which was discussed at the end of the section dedicated to the 2-dimensional Ising model.

C.5 The 3-dimensional lattice Higgs model

After having familiarized ourselves with the lattice XY model, it is now time for us to deal with the model which we introduced in Chapter 2. In this model we have as variables phase $\varphi_{\vec{r}}$ of the scalar field (as in the XY model), coupled to an Abelian gauge field $A_{\mu,\vec{r}}$. This coupling is made in the standard way: by substitution of the (discrete) partial derivative by the (discrete) covariant derivative; since the scalar field's length $|\psi|$ is maintained constant in this model, the covariant derivative is simplified into

$$\Delta_{\mu}\varphi_{\vec{r}} \rightarrow D_{\mu}\varphi_{\vec{r}} = \Delta_{\mu}\varphi_{\vec{r}} - qA_{\mu,\vec{r}}, \quad (\text{C.241})$$

expresion which is invariant under a local phase change

$$\begin{cases} \varphi_{\vec{r}} & \rightarrow \varphi'_{\vec{r}} & = \varphi_{\vec{r}} - q\chi_{\vec{r}}, \\ A_{\mu,\vec{r}} & \rightarrow A'_{\mu,\vec{r}} & = A_{\mu,\vec{r}} - (\chi_{\vec{r}+\hat{\mu}} - \chi_{\vec{r}}). \end{cases} \quad (\text{C.242})$$

For the moment, and using a common notation of Particle Physics, integer q in (C.241) will be called the scalar field's charge, which measures, in units of coupling constant e , the strength of the coupling between scalar field $\psi_{\vec{r}}$ and gauge field $A_{\mu,\vec{r}}$; remember that lattice gauge field $A_{\mu,\vec{r}}$ has been defined in Chapter 2 with coupling constant e already included in it: $A_{\mu,\vec{r}} \equiv aeA_{\mu}(\vec{r})$, with $A_{\mu}(\vec{r})$ the continuous gauge field at site $\vec{r}$. As we shall see, if the gauge field's range is bounded, q has a very specific meaning.

In the XY model, we defined in a 2-dimensional internal space an “arrow” $S_{\vec{r}}^{\text{scalar}}$ to describe scalar field $\psi_{\vec{r}} = |\psi|S_{\vec{r}}^{\text{scalar}}$; such an arrow was given by $S_{\vec{r}}^{\text{scalar}} = e^{i\varphi_{\vec{r}}}$ or equivalently by $S_{\vec{r}}^{\text{scalar}} = \begin{pmatrix} \cos \varphi_{\vec{r}} \\ \sin \varphi_{\vec{r}} \end{pmatrix}$. Since $\varphi_{\vec{r}}$ is an angular variable, its range is $[-\pi, \pi)$, and $S_{\vec{r}}^{\text{scalar}}$ is an element of the (fundamental) representation of Lie group $U(1) \equiv SO(2)$ (see Appendix B for an introduction to Lie algebras and groups).

Similarly, we could define an internal $U(1)$ symmetry space for gauge field $A_{\mu,\vec{r}}$ if its character were that of an angular variable, i.e., if its range were bounded. The range of gauge field $A_{\mu,\vec{r}}$ in the usual Magnetism, for instance, covers the whole real line $T(1)$ and therefore is not bounded. However, we have already seen a case where the range of an Abelian gauge field A_{μ} is bounded: the “ $SO(3)$ ”-superconductor in Appendix B. When the scalar field draws two complete rotations in its internal symmetry space, we have a double magnetic flux for the Abelian gauge field (this means here that the sum of $A_{\mu,\vec{r}}$ along a closed path is equal to 4π). In the “ $SO(3)$ ”-superconductor, nevertheless, such a double flux is topologically equivalent to no flux, i.e., through a $SO(3)$ gauge transformation any double flux configuration can be taken into a configuration of zero flux without modifying the system's energy (this does not occur in the ordinary superconductor, where double flux is stable and not gauge-related to no flux); in the moment where the difference in $A_{\mu,\vec{r}}$ between two neighbour points $\vec{r}$ and $\vec{r} + \hat{\mu}$ reaches 4π , the difference is “set” equal to zero: in the “ $SO(3)$ ”-superconductor, the range of Abelian gauge field is $A_{\mu,\vec{r}} \in [0, 4\pi)$ ¹⁷. The boundeness of $A_{\mu,\vec{r}}$ result from the fact that Abelian

¹⁷Although we shall deal later with the question about q , let us now advance its function. For the “ $SO(3)$ ”-supercondutor — a special case of a compact $U(1)$ superconductor where the Abelian $U(1)$

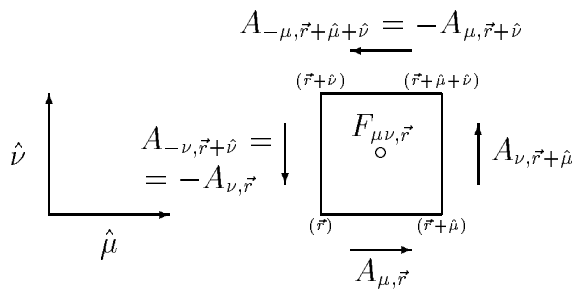
gauge symmetry $U(1)$ is embedded in compact Lie group $SO(3)$ and from the fact that its covering group, $SU(2)$, is simply-connected. Further, the non-stability of double flux is the signal of existence of magnetic monopoles carrying two units of flux. We obtain therefore the following equivalences in meaning

$$A_{\mu,\vec{r}} \text{ as angular variable} \iff \text{compact } A_{\mu,\vec{r}} \text{ range} \iff \text{existence of monopoles},$$

which, of course, holds also if the fields are defined in a discrete lattice.

For the gauge invariant model, in addition to the covariant derivative, we need include a term depending exclusively on gauge invariant “curl” $\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}}$ to take into account energy contributions due to inhomogeneous distributions of the gauge field

$$F_{\mu\nu,\vec{r}} = \Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}} = A_{\nu,\vec{r}+\hat{\mu}} - A_{\nu,\vec{r}} - A_{\mu,\vec{r}+\hat{\nu}} + A_{\mu,\vec{r}}. \quad (\text{C.243})$$



Field $F_{\mu\nu,\vec{r}}$ measures the sum of gauge field $A_{\mu,\vec{r}}$ along a unit plaquette beginning and ending at site $\vec{r}$: $F_{\mu\nu,\vec{r}}$ is the discrete formulation of the magnetic field crossing the plaquette. The functional form of the term containing $F_{\mu\nu,\vec{r}}$ depends on whether $A_{\mu,\vec{r}}$ has angular character or not, i.e., whether the Abelian gauge symmetry describing the

(effective) model's gauge invariance is or not embedded in a larger non-Abelian compact group. In the case where the Abelian symmetry is the actual underlying model's symmetry, the group is the real line $T(1)$ with $A_{\mu,\vec{r}} \in (-\infty, +\infty)$ unbounded and

$$E_{\text{gauge non-compact}} = \frac{1}{2ae^2} \sum_{\vec{r}} \sum_{\mu < \nu=1,2,3} (\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})^2, \quad (\text{C.244})$$

where the e^2 in the denominator is because we have included this e into the definition of $A_{\mu,\vec{r}}$. We keep the lattice spacing a in order to follow the notation of Chapter 2. The above model, which we shall call non-compact, contains no magnetic monopole since $A_{\mu,\vec{r}}$ has no angular character, and thus flux of any winding number is stable. For the case of embedding the Abelian symmetry into a compact Lie group, for example $SO(3)$, the gauge field is an angular variable and we can define a 2-dimensional internal symmetry

group results from the partial breakdown of $SO(3)$ (the actual underlying symmetry) by a scalar field in the adjoint representation — instead of considering the covariant derivative as $\Delta_\mu \varphi_{\vec{r}} - A_{\mu,\vec{r}}$ with $\varphi \in [-\pi, \pi)$ and $A_{\mu,\vec{r}} \in [-2\pi, 2\pi)$, we can equivalently define the bounded range of the gauge field as $A_{\mu,\vec{r}} \in [-\pi, \pi)$ always and include the constant $q = 2$ into the covariant derivative as in (C.241). Larger values of q are obtained by breaking group $SO(3)$ (and $SU(2)$, the other compact group which shares with $SO(3)$ the same complete antisymmetric structure constants) down to $U(1)$ through a scalar field in representations of higher rank (see comments previous to (B.242)). “Breaking a symmetry group” within the framework of a gauge invariant model is only an abbreviated form of saying that the set of *representative* configurations left invariant by the reduced broken symmetry is smaller. A representative is one configuration chosen (by imposing certain boundary conditions which removes the gauge invariance) from the infinite many configurations related among each other by means of a gauge transformation, and this is the reason because speaking of reduced symmetries has only a sense when working with representative configurations: by working with all configurations one would obtain always an infinite number of invariant field arrangements due to the gauge invariance.

space as in the case of $\psi_{\vec{r}}$: $S_{\mu,\vec{r}}^{\text{gauge}} = e^{iA_{\mu,\vec{r}}}$ or $S_{\mu,\vec{r}}^{\text{gauge}} = \begin{pmatrix} \cos A_{\mu,\vec{r}} \\ \sin A_{\mu,\vec{r}} \end{pmatrix}$. The necessary energy term due to the gradient in the gauge field and which takes account of $A_{\mu,\vec{r}}$'s angular character is

$$E_{\text{gauge compact}} = \frac{1}{ae^2} \sum_{\vec{r}} \sum_{\mu < \nu=1,2,3} [1 - \cos(\Delta_{\mu}A_{\nu,\vec{r}} - \Delta_{\nu}A_{\mu,\vec{r}})] . \quad (\text{C.245})$$

By expanding the above expression in powers of $(\Delta_{\mu}A_{\nu,\vec{r}} - \Delta_{\nu}A_{\mu,\vec{r}})$, we see that the first term is just (C.244), plus other terms responsible for the difference between the two models: this difference is the existence of magnetic monopoles in the compact model (C.245). It is important to note that this embedding fixes the value of q .

In the following we shall study the compact model: a scalar field defined by a phase $\varphi_{\vec{r}} \in [-\pi, \pi)$, which is placed on each lattice site $\vec{r}$, is coupled to an Abelian gauge field $A_{\mu,\vec{r}} \in [-\pi, \pi)$ located on each bond $(\vec{r}, \mu)$. In the dislocation model of Chapter 2, phase $\varphi_{\vec{r}}$ describes the deformation of the atomic cell, and gauge field $A_{\mu,\vec{r}}$ defines $F_{\mu\nu,\vec{r}}$, which is the density of gliding dislocations. This model is called the lattice (compact) Higgs model and is described by the following periodic energy

$$E_{\text{compact}} = \frac{1}{ae^2} \sum_{\vec{r}} \sum_{\mu < \nu=1,2,3} [1 - \cos(\Delta_{\mu}A_{\nu,\vec{r}} - \Delta_{\nu}A_{\mu,\vec{r}})] + a|\psi|^2 \sum_{\vec{r}} \sum_{\mu=1,2,3} [1 - \cos(\Delta_{\mu}\varphi_{\vec{r}} - qA_{\mu,\vec{r}})] \quad \text{with } q \text{ integer} . \quad (\text{C.246})$$

Note that since the integration range for gauge field $A_{\mu,\vec{r}}$ is compact, there is no problem with the infinite number of gauge transformations with the same contribution to the partition. We don't have to include a term like $\det(-\Delta\Delta) \prod_{\vec{r},\mu} \delta(\sum_{\mu} \Delta_{\mu}A_{\mu,\vec{r}})$ as in (C.194) for the previous 3-dimensional XY model.

A magnetic monopole is a point where magnetic flux is absorbed (or created). In the language of the dislocation model of Chapter 2, a point where there is a net decrease (or production) of the density of gliding dislocations means that at this site some gliding dislocation has become stopped (or liberated) ¹⁸.

By means of the above energy, we shall study correlation $W(C) = \left\langle e^{ip \sum_{\text{loop } C} A_{\mu,\vec{r}}} \right\rangle$, with $p \leq q$ an integer and C a closed path. This correlation function is called the Wilson loop. Path C is defined by integer field $J_{\mu,\vec{r}}$, which is the unit vector tangent to C

$$J_{\mu,\vec{r}} = \begin{cases} +1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{coincides with the direction of } C, \\ -1 & \text{if bond } (\vec{r}, \mu) \text{ belongs to } C \text{ and the direction } \vec{r} \rightarrow \vec{r} + \hat{\mu} \\ & \text{is the opposite to the direction of } C \\ 0 & \text{on all other bonds and for all other directions,} \end{cases} \quad (\text{C.247})$$

¹⁸In the following discussion, in order to use a more common name, one will only speak of magnetic field and magnetic monopoles instead of density of gliding (and able-to glide) dislocations and points where gliding dislocations are stopped due to sessile dislocations.

similar to (C.95), but here $J_{\mu,\vec{r}}$ fulfils

$$\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} = 0, \quad (\text{C.248})$$

since C is a closed path. With these definitions, correlation $W(C)$ is given by

$$\begin{aligned} W(C) &= \left\langle e^{ip \sum_{\text{loop } C} A_{\mu,\vec{r}}} \right\rangle = \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r},\mu} \left(\int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} \right) \times \\ &\times \exp \left[ip \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r},\mu<\nu} [1 - \cos(\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})] \right] \times \\ &\times \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}})] \right], \end{aligned} \quad (\text{C.249})$$

with partition function $\mathcal{Z}$ equal to

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r},\mu} \left(\int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r},\mu<\nu} [1 - \cos(\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})] \right] \times \\ &\times \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}})] \right]. \end{aligned} \quad (\text{C.250})$$

The strategy to study this correlation function will be similar to that in the XY model: use of the “high temperature” power expansion for small $a|\psi|^2/2$ and $1/ae^2\eta$, and the Villain approximation when one the parameters $a|\psi|^2/2$, $1/ae^2\eta$ is large. As we have seen in the XY model, even for small $a|\psi|^2/2$ the Villain approximation is able to reproduce the result calculated by power expansion; we shall see later that this holds also for the gauge model. Hence, even though the validity of the Villain approximation lies at large $a|\psi|^2/2$ and small $ae^2\eta$, it will be also used in the range where it does not work so well, small $a|\psi|^2/2$ or large $ae^2\eta$, to discuss the model qualitatively.

Let us introduce now the Villain approximation, although it will be studied later. Partition function (C.250) formulated in the Villain approximation reads

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r},\mu} \int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \sum_{\{g_{\mu\nu,\vec{r}}\}} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\mu<\nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}})^2 \right] \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r},\mu} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right], \end{aligned} \quad (\text{C.251})$$

with notation $\prod_{\vec{r}} \prod_{\mu<\nu} \sum_{g_{\mu\nu,\vec{r}}=-\infty}^{+\infty} \equiv \sum_{\{g_{\mu\nu,\vec{r}}=-\infty, \dots, +\infty\}} = \sum_{g_{\mu\nu,\vec{r}}=-\infty}^{+\infty} \sum_{g_{\mu\nu,\vec{r}}=-\infty}^{+\infty} \dots \sum_{g_{\mu\nu,\vec{r}_N}=-\infty}^{+\infty}$, and the same for $\sum_{\{m_{\mu,\vec{r}}\}}$. Integer field $m_{\mu,\vec{r}}$ carries the periodicity of the covariant derivative term

and measures change of 2π in phase $\varphi_{\vec{r}}$ between two neighbour sites. On the other hand, integer antisymmetric tensor $g_{\mu\nu,\vec{r}}$ carries the periodicity of term $\cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})$. This last periodicity was originated by the angular character of gauge field $A_{\mu,\vec{r}}$: jumps of $2\pi k$ with k integer are allowed and these jumps correspond to magnetic monopoles. Thus, $g_{\mu\nu,\vec{r}}$ carries the information about the monopoles. More directly, if all $g_{\mu\nu,\vec{r}}$ were zero, the pure gauge part of (C.251) would reduce to the non-compact model (C.244), which as already seen, has no monopoles.

Let us consider a unit closed surface (a unit cube), with field $g_{\mu\nu,\vec{r}}$ placed on every cube's plaquette. Taking account of the antisymmetry of $g_{\mu\nu,\vec{r}}$, the monopole density within this closed surface is defined as (minus) the sum of integers $g_{\mu\nu,\vec{r}}$ at the surface (the minus is due to taking outwards as positive direction)

$$G_{\vec{r}} = -g_{32,\vec{r}+\hat{1}} - g_{13,\vec{r}+\hat{2}} - g_{21,\vec{r}+\hat{3}} - g_{23,\vec{r}} - g_{31,\vec{r}} - g_{12,\vec{r}} = \sum_{\lambda,\mu<\nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda g_{\mu\nu,\vec{r}}. \quad (\text{C.252})$$

In the Villain approximation, the meaning of constant q in the covariant derivative becomes specially clear. Let us consider again the “ $SO(3)$ ”-superconductor of Appendix B (we shall return to this model several times along this section). In this model, the jumps in gauge field $A_{\mu,\vec{r}}$ are of 4π , which are absorbed by magnetic monopoles of double flux $\Phi_0 = 4\pi$. We can change the $A_{\mu,\vec{r}}$ range in (C.251) from $[-\pi, \pi)$ to cover the range between jumps: $[0, 4\pi)$ or $[-2\pi, 2\pi)$. This can be done by including $q = 2$ into the gauge field, defining thus a new gauge field $\bar{A}_{\mu,\vec{r}} = qA_{\mu,\vec{r}}$. This model is described by the following partition function

$$\mathcal{Z} = \prod_{\vec{r},\mu} \int_{-2\pi}^{2\pi} \frac{d\bar{A}_{\mu,\vec{r}}}{4\pi} \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r},\mu} (\Delta_\mu \varphi_{\vec{r}} - \bar{A}_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right] \times \sum_{\{g_{\mu\nu,\vec{r}}\}} \exp \left[-\frac{1}{2a(2^2)e^2\eta} \sum_{\vec{r},\mu<\nu} (\Delta_\mu \bar{A}_{\nu,\vec{r}} - \Delta_\nu \bar{A}_{\mu,\vec{r}} - 4\pi g_{\mu\nu,\vec{r}})^2 \right], \quad (\text{C.253})$$

where it should be noted factor 4π , instead of 2π , which multiplies $g_{\mu\nu,\vec{r}}$: this means that we are measuring monopole charges in units of 4π .

Hence, the compact $U(1)$ gauge symmetry embedded in the $SO(3)$ model of Appendix B corresponds to a scalar charge $q = 2$. At first sight, this could seem strange; however, the $U(1)$ coupling constant e of our model is just the same as in the coupling between isovector Ψ^a (the scalar field in the adjoint representation which breaks $SO(3)$ down to $U(1)$) and the non-Abelian $SO(3)$ gauge field W_μ^a , whose a component in isospace coinciding with the isovector's orientation after breaking $SO(3)$ corresponds to the Abelian gauge field A_μ . Each (irreducible) matrix representation of a Lie algebra is labelled by an integer constant, called charge or coupling constant [35]. For the Lie algebra defined by the structure constants ϵ_{abc} , the fundamental or isospinor representation carries the smallest charge $q = 1$ (in units of e); the adjoint or isovector representation,

$q = 2$ ¹⁹. Since in the underlying $SO(3)$ model containing the compact $U(1)$, Ψ^a is in the adjoint representation, the constant for the coupling of Abelian gauge field A_μ to the scalar field ψ , which breaks the remaining symmetry completely, is $q = 2$.

If partition function (C.253), with q included into the gauge field, is preferred, the correlation function is given by $W(C) = \left\langle e^{i \frac{p}{q} \sum_{\text{loop } C} \bar{A}_{\mu, \vec{r}}} \right\rangle$. As we shall see, the behaviour of $W(C)$ depends very strongly on the quotient p/q , whether it is fractionary or integer. For instance, in the $SO(3)$ model the only p value different from q ($q = 2$) is $p = 1$. This $p = 1 \neq q$ belongs to a scalar field ψ representing rotations about the fixed orientation of scalar field Ψ (used to break to break down the non-Abelian symmetry). However, in this case the breaking scalar field was in the fundamental representation, i.e., $SU(2)$ (and not $SO(3)$) was the symmetry group broken down to $U(1)$. The result is the “ $SU(2)$ ”-superconductor, which does not have any topologically stable vortices (since $SU(2)$ has the topology of a sphere’s surface S^3) and has a different behaviour from the “ $SO(3)$ ”-superconductor. Hence, we should expect a behaviour of the correlation function for $p \neq q$ different from the case of $p = q$.

The discussion of the Higgs model will be made in three parts. Firstly, the case $|\psi| = 0$, where gauge field $A_{\mu, \vec{r}}$ is not coupled to a scalar field. This model is called the pure $U(1)$ gauge model. After studying this case, we shall consider another related pure gauge model where the $A_{\mu, \vec{r}}$ range is not the complete interval $[-\pi, \pi)$ but only the values 0 and π . This is called the pure $Z(2)$ gauge model, and as we shall see later, corresponds to the limit $|\psi| = \infty$. At the third part, the complete Higgs model will be studied.

C.5.1 The 3-dimensional pure compact $U(1)$ gauge model

For the discussion of the pure gauge case, we shall take $p = 1$; two separated Notes about the case $p \neq 1$ will show that the correlation behaviour does not depend on whether $p = 1$ or $p \neq 1$. In fact, without coupling the gauge field to a scalar field ψ , it makes no sense to speak about a charge $p < q$ or $p = q$ since q is not defined.

¹⁹This can be better seen by noting that the adjoint representation is obtained from the direct product of the fundamental representation with itself. This is a well-known result, for example, as the spin is studied in Quantum Mechanics: direct product of spin 1/2 with spin 1/2 produces representation of spin 1 (plus the scalar spin 0), which is the adjoint representation.

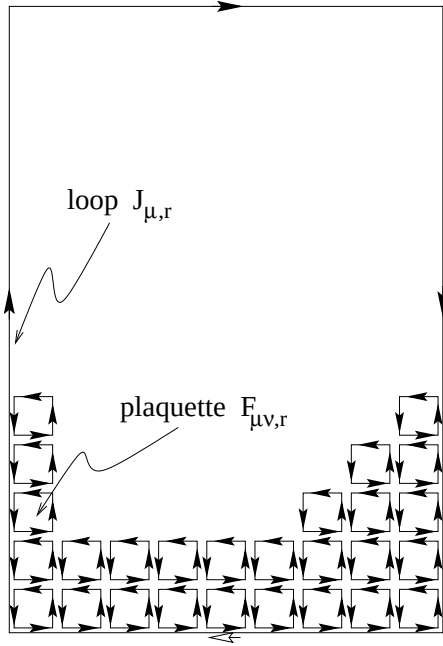
However, one should be careful and not mix concepts. Factor $q = 2$ appearing in the Abrikosov model of the superconductivity (see (A.1)), has another origin. The Abrikosov model is an effective theory for a condensation state of particles with spin, whose behaviour is described by Quantum Electrodynamics. The spin (not isospin) group, although defined by means of $SU(2)$ matrices, is a representation of the Poincaré group (the group of the Lorentz transformations). Spin is not related to an internal symmetry which we could make local: the theory with local Lorentz transformations is the General Relativity, not Quantum Electrodynamics. On the contrary, a gauge group applied to isospin denotes the freedom to choose in any point of external space an arbitrary direction of the isospinor in its internal symmetry space. This is directly seen by noting that in Quantum Electrodynamics, a spinor is coupled to a sole gauge vector A_μ , whereas both an isospinor and an isovector are coupled to a isovector of gauge fields W_μ^a , $a = 1, 2, 3$. The gauge group for the Abrikosov model, and for spin fields in Quantum Electrodynamics, is the non-compact $U(1) \equiv T(1)$ group of the usual Electromagnetism.

As for the XY model, we shall study the correlation by means of the power expansion for large $ae^2\eta$, and using the Villain approximation for small values of $ae^2\eta$ (note that the factor multiplying the term $\sum_{\vec{r},\mu,\nu}(1 - \cos F_{\mu\nu,\vec{r}})$ is $1/ae^2\eta$). For large values of $ae^2\eta$, $\exp\left(-\frac{1}{ae^2\eta}[1 - \cos(\cdot)]\right)$ can be approximated by $1 - \frac{1}{ae^2\eta}[1 - \cos(\cdot)]$, obtaining thus

$$W(C) \approx \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{(\vec{r}',\mu') \in \text{loop}} e^{iA_{\mu',\vec{r}'}} \times \\ \times \prod_{\vec{r},\mu} \left[\left(1 - \frac{1}{ae^2\eta}\right) + \frac{1}{2ae^2\eta} \left(e^{-i \underbrace{(A_{\mu,\vec{r}} + A_{\nu,\vec{r}+\hat{\mu}} + A_{-\mu,\vec{r}+\hat{\mu}+\hat{\nu}} + A_{-\nu,\vec{r}+\hat{\nu}})}_{\text{plaquette}}} + \text{c.c.} \right) \right], \quad (\text{C.254})$$

where c.c. means complex conjugated. Considering the relations

$$\int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} e^{iA_{\mu,\vec{r}}} = 0 \\ \int_{-\pi}^{\pi} \frac{dA_{\mu,\vec{r}}}{2\pi} e^{iA_{\mu,\vec{r}}} e^{-iA_{\mu,\vec{r}}} = 1 \quad (\text{C.255})$$



which are similar to equation (C.72) for the XY model (or to (C.33) for the Ising model), we can identify the products contained in (C.254) which produce a nonvanishing contribution. Only the products, where the area delimited by the loop $J_{\mu,\vec{r}}$ is completely covered by plaquettes, do not vanish since each $A_{\mu,\vec{r}}$ appears twice, once in a direction and once in the opposite one. The leading contribution for large $ae^2\eta$ comes from the product covering the minimal area $\mathcal{A}$ enclosed by the loop C . Higher powers in $1/ae^2\eta$ are produced by surfaces with “lumps” which cover the loop area

$$W(C) = \left(\frac{1}{2ae^2\eta} \right)^{\mathcal{A}} + \dots, \quad (\text{C.256})$$

where in $\dots$ the higher powers are included.

Note. Correlation $W(C) = \left\langle e^{ip \sum_{\text{loop } C} A_{\mu,\vec{r}}} \right\rangle$ for $p \neq 1$ at strong $ae^2\eta$.

Due to the relations (C.255), in the case of $p > 1$ we have to expand $\exp\left(-\frac{1}{ae^2\eta}[1 - \cos(\cdot)]\right)$ up to reach the term $\left(\frac{1}{ae^2\eta}[1 - \cos(\cdot)]\right)^p$. In this way we obtain at each site $\vec{r}$ powers of the kind $e^{-ipA_{\mu,\vec{r}}}$, which produce a nonvanishing contribution when integrated with the $pA_{\mu,\vec{r}}$ placed along the closed path C : the leading contribution is that where each plaquette inside C are covered p times by the plaquette term $1 - \cos(\cdot)$ arising from the power expansion of $\exp\left(-\frac{1}{ae^2\eta}[1 - \cos(\cdot)]\right)$

$$W(C) \stackrel{\text{large } ae^2\eta}{\approx} \left[\left(\frac{1}{2ae^2\eta} \right)^p \frac{1}{p!} \right]^{\mathcal{A}} \approx \exp[-p\mathcal{A} \ln(2ae^2\eta)]. \quad (\text{C.257})$$

Result (C.256) is general for any lattice dimension: it works for two dimensions as well as for dimension 4. On the other hand, the behaviour of $W(C)$ for small $ae^2\eta$ is strongly dependent on the lattice dimensionality, as we shall see later. To study this range, the Villain approximation is used, as already mentioned

$$W(C) = \frac{1}{\mathcal{Z}} \prod_{\vec{r}, \mu} \int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \times \\ \times \prod_{\vec{r}, \mu < \nu} \left\{ \sum_{g_{\mu\nu, \vec{r}} = -\infty}^{+\infty} \exp \left[-\frac{1}{2ae^2\eta} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}})^2 \right] \right\}, \quad (\text{C.258})$$

with $\mathcal{Z}$ equal to

$$\prod_{\vec{r}, \mu} \int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \sum_{\{g_{\mu\nu, \vec{r}} = -\infty\}}^{\{+\infty\}} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}})^2 \right]. \quad (\text{C.259})$$

As in the XY model (C.77), the partition invariance under the following transformation, with $N_{\mu, \vec{r}}$ an integer vector field,

$$\begin{cases} A_{\mu, \vec{r}} \rightarrow A_{\mu, \vec{r}} + 2\pi N_{\mu, \vec{r}}, \\ g_{\mu\nu, \vec{r}} \rightarrow g_{\mu\nu, \vec{r}} + (N_{\nu, \vec{r}+\hat{\mu}} - N_{\nu, \vec{r}} - N_{\mu, \vec{r}+\hat{\nu}} + N_{\mu, \vec{r}}), \end{cases} \quad (\text{C.260})$$

is used to extend the integration range in $A_{\mu, \vec{r}}$ from $[-\pi, +\pi)$ to $(-\infty, +\infty)$ ²⁰. It is important to note that this extension does not mean that we have lost the compact character of the gauge field and thus the magnetic monopoles existing in the model: integer field $g_{\mu\nu, \vec{r}}$ remains. By means of the above transformation, all $g_{\mu\nu, \vec{r}}$'s cannot be made simultaneously equal to zero. In fact, this transformation does not change the magnetic monopole located at $\vec{r}$, as one can see by looking at (C.252). As in the XY model (C.78), such an extension in the range of integration must be accompanied by restricting the sum on $g_{\mu\nu, \vec{r}}$ to avoid redundancy; such a restriction can be achieved, for instance, by means of the following constraint

$$\sum_{\mu} (g_{\mu\nu, \vec{r}+\hat{\mu}} - g_{\mu\nu, \vec{r}}) = 0. \quad (\text{C.261})$$

The final result of the partition function will depend only on $G_{\vec{r}} = \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$ but no longer on $g_{\mu\nu, \vec{r}}$. Thus, without constraining the sum on the latter, we would obtain many different $g_{\mu\nu, \vec{r}}$ configurations corresponding to the same $G_{\vec{r}}$. Although not explicitly written, a constraint on $\sum_{\{g_{\mu\nu, \vec{r}}\}}$ must be understood for the following.

In addition to this redundancy in the sum on the $g_{\mu\nu, \vec{r}}$'s, there is another source of infinities in the partition function: its gauge invariance under the transformation $A_{\mu, \vec{r}} \rightarrow A'_{\mu, \vec{r}} - \Delta_{\mu} \chi(\vec{r})$ (the difference with the gauge invariance in the continuous formulation $A_{\mu}(\vec{r}) \rightarrow A'_{\mu}(\vec{r}) = A_{\mu}(\vec{r}) - \frac{1}{e} \partial_{\mu} \chi(\vec{r})$) is the result of including the coupling constant e into

²⁰Since $J_{\mu, \vec{r}} = 0, \pm 1$, term $\exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right]$ and therefore correlation function $W(C)$ is also invariant under this transformation.

lattice gauge field $A_{\mu,\vec{r}}$). Since the Villain approximation allows for an infinite integration range in $A_{\mu,\vec{r}}$, there are infinite many $A_{\mu,\vec{r}}$ configurations corresponding to the same state. Instead of introducing a gauge fixing term, which would render some integrals more difficult to carry out, the calculation will be carried through without worrying about the infinities arising from the gauge invariance: only at the last integration, the gauge fixing term, if necessary, will be considered; nevertheless, the easy infinities to identify, such as products of Dirac deltas $\delta(0)$, will be directly removed during the calculations.

In order to integrate $A_{\mu,\vec{r}}$ out, we introduce the auxiliary field $B_{\mu\nu,\vec{r}}$ by means of a Gaussian integration, obtaining thus

$$W(C) = \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r},\mu<\nu} \left(\int_{-\infty}^{\infty} \sqrt{\frac{ae^2\eta}{2\pi}} dB_{\mu\nu,\vec{r}} \right) \sum_{\{g_{\mu\nu,\vec{r}}\}} \times$$

$$\times \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r},\mu<\nu} B_{\mu\nu,\vec{r}}^2 + i \sum_{\vec{r},\mu<\nu} B_{\mu\nu,\vec{r}} (\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}}) + i \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} \right], \quad (\text{C.262})$$

with the partition function given by

$$\mathcal{Z} = \left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{3|\Lambda|} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r},\mu<\nu} \int_{-\infty}^{\infty} dB_{\mu\nu,\vec{r}} \sum_{\{g_{\mu\nu,\vec{r}}\}} \times$$

$$\times \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r},\mu<\nu} B_{\mu\nu,\vec{r}}^2 + i \sum_{\vec{r},\mu<\nu} B_{\mu\nu,\vec{r}} (\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}}) \right]. \quad (\text{C.263})$$

Now we carry out the same steps as in the XY model to integrate out $A_{\mu,\vec{r}}$ and leave only the topological fields $g_{\mu\nu,\vec{r}}$.

1. The integrand in (C.262) contains the following terms

$$\sum_{\vec{r},\mu<\nu} B_{\mu\nu,\vec{r}} (\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}}) = \sum_{\vec{r}} B_{12,\vec{r}} (\Delta_1 A_{2,\vec{r}} - \Delta_2 A_{1,\vec{r}} - 2\pi g_{12,\vec{r}}) +$$

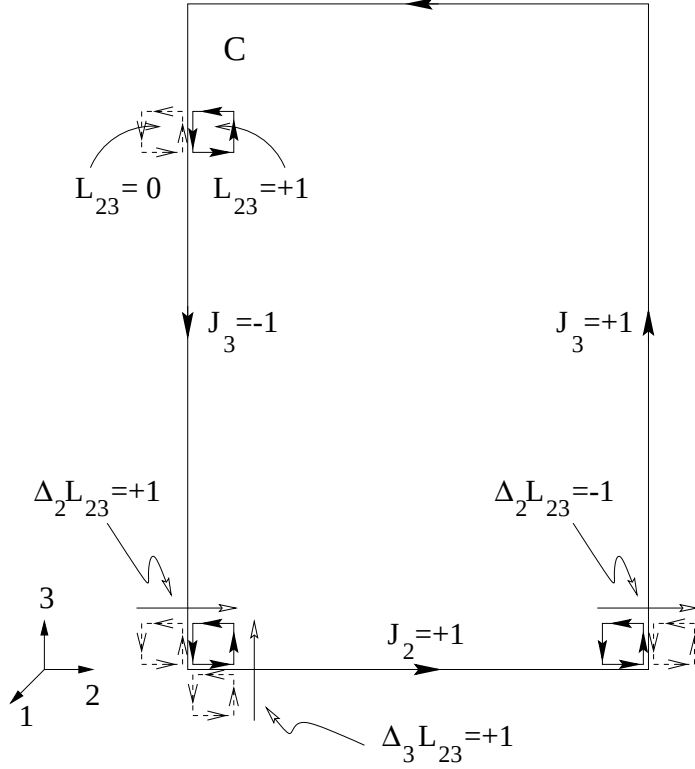
$$\sum_{\vec{r}} B_{13,\vec{r}} (\Delta_1 A_{3,\vec{r}} - \Delta_3 A_{1,\vec{r}} - 2\pi g_{13,\vec{r}}) + B_{23,\vec{r}} (\Delta_2 A_{3,\vec{r}} - \Delta_3 A_{2,\vec{r}} - 2\pi g_{23,\vec{r}}),$$

and the integration over $dA_{\mu,\vec{r}}$ produces a constraint on $B_{\mu\nu,\vec{r}}$ given by

$$\sum_{\mu} \Delta_\mu B_{\mu\nu,\vec{r}-\hat{\mu}} = J_{\nu,\vec{r}}. \quad (\text{C.264})$$

Let us define the following integer field $L_{\mu\nu,\vec{r}}$, antisymmetric in the indices $\mu\nu$

$$L_{\mu\nu,\vec{r}} = \begin{cases} +1 & \text{if plaquette } (\vec{r}, \mu\nu) \text{ lies inside } C \text{ and the direction } \vec{r} \rightarrow \\ & \rightarrow \vec{r} + \hat{\mu} \rightarrow \vec{r} + \hat{\mu} + \hat{\nu} \text{ coincides with the direction of } C, \\ -1 & \text{if plaquette } (\vec{r}, \mu\nu) \text{ lies inside } C \text{ and the direction } \vec{r} \rightarrow \\ & \rightarrow \vec{r} + \hat{\mu} \rightarrow \vec{r} + \hat{\mu} + \hat{\nu} \text{ is the opposite to the direction of } C, \\ 0 & \text{on all other plaquettes and for all other directions,} \end{cases} \quad (\text{C.265})$$



Integer field $L_{\mu\nu, \vec{r}}$ fulfils the following equation

$$\sum_{\mu} \Delta_{\mu} L_{\mu\nu, \vec{r}-\hat{\mu}} = -J_{\nu, \vec{r}}. \quad (\text{C.266})$$

As an example, let us suppose path C is given by the $J_{\mu, \vec{r}}$ of the picture. In this case, field $L_{\mu\nu, \vec{r}}$ is zero for other indices different from 23 (or 32), and equal to ± 1 inside the loop, $L_{23, \vec{r}} = 1$, and zero outside. If both $L_{\mu\nu, \vec{r}-\hat{\mu}}$ and $L_{\mu\nu, \vec{r}}$ are inside the loop, $\sum_{\mu} \Delta_{\mu} L_{\mu\nu, \vec{r}-\hat{\mu}} = 0$; at the border, however, $L_{\mu\nu, \vec{r}}$ differs from zero only at one side of the loop: at the right corner of the picture, for example, $\Delta_2 L_{23, \vec{r}-\hat{3}} = L_{23, \vec{r}} - L_{23, \vec{r}-\hat{3}} = 0 - 1$ coinciding with (minus) the value of J_3 at this site.

The general solution of the constraint (C.264) is

$$B_{\mu\nu, \vec{r}} = \sum_{\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} - L_{\mu\nu, \vec{r}}. \quad (\text{C.267})$$

Substituting into the correlation function,

$$W(C) = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 \right] \frac{1}{\mathcal{Z}} \left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{3|\Lambda|} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{dC_{\vec{r}}}{2\pi} \sum_{\{g_{\mu\nu, \vec{r}}\}} \times \quad (\text{C.268})$$

$$\exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \lambda} \underbrace{(\Delta_{\lambda} C_{\vec{r}-\hat{\lambda}})^2}_{\sum_{\vec{r}} C_{\vec{r}} (-\sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda}) C_{\vec{r}}} - ae^2\eta \sum_{\vec{r}} C_{\vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}} - i2\pi \sum_{\vec{r}, \lambda, \mu < \nu} \underbrace{g_{\mu\nu, \vec{r}} \epsilon_{\mu\nu\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}}_{+i2\pi \sum_{\vec{r}} C_{\vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}} \right].$$

Term $\exp \left[i2\pi \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}} g_{\mu\nu, \vec{r}} \right]$ is equal to 1 since both $L_{\mu\nu, \vec{r}}$ and $g_{\mu\nu, \vec{r}}$ are integers. The last term in the exponent is equal to $i2\pi \sum_{\vec{r}} C_{\vec{r}} G_{\vec{r}}$ according to the definition of $G_{\vec{r}}$. Field $G_{\vec{r}}$ describes the magnetic monopole at the center of the cube formed by the three plaquettes attached to site $\vec{r}$ and their corresponding parallel plaquettes which close the cube. Partition function $\mathcal{Z}$ is given by (C.268) but with $L_{\mu\nu, \vec{r}} = 0$.

2. The propagator is the same as in the 3-dimensional XY model (C.206)

$$D_{\vec{r}, \vec{r}'} = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} \frac{e^{i\vec{k} \cdot (\vec{r}-\vec{r}')}}{6 - 2\cos k_x - 2\cos k_y - 2\cos k_z}, \quad (\text{C.269})$$

well approximated for distances larger than the lattice spacing by the continuum Coulomb propagator

$$D_{\vec{r},\vec{r}'} = D(0)\delta_{\vec{r},\vec{r}'} + D'_{\vec{r},\vec{r}'} \quad \text{with} \quad \begin{cases} D(0) \approx 0.253 \\ D'_{\vec{r},\vec{r}} = 0 \\ D'_{\vec{r},\vec{r}'} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}, \quad |\vec{r} - \vec{r}'| \neq 0 \end{cases} \quad (\text{C.270})$$

After integrating over $dC_{\vec{r}}$, the partition function takes the form

$$\mathcal{Z} = \mathcal{Z}_0 \sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{\pi}{2ae^2\eta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} G_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} G_{\vec{r}'} \right], \quad (\text{C.271})$$

where $\mathcal{Z}_0$ is given by

$$\left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{3|\Lambda|} \left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|} \frac{1}{\sqrt{\det(-\Delta\Delta)}} = \left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{2|\Lambda|} \frac{1}{\sqrt{\det(-\Delta\Delta)}}. \quad (\text{C.272})$$

The physical meaning of $\mathcal{Z}_0$ becomes clearer if we consider the partition function of gauge field $A_{\mu,\vec{r}}$ without monopoles, i.e., the non-compact gauge model (C.244)

$$\begin{aligned} \mathcal{Z}_{\text{no monopoles}} &= \prod_{\vec{r},\mu} \left(\int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\mu < \nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})^2 \right] = \\ &= \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} dA_{\mu,\vec{r}} \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r},\nu} A_{\nu,\vec{r}} \left(-\sum_{\mu} \Delta_{\mu} \Delta_{\mu} A_{\nu,\vec{r}-\hat{\mu}} + \Delta_{\nu} \sum_{\mu} \Delta_{\mu} A_{\mu,\vec{r}-\hat{\mu}} \right) \right]. \end{aligned} \quad (\text{C.273})$$

This model, in the language of the dislocation model developed in Chapter 2, corresponds to the partition function for gliding dislocations which are not impeded by any obstacle. According to the Faddeev-Popov method, we have to impose a gauge fixing condition to avoid infinities originated in (C.273) by the integrand invariance under the gauge transformation $A_{\mu,\vec{r}} \rightarrow A_{\mu,\vec{r}} - \Delta_{\mu} \chi_{\vec{r}}$. As for the case in (C.200), a possible choice is to include the factor

$$\left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|} \det(-\Delta\Delta) \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r},\nu} A_{\nu,\vec{r}} \left(-\Delta_{\nu} \sum_{\mu} \Delta_{\mu} A_{\mu,\vec{r}-\hat{\mu}} \right) \right],$$

which cancels the term $A_{\nu,\vec{r}} \Delta_{\nu} \Delta_{\mu} A_{\mu,\vec{r}-\hat{\mu}}$ in (C.273) and to carry out the integral over $dA_{\mu,\vec{r}}$ without being concerned about infinities.

$$\left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|} \det(-\Delta\Delta) \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} dA_{\mu,\vec{r}} \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r},\mu} A_{\mu,\vec{r}} \left(-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} A_{\mu,\vec{r}-\hat{\nu}} \right) \right] =$$

$$= \left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|-3|\Lambda|} \det(-\Delta\Delta) \left(\frac{1}{\det(-\Delta\Delta)} \right)^{3/2} = \left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{2|\Lambda|} \frac{1}{\sqrt{\det(-\Delta\Delta)}}, \quad (\text{C.274})$$

obtaining again the result (C.272) and therefore $\mathcal{Z}_0 = \mathcal{Z}_{\text{no monopoles}} = \mathcal{Z}_{\text{non-compact}}$. As for $\mathcal{Z}$ and after integrating over $dC_{\vec{r}}$, correlation $W(C)$ becomes

$$W(C) = \frac{1}{\mathcal{Z}} \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 \right] \mathcal{Z}_0 \sum_{\{G_{\vec{r}}\}} \times \quad (\text{C.275})$$

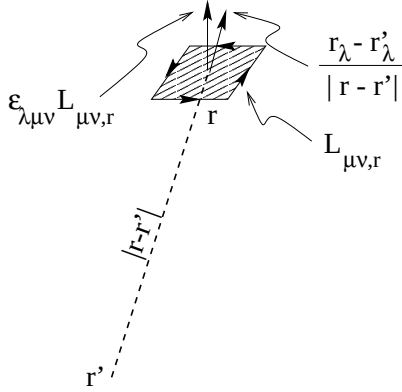
$$\exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} \left(G_{\vec{r}} + i \frac{ae^2\eta}{2\pi} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(G_{\vec{r}'} + i \frac{ae^2\eta}{2\pi} \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{r}')}^{\lambda'} L_{\mu'\nu', \vec{r}'} \right) \right],$$

with $\sum_{\lambda, \mu < \nu} \equiv \sum_{\lambda=1,2,3} \sum_{\mu < \nu}$.

3. One can obtain a concise expression for sum $\sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} (\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'}$ as the solid angle under which the monopole at $\vec{r}'$ sees the area enclosed by path C of correlation $W(C)$. Approximating the propagator $D_{\vec{r}, \vec{r}'}$ by the continuum Green's function if the distance $|\vec{r} - \vec{r}'|$ is larger than the lattice spacing

$$\sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} (\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'} = - \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \Delta_{(\vec{r})}^\lambda D_{\vec{r}, \vec{r}'} \approx$$

$$\approx \frac{1}{4\pi} \sum_{\vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \frac{r_\lambda - r'_\lambda}{|\vec{r} - \vec{r}'|} \frac{1}{|\vec{r} - \vec{r}'|^2}. \quad (\text{C.276})$$



Vector $\sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}}$ is the area unit vector of the plaquette belonging to the area stretched by the path C . The scalar product of this area vector and the unit vector $(r_\lambda - r'_\lambda)/|\vec{r} - \vec{r}'|$ divided by $|\vec{r} - \vec{r}'|^2$ is the solid angle of the plaquette seen from $\vec{r}'$. Summing up all the plaquette enclosed by C we have the solid parallax angle $\Omega_{\vec{r}}$ for the the whole area

$$\sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} (\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'} = \frac{1}{4\pi} \Omega_{\vec{r}'} . \quad (\text{C.277})$$

4. The other sum $\sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \mu < \nu} \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda\mu\nu} (\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'} \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{r}')}^{\lambda'} L_{\mu'\nu', \vec{r}'}$ is calculated as in the 2-dimensional XY case, by using the derivative perpendicular to the area enclosed by C

$$\Delta_{(\vec{r})}^\perp = \sum_{\lambda} \sum_{\mu < \nu} L_{\mu\nu, \vec{r}} \epsilon_{\mu\nu\lambda} \Delta_{(\vec{r})}^\lambda,$$

and the parallel derivatives along the two directions on the surface where is resting the path C

$$\Delta_{(\vec{r})}(\mu) = \sum_{\nu} L_{\mu\nu, \vec{r}} \Delta_{(\vec{r})}^{\nu}.$$

For the following $\Delta_{(\vec{r}')}^{\nu}$ will be abbreviated as Δ'_{ν} . With these definitions, the sum we want to evaluate takes the following form

$$+ \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \mu < \nu} \sum_{\lambda', \mu' < \nu'} L_{\mu\nu, \vec{r}} L_{\mu'\nu', \vec{r}'} \epsilon_{\lambda\mu\nu} \epsilon_{\lambda'\mu'\nu'} \Delta_{\lambda} \Delta'_{\lambda'} D_{\vec{r}-\hat{\lambda}, \vec{r}'-\hat{\lambda}'} = \sum_{\vec{r}, \vec{r}'} \Delta_{\perp} \Delta'_{\perp} D_{\vec{r}, \vec{r}'} . \quad (\text{C.278})$$

Completely analogously to the equation (C.124), one can show

$$\Delta_{\perp} \Delta'_{\perp} D_{\vec{r}, \vec{r}'} + \sum_{\mu, \mu'} \Delta_{\parallel(\mu)} \Delta'_{\parallel(\mu')} \delta_{\mu\mu'} D_{\vec{r}, \vec{r}'} = \sum_{\mu < \nu} L_{\mu\nu, \vec{r}} L_{\mu\nu, \vec{r}'} \delta_{\vec{r}, \vec{r}'} . \quad (\text{C.279})$$

This result can be easily checked for the case where the surface enclosed by C is resting on the x plane and the only $L_{\mu\nu, \vec{r}}$ distinct from zero are L_{23} : $\Delta_{\perp} = \Delta_1$, $\Delta_{\parallel(2)} = \Delta_3$ and $\Delta_{\parallel(3)} = -\Delta_2$ for sites $\vec{r}$ inside the enclosed area, and therefore

$$\Delta_{\perp} \Delta'_{\perp} D_{\vec{r}, \vec{r}'} + \sum_{\mu, \mu'} \Delta_{\parallel(\mu)} \Delta'_{\parallel(\mu')} \delta_{\mu\mu'} D_{\vec{r}, \vec{r}'} = (-\Delta\Delta) D_{\vec{r}, \vec{r}'} .$$

Substitution of (C.279) into (C.278), followed by the regrouping of the terms in the sum and by noting the definition of parallel derivative $\Delta_{\parallel(\mu)} = \sum_{\nu} L_{\mu\nu, \vec{r}} \Delta_{\nu}$, leads to

$$\begin{aligned} & \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \mu < \nu} \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda\mu\nu} (\Delta_{\lambda} L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'} \epsilon_{\lambda'\mu'\nu'} \Delta'_{\lambda'} L_{\mu'\nu', \vec{r}'} = \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 - \\ & - \sum_{\vec{r}, \vec{r}'} \sum_{\mu, \mu'} \delta_{\mu\mu'} \underbrace{\left(\sum_{\nu} \Delta_{\nu} L_{\mu\nu, \vec{r}-\hat{\nu}} \right)}_{J_{\mu, \vec{r}}} D_{\vec{r}, \vec{r}'} \underbrace{\sum_{\nu'} \Delta_{\nu'} L_{\mu'\nu', \vec{r}'-\hat{\nu}'}}_{J_{\mu', \vec{r}'}} , \end{aligned} \quad (\text{C.280})$$

where in the last term the equation defining $L_{\mu\nu, \vec{r}}$, $\sum_{\mu} \Delta_{\mu} L_{\mu\nu, \vec{r}-\hat{\mu}} = -J_{\nu, \vec{r}}$, was used.

5. With these previous results, correlation $W(C)$ becomes

$$\begin{aligned} W(C) = & \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] \frac{Z_0}{Z} \sum_{\{G_{\vec{r}}\}} \times \\ & \times \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} - \frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{\pi}{2ae^2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \frac{G_{\vec{r}} G_{\vec{r}'}}{|\vec{r} - \vec{r}'|} \right] , \end{aligned} \quad (\text{C.281})$$

with $\mathcal{Z}$ equal to equation (C.271). As in the XY model, we can rewrite this equation as follows

$$W(C) = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] \left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G, \quad (\text{C.282})$$

with the expectation value $\langle \rangle_G$ evaluated by means of the partition function $\mathcal{Z}_G$ given by

$$\mathcal{Z}_G = \sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{\pi}{2ae^2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} G_{\vec{r}'} \right]. \quad (\text{C.283})$$

Note. About the topological fields (II): the Polyakov model.

As it was done for the XY model, we shall re-obtain the above partition function by means of the Polyakov's argument [81] (see also [86]) instead of the Villain approximation. This another line of argumentation leads to a clearer meaning of the topological integer fields $G_{\vec{r}}$'s. Let us consider again the original partition function for the (compact) pure gauge field

$$\mathcal{Z} = \prod_{\vec{r}, \mu} \int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (1 - \cos F_{\mu\nu, \vec{r}}) \right], \quad (\text{C.284})$$

with $F_{\mu\nu, \vec{r}} = \Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}$. The periodic energy function (as we know, indication of the angular character of Abelian gauge field $A_{\mu, \vec{r}}$) can be approximated, specially for small $ae^2\eta$, by its quadratic term and simultaneously, one extends the $A_{\mu, \vec{r}}$ integration range from $-\infty$ to ∞ . Thus, the energy becomes as in the continuum formulation

$$E[A_{\mu}] = \frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (1 - \cos F_{\mu\nu, \vec{r}}) \approx \frac{1}{2ae^2\eta} \int d^3\vec{r} F_{\mu\nu}^2(\vec{r}). \quad (\text{C.285})$$

But we should bear in mind that $A_{\mu, \vec{r}}$ has angular character and therefore jumps of $2\pi k$ (with k an integer) are also allowed. Following the same strategy as for the XY model, the main contribution to the partition arises from the minimum of $E[A_{\mu}]$, the solution of

$$\partial_{\mu} F_{\mu\nu} = 0 \pmod{2\pi}, \quad (\text{C.286})$$

where $\pmod{2\pi}$ takes account of the $2\pi k$ jumps. As already seen in the introduction of this section, a case where such jumps take place is the “ $SO(3)$ ”-superconductor discussed in Appendix B. The jumps in the gauge field of this model are over 4π , and the solution of the above equation is

$$F_{\text{cl}\mu\nu} = \frac{1}{2} \sum_{\vec{r}_m} \sum_{\lambda} G_{\vec{r}_m} \epsilon_{\mu\nu\lambda} \frac{(\vec{r} - \vec{r}_m)_{\lambda}}{|\vec{r} - \vec{r}_m|^3} \quad \text{with} \quad G_{\vec{r}_m} = \pm 2, \quad (\text{C.287})$$

which describes the magnetic field created positive(negative) 't Hooft-Polyakov monopoles of strength $g_{\text{mg}} = 4\pi$ ²¹. Let us call $A_{\text{cl}\mu}$ the gauge potential giving rise to solution (C.287). Of course, such a gauge potential is multi-valued, as its “jumps” indicate.

The role of integers $G_{\vec{r}_m}$ is completely analogous to that of $M_{\vec{r}_m}$ in the XY model (see equation (C.133)): there, we had partition function $\mathcal{Z} = \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left[-\frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu} (1 - \cos \Delta_{\mu} \varphi_{\vec{r}}) \right]$ and

²¹One should remember that A_{μ} and $F_{\mu\nu}$ already contains e and therefore $g_{\text{mg}} = 4\pi$ is equivalent to relation $g_{\text{mg}} = 4\pi/e$ (a special case of Dirac relation $g_{\text{mg}}e = n2\pi$) in (B.184), when e was not absorbed into the gauge field.

the main contribution was due to the solutions of $\nabla^2 \varphi_{\text{cl}} = 0 \bmod 2\pi$. It held for those solutions $\oint_{\text{closed path around } \vec{r}_m} \partial_\mu \varphi_{\text{cl}} dl^\mu = 2\pi M_{\vec{r}_m}$. Here, solution (C.287) fulfils $\oint_{\text{closed surface around } \vec{r}_m} F_{\text{cl}\mu\nu} dS^{\mu\nu} = 2\pi G_{\vec{r}_m}$. The analogy can be summarized as follows

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r}} \int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left[-\frac{|\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu} (1 - \cos \Delta_{\mu} \varphi_{\vec{r}}) \right] : \quad \sum_{\substack{\text{along a} \\ \text{closed path} \\ \text{around } \vec{r}_m}} \Delta_{\mu} \varphi_{\vec{r}} \neq 0 \quad \Leftrightarrow \quad M_{\vec{r}_m} \neq 0, \\ \mathcal{Z} &= \prod_{\vec{r}, \mu} \int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{ae^2 \eta} \sum_{\vec{r}} \sum_{\mu < \nu} (1 - \cos F_{\mu\nu, \vec{r}}) \right] : \quad \sum_{\substack{\text{on a closed} \\ \text{surface} \\ \text{around } \vec{r}_m}} F_{\mu\nu, \vec{r}} \neq 0 \quad \Leftrightarrow \quad G_{\vec{r}_m} \neq 0. \end{aligned} \quad (\text{C.288})$$

Let us continue with the $SO(3)$ model, also called the Polyakov model. At small $ae^2\eta$, the most probable A_μ configurations are given by $A_\mu = A_{\text{cl}\mu} + \Gamma_\mu$, where Γ_μ represents a small perturbation on $A_{\text{cl}\mu}$. Hence, the energy of such a configuration becomes

$$E[A_\mu] \approx E[A_{\text{cl}\mu}] + \underbrace{\int d^3\vec{r} \frac{\delta E[A_\mu]}{\delta A_\mu(\vec{r})} \Big|_{\text{cl}}}_0 \Gamma_\mu(\vec{r}) + \frac{1}{2} \int d^3\vec{r} d^3\vec{r'} \underbrace{\frac{\delta^2 E[A_\mu]}{\delta A_\mu(\vec{r}) \delta A_\nu(\vec{r'})} \Big|_{\text{cl}}}_{-\frac{1}{ae^2\eta} (\delta_{\mu\nu} \partial_\rho \partial_\rho \delta^3(\vec{r} - \vec{r'}) - \partial_\mu \partial_\nu \delta^3(\vec{r} - \vec{r'}))} \Gamma_\mu(\vec{r}) \Gamma_\nu(\vec{r'}),$$

and partition function (C.284) is well approximated by

$$\mathcal{Z} \approx e^{-E[A_{\text{cl}\mu}]} \prod_{\vec{r}, \mu} \int_{-\infty}^{\infty} \frac{d\Gamma_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \Gamma_{\mu, \vec{r}} \left(-\delta_{\mu\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} + \Delta_{\mu} \Delta_{\nu} \right) \Gamma_{\nu, \vec{r}} \right], \quad (\text{C.289})$$

with

$$E[A_{\text{el}\mu}] = \frac{1}{8ae^2\eta} \int d^3\vec{r} \left(\sum_{\vec{r}_m} \sum_{\lambda} G_{\vec{r}_m} \epsilon_{\mu\nu\lambda} \frac{(\vec{r} - \vec{r}_m)_\lambda}{|\vec{r} - \vec{r}_m|^3} \right)^2. \quad (\text{C.290})$$

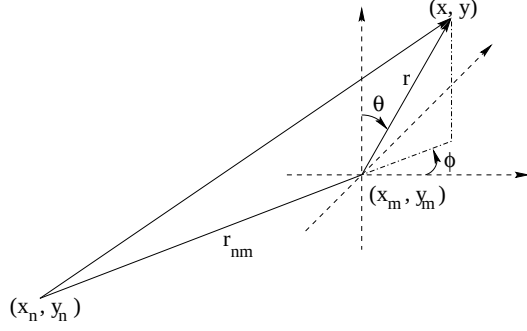
The above integral over $d\Gamma_{\mu, \vec{r}}$, with the integrand rewritten in discrete form, is partition $\mathcal{Z}_0$ (C.273) due to the gauge part without monopoles. In the dislocation model, $\mathcal{Z}_0$ represents the contribution of gliding dislocation lines which are not stopped at any of their points.

To complete the partition function, we need to calculate $E[A_{\text{cl}\mu}]$

$$E[A_{cl\mu}] = \frac{1}{8ae^2\eta} \sum_{\vec{r}_m} \sum_{\vec{r}_n} G_{\vec{r}_m} G_{\vec{r}_n} \int d^3\vec{r} \frac{(\vec{r} - \vec{r}_m) \cdot (\vec{r} - \vec{r}_n)}{|\vec{r} - \vec{r}_m|^3 |\vec{r} - \vec{r}_n|^3}.$$

To carry out the integral, we firstly fix $\vec{r}_m$ and sum over all $\vec{r}_n$, choosing for each $\vec{r}_n$ the coordinates of the following figure

$$\begin{aligned} & \sum_{\vec{r}_n} G_{\vec{r}_n} \int d^3\vec{r} \frac{(x-x_m)(x-x_n) + (y-y_m)(y-y_n) + (z-z_m)(z-z_n)}{[(x-x_m)^2 + (y-y_m)^2 + (z-z_m)^2]^{\frac{3}{2}} [(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2]^{\frac{3}{2}}} = \\ & = G_{\vec{r}_m} \int_c^{\sqrt[3]{|\Lambda|}} dr \frac{4\pi}{r^2} + \sum_{\vec{r}_n \neq \vec{r}_m} G_{\vec{r}_n} \int_0^{\sqrt[3]{|\Lambda|}} 2\pi dr \int_0^\pi d\theta \sin\theta \frac{r + r_{nm} \cos\theta}{(r^2 + r_{nm}^2 + 2rr_{nm} \cos\theta)^{\frac{3}{2}}}, \end{aligned}$$



where $\sqrt[3]{|\Lambda|}$ is the length dimension of the whole lattice and c is the cut-off to avoid the divergence in the first r integral at very short distances, where the continuum approximation is not longer valid. The angular integral of the second summand is equal to

$$\int_0^\pi \frac{\sin \theta (r + r_{nm} \cos \theta) d\theta}{(r^2 + r_{nm}^2 + 2rr_{nm} \cos \theta)^{\frac{3}{2}}} = \begin{cases} 0 & r < r_{nm} \\ \frac{2}{r^2} & r > r_{nm} \end{cases}$$

as one can easily prove²². Hence, $E[A_{cl\mu}]$ becomes

$$E[A_{cl\mu}] = \frac{\pi}{2ae^2\eta} \sum_{\vec{r}_m} \frac{G_{\vec{r}_m}^2}{c} + \frac{\pi}{2ae^2\eta} \sum_{\vec{r}_m, \vec{r}_n \neq \vec{r}_m} \frac{G_{\vec{r}_m} G_{\vec{r}_n}}{r_{nm}}. \quad (C.291)$$

As already discussed in the “ $SO(3)$ ”-superconductor (see equation (B.134)), the monopole solution

$$F_{\mu\nu} = \frac{1}{2} \sum_{\vec{r}_m} \sum_{\lambda} G_{\vec{r}_m} \epsilon_{\mu\nu\lambda} \frac{(\vec{r} - \vec{r}_m)_\lambda}{|\vec{r} - \vec{r}_m|^3},$$

holds no longer at distances smaller than range ξ_w of the $SO(3)$ gauge fields. Hence, the above cut-off ac (restoring the length dimension), which in the propagator's approximation (C.270) has been taken approximately equal to the lattice spacing, is determined by the parameters of the underlying $SO(3)$ model, where the compact $U(1)$ symmetry is embedded. Note the similarity of the monopole self-energy $2\pi/e^2c$ (c with length dimension) to equation (B.196).

The above result (C.291) for $E[A_{cl\mu}]$ and equation (C.289) are the same as the result of the Villain approximation (C.283), provided the system has only monopoles of double magnetic flux $G_{\vec{r}} = \pm 2$. This was already expected, after the discussion following equation (C.253); for this case of double magnetic flux monopoles, we could work with $\mathcal{Z} = \dots \sum_{\{g_{\mu\nu}, \vec{r}\}} e^{-\frac{1}{2a(2^2)e^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_\mu \bar{A}_{\nu, \vec{r}} - \Delta_\nu \bar{A}_{\mu, \vec{r}} - 4\pi g_{\mu\nu, \vec{r}})^2}$

and $F_{cl\mu\nu} = \sum_{\vec{r}_m} \sum_{\lambda} G_{\vec{r}_m} \epsilon_{\mu\nu\lambda} \frac{(\vec{r} - \vec{r}_m)_\lambda}{|\vec{r} - \vec{r}_m|^3}$ with $G_{\vec{r}} = \pm 1$, obtaining the same result.

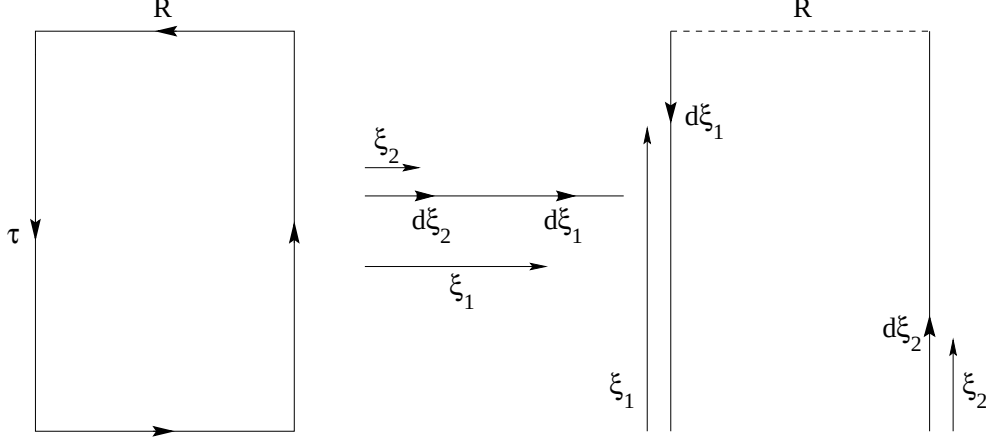
Before continuing with the discussion of the correlation behaviour at low and high $ae^2\eta$, let us obtain a more manageable form for $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}$ in the case of a large closed path C

$$\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} = D(0) \mathcal{P} + \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\mu} J_{\mu, \vec{r}} \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} J_{\mu, \vec{r}'}, \quad (C.292)$$

where $\mathcal{P} = \sum_{\vec{r}, \mu} J_{\mu, \vec{r}}^2$ is the loop's perimeter.

²²By means of

$$\int_{-1}^{+1} dx \frac{r + r_{nm}x}{(r^2 + r_{nm}^2 + 2rr_{nm}x)^{\frac{3}{2}}} = -\frac{1}{r_{nm}} \left(\frac{1}{r + r_{nm}} - \frac{1}{|r - r_{nm}|} \right) - \frac{1}{r} \left(\frac{1}{r + r_{nm}} + \frac{1}{|r - r_{nm}|} \right) - \frac{r + r_{nm} - |r - r_{nm}|}{r^2 r_{nm}}.$$



Let C be the closed path of the figure, with $\tau \gg R \gg a$ and a the lattice spacing. In this case, we can approximate the second sum in (C.292) by an integral. There is only interaction inside the same side or between parallel sides (but not between adjacent sides, due to the scalar product $\dots \sum_{\mu} J_{\mu, \vec{r}} J_{\mu, \vec{r}'} \dots$)

$$I_R = 2 \int_a^R d\xi_1 \int_0^{\xi_1-a} d\xi_2 \frac{1}{4\pi(\xi_1 - \xi_2)} = \frac{1}{2\pi} \left(R \ln \frac{R}{a} - R + a \right),$$

$$\begin{aligned} I_{\tau\tau} &= - \int_0^\tau d\xi_1 \int_0^\tau d\xi_2 \frac{1}{4\pi\sqrt{R^2 + (\xi_1 - \xi_2)^2}} = \\ &= \frac{1}{2\pi} \left(-\frac{\tau}{2} \ln \left(\frac{\sqrt{\tau^2 + R^2} + \tau}{\sqrt{\tau^2 + R^2} - \tau} \right) + \sqrt{\tau^2 + R^2} - R \right). \end{aligned}$$

The interaction of every segment in C with all others is given by $2(I_R + I_\tau + I_{\tau\tau} + I_{RR})$, and for $\tau \gg R$, one obtains

$$\sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\mu} \frac{J_{\mu, \vec{r}} J_{\mu, \vec{r}'}}{4\pi|\vec{r} - \vec{r}'|} = 2(I_R + I_\tau + I_{\tau\tau} + I_{RR}) \approx \frac{1}{\pi} \left(\tau - \tau \ln \frac{R}{2a} \right) \approx \frac{\tau}{\pi} \ln R.$$

$$\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \approx \frac{\tau}{\pi} \ln R, \quad (\text{C.293})$$

since $\tau \ln R$ is stronger than perimeter $\mathcal{P} = 2(\tau + R) \approx 2\tau$.

Now let us study in detail the correlation function (C.282) and (C.283). At first glance, it is similar to the 2-dimensional XY model since both models contain point topological defects. But in our case, the interaction between these defects decreases with the distance, $D_{\vec{r}, \vec{r}'} \sim |\vec{r} - \vec{r}'|^{-1}$, whereas in the XY model increases, $D_{\vec{r}, \vec{r}'} \sim \ln |\vec{r} - \vec{r}'|$. Therefore, in the 3-dimensional pure gauge model, the topological defects are not bound in close dipoles but they are acting as isolated defects even at low $ae^2\eta$ (small $ae^2\eta$ implies only low monopole density). With the estimation method of Peierls' contours, we always obtain an exponential decay with the enclosed area. Each time an isolated monopole crosses the area delimited by C , $\Omega_{\vec{r}}$ is equal to 2π and we obtain a negative contribution in the correlation denominator; the number of sites available for an isolated

monopole which crosses this area is proportional to the area itself. For monopoles which are approximately isolated, we can estimate

$$\left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G = \frac{1}{\mathcal{Z}'} \sum_{\text{isolated monopoles}} \frac{1}{k!} \Sigma_k \zeta^k ,$$

with

$$\Sigma_k = [(|\Lambda| - \mathcal{A}) \cdot 1 + \mathcal{A} \cdot (-1)]^k , \quad (\text{C.294})$$

obtaining thus

$$\left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G \approx e^{-2\mathcal{A}\zeta} . \quad (\text{C.295})$$

Correlation $W(C) = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] \left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G$ decays therefore with an area law for both $ae^2\eta$ ranges

$$W(C) \sim \exp \left[-\frac{1}{\pi} \tau \ln R \right] \exp [-\mathcal{A}] \approx e^{-\mathcal{A}} , \quad (\text{C.296})$$

since the decay like the loop area $\mathcal{A} = R\tau$ is stronger than decay like $\tau \ln R$. Neither at small $ae^2\eta$, we can neglect the effect of monopoles (if they had been neglected, then we would have obtained a decay like $\tau \ln R$).

Following the paper by Polyakov [81] (see also [18]), we shall calculate in detail the correlation $W(C)$ for the low $ae^2\eta$ range and prove thus that the above estimation is correct. Let us start with $\left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G$

$$\frac{1}{\mathcal{Z}_G} \sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} - \frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} G_{\vec{r}'} \right] . \quad (\text{C.297})$$

According to equation (C.277), the Laplace's operator applied on $\Omega_{\vec{r}}$ is equal to

$$\left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} \right) \Omega_{\vec{r}} = 4\pi \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \frac{\Delta}{(\vec{r})} \lambda L_{\mu\nu, \vec{r}} . \quad (\text{C.298})$$

Further by considering (C.280), the following useful relation is proved

$$\underbrace{\sum_{\vec{r}} \Omega_{\vec{r}} \left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} \right) \Omega_{\vec{r}}}_{\sum_{\vec{r}, \rho} (\Delta_{\rho} \Omega_{\vec{r}})^2} = 16\pi^2 \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \mu < \nu} \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda\mu\nu} \frac{\Delta}{(\vec{r})} \lambda L_{\mu\nu, \vec{r}} D_{\vec{r}, \vec{r}'} \epsilon_{\lambda'\mu'\nu'} \frac{\Delta}{(\vec{r}')} \lambda' L_{\mu'\nu', \vec{r}'} ,$$

$$\sum_{\vec{r}, \rho} (\Delta_{\rho} \Omega_{\vec{r}})^2 = 16\pi^2 \left[\sum_{\vec{r}} \sum_{\mu < \nu} L_{\mu\nu, \vec{r}}^2 - \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] . \quad (\text{C.299})$$

Since the closed path C is much larger than the lattice spacing, it is easier to evaluate (C.297) in the continuum formulation: as first step we rewrite the Coulomb propagator

$|\vec{r} - \vec{r}'|^{-1}$ as a Gaussian integral over an scalar field $\phi_{\vec{r}}$ and then we change to the continuum limit with the following associations

$$a^{-1}\Delta_\mu \rightarrow \vec{\nabla}, \quad a^3 \sum_{\vec{r}} \rightarrow \int d^3\vec{r}, \quad \prod_{\vec{r}} \left(\int_{-\infty}^{\infty} d\phi_{\vec{r}} \right) \rightarrow \int_{-\infty}^{\infty} \mathcal{D}\phi(\vec{r}),$$

where $\int \mathcal{D}\phi$ is a functional integral. The first step gives

$$\begin{aligned} \left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G &= \frac{1}{\mathcal{Z}'_G} \prod_{\vec{r}} \int_{-\infty}^{\infty} d\phi_{\vec{r}} \exp \left[-\frac{ae^2\eta}{8\pi^2} \sum_{\vec{r}, \mu} (\Delta_\mu \phi_{\vec{r}})^2 \right] \times \\ &\times \prod_{\vec{r}} \sum_{G_{\vec{r}}} \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} - \frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 + i \sum_{\vec{r}} \phi_{\vec{r}} G_{\vec{r}} \right], \end{aligned} \quad (\text{C.300})$$

where the factor $\left(\sqrt{\frac{ae^2\eta}{8\pi^3}} \right)^{|\Lambda|} \sqrt{\det(-\Delta\Delta)}$ arising from the inclusion of the $\phi_{\vec{r}}$ integration has been absorbed into $\mathcal{Z}_G$, changing thus into $\mathcal{Z}'_G$. At small $ae^2\eta$ the factor $\exp[-2\pi^2/ae^2\eta D(0) \sum_{\vec{r}} G_{\vec{r}}^2]$ makes the monopole density be low, and therefore we only need sum over $G_{\vec{r}} = -1, 0, 1$

$$\begin{aligned} \prod_{\vec{r}} \left[1 + e^{i\left(\phi_{\vec{r}} - \frac{\Omega_{\vec{r}}}{2}\right) - \frac{2\pi^2}{ae^2\eta} D(0)} + e^{-i\left(\phi_{\vec{r}} - \frac{\Omega_{\vec{r}}}{2}\right) - \frac{2\pi^2}{ae^2\eta} D(0)} \right] &= \prod_{\vec{r}} \left[1 + 2 \cos \left(\phi_{\vec{r}} - \frac{\Omega_{\vec{r}}}{2} \right) e^{-\frac{2\pi^2}{ae^2\eta} D(0)} \right] \approx \\ &\stackrel{\text{small } ae^2\eta}{\approx} \exp \left[\sum_{\vec{r}} 2 \cos \left(\phi_{\vec{r}} - \frac{\Omega_{\vec{r}}}{2} \right) e^{-\frac{2\pi^2}{ae^2\eta} D(0)} \right]. \end{aligned} \quad (\text{C.301})$$

In the continuum formulation, correlation $\left\langle \exp \left[-\frac{i}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} \right] \right\rangle_G$ becomes

$$\frac{1}{\mathcal{Z}'_G} \int_{-\infty}^{\infty} \mathcal{D}\phi \exp \left\{ -\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi)^2 - \frac{8\pi^2}{a^3e^2\eta} e^{-\frac{2\pi^2}{ae^2\eta} D(0)} \cos \left(\phi - \frac{\Omega}{2} \right) \right] \right\}. \quad (\text{C.302})$$

The partition function is given by

$$\mathcal{Z}'_G = \int_{-\infty}^{\infty} \mathcal{D}\phi \exp \left\{ -\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi)^2 - M^2 \cos \phi(\vec{r}) \right] \right\}, \quad (\text{C.303})$$

where M^2 has dimension of length^{-2} ($ae^2\eta$ is dimensionless) and is equal to

$$M^2 = \frac{8\pi^2}{a^3e^2\eta} e^{-\frac{2\pi^2}{ae^2\eta} D(0)}. \quad (\text{C.304})$$

Let us suppose that the path C is a large rectangle resting on the $x = 0$ plane, with dimensions $R \times \tau$ and $R \ll \tau$: the field $L_{\mu\nu}(\vec{r})$ characterizing C is therefore given by

$$L_{\mu\nu} = L_{23}(x, y, z) = \Theta_C(y, z) \delta(x), \quad (\text{C.305})$$

with $\Theta_C(y, z) = 1$ if y and z are inside the path C and zero otherwise. In the numerator of $W(C)$ we change integration variables from ϕ to $\phi' = \phi - \frac{1}{2}\Omega$. Using the result (C.298) and (C.299) ²³, we obtain (after renaming ϕ' into ϕ)

$$e^{-\frac{e^2\eta}{2}a^2\mathcal{A}\delta(0)} \frac{\int_{-\infty}^{\infty} \mathcal{D}\phi \exp\left\{-\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi)^2 - \phi 2\pi\Theta_C \partial_x \delta(x) - M^2 \cos \phi\right]\right\}}{\int_{-\infty}^{\infty} \mathcal{D}\phi \exp\left\{-\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi)^2 - M^2 \cos \phi\right]\right\}}. \quad (\text{C.306})$$

We need not to be worried about the divergence in the first exponent since it will be canceled out. Factor $\exp\left[+\frac{e^2\eta}{2} \oint_C \oint_C d\vec{r} d\vec{r}' J_\mu(\vec{r}) D(|\vec{r} - \vec{r}'|) J_\mu(\vec{r}')\right]$ arising from term $\int (\vec{\nabla}\Omega)^2$ is compensated by the first factor $\exp\left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_\mu J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}\right]$ of the correlation $W(C)$ (C.282).

Let us focus our attention on the correlation's exponent: the interaction terms contained in the different ϕ powers of the cosine's expansion are all multiplied by M^2 , which is very small at small $ae^2\eta$. The main contribution arises from ϕ_{cl} , for which the exponent is extremal; but it is not the unique contribution and therefore we have to consider also all possible fluctuations about this extremum. Substitution of $\phi = \phi_{\text{cl}} + \phi_{\text{fluct}}$ into the exponent allows the expansion in powers of fluctuation ϕ_{fluct} . However, the terms which couple ϕ_{fluct} to ϕ_{cl} are multiplied by M^2 , i.e., fluctuations are very weakly coupled to the energy's extremum ϕ_{cl} and we can approximate the correlation only by the contribution due to ϕ_{cl} . This approximation is called the mean field approximation. The same argument can be applied to the partition function in the denominator. One obtains thus

$$W(C) \approx e^{-\frac{e^2\eta}{2}a^2\mathcal{A}\delta(0)} \frac{\exp\left\{-\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi_{\text{cl}})^2 + \phi_{\text{cl}} 2\pi\Theta_C \partial_x \delta(x) - M^2 \cos \phi_{\text{cl}}\right]\right\}}{\exp\left\{-\frac{e^2\eta}{4\pi^2} \int d^3\vec{r} \left[\frac{1}{2} (\vec{\nabla}\phi_{\text{cl}})^2 - M^2 \cos \phi_{\text{cl}}\right]\right\}}. \quad (\text{C.307})$$

Configuration ϕ_{cl} in the numerator is obtained by solving the following equation

$$\nabla^2 \phi_{\text{cl}} = 2\pi\Theta_C(y, z) \partial_x \delta(x) + M^2 \sin \phi_{\text{cl}}. \quad (\text{C.308})$$

Neglecting the effects from the boundary of C , i.e. for sites inside C and far from the boundaries, this equation becomes one dimensional

$$\partial_x^2 \phi_{\text{cl}} = 2\pi\Theta_C(y, z) \partial_x \delta(x) + M^2 \sin \phi_{\text{cl}}, \quad (\text{C.309})$$

$$\begin{aligned} \int d^3\vec{r} \phi (\nabla^2 \Omega) &\rightarrow a \sum_{\vec{r}} \phi_{\vec{r}} (\Delta \Omega_{\vec{r}}) = -4\pi a \sum_{\vec{r}} \phi_{\vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda L_{\mu\nu, \vec{r}} \rightarrow -4\pi \frac{a}{a^3} \int d^3\vec{r} \phi \Theta_C a \partial_x (a\delta(x)), \\ \int d^3\vec{r} (\vec{\nabla}\Omega)^2 &\rightarrow a \sum_{\vec{r}, \rho} (\Delta_\rho \Omega_{\vec{r}})^2 = 16\pi^2 a \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 - 16\pi^2 a \sum_{\vec{r}, \vec{r}', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \rightarrow \\ &\rightarrow 16\pi^2 a^2 \mathcal{A}\delta(0) - 16\pi^2 \oint_C \oint_C d\vec{r} d\vec{r}' J_\mu(\vec{r}) D(|\vec{r} - \vec{r}'|) J_\mu(\vec{r}') \quad \text{with } \int \int dy dz \Theta_C = \mathcal{A} a^2 \text{ the enclosed area.} \end{aligned}$$

whose solution is given ²⁴ by

$$\phi_{\text{cl}}(x, y, z) = \Theta_C(y, z) [\vartheta(x)\phi_0(x) - \vartheta(-x)\phi_0(-x)] , \quad (\text{C.310})$$

with $\vartheta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$, and where ϕ_0 is the solution of equation $\partial_\xi^2 \phi_0(\xi) = M^2 \sin \phi_0(\xi)$ with $\phi_0(0) = \pi$

$$\phi_0(\xi) = 4 \arctan \left(e^{-M\xi} \right) . \quad (\text{C.311})$$

This solution can be easily proved if we consider the relation $\sin(4\alpha) = \frac{4 \tan \alpha (1 - \tan^2 \alpha)}{(1 + \tan^2 \alpha)^2}$. The same steps applied to the denominator has as result that its corresponding ϕ_{cl} is ϕ_0 of equation (C.311). Outside the area $a^2 \mathcal{A}$ enclosed by C , both numerator and denominator are equal and therefore the correlation takes the following form

$$e^{-\frac{e^2 \eta}{2} a^2 \mathcal{A} \delta(0)} \frac{\exp \left\{ -\frac{e^2 \eta}{4\pi^2} a^2 \mathcal{A} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{d\phi_{\text{cl}}(x)}{dx} \right)^2 + 2\pi \frac{d\phi_{\text{cl}}(x)}{dx} \delta(x) - M^2 \cos \phi_{\text{cl}}(x) \right] \right\}}{\exp \left\{ -\frac{e^2 \eta}{4\pi^2} a^2 \mathcal{A} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{d\phi_0(x)}{dx} \right)^2 - M^2 \cos \phi_0(x) \right] \right\}} .$$

Let us substitute the obtained solution ϕ_{cl} into the numerator exponent, using the relation $\vartheta(x)\delta(x) = \vartheta(x) \frac{d\vartheta(x)}{dx} = \frac{1}{2} \frac{d\vartheta^2(x)}{dx} = \frac{1}{2} \frac{d\vartheta(x)}{dx} = \frac{1}{2} \delta(x)$. We obtaining thus

$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} dx \frac{d\phi_{\text{cl}}(x)}{dx} \delta(x) &= -4\pi^2 \int dx \delta(x) \delta(x) + 2 \times 8\pi M \int dx \frac{1}{2} \delta(x) \frac{e^{-Mx}}{1 + e^{-2Mx}} = \\ &= -4\pi^2 \delta(0) + 4\pi M , \end{aligned} \quad (\text{C.312})$$

and

$$\int_{-\infty}^{\infty} dx \frac{1}{2} \left(\frac{d\phi_{\text{cl}}(x)}{dx} \right)^2 = 2\pi^2 \delta(0) - 2\pi M + \dots , \quad (\text{C.313})$$

where $\dots$ contains the same terms as the denominator. Term $-2\pi^2 \delta(0)$ cancels the (infinite) factor $e^{-\frac{e^2 \eta}{2} a^2 \mathcal{A} \delta(0)}$ out. Finally, the correlation function at small $a e^2 \eta$ is given by

$$W(C) = \exp \left[-\frac{e^2 \eta}{2\pi} M a^2 \mathcal{A} \right] . \quad (\text{C.314})$$

²⁴By considering the equation $\frac{d}{dx} \vartheta(x) = \delta(x)$ and $\delta(-x) = \delta(x)$, we obtain

$$\partial_x \phi_{\text{cl}} = \Theta_C \left[\underbrace{\delta(x)\phi_0(x)}_{\delta(x)\phi_0(0)} + \vartheta(x) \partial_x \phi_0(x) + \delta(x)\phi_0(0) + \vartheta(-x) \partial_{(-x)} \phi_0(-x) \right] ,$$

and since in general $\lim_{x \rightarrow 0} \partial_{(-x)} \phi_0(-x) = \lim_{x \rightarrow 0} \partial_x \phi_0(x)$ holds,

$$\partial_x^2 \phi_{\text{cl}} = \Theta_C \left[2\partial_x \delta(x)\phi_0(0) + \vartheta(x) \partial_x^2 \phi_0(x) - \vartheta(-x) \partial_{(-x)}^2 \phi_0(-x) \right] ,$$

provided that $\phi_0(0) = \pi$.

As we have expected according to our estimation with the estimation method based on disordering contours (C.295), the correlation decays with an area law also for the small $ae^2\eta$ range.

As in the 2-dimensional XY model, the decay for large $ae^2\eta$ obtained by the “high temperature” power expansion (C.256) will be re-obtained by means of the Villain-approximated correlation (C.282). At this range, we can suppose that the interactions between monopoles are cancelled out against each other due to their highly random distribution. Therefore the term $\sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} G_{\vec{r}'} |\vec{r} - \vec{r}'|^{-1}$ will be neglected, and the partition function becomes

$$\mathcal{Z}_G \approx \prod_{\vec{r}} \sum_{G_{\vec{r}}} \exp \left[-\frac{2\pi^2}{ae^2\eta} D(0) G_{\vec{r}}^2 \right] = \left[\sum_{G=-\infty}^{+\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0) G^2} \right]^{|\Lambda|}. \quad (\text{C.315})$$

If an isolated monopole crosses (or lies very near to) the surface delimited by C , the value of $\Omega_{\vec{r}}$ is approximately 2π ; if it is distant, the solid angle would become very small. Then the correlation is approximately given by

$$\begin{aligned} \left\langle \prod_{\vec{r}} \exp \left(-\frac{i}{2} \Omega_{\vec{r}} G_{\vec{r}} \right) \right\rangle_G &\approx \frac{1}{\mathcal{Z}_G} \prod_{\vec{r}} \sum_{G_{\vec{r}}} \exp \left[-\frac{i}{2} \Omega_{\vec{r}} G_{\vec{r}} - \frac{2\pi^2}{ae^2\eta} D(0) G_{\vec{r}}^2 \right] \approx \\ &\approx \frac{\left[2 \sum_{G=0}^{\infty} (-1)^G e^{-\frac{2\pi^2}{ae^2\eta} D(0) G^2} - 1 \right]^{\mathcal{A}} \left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0) G^2} - 1 \right]^{|\Lambda| - \mathcal{A}}}{\left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0) G^2} - 1 \right]^{\mathcal{A}} \left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0) G^2} - 1 \right]^{|\Lambda| - \mathcal{A}}} = \exp [-\kappa \mathcal{A}]. \end{aligned} \quad (\text{C.316})$$

This last equation is equivalent to (C.156) for the XY model, with $y = \exp \left[-\frac{2\pi^2}{ae^2\eta} D(0) \right]$. Using the same argumentation which leads to (C.159), we obtain

$$\kappa \stackrel{\text{large } ae^2\eta}{\approx} \ln(2ae^2\eta), \quad (\text{C.317})$$

as expected. The only difference between the area decay at low $ae^2\eta$ and that for large $ae^2\eta$ is the factor multiplying area $\mathcal{A}$: a very small factor, $\sqrt{ae^2\eta} \exp(-1/ae^2\eta)$, at low $ae^2\eta$ and a large factor, $\ln(ae^2\eta)$, for the large range.

Note. The pure compact $U(1)$ gauge model in two and in four dimensions.

The different phases of the pure gauge model are strongly dependent on the lattice dimensionality. Hence, the 4-dimensional pure gauge model has two different phases whilst the 3-dimensional has only one. In the 2-dimensional model, on the other hand, the correlation function decays always with an area law as in three dimensions, although this fact has nothing to do with monopoles.

In the 2-dimensional case ($\mu, \nu = 1, 2$) we have only one $B_{\mu\nu, \vec{r}} = B_{12, \vec{r}}$, being the correlation $W(C)$ equal to

$$W(C) = \frac{1}{\mathcal{Z}} \left(\sqrt{\frac{a^2 e^2 \eta}{2\pi}} \right)^{|\Lambda|} \prod_{\vec{r}, \mu=1,2} \int_{-\infty}^{\infty} \frac{dA_{\mu, \vec{r}}}{2\pi} \prod_{\vec{r}} \int_{-\infty}^{\infty} dB_{12, \vec{r}} \sum_{\{g_{12, \vec{r}}\}} \times$$

$$\exp \left[-\frac{a^2 e^2 \eta}{2} \sum_{\vec{r}} B_{12,\vec{r}}^2 + i \sum_{\vec{r}} B_{12,\vec{r}} (\Delta_1 A_{2,\vec{r}} - \Delta_2 A_{1,\vec{r}} - 2\pi g_{12,\vec{r}}) + i \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} \right]. \quad (\text{C.318})$$

The constraint on $B_{12,\vec{r}}$

$$\sum_{\mu=1,2} \Delta_\mu B_{\mu\nu,\vec{r}} = \begin{cases} \Delta_1 B_{12,\vec{r}} = J_{2,\vec{r}} \\ -\Delta_2 B_{12,\vec{r}} = J_{1,\vec{r}} \end{cases},$$

produced by the integration over $dA_{\mu,\vec{r}}$ has as solution

$$B_{12,\vec{r}} = -L_{12,\vec{r}}, \quad (\text{C.319})$$

which is an integer field. Substituting into the correlation, we note that the effect due to the topological fields $g_{12,\vec{r}}$ disappears completely from the correlation, since $L_{12,\vec{r}} g_{12,\vec{r}}$ is an integer. The correlation becomes trivial

$$W(C) = \frac{\left(\sqrt{\frac{a^2 e^2 \eta}{2\pi}} \right)^{|\Lambda|} \exp \left[-\frac{a^2 e^2 \eta}{2} \sum_{\vec{r}} L_{12,\vec{r}}^2 \right] \sum_{\{g_{12,\vec{r}}\}} 1}{\left(\sqrt{\frac{a^2 e^2 \eta}{2\pi}} \right)^{|\Lambda|} \sum_{\{g_{12,\vec{r}}\}} 1} = \exp \left[-\frac{a^2 e^2 \eta}{2} \mathcal{A} \right], \quad (\text{C.320})$$

with $\mathcal{A}$ the area enclosed by C . The compact character of the gauge field is completely irrelevant and the same result is obtained if we had started with the non-compact model (C.244): in this case,

we would have obtained $\exp \left[-\frac{a^2 e^2 \eta}{2} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\mu} J_{\mu,\vec{r}} D'_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} \right]$ alone, which together with $D'_{\vec{r},\vec{r}'} = -1/2\pi \ln |\vec{r}' - \vec{r}|$ for $\vec{r}' \neq \vec{r}$ leads to the above result. The calculation of $-\frac{1}{2\pi} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\mu} J_{\mu,\vec{r}} \ln |\vec{r}' - \vec{r}| J_{\mu,\vec{r}'}$

is similar to equation (C.293) and for $\tau \gg R \gg a$, we obtain

$$\begin{aligned} I_R &= 2 \int_a^R d\xi_1 \int_0^{\xi_1 - a} d\xi_2 \left(-\frac{1}{2\pi} \ln (\xi_1 - \xi_2) \right) \approx \frac{1}{2\pi} \left(\frac{3}{2} R^2 - R^2 \ln R \right), \\ I_{\tau\tau} &= - \int_0^\tau d\xi_1 \int_0^\tau d\xi_2 \left(-\frac{1}{2\pi} \ln \sqrt{R^2 + (\xi_1 - \xi_2)^2} \right) = \frac{1}{2\pi} \left(2R\tau \arctan \left(\frac{\tau}{R} \right) - \frac{3}{2} \tau^2 + \right. \\ &\quad \left. + \frac{\tau^2 - R^2}{2} \ln(R^2 + \tau^2) + R^2 \ln R \right), \end{aligned}$$

and therefore

$$\sum_{\vec{r}, \vec{r}', \mu} J_{\mu,\vec{r}} \left(-\frac{1}{2\pi} \ln |\vec{r}' - \vec{r}| \right) J_{\mu,\vec{r}'} = 2(I_R + I_\tau + I_{\tau\tau} + I_{RR}) = R\tau = \mathcal{A}. \quad (\text{C.321})$$

For the 4-dimensional model ($\mu, \nu = 1, 2, 3, 4$), the correlation function is calculated in a similar way as in the 3-dimensional case

$$W(C) = \frac{1}{\mathcal{Z}} \left(\sqrt{\frac{e^2 \eta}{2\pi}} \right)^{6|\Lambda|} \prod_{\vec{r}, \mu=1,2,3,4} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r}, \mu < \nu} \int_{-\infty}^{\infty} dB_{\mu\nu,\vec{r}} \sum_{\{g_{\mu\nu,\vec{r}}\}} \times$$

$$\exp \left[-\frac{e^2 \eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}}) + i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right]. \quad (\text{C.322})$$

The solution of the constraint $\sum_{\mu=1,2,3,4} \Delta_\mu B_{\mu\nu, \vec{r}-\hat{\mu}} = J_{\nu, \vec{r}}$ is here $B_{\mu\nu, \vec{r}} = \sum_{\lambda, \rho} \epsilon_{\mu\nu\lambda\rho} \Delta_\lambda C_{\rho, \vec{r}-\hat{\lambda}} - L_{\mu\nu, \vec{r}}$.

The integration over $dC_{\rho, \vec{r}}$ requires a gauge choice, made for instance by the inclusion of the following gauge fixing factor

$$\left(\sqrt{\frac{e^2 \eta}{2\pi}} \right)^{|\Lambda|} \det(-\Delta\Delta) \exp \left[-\frac{e^2 \eta}{2} \sum_{\vec{r}, \rho} C_{\rho, \vec{r}} \left(-\Delta_\rho \sum_{\mu} \Delta_\mu C_{\mu, \vec{r}-\hat{\mu}} \right) \right].$$

Monopole integer field $G_{\rho, \vec{r}} = -\sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_\lambda g_{\mu\nu, \vec{r}}$ (with $\epsilon_{1234} = 1$) has vector character and fulfils the following relation

$$\sum_{\rho} \Delta_\rho G_{\rho, \vec{r}} = 0. \quad (\text{C.323})$$

The volume element in 4 dimensions is equal to $r^3 \sin \theta_1 \sin^2 \theta_2 d\phi d\theta_1 d\theta_2 dr$ with $\phi \in [0, 2\pi]$ and $\theta_1, \theta_2 \in [0, \pi]$; θ_2 describes the fourth dimension. The “surface” of a 3-dimensional sphere is $r^3 \int \int \int d\phi d\theta_1 d\theta_2 = 2\pi^2 r^3$. Hence, $\delta^4(\vec{r}) = \frac{\delta(r)}{2\pi r^3}$ and the (continuum) propagator in four dimensions, which is also the approximation for the lattice propagator at large distances, is equal to $\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|^2}$. The final result for the correlation is similar to (C.281) with the difference that here the monopoles have to form closed lines

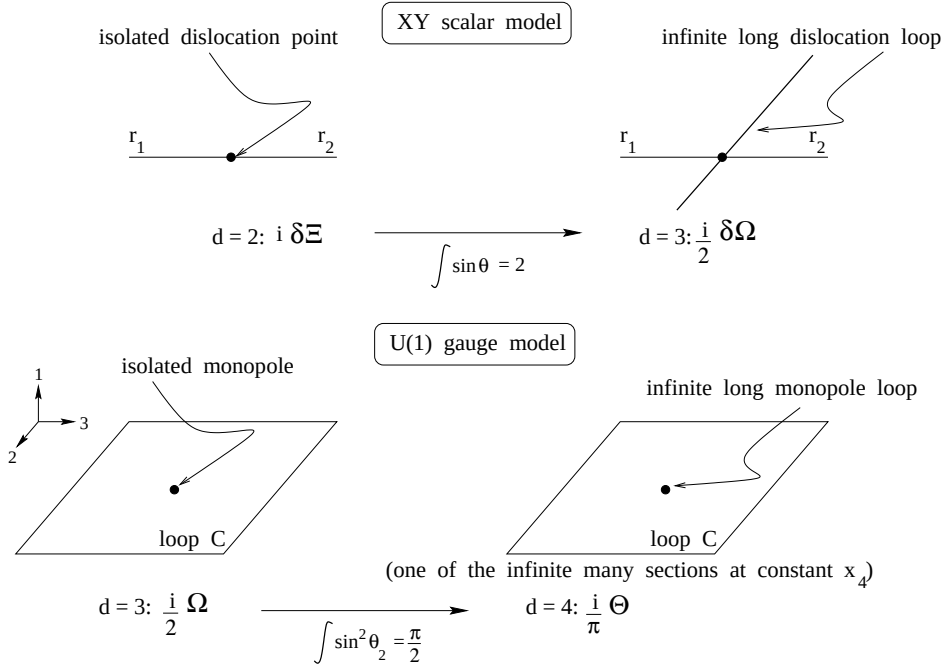
$$W(C) = \exp \left[-\frac{e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] \frac{\mathcal{Z}_0}{\mathcal{Z}} \sum_{\{G_{\lambda, \vec{r}}\}} \delta \left(\sum_{\rho} \Delta_\rho G_{\rho, \vec{r}}, 0 \right) \times \\ \exp \left[i2\pi \sum_{\vec{r}', \vec{r}, \rho} G_{\rho, \vec{r}'} D_{\vec{r}', \vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} - \frac{2\pi^2}{e^2 \eta} D(0) \sum_{\vec{r}, \rho} G_{\rho, \vec{r}}^2 - \frac{1}{2e^2 \eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}, \rho} \frac{G_{\rho, \vec{r}} G_{\rho, \vec{r}'}}{|\vec{r} - \vec{r}'|^2} \right], \quad (\text{C.324})$$

with $D(0) = 0.155$ and $\mathcal{Z}_0$ equal to $\left(\sqrt{\frac{e^2 \eta}{2\pi}} \right)^{3|\Lambda|} \det(-\Delta\Delta)^{-1}$.

For a monopole loop, the imaginary term in the exponential becomes

$$2\pi \sum_{\vec{r}', \vec{r}, \rho} G_{\rho, \vec{r}'} D_{\vec{r}', \vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} = \frac{1}{\pi} \sum_{\vec{r}', \vec{r}} \sum_{\rho, \lambda} \frac{1}{|\vec{r} - \vec{r}'|^3} \sum_{\mu < \nu} \epsilon_{\rho\lambda\mu\nu} G_{\rho, \vec{r}'} \frac{(r_\lambda - r'_\lambda)}{|\vec{r} - \vec{r}'|} L_{\mu\nu, \vec{r}} = \frac{|G|}{\pi} \Theta,$$

where Θ is the 4-dimensional “solid angle”, under which the monopole loop observes loop C . If the monopole loop encircles C , $\Theta = \pi^2$ and the contribution to the correlation’s numerator is negative. This value of Θ can be “visualized” by considering how the extension from dimension 2 to dimension 3 was made for the correlation of the XY model (see equations (C.121) and (C.212)): in $d = 2$, a parallax difference $\delta\Xi = \Xi_{\vec{r}_2} - \Xi_{\vec{r}_1}$ equal to π becomes in the $d = 3$ case a difference in solid angle $\delta\Omega = \Omega_{\vec{r}_2} - \Omega_{\vec{r}_1}$ of π times the integral over the new coordinate: $\pi \times \int_0^\pi \sin \theta = 2\pi$. For the $U(1)$ gauge model the extension is similar: a monopole located in the interior of C corresponds to a solid angle $\Omega = 2\pi$, under which the monopole sees C ; in the 4-dimensional model, this solid angle’s value becomes $2\pi \times \int_0^\pi \sin^2 \theta_2 = \pi^2$. In the following figure, one section of loop C croseed by an infinite long monopole loop (stretched along the fourth direction) is represented.



Note that since monopoles have to form closed loops the energy of such a loop grows like R , similar to the energy of dislocation loop in the 3-dimensional XY model (C.221)

$$\begin{aligned}
 & \frac{2\pi^2}{e^2\eta} |G|^2 D(0) 2\pi R + \frac{1}{2e^2\eta} |G|^2 \oint_{\phi=0}^{\phi=2\pi} \int_{\alpha=\frac{\pi}{R}}^{\alpha=2\pi-\frac{\pi}{R}} d\alpha d\phi \frac{R^2 \cos \alpha}{R^2(2-2\cos \alpha)} = \\
 & = \frac{2\pi^2}{e^2\eta} |G|^2 \left[R \left(2\pi D(0) + \frac{1}{\pi} \right) - \frac{1}{2} \right]. \quad (C.325)
 \end{aligned}$$

Therefore, at low $e^2\eta$ the monopoles are tightly bound in small loops and cannot disorder the system: their effect is not relevant and the correlation $W(C)$ (C.324) can be approximated by

$$W(C)_{\text{small } e^2\eta} \approx \exp \left[-\frac{e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right]. \quad (C.326)$$

This approximation cannot be carried out in the 3-dimensional model since there monopoles are almost isolated. The calculation of the exponent in (C.326) for $\tau \gg R \gg a$ gives

$$\begin{aligned}
 I_R &= 2 \int_a^R d\xi_1 \int_0^{\xi_1-a} d\xi_2 \frac{1}{4\pi^2 (\xi_1 - \xi_2)^2} \approx \frac{1}{2\pi^2} \left(\frac{R}{a} - \ln \frac{R}{a} \right), \\
 I_{\tau\tau} &= - \int_0^\tau d\xi_1 \int_0^\tau d\xi_2 \frac{1}{4\pi^2 (R^2 + (\xi_1 - \xi_2)^2)} = -\frac{1}{2\pi^2} \left(\arctan \left(\frac{\tau}{R} \right) - \frac{1}{2} \ln \left(\frac{R^2 + \tau^2}{R^2} \right) \right),
 \end{aligned}$$

and thus

$$\sum_{\vec{r}, \vec{r}', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} = D(0) \mathcal{P} + \frac{1}{\pi^2} \left(\frac{\mathcal{P}}{2a} - \frac{\pi}{2} \frac{\tau}{R} - 2 \ln \frac{R}{a} \right) \approx \mathcal{P} \left(D(0) + \frac{1}{2a\pi^2} \right) - \frac{1}{2\pi} \frac{\tau}{R}. \quad (C.327)$$

Correlation $W(C)$ decays with the perimeter of C at low $e^2\eta$, unlike the 3-dimensional case, where $W(C)$ also decays with the area at this range. "Correction" term τ/R is much weaker than the perimeter $\mathcal{P}$, as different from the three dimensional model, where term $\tau \ln R \gg \mathcal{P}$ occurred. On the other hand, at

strong $e^2\eta$ the “high temperature” expansion produces the typical area law. These different behaviours in $W(C)$ indicates a phase transition as we shall see in the next section when the pure $Z(2)$ gauge model will be studied. Let us for the present discussion accept that this phase transition takes place.

The transition point separating both behaviours in the 4-dimensional model is estimated by an argument similar to that leading to relation (C.226) in the 3-dimensional XY model. The fact that this point is a second-order phase transition point, is supported by Monte Carlo simulations [29]. As the transition point is approached, we can neglect the mutual interaction between the monopole loops and approximate the system energy by $\frac{2\pi^2}{e^2\eta}D(0)\sum_{\vec{r},\rho}G_{\rho,\vec{r}}^2 \sim \frac{2\pi^2}{e^2\eta}D(0)L$, with L the length of the monopole loops (we suppose that all have approximately the same large length). The number of closed loops of length L is bounded by 7^L , where 7 are the available directions for a new step of a walk in four dimensions. Therefore the contribution of monopole loops of length L is estimated in

$$7^L \exp \left[-\frac{2\pi^2}{e^2\eta}D(0)\sum_{\vec{r},\rho}G_{\rho,\vec{r}}^2 \right] = \exp \left[L \left(\ln 7 - \frac{2\pi^2}{e^2\eta}D(0) \right) \right],$$

with $D(0) = 0.155$. The transition takes place when contribution due to very large monopole loops becomes nonvanishing, i.e., if $\ln 7 - \frac{2\pi^2}{e^2\eta}D(0) \geq 0$

$$(e^2\eta)_{\text{critical}} \approx \frac{2\pi^2 D(0)}{\ln 7} = 1.572. \quad (\text{C.328})$$

The $W(C)$'s decay like the area can be also obtained as follows: each time a monopole loop encircles C and crosses thus the area enclosed by C , we obtain a negative contribution in the correlation's numerator ($\Theta/\pi = \pi$). Such a negative value can be maintained for the whole area when the size of monopole loops is large, i.e., for large $e^2\eta$. Hence, one has not only estimated the critical point but also obtained a picture of how the phase transition occurs: by the appearance at large $e^2\eta$ of large monopole loops.

Summarising, the Villain approximation of the pure gauge model is able to reproduce both the “high temperature” power expansion result and the expected behaviour at low $ae^2\eta$. In the following Note, we shall consider briefly the question of the independence of the correlation behaviour on the integer p , which will be argued using the Villain approximation.

Note. The pure gauge model for $p \neq 1$.

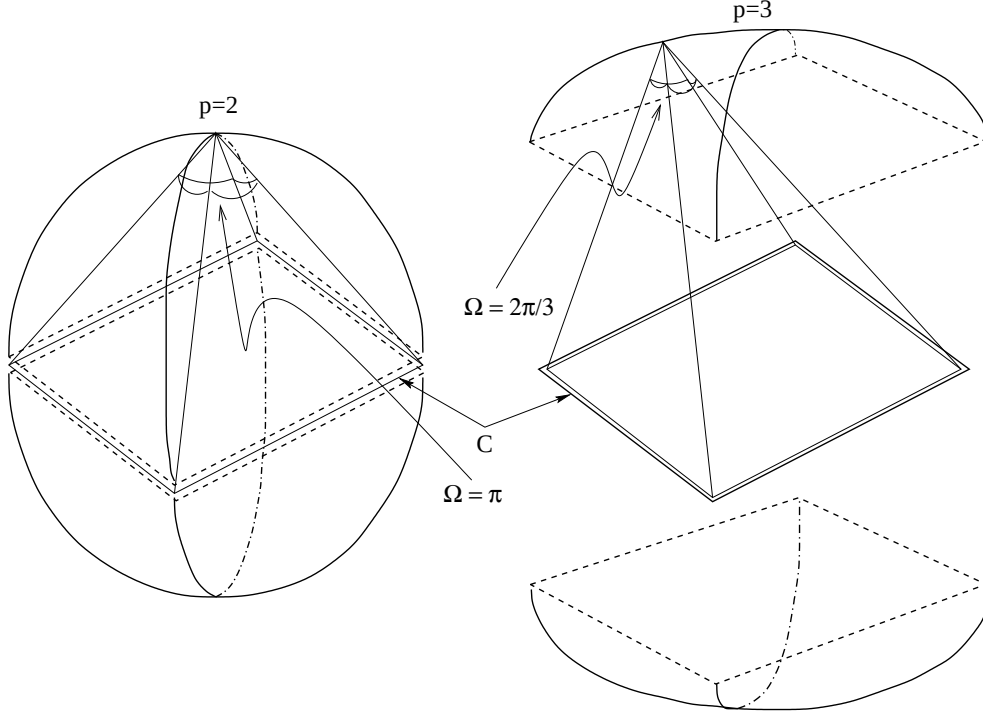
Let us consider the correlation with $p > 1$ in the Villain approximation

$$W(C) = \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \sum_{\{g_{\mu\nu,\vec{r}}\}} \exp \left[ip \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} - \frac{1}{2ae^2\eta} \sum_{\vec{r},\mu < \nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}})^2 \right].$$

Repeating the same steps which have led us to (C.281), we obtain

$$\begin{aligned} W(C) = & \exp \left[-\frac{ap^2e^2\eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} \right] \frac{\mathcal{Z}_0}{\mathcal{Z}} \times \\ & \times \sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{ip}{2} \sum_{\vec{r}} \Omega_{\vec{r}} G_{\vec{r}} - \frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{\pi}{2ae^2\eta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} G_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} G_{\vec{r}'} \right]. \end{aligned} \quad (\text{C.329})$$

Let us consider the case of $p = 2$. In order to repeat the calculation for high $ae^2\eta$ similar to (C.316), we need to find the number of lattice sites from which the path C is observed under a solid angle of $\Omega_{\vec{r}} = 2\pi/p = \pi$. These sites are placed on a sort of “cap” above and below C . The surface of each cap is of the order of the area $\mathcal{A}$ enclosed by C .



For an odd p , we have $\frac{p-1}{2}$ caps placed above C , the same number below, and one cap coinciding with the surface inside C , since $e^{-ip\pi} = -1$ for odd p . Therefore, the number of sites from which the angle $\Omega_{\vec{r}}$ is equal to $\frac{2\pi}{p}$ is approximately $p\mathcal{A}$ and they are located on these caps

$$\left\langle \prod_{\vec{r}} \exp \left(-\frac{ip}{2} \Omega_{\vec{r}} G_{\vec{r}} \right) \right\rangle_G \approx \frac{\left[2 \sum_{G=0}^{\infty} (-1)^G e^{-\frac{2\pi^2}{ae^2\eta} D(0)G^2} - 1 \right]^{p\mathcal{A}} \left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0)G^2} - 1 \right]^{|A|-p\mathcal{A}}}{\left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0)G^2} - 1 \right]^{p\mathcal{A}} \left[2 \sum_{G=0}^{\infty} e^{-\frac{2\pi^2}{ae^2\eta} D(0)G^2} - 1 \right]^{|A|-p\mathcal{A}}} =$$

$$= \exp \left[-p\mathcal{A} \ln(2ae^2\eta) \right]. \quad (\text{C.330})$$

This result has been already shown in equation (C.257). For low $ae^2\eta$, we can follow Polyakov's argument. Firstly the angle $p\Omega_{\vec{r}} = p4\pi \sum_{\vec{r}'} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}'} D_{\vec{r}', \vec{r}}$ will be written as follows

$$p\Omega_{\vec{r}} = 4\pi \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} \bar{L}_{\mu\nu, \vec{r}'} D_{\vec{r}', \vec{r}}, \quad (\text{C.331})$$

where $\bar{L}_{\mu\nu, \vec{r}'}$ denotes the plaquette at site $\vec{r}'$ resting on one of the above defined caps, with

$$\sum_{\vec{r}'} \sum_{\mu < \nu} \bar{L}_{\mu\nu, \vec{r}'}^2 = p\mathcal{A}. \quad (\text{C.332})$$

Now the equations (C.298) and (C.299) are slightly modified into

$$\left(-\Delta_{(\vec{r})} \Delta_{(\vec{r})} \right) p\Omega_{\vec{r}} = 4\pi \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} \bar{L}_{\mu\nu, \vec{r}} \quad (\text{C.333})$$

and ²⁵

$$\sum_{\vec{r}, \rho} (\Delta_\rho p \Omega_{\vec{r}})^2 = 16\pi^2 a \left[p a \mathcal{A} \delta(0) - p^2 \sum_{\vec{x}, \vec{x}'} \sum_{\nu} J_{\nu, \vec{x}} D_{\vec{x}, \vec{x}'} J_{\nu, \vec{x}'} \right]. \quad (\text{C.334})$$

As in Polyakov's argument, let us go into the continuum formulation, defining the coordinates ζ_y, ζ_z as tangent to the cap surface and the coordinate ζ_x normal to it: expressed in these coordinates, equation (C.333) reads

$$-\nabla^2 (p\Omega) = 4\pi\Theta(\zeta_y, \zeta_z) \partial_{\zeta_x} \delta(\zeta_x). \quad (\text{C.335})$$

Finally, as in (C.314), we obtain that $W(C)$ at low $ae^2\eta$ decays according to the following equation

$$\exp \left\{ -\frac{p^2 e^2 \eta}{2} a^2 \mathcal{A} \delta(0) - \underbrace{\frac{e^2 \eta}{4\pi^2} \iint d\zeta_y d\zeta_z \Theta(\zeta_y, \zeta_z)}_{p\mathcal{A}a^2} M \int d\zeta_x \left[\frac{(\partial_{\zeta_x} \phi_{\text{cl}})^2}{2} + \phi 2\pi\Theta_C \partial_{\zeta_x} \delta(\zeta_x) - \cos \phi_{\text{cl}} \right] \right\}, \quad (\text{C.336})$$

where $\phi(x)_{\text{cl}}$ inside C is given by $[\vartheta(\zeta_x)\phi_o(\zeta_x) - \vartheta(-\zeta_x)\phi_o(-\zeta_x)]$ (and zero outside); and $\phi_o(\zeta_x)$ is the solution of equation $\partial_\xi^2 \phi_o(\xi) = \sin \phi_o(\xi)$ with $\phi_o(0) = \pi$ as in (C.311) ²⁶. By substituting this solution into the integral in (C.336), we obtain a term containing a $\delta(0)$, cancelled with $\exp[-p^2 e^2 \eta / 2 a^2 \mathcal{A} \delta(0)]$, plus a constant. Hence, correlation $W(C)$ for $p \neq 1$ also decays with the area $\mathcal{A}$ at small $ae^2\eta$.

Before closing this part, let us discuss another gauge model which, as different from the (compact) Abelian pure gauge model does present a phase transition in three dimensions. This model can appear rather artificial now, but it will play an important role in the discussion of the Higgs model phase diagram.

C.5.2 The pure $Z(2)$ gauge model

Let us return to the original equation of the energy for the compact gauge field

$$\frac{E_{\text{compact gauge}}}{\eta} = \beta_e \sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})],$$

where we have written a general coupling strength β_e instead of $1/ae^2\eta$, since in the following we shall consider analogies with the Ising model and this is the usual notation in this model. This energy can be rewritten as follows

$$\frac{E_{\text{gauge compact}}}{\eta} = \beta_e \sum_{\vec{r}} \sum_{\mu < \nu} \left[1 - \frac{1}{2} \left(S_{\mu, \vec{r}}^{\text{gauge}} S_{\nu, \vec{r} + \hat{\mu}}^{\text{gauge}} S_{-\mu, \vec{r} + \hat{\mu} + \hat{\nu}}^{\text{gauge}} S_{-\nu, \vec{r} + \hat{\nu}}^{\text{gauge}} + \text{c.c.} \right) \right], \quad (\text{C.337})$$

where c.c. means complex conjugated, and field $S_{\mu, \vec{r}}^{\text{gauge}}$ is given by

$$S_{\mu, \vec{r}}^{\text{gauge}} = e^{iA_{\mu, \vec{r}}}, \quad (\text{C.338})$$

²⁵Note that for $\sum_{\mu(\vec{r})} \bar{\Delta}_{\mu\nu, \vec{r}}$ inside each cap is zero and doesn't vanish only on the cap's boundary, where it is equal to (minus) the $J_{\nu, \vec{r}}$ defining C . Therefore for *each* of the p caps, the following holds $\sum_{\mu} \bar{\Delta}_{\mu\nu, \vec{r}} = -J_{\nu, \vec{r}}$.

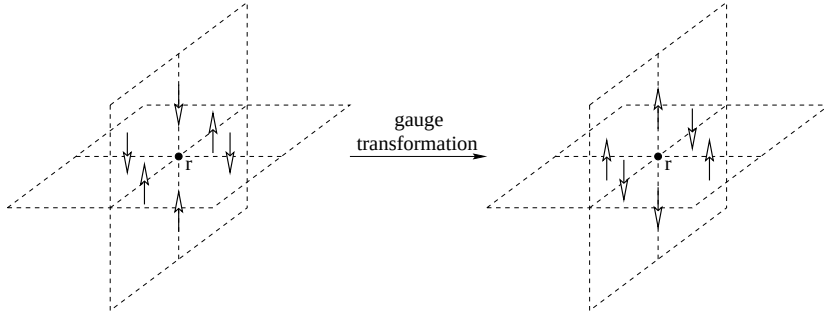
²⁶For the careful reader, the ζ 's are in fact curvilinear coordinates and therefore in (C.336) we should introduce the corresponding tensor elements in the gradient and in the differential volume element. However, in order to avoid an overly complicated notation, the equation has been written as if the ζ coordinates were Cartesian.

with $A_{-\mu, \vec{r} + \hat{\mu}} = -A_{\mu, \vec{r}}$. The $A_{\mu, \vec{r}}$ range is $[0, 2\pi)$ and the model is invariant under a local change in the orientation of $S_{\mu, \vec{r}}^{\text{gauge}}$ in the internal symmetry space; this orientation can be better visualize by using the $SO(2)$ notation $S_{\mu, \vec{r}}^{\text{gauge}} = \begin{pmatrix} \cos A_{\mu, \vec{r}} \\ \sin A_{\mu, \vec{r}} \end{pmatrix}$.

Now, let us construct another gauge model where the range of $A_{\mu, \vec{r}}$ is only 0 and π , i.e., $S_{\mu, \vec{r}}^{\text{gauge}}$ can only point upward and downward. Such a model is called the (pure) $Z(2)$ gauge model and is described by the energy

$$\frac{E}{\eta} = \beta_e \sum_{\vec{r}} \sum_{\mu < \nu} \left[1 - S_{\mu, \vec{r}}^{\text{gauge}} S_{\nu, \vec{r} + \hat{\mu}}^{\text{gauge}} S_{\mu, \vec{r} + \hat{\nu}}^{\text{gauge}} S_{\nu, \vec{r}}^{\text{gauge}} \right], \quad (\text{C.339})$$

where it not necessary to distinguish between $S_{\mu, \vec{r}}^{\text{gauge}}$ and its complex conjugated ($e^{i\pi} = e^{-i\pi}$) and thus $S_{-\mu, \vec{r} + \hat{\mu} + \hat{\nu}}^{\text{gauge}} = S_{\mu, \vec{r} + \hat{\nu}}^{\text{gauge}}$. Field $S_{\mu, \vec{r}}^{\text{gauge}}$ is located at the bond connecting $\vec{r}$ with $\vec{r} + \hat{\mu}$; the energy contains the product of the four $S_{\mu, \vec{r}}^{\text{gauge}}$'s placed at the bonds of an unit plaquette.



The $Z(2)$ gauge model is invariant under the following local change: at all bonds beginning at site $\vec{r}$ (but only at these bonds), field $S_{\mu, \vec{r}}^{\text{gauge}}$ is reversed. Under such a transformation

$$\begin{cases} S_{\mu, \vec{r}}^{\text{gauge}} \rightarrow S_{\mu, \vec{r}}^{\prime \text{gauge}} = -S_{\mu, \vec{r}}^{\text{gauge}} \\ S_{\mu, \vec{r}'}^{\text{gauge}} \rightarrow S_{\mu, \vec{r}'}^{\prime \text{gauge}} = S_{\mu, \vec{r}'}^{\text{gauge}} \quad \vec{r}' \neq \vec{r} \end{cases} \quad (\text{C.340})$$

the product of the four $S_{\mu, \vec{r}}^{\text{gauge}}$'s on a plaquette remains unchanged since the transformation affects to two bonds of each plaquette.

By means of the partition function corresponding to energy (C.339), let us calculate the value of the correlation function $W(C)$. In this model, $W(C)$ is given by the product of the gauge “arrows” $S_{\mu, \vec{r}}^{\text{gauge}}$ along path C

$$W(C) = \left\langle \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \right\rangle = \left\langle \prod_C S_{\mu, \vec{r}}^{\text{gauge}} \right\rangle. \quad (\text{C.341})$$

Of course, in this case it has no meaning to consider $p \neq 1$ since for $p = 2$, one would study the product along C of $(S_{\mu, \vec{r}}^{\text{gauge}})^2 = 1$ and $W(C)$ would be equal to 1 trivially. Hence, the correlation function we shall study is

$$W(C) = \frac{1}{\mathcal{Z}} \sum_{\{S_{\mu, \vec{r}} = \pm 1\}} \left(\prod_C S_{\mu, \vec{r}} \right) \exp \left[-\beta_e \sum_{\vec{r}} \sum_{\mu < \nu} (1 - S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}}) \right], \quad (\text{C.342})$$

where $\sum_{\{S_{\mu,\vec{r}}=\pm 1\}} = \prod_{\vec{r},\mu} \sum_{S_{\mu,\vec{r}}=\pm 1}$, and $S_{\mu,\vec{r}}$ has been written instead of $S_{\mu,\vec{r}}^{\text{gauge}}$ to avoid an overly complicated notation. Partition $\mathcal{Z}$ is given by the above numerator without term $(\prod_C S_{\mu,\vec{r}})$. As in the Ising model (see (C.32)), one can formulate (C.342) as

$$W(C) = \frac{1}{\mathcal{Z}} \sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \left(\prod_C S_{\mu,\vec{r}} \right) \prod_{\vec{r},\mu} \left[(1 + S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e) (e^{-\beta_e} \cosh \beta_e) \right].$$

Starting with this relation, the $W(C)$ behaviour at small β_e is obtained by the “high temperature” power expansion method. Since $\sum_{\{S_{\mu,\vec{r}}=\pm 1\}} S_{\mu,\vec{r}} = 0$, the leading contribution to $W(C)$ arises when all plaquettes of the minimal area enclosed by C are covered by the plaquette factors $S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e$: thus each gauge arrow $S_{\mu,\vec{r}}$ lying on C appears as quadratic power and $\sum_{\{S_{\mu,\vec{r}}\}}$ applied to them does not vanish. This produces the already well-known decay with the area $\mathcal{A}$ enclosed by C

$$W(C)_{\text{small } \beta_e} \approx (2 \tanh \beta_e)^{\mathcal{A}} \approx (\beta_e)^{\mathcal{A}} \approx \exp \left[-\ln \left(\frac{1}{\beta_e} \right) \mathcal{A} \right]. \quad (\text{C.343})$$

For $\beta_e \rightarrow \infty$, the energy term in the partition function forces all gauge arrows to point in the same direction (as in a magnet at large $\beta = J/k_B T$): such a configuration has the lowest possible energy and thus still produces a contribution to the partition function for large β_e . Of course, not only the configuration with all $S_{\mu,\vec{r}}$ “up” has the lowest energy: any configuration, obtained from that by reversing the $S_{\mu,\vec{r}}$ ’s at all bonds attached to a certain point, has the same energy since the system is invariant under gauge transformations (C.340). Any arbitrary configuration, nevertheless, cannot be obtained from the configuration with all $S_{\mu,\vec{r}}$ pointing upward by means of gauge transformations, since each of them affects simultaneously to several bonds.

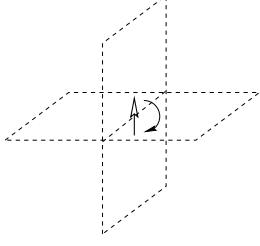
In order to obtain the partition function, we shall choose a representative configuration with lowest energy (in our case, with all arrows “up”) and the perturbations about such a configuration. The contributions produced by these considered configurations must be multiplied by an (infinite) factor arising from the gauge equivalent configurations. However, such an infinite factor occurs in both numerator and denominator of the correlation function and we need not worry about it. If we were interested in obtaining the partition function, this infinite factor could be removed at the end of the calculations, as we have made for the XY model by means of the gauge fixing term.

Considering that in d -dimensional lattice with $|\Lambda|$ sites, the number of bonds (or links) and of plaquettes are given by

$$\text{number of bonds} = d|\Lambda| \quad \text{number of plaquettes} = \frac{1}{2}d(d-1)|\Lambda|, \quad (\text{C.344})$$

the main contributions to the partition function arise from

- All gauge arrows $S_{\mu,\vec{r}}$ pointing “up”: $\prod_{\vec{r}} \prod_{\mu < \nu} e^{-\beta_e(1-S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}})} = 1$. There is only one configuration (aside the gauge equivalents) which produces this contribution.
- One of the arrows is pointing downward. This down arrow can be placed in any of the $d|\Lambda|$ available bonds and affects to $2(d-1)$ plaquettes (in the figure, 4 pla-



quettes for a 3-dimensional lattice). The value of $S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}}$ is -1 for the $2(d-1)$ affected plaquettes, and 1 for the other $\frac{1}{2}d(d-1)|\Lambda| - 2(d-1)$ plaquettes: $\prod_{\vec{r}} \prod_{\mu < \nu} e^{-\beta_e(1-S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}})} = e^{-\beta_e 4(d-1)}$. The contribution to the partition function is then equal to

$$\left(\prod_{\vec{r}} \prod_{\mu < \nu} \sum_{S_{\mu,\vec{r}}=\pm 1} e^{-\beta_e(1-S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}})} \right)_{\text{one flipped arrow}} = d|\Lambda| e^{-\beta_e 4(d-1)}.$$

- Two of the arrows are pointing downward. In the approximation of treating these down arrows as independent of each other, each flipped arrow can be located in one of $d|\Lambda|$ lattice bonds. The contribution of both flipped arrows must be multiplied by $1/2$ in order to compensate for the double overcounting (the two arrows cannot be distinguished)

$$\left(\prod_{\vec{r}} \prod_{\mu < \nu} \sum_{S_{\mu,\vec{r}}=\pm 1} e^{-\beta_e(1-S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}})} \right)_{\text{two flipped arrows}} = \frac{1}{2} (d|\Lambda|)^2 e^{-\beta_e 8(d-1)}.$$

The partition function becomes thus

$$\mathcal{Z} = \left(\left[1 + d|\Lambda| e^{-\beta_e 4(d-1)} + \frac{1}{2} (d|\Lambda|)^2 e^{-\beta_e 8(d-1)} + \dots \right] \right) \approx \left(\exp[d|\Lambda| e^{-\beta_e 4(d-1)}] \right), \quad (\text{C.345})$$

where $\left(\right)$ represents the infinite number of equivalent configurations producing the same contribution as the chosen one.

For the correlation function's denominator, the argument runs in a parallel way

- If no arrow is flipped, the product of arrows along path C is equal to $\prod_C S_{\mu,\vec{r}} = 1$.
- For the case of only one flipped arrow, this arrow can be located on one of the $\mathcal{P}$ bonds of path C ($\mathcal{P}$ is the perimeter of C), producing thus a negative value for $\prod_C S_{\mu,\vec{r}}$. Its contribution to the correlation numerator is

$$\begin{aligned} & \left(\prod_{\vec{r}} \prod_{\mu < \nu} \sum_{S_{\mu,\vec{r}}=\pm 1} \prod_C S_{\mu,\vec{r}} e^{-\beta_e(1-S_{\mu,\vec{r}}S_{\nu,\vec{r}+\hat{\mu}}S_{\mu,\vec{r}+\hat{\nu}}S_{\nu,\vec{r}})} \right)_{\text{one flipped arrow}} = \\ & = e^{-\beta_e 4(d-1)} [\mathcal{P} \cdot (-1) + (d|\Lambda| - \mathcal{P}) \cdot (1)]. \end{aligned}$$

- If there are two arrows pointing down, we have three possibilities: both arrows are not placed on C , both are placed on it (in these two cases, the contribution must be multiplied by $1/2$); and one of the flipped arrows is lying on the path but the

other not (here, the factor $1/2$ is not necessary since we can distinguish the arrow on C from that sitting outside). Only in the last case, $\prod_C S_{\mu,\vec{r}}$ is negative ²⁷

$$\left(\prod_{\vec{r}} \prod_{\mu < \nu} \sum_{S_{\mu,\vec{r}} = \pm 1} \prod_C S_{\mu,\vec{r}} e^{-\beta_e (1 - S_{\mu,\vec{r}} S_{\nu,\vec{r} + \hat{\mu}} S_{\mu,\vec{r} + \hat{\nu}} S_{\nu,\vec{r}})} \right)_{\text{two flipped arrows}} =$$

$$= e^{-\beta_e 8(d-1)} \left[\frac{1}{2} (d|\Lambda| - \mathcal{P})^2 \cdot (1) + \frac{1}{2} \mathcal{P}^2 \cdot (-1)^2 + \mathcal{P} (d|\Lambda| - \mathcal{P}) \cdot (-1) \right].$$

The correlation function at large β_e is therefore given by

$$W(C)_{\text{large } \beta_e} \approx \frac{\binom{d|\Lambda|}{\mathcal{P}} \exp \left[(d|\Lambda| - 2\mathcal{P}) e^{-\beta_e 4(d-1)} \right]}{\binom{d|\Lambda|}{0} \exp \left[d|\Lambda| e^{-\beta_e 4(d-1)} \right]} = \exp \left[-2e^{-\beta_e 4(d-1)} \mathcal{P} \right]. \quad (\text{C.346})$$

A plaquette with $S_{\mu,\vec{r}} S_{\nu,\vec{r} + \hat{\mu}} S_{\mu,\vec{r} + \hat{\nu}} S_{\nu,\vec{r}} = -1$ will be called a *frustrated* plaquette. Although the argument leading to the area decay at small β_e holds always, the just discussed method to obtain the perimeter decay at large β_e does not hold for a 2-dimensional lattice [59]. The base of the above method was that producing more flipped arrows, which disorder the lowest energy configuration, should cost energy because of the higher number of frustrated plaquettes. However, in two dimensions one can construct an arbitrary long line of reversed arrows with only two frustrated plaquettes (Fig. C.4). This line disorder the system very effectively. We have already seen a similar case in the 1-dimensional Ising model (see the discussion after Table C.1).

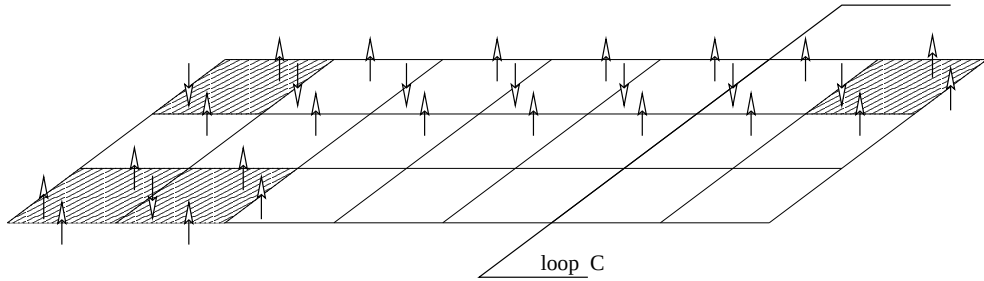


Figure C.4: In a 2-dimensional lattice, a long line of flipped arrows costs energetically only the first and the last frustrated plaquette.

In a 2-dimensional lattice thus, two frustrated plaquettes, at which a long line of flipped arrows begins and ends, can be further separated from each other and considered as isolated with no further increase in the energy. In a configuration where most of the arrows are oriented in the same direction, each time a line of flipped arrows crosses C , $\prod_C S_{\mu,\vec{r}} = -1$ is produced. If one of the isolated frustrated plaquettes is located inside the area enclosed by C , the line of reversed arrows emerging from the frustrated plaquette crosses C an odd number of times before going to spatial infinity and therefore

²⁷The reader has probably notice the similarity with the method based on contour method, as for instance, in equation (C.53).

$\prod_C S_{\mu,\vec{r}} = -1$. This effect is produced if the reversed arrow of the frustated plaquette is placed at any of the bonds contained inside C ; let us call N_C the number of these bonds: $N_C \sim 2\mathcal{A}$. If, on the contrary, the isolated frustated plaquette is located outside the area, either does not cross C or crosses it an even number of times: in both cases, $\prod_C S_{\mu,\vec{r}} = 1$. Since $2|\Lambda|$ is the number of bonds in two dimensions, the contributions to $W(C)$'s numerator arising from these isolated frustated plaquettes are

$$\underbrace{1}_{\text{no plaquettes}} + \underbrace{[(2|\Lambda| - N_C) \cdot (1) + N_C \cdot (-1)] e^{-2\beta_e}}_{\text{1 plaquette}} + \dots$$

Considering that the frustated plaquettes are not correlated to each other (this approximation holds since the mutual interaction between frustated plaquettes is very weak as we have seen) the effect due to all possible arrangements of isolated frustated plaquettes can be approximated by the exponential of the previous result; the same argument holds true for the correlation's denominator, the partition function. This reasoning is analogous to that leading to equation (C.164) in the 2-dimensional XY model: here the frustated isolated plaquettes play the role of the disordering contours, which in the 2-dimensional XY model was played by large dislocation pairs considered as two isolated dislocation points. Hence, we obtain that $W(C)$ decays with the area also at the range where we could expect a perimeter decay

$$\left\langle \prod_C S_{\mu,\vec{r}} \right\rangle_{\text{in 2 dim. at large } \beta_e} \approx \frac{\exp \left[(2|\Lambda| - 2N_C) e^{-2\beta_e} \right]}{\exp \left[2|\Lambda| e^{-2\beta_e} \right]} = \exp \left[-2N_C e^{-2\beta_e} \right],$$

$$W(C)_{\text{large } \beta_e} \approx \exp \left[-4e^{-2\beta_e} \mathcal{A} \right], \quad (\text{C.347})$$

an indication of the disordered state even for β_e large. The 2-dimensional $Z(2)$ gauge model has no phase transtion. Further, the previous argument allows us to “visualize” how the area law comes about at small β_e also.

For the 3-dimensional $Z(2)$ gauge model, a phase transition separates the decay with the area of C at the disordered state β_e small, from the decay with perimeter of C at large β_e . To close the study of this model, we shall obtain the value of the critical value of $(\beta_e)_{\text{crit}}$ at which this transition occurs. Let us consider the following extension of the $Z(2)$ gauge model, described by the partition function

$$\mathcal{Z} = \sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \exp \left[-\beta_e \sum_{\vec{r}} \sum_{\mu < \nu} (1 - S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}}) - \beta_\psi \sum_{\vec{r}} \sum_{\mu} (1 - S_{\mu,\vec{r}}) \right]. \quad (\text{C.348})$$

This model is called the $Z(2)$ Higgs model for a reason which will become clear when the phase diagram of the compact Higgs model will be discussed. The $Z(2)$ gauge model is the above equation with $\beta_\psi = 0$ ²⁸. The partition function can be rewritten as follows

$$\mathcal{Z} = \sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \underbrace{\prod_{\vec{r}, \mu < \nu} \left[(1 + S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e) (e^{-\beta_e} \cosh \beta_e) \right]}_{\text{plaquettes}} \times$$

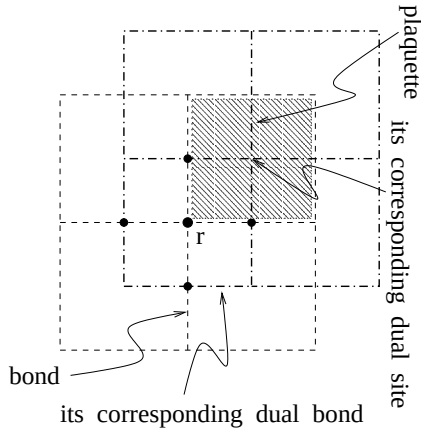
²⁸We shall maintain $\beta_\psi \neq 0$ until the final result, since it allows us to re-obtain the nonexistence of a phase transition in the 2-dimensional $Z(2)$ gauge model.

$$\begin{aligned}
& \times \underbrace{\prod_{\vec{r}, \mu} \left[(1 + S_{\mu, \vec{r}} \tanh \beta_\psi) (e^{-\beta_\psi} \cosh \beta_\psi) \right]}_{\text{bonds}} = \left(e^{-\beta_e} \cosh \beta_e \right)^{\frac{1}{2}d(d-1)|\Lambda|} \left(e^{-\beta_\psi} \cosh \beta_\psi \right)^{d|\Lambda|} \times \\
& \times \sum_{\{S_{\mu, \vec{r}} = \pm 1\}} \left[\prod_{\vec{r}} \prod_{\mu < \nu} (1 + S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}} \tanh \beta_e) \right] \left[\prod_{\vec{r}} \prod_{\mu} (1 + S_{\mu, \vec{r}} \tanh \beta_\psi) \right]. \quad (\text{C.349})
\end{aligned}$$

As we know from the power expansion method, the only terms which contribute when summed $\sum_{\{S_{\mu, \vec{r}} = \pm 1\}}$ are those containing only quadratic powers of $S_{\mu, \vec{r}}$. Each contributing configuration corresponds to an open surface whose interior is completely covered by plaquettes arising from the expansion of $\prod_{\vec{r}} \prod_{\mu < \nu} (1 + S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}} \tanh \beta_e)$, and whose boundary is completely covered by bonds coming from the expansion of $\prod_{\vec{r}} \prod_{\mu} (1 + S_{\mu, \vec{r}} \tanh \beta_\psi)$; the contribution of such a configuration is $(\tanh \beta_e)^P (\tanh \beta_\psi)^B$, where P the number of plaquettes to tile the surface and B is the number of bonds covering the surface's boundary (a closed path). A closed surface of plaquettes (if the lattice dimensionality is high enough) is a special case of the above contribution. Thus, the partition function can be written as sum over all possible configurations with surfaces completely filled with plaquettes and borders completely covered by bonds

$$\mathcal{Z} = \left(e^{-\beta_e} \cosh \beta_e \right)^{\frac{1}{2}d(d-1)|\Lambda|} \left(e^{-\beta_\psi} \cosh \beta_\psi \right)^{d|\Lambda|} \sum_{\substack{\text{surfaces tiled by plaquettes with} \\ \text{boundary covered by bonds}}} (\tanh \beta_e)^P (\tanh \beta_\psi)^B. \quad (\text{C.350})$$

Each of these nonvanishing contributions can be formulated in a simple form by using the dual lattice [17]. The dual lattice of a square lattice is defined as the square lattice displaced from the original by a half of lattice spacing in the $[1, 1, \dots, 1]$ direction. Thus, for a 2-dimensional lattice, the dual lattice (see figure) elements are related to those of the original lattice as follows:



$$\begin{aligned}
\text{site} & \leftrightarrow \text{dual plaquette} \\
\text{bond} & \leftrightarrow \text{dual bond} \\
\text{plaquette} & \leftrightarrow \text{dual site}
\end{aligned} \quad (\text{C.351})$$

A lattice site is the center of only a dual plaquette, a lattice bond crosses only one dual bond and a lattice plaquette contains only a dual site.

Let us denote $\vec{r}^*$ the sites of the dual lattice; fields defined at the dual sites are denoted by $\zeta_{\vec{r}^*}$, and those located at dual bonds by $\zeta_{\mu, \vec{r}^*}$. Now, let us consider the above configuration with nonvanishing contribution $(\tanh \beta_e)^P (\tanh \beta_\psi)^B$, which can be defined by means of the dual lattice as follows:

- Each of the plaquettes contained in this configuration is associated to a field $\zeta_{\vec{r}^*} = -1$ located at the dual site inside the plaquette; $\zeta_{\vec{r}^*} = 1$ if the dual site does not correspond to a plaquette of the considered configuration.

- Each of the bonds at the boundary is associated to the product $\zeta_{\vec{r}^*+\hat{\mu}}\zeta_{\vec{r}^*} = -1$, since one dual site adjacent to the boundary bond is placed in a plaquette of the configuration and the other dual site not.

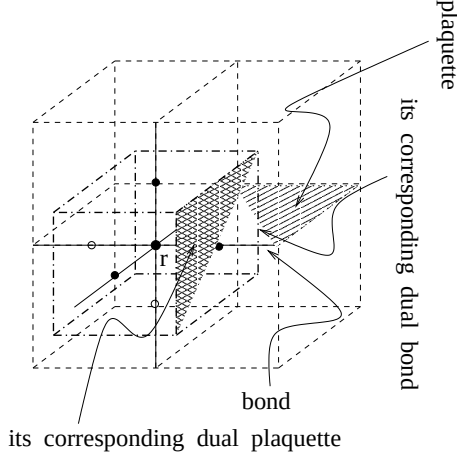
The given configuration, with B bonds forming a closed path and with P plaquettes tiling the interior of the path, is characterized by

$$B = \sum_{\vec{r}^*} \sum_{\mu} \frac{1}{2} (1 - \zeta_{\vec{r}^*+\hat{\mu}}\zeta_{\vec{r}^*}) , \quad (\text{C.352})$$

and

$$P = \sum_{\vec{r}^*} \frac{1}{2} (1 - \zeta_{\vec{r}^*}) , \quad (\text{C.353})$$

for the case of a 2-dimensional lattice.



For a 3-dimensional lattice, the corresponding dual lattice elements are:

site	$\leftrightarrow$	dual cube	
bond	$\leftrightarrow$	dual plaquette	
plaquette	$\leftrightarrow$	dual bond	(C.354)
cube	$\leftrightarrow$	dual site	

where the dual cube is the dashed-dotted cube of the figure whose center is the lattice site $\vec{r}$. As before, let us consider again a configuration which produces a nonvanishing contribution $(\tanh \beta_e)^P (\tanh \beta_\psi)^B$ when summed over $S_{\mu, \vec{r}} = \pm 1$, and let us define the following correspondences

- Each plaquette of this configuration is associated to the dual-bond field $\zeta_{\mu, \vec{r}^*} = -1$. Plaquettes not contained in the configuration are characterized by $\zeta_{\mu, \vec{r}^*} = 1$.
- Each bond at the surface's boundary is associated to $\zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^*+\hat{\mu}} \zeta_{\mu, \vec{r}^*+\hat{\nu}} \zeta_{\nu, \vec{r}^*} = -1$, since only one $\zeta_{\mu, \vec{r}^*}$ crosses a plaquette of the configuration.

The number of bonds along the border and of plaquettes inside for this configuration in a 3-dimensional lattice are given by

$$B = \sum_{\vec{r}^*} \sum_{\mu < \nu} \frac{1}{2} (1 - \zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^*+\hat{\mu}} \zeta_{\mu, \vec{r}^*+\hat{\nu}} \zeta_{\nu, \vec{r}^*}) , \quad (\text{C.355})$$

and

$$P = \sum_{\vec{r}^*} \sum_{\mu} \frac{1}{2} (1 - \zeta_{\mu, \vec{r}^*}) . \quad (\text{C.356})$$

Considering (C.350) and the previous results, the partition function for the 2-dimensional lattice is equal to

$$\mathcal{Z}_{2\text{-dim.lattice}} = \left(e^{-\beta_e} \cosh \beta_e \right)^{|\Lambda|} \left(e^{-\beta_\psi} \cosh \beta_\psi \right)^{2|\Lambda|} \sum_{\{\zeta_{\vec{r}^*} = \pm 1\}} \times \\ \times \exp \left[\frac{1}{2} \ln \tanh \beta_\psi \sum_{\vec{r}^*} \sum_{\mu} (1 - \zeta_{\vec{r}^* + \hat{\mu}} \zeta_{\vec{r}^*}) + \frac{1}{2} \ln \tanh \beta_e \sum_{\vec{r}^*} (1 - \zeta_{\vec{r}^*}) \right], \quad (\text{C.357})$$

whilst for three dimensions

$$\mathcal{Z}_{3\text{-dim.lattice}} = \left(e^{-\beta_e} \cosh \beta_e \right)^{3|\Lambda|} \left(e^{-\beta_\psi} \cosh \beta_\psi \right)^{3|\Lambda|} \sum_{\{\zeta_{\mu, \vec{r}^*} = \pm 1\}} \times \\ \times \exp \left[\frac{\ln \tanh \beta_\psi}{2} \sum_{\vec{r}^*, \mu < \nu} (1 - \zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^* + \hat{\mu}} \zeta_{\mu, \vec{r}^* + \hat{\nu}} \zeta_{\nu, \vec{r}^*}) + \frac{\ln \tanh \beta_e}{2} \sum_{\vec{r}^*, \mu} (1 - \zeta_{\mu, \vec{r}^*}) \right]. \quad (\text{C.358})$$

Up to a factor, one recognizes result (C.357) as the partition function of the 2-dimensional Ising model in the presence of an external magnetic field h

$$\mathcal{Z}_{\text{Ising}} = \sum_{\{\sigma_{\vec{r}} = \pm 1\}} \exp \left[-\beta_{\text{Ising}} \sum_{\vec{r}} \sum_{\mu} (1 - \sigma_{\vec{r} + \hat{\mu}} \sigma_{\vec{r}}) - h \sum_{\vec{r}} (1 - \sigma_{\vec{r}}) \right],$$

with interspin coupling strength β_{Ising}^* and external magnetic field h^* (the scale η has been absorbed into β_{Ising} and h) given by

$$\begin{cases} \beta_{\text{Ising}}^* &= -\frac{1}{2} \ln \tanh \beta_\psi, \\ h^* &= -\frac{1}{2} \ln \tanh \beta_e. \end{cases} \quad (\text{C.359})$$

On the other hand, result (C.358) is again the original 3-dimensional $Z(2)$ Higgs model

$$\mathcal{Z} = \sum_{\{S_{\mu, \vec{r}} = \pm 1\}} \exp \left[-\beta_e \sum_{\vec{r}} \sum_{\mu < \nu} (1 - S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}}) - \beta_\psi \sum_{\vec{r}} \sum_{\mu} (1 - S_{\mu, \vec{r}}) \right],$$

with coupling strength β_e^* and β_ψ^* given by

$$\begin{cases} \beta_e^* &= -\frac{1}{2} \ln \tanh \beta_\psi, \\ \beta_\psi^* &= -\frac{1}{2} \ln \tanh \beta_e. \end{cases} \quad (\text{C.360})$$

Such a model is called self-dual. Note that the low (high) range of β_e corresponds to the high (low) range of β_ψ in the dual model, and the same holds for the other coupling strength β_ψ related to β_e^* .

We can now discuss the transition point for the $Z(2)$ gauge model by setting $\beta_\psi = 0$. In two dimensions, the Ising model has a critical point (separating an ordered from a disordered phase) only for a vanishing external magnetic field. Here, the dual of the 2-dimensional $Z(2)$ gauge model is the Ising model with $\beta_{\text{Ising}} \rightarrow \infty$ and a nonvanishing external magnetic field, and therefore, the system has no critical point. This can be directly seen by calculating the partition function. Limit $\beta_{\text{Ising}} = -1/2 \ln \tanh \beta_\psi \rightarrow \infty$ in (C.357) forces all dual fields $\zeta_{\vec{r}^*}$ to point in the same direction ($\zeta_{\vec{r}^*+\hat{\mu}}\zeta_{\vec{r}^*} = 1$). The partition function becomes

$$\mathcal{Z}_{2\text{-dim. lattice}} = \left(e^{-\beta_e} \cosh \beta_e \right)^{|\Lambda|} \left[1 + (\tanh \beta_e)^{|\Lambda|} \right],$$

and the free energy density $\mathcal{F} = - \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda|} \ln \mathcal{Z}$ is an analytical function, without singularities

$$\mathcal{F}_{2\text{-dim. lattice}} = \beta_e - \ln \cosh \beta_e - \underbrace{\lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda|} \ln \left[1 + (\tanh \beta_e)^{|\Lambda|} \right]}_{=0 \quad \text{since} \quad 0 \leq \tanh \beta_e \leq 1}. \quad (\text{C.361})$$

Note that the sole phase of the $Z(2)$ gauge model, a disordered state with long lines of flipped arrows, corresponds to the ordered phase of its dual model.

In three dimensions, $\beta_\psi = 0 \Leftrightarrow \beta_e^* = \infty$ forces term $1 - \zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^*+\hat{\mu}} \zeta_{\mu, \vec{r}^*+\hat{\nu}} \zeta_{\nu, \vec{r}^*}$ in (C.358) to be equal to zero

$$\zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^*+\hat{\mu}} \zeta_{\mu, \vec{r}^*+\hat{\nu}} \zeta_{\nu, \vec{r}^*} = 1 \quad \implies \quad \zeta_{\mu, \vec{r}^*} = \sigma_{\vec{r}^*} \sigma_{\vec{r}^*+\hat{\mu}} \quad \text{with} \quad \sigma_{\vec{r}^*} = \pm 1. \quad (\text{C.362})$$

This relation is easily shown by noting that with $\zeta_{\mu, \vec{r}^*} = \sigma_{\vec{r}^*+\hat{\mu}} \sigma_{\vec{r}^*}$, each $\sigma_{\vec{r}^*}$ appears twice in the plaquette term

$$\zeta_{\mu, \vec{r}^*} \zeta_{\nu, \vec{r}^*+\hat{\mu}} \zeta_{\mu, \vec{r}^*+\hat{\nu}} \zeta_{\nu, \vec{r}^*} = \left(\sigma_{\vec{r}^*} \sigma_{\vec{r}^*+\hat{\mu}} \right) \left(\sigma_{\vec{r}^*+\hat{\mu}} \sigma_{\vec{r}^*+\hat{\mu}+\hat{\nu}} \right) \left(\sigma_{\vec{r}^*+\hat{\mu}+\hat{\nu}} \sigma_{\vec{r}^*+\hat{\nu}} \right) \left(\sigma_{\vec{r}^*+\hat{\nu}} \sigma_{\vec{r}^*} \right).$$

Therefore, the dual model corresponding to the 3-dimensional $Z(2)$ gauge model is, except for a factor, the 3-dimensional Ising model *without* an external magnetic field

$$\mathcal{Z}_{3\text{-dim. lattice}} = \left(e^{-\beta_e} \cosh \beta_e \right)^{3|\Lambda|} \sum_{\{\sigma_{\vec{r}^*} = \pm 1\}} \exp \left[-\beta_{\text{Ising}}^* \sum_{\vec{r}^*, \mu} (1 - \sigma_{\vec{r}^*+\hat{\mu}} \sigma_{\vec{r}^*}) \right], \quad (\text{C.363})$$

with $\beta_{\text{Ising}}^* = -\frac{1}{2} \ln \tanh \beta_e$. The 3-dimensional Ising model has a second order phase transition at a critical value $(\beta_{\text{Ising}})_{\text{critical}}$, which must be calculated numerically and is equal to [17] $(\beta_{\text{Ising}})_{\text{critical}} = 0.222$. Thus, the critical value for the 3-dimensional $Z(2)$ gauge model is

$$(\beta_e)_{\text{critical}} = 0.761. \quad (\text{C.364})$$

In this model, it is specially clear the correspondence between the existence of a critical point (which we know to exist from a model external to the pure $Z(2)$ gauge model) and the existence of two different behaviours in $W(C)$. Hence, $W(C)$ is a good correlation to

distiguish phases; the complete correspondence between phases and $W(C)$'s behaviours will be dealt with in a next Note. As in the 2-dimensional $Z(2)$ gauge model, the disordered phase at $\beta_e < (\beta_e)_{\text{critical}}$ (characterized by the area law decay) corresponds to the ordered of the 3-dimensional Ising model $\beta_{\text{Ising}} > (\beta_{\text{Ising}})_{\text{critical}}$.

As a by-product of this discussion, we can calculate the critical point of the 2-dimensional Ising model in absence of an external magnetic field. Returning to the $Z(2)$ Higgs model

$$\mathcal{Z} = \sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \exp \left[-\beta_e \sum_{\vec{r}} \sum_{\mu < \nu} (1 - S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}}) - \beta_\psi \sum_{\vec{r}} \sum_{\mu} (1 - S_{\mu,\vec{r}}) \right],$$

let us study the limit $\beta_e \rightarrow \infty$, which forces $S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} = 1$ whose solution as in (C.362) is $S_{\mu,\vec{r}} = \bar{\sigma}_{\vec{r}+\hat{\mu}} \bar{\sigma}_{\vec{r}}$ with $\bar{\sigma}_{\vec{r}} = \pm 1$. Hence, the $Z(2)$ Higgs model becomes in this limit the 2-dimensional Ising model

$$\mathcal{Z} = \sum_{\{\bar{\sigma}_{\vec{r}}=\pm 1\}} \exp \left[-\beta_{\text{Ising}} \sum_{\vec{r}} \sum_{\mu} (1 - \bar{\sigma}_{\vec{r}+\hat{\mu}} \bar{\sigma}_{\vec{r}}) \right] = \mathcal{Z}_{\text{Ising}}, \quad \beta_{\text{Ising}} \equiv \beta_\psi. \quad (\text{C.365})$$

However, the $Z(2)$ Higgs model in this limit $\beta_e = \infty$ is dual to model (C.357) with coupling strength $\beta_{\text{Ising}}^* = -1/2 \ln \tanh \beta_\psi = -1/2 \ln \tanh \beta_{\text{Ising}}$ and $h^* = -1/2 \ln \tanh \beta_e = 0$, i.e., it is a self-dual model in which the phase at low β_{Ising} for the original spins $\bar{\sigma}_{\vec{r}}$ is described by $\sigma_{\vec{r}^*}$ -configurations belonging to the phase at large $\beta_{\text{Ising}}^* = -1/2 \ln \tanh \beta_{\text{Ising}}$. As in the discussion about the critical point in the 3-dim. XY model (C.230), critical value $(\beta_{\text{Ising}})_{\text{critical}}$ is obtained by solving the equation $(\beta_{\text{Ising}})_{\text{crit}} = -1/2 \ln \tanh (\beta_{\text{Ising}})_{\text{crit}}$, or equivalently $\sinh^2 2 (\beta_{\text{Ising}})_{\text{crit}} = 1$. The solution is

$$(\beta_{\text{Ising}})_{\text{critical}} = \frac{1}{2} \ln (\sqrt{2} + 1) \approx 0.441. \quad (\text{C.366})$$

Curiously, a similar structure is shown by the 4-dimensional pure $Z(2)$ gauge model [22]. As for the 2-dim. Ising model, let us return to the $Z(2)$ Higgs model. In a 4-dimensional lattice, the dual of a lattice plaquette is the plaquette in the 4-dim. dual lattice which encircles or “pierces” the plaquette of the original lattice; the dual of a lattice bond is the cube in the 4-dim. dual lattice crossed by this lattice bond. The energy in the dual model is formed by the following two terms: $-1/2 \ln \tanh \beta_e \sum_{\text{dual plaquettes}} (1 - \zeta_{\mu\nu,\vec{r}^*})$ arising from $(\tanh \beta_e)^P$, and $-1/2 \ln \tanh \beta_\psi \sum_{\text{dual cubes}} (1 - \zeta_{\mu\nu,\vec{r}^*} \zeta_{\nu\lambda,\vec{r}^*} \zeta_{\lambda\mu,\vec{r}^*} \zeta_{\mu\nu,\vec{r}^*+\hat{\lambda}} \zeta_{\nu\lambda,\vec{r}^*+\hat{\mu}} \zeta_{\lambda\mu,\vec{r}^*+\hat{\nu}})$

coming from $(\tanh \beta_\psi)^B$ and where we have the interaction of the 6 dual plaquettes forming the dual cube. The limit $\beta_\psi \rightarrow 0$ defining the pure $Z(2)$ gauge model forces in the term of the dual cubes that $\zeta_{\mu\nu,\vec{r}^*}$ becomes $\zeta_{\mu\nu,\vec{r}^*} = \bar{S}_{\mu,\vec{r}^*} \bar{S}_{\nu,\vec{r}^*+\hat{\mu}} \bar{S}_{\mu,\vec{r}^*+\hat{\nu}} \bar{S}_{\nu,\vec{r}^*}$. Hence, the dual model of the 4-dimensional pure $Z(2)$ gauge model is the original model but with coupling strength $\beta_e^* = -1/2 \ln \tanh \beta_e$; its partition function is given by

$$\mathcal{Z} = \left(e^{-\beta} \cosh \beta_e \right)^{6|\Lambda|} \underbrace{\sum_{\{\bar{S}_{\mu,\vec{r}^*}=\pm 1\}} \exp \left[-\beta_e^* \sum_{\vec{r}^*, \mu < \nu} (1 - \bar{S}_{\mu,\vec{r}^*} \bar{S}_{\nu,\vec{r}^*+\hat{\mu}} \bar{S}_{\mu,\vec{r}^*+\hat{\nu}} \bar{S}_{\nu,\vec{r}^*}) \right]}_{\mathcal{Z}^*}. \quad (\text{C.367})$$

The transition point is exactly the same as in the 2-dimensional Ising model,

$$(\beta_e)_{\text{critical}} \approx 0.441, \quad (\text{C.368})$$

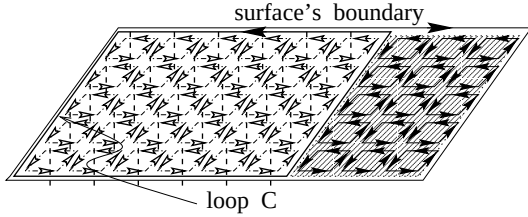
although this does not imply that the phase transition is of the same order as in the Ising model. In fact, Monte Carlo simulations give that the phase transition of the 4-dim. $Z(2)$ gauge model is of the first-order type [29].

Note. Evaluation of $W(C)$ in the dual model.

Up to this point, it has been shown that $W(C)$ shows two different behaviours in the 3-dimensional $Z(2)$ gauge model, and that the model has two phases. Now, it can be interesting to obtain how the low β_{Ising} phase of the 3-dim. Ising model produces the perimeter law decay at large β_e in the 3-dim. $Z(2)$ gauge model. Let us temporarily return to the general $Z(2)$ Higgs model and obtain the form adquired by the correlation function

$$W(C) = \frac{\sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \prod_C S_{\mu,\vec{r}} \left[\prod_{\vec{r}} \prod_{\mu < \nu} (1 + S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e) \right] \left[\prod_{\vec{r}} \prod_{\mu} (1 + S_{\mu,\vec{r}} \tanh \beta_\psi) \right]}{\sum_{\{S_{\mu,\vec{r}}=\pm 1\}} \left[\prod_{\vec{r}} \prod_{\mu < \nu} (1 + S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e) \right] \left[\prod_{\vec{r}} \prod_{\mu} (1 + S_{\mu,\vec{r}} \tanh \beta_\psi) \right]},$$

in the dual model. In the numerator of $W(C)$ we can repeat the same steps as above to obtain the corresponding dual model defined. The contributing configuration are again a closed path as boundary of an open surface; but of its interior, only the part not included inside the path C defining $W(C)$ must be tiled with plaquette-factors arising from the expansion of $\prod_{\vec{r}} \prod_{\mu < \nu} (1 + S_{\mu,\vec{r}} S_{\nu,\vec{r}+\hat{\mu}} S_{\mu,\vec{r}+\hat{\nu}} S_{\nu,\vec{r}} \tanh \beta_e)$.



The reason is the factor $\prod_C S_{\mu,\vec{r}}$ in the numerator: by surrounding C *either* with the bonds of the surface's boundary *or* with plaquette's edges (if the surface's border contains completely C), the gauge arrows contained in $\prod_C S_{\mu,\vec{r}}$ already occur twice. In the figure, the only necessary plaquettes for a nonzero contribution are those in the shaded part.

Hence, the configuration of the figure produces $(\tanh \beta_e)^{P-\mathcal{A}} (\tanh \beta_\psi)^B$ when summed over $S_{\mu,\vec{r}} = \pm 1$, where P is the number of plaquettes inside the open surface characterizing this configuration, B is the number of bonds along the surface's boundary and $\mathcal{A}$ is the number of plaquettes inside C . Defining again field $\zeta_{\mu,\vec{r}} = -1$ for the dual bonds crossing the plaquettes inside the surface's boundary characterizing a contributing configuration, the number of plaquette terms which appear is given by

$$P' = \sum_{\substack{\text{dual bonds crossing} \\ \text{interior of } C}} \frac{1}{2} (1 + \zeta_{\mu,\vec{r}}) + \sum_{\substack{\text{dual bonds not crossing} \\ \text{interior of } C}} \frac{1}{2} (1 - \zeta_{\mu,\vec{r}}), \quad (\text{C.369})$$

whilst the number of bonds remains equation (C.355). The correlation's numerator takes form similar to that in the denominator

$$\exp \left[\frac{\ln \tanh \beta_\psi}{2} \sum_{\text{dual plaquettes}} (1 - \zeta_{\mu,\vec{r}} \zeta_{\nu,\vec{r}+\hat{\mu}} + \zeta_{\mu,\vec{r}+\hat{\nu}} \zeta_{\nu,\vec{r}}) + \frac{\ln \tanh \beta_e}{2} \sum_{\text{dual bonds}} (1 - \zeta_{\mu,\vec{r}}) \right],$$

but with term $\sum_{\text{dual bonds}} (1 - \zeta_{\mu,\vec{r}})$ substituted by $\sum_{\substack{\text{dual bonds crossing} \\ \text{interior of } C}} (1 + \zeta_{\mu,\vec{r}}) + \sum_{\substack{\text{dual bonds not crossing} \\ \text{interior of } C}} (1 - \zeta_{\mu,\vec{r}})$. The correlation function is therefore given by

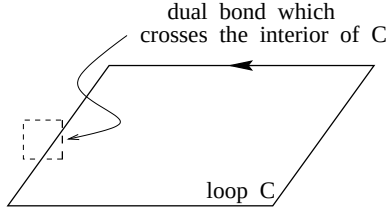
$$W(C) = \frac{\mathcal{Z}_{- \rightarrow + \text{ in dual bonds crossing}}^*}{\mathcal{Z}_{\text{Ising}}}, \quad (\text{C.370})$$

where $Z_{- \rightarrow +}^*$ is the dual partition function Z_{Ising} but with the change $1 - \zeta_{\mu, \tilde{\sigma}^*} \rightarrow 1 + \zeta_{\mu, \tilde{\sigma}^*}$ at the dual bonds (represented by dashed lines in the figure) crossing the surface enclosed by C [17].

The above result holds for the general $Z(2)$ Higgs model. For the pure $Z(2)$ gauge model ($\beta_\psi = 0$ and field $\zeta_{\mu, \tilde{\sigma}^*}$ forced to be $\sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*}$), $W(C)$ becomes

$$\frac{\sum_{\{\sigma_{\tilde{\sigma}^*}\}} \left[\prod_{\substack{\text{dual bonds crossing} \\ \text{interior of } C}} (1 + \sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*} \tanh \beta_{\text{Ising}}^*) \right] \left[\prod_{\substack{\text{dual bonds not crossing} \\ \text{interior of } C}} (1 - \sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*} \tanh \beta_{\text{Ising}}^*) \right]}{\sum_{\{\sigma_{\tilde{\sigma}^*}\}} \left[\prod_{\text{all dual bonds}} (1 + \sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*} \tanh \beta_{\text{Ising}}^*) \right]},$$

with $\beta_{\text{Ising}}^* = -1/2 \ln \tanh \beta_e$. The phase at large β_e corresponds to small β_{Ising}^* and therefore, we can use the “high temperature” expansion as in the 2-dimensional Ising model (C.34): the two principal contributions is 1 once, and $(\tanh \beta_{\text{Ising}}^*)^4$ for the smallest rectangular path (a 1×1 square) containing four (dual) bonds. The above denominator is given by $1 + 3|\Lambda| (\tanh \beta_{\text{Ising}}^*)^4 + \dots$, where $|\Lambda|$ is the number of dual sites (equal to the number of lattice sites) and $\dots$ contains higher powers in $\tanh \beta_{\text{Ising}}^*$. For the numerator, nevertheless, this 1×1 square can be located as follows:



- In $3|\Lambda| - \mathcal{P}$ dual plaquettes without being crossed by C . Its contribution is equal to $(\tanh \beta_{\text{Ising}}^*)^4$ and arises from the factors $(1 + \sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*} \tanh \beta_{\text{Ising}}^*)$.
- In $\mathcal{P}$ dual plaquettes which intersect C . Its contribution is $-(\tanh \beta_{\text{Ising}}^*)^4$, which comes from the bond-factors with the sign changed, since one of the dual bonds (in the figure, drawn in bold) of the square crosses the area enclosed by C .

Therefore, the correlation function at large β_e shows the expected perimeter decay

$$W(C)_{\text{large } \beta_e} = \frac{1 + (3|\Lambda| - 2\mathcal{P})(\tanh \beta_{\text{Ising}}^*)^4 + \dots}{1 + 3|\Lambda| (\tanh \beta_{\text{Ising}}^*)^4 + \dots} \approx \frac{\exp[(3|\Lambda| - 2\mathcal{P}) \beta_{\text{Ising}}^{*4}]}{\exp[3|\Lambda| \beta_{\text{Ising}}^{*4}]} = \exp \left[-2\mathcal{P} \left(-\frac{\ln \tanh \beta_e}{2} \right)^4 \right],$$

which for large β_e is equal to (C.346): $\lim_{\beta_e \rightarrow \infty} -\frac{\ln \tanh \beta_e}{2e^{-2\beta_e}} = 1$.

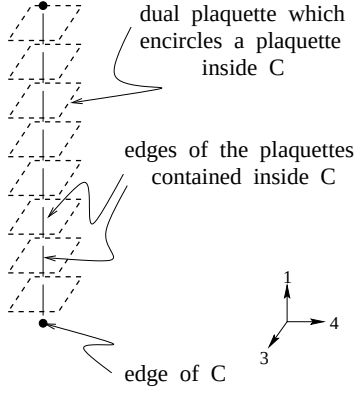
Similarly, the small β_e range in the $Z(2)$ gauge model corresponds to the ordered state (large β_{Ising}^*) in the dual model, where (approximately) all spins $\sigma_{\tilde{\sigma}^*}$ are oriented in the same direction. Such a configuration produces 1 in the denominator; but in the numerator and because of the term $\exp \left[-\beta_{\text{Ising}}^* \sum_{\substack{\text{dual bonds crossing} \\ \text{interior of } C}} (1 + \sigma_{\tilde{\sigma}^* + \hat{\mu}} \sigma_{\tilde{\sigma}^*}) \right]$, this configuration produces a factor $e^{-2\beta_{\text{Ising}}^*}$ at each dual

bond which crosses the area delimited by C

$$W(C)_{\text{small } \beta_e} = \frac{(e^{-2\beta_{\text{Ising}}^*})^{\mathcal{A}} + \dots}{1 + \dots} \approx \exp[-\mathcal{A}(-\ln \tanh \beta_e)],$$

equal, since β_e is small, to equation (C.343).

In general, it is characteristic of the dual models to describe the disordered phase of the original system by means of its own ordered phase and viceversa. This can be at best seen in a self-dual model, like the 4-dimensional $Z(2)$ pure gauge model.



In this model 4-dim. $Z(2)$ pure gauge model, let us consider the Wilson loop correlation $W(C)$ with loop C on the 1-2 plane. In the figure is represented a section of the 4-dimensional space for constant x_2 with the two points of the loop C . By repeating a similar discussion which has led us to equation (C.370), we obtain that the Wilson loop correlation $W(C)$ is given by

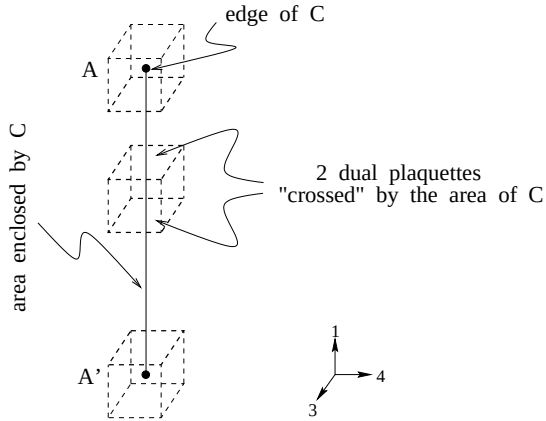
$$W(C) = \frac{\mathcal{Z}_{- \rightarrow +}^* \text{ in dual plaquettes encircling } C}{\mathcal{Z}^*}, \quad (\text{C.371})$$

where $\mathcal{Z}^*$ is the dual (in this case self-dual) partition function (C.367); $\mathcal{Z}_{- \rightarrow +}^*$ is equal to $\mathcal{Z}^*$ but with the change

$$1 - \bar{S}_{\mu, \vec{r}^*} \bar{S}_{\nu, \vec{r}^* + \hat{\mu}} \bar{S}_{\mu, \vec{r}^* + \hat{\nu}} \bar{S}_{\nu, \vec{r}^*} \rightarrow 1 + \bar{S}_{\mu, \vec{r}^*} \bar{S}_{\nu, \vec{r}^* + \hat{\mu}} \bar{S}_{\mu, \vec{r}^* + \hat{\nu}} \bar{S}_{\nu, \vec{r}^*},$$

at the dual plaquettes (whose dual bonds are represented by dashed lines in the figure) encircling the plaquettes contained inside C , i.e., at the plaquettes “crossed” by the area enclosed by C . The numerator of C takes the following form

$$\left[\prod_{\text{dual plaquettes not encircling plaquettes inside } C} (1 + \zeta_{\text{dual plaquette}} \tanh \beta_e^*) \right] \left[\prod_{\text{dual bonds encircling plaquettes inside } C} (1 - \zeta_{\text{dual plaquette}} \tanh \beta_e^*) \right],$$



with $\zeta_{\text{dual plaq.}} = \bar{S}_{\mu, \vec{r}^*} \bar{S}_{\nu, \vec{r}^* + \hat{\mu}} \bar{S}_{\mu, \vec{r}^* + \hat{\nu}} \bar{S}_{\nu, \vec{r}^*}$. For small β_e^* (large β_e), the leading contribution is 1. The next is obtained by expanding the above expression up to the first nonvanishing power of $\tanh \beta_e^*$, which corresponds to the smallest closed surface tiled by dual plaquettes: a $1 \times 1 \times 1$ cube. A unit cube in a d -dimensional lattice can be located in $d(d-1)(d-2)|\Lambda|/6!$ different forms. In this cube surrounds C (represented by points A and A' in the section of loop C of the figure), it contains one dual plaquette at which we have changed $1 + \zeta_{\text{dual plaquette}} \rightarrow 1 - \zeta_{\text{dual plaquette}}$. In the rest of the location of the cube, either contains no of such dual plaquettes if the cube lies completely outside C , or contains an even number of them if the cube is placed completely inside C .

Hence, the contribution of the cube to the $W(C)$'s numerator is $-\mathcal{P} (\tanh \beta_e^*)^6$ when the cube surrounds C , and $(4|\Lambda| - \mathcal{P}) (\tanh \beta_e^*)^6$ otherwise

$$W(C)_{\text{large } \beta_e} = \frac{1 + (4|\Lambda| - 2\mathcal{P}) (\tanh \beta_e^*)^6 + \dots}{1 + 4|\Lambda| (\tanh \beta_e^*)^6 + \dots} \approx \frac{\exp [(4|\Lambda| - 2\mathcal{P}) \beta_e^{*6}]}{\exp [4|\Lambda| \beta_e^{*6}]} = \exp \left[-2\mathcal{P} \left(-\frac{\ln \tanh \beta_e}{2} \right)^6 \right].$$

As expected, at large β_e (small β_e^*), $W(C)$ decays like the perimeter and is equal to (C.346).

In the dual lattice one can also define a closed loop C^* , lying on the 3-4 plane for example, and study the correlation of the dual-bond fields $\bar{S}_{\mu, \vec{r}^*}$ along this path: $H(C^*) = \left\langle \prod_{C^*} \bar{S}_{\mu, \vec{r}^*} \right\rangle$, called the 't-Hooft loop correlation. $H(C^*)$ decays like the area for β_e^* small (through an analogous calculation to that of $W(C)$ in the original model for β_e small, since the model is self-dual). Thus, at large β_e , $W(C)$ shows a perimeter law whereas $H(C^*)$ has an area decay. We can continue the discussion by calculating $H(C^*)$ at large β_e^* . If we repeat the same steps leading to (C.371) but this time starting from the dual model and finishing at the original model as dual of the dual model, we shall obtain that at large

β_e^* (equivalently, small β_e), $H(C^*)$ shows a perimeter decay: $H(C^*)_{\text{large } \beta_e^*} = \exp[-2\beta_e^6 \mathcal{P}^*]$, with $\mathcal{P}^*$ the perimeter of C^* . For this range, $W(C)$ decays like the area inside C .

The phase diagram for the 4-dimensional pure $Z(2)$ gauge model can be described in two different but equivalent ways. Phase at large β_e can be characterized as the disordered phase $H(C^*) \sim e^{-\text{Area}}$ of the dual partition function, or as the “ordered” phase $W(C) \sim e^{-\text{Perimeter}}$ of the original partition function. “Ordered” has been put between quotation marks not because $W(C)$ does not vanish for a large loop C , but because the phase has a weaker decay than in the phase with many disordering lines of flipped gauge “arrows”. On the other hand, phase at small β_e is described as the disordered phase of the original partition function, or as the “ordered” phase of the dual partition function

Phase I: large β_e “ordered” phase $W(C) \sim e^{-\text{Perimeter}}$ disordered dual phase $H(C^*) \sim e^{-\text{Area}}$	Phase II: small β_e disordered phase $W(C) \sim e^{-\text{Area}}$ “ordered” dual phase $H(C^*) \sim e^{-\text{Perimeter}}$	(C.372)

Now we can consider a generalization of the 3-dimensional $Z(2)$ gauge model, where the $A_{\mu, \vec{r}}$ values in $\psi_{\mu, \vec{r}} = e^{iA_{\mu, \vec{r}}}$ are not only 0 and π but the following discrete set: $A_{\mu, \vec{r}} = \{0, \frac{2\pi}{N}, \dots, \frac{2\pi(N-1)}{N}\}$. Thus gauge “arrow” $\psi_{\mu, \vec{r}}$ can point in N different directions in the 2-dimensional internal symmetry space. This model is called the $Z(N)$ gauge model. Again, we can study the behaviour of correlation $W(C) = \langle \prod_C (\psi_{\mu, \vec{r}})^p \rangle$ (with $p < N$, since $(\psi_{\mu, \vec{r}})^p = 1$ for $p = N$) and calculate critical point $(\beta_e)_{\text{critical}}$ for the transition between the phase $\beta_e > (\beta_e)_{\text{critical}}$, where $W(C)$ shows a perimeter decay, and the phase $\beta_e < (\beta_e)_{\text{critical}}$, where the correlation has the stronger area decay. We shall not calculate this critical value, but we can guess its form when N increases (the exact calculation will be treated in the Note on page 576). The case $N \rightarrow \infty$ corresponds to the pure gauge $U(1)$ model since the gauge arrow can point in any direction in the 2-dimensional internal space. However, the pure $U(1)$ gauge model has no phase transition and the sole existing phase is the disordered one with $W(C) \sim e^{-\text{Area}}$. Therefore, $(\beta_e)_{\text{critical}} \rightarrow \infty$ when N increases, which can be expected since an increase in the possible $\psi_{\mu, \vec{r}}$ ’s orientations makes easier to disorder the system.

This argument, although it works for the 3-dimensional case, is not correct for any dimension. Before we consider the 4-dimensional pure gauge model to see the difference, let us firstly study the 2-dimensional scalar model; with a discrete $Z(2)$ symmetry it is the Ising model and with a continuous $U(1)$ symmetry it is the XY model. The 2-dimensional $Z(2)$ scalar model is, like the 4-dimensional $Z(2)$ pure gauge model, self-dual. As we know, the Ising model has an ordered phase for $\beta_\psi > (\beta_\psi)_{\text{crit}}$ where correlation $\langle \psi_{\vec{r}_1} \psi_{\vec{r}_2}^* \rangle$ is nonzero at large distances, and a disordered phase with an exponential decay in the correlation for β_ψ values smaller than the self-dual critical point. We expect that by increasing the number of possible $\psi_{\vec{r}}$ ’s orientations, the available β_ψ range for the ordered phase will decrease, i.e., the self-dual critical point $(\beta_\psi)_{\text{crit}}$ increases for $N \rightarrow \infty$ ²⁹. In fact, in the 2-dimensional XY (limit $N \rightarrow \infty$ of the scalar $Z(N)$ model), there is no phase with a nonvanishing value of the correlation at $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$. However,

²⁹More exactly, $(\beta_\psi)_{\text{crit}} \sim N^2$ [95]. Although not rigorous, because the dual transform of the $Z(N)$ model does not have, for large N , the same energy form as the original model, the following argument al-

this does not mean that the whole β_ψ range is “filled” with the disordered phase with exponential decay. In the XY model, in addition to the phase with exponential decay in the correlation, there exists another phase which shows a power decay.

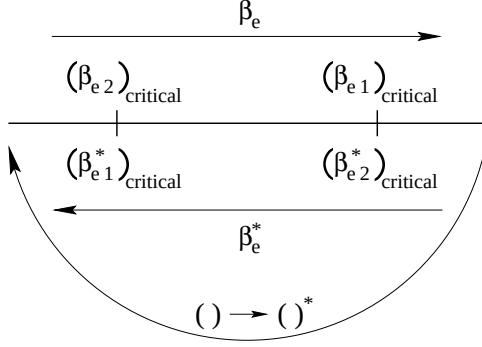
Therefore, one expects that at a certain N value, the self-dual critical point bifurcates into two different critical points $(\beta_{\psi 1})_{\text{crit}}$ and $(\beta_{\psi 2})_{\text{crit}} < (\beta_{\psi 1})_{\text{crit}}$, which are dual to each other: $(\beta_{\psi 1})_{\text{crit}}^* = (\beta_{\psi 2})_{\text{crit}}$. The new phase between these two critical values shows a decay softer than the exponential of phase $\beta_\psi < (\beta_{\psi 2})_{\text{crit}}$. The ordered phase with nonvanishing correlation’s value takes place at $\beta_\psi > (\beta_{\psi 1})_{\text{crit}}$. At $N \rightarrow \infty$, $(\beta_{\psi 2})_{\text{crit}}$ remains finite (not far from the self-dual critical point $(\beta_\psi)_{\text{crit}} = 0.441$ of the Ising model) whilst $(\beta_{\psi 1})_{\text{crit}}$ increases with N and thus the ordered phase disappears in the limit $N \rightarrow \infty$. One can rule out that the intermediate phase only occurs at $N = \infty$: if we were supposing that for any finite N we have only the self-dual critical point, one would obtain that for a large N we have the disordered phase with the exponential decay covering almost the whole β_ψ range (remember that the self-dual critical value increases with N); when $N = \infty$ were reached, then for a range of β_ψ values the correlation would change from an exponential into a power decay with the distance, a much weaker decay. But one can hardly expect that by increasing the possible orientations of the scalar field (and thus allowing for a more efficient disordering), the correlation should decay more slowly. Therefore, for N smaller than a certain value $\bar{N}$, we have a disordered and an ordered phase separated by the self-dual critical value; for $N > \bar{N}$, the disordered phase keeps approximately the same β_ψ range as before, whereas the ordered phase gives gradually place to the phase with the soft decay (weaker than the exponential of the disordered phase) in the correlation function ³⁰.

The same discussion can be followed for the 4-dimensional pure gauge model; here, only the results will be introduced and the detailed discussion will be postponed for later (see page 483). From the previous section (C.327), we know that the $U(1)$ model shows two different phases: at large $\beta_e = 1/e^2\eta$, $W(C)$ decays like the perimeter minus term τ/R ; at small β_e , $W(C)$ shows the typical area law. On the other hand, the pure $Z(2)$ gauge model in four dimensions has also two phases: the disordered phase characterized by $W(C) \sim e^{-\text{Area}}$ at small β_e and the “ordered” phase with $W(C) \sim e^{-\text{Perimeter}}$

allows us to gain some insight into this question. The partition function for the scalar 2-dimensional $Z(N)$ model is given by $\sum_{\{n_{\vec{r}}=0,\dots,N-1\}} \exp \left[-\beta_\psi \sum_{\text{bonds}} 1 - \cos \left(\frac{2\pi}{N} (n_{\vec{r}+\hat{\mu}} - n_{\vec{r}}) \right) \right]$. By expanding in powers of $\cos()$ or equivalently in complex exponentials of the bond-terms $e^{i2\pi(n_{\vec{r}+\hat{\mu}} - n_{\vec{r}})/N}$, we see that the nonvanishing contributions, which in the $Z(2)$ model were open surfaces with borders completely covered by bonds-terms (in order to ensure that each $\sigma_{\vec{r}}$ field appeared twice), are open surfaces with boundaries covered $N - 1$ times by bond-terms (so that when each field $e^{i2\pi n_{\vec{r}}/N}$ at the sites of these boundaries is summed over $n_{\vec{r}} = 0, \dots, N - 1$, it produces a nonzero). Hence, instead of a $Z(2)$ contribution of the kind $(\beta_\psi)^B$ with B the number of bonds in the considered surface’s boundary, we obtain $(\beta_\psi^{N-1})^B$. The dual model, as the original model, contains again only interactions between discrete fields located at the extremes of a dual bond (the model is self-dual); its coupling strength is $\beta_\psi^* \sim (N - 1) \ln \beta_\psi$, to be compared with $\beta_\psi^* \sim \ln \beta_\psi$ for the $Z(2)$ model. This means that the dual coupling strength in the 2-dim. $Z(N)$ model has been pushed toward larger values than in the $Z(2)$ model. The same holds for the self-dual 4-dim. $Z(2) - Z(N)$ pure gauge model: see page 589.

³⁰All these handwaving arguments can be put on a solid (mathematical) ground: see the Note on page 576 or References [37] and [95]. The conclusions, nevertheless, are correct.

at large β_e . As in the 2-dimensional Ising model, if for large N the only critical point were the self-dual critical value $(\beta_e)_{\text{critical}}$ (which increases with N), then the phase with $W(C) \sim e^{-\text{Area}}$ would occupy the whole β_e range and the $U(1)$ model would have only the disordered phase. Since this does not occur, at a certain $\tilde{N}$ an intermediate phase must appear for a β_e value between the two critical values, $(\beta_{e1})_{\text{critical}}$ and $(\beta_{e2})_{\text{critical}} < (\beta_{e1})_{\text{critical}}$, which are dual to each other: $-1/2 \ln \tanh(\beta_{e1})_{\text{critical}} = (\beta_{e2})_{\text{critical}}$.



This intermediate phase is self-dual, as one can easily see: let us imagine a map which changes β_e , C and gauge arrow $S_{\mu, \vec{r}}$ into the dual variables β_e^* , C^* and $\tilde{S}_{\mu, \vec{r}^*}$. Such a correspondence transforms large β_e into large β_e^* ; this means, it transforms Phase I of (C.372) into Phase II

Phase I: large $\beta_e \rightarrow$ Phase II: large β_e^*

$$\begin{aligned} W(C) \sim e^{-\text{Perimeter}} &\rightarrow H(C^*) \sim e^{-\text{Perimeter}} \\ H(C^*) \sim e^{-\text{Area}} &\rightarrow W(C) \sim e^{-\text{Area}} \end{aligned}$$

Hence, phase with very large β_e (i.e., $\beta_e > (\beta_{e1})_{\text{crit}}$ or Phase I) is mapped into phase $\beta_e^* > (\beta_{e1})_{\text{crit}}^*$ or equivalently $\beta_e < (\beta_{e2})_{\text{crit}}$ (Phase II). The intermediate phase, on the other hand, is mapped onto itself: phase $(\beta_{e2})_{\text{crit}} < \beta_e < (\beta_{e1})_{\text{crit}}$ into phase $(\beta_{e2}^*)_{\text{crit}} < \beta_e^* < (\beta_{e1}^*)_{\text{crit}} \Leftrightarrow (\beta_{e1})_{\text{crit}} > \beta_e > (\beta_{e2})_{\text{crit}}$. Therefore, Wilson loop $W(C)$ and 't-Hooft loop $H(C^*)$ shows the same decay in the intermediate phase. Since we know from the $U(1)$ model that $W(C) \sim e^{-\text{Perimeter} + \tau/R} \approx e^{-\text{Perimeter}}$ at this phase, $H(C^*)$ also decays like the perimeter (of C^*). This is the difference between this third phase and the “ordered” phase (Phase I): although in both phases $W(C)$ decays like the perimeter, the 't-Hooft loop shows an area law in the “ordered” phase whilst its decay is with the perimeter for the third phase.

For $N \rightarrow \infty$, the “ordered” phase disappears gradually ($(\beta_{e1})_{\text{crit}} \rightarrow \infty$) and its place is occupied by the third phase; in this limit, $(\beta_{e2})_{\text{crit}}$ tends to the critical value of the pure compact $U(1)$ gauge model. According to the work of Creutz, Jacobs and Rebbi [29], both transition points $(\beta_{e1})_{\text{crit}}$ and $(\beta_{e2})_{\text{crit}}$ are second-order phase-transition points.

It is worth noting that the origin of the intermediate phase is the existence of the two phases in the pure $U(1)$ gauge model in four dimensions; since in three dimensions the $U(1)$ only shows one phase, the intermediate phase does not occur in the pure $Z(N)$ gauge model.

C.5.3 The lattice Higgs model in the Villain approximation

After the study of the pure gauge model, the gauge field and the scalar field of the XY model will be coupled to each other in a gauge invariant way. The steps we are following are the same as in the study of topological defects in the continuous model for a superconductor in Appendix B. There, the gauge field of the ordinary $U(1)$ Magnetism (without magnetic monopoles) or of the compact $U(1)$ Magnetism (with monopoles) is

coupled to a scalar field, whose phase gives rise to topological line defects.

Let us start the discussion of the lattice Higgs model by writing the correlation function we want to evaluate

$$W(C) = \left\langle e^{ip \sum_{\text{loop } C} A_{\mu, \vec{r}}} \right\rangle = \frac{1}{\mathcal{Z}} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r}, \mu} \left(\int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[ip \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \times \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] - \frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})] \right]. \quad (\text{C.373})$$

At large $ae^2\eta$ and small $a|\psi|^2/\eta$, the expansion of the exponential in cosine terms, or equivalently, in powers of complex exponentials of $\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}$ and of $\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}}$ can be broken at the first power which produces a nonvanishing contribution (the “high temperature” power expansion). Due to the relations

$$\int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} e^{iA_{\mu, \vec{r}}} = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} e^{iA_{\mu, \vec{r}}} e^{-iA_{\mu, \vec{r}}} = 1, \quad (\text{C.374})$$

the terms $e^{-\frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})]}$ and $e^{-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})]}$ must be expanded up to the power $e^{-ipA_{\mu, \vec{r}}}$ in order to produce a nonvanishing contribution when integrated with the $A_{\mu, \vec{r}}$ contained in $e^{ip \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}}}$. For the first term there is no difficulty: this p power $e^{-ipA_{\mu, \vec{r}}}$ is reached when each plaquette of the surface enclosed by C is tiled p times by the plaquette-factor $e^{i(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})}$. On the contrary, for the second energy part $e^{-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})]}$, the power $e^{-ipA_{\mu, \vec{r}}}$ can only be achieved if $p = q$: when each segment or bond of closed path C is covered by the bond-factor $e^{i(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})}$. Therefore, the contributions to $W(C)$ for large $ae^2\eta$ and small $a|\psi|^2/\eta$, arising from tiling p times the complete interior with plaquette-factors and from surrounding C with bond-factors, are the following ³¹

$$W(C) = \left[\left(\frac{1}{2ae^2\eta} \right)^p \frac{1}{p!} \right]^{\mathcal{A}} + \left[\frac{a|\psi|^2}{2\eta} \right]^{\mathcal{P}} + \dots \approx \left[\frac{a|\psi|^2}{2\eta} \right]^{\mathcal{P}} = \exp \left[-\mathcal{P} \ln \left(\frac{2\eta}{a|\psi|^2} \right) \right],$$

where the approximation holds as a result of considering a large loop C (the perimeter is much smaller than the area) and of the range we are studying. However, the above result only occurs if $p = q$. For the case of a fractionary quotient p/q , the perimeter contribution vanishes due to the fact that there is no bond-factor $e^{i(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})}$ which

³¹The case with $p > q$ reduces to the discussed case by applying the argument to p' with $p = nq + p'$ (n integer and $p' \leq q$). The integer nq power is obtained by the n times product of bond-factors $e^{i(\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})}$, and the argumentation follows in a similar way as above by studying if $p' = q$ or not. Thus for $p' = q$, $W(C)$ becomes

$$W(C) = \left[\left(\frac{1}{2ae^2\eta} \right)^p \frac{1}{p!} \right]^{\mathcal{A}} + \left[\left(\frac{a|\psi|^2}{2\eta} \right)^n \frac{1}{n!} \right]^{\mathcal{P}} + \dots$$

could compensated power $e^{ipA_{\mu,\vec{r}}}$ arising from C ; therefore, the only remaining contribution, $\left[\left(\frac{1}{2ae^2\eta}\right)^p \frac{1}{p!}\right]^{\mathcal{A}}$, decays with the enclosed area $\mathcal{A}$ [35]. Summarizing, for a large loop C the correlation function $W(C)$ at large $ae^2\eta$ and small $a|\psi|^2/\eta$ decays as follows

$$W(C)_{\substack{p=q \\ \text{large } ae^2\eta, \text{ small } a|\psi|^2/\eta}} \approx \exp\left[-\mathcal{P} \ln\left(\frac{2\eta}{a|\psi|^2}\right)\right], \quad (\text{C.375})$$

$$W(C)_{\substack{p/q \text{ non integer} \\ \text{large } ae^2\eta, \text{ small } a|\psi|^2/\eta}} \approx \exp\left[-p\mathcal{A} \ln(ae^2\eta)\right]. \quad (\text{C.376})$$

For the other ranges of $ae^2\eta$ and $a|\psi|^2/\eta$, the Villain approximation of the lattice Higgs model is used

$$\begin{aligned} W(C) &= \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \exp\left[ip \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}\right] \times \\ &\times \sum_{\{g_{\mu\nu,\vec{r}}\}} \exp\left[-\frac{1}{2ae^2\eta} \sum_{\vec{r},\mu<\nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}})^2\right] \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp\left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r},\mu} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})^2\right], \end{aligned} \quad (\text{C.377})$$

or if one prefers to include q into gauge field

$$\begin{aligned} W(C) &= \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{d\bar{A}_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \exp\left[i\frac{p}{q} \sum_{\vec{r},\mu} J_{\mu,\vec{r}} \bar{A}_{\mu,\vec{r}}\right] \times \\ &\times \sum_{\{\bar{g}_{\mu\nu,\vec{r}}=0,\pm q,\pm 2q,\dots\}} \exp\left[-\frac{1}{2aq^2e^2\eta} \sum_{\vec{r},\mu<\nu} (\Delta_{\mu} \bar{A}_{\nu,\vec{r}} - \Delta_{\nu} \bar{A}_{\mu,\vec{r}} - 2\pi \bar{g}_{\mu\nu,\vec{r}})^2\right] \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp\left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r},\mu} (\Delta_{\mu} \varphi_{\vec{r}} - \bar{A}_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})^2\right]. \end{aligned} \quad (\text{C.378})$$

The partition function is similar to these equations but with $J_{\mu,\vec{r}}$ set to zero.

As already known from the XY model or the pure gauge model, due to the periodic character of the original variables, we cannot sum independently over each $m_{\mu,\vec{r}}$ and $g_{\mu\nu,\vec{r}}$ but both sums have to be constrained in order to avoid repetition of the same $m_{\mu,\vec{r}}$ and $g_{\mu\nu,\vec{r}}$ -contribution. Such a constraint must be understood for the following. In addition to the redundancy in the sum over the integer fields, the gauge invariance of the partition and correlation function under transformations as

$$\begin{cases} \varphi_{\vec{r}} & \rightarrow & \varphi'_{\vec{r}} & = & \varphi_{\vec{r}} - q\chi_{\vec{r}}, \\ A_{\mu,\vec{r}} & \rightarrow & A'_{\mu,\vec{r}} & = & A_{\mu,\vec{r}} - (\chi_{\vec{r}+\hat{\mu}} - \chi_{\vec{r}}), \end{cases} \quad (\text{C.379})$$

produces other infinities if we do not choose a special gauge fixing condition ³². The integration over $dA_{\mu,\vec{r}}$ and $d\varphi_{\vec{r}}$ will be carried out after introducing auxiliary variables $dB_{\mu\nu,\vec{r}}$ and $db_{\mu,\vec{r}}$

$$\begin{aligned}
& \frac{1}{\mathcal{Z}} \prod_{\vec{r},\mu} \int_{-\infty}^{\infty} \frac{dA_{\mu,\vec{r}}}{2\pi} \prod_{\vec{r},\mu < \nu} \left(\int_{-\infty}^{\infty} \sqrt{\frac{ae^2\eta}{2\pi}} dB_{\mu\nu,\vec{r}} \right) \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\varphi_{\vec{r}}}{2\pi} \prod_{\vec{r},\mu} \left(\int_{-\infty}^{\infty} \sqrt{\frac{\eta}{2\pi a|\psi|^2}} db_{\mu,\vec{r}} \right) \times \\
& \times \exp \left[ip \sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] \times \\
& \times \sum_{\{g_{\mu\nu,\vec{r}}\}} \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r},\mu < \nu} B_{\mu\nu,\vec{r}}^2 + i \sum_{\vec{r},\mu < \nu} B_{\mu\nu,\vec{r}} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}} - 2\pi g_{\mu\nu,\vec{r}}) \right] \times \\
& \times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}}) \right], \quad (C.380)
\end{aligned}$$

such that the result of the $dA_{\mu,\vec{r}}$ and $d\varphi_{\vec{r}}$ integration is only products of Dirac deltas; we shall not include the gauge fixing term in the integration over $A_{\mu,\vec{r}}$ and $\varphi_{\vec{r}}$ because the infinities arising from the gauge invariance are in this case easy to identify and to remove.

Hence, the $d\varphi_{\vec{r}}$ integration produces the following product of Dirac deltas as constraint on $b_{\mu,\vec{r}}$

$$\prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} \right) \Rightarrow b_{\mu,\vec{r}} = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}. \quad (C.381)$$

On the other hand, the $dA_{\mu,\vec{r}}$ integration constrains the $B_{\mu\nu,\vec{r}}$ as follows

$$\prod_{\vec{r},\nu} \delta \left(\sum_{\mu} \Delta_{\mu} B_{\mu\nu,\vec{r}-\hat{\mu}} - p J_{\nu,\vec{r}} + q b_{\nu,\vec{r}} \right) \Rightarrow B_{\mu\nu,\vec{r}} = \sum_{\lambda} \epsilon_{\mu\nu\lambda} \left(\Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} + q a_{\lambda,\vec{r}} \right) + p L_{\mu\nu,\vec{r}}, \quad (C.382)$$

coupling field $C_{\vec{r}}$ with $a_{\lambda,\vec{r}}$. Note that constraint (C.381) $\Delta_{\mu} b_{\mu,\vec{r}} = 0$ is already contained in constraint (C.382) as a result of the antisymmetry of $B_{\mu\nu,\vec{r}}$ and of the $J_{\mu,\vec{r}}$'s definition $\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} = 0$. Therefore, one obtains

$$\begin{aligned}
& \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} \right) \prod_{\vec{r},\nu} \delta \left(\sum_{\mu} \Delta_{\mu} B_{\mu\nu,\vec{r}-\hat{\mu}} - p J_{\nu,\vec{r}} + q b_{\nu,\vec{r}} \right) = \\
& = \left(\prod_{\vec{r}} \delta(0) \right) \prod_{\vec{r},\nu} \delta \left(\sum_{\mu} \Delta_{\mu} B_{\mu\nu,\vec{r}-\hat{\mu}} - p J_{\nu,\vec{r}} + q b_{\nu,\vec{r}} \right), \quad (C.383)
\end{aligned}$$

³²The definition of $J_{\mu,\vec{r}}$, $\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} = 0$, makes also $\sum_{\vec{r},\mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}}$ remain invariant under a gauge transformation $\sum_{\vec{r},\mu} J_{\mu,\vec{r}} (\chi_{\vec{r}+\hat{\mu}} - \chi_{\vec{r}}) = \sum_{\vec{r},\mu} J_{\mu,\vec{r}-\hat{\mu}} \chi_{\vec{r}} - \sum_{\vec{r},\mu} J_{\mu,\vec{r}} \chi_{\vec{r}} = - \sum_{\vec{r}} \left(\sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} \right) \chi_{\vec{r}} = 0$.

where the infinities due to the not suppressed gauge invariance are easily isolated: by removing the infinite term $\prod_{\vec{r}} \delta(0)$.

Substitution of (C.381) gives

$$-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 = -\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}} \left(-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} a_{\lambda, \vec{r}-\hat{\nu}} + \sum_{\nu} \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r}-\hat{\lambda}} \right), \quad (\text{C.384})$$

whilst substitution of (C.382) produces the following terms

$$\begin{aligned} -\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 = & -\frac{ae^2\eta}{2} \left[\sum_{\vec{r}} C_{\vec{r}} \left(-\sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} \right) + \underbrace{p^2 \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 + q^2 \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}}^2 +}_{\substack{+2q \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} + 2pq \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} a_{\lambda, \vec{r}} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} - 2p \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} C_{\vec{r}} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}}} \right]. \quad (\text{C.385}) \end{aligned}$$

After these two substitutions, the coupling between the fields $a_{\lambda, \vec{r}}$, $C_{\vec{r}}$ and the integer fields $m_{\mu, \vec{r}}$, $g_{\mu\nu, \vec{r}}$ looks as follows

$$-i2\pi \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}} (M_{\lambda, \vec{r}} + q\epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}}) + i2\pi \sum_{\vec{r}} \underline{C_{\vec{r}} G_{\vec{r}}}, \quad (\text{C.386})$$

where fields $M_{\lambda, \vec{r}}$ and $G_{\vec{r}}$ are given by

$$M_{\lambda, \vec{r}} = \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu, \vec{r}} \quad \text{with} \quad \sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} = 0, \quad (\text{C.387})$$

and

$$G_{\vec{r}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}. \quad (\text{C.388})$$

The best form to obtain the physical meaning of these integer fields is to discuss the model as Polyakov did for the pure gauge model [34]. At the range we are working, the energy functional in the exponent of the partition function can be well-approximated by the continuous formulation

$$E[\varphi, A_{\mu}] \approx \int d^3\vec{r} \left[\frac{1}{4e^2} F_{\mu\nu}^2 + \frac{|\psi|^2}{2} (\partial_{\mu}\varphi - qA_{\mu})^2 \right], \quad (\text{C.389})$$

but without forgetting the angular character of both φ and A_{μ} .

If phase φ had no “jumps”, one could choose to absorb φ into gauge field A_{μ} by means of the gauge transformation defined by $q\chi = \varphi$. After covering a closed path in ordinary space, nevertheless, the scalar field’s phase can change by a multiple of 2π due to its angular character. Such a jump in φ cannot be absorbed by means of a regular (allowed) gauge transformation. This leads us to the classification of the possible configurations in the model:

- Vortices, where jumps of $2\pi k$ take place in φ . These jumps forces A_{μ} and $F_{\mu\nu}$ to carry a multiple of flux $2\pi/q$ to ensure the finiteness of the energy at large distances.

- Plus any local deformation of the vortices which is represented by any allowed gauge transformation applied on them.

Equivalently, we can reach the same classification by absorbing φ into A_μ in any case, and then by classifying the line singularities produced in the field strength $F_{\mu\nu}$ by the singular gauge transformations which have been necessary to remove the $2\pi k$ changes in the phase (remember equation (B.34) in Appendix B). We have transferred the topological content from φ into the vortices produced in the gauge field. If the gauge field is non-compact, as in the ordinary $U(1)$ Magnetism, each of these singular lines (vortices or dislocations) cannot be removed by a regular gauge transformation, which only represents a local deformation of the singular line (in the language of the dislocation model, a gauge transformation describes a local displacement of the dislocation). Therefore, the lines have to form closed lines.

In our model, however, A_μ is compact, has angular character too, and jumps in its range are allowed: there exist regular gauge transformations which starting from a state with $\oint \partial_\mu \varphi dl^\mu = 0$ can produce $\oint \partial_\mu \varphi dl^\mu \neq 0$. For instance, in Appendix B and for the “ $SO(3)$ ”-superconductor we have constructed the transformation connecting zero winding number with double winding number (see (B.159)). Hence, by means of a gauge transformation, we can absorb into A_μ certain jumps in φ without producing singular lines. The main contribution to the partition function arises from the solutions of the following equation

$$\partial_\mu F_{\mu\nu} - q^2 e^2 |\psi|^2 A_\nu = 0 \mod 2\pi. \quad (\text{C.390})$$

The above equation is similar to that obtained in the pure gauge model (see (C.286)) $\partial_\mu F_{\mu\nu} = 0 \mod 2\pi$, and therefore, we could expect isolated monopoles as topological objects; these monopoles carry multiples of 2π as magnetic flux. However, the new term $q^2 e^2 |\psi|^2 A_\nu$ makes the magnetic field be short-range. This can be easily seen by noting that $\partial_\mu F_{\mu\nu} - q^2 e^2 |\psi|^2 A_\nu = 0$ is just Ampere’s law $\partial_\mu F_{\mu\nu} = -j_\nu$ with a current j_ν proportional to (minus) gauge field A_ν . Such a current defines the London equation of superconductivity $(j_{\text{London}})_\nu = -\frac{1}{\lambda^2} A_\nu$, which allows the magnetic field to spread only until a distance $\lambda = 1/qe|\psi|$: the Meissner effect. Therefore, the magnetic field created by the isolated monopoles must be confined into certain regions: these regions are the vortices originated in the φ jumps³³. Each vortex carries a flux of $\oint A_\mu dl^\mu = 2\pi/q$ and the monopoles have a magnetic charge of 2π , which is pushed into q vortices. If one prefers to include q into A_μ , vortices carry a flux of 2π and the monopoles a magnetic charge of $2\pi q$.

We arrive thus at the meaning of integer fields $M_{\lambda,\vec{r}}$ and $G_{\vec{r}}$: in the system there are closed vortices/dislocations (or infinitely long vortices) plus open vortices ending on monopoles. The topological content of the closed vortices is carried by $M_{\lambda,\vec{r}}$, whereas

³³A special case of the above argument is when the allowed transformations for the compact gauge field are able to connect configurations of any flux with zero flux configurations: in this case there are no stable vortices where the monopole’s field could be confined. An example of this situation is the “ $SU(2)$ ”-superconductor, which corresponds to $p = q = 1$.

that of the open vortices by $q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}}$. The total vorticity is given by

$$V_{\lambda, \vec{r}} = M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}}, \quad (\text{C.391})$$

whose divergence does not vanish and is proportional to the monopole density $G_{\vec{r}}$

$$\sum_{\lambda} \Delta_{\lambda} V_{\lambda, \vec{r}} = \sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} + q \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}} = q G_{\vec{r}}. \quad (\text{C.392})$$

Factor q means that the number of open vortices which end on a monopole is q ; for example, in the “ $SO(3)$ ”-superconductor, we already know that $q = 2$ and that two vortices form a monopole where they meet.

Those configurations, which produce the leading contributions to the partition function, are the above topological objects (closed vortices and open vortices ending on monopoles) described by $\{A_{\text{topol}, \mu, \vec{r}}, \varphi_{\text{topol}, \vec{r}}\}$, or by $\{M_{\lambda, \vec{r}}, \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}}\}$, plus perturbations $\{\Gamma_{\mu, \vec{r}}, \gamma_{\vec{r}}\}$ without topological content about the former. The partition function becomes thus

$$\begin{aligned} \mathcal{Z} = & \mathcal{Z}_{\text{topol}} \prod_{\vec{r}, \mu} \int_{-\infty}^{\infty} \frac{d\Gamma_{\mu, \vec{r}}}{2\pi} \prod_{\vec{r}} \int_{-\infty}^{\infty} \frac{d\gamma_{\vec{r}}}{2\pi} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} \Gamma_{\nu, \vec{r}} - \Delta_{\nu} \Gamma_{\mu, \vec{r}})^2 \right] \times \\ & \times \exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \gamma_{\vec{r}} - q\Gamma_{\mu, \vec{r}})^2 \right], \end{aligned} \quad (\text{C.393})$$

where the part containing fields $\{\Gamma_{\mu, \vec{r}}, \gamma_{\vec{r}}\}$ will be called $\mathcal{Z}_0$. The form of $\mathcal{Z}_{\text{topol}}$ will be calculated in the following.

Continuing with the calculation of the correlation function, we have to integrate now $da_{\lambda, \vec{r}}$ and $dC_{\vec{r}}$ out. Let us start with the first set of variables. We face again the technical problem of having many $a_{\lambda, \vec{r}}$ configurations which correspond to the same value in the partition. Now, to isolate the infinities is a bit more involved than in (C.383)³⁴, and it is convenient to include a gauge fixing factor which compensates for the infinite number of transformations $a_{\lambda, \vec{r}} \rightarrow a_{\lambda, \vec{r}} - \Delta_{\lambda} \chi_{\vec{r} - \hat{\lambda}}$ giving the same contribution. This factor is the same used in the 3-dimensional XY model

$$\det(-\Delta\Delta) \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{|\Lambda|} \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} \sum_{\lambda, \nu} \Delta_{\lambda} a_{\lambda, \vec{r}} \Delta_{\nu} a_{\nu, \vec{r}} \right], \quad (\text{C.394})$$

which cancels out term $\sum_{\vec{r}} \sum_{\lambda, \nu} a_{\lambda, \vec{r}} \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r} - \hat{\lambda}}$ in (C.384). This choice is not the most comfortable for the Higgs model, but it has been used here to follow the same steps as in the XY model. In the last section of this Appendix it will be shown how for different choices the final result of the correlation is always the same.

³⁴The origin of the infinities, though, has been isolated in the 3-dimensional XY model: see (C.183).

Considering (C.384), (C.385) and the gauge fixing factor, the quadratic term in $a_{\lambda,\vec{r}}$ takes the following form

$$-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(-\Delta\Delta + m^2 \right) a_{\lambda,\vec{r}} \quad \text{with} \quad \lambda^2 = \frac{1}{a^2 q^2 e^2 |\psi|^2}. \quad (\text{C.395})$$

We have just met a similar equation (in the continuum formulation) in (C.390). It describes a short-range vector field. The origin of the short range is the coupling between scalar field's phase $\varphi_{\vec{r}}$ and gauge field $A_{\mu,\vec{r}}$. Such a coupling, although $\varphi_{\vec{r}}$ and $A_{\mu,\vec{r}}$ have been already integrated out, produces the constraint (C.382) and leads to equation (C.395). Thus, the original short-range in the gauge field has left its “footprint”; note that the range is the same (except for the factor a^2 in the discrete formulation). In order to avoid confusion between the range and the subindex λ , and to follow the usual notation in lattice gauge theories, we shall use m^2 instead of λ^{-2} to denote the quadratic inverse of the range.

The Green's function or propagator of discrete differential operator $(-\Delta\Delta + m^2)$ is defined by the equation

$$\left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} + m^2 \right) D_{\vec{r},\vec{r}'}^{(m^2)} \equiv \left(-\sum_{\nu} \frac{\Delta}{(\vec{r})} \nu \frac{\Delta}{(\vec{r})} D_{\vec{r}-\hat{\nu},\vec{r}'}^{(m^2)} + m^2 D_{\vec{r},\vec{r}'}^{(m^2)} \right) = \delta_{\vec{r},\vec{r}'}, \quad (\text{C.396})$$

whose solution is

$$D_{\vec{r},\vec{r}'}^{(m^2)} = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} \frac{e^{i\vec{k} \cdot (\vec{r}-\vec{r}')}}{m^2 + 6 - 2\cos k_x - 2\cos k_y - 2\cos k_z}. \quad (\text{C.397})$$

For large distance $|\vec{r} - \vec{r}'|$, the 3-dimensional propagator is approximately given by the Coulomb propagator multiplied by an exponential decay with characteristic length equal to range m^{-1}

$$D_{\vec{r},\vec{r}'}^{(m^2)} \approx \frac{e^{-m|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \quad \text{for } |\vec{r}-\vec{r}'| \text{ large}, \quad (\text{C.398})$$

as one can easily shown³⁵. The result of the Gaussian integration over $da_{\lambda,\vec{r}}$ produces the following factor multiplying the integral over $dC_{\vec{r}}$

$$\left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{3|\Lambda|} \left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{(3+1-3)|\Lambda|} \frac{\det(-\Delta\Delta)}{\left(\sqrt{\det(-\Delta\Delta + m^2)} \right)^3}, \quad (\text{C.399})$$

³⁵For large distances one can use the continuum formulation, and by noting that $\vec{\nabla}r = \frac{\vec{r}}{r}$ and $2\left(\vec{\nabla}r\right)\left(\vec{\nabla}\frac{1}{r}\right) + \frac{1}{r}\vec{\nabla}\vec{\nabla}r = 0$ ($\vec{\nabla} \cdot \vec{r} = 3$), one obtains

$$\vec{\nabla}\vec{\nabla} D_{\vec{r},\vec{r}'}^{(m^2)} = \frac{e^{-mr}}{4\pi} \underbrace{\left(\vec{\nabla}\vec{\nabla}\frac{1}{r} \right)}_{-4\pi\delta^3(\vec{r})} + m^2 \frac{e^{-mr}}{4\pi r} \underbrace{\left(\vec{\nabla}r \right)^2}_1 - \frac{me^{-mr}}{4\pi} \left(2\left(\vec{\nabla}r\right)\left(\vec{\nabla}\frac{1}{r}\right) + \frac{1}{r}\vec{\nabla}\vec{\nabla}r \right) = -\delta^3(\vec{r}) + m^2 D_{\vec{r},\vec{r}'}^{(m^2)}.$$

arising from the choice of gauge (C.394) and from the integrations carried out up to this moment; and the following exponent inside the integral

$$-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left[2\pi \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right) - ipqa e^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} - iqa e^2 \eta \Delta_{(\vec{r})\lambda} C_{\vec{r}-\hat{\lambda}} \right] D_{\vec{r}, \vec{r}'}^{(m^2)} \Bigg]_{\lambda, \vec{r}'}$$

which has to be added to the terms depending on $C_{\vec{r}}$, underlined in (C.385) and (C.386). Sum by parts, $\sum_{\vec{r}} g_{\vec{r}} \Delta_{\lambda} f_{\vec{r}-\hat{\lambda}} = \sum_{\vec{r}} g_{\vec{r}} f_{\vec{r}} - \sum_{\vec{r}} g_{\vec{r}} f_{\vec{r}-\hat{\lambda}} = \sum_{\vec{r}} g_{\vec{r}} f_{\vec{r}} - \sum_{\vec{r}} g_{\vec{r}+\hat{\lambda}} f_{\vec{r}} = -\sum_{\vec{r}} (\Delta_{\lambda} g_{\vec{r}}) f_{\vec{r}}$ in the above exponent produces

$$\begin{aligned} & -\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left[2\pi \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right) - ipqa e^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right] D_{\vec{r}, \vec{r}'}^{(m^2)} \Bigg]_{\lambda, \vec{r}'} - \\ & -iqa^2 e^2 |\psi|^2 \sum_{\vec{r}, \vec{r}', \lambda} \left[2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqa e^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right] C_{\vec{r}'} \underbrace{\Delta_{(\vec{r}')}_{\lambda} D_{\vec{r}, \vec{r}'}^{(m^2)}}_{-\Delta_{(\vec{r})\lambda} D_{\vec{r}-\hat{\lambda}, \vec{r}'}^{(m^2)}} + \\ & + \frac{ae^2 \eta}{2} m^2 \sum_{\vec{r}, \vec{r}'} C_{\vec{r}} \sum_{\lambda} \Delta_{(\vec{r})\lambda} \Delta_{(\vec{r}')\lambda} D_{\vec{r}, \vec{r}'}^{(m^2)}. \end{aligned} \quad (C.400)$$

In the previous relation, the property $\Delta_{(\vec{r}')\lambda} D_{\vec{r}, \vec{r}'}^{(m^2)} = -\Delta_{(\vec{r})\lambda} D_{\vec{r}-\hat{\lambda}, \vec{r}'}^{(m^2)}$ has been used, property which can be easily shown by noting that $\Delta_{(\vec{r}')\lambda} D_{\vec{r}, \vec{r}'}^{(m^2)} = D_{\vec{r}, \vec{r}'+\hat{\lambda}}^{(m^2)} - D_{\vec{r}, \vec{r}'}^{(m^2)}$ and that $D_{\vec{r}, \vec{r}'+\hat{\lambda}}^{(m^2)} = D_{\vec{r}-\hat{\lambda}, \vec{r}'}^{(m^2)}$ since the Green's function depends on distance $|\vec{r}' + \hat{\lambda} - \vec{r}|$ only. Using again the summation by parts, the second line of (C.400) takes the form

$$\begin{aligned} & -iqa^2 e^2 |\psi|^2 \sum_{\vec{r}, \vec{r}'} C_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(m^2)} \left[2\pi \sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} + 2\pi q \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}} - ipqa e^2 \eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}} \right] = \\ & = -iq^2 a^2 e^2 |\psi|^2 \sum_{\vec{r}, \vec{r}'} C_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(m^2)} \left[2\pi G_{\vec{r}} - ipa e^2 \eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}} \right], \end{aligned} \quad (C.401)$$

where equations (C.387) and (C.388) have been used.

It only remains the integral over $dC_{\vec{r}}$, which is a bit involved: the term quadratic in $C_{\vec{r}}$ is given by

$$-\frac{ae^2 \eta}{2} \sum_{\vec{r}} C_{\vec{r}} \left(-\sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} \right) + \frac{ae^2 \eta}{2} m^2 \sum_{\vec{r}, \vec{r}', \lambda} C_{\vec{r}} C_{\vec{r}'} \Delta_{(\vec{r})\lambda} \Delta_{(\vec{r}')\lambda} D_{\vec{r}, \vec{r}'}^{(m^2)}, \quad (C.402)$$

whereas the lineal terms are the following

$$i2\pi \sum_{\vec{r}} C_{\vec{r}} G_{\vec{r}} + pa e^2 \eta \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} C_{\vec{r}} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu, \vec{r}} +$$

$$\begin{aligned}
& -im^2 \sum_{\vec{r}, \vec{r}'} C_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(m^2)} \left[2\pi G_{\vec{r}} - ipae^2 \eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda L_{\mu\nu, \vec{r}} \right] = \\
& = i \sum_{\vec{r}, \vec{r}'} C_{\vec{r}} \left(\delta_{\vec{r}, \vec{r}'} - m^2 D_{\vec{r}, \vec{r}'}^{(m^2)} \right) \left[2\pi G_{\vec{r}'} - ipae^2 \eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta'_\lambda L_{\mu\nu, \vec{r}'} \right]. \quad (C.403)
\end{aligned}$$

Let $\bar{D}_{\vec{x}, \vec{x}'}^{(m^2)}$ be the Green's function of the discrete differential operator (C.402). According to definition (C.396) of $D_{\vec{r}, \vec{r}'}^{(m^2)}$, we obtain

$$\Delta_{(\vec{x})} \Delta_{(\vec{x}')} \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} - m^2 \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} + m^4 \sum_{\vec{r}'} D_{\vec{r}', \vec{x}'}^{(m^2)} \bar{D}_{\vec{r}', \vec{x}'}^{(m^2)} = \delta_{\vec{x}, \vec{x}'}, \quad (C.404)$$

where, since $\bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} = \bar{D}_{\vec{x}', \vec{x}}^{(m^2)}$, the term $\sum_{\vec{r}'} D_{\vec{r}', \vec{x}'}^{(m^2)} \bar{D}_{\vec{r}', \vec{x}'}^{(m^2)}$ in the previous equation can also be written as $\sum_{\vec{r}'} \bar{D}_{\vec{x}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{x}}^{(m^2)}$. Before continuing with the integration over $dC_{\vec{r}}$, we shall prove in the next Note the following useful relation

$$\Pi_{\vec{x}, \vec{x}'} = \sum_{\vec{r}, \vec{r}'} \left(\delta_{\vec{x}, \vec{r}} - m^2 D_{\vec{x}, \vec{r}}^{(m^2)} \right) \bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} \left(\delta_{\vec{r}', \vec{x}'} - m^2 D_{\vec{r}', \vec{x}'}^{(m^2)} \right) = D_{\vec{x}, \vec{x}'}^{(m^2)}. \quad (C.405)$$

Thus the integral over $C_{\vec{r}}$, which contains the lineal term (C.403), produces thus

$$\sum_{\vec{x}, \vec{x}'} \sum_{\vec{r}, \vec{r}'} \left[\left[\vec{x} \left(\delta_{\vec{x}, \vec{r}} - m^2 D_{\vec{x}, \vec{r}}^{(m^2)} \right) \bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} \left(\delta_{\vec{r}', \vec{x}'} - m^2 D_{\vec{r}', \vec{x}'}^{(m^2)} \right) \right] \right]_{\vec{x}'} = \left[\left[\vec{x} D_{\vec{x}, \vec{x}'}^{(m^2)} \right] \right]_{\vec{x}'}.$$

Note. Calculation of $\Pi_{\vec{x}, \vec{x}'}$.

We have to show

$$\sum_{\lambda} \Delta_{(\vec{x})} \Delta_{(\vec{x}')} \Pi_{\vec{x}, \vec{x}'} + m^2 \Pi_{\vec{x}, \vec{x}'} = \delta_{\vec{x}, \vec{x}'}, \quad (C.406)$$

which can be carried out using the summation by parts $\sum_{\vec{r}} g_{\vec{r}} \Delta_\lambda f_{\vec{r}-\hat{\lambda}} = -\sum_{\vec{r}} (\Delta_\lambda g_{\vec{r}}) f_{\vec{r}}$, the property $\Delta_{(\vec{r}')}^\lambda D_{\vec{r}, \vec{r}'}^{(m^2)} = -\Delta_{(\vec{r})}^\lambda D_{\vec{r}-\hat{\lambda}, \vec{r}'}^{(m^2)}$ (for both propagators $D_{\vec{r}, \vec{r}'}^{(m^2)}$ and $\bar{D}_{\vec{r}, \vec{r}'}^{(m^2)}$) and the definition of $D_{\vec{x}, \vec{r}}^{(m^2)}$: $\Delta_{(\vec{x})} \Delta_{(\vec{r})} D_{\vec{x}, \vec{r}}^{(m^2)} = \delta_{\vec{x}, \vec{r}} - m^2 D_{\vec{x}, \vec{r}}^{(m^2)}$.

$$\begin{aligned}
\sum_{\lambda} \Delta_{(\vec{x})} \Delta_{(\vec{x}')} \Pi_{\vec{x}, \vec{x}'} &= \sum_{\lambda} \Delta_{(\vec{x})} \Delta_{(\vec{x}')} \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} - m^2 \sum_{\vec{r}} \sum_{\lambda} \Delta_{(\vec{x})}^\lambda D_{\vec{x}, \vec{r}}^{(m^2)} \underbrace{\Delta_{(\vec{x}')}^\lambda \bar{D}_{\vec{r}, \vec{x}'}^{(m^2)}}_{-\Delta_{(\vec{r})}^\lambda \bar{D}_{\vec{r}-\hat{\lambda}, \vec{x}'}^{(m^2)}} - \\
&\quad \sum_{\vec{r}} \left(\sum_{\lambda} \Delta_{(\vec{x})}^\lambda \Delta_{(\vec{r})}^\lambda D_{\vec{x}, \vec{r}}^{(m^2)} \right) \bar{D}_{\vec{r}, \vec{x}'}^{(m^2)} \\
&= -m^2 \sum_{\vec{r}'} \bar{D}_{\vec{x}, \vec{r}'}^{(m^2)} \left(\delta_{\vec{r}', \vec{x}'} - m^2 D_{\vec{r}', \vec{x}'}^{(m^2)} \right) + m^4 \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} \left(\Delta_{(\vec{x})}^\lambda D_{\vec{x}, \vec{r}}^{(m^2)} \right) \underbrace{\bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} \left(\Delta_{(\vec{x}')}^\lambda D_{\vec{r}', \vec{x}'}^{(m^2)} \right)}_{-\Delta_{(\vec{r})}^\lambda \bar{D}_{\vec{r}-\hat{\lambda}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{x}'}^{(m^2)}} = \\
&\quad + \sum_{\vec{r}, \vec{r}'} \left(\sum_{\lambda} \Delta_{(\vec{x})}^\lambda \Delta_{(\vec{r})}^\lambda D_{\vec{x}, \vec{r}}^{(m^2)} \right) \bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{x}'}^{(m^2)}
\end{aligned}$$

$$= \sum_{\lambda} \Delta_{(\vec{x})}^{\lambda} \Delta_{(\vec{x}')}^{\lambda} \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} - 2m^2 \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} + 3m^4 \sum_{\vec{r}} D_{\vec{x}, \vec{r}}^{(m^2)} \bar{D}_{\vec{r}, \vec{x}'}^{(m^2)} - m^6 \sum_{\vec{r}, \vec{r}'} D_{\vec{x}, \vec{r}}^{(m^2)} \bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{x}'}^{(m^2)}. \quad (\text{C.407})$$

On the other side

$$m^2 \Pi_{\vec{x}, \vec{x}'} = m^2 \bar{D}_{\vec{x}, \vec{x}'}^{(m^2)} - 2m^4 \sum_{\vec{r}} D_{\vec{x}, \vec{r}}^{(m^2)} \bar{D}_{\vec{r}, \vec{x}'}^{(m^2)} + m^6 \sum_{\vec{r}, \vec{r}'} D_{\vec{x}, \vec{r}}^{(m^2)} \bar{D}_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{x}'}^{(m^2)}. \quad (\text{C.408})$$

Looking at the equation (C.404) which defines $\bar{D}_{\vec{x}, \vec{x}'}^{(m^2)}$, we obtain the desired result (C.406).

As explained, the integration over $dC_{\vec{r}}$ produces in the numerator's exponent

$$\sum_{\vec{x}, \vec{x}'} \left(2\pi G_{\vec{x}} - ipae^2 \eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{x})}^{\lambda} L_{\mu\nu, \vec{x}} \right) D_{\vec{x}, \vec{x}'}^{(m^2)} \left(2\pi G_{\vec{x}'} - ipae^2 \eta \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{x}')}^{\lambda'} L_{\mu'\nu', \vec{x}'} \right), \quad (\text{C.409})$$

similar to (C.275) in the pure gauge model, although here the propagator is short ranged. The partition function is given by

$$\begin{aligned} \mathcal{Z} = \mathcal{Z}_0 \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \sum_{\{G_{\vec{r}}\}} \times \\ \times \left\{ \begin{aligned} & \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(M_{\lambda, \vec{r}'} + q \sum_{\mu' < \nu'} \epsilon_{\lambda\mu'\nu'} g_{\mu'\nu', \vec{r}'} \right) \right] \\ & \exp \left[-\frac{2\pi^2}{ae^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} G_{\vec{r}'} \right] \end{aligned} \right\} \end{aligned} \quad (\text{C.410})$$

where, considering (C.399) and (C.404), $\mathcal{Z}_0$ is equal to

$$\begin{aligned} \mathcal{Z}_0 = \left(\sqrt{\frac{ae^2 \eta}{2\pi}} \right)^{(3-1)|\Lambda|} \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{|\Lambda|} \frac{\det(-\Delta \Delta)}{\left(\sqrt{\det(-\Delta \Delta + m^2)} \right)^3} \times \\ \times \frac{1}{\sqrt{\det \left(-\Delta \Delta - m^2 + \frac{m^4}{-\Delta \Delta + m^2} \right)}} = \\ = \left(\sqrt{\frac{ae^2 \eta}{2\pi}} \right)^{2|\Lambda|} \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{|\Lambda|} \frac{1}{\det(-\Delta \Delta + m^2)}. \quad (\text{C.411}) \end{aligned}$$

This partition is that resulting from the fields without topological defects in (C.393) ³⁶.

³⁶Thus, after including factor 2π into the integration variables of (C.393), one obtains

$$\mathcal{Z}_0 = \prod_{\vec{r}, \mu} \int_{-\infty}^{\infty} d\Gamma_{\mu, \vec{r}} \prod_{\vec{r}} \int_{-\infty}^{\infty} d\gamma_{\vec{r}} \exp \left[-\frac{2\pi^2}{ae^2 \eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} \Gamma_{\nu, \vec{r}} - \Delta_{\nu} \Gamma_{\mu, \vec{r}})^2 - 2\pi^2 a |\psi|^2 \sum_{\vec{r}, \mu} (\Delta_{\mu} \gamma_{\vec{r}} - q \Gamma_{\mu, \vec{r}})^2 \right],$$

The correlation function $W(C)$, for its part, takes the following form

$$W(C) = \frac{\mathcal{Z}_0}{\mathcal{Z}} \exp \left[-\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 \right] \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \sum_{\{G_{\vec{r}}\}} \times \quad (C.412)$$

$$\times \left\{ \begin{aligned} & \exp \left[-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left(2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqa e^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\begin{pmatrix} \\ \end{pmatrix} \right)_{\lambda, \vec{r}'} \right] \\ & \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \vec{r}'} \left(2\pi G_{\vec{r}} - ipae^2\eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \underline{\Delta_{\lambda} L_{\mu\nu, \vec{r}}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\begin{pmatrix} \\ \end{pmatrix} \right)_{\vec{r}'} \right] \end{aligned} \right\}$$

The exponent's summand arising from the quadratic power of the underlined term

$$\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \lambda'} \sum_{\mu < \nu, \mu' < \nu'} \epsilon_{\lambda\mu\nu} (\Delta_{\vec{r}} L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'}^{(m^2)} \epsilon_{\lambda'\mu'\nu'} \Delta_{\vec{r}'} L_{\mu'\nu', \vec{r}'},$$

can, by repeating the argumentation leading to (C.280), be rewritten as follows

$$\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu < \nu} L_{\mu\nu, \vec{r}} \left(-\Delta \Delta D_{\vec{r}, \vec{r}'}^{(m^2)} \right) L_{\mu\nu, \vec{r}'} - \sum_{\vec{r}, \vec{r}'} \sum_{\mu, \mu'} \delta_{\mu\mu'} \underbrace{\left(\sum_{\nu} \Delta_{\nu} L_{\mu\nu, \vec{r}-\hat{\nu}} \right)}_{J_{\mu, \vec{r}}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu', \vec{r}'}.$$

Hence, considering this result and relation $-\Delta \Delta D_{\vec{r}, \vec{r}'}^{(m^2)} = \delta_{\vec{r}, \vec{r}'} - m^2 D_{\vec{r}, \vec{r}'}^{(m^2)}$, the correlation function can be further simplified in the terms containing quadratic powers in $L_{\mu\nu, \vec{r}}$

$$\begin{aligned} & -\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 + \frac{p^2 a e^2 \eta}{2} m^2 \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} \sum_{\mu < \nu, \mu' < \nu'} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \epsilon_{\lambda'\mu'\nu'} L_{\mu'\nu', \vec{r}'} + \\ & + \frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \lambda'} \sum_{\mu < \nu, \mu' < \nu'} \epsilon_{\lambda\mu\nu} (\Delta_{\vec{r}} L_{\mu\nu, \vec{r}}) D_{\vec{r}, \vec{r}'}^{(m^2)} \epsilon_{\lambda'\mu'\nu'} \Delta_{\vec{r}'} L_{\mu'\nu', \vec{r}'} = \\ & = -\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} . \end{aligned} \quad (C.413)$$

where we include the gauge fixing $\det(-\Delta \Delta) \left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|} \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}} \left(\sum_{\mu} \Delta_{\mu} \Gamma_{\mu, \vec{r}} \right)^2 \right]$. The integration over $d\Gamma_{\mu, \vec{r}}$ produces factor $\left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{3|\Lambda|} 1 / \left(\sqrt{\det(-\Delta \Delta + m^2)} \right)^3$ outside the integral and term $-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} \gamma_{\vec{r}} \left(-m^2 \delta_{\vec{r}, \vec{r}'} + m^4 D_{\vec{r}, \vec{r}'}^{(m^2)} \right) \gamma_{\vec{r}'}$ in the integrand's exponent. On the other hand, the $d\gamma_{\vec{r}}$ integration gives $\left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}} \right)^{|\Lambda|} 1 / \sqrt{\det \left(-\Delta \Delta - m^2 + \frac{m^4}{-\Delta \Delta + m^2} \right)}$, which together with the previous result and the gauge fixing, results in equation (C.411).

$W(C)$ thus simplified, is equal to

$$\begin{aligned}
W(C) = & \frac{\mathcal{Z}_0}{\mathcal{Z}} \exp \left[-\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} \right] \sum_{\{M_{\lambda, \vec{r}}, G_{\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} V_{\lambda, \vec{r}} - q G_{\vec{r}} \right) \times \\
& \times \exp \left[i 2\pi \frac{p}{q} m^2 \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} V_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \sum_{\mu < \nu} \epsilon_{\lambda \mu \nu} L_{\mu \nu, \vec{r}'} + i 2\pi p \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda \mu \nu} \Delta_{(\vec{r}')}^{\lambda} L_{\mu \nu, \vec{r}'} \right] \times \\
& \times \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} V_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} V_{\lambda, \vec{r}'} - \frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} G_{\vec{r}'} \right], \quad (\text{C.414})
\end{aligned}$$

with $V_{\lambda, \vec{r}}$ the total vorticity: $V_{\lambda, \vec{r}} = M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda \mu \nu} g_{\mu \nu, \vec{r}}$ with $\sum_{\lambda} \Delta_{\lambda} V_{\lambda, \vec{r}} = q G_{\vec{r}}$. The partition function is given by (C.414) with $J_{\mu, \vec{r}} = 0 = L_{\mu \nu, \vec{r}}$. As expected, the partition is formed by the interactions between closed and open vortices/dislocations, described by $V_{\lambda, \vec{r}}$, and by the interactions between the monopoles at the end of the open vortices. This interaction is carried by the short-range $D_{\vec{r}, \vec{r}'}^{(m^2)}$.

Let us now prove that first correlation's term $\exp \left[-\frac{p^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} \right]$ decays like the loop perimeter. This can be easily obtained by approximating short-range propagator $D_{\vec{r}, \vec{r}'}^{(m^2)}$ by the infinite-range $1/4\pi |\vec{r} - \vec{r}'|$ for distances smaller than range m^{-1} and by zero outside this range. Thus, for a large path C with $\tau \gg R \gg m^{-1}$, equation (C.293) changes into

$$\begin{aligned}
I_R^{(m^2)} &= 2 \int_a^R d\xi_1 \int_{\xi_1 - m^{-1}}^{\xi_1 - a} d\xi_2 \frac{1}{4\pi (\xi_1 - \xi_2)} = \frac{1}{2\pi} R \ln \frac{m^{-1}}{a}, \\
I_{\tau\tau}^{(m^2)} &= - \int_0^\tau d\xi_1 \int_{\xi_1 - m^{-1}}^{\xi_1 + m^{-1}} d\xi_2 \frac{1}{4\pi \sqrt{R^2 + (\xi_1 - \xi_2)^2}} = -\frac{1}{4\pi} \tau \ln \left(\frac{\sqrt{R^2 + m^{-2}} + m^{-1}}{\sqrt{R^2 + m^{-2}} - m^{-1}} \right),
\end{aligned}$$

and therefore

$$\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} = 2 \left(I_R^{(m^2)} + I_{\tau}^{(m^2)} + I_{\tau\tau}^{(m^2)} + I_{RR}^{(m^2)} \right) \approx \frac{1}{2\pi} \mathcal{P} \ln \frac{m^{-1}}{a}, \quad (\text{C.415})$$

with $\mathcal{P} = 2\tau + 2R$ the loop's perimeter. Note that the propagator's short range makes term $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'}$ be proportional only to the perimeter of C , as different from the 3-dim. pure gauge model (C.293) where $\tau \ln R$ occurred, stronger than the perimeter.

The correlation function which we shall use in the discussion of the model phases, is given by

$$\begin{aligned}
W(C) = & \exp \left[-\frac{p^2 a e^2 \eta}{4\pi} \mathcal{P} \ln \frac{m^{-1}}{a} \right] \times \\
& \times \left\langle \exp \left[i 2\pi \frac{p}{q} m^2 \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} V_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \sum_{\mu < \nu} \epsilon_{\lambda \rho \sigma} L_{\rho \sigma, \vec{r}'} + i 2\pi p \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda \mu \nu} \Delta_{(\vec{r}')}^{\lambda} L_{\mu \nu, \vec{r}'} \right] \right\rangle_{V \text{ and } G}
\end{aligned} \quad (\text{C.416})$$

with the expectation value $\langle \rangle_{V \text{ and } G}$ evaluated by means of the following partition function

$$\begin{aligned} \mathcal{Z}_{V \text{ and } G} = & \sum_{\{M_{\lambda, \vec{r}}, G_{\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} V_{\lambda, \vec{r}} - q G_{\vec{r}} \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} V_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} V_{\lambda, \vec{r}'} \right] \times \\ & \times \exp \left[-\frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} G_{\vec{r}'} \right]. \end{aligned} \quad (\text{C.417})$$

The question now is whether the topological objects of $\mathcal{Z}_{V \text{ and } G}$ can or not produce the more stronger decay with the loop area.

Before we evaluate the correlation function $W(C)$ for the different ranges of $a e^2 \eta$ and $a |\psi|^2 / \eta$, let us discuss qualitatively by means of the estimation contour method the results we have to expect (see [22] and [60]). As already known, a vortex is a topological line object characterized by $\sum_{\text{path } C'} A_{\mu, \vec{r}} = \frac{2\pi}{q}$ when a closed path C' is covered around the vortex³⁷. Each time a vortex (or a dislocation if we are studying the model of Chapter 2) crosses the interior of path C , $\sum_{\text{loop } C} A_{\mu, \vec{r}}$ changes by an amount of $2\pi/q$. In the case where there are many vortices, such that we can neglect their mutual interactions, the expectation value $W(C)$ can be estimated by the relation

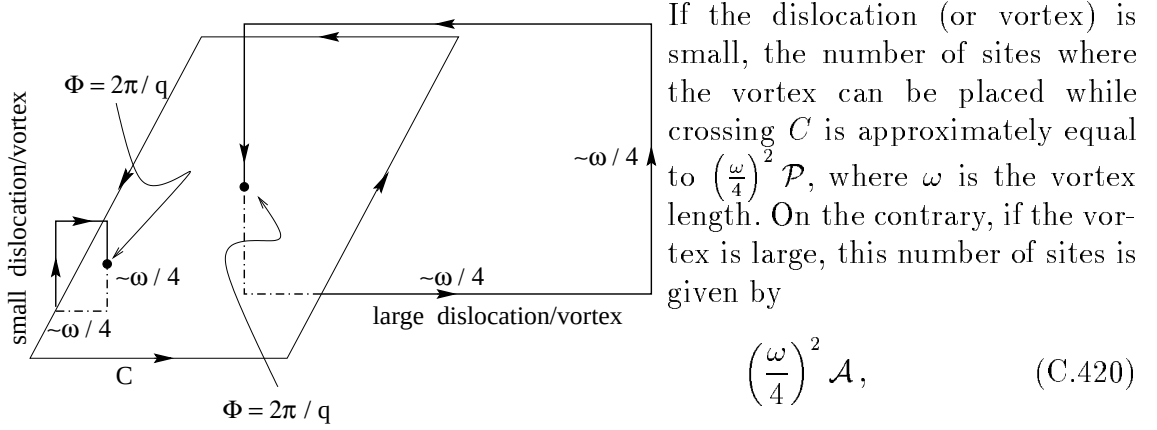
$$W(C) \approx \frac{1}{\mathcal{Z}'} \sum_{k \text{ vortices}} \frac{1}{k!} \Sigma_k \zeta^k, \quad (\text{C.418})$$

with Σ_k all possible values of $e^{ip \sum_{\text{loop } C} A_{\mu, \vec{r}}}$ due to all possible distributions of the k vortices

$$\Sigma_k = \sum_{\substack{\text{all distributions} \\ \text{of } k \text{ dislocations}}} \left(e^{ip \sum_{\text{loop } C} A_{\mu, \vec{r}}} \right)_{\text{each distribution}}. \quad (\text{C.419})$$

If $p = q$, the contribution to $W(C)$'s numerator made by a crossing vortex/dislocation is equal to $e^{i2\pi} = 1$, not different from the contributions to the partition function of the denominator. Only for the case $p \neq q$, the contribution of a certain vortex configuration to the correlation's numerator differs from that of the denominator. As it is typical in this kind of argument, the number of lattice sites for which this different contribution can be maintained, is very relevant.

³⁷This condition arises either from the formulation that the covariant derivative must vanish at large distances for a configuration where $\varphi_{\vec{r}}$ presents a jump of 2π . Or if one prefers to absorb φ into A_{μ} by means of the singular gauge transformation $q\chi_{\vec{r}} = \varphi_{\vec{r}}$, the condition is due to the flux carried by the line singularity which has been created by the gauge transformation. One should remember that in a model with $q \neq 1$, q units of elementary flux are not topologically stable since they coincide with the magnetic charge of a monopole, as for instance in the “ $SO(3)$ ”-superconductor where $q = 2$.



with $\mathcal{A}$ the area enclosed by C . For the case of two large vortices, the possible arrangements are:

- Both vortices lie completely outside C , with the number of possible sites to place both vortices being equal to $\left(|\Lambda| - \frac{\omega^2}{16}\mathcal{A}\right)^2$ where $|\Lambda|$ is the total number of lattice sites.
- One of the vortices surrounds C and thus the number of sites for this configuration is given by $2\left(|\Lambda| - \frac{\omega^2}{16}\mathcal{A}\right)\left(\frac{\omega^2}{16}\mathcal{A}\right)$, where the factor 2 arises from the two different cases where the first or the second of the vortices passes through the enclosed area. Since the vortex is an oriented object, half of these contributions are equal to $e^{i\frac{p}{q}2\pi}$ and the other half to $e^{-i\frac{p}{q}2\pi}$.
- Finally, if both vortices surround C , the number of sites to be placed is $\left(\frac{\omega^2}{16}\mathcal{A}\right)^2$, and the contributions are $e^{i\frac{p}{q}4\pi}$ or $e^{-i\frac{p}{q}4\pi}$ if both vortices have the same orientation, and 1 for the other cases.

Therefore, the value of $\Sigma_{k=2}$ for two vortices is given by

$$\left(|\Lambda| - \frac{\omega^2}{16}\mathcal{A}\right)^2 e^{i0} + 2\left(|\Lambda| - \frac{\omega^2}{16}\mathcal{A}\right)\left(\frac{\omega^2}{16}\mathcal{A}\right)\frac{e^{i\frac{p}{q}2\pi} + e^{-i\frac{p}{q}2\pi}}{2} + \left(\frac{\omega^2}{16}\mathcal{A}\right)^2\frac{e^{i\frac{p}{q}4\pi} + e^{-i\frac{p}{q}4\pi} + 2}{4},$$

$$\Sigma_2 = \left[|\Lambda| - \frac{\omega^2}{16}\mathcal{A}\left(1 - \cos 2\pi\frac{p}{q}\right)\right]^2. \quad (\text{C.421})$$

Thus the estimation for $W(C)$ for the case of many large vortices/dislocations looks as follows

$$W(C)_{\text{large vortices}} \approx \exp\left[-\frac{\omega^2}{16}\zeta\mathcal{A}\left(1 - \cos 2\pi\frac{p}{q}\right)\right], \quad (\text{C.422})$$

where ζ contains the information about the energy cost of a large vortex. If there are only small vortices/dislocations, the result is similar but with the enclosed area $\mathcal{A}$ replaced by the perimeter $\mathcal{P}$

$$W(C)_{\text{small vortices}} \approx \exp\left[-\frac{\omega^2}{16}\zeta\mathcal{P}\left(1 - \cos 2\pi\frac{p}{q}\right)\right]. \quad (\text{C.423})$$

Note that if $p = q$ the above estimation results 1³⁸. From here on, we shall concentrate ourselves in $p = 1$ and study the cases $q = 1$ ($p = q$) and $q \neq 1$ ($p \neq q$). In principle, to the Abelian $U(1)$ gauge field $A_{\mu,\vec{r}}$ any field could be coupled with charge q such that $p/q = 1/q$ were fractionary. But if $U(1)$ is compact this procedure is out: the possible q values are not arbitrary but fixed by the underlying gauge group in which $U(1)$ is embedded [41]. Let us return again to the compact superconductor of Appendix B: depending on the representation (fundamental or adjoint) of the non-Abelian scalar field Ψ which breaks the non-Abelian symmetry down to $U(1)$, the scalar field $\psi_{\vec{r}}$ (which represents the rotations about the fixed orientation of Ψ) has charge $q = 1$ or $q = 2$. The set of possible q values is fixed by the compact nature of $U(1)$.

The reader probably wonders why we need a compact $A_{\mu,\vec{r}}$ (and all the lengthy introduction in Appendix B to arrive at the “ $SO(3)$ ”-superconductor), since during the above estimation based on the contour method one has not used the fact of the compactness of the gauge field. Thus, one could expect that the above argumentation should apply also to the non-compact (ordinary) superconductor, in which therefore, a phase with an area decay for $W(C)$ should exist. However, this is not correct. The idea bases on Polyakov’s original argument of comparing the behaviour of $W(C)$ to the two-point correlation function in the scalar model [81] (see also [10]). Thus, let us return to the pure scalar model, where the energy periodicity (term $\cos \Delta_\mu \varphi$ or equivalently, the angular character of phase φ) allows for several states where all scalar fields are pointing in the same direction for a certain range of parameter $|\psi|^2/\eta$. A similar reasoning applies to the case of a discrete symmetry: for instance in the Ising model with an energy term $\sigma_{\vec{r}+\hat{\mu}}\sigma_{\vec{r}}$, where we obtain two possible ground states: all spins “up” or all spins “down”. The system is invariant under the global rotations of the scalar field in its internal space and thus all these states have the same energy. Depending on the value of $|\psi|^2/\eta$, this symmetry can or not be broken down spontaneously, i.e., if choosing one of the states where all the scalar fields are aligned in the same direction can or not be energetically favourable for the system. If the symmetry breakdown takes place, the system acquires a spontaneous magnetization characterized by a nonvanishing two-point correlation at large distances. On the contrary, if the symmetry is not broken (absence of spontaneous magnetization) the correlation decays exponentially with the distance³⁹. A broken symmetry is restored when, at certain values of the model’s parameters, there exist certain domain walls which without a large energy cost can connect one zone of the system, where all scalar fields are aligned, to another part of the system where all scalar fields are aligned in another quite different direction. The average of the scalar field’s orientation over the whole material becomes zero and there is no spontaneous magnetization. Of course, this argument is only qualitative, the exact details depend on the lattice dimensionality and on whether the symmetry is continuous or discrete; thus, in the 2-dimensional XY model instead of a spontaneous symmetry breakdown

³⁸One should bear in mind that the result obtained from the estimation contour method cannot reproduce the factor in front of the expectation value for the topological defects, factor given by $\exp\left[-\frac{\eta}{a|\psi|^2}D_{\vec{r}_2,\vec{r}_1}\right]$ for the XY model (C.219) or $\exp\left[-\frac{p^2ae^2\eta}{2}\sum_{\vec{r},\vec{r}'}\sum_{\mu}J_{\mu,\vec{r}}D_{\vec{r},\vec{r}'}^{(m^2)}J_{\mu,\vec{r}'}\right]$ for the Higgs model (C.413). The method only produces an estimation for the expectation value $\langle \rangle_{V \text{ and } G}$.

³⁹It is worth noting that the phase with symmetry breakdown corresponds to the ordered phase, whereas the absence of symmetry breakdown is the main characteristic of the disordered phase.

with a nonvanishing correlation, one obtains a decay with the distance weaker than in the disordered phase.

Let us apply this argument to our gauge model by starting with the pure $Z(2)$ gauge model of the previous section

$$E_{\text{gauge}} = \frac{1}{ae^2} \sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})] \quad \text{with} \quad A_{\mu, \vec{r}} = 0, \pi,$$

or equivalently

$$E_{\text{gauge}} = \frac{1}{ae^2} \sum_{\vec{r}} \sum_{\mu < \nu} [1 - S_{\mu, \vec{r}}^{\text{gauge}} S_{\nu, \vec{r} + \hat{\mu}}^{\text{gauge}} S_{\mu, \vec{r} + \hat{\nu}}^{\text{gauge}} S_{\nu, \vec{r}}^{\text{gauge}}] \quad \text{with} \quad S_{\mu, \vec{r}}^{\text{gauge}} = e^{iA_{\mu, \vec{r}}} = 1, -1.$$

At very low $e^2\eta$, the lowest-energy state is represented by the configuration with all gauge “arrows” $S_{\mu, \vec{r}}^{\text{gauge}}$ pointing in the same direction. Here “represented” means finding among the many gauge equivalent configurations a representative configuration where all arrows are oriented in the same direction. As in the pure scalar model, depending on the model’s parameter $e^2\eta$, the system can prefer to choose the state characterized by the representative where all gauge arrows are aligned, or to choose a configuration with randomly distributed gauge arrows (note that this last random configuration cannot be obtained by means of any gauge transformation from the aligned state). In the case where the choice has been the “representative” with the aligned arrows, the symmetry “up/down” is broken and correlation $W(C)$ shows the weaker decay like the perimeter of C (see equation (C.346)). If the symmetry is not broken, $W(C) \sim e^{-\text{Area}}$. The change from one phase to the other is due to the appearance of long lines of reversed arrows which causes the area decay in $W(C)$. The case of a continuous $U(1)$ symmetry is similar: here the existence of more possible orientations for the gauge arrow allows for a lesser energy cost for the disordering, i.e., avoids the symmetry breakdown in a more strongly way than a discrete symmetry.

Continuing with this discussion, let us consider the compact Higgs model. Since the model is invariant under any local transformation of the scalar field’s phase, one can choose to absorb $\varphi_{\vec{r}}$ into the gauge field by means of the following transformation [41]

$$\begin{cases} \varphi_{\vec{r}} & \rightarrow \varphi'_{\vec{r}} = 0, \\ A_{\mu, \vec{r}} & \rightarrow A'_{\mu, \vec{r}} = A_{\mu, \vec{r}} - \Delta_\mu \chi_{\vec{r}} \quad \text{with} \quad q\chi_{\vec{r}} = \varphi_{\vec{r}}, \end{cases} \quad (\text{C.424})$$

as it was done for the discussion of the topological configurations in the compact Higgs model (see comments previous to equation (C.390))⁴⁰. Working with this choice, which *removes* the gauge invariance, the partition function is given by

$$\begin{aligned} \mathcal{Z} = & \prod_{\vec{r}, \mu} \left(\int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})] \right] \times \\ & \times \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos qA_{\mu, \vec{r}}] \right]. \end{aligned} \quad (\text{C.425})$$

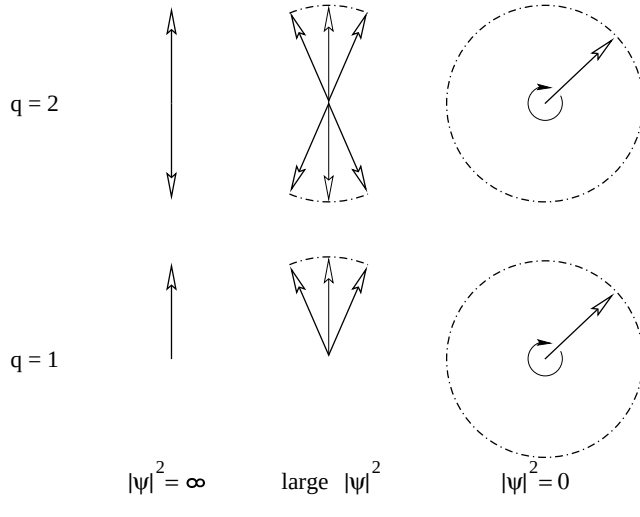
⁴⁰Removing *explicitly* (not spontaneously) the gauge invariance by means of the boundary condition, or gauge choice, $\varphi_{\vec{r}} = 0 \ \forall \vec{r}$ is called to work in the unitary gauge.

Limit $|\psi|^2 = \infty$ and the underlined term in the above partition function constrains the gauge field to take on, of its whole range $A_{\mu,\vec{r}} \in [0, 2\pi)$, only the following values

$$A_{\mu,\vec{r}} = \frac{2\pi}{q} n_{\mu,\vec{r}} \quad n_{\mu,\vec{r}} = 0, 1, \dots, (q-1) \quad \text{an integer valued field..} \quad (\text{C.426})$$

In this limit, the compact Higgs model has been reduced to the pure $Z(q)$ gauge model. As we have seen in the previous section, for small $e^2\eta$ and $q \neq 1$, the symmetry (always referred to the above representatives) is broken and $W(C)$ shows a decay like the perimeter; at large $e^2\eta$ the symmetry is restored and $W(C) \sim e^{-\text{Area}}$.

If we decrease $|\psi|^2$, orientations close to those of the limit $|\psi|^2 = \infty$ becomes available to $S_{\mu,\vec{r}}^{\text{gauge}}$. At $q > 1$, the configurations of reversed arrows in the $|\psi|^2 = \infty$ case which connect zones with nearly opposite arrow orientations occur also in this case (they are energetically even a bit more favourable since the arrow's orientations to be connected are a bit closer). Hence, when these “connecting configurations” become energetically possible, the broken symmetry can be restored. In the $q = 1$ case, the symmetry is completely broken since there are no opposite orientations which can allow a “connecting configuration” of flipped arrows.



This is the “footprint” of the trivial breakdown at $|\psi|^2 = \infty$, where in the representative configuration the gauge arrow only points upward. For $q = 1$, only when we reach the limit $|\psi|^2 = 0$, we can have a restoration of the broken symmetry and thus $W(C)$ can show the area decay. This is schematically represented in the figure. For the $q > 1$ case, nevertheless, the “connecting configurations” of flipped arrows can allow us to restore the broken symmetry before we reach the limit $|\psi|^2 = 0$.

The difference between the case $q \neq 1$ and the case $q = 1$ can be explained again by means of the compact superconductor, where the $U(1)$ symmetry arises, for instance, from the breakdown of the compact gauge symmetry described by the structure constants ϵ_{abc} . If it is $SU(2)$ (the largest group with these structure constants) the group which is broken down to $U(1)$ ($q = 1$ case), when $U(1)$ for its part is broken by scalar field ψ , it remains no symmetry left. On the other hand, if $SO(3) = SU(2)/Z(2)$ was broken down to $U(1)$ ($q = 2$ case) and then this $U(1)$ was also broken, discrete symmetry $Z(2)$ can remain intact; in this case, symmetry “up/down” exists and $W(C) \sim e^{-\text{Area}}$. If even this $Z(2)$ symmetry is broken, we have the same case as when $U(1)$ was embedded into $SU(2)$: the complete symmetry breakdown, and as we know, this is equivalent to the weaker decay of $W(C)$ like the perimeter.

Now let us consider the non-compact model, where the energy part depending on the gauge field is $E_{\text{gauge}} = \frac{1}{2ae^2} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})^2$. Here the energy contains no periodic function ($A_{\mu, \vec{r}}$ has lost its angular character) and there is *only* one lowest-energy state at small $e^2\eta$. Therefore, the symmetry is from the very beginning already broken and no area decay appears in this model.

A more direct proof of the absence of a phase with $W(C) \sim e^{-\text{Area}}$ can be obtained by considering partition function (C.417) for the non-compact case (all $g_{\mu\nu, \vec{r}}$ set equal to zero)

$$\mathcal{Z}_{\text{non-compact}} = \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0\right) \exp\left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'}\right], \quad (\text{C.427})$$

which is the partition function of the XY model but with a short-range propagator (due to the Meissner effect (C.390)). Let us consider our model for fixed $ae^2\eta$ and decrease $|\psi|^2$ from a very large value down to 0; at the limit $|\psi|^2 = 0$ we obtain the pure gauge model. By decreasing $|\psi|^2$, it becomes easier to have large vortices, as one can see by repeating the estimation of the critical value for the XY model (C.226): for $\frac{a|\psi|^2}{\eta} < \left(\frac{a|\psi|^2}{\eta}\right)_{\text{critical}}$ with $\left(\frac{a|\psi|^2}{\eta}\right)_{\text{critical}} \approx \frac{\ln 5}{2\pi^2 D^{(m^2)}(0)}$, large vortices dominates in the partition function. Thus, correlation $W(C)$ decays more strongly for small values of $|\psi|^2$ than for larger values; the strongest decay takes place in the pure gauge model. According to equation (C.293), nevertheless, $W(C)$ decays like $\tau \ln R$ in absence of monopoles (i.e., for the non-compact model), a decay weaker than the area decay of the compact model. Therefore, $W(C)$ never reaches the area law decay in the non-compact model. Later in this section (see equation (C.471)), we shall see in an explicit case how the area decay is cancelled out in the non-compact model.

It is finally time to study the different phases shown by the compact Higgs model. Let us start with the limit m^2 large, i.e., when the propagator's range is very small. In this limit, the propagator takes a very simple form

$$D_{\vec{r}, \vec{r}'}^{(m^2)} = \int_{-\pi}^{\pi} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{m^2 + 6 - 2 \sum_{\rho} \cos k_{\rho}} \approx \frac{1}{m^2} \delta_{\vec{r}, \vec{r}'}, \quad (\text{C.428})$$

and by considering (C.414), correlation function $W(C)$ with $p = 1$ becomes

$$\begin{aligned} W(C) &= \frac{\mathcal{Z}_0}{\mathcal{Z}} \exp\left[-\frac{\eta}{2a|\psi|^2} \frac{1}{q^2} \underbrace{\sum_{\vec{r}, \mu} J_{\mu, \vec{r}}^2}_{\mathcal{P}}\right] \sum_{\{M_{\lambda, \vec{r}}, G_{\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\lambda} \Delta_{\lambda} V_{\lambda, \vec{r}} - qG_{\vec{r}}\right) \times \\ &\times \exp\left[i \frac{2\pi}{q} \sum_{\vec{r}} \sum_{\lambda, \mu < \nu} M_{\lambda, \vec{r}} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}}\right] \times \end{aligned}$$

$$\begin{aligned}
& \times \exp \left[\underbrace{i 2\pi \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \sum_{\rho < \sigma} \epsilon_{\lambda\rho\sigma} L_{\rho\sigma, \vec{r}}}_{\text{integer}} + i \underbrace{\frac{2}{m^2} \sum_{\vec{r}} G_{\vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda L_{\mu\nu, \vec{r}}}_{\approx 0} \right] \times \\
& \times \exp \left[-\frac{2\pi^2}{a q^2 e^2 \eta} \sum_{\vec{r}, \lambda} \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right)^2 - \frac{2\pi^2}{a e^2 \eta} \frac{1}{m^2} \sum_{\vec{r}} G_{\vec{r}}^2 \right]. \quad (\text{C.429})
\end{aligned}$$

The complex exponent of the second and third lines is the responsible for contributions which can lead to an area law in the correlation's decay. It is worth noting that the effect due to monopoles disappears in the $m^2 \rightarrow \infty$ limit for the Higgs model, completely different from the pure gauge model, where just the monopoles caused the decay with the loop area. One could therefore think (again) that the compactness of the gauge model is unnecessary to obtain here an area law. However, as already noted, the compactness is “hidden” not only in the fact that q can take on only integer values (if the gauge group were not compact, the q values could be arbitrary), but also in the possibility of restoring the broken gauge symmetry.

In the above correlation function and since integer field $L_{\mu\nu, \vec{r}}$ differs from zero only at the plaquettes of the area enclosed by C , $\sum_{\vec{r}} \sum_{\lambda, \mu < \nu} M_{\lambda, \vec{r}} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}}$ produces a ± 1 each time a closed dislocation/vortex $M_{\lambda, \vec{r}}$ crosses this area. Of course, if $q = p = 1$, all these contribution are equal to 1 and from the correlation function only remains the perimeter term $\exp \left[-\frac{\eta}{2a|\psi|^2} \frac{1}{q^2} \mathcal{P} \right]$. For $q \neq 1$, a vortex configuration where the dislocation/vortex loop crosses the loop area while encircling C , produces a contribution to the correlation's numerator different from that in the denominator (if the dislocation loop does not encircle C , then it crosses twice the interior of C , once in the opposite direction to the other time and therefore, the contribution is 1). Depending on the dislocation size, factor $e^{i2\pi/q} \neq 1$ can be maintained for the whole area enclosed by C in the case of large dislocation loops, or only for a strip near to the border C if the dislocations are small. Hence, we can return to the estimations calculated previous to the discussion about the compactness of the model: relations (C.422) and (C.423) (with $p = 1$).

Term $\frac{2\pi^2}{a q^2 e^2 \eta} \sum_{\vec{r}, \lambda} \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right)^2$ forces, at very large $|\psi|^2$ and $e^2 \approx 0$ (with $m^2 = a^2 q^2 e^2 |\psi|^2$ large), that

$$M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \approx 0 \quad \implies \quad M_{\lambda, \vec{r}} \approx 0, \quad \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \approx 0,$$

since $M_{\lambda, \vec{r}}$ describes closed vortices whereas $\sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}}$ open vortices. The length $\omega \sim \sum_{\lambda, \vec{r}} M_{\lambda, \vec{r}}^2$ of the vortices/dislocations is very small and their contributions, which only produce a perimeter decay, can be neglected. Correlation function $W(C)$ is then approximated by the perimeter factor

$$W(C)_{\text{small } a e^2 \eta, \text{ large } a |\psi|^2 / \eta} \approx \exp \left[-\frac{\eta}{2a|\psi|^2} \frac{1}{q^2} \mathcal{P} \right]. \quad (\text{C.430})$$

If e^2 is increased (without leaving the case of large m^2), the dislocations become longer and eventually the number of large dislocations begins to explode: according to (C.422), the term with the area law decay dominates in the correlation function.

The description of the different $W(C)$ behaviours in the 3-dimensional compact Higgs model is completed by considering the following regions of the $a|\psi|^2/\eta - ae^2\eta$ diagram Fig. C.5. Phases along a line of large m^2 has been just discussed.

- The pure (compact) $U(1)$ gauge model: $|\psi|^2 = 0$.

In this case the scalar field's part disappears from the original partition function

$$\mathcal{Z} = \prod_{\vec{r}, \mu} \left(\int_{-\pi}^{\pi} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \times \\ \times \exp \left[- \frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] - \underbrace{\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - qA_{\mu, \vec{r}})]}_{0} \right],$$

obtaining thus the pure gauge model, where $W(C)$ in three dimensions always decays like the loop area.

- The XY model: $e^2 = 0$.

This limit can be directly obtained from equation (C.417): $1/e \rightarrow \infty$ forces $G_{\vec{r}} = 0 \forall \vec{r}$ (there are no monopoles) and so we obtain the partition function of the XY model $\mathcal{Z}_M = \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[- \frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'} M_{\lambda, \vec{r}'} \right]$. The (infinite-range) Coulomb propagator $D_{\vec{r}, \vec{r}'}$ fulfils $-\Delta \Delta D_{\vec{r}, \vec{r}'} = \delta_{\vec{r}, \vec{r}'}$. Further, integer field $g_{\mu\nu, \vec{r}}$ can be absorbed into $M_{\lambda, \vec{r}}$ since $G_{\vec{r}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}} = 0$ implies $\sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} = \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\mu} l_{\nu, \vec{r} - \hat{\mu} - \hat{\lambda}}$ and therefore

$$V_{\lambda, \vec{r}} = \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\mu} m_{\nu, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\mu} l_{\nu, \vec{r} - \hat{\mu} - \hat{\lambda}} \equiv \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\mu} m'_{\nu, \vec{r}} = M'_{\lambda, \vec{r}},$$

with $\sum_{\lambda} \Delta_{\lambda} M'_{\lambda, \vec{r}} = 0$. The calculation of $W(C)$ is specially simple by looking at (C.416): $e^2 = 0 = m^2$ and $G_{\vec{r}} = 0$ produces

$$W(C)_{XY \text{ model}} = 1. \quad (\text{C.431})$$

The same result can be obtained starting from the original partition function instead of the Villain approximation. Limit $e^2 = 0$ in the original partition forces gauge field $A_{\mu, \vec{r}}$ to take the following form ⁴¹

$$A_{\mu, \vec{r}} = \Delta_{\mu} \chi_{\vec{r}}, \quad (\text{C.432})$$

⁴¹The gauge field $A_{\mu, \vec{r}}$ is equal to 0 except for a gauge transformation:

$$\Delta_{\nu} \chi_{\vec{r} + \hat{\mu}} - \Delta_{\nu} \chi_{\vec{r}} - \Delta_{\mu} \chi_{\vec{r} + \hat{\nu}} + \Delta_{\mu} \chi_{\vec{r}} = (\chi_{\vec{r} + \hat{\mu} + \hat{\nu}} - \chi_{\vec{r} + \hat{\mu}}) - (\chi_{\vec{r} + \hat{\nu}} - \chi_{\vec{r}}) - (\chi_{\vec{r} + \hat{\nu} + \hat{\mu}} - \chi_{\vec{r} + \hat{\nu}}) + (\chi_{\vec{r} + \hat{\mu}} - \chi_{\vec{r}}) = 0.$$

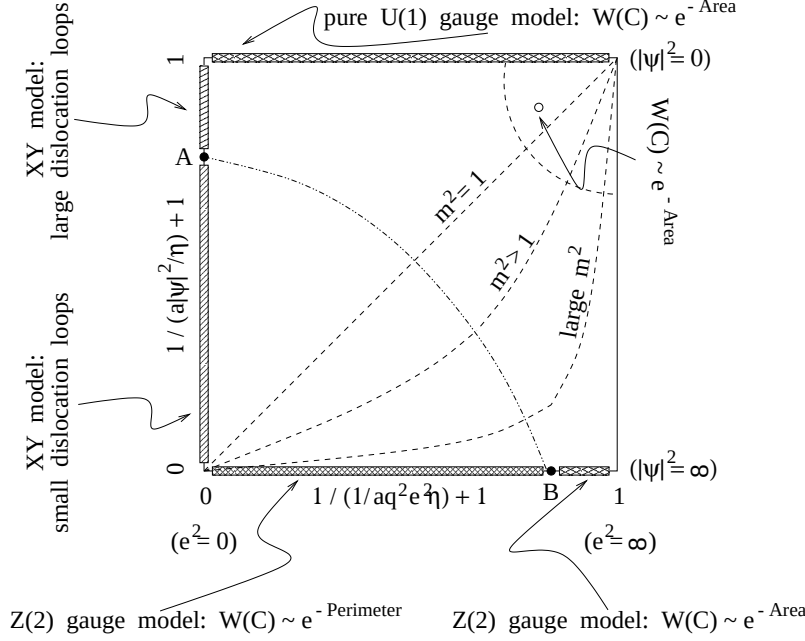


Figure C.5: $a|\psi|^2/\eta - ae^2\eta$ diagram for $q = 2$ and a 3-dimensional system. Axis $|\psi|^2 = 0$ corresponds to the (compact) pure $U(1)$ gauge model, axis $e^2 = 0$ to the XY model and axis $|\psi|^2 = \infty$ is the pure $Z(2)$ gauge model. If $q = p = 1$, the value of the correlation function in the $Z(2)$ gauge model is exactly 1. The diagonal line from the lower left corner to the upper right corner corresponds to $m^2 = a^2 q^2 e^2 |\psi|^2 = 1$. Point A is the critical value (or second-order transition point) of the 3-dimensional XY model: $(a|\psi|^2/\eta)_{\text{crit}} = 0.322$; point B denotes the critical value of the 3-dimensional $Z(2)$ gauge model: $(ae^2\eta)_{\text{crit}} = 1.314$.

where $\chi_{\vec{r}}$ presents no singularity. The phase $\chi_{\vec{r}}$ can be absorbed into $\varphi_{\vec{r}}$ as follows: let us call $\varphi_{\vec{r}}^{\pm} = \varphi_{\vec{r}} \pm q\chi_{\vec{r}}$, then the partition function only depends on $\varphi_{\vec{r}}^{\pm}$ and it is equal to the XY partition (up to a infinite constant $\mathcal{N}$ coming from the integration over $\varphi_{\vec{r}}^{-}$, which has to be removed)

$$\begin{aligned} \mathcal{Z} &= \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\chi_{\vec{r}}}{2\pi} \right) \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu}\varphi_{\vec{r}} - q\Delta\chi_{\vec{r}})] \right] = \\ &= \mathcal{N} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}^{\pm}}{2\pi} \right) \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} (1 - \cos \Delta_{\mu}\varphi_{\vec{r}}^{\pm}) \right]. \end{aligned}$$

The local (gauge) invariance exists no longer. The partition function remains invariant only under global $U(1)$ transformations as $\varphi_{\vec{r}}^{\pm} \rightarrow \varphi_{\vec{r}}^{\pm} - \chi_0$. As a result of

(C.432) and of the equation defining $J_{\mu,\vec{r}}, \sum_{\mu} \Delta_{\mu} J_{\mu,\vec{r}-\hat{\mu}} = 0$, $W(C)$ becomes

$$W(C) = \frac{\mathcal{N} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}^+}{2\pi} \right) \exp \left[i \sum_{\vec{r},\mu} J_{\mu,\vec{r}} \Delta_{\mu} \chi_{\vec{r}} \right] \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\mu} (1 - \cos \Delta_{\mu} \varphi_{\vec{r}}^+) \right]}{\mathcal{N} \prod_{\vec{r}} \left(\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}^+}{2\pi} \right) \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r},\mu} (1 - \cos \Delta_{\mu} \varphi_{\vec{r}}^+) \right]},$$

which is trivially 1.

The 3-dimensional XY model has two phases separated by a second-order phase-transition point: one phase for $a|\psi|^2/\eta > (a|\psi|^2/\eta)_{\text{crit}} = 0.322$ (C.226), where the global $U(1)$ symmetry is spontaneously broken and correlation function $\langle \psi_{\vec{r}_1} \psi_{\vec{r}_2}^* \rangle$ does not vanish at large distances. In this phase there are only small closed vortices/dislocations (open vortices are not allowed because no monopoles exist). And a disordered phase for $a|\psi|^2/\eta < (a|\psi|^2/\eta)_{\text{crit}}$ with no spontaneous symmetry breakdown and characterized by large closed dislocation loops.

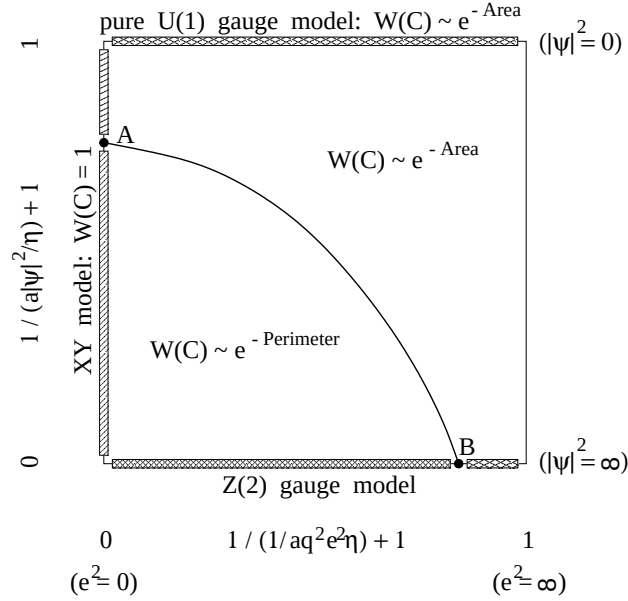


Figure C.6: Expected phase diagram for the 3-dimensional compact Higgs model with $q = 2$. Solid line AB is a line of second-order phase transitions.

- The pure $Z(q)$ gauge model: $|\psi|^2 = \infty$.

As already seen, limit $|\psi|^2 = \infty$ constrains gauge field $A_{\mu,\vec{r}}$ to take on values of a discrete set: $A_{\mu,\vec{r}} = 0, \frac{2\pi}{q}, \dots, (q-1)\frac{2\pi}{q}$. Thus, the compact Higgs model in this

limit becomes the pure $Z(q)$ gauge model. Let us continue the discussion for $q = 2$. From the end of the previous section, we know that the 3-dimensional $Z(2)$ gauge model shows a second-order phase-transition point at $(\beta_e)_{\text{crit}} = (ae^2\eta)_{\text{crit}}^{-1} = 0.716$ (C.364). Correlation $W(C) = \langle \prod_C e^{iA_\mu, \tau} \rangle$ decays with the perimeter of C for $ae^2\eta < 1.314$, whilst it shows an area law for $ae^2\eta > 1.314$. For larger values of q , the phase structure is the same but with critical point $(\beta_e)_{\text{crit}}$ shifted towards smaller values. In the 4-dimensional model, this region is a bit different since for q large enough, an intermediate phase appears, where $W(C)$ decays like the perimeter plus the term τ/R , but this will be seen later.

- The region of large e^2 and small $|\psi|^2$.

This is the diagram's region studied by means of the “high temperature” power expansion: equations (C.375) and (C.376). For $q = 1$ ($q = p$), $W(C)$ decays like the perimeter of C , whereas for $q \neq 1$, $W(C)$ shows an area law decay.

Summarizing, the 3-dimensional compact $U(1)$ Higgs model for $q = 2$ shows two phases. Its diagram is expected to be as in Fig. C.6 [35], [41], although the exact form cannot be calculated analytically. Both points A and point B denote a second-order phase-transition point. The nature of the transition (first-order or second-order) across the line connecting A and B will be discussed later, after studying the non-compact Higgs model (see (C.484)).

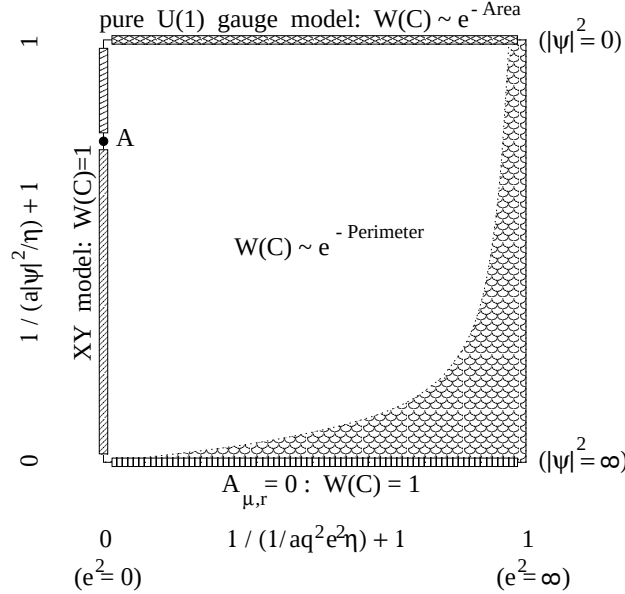


Figure C.7: Expected phase diagram for the 3-dimensional compact Higgs model with $q = p = 1$. Inside the shaded region, no transition occurs, unlike the $q \neq p = 1$ case.

If $q = p = 1$, the model shows only one phase characterized by the $W(C)$'s decay like the perimeter (except for the limit $|\psi|^2 = 0$), although this does not imply that the coefficient multiplying the perimeter must be same for all values of e^2 or $|\psi|^2$. Its

expected phase diagram is shown in Fig. C.7. As in the previous case, the whole diagram cannot be calculated; it can occur that a line of transitions begin at point A and ends at an interior point of the diagram. However, one can prove that not only along a line of large m^2 but inside a complete strip (the shaded region in the figure) the system shows no phase transition, unlike for $q \neq 1$ case. As we saw at the beginning of this long Appendix, a phase transition occurs when the free energy density $\mathcal{F}_\Lambda = -\eta/|\Lambda| \ln \mathcal{Z}_\Lambda$ ⁴² or its derivatives, which define correlation functions, becomes nonanalytic in the thermodynamic limit $|\Lambda| \rightarrow \infty$. Therefore the absence of a phase transition in the shaded strip must be proved in the limit of an infinite lattice. The rigorous proof of the analyticity of the free energy, and of any correlation derived from it, is a bit involved and will be given separately in the next Note; the main tool in it is a refinement of the “high temperature” expansion.

Note. Absence of a phase transition between the region of large e^2 , and the region of large $|\psi|^2$ and small e^2 when $q = 1$ ([41], [74] and [75]).

Let us consider a general *gauge invariant* function K which only depends on bonds belonging to plaquettes included in Q_0 . We shall call Q_0 the *support* of K . Let us study expectation value $\langle K \rangle_\Lambda$ (in a finite lattice Λ) and obtain an upper bound for it which is independent of the lattice size. This bound will lead us to the proof of the thermodynamic limit’s existence, $|\Lambda| \rightarrow \infty$, and of the analyticity of $\langle K \rangle$ for an infinite lattice and for a certain range (the shaded region of Fig. C.7) of the model’s parameters. An example of K is $\prod_C e^{iA_{\mu,\vec{r}}}$ which depends on the bonds along path C , whose support is any surface whose boundary is C .

Using the gauge invariance of the system, we shall choose to work in the unitary gauge, i.e., by absorbing the scalar field’s phase into the gauge field. Hence, the system’s energy depends only on $A_{\mu,\vec{r}}$

$$\exp \left[-\frac{E_\Lambda}{\eta} \right] = \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r},\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})] - \frac{a|\psi|^2}{\eta} \sum_{\vec{r},\mu} [1 - \cos A_{\mu,\vec{r}}] \right].$$

The second part of the energy does not contain any interaction between neighbour bond variables, what allows us to write the expectation value of K in a more manageable form

$$\langle K \rangle_\Lambda = \frac{1}{\mathcal{Z}_\Lambda} \int_{-\pi}^{\pi} \left[\prod_{\vec{r},\mu} \left(\frac{dA_{\mu,\vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu,\vec{r}}]} \right) \right] K \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r},\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})] \right].$$

In order to abbreviate the notation, we can define the following probability measure

$$d\mu = \frac{\prod_{\vec{r},\mu} \left(\frac{dA_{\mu,\vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu,\vec{r}}] \right] \right)}{\int_{-\pi}^{\pi} \prod_{\vec{r},\mu} \left(\frac{dA_{\mu,\vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu,\vec{r}}] \right] \right)}, \quad (\text{C.433})$$

which is the product of elements depending only on bonds. This measure fulfils $\int_{-\pi}^{\pi} d\mu = 1$, where the integration must be carried out over all $dA_{\mu,\vec{r}}$.

The energy’s part containing the interaction between neighbour bonds can be rewritten as follows

$$\sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})] = \frac{1}{2} \sum_{\vec{r}} \sum_{\mu,\nu} \left[1 - e^{iA_{\mu,\vec{r}}} e^{iA_{\nu,\vec{r}+\hat{\mu}}} e^{-iA_{\mu,\vec{r}+\hat{\nu}}} e^{-iA_{\nu,\vec{r}}} \right],$$

⁴²We have written again the index Λ to denote that we are working with a finite lattice.

and thus

$$\exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})] \right] = \prod_{\vec{r}} \prod_{\mu, \nu} [1 + \varrho_{\mu\nu, \vec{r}}] ,$$

with

$$\varrho_{\mu\nu, \vec{r}} = \exp \left[\frac{1}{2ae^2\eta} (e^{iA_{\mu, \vec{r}}} e^{iA_{\nu, \vec{r}+\hat{\mu}}} e^{-iA_{\mu, \vec{r}+\hat{\nu}}} e^{-iA_{\nu, \vec{r}}} - 1) \right] - 1 . \quad (\text{C.434})$$

Since $q = 1$, the limit $|\psi|^2 \rightarrow \infty$ breaks the symmetry completely by forcing $e^{iA_{\mu, \vec{r}}} = 1$. For $|\psi|^2$ large, configurations with $e^{iA_{\mu, \vec{r}}} \approx 1$ are strongly favoured and in this case $\varrho_{\mu\nu, \vec{r}}$ is small. On the other hand and according to (C.434), if e^2 is large $\varrho_{\mu\nu, \vec{r}}$ is also small. In both cases we can follow the same strategy as in the “high temperature” expansion: we expand any expectation value in powers of plaquette-factor $\varrho_{\mu\nu, \vec{r}}$, obtaining a sequence of summands which correspond to larger and larger blocks of plaquettes.

By using $\prod_{\vec{r}} \prod_{\mu, \nu} [1 + \varrho_{\mu\nu, \vec{r}}] \equiv \prod_{\text{plaquette } P \subset \Lambda} [1 + \varrho_P] = \sum_{Q \subset \Lambda} \prod_{P \in Q} \varrho_P$ where Q is any collection of plaquettes in lattice Λ , the expectation value of K in abbreviated notation is given by

$$\langle K \rangle_\Lambda = \sum_{Q \subset \Lambda} \frac{1}{\mathcal{Z}_\Lambda} \int_{-\pi}^{\pi} d\mu K \prod_{P \in Q} \varrho_P . \quad (\text{C.435})$$

In general, each Q is formed by the union of several connected parts

$$Q = Q_{\text{conn. } 1} \cup Q_{\text{conn. } 2} \cup \dots \cup Q_{\text{conn. } k} \cup Q_{\text{conn. } k+1} \cup \dots \cup Q_{\text{conn. } n} ,$$

where some of these connected components, let us suppose the first k components, have bonds in common with the support of K , whilst the rest have no bonds in common with Q_0

$$Q_{\text{common}} \equiv Q_{\text{conn. } 1} \cup Q_{\text{conn. } 2} \cup \dots \cup Q_{\text{conn. } k} \quad \text{and} \quad Q_{\text{not}} \equiv Q \setminus Q_{\text{common}} .$$

Hence, the product of plaquettes contained in $\langle K \rangle_\Lambda$ can be separated as follows

$$\langle K \rangle_\Lambda = \sum_{Q \subset \Lambda} \frac{1}{\mathcal{Z}_\Lambda} \int_{-\pi}^{\pi} d\mu \left(K \prod_{P \in Q_{\text{common}}} \varrho_P \right) \left(\prod_{P \in Q_{\text{not}}} \varrho_P \right) ,$$

and since integration measure $d\mu$ only depends on bond variables and the two factors above do not contain bonds in common, then

$$\langle K \rangle_\Lambda = \sum_{Q \subset \Lambda} \frac{1}{\mathcal{Z}_\Lambda} \left(\int_{-\pi}^{\pi} d\mu K \prod_{P \in Q_{\text{common}}} \varrho_P \right) \left(\int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{not}}} \varrho_P \right) \left(\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \left[\frac{dA_{\mu, \vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu, \vec{r}}]} \right] \right) .$$

Now we fix Q_{common} (or $Q_{\text{conn. } 1}, \dots, Q_{\text{conn. } k}$) and sum over all possible Q_{not} having no bonds in common with either support Q_0 or Q_{common} , obtaining thus

$$\sum_{Q_{\text{not}} \subset \Lambda} \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{not}}} \varrho_P = \frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \left[\frac{dA_{\mu, \vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu, \vec{r}}]} \right]} , \quad (\text{C.436})$$

where $\overline{Q_{\text{common}} \cup Q_0}$ denotes the set of all plaquettes which have bonds in common with the plaquettes in $Q_{\text{common}} \cup Q_0$. Therefore, the expectation value of K becomes

$$\langle K \rangle_\Lambda = \sum_{Q_{\text{common}} \subset \Lambda} \int_{-\pi}^{\pi} d\mu K \prod_{P \in Q_{\text{common}}} \varrho_P \frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{\mathcal{Z}_\Lambda} . \quad (\text{C.437})$$

This is called the “high temperature” cluster expansion. In order to show the analyticity of the above expression for an arbitrary K , one has to prove that the sum inside $\langle K \rangle_\Lambda$ converges absolutely and that this result is maintained in the thermodynamic limit. This will be carried out in two steps.

Step one

Firstly, we shall obtain an upper bound for $\langle K \rangle_\Lambda$ in the form of a geometric serie. Let us take the absolute value of equation (C.437)

$$|\langle K \rangle_\Lambda| \leq |K|_\infty \sum_{Q_{\text{common}} \subset \Lambda} \left\{ \left| \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} \varrho_P \right| \frac{Z_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{Z_\Lambda} \right\}, \quad (\text{C.438})$$

where K is assumed to be bounded and thus $|K|_\infty$ is a constant which does not depend on the lattice’s size or on the measure.

$$\text{In } \left| \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} \varrho_P \right| \leq \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} |\varrho_P| \text{ we have to re-group bond factors } e^{iA_{\mu,\vec{r}}} \text{ since mea-}$$

sure $d\mu$ only contains products of bonds variables, $d\mu \sim \prod_{\vec{r},\mu} \left(dA_{\mu,\vec{r}} e^{-\frac{a|\psi|^2}{\eta}[1-\cos A_{\mu,\vec{r}}]} \right)$. Each bond is contained in at most $b = 2(d-1)$ plaquettes and thus each bond-factor $e^{iA_{\mu,\vec{r}}}$ appears in at most b plaquette-factors ϱ_P :

$$\begin{aligned} \int_{-\pi}^{\pi} d\mu \prod_{P \in Q} |\varrho_P| &= \prod_{\text{all different bond}_i \in Q} \left[\frac{\int_{-\pi}^{\pi} dA_{\text{bond}_i} e^{-\frac{a|\psi|^2}{\eta}[1-\cos A_{\text{bond}_i}]} |\varrho_P|^{\text{number plaq. contain. bond}_i}}{\int_{-\pi}^{\pi} dA_{\text{bond}_i} e^{-\frac{a|\psi|^2}{\eta}[1-\cos A_{\text{bond}_i}]}} \right] \leq \\ &\quad \underbrace{\int_{-\pi}^{\pi} d\mu |\varrho_P|^{\text{number plaq. contain. bond}_i}}_{\leq \int_{-\pi}^{\pi} d\mu |\varrho_P|^b} \leq \int_{-\pi}^{\pi} d\mu |\varrho_P|^b \\ &\leq \prod_{P \in Q} \left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b}, \end{aligned}$$

and thus

$$\left| \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} \varrho_P \right| \leq \prod_{P \in Q_{\text{common}}} \left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b}. \quad (\text{C.439})$$

In $\left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b}$, let us write $\varrho_P = \exp \left[\frac{1}{2ae^2\eta} (e^{iA_{\mu,\vec{r}}} e^{iA_{\nu,\vec{r}+\vec{\mu}}} e^{-iA_{\mu,\vec{r}+\vec{\nu}}} e^{-iA_{\nu,\vec{r}}} - 1) \right] - 1$ as follwos

$\varrho_P = \int_0^1 S_P e^{tS_P} dt$ with $S_P = \frac{1}{2ae^2\eta} (e^{iA_{\mu,\vec{r}}} e^{iA_{\nu,\vec{r}+\vec{\mu}}} e^{-iA_{\mu,\vec{r}+\vec{\nu}}} e^{-iA_{\nu,\vec{r}}} - 1)$. The Schwarz’s inequality ap-

plied to this case, i.e. $\left[\sum_i |A_i B_i|^b \right]^{1/b} \leq \left[\sum_i |A_i A_i|^b \right]^{1/2b} \left[\sum_i |B_i B_i|^b \right]^{1/2b}$, produces

$$\begin{aligned} \left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b} &= \left[\int_{-\pi}^{\pi} d\mu \left| \int_0^1 S_P e^{tS_P} dt \right|^b \right]^{1/b} \leq \sup_{0 \leq t \leq 1} \left[\int_{-\pi}^{\pi} d\mu |S_P e^{tS_P}|^b \right]^{1/b} \stackrel{\text{Schwarz}}{\leq} \\ &\leq \left[\int_{-\pi}^{\pi} d\mu |S_P|^{2b} \right]^{1/2b} \sup_{0 \leq t \leq 1} \left[\int_{-\pi}^{\pi} d\mu |e^{tS_P}|^{2b} \right]^{1/2b} \leq \text{constant} \left[\int_{-\pi}^{\pi} d\mu |S_P|^{2b} \right]^{1/2b}, \quad (\text{C.440}) \end{aligned}$$

where the last inequality is due to $S_P \leq 0$. Let us work further on this expression. First of all, we shall obtain an upper bound for $|1 - e^{iA_{\mu,\vec{r}}} e^{iA'_{\mu',\vec{r}'}}|$ which will be applied to $|S_P|$

$$\left| 1 - e^{iA_{\mu,\vec{r}}} e^{iA'_{\mu',\vec{r}'}} \right| = \left| \left(1 - e^{iA_{\mu,\vec{r}}} \right) + \left(1 - e^{iA'_{\mu',\vec{r}'}} \right) - \left(1 - e^{iA_{\mu,\vec{r}}} \right) \left(1 - e^{iA'_{\mu',\vec{r}'}} \right) \right| \leq$$

$$\leq \left| 1 - e^{iA_{\mu,\vec{r}}} \right| + \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right| + \sqrt{\left(1 - e^{iA_{\mu,\vec{r}}} \right) \left(1 - e^{-iA_{\mu,\vec{r}}} \right) \left(1 - e^{iA'_{\mu',\vec{r}'}} \right) \left(1 - e^{-iA'_{\mu',\vec{r}'}} \right)}$$

and using $\left| \left(1 - e^{iA_{\mu,\vec{r}}} \right) \left(1 - e^{-iA_{\mu,\vec{r}}} \right) \right| \leq 2 \left| 1 - e^{iA_{\mu,\vec{r}}} \right|$, since $\left| 1 - e^{\pm iA_{\mu,\vec{r}}} \right| \leq 2$, the square root of the last equation is less or equal than

$$2\sqrt{\left| 1 - e^{iA_{\mu,\vec{r}}} \right| \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right|} \leq \left| 1 - e^{iA_{\mu,\vec{r}}} \right| + \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right| ,$$

where $\left(\left| 1 - e^{iA_{\mu,\vec{r}}} \right| - \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right| \right)^2 \geq 0$ and $\left(\left| 1 - e^{iA_{\mu,\vec{r}}} \right| + \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right| \right)^2 \geq 4 \left| 1 - e^{iA_{\mu,\vec{r}}} \right| \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right|$ have been used. Thus, we obtain

$$\left| 1 - e^{iA_{\mu,\vec{r}}} e^{iA'_{\mu',\vec{r}'}} \right| \leq 2 \left| 1 - e^{iA_{\mu,\vec{r}}} \right| + 2 \left| 1 - e^{iA'_{\mu',\vec{r}'}} \right| . \quad (\text{C.441})$$

The same argument applied to $|S_P| = \left| 1 - e^{i(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})} \right|$ produces

$$\left| 1 - e^{iA_{\mu,\vec{r}}} e^{iA_{\nu,\vec{r}+\vec{\mu}}} e^{-iA_{\mu,\vec{r}+\vec{\nu}}} e^{-iA_{\nu,\vec{r}}} \right| \leq 4 \left| 1 - e^{iA_{\mu,\vec{r}}} \right| + 4 \left| 1 - e^{iA_{\nu,\vec{r}+\vec{\mu}}} \right| + 4 \left| 1 - e^{-iA_{\mu,\vec{r}+\vec{\nu}}} \right| + 4 \left| 1 - e^{-iA_{\nu,\vec{r}}} \right| ,$$

which together with (C.440) and the definition of $d\mu$ (C.433) gives

$$\begin{aligned} \left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b} &\leq \text{constant} \frac{1}{ae^2\eta} \left[\int_{-\pi}^{\pi} d\mu |1 - \cos A_{\mu,\vec{r}}|^{2b} \right]^{1/2b} = \\ &= \text{constant} \frac{1}{ae^2\eta} \left[\frac{\int_{-\pi}^{\pi} dA_{\mu,\vec{r}} e^{-\frac{a|\psi|^2}{\eta}[1 - \cos A_{\mu,\vec{r}}]} |1 - \cos A_{\mu,\vec{r}}|^{2b}}{\int_{-\pi}^{\pi} dA_{\mu,\vec{r}} e^{-\frac{a|\psi|^2}{\eta}[1 - \cos A_{\mu,\vec{r}}]}} \right]^{1/2b} , \end{aligned} \quad (\text{C.442})$$

where the integration over the other bond elements not containing $A_{\mu,\vec{r}}$ has produced 1.

We are interested in the range of the model's parameters such that (C.442) is bounded from above. This can be reached for large e^2 ; if e^2 is small, we can still obtain a bound when $|\psi| \rightarrow \infty$, since in this case the main contribution comes from $A \approx 0$ ($1 - \cos A \approx A^2/2$) and one can extend the integration range from $[-\pi, \pi]$ to $(-\infty, \infty)$

$$\frac{1}{ae^2\eta} \left[\frac{\int_{-\infty}^{\infty} dA e^{-\frac{a|\psi|^2}{2\eta}A^2} \left(\frac{A^2}{2} \right)^{2b}}{\int_{-\infty}^{\infty} dA e^{\frac{a|\psi|^2}{2\eta}A^2}} \right]^{1/2b} < \text{constant} \iff \frac{1}{a^2e^2|\psi|^2} < \text{constant} . \quad (\text{C.443})$$

Hence, for e^2 or m^2 large enough (the shaded strip in Fig. C.7),

$$\left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b} \leq \left[\int_{-\pi}^{\pi} d\mu |S_P|^{2b} \right]^{1/2b} \leq c_1 ,$$

with constant c_1 as small as we need for the complete bound of $\langle K \rangle_\Lambda$. According to (C.439), we obtain

$$\left| \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} \varrho_P \right| \leq \prod_{P \in Q_{\text{common}}} \left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b} \leq (c_1)^{Q_{\text{common}}} . \quad (\text{C.444})$$

Returning to equation (C.438), we need a bound for the number of all possible $Q_{\text{common}} \subset \Lambda$ and a bound for $\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}} / \mathcal{Z}_\Lambda$. Following a similar argument as in estimation (C.225) for a path, the

number of all possible collections Q of plaquettes in Λ such that the number of its plaquettes is $||Q|| = p$ and such that all components of Q have bonds in common with support Q_0 , is bounded by

$$N(p) \leq c_2 c_3^p \quad \text{with} \quad c_3 \lesssim 2d - 1, \quad (\text{C.445})$$

where c_2 depends on the number of plaquettes in the support of K : $c_2 = c_2(||Q_0||)$; $2d - 1$ is the number of different ways in a d -dimensional lattice to attach a unit plaquette to a given surface. On the other hand, the bound for $\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}} / \mathcal{Z}_\Lambda$ is directly obtained by noting that $\mathcal{Z}_\Lambda$ contains the integration over all bonds of $\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}$ *plus* the integration over other bonds

$$0 \leq \frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{\mathcal{Z}_\Lambda} \leq 1. \quad (\text{C.446})$$

This result holds for any size of lattice Λ . By considering results (C.445), (C.444) and (C.446) substituted into (C.438), one gets that $\langle K \rangle_\Lambda$ is bounded by a geometric serie

$$\begin{aligned} |\langle K \rangle_\Lambda| &\leq |K|_\infty \sum_{Q_{\text{common}} \subset \Lambda} \left\{ \left| \int_{-\pi}^{\pi} d\mu \prod_{P \in Q_{\text{common}}} \varrho_P \right| \frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{\mathcal{Z}_\Lambda} \right\} \leq |K|_\infty \sum_{p \geq R} N(p) (c_1)^p \leq \\ &\leq |K|_\infty c_2 \frac{(c_1 c_3)^R}{1 - c_1 c_3} \quad \text{with } e^2 \text{ or } m^2 \text{ large enough such that } c_1 \text{ fulfils } c_1 c_3 < 1, \end{aligned} \quad (\text{C.447})$$

where R is the number of plaquettes of the smallest Q_{common} ⁴³.

The above result is independent of the lattice's size and proves the analyticity of any (gauge invariant) correlation function and of the partition function itself ($\mathcal{Z}_\Lambda$ is a case of a correlation function with the whole lattice as support).

Step two

Let us consider a function K of finitely many bond variables $A_{\mu, \vec{r}}$ and a lattice vector $\vec{v}$: we define the translate function $K^{\vec{v}}$ as $K^{\vec{v}}[A_{\mu, \vec{r}}] = K[A_{\mu, \vec{r} + \vec{v}}]$. Now let K_1 and K_2 be two functions depending on a finite number of bonds. For e^2 or m^2 are large enough, the following result holds

$$|\langle K_1 K_2^{\vec{v}} \rangle_\Lambda - \langle K_1 \rangle_\Lambda \langle K_2 \rangle_\Lambda| \leq \text{constant } e^{-|\vec{v}|/\xi}, \quad (\text{C.448})$$

where the bound is independent of the lattice and decay length ξ fulfils $\xi^{-1} \geq 4 \ln(1/c_1)$.

To prove this result, let us introduce an artificial symmetry as follows: we associate a second bond variable $\bar{A}_{\mu, \vec{r}}$ to each bond $(\vec{r}, \mu)$; in the same way, let $\bar{K}$ denote the function K but depending on the $\bar{A}_{\mu, \vec{r}}$ instead of $A_{\mu, \vec{r}}$, and let us associate with each plaquette the following "double" plaquette-factor

$$(\vec{r}, \mu\nu) \rightarrow \left(e^{iA_{\mu, \vec{r}}} e^{iA_{\nu, \vec{r} + \vec{\mu}}} e^{-iA_{\mu, \vec{r} + \vec{\nu}}} e^{-iA_{\nu, \vec{r}}} \right) \left(e^{i\bar{A}_{\mu, \vec{r}}} e^{i\bar{A}_{\nu, \vec{r} + \vec{\mu}}} e^{-i\bar{A}_{\mu, \vec{r} + \vec{\nu}}} e^{-i\bar{A}_{\nu, \vec{r}}} \right). \quad (\text{C.449})$$

We define a new expectation value

$$\begin{aligned} \langle\langle \cdot \rangle\rangle_\Lambda &= \frac{1}{\mathcal{Z}_\Lambda^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\prod_{\vec{r}, \mu} \left(\frac{dA_{\mu, \vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu, \vec{r}}]} \right) \right] \left[\prod_{\vec{r}, \mu} \left(\frac{d\bar{A}_{\mu, \vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos \bar{A}_{\mu, \vec{r}}]} \right) \right] \times \\ &\times \left(\cdot \right) \exp \left[-\frac{1}{ae^2 \eta} \sum_{\vec{r}, \mu < \nu} \left\{ [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})] + [1 - \cos(\Delta_\mu \bar{A}_{\nu, \vec{r}} - \Delta_\nu \bar{A}_{\mu, \vec{r}})] \right\} \right]. \end{aligned} \quad (\text{C.450})$$

⁴³The argument can be slightly modified for the case of a complex e . The final result is similar to (C.447) but with e^2 substituted by $|e|^2$, and with $\left[\int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right]^{1/b}$ and c_1 substituted by $\left| \int_{-\pi}^{\pi} d\mu |\varrho_P|^b \right|^{1/b}$ and by $|c_1|$.

The theory is now symmetric under the change $A_{\mu,\vec{r}} \leftrightarrow \bar{A}_{\mu,\vec{r}}$. By means of this new expectation value, one can the left hand side of inequality (C.448) as follows ⁴⁴

$$\langle K_1 K_2^{\vec{v}} \rangle_{\Lambda} - \langle K_1 \rangle_{\Lambda} \langle K_2 \rangle_{\Lambda} = \frac{1}{2} \left\langle \left\langle (K_1 - \bar{K}_1) (K_2 - \bar{K}_2)^{\vec{v}} \right\rangle \right\rangle_{\Lambda}. \quad (\text{C.451})$$

To this last expression we can apply the cluster expansion, with the following slight modifications

$$\begin{aligned} & -\frac{1}{ae^2\eta} \sum_{\vec{r}, \mu < \nu} \{ [1 - \cos(\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})] + [1 - \cos(\Delta_{\mu} \bar{A}_{\nu,\vec{r}} - \Delta_{\nu} \bar{A}_{\mu,\vec{r}})] \} \\ & = \prod_{\vec{r}} \prod_{\mu, \nu} [1 + \varrho_{\mu\nu,\vec{r}}] [1 + \bar{\varrho}_{\mu\nu,\vec{r}}] \equiv \prod_{\vec{r}} \prod_{\mu, \nu} [1 + \rho_{\mu\nu,\vec{r}}], \end{aligned}$$

and

$$\sum_{Q_{\text{not}} \subset \Lambda} \int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} \prod_{P \in Q_{\text{not}}} \rho_P = \frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}^2}{\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\prod_{\vec{r}, \mu} \left(\frac{dA_{\mu,\vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu,\vec{r}}]} \right) \right] \left[\prod_{\vec{r}, \mu} \left(\frac{d\bar{A}_{\mu,\vec{r}}}{2\pi} e^{-\frac{a|\psi|^2}{\eta} [1 - \cos \bar{A}_{\mu,\vec{r}}]} \right) \right]}.$$

Moreover, for e^2 or m^2 large enough, a relation similar to (C.444) can be obtained for $\rho_P = \varrho_P + \bar{\varrho}_P + \varrho_P \bar{\varrho}_P$. Thus ⁴⁵

$$\left[\int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} |\rho_P|^b \right]^{1/b} \leq c'_1,$$

and

$$\left| \int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} \prod_{P \in Q} \rho_P \right| \leq \prod_{P \in Q} \left[\int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} |\rho_P|^b \right]^{1/b} \leq (c'_1)^Q. \quad (\text{C.452})$$

The cluster expansion of (C.451) becomes

$$\begin{aligned} \langle K_1 K_2^{\vec{v}} \rangle_{\Lambda} - \langle K_1 \rangle_{\Lambda} \langle K_2 \rangle_{\Lambda} &= \frac{1}{2} \sum_{Q_{\text{common}} \subset \Lambda} \int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} \times \\ &\times (K_1 - \bar{K}_1)(K_2 - \bar{K}_2)^{\vec{v}} \prod_{P \in Q_{\text{common}}} \varrho_P \left(\frac{\mathcal{Z}_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{\mathcal{Z}_{\Lambda}} \right)^2, \end{aligned} \quad (\text{C.453})$$

where Q_0 is the support of $(K_1 - \bar{K}_1)(K_2 - \bar{K}_2)^{\vec{v}}$ and Q_{common} is any subset of plaquettes which have bonds in common with Q_0 . The upper bound for this expression will be calculated as in *Step one*: by

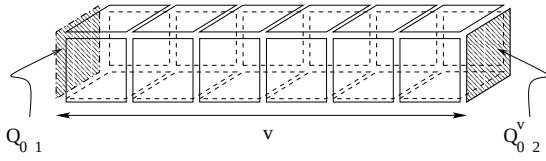
⁴⁴Proved by noting that $\langle \langle K_1 \bar{K}_2^{\vec{v}} \rangle \rangle_{\Lambda} = \langle K_1 \rangle_{\Lambda} \langle \bar{K}_2^{\vec{v}} \rangle_{\Lambda} = \langle K_1 \rangle_{\Lambda} \langle K_2 \rangle_{\Lambda}$ since K_1 only depends on $A_{\mu,\vec{r}}$, whilst $\bar{K}_2$ does on $\bar{A}_{\mu,\vec{r}}$ only.

⁴⁵By considering $\left[\sum_i |A_i + B_i|^b \right] \leq \left[\sum_i (|A_i| + |B_i|)^b \right] \leq \left[\sum_i (|A_i| + |B_i|) \right]^b$ together with (C.441) and Schwarz's inequality (C.440), one obtains

$$\left[\int_{-\pi}^{\pi} d\mu \int_{-\pi}^{\pi} d\bar{\mu} |\rho_P|^b \right]^{1/b} \leq 2 \int_{-\pi}^{\pi} d\mu |\varrho_P| + \left[\int_{-\pi}^{\pi} d\mu |\varrho_P| \right]^2 \leq \text{const} \left[\int_{-\pi}^{\pi} d\mu |S_P|^2 \right]^{1/2} + \text{const} \int_{-\pi}^{\pi} d\mu |S_P|^2,$$

which can be made as small as we want by choosing e^2 or $e^2|\psi|^2$ large enough.

using (C.452) and by noting that bound $\frac{Z_{\Lambda \setminus \overline{Q_{\text{common}} \cup Q_0}}}{Z_{\Lambda}} \leq 1$ holds independent of the lattice's size. We only need to obtain the size of the different Q_{common} 's. Let Q_{01} and Q_{02} denote the supports of K_1 and K_2 respectively, or equivalently, the supports of $K_1 - \bar{K}_1$ and $K_2 - \bar{K}_2$. Therefore, the support of $(K_1 - \bar{K}_1)(K_2 - \bar{K}_2)^{\vec{v}}$ is equal to $Q_0 = Q_{01} \cup Q_{02}^{\vec{v}}$. If lattice vector $\vec{v}$ is sufficiently large, Q_0 consists of two disjoint supports. Let us consider one of the possible Q_{common} : if all its connected components have bonds in common only with one of the two separated supports (for instance Q_{01}), then we can apply the symmetry $A_{\mu, \vec{r}} \leftrightarrow \bar{A}_{\mu, \vec{r}}$ to all the bonds in Q_{01} and in Q_{common} (but only to these bonds). Since $(K_1 - \bar{K}_1)(K_2 - \bar{K}_2)^{\vec{v}}$ is odd under this transformation but measure $d\mu d\bar{\mu}$ is invariant, such a Q_{common} produces no contribution to (C.453). The nonvanishing terms are produced by Q_{common} 's which connect Q_{01} and $Q_{02}^{\vec{v}}$, and containing thus at least $(|\vec{v}| - \text{constant})$ plaquettes. The constant depends on the size and form of the two separated supports. A more careful estimation gives that the smallest possible Q_{common} must contain $4(|\vec{v}| - \text{constant})$ plaquettes:



factor 4 arises from the fact that when integrating over $\int_{-\pi}^{\pi}$ over each lattice's bonds in $d\mu$ and in $d\bar{\mu}$, the only nonvanishing contributions come from configurations where each bond appears twice, once covered in the opposite direction to other time. This can be seen in the following figure for the case of supports Q_{01} and $Q_{02}^{\vec{v}}$ formed by one plaquette.

This argument has been already used in the “high temperature” expansion (like for instance in (C.376)). Collecting results (C.445) and (C.452), we obtain the following bound for e^2 or m^2 large enough (such that $c'_1 c_3 < 1$)

$$|\langle K_1 K_2^{\vec{v}} \rangle_{\Lambda} - \langle K_1 \rangle_{\Lambda} \langle K_2 \rangle_{\Lambda}| \leq \frac{c_4}{2} (c'_1 c_3)^{4|\vec{v}| - \text{constant}}, \quad (\text{C.454})$$

since configurations of plaquettes, which connect Q_{01} and $Q_{02}^{\vec{v}}$ but not along the shortest path, produce a smaller contribution ($c'_1 c_3 < 1$). The above result, called *clustering*, is equation (C.448) with correlation length $\xi \leq \frac{1}{4} \ln \left(\frac{1}{c'_1 c_3} \right)$.

Finally, we can prove the existence of the infinite lattice limit for any expectation value in the range of large e^2 or large m^2 [46]. As before, let K be a function with support Q_0 (for instance, partition function Z_{Λ} with $Q_0 = \Lambda$ a finite lattice). For the infinite d -dimensional cubic lattice, denoted by $\mathbf{Z}^d$, we order the plaquettes $\{P_i\}$ in $\mathbf{Z}^d \setminus Q_0$ such that the distance of P_i from Q_0 is a nondecreasing function of i . Let us define $Q_k = Q_0 \cup \{P_1, \dots, P_k\}$ and the following expectation value

$$\langle K \rangle_{Q_k} = \frac{1}{Z_{Q_k}} \int_{-\pi}^{\pi} d\mu_{\infty} K \exp \left[-\frac{1}{2ae^2\eta} \sum_{\text{all plaquettes } \subset Q_k} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] \right], \quad (\text{C.455})$$

with

$$Z_{Q_k} = \int_{-\pi}^{\pi} d\mu_{\infty} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\text{all plaquettes } \subset Q_k} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] \right], \quad (\text{C.456})$$

and

$$d\mu_{\infty} = \frac{\prod_{\text{all bonds } \in \mathbf{Z}^d} \left(\frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu, \vec{r}}] \right] \right)}{\int_{-\pi}^{\pi} \prod_{(\vec{r}, \mu) \in \mathbf{Z}^d} \left(\frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} [1 - \cos A_{\mu, \vec{r}}] \right] \right)}, \quad (\text{C.457})$$

and now we consider

$$\mathcal{D}_k = \langle K \rangle_{Q_{k+1}} - \langle K \rangle_{Q_k} , \quad (\text{C.458})$$

in order to study

$$\langle K \rangle_{\Lambda=\mathbf{Z}^d} = \langle K \rangle_{Q_0} + \sum_{k=0}^{\infty} \mathcal{D}_k . \quad (\text{C.459})$$

By defining

$$J_{k+1} = \exp \left[-\frac{1}{2ae^2\eta} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})]_{\text{plaquette } P_k} \right] , \quad (\text{C.460})$$

and looking at (C.456), we can write

$$\mathcal{Z}_{Q_{k+1}} = \int_{-\pi}^{\pi} d\mu_\infty J_{k+1} \exp \left[-\frac{1}{2ae^2\eta} \sum_{P \subset Q_k} [1 - \cos(\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})] \right] = \langle J_{k+1} \rangle_{Q_k} \mathcal{Z}_{Q_k} ,$$

and

$$\langle K \rangle_{Q_{k+1}} = \frac{\langle K J_{k+1} \rangle_{Q_k} \mathcal{Z}_{Q_k}}{\mathcal{Z}_{Q_{k+1}}} = \frac{\langle K J_{k+1} \rangle_{Q_k}}{\langle J_{k+1} \rangle_{Q_k}} .$$

With these definitions, one obtains the following bound for $\mathcal{D}_k$

$$|\mathcal{D}_k| = \frac{1}{|\langle J_{k+1} \rangle_{Q_k}|} \left| \langle K J_{k+1} \rangle_{Q_k} - \langle K \rangle_{Q_k} \langle J_{k+1} \rangle_{Q_k} \right| ,$$

where $|\langle J_{k+1} \rangle_{Q_k}|^{-1} < 1$, since $\frac{\mathcal{Z}_{Q_k}}{\mathcal{Z}_{Q_{k+1}}} < 1$, whilst the second factor decays exponentially with the distance between Q_0 and Q_{k+1} , according to (C.454). Therefore, $\langle K \rangle_{\Lambda=\mathbf{Z}^d}$ in (C.459) is well defined since the sum over k converges absolutely.

Summarizing the results of this Note, when e^2 or m^2 are large enough, every correlation function and the free energy itself are analytic even in the thermodynamic limit, and thus, there exist no phase transition. These results are general and hold also for any lattice dimension $d \geq 2$.

Before finishing the discussion of the 3-dimensional Higgs model, let us deal with the phase diagram of the non-compact $U(1)$ Higgs model. In the Villain approximation of this model, $W(C)$ (with $p = 1$) is given by

$$\begin{aligned} W(C)_{\text{non-compact}} &= \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu,\vec{r}}}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left[i \sum_{\vec{r}, \mu} J_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] \sum_{\{m_{\mu,\vec{r}}\}} \times \\ &\times \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_\mu A_{\nu,\vec{r}} - \Delta_\nu A_{\mu,\vec{r}})^2 - \frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_\mu \varphi_{\vec{r}} - qA_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right] , \quad (\text{C.461}) \end{aligned}$$

where partition function $\mathcal{Z}$ equal to the numerator of the above equation but with $J_{\mu,\vec{r}} = 0$. In a non-compact model as this, q is not related to the representation which breaks a compact gauge group down to $U(1)$ and can therefore take on an arbitrary value. In the same way as limit $|\psi|^2 = \infty$ in the compact model was the starting point

to prove the existence of one or two phases, let us study also this limit in the non-compact model. It will result that this limit is dual to the XY model.

Following Peskin [79], let us start calculating the following correlation function in the Villain approximation of the 3-dimensional XY model

$$K(C)_{XY} = \frac{1}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \sum_{\{m_{\mu,\vec{r}}^*\}} \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r},\mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^* - 2\pi m_{\mu,\vec{r}}^* \right)^2 \right]} \times \quad (C.462)$$

$$\times \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \sum_{\{m_{\mu,\vec{r}}^*\}} \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r},\mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma,\vec{r}-\hat{\rho}} - 2\pi m_{\mu,\vec{r}}^* \right)^2 \right],$$

where integer field $L_{\rho\sigma,\vec{r}-\hat{\rho}}$ is the same field as in (C.265) and fulfils $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma,\vec{r}-\hat{\rho}} = -J_{\sigma,\vec{r}}$, with $J_{\sigma,\vec{r}}$ the tangent unit vector to closed path C . In order to distinguish the will-become dual model from original model (C.461), we shall use φ^* and $|\psi^*|$ to denote the phase and length of the scalar field. Now the calculation of $K(C)_{XY}$ runs along the same line as in the 3-dimensional XY model: we introduce an auxiliar field $b_{\mu,\vec{r}}^*$

$$\left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \left(\prod_{\vec{r},\mu} \frac{db_{\mu,\vec{r}}^*}{2\pi} \right) e^{-\frac{\eta}{2a|\psi^*|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^{*2} + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^* \left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma,\vec{r}-\hat{\rho}} - 2\pi m_{\mu,\vec{r}}^* \right)},$$

and the integration over all $d\varphi_{\vec{r}}^*$ produces the constraint $b_{\mu,\vec{r}}^* = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}^*$ which substituted into $K(C)_{XY}$ gives

$$K(C)_{XY} = \frac{1}{\mathcal{Z}_{XY}} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\lambda} \frac{da_{\lambda,\vec{r}}^*}{2\pi} \right) \sum_{\{M_{\lambda,\vec{r}}^*\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}}, 0 \right) e^{-i \frac{2\pi}{q} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^* \overbrace{\sum_{\rho} \Delta_{\rho} L_{\rho\lambda,\vec{r}-\hat{\rho}}}^{-J_{\lambda,\vec{r}}}} \times$$

$$\times \exp \left[-i 2\pi \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^* M_{\lambda,\vec{r}}^* - \frac{\eta}{2a|\psi^*|^2} \sum_{\vec{r}} \sum_{\nu < \lambda} \left(\Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}}^* - \Delta_{\lambda} a_{\nu,\vec{r}-\hat{\lambda}}^* \right)^2 \right], \quad (C.463)$$

where $M_{\lambda,\vec{r}}^* = \sum_{\nu,\mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu,\vec{r}}^*$ fulfils $\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}}^* = 0$. This last constraint can be implemented by introducing an auxiliary field $\theta_{\vec{r}}^*$

$$\prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}}, 0 \right) = \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\theta_{\vec{r}}^*}{2\pi} \right) \exp \left[-i \overbrace{\sum_{\vec{r},\lambda} \theta_{\vec{r}+\hat{\lambda}}^* \Delta_{\lambda} M_{\lambda,\vec{r}}^*}^{-\sum_{\vec{r},\lambda} M_{\lambda,\vec{r}}^* \Delta_{\lambda} \theta_{\vec{r}}^*} \right]. \quad (C.464)$$

By including this part into (C.463), we can sum over all $M_{\lambda,\vec{r}}^*$ without being worried about the constraint of vanishing divergence for $M_{\lambda,\vec{r}}^*$. Further, since configurations with

large vorticity cost a large amount of energy, we expect that $|M_{\lambda,\vec{r}}^*|$ remains small and therefore we can introduce into (C.463) the following factor

$$1 = \lim_{\mathcal{T} \rightarrow 0} \exp \left[-\frac{\mathcal{T}}{2} \sum_{\vec{r},\lambda} M_{\lambda,\vec{r}}^{*2} \right]. \quad (\text{C.465})$$

The sum over $M_{\lambda,\vec{r}}^*$ can be calculated directly⁴⁶

$$\begin{aligned} \lim_{\mathcal{T} \rightarrow 0} \sum_{\{M_{\lambda,\vec{r}}^*\}} \exp \left[i \sum_{\vec{r},\lambda} M_{\lambda,\vec{r}}^* (\Delta_\lambda \theta_{\vec{r}}^* - 2\pi a_{\lambda,\vec{r}}^*) - \frac{\mathcal{T}}{2} \sum_{\vec{r},\lambda} M_{\lambda,\vec{r}}^{*2} \right] = \\ = \lim_{\mathcal{T} \rightarrow 0} \left(\sqrt{\frac{2\pi}{\mathcal{T}}} \right)^{3|\Lambda|} \sum_{\{l_{\lambda,\vec{r}}^*\}} \exp \left[-\frac{1}{2\mathcal{T}} \sum_{\vec{r},\lambda} (\Delta_\lambda \theta_{\vec{r}}^* - 2\pi a_{\lambda,\vec{r}}^* - 2\pi l_{\lambda,\vec{r}}^*)^2 \right], \end{aligned}$$

and since we shall not carry out the integration over $da_{\lambda,\vec{r}}^*$, we need not to include a gauge fixing factor (to remove the gauge invariance in $a_{\lambda,\vec{r}}^*$): correlation $K(C)_{XY}$ becomes thus

$$\begin{aligned} K(C)_{XY} = \frac{1}{\mathcal{Z}^*} \lim_{\mathcal{T} \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{d\Gamma_{\mu,\vec{r}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\theta_{\vec{r}}^*}{2\pi} \right) \exp \left[i \sum_{\vec{r},\mu} J_{\mu,\vec{r}} \Gamma_{\mu,\vec{r}}^* \right] \sum_{\{l_{\mu,\vec{r}}^*\}} \times \\ \exp \left[-\frac{q^2 \eta}{8\pi^2 a |\psi^*|^2} \sum_{\vec{r},\mu < \nu} (\Delta_\mu \Gamma_{\nu,\vec{r}-\hat{\mu}}^* - \Delta_\nu \Gamma_{\mu,\vec{r}-\hat{\nu}}^*)^2 - \frac{1}{2\mathcal{T}} \sum_{\vec{r},\mu} (\Delta_\mu \theta_{\vec{r}}^* - q \Gamma_{\mu,\vec{r}}^* - 2\pi l_{\mu,\vec{r}}^*)^2 \right], \quad (\text{C.466}) \end{aligned}$$

with $\Gamma_{\mu,\vec{r}}^* = \frac{2\pi}{q} a_{\mu,\vec{r}}^*$. Partition function $\mathcal{Z}^*$ is given by

$$\begin{aligned} \mathcal{Z}^* = \lim_{\mathcal{T} \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{d\Gamma_{\mu,\vec{r}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\theta_{\vec{r}}^*}{2\pi} \right) \sum_{\{l_{\mu,\vec{r}}^*\}} \times \\ \exp \left[-\frac{q^2 \eta}{8\pi^2 a |\psi^*|^2} \sum_{\vec{r},\mu < \nu} (\Delta_\mu \Gamma_{\nu,\vec{r}-\hat{\mu}}^* - \Delta_\nu \Gamma_{\mu,\vec{r}-\hat{\nu}}^*)^2 - \frac{1}{2\mathcal{T}} \sum_{\vec{r},\mu} (\Delta_\mu \theta_{\vec{r}}^* - q \Gamma_{\mu,\vec{r}}^* - 2\pi l_{\mu,\vec{r}}^*)^2 \right], \quad (\text{C.467}) \end{aligned}$$

directly obtained and equal to the partition function of the XY model (up to factor $(\sqrt{\frac{2\pi}{\mathcal{T}}})^{3|\Lambda|}$). However, this is just the partition function of the non-compact Higgs model with $ae^2\eta = \frac{4\pi^2 a |\psi^*|^2}{q^2 \eta}$ and in the limit $\frac{a |\psi^*|^2}{\eta} = \frac{1}{\mathcal{T}} \rightarrow \infty$, i.e., the limit we want to study (and the limit we shall call the Peskin model): by comparing (C.467) with (C.461), we

⁴⁶By using the Poisson summation formula $\sum_{M=-\infty}^{+\infty} f(M) = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{\infty} d\phi f(\phi) e^{-i2\pi l \phi}$ which applied to our case gives $\sum_{M=-\infty}^{+\infty} e^{i\zeta M} e^{-\frac{\mathcal{T}}{2} M^2} = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{\infty} d\phi e^{i\zeta \phi} e^{-\frac{\mathcal{T}}{2} \phi^2} e^{-i2\pi l \phi} = \sqrt{\frac{2\pi}{\mathcal{T}}} \sum_{l=-\infty}^{+\infty} e^{-\frac{1}{2\mathcal{T}} (\zeta - 2\pi l)^2}$.

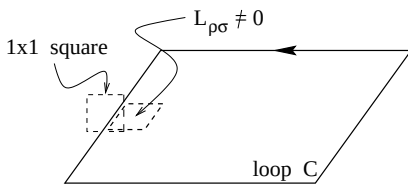
obtain that $K(C)_{XY}$ is $W(C)_{\text{non-compact}}$ in the limit of $|\psi|^2 = \infty$. According to (C.226), the 3-dimensional XY model has a second-order phase transition with a critical value equal to $(a|\psi^*|^2/\eta)_{\text{crit}} = 0.322$ and therefore, the 3-dimensional non-compact Higgs model at $|\psi|^2 = \infty$ shows also a critical point at $(ae^2\eta)_{\text{crit}} = 12.71/q^2$.

Let us calculate $W(C)$ in the $|\psi|^2 = \infty$ case by using the dual model of equation (C.462). For small $|\psi^*|^2$ (or equivalently $e^2 < e_{\text{crit}}^2$), the Villain formulation is a good approximation for

$$K(C)_{XY} = \frac{\int_{-\pi}^{\pi} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \exp \left[-\frac{a|\psi^*|^2}{\eta} \sum_{\vec{r}, \mu} \left[1 - \cos \left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right) \right] \right]}{\int_{-\pi}^{\pi} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \exp \left[-\frac{a|\psi^*|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}}^*)] \right]} \quad (\text{C.468})$$

which for small $|\psi^*|^2$ can be directly evaluated by the “high temperature” expansion of $\prod_{\vec{r}, \mu} \left[1 - \frac{a|\psi^*|^2}{\eta} \frac{e^{i(\Delta_{\mu} \varphi_{\vec{r}}^* + \dots)} + e^{-i(\Delta_{\mu} \varphi_{\vec{r}}^* + \dots)}}{2} \right]$, in bond-factors $\frac{a|\psi^*|^2}{2\eta} e^{\pm i(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}})}$ in the numerator and in terms of $\frac{a|\psi^*|^2}{2\eta} e^{\pm i\Delta_{\mu} \varphi_{\vec{r}}^*}$ in the denominator. The two leading con-

tributions, after integrating $\int_{-\pi}^{\pi} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right)$, are: 1 with no bond-factors, and $\left(\frac{a|\psi^*|^2}{2\eta} \right)^4$ for four bonds forming a 1×1 square. Each time this 1×1 square encircles path C , it crosses only one plaquette with $L_{\rho\sigma, \vec{r}} = \pm 1$; if the 1×1 square is placed inside C , it crosses two plaquettes with $L_{\rho\sigma, \vec{r}} \neq 0$ but their effects are compensated against each other since the square crosses in opposite directions each plaquette. Note that the square crosses $L_{\rho\sigma, \vec{r}}$, but does not contain it, due to tensor $\epsilon_{\mu\rho\sigma}$ in $\left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right)$.



In the correlation's numerator, the 1×1 square can be placed in $3|\Lambda| - \mathcal{P}$ plaquettes without encircling C . The remaining $\mathcal{P}$ plaquettes produces a contribution $\left(\frac{a|\psi^*|^2}{2\eta} \right)^4 e^{\pm i\frac{2\pi}{q}}$ to the numerator (one half of the contributions produce $e^{i\frac{2\pi}{q}}$ and the other half $e^{-i\frac{2\pi}{q}}$) whereas in the denominator the contribution is 1. Hence, the correlation function at small $|\psi^*|^2$ becomes

$$K(C)_{XY}^{\text{small } |\psi^*|^2} = \frac{1 + (3|\Lambda| - \mathcal{P}) \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 + \mathcal{P} \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 \left(\frac{e^{i\frac{2\pi}{q}}}{2} + \frac{e^{-i\frac{2\pi}{q}}}{2} \right) + \dots}{1 + 3|\Lambda| \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 + \dots} \approx$$

$$\approx \frac{\exp \left[3|\Lambda| \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 - \mathcal{P} \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 \left(1 - \cos \frac{2\pi}{q} \right) \right]}{\exp \left[3|\Lambda| \left(\frac{a|\psi^*|^2}{2\eta} \right)^4 \right]},$$

$$K(C)_{XY}^{\text{small } |\psi^*|^2} = W(C)_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2 = \infty} = \exp \left[-\mathcal{P} \left(\frac{aq^2 e^2 \eta}{8\pi^2} \right)^4 \left(1 - \cos \frac{2\pi}{q} \right) \right]. \quad (\text{C.469})$$

On the other hand, from the discussion of the 3-dimensional XY model we can neglect the effect of $m_{\mu, \vec{r}}^*$ at large $|\psi^*|^2$ ($e^2 > e_{\text{crit}}^2$). The correlation function is then given by

$$K(C)_{XY}^{\text{large } |\psi^*|^2} = \frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right)^2 \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \varphi_{\vec{r}}^*)^2 \right]}, \quad (\text{C.470})$$

where the Gaussian integration over $d\varphi_{\vec{r}}^*$ is calculated directly

$$\begin{aligned} & -\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} \left(\Delta_{\mu} \varphi_{\vec{r}}^* + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right)^2 = -\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} \varphi_{\vec{r}}^* (-\Delta_{\mu}) \varphi_{\vec{r}}^* \\ & -\frac{2\pi^2 a |\psi^*|^2}{q^2 \eta} \sum_{\vec{r}} \sum_{\rho < \sigma} L_{\rho\sigma, \vec{r}}^2 + \frac{2\pi a |\psi^*|^2}{q\eta} \sum_{\vec{r}} \varphi_{\vec{r}}^* \sum_{\mu, \rho < \sigma} \epsilon_{\mu\rho\sigma} \Delta_{\mu} L_{\rho\sigma, \vec{r}-\hat{\rho}-\hat{\mu}}, \end{aligned}$$

which produces in the numerator the following exponent

$$\begin{aligned} & -\frac{2\pi^2 a |\psi^*|^2}{q^2 \eta} \underbrace{\sum_{\vec{r}, \rho < \sigma} L_{\rho\sigma, \vec{r}}^2}_{\mathcal{A}} + \frac{2\pi^2 a |\psi^*|^2}{q^2 \eta} \underbrace{\sum_{\vec{r}, \vec{r}'} \sum_{\mu, \rho < \sigma} \sum_{\mu', \rho' < \sigma'} \epsilon_{\mu\rho\sigma} (\Delta_{\vec{r}})_{\mu} L_{\rho\sigma, \vec{r}-\hat{\rho}-\hat{\mu}} D_{\vec{r}, \vec{r}'} \epsilon_{\mu'\rho'\sigma'} (\Delta_{\vec{r}'}^{\mu'}) L_{\rho'\sigma', \vec{r}'-\hat{\rho}'-\hat{\mu}'} }_{\sum_{\vec{r}, \rho < \sigma} L_{\rho\sigma, \vec{r}}^2 - \sum_{\vec{r}, \vec{r}', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}} \\ & = -\frac{ae^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}, \end{aligned} \quad (\text{C.471})$$

by considering equation (C.280)⁴⁷. It is worth noting that the term proportional to the loop area is cancelled, remaining only the weaker term $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}$, which ac-

⁴⁷The same result can be obtained considering a special case: let us suppose that C lies on the 2-3 plane, i.e., $L_{23, \vec{r}} \neq 0$ for the plaquettes inside C . In this case $\sum_{\vec{r}, \vec{r}'} \sum_{\mu, \rho < \sigma} \sum_{\mu', \rho' < \sigma'} \epsilon_{\mu\rho\sigma} (\Delta_{\vec{r}})_{\mu} L_{\rho\sigma, \vec{r}-\hat{\rho}-\hat{\mu}} D_{\vec{r}, \vec{r}'} \epsilon_{\mu'\rho'\sigma'} (\Delta_{\vec{r}'}^{\mu'}) L_{\rho'\sigma', \vec{r}'-\hat{\rho}'-\hat{\mu}'} = \sum_{\vec{r}, \vec{r}'} L_{23, \vec{r}} \left(\Delta_{\vec{r}}^1 \Delta_{\vec{r}'}^1 D_{\vec{r}, \vec{r}'} \right) L_{23, \vec{r}'}$ and, by taking into account the following relation $\Delta_{\vec{r}}^1 \Delta_{\vec{r}'}^1 D_{\vec{r}, \vec{r}'} = \delta_{\vec{r}, \vec{r}'} - \Delta_{\vec{r}}^2 \Delta_{\vec{r}'}^2 D_{\vec{r}, \vec{r}'} - \Delta_{\vec{r}}^3 \Delta_{\vec{r}'}^3 D_{\vec{r}, \vec{r}'}$ together with $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r}-\hat{\rho}} = -J_{\sigma, \vec{r}}$, result (C.471) follows.

cording to (C.293), behaves like $\tau \ln R$, where τ and R are the side lengths of rectangular path C . Therefore

$$K(C)_{XY}^{\text{large } |\psi^*|^2} = W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2 = \infty} = \exp \left[-\frac{ae^2\eta}{2\pi} \tau \ln R \right]. \quad (\text{C.472})$$

According to [79], the phase with $W(C) \sim e^{-\text{Perimeter}}$ corresponds to the superconducting phase of the system (a very rigid superconductor since $|\psi|^2 = \infty$). In the language of the dislocation model, it corresponds to the easy glide stage in a very rigid crystal. Let us consider a pair formed by a negative and a positive monopole located at $\vec{r}_1$ and $\vec{r}_2$ respectively; they carry the magnetic flux of one vortex. Their magnetic field is described by ⁴⁸ $(F_{\mu\nu, \vec{r}})_{\text{mono. pair}} = -\frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}$, where integer field $L_{\lambda, \vec{r}-\hat{\lambda}}$ is equal to 1 for a line connecting $\vec{r}_1$ to $\vec{r}_2$ (as in equation (C.95)) and equal to zero otherwise: $\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{r}-\hat{\lambda}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}$. The divergence of the magnetic field is thus $\frac{1}{2} \sum_{\rho, \mu, \nu} \epsilon_{\rho\mu\nu} \Delta_{\rho} (F_{\mu\nu, \vec{r}})_{\text{mono. pair}} = -\frac{2\pi}{q} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})$. Now, in the non-compact Higgs model at $|\psi|^2 = \infty$, we study the following correlation function

$$H(|\vec{r}_2 - \vec{r}_1|) = \frac{\mathcal{Z}_{|\psi|^2=\infty} \left[\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi/q \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}} \right]}{\mathcal{Z}_{|\psi|^2=\infty} [\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}]}, \quad (\text{C.473})$$

with

$$\begin{aligned} \mathcal{Z}_{|\psi|^2=\infty} \left[\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi/q \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}} \right] &= \lim_{|\psi|^2 \rightarrow \infty} \int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \sum_{\{m_{\mu, \vec{r}}\}} \times \\ &\times \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}} \right)^2 - \frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \varphi_{\vec{r}} - qA_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}})^2 \right]. \end{aligned}$$

Since in general $\mathcal{Z} = e^{-\frac{1}{\eta} \sum_{\vec{r}} \mathcal{F}_{\vec{r}}}$ with $\mathcal{F}_{\vec{r}}$ the free energy density, correlation $H(|\vec{r}_2 - \vec{r}_1|)$ is equal to the exponential of the increase in the free energy due to the monopole pair. The calculation of this correlation is very easy in this model, since it results to be equal to $\left\langle e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \right\rangle$ in the 3-dimensional XY model: if we begin with

$$\left\langle e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \right\rangle = \frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \sum_{\{m_{\mu, \vec{r}}^*\}} \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \varphi_{\vec{r}}^* - 2\pi m_{\mu, \vec{r}}^*)^2 \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \sum_{\{m_{\mu, \vec{r}}^*\}} \exp \left[-\frac{a|\psi^*|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \varphi_{\vec{r}}^* - 2\pi m_{\mu, \vec{r}}^*)^2 \right]},$$

⁴⁸Remember that from the discussion in equation (C.253), it is equivalent to work with $qA_{\mu, \vec{r}}$, $A_{\mu, \vec{r}} \in [0, 2\pi)$, vortex flux $\Phi_0 = \frac{2\pi}{q}$ and monopoles of charge multiple of 2π , or to absorb q into the range of $A_{\mu, \vec{r}}$ ($A_{\mu, \vec{r}} \in [0, q2\pi)$) and thus to study monopoles of charge $n \times q2\pi$. In both cases, a monopole in a superconductor needs q vortices to put in its magnetic field.

after introducing auxiliary field $b_{\mu,\vec{r}}^*$

$$\left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}^*}{2\pi} \right) \left(\prod_{\vec{r},\mu} \frac{db_{\mu,\vec{r}}^*}{2\pi} \right) e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} e^{-\frac{\eta}{2a|\psi^*|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^{*2} + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^* (\Delta_\mu \varphi_{\vec{r}}^* - 2\pi m_{\mu,\vec{r}}^*)},$$

and integrating over $d\varphi_{\vec{r}}^*$, one obtains $b_{\mu,\vec{r}}^* = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_\nu a_{\lambda,\vec{r}-\hat{\nu}}^* - L_{\mu,\vec{r}}$. Now (by remembering that $\exp(i2\pi L_{\mu,\vec{r}} m_{\mu,\vec{r}}^*) = 1$ since both $L_{\mu,\vec{r}}$ and $m_{\mu,\vec{r}}^*$ are integers), we only need to repeat steps (C.464) and (C.465) and we finally arrive at the correlation function which we want to study

$$\frac{1}{\mathcal{Z}^*} \lim_{T \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{d\Gamma_{\mu,\vec{r}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\theta_{\vec{r}}^*}{2\pi} \right) \sum_{\{l_{\mu,\vec{r}}^*\}} \exp \left[-\frac{1}{2\mathcal{T}} \sum_{\vec{r},\mu} (\Delta_\mu \theta_{\vec{r}}^* - q\Gamma_{\mu,\vec{r}}^* - 2\pi l_{\mu,\vec{r}}^*)^2 \right] \times$$

$$\times \exp \left[-\frac{q^2 \eta}{8\pi^2 a |\psi^*|^2} \sum_{\vec{r},\mu < \nu} \left(\Delta_\mu \Gamma_{\nu,\vec{r}-\hat{\mu}}^* - \Delta_\nu \Gamma_{\mu,\vec{r}-\hat{\nu}}^* - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda,\vec{r}-\hat{\lambda}} \right)^2 \right],$$

with $\mathcal{Z}^*$ equal to the above numerator but with $L_{\lambda,\vec{r}} = 0$. From the behaviour of $\left\langle e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \right\rangle$ at large and small $|\psi^*|^2$ (small and large e^2 in the non-compact Higgs model at $|\psi|^2 = \infty$), we obtain the following relations

$$\begin{aligned} \left\langle e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \right\rangle &= e^{-|\vec{r}_2 - \vec{r}_1| \ln(2\eta/a|\psi^*|^2)} && \text{small } |\psi^*|^2 \\ \implies \text{Energy}_{\text{monopole pair}} &= \eta \ln \left(\frac{8\pi^2}{aq^2 e^2 \eta} \right) |\vec{r}_2 - \vec{r}_1| && \text{small } e^2, \\ \left\langle e^{i(\varphi_{\vec{r}_2}^* - \varphi_{\vec{r}_1}^*)} \right\rangle &= e^{\frac{\eta}{4\pi a |\psi^*|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|}} && \text{large } |\psi^*|^2 \\ \implies \text{Energy}_{\text{monopole pair}} &= -\frac{1}{4\pi a e^2} \left(\frac{2\pi}{q} \right)^2 \frac{1}{|\vec{r}_2 - \vec{r}_1|} && \text{large } e^2. \end{aligned} \tag{C.474}$$

The magnetic field of a monopole pair inside a superconductor becomes confined into a vortex, and since the vortex' energy is proportional to its length, the energy of a monopole pair in a superconductor grows linearly with the distance between the monopoles. The phase at small e^2 , in which $W(C) \sim e^{-\text{Perimeter}}$, corresponds thus to the superconducting state. On the other hand, in the normal conducting state, the interaction between monopoles is like that between electric charges, i.e., a Coulomb potential which decreases with the distance; this corresponds to the phase at large e^2 , which shows a $W(C)$'s decay like $\tau \ln R$.

It is curious to note that limit $|\psi|^2 = \infty$ in the 3-dim. compact Higgs model is dual to the 3-dim. Ising model, whilst the same limit in the non-compact model (limit called the Peskin model) is dual to the 3-dim. XY model, where there is a phase transition independent of the value of q . This is another signal of the loss of the compact character

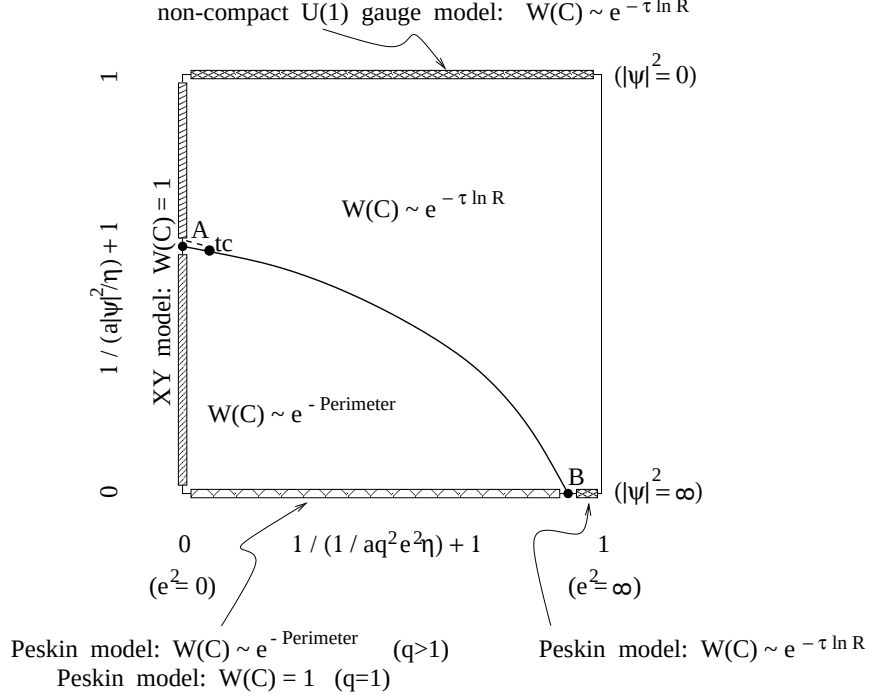


Figure C.8: Expected phase diagram for the 3-dimensional non-compact Higgs model for any q . Point A is the critical value of the 3-dimensional XY model: $(a|\psi|^2/\eta)_{\text{crit}} = 0.322$; point B denotes the critical value of the 3-dimensional Peskin model: $(aq^2e^2\eta)_{\text{crit}} = 12.71$. The short solid-dashed part of line AB near to A denotes first order-transitions; the rest of AB is expected to be of second-order kind. Point tc denotes the tricritical point separating the segment of first-order transitions from the line of transitions of second-order type.

in the gauge field. The expected phase diagram for the 3-dimensional non-compact Higgs model is represented in Fig. C.8. The pure gauge model (limit $|\psi|^2 = 0$) shows no transition. Correlation $W(C)$ is given by

$$W(C)_{|\psi|^2=0, \text{non-comp.}} = \frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 \right]},$$

which after introducing a gauge choice to remove the gauge invariance, for instance by means of $\left(\sqrt{\frac{1}{2\pi e^2\eta}} \right)^{|\Lambda|} \det(-\Delta\Delta) \exp \left[-\frac{1}{2e^2\eta} \sum_{\vec{r}, \nu} A_{\nu, \vec{r}} \left(-\Delta_{\nu} \sum_{\mu} \Delta_{\mu} A_{\mu, \vec{r}-\hat{\mu}} \right) \right]$, can be directly carried out

$$W(C)_{|\psi|^2=0, \text{non-comp.}} = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right], \quad (\text{C.475})$$

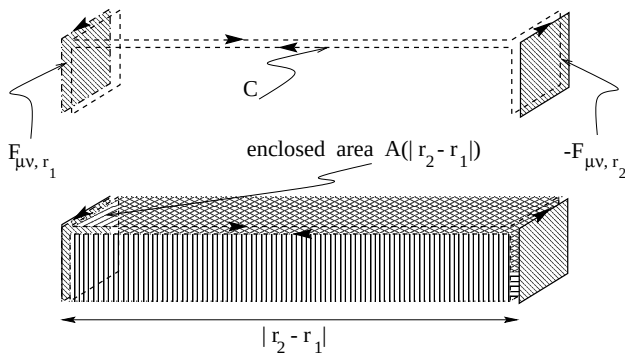
result which holds in any lattice dimension for the pure non-compact gauge model. Moreover, in this case $W(C)_{|\psi|^2=0, \text{non-comp.}}$ results to be equal to $W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2=\infty}$ (C.472).

Line AB divides the diagram into a phase with $W(C) \sim e^{-\tau \ln R}$ and another phase with a weaker decay $W(C) \sim e^{-\text{Perimeter}}$. By comparing this phase diagram with those for the 3-dimensional compact model Fig. C.6 and Fig. C.7, one sees that $W(C)$ behaves in a similar way in both models only in the lower left region of the diagram (although the factor which multiplies the loop perimeter is different). Further, along a line of large $m^2 = a^2 q^2 e^2 |\psi|^2$ and for $q = 1$, a phase transition occurs in the non-compact whereas in the compact model such a transition does not take place.

Since the XY model shows a second order phase-transition point (point A) and the Peskin model (limit $|\psi|^2 = \infty$) is dual to the XY model, both points A and B denote a second-order transition. However, this does not necessarily mean that the line between A and B is a line of second-order phase transitions. Let us consider a point near to A and lying on this line. Above the line (in the normal conducting phase), the gauge field is long-range and the Coulomb propagator is given at large distances by $1/|\vec{r} - \vec{r}'|$, which is the origin of $W(C) \sim e^{-\tau \ln R}$. This long range disrupts the ordering in the scalar field and there is no “spontaneous magnetization”⁴⁹. Under the line (in the superconducting phase), an infinitesimal magnetization appears; the gauge field becomes short-range due to the Meissner effect and is not able to disturb at large distances the ordering in the scalar field: the coupling between two distant scalar fields increases, the magnetization becomes larger and the gauge field’s range becomes shorter. Hence, it can occur that the system, in a state near to A and just under the line AB , acquires with a “jump” a finite magnetization, unlike at A where the transition between zero and an infinitesimal nonvanishing magnetization is continuous. Whether this transition is continuous (of second order type or of Kosterlitz-Thouless kind) or with jumps (of first order type) depends on how large is the difference in the gauge field’s range across the line AB [79]. And this for its part largely depends on whether the scalar field is coupled to a compact or to a non-compact gauge field [93].

In order to make more concrete this statement about the range, let us study the following connected (and gauge invariant) correlation function [41]

$$\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) = \langle e^{iF_{\mu\nu, \vec{r}_1}} e^{-iF_{\mu\nu, \vec{r}_2}} \rangle - \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle \langle e^{iF_{\mu\nu, \vec{r}_2}} \rangle, \quad (\text{C.476})$$



with $F_{\mu\nu, \vec{r}} = \Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}}$. For a translation invariant system as the ours, it holds $\langle e^{iF_{\mu\nu, \vec{r}_2}} \rangle = \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle$. For the range in e^2 where $W(C)$ displays the area decay, correlation $\langle e^{iF_{\mu\nu, \vec{r}_1}} e^{-iF_{\mu\nu, \vec{r}_2}} \rangle$ can be immediately calculated by noting that it can be rewritten as a special case of $W(C)$ with path C as in the figure.

⁴⁹Here “spontaneous magnetization” refers to the expectation value $\langle \psi \rangle$ once we have chosen a gauge fixing condition which removes the gauge invariance, since the expectation value of a field not invariant under a gauge transformation is always zero according to the Elitzur theorem (see page 212). This gauge fixing, which is actually a boundary condition imposed on the system, must be strong enough to permit $\langle \psi \rangle \neq 0$: for instance, Coulomb gauge $\sum_\mu \Delta_\mu A_{\mu, \vec{r} - \vec{\mu}} = 0$.

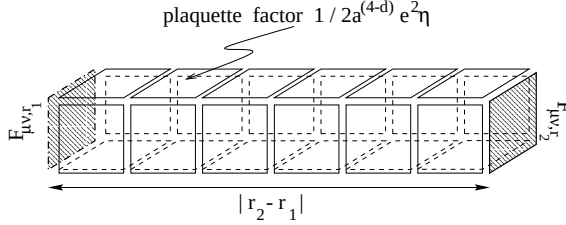
However there are two possible areas enclosed by such a C : area $\mathcal{A}' = 2$, sum of the two side covers, and area $\mathcal{A}(|\vec{r}_2 - \vec{r}_1|)$ wrapping the space between the two plaquettes. Thus, for the range of the model's parameters where $W(C)$ decays like the area, one obtains $\langle e^{iF_{\mu\nu,\vec{r}_1}} e^{-iF_{\mu\nu,\vec{r}_2}} \rangle \sim e^{-\mathcal{A}'} + e^{-\mathcal{A}(|\vec{r}_2 - \vec{r}_1|)}$. Since in this range $\langle e^{iF_{\mu\nu,\vec{r}_1}} \rangle \sim e^{-\text{side's area}} = e^{-\mathcal{A}'/2}$, the connected correlation only decays like the area *enclosed by C between the two plaquettes*

$$\langle e^{iF_{\mu\nu,\vec{r}_1}} e^{-iF_{\mu\nu,\vec{r}_2}} \rangle - \langle e^{iF_{\mu\nu,\vec{r}_1}} \rangle \langle e^{-iF_{\mu\nu,\vec{r}_2}} \rangle \sim \text{const.} \times e^{-\mathcal{A}(|\vec{r}_2 - \vec{r}_1|)},$$

$$\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \exp[-4|\vec{r}_2 - \vec{r}_1| \ln(2a^{4-d}e^2\eta)],$$

for the same range where $W(C)$ shows the area decay. The same result could have been obtained by direct application of the “high temperature” expansion: in order to obtain a nonvanishing contribution when the compact gauge field is integrated out, each bond-power of the two isolated plaquette-powers $e^{iF_{\mu\nu,\vec{r}_1}}$ and $e^{-iF_{\mu\nu,\vec{r}_2}}$ must be covered by one

bond-factor arising from the expansion of $\prod_{\vec{r},\mu < \nu} \left[1 - \frac{1}{a^{4-d}e^2\eta} \frac{e^{i(\Delta_\mu A_\nu, \vec{r} - \Delta_\nu A_\mu, \vec{r})} + e^{-iF_{\mu\nu,\vec{r}}}}{2} \right]$ in plaquette-powers $\frac{e^{\pm i(\Delta_\mu A_\nu, \vec{r} - \Delta_\nu A_\mu, \vec{r})}}{2a^{4-d}e^2\eta}$.



This is reached by covering with expansion's plaquettes the area which wraps the space between the plaquettes located at $\vec{r}_1$ and $\vec{r}_2$.

For those ranges of the model's parameters where, for a rectangular $\tau \times R$ closed path C , $W(C)$ displays the perimeter decay (with or without corrections of the kind τ/R)

or decays like $\tau \ln R$, then one can write $W(C) \approx \exp \left[-\frac{a^{4-d}e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(0, m^2)} J_{\mu, \vec{r}'} \right]$,

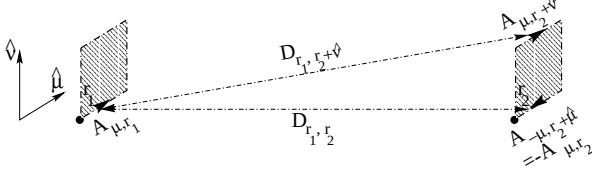
where $J_{\mu, \vec{r}}$ is the integer (unit) vector field tangent to C , and $D_{\vec{r}, \vec{r}'}^{(0)}$ and $D_{\vec{r}, \vec{r}'}^{(m^2)}$ are the long-range and the short-range propagators respectively⁵⁰. The exponent of $\langle e^{iF_{\mu\nu,\vec{r}_1}} e^{-iF_{\mu\nu,\vec{r}_2}} \rangle$

⁵⁰The pure perimeter law is characteristic of the phase at small $a^{4-d}e^2\eta$ and large $a^{d-2}|\psi|^2/\eta$ for the 3-dim. and the 4-dim. Higgs model, both compact and non-compact. Such a behaviour is obtained from $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'}$ where for $m^2 = (a^{d-2}|\psi|^2/\eta) (a^{4-d}e^2\eta)$ large, the short-range propagator can be approximated by $D_{\vec{r}, \vec{r}'}^{(m^2)} \approx \frac{1}{m^2} \delta_{\vec{r}, \vec{r}'}$ and thus $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} \approx \frac{1}{m^2} \sum_{\vec{r}, \mu} J_{\mu, \vec{r}}^2 = \frac{1}{m^2} \text{Perimeter}_C$.

In the range of small $a^{4-d}e^2\eta$ but at $a^{d-2}|\psi|^2/\eta$ small too, $W(C)$ behaves like $e^{-\tau \ln R}$ for the 3-dimensional non-compact model, or like $e^{-\mathcal{P} + \tau/R}$ for both the compact and the non-compact 4-dimensional Higgs model; this is the result of $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(0)} J_{\mu, \vec{r}'}$ with long-range propagator given

by $D_{\vec{r}, \vec{r}'}^{(0)} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|^{d-2}}$ for large distances $|\vec{r} - \vec{r}'|$ and $d = 3, 4$. In fact, if in the compact model one wants to obtain the behaviour of $W(C)$ for small $a^{d-2}|\psi|^2/\eta$ a bit more exactly, one has to substitute $\frac{1}{a^{4-d}e^2\eta}$ with $\frac{1}{a^{4-d}e^2\eta} + \frac{1}{4} \left(\frac{a^{d-2}|\psi|^2}{\eta} \right)^4$ as it will be explained later in the footnote on page 483.

contains: *a*) interactions between the bonds of the same plaquette, which produce a term equal to $\langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle$, and *b*) interactions between the separated plaquettes carried by



$D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}$. By noting that $F_{\mu\nu, \vec{r}}$ is the sum of the gauge fields along an unit plaquette, $F_{\mu\nu, \vec{r}_1} = A_{\mu, \vec{r}_1} + A_{\nu, \vec{r}_1 + \hat{\mu}} + A_{-\mu, \vec{r}_1 + \hat{\mu} + \hat{\nu}} + A_{-\nu, \vec{r}_1 + \hat{\nu}}$, and from the contiguous figure, one obtains that the interactions between the two isolated plaquettes produce

$$\begin{aligned}
\langle e^{iF_{\mu\nu, \vec{r}_1}} e^{-iF_{\mu\nu, \vec{r}_2}} \rangle &= \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle^2 \exp \left\{ -\frac{a^{4-d} e^2 \eta}{2} \left[-D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + D_{\vec{r}_1, \vec{r}_2 + \hat{\nu}}^{(0, m^2)} - D_{\vec{r}_1 + \hat{\mu}, \vec{r}_2 + \hat{\mu}}^{(0, m^2)} + D_{\vec{r}_1 + \hat{\mu}, \vec{r}_2}^{(0, m^2)} - \right. \right. \\
&\quad \left. \left. - D_{\vec{r}_1 + \hat{\nu}, \vec{r}_2 + \hat{\nu}}^{(0, m^2)} + D_{\vec{r}_1 + \hat{\nu}, \vec{r}_2}^{(0, m^2)} - D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + D_{\vec{r}_1, \vec{r}_2 + \hat{\mu}}^{(0, m^2)} \right] \right\} = \\
&= \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle^2 \exp \left[-\frac{a^{4-d} e^2 \eta}{2} \left(\underbrace{\Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}}_{D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}} - \underbrace{\Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1 + \hat{\mu}, \vec{r}_1}^{(0, m^2)}}_{D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}} - \underbrace{\Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1 + \hat{\nu}, \vec{r}_2}^{(0, m^2)}}_{D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + \Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}} + \underbrace{\Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}}_{D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} + \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}} \right) \right], \\
\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) &= \langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle^2 \left(\exp \left[-\frac{a^{4-d} e^2 \eta}{2} \left(-\Delta_{(\vec{r}_1)}^\mu \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} - \Delta_{(\vec{r}_1)}^\nu \Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} \right) \right] - 1 \right) \approx \\
&\stackrel{a^{4-d} e^2 \eta \text{ small}}{\approx} -\text{const.} \times \frac{a^{4-d} e^2 \eta}{2} \left(-\Delta_{(\vec{r}_1)}^\mu \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} - \Delta_{(\vec{r}_1)}^\nu \Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} \right), \quad (\text{C.477})
\end{aligned}$$

without summing on μ or on ν . In the last step of the above result, one has used that the ranges where $W(C)$ can be approximated by $\exp \left[-\frac{a^{4-d} e^2 \eta}{2} \sum_{\vec{r}', \vec{r}'', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(0, m^2)} J_{\mu, \vec{r}''} \right]$ always lie in the range of small $a^{4-d} e^2 \eta$. Now, one only needs to calculate explicitly result (C.477) in the d -dimensional Higgs model ($d = 3, 4$), where the propagators at large distances are given by $D_{\vec{r}_1, \vec{r}_2}^{(0)} = \frac{1}{4\pi |\vec{r}_1 - \vec{r}_2|^{d-2}}$ and $D_{\vec{r}_1, \vec{r}_2}^{(m^2)} = \frac{1}{2m} \left(\frac{m}{2\pi |\vec{r}_1 - \vec{r}_2|} \right)^{(d-1)/2} e^{-m|\vec{r}_1 - \vec{r}_2|}$ for the long-range and the short-range case respectively⁵¹: since we are studying the be-

⁵¹The general form of the short-range propagator at large distances is obtained as follows: from its definition $D_{\vec{r}_1, \vec{r}_2}^{(m^2)} = \int_{-\pi}^{\pi} \frac{d^d \vec{k}}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}}{m^2 + 2d - 2 \sum_{\mu=1}^d \cos k_\mu}$ for an infinite large lattice we can approximate it by $D_{\vec{r}_1, \vec{r}_2}^{(m^2)} = \int_{-\infty}^{\infty} \frac{d^d \vec{k}}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}}{m^2 + \sum_{\mu} k_\mu^2}$; this is because we are interested in the behaviour

at large $|\vec{r}_1 - \vec{r}_2|$, where the discreteness of the lattice gets lost and where the main contribution to the otherwise rapid oscillating $e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}$ comes from the range of $k_\mu \approx 0$ (what allows us to

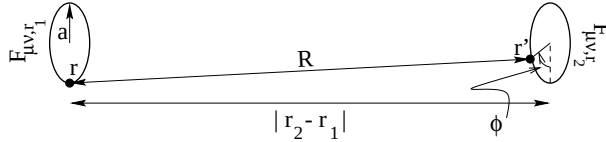
haviour at large distances, discrete derivatives can be approximated by the usual derivatives and this leads to the result that the leading terms in $-\Delta_{(\vec{r}_1)}^\mu \Delta_{(\vec{r}_2)}^\mu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)} - \Delta_{(\vec{r}_1)}^\nu \Delta_{(\vec{r}_2)}^\nu D_{\vec{r}_1, \vec{r}_2}^{(0, m^2)}$ are ⁵²

$$\begin{cases} \frac{1}{|\vec{r}_1 - \vec{r}_2|^d} & \text{for } D_{\vec{r}_1, \vec{r}_2}^{(0)} \sim \frac{1}{|\vec{r}_1 - \vec{r}_2|^{d-2}}, \\ \frac{m^{(d-1)/2}}{|\vec{r}_1 - \vec{r}_2|^{(d+1)/2}} e^{-m|\vec{r}_1 - \vec{r}_2|} & \text{for } D_{\vec{r}_1, \vec{r}_2}^{(m^2)} \sim \frac{1}{m} \left(\frac{m}{|\vec{r}_1 - \vec{r}_2|} \right)^{(d-1)/2} e^{-m|\vec{r}_1 - \vec{r}_2|}. \end{cases} \quad (\text{C.478})$$

Finally, connected correlation $\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|)$ for the 3-dimensional case is thus given

by extending the integration range on $d^d \vec{k}$. The propagator can be further rewritten ([99], page 133-134) as $D_{\vec{r}_1, \vec{r}_2}^{(m^2)} = \int_{-\infty}^{\infty} \frac{d^d \vec{k}}{(2\pi)^d} \int_0^{\infty} dt e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} e^{-t(m^2 + \vec{k} \cdot \vec{k})} = \frac{\pi^{d/2}}{(2\pi)^d} \int_0^{\infty} \frac{dt}{t^{d/2}} \exp\left[-\left(tm^2 + \frac{|\vec{r}_1 - \vec{r}_2|^2}{4t}\right)\right]$. The main contribution to the exponential (the so-called ‘‘saddle point’’) is that t -value for which $f(t) = tm^2 + \frac{|\vec{r}_1 - \vec{r}_2|^2}{4t}$ has a minimum: $t_0 = \frac{|\vec{r}_1 - \vec{r}_2|^2}{4m^2}$, with $\frac{d^2 f}{dt^2}(t=t_0) > 0$. Changing the integration variable to $s = t \frac{2m}{|\vec{r}_1 - \vec{r}_2|}$, the short-range propagator takes the following form: $\frac{1}{4\pi} \left(\frac{m}{2\pi|\vec{r}_1 - \vec{r}_2|} \right)^{d/2-1} \int_0^{\infty} \frac{ds}{s^{d/2}} \exp\left[-\frac{m|\vec{r}_1 - \vec{r}_2|}{2} \left(s + \frac{1}{s}\right)\right]$. And for $|\vec{r}_1 - \vec{r}_2| \rightarrow \infty$, the previous integrand works as a Gaussian curve which picks the value for which $\left(s + \frac{1}{s}\right)$ is minimum, as one can easily obtain by Taylor-expanding, $\exp\left[-\frac{m|\vec{r}_1 - \vec{r}_2|}{2} \left(s + \frac{1}{s}\right)\right] \stackrel{|\vec{r}_1 - \vec{r}_2| \rightarrow \infty}{\approx} \exp\left[-\frac{m|\vec{r}_1 - \vec{r}_2|}{2} (2 + (s-1)^2)\right]$; hence, from relation between the Dirac delta and a Gaussian curve, $\delta(s-a) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\pi\varepsilon}} e^{-(s-a)^2/\varepsilon^2}$, one obtains for $\varepsilon \equiv \sqrt{\frac{2}{m|\vec{r}_1 - \vec{r}_2|}}$ the final result $D_{\vec{r}_1, \vec{r}_2}^{(m^2)} \approx \frac{1}{4\pi} \left(\frac{m}{2\pi|\vec{r}_1 - \vec{r}_2|} \right)^{d/2-1} \sqrt{\frac{2\pi}{m|\vec{r}_1 - \vec{r}_2|}} e^{-m|\vec{r}_1 - \vec{r}_2|} \times \int_0^{\infty} \frac{ds}{s^{d/2}} \delta(s-1) = \frac{1}{2m} \left(\frac{m}{2\pi|\vec{r}_1 - \vec{r}_2|} \right)^{(d-1)/2} e^{-m|\vec{r}_1 - \vec{r}_2|}$.

⁵²These results can be also obtained by considering in a continuum model the interaction between infinitesimal length elements belonging to 2 small parallel rings of radius a (the lattice spacing, which



in the discrete model has been taken as 1) separated by a large distance $|\vec{r}_2 - \vec{r}_1| \gg a$; the interaction is carried by propagator $D^{(0, m^2)}(|\vec{r} - \vec{r}'|)$. Hence in the range of small $a^{d-4} \varepsilon^2 \eta$

$$\Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \approx -\langle e^{iF_{\mu\nu, \vec{r}_1}} \rangle^2 \times \frac{a^{4-d} \varepsilon^2 \eta}{2} 2\pi a \int_{\phi=0}^{\phi=2\pi} d\phi D^{(0, m^2)}(R) \cos \phi,$$

with $R = \sqrt{|\vec{r}_2 - \vec{r}_1|^2 + a^2(1 - \cos \phi)^2 + a^2 \sin^2 \phi} \stackrel{a \ll |\vec{r}_2 - \vec{r}_1|}{\approx} |\vec{r}_2 - \vec{r}_1| \left(1 + \frac{a^2(1 - \cos \phi)}{|\vec{r}_2 - \vec{r}_1|^2}\right)$. For long-range propagator $D^{(0)}(R) = \frac{1}{4\pi R^{d-2}}$ ($d = 3, 4$), the above integral becomes $\frac{(d-2)a^2}{4|\vec{r}_2 - \vec{r}_1|^d}$.

And for short-range propagator $D^{(m^2)}(R) = \frac{1}{2m} \left(\frac{m}{2\pi R} \right)^{(d-1)/2} e^{-mR}$, the integral is equal to $\frac{1}{2m} \left(\frac{m}{2\pi} \right)^{(d-1)/2} \int_0^{2\pi} d\phi \cos \phi e^{-m|\vec{r}_2 - \vec{r}_1|} \left(1 - \frac{(d-1)a^2(1 - \cos \phi)}{2|\vec{r}_2 - \vec{r}_1|^2} - \frac{ma^2(1 - \cos \phi)}{|\vec{r}_2 - \vec{r}_1|}\right)$, whose leading term at $|\vec{r}_2 - \vec{r}_1| \rightarrow \infty$ is $\frac{\pi a^2}{4} \left(\frac{m}{2\pi} \right)^{(d-1)/2} \frac{e^{-m|\vec{r}_2 - \vec{r}_1|}}{|\vec{r}_2 - \vec{r}_1|^{(d+1)/2}}$.

by [41]

$$W(C) \sim e^{-\mathcal{A}} \Rightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \exp \left[-4 \ln(2ae^2\eta) |\vec{r}_2 - \vec{r}_1| \right], \quad (\text{C.479})$$

$$W(C) \sim e^{-\mathcal{P}} \Rightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \frac{ae^2m\eta}{|\vec{r}_2 - \vec{r}_1|^2} \exp \left[-m|\vec{r}_2 - \vec{r}_1| \right],$$

and

$$W(C) \sim e^{-\tau \ln R} \Rightarrow \Gamma_{\text{gauge}}(|\vec{r}_2 - \vec{r}_1|) \sim \frac{ae^2\eta}{|\vec{r}_2 - \vec{r}_1|^3}, \quad (\text{C.480})$$

with $m = \sqrt{a^2e^2|\psi|^2}$.

According to (C.479), the gauge field in the compact model Fig. C.6 is short-range at both sides of line AB and thus one can expect that the transition remains the same as at point A , second-order, in the whole line. In the non-compact model, on the other hand, there is a large change in the gauge field's range when one crosses the part of AB near to A : from long range above the line to short range under AB . This is an indication of a change in the character of the phase transition when compared with the second-order type at A . The calculations by Halperin, Lubensky and Ma [47] in the case of a non-compact gauge field suggest that the transition becomes first-order for a lattice of dimensionality $d = 3$ and $d = 4$. However, this does not mean that by leaving A and by approaching B along AB , the transition remains first-order: in fact, Monte Carlo simulations carried out by Dasgupta and Halperin [30] show that the extension along AB of the first-order character for $d = 3$ is very short; and further, the part of AB near to B is a line of second-order transitions for both $d = 3$ and $d = 4$. Following these last authors, the argument which proves the recovering of the second-order character when point B is approached runs as follows: let us start with the XY model, in the Villain approximation, coupled to a non-compact $U(1)$ gauge field, where after introducing

auxiliary field $b_{\mu,\vec{r}}$, $\left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \left(\prod_{\vec{r},\mu} \frac{db_{\mu,\vec{r}}}{2\pi} \right) e^{-\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu,\vec{r}} - 2\pi m_{\mu,\vec{r}})}$, one integrates over $d\varphi_{\vec{r}}$ producing $\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} = 0$ and the partition function becomes

$$\mathcal{Z}_{\text{non-compact}} = \left(\sqrt{\frac{2\pi\eta}{a^{d-2}|\psi|^2}} \right)^{d|\Lambda|} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{dA_{\mu,\vec{r}}}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{db_{\mu,\vec{r}}}{2\pi} \right) \exp \left[-\frac{a^{d-4}}{2e^2\eta} \sum_{\vec{r},\mu < \nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})^2 \right] \times$$

$$\times \sum_{\{m_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 - iq \sum_{\vec{r},\mu} b_{\mu,\vec{r}} A_{\mu,\vec{r}} - i2\pi \sum_{\vec{r},\mu} b_{\mu,\vec{r}} m_{\mu,\vec{r}} \right].$$

Now, instead of following the techniques developed in previous sections as one changed the integration over $db_{\mu,\vec{r}}$ by $da_{\lambda,\vec{r}}$ (after choicing a gauge fixing factor) and integrate

it, one applies the Poisson summation $\sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} db f(b) e^{-i2\pi bm} = \sum_{l=-\infty}^{+\infty} f(l)$ and changes variables $A_{\mu,\vec{r}}/2\pi \rightarrow A_{\mu,\vec{r}}$

$$\mathcal{Z}_{\text{non-compact}} = \left(\sqrt{\frac{\eta}{2\pi a^{d-2}|\psi|^2}} \right)^{d|\Lambda|} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} dA_{\mu,\vec{r}} \right) \exp \left[-\frac{2\pi^2 a^{d-4}}{e^2\eta} \sum_{\vec{r},\mu < \nu} (\Delta_{\mu} A_{\nu,\vec{r}} - \Delta_{\nu} A_{\mu,\vec{r}})^2 \right] \times$$

$$\times \sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2 - i2\pi q \sum_{\vec{r},\mu} l_{\mu,\vec{r}} A_{\mu,\vec{r}} \right], \quad (\text{C.481})$$

and finally integrates over the non-compact gauge field, after including a gauge fixing factor as for example $\left(\sqrt{\frac{2\pi a^{d-4}}{e^2 \eta}} \right)^{|\Lambda|} \det(-\Delta \Delta) e^{-\frac{2\pi^2 a^{d-4}}{e^2 \eta} \sum_{\vec{r},\nu} A_{\nu,\vec{r}} (-\Delta_{\nu} \sum_{\mu} \Delta_{\mu} A_{\mu,\vec{r}-\hat{\mu}})}$

$$\mathcal{Z}_{\text{non-comp.}} = \left(\sqrt{\frac{\eta}{2\pi a^{d-2}|\psi|^2}} \right)^{d|\Lambda|} \left(\sqrt{\frac{e^2 \eta}{2\pi a^{d-4}}} \right)^{(d-1)|\Lambda|} \frac{1}{\sqrt{\det(-\Delta \Delta)}} \times \quad (\text{C.482})$$

$$\times \sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{q^2 e^2 \eta}{2a^{d-4}} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} - \frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2 \right].$$

For $d = 3$, and except for a constant factor, this partition function is that of the

$$XY \text{ model } \mathcal{Z}_{XY} = \left(\right) \sum_{\{M_{\lambda,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r},\vec{r}'} \sum_{\lambda} M_{\lambda,\vec{r}} D_{\vec{r},\vec{r}'} M_{\lambda,\vec{r}'} \right] \quad (\text{C.208})$$

with a modification in the interaction at short distances: instead of

$$\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r},\vec{r}'} \sum_{\lambda} M_{\lambda,\vec{r}} D_{\vec{r},\vec{r}'} M_{\lambda,\vec{r}'} = \frac{2\pi^2 a |\psi|^2}{\eta} \left[\sum_{\vec{r},\vec{r}' \neq \vec{r}} \sum_{\lambda} \frac{M_{\lambda,\vec{r}} M_{\lambda,\vec{r}'}}{4\pi |\vec{r} - \vec{r}'|} + D(0) \sum_{\vec{r},\lambda} M_{\lambda,\vec{r}}^2 \right],$$

as in the XY model, we obtain in the non-compact Higgs model

$$\begin{aligned} & \frac{q^2 e^2 \eta}{2a^{d-4}} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} + \frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2 = \\ & = \frac{q^2 e^2 \eta}{2a^{d-4}} \left[\sum_{\vec{r},\vec{r}' \neq \vec{r}} \sum_{\mu} \frac{l_{\mu,\vec{r}} l_{\mu,\vec{r}'}}{4\pi |\vec{r} - \vec{r}'|} + \left(D(0) + \frac{1}{m^2} \right) \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2 \right]. \end{aligned} \quad (\text{C.483})$$

For $d = 4$, (C.482) is similar (up to a constant factor) to the partition of the pure compact $U(1)$ model $\mathcal{Z}_{\text{comp.}} = \left(\right) \sum_{\{G_{\rho,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2}{e^2 \eta} \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'} \right] \quad (\text{C.324})$ with the same modification for short distances as in three dimensions.

For $|\psi|^2 = \infty = m^2$ in the non-compact Higgs model (the Peskin model), the 3-dim. case is dual to the 3-dim. XY model with $(e^2 \eta)_{\text{non-comp.}} = (|\psi|^2 / \eta)_{XY} \frac{4\pi^2}{q^2}$; and the 4-dim. case is dual to the 4-dim. pure compact $U(1)$ model with $(e^2 \eta)_{\text{non-comp.}} (e^2 \eta)_{\text{comp.}} = \frac{4\pi^2}{q^2}$.

This is another (shorter) form to obtain the model dual to the Peskin model. Starting at the critical point of the Peskin model (point B), a large but finite $|\psi|^2$ only changes the short-distance behaviour: since the long-distance part (coupling of a macroscopic number of fields) is what determines the nature of the transition, the phase transition across the line AB at points near to B remains therefore as at B , i.e., of the second order type, both for $d = 3$ and $d = 4$. Moreover, by studying equation (C.482) in the neighbourhood

of the XY 's critical point, one can also argue that the nature of the transition across AB for points near to A cannot be the same as at A : leaving A means passing from $e^2 = 0$, where the exponent in (C.482) contains only the very short-range (contact) interaction $\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$, to a state where long-distance interactions, carried by $D_{\vec{r}, \vec{r}'}$, take place. Thus, by leaving point A , the transition becomes first-order, result which coincides with that by Halperin, Lubensky and Ma. Summarizing, line AB in the 3-dimensional Higgs model has different characteristics for points near to the critical point of the XY model depending on the nature of the gauge field

compact model: the whole line AB is a line of second-order transitions ,

non-compact model: very near to A , the line is of first-order kind;
the rest is a line of second-order transitions .

Now and as the last point of this section, let us consider in the following Note the phase diagram of the Higgs model, both compact and non-compact, in four and two dimensions. These models show properties quite different from those of the 3-dim. case.

Note. The phase diagram for the 4-dimensional and the 2-dimensional Higgs model, both compact and non-compact.

In the 4-dim. compact Higgs model, we consider the following expectation value

$$H(C^*) = \frac{\int_{-\pi}^{\pi} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{e^2 \eta} \sum_{\vec{r}, \mu < \nu} \left[1 - \cos \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r} - \hat{\rho}}^* \right) \right] \right]}{\int_{-\pi}^{\pi} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{e^2 \eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] \right]} \times$$

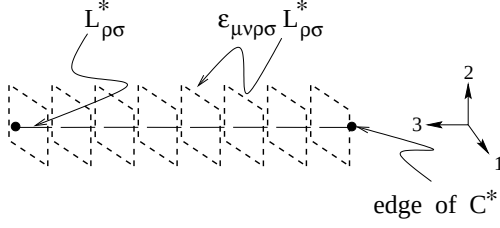
$$\times \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left[-\frac{a^2 |\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})] \right]$$

$$\times \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left[-\frac{a^2 |\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}})] \right], \quad (C.485)$$

with $\epsilon_{1234} = 1$. Integer field $L_{\rho\sigma, \vec{r} - \hat{\rho}}^*$ is only different from zero and equal to ± 1 for plaquettes inside the closed path C^* : $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r} - \hat{\rho}}^* = -J_{\sigma, \vec{r}}^*$ with $J_{\sigma, \vec{r}}^*$ the unit vector tangent to C^* .

We have already met a special case of the above equation: the 't-Hooft loop correlation in the pure $Z(2)$ gauge model (see (C.372)). This correlation function shows a behaviour complementary to $W(C)$ in the $Z(2)$ gauge model: when $W(C)_{\text{pure } Z(2)}$ decays like the area, $H(C^*)_{\text{pure } Z(2)}$ does with the perimeter, and viceversa. From a discussion similar to that in equation (C.371), correlation function

$$H(C^*)_{\text{pure } Z(2)} = \frac{\sum_{\{\bar{S}_{\mu, \vec{r}^*} = \pm 1\}} \left(\prod_{C^*} \bar{S}_{\mu, \vec{r}^*} \right) \exp \left[-\beta_e^* \sum_{\vec{r}^*} \sum_{\mu < \nu} (1 - \bar{S}_{\mu, \vec{r}^*} \bar{S}_{\nu, \vec{r}^* + \hat{\mu}} \bar{S}_{\mu, \vec{r}^* + \hat{\nu}} \bar{S}_{\nu, \vec{r}^*}) \right]}{\underbrace{\sum_{\{\bar{S}_{\mu, \vec{r}^*} = \pm 1\}} \exp \left[-\beta_e^* \sum_{\vec{r}^*} \sum_{\mu < \nu} (1 - \bar{S}_{\mu, \vec{r}^*} \bar{S}_{\nu, \vec{r}^* + \hat{\mu}} \bar{S}_{\mu, \vec{r}^* + \hat{\nu}} \bar{S}_{\nu, \vec{r}^*}) \right]}_{Z^*}}, \quad (C.486)$$



can be written in its dual model (again the pure $Z(2)$ gauge model) as quotient $\frac{\mathcal{Z}_{- \rightarrow +}}{\mathcal{Z}}$ where $\mathcal{Z}$ is equal to $\mathcal{Z}^*$ but described by variables without $*$ and by $\beta_e = -1/2 \ln \tanh \beta_e^*$; $\mathcal{Z}_{- \rightarrow +}$ is equal to $\mathcal{Z}$ but with the change $1 - S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}} \rightarrow 1 + S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}}$ at the plaquettes, denoted by $\sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r} - \hat{\rho}}^*$, which are crossed by plaquettes $L_{\rho\sigma, \vec{r} - \hat{\rho}}^*$ (located in the 3-4 plane in the figure) inside C^* .

With these definitions, $\frac{\mathcal{Z}_{- \rightarrow +}}{\mathcal{Z}}$ is thus limit $|\psi|^2 = \infty$ of equation (C.485) for $q = 2$ (as one can rapidly check by remembering that this limit forces $S_{\mu, \vec{r}} = e^{iA_{\mu, \vec{r}}} = \pm 1$), and therefore

$$1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}}) = 1 - S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}},$$

$$1 - \cos\left(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - \pi \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r} - \hat{\rho}}^*\right) = 1 + S_{\mu, \vec{r}} S_{\nu, \vec{r} + \hat{\mu}} S_{\mu, \vec{r} + \hat{\nu}} S_{\nu, \vec{r}},$$

when $L_{\rho\sigma, \vec{r} - \hat{\rho}}^* \neq 0$. Hence, equation (C.485) is more general definition of the 't-Hooft loop correlation. Comparing this correlation with equation (C.473), one obtains that the logarithm of $H(C^*)$ is equal to (minus) the free energy of a magnetic loop carried by closed path C^* (in the 3-dimensional model, we only considered the negative and positive monopoles of the section of this loop along the fourth axis).

After this introduction to correlation function $H(C^*)$, let us start with the study of the phase diagram for the 4-dim. non-compact Higgs model. As in the 3-dimensional case and again following Peskin [79], limit $|\psi|^2 = \infty$ of the non-compact Higgs model will result to be dual to an already known model, this time not the XY model but the pure compact $U(1)$ gauge model. Thus, let us consider in the Villain approximation of the pure compact $U(1)$ model the following correlation function

$$H(C)_{\text{compact}} = \frac{1}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}^*}{2\pi} \right) \sum_{\{g_{\mu\nu, \vec{r}}^*\}} \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - 2\pi g_{\mu\nu, \vec{r}}^*)^2 \right]} \times \quad (\text{C.487})$$

$$\times \int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}^*}{2\pi} \right) \sum_{\{g_{\mu\nu, \vec{r}}^*\}} \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r} - \hat{\rho}} - 2\pi g_{\mu\nu, \vec{r}}^* \right)^2 \right],$$

where the variables have been written with $*$ since they will become the dual variables of those describing the non-compact Higgs model. Now we can repeat the same steps as in (C.462) by introducing an antisymmetric auxiliar field $B_{\mu\nu, \vec{r}}^*$

$$\left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}^*}{2\pi} \right) \left(\prod_{\vec{r}, \mu < \nu} \frac{dB_{\mu\nu, \vec{r}}^*}{2\pi} \right) e^{-\frac{e^{*2}\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^{*2} + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^* \left(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r} - \hat{\rho}} - 2\pi g_{\mu\nu, \vec{r}}^* \right)}$$

and then integrating over $dA_{\mu, \vec{r}}^*$, which produces the constraint $B_{\mu\nu, \vec{r}}^* = -\sum_{\tau, \lambda} \epsilon_{\mu\nu\tau\lambda} \Delta_\tau C_{\lambda, \vec{r} - \hat{\tau}}^*$. Summing by parts

$$i2\pi \sum_{\vec{r}} \sum_{\mu < \nu} B_{\mu\nu, \vec{r}}^* g_{\mu\nu, \vec{r}}^* + i2\pi \sum_{\vec{r}, \lambda} C_{\lambda, \vec{r}}^* \left(\sum_{\mu < \nu} -\epsilon_{\lambda\tau\mu\nu} \Delta_\tau g_{\mu\nu, \vec{r}}^* \right) = i2\pi \sum_{\vec{r}, \lambda} C_{\lambda, \vec{r}}^* G^{*}_{\lambda, \vec{r}},$$

where $G_{\lambda,\vec{\tau}}^*$ fulfils $\sum_{\lambda} \Delta_{\lambda} G_{\lambda,\vec{\tau}} = 0$, $H(C)_{\text{comp.}}$ becomes

$$H(C)_{\text{comp.}} = \frac{1}{\mathcal{Z}_{\text{pure compact}}} \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\lambda} \frac{dC_{\lambda,\vec{\tau}}^*}{2\pi} \right) \sum_{\{G_{\lambda,\vec{\tau}}^*\}} \prod_{\vec{\tau}} \delta \left(\sum_{\lambda} \Delta_{\lambda} G_{\lambda,\vec{\tau}}, 0 \right) e^{-i \frac{2\pi}{q} \sum_{\vec{\tau},\lambda} C_{\lambda,\vec{\tau}}^* \sum_{\rho} \overbrace{\Delta_{\rho} L_{\rho\lambda,\vec{\tau}-\hat{\rho}}}^{-J_{\lambda,\vec{\tau}}}} \times$$

$$\times \exp \left[-i 2\pi \sum_{\vec{\tau},\lambda} C_{\lambda,\vec{\tau}}^* G_{\lambda,\vec{\tau}}^* - \frac{e^{*2} \eta}{2} \sum_{\vec{\tau}} \sum_{\nu < \lambda} \left(\Delta_{\nu} C_{\lambda,\vec{\tau}-\hat{\nu}}^* - \Delta_{\lambda} C_{\nu,\vec{\tau}-\hat{\lambda}}^* \right)^2 \right]. \quad (\text{C.488})$$

As in the 3-dimensional case (C.464), one firstly introduces auxiliary field $\theta_{\vec{\tau}}^*$

$$\prod_{\vec{\tau}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{\tau}}, 0 \right) = \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau}} \frac{d\theta_{\vec{\tau}}^*}{2\pi} \right) \exp \left[i \sum_{\vec{\tau},\lambda} G_{\lambda,\vec{\tau}}^* \Delta_{\lambda} \theta_{\vec{\tau}}^* \right],$$

to implement the constraint on $G_{\lambda,\vec{\tau}}^*$, and then convergence factor $1 = \lim_{\mathcal{T} \rightarrow 0} \exp \left[-\frac{\mathcal{T}}{2} \sum_{\vec{\tau},\lambda} G_{\lambda,\vec{\tau}}^{*2} \right]$. Hence, after summing over $G_{\lambda,\vec{\tau}}^*$, we obtain

$$H(C)_{\text{comp.}} = \frac{1}{\mathcal{Z}^*} \lim_{\mathcal{T} \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\mu} \frac{d\Gamma_{\mu,\vec{\tau}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau}} \frac{d\theta_{\vec{\tau}}^*}{2\pi} \right) \exp \left[i \sum_{\vec{\tau},\mu} J_{\mu,\vec{\tau}} \Gamma_{\mu,\vec{\tau}}^* \right] \sum_{\{l_{\mu,\vec{\tau}}^*\}} \times$$

$$\times \exp \left[-\frac{q^2 e^{*2} \eta}{8\pi^2} \sum_{\vec{\tau},\mu < \nu} \left(\Delta_{\mu} \Gamma_{\nu,\vec{\tau}-\hat{\mu}}^* - \Delta_{\nu} \Gamma_{\mu,\vec{\tau}-\hat{\nu}}^* \right)^2 - \frac{1}{2\mathcal{T}} \sum_{\vec{\tau},\mu} \left(\Delta_{\mu} \theta_{\vec{\tau}}^* - q \Gamma_{\mu,\vec{\tau}}^* - 2\pi l_{\mu,\vec{\tau}}^* \right)^2 \right], \quad (\text{C.489})$$

with $\Gamma_{\mu,\vec{\tau}}^* = \frac{2\pi}{q} C_{\mu,\vec{\tau}}^*$. Partition function $\mathcal{Z}^*$ is given by

$$\mathcal{Z}^* = \lim_{\mathcal{T} \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\mu} \frac{d\Gamma_{\mu,\vec{\tau}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau}} \frac{d\theta_{\vec{\tau}}^*}{2\pi} \right) \sum_{\{l_{\mu,\vec{\tau}}^*\}} \times$$

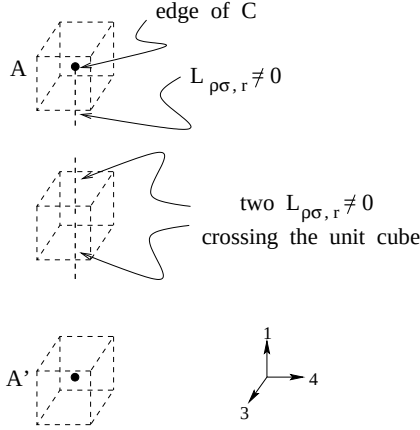
$$\times \exp \left[-\frac{q^2 e^{*2} \eta}{8\pi^2} \sum_{\vec{\tau},\mu < \nu} \left(\Delta_{\mu} \Gamma_{\nu,\vec{\tau}-\hat{\mu}}^* - \Delta_{\nu} \Gamma_{\mu,\vec{\tau}-\hat{\nu}}^* \right)^2 - \frac{1}{2\mathcal{T}} \sum_{\vec{\tau},\mu} \left(\Delta_{\mu} \theta_{\vec{\tau}}^* - q \Gamma_{\mu,\vec{\tau}}^* - 2\pi l_{\mu,\vec{\tau}}^* \right)^2 \right], \quad (\text{C.490})$$

and is, up to factor $\left(\sqrt{\frac{2\pi}{\mathcal{T}}} \right)^{4|\Lambda|}$, equal to the partition function of the non-compact Higgs model with $e^2 \eta = \frac{4\pi^2}{q^2 e^{*2} \eta}$ in the limit $\frac{a^2 |\psi|^2}{\eta} = \frac{1}{\mathcal{T}} \rightarrow \infty$ (limit which we have called the Peskin model). Equation (C.489) for its part, is equal to correlation function $W(C)$ in the Peskin model. Since the 4-dimensional pure compact $U(1)$ model shows a transition at $(e^{*2} \eta)_{\text{crit}} = 1.572$ (C.328), the critical point in the Peskin model is given by $(e^2 \eta)_{\text{crit}} = 25.11/q^2$.

The evaluation of $H(C)_{\text{comp.}}$ is straightforward. For large e^{*2} (small e^2 in the limit $|\psi|^2 = \infty$ of the non-compact Higgs model), we can write $H(C)_{\text{comp.}}$ (C.487) as follows

$$\frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\mu} \frac{dA_{\mu,\vec{\tau}}^*}{2\pi} \right) \exp \left[-\frac{1}{e^{*2} \eta} \sum_{\vec{\tau}} \sum_{\mu < \nu} \left[1 - \cos \left(\Delta_{\mu} A_{\nu,\vec{\tau}}^* - \Delta_{\nu} A_{\mu,\vec{\tau}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma,\vec{\tau}-\hat{\rho}} \right) \right] \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\mu} \frac{dA_{\mu,\vec{\tau}}^*}{2\pi} \right) \exp \left[-\frac{1}{e^{*2} \eta} \sum_{\vec{\tau}} \sum_{\mu < \nu} \left[1 - \cos \left(\Delta_{\mu} A_{\nu,\vec{\tau}}^* - \Delta_{\nu} A_{\mu,\vec{\tau}}^* \right) \right] \right]},$$

and evaluate it by expanding $\prod_{\vec{r}, \mu < \nu} \left[1 - \frac{1}{e^{*2}\eta} \frac{e^{i(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^*)} - \dots + e^{-i(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^*)} - \dots \right]$ in plaquette-factors $\frac{1}{2e^{*2}\eta} e^{i(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}})}$ (in the denominator, with $L_{\rho\sigma, \vec{r}} = 0$). The leading contributions are 1, which does not contain any plaquette-factor, and $\left(\frac{1}{2e^{*2}\eta}\right)^6$ for six plaquettes forming an unit cube.



Note that due to tensor $\epsilon_{\mu\nu\rho\sigma}$, which multiplies $L_{\rho\sigma, \vec{r}}$ in $\left(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}}\right)$, the unit cube is crossed by $L_{\rho\sigma, \vec{r}}$ instead of containing it. Let us suppose that C lies on the 1-2 plane, i.e., $L_{\rho\sigma, \vec{r}} = L_{12, \vec{r}}$. Each time this unit cube encircles a bond of C (points A and A' in the figure, where a section along axis 2 is represented), only one $L_{\rho\sigma, \vec{r}} \neq 0$ crosses one of the cube's plaquettes. If the cube is placed inside C , two non-zero $L_{\rho\sigma, \vec{r}}$ crossed the cube but one in the opposite direction to the other and thus the effect of $L_{\rho\sigma, \vec{r}}$ is compensated. In $H(C)_{\text{comp.}}$'s numerator, the cube can be placed in $4|\Lambda| - \mathcal{P}$ different forms without encircling C , and each of these positions produces 1. The remaining $\mathcal{P}$ positions produces a contribution $\left(\frac{1}{2e^{*2}\eta}\right)^6 e^{\pm i \frac{2\pi}{q}}$. Hence, correlation function $H(C)_{\text{comp.}}$ takes the following form

$$H(C)_{\text{comp.}}^{\text{large } e^{*2}} = \frac{1 + (4|\Lambda| - \mathcal{P}) \left(\frac{1}{2e^{*2}\eta}\right)^6 + \mathcal{P} \left(\frac{1}{2e^{*2}\eta}\right)^6 \left(\frac{e^{i \frac{2\pi}{q}}}{2} + \frac{e^{-i \frac{2\pi}{q}}}{2}\right) + \dots}{1 + 4|\Lambda| \left(\frac{1}{2e^{*2}\eta}\right)^6 + \dots},$$

$$H(C)_{\text{comp.}}^{\text{large } e^{*2}} = W(C)_{\text{comp.}}^{\text{small } e^2, |\psi|^2 = \infty} = \exp \left[-\mathcal{P} \left(\frac{q^2 e^2 \eta}{8\pi^2}\right)^6 \left(1 - \cos \frac{2\pi}{q}\right) \right]. \quad (\text{C.491})$$

On the other hand, we already know that at small e^{*2} (large e^2) the effect of $g_{\mu\nu, \vec{r}}^*$ can be neglected in the 4-dimensional pure compact model, and therefore, $H(C)_{\text{comp.}}$ is given by

$$H(C)_{\text{comp.}}^{\text{small } e^{*2}} = \frac{1}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}^*}{2\pi} \right) \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^*)^2 \right]} \times \int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}^*}{2\pi} \right) \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right)^2 \right]. \quad (\text{C.492})$$

After including term $\left(\sqrt{\frac{1}{2\pi e^{*2}\eta}}\right)^{|\Lambda|} \det(-\Delta\Delta) \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}, \nu} A_{\nu, \vec{r}}^* \left(-\Delta_\nu \sum_{\mu} \Delta_\mu A_{\mu, \vec{r}-\hat{\mu}}^* \right) \right]$ to remove the gauge invariance (and its accompanying infinities) in both numerator and denominator, the Gaussian integration over $dA_{\vec{r}}^*$ is calculated directly

$$-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}}^* - \Delta_\nu A_{\mu, \vec{r}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right)^2 - \frac{1}{2e^{*2}\eta} \sum_{\vec{r}, \nu} A_{\nu, \vec{r}}^* \left(-\Delta_\nu \sum_{\mu} \Delta_\mu A_{\mu, \vec{r}-\hat{\mu}}^* \right) =$$

$$= -\frac{1}{2e^{*2}\eta} \sum_{\vec{r},\mu} A_{\mu,\vec{r}}^* (-\Delta\Delta) A_{\mu,\vec{r}}^* - \frac{2\pi^2}{q^2 e^{*2}\eta} \sum_{\vec{r}} \sum_{\rho < \sigma} L_{\rho\sigma,\vec{r}}^2 + \frac{2\pi}{qe^{*2}\eta} \sum_{\vec{r},\mu} A_{\mu,\vec{r}}^* \sum_{\nu,\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} \Delta_\mu L_{\rho\sigma,\vec{r}-\hat{\rho}-\hat{\mu}} ,$$

which produces in the numerator the following exponent ⁵³

$$\begin{aligned} & -\frac{2\pi^2}{q^2 e^{*2}\eta} \underbrace{\sum_{\vec{r},\rho < \sigma} L_{\rho\sigma,\vec{r}}^2}_A + \frac{2\pi^2}{q^2 e^{*2}\eta} \underbrace{\sum_{\vec{r},\vec{r}'} \sum_{\mu,\nu} \sum_{\rho < \sigma} \sum_{\mu',\nu'} \sum_{\rho' < \sigma'} \epsilon_{\mu\nu\rho\sigma} \left(\Delta_{\vec{r}}^\mu L_{\rho\sigma,\vec{r}-\hat{\rho}-\hat{\mu}} \right) D_{\vec{r},\vec{r}'} \epsilon_{\mu'\nu'\rho'\sigma'} \left(\Delta_{\vec{r}'}^{\mu'} L_{\rho'\sigma',\vec{r}'-\hat{\rho}'-\hat{\mu}'} \right)}_{\sum_{\vec{r},\rho < \sigma} L_{\rho\sigma,\vec{r}}^2 - \sum_{\vec{r},\vec{r}',\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'}} \\ & = -\frac{e^2\eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} . \end{aligned} \quad (\text{C.493})$$

According to (C.327), $\sum_{\vec{r},\vec{r}',\mu} J_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} J_{\mu,\vec{r}'} \sim \mathcal{P} \left(D(0) + \frac{1}{2a\pi^2} \right) - \frac{1}{2\pi} \frac{\tau}{R}$ and thus

$$H(C)_{\text{comp.}}^{\text{small } e^{*2}} = W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2 = \infty} = \exp \left[-\frac{e^2\eta}{2} \mathcal{P} \left(D(0) + \frac{1}{2a\pi^2} \right) + \frac{e^2\eta}{4\pi} \frac{\tau}{R} \right] , \quad (\text{C.494})$$

with $D(0) = 0.155$.

Correlation function $W(C)_{\text{non-comp.}}$ fails to distinguish clearly between the two phases of the Peskin model, since it mainly decays like the perimeter in both phases. Since the model under study and its dual are gauge invariant model and the correlation functions are defined for a closed loop, let us consider in the Peskin model the correlation $H(\bar{C})_{\text{non-comp.}}$ similar to that in (C.487) for the pure compact model. The numerator of $H(\bar{C})_{\text{non-comp.}}$ is given by

$$\begin{aligned} & \lim_{\mathcal{T} \rightarrow 0} \int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{d\Gamma_{\mu,\vec{r}}^*}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\theta_{\vec{r}}^*}{2\pi} \right) \sum_{\{l_{\mu,\vec{r}}^*\}} \exp \left[-\frac{1}{2\mathcal{T}} \sum_{\vec{r},\mu} (\Delta_\mu \theta_{\vec{r}}^* - q\Gamma_{\mu,\vec{r}}^* - 2\pi l_{\mu,\vec{r}}^*)^2 \right] \times \\ & \times \exp \left[-\frac{1}{2e^2\eta} \sum_{\vec{r},\mu < \nu} \left(\Delta_\mu \Gamma_{\nu,\vec{r}-\hat{\mu}}^* - \Delta_\nu \Gamma_{\mu,\vec{r}-\hat{\nu}}^* - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} \mathcal{L}_{\rho\sigma,\vec{r}-\hat{\rho}} \right)^2 \right] , \end{aligned} \quad (\text{C.495})$$

where integer field $\mathcal{L}_{\rho\sigma,\vec{r}}$ defines the closed path $\bar{C}$, i.e., $\sum_{\rho} \Delta_\rho \mathcal{L}_{\rho\sigma,\vec{r}-\hat{\rho}} = \mathcal{J}_{\sigma,\vec{r}}$ with $\mathcal{J}_{\sigma,\vec{r}}$ the unit vector tangent to path $\bar{C}$. This correlation function is equal to $W(C)$ in the pure compact model as one can easily prove by repeating the same steps from (C.487) to (C.489): starting with

$$W(\bar{C})_{\text{compact}} = \frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{dA_{\mu,\vec{r}}^*}{2\pi} \right) \exp \left[i \sum_{\vec{r},\mu} \mathcal{J}_{\mu,\vec{r}} A_{\mu,\vec{r}}^* \right] \sum_{\{g_{\mu\nu,\vec{r}}^*\}} \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu,\vec{r}}^* - \Delta_\nu A_{\mu,\vec{r}}^* - 2\pi g_{\mu\nu,\vec{r}}^*)^2 \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r},\mu} \frac{dA_{\mu,\vec{r}}^*}{2\pi} \right) \sum_{\{g_{\mu\nu,\vec{r}}^*\}} \exp \left[-\frac{1}{2e^{*2}\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu,\vec{r}}^* - \Delta_\nu A_{\mu,\vec{r}}^* - 2\pi g_{\mu\nu,\vec{r}}^*)^2 \right]} ,$$

⁵³For instance, let us suppose that C lies on the 1-2 plane, i.e., $L_{12,\vec{r}} \neq 0$ for the plaquettes inside C . In this case

$$\sum_{\vec{r},\vec{r}'} \sum_{\mu,\nu} \sum_{\rho < \sigma} \sum_{\mu',\nu'} \sum_{\rho' < \sigma'} \epsilon_{\mu\nu\rho\sigma} \left(\Delta_{\vec{r}}^\mu L_{\rho\sigma,\vec{r}-\hat{\rho}-\hat{\mu}} \right) D_{\vec{r},\vec{r}'} \epsilon_{\mu'\nu'\rho'\sigma'} \left(\Delta_{\vec{r}'}^{\mu'} L_{\rho'\sigma',\vec{r}'-\hat{\rho}'-\hat{\mu}'} \right) = \sum_{\vec{r},\vec{r}'} L_{12,\vec{r}} \left(\Delta_{\vec{r}}^1 \Delta_{\vec{r}'}^1 + \Delta_{\vec{r}}^2 \Delta_{\vec{r}'}^2 \right) D_{\vec{r},\vec{r}'} L_{12,\vec{r}'} .$$

Since $\left(\Delta_{\vec{r}}^1 \Delta_{\vec{r}'}^1 + \Delta_{\vec{r}}^2 \Delta_{\vec{r}'}^2 \right) D_{\vec{r},\vec{r}'} = \delta_{\vec{r},\vec{r}'} - \left(\Delta_{\vec{r}}^3 \Delta_{\vec{r}'}^3 + \Delta_{\vec{r}}^4 \Delta_{\vec{r}'}^4 \right) D_{\vec{r},\vec{r}'}$, together with equation $\sum_{\rho} \Delta_\rho L_{\rho\sigma,\vec{r}-\hat{\rho}} = -J_{\sigma,\vec{r}}$, one obtains result (C.493).

and after introducing the auxiliary field $B_{\mu\nu,\vec{\tau}}^*$, the integration over $dA_{\mu,\vec{\tau}}^*$ produces the constraint $B_{\mu\nu,\vec{\tau}}^* = -\sum_{\tau,\lambda} \epsilon_{\mu\nu\tau\lambda} \Delta_\tau C_{\lambda,\vec{\tau}-\hat{\tau}}^* - \mathcal{L}_{\mu\nu,\vec{\tau}}$. Then (noting that $\exp(i2\pi \mathcal{L}_{\mu\nu,\vec{\tau}} g_{\mu\nu,\vec{\tau}}^*) = 1$ since $\mathcal{L}_{\mu\nu,\vec{\tau}}$ and $g_{\mu\nu,\vec{\tau}}^*$ are integers), by including $\int_{-\infty}^{\infty} \left(\prod_{\vec{\tau}} \frac{d\theta_{\vec{\tau}}^*}{2\pi} \right) \exp \left[i \sum_{\vec{\tau},\lambda} G_{\lambda,\vec{\tau}}^* \Delta_\lambda \theta_{\vec{\tau}}^* \right]$ and $1 = \lim_{\mathcal{T} \rightarrow 0} \exp \left[-\frac{\mathcal{T}}{2} \sum_{\vec{\tau},\lambda} G_{\lambda,\vec{\tau}}^{*2} \right]$, the unconstrained summation over $G_{\lambda,\vec{\tau}}^*$ gives (C.495) with $e^2 \eta = \frac{4\pi^2}{q^2 e^{*2} \eta}$. From the behaviour of $W(\bar{C})$ in the 4-dimensional pure compact model at large e^{*2} (area decay (C.256)) and small e^{*2} (perimeter decay plus correction τ/R (C.327)), we obtain two different behaviours for $H(\bar{C})_{\text{non-comp.}}$

$$\begin{aligned} H(\bar{C})_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2=\infty} &= \exp \left[-\bar{\mathcal{A}} \ln \left(\frac{8\pi^2}{q^2 e^2 \eta} \right) \right], \\ H(\bar{C})_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2=\infty} &= \exp \left[-\bar{\mathcal{P}} \frac{2\pi^2}{q^2 e^2 \eta} \left(D(0) + \frac{1}{2a\pi^2} \right) + \frac{\pi}{q^2 e^2 \eta} \frac{\bar{\tau}}{R} \right], \end{aligned} \quad (\text{C.496})$$

which permits us to distinguish between the two phases of limit $|\psi|^2 = \infty$ in the 4-dim. non-compact Higgs model.

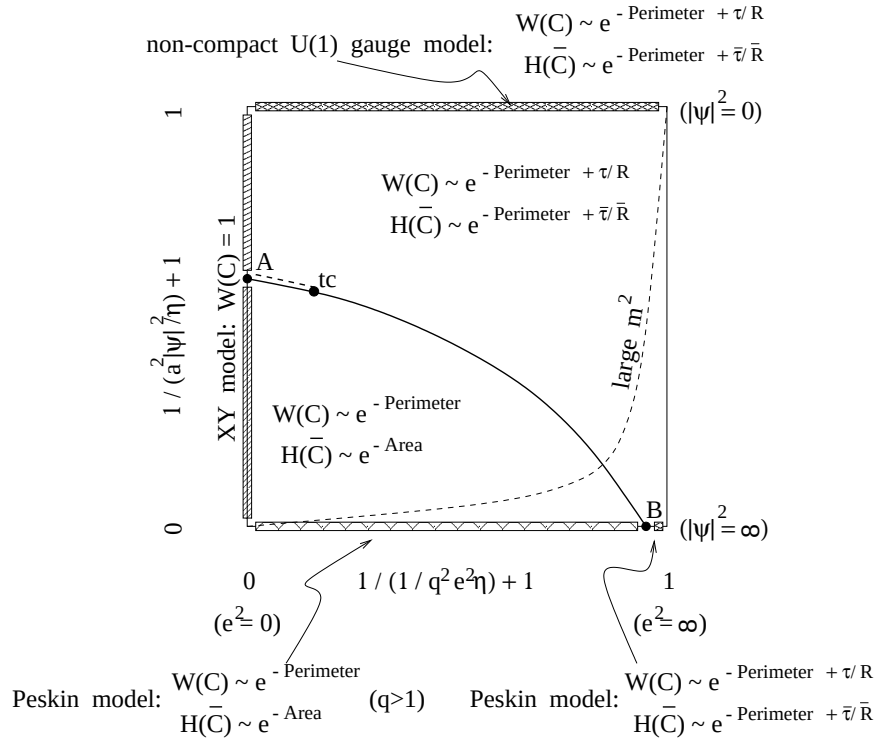


Figure C.9: Expected phase diagram for the 4-dimensional non-compact Higgs model for $q > 1$. Point A is the critical value of the 4-dimensional XY model: $(a^2|\psi|^2/\eta)_{\text{crit}} = 0.526$; point B denotes the critical value of the 4-dimensional Peskin model: $(q^2 e^2 \eta)_{\text{crit}} = 25.11$. The short solid-dashed part of line AB near to A denotes first order-transitions; the rest of AB is expected to be of second-order type. Point tc denotes the tricritical point separating the segment of first-order transitions from the line of transitions of second-order type. For $q = 1$ there is the minor change of $W(C) = 1$ at small e^2 and $|\psi|^2 = \infty$.

In the pure non-compact model (limit $|\psi|^2 = 0$), $W(C)_{\text{non-comp.}}$ is given by (C.475)

$$W(C)_{\text{non-comp.}}^{|\psi|^2=0} = \exp \left[-\frac{e^2 \eta}{2} \sum_{\vec{r}, \vec{r}', \mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] = \exp \left[-\frac{e^2 \eta}{2} \mathcal{P} \left(D(0) + \frac{1}{2a\pi^2} \right) + \frac{e^2 \eta}{4\pi} \frac{\tau}{R} \right], \quad (\text{C.497})$$

whilst $H(\bar{C})_{\text{non-comp.}}$ is calculated as in equation (C.492) (but for variables without $*$)

$$H(\bar{C})_{\text{non-comp.}}^{|\psi|^2=0} = \frac{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{2e^2 \eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} \mathcal{L}_{\rho\sigma, \vec{r} - \hat{\rho}} \right)^2 \right]}{\int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{2e^2 \eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 \right]},$$

and one obtains

$$H(\bar{C})_{\text{non-comp.}}^{|\psi|^2=0} = \exp \left[-\frac{2\pi^2}{q^2 e^2 \eta} \sum_{\vec{r}, \vec{r}', \mu} \mathcal{J}_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} \mathcal{J}_{\mu, \vec{r}'} \right] = \exp \left[-\frac{2\pi^2}{q^2 e^2 \eta} \bar{\mathcal{P}} \left(D(0) + \frac{1}{2a\pi^2} \right) + \frac{\pi}{q^2 e^2 \eta} \frac{\bar{\tau}}{R} \right]. \quad (\text{C.498})$$

Limits $|\psi|^2 = \infty$ and $|\psi|^2 = 0$ of the non-compact Higgs model allow us to characterize the expected model's phase diagram (Fig. C.9)

limit $|\psi|^2 = \infty$ in the non-compact Higgs model:

$$H(\bar{C}) = \exp \left[-\frac{2\pi^2}{q^2 e^2 \eta} \left(D(0) + \frac{1}{2a\pi^2} \right) \bar{\mathcal{P}} + \frac{\pi}{q^2 e^2 \eta} \frac{\bar{\tau}}{R} \right] \quad \text{large } e^2 \quad W(C) = \exp \left[-\frac{e^2 \eta}{2} \left(D(0) + \frac{1}{2a\pi^2} \right) \mathcal{P} + \frac{e^2 \eta}{4\pi} \frac{\tau}{R} \right], \quad (\text{C.499})$$

$$H(\bar{C}) = \exp \left[-\ln \left(\frac{8\pi^2}{q^2 e^2 \eta} \right) \bar{\mathcal{A}} \right] \quad \text{small } e^2 \quad W(C) = \exp \left[-\left(\frac{q^2 e^2 \eta}{8\pi^2} \right)^6 \left(1 - \cos \frac{2\pi}{q} \right) \mathcal{P} \right],$$

pure non-compact $U(1)$ model:

$$H(\bar{C}) = \exp \left[-\frac{2\pi^2}{q^2 e^2 \eta} \left(D(0) + \frac{1}{2a\pi^2} \right) \bar{\mathcal{P}} + \frac{\pi}{q^2 e^2 \eta} \frac{\bar{\tau}}{R} \right] \quad \text{any } e^2 \quad W(C) = \exp \left[-\frac{e^2 \eta}{2} \left(D(0) + \frac{1}{2a\pi^2} \right) \mathcal{P} + \frac{e^2 \eta}{4\pi} \frac{\tau}{R} \right]. \quad (\text{C.500})$$

The line connecting the critical point of the XY model (C.239) with the transition point of the Peskin model separates the phase where $H(\bar{C})$ decays like the area of $\bar{C}$ from the phase where $H(\bar{C})$ shows a weaker decay. From the interpretation of $H(\bar{C})$ as the energy (in this case, in the non-compact model) of a monopole loop, the state with $H(\bar{C}) \sim e^{-\text{Area}}$ corresponds to the superconducting phase: in the 3-dim. non-compact model we have shown that the energy of two point-monopoles inside a superconductor grows with the distance between them, and for a monopole loop in a 4-dim. system therefore, the loop's energy grows with the quadratic power of its radius, i.e., with the area enclosed by the loop. On the other hand, if the magnetic field created by the monopole loop can spread out, the interaction between two loop's elements decreases with the distance according to the Coulomb potential: the energy of the loop is given by equation (C.325) and is proportional to the loop's length. Note that the confinement of the magnetic field into a limited region is related to a $H(\bar{C})$'s decay stronger than in the case where the magnetic field can spread out [52]; since the behaviour of $W(C)$ is complementary to that of $H(\bar{C})$, we can expect a perimeter decay in $W(C)$ in the state where the magnetic field has a finite penetration length.

According to the discussion following equation (C.482), line AB is a line of second-order phase transitions except for the short segment near to A , where the transitions are of first-order type. This last characteristic is due to the difference in the interaction between the integer fields of the partition function

$$\mathcal{Z}_{\text{non-comp.}} = (\text{const.}) \sum_{\{l_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r} - \hat{\mu}}, 0 \right) \exp \left[-\frac{q^2 e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} l_{\mu, \vec{r}'} - \frac{\eta}{2a^2 |\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 \right].$$

resulting from the Villain approximation: very short-range for $e^2 = 0$, and long-range interactions for $e^2 \neq 0$. As in the 3-dimensional case, the existence of the phase transition along a line of large $m^2 = a^2 q^2 e^2 |\psi|^2$ even for $q = 1$ is the “footprint” of the non-compact character of the gauge field.

The compact Higgs model *at small e^2* shows a phase diagram similar to that of the non-compact model. This is due to the fact that in the pure $U(1)$ gauge model, monopole loops can be neglected at small e^2 and thus, the system can be approximated by the non-compact model for this e^2 range. It is worth noting that this is a property only of the 4-dimensional model, since in three dimensions, monopoles in the pure $U(1)$ model are not bounded in pairs either at small e^2 and their effect cannot be neglected (see the Polyakov’s argument in (C.314) or the more intuitive estimation in (C.296)); at small e^2 , the 3-dim. pure compact $U(1)$ model shows a behaviour different from the pure non-compact model. The expected phase diagram for the 4-dim. compact Higgs model is represented in Fig. C.10. The pure compact $U(1)$ model shows two phases, whose characteristics are easily obtained by noting that this model is dual to the Peskin model (limit $|\psi|^2 = \infty$ in the non-compact Higgs model). Thus, one only needs to change in (C.499) $H(\bar{C}) \leftrightarrow W(C)$ and to substitute $(e^2 \eta)_{\text{Peskin}}$ by $(e^2 \eta)_{\text{pure compact}}$ with $(e^2 \eta)_{\text{pure compact}} (e^2 \eta)_{\text{Peskin}} = 4\pi^2 / q^2$

pure compact $U(1)$ gauge model:

$$W(C) = \exp \left[-\frac{e^2 \eta}{2} \left(D(0) + \frac{1}{2a\pi^2} \right) \mathcal{P} + \frac{e^2 \eta}{4\pi} \frac{\tau}{R} \right] \quad \text{small } e^2 \quad H(\bar{C}) = \exp \left[-\frac{2\pi^2}{q^2 e^2 \eta} \left(D(0) + \frac{1}{2a\pi^2} \right) \bar{\mathcal{P}} + \frac{\pi}{q^2 e^2 \eta} \frac{\bar{\tau}}{R} \right], \quad (\text{C.501})$$

$$W(C) = \exp \left[-\ln(2e^2 \eta) \mathcal{A} \right] \quad \text{large } e^2 \quad H(\bar{C}) = \exp \left[-\left(\frac{1}{2e^2 \eta} \right)^6 \left(1 - \cos \frac{2\pi}{q} \right) \bar{\mathcal{P}} \right].$$

The critical point is given by $(e^2 \eta)_{\text{crit}} = 1.572$ (C.328). Limit $|\psi|^2 = \infty$ is the pure $Z(q)$ gauge model: for $q = 2$, it is self-dual (and also for $q = 3$ and $q = 4$ as we shall see later), and $W(C)$ and $H(\bar{C})$ show the following behaviours

pure $Z(2)$ gauge model:

$$W(C) = \exp \left[-2e^{-12/e^2 \eta} \mathcal{P} \right] \quad \text{small } e^2 \quad H(\bar{C}) = \exp \left[-\frac{2}{e^2 \eta} \bar{\mathcal{A}} \right], \quad (\text{C.502})$$

$$W(C) = \exp \left[-\ln(e^2 \eta) \mathcal{A} \right] \quad \text{large } e^2 \quad H(\bar{C}) = \exp \left[-2 \left(\frac{1}{e^2 \eta} \right)^6 \bar{\mathcal{P}} \right],$$

whilst its transition point is given by $(e^2 \eta)_{\text{crit}} = 0.441^{-1} = 2.268$ (C.368), denoted by point B in the right diagram of Fig. C.10, and is a first-order transition-point. For large q , the pure $Z(q)$ gauge model has an intermediate phase with characteristics similar to the phase of the pure $U(1)$ gauge model at small e^2 ; this corresponds to e^2 values between points B_1 and B_2 in the left diagram of Fig. C.10 (note that q^2 is also contained in the horizontal axis of this diagram and therefore, B_2 lies very near to B_1 for large q^2 although their corresponding transition points are distant from each other). Both points B_1 and B_2 are second-order transition-points for large q [29]. For $q \rightarrow \infty$, the pure $Z(q)$ model becomes the pure compact $U(1)$ model and the phase with $H(\bar{C}) \sim e^{-\text{Area}}$ disappears.

In addition to the phases at small e^2 , which are similar to those in the non-compact model⁵⁴, at large e^2 the compact nature of the gauge field becomes relevant and configurations which restore the “broken” gauge symmetry are possible: $W(C)$ decays therefore like the area⁵⁵. For $q = 1$, the free energy density and all correlation functions along a line of large $m^2 = a^2 q^2 e^2 |\psi|^2$ are analytic and therefore, the separating line DB disappears. Aside from the phase in the upper left corner of the diagram, where both $W(C)$ and $H(\bar{C})$ decays like the perimeter (plus correction τ/R), the rest of the

⁵⁴ Aside the use of $H(\bar{C})$ to distinguish between the two different phases of the compact model with a rather similar $W(C)$ ’s behaviour, there is another form to characterize these two phases, due to Münster [70].

⁵⁵ The critical point of the pure $Z(q)$ model, which separates the phase characterized by $W(C) \sim e^{-\text{Perimeter}}$ from the phase with $W(C) \sim e^{-\text{Area}}$, is always larger than the critical point of the pure compact $U(1)$ gauge model, since in the limit $|\psi|^2 \rightarrow 0$ there are more possible orientations for $S_{\mu, \vec{r}} = e^{i\vec{A}_{\mu, \vec{r}}}$ and thus configurations which can restore the “broken” symmetry are more easily produced. Here “broken” means that after choicing the configuration with all $S_{\mu, \vec{r}}$ -arrows aligned as representative for all possible gauge equivalent configurations at $|\psi|^2 = \infty$, the energy cost to produce many flipped arrows is too high.

diagram is characterized by $W(C) \sim e^{-\text{Perimeter}}$ (except for the pure $U(1)$ model where $W(C)$ decays like the area) and $H(\bar{C}) = 1$. This last result can be seen by noting that $H(\bar{C})$, (as well as $W(C)$ ⁵⁶) presents no phase transition along a line of large m^2 and in the upper right corner, where the compact character of the gauge field is relevant, the integer field $\mathcal{L}_{\rho\sigma,\vec{\tau}}$ in the numerator of $H(\bar{C})$ can be absorbed into the integer field $g_{\mu\nu,\vec{\tau}}$

$$H(\bar{C}) = \frac{\dots \sum_{\{g_{\mu\nu,\vec{\tau}}=-\infty\}}^{\{\infty\}} \exp \left[-\frac{1}{2e^2\eta} \sum_{\vec{\tau},\mu<\nu} \left(\Delta_\mu A_{\nu,\vec{\tau}} - \Delta_\nu A_{\mu,\vec{\tau}} - \overbrace{\frac{2\pi}{1} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} \mathcal{L}_{\rho\sigma,\vec{\tau}-\hat{\rho}} - 2\pi g_{\mu\nu,\vec{\tau}}}^{-2\pi g'_{\mu\nu,\vec{\tau}}} \right) \right] \dots}{\dots \sum_{\{g_{\mu\nu,\vec{\tau}}=-\infty\}}^{\{\infty\}} \exp \left[-\frac{1}{2e^2\eta} \sum_{\vec{\tau},\mu<\nu} \left(\Delta_\mu A_{\nu,\vec{\tau}} - \Delta_\nu A_{\mu,\vec{\tau}} - 2\pi g_{\mu\nu,\vec{\tau}} \right) \right] \dots} = 1$$

without changing the main property of $G_{\lambda,\vec{\tau}} = -\sum_{\tau} \sum_{\mu<\nu} \epsilon_{\lambda\tau\mu\nu} \Delta_\tau g_{\mu\nu,\vec{\tau}}$: $\sum_{\lambda} \Delta_\lambda G_{\lambda,\vec{\tau}} = 0$

$$G'_{\lambda,\vec{\tau}} = -\sum_{\tau} \sum_{\mu<\nu} \epsilon_{\lambda\tau\mu\nu} \Delta_\tau g'_{\mu\nu,\vec{\tau}} = -\underbrace{\sum_{\tau} \Delta_\tau \mathcal{L}_{\lambda\tau,\vec{\tau}-\hat{\tau}}}_{\mathcal{J}_{\lambda,\vec{\tau}}} - \sum_{\tau} \sum_{\mu<\nu} \epsilon_{\lambda\tau\mu\nu} \Delta_\tau g_{\mu\nu,\vec{\tau}},$$

$$\sum_{\lambda} \Delta_\lambda G'_{\lambda,\vec{\tau}} = -\sum_{\lambda} \Delta_\lambda \mathcal{J}_{\lambda,\vec{\tau}} + \sum_{\lambda} \Delta_\lambda G_{\lambda,\vec{\tau}} = 0.$$

The physical origin of $H(\bar{C}) = 1$ when $q = 1$ is the fact that the compact superconductor with $q = 1$ (the “ $SU(2)$ ”-superconductor) has no stable vortices; the monopoles introduced by hand (and whose energy measures $H(\bar{C}) = 1$) are for $q = 1$ equal to the monopoles which the system itself can produce and therefore the monopoles defining $H(\bar{C})$ are immediately surrounded by the system’s monopoles and the magnetic field of the former “disappears”. As we already know, the weaker is the $H(\bar{C})$ ’s decay, the easier is the spreading out of the magnetic field of its monopoles; in our case, this magnetic field gets completely diluted and therefore we can expect $H(\bar{C}) = 1$. Of course, this argument does not apply to the non-compact model, where all vortices of any winding number are stable and thus the magnetic field, created by the monopoles defining $H(\bar{C})$, is always confined in the superconducting phase, even for $q = 1$.

To finish the discussion on the 4-dim. compact Higgs model, one has to determine the nature of the transition across the lines which separate the different phases. Let us start with the line which begins at the transition point of the pure compact $U(1)$ gauge model (point E). This point is a second-order transition-point and as we shall now see, the second-order character remains even after being coupled to a scalar field ($|\psi|^2 \neq 0$). The analysis runs along the same line as the argument by Dasgupta and Halperin for the non-compact Higgs model: let us return to (C.481) (without changing $A_{\mu,\vec{\tau}}/2\pi \rightarrow A_{\mu,\vec{\tau}}$) and formulate it with a compact gauge field

$$\mathcal{Z} = \left(\sqrt{\frac{2\pi\eta}{a^2|\psi|^2}} \right)^{4|\Lambda|} \int_{-\infty}^{\infty} \left(\prod_{\vec{\tau},\mu} \frac{dA_{\mu,\vec{\tau}}}{2\pi} \right) \sum_{\{g_{\mu\nu,\vec{\tau}}\}} \exp \left[-\frac{1}{2e^2\eta} \sum_{\vec{\tau},\mu<\nu} (\Delta_\mu A_{\nu,\vec{\tau}} - \Delta_\nu A_{\mu,\vec{\tau}} - 2\pi g_{\mu\nu,\vec{\tau}})^2 \right] \times$$

$$\times \sum_{\{l_{\mu,\vec{\tau}}\}} \prod_{\vec{\tau}} \delta \left(\sum_{\mu} \Delta_\mu l_{\mu,\vec{\tau}-\hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a^2|\psi|^2} \sum_{\vec{\tau},\mu} l_{\mu,\vec{\tau}}^2 - iq \sum_{\vec{\tau},\mu} l_{\mu,\vec{\tau}} A_{\mu,\vec{\tau}} \right]. \quad (\text{C.503})$$

⁵⁶ And since the leading contribution to $W(C)$ for $q = 1$ in the upper right corner is, according to the “high temperature” expansion, $W(C) \sim e^{-\text{Perimeter}} + e^{-\text{Area}} \approx e^{-\text{Perimeter}}$ (for a large loop C), this is the $W(C)$ ’s decay in a whole strip covering the zone of large m^2 .

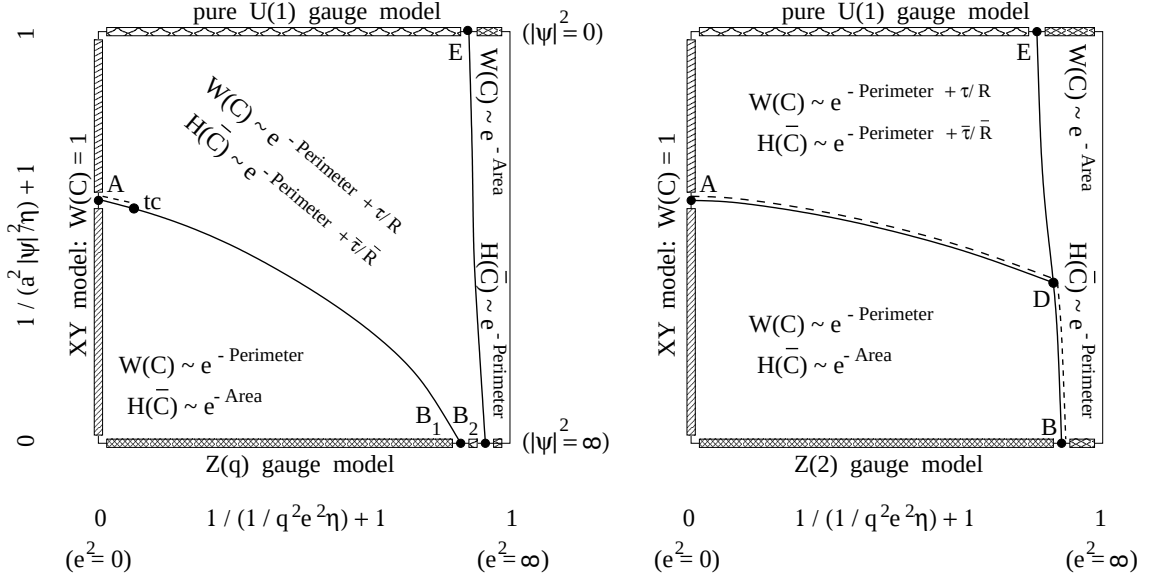


Figure C.10: Expected phase diagram for the 4-dimensional compact Higgs model for large q (left) and $q = 2$ (right). Point A is the critical value of the 4-dimensional XY model: $(a^2|\psi|^2/\eta)_{\text{crit}} = 0.526$; point B denotes the first-order transition point of the pure $Z(2)$ gauge model: $(e^2\eta)_{\text{crit}} = 2.268$; point E represents the second-order transition point of the pure compact $U(1)$ gauge model: $(e^2\eta)_{\text{crit}} = 1.572$. Point B_1 decreases like $1/q^2$ and point B_2 tends to the critical value of the pure compact $U(1)$ model when $q \rightarrow \infty$. Solid-dashed lines are lines of first-order phase-transitions and solid lines of second-order type. Point tc or D denotes the tricritical point separating the segment of first-order transitions from the line of transitions of second-order type.

The integration over $dA_{\mu,\vec{r}}$ can be done as follows: $\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\hat{\mu}} = 0$ tell us that integer field $l_{\mu,\vec{r}}$ forms closed loops, and thus the part depending on $A_{\mu,\vec{r}}$ in the above partition function is similar to the numerator of a correlation function $W(C)$ with closed path C defined by $q l_{\mu,\vec{r}}$. This correlation function would be evaluated by means of the partition function of the 4-dim. pure compact $U(1)$ gauge model, and looking at the result for $W(C)$ in (C.324), the partition function of the compact Higgs model can be also written as

$$\begin{aligned}
\mathcal{Z} = & \left(\right) \sum_{\{G_{\lambda,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2}{e^2\eta} \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'} \right] \times \\
& \times \sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{q^2 e^2 \eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} - \underbrace{\frac{\eta}{2a^2|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2}_{\text{underlined}} \right] \times \\
& \times \exp \left[-iq2\pi \sum_{\vec{r}',\vec{r},\rho} G_{\rho,\vec{r}'} D_{\vec{r}',\vec{r}} \sum_{\lambda,\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{\vec{r}}^{\lambda} l_{\mu\nu,\vec{r}} \right], \tag{C.504}
\end{aligned}$$

where integer field $l_{\mu\nu,\vec{r}}$ is related to $l_{\mu,\vec{r}}$ by $\sum_{\mu} \Delta_{\mu} l_{\mu\nu,\vec{r}-\hat{\mu}} = -l_{\nu,\vec{r}}$. A similar result can be found in [19].

For $|\psi|^2$ exactly zero, the underlined term in (C.504) forces $l_{\mu,\vec{r}} = 0 \quad \forall \vec{r}$ and we recover the partition

function of the pure gauge model. Term $2\pi \sum_{\vec{\tau}, \vec{\tau}'} G_{\rho, \vec{\tau}} D_{\vec{\tau}', \vec{\tau}} \sum_{\lambda, \mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{\tau})}^{\lambda} l_{\mu\nu, \vec{\tau}}$ is equal to $\pi |G_{\rho}| |l_{\mu}|$ each time monopole loop $G_{\rho, \vec{\tau}}$ is crossed by loop $l_{\mu, \vec{\tau}}$ (or $G_{\rho, \vec{\tau}}$ “encircles” in four dimensions loop $l_{\mu, \vec{\tau}}$; see discussion following equation (C.324)). Equivalently, this exponent can be also written as ⁵⁷

$$- \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu < \nu} g_{\mu\nu, \vec{\tau}} l_{\mu\nu, \vec{\tau}} + \sum_{\vec{\tau}, \vec{\tau}', \mu} l_{\mu, \vec{\tau}} D_{\vec{\tau}', \vec{\tau}} \sum_{\nu, \lambda < \rho} \epsilon_{\mu\nu\lambda\rho} \Delta_{(\vec{\tau})}^{\nu} G_{\lambda\rho, \vec{\tau}}, \quad (\text{C.505})$$

where integer field $G_{\lambda\rho, \vec{\tau}} = \sum_{\mu' < \nu'} \epsilon_{\lambda\rho\mu'\nu'} g_{\mu'\nu', \vec{\tau}}$ fulfils $\sum_{\lambda} \Delta_{\lambda} G_{\lambda\rho, \vec{\tau}} = G_{\rho, \vec{\tau}}$. Since both $g_{\mu\nu, \vec{\tau}}$ and $l_{\mu\nu, \vec{\tau}}$ are integer fields, term $2\pi \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu < \nu} g_{\mu\nu, \vec{\tau}} l_{\mu\nu, \vec{\tau}}$ does not contribute to the exponent of the last line of equation

(C.504), and thus, this exponent also becomes equal to $\pi |G_{\rho}| |l_{\mu}|$ each time loop $l_{\mu, \vec{\tau}}$ gets crossed by monopole loop $G_{\rho, \vec{\tau}}$. Let us study this option by considering a certain monopole loop $G_{\rho, \vec{\tau}}$. For small $|\psi|^2$, the only possible $l_{\mu, \vec{\tau}}$ ’s have to build *very small* loops. Therefore, contribution $e^{\pm i q \pi |G_{\rho}| |l_{\mu}|}$ can only be maintained for the perimeter of monopole loop $G_{\rho, \vec{\tau}}$, but not for its area, since the $l_{\mu, \vec{\tau}}$ ’s size is not sufficient large to cross plaquettes in the interior of the monopole loop and, simultaneously, to encircle $G_{\rho, \vec{\tau}}$. The leading contributions to sum $\sum_{\{l_{\mu, \vec{\tau}}\}}$ are produced by $l_{\mu, \vec{\tau}} = 0$ and by the smallest loops

(of length $\sum_{\vec{\tau}, \mu} l_{\mu, \vec{\tau}}^2 = 4 |l_{\mu}|^2$), which can at most encircle $G_{\rho, \vec{\tau}}$. Assuming that $|G_{\rho}| = 1$ and $|l_{\mu}| = 1$, the sum on $l_{\mu, \vec{\tau}}$ will give a term proportional to $\mathcal{P}_{G_{\rho}}$, the perimeter of $G_{\rho, \vec{\tau}}$

$$\begin{aligned} & 1 + \underbrace{\left(6|\Lambda| - \mathcal{P}_{G_{\rho}}\right) e^{-4\left(\frac{q^2 e^2 \eta D(0)}{2} + \frac{\eta}{2a^2 |\psi|^2}\right)}}_{\text{without encircling } G_{\rho}} + \underbrace{\mathcal{P}_{G_{\rho}} e^{-4\left(\frac{q^2 e^2 \eta D(0)}{2} + \frac{\eta}{2a^2 |\psi|^2}\right)} \left(\frac{e^{-iq\pi} + e^{+iq\pi}}{2}\right)}_{\text{encircling } G_{\rho}} + \dots \approx \\ & \approx (\text{constant}) \exp \left[-e^{-4\left(\frac{q^2 e^2 \eta D(0)}{2} + \frac{\eta}{2a^2 |\psi|^2}\right)} (1 - \cos(q\pi)) \mathcal{P}_{G_{\rho}} \right]. \end{aligned}$$

Since in four dimensions $\mathcal{P}_{G_{\rho}} \sim \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\rho} G_{\rho} D_{\vec{\tau}', \vec{\tau}} G_{\rho, \vec{\tau}}$ (see for instance equation (C.325)), the partition function of the compact Higgs model at a very small $|\psi|^2$ is, up to a factor, approximately equal to

$$\mathcal{Z} = \left(\right)_{\{G_{\lambda, \vec{\tau}}\}} \sum_{\vec{\tau}} \prod_{\vec{\tau}'} \delta \left(\sum_{\rho} \Delta_{\rho} G_{\rho, \vec{\tau}}, 0 \right) \exp \left[- \left(\frac{2\pi^2}{e^2 \eta} + k (1 - \cos q\pi) \right) \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\rho} G_{\rho, \vec{\tau}} D_{\vec{\tau}', \vec{\tau}} G_{\rho, \vec{\tau}'} \right], \quad (\text{C.506})$$

with $k \sim \exp(-2q^2 e^2 \eta D(0) - 2\eta/a^2 |\psi|^2)$ a very small value for $|\psi|^2 \approx 0$. Hence, the effect of coupling the pure gauge model to a scalar field is only a slight change in coupling constant e^2 for the *pure*

⁵⁷Remembering that $G_{\rho, \vec{\tau}} = - \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{\tau}}$, we consider the case where both $l_{\mu\nu, \vec{\tau}}$ and $g_{\mu\nu, \vec{\tau}}$ lie on the 1-2 plane: $\sum_{\vec{\tau}', \vec{\tau}, \rho} G_{\rho, \vec{\tau}'} D_{\vec{\tau}, \vec{\tau}'} \sum_{\lambda, \mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{\tau})}^{\lambda} l_{\mu\nu, \vec{\tau}} = \sum_{\vec{\tau}, \vec{\tau}'} \left[- \left(\Delta_{(\vec{\tau})}^4 g_{12, \vec{\tau}'} \right) D_{\vec{\tau}', \vec{\tau}} \left(\Delta_{(\vec{\tau})}^4 l_{12, \vec{\tau}} \right) - \left(\Delta_{(\vec{\tau})}^3 g_{12, \vec{\tau}'} \right) D_{\vec{\tau}', \vec{\tau}} \left(\Delta_{(\vec{\tau})}^3 l_{12, \vec{\tau}} \right) \right]$. From the definition of the propagator $\Delta_{(\vec{\tau})}^{\lambda} D_{\vec{\tau}', \vec{\tau}} = \delta_{\vec{\tau}', \vec{\tau}}$, the previous sum becomes

$$\begin{aligned} & - \sum_{\vec{\tau}, \vec{\tau}'} g_{12, \vec{\tau}'} l_{12, \vec{\tau}} + \sum_{\vec{\tau}, \vec{\tau}'} \left[\left(\Delta_{(\vec{\tau})}^1 g_{12, \vec{\tau}'} \right) D_{\vec{\tau}', \vec{\tau}} \left(\Delta_{(\vec{\tau})}^1 l_{12, \vec{\tau}} \right) + \left(\Delta_{(\vec{\tau})}^2 g_{12, \vec{\tau}'} \right) D_{\vec{\tau}', \vec{\tau}} \left(\Delta_{(\vec{\tau})}^2 l_{12, \vec{\tau}} \right) \right] = \\ & = - \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu < \nu} g_{\mu\nu, \vec{\tau}} l_{\mu\nu, \vec{\tau}} + \sum_{\vec{\tau}, \vec{\tau}', \mu} l_{\mu, \vec{\tau}} D_{\vec{\tau}', \vec{\tau}} \sum_{\nu, \lambda < \rho} \epsilon_{\mu\nu\lambda\rho} \Delta_{(\vec{\tau})}^{\nu} \left(\sum_{\mu' < \nu'} \epsilon_{\lambda\rho\mu'\nu'} g_{\mu'\nu', \vec{\tau}'} \right). \end{aligned}$$

compact $U(1)$ gauge model: the transition across the line which starts at the pure gauge model (point E) has the same character as at E ⁵⁸.

Case $|\psi|^2 = \infty$ (the pure $Z(q)$ gauge model) is described by the following partition function

$$\begin{aligned} \mathcal{Z}_{Z(q)} = & \left(\right) \sum_{\{G_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\rho} \Delta_{\rho} G_{\rho, \vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2}{e^2 \eta} \sum_{\vec{r}, \vec{r}'} \sum_{\rho} G_{\rho, \vec{r}} D_{\vec{r}, \vec{r}'} G_{\rho, \vec{r}'} \right] \times \\ & \times \sum_{\{l_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}}, 0 \right) \exp \left[-\frac{q^2 e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} l_{\mu, \vec{r}'} \right] \times \\ & \times \exp \left[-iq2\pi \sum_{\vec{r}', \vec{r}, \rho} G_{\rho, \vec{r}'} D_{\vec{r}', \vec{r}} \sum_{\lambda, \mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{r})}^{\lambda} l_{\mu\nu, \vec{r}} \right], \end{aligned} \quad (\text{C.507})$$

and since as a result of (C.505) the last exponent can be also written $q2\pi \sum_{\vec{r}, \vec{r}', \mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} \sum_{\nu, \lambda < \rho} \epsilon_{\mu\nu\lambda\rho} \Delta_{(\vec{r}')}^{\nu} G_{\lambda\rho, \vec{r}'}$,

the above model is self-dual

$$\left\{ \begin{array}{l} G_{\mu, \vec{r}} \\ l_{\mu, \vec{r}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} l_{\mu, \vec{r}} \\ G_{\mu, \vec{r}} \end{array} \right\} \quad \text{and} \quad e^2 \eta \longleftrightarrow \frac{4\pi^2}{q^2 e^2 \eta},$$

⁵⁸ Another argument to show that the second-order transition of the pure gauge model remains second-order for small but nonzero $|\psi|^2$ is due to Fradkin and Shenker [41], following an original idea by Wegner [97]. Let us return to the original

partition function for the case $q = 1$: $\mathcal{Z} = \int_{-\pi}^{\pi} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \int_{-\pi}^{\pi} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{e^2 \eta} \sum_{\vec{r}, \mu < \nu} [1 - \cos F_{\mu\nu, \vec{r}}] \right] \times$
 $\times \exp \left[-\frac{a^2 |\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos (\Delta_{\mu} \varphi_{\vec{r}} - A_{\mu, \vec{r}})] \right]$ with $F_{\mu\nu, \vec{r}} = \Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}$. At the lowest order of $|\psi|^2$, the

scalar field's phase can be integrated out, obtaining an effective energy for the gauge field: the “high temperature” expansion of the underlined part of the partition is, up to factor $\exp \left[-a^2 |\psi|^2 / \eta \sum_{\vec{r}, \mu} 1 \right]$, approximated

by $\int_{-\pi}^{\pi} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \prod_{\vec{r}, \mu} \left[1 + \frac{a^2 |\psi|^2}{\eta} \frac{e^{i(\Delta_{\mu} \varphi_{\vec{r}} - A_{\mu, \vec{r}})} + e^{-i(\Delta_{\mu} \varphi_{\vec{r}} - A_{\mu, \vec{r}})}}{2} \right]$ at small $|\psi|^2$. In the expansion of this product, aside 1, the leading contribution arises from the product of four bounds forming a 1×1 plaquette: $e^{i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - A_{\mu, \vec{r}})} \times e^{i(\varphi_{\vec{r}+\hat{\nu}} - \varphi_{\vec{r}} - A_{\nu, \vec{r}})} \times e^{i(\varphi_{\vec{r}+\hat{\mu}+\hat{\nu}} - \varphi_{\vec{r}+\hat{\mu}} - A_{\nu, \vec{r}+\hat{\mu}})} \times e^{i(\varphi_{\vec{r}+\hat{\mu}+\hat{\nu}} - \varphi_{\vec{r}+\hat{\nu}} - A_{\mu, \vec{r}+\hat{\nu}})}$, and thus $\prod_{\vec{r}} \prod_{\mu, \nu} \left[1 + \frac{1}{2^4} \left(\frac{a^2 |\psi|^2}{\eta} \right)^4 \left(e^{-i(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})} + e^{i(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})} \right) \right] \approx \exp \left[-\frac{1}{4} \left(\frac{a^2 |\psi|^2}{\eta} \right)^4 \sum_{\vec{r}, \mu < \nu} [1 - \cos F_{\mu\nu, \vec{r}}] \right]$.

Therefore, evaluating an expectation value with the whole (gauge+scalar) energy for small $|\psi|^2$ is the same as evaluating

it by means of only the pure gauge part but with $\left(\frac{1}{e^2 \eta} \right)_{\text{eff}} \approx \frac{1}{e^2 \eta} + \frac{1}{4} \left(\frac{a^2 |\psi|^2}{\eta} \right)^4$ instead of $\frac{1}{e^2 \eta}$. If we go to terms with

higher powers of $|\psi|^2$, we shall obtain not only interactions between the bonds of an unit plaquette but also interactions between many plaquettes. And these interactions between distant plaquettes may change the behaviour of the model at large distances. However, such terms are produced by contributions of the above “high temperature” expansion in which 1×1 squares are *linked* to each other by at least one vertex (otherwise, one of the $e^{i\varphi_{\vec{r}'}}$'s would appear only once and after integrating over it, the contribution would vanish); they contribute with $(|\psi|^2)^L$, with L the length of the loop which encloses these linked plaquettes, and the interaction term between distant plaquettes is exponentially damped by a factor $e^{-L \ln(1/|\psi|^2)}$ ($|\psi|^2$ small).

According to Wegner, if the transition at $|\psi|^2 = 0$ (point E) is of *second-order* kind, the nonanalytic part of the free energy density in the neighbourhood of the line which starts at E is given by $\mathcal{F}_{\text{sing}}(e^2, |\psi|^2) = \mathcal{F}_{\text{sing}}(e^2 - e_{\text{crit}}^2, |\psi|^2)$. Since for small $|\psi|^2$ the whole model can be approximated by the pure gauge model with the above effective coupling and the part of the free energy responsible for the transition depends only on this coupling, the curve starting at point E

is given by $\left(\frac{1}{e^2 \eta} \right)_{\text{crit}} (|\psi|^2) = \left(\frac{1}{e^2 \eta} \right)_{\text{crit}} (|\psi|^2 = 0) + \frac{1}{4} \left(\frac{a^2 |\psi|^2}{\eta} \right)^4$: the transition keeps the same character as in the pure gauge model (where it was a second-order phase-transition).

$$\mathcal{Z}_{\text{compact Higgs}}(|\psi|^2 = \infty, e^2\eta) = \mathcal{Z}_{\text{compact Higgs}}(|\psi|^2 = \infty, 4\pi^2/q^2 e^2\eta) . \quad (\text{C.508})$$

Therefore, if there is only one transition point, this must be given by $(e^2\eta)_{\text{crit}} = \left(\frac{4\pi^2}{q^2 e^2\eta}\right)_{\text{crit}}$, or equivalently, $(e^2\eta)_{\text{crit}} = 2\pi/q$. This last result is only a (good) estimation, since we are working with the Villain approximation: for example, we already know that the transition point of the 4-dimensional $Z(2)$ gauge model is $(e^2\eta)_{\text{crit}} \approx 2.7$, which is a bit smaller than result $(e^2\eta)_{\text{crit}} = \pi$. The property of self-duality, nevertheless, is correct.

However, from the discussion of the $Z(N)$ gauge model (see page 426), the model shows three phases for N large enough (actually $N \geq 5$). The two transition points which separates the three phases can be estimated as it was done for the 4-dim. pure compact $U(1)$ model. As one approaches the transition point where large $G_{\rho,\vec{r}}$ loops occur, we can neglect the mutual interaction between the monopole loops and approximate their energy by $\frac{2\pi^2}{e^2\eta} D(0) \sum_{\vec{r},\rho} G_{\rho,\vec{r}}^2 \sim \frac{2\pi^2}{e^2\eta} D(0) L$, with L the length

of the monopole loops (assuming that all of them have approximately the same large length). Since the number of closed loops of length L in four dimensions is bounded by 7^L , this transition takes place when $7^L \exp \left[-\frac{2\pi^2}{e^2\eta} D(0) \sum_{\vec{r},\rho} G_{\rho,\vec{r}}^2 \right] = \exp \left[L \left(\ln 7 - \frac{2\pi^2}{e^2\eta} D(0) \right) \right]$ becomes nonvanishing for large

L , and this holds for $e^2\eta \geq (e^2\eta)_{G_{\rho,\vec{r}} \text{ crit}} \approx \frac{2\pi^2 D(0)}{\ln 7} = 1.572$ (as in equation (C.328)⁵⁹). Similarly and by looking at (C.507), the transition where the size of $l_{\mu,\vec{r}}$ loops becomes large occurs when $e^2\eta \leq (e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}} \approx \frac{2 \ln 7}{q^2 D(0)} = \frac{25.11}{q^2}$; as expected, this transition point (denoted by B_1 in Fig. C.10) goes to zero for $q \rightarrow \infty$, since in this limit, the $Z(q)$ becomes the pure compact $U(1)$ model. The pure $Z(q)$ gauge model presents the following phases: a) $e^2\eta < (e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}}$ characterized by large $l_{\mu,\vec{r}}$ and small $G_{\rho,\vec{r}}$ loops; b) $(e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}} < e^2\eta < (e^2\eta)_{G_{\rho,\vec{r}} \text{ crit}}$ with both $l_{\mu,\vec{r}}$ and $G_{\rho,\vec{r}}$ small; and c) $e^2\eta > (e^2\eta)_{G_{\rho,\vec{r}} \text{ crit}}$ where the monopole loops are large but the $l_{\mu,\vec{r}}$ loops are small. Note that for $q \leq 4$, transition point $(e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}}$ passes over transition point $(e^2\eta)_{G_{\rho,\vec{r}} \text{ crit}}$ and the intermediate phase does not take place: the system shows only the self-dual transition point $(e^2\eta)_{\text{crit}} = 2\pi/q$ [95]. These results are also supported by the Monte Carlo simulations studied by Creutz, Jacobs and Rebbi [29].

Let us now study the stability of $Z(2)$ -model's transition point $(e^2\eta)_{\text{crit}} = 0.441^{-1}$ and of point $(e^2\eta)_{\text{crit}} = (e^2\eta)_{G_{\rho,\vec{r}} \text{ crit}}$ (points B and B_2 in Fig. C.10) when $|\psi|^2$ is very large but finite: $\frac{\eta}{2a^2|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2$

does not longer forbid a large size in the $l_{\mu,\vec{r}}$ loops and the only term which controls this size is $q^2 e^2\eta/2 \sum_{\vec{r},\vec{r}',\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'}$. At these two points, monopole loops $G_{\rho,\vec{r}}$ are becoming large and thus

$$\sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{q^2 e^2\eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} - i q 2\pi \sum_{\vec{r}',\vec{r},\rho} G_{\rho,\vec{r}'} D_{\vec{r}',\vec{r}} \sum_{\lambda,\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{\vec{r}}^{\lambda} l_{\mu\nu,\vec{r}} \right]$$

will give an exponential of area $\mathcal{A}_{G_{\rho}}$ enclosed by $G_{\rho,\vec{r}}$ or of $G_{\rho,\vec{r}}$'s perimeter $\mathcal{P}_{G_{\rho}}$ depending on whether $l_{\mu,\vec{r}}$ loops are large or not, respectively. Since at point B in the pure $Z(2)$ model $(e^2\eta)_{\text{crit}} < (e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}}$,

the result of this sum is $\exp \left[- \left(\right) \mathcal{A}_{G_{\rho}} \right] = \exp \left[- \left(\right) \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} \mathcal{D}'_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'} \right]$ with $\mathcal{D}'_{\vec{r},\vec{r}'}$ a propaga-

tor which *increases* with distance $|\vec{r}'' - \vec{r}'|$, as for instance $\ln |\vec{r}'' - \vec{r}'|$ (see (C.321)). This means that the large distance behaviour is not longer carried by the 4-dim. Coulomb propagator $1/|\vec{r}'' - \vec{r}'|^2$, and such a modification at long distances indicates that the transition point at $|\psi|^2 = \infty$ becomes unstable and changes its character when $|\psi|^2 < \infty$. This was actually expected since the transition

⁵⁹ Again, this is only an estimation since the transition point where $W(C) \sim e^{-\text{Area}}$ begins must be a bit larger in the $Z(q)$ model than in the $U(1)$ model. The reader may remember that an area decay in $W(C)$ indicates the restoring of the "broken" gauge symmetry, and since an $U(1)$ symmetry has more possible orientations for $e^{iA_{\mu,\vec{r}}}$ than a $Z(q)$ symmetry, this restoring occurs in the $U(1)$ model for a smaller value.

at B is already of the first-order type. For large q , and for e^2 between B_1 and ∞ , the pure $Z(q)$ gauge model shows a behaviour very similar to the pure compact $U(1)$ model and therefore is approximately described by $\sum_{\{G_{\lambda,\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}}, 0\right) \exp\left[-\frac{2\pi^2}{e^2\eta} \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'}\right]$. Further, since at point B_2 , $(e^2\eta)_{\text{crit}} > (e^2\eta)_{l_{\mu,\vec{r}} \text{ crit}}$ for large q , the $l_{\mu,\vec{r}}$ loops are small, and for $|\psi|^2 < \infty$ the result of sum $\sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\vec{\mu}}, 0\right) \exp\left[-\frac{q^2 e^2 \eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} - iq2\pi \sum_{\vec{r}',\vec{r},\rho} G_{\rho,\vec{r}'} D_{\vec{r},\vec{r}'} \sum_{\lambda,\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{r})}^{\lambda} l_{\mu\nu,\vec{r}}\right]$ is $\exp\left[-\left(\right) \mathcal{P}_{G_{\rho}}\right] = \exp\left[-\left(\right) \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'}\right]$: the long distance interactions remain as in the pure $Z(q)$ gauge model and the transition near to B_2 keeps the character of the transition at B_2 (point which like B_1 is a second-order transition point).

On the other hand, the pure $Z(q)$ model for large q and with e^2 between 0 and B_1 is approximately described by $\sum_{\{l_{\mu,\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\mu} \Delta_{\mu} l_{\mu,\vec{r}-\vec{\mu}}, 0\right) \exp\left[-\frac{q^2 e^2 \eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'}\right]$ since at small e^2 , monopole fields $G_{\rho,\vec{r}}$ play no role because they form small loops. Noting that at point B_1 the $l_{\mu,\vec{r}}$ loops start becoming large, we return to (C.504) and for $|\psi|^2 < \infty$, we can sum over the small sized $G_{\rho,\vec{r}}$ for a certain fixed large $l_{\mu,\vec{r}}$. We obtain thus an exponential decay like the perimeter of $l_{\mu,\vec{r}}$, since if a small $G_{\rho,\vec{r}}$ “wants” to cross the interior of $l_{\mu,\vec{r}}$ (in order to produce a $e^{\pm iq\pi|G_{\rho}||l_{\mu}|}$ contribution) and simultaneously encircle loop $l_{\mu,\vec{r}}$, it can at most achieve this for the plaquettes located just at the $l_{\mu,\vec{r}}$ ’s boundary, i.e., for a number of plaquettes proportional to the perimeter of $l_{\mu,\vec{r}}$:

$$\sum_{\{G_{\rho,\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}}, 0\right) \exp\left[-\frac{2\pi^2}{e^2\eta} \sum_{\vec{r},\vec{r}'} \sum_{\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'} - iq2\pi \sum_{\vec{r}',\vec{r},\rho} G_{\rho,\vec{r}'} D_{\vec{r},\vec{r}'} \sum_{\lambda,\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{(\vec{r})}^{\lambda} l_{\mu\nu,\vec{r}}\right] \sim \sim \exp\left[-\left(\right) \mathcal{P}_{l_{\mu}}\right] = \exp\left[-\left(\right) \sum_{\vec{r},\vec{r}'} \sum_{\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'}\right].$$

Hence, the effect of a large but finite $|\psi|^2$ at B_1 is only to modify the value of e^2 but the long distance behaviour is still carried by Coulomb propagator $D_{\vec{r},\vec{r}'}$: the transition point is of the same order as at point B_1 .

As in the non-compact model, the transition point of the XY model changes its character when the gauge field is switched on: limit $e^2 = 0$ in the partition function of the compact Higgs model $\dots \exp\left[-\frac{2\pi^2}{e^2\eta} \sum_{\vec{r},\vec{r}',\rho} G_{\rho,\vec{r}} D_{\vec{r},\vec{r}'} G_{\rho,\vec{r}'} - \frac{q^2 e^2 \eta}{2} \sum_{\vec{r},\vec{r}',\mu} l_{\mu,\vec{r}} D_{\vec{r},\vec{r}'} l_{\mu,\vec{r}'} - \frac{\eta}{2a^2|\psi|^2} \sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2\right] \dots$ forces $G_{\rho,\vec{r}} = 0 \forall \vec{r}$ and therefore, only term $\sum_{\vec{r},\mu} l_{\mu,\vec{r}}^2$ remains. This represents an interaction of $l_{\mu,\vec{r}}$ with $l_{\mu,\vec{r}'}$ only when $\vec{r}' = \vec{r}$ (a very short range interaction), whereas for $e^2 \neq 0$, there exist long range interactions carried by $D_{\vec{r},\vec{r}'}$. Summarizing, in the 4-dimensional compact Higgs model

$$\begin{aligned} &\text{for the pure gauge models (the } Z(q) \text{ and the compact } U(1)), \text{ the transition points} \\ &\text{which are of } \textit{second-order} \text{ kind, keep their second-order character also} \\ &\text{for a large } |\psi|^2 < \infty \text{ and for a small } |\psi|^2 > 0. \end{aligned} \tag{C.509}$$

This closes the discussion about the phase diagram of Fig. C.10.

Finally, we shall briefly consider the 2-dimensional case, which shows properties different from those of the 3-dimensional and 4-dimensional model and whose expected phase diagram is given in Fig. C.11. Beginning with the compact model, the pure $U(1)$ model has no phase transition point and always shows an area law, not due to the compact character of the gauge field but to the Coulomb propagator, which increases with the distance (see equations (C.320) and (C.321)). Also the pure $Z(2)$ gauge model

(limit $|\psi|^2 = \infty$ of the compact Higgs model) shows no phase transition (see equations (C.343) and (C.347)) and $W(C) \sim e^{-\text{Area}}$

pure $U(1)$ gauge model:

$$W(C) = \exp \left[-\frac{a^2 e^2 \eta}{2} \mathcal{A} \right] \quad \text{small } e^2, \quad (\text{C.510})$$

$$W(C) = \exp \left[-\ln(2a^2 e^2 \eta) \mathcal{A} \right] \quad \text{large } e^2,$$

pure $Z(2)$ gauge model:

$$W(C) = \exp \left[-\ln(a^2 e^2 \eta) \mathcal{A} \right] \quad \text{small } e^2, \quad (\text{C.511})$$

$$W(C) = \exp \left[-4e^{-2/a^2 e^2 \eta} \mathcal{A} \right] \quad \text{large } e^2.$$

Since by decreasing $|\psi|^2$, the number of possible $S_{\mu, \vec{r}} = e^{i A_{\mu, \vec{r}}}$'s orientations increases and this makes easier to disorder the system, the decay above the line of $|\psi|^2 = \infty$ must be stronger than that in the $Z(2)$ gauge model and weaker than in the pure $U(1)$ gauge model. This leads to the fact that for $q \neq 1$ (for instance, $q = 2$), the whole diagram (except for the XY limit) is filled with an unique phase characterized by $W(C) \sim e^{-\text{Area}}$. For $q = 1$ and since limit $|\psi|^2 = \infty$ has no transition, any correlation function is analytic and $W(C) \sim e^{-\text{Perimeter}}$ in the whole diagram (except for the pure $U(1)$ gauge model, where $W(C) \sim e^{-\text{Area}}$ remains). This can be easily proved by noting that in the upper right corner, the “high temperature” expansion gives an perimeter law for $W(C)$ (the bonds of C are covered by bond-factors $e^{i(\Delta_\mu \varphi_{\vec{r}} - A_{\mu, \vec{r}})}$), whereas near to $|\psi|^2 = \infty$ the behaviour of $W(C)$ is dominated

by $\exp \left[-\frac{a^2 e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} \right]$. Short-range propagator $D_{\vec{r}, \vec{r}'}^{(m^2)}$ is approximately [35] given by

$$D_{\vec{r}, \vec{r}'}^{(m^2)} \approx \frac{e^{-m|\vec{r} - \vec{r}'|}}{2\pi \sqrt{m|\vec{r} - \vec{r}'|}} \quad \text{for large } |\vec{r} - \vec{r}'|. \quad (\text{C.512})$$

In order to calculate $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'}$, we proceed as in the 3-dimensional case (C.415) and approximate $D_{\vec{r}, \vec{r}'}^{(m^2)}$ by $1/2\pi \sqrt{m|\vec{r} - \vec{r}'|}$ for distances smaller than range m^{-1} and by zero for distances larger than the range, obtaining thus for a large path C with $\tau \gg R \gg m^{-1}$

$$I_R^{(m^2)} = 2 \int_a^R d\xi_1 \int_{\xi_1 - m^{-1}}^{\xi_1 - a} d\xi_2 \frac{1}{2\pi \sqrt{m^{-1}(\xi_1 - \xi_2)}} \approx \frac{1}{2\pi} R \sqrt{m^{-1}},$$

$$I_{\tau\tau}^{(m^2)} = - \int_0^\tau d\xi_1 \int_{\xi_1 - m^{-1}}^{\xi_1 + m^{-1}} d\xi_2 \frac{1}{2\pi [R^2 + (\xi_1 - \xi_2)]^{1/4}} \sim \frac{\tau}{R},$$

where the last integral involves elliptic integrals of the second kind but whose leading contribution for $R \gg m^{-1}$ is ⁶⁰ of the order of $\tau/\sqrt{R}$ and therefore, much smaller than the contributions arising from $I_R^{(m^2)}$ or $I_\tau^{(m^2)}$

$$\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} J_{\mu, \vec{r}'} = 2 \left(I_R^{(m^2)} + I_\tau^{(m^2)} + I_{\tau\tau}^{(m^2)} + I_{RR}^{(m^2)} \right) \approx \frac{1}{2\pi} \mathcal{P} \sqrt{m^{-1}}, \quad (\text{C.513})$$

also a decay like the perimeter $\mathcal{P} = 2\tau + 2R$.

$$\int_{-m^{-1}}^{m^{-1}} \frac{dy}{(R^2 + y^2)^{1/4}} = \sqrt{R} \int_{-m^{-1}/R}^{m^{-1}/R} \frac{dx}{\sqrt{1 + x^2}} \approx \sqrt{R} \int_{-m^{-1}/R}^{m^{-1}/R} dx \left[1 - \frac{1}{4} x^2 \right] \approx 2 \frac{m^{-1}}{\sqrt{R}}$$

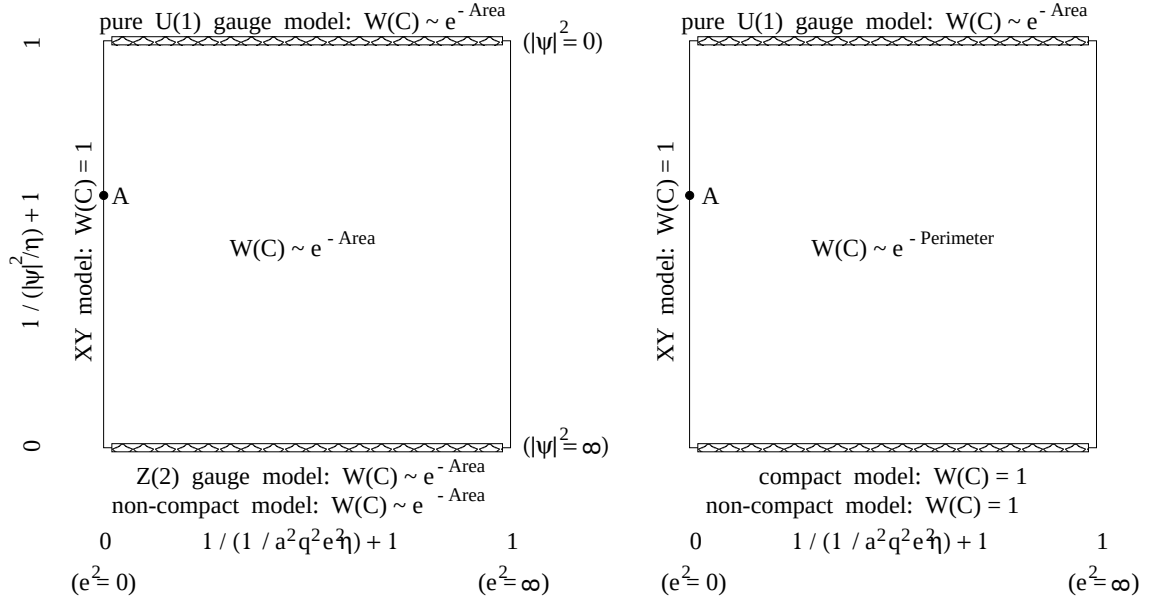


Figure C.11: Expected phase diagram for the 2-dimensional compact and non-compact Higgs model: left, for $q = 2 \neq p = 1$, and right, for $q = 1$. Point A denotes the Kosterlitz-Thouless transition of the 2-dimensional XY model: $(|\psi|^2/\eta)_{\text{crit}} = 0.637$.

Since in two dimensions the compact character of the gauge field plays no role in the pure $U(1)$ gauge model, and it is just this model what determines many of the characteristics of the phase diagram, we can expect that for the non-compact model a phase structure similar to that of the compact case. Limit $|\psi|^2 = 0$ is the same as in the compact model (C.510). Let us study limit $|\psi|^2 = \infty$, i.e., the Peskin model

$$W(C)_{\text{non-comp.}} = \lim_{|\psi|^2 \rightarrow \infty} \frac{1}{Z_{\text{non-comp.}}} \int_{-\infty}^{\infty} \left(\prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) e^{i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}}} \sum_{\{m_{\mu, \vec{r}}\}} \times \\ \times \exp \left[-\frac{1}{2a^2 e^2 \eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 - \frac{|\psi|^2}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}})^2 \right]. \quad (\text{C.514})$$

Specific of the 2-dimensional case, there is only one area enclosed by C (in higher dimensions, C encloses any open surface whose border is C) and this allows us to write

$$\sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} = \sum_{\vec{r}} \left[\left(-\Delta_2 L_{21, \vec{r}-\hat{2}} \right) A_{1, \vec{r}} + \left(-\Delta_1 L_{12, \vec{r}-\hat{1}} \right) A_{\vec{r}} \right] = \sum_{\vec{r}} L_{12, \vec{r}} \left(\underbrace{\Delta_1 A_{2, \vec{r}} - \Delta_2 A_{1, \vec{r}}}_{F_{12, \vec{r}}} \right).$$

Correlation $W(C)$ is calculated by means of the same techniques as in the Peskin model in other dimensions: firstly, by introducing an auxiliary field $b_{\mu, \vec{r}}$

$$\left(\prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \right) \left(\prod_{\vec{r}, \mu} \frac{db_{\mu, \vec{r}}}{2\pi} \right) e^{-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}})},$$

secondly by integrating over $d\varphi_{\vec{r}}$ which produces $b_{\mu, \vec{r}} = \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}}$, and then followed by the integration over $dA_{\mu, \vec{r}}$, which in this 2-dimensional case can be substituted by the integration over $dF_{12, \vec{r}}$

$$\left(\prod_{\vec{r}} \frac{da_{\vec{r}}}{2\pi} \right) e^{-\frac{\eta}{2|\psi|^2} \sum_{\vec{r}} a_{\vec{r}} (-\Delta \Delta) a_{\vec{r}} - \frac{a^2 e^2 \eta}{2} \sum_{\vec{r}} (q a_{\vec{r}} - L_{12, \vec{r}})^2 - i \sum_{\vec{r}} a_{\vec{r}} M_{\vec{r}}},$$

with $M_{\vec{r}} = \epsilon_{\nu\mu} \Delta_\nu m_{\mu,\vec{r}}$ the density of (point) dislocations/vortices. After integrating over $da_{\vec{r}}$, one obtains

$$W(C)_{\text{non-comp.}} = \lim_{|\psi|^2 \rightarrow \infty} \frac{\exp \left[-\frac{a^2 e^2 \eta}{2} \sum_{\vec{r}} L_{12,\vec{r}}^2 \right]}{\sum_{\{M_{\vec{r}}\}} \exp \left[-\frac{2\pi^2 |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} M_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\vec{r}'} \right]} \times$$

$$\times \sum_{\{M_{\vec{r}}\}} \exp \left[-\frac{2\pi^2 |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \left(M_{\vec{r}} + iq \frac{a^2 e^2 \eta}{2\pi} L_{12,\vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(M_{\vec{r}'} + iq \frac{a^2 e^2 \eta}{2\pi} L_{12,\vec{r}'} \right) \right],$$

and since $|\psi|^2 = \infty = m^2$, one can approximate $D_{\vec{r}, \vec{r}'}^{(m^2)} \approx \frac{1}{m^2} \delta_{\vec{r}, \vec{r}'}$

$$W(C)_{\text{non-comp.}}^{|\psi|^2=\infty} = \frac{\sum_{\{M_{\vec{r}}\}} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} \sum_{\vec{r}} M_{\vec{r}}^2 - i \frac{2\pi}{q} \sum_{\vec{r}} M_{\vec{r}} L_{12,\vec{r}} \right]}{\sum_{\{M_{\vec{r}}\}} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} \sum_{\vec{r}} M_{\vec{r}}^2 \right]} =$$

$$= \frac{\left(\sum_{M=-\infty}^{\infty} e^{i \frac{2\pi}{q} M} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} M^2 \right] \right)^{\mathcal{A}} \left(\sum_{M=-\infty}^{\infty} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} M^2 \right] \right)^{|\Lambda| - \mathcal{A}}}{\left(\sum_{M=-\infty}^{\infty} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} M^2 \right] \right)^{\mathcal{A}} \left(\sum_{M=-\infty}^{\infty} \exp \left[-\frac{2\pi^2}{a^2 q^2 e^2 \eta} M^2 \right] \right)^{|\Lambda| - \mathcal{A}}} \quad (\text{C.515})$$

According to the denominator, the point dislocations are not bounded by a logarithmic potential (there is only a contact interaction) and therefore all $M_{\vec{r}}$'s are isolated and no transition occurs between a phase where dislocations are bounded into pairs to a phase with free dislocation points. This can be also obtained by calculating (C.515): for large e^2 , and for instance $q = 2$, one can repeat the same steps as in (C.156) obtaining thus

$$W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2=\infty} = \exp[-\kappa \mathcal{A}] \quad \text{with} \quad \kappa \approx \ln(2a^2 e^2 \eta). \quad (\text{C.516})$$

On the other hand, large $|M_{\vec{r}}|$ are suppressed for small e^2 and thus, $W(C)$ for $M_{\vec{r}} = -1, 0, 1$ takes the following form

$$W(C)_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2=\infty} \approx \frac{\left(1 + \left(e^{i \frac{2\pi}{q}} + e^{-i \frac{2\pi}{q}} \right) e^{-2\pi^2/a^2 q^2 e^2 \eta} \right)^{\mathcal{A}}}{\left(1 + 2e^{-2\pi^2/a^2 q^2 e^2 \eta} \right)^{\mathcal{A}}} \approx \frac{\exp \left[2e^{-2\pi^2/a^2 q^2 e^2 \eta} \cos \frac{2\pi}{q} \mathcal{A} \right]}{\exp \left[2e^{-2\pi^2/a^2 q^2 e^2 \eta} \mathcal{A} \right]},$$

$$W(C)_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2=\infty} \approx \exp \left[-2e^{-2\pi^2/a^2 q^2 e^2 \eta} \left(1 - \cos \frac{2\pi}{q} \right) \mathcal{A} \right]. \quad (\text{C.517})$$

According to (C.515), it is clear that for $q = 1$, $W(C)_{\text{non-comp.}} = 1$. It is worth noting that the 2-dimensional case for $q \neq 1$ shows no phase with a perimeter decay in $W(C)$, unlike in the higher dimensional models (see also [26]).

C.6 The lattice compact Villain-Higgs model in several different gauges

To finish this Appendix on lattice gauge models and for the sake of completeness, let us consider the compact Higgs model in the Villain approximation and under another gauge choice for $a_{\lambda,\vec{r}}$ different from (C.394). It will be shown that the final result (C.412) obtained in the previous section is completely independent of the choice which we have made.

Let us come back to general result (C.194)

$$\left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}}\right)^{3|\Lambda|} \left(\sqrt{\frac{ae^2\eta}{2\pi}}\right)^{3|\Lambda|} \prod_{\vec{r},\lambda} \int da_{\lambda,\vec{r}} \prod_{\vec{r}} \int dC_{\vec{r}} \exp \left[-\frac{aq^2e^2\eta}{2} \sum_{\vec{r},\mu<\nu} L_{\mu\nu,\vec{r}}^2 \right] \times \\ \times \sum_{\{g_{\mu\nu,\vec{r}}\}} \sum_{\{M_{\lambda,\vec{r}}\}} \prod_{\vec{r}} \det \left(\frac{\partial F_{\vec{r}}(a_{\lambda}, C)}{\partial \chi_{\vec{r}}} \right) \prod_{\vec{r}} \delta(F_{\vec{r}}(a_{\lambda}, C)) \exp \left[-\frac{\eta}{2a|\psi|^2} E[a_{\lambda,\vec{r}}, C_{\vec{r}}] \right], \quad (\text{C.518})$$

where $-\frac{\eta}{2a|\psi|^2} E[a_{\lambda,\vec{r}}, C_{\vec{r}}]$ for the Higgs model, according to equations (C.384), (C.385) and (C.386), is given by

$$-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \left(-\sum_{\nu} \Delta_{\nu} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + \sum_{\nu} \Delta_{\lambda} \Delta_{\nu} a_{\nu,\vec{r}-\hat{\lambda}} \right) - i2\pi \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} M_{\lambda,\vec{r}} - \\ -\frac{aq^2e^2\eta}{2} \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}}^2 - \frac{ae^2\eta}{2} \sum_{\vec{r}} C_{\vec{r}} \left(-\sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} \right) - \underline{aqe^2\eta \sum_{\vec{r},\lambda} a_{\lambda,\vec{r}} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}} - \\ -ape^2\eta \sum_{\vec{r},\lambda} \sum_{\mu<\nu} (qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}) \epsilon_{\lambda\mu\nu} L_{\mu\nu,\vec{r}} - i2\pi \sum_{\vec{r},\lambda} \sum_{\mu<\nu} (qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}) \epsilon_{\lambda\mu\nu} g_{\mu\nu,\vec{r}}. \quad (\text{C.519})$$

This energy is invariant under the following transformation

$$\begin{cases} a_{\lambda,\vec{r}} \rightarrow a_{\lambda,\vec{r}} - \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}} \\ C_{\vec{r}} \rightarrow C_{\vec{r}} + q\chi_{\vec{r}} \end{cases} \Rightarrow qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} \rightarrow qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}, \quad (\text{C.520})$$

as it can be easily shown: the first line of (C.519) is invariant since $\sum_{\lambda} \Delta_{\lambda} M_{\lambda,\vec{r}} = 0$; the second line can be rewritten as $-\frac{ae^2\eta}{2} \sum_{\vec{r}} \sum_{\lambda} (qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}})^2$ and together with the third line, depends only on $qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}$, which is gauge invariant. This is not surprising, since the terms in the second and third line arise from $B_{\mu\nu,\vec{r}}^2$ and this field depends on $qa_{\lambda,\vec{r}} + \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}}$ (C.382). The functional $F_{\vec{r}}(a_{\lambda}, C)$ in (C.518) is chosen as follows

$$F_{\vec{r}}(a_{\lambda}, C) = \sum_{\lambda} \Delta_{\lambda} a_{\lambda,\vec{r}} + \alpha a^2 q e^2 |\psi|^2 C_{\vec{r}}, \quad (\text{C.521})$$

which under a gauge transformation of the kind (C.520) transforms as

$$F_{\vec{r}}(a_\lambda, C) \rightarrow F_{\vec{r}}(a_\lambda, C) + \left(-\Delta\Delta + \alpha a^2 q^2 e^2 |\psi|^2 \right) \chi_{\vec{r}},$$

with α an arbitrary parameter, on which the final result does not depend, as we shall see. The reason for this choice will become clear in equation (C.523): it cancels the underlined term in equation (C.519) which couples the integration variable $a_{\lambda, \vec{r}}$ with $C_{\vec{r}}$. Hence the integration over $da_{\lambda, \vec{r}}$ can be carried out independent of $dC_{\vec{r}}$, as distinct from the previous section, where the choice has not cancelled this mixing term and the propagator calculation of (C.402) was rather complicated. We obtain then

$$\prod_{\vec{r}} \det \left(\frac{\partial F_{\vec{r}}(a_\lambda, C)}{\partial \chi_{\vec{r}}} \right) = \det \left(-\Delta\Delta + \alpha m^2 \right) \quad \text{with} \quad m^2 = a^2 q^2 e^2 |\psi|^2. \quad (\text{C.522})$$

In order to implement the product of Dirac's deltas $\prod_{\vec{r}} \delta(F_{\vec{r}}(a_\lambda, C))$ we shall use the same method as in (C.197): since the correlation is independent of the choice made, we introduce a function $f_{\vec{r}}$ inside the deltas, multiply the correlation by an arbitrary functional $G[f_{\vec{r}}]$, integrate over $\prod_{\vec{r}} df_{\vec{r}}$ and finally divide the result by $\mathcal{N} = \prod_{\vec{r}} \int df_{\vec{r}} G[f_{\vec{r}}]$.

In the Villain approximation, we are dealing with Gaussian integrals and therefore the functional $G[f_{\vec{r}}]$ is the same as in equation (C.201): $G[f_{\vec{r}}] = \exp \left[-\frac{1}{\alpha} \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} f_{\vec{r}}^2 \right]$, with

$\mathcal{N} = \left(\sqrt{\frac{\alpha 2\pi a |\psi|^2}{\eta}} \right)^{|\Lambda|}$. The correlation $W(C)$ with this general gauge fixing factor takes thus the following form

$$\begin{aligned} W(C) &= \frac{1}{\mathcal{Z}} \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{4|\Lambda|} \left(\sqrt{\frac{a e^2 \eta}{2\pi}} \right)^{3|\Lambda|} \frac{1}{(\sqrt{\alpha})^{|\Lambda|}} \det \left(-\Delta\Delta + \alpha m^2 \right) \times \\ &\times \prod_{\vec{r}, \lambda} \int da_{\lambda, \vec{r}} \prod_{\vec{r}} \int dC_{\vec{r}} \exp \left[-\frac{a q^2 e^2 \eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 \right] \sum_{\{g_{\mu\nu, \vec{r}}\}} \sum_{\{M_{\lambda, \vec{r}}\}} \times \\ &\times \exp \left[-\frac{\eta}{2a|\psi|^2} E[a_{\lambda, \vec{r}}, C_{\vec{r}}] - \frac{1}{\alpha} \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} \left(\sum_{\lambda} \Delta_{\lambda} a_{\lambda, \vec{r}} + \alpha a^2 q e^2 |\psi|^2 C_{\vec{r}} \right)^2 \right]. \quad (\text{C.523}) \end{aligned}$$

The partition function $\mathcal{Z}$ is similar but with all $L_{\mu\nu, \vec{r}}$ equal to zero. Note that the term $-a q e^2 \eta \sum_{\vec{r}} \sum_{\lambda} \Delta_{\lambda} a_{\lambda, \vec{r}} C_{\vec{r}} = +a q e^2 \eta \sum_{\vec{r}} \sum_{\lambda} a_{\lambda, \vec{r}} \Delta_{\lambda} C_{\vec{r} - \hat{\lambda}}$ arising from the last sum, cancels the underlined term in (C.519).

Looking at equations (C.519) and (C.523), the term quadratic in $a_{\lambda, \vec{r}}$ is equal to

$$-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}} \sum_{\lambda, \nu} a_{\lambda, \vec{r}} \left[-\sum_{\rho} \Delta_{\rho} \Delta_{\rho} \delta_{\lambda\nu} a_{\nu, \vec{r} - \hat{\rho}} + \left(1 - \frac{1}{\alpha} \right) \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r} - \hat{\lambda}} + m^2 \delta_{\lambda\nu} a_{\nu, \vec{r}} \right]. \quad (\text{C.524})$$

In order to carry out the (Gaussian) integration over $da_{\lambda, \vec{r}}$, it is necessary to obtain the propagator and the determinant of the discrete differential operator in (C.524): to

achieve this, we shall work with the Fourier transforms. Hence, when applied to the Fourier components $\tilde{a}_\lambda$, the differential operator takes the form

$$\left(6 - 2 \sum_\rho \cos k_\rho\right) \delta_{\lambda\nu} + \left(1 - \frac{1}{\alpha}\right) e^{-ik_\lambda} (e^{ik_\lambda} - 1)(e^{ik_\nu} - 1) + m^2 \delta_{\lambda\nu}. \quad (\text{C.525})$$

For the following, we shall abbreviate $6 - 2 \sum_\rho \cos k_\rho$ by Σ . Let us firstly evaluate the determinant of the matrix (C.525)

$$\begin{aligned} & \begin{vmatrix} m^2 + \Sigma + \frac{\alpha-1}{\alpha} e^{-ik_x} (e^{ik_x} - 1)^2 & \frac{\alpha-1}{\alpha} e^{-ik_x} (e^{ik_x} - 1)(e^{ik_y} - 1) & \frac{\alpha-1}{\alpha} e^{-ik_x} (e^{ik_x} - 1)(e^{ik_z} - 1) \\ \frac{\alpha-1}{\alpha} e^{-ik_y} (e^{ik_y} - 1)(e^{ik_x} - 1) & m^2 + \Sigma + \frac{\alpha-1}{\alpha} e^{-ik_y} (e^{ik_y} - 1)^2 & \frac{\alpha-1}{\alpha} e^{-ik_y} (e^{ik_y} - 1)(e^{ik_z} - 1) \\ \frac{\alpha-1}{\alpha} e^{-ik_z} (e^{ik_z} - 1)(e^{ik_x} - 1) & \frac{\alpha-1}{\alpha} e^{-ik_z} (e^{ik_z} - 1)(e^{ik_y} - 1) & m^2 + \Sigma + \frac{\alpha-1}{\alpha} e^{-ik_z} (e^{ik_z} - 1)^2 \end{vmatrix} \\ &= \frac{1}{\alpha} \left(m^2 + 6 - 2 \sum_\rho \cos k_\rho\right)^2 \left(\alpha m^2 + 6 - 2 \sum_\rho \cos k_\rho\right). \end{aligned} \quad (\text{C.526})$$

The determinant of the differential operator in (C.524) is the product over all $\vec{k}$ of the determinant (C.526), $\prod_{\vec{k}} \left[\frac{1}{\alpha} \left(m^2 + 6 - 2 \sum_\rho \cos k_\rho\right)^2 \left(\alpha m^2 + 6 - 2 \sum_\rho \cos k_\rho\right) \right]$, equal to

$$\left(\frac{1}{\alpha}\right)^{|\Lambda|} \left[\det(-\Delta\Delta + m^2) \right]^2 \left[\det(-\Delta\Delta + \alpha m^2) \right]. \quad (\text{C.527})$$

The propagator is obtained by calculating the inverse of the matrix, whose determinant we have calculated. For instance, the inverse of the (1, 1) and (1, 2) elements are

$$\frac{\Sigma + \alpha m^2 + (1 - \alpha) e^{-ik_x} (e^{ik_x} - 1)^2}{(\Sigma + \alpha m^2)(\Sigma + m^2)} \quad \text{and} \quad \frac{(1 - \alpha)(e^{ik_x} - 1)e^{-ik_y}(e^{ik_y} - 1)}{(\Sigma + \alpha m^2)(\Sigma + m^2)},$$

and the propagator of the discrete differential equation (C.524) is given by ⁶¹

$$\delta_{\lambda\rho} D_{\vec{r},\vec{x}}^{(m^2)} + (1 - \alpha) \sum_{\vec{r}'} D_{\vec{r},\vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r}')}^\lambda \Delta_{(\vec{r})}^\rho D_{\vec{r}'-\hat{\lambda},\vec{x}}^{(m^2)}. \quad (\text{C.528})$$

⁶¹One can also show it by direct calculation (where the $-\hat{\lambda}$ in the site-index of the propagators has been eliminated in order to simplify the notation)

$$\begin{aligned} & \left[\delta_{\lambda\rho} \left(-\Delta_{(\vec{r})}^\lambda \Delta_{(\vec{r})}^\rho + m^2 \right) + \left(1 - \frac{1}{\alpha} \right) \Delta_{(\vec{r})}^\lambda \Delta_{(\vec{r})}^\rho \right] \left(\delta_{\lambda\rho} D_{\vec{r},\vec{x}}^{(m^2)} + (1 - \alpha) \sum_{\vec{r}'} D_{\vec{r},\vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r}')}^\lambda \Delta_{(\vec{r})}^\rho D_{\vec{r}',\vec{x}}^{(m^2)} \right) = \\ &= \delta_{\nu\rho} \delta_{\vec{r},\vec{x}} + \left(1 - \frac{1}{\alpha} \right) \Delta_{(\vec{r})}^\rho \Delta_{(\vec{r})}^\nu D_{\vec{r},\vec{x}}^{(m^2)} + (1 - \alpha) \sum_{\vec{r}'} \underbrace{\left(-\sum_\rho \Delta_{(\vec{r})}^\rho \Delta_{(\vec{r})}^\rho + m^2 \right)}_{\delta_{\vec{r},\vec{r}'} + (1 - \alpha) m^2 D_{\vec{r},\vec{r}'}^{(\alpha m^2)}} D_{\vec{r},\vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r}')}^\nu \Delta_{(\vec{r})}^\rho D_{\vec{r}',\vec{x}}^{(m^2)} - \\ & \quad - \frac{(1 - \alpha)^2}{\alpha} \sum_{\vec{r}'} \underbrace{\left(-\sum_\lambda \Delta_{(\vec{r})}^\lambda \Delta_{(\vec{r}')}^\lambda D_{\vec{r},\vec{r}'}^{(\alpha m^2)} \right)}_{-\delta_{\vec{r},\vec{r}'} + \alpha m^2 D_{\vec{r},\vec{r}'}^{(\alpha m^2)}} \left(+\Delta_{(\vec{r}')}^\nu \Delta_{(\vec{r})}^\rho D_{\vec{r}',\vec{x}}^{(m^2)} \right) = \delta_{\nu\rho} \delta_{\vec{r},\vec{x}}. \end{aligned}$$

The integration over $da_{\lambda,\vec{r}}$ produces then a factor outside the integral over $dC_{\vec{r}}$

$$\left(\sqrt{\frac{2\pi a|\psi|^2}{\eta}}\right)^{3|\Lambda|} (\sqrt{\alpha})^{|\Lambda|} \frac{1}{\det(-\Delta\Delta + m^2)\sqrt{\det(-\Delta\Delta + \alpha m^2)}}, \quad (\text{C.529})$$

which multiplies the already existing factor $\left(\sqrt{\frac{\eta}{2\pi a|\psi|^2}}\right)^{4|\Lambda|} \left(\sqrt{\frac{a\epsilon^2\eta}{2\pi}}\right)^{3|\Lambda|} \frac{\det(-\Delta\Delta + \alpha m^2)}{(\sqrt{\alpha})^{|\Lambda|}}$ in equation (C.523). Moreover, this integration produces the following exponent

$$-\frac{a|\psi|^2}{2\eta} \left\{ \sum_{\vec{r}, \vec{x}, \lambda} \left[2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqa\epsilon^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right] D_{\vec{r}, \vec{x}}^{(m^2)} \right]_{\lambda, \vec{x}} + \quad (\text{C.530})$$

$$(\alpha - 1) \sum_{\vec{r}, \vec{x}, \lambda, \rho} \left[2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqa\epsilon^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right] \sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{x})}^{\rho} D_{\vec{r}' - \hat{\lambda}, \vec{x} - \hat{\rho}}^{(m^2)} \left]_{\rho, \vec{x}} \right\}$$

where the property $\Delta_{(\vec{r})}^{\nu} D_{\vec{r}, \vec{r}'} = -\Delta_{(\vec{r}')}^{\nu} D_{\vec{r}, \vec{r}' - \hat{\nu}}$ was used, relation which holds for every propagator $D_{\vec{r}, \vec{r}'}$ depending on the distance $|\vec{r} - \vec{r}'|$. The last term in (C.530) containing the product of two propagators can be rewritten in a more useful form if we consider

$$\sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{x})}^{\rho} D_{\vec{r}' - \hat{\lambda}, \vec{x} - \hat{\rho}}^{(m^2)} = + \sum_{\vec{r}'} \left(\Delta_{(\vec{r})}^{\lambda} D_{\vec{r} - \hat{\lambda}, \vec{r}'}^{(\alpha m^2)} \right) \Delta_{(\vec{r}')}^{\rho} D_{\vec{r}', \vec{x} - \hat{\rho}}^{(m^2)}$$

obtained after summation by parts, and we use the following property

$$D_{\vec{r}, \vec{x}}^{(m^2)} - D_{\vec{r}, \vec{x}}^{(\alpha m^2)} = (\alpha - 1)m^2 \sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(\alpha m^2)} D_{\vec{r}', \vec{x}}^{(m^2)}, \quad (\text{C.531})$$

which is proved by noting that the respective Fourier transforms of $D_{\vec{r}, \vec{x}}^{(m^2)}$ and $D_{\vec{r}, \vec{x}}^{(\alpha m^2)}$ are $(\Sigma + m^2)^{-1}$ and $(\Sigma + \alpha m^2)^{-1}$. With these results, the second line of (C.530) becomes as follows

$$\sum_{\vec{r}, \vec{x}} \left[\right]_{\lambda, \vec{r}} \sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(\alpha m^2)} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{x})}^{\rho} D_{\vec{r}' - \hat{\lambda}, \vec{x} - \hat{\rho}}^{(m^2)} \left[\right]_{\rho, \vec{x}} = + \sum_{\vec{r}, \vec{x}} \Delta_{(\vec{r})}^{\lambda} \left[\right]_{\lambda, \vec{r}} \underbrace{\sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(\alpha m^2)} D_{\vec{r}', \vec{x}}^{(m^2)} \Delta_{(\vec{x})}^{\rho}}_{\frac{D_{\vec{r}, \vec{x}}^{(m^2)} - D_{\vec{r}, \vec{x}}^{(\alpha m^2)}}{(\alpha - 1)m^2}} \left[\right]_{\rho, \vec{x}}. \quad (\text{C.532})$$

The content of $\Delta_{(\vec{r})}^{\lambda} \left[\right]_{\lambda, \vec{r}}$ can be further simplified by considering the definition of field $M_{\lambda, \vec{r}} = \epsilon_{\lambda\mu\nu} \Delta_{\nu} m_{\mu, \vec{r}}$, with $\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} = 0$, and of field $G_{\vec{r}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$. The exponent's terms arising from the integration over $da_{\lambda, \vec{r}}$ are finally given in a concise form by

$$-\frac{a|\psi|^2}{2\eta} \sum_{\vec{r}, \vec{x}, \lambda} \left(2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqa\epsilon^2 \eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{x}}^{(m^2)} \left(\right)_{\lambda, \vec{x}} -$$

$$-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \vec{x}} \left(2\pi G_{\vec{r}} - ipae^2\eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} \right) \left(D_{\vec{r}, \vec{x}}^{(m^2)} - D_{\vec{r}, \vec{x}}^{(\alpha m^2)} \right) \left(\right)_{\vec{x}}. \quad (\text{C.533})$$

Looking back at (C.519) and (C.523), the term quadratic in $C_{\vec{r}}$ is equal to

$$-\frac{ae^2\eta}{2} \sum_{\vec{r}} C_{\vec{r}} \left(-\sum_{\lambda} \Delta_{\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} + m^2 C_{\vec{r}} \right). \quad (\text{C.534})$$

Therefore, the integration over $dC_{\vec{r}}$ produces for its part the following exponent

$$-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \vec{x}} \left(-2\pi G_{\vec{r}} - ipae^2\eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{x}}^{(\alpha m^2)} \left(\right)_{\vec{x}}, \quad (\text{C.535})$$

and a factor equal to

$$\left(\sqrt{\frac{2\pi}{ae^2\eta}} \right)^{|\Lambda|} \frac{1}{\sqrt{\det(-\Delta\Delta + \alpha m^2)}}, \quad (\text{C.536})$$

multiplying the whole result. The partition function has the following aspect

$$\begin{aligned} \mathcal{Z} = \mathcal{Z}_0 \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \sum_{\{G_{\vec{r}}\}} \times \\ \times \left\{ \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left(M_{\lambda, \vec{r}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(M_{\lambda, \vec{r}'} + q \sum_{\mu' < \nu'} \epsilon_{\lambda\mu'\nu'} g_{\mu'\nu', \vec{r}'} \right) \right] \right. \\ \left. \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} G_{\vec{r}'} \right] \right\} \end{aligned} \quad (\text{C.537})$$

According to equations (C.523), (C.529) and (C.536), $\mathcal{Z}_0$ is given by

$$\mathcal{Z}_0 = \left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{|\Lambda|} \left(\sqrt{\frac{ae^2\eta}{2\pi}} \right)^{2|\Lambda|} \frac{1}{\det(-\Delta\Delta + m^2)}, \quad (\text{C.538})$$

which corresponds to the partition function of the Higgs model for a gauge field without compact character. Finally, correlation function $W(C)$ is equal to

$$\begin{aligned} W(C) = \frac{\mathcal{Z}_0}{\mathcal{Z}} \exp \left[-\frac{p^2 ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} L_{\mu\nu, \vec{r}}^2 \right] \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \sum_{\{G_{\vec{r}}\}} \times \\ \times \left\{ \exp \left[-\frac{a |\psi|^2}{2\eta} \sum_{\vec{r}, \vec{r}', \lambda} \left(2\pi M_{\lambda, \vec{r}} + 2\pi q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu, \vec{r}} - ipqae^2\eta \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\right)_{\lambda, \vec{r}'} \right] \right. \\ \left. \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \vec{r}'} \left(2\pi G_{\vec{r}} - ipae^2\eta \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\right)_{\vec{r}'} \right] \right\} \end{aligned} \quad (\text{C.539})$$

These three last results are equal respectively to equations (C.410), (C.411) and (C.412) of the previous section, although we have started from another different choice; such a choice was even dependent on a parameter α which do not appear in the final results. This was to expect, since the partition and correlation function are not dependent on the gauge chosen.

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 Numbers in *italic* refer to footnotes.

Appendix D

Finite Temperature Effects

In the previous Appendix we have seen the different phases shown by the system under study in dependence on the lattice dimensions, on the character compact/noncompact of the gauge field and on the two parameters of the model, $|\psi|^2$ and e^2 . All those phase diagrams hold for temperature $T = 0$. Now we shall see how these diagrams change and which phases disappear when the temperature is increased. We shall follow some chapters of three classical books on Statistical Physics, chapter 12 of G. Parisi's book [78], chapter 20 of J. Zinn-Justin's book [99] and chapter 15 of M. Creutz' book [28], as well as the report by J. B. Kogut [59]. In the first part of this Appendix, the technique of the transfer matrix will be introduced, which allows us to explain in our dislocation model in an easy way the disappearance of Stage II for high temperatures. In the second and last part, the effect of a finite temperature in the model with non-compact gauge field will be considered, following the works by Dolan & Jackiw [31], by Kirzhnits & Linde [58], and by Kapusta [56].

D.1 The Transfer Matrix $\hat{\mathcal{T}}$ and the Hamiltonian formulation in Statistical Mechanics

Let us start by considering a 1-dimensional model in a *periodic* lattice of N sites with lattice spacing a whose energy is due to the interaction of neighbour sites

$$\frac{E}{\eta} = \frac{1}{\eta} \left[\frac{J}{2a} \sum_{j=1}^N (\sigma_{j+1} - \sigma_j)^2 + a \sum_{j=1}^N V(\sigma_j) \right], \quad (\text{D.1})$$

where σ_j is in general a continuous variable at the site denoted by j , and the periodicity condition means $\sigma_{N+1} = \sigma_1$. With this energy one can calculate the partition function as it has been explained in Appendix C. The method of the transfer matrix converts this calculation into the computation of eigenvalues of a linear operator acting on a Hilbert space. Starting from the above energy (D.1), let us define the integral operator $\hat{\mathcal{T}}$ acting on an arbitrary function $\psi(\sigma)$

$$\hat{\mathcal{T}}\psi(\sigma) \stackrel{\text{def}}{=} \int d\rho \mathcal{T}(\sigma, \rho) \psi(\rho), \quad (\text{D.2})$$

with integral (symmetrical) kernel $\mathcal{T}(\sigma, \rho)$ given by

$$\mathcal{T}(\sigma, \rho) = \mathcal{T}(\rho, \sigma) = \exp \left[-\frac{J}{2a\eta}(\sigma - \rho)^2 - \frac{a}{2\eta}V(\sigma) - \frac{a}{2\eta}V(\rho) \right]. \quad (\text{D.3})$$

This operator $\hat{\mathcal{T}}$ is called the *transfer matrix*, where the name matrix arises from the matrix representation for the case of a variable σ with discrete values. It fulfils that the trace of $\hat{\mathcal{T}}^2$ is finite, where $\hat{\mathcal{T}}^2$ has as kernel $\int d\sigma' \mathcal{T}(\sigma, \sigma') \mathcal{T}(\sigma', \rho)$:

$$\text{Tr} \hat{\mathcal{T}}^2 = \int d\rho \left(\int d\sigma' \mathcal{T}(\rho, \sigma') \mathcal{T}(\sigma', \rho) \right) = \int d\rho d\sigma \mathcal{T}^2(\rho, \sigma) < \infty. \quad (\text{D.4})$$

From this property it follows that $\hat{\mathcal{T}}$ acting on square integrable functions, has only a discrete set of eigenvalues [78]. Let us obtain the complete set of eigenvectors and eigenvalues of the transfer matrix $\hat{\mathcal{T}}\psi_m(\sigma) = \lambda_m\psi_m(\sigma)$, which form a Hilbert space

$$\text{Tr} \hat{\mathcal{T}}^2 = \int d\sigma d\rho_1 d\rho_2 \mathcal{T}(\sigma, \rho_1) \mathcal{T}(\rho_1, \rho_2) \delta(\rho_2 - \sigma) = \sum_n \int d\sigma d\rho_1 d\rho_2 \psi_n(\sigma) \mathcal{T}(\sigma, \rho_1) \mathcal{T}(\rho_1, \rho_2) \psi_n(\rho_2)$$

where the last expression can be rewritten as $\sum_n \langle \psi_n | \hat{\mathcal{T}}^2 | \psi_n \rangle = \sum_n \lambda_n^2$, using the usual bra-ket notation. One should stress that we are not doing any calculation in Quantum Physics, only reformulating the computation of the Statistical Physics in a language of linear operators. By means of the eigenvectors, one obtain the following expression for $\mathcal{T}(\sigma, \rho) = \sum_n \lambda_n \psi_n(\sigma) \psi_n(\rho)$ which fulfils $\hat{\mathcal{T}}\psi_m(\sigma) = \sum_n \lambda_n \int d\rho \psi_n(\sigma) \psi_n(\rho) \psi_m(\rho) = \sum_n \lambda_n \psi_n(\sigma) \delta_{nm} = \lambda_m \psi_m(\sigma)$.

The free energy density at the thermodynamic limit of an infinite lattice ($N \rightarrow \infty$) is given by $\mathcal{F} = -\eta \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathcal{Z}_N$ with the partition function equal to

$$\mathcal{Z}_N = \prod_{j=1}^N \left(\int_{\text{range of } \sigma} d\sigma_j \right) \exp \left[-\frac{J}{2a\eta} \sum_{j=1}^N (\sigma_{j+1} - \sigma_j)^2 - \frac{a}{\eta} \sum_{j=1}^N V(\sigma_j) \right]. \quad (\text{D.5})$$

Note that in the limit of a lattice of infinite size (limit $Na \rightarrow \infty$), the periodicity condition is irrelevant. Product $\hat{\mathcal{T}} \cdot \hat{\mathcal{T}}$ acts as $(\hat{\mathcal{T}} \cdot \hat{\mathcal{T}})\psi(\sigma) = \int d\rho \left(\int d\sigma' \mathcal{T}(\sigma, \sigma') \mathcal{T}(\sigma', \rho) \right) \psi(\rho)$, and partition function $\mathcal{Z}_N = \int d\sigma_1 \int d\sigma_2 \dots \int d\sigma_N e^{-\frac{J}{2a\eta}(\sigma_1 - \sigma_2)^2 - \frac{a}{\eta}V(\sigma_1)} e^{-\frac{J}{2a\eta}(\sigma_2 - \sigma_3)^2 - \frac{a}{\eta}V(\sigma_2)} \times \dots \times e^{-\frac{J}{2a\eta}(\sigma_N - \sigma_{N+1})^2 - \frac{a}{\eta}V(\sigma_N)}$ can be rewritten as follows after having considered the periodicity in the 1-dim. lattice $\sigma_{N+1} = \sigma_1$

$$\begin{aligned} \mathcal{Z}_N &= \int d\sigma_1 \dots \int d\sigma_N e^{-\frac{J}{2a\eta}(\sigma_1 - \sigma_2)^2 - \frac{a}{2\eta}V(\sigma_1) - \frac{a}{2\eta}V(\sigma_2)} e^{-\frac{J}{2a\eta}(\sigma_2 - \sigma_3)^2 - \frac{a}{2\eta}V(\sigma_2) - \frac{a}{2\eta}V(\sigma_3)} \times \\ &\times \dots \times e^{-\frac{J}{2a\eta}(\sigma_N - \sigma_1)^2 - \frac{a}{2\eta}V(\sigma_N) - \frac{a}{2\eta}V(\sigma_1)} = \text{Tr} \left[\underbrace{\hat{\mathcal{T}} \cdot \hat{\mathcal{T}} \dots \hat{\mathcal{T}}}_{N \text{ times}} \right] = \sum_m \lambda_m^N. \end{aligned} \quad (\text{D.6})$$

It is important to note that the above result can only be obtained if the energy, which defines the contribution of a given configuration to the partition function, only contains

interaction between neighbour σ variables, i.e., if the configuration energy describes a *local* interaction.

Result (D.6) has reduced the evaluation of the partition function to the computation of the eigenvalues of the transfer matrix. Also generalized correlation functions¹ as

$$\langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle = \lim_{N \rightarrow \infty} \frac{\prod_{j=1}^N \left(\int d\sigma_j \right) g_1(\sigma_i)g_2(\sigma_{i+r}) \exp \left[-\frac{J}{2a\eta} \sum_j (\sigma_{j+1} - \sigma_j)^2 - \frac{a}{\eta} \sum_j V(\sigma_j) \right]}{\prod_{j=1}^N \left(\int d\sigma_j \right) \exp \left[-\frac{J}{2a\eta} \sum_j (\sigma_{j+1} - \sigma_j)^2 - \frac{a}{\eta} \sum_j V(\sigma_j) \right]},$$

can be expressed by means of transfer matrix. The kernel inside $\int d\sigma_1 \int d\sigma_2 \dots \int d\sigma_N$ in the numerator of the correlation function is

$$\overbrace{\mathcal{T}(\sigma_1, \sigma_2) \dots \mathcal{T}(\sigma_{i-1}, \sigma_i)}^{(i-1) \text{ times}} g_1(\sigma_i) \overbrace{\mathcal{T}(\sigma_i, \sigma_{i+1}) \dots \mathcal{T}(\sigma_{i+r-1}, \sigma_{i+r})}^{r \text{ times}} g_2(\sigma_{i+r}) \overbrace{\mathcal{T}(\sigma_{i+r}, \sigma_{i+r+1}) \dots \mathcal{T}(\sigma_N, \sigma_1)}^{(N-r-i+1) \text{ times}},$$

and thus, using the property $\text{Tr} [\hat{A}\hat{B}] = \text{Tr} [\hat{B}\hat{A}]$, correlation $\langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle$ in the infinite lattice limit takes the following form

$$\begin{aligned} \langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle &= \lim_{N \rightarrow \infty} \frac{\text{Tr} [\hat{\mathcal{T}}^{i-1} \cdot \hat{g}_1 \cdot \hat{\mathcal{T}}^r \cdot \hat{g}_2 \cdot \hat{\mathcal{T}}^{N-r-i+1}]}{\mathcal{Z}_N} = \\ &= \lim_{N \rightarrow \infty} \frac{\text{Tr} [\hat{g}_1 \cdot \hat{\mathcal{T}}^r \cdot \hat{g}_2 \cdot \hat{\mathcal{T}}^{N-r}]}{\text{Tr} [\hat{\mathcal{T}}^N]}, \end{aligned} \quad (\text{D.7})$$

where $\hat{g}_{1,2}$ are multiplicative operators $\hat{g}_{1,2}\psi(\sigma) = g_{1,2}(\sigma)\psi(\sigma)$; the integral kernel for $\hat{g}_{1,2}$ is given by $\delta(\sigma - \rho)g_{1,2}(\rho)$ as one easily shows $\int d\rho \delta(\sigma - \rho)g_{1,2}(\rho)\psi(\rho) = g_{1,2}(\sigma)\psi(\sigma)$.

The introduction of the operator $\hat{\mathcal{T}}^r$ between $\hat{g}_1$ and $\hat{g}_2$ makes that $\hat{g}_2$ acts on a variable displaced r steps with respect to the variable on which $\hat{g}_1$ is acting. Operator $\hat{\mathcal{T}}$ produces a translation of distance equal to the lattice spacing a along the unique lattice's direction.

Now we are interested in the so-called continuum limit, i.e., when the spacing of the 1-dim. lattice goes to zero. As we shall now see in the following Note, the previous result (D.7) for the correlation function in such a continuum limit $a \rightarrow 0$, but maintaining fixed the distance $\tau = ra$, can be expressed for a large lattice ($N \rightarrow \infty$) as follows

$$\langle g_1(\sigma(0))g_2(\sigma(\tau)) \rangle = \sum_m \langle \psi_0 | \hat{g}_1 | \psi_m \rangle \langle \psi_m | \hat{g}_2 | \psi_0 \rangle e^{-\tau(E_m - E_0)/\eta}, \quad (\text{D.8})$$

where $\{E_m\}$ are the eigenvalues of a certain operator $\hat{\mathcal{H}}$ related to the transfer matrix by means of $\hat{\mathcal{T}} = \exp \left[-a \frac{\hat{\mathcal{H}}}{\eta} \right]$ (with a infinitesimal), and thus the eigenvectors $\{|\psi_m\rangle\}$ of the

¹A special case of such a correlation function is $\langle \sigma_i \sigma_{i+r} \rangle$, as already studied for the Ising model or for the XY-model, where $g_1(x) = g_2(x) = x$.

transfer matrix are also eigenvectors of $\hat{\mathcal{H}}$. Eigenvalue E_0 is the smallest of all the eigenvalues of $\hat{\mathcal{H}}$, and $|\psi_0\rangle$ its corresponding eigenvector. Operator $\hat{\mathcal{H}}$ for the configuration energy $E = \frac{aJ}{2} \sum_{j=1}^N \left(\frac{\sigma_{j+1} - \sigma_j}{a} \right)^2 + a \sum_{j=1}^N V(\sigma_j)$ takes the following form

$$\hat{\mathcal{H}} = -\frac{\eta^2}{2J} \frac{\hat{d}^2}{d\sigma^2} + \hat{V}, \quad (\text{D.9})$$

where $\hat{V}$ is the multiplicative operator obtained from $V(\sigma)$ in the configuration energy: $\hat{V}\psi(\sigma) = V(\sigma)\psi(\sigma)$. Since operator $\hat{\mathcal{H}}$ has the same form as the quantum Hamiltonian operator for a 1-dim. problem with potential $V(\sigma)$, this approach is called the Hamiltonian formulation of the statistical calculation with the above configuration energy.

Note. Connection between Statistical and Quantum Physics.

Let us first of all consider an easy case, the 1-dim. Ising model, in order to see how the transfer matrix method works and how the eigenvalues of $\hat{\mathcal{T}}$ are related to the free energy density and to the correlation functions. Although we have chosen the Ising model, where variable σ_i only takes on a discrete set of values and thus $\mathcal{T}(\sigma, \rho)$ is a finite matrix, the argument which will follow now can be generalized to a continuous variable. The 1-dim. Ising model is defined by energy $\frac{E}{\eta} = \frac{J}{2a\eta} \sum_{j=1}^N (\sigma_{j+1} - \sigma_j)^2 - \frac{ah}{\eta} \sum_{j=1}^N \sigma_j$, with $\sigma_i = +1, -1$ and where h is the external magnetic field. Integral $\int d\rho \mathcal{T}(\sigma, \rho) \psi(\rho) = \hat{\mathcal{T}}\psi(\sigma)$ is simplified into a sum which can be represented by a matrix product with the integral kernel given by the matrix

$$\begin{aligned} \mathcal{T}(+1, +1) &= e^{+ah/\eta}, & \mathcal{T}(+1, -1) &= e^{-2J/a\eta + ah/2\eta - ah/2\eta}, & \mathcal{T}(-1, -1) &= e^{-ah/\eta}, \\ \mathcal{T}(\sigma, \rho) &= \begin{pmatrix} e^{ah/\eta} & e^{-2J/a\eta} \\ e^{-2J/a\eta} & e^{-ah/\eta} \end{pmatrix}, \end{aligned} \quad (\text{D.10})$$

in the basis $\psi(\sigma = +1) \equiv |\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\psi(\sigma = -1) \equiv |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since $\mathcal{T}(\sigma, \rho)$ is real and symmetric, all eigenvalues are real. Let λ_0 be the eigenvalue with the largest modulus and $|0\rangle$ its corresponding eigenvector; and let $\mathcal{T}_{ab}$ denote the elements of the transfer matrix, which are all strictly positive, and v_a^0 the components of eigenvector $|0\rangle$. Since $\hat{\mathcal{T}}|0\rangle = \lambda_0|0\rangle$ and $\lambda_0 = \frac{\langle 0|\hat{\mathcal{T}}|0\rangle}{\langle 0|0\rangle} = \frac{v_a^0 \mathcal{T}_{ab} v_b^0}{v_a^0 v_a^0}$ (with sum over repeated indices), one obtains

$$|\lambda_0| \leq \frac{|v_a^0| \mathcal{T}_{ab} |v_b^0|}{|v_a^0| |v_a^0|}.$$

But the strict inequality is excluded since $|\lambda_0| < \frac{|v_a^0| \mathcal{T}_{ab} |v_b^0|}{|v_a^0| |v_a^0|}$ would lead to the existence of an eigenvalue t'_0 such that $|\lambda_0| < |t'_0| \leq \frac{|v_a^0| \mathcal{T}_{ab} |v_b^0|}{|v_a^0| |v_a^0|}$, in contradiction with the hypothesis on λ_0 . Due to the fact that the transfer matrix has strictly positive coefficients and because of the property $|\lambda_0| = \frac{|v_a^0| \mathcal{T}_{ab} |v_b^0|}{|v_a^0| |v_a^0|}$, being $\lambda_0 = \frac{v_a^0 \mathcal{T}_{ab} v_b^0}{v_a^0 v_a^0}$, one obtains that all components v_a^0 must have the same sign, which we can choose as the positive sign, and that eigenvalue λ_0 is positive. Moreover, this eigenvalue cannot be degenerate

because two vectors with strictly positive components cannot be orthogonal to each other (the scalar product of such vectors cannot be zero). Summarizing, since the transfer matrix is real symmetric and with strictly positive elements, the eigenvalue with the largest modulus is positive ² and its eigenvector is unique.

According to (D.6), the free energy density at the thermodynamic limit $\mathcal{F} = -\eta \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathcal{Z}_N$ for a general 1-dim. model with a discrete valued σ is equal to

$$\mathcal{F} = -\eta \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left[\lambda_0^N + \sum_{m=1}^{\infty} \lambda_m^N \right] = -\eta \ln \lambda_0 - \eta \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left[1 + \left(\frac{\lambda_1}{\lambda_0} \right)^N + \dots \right] = -\eta \ln \lambda_0. \quad (\text{D.11})$$

For the special case of the 1-dim. Ising model (D.10), eigenvalue λ_0 is directly calculated

$$\lambda_0 = \cosh \left(\frac{ah}{\eta} \right) + \sqrt{\sinh^2 \left(\frac{ah}{\eta} \right) + e^{-4J/a\eta}}, \quad (\text{D.12})$$

and we obtain thus the free energy density for this model.

Returning to the correlation function of the general 1-dim. case and by considering the limit $N \rightarrow \infty$ in (D.7), we obtain

$$\text{Tr} \left[\hat{g}_1 \hat{T}^r \hat{g}_2 \hat{T}^{N-r} \right] = \sum_n \langle \psi_n | \hat{g}_1 \hat{T}^r \hat{g}_2 | \psi_n \rangle \lambda_n^{N-r} = \langle 0 | \hat{g}_1 \hat{T}^r \hat{g}_2 | 0 \rangle \lambda_0^{N-r} + \dots = \sum_m \langle 0 | \hat{g}_1 \hat{T}^r | \psi_m \rangle \langle \psi_m | \hat{g}_2 | 0 \rangle \lambda_0^{N-r} + \dots$$

where $\{\psi_n\}$ are the complete set of eigenfunctions of the transfer matrix. Hence,

$$\langle g_1(\sigma_i) g_2(\sigma_{i+r}) \rangle = \lim_{N \rightarrow \infty} \frac{\sum_m \langle 0 | \hat{g}_1 | \psi_m \rangle \langle \psi_m | \hat{g}_2 | 0 \rangle \lambda_0^{N-r} \lambda_m^r + \dots}{\lambda_0^N + \dots} = \sum_m \langle 0 | \hat{g}_1 | \psi_m \rangle \langle \psi_m | \hat{g}_2 | 0 \rangle \left(\frac{\lambda_m}{\lambda_0} \right)^r,$$

and

$$\langle g_1(\sigma_i) \rangle = \lim_{N \rightarrow \infty} \frac{\text{Tr} \left[\hat{T}^{i-1} \cdot \hat{g}_1 \cdot \hat{T}^{N-i+1} \right]}{Z_N} = \lim_{N \rightarrow \infty} \frac{\text{Tr} \left[\hat{g}_1 \cdot \hat{T}^N \right]}{\text{Tr} \left[\hat{T}^N \right]} = \langle 0 | \hat{g}_1 | 0 \rangle.$$

Summarizing

$$\langle g_1(\sigma_i) g_2(\sigma_{i+r}) \rangle \xrightarrow{N \rightarrow \infty} \langle 0 | \hat{g}_1 \hat{R}^r \hat{g}_2 | 0 \rangle \quad \text{with} \quad \hat{R} = \frac{\hat{T}}{\lambda_0}. \quad (\text{D.13})$$

It is clear that for $r \rightarrow \infty$, $\hat{R}^r$ becomes the projector on $|0\rangle$: $\langle g_1(\sigma_i) g_2(\sigma_{i+r}) \rangle \xrightarrow{r \rightarrow \infty} \langle g_1(\sigma) \rangle \langle g_2(\sigma) \rangle$ and

$$\langle 0 | \hat{g}_1 \hat{R}^r \hat{g}_2 | 0 \rangle \xrightarrow{r \rightarrow \infty} \langle 0 | \hat{g}_1 | 0 \rangle \langle 0 | \hat{g}_2 | 0 \rangle \Rightarrow \lim_{r \rightarrow \infty} \hat{R}^r = |0\rangle \langle 0|. \quad (\text{D.14})$$

²In fact, all eigenvalues of $\hat{T}$ are positive since $\langle \psi | \hat{T} | \psi \rangle > 0$ for any nonzero ψ , as one can easily show:

$$\begin{aligned} \langle \psi | \hat{T} | \psi \rangle &= \int d\sigma d\rho \psi(\sigma) \mathcal{T}(\sigma, \rho) \psi(\rho) = \int d\sigma d\rho \underbrace{\left(\psi(\sigma) e^{-aV(\sigma)/2\eta} \right)}_{\int dk_1 e^{ik_1\sigma} \tilde{\psi}_v(k_1)} e^{-J(\sigma-\rho)^2/2a\eta} \underbrace{\left(e^{-aV(\rho)/2\eta} \psi(\rho) \right)}_{\int dk_2 e^{ik_2\rho} \tilde{\psi}_v(k_2)} = \\ &= \sqrt{\frac{2\pi\eta}{J}} \int d\rho dk_1 dk_2 e^{i\rho(k_1+k_2)} e^{-ak_1^2\eta/2J} \tilde{\psi}_v(k_1) \tilde{\psi}_v(k_2) = 2\pi \sqrt{\frac{2\pi\eta}{J}} \int dk |\tilde{\psi}_v(k)|^2 e^{-ak^2\eta/2J} > 0, \end{aligned}$$

where $\tilde{\psi}_v(-k) = \tilde{\psi}_v^*(k)$ since $\psi(\sigma)$ is a real function.

Moreover, the correlation length ξ can be also obtained directly when the connected correlation function given by $\langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle_{\text{conn.}} = \langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle - \langle g_1(\sigma) \rangle \langle g_2(\sigma) \rangle$ is calculated for a large distance r , $\langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle_{\text{conn.}} = \sum_m \langle 0|\hat{g}_1|\psi_m\rangle\langle\psi_m|\hat{g}_2|0\rangle \left(\frac{\lambda_m}{\lambda_0}\right)^r - \langle 0|\hat{g}_1|0\rangle\langle 0|\hat{g}_2|0\rangle \stackrel{\text{large } r}{\approx} \langle 0|\hat{g}_1|\psi_1\rangle\langle\psi_1|\hat{g}_2|0\rangle \left(\frac{\lambda_1}{\lambda_0}\right)^r$ where λ_1 is the next greatest eigenvalue

$$\langle g_1(\sigma_i)g_2(\sigma_{i+r}) \rangle_{\text{conn.}} \approx \langle 0|\hat{g}_1|\psi_1\rangle\langle\psi_1|\hat{g}_2|0\rangle e^{-r/\xi} \quad \text{with} \quad \xi^{-1} = \ln \left(\frac{\lambda_0}{\lambda_1} \right). \quad (\text{D.15})$$

For the 1-dim. Ising model (D.10), and by considering the relation $\tanh^{-1} y = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right)$, we obtain

$$\xi^{-1} = 2 \tanh^{-1} \left(\sqrt{\tanh^2 \left(\frac{ah}{\eta} \right) + e^{-4J/a\eta} / \cosh^2 \left(\frac{ah}{\eta} \right)} \right), \quad (\text{D.16})$$

which remains finite except for η zero, sign that there is no phase transition for this model. With this example it has been shown how a statistical calculation can be formulated as a typical eigenvalue problem in a Hilbert space. The further connection to Quantum Physics arises from the fact that the transfer matrix plays the same role in the equations as the quantum-mechanical evolution operator $e^{-\Delta\tau\hat{H}/\hbar c}$ for a very small time interval $\Delta\tau$ in the formulation with the imaginary (or Euclidean) time $\tau = ict$ and c the velocity of the light; $\hat{H}$ is the quantum Hamiltonian operator. Let us see how this connection comes about.

For the energy in the partition function $\frac{E}{\eta} = \frac{J}{2a\eta} \sum_{j=1}^N (\sigma_{j+1} - \sigma_j)^2 + \frac{a}{\eta} \sum_j V(\sigma_j)$, one can obtain

a more concise form for operator $\hat{\mathcal{T}}$

$$\hat{\mathcal{T}} \equiv \hat{\mathcal{T}}_a = \sqrt{\frac{2\pi a\eta}{J}} \exp \left[-\frac{a}{2\eta} \hat{V} \right] \exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right] \exp \left[-\frac{a}{2\eta} \hat{V} \right], \quad (\text{D.17})$$

where $\hat{V}$ is only a multiplicative operator: $\hat{V}\psi(\sigma) = V(\sigma)\psi(\sigma)$. By taking into account that operator $\exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right]$ applied to any arbitrary function $f(\sigma) = \int dk e^{ik\sigma} \tilde{f}(k)$ produces

$$\left[1 + \frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} + \frac{1}{2} \left(\frac{a\eta}{2J} \right)^2 \frac{\hat{d}^4}{d\sigma^4} + \dots \right] \left(\int dk e^{ik\sigma} \tilde{f}(k) \right) = \int dk e^{ik\sigma} e^{-a\eta k^2/2J} \tilde{f}(k),$$

and by noticing that $\int d\rho \sqrt{\frac{J}{2\pi a\eta}} \exp \left[-\frac{J}{2a\eta} (\sigma - \rho)^2 \right] \int dk e^{ik\sigma} \tilde{f}(k) = \int dk e^{ik\sigma} e^{-a\eta k^2/2J} \tilde{f}(k)$, one

obtains that the integral kernel for $\exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right]$ is $\sqrt{\frac{J}{2\pi a\eta}} \exp \left[-\frac{J}{2a\eta} (\sigma - \rho)^2 \right]$. Hence

$$\begin{aligned} \hat{\mathcal{T}}_a \psi(\sigma) &= \sqrt{\frac{2\pi a\eta}{J}} \exp \left[-\frac{a}{2\eta} \hat{V} \right] \exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right] \exp \left[-\frac{a}{2\eta} \hat{V} \right] \psi(\sigma) = \\ &= \sqrt{\frac{2\pi a\eta}{J}} \exp \left[-\frac{a}{2\eta} \hat{V} \right] \exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right] \left(e^{-aV(\sigma)/2\eta} \psi(\sigma) \right) = \\ &= \exp \left[-\frac{a}{2\eta} \hat{V} \right] \int d\rho \exp \left[-\frac{J}{2a\eta} (\sigma - \rho)^2 \right] e^{-aV(\rho)/2\eta} \psi(\rho) = \\ &= \int d\rho e^{-aV(\sigma)/2\eta} e^{-J(\sigma-\rho)^2/2a\eta} e^{-aV(\rho)/2\eta} \psi(\rho). \end{aligned} \quad (\text{D.18})$$

Let us now define the operator

$$\hat{\mathcal{T}}_\tau = \lim_{a \rightarrow 0} \left[c(a) \hat{\mathcal{T}}_a \right]^{\tau/a}, \quad (\text{D.19})$$

where $c(a)$ is only a normalization factor equal to the factor resulting from the Gaussian integration, $c(a) = \sqrt{\frac{J}{2\pi a\eta}}$, in order to obtain that $\lim_{a \rightarrow 0} c(a) \hat{\mathcal{T}}_a = 1$; this factor $c(a)$ only contributes with a multiplicative factor in the partition function and therefore any correlation function remains unchanged since the same power of $c(a)$ appears in both numerator and denominator of the correlation function.

We define also operator $\hat{H}$ by means of the relation $\hat{\mathcal{T}}_\tau = \exp \left[-\tau \frac{\hat{H}}{\eta} \right]$, or equivalently,

$$\eta \frac{d}{d\tau} \hat{\mathcal{T}}_\tau = -\hat{H} \hat{\mathcal{T}}_\tau. \quad (\text{D.20})$$

For a small but finite a we have $\hat{\mathcal{T}}_{\tau+a} = c(a) \hat{\mathcal{T}}_a \hat{\mathcal{T}}_\tau$ and therefore

$$\frac{\hat{\mathcal{T}}_{\tau+a} - \hat{\mathcal{T}}_\tau}{a} = \frac{c(a) \hat{\mathcal{T}}_a - 1}{a} \hat{\mathcal{T}}_\tau \quad \Rightarrow \quad \hat{H} = -\eta \lim_{a \rightarrow 0} \frac{c(a) \hat{\mathcal{T}}_a - 1}{a}.$$

By applying the Baker-Hausdorff-Campbell formula, we obtain

$$\begin{aligned} c(a) \hat{\mathcal{T}}_a &= \exp \left[-\frac{a}{2\eta} \hat{V} \right] \exp \left[\frac{a\eta}{2J} \frac{\hat{d}^2}{d\sigma^2} \right] \exp \left[-\frac{a}{2\eta} \hat{V} \right] = 1 - \frac{a}{\eta} \left[-\frac{\eta^2}{2J} \frac{\hat{d}^2}{d\sigma^2} + \hat{V} \right] + O(a^2), \\ \hat{H} &= -\eta \lim_{a \rightarrow 0} \frac{c(a) \hat{\mathcal{T}}_a - 1}{a} = -\frac{\eta^2}{2J} \frac{\hat{d}^2}{d\sigma^2} + \hat{V}, \end{aligned} \quad (\text{D.21})$$

which, up to the constant multiplying $\hat{d}^2/d\sigma^2$, is the Hamiltonian operator $\hat{\mathcal{H}}$ of the 1-dimensional quantum system with potential energy $V(\sigma)$.

Since $\hat{\mathcal{T}}_\tau = \exp \left[-\tau \frac{\hat{H}}{\eta} \right]$, then $(c(a) \hat{\mathcal{T}}_a)^{\tau/a} = \exp \left[-\tau \frac{\hat{H}}{\eta} + O(a) \right]$ and one obtains that the transfer

matrix defined at the beginning of this Note is equal to $\hat{\mathcal{T}} \equiv \hat{\mathcal{T}}_a = c(a)^{-1} \exp \left[-\frac{a}{\eta} \hat{H} + O(a^2) \right]$. As already mentioned, the transfer matrix $\hat{\mathcal{T}} = \hat{\mathcal{T}}_a$ which “moves” operators a distance a along the lattice direction (on which we are studying the continuum limit) have the same form as the quantum operator $e^{-\Delta\tau \hat{\mathcal{H}}/\hbar c}$ defining a infinitesimal translation in (imaginary) time $a \leftrightarrow \Delta\tau = ic\Delta t$, where the scale η in Statistical Physics becomes the quantum scale $\hbar c$ in Quantum Physics. If $V(\sigma)$ grows fast enough when $\sigma \rightarrow \infty$, operator $\hat{H}$ has a discrete set of eigenvalues $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$. The largest eigenvalue of the transfer matrix is $\lambda_0 = c(a)^{-1} e^{-aE_0/\eta}$ and the projector operator of equation (D.14) becomes

$$\hat{R}^r = \left(\frac{\hat{\mathcal{T}}_a}{\lambda_0} \right)^r = e^{\tau E_0/\eta} \exp \left[-\tau \frac{\hat{H}}{\eta} + O(a) \right] \quad \text{with} \quad r = \frac{\tau}{a}. \quad (\text{D.22})$$

According to (D.13) and (D.22) and for a large lattice ($N \rightarrow \infty$) in the continuum limit ($a \rightarrow 0$), correlation function $\langle g_1(\sigma_i) g_2(\sigma_{i+r}) \rangle \rightarrow \langle g_1(\sigma(0)) g_2(\sigma(\tau)) \rangle$ with $\tau = ra$ maintained fixed in the limit³, takes the following form

$$\langle g_1(\sigma(0)) g_2(\sigma(\tau)) \rangle = \langle 0 | \hat{g}_1 \hat{R}^{\tau/a} \hat{g}_2 | 0 \rangle = \sum_m \langle 0 | \hat{g}_1 | \psi_m \rangle \langle \psi_m | \hat{g}_2 | 0 \rangle e^{-\tau(E_m - E_0)/\eta}, \quad (\text{D.23})$$

³In the limit considered, thermodynamic limit $N \rightarrow \infty$ followed by continuum limit $a \rightarrow 0$, lattice's size Na can tend to infinity too or can be set equal to a finite value: in the first case, the lattice's periodicity becomes irrelevant and we are studying then only the bulk of the system, whereas in the case of Na finite, the periodicity is maintained. This last case will play a very important role in the discussion of high temperature effects.

as in equation (D.8). Hence, given the energy $E = \frac{aJ}{2} \sum_j \left(\frac{\sigma_{j+1} - \sigma_j}{a} \right)^2 + a \sum_j V(\sigma_j)$ of a 1-dimensional model from which correlation functions have to be calculated, the problem is reduced to the solution of the time-independent Schrödinger equation for the corresponding Hamiltonian operator, where this correspondence is between the energy of a statistical configuration

$$E = \frac{aJ}{2} \sum_{j=1}^N \left(\frac{\sigma_{j+1} - \sigma_j}{a} \right)^2 + a \sum_{j=1}^N V(\sigma_j) \xrightarrow{a \rightarrow 0} \int d\tau \left[\frac{J}{2} \left(\frac{d\sigma}{d\tau} \right)^2 + V(\sigma(\tau)) \right],$$

and the action (times $-ic$) in Quantum Physics, obtained from the last equation by substituting $\tau = ict$, $-i \int dt \left[\frac{J}{2} \left(\frac{d\sigma}{dt} \right)^2 - V(\sigma(t)) \right]$, with t the time. The quantum ground state E_0 gives the free energy density in the infinite lattice limit. The result can be generalized for lattices of higher dimensionality.

In general we shall have a (classical) statistical model in a d -dimensional lattice in which we are interested to study the continuum limit in *one* (called the τ -direction) of the spatial directions $a_\tau \rightarrow 0$; the lattice spacing a in the other spatial directions remains finite in such limit. The energy term depending on the gradients in the orientations of $\sigma_{\vec{r}}$ between two neighbour sites is

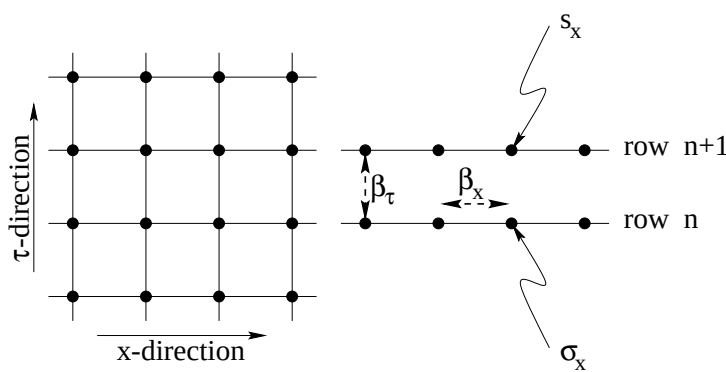
$$\frac{E}{\eta} = \frac{\beta_\tau}{2\eta} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{\tau}} - \sigma_{\vec{r}})^2 + \frac{\beta_x}{2\eta} \sum_{\vec{r}} \sum_{k=1}^{d-1} (\sigma_{\vec{r}+\hat{k}} - \sigma_{\vec{r}})^2, \quad (\text{D.24})$$

where the coupling strength between neighbour spins along the τ -direction has been taken in general different from the coupling along the other spatial direction. We have not written explicitly the dependence on a_τ , which has been absorbed in both coupling strengths; how β_τ and β_x vary with a_τ will be obtained later. Sum $\sum_k$ extends over the spatial directions other than the direction τ along which the continuum limit is taken.

Let us begin with a special case of the above equation: the 2-dimensional Ising model without magnetic field, whose energy is given by

$$\frac{E}{\eta} = \frac{\beta_\tau}{2\eta} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{\tau}} - \sigma_{\vec{r}})^2 + \frac{\beta_x}{2\eta} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{x}} - \sigma_{\vec{r}})^2, \quad (\text{D.25})$$

with $\sigma_{\vec{r}} = \pm 1$. Because of the local interaction only between neighbour sites, this energy (as well as the more general (D.24)) can be rewritten as



$$\frac{E}{\eta} = \sum_n L(n+1, n), \quad (\text{D.26})$$

where sum $\sum_n$ runs over the $(d-1)$ -dimensional planes normal to the τ -direction (for the 2-dim. Ising model in the figure, n denotes the rows of sites perpendicular to the τ -direction), and $L(n+1, n)$ is given by

$$L(n+1, n) = \frac{\beta_\tau}{2\eta} \sum_{\vec{x} \in \text{row}_n} (s_{\vec{x}} - \sigma_{\vec{x}})^2 + \frac{\beta_x}{2\eta} \sum_{\vec{x} \in \text{row}_n} \left[\frac{(\sigma_{\vec{x}+\hat{1}} - \sigma_{\vec{x}})^2}{2} + \frac{(s_{\vec{x}+\hat{1}} - s_{\vec{x}})^2}{2} \right]. \quad (\text{D.27})$$

In the last equation and for the following $\vec{x}$ denotes a site on the $(d-1)$ -dim. plane or in the row normal to the τ -direction. Assuming that there are N sites on each $\vec{x}$ -row, there are 2^N different spin configurations of the sort $|\underbrace{\uparrow \dots \uparrow \downarrow \downarrow \dots \uparrow \dots \downarrow}_{N \text{ spins}}\rangle$ which form

the basis for the matrix representation of the $2^N \times 2^N$ transfer matrix. The symmetrical matrix $\hat{\mathcal{T}}$ connects two neighbour rows and its elements $\mathcal{T}_{ab}$ are given by the exponential of $-L$ (upper row, lower row) evaluated with all possible row-configurations. The diagonal elements of $\hat{\mathcal{T}}$ belong to the a spin configuration in the upper arrow equal to that in the preceding row ($s_{\vec{x}} = \sigma_{\vec{x}}$)

$$\mathcal{T}_{\text{diagonal}} = \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x} \in \text{row}} (\sigma_{\vec{x}+\hat{1}} - \sigma_{\vec{x}})^2 \right], \quad (\text{D.28})$$

The off-diagonal elements are formed by configurations which in the upper row contain 1, 2, 3, ... flipped spins with the respect to the configuration in the lower row: for instance configuration $|\uparrow \dots \uparrow \underline{\downarrow} \downarrow \dots \uparrow \dots \downarrow\rangle$ above and $|\uparrow \dots \uparrow \uparrow \underline{\downarrow} \downarrow \dots \uparrow \dots \downarrow\rangle$ below

$$\begin{aligned} \mathcal{T}_{1 \text{ flip}} &= e^{-2\beta_\tau/\eta} \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}} \frac{(\sigma_{\vec{x}+\hat{1}} - \sigma_{\vec{x}})^2}{2} + \frac{(s_{\vec{x}+\hat{1}} - s_{\vec{x}})^2}{2} \right], \\ \mathcal{T}_{2 \text{ flips}} &= e^{-4\beta_\tau/\eta} \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}} \frac{(\sigma_{\vec{x}+\hat{1}} - \sigma_{\vec{x}})^2}{2} + \frac{(s_{\vec{x}+\hat{1}} - s_{\vec{x}})^2}{2} \right], \\ \mathcal{T}_{m \text{ flips}} &= e^{-2m\beta_\tau/\eta} \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}} \frac{(\sigma_{\vec{x}+\hat{1}} - \sigma_{\vec{x}})^2}{2} + \frac{(s_{\vec{x}+\hat{1}} - s_{\vec{x}})^2}{2} \right], \end{aligned} \quad (\text{D.29})$$

where the sum over $\vec{x}$ should be intended as the sum over the sites of a row ⁴.

As already seen in the previous 1-dim. model, the operator $\hat{\mathcal{H}}$ (similar to the Hamiltonian operator in Quantum Physics) is defined through relation $\hat{\mathcal{T}} = \exp \left[-a_\tau \hat{\mathcal{H}}/\eta \right]$ for an infinitesimal a_τ , i.e., $\hat{\mathcal{T}} \approx \mathbf{1} - \frac{a_\tau}{\eta} \hat{\mathcal{H}}$ with $\mathbf{1}$ the $2^N \times 2^N$ unit matrix. Therefore $\mathcal{T}_{\text{diagonal}} \rightarrow \mathbf{1}$ for $a_\tau \rightarrow 0$ and $\mathcal{T}_{m \geq 1 \text{ flips}}$ goes to zero for $a_\tau \rightarrow 0$ at least as fast as a_τ . Comparing these relations with equation (D.28) and (D.29), one obtain how the two coupling strengths β_τ, β_x behave for $a_\tau \rightarrow 0$

$$\begin{cases} a_\tau \sim \beta_x & \text{and} & \beta_x \xrightarrow{a_\tau \rightarrow 0} 0, \\ a_\tau \sim e^{-2\beta_\tau/\eta} & \text{and} & \beta_\tau \xrightarrow{a_\tau \rightarrow 0} \infty, \end{cases} \quad (\text{D.30})$$

and therefore, in the 2-dim. Ising model β_x and $e^{-2\beta_\tau/\eta}$ are proportional to each other

$$\beta_x = \kappa e^{-2\beta_\tau/\eta}, \quad (\text{D.31})$$

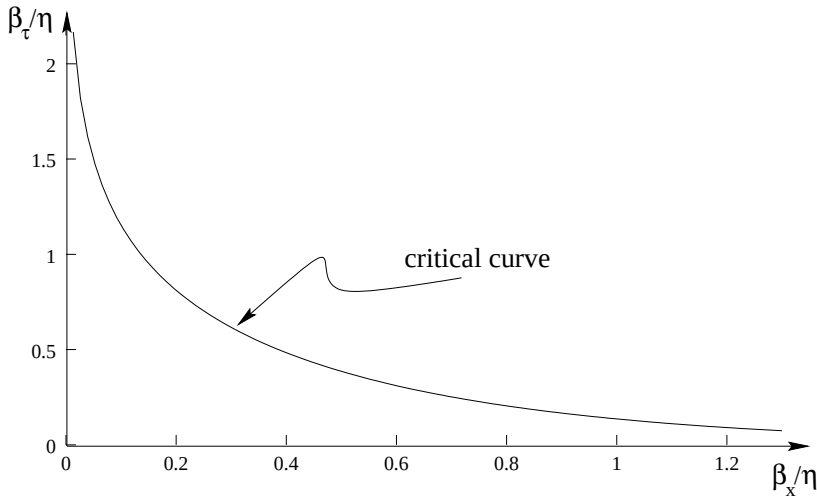
⁴Also in this model one can study the correlation function and look for whether there is a phase where ξ diverges, pointing thus to an ordered phase. In this Appendix we shall not consider this case, since for a model with dimensions larger than 1 it has been studied qualitatively in the previous Appendix: see the argument in page 383 based on the energy cost of a large domain wall depending on the dimensionality when the lattice's size grows toward infinity.

with κ the proportionality constant. We can identify a_τ with $e^{-2\beta_\tau/\eta}$ ⁵ and thus $\beta_x = \kappa a_\tau$ (on the other hand, we could always factorize out a proportionality constant between a_τ and $e^{-2\beta_\tau/\eta}$ in the Hamiltonian operator we are looking for). The meaning of κ : depending on the strength of β_τ and β_x the system is in an ordered or in a disordered phase, and there is a critical curve in the (β_τ, β_x) plane separating both phases. For the case of the 2-dim. Ising model with two coupling strengths β_τ, β_x in absence of an external magnetic field, we obtain that the model is self-dual by repeating the arguments preceding equation (C.365) (it should be noticed that in that equation of Appendix C, scale η is already included into β): coupling $\sigma_{\vec{r}+\hat{\mu}}\sigma_{\vec{r}}$ along a bond is, for a 2-dim. lattice, dual to a coupling along a bond again; but a bond in the τ -direction is crossed by a bond in the x -direction in the dual lattice, and viceversa. Hence we obtain the following relation between the original and the dual model

$$\frac{\beta_\tau^*}{\eta} = -\frac{1}{2} \ln \tanh \left(\frac{\beta_x}{\eta} \right) \quad \text{and} \quad \frac{\beta_x^*}{\eta} = -\frac{1}{2} \ln \tanh \left(\frac{\beta_\tau}{\eta} \right).$$

The phase characterized by large β_x and β_τ in the original model is described in the dual model by the phase of small β_τ^* and β_x^* and viceversa. The critical curve is thus given when both phases coincide, i.e., $(\beta_\tau^*)_{\text{crit}} = (\beta_\tau)_{\text{crit}}$ and $(\beta_x^*)_{\text{crit}} = (\beta_x)_{\text{crit}}$, and therefore⁶

$$\sinh \left(2 \frac{(\beta_\tau)_{\text{crit}}}{\eta} \right) \sinh \left(2 \frac{(\beta_x)_{\text{crit}}}{\eta} \right) = 1, \quad (\text{D.32})$$



which determines the critical curve in the (β_τ, β_x) plane. Let us study the part of this curve for the values of interest in the continuum limit $a_\tau \rightarrow 0$, i.e., for $\beta_\tau \rightarrow \infty$ and thus $\sinh \left(\frac{2(\beta_\tau)_{\text{cr}}}{\eta} \right) \approx \frac{e^{2(\beta_\tau)_{\text{cr}}/\eta}}{2}$, and $\beta_x \rightarrow 0$ with $\sinh \left(\frac{2(\beta_x)_{\text{cr}}}{\eta} \right) \approx \frac{(\beta_x)_{\text{cr}}}{\eta}$:

$$\frac{(\beta_x)_{\text{cr}}}{\eta} = e^{-2(\beta_\tau)_{\text{cr}}/\eta} \implies \frac{\kappa_{\text{crit}}}{\eta} = 1, \quad (\text{D.33})$$

where the last result arises by comparing the values on the critical curve at $a_\tau \rightarrow 0$ with the general relation (D.31), $\frac{\beta_x}{\eta} = \frac{\kappa}{\eta} e^{-2\beta_\tau/\eta}$. Hence, depending whether κ is greater or

⁵If a_τ is considered with length dimensions, the exact identification is $\frac{a_\tau}{a} \equiv e^{-2\beta_\tau/\eta}$; for the following, nevertheless, a will be taken equal to 1 and only when necessary the lattice dimension will be restored.

⁶By substituting: $\sinh \left(2 \frac{(\beta_\tau)_{\text{crit}}}{\eta} \right) = \frac{e^{-\ln \tanh((\beta_x)_{\text{cr}}/\eta)} - e^{\ln \tanh((\beta_x)_{\text{cr}}/\eta)}}{2} = 1 / \sinh \left(2 \frac{(\beta_x)_{\text{crit}}}{\eta} \right)$.

smaller than η , the system is at one side of the critical curve or at the other one: the value of κ characterizes the phase in which the model finds itself. As we shall see in the next Note on page 521, $\kappa > 1$ corresponds to the ordered or magnetized phase.

For the sake of completeness, let us obtain the Hamiltonian operator corresponding to the 2-dim. Ising model without an external magnetic field. From (D.28) and (D.29)

$$\begin{aligned}\mathcal{T}_{0 \text{ flips}} &\approx 1 - \frac{\beta_x}{\eta} \sum_{\vec{x} \in \text{row}} (1 - \sigma_{\vec{x}+\hat{1}} \sigma_{\vec{x}}) = 1 - \frac{a_\tau}{\eta} \kappa \sum_{\vec{x} \in \text{row}} (1 - \sigma_{\vec{x}+\hat{1}} \sigma_{\vec{x}}) \approx 1 - \frac{a_\tau}{\eta} \mathcal{H}_{0 \text{ flips}}, \\ \mathcal{T}_{1 \text{ flip}} &\approx a_\tau \binom{\cdot}{\cdot} \approx -\frac{a_\tau}{\eta} \mathcal{H}_{1 \text{ flip}}, \\ \mathcal{T}_{m \text{ flips}} &\approx (a_\tau)^m \binom{\cdot}{\cdot} = a_\tau (a_\tau)^{m-1} \binom{\cdot}{\cdot} \approx -\frac{a_\tau}{\eta} \mathcal{H}_{m \text{ flips}}.\end{aligned}$$

The two last equations imply that at the limit $a_\tau \rightarrow 0$ only $\mathcal{H}_{0 \text{ flips}}$ and $\mathcal{H}_{1 \text{ flip}}$ remain. Using the basis $|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at each row site, which are the eigenvectors of Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the form of operator $\hat{\mathcal{H}}$ can be directly read off ⁷

$$\frac{\hat{\mathcal{H}}}{\eta} = - \sum_{\vec{x} \in \text{row} \perp \tau\text{-direct.}} (\hat{\sigma}_1)_{\vec{x}} + \frac{\kappa}{\eta} \sum_{\vec{x} \in \text{row}} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]. \quad (\text{D.34})$$

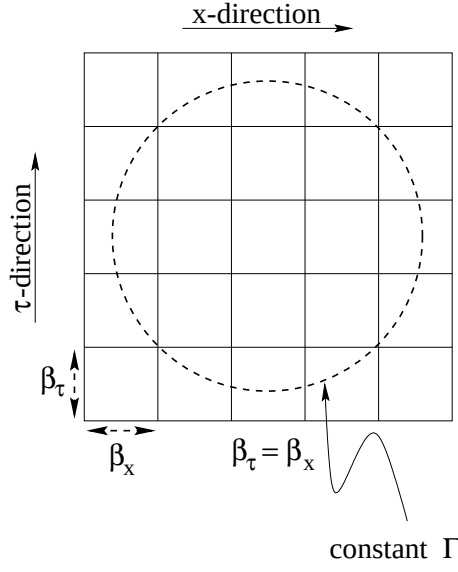
Note that κ plays in the role of the coupling strength in a (usual) 1-dim. Ising model; but in $\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in \text{row} \perp a_\tau} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \text{row}} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]$ there appears also the $\hat{\sigma}_1$ -term, which has no equivalence in the classical Ising model and is proportional to scale η , scale which in the already discussed “translation” between the Statistical and the Quantum Physics is analogous to the quantum scale $\hbar$. Hence, solving the 2-dim. Ising model in a lattice whose spacing along one of the dimensions goes to zero is equivalent to solve the above Hamiltonian operator in a 1-dim. system, model which is called the 1-dim.

⁷To $\mathcal{H}_{0 \text{ flips}} = \kappa \sum_{\vec{x} \in \text{row}} (1 - \sigma_{\vec{x}+\hat{1}} \sigma_{\vec{x}})$ corresponds operator $\hat{\mathcal{H}}_{0 \text{ flips}} = \kappa \sum_{\vec{x} \in \text{row}} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]$ where $(\hat{\sigma}_3)_{\vec{x}}$ is the operator defined by the action of Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the spin at row site $\vec{x}$:

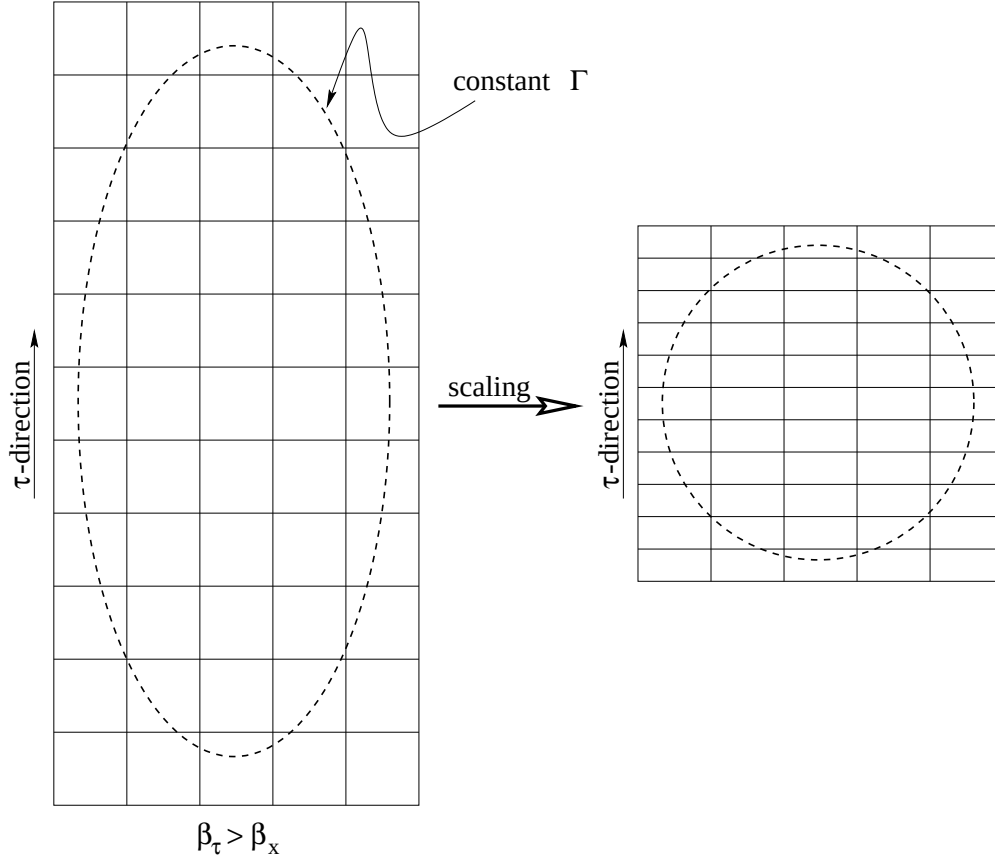
$$\begin{aligned}(\hat{\sigma}_3)_{\vec{x}} |\dots \uparrow_{\vec{x}} \dots\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\vec{x}} |\dots \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\vec{x}} \dots\rangle = |\dots \uparrow_{\vec{x}} \dots\rangle, \\ (\hat{\sigma}_3)_{\vec{x}} |\dots \downarrow_{\vec{x}} \dots\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\vec{x}} |\dots \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\vec{x}} \dots\rangle = -1 |\dots \downarrow_{\vec{x}} \dots\rangle.\end{aligned}$$

And operator $\hat{\mathcal{H}}_{1 \text{ flip}} = - \sum_{\vec{x} \in \text{row}} (\hat{\sigma}_1)_{\vec{x}}$ corresponds to the energy's term with a flipped spin between the neighbour rows. The effect of $\hat{\mathcal{H}}_{1 \text{ flip}}$ is to flip spins (where “spin” is understood as eigenvector of operator $(\hat{\sigma}_3)_{\vec{x}}$), since operator $(\hat{\sigma}_1)_{\vec{x}}$ is Pauli matrix $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acting on site $\vec{x}$ with the following effect: $(\hat{\sigma}_1)_{\vec{x}} |\dots \uparrow_{\vec{x}} \dots\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\vec{x}} |\dots \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\vec{x}} \dots\rangle = |\dots \downarrow_{\vec{x}} \dots\rangle$ and $(\hat{\sigma}_1)_{\vec{x}} |\dots \downarrow_{\vec{x}} \dots\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\vec{x}} |\dots \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\vec{x}} \dots\rangle = |\dots \uparrow_{\vec{x}} \dots\rangle$.

quantum Ising model.



Let us conclude the discussion on the 2-dim. Ising model by considering in more detail ([42] and [59]) the relations $\begin{cases} \beta_x & \xrightarrow{a_\tau \rightarrow 0} 0 \\ \beta_\tau & \xrightarrow{a_\tau \rightarrow 0} \infty \end{cases}$, which as we shall see later hold in general for every model (the more specific relation $\beta_x \sim e^{-2\beta_\tau/\eta}$ is characteristic for the Ising model). Let us consider the correlation function Γ between two separated spins. If $\beta_\tau = \beta_x$, the configuration energy $E = \frac{\beta_\tau}{2} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{\tau}} - \sigma_{\vec{r}})^2 + \frac{\beta_x}{2} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{x}} - \sigma_{\vec{r}})^2$ is symmetric under rotations through 90° which interchanges the τ and x directions. Therefore the points which share the same value of correlation function Γ form a circumference.



Now let us increase β_τ : spins along the τ -direction are stronger correlated, the correlation's range along the τ -direction is longer and thus the curves of constant Γ becomes distorted into ellipses (as in the figure above). Suppose we demand that the physics must remain the same in the case $\beta_\tau > \beta_x$: the distortion in the curves of constant Γ

must be compensated by adjusting the lattice spacing along the τ -direction to a value smaller than the lattice spacing along the x direction. Hence, in the limit $\beta_\tau \rightarrow \infty$ lattice spacing a_τ is compressed to zero in order to leave the form of the correlation functions and the physics invariant. And viceversa, the continuum limit $a_\tau \rightarrow 0$ in the model we are studying should not change the physics and therefore $a_\tau \ll a$ is equivalent to $\beta_\tau \gg \beta_x$.

Note. Characterizing the phases of the 2-dim. Ising model in the Hamiltonian formulation [59].

Firstly let us remember well-known results of the perturbation theory: let us suppose we have an operator $\hat{\mathcal{H}}_0$, whose complete set of normalized eigenvectors is denoted by $\{|n\rangle\}$ (which are orthogonal to each other) and their eigenvalues by $\{E_n^{(0)}\}$,

$$\hat{\mathcal{H}}_0|n\rangle = E_n^{(0)}|n\rangle,$$

and we are looking for the eigenvectors of operator $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{\mathcal{H}}_1$ where $\lambda \ll 1$ is a small parameter and thus $\lambda\hat{\mathcal{H}}_1$ is considered as a perturbation on $\hat{\mathcal{H}}_0$. Let $|n_0\rangle$ be the eigenvector of $\hat{\mathcal{H}}_0$ for a certain state and $E_{n_0}^{(0)} \equiv E_a^{(0)}$ its corresponding eigenvalue, and we want to calculate the eigenvector and the eigenvalue, $|a\rangle$ and E_a , in the perturbed case as an expansion in powers of λ

$$\left. \begin{aligned} |a\rangle &= |n_0\rangle + \lambda \sum_{n \neq n_0} c_n^{(1)}|n\rangle + \lambda^2 \sum_{n \neq n_0} c_n^{(2)}|n\rangle + \lambda^3 \sum_{n \neq n_0} c_n^{(3)}|n\rangle + \dots \\ E_a &= E_a^{(0)} + \lambda E_a^{(1)} + \lambda^2 E_a^{(2)} + \lambda^3 E_a^{(3)} + \lambda^4 E_a^{(4)} + \dots \end{aligned} \right\} \left(\hat{\mathcal{H}}_0 + \lambda\hat{\mathcal{H}}_1 \right) |a\rangle = E_a |a\rangle,$$

or equivalently

$$\begin{aligned} & \hat{\mathcal{H}}_0|n_0\rangle + E_{n_0}^{(0)}|n_0\rangle + \\ & \lambda \left[\hat{\mathcal{H}}_0 \sum_{n \neq n_0} c_n^{(1)}|n\rangle + \hat{\mathcal{H}}_1|n_0\rangle \right] + \lambda \left[E_a^{(0)} \sum_{n \neq n_0} c_n^{(1)}|n\rangle + E_a^{(1)}|n_0\rangle \right] + \\ & \lambda^2 \left[\hat{\mathcal{H}}_0 \sum_{n \neq n_0} c_n^{(2)}|n\rangle + \hat{\mathcal{H}}_1 \sum_{n \neq n_0} c_n^{(1)}|n\rangle \right] + \lambda^2 \left[E_a^{(0)} \sum_{n \neq n_0} c_n^{(2)}|n\rangle + E_a^{(1)} \sum_{n \neq n_0} c_n^{(1)}|n\rangle + E_a^{(2)}|n_0\rangle \right] + \\ & \lambda^3 \left[\hat{\mathcal{H}}_0 \sum_{n \neq n_0} c_n^{(3)}|n\rangle + \hat{\mathcal{H}}_1 \sum_{n \neq n_0} c_n^{(2)}|n\rangle \right] + \lambda^3 \left[E_a^{(0)} \sum_{n \neq n_0} c_n^{(3)}|n\rangle + E_a^{(1)} \sum_{n \neq n_0} c_n^{(2)}|n\rangle + \right. \\ & \quad \left. + E_a^{(2)} \sum_{n \neq n_0} c_n^{(1)}|n\rangle + E_a^{(3)}|n_0\rangle \right] \\ & \lambda^4 \left[\hat{\mathcal{H}}_0 \sum_{n \neq n_0} c_n^{(4)}|n\rangle + \hat{\mathcal{H}}_1 \sum_{n \neq n_0} c_n^{(3)}|n\rangle \right] + \dots \quad \lambda^4 \left[E_a^{(0)} \sum_{n \neq n_0} c_n^{(4)}|n\rangle + E_a^{(1)} \sum_{n \neq n_0} c_n^{(3)}|n\rangle + \right. \\ & \quad \left. + E_a^{(2)} \sum_{n \neq n_0} c_n^{(2)}|n\rangle + E_a^{(3)} \sum_{n \neq n_0} c_n^{(1)}|n\rangle + E_a^{(4)}|n_0\rangle \right] + \dots \end{aligned}$$

By multiplying by left with $\langle n_0|$ one obtains that the contribution to j th λ -order in the eigenvalue E_a is given by

$$E_a^{(j)} = \sum_{n \neq n_0} c_n^{(j)} \underbrace{\langle n_0|\hat{\mathcal{H}}_0|n\rangle}_0 + \sum_{n \neq n_0} c_n^{(j-1)} \langle n_0|\hat{\mathcal{H}}_1|n\rangle \quad j > 0,$$

and therefore we need the $c_n^{(j)}$'s in order to calculate E_a . This is obtained order by order in λ by multiplying by left with $\langle m \neq n_0 |$ the two above columns:

$$\begin{aligned}
\lambda^1 : c_m^{(1)} E_m^{(0)} + \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle &= E_{n_0}^{(0)} c_m^{(1)} \Rightarrow c_m^{(1)} = -\frac{\langle m | \hat{\mathcal{H}}_1 | n_0 \rangle}{E_m^{(0)} - E_{n_0}^{(0)}}, \\
\lambda^2 : c_m^{(2)} E_m^{(0)} + \sum_{n \neq n_0} c_n^{(1)} \langle m | \hat{\mathcal{H}}_1 | n \rangle &= \Rightarrow c_m^{(2)} = \sum_{n \neq n_0} \frac{\langle m | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)}) (E_n^{(0)} - E_{n_0}^{(0)})} \\
&= E_a^{(1)} c_m^{(1)} + E_{n_0}^{(0)} c_m^{(2)} - \frac{\langle m | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)})^2}, \\
\lambda^3 : c_m^{(3)} E_m^{(0)} + \sum_{n \neq n_0} c_n^{(2)} \langle m | \hat{\mathcal{H}}_1 | n \rangle &= \Rightarrow c_m^{(3)} = -\sum_{n \neq n_0} \sum_{k \neq n_0} \frac{\langle m | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | k \rangle \langle k | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)}) (E_n^{(0)} - E_{n_0}^{(0)}) (E_k^{(0)} - E_{n_0}^{(0)})} \\
&= E_a^{(2)} c_m^{(1)} + E_a^{(1)} c_m^{(2)} + E_{n_0}^{(0)} c_m^{(3)} + \sum_{n \neq n_0} \frac{\langle m | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)})^2 (E_n^{(0)} - E_{n_0}^{(0)})} \\
&\quad + \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \sum_{n \neq n_0} \frac{\langle m | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)})^2 (E_n^{(0)} - E_{n_0}^{(0)})} \\
&\quad + \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \sum_{n \neq n_0} \frac{\langle m | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)}) (E_n^{(0)} - E_{n_0}^{(0)})^2} \\
&\quad - \frac{(\langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle)^2 \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)})^3}.
\end{aligned}$$

By introducing the resolvent operator $\hat{g} = -\frac{1 - |n_0\rangle\langle n_0|}{\hat{\mathcal{H}}_0 - E_{n_0}^{(0)}} = -\sum_{n \neq n_0} \frac{|n\rangle\langle n|}{\hat{\mathcal{H}}_0 - E_{n_0}^{(0)}}$, the above results substituted into $E_a^{(j)} = \sum_{m \neq n_0} c_m^{(j-1)} \langle n_0 | \hat{\mathcal{H}}_1 | m \rangle$ can be simplified into ⁸

$$\begin{aligned}
E_a^{(1)} &= \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle, \\
E_a^{(2)} &= \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle, \\
E_a^{(3)} &= \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 | n_0 \rangle, \\
E_a^{(4)} &= \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle - \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 | n_0 \rangle \\
&\quad - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 | n_0 \rangle \\
&\quad + \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \langle n_0 | \hat{\mathcal{H}}_1 \hat{g}^3 \hat{\mathcal{H}}_1 | n_0 \rangle,
\end{aligned} \tag{D.35}$$

⁸The resolvent operator satisfies $\hat{g}|m \neq n_0\rangle = \frac{1}{E_{n_0}^{(0)}} \sum_{j=0}^{\infty} \left(\frac{\hat{\mathcal{H}}_0}{E_{n_0}^{(0)}} \right)^j |m\rangle = \frac{1}{E_{n_0}^{(0)}} \sum_{j=0}^{\infty} \left(\frac{E_m^{(0)}}{E_{n_0}^{(0)}} \right)^j |m\rangle = -\frac{|m\rangle}{E_m^{(0)} - E_{n_0}^{(0)}}$ and $\hat{g}|n_0\rangle = 0$. Thus $\hat{g}$ is a projector onto the subspace normal to eigenvector $|n_0\rangle$ which further "reads" the eigenvalue corresponding to operator $\hat{\mathcal{H}}_0$. Since the set of eigenvectors $\{|n\rangle\}$ is complete, one obtains for instance $\langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle = \sum_m \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} | m \rangle \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle = -\sum_{m \neq n_0} \frac{\langle n_0 | \hat{\mathcal{H}}_1 | m \rangle \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle}{E_m^{(0)} - E_{n_0}^{(0)}}$ and $\langle n_0 | \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 | n_0 \rangle = \sum_{m \neq n_0} \frac{\langle n_0 | \hat{\mathcal{H}}_1 | m \rangle \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)})^2}$, and $\langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle = \sum_m \sum_n \langle n_0 | \hat{\mathcal{H}}_1 \hat{g} | m \rangle \langle m | \hat{\mathcal{H}}_1 \hat{g} | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle = \sum_{m \neq n_0} \sum_{n \neq n_0} \frac{\langle n_0 | \hat{\mathcal{H}}_1 | m \rangle \langle m | \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{\mathcal{H}}_1 | n_0 \rangle}{(E_m^{(0)} - E_{n_0}^{(0)}) (E_n^{(0)} - E_{n_0}^{(0)})}$.

and eigenvector $|a\rangle$ takes the form⁹

$$|a\rangle = |n_0\rangle + \lambda \hat{g} \hat{\mathcal{H}}_1 |n_0\rangle + \lambda^2 \left(\hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 |n_0\rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \hat{g}^2 \hat{\mathcal{H}}_1 |n_0\rangle \right) + \lambda^3 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 |n_0\rangle + \dots$$

$$+ \lambda^3 \left(-\langle n_0 | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle \hat{g}^2 \hat{\mathcal{H}}_1 |n_0\rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \left(\hat{g}^2 \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 + \hat{g} \hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1 \right) |n_0\rangle + \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle^2 \hat{g}^3 \hat{\mathcal{H}}_1 |n_0\rangle \right) + \dots$$

Let us apply these results to the discussion of which value of κ characterizes the ordered or magnetized state of the 2-dim. Ising model in the continuum limit $a_\tau \rightarrow 0$. Returning to 1-dim. Hamiltonian operator which describes such a limit $\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in 1\text{-dim. row} \perp a_\tau} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \text{row}} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]$, it will be shown that the magnetization for $\kappa > 1$ does not vanish while for $\kappa < 1$ is zero. In order to calculate magnetization $\mathcal{M} = \langle (\hat{\sigma}_3)_{\vec{x}_0} \rangle$, which is defined as the derivative of the free energy density with respect to an external magnetic field h , the Hamiltonian operator is extended into

$$\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}] - h \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}}, \quad (\text{D.37})$$

where $|\Lambda_\perp|$ is the number of lattice sites in the direction normal to the continuum τ -axis; the lattice's size along the τ -direction, Na_τ (which here is considered infinite in order to remove the periodicity along such a dimension), should not be confused with the lattice's size along the perpendicular direction $|\Lambda_\perp|a$ (which is still maintained finite). For κ large one can separate the Hamiltonian operator in the

following pieces $\frac{\hat{\mathcal{H}}}{\kappa} = \left(\sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}] - \frac{h}{\kappa} \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}} \right) + \frac{\eta}{\kappa} \left(- \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}} \right)$ and thus

consider the last term as a perturbation of order $\lambda = \eta/\kappa$. Since along the continuum direction the limit of an infinite large lattice is taken, the free energy is dominated by the largest eigenvalue of transfer matrix operator $\hat{T} = \exp[-a_\tau \hat{\mathcal{H}}/\eta]$, or equivalently, by the smallest eigenvalue of $\hat{\mathcal{H}}$: for $h \neq 0$, operator

$\frac{\hat{\mathcal{H}}_0}{\kappa} \equiv \sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}] - \frac{h}{\kappa} \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}}$ has only one eigenvector with the smallest possible

eigenvalue. Denoting by $|\uparrow_{\vec{x}}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\vec{x}}$ and $|\downarrow_{\vec{x}}\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\vec{x}}$ the eigenvectors of $(\hat{\sigma}_3)_{\vec{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\vec{x}}$, the

effect of $(\hat{\sigma}_1)_{\vec{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\vec{x}}$ contained in $\frac{\hat{\mathcal{H}}_1}{\kappa} \equiv - \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}}$ is to reverse the $\hat{\mathcal{H}}_0$ -eigenvector at $\vec{x}$

and thus the ground state of $\hat{\mathcal{H}}_0$ is written as¹⁰

$$|n_0\rangle \equiv |0\rangle_{\kappa=\infty} = \underbrace{|\uparrow \dots \uparrow\rangle}_{|\Lambda_\perp| \text{ sites}}. \quad (\text{D.38})$$

⁹Starting at the λ^1 -order contribution $\sum_{m \neq n_0} c_m^{(1)} |m\rangle$ and substituting the value of $c_m^{(1)}$ into it, one gets

$$\sum_{m \neq n_0} c_m^{(1)} |m\rangle = - \sum_{m \neq n_0} \frac{|m\rangle \langle m | \hat{\mathcal{H}}_1 | n_0 \rangle}{E_m^{(0)} - E_{n_0}^{(0)}} = \sum_{m \neq n_0} |m\rangle \langle m | \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle. \text{ Since } \hat{g} \sim \sum_{n \neq n_0} |n\rangle \langle n| \text{ then } \sum_{m \neq n_0} |m\rangle \langle m | \hat{g} = \sum_{\text{all } m} |m\rangle \langle m | \hat{g} = \hat{g}$$

and thus $\sum_{m \neq n_0} c_m^{(1)} |m\rangle = \hat{g} \hat{\mathcal{H}}_1 |n_0\rangle$. In the same way, $\sum_{m \neq n_0} c_m^{(2)} |m\rangle = \sum_{m \neq n_0} \sum_{n \neq n_0} |m\rangle \langle m | \hat{g} \hat{\mathcal{H}}_1 | n \rangle \langle n | \hat{g} \hat{\mathcal{H}}_1 | n_0 \rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \sum_{m \neq n_0} |m\rangle \langle m | \hat{g}^2 \hat{\mathcal{H}}_1 | n_0 \rangle = \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 |n_0\rangle - \langle n_0 | \hat{\mathcal{H}}_1 | n_0 \rangle \hat{g}^2 \hat{\mathcal{H}}_1 |n_0\rangle$.

¹⁰Without the external magnetic field, unperturbed Hamiltonian $\hat{\mathcal{H}}_0 = \sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]$ has two possible

eigenvectors of lowest eigenvalues, $|\uparrow \dots \uparrow\rangle$ and $|\downarrow \dots \downarrow\rangle$, whose $\hat{\mathcal{H}}_0$ -eigenvalue is zero; the existence of more than one possible lowest-energy eigenvectors at $\kappa = \infty$ is due to the global $Z(2)$ symmetry of $\hat{\mathcal{H}}$. After switching on the external magnetic field, only one of these two eigenvectors is chosen as the ground state, which here is taken to be that with all $(\hat{\sigma}_3)_{\vec{x}}$ -eigenvectors up.

Its $\hat{\mathcal{H}}_0$ -eigenvalue is $E_{n_0}^{(0)} \equiv \kappa|\Lambda_\perp| \cdot 0 - h|\Lambda_\perp|$ and thus

$$\begin{aligned} \frac{1}{\hat{\mathcal{H}}_0 - E_{n_0}^{(0)}} |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow\rangle &= \frac{|\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow\rangle}{[\kappa(|\Lambda_\perp| - 2) \cdot 0 + 2\kappa \cdot 2 - h(|\Lambda_\perp| - 1) + h] - [-h|\Lambda_\perp|]}, \\ -\frac{1}{\hat{\mathcal{H}}_0 - E_{n_0}^{(0)}} \frac{\hat{\mathcal{H}}_1}{\kappa} |\uparrow \dots \uparrow\rangle &= \frac{1}{4\kappa + 2h} \sum_{\vec{x}' \in |\Lambda_\perp|} |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow\rangle, \end{aligned} \quad (\text{D.39})$$

where 4κ arises from the two bonds “touched” by the flipped spin: $\dots \uparrow_{\vec{x}'-1} \downarrow_{\vec{x}'} \dots$ and $\dots \downarrow_{\vec{x}'} \uparrow_{\vec{x}'+1} \dots$.

Further, since $\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle$ is orthogonal to $|\uparrow \dots \uparrow\rangle$, then the projection’s effect of $\hat{g}$ on $\hat{\mathcal{H}}_0$ -eigenvectors orthogonal to this last state is not necessary to first perturbative order, or equivalently, $\hat{g}\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle = -\frac{1}{\hat{\mathcal{H}}_0 - E_{n_0}^{(0)}} \hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle$. Element $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1^p | \uparrow \dots \uparrow \rangle$ vanishes for integer p equal to

an odd value; the same holds for any combination of $\hat{g}$ and $\hat{\mathcal{H}}_1$ which contains an odd number of $\hat{\mathcal{H}}_1$. The leading contributions to the perturbation at large κ arise from $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g}^p \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle$ and $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle$: in the first of these two elements, $\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle$ contains only one flipped spin (where here “spin” is understood as the eigenvector of operator $(\hat{\sigma}_3)_{\vec{x}}$) and thus, according to (D.39), one obtains

$$\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g}^p \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = \kappa^2 \left(\frac{-1}{4\kappa + 2h} \right)^p \sum_{\vec{x}, \vec{x}' \in |\Lambda_\perp|} \underbrace{\langle \uparrow \dots \uparrow \downarrow_{\vec{x}} \uparrow \dots \uparrow | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \rangle}_{\delta_{\vec{x}, \vec{x}'}} = \frac{(-1)^p \kappa^2}{(4\kappa + 2h)^p} |\Lambda_\perp|,$$

while the two flipped spins in $\hat{g}\hat{\mathcal{H}}_1 \hat{g}\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle$ can be placed as follows¹¹: as nearest neighbours, $\left(\frac{\kappa}{4\kappa + 4h}\right) \left(\frac{\kappa}{4\kappa + 2h}\right) |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+1} \uparrow \dots \uparrow\rangle$ (the 4κ from the second application of $\hat{g}$ comes from the two bonds located touching the flipped spins, $\uparrow_{\vec{x}'-1} \downarrow_{\vec{x}'}$ and $\downarrow_{\vec{x}'+1} \uparrow_{\vec{x}'+2}$), or separated from each other, $\left(\frac{\kappa}{8\kappa + 4h}\right) \left(\frac{\kappa}{4\kappa + 2h}\right) |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \downarrow_{\vec{x}'+j} \uparrow \dots \uparrow\rangle$, where 8κ is produced by the four bonds affected by the flipped spins. Hence

$$\begin{aligned} \hat{g}\hat{\mathcal{H}}_1 \hat{g}\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle &= 2 \left(\frac{\kappa}{4\kappa + 4h} \right) \left(\frac{\kappa}{4\kappa + 2h} \right) \sum_{\vec{x}' \in |\Lambda_\perp|} |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+1} \uparrow \dots \uparrow\rangle + \\ &+ \left(\frac{\kappa}{8\kappa + 4h} \right) \left(\frac{\kappa}{4\kappa + 2h} \right) \sum_{\vec{x}' \in |\Lambda_\perp|} \sum_{j' \in (|\Lambda_\perp| - 3) \text{ sites } \neq \vec{x}', \vec{x}' \pm 1} |\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \downarrow_{\vec{x}'+j'} \uparrow \dots \uparrow\rangle, \end{aligned} \quad (\text{D.40})$$

where the factor 2 in the first term takes into account that state $|\uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+1} \uparrow \dots \uparrow\rangle$ is produced by the action of the operator $(\hat{\sigma}_1)_{\vec{x}'}$ (contained in $\hat{\mathcal{H}}_1$) on $|\uparrow \dots \uparrow \uparrow_{\vec{x}'} \downarrow_{\vec{x}'+1} \uparrow \dots \uparrow\rangle$ as well as by the action of $(\hat{\sigma}_1)_{\vec{x}'+1}$ on $|\uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow_{\vec{x}'+1} \uparrow \dots \uparrow\rangle$. And $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle$ becomes equal to

$$\begin{aligned} &\frac{-\kappa^4}{(4\kappa + 2h)^2} \left[\frac{1}{8\kappa + 4h} \sum_{\vec{x}, \vec{x}' \in |\Lambda_\perp|} \sum_{j, j' \in (|\Lambda_\perp| - 3) \text{ sites}} \langle \uparrow \dots \uparrow \downarrow_{\vec{x}-j} \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \downarrow_{\vec{x}'+j'} \uparrow \dots \uparrow \rangle + \right. \\ &\left. + \frac{4}{4\kappa + 4h} \sum_{\vec{x}, \vec{x}' \in |\Lambda_\perp|} \langle \uparrow \dots \uparrow \downarrow_{\vec{x}} \downarrow_{\vec{x}+1} \uparrow \dots \uparrow | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+j'} \uparrow \dots \uparrow \rangle \right] = \frac{-\kappa^4 |\Lambda_\perp|}{(4\kappa + 2h)^2} \left[\frac{2(|\Lambda_\perp| - 3)}{8\kappa + 4h} + \frac{4}{4\kappa + 4h} \right], \end{aligned}$$

¹¹Of course, the projector part in $\hat{g} \sim \sum_{\text{states } \perp |\uparrow \dots \uparrow\rangle} |n\rangle \langle n|$ impedes that the consecutive application of the two $\hat{\mathcal{H}}_1$ ’s acts on the same spin, i.e., $\hat{g}\hat{\mathcal{H}}_1 \hat{g}\hat{\mathcal{H}}_1 |\uparrow \dots \uparrow\rangle$ cannot contain $|\uparrow \dots \uparrow\rangle$.

without forgetting all possible spin arrangements. Since in the limit of an infinite lattice's length Na_τ along the continuum periodic direction (but keeping finite the lattice's size along the other directions), the free energy density, $\mathcal{F}_{\Lambda_\perp} = -\frac{1}{|\Lambda_\perp|} \lim_{(Na_\tau) \rightarrow \infty} \frac{\eta}{Na_\tau} \ln \left[e^{-(Na_\tau) E_{\text{lowest}}/\eta} + \dots \right]$, is determined by the lowest eigenvalue of $\hat{\mathcal{H}}$, and by using (D.35)¹²

$$(\mathcal{F}_{\Lambda_\perp})_{\text{large } \kappa} = -h - \frac{\eta^2}{4\kappa + 2h} - \frac{\eta^4}{(4\kappa + 2h)^3} \frac{\kappa - h}{\kappa + h} - \dots \quad (\text{D.41})$$

By definition $\mathcal{M}_{\Lambda_\perp}(h) \equiv \langle (\hat{\sigma}_3)_{\vec{x}_0} \rangle_{\Lambda_\perp, h} = -\frac{\partial \mathcal{F}_{\Lambda_\perp}}{\partial h}$ (where the lattice site $\vec{x}_0$ is irrelevant since the model's energy is invariant under translations), and thus the magnetization which remains after removing the external magnetic field is, for large κ , equal to

$$m_{\text{sp}} = \lim_{h \rightarrow 0} \mathcal{M}_{\Lambda_\perp}(h) = 1 - \frac{\eta^2}{8\kappa^2} - \frac{7\eta^4}{128\kappa^4} - \dots, \quad (\text{D.42})$$

which is independent of the lattice's size along the direction normal to the continuum dimension¹³. From this result, the spontaneous magnetization decreases with decreasing κ and thus we can suppose that for κ small enough, a phase transition to a disordered state with $m_{\text{sp}} = 0$ will occur. At $\kappa = 0$,

¹²Note that contribution $+\frac{\eta^4|\Lambda_\perp|^2}{(4\kappa + 2h)^3}$, arising from $-\left(\frac{\eta}{\kappa}\right)^4 \langle \uparrow \dots \uparrow |\hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1| \uparrow \dots \uparrow \rangle \langle \uparrow \dots \uparrow |\hat{\mathcal{H}}_1 \hat{g}^2 \hat{\mathcal{H}}_1| \uparrow \dots \uparrow \rangle$, cancels the term depending on $|\Lambda_\perp|^2$ in the $\left(\frac{\eta}{\kappa}\right)^4 \langle \uparrow \dots \uparrow |\hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1| \uparrow \dots \uparrow \rangle$ -contribution $-\frac{\eta^4}{(4\kappa + 2h)^2} \frac{2|\Lambda_\perp|^2}{8\kappa + 4h}$.

$$\begin{aligned} (E_{\text{lowest}})_{\text{large } \kappa} &\approx -h|\Lambda_\perp| - \frac{\eta^2|\Lambda_\perp|}{4\kappa + 2h} - \frac{\eta^4}{(4\kappa + 2h)^2} \left(\frac{4|\Lambda_\perp|}{4\kappa + 4h} + \frac{2|\Lambda_\perp|(|\Lambda_\perp| - 3)}{8\kappa + 4h} \right) + \frac{\eta^4|\Lambda_\perp|^2}{(4\kappa + 2h)^3} = \\ &= |\Lambda_\perp| \left[-h - \frac{\eta^2}{4\kappa + 2h} - \frac{\eta^4}{(4\kappa + 2h)^3} \frac{\kappa - h}{\kappa + h} \right]. \end{aligned}$$

¹³The same result can be obtained in the following way [42]: without considering an external magnetic field ($h = 0$), one imposes as in the Peierls argument that the spins at the boundary (of lattice $\Lambda_\perp$) must be up and then one studies, in dependence on coupling strength κ , whether the strength of such a boundary condition is able to maintain a state in the whole system with almost all spins aligned. At $\kappa = \infty$, the boundary together with Hamiltonian $\hat{\mathcal{H}}|_{\kappa=\infty} \equiv \hat{\mathcal{H}}_0 = \kappa \sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}]$ forces all spins to be up. Since

$\hat{\mathcal{H}}_1 | \uparrow \dots \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+\hat{1}} \uparrow \dots \uparrow \rangle = | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+\hat{1}} \uparrow \dots \uparrow \rangle + | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow_{\vec{x}'+\hat{1}} \uparrow \dots \uparrow \rangle + \dots$, where $\dots$ means states with 3 spins down, and $\hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \downarrow_{\vec{x}'+j\hat{1}} \uparrow \dots \uparrow \rangle = | \uparrow \dots \uparrow \downarrow_{\vec{x}'+j\hat{1}} \uparrow \dots \uparrow \rangle + | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \rangle + \dots$, then according to (D.40) and with $h = 0$, one obtains $\left(\frac{\eta}{\kappa}\right)^3 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = \eta^3 \left(\frac{4}{(4\kappa)^3} + \frac{2(|\Lambda_\perp| - 3)}{8\kappa(4\kappa)^2} \right) \sum_{\vec{x}' \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \rangle$. Sub-

stituting this result into (D.36), and remembering $\langle \uparrow \dots \uparrow |\hat{\mathcal{H}}_1| \uparrow \dots \uparrow \rangle = 0$, the perturbative contributions at large but finite κ to the ground state with the above boundary condition is equal to

$$\begin{aligned} |0\rangle_{\text{large } \kappa} &\approx | \uparrow \dots \uparrow \rangle + \left(\frac{\eta}{\kappa}\right) \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle + \left(\frac{\eta}{\kappa}\right)^3 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle - \left(\frac{\eta}{\kappa}\right)^3 \langle \uparrow \dots \uparrow |\hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1| \uparrow \dots \uparrow \rangle \hat{g}^2 \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle + \\ &+ \left(\frac{\eta}{\kappa}\right)^2 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = | \uparrow \dots \uparrow \rangle + \frac{\eta}{4\kappa} \left(1 + \frac{4\eta^2}{(4\kappa)^2} + \frac{\eta^2(|\Lambda_\perp| - 3)}{(4\kappa)^2} - \frac{\eta^2|\Lambda_\perp|}{(4\kappa)^2} \right) \sum_{\vec{x}' \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \rangle + \dots + \\ &+ \frac{2\eta^2}{(4\kappa)^2} \sum_{\vec{x}' \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \downarrow_{\vec{x}'+\hat{1}} \uparrow \dots \uparrow \rangle + \frac{\eta^2}{2(4\kappa)^2} \sum_{\vec{x}' \in |\Lambda_\perp|} \sum_{j' \in (|\Lambda_\perp| - 3) \text{ sites } \neq \vec{x}', \vec{x}' \pm \hat{1}} | \uparrow \dots \uparrow \downarrow_{\vec{x}'} \uparrow \dots \uparrow \downarrow_{\vec{x}'+j'\hat{1}} \uparrow \dots \uparrow \rangle, \end{aligned}$$

where it has been kept terms only up to order $(\eta/\kappa)^4$. Hence, up to the same order in κ , one obtains

$$|0\rangle_{\text{large } \kappa} \langle 0|0\rangle_{\text{large } \kappa} = 1 + \frac{\eta^2|\Lambda_\perp|}{(4\kappa)^2} \left(1 + \frac{\eta^2}{(4\kappa)^2} \right)^2 + \frac{4\eta^4|\Lambda_\perp|}{(4\kappa)^4} + \frac{2\eta^4|\Lambda_\perp|(|\Lambda_\perp| - 3)}{4(4\kappa)^4},$$

$\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}}$: if $|\uparrow\vec{x}\rangle$ denotes the eigenvector of operator $(\hat{\sigma}_1)_{\vec{x}}$ with eigenvalue $+1$, then the state with the smallest $\hat{\mathcal{H}}$ -eigenvalue is $|0\rangle_{\kappa=0} \equiv |\uparrow \dots \uparrow\rangle$

$$(\hat{\sigma}_1)_{\vec{x}} |0\rangle_{\kappa=0} = |0\rangle_{\kappa=0} \quad \forall \vec{x},$$

$$\left(-\eta \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}} \right) |0\rangle_{\kappa=0} = -\eta |\Lambda_\perp| |0\rangle_{\kappa=0}. \quad (\text{D.43})$$

In this eigenvector basis, the representation of matrix operator $(\hat{\sigma}_3)_{\vec{x}}$ is not diagonal but $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\vec{x}}$ and

therefore $m_{sp} = \frac{\kappa=0 \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\kappa=0}}{\kappa=0 \langle 0 | 0 \rangle_{\kappa=0}} = \frac{\langle \uparrow \dots \uparrow \vec{x}_0 \dots \uparrow | \uparrow \dots \downarrow \vec{x}_0 \dots \uparrow \rangle}{\langle \uparrow \dots \uparrow | \uparrow \dots \uparrow \rangle} = 0$; the same result can be ob-

tained by using the algebraic properties of the Pauli matrices, $\begin{cases} (\hat{\sigma}_1)_{\vec{x}'} (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_1)_{\vec{x}'} = -(\hat{\sigma}_3)_{\vec{x}}, & \vec{x}' = \vec{x} \\ (\hat{\sigma}_1)_{\vec{x}'} (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_1)_{\vec{x}'} = (\hat{\sigma}_3)_{\vec{x}}, & \vec{x}' \neq \vec{x} \end{cases}$, which leads to $\kappa=0 \langle 0 | (\hat{\sigma}_1)_{\vec{x}_0} (\hat{\sigma}_3)_{\vec{x}_0} (\hat{\sigma}_1)_{\vec{x}_0} | 0 \rangle_{\kappa=0} = -\kappa=0 \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\kappa=0}$, and by considering on the other hand that $(\hat{\sigma}_1)_{\vec{x}} | 0 \rangle_{\kappa=0} = | 0 \rangle_{\kappa=0} \quad \forall \vec{x}$ and thus $\kappa=0 \langle 0 | (\hat{\sigma}_1)_{\vec{x}_0} (\hat{\sigma}_3)_{\vec{x}_0} (\hat{\sigma}_1)_{\vec{x}_0} | 0 \rangle_{\kappa=0} = \kappa=0 \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\kappa=0}$. At small $\kappa \neq 0$, one proceeds as at large κ by switching on an external magnetic field and by separating

now the Hamiltonian into $\hat{\mathcal{H}}_0 + \frac{\kappa}{\eta} \hat{\mathcal{H}}_1 = \left(-\eta \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_1)_{\vec{x}} \right) + \frac{\kappa}{\eta} \left(\eta \sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_3)_{\vec{x}+\hat{1}}] - \frac{h\eta}{\kappa} \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}} \right)$

where one can carry out the perturbative expansion in powers of $\lambda = \kappa/\eta$: the result is the same as at $\kappa = 0$, namely, $m_{sp} = 0$ as one can easily seen by considering the following partial results

$$\hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = \eta |\Lambda_\perp| | \uparrow \dots \uparrow \rangle - \eta \sum_{\vec{x} \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}} \downarrow_{\vec{x}+\hat{1}} \uparrow \dots \uparrow \rangle - \frac{h\eta}{\kappa} \sum_{\vec{x} \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}} \uparrow \dots \uparrow \rangle,$$

$$\hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = \frac{\eta}{-\eta(|\Lambda_\perp| - 2 - 2) + \eta |\Lambda_\perp|} \sum_{\vec{x} \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}} \downarrow_{\vec{x}+\hat{1}} \uparrow \dots \uparrow \rangle + \frac{h\eta}{\kappa 2\eta} \sum_{\vec{x} \in |\Lambda_\perp|} | \uparrow \dots \uparrow \downarrow_{\vec{x}} \uparrow \dots \uparrow \rangle,$$

without forgetting the projective property of operator $\hat{g}$ onto the subspace orthogonal to $| \uparrow \dots \uparrow \rangle$.

Hence $\frac{\kappa}{\eta} \langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = \kappa |\Lambda_\perp|$, $\frac{\kappa^2}{\eta^2} \langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle = -\frac{\kappa^2 |\Lambda_\perp|}{4\eta} - \frac{h^2 |\Lambda_\perp|}{2\eta}$ and therefore

$$\mathcal{F}_{\Lambda_\perp}(h) = \frac{1}{|\Lambda_\perp|} |\Lambda_\perp| \left(-\eta + \kappa - \frac{\kappa^2}{4\eta} - \frac{h^2}{2\eta} \right), \quad (\text{D.44})$$

where in the considered κ -order, $\frac{1}{(4\kappa)^2} \left(1 + \frac{\eta^2}{(4\kappa)^2} \right)^2$ can be approximated by $\frac{1}{(4\kappa)^2} \left(1 + \frac{2\eta^2}{(4\kappa)^2} \right)$; and according to the expansion of $\frac{1}{1+x} = 1 - x + x^2 + \dots$,

$$\frac{1}{\text{large } \kappa \langle 0 | 0 \rangle_{\text{large } \kappa}} \approx 1 - \left[\frac{\eta^2 |\Lambda_\perp|}{(4\kappa)^2} \left(1 + \frac{2\eta^2}{(4\kappa)^2} \right) + \frac{4\eta^4 |\Lambda_\perp|}{(4\kappa)^4} + \frac{2\eta^4 |\Lambda_\perp| (|\Lambda_\perp| - 3)}{4(4\kappa)^4} \right] + \left[\frac{\eta^2 |\Lambda_\perp|}{(4\kappa)^2} \right]^2.$$

The numerator of $\frac{\text{large } \kappa \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\text{large } \kappa}}{\text{large } \kappa \langle 0 | 0 \rangle_{\text{large } \kappa}}$ is, up to order κ^{-4} , given by

$$\text{large } \kappa \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\text{large } \kappa} = 1 + \frac{\eta^2 (|\Lambda_\perp| - 1 - 1)}{(4\kappa)^2} \left(1 + \frac{2\eta^2}{(4\kappa)^2} \right) + \frac{4\eta^4 (|\Lambda_\perp| - 2 - 2)}{(4\kappa)^4} + \frac{2\eta^4 (|\Lambda_\perp| - 2 - 2) (|\Lambda_\perp| - 3)}{4(4\kappa)^4},$$

and finally the (spontaneous) magnetization is equal to

$$\frac{\text{large } \kappa \langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle_{\text{large } \kappa}}{\text{large } \kappa \langle 0 | 0 \rangle_{\text{large } \kappa}} = 1 - \frac{2\eta^2}{(4\kappa)^2} - \frac{(4 + 16 - 6)\eta^4}{(4\kappa)^4} + \frac{\eta^4 |\Lambda_\perp| (-2 + 2)}{(4\kappa)^4} + \frac{\eta^4 |\Lambda_\perp|^2 (-1 + 1)}{(4\kappa)^4} = 1 - \frac{\eta^2}{8\kappa^2} - \frac{7\eta^4}{128\kappa^4}.$$

keeping terms until order κ^2 . Therefore $m_{\text{sp}} = \lim_{h \rightarrow 0} \frac{\partial \mathcal{F}_{\Lambda_\perp}}{\partial h} = 0$. In fact all terms in the expansion of $\mathcal{F}_{\Lambda_\perp}(h)$ for $\lambda = \kappa/\eta < 1$ will contain only even powers of h , what leads to $m_{\text{sp}} = 0$ at any order of $\kappa/\eta < 1$. In order to show that $\mathcal{F}_{\Lambda_\perp}(h)$ is an even function of h , let us consider a general term in its perturbative expansion¹⁴ $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g}^{n_k} \hat{\mathcal{H}}_1 \dots \hat{\mathcal{H}}_1 \hat{g}^{n_1} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle$: since the part of $\hat{\mathcal{H}}_1$ which contains h flips only one $\uparrow$ -spin, we obtain the following schematic table

$$\begin{aligned}
\hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle &= \eta \underbrace{(0 \text{ flip})}_{(2 \text{ flips})} + h \overbrace{(1 \text{ flip})}^{h \downarrow} + \\
\hat{\mathcal{H}}_1 \hat{g}^{n_1} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle &= \eta^2 \underbrace{(0 \text{ flip})}_{(2 \text{ flips})} + h\eta \overbrace{(1 \text{ flip})}^{h \downarrow} + h^2 \overbrace{(2 \text{ flips})}^{\eta \downarrow \downarrow} + \\
&\quad \eta^2 \underbrace{(2 \text{ flips})}_{(4 \text{ flips})} + h\eta \underbrace{(3 \text{ flips})}_{(4 \text{ flips})} + \underline{\underline{h^2 (0 \text{ flip})}} + \\
\hat{\mathcal{H}}_1 \hat{g}^{n_2} \hat{\mathcal{H}}_1 \hat{g}^{n_1} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle &= \eta^3 \underbrace{(0 \text{ flip})}_{(2 \text{ flips})} + h\eta^2 \overbrace{(1 \text{ flip})}^{h \downarrow} + h^2 \eta \overbrace{(2 \text{ flips})}^{\eta \downarrow \downarrow} + h^3 \overbrace{(1 \text{ flip})}^{h \downarrow} + \\
&\quad \eta^3 \underbrace{(2 \text{ flips})}_{(4 \text{ flips})} + h\eta^2 \underbrace{(3 \text{ flips})}_{(4 \text{ flips})} + h^2 \eta \underbrace{(4 \text{ flips})}_{(4 \text{ flips})} + h^3 \underbrace{(3 \text{ flips})}_{(4 \text{ flips})} + \\
&\quad \eta^3 \underbrace{(4 \text{ flips})}_{(6 \text{ flips})} + h\eta^2 \underbrace{(5 \text{ flips})}_{(6 \text{ flips})} + \underline{\underline{h^2 \eta (0 \text{ flip})}} + \\
&\quad \eta^3 \underbrace{(6 \text{ flips})}_{(6 \text{ flips})}
\end{aligned}$$

where the double underlined term is the only contribution depending on h which remains when, at a certain κ -order, element $\langle \uparrow \dots \uparrow | \hat{\mathcal{H}}_1 \hat{g}^{n_k} \hat{\mathcal{H}}_1 \dots \hat{\mathcal{H}}_1 \hat{g}^{n_1} \hat{\mathcal{H}}_1 | \uparrow \dots \uparrow \rangle$ is calculated; and this underlined part is also the term eliminated by $\hat{g}^{n_{k+1}}$ when we go to next perturbation order. On the other hand,

$\overbrace{(1 \text{ flip})}^{h \downarrow}$ denotes the term which will produce one of the $\underbrace{(0 \text{ flip})}_{(2 \text{ flips})}$ -contributions (depending on h) in the next order by the action of $\hat{\mathcal{H}}_1$'s part $-\frac{h\eta}{\kappa} \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}}$ on it; analogously, $\overbrace{(2 \text{ flips})}^{\eta \downarrow \downarrow}$ denotes the same but for the action of $\hat{\mathcal{H}}_1$'s part $-\kappa \sum_{\vec{x} \in |\Lambda_\perp|} (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_3)_{\vec{x}+\hat{1}}$. Note that the action of $\kappa \sum_{\vec{x} \in |\Lambda_\perp|} 1$ on $\underline{\underline{(0 \text{ flip})}}$ does not appear in the next κ -order since this last term is filtered out by $\hat{g}$. From all the above discussion, at a certain perturbation order the only terms will be contributed in the next order with a h -power in $\underbrace{(0 \text{ flip})}_{(2 \text{ flips})}$ are: $\overbrace{(1 \text{ flip})}^{h \downarrow}$, where $\underbrace{(1 \text{ flip})}_{(2 \text{ flips})}$ must be multiplied by an odd power of h since $\sum_{\vec{x} \in |\Lambda_\perp|} [1 - (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_3)_{\vec{x}+\hat{1}}]$

only produces an even number of flips, and $\overbrace{(2 \text{ flips})}^{\eta \downarrow \downarrow}$, where this $\underbrace{(2 \text{ flips})}_{(4 \text{ flips})}$ is multiplied by h^{2n} with n strictly positive. Hence at any order in $\kappa/\eta < 1$, $\underline{\underline{(0 \text{ flip})}}$ is always multiplied by an even power of h , and therefore $\mathcal{F}_{\Lambda_\perp}(h)$ is an even function of h as long as $\kappa < \eta$.

Returning to equation (D.42), for large κ the spontaneous magnetization is given by an expansion in powers of $\zeta = \eta^2/\kappa^2$, $m_{\text{sp}} = \mathcal{M}(h=0) = \sum_n a_n \zeta^n$, and on the other hand $m_{\text{sp}} = 0$ at small κ . Near to critical point ζ_{crit} , $\zeta \lesssim \zeta_{\text{crit}}$, can the magnetization be described by a power behaviour of

¹⁴It is noteworthy that in such an element cannot stay contiguously two operators $\hat{\mathcal{H}}_1$ without being separated by some power of projection operator $\hat{g}$. This is due to the very form of the perturbation expansion: $E_a^{(j)} = \sum_{m \neq n_0} c_m^{(j-1)} \langle n_0 | \hat{\mathcal{H}}_1 | m \rangle$

and further $\underline{\underline{c_m^{(k)} E_m^{(0)}}} + \sum_{n \neq n_0} c_n^{(k-1)} \langle m | \hat{\mathcal{H}}_1 | n \rangle = E_a^{(k-1)} c_m^{(1)} + \dots + E_a^{(1)} c_m^{(k-1)} + \underline{\underline{E_{n_0}^{(0)} c_m^{(k)}}}$, where the two underlined terms

secure the appearance of $\hat{g}$ multiplying the contributions of lower perturbative order $\{c_m^{(1)}, \dots, c_m^{(k-1)}\}$.

the kind $m_{sp} \xrightarrow{\zeta \rightarrow \zeta_{crit}} b(\zeta_{crit} - \zeta)^{-\gamma}$? In order to know whether this is possible [59], ratio $R_n = \frac{a_n}{a_{n-1}}$ is calculated, since according to $\left(1 - \frac{\zeta}{\zeta_{cr}}\right)^{-\gamma} = 1 + \gamma\left(\frac{\zeta}{\zeta_{cr}}\right) + \dots + \frac{\gamma(\gamma+1)\dots(\gamma+n-1)}{n!}\left(\frac{\zeta}{\zeta_{cr}}\right)^n + \dots$ one should obtain $R_n = \frac{1}{\zeta_{cr}} \frac{1}{n}(\gamma+n-1)$ if $m_{sp} \sim b\zeta_{cr}^{-\gamma} \left(1 - \frac{\zeta}{\zeta_{cr}}\right)^{-\gamma}$. This $R_n = \frac{1}{\zeta_{cr}} \left(1 + \frac{\gamma-1}{n}\right)$, when represented as a function of $1/n$, is a straight line of slope $(\gamma-1)/\zeta_{cr}$ intersecting the vertical coordinate-axis at $1/\zeta_{cr}$. Applying this calculation to (D.42), $m_{sp} = 1\left(\frac{\eta^2}{\kappa^2}\right)^0 - \frac{1}{8}\left(\frac{\eta^2}{\kappa^2}\right)^1 - \frac{7}{128}\left(\frac{\eta^2}{\kappa^2}\right)^2 - \dots$, one obtains $R_1 = -\frac{1}{8}$, $R_2 = +\frac{7}{16}$: the slope is equal to $(\gamma-1)\frac{\kappa_{cr}^2}{\eta^2} = \frac{R_2 - R_1}{\frac{1}{2} - 1} = -\frac{9}{8}$ and thus

$$\begin{cases} \frac{\kappa_{cr}^2}{\eta^2} = R_1 - \text{slope} \cdot 1 = 1, \\ -\gamma = -1 - \frac{\text{slope}}{\kappa_{cr}^2/\eta^2} = \frac{1}{8}. \end{cases} \quad (\text{D.45})$$

The thus obtained critical value is, as expected, the same as that calculated by self-duality in (D.33). Exponent $-\gamma$ is called the magnetization critical exponent.

This finishes the discussion that above critical value $\kappa_{crit} = \eta$, the 2-dim. Ising model in a lattice whose spacing along one direction goes to zero (and whose size along this continuous dimension is infinite), shows the ordered phase characterized by a non-vanishing magnetization.

After having seen the general equivalence between the continuum limit in a spatial direction and the divergence of the coupling strength between variables along such a direction, let us consider the more general case of scalar fields $\sigma_{\vec{r}}$ with a continuous range and described by an energy containing local interactions

$$\frac{E}{\eta} = \frac{\beta_\tau}{2\eta} \sum_{\vec{r}} (\sigma_{\vec{r}+\hat{\tau}} - \sigma_{\vec{r}})^2 + \frac{\beta_x}{2\eta} \sum_{\vec{r}} \sum_{k=1}^{d-1} (\sigma_{\vec{r}+\hat{k}} - \sigma_{\vec{r}})^2 + \frac{\mu_0}{\eta} \sum_{\vec{r}} V(\sigma_{\vec{r}}), \quad (\text{D.46})$$

with μ_0 a constant giving the strength scale for potential $V(\sigma_{\vec{r}})$. The XY model belongs also to this class of models, but with the continuous range taking on complex values and thus term $(\sigma_{\vec{r}+\hat{\tau},\hat{k}} - \sigma_{\vec{r}})$ must be substituted by $|\sigma_{\vec{r}+\hat{\tau},\hat{k}} - \sigma_{\vec{r}}|$. The integral kernel of the transfer matrix, which “propagates” the field variable from a $\sigma_{\vec{x}}$ -row (more exactly, a $(d-1)$ -dim. plane normal to the τ -direction) to the neighbour $s_{\vec{x}}$ -row along the τ -direction, is given by

$$\begin{aligned} \mathcal{T}(s_{\vec{x}}, \sigma_{\vec{x}}) = \exp & \left[-\frac{\beta_\tau}{2\eta} \sum_{\vec{x} \in \text{row}} (s_{\vec{x}} - \sigma_{\vec{x}})^2 - \frac{\beta_x}{2\eta} \sum_{\vec{x} \in \text{row}} \sum_{k=1}^{d-1} \left[\frac{(\sigma_{\vec{x}+\hat{k}} - \sigma_{\vec{x}})^2}{2} + \frac{(s_{\vec{x}+\hat{k}} - s_{\vec{x}})^2}{2} \right] - \right. \\ & \left. - \frac{\mu_0}{\eta} \sum_{\vec{x} \in \text{row}} \left[\frac{V(\sigma_{\vec{x}})}{2} + \frac{V(s_{\vec{x}})}{2} \right] \right], \end{aligned} \quad (\text{D.47})$$

and operator $\hat{\mathcal{T}}$ acting on a function $\psi(\{\sigma_{\vec{x}}\})$ of all variables $\sigma_{\vec{x}}$ of a row is equal to

$$\hat{\mathcal{T}}\psi(\{\sigma_{\vec{x}}\}) = \int \prod_{\vec{x}} ds_{\vec{x}} \mathcal{T}(\sigma_{\vec{x}}, s_{\vec{x}}) \psi(\{s_{\vec{x}}\}).$$

The corresponding operator $\hat{\mathcal{T}}$ takes a form similar to that in the 1-dim. model (D.17)

$$\begin{aligned} \hat{\mathcal{T}} \equiv \hat{\mathcal{T}}_{a_\tau} = & \left(\sqrt{\frac{2\pi\eta}{\beta_\tau}} \right)^N \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x},k} \frac{(\hat{\sigma}_{\vec{x}+\hat{k}} - \hat{\sigma}_{\vec{x}})^2}{2} - \frac{\mu_0}{2\eta} \hat{V} \right] \times \\ & \times \exp \left[\frac{\eta}{2\beta_\tau} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \sigma_{\vec{x}}^2} \right] \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x},k} \frac{(\hat{\sigma}_{\vec{x}+\hat{k}} - \hat{\sigma}_{\vec{x}})^2}{2} - \frac{\mu_0}{2\eta} \hat{V} \right], \end{aligned} \quad (\text{D.48})$$

where $\hat{\partial}^2/\partial \sigma_{\vec{x}}^2$ is the partial derivative acting on the field variable at row site $\vec{x}$ ¹⁵. As in the 2-dim. Ising model, the explicit dependence of β_τ, β_x on the lattice spacing a_τ can be directly read off the previous equation: $\hat{\mathcal{T}} = \exp[-a_\tau \hat{\mathcal{H}}/\eta]$ (with a_τ infinitesimal) leads to the behaviour

$$\begin{cases} \beta_x & \sim & a_\tau, \\ \beta_\tau & \sim & \frac{1}{a_\tau}, \end{cases} \quad (\text{D.49})$$

plus $\mu_0 \sim a_\tau$. This behaviour at $a_\tau \rightarrow 0$, $\begin{cases} \beta_x \rightarrow 0 \\ \beta_\tau \rightarrow \infty \end{cases}$, is similar (not the exact dependence form) to that in the 2-dim. Ising model discussed above by means of the curves of equal value of a certain correlation function. Equation (D.49) can be also expressed as follows

$$\beta_\tau \beta_x \sim \text{constant}. \quad (\text{D.50})$$

Further, the connection with the Quantum Field Theory appears again straightforward: we identify $-i\eta \frac{\partial}{\partial \sigma_{\vec{x}}}$ as the momentum density operator conjugate to field operator $\hat{\sigma}_{\vec{x}}$

$$\hat{\pi}_{\vec{x}} \equiv -i\eta \frac{\partial}{\partial \sigma_{\vec{x}}} \implies [\hat{\pi}_{\vec{x}'}, \hat{\sigma}_{\vec{x}}] = -i\eta \delta_{\vec{x}', \vec{x}}. \quad (\text{D.51})$$

By setting $\beta_\tau = 1/a_\tau$ and calling J the constant in (D.50), the Hamiltonian operator corresponding to transfer matrix (D.48) becomes

$$\hat{\mathcal{T}} = \exp[-a_\tau \hat{\mathcal{H}}/\eta] \implies \hat{\mathcal{H}} = \frac{1}{2} \sum_{\vec{x}} \hat{\pi}_{\vec{x}}^2 + \frac{J}{2} \sum_{\vec{x},k} (\hat{\sigma}_{\vec{x}+\hat{k}} - \hat{\sigma}_{\vec{x}})^2 + \mu_0 \hat{V}, \quad (\text{D.52})$$

after applying the Baker-Hausdorff-Campbell relation in the limit of an infinitesimal a_τ .

In order to finish this introduction in the method of the transfer matrix, we shall see a further connection between the Hamiltonian formulation of the (classical) Statistical Mechanics and the *Quantum* Statistical Mechanics [28]. Let us denote the field variable at the m th $((d-1)$ -dimensional) plane normal to the τ -direction and at site $\vec{x}$ on this plane by $\sigma_{ma_\tau, \vec{x}}$. We return to our partition function calculated as the trace over all steps

¹⁵Expression $\frac{\hat{\partial}^2}{\partial \sigma_{\vec{x}}^2}$ is only the discrete formulation of functional derivative $\frac{\delta^2}{\delta \sigma(\vec{x})^2}$.

in the τ -direction for a *periodic* lattice along this direction: $\sigma_{(N+1)a_\tau, \vec{x}} = \sigma_{1a_\tau, \vec{x}}$. For the 1-dim. lattice we have equation (D.6)

$$\mathcal{Z}_N = \int d\sigma_{a_\tau} \dots \int d\sigma_{Na_\tau} e^{-\frac{\beta\tau}{2\eta}(\sigma_{2a_\tau} - \sigma_{a_\tau})^2 - \frac{\mu_0}{2\eta}[V(\sigma_{a_\tau}) + V(\sigma_{2a_\tau})]} e^{-\frac{\beta\tau}{2\eta}(\sigma_{3a_\tau} - \sigma_{2a_\tau})^2 - \frac{\mu_0}{2\eta}[V(\sigma_{2a_\tau}) + V(\sigma_{3a_\tau})]} \times \\ \times \dots e^{-\frac{\beta\tau}{2\eta}(\sigma_{Na_\tau} - \sigma_{(N-1)a_\tau})^2 - \frac{\mu_0}{2\eta}[V(\sigma_{(N-1)a_\tau}) + V(\sigma_{Na_\tau})]} = \text{Tr} \left[\left(\hat{\mathcal{T}}_{a_\tau} \right)^N \right],$$

or for a d -dimensional lattice in which one of the directions has a spacing a_τ different from the spacing along the other $(d-1)$ spatial directions

$$\mathcal{Z}_N = \int \prod_{\vec{x}} d\sigma_{a_\tau, \vec{x}} \int \prod_{\vec{x}} d\sigma_{2a_\tau, \vec{x}} \dots \int \prod_{\vec{x}} d\sigma_{Na_\tau, \vec{x}} \times \\ \times e^{-\frac{\beta\tau}{2\eta} \sum_{\vec{x}} (\sigma_{2a_\tau, \vec{x}} - \sigma_{a_\tau, \vec{x}})^2 - \frac{\beta\tau}{2\eta} \sum_{\vec{x}, k} \left[\frac{(\sigma_{a_\tau, \vec{x}+\vec{k}} - \sigma_{a_\tau, \vec{x}})^2}{2} + \frac{(\sigma_{2a_\tau, \vec{x}+\vec{k}} - \sigma_{2a_\tau, \vec{x}})^2}{2} \right] - \frac{\mu_0}{2\eta} \sum_{\vec{x}} [V(\sigma_{a_\tau, \vec{x}}) + V(\sigma_{2a_\tau, \vec{x}})]} \times \\ \vdots \\ \times e^{-\frac{\beta\tau}{2\eta} \sum_{\vec{x}} (\sigma_{Na_\tau, \vec{x}} - \sigma_{(N-1)a_\tau, \vec{x}})^2 - \frac{\beta\tau}{2\eta} \sum_{\vec{x}, k} \left[\frac{(\sigma_{(N-1)a_\tau, \vec{x}+\vec{k}} - \sigma_{(N-1)a_\tau, \vec{x}})^2}{2} + \frac{(\sigma_{Na_\tau, \vec{x}+\vec{k}} - \sigma_{Na_\tau, \vec{x}})^2}{2} \right] - \frac{\mu_0}{2\eta} \sum_{\vec{x}} [V(\sigma_{(N-1)a_\tau, \vec{x}}) + V(\sigma_{Na_\tau, \vec{x}})]} = \\ = \text{Tr} \left[\left(\hat{\mathcal{T}}_{a_\tau} \right)^N \right], \quad (\text{D.53})$$

The partition function in Quantum Statistical Mechanics is defined by

$$\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{\hat{\mathcal{H}}}{k_B T} \right) \right], \quad (\text{D.54})$$

with $\hat{\mathcal{H}}$ the *quantum* Hamiltonian operator and T the temperature. Comparing this equation with the above result (D.53) and with the definition of *quantum* operator $\hat{\mathcal{H}}$ in the transfer matrix method, $\hat{\mathcal{T}}_{a_\tau} = \exp \left(-\frac{a_\tau}{\eta} \hat{\mathcal{H}} \right)$ with $\eta = \hbar c$, one obtains (the constant factor $c(a_\tau)$ has been neglected in this discussion; it only contributes with a constant term to the free energy)

$$\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{Na_\tau}{\hbar c} \hat{\mathcal{H}} \right) \right], \quad (\text{D.55})$$

where Na_τ is the lattice length in the spacelike τ -direction. If simultaneously with the continuum limit $a_\tau \rightarrow 0$ we go to the large time limit $Na_\tau \rightarrow \infty$, limit in which the lattice's periodicity gets lost, and then change from the imaginary (Euclidean) time

$\tau = ict$ to the real time t over ¹⁶, we have in $\mathcal{Z}$ the sum over all configurations of $\exp \left[\frac{i}{\hbar} \int_{t=-\infty}^{t=\infty} dt \sum_{\vec{x} \in \mathbf{R}^{(d-1)}} \ell_{\text{config}} \right]$, where $\int_{t=-\infty}^{t=\infty} dt \sum_{\vec{x} \in \mathbf{R}^{(d-1)}} \ell_{\text{config}}$ is the action for each configuration. If on the other hand we do not go to the limit of a infinitely large length in the τ -direction, and keep thus the lattice's periodicity, we obtain that calculating the classical partition of a d -dimensional model with a *periodicity* along the direction in which the lattice spacing is going to the continuum, is equivalent to evaluate the $(d-1)$ -dim. quantum problem described by Hamiltonian operator $\hat{\mathcal{H}}$ at a temperature T related to the lattice's length Na_τ along the periodic dimension by

$$\frac{1}{k_B T} = \frac{Na_\tau}{\hbar c}. \quad (\text{D.56})$$

This relation is similar to the ansatz in Chapter 2 (2.93) which relates the temperature with the height of the crystal zone where Stage II is maintained without being destroyed thorough cross-slip; this zone repeats itself periodically along one direction of the crystal lattice.

Phase diagram of the d -dim. Ising model in a lattice periodic along one direction.

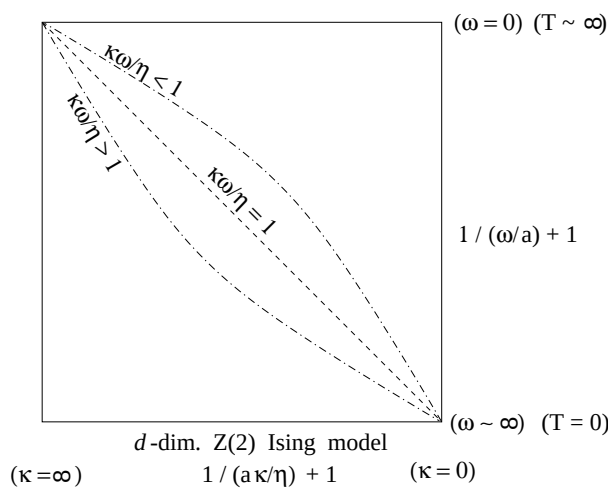
Before starting the discussion of the gauge model, let us consider briefly the phase diagram of the d -dim. Ising model, not in the case where the lattice length along the continuous τ -direction is infinite, but in the case where this length is finite and equal to ω . Repeating the derivation of equation (D.34) for a dimension $d \geq 2$ one obtains that, when $a_\tau \rightarrow 0$, the partition function of the Ising model in a d -dim. lattice periodic along one direction can be written as $\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}} \right) \right]$ with

$\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in \Lambda_\perp} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{k}} (\hat{\sigma}_3)_{\vec{x}})$, where $\Lambda_\perp$ denotes the $(d-1)$ -dim. sublattice normal to the τ -dimension.

We want to study the dependence on ω of the extension of the different phases which the model shows at $\omega = \infty$, i.e., the ordered (or magnetized) and the disordered phase. The diagram will be represented in the coordinate axes of the next figure, where also temperature T has been depicted, always assuming a relation between ω and T of the just discussed kind $\omega \sim \frac{1}{T}$.

Although at $\omega = \infty$ the smallest eigenvalue of $\hat{\mathcal{H}}$ determines the free energy density, as it has been seen for the 2-dim. Ising model in equations (D.41) and (D.44) (results which can be readily extended to any d -dim. Ising model with $d \geq 2$), the same does not hold at a finite length along the periodic dimension.

¹⁶By means of this change of time, and since $\beta_\tau \sim a_\tau^{-1}$, term $\frac{\beta_\tau}{2\eta} \sum_{\vec{x}} (\sigma_{(j+1)a_\tau, \vec{x}} - \sigma_{ja_\tau, \vec{x}})^2$ in (D.53) becomes in the continuum limit $\frac{a_\tau}{2\eta} \sum_{\vec{x}} \frac{(\sigma_{(j+1)a_\tau, \vec{x}} - \sigma_{ja_\tau, \vec{x}})^2}{a_\tau^2} \xrightarrow{a_\tau \rightarrow 0} -i \frac{c}{\eta} \sum_{\vec{x}} \frac{1}{2} \left(\frac{\partial \sigma_{\vec{x}}(t)}{\partial t} \right)^2 dt$, whilst term $\frac{\beta_x}{2\eta} \sum_{\vec{x}, k} (\sigma_{a_\tau, \vec{x}+\hat{k}} - \sigma_{a_\tau, \vec{x}})^2$ takes the following form: $+i \frac{c}{\eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} \frac{1}{2} (\Delta_k \sigma_{\vec{x}}(t))^2 dt$, since $\beta_x \sim a_\tau$. Note that in the other $(d-1)$ dimensions different from the τ -direction $\vec{x} \in \mathbf{R}^{(d-1)}$, the lattice's discreteness remains.



With decreasing $Na_\tau \equiv \omega$, higher $\hat{\mathcal{H}}$ -eigenvalues play also a relevant role and for a small enough ω , one can expect that all eigenvectors contribute in a similar way to the partition function: the effect is the same as the breakdown of the perturbation expansion, where instead of having a leading eigenvector plus small contributions coming from all the other possible eigenvectors, all eigenvectors must be dealt with on an equal footing. Therefore, when ω decreases below a certain value, the breaking of the perturbation expansion at large κ considered in (D.42) should lead to a phase transition towards a state with vanishing spontaneous magnetization.

In the range of large κ , the Hamiltonian can be approximated by

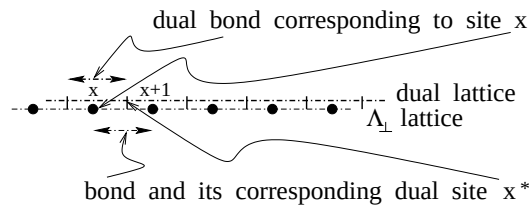
$$\hat{\mathcal{H}}_{\text{large } \kappa} \approx \kappa \sum_{\vec{x} \in (d-1)\text{-dim. } \Lambda_\perp} \sum_{k=1}^{d-1} (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{k}} (\hat{\sigma}_3)_{\vec{x}}), \quad (\text{D.57})$$

and thus partition function $\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}} \right) \right] = \prod_{\vec{x} \in \Lambda_\perp} \sum_{\sigma_{\vec{x}} = \pm 1} \exp \left[-\frac{\omega \kappa}{\eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} (1 - \sigma_{\vec{x}+\hat{k}} \sigma_{\vec{x}}) \right]$

takes the form of that of the (classical) Ising model in a non-periodic $(d-1)$ -dim. discrete lattice¹⁷ with a coupling strength given by $\beta_{\text{Ising}} = \kappa\omega$. Since the $(d-1)$ -dim. Ising model for $d \geq 3$ shows an ordered phase at large β_{Ising} but it finds itself in the disordered state at small β_{Ising} , then the d -dim. Ising model in a periodic lattice at large κ loses its ordered state when the lattice length along the periodic direction is small enough. A special case is $d = 2$: the 1-dim. Ising model in a non-periodic lattice is in the disordered phase for any $\beta_{\text{Ising}} < \infty$ (see the argument on page 329, where it is shown that the energy-cost of a large disordering Peierls contour is only two flipped spins), and therefore the 2-dim. periodic Ising model will show at large κ only one phase for any finite ω , namely, the disordered one.

At small κ , on the other hand, Hamiltonian's term $\kappa \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{k}} (\hat{\sigma}_3)_{\vec{x}})$ can be neglected;

the resulting model can be more easily studied if formulated in the dual variables (technique already introduced for the $Z(2)$ gauge model on page 417). In order to achieve this dual formulation, let us



start with the 2-dim. Ising model and show again its self-duality, by this time by means of the Hamiltonian formulation. The lattice dual to the 1-dim. lattice which supports the Hamiltonian of the 2-dim. Ising model is represented in the figure, where it becomes clear that bond and site variables are interchanged when one goes to the dual lattice. Let us define the dual site-operators $(\hat{\mu}_3)_{\vec{x}^*}$ and $(\hat{\mu}_1)_{\vec{x}^*}$ through

$$\begin{aligned} (\hat{\mu}_1)_{\vec{x}^*} &= (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}} \implies (\hat{\sigma}_3)_{\vec{x}} = \prod_{\vec{x}^* < \vec{x}} (\hat{\mu}_1)_{\vec{x}^*}, \\ (\hat{\sigma}_1)_{\vec{x}} &= (\hat{\mu}_3)_{\vec{x}^*+\hat{1}} (\hat{\mu}_3)_{\vec{x}^*} \implies (\hat{\mu}_3)_{\vec{x}^*} = \prod_{\vec{x} < \vec{x}^*} (\hat{\sigma}_1)_{\vec{x}}, \end{aligned} \quad (\text{D.58})$$

¹⁷For the models in a periodic lattice is characteristic this reduction in one dimension in the resulting non-periodic model (just the periodic dimension).

where $\vec{x}$, as well as $\vec{x}^*$, denotes sites along a 1-dim. discrete lattice; further, dual site $\vec{x}^*$ lies between sites $\vec{x}$ and $\vec{x} + \hat{1}$. By using property $(\hat{\sigma}_3)_{\vec{x}}^2 = \mathbf{1} = (\hat{\sigma}_1)_{\vec{x}}^2 \quad \forall \vec{x}$ of the Pauli matrices, one can easily prove the second equalities in definitions (D.58), $\prod_{\vec{x}^* < \vec{x}} (\hat{\mu}_1)_{\vec{x}^*} = (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_3)_{\vec{x}-\hat{1}}^2 (\hat{\sigma}_3)_{\vec{x}-2\hat{1}}^2 \cdots = (\hat{\sigma}_3)_{\vec{x}}$ and

$$(\hat{\mu}_3)_{\vec{x}^*+\hat{1}} (\hat{\mu}_3)_{\vec{x}^*} = \left[\prod_{\vec{x} < \vec{x}^*+\hat{1}} (\hat{\sigma}_1)_{\vec{x}} \right] \left[\prod_{\vec{x} < \vec{x}^*} (\hat{\sigma}_1)_{\vec{x}} \right] = (\hat{\sigma}_1)_{\vec{x}} (\hat{\sigma}_1)_{\vec{x}-\hat{1}}^2 (\hat{\sigma}_1)_{\vec{x}-2\hat{1}}^2 \cdots = (\hat{\sigma}_1)_{\vec{x}}; \text{ moreover, property}$$

$$\begin{cases} (\hat{\sigma}_1)_{\vec{x}} (\hat{\sigma}_3)_{\vec{y}} = -(\hat{\sigma}_3)_{\vec{y}} (\hat{\sigma}_1)_{\vec{x}}, & \vec{x} = \vec{y} \\ (\hat{\sigma}_1)_{\vec{x}} (\hat{\sigma}_3)_{\vec{y}} = (\hat{\sigma}_3)_{\vec{y}} (\hat{\sigma}_1)_{\vec{x}}, & \vec{x} \neq \vec{y} \end{cases} \text{ produces } (\hat{\mu}_1)_{\vec{x}^*} (\hat{\mu}_3)_{\vec{x}^*} = (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}} \left[\underline{(\hat{\sigma}_1)_{\vec{x}}} (\hat{\sigma}_1)_{\vec{x}-\hat{1}} \cdots \right] =$$

$$= - \left[(\hat{\sigma}_1)_{\vec{x}} (\hat{\sigma}_1)_{\vec{x}-\hat{1}} \cdots \right] (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}} = -(\hat{\mu}_3)_{\vec{x}^*} (\hat{\mu}_1)_{\vec{x}^*} \text{ and } (\hat{\mu}_1)_{\vec{x}^*} (\hat{\mu}_3)_{\vec{y}^* \neq \vec{x}^*} = (\hat{\mu}_3)_{\vec{y}^*} (\hat{\mu}_1)_{\vec{x}^*}^{18}. \text{ Therefore, operators } (\hat{\mu}_1)_{\vec{x}^*} \text{ and } (\hat{\mu}_3)_{\vec{x}^*} \text{ act as Pauli-matrices operators } (\hat{\sigma}_{1,3})_{\vec{x}} \text{ and the 1-dim. Hamiltonian corresponding to the 2-dim. Ising model can be written, up to an additive constant } (\eta - \kappa)|\Lambda_{\perp}|, \text{ as}$$

$$\begin{aligned} \hat{\mathcal{H}} &= -\eta \sum_{\vec{x} \in |\Lambda_{\perp}|} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in |\Lambda_{\perp}|} (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{1}} (\hat{\sigma}_3)_{\vec{x}}) \rightarrow \\ &\rightarrow \frac{\kappa}{\eta} \left[\eta \frac{\eta}{\kappa} \sum_{\vec{x}^* \in |\Lambda_{\perp}|} (1 - (\hat{\mu}_3)_{\vec{x}^*+\hat{1}} (\hat{\mu}_3)_{\vec{x}^*}) - \eta \sum_{\vec{x}^* \in |\Lambda_{\perp}|} (\hat{\mu}_1)_{\vec{x}^*} \right], \end{aligned} \quad (\text{D.59})$$

Hamiltonian which has the same form as the original one, and thus the self-duality is shown by means of the Hamiltonian formulation¹⁹. Equation (D.59) has the appropriate form for the study in the range of small κ , where $\hat{\mathcal{H}}$ can be approximated by $\eta \sum_{\vec{x}^* \in |\Lambda_{\perp}|} (1 - (\hat{\mu}_3)_{\vec{x}^*+\hat{1}} (\hat{\mu}_3)_{\vec{x}^*})$, and thus the partition function

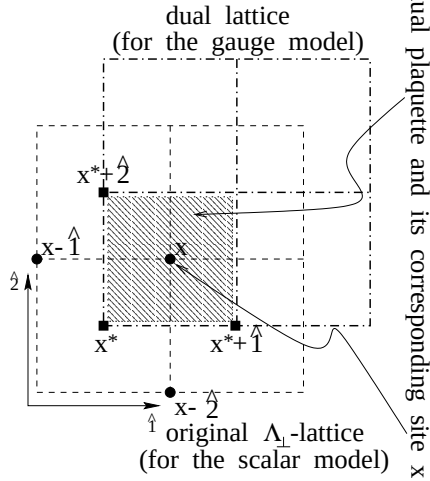
of the 2-dim. Ising model in a periodic lattice at small κ is again equivalent to the classical 1-dim. Ising model with coupling strength $\beta_{\text{Ising}} = \omega\eta$. Since the critical value of β_{Ising} for the 1-dim. Ising model is ∞ , then the Ising model in a 2-dim. lattice which is periodic along one direction shows only one phase for any finite lattice's length along this periodic dimension, at large as well as at small κ .

For the 3-dimensional case in a lattice periodic along one direction, we have already seen that at large κ a phase transition towards the disordered state occurs when ω is decreased. At small κ , as we shall now see, there is no phase transition: the model is in the disordered phase at any finite ω (just as in

the 2-dimensional model). The 2-dim. Hamiltonian $\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in \Lambda_{\perp}} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{k=1}^2 (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{k}} (\hat{\sigma}_3)_{\vec{x}})$ corresponding to the 3-dim. Ising model, when expressed using the dual variables, does not take the form of an Ising model: site-interaction $\sum_{\vec{x} \in \Lambda_{\perp}} (\hat{\sigma}_1)_{\vec{x}}$ in the dual lattice corresponds to a plaquette-interaction, as one can see in the next figure.

¹⁸It can also occur that $\vec{y}^*$ coincides with $\vec{x}^* + \hat{1}$: in this case the two first $\hat{\sigma}_1$'s contained in $(\hat{\mu}_3)_{\vec{y}^*}$ are $(\hat{\sigma}_1)_{\vec{x}+\hat{1}}$ and $(\hat{\sigma}_1)_{\vec{x}}$ and thus $(\hat{\mu}_1)_{\vec{x}^*} (\hat{\mu}_3)_{\vec{x}^*+\hat{1}} = \underline{(\hat{\sigma}_3)_{\vec{x}+\hat{1}}} \underline{(\hat{\sigma}_3)_{\vec{x}}} \left[\underline{(\hat{\sigma}_1)_{\vec{x}+\hat{1}}} (\hat{\sigma}_1)_{\vec{x}} (\hat{\sigma}_1)_{\vec{x}-\hat{1}} \cdots \right] = +(\hat{\mu}_3)_{\vec{x}^*+\hat{1}} (\hat{\mu}_1)_{\vec{x}^*}$.

¹⁹At $\omega = Na_{\tau} = \infty$, when the lattice periodicity get lost, the phases of the 2-dim. Ising model can be also described by the vanishing or non-vanishing value of the expectation value of operator $(\hat{\mu}_3)_{\vec{x}_0^*}$ for the $\hat{\mathcal{H}}$ -eigenvector with smallest eigenvalue. Hence, at a very large κ the leading contribution to the free energy (for $\omega = \infty$) arises from the eigenstate of $(\hat{\sigma}_3)_{\vec{x}}$ of eigenvalue $+1$, $(\hat{\sigma}_3)_{\vec{x}}|0\rangle_{\kappa=\infty} = |0\rangle_{\kappa=\infty} \quad \forall \vec{x}$, whilst operator $(\hat{\sigma}_1)_{\vec{x}}$ changes this eigenstate into states of higher $\hat{\mathcal{H}}$ -eigenvalues: therefore $\kappa=\infty \langle 0 | (\hat{\mu}_3)_{\vec{x}_0^*} | 0 \rangle_{\kappa=\infty} = \kappa=\infty \langle 0 | (\hat{\sigma}_1)_{\vec{x}_0^*+\hat{1}} (\hat{\sigma}_1)_{\vec{x}_0^*} | 0 \rangle_{\kappa=\infty} = 0$. On the other side, at $\kappa = 0$ the leading state at $\omega = \infty$ is eigenvector of operator $(\hat{\sigma}_1)_{\vec{x}}$, $(\hat{\sigma}_1)_{\vec{x}}|0\rangle_{\kappa=0} = |0\rangle_{\kappa=0} \quad \forall \vec{x}$ and therefore $\kappa=0 \langle 0 | (\hat{\mu}_3)_{\vec{x}_0^*} | 0 \rangle_{\kappa=0} = \kappa=0 \langle 0 | (\hat{\sigma}_1)_{\vec{x}_0^*+\hat{1}} (\hat{\sigma}_1)_{\vec{x}_0^*} | 0 \rangle_{\kappa=0} = 1 \neq 0$. Note that the behaviour of $\langle 0 | (\hat{\mu}_3)_{\vec{x}_0^*} | 0 \rangle$ in the two phases of the model is just the opposite of $\langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle$: the ordered phase at large κ (and always at $\omega = \infty$) corresponds to $\langle 0 | (\hat{\mu}_3)_{\vec{x}_0^*} | 0 \rangle = 0$ and $\langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle \neq 0$, whereas the disordered phase of κ small is described by $\langle 0 | (\hat{\sigma}_3)_{\vec{x}_0} | 0 \rangle = 0$ and $\langle 0 | (\hat{\mu}_3)_{\vec{x}_0^*} | 0 \rangle \neq 0$. This behaviour's interchange is characteristic when the model is formulated by means of the dual variables. This fact will be especially useful when one will study the different phases of the gauge model in a periodic lattice of finite ω .



However, we already know from equation (C.363) that the partition function of the 3-dim. $Z(2)$ gauge model is dual to the 3-dim. Ising model. Now we shall obtain the same result but by means of the Hamiltonian formulation: starting at the end, at the 3-dimensional $Z(2)$ gauge model in the Hamiltonian form, we shall obtain that this last model is dual to the 3-dimensional scalar model. Although one will obtain later (see equation (D.168)) the Hamiltonian operator at $a_\tau \rightarrow 0$ which corresponds to the gauge model described by the energy

$$\text{given by } \sum_n \left\{ \beta_\tau \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^2 \left[1 - S_{j,(na_\tau, \vec{x}^*)}^{\text{gauge}} S_{j,((n+1)a_\tau, \vec{x}^*)}^{\text{gauge}} \cdots \right] + \right. \\ \left. + \beta_x \sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - S_{1,(na_\tau, \vec{x}^*)}^{\text{gauge}} S_{2,(na_\tau, \vec{x}^* + \hat{1})}^{\text{gauge}} S_{1,(na_\tau, \vec{x}^* + \hat{2})}^{\text{gauge}} S_{2,(na_\tau, \vec{x}^*)}^{\text{gauge}} \right] \right\},$$

where $\sum_n$ denotes sum along the will-become-continuous τ -direction and $S_{j,(na_\tau, \vec{x}^*)}^{\text{gauge}} = e^{iA_{j,(na_\tau, \vec{x}^*)}} = \pm 1$,

one can nevertheless guess the aspect of the Hamiltonian operator from the experience already gained with the Ising model: the interaction between fields located on the same n -plane and contained in the second term becomes operator $\sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right]$, while the interaction between two fields displaced along the τ -direction produces the operator which flips the $(\hat{\sigma}_3)_{k, \vec{x}^*}$ -eigenstate:

$$\hat{\mathcal{H}} = -\eta \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^2 (\hat{\sigma}_1)_{j, \vec{x}^*} + \kappa_e \sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right], \quad (\text{D.60})$$

where now $(\hat{\sigma}_{1,3})_{k, \vec{x}^*}$ are Pauli matrices acting as operators on the bond connecting site $\vec{x}^*$ and site $\vec{x}^* + \hat{k}$. However, the gauge invariance of the dual model causes that there exist infinite many different configurations with the same energy. This, in the Hamiltonian formulation, is stated by the fact that operator (D.60) is invariant under the action of the following operator $\hat{\mathcal{G}}_{\vec{y}^*} = (\hat{\sigma}_1)_{1, \vec{y}^*} (\hat{\sigma}_1)_{-1, \vec{y}^*} (\hat{\sigma}_1)_{2, \vec{y}^*} (\hat{\sigma}_1)_{-2, \vec{y}^*}$, the product of all bond-operators $(\hat{\sigma}_1)_{k, \vec{x}^*}$ attached to site $\vec{y}^*$ in a 2-dim. lattice. This invariance can be

easily shown by using properties $\begin{cases} (\hat{\sigma}_1)_{k, \vec{x}^*} (\hat{\sigma}_3)_{j, \vec{y}^*} (\hat{\sigma}_1)_{k, \vec{x}^*} = -(\hat{\sigma}_3)_{j, \vec{y}^*}, & \vec{x}^* = \vec{y}^* \text{ and } k = j \\ (\hat{\sigma}_1)_{k, \vec{x}^*} (\hat{\sigma}_3)_{j, \vec{y}^*} (\hat{\sigma}_1)_{k, \vec{x}^*} = (\hat{\sigma}_3)_{j, \vec{y}^*}, & \vec{x}^* \neq \vec{y}^* \text{ or } k \neq j \end{cases}$: term

$\sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^2 (\hat{\sigma}_1)_{j, \vec{x}^*}$ satisfies trivially $\hat{\mathcal{G}}_{\vec{y}^*} \left(\sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^2 (\hat{\sigma}_1)_{j, \vec{x}^*} \right) \hat{\mathcal{G}}_{\vec{y}^*}^{-1} = \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^2 (\hat{\sigma}_1)_{j, \vec{x}^*} \quad \forall \vec{y}^*$ ²⁰; on the

other hand, each of the four plaquettes contained in term $\sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right]$

and which are touching site $\vec{y}^*$, have two bond-operators in common with $\hat{\mathcal{G}}_{\vec{y}^*}$, leading to the result

$$\hat{\mathcal{G}}_{\vec{y}^*} \left(\sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right] \right) \hat{\mathcal{G}}_{\vec{y}^*} = \sum_{\vec{x}^* \in \Lambda_\perp} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right]$$

$\forall \vec{y}^*$. Invariance $\hat{\mathcal{G}}_{\vec{y}^*}^{-1} \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{y}^*} = \hat{\mathcal{H}} \quad \forall \vec{y}^*$ has to be removed in order to avoid the infinite repetition of the same contribution in the partition function: this is obtained by imposing on the states on which the Hamiltonian operator is acting a constraint, called the gauge fixing, of the form

$$(\hat{\sigma}_1)_{1, \vec{y}^*} (\hat{\sigma}_1)_{-1, \vec{y}^*} (\hat{\sigma}_1)_{2, \vec{y}^*} (\hat{\sigma}_1)_{-2, \vec{y}^*} = 1 \quad \forall \vec{y}^*. \quad (\text{D.61})$$

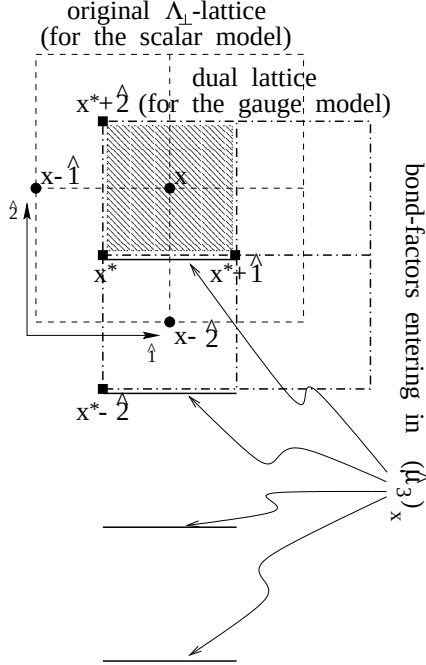
This condition allows us to solve $(\hat{\sigma}_1)_{2, \vec{y}^*}$ in terms of the other bond-operators attached to site $\vec{y}^*$

$$(\hat{\sigma}_1)_{2, \vec{y}^*} = (\hat{\sigma}_1)_{1, \vec{y}^*} (\hat{\sigma}_1)_{-1, \vec{y}^*} (\hat{\sigma}_1)_{-2, \vec{y}^*},$$

²⁰Note that $\hat{\mathcal{G}}_{\vec{y}^*}^{-1} = \hat{\mathcal{G}}_{\vec{y}^*}$ since $\hat{\mathcal{G}}_{\vec{y}^*} \hat{\mathcal{G}}_{\vec{y}^*} = 1$.

and analogously, $(\hat{\sigma}_1)_{-2,\vec{x}^*} = (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{2}}(\hat{\sigma}_1)_{-1,\vec{x}^*-\hat{2}}(\hat{\sigma}_1)_{-2,\vec{x}^*-\hat{2}\cdot\hat{2}}$. By applying the same substitution strategy to all sites along the $-\hat{2}$ -direction, any $(\hat{\sigma}_1)_{\pm 2,\vec{x}^*}$ can be written as function ²¹ of $(\hat{\sigma}_1)_{\pm 1,\vec{x}^*}$'s

$$(\hat{\sigma}_1)_{2,\vec{x}^*} = [(\hat{\sigma}_1)_{1,\vec{x}^*}(\hat{\sigma}_1)_{-1,\vec{x}^*}] [(\hat{\sigma}_1)_{1,\vec{x}^*-\hat{2}}(\hat{\sigma}_1)_{-1,\vec{x}^*-\hat{2}}] [(\hat{\sigma}_1)_{1,\vec{x}^*-\hat{2}\cdot\hat{2}}(\hat{\sigma}_1)_{-1,\vec{x}^*-\hat{2}\cdot\hat{2}}] \times \cdots \quad (\text{D.62})$$



Now it is easy to obtain the model dual to the 3-dim. $Z(2)$ gauge model in the Hamiltonian formulation, which will result to be the 3-dim. Ising model [59]: let us define the following site-operators located in the lattice which is dual to that where the bond-operators of the gauge model resides

$$\begin{aligned} (\hat{\mu}_1)_{\vec{x}} &= (\hat{\sigma}_3)_{1,\vec{x}^*} (\hat{\sigma}_3)_{2,\vec{x}^*+\hat{1}} \underbrace{(\hat{\sigma}_3)_{-1,\vec{x}^*+\hat{1}+\hat{2}} (\hat{\sigma}_3)_{-2,\vec{x}^*+\hat{2}}}_{= (\hat{\sigma}_3)_{1,\vec{x}^*+\hat{2}} = (\hat{\sigma}_3)_{2,\vec{x}^*}} \quad (\text{D.63}) \\ (\hat{\mu}_3)_{\vec{x}} &= \prod_{n' \geq 0} (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{n}\cdot\hat{2}}. \end{aligned}$$

Site $\vec{x}$ is located in relation to $\vec{x}^*$ as depicted in the figure. It is clear that $(\hat{\mu}_1)_{\vec{x}}^2 = 1 = (\hat{\mu}_3)_{\vec{x}}^2$; further, $(\hat{\mu}_1)_{\vec{x}} (\hat{\mu}_3)_{\vec{x}} =$

$$= \left[(\hat{\sigma}_3)_{2,\vec{x}^*} (\hat{\sigma}_3)_{2,\vec{x}^*+\hat{1}} (\hat{\sigma}_3)_{1,\vec{x}^*+\hat{2}} (\hat{\sigma}_3)_{1,\vec{x}^*} \right] \left[(\hat{\sigma}_1)_{1,\vec{x}^*} \prod_{n' > 0} (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{n}\cdot\hat{2}} \right] =$$

$$= -(\hat{\mu}_3)_{\vec{x}} (\hat{\mu}_1)_{\vec{x}}, \text{ and } (\hat{\mu}_1)_{\vec{x}} (\hat{\mu}_3)_{\vec{y}} = (\hat{\mu}_3)_{\vec{y}} (\hat{\mu}_1)_{\vec{x}} \text{ if } \vec{x} \neq \vec{y} \text{ since}$$

in this last case it can only occur that either no bond-operator contained in the “ladder” $\prod_{n' \geq 0} (\hat{\sigma}_1)_{1,\vec{y}^*-\hat{n}\cdot\hat{2}}$ meets

a bond-operator of the plaquette whose center is $\vec{x}$ or that two of the $(\hat{\sigma}_1)_{1,\vec{y}^*-\hat{n}\cdot\hat{2}}$'s act on two parallel bonds of such a plaquette. And these are the properties of operators working as the Pauli matrices.

From its very definition, $(\hat{\mu}_3)_{\vec{x}}$ fulfils $(\hat{\mu}_3)_{\vec{x}} (\hat{\mu}_3)_{\vec{x}-\hat{2}} = (\hat{\sigma}_1)_{1,\vec{x}^*} \prod_{n' > 0} (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{n}\cdot\hat{2}}$, and according to (D.62), $(\hat{\mu}_3)_{\vec{x}} (\hat{\mu}_3)_{\vec{x}-\hat{1}} = (\hat{\sigma}_1)_{1,\vec{x}^*} (\hat{\sigma}_1)_{-1,\vec{x}^*} (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{2}} (\hat{\sigma}_1)_{-1,\vec{x}^*-\hat{2}} (\hat{\sigma}_1)_{1,\vec{x}^*-\hat{2}\cdot\hat{2}} (\hat{\sigma}_1)_{-1,\vec{x}^*-\hat{2}\cdot\hat{2}} \cdots = (\hat{\sigma}_1)_{2,\vec{x}^*}$

$$\begin{cases} (\hat{\mu}_3)_{\vec{x}-\hat{2}} (\hat{\mu}_3)_{\vec{x}} &= (\hat{\sigma}_1)_{1,\vec{x}^*}, \\ (\hat{\mu}_3)_{\vec{x}-\hat{1}} (\hat{\mu}_3)_{\vec{x}} &= (\hat{\sigma}_1)_{2,\vec{x}^*}. \end{cases} \quad (\text{D.64})$$

By using these relations, the Hamiltonian operator which describes the 3-dimensional $Z(2)$ gauge model,

$$\begin{aligned} \hat{\mathcal{H}}^{\text{gauge}} &= -\eta \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j=1}^2 (\hat{\sigma}_1)_{j,\vec{x}^*} + \kappa_e \sum_{\vec{x}^* \in \Lambda_{\perp}} \left[1 - (\hat{\sigma}_3)_{1,\vec{x}^*} (\hat{\sigma}_3)_{2,\vec{x}^*+\hat{1}} (\hat{\sigma}_3)_{1,\vec{x}^*+\hat{2}} (\hat{\sigma}_3)_{2,\vec{x}^*} \right], \text{ can be written as} \\ \frac{\eta}{\kappa_e} \hat{\mathcal{H}}^{\text{gauge}} &= \eta \frac{\eta}{\kappa_e} \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j=1}^2 \left(1 - (\hat{\mu}_3)_{\vec{x}-j} (\hat{\mu}_3)_{\vec{x}} \right) - \eta \sum_{\vec{x} \in \Lambda_{\perp}} (\hat{\mu}_1)_{\vec{x}}, \end{aligned} \quad (\text{D.65})$$

²¹In this way one has eliminated the $(\hat{\sigma}_1)_{\pm 2,\vec{x}^*}$'s as actual field variables in the Hamiltonian and in transfer matrix operator $\hat{\mathcal{T}}_{a\tau} = \exp\left(-\frac{a\tau}{\eta} \hat{\mathcal{H}}\right)$. The disappearance of $(\hat{\sigma}_1)_{\pm 2,\vec{x}^*}$ as a variable in the Hamiltonian leads to the fact that any $(\hat{\sigma}_3)_{\pm 2,\vec{x}^*}$ commutes with $\hat{\mathcal{H}}$ and with $\hat{\mathcal{T}}_{a\tau}$. From the very definition of the transfer matrix, which propagates any operator a lattice spacing along the continuous τ -direction (see equation (D.7)), it follows that the value of $(\hat{\sigma}_3)_{\pm 2,\vec{x}^*}$ remains constant along the whole τ -direction, not playing any role in the calculation of correlation functions or of partition function $\mathcal{Z} = \text{Tr}[\hat{\mathcal{T}}_{a\tau}^N]$. One therefore can fix $(\hat{\sigma}_3)_{\pm 2,\vec{x}^*}$ to be a constant [59], remaining finally thus as actual field variables only $(\hat{\sigma}_1)_{\pm 1,\vec{x}^*}$ and $(\hat{\sigma}_3)_{\pm 1,\vec{x}^*}$. The fact that the relevant operators after removing the gauge freedom by means of $(\hat{\sigma}_1)_{1,\vec{x}^*} (\hat{\sigma}_1)_{-1,\vec{x}^*} (\hat{\sigma}_1)_{2,\vec{x}^*} (\hat{\sigma}_1)_{-2,\vec{x}^*} = 1 \quad \forall \vec{x}^*$ are two scalar operators is the reason for the duality relation to the Ising (scalar) model.

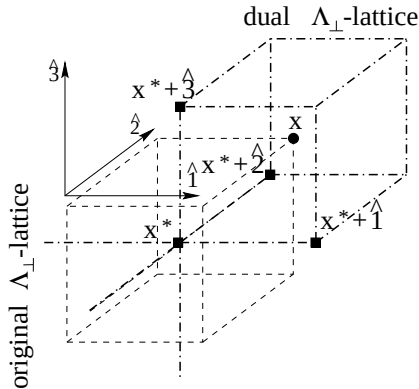
which, as expected, is equal to the Hamiltonian describing the 3-dim. Ising model (up to an additive constant): $\hat{\mathcal{H}}^{\text{Ising}} = -\eta \sum_{\vec{x} \in \Lambda_{\perp}} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j=1}^2 \left(1 - (\hat{\sigma}_3)_{\vec{x}+\hat{j}} (\hat{\sigma}_3)_{\vec{x}}\right)$. Ising model's κ corresponds to gauge model's $\eta \frac{\eta}{\kappa_e}$, and therefore, the Ising model in a 3-dim. lattice periodic along one direction is described in the range of small κ (large κ_e) by $\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}}_{\text{small } \kappa}^{\text{Ising}} \right) \right] = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \frac{\eta}{\kappa_e} \hat{\mathcal{H}}_{\text{large } \kappa_e}^{\text{gauge}} \right) \right]$

$$\begin{aligned} \mathcal{Z}_{\text{small } \kappa}^{\text{Ising}} &= \text{Tr} \left[\exp \left(-\omega \sum_{\vec{x}^* \in \Lambda_{\perp}} \left[1 - (\hat{\sigma}_3)_{1, \vec{x}^*} (\hat{\sigma}_3)_{2, \vec{x}^* + \hat{1}} (\hat{\sigma}_3)_{1, \vec{x}^* + \hat{2}} (\hat{\sigma}_3)_{2, \vec{x}^*} \right] \right) \right] = \\ &= \prod_{\vec{x}^* \in 2\text{-dim. } \Lambda_{\perp}} \sum_{\sigma_{1, \vec{x}^*} = \pm 1} \exp \left[-\frac{\omega \eta}{\eta} \sum_{\vec{x}^* \in \Lambda_{\perp}} \left(1 - \sigma_{1, \vec{x}^*} \sigma_{2, \vec{x}^* + \hat{1}} \sigma_{1, \vec{x}^* + \hat{2}} \sigma_{2, \vec{x}^*} \right) \right]. \end{aligned} \quad (\text{D.66})$$

But this last equation is the partition function of the (classical) 2-dim. $Z(2)$ gauge model in a non-periodic lattice with coupling strength $\beta_e = \omega \eta$ and such a model shows no phase transition: it finds itself always in the state described by the strong area decay of the gauge correlation function in at small as well as at large β_e (see figure on page 415 and equation (C.347)). Therefore at small κ and for any ω , the 3-dim. Ising model in a periodic lattice shows only one phase.

In fact, this is a property which holds also in the 4-dimensional Ising model at small κ . Following an argument based on the idea of figure on page 415, let us sketch the proof: if we repeat the above steps in order to obtain the Hamiltonian operator which describes the model dual to the 4-dim. Ising model, we shall obtain an operator of the following form:

- Since the support of Hamiltonian is the 3-dim. sublattice $\Lambda_{\perp}$ normal to the τ -direction, and since to a site $\vec{x}$ of the original 3-dim. lattice corresponds in the dual (3-dim.) lattice the cube which surrounds $\vec{x}$, then the dual of part $-\eta \sum_{\vec{x} \in \Lambda_{\perp}} (\hat{\sigma}_1)_{\vec{x}}$, which contains site-interactions, will become $\eta \sum_{\vec{x}^* \in \Lambda_{\perp}} \left(1 - (\hat{\mu}_3)_{12, \vec{x}^*} (\hat{\mu}_3)_{21, \vec{x}^* + \hat{3}} (\hat{\mu}_3)_{23, \vec{x}^*} (\hat{\mu}_3)_{32, \vec{x}^* + \hat{1}} (\hat{\mu}_3)_{31, \vec{x}^*} (\hat{\mu}_3)_{13, \vec{x}^* + \hat{2}} \right)$, sum of the local interaction between Pauli-matrix operators $(\hat{\mu}_3)_{jk, \vec{x}^*}$ located at the center of a each face of the unit cube dual to site $\vec{x}$.



- The term dual to the Hamiltonian part containing the bond-interactions, $\kappa \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{i=1}^3 \left(1 - (\hat{\sigma}_3)_{\vec{x}} (\hat{\sigma}_3)_{\vec{x} + \hat{i}} \right)$, is equal to $-\kappa \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j < k=1}^3 (\hat{\mu}_1)_{jk, \vec{x}^*}$ where plaquette-operator $(\hat{\mu}_1)_{jk, \vec{x}^*}$ flips the $(\hat{\mu}_3)_{jk, \vec{x}^*}$ -eigenstate on the plaquette which is crossed by the bond from $\vec{x}$ to $\vec{x} + \epsilon_{jk} \hat{i}$.

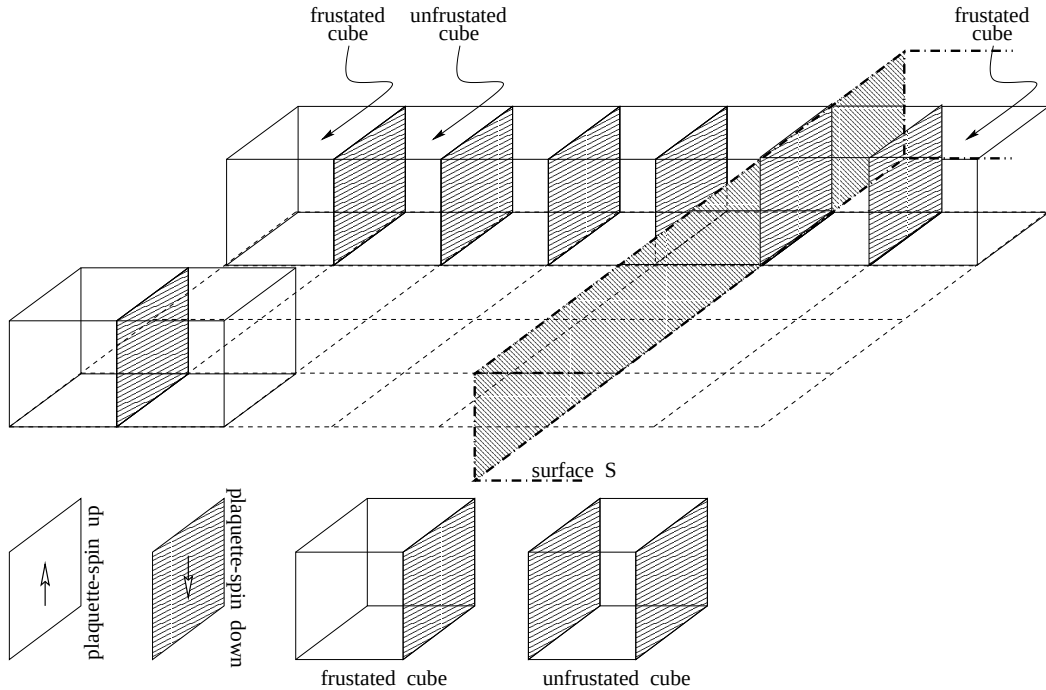
Now, at small κ the partition function of the 4-dim. Ising model in a periodic lattice along the τ -direction of period ω is equal to the partition function of the following (classical) 3-dimensional model

$$\mathcal{Z}_{\text{small } \kappa}^{\text{Ising}} \equiv \prod_{\vec{x}^* \in 3\text{-dim. } \Lambda_{\perp}} \prod_{j < k=1}^3 \sum_{\sigma_{jk, \vec{x}^*} = \pm 1} \exp \left[-\omega \sum_{\vec{x}^* \in \Lambda_{\perp}} \left(1 - \sigma_{12, \vec{x}^*} \sigma_{21, \vec{x}^* + \hat{3}} \sigma_{23, \vec{x}^*} \sigma_{32, \vec{x}^* + \hat{1}} \sigma_{31, \vec{x}^*} \sigma_{13, \vec{x}^* + \hat{2}} \right) \right].$$

In order to study whether the above model has more than one phase, let us calculate the expectation value of the product of the plaquette-variables $\sigma_{jk,\vec{x}^*}$ distributed on a closed surface S , product which defines correlation function $K(S) = \left\langle \prod_{\text{plaquettes } \subset \text{ closed surface } S} \sigma_{jk,\vec{x}^*} \right\rangle$. Remaining at small κ , if ω is small $K(S)$

can be directly obtained by means of the “high-temperature” expansion: sum $\prod_{\vec{x}^* \in \Lambda_\perp} \prod_{j < k=1}^3 \sum_{\sigma_{jk,\vec{x}^*} = \pm 1}$ makes that the first non-vanishing contribution to $K(S)$ of lowest order in ω arises from the expansion in powers of $\tanh \omega$ of $\prod_{\vec{x}^* \in \Lambda_\perp} \prod_{j < k=1}^3 \left[1 + \sigma_{12,\vec{x}^*} \sigma_{21,\vec{x}^*+\hat{3}} \sigma_{23,\vec{x}^*} \sigma_{32,\vec{x}^*+\hat{1}} \sigma_{31,\vec{x}^*} \sigma_{13,\vec{x}^*+\hat{2}} \tanh \omega \right]$ when the elementary cube-factors $\sigma_{12,\vec{x}^*} \sigma_{21,\vec{x}^*+\hat{3}} \sigma_{23,\vec{x}^*} \sigma_{32,\vec{x}^*+\hat{1}} \sigma_{31,\vec{x}^*} \sigma_{13,\vec{x}^*+\hat{2}}$ fill completely the volumen enclosed by the closed surface S

$$\left\langle \prod_{\text{plaq. } \subset S} \sigma_{jk,\vec{x}^*} \right\rangle_{\text{small } \omega} \sim (\tanh \omega)^{\text{Volumen}_S} \approx \exp \left[-\text{Volumen}_S \ln \left(\frac{1}{\omega} \right) \right]. \quad (\text{D.67})$$



On the other hand, at large ω the main contributions to the correlation function come from configuration with all or almost all plaquette-spins $\sigma_{jk,\vec{x}^*}$ “up”²²; a unit cube, in which an odd number of $\sigma_{jk,\vec{x}^*}$ ’s on its faces are pointing “down”, will be called a frustated cube. A line formed by unit cubes with two $\sigma_{jk,\vec{x}^*}$ ’s “down” starts at a frustated cube with one $\sigma_{jk,\vec{x}^*} = -1$, line which finishes at another frustated cube (see figure). Each time such a line crosses the closed surface S , one of the plaquettes contained into $\prod_{\text{plaq. } \subset S} \sigma_{jk,\vec{x}^*}$ fulfils $\sigma_{jk,\vec{x}^*} = -1$, and therefore a negative contribution in the numerator

of $K(S)$ will be produced if the ending frustated cube is located inside the volumen enclosed by S , i.e., if the cube-line stretching from one frustated cube to the other crosses an odd number of times surface S . It will depend on the energy cost of these long cube-strips wheter the frustated cubes can be largely separated from or must be very near to each other, and this for its part, whether the negative

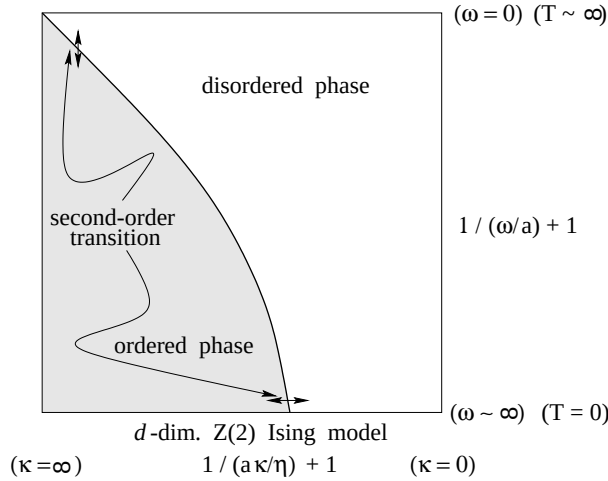
²²Cube-interactions $\sigma_{12,\vec{x}^*} \sigma_{21,\vec{x}^*+\hat{3}} \sigma_{23,\vec{x}^*} \sigma_{32,\vec{x}^*+\hat{1}} \sigma_{31,\vec{x}^*} \sigma_{13,\vec{x}^*+\hat{2}}$ allow for local (gauge) transformations too, and thus, when one speaks about a configuration with all $\sigma_{jk,\vec{x}^*}$ “up” it should be understood that we are working with a representative of all possible configurations which can be obtained from a configuration with $\sigma_{jk,\vec{x}^*} = 1 \quad \forall \vec{x}^*, j, k$ by means of any local transformation.

contributions to $K(S)$ produced by a frustated cube are proportional to the volumen enclosed by S (many long strips of cubes with two $\sigma_{jk,\vec{x}} = -1$) or to the area spanned by surface S (short strips).

Analogously to the argument leading to result (C.347), since $\Lambda_\perp$ is a 3-dimensional lattice for the case under study, a long line containing unit cubes with two flipped $\sigma_{jk,\vec{x}}$'s costs energetically the same amount as if the starting and ending frustated cube were touching each other. This would not hold in $\Lambda_\perp$ were of higher dimensionality, since then the two down $\sigma_{jk,\vec{x}}$'s on each cube of the line would belong also to another cubes extending through the fourth, fifth, ... dimensions and then the energy of the considered line would become proportional to the line's length. For the 3-dim. $\Lambda_\perp$ each isolated frustated cube can be placed in $|\Lambda_\perp|$ sites, sites from which a long line of cubes with two $\sigma_{jk,\vec{x}} = -1$ can start with minimal energy cost: if the frustated cube is located in one of the N_S sites enclosed by surface S , then $\prod_{\text{plaq.} \subset S} \sigma_{jk,\vec{x}}$ will be negative for such a configuration. Assuming that the frustated cubes are not correlated among them, something which is satisfied since two frustated can be separated a large distance without an energy increase, one obtains

$$\begin{aligned} \left\langle \prod_{\text{plaq.} \subset S} \sigma_{jk,\vec{x}} \right\rangle_{\text{large } \omega} &\approx \frac{\overbrace{1}^{\text{no frust. cubes}} + \overbrace{\left[(|\Lambda_\perp| - N_S) \cdot (1) + N_S \cdot (-1) \right] e^{-2\omega} + \frac{1}{2} \left[(|\Lambda_\perp| - 2N_S) e^{-2\omega} \right]^2 + \dots}^{\text{1 frust. cube}}}{1 + |\Lambda_\perp| e^{-2\omega} + \frac{1}{2} \left[|\Lambda_\perp| e^{-2\omega} \right]^2 + \dots} = \\ &= \frac{\exp \left[(|\Lambda_\perp| - 2N_S) e^{-2\omega} \right]}{\exp [|\Lambda_\perp| e^{-2\omega}]} = \exp [-2 \text{Volumen}_S e^{-2\omega}] , \end{aligned} \quad (\text{D.68})$$

where combinatorial factor 1/2 has been introduced to avoid overcounting in the order of considering the two frustated cubes. Result (D.68) is just the same decay like the volumen as that at ω small of equation (D.67). This proves, for the 4-dimensional Ising model in a periodic lattice along one dimension at small κ , the absence of a phase transition when ω decreases.

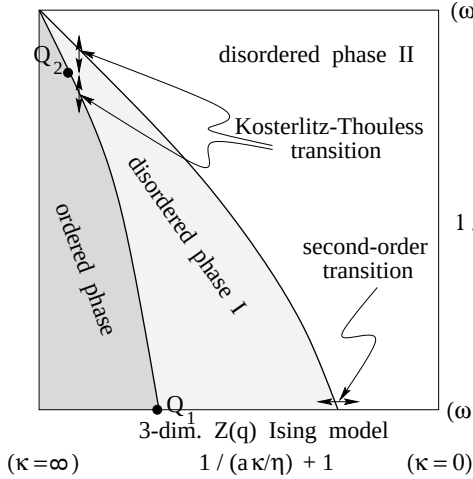


($\omega = 0$) ($T \sim \infty$) The phase diagram for the $Z(2)$ Ising model in a d -dimensional lattice (with $d = 3, 4$) periodic along one dimension is represented in the figure, where the line separating the two phases lies in the region of large κ and is given by $\frac{\omega}{a} \frac{a\kappa}{\eta} = \left(\frac{\beta_{(d-1)\text{-dim Ising}}}{\eta} \right)_{\text{crit}}$, where $(\beta_{(d-1)\text{-dim Ising}})_{\text{cr}}$ is the critical value of the $(d-1)$ -dim. (non-periodic) Ising model. The phase transition across this line is, for the $Z(2)$ scalar model, of second-order kind. The ending point of such a line lies on the axis of $\omega = \infty$ at the critical point of the d -dim. non-periodic Ising model, which is also a second-order transition point.

The special case of $d = 2$ has a more simplified phase diagram, without any phase-transition line and all range of κ , for any ω , occupied by the disordered phase: $(\beta_{1\text{-dim Ising}})_{\text{cr}} = \infty$.

Before closing this Note, let us discuss briefly the phase diagram of the $Z(q)$ scalar model in a 3-dimensional lattice (periodic along the τ -direction) for large q . At small κ , the non-existence of a phase transition when ω is decreased holds as for the $Z(2)$ (Ising) model: the reason is that the 2-dim. $Z(q)$ gauge model in a non-periodic lattice has no phase transition, since at large q the $Z(q)$ gauge model tends towards the $U(1)$ gauge model, which shows trivially only one phase (see equation (C.320) and the discussion which follows there). At κ large, similarly as in the $Z(2)$ scalar model, the resulting

partition function constructed with the Hamiltonian at such a κ -value corresponds to the non-periodic 2-dim. $Z(q)$ scalar model, whose coupling strength becomes to be proportional to ω : thus, by decreasing ω , the 3-dim. periodic $Z(q)$ scalar model at large κ will show a phase transition. However, at a q -value large enough, the $Z(q)$ scalar model shows not only the two characteristic states as for $q = 2$ but also an intermediate phase between these two, i.e., a third state besides the magnetized phase characterized²³ by $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim \text{const.}$ and the disordered state with $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-|\vec{r}_2-\vec{r}_1|}$. This intermediate state corresponds to phase $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-\ln|\vec{r}_2-\vec{r}_1|}$ of the 2-dim. $U(1)$ scalar model (the 2-dim. XY model); with increasing q , ordered phase $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim \text{const.}$ gradually gives place to this intermediate phase in such a way that at $q \rightarrow \infty$, limit which corresponds to the XY model, only the ordered and the intermediate states remain.



The two resulting phase transitions at large κ and small ω are of Kosterlitz-Thouless type, since this is the kind of the transition in the $Z(q)$ scalar model in a 2-dimensional non-periodic lattice (one should remember that the phase transition of the 2-dim. XY model is Kosterlitz-Thouless and that the two phase transition points of 2-dim. $Z(q)$ are dual to each other). The phase diagram is depicted in the figure, where “disordered phase I & II” corresponds, respectively, to $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-\ln|\vec{r}_2-\vec{r}_1|}$ and $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-|\vec{r}_2-\vec{r}_1|}$.

Point Q_2 (range of large κ) tends towards the bottom part of the diagram when $q \rightarrow \infty$, since the 2-dim. non-periodic XY model shows only one phase transition: from $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-\ln|\vec{r}_2-\vec{r}_1|}$ to $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle \sim e^{-|\vec{r}_2-\vec{r}_1|}$.

Three different phases occur also at $\omega = \infty$, since the non-periodic 3-dim. $Z(q)$ scalar model show so many phases when q is large enough. The Kosterlitz-Thouless character of the phase transition across the two lines of the above diagram is not, nevertheless, kept when $\omega = \infty$ is approached: for instance, the right line is of second-order type in the range of $\omega \approx \infty$, since this is the phase-transition kind of the 3-dim. XY model in a non-periodic lattice. With increasing q , point Q_1 tend towards the left since the 3-dim. (non-periodic) XY model shows only two different phases. This fact, together with the tendence of point Q_2 when q increases, makes the ordered phase disappear completely at $q \rightarrow \infty$: the 3-dim. XY model in a lattice periodic along one dimension has no transition at small κ for any ω and it shows, when ω is decreased, only one phase transition in the range of κ large.

²³As it was discussed during the Peierls proof of the existence of a non-vanishing spontaneous magnetization in the 2-dim. Ising model (in a non-periodic lattice), the magnetized state at large coupling strength can be also characterized by a non-vanishing of correlation function $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle$ even at large distances $|\vec{r}_2-\vec{r}_1|$: see relations (C.39) and (C.62).

On the contrary, to the disordered state of the model corresponds a $\langle e^{i\frac{2\pi}{q}(n_{\vec{r}_1}-n_{\vec{r}_2})} \rangle$ which decays exponentially like the distance and therefore vanishes at $|\vec{r}_2-\vec{r}_1| \rightarrow \infty$.

D.2 The “deconfinement” transition

In the previous section we have seen that by starting from an energy defined on a d -dimensional discrete lattice and studying the continuum limit along only one of the lattice dimensions, one obtains that solving the model can also be formulated as the solution of a quantum Hamiltonian operator; the potential energy of such a Hamiltonian depends on field variables placed in the $(d - 1)$ -dimensional (sub)lattice which is maintained discrete, whereas the momentum operator arises from the axis along which the continuum limit is considered. We have already studied the case of a scalar field; the same discussion will be now undertaken for the gauge field, where a new problem arises due just to the invariance of the energy under a transformation in gauge potential $A_{\mu, \vec{r}}$ by the gradient of an arbitrary scalar function. Since we are calculating partition functions and all the field configurations related to each other by means of such a (gauge) transformation produce the same contribution to the partition function, we have to remove this gauge freedom in order to avoid the overcounting of the same contribution an infinite number of times. This important point will be discussed firstly; the term “deconfinement” will be explained later.

D.2.1 The Hamiltonian formulation of the pure compact $U(1)$ gauge model

As in the previous section, greek indices for the vector fields will be used to denote directions in the complete d -dimensional lattice, whilst the latin indices correspond to the coordinates on the $(d - 1)$ -dimensional plane normal to the direction (the τ direction) which is continuized. In the same manner, $\vec{r}$ denotes a site in the d -dim. lattice and $\vec{x}$ in the $(d - 1)$ -dim. sublattice. According to equation (C.243), we define field $F_{jk, \vec{x}}$ residing at the center of the plaquette starting at $\vec{x}$

$$F_{jk, (na_\tau, \vec{x})} = A_{j, (na_\tau, \vec{x})} + A_{k, (na_\tau, \vec{x} + \hat{j})} - A_{j, (na_\tau, \vec{x} + \hat{k})} - A_{k, (na_\tau, \vec{x})}, \quad (D.69)$$

$A_{-j, (na_\tau, \vec{x} + \hat{j} + \hat{k})} = -A_{j, (na_\tau, \vec{x} + \hat{k})}$
 $A_{-k, (na_\tau, \vec{x} + \hat{k})} = -A_{k, (na_\tau, \vec{x})}$

where na_τ denotes the value along the τ direction for the subplane where $\vec{x}$ lies and $j, k = 1, \dots, d - 1$. For the following, the notation $(na_\tau, \vec{x})$ will be abbreviated by $\vec{x}$ and only when the considered plaquette does not reside on a plane normal to the τ direction, the more explicit notation will be used.

In the long Appendix C we have learnt that in the phase diagram for the *Higgs* model with a *compact* gauge field, the characteristics which makes it different from the *non-compact* model are controlled by the *pure compact* gauge model. Therefore in this section we shall study this pure model, i.e., without coupling the compact gauge field to any scalar field; the model with a non-compact gauge field coupled to a scalar field will

be considered in the next section of this Appendix. Only as a reminder, in a compact gauge theory gauge field $A_{\mu,\vec{r}}$ has angular character.

The energy in a d -dimensional lattice for the pure $U(1)$ compact gauge theory is given by

$$E = \sum_n \left\{ \beta_\tau \sum_{\vec{x} \in \text{plane}_n} \sum_{j=1}^{d-1} \left[1 - \cos \left(A_{j,(na_\tau,\vec{x})} + A_{\tau,(na_\tau,\vec{x}+\hat{j})} - A_{j,((n+1)a_\tau,\vec{x})} - A_{\tau,(na_\tau,\vec{x})} \right) \right] + \right. \\ \left. + \beta_x \sum_{\vec{x} \in \text{plane}_n} \sum_{j < k=1}^{d-1} \left[1 - \cos \left(A_{j,(na_\tau,\vec{x})} + A_{k,(na_\tau,\vec{x}+\hat{j})} - A_{j,(na_\tau,\vec{x}+\hat{k})} - A_{k,(na_\tau,\vec{x})} \right) \right] \right\}. \quad (\text{D.70})$$

As the lattice spacing equal in all directions was, it held $\beta_\tau = \beta_x = \frac{a^{d-4}}{e^2}$, with e^2 the coupling constant; now, where the lattice spacing along the τ direction goes to zero whereas the lattice spacing along the other directions remains finite, the dependence of β_τ and β_x on a_τ must be obtained from the relation $\hat{\mathcal{T}} = \exp \left[-a_\tau \hat{\mathcal{H}}/\eta \right]$ for an infinitesimal a_τ .

From the gauge freedom we have to our disposal, we reduce it by choosing the following gauge

$$A_{\tau,(na_\tau,\vec{x})} = 0 \quad \forall n, \vec{x}, \quad (\text{D.71})$$

obtained as follows (see J. D. Bjorken, S. D. Drell, *Relativistic Quantum Fields* (1965), chapter 14): starting from a general $A_{\mu,\vec{r}} \equiv \{A_{\tau,(na_\tau,\vec{x})}, A_{j,(na_\tau,\vec{x})}\}$, one carries out the gauge transformation $A_{\mu,\vec{r}} \rightarrow A'_{\mu,\vec{r}} = A_{\mu,\vec{r}} - \Delta_\mu \chi'_{\vec{r}}$ with $\chi'_{\vec{r}} \equiv \chi'_{(na_\tau,\vec{x})} = \sum_{m=0}^{m=n-1} A_{\tau,(ma_\tau,\vec{x})}$ and $A'_{\tau,(na_\tau,\vec{x})} = A_{\tau,(na_\tau,\vec{x})} - \left(\sum_{m=0}^{m=n} A_{\tau,(ma_\tau,\vec{x})} - \sum_{m=0}^{m=n-1} A_{\tau,(ma_\tau,\vec{x})} \right) = A_{\tau,(na_\tau,\vec{x})} - A_{\tau,(na_\tau,\vec{x})} = 0$. The above choice still allows for a restricted gauge invariance in the energy: that where the gauge potential changes by a gradient of a function independent of the τ coordinate

$$A'_{k,(na_\tau,\vec{x})} \rightarrow A''_{k,(na_\tau,\vec{x})} = A'_{k,(na_\tau,\vec{x})} - \Delta_{(\vec{x})k} \chi''_{\vec{x}}, \quad (\text{D.72})$$

since such a transformation does not change $A'_\tau = 0$. To this question we shall return soon.

Under gauge (D.71), the energy becomes

$$E = \sum_n \sum_{\vec{x} \in \text{plane}_n} \left\{ \beta_\tau \sum_{j=1}^{d-1} \left[1 - \cos \left(A_{j,(na_\tau,\vec{x})} - A_{j,((n+1)a_\tau,\vec{x})} \right) \right] + \beta_x \sum_{j < k=1}^{d-1} \left[1 - \cos F_{jk,\vec{x}} \right] \right\}. \quad (\text{D.73})$$

As for the scalar model in equation (D.47) one can define the transfer matrix operator which acts on a function of the gauge fields residing on the na_τ plane and “transfers” this function to the next plane along the τ direction. Similar to relation (D.49), coupling strengths β_τ and β_x behave as $\begin{cases} \beta_x \rightarrow 0 \\ \beta_\tau \rightarrow \infty \end{cases}$ at the continuum limit a_τ . Therefore, the

underlined term in the previous equation forces the gradient along the τ direction, $(A_{j,((n+1)a_\tau, \vec{x})} - A_{j,(na_\tau, \vec{x})})$, to be small and thus we can approximate (D.73) as follows

$$E = \sum_n \sum_{\vec{x} \in \text{plane}_n} \left\{ \frac{\beta_\tau}{2} \sum_{j=1}^{d-1} (A_{j,((n+1)a_\tau, \vec{x})} - A_{j,(na_\tau, \vec{x})})^2 + \beta_x \sum_{j < k=1}^{d-1} (1 - \cos F_{jk, \vec{x}}) \right\}. \quad (\text{D.74})$$

The kernel of the transfer matrix corresponding to (D.74) takes the form

$$\begin{aligned} \mathcal{T}(A_{k,(na_\tau, \vec{x})}, A_{k,((n+1)a_\tau, \vec{x})}) = \exp \left[-\frac{\beta_\tau}{2\eta} \sum_{\vec{x} \in \text{plane}} \sum_{j=1}^{d-1} (A_{j,((n+1)a_\tau, \vec{x})} - A_{j,(na_\tau, \vec{x})})^2 - \right. \\ \left. -\frac{\beta_x}{\eta} \sum_{\vec{x} \in \text{plane}} \sum_{j < k=1}^{d-1} \left(\frac{1 - \cos(F_{jk,(na_\tau, \vec{x})})}{2} + \frac{1 - \cos(F_{jk,((n+1)a_\tau, \vec{x})})}{2} \right) \right], \quad (\text{D.75}) \end{aligned}$$

and the corresponding operator defined by means of

$$\hat{\mathcal{T}}\psi(\{A_{k,((n+1)a_\tau, \vec{x})}\}) = \int \prod_{\vec{x}} \prod_k dA_{k,(na_\tau, \vec{x})} \mathcal{T}(A_{k,((n+1)a_\tau, \vec{x})}, A_{k,(na_\tau, \vec{x})}) \psi(\{A_{k,(na_\tau, \vec{x})}\}) ,$$

is equal to

$$\begin{aligned} \hat{\mathcal{T}} \equiv \hat{\mathcal{T}}_{a_\tau} = \left(\sqrt{\frac{2\pi\eta}{\beta_\tau}} \right)^{N(d-1)} \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} (1 - \cos(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}})) \right] \times \\ \times \exp \left[\frac{\eta}{2\beta_\tau} \sum_{\vec{x},j} \frac{\hat{\partial}^2}{\partial A_{j,\vec{x}}^2} \right] \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} (1 - \cos(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}})) \right]. \quad (\text{D.76}) \end{aligned}$$

Obtaining the Hamiltonian operator, which corresponds to the above operator $\hat{\mathcal{T}}$ according to relation $\hat{\mathcal{T}} = \exp[-a_\tau \hat{\mathcal{H}}/\eta]$ for an infinitesimal a_τ , leads to the expected behaviour of the coupling strengths

$$\beta_x, \frac{1}{\beta_\tau} \sim a_\tau, \quad (\text{D.77})$$

and hence we introduce coupling constant $1/e^2$ as the proportionality constant between the above coupling strengths (β_x, β_τ) and (a_τ, a_τ^{-1})

$$\beta_x = \frac{a_\tau}{e^2} a^{d-5} \quad \text{and} \quad \beta_\tau = \frac{1}{a_\tau e^2} a^{d-3}, \quad (\text{D.78})$$

where a is the lattice spacing along all directions other than the τ direction; terms a^{d-5} and a^{d-3} give the expected dimensions to the coupling strengths β_x and β_τ . The Hamiltonian operator can be immediately obtained from (D.76) taking into account limit $a_\tau \rightarrow 0$

$$\hat{\mathcal{H}} = \frac{a^{3-d} e^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k,\vec{x}}^2 + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} [1 - \cos(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}})], \quad (\text{D.79})$$

with operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ defined as

$$\hat{\mathcal{L}}_{k',\vec{x}'} = -i\eta \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}}. \quad (\text{D.80})$$

An interesting point: since gauge field $A_{k,\vec{x}}$ has *angular* (or compact) character, operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ works as an angular momentum operator, in the sense that it is the generator of a infinitesimal change in a general functional depending on the compact gauge field $A_{k,\vec{x}}$:

$$\psi[A_{k,\vec{x}} + dA_{k,\vec{x}}] = \psi[A_{k,\vec{x}}] + \sum_{\vec{x}'} \sum_{k'} \frac{\partial \psi[\cdot]}{\partial A_{k',\vec{x}'}} dA_{k',\vec{x}'} = \left[1 + \frac{i}{\eta} \sum_{\vec{x}'} \sum_{k'} dA_{k',\vec{x}'} \hat{\mathcal{L}}_{k',\vec{x}'} \right] \{ \psi[A_{k,\vec{x}}] \},$$

where $\hat{\mathcal{L}}_{k',\vec{x}'} \{ \cdot \}$ means operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ acting on the term inside the brackets $\{ \cdot \}$. Therefore, the set of eigenvalues of operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ is discrete and equal to an integer number multiplied by η , as for instance the angular momentum operator in Quantum Mechanics.

Considering relation (D.78) and limit $a_\tau \rightarrow 0$, let us return to the question of the remaining gauge freedom by writing the energy without choosing any gauge

$$E = \frac{a_\tau a^{d-3}}{e^2} \sum_n \sum_{\vec{x} \in \text{plane}_n} \left\{ \frac{1}{2a_\tau^2} \sum_{j=1}^{d-1} \left(A_{j,(na_\tau,\vec{x})} + A_{\tau,(na_\tau,\vec{x}+\hat{j})} - A_{j,((n+1)a_\tau,\vec{x})} - A_{\tau,(na_\tau,\vec{x})} \right)^2 + \right. \\ \left. + \frac{1}{2a^2} \sum_{j < k=1}^{d-1} \left(1 - \cos F_{jk,(na_\tau,\vec{x})} \right) \right\}, \quad (\text{D.81})$$

where because of the dependence $\beta_\tau \sim 1/a_\tau$, the first term in the energy (D.70) has been approximated by the quadratic power. The field equations deduced from the Lagrangian function, from which the above energy (and therefore also Hamiltonian operator (D.79)) is derived, are to leading order in $a_\tau \approx 0$ the following

$$a_\tau^{-2} \Delta_\tau \left(\Delta_\tau A_{k,(na_\tau,\vec{x})} - \Delta_{(\vec{x})}^k A_{\tau,(na_\tau,\vec{x})} \right) + a^{-2} \sum_{j=1}^{d-1} \Delta_{(\vec{x})}^j \left(\sin F_{jk,(na_\tau,\vec{x})} \right) = 0, \quad (\text{D.82})$$

$$a_\tau^{-2} \sum_{j=1}^{d-1} \Delta_{(\vec{x})}^j \left(-\Delta_\tau A_{j,(na_\tau,\vec{x}-\hat{j})} + \Delta_{(\vec{x})}^j A_{\tau,(na_\tau,\vec{x}-\hat{j})} \right) = 0. \quad (\text{D.83})$$

These equations are the formulation with discrete derivatives of the continuum equations $\begin{cases} \partial_\tau (\partial_\tau A_k - \partial_k A_\tau) + \partial_j (\sin F_{jk}) = 0 \\ \partial_j (-\partial_\tau A_j + \partial_j A_\tau) = 0 \end{cases}$ arising from the Lagrangian density equal

to $\ell = \int d\tau d^{d-1} \vec{x} \left[-\frac{1}{2} (\partial_j A_\tau(\tau, \vec{x}) - \partial_\tau A_j(\tau, \vec{x}))^2 - (1 - \cos F_{jk}(\tau, \vec{x})) \right]$. As already mentioned, after carrying out a gauge transformation with produces $A_{\tau,(na_\tau,\vec{x})} = 0$, one has to remove the remaining gauge freedom of changing $A_{j,(na_\tau,\vec{x})}$ by the gradient of a function independent of the τ coordinate. As long as the energy does not contain coupling term of gauge field $A_{\mu,\vec{r}}$ other than those in (D.81), the gauge freedom can be removed as follows: after the transformation $A_{\mu,\vec{r}} \rightarrow A'_{\mu,\vec{r}} = A_{\mu,\vec{r}} - \Delta_\mu \left(\sum_{m=0}^{n-1} A_{\tau,(ma_\tau,\vec{x})} \right)$ which fixes

$A'_{\tau,(na_\tau,\vec{x})} = 0$, we carry out a second gauge transformation which produces a second condition on the gauge field: $\sum_k \Delta_k A''_{k,(na_\tau,\vec{x}-\hat{k})} = 0$, without disturbing the previous choice

$A''_{k,(na_\tau,\vec{x})} = 0$. Such a transformation $A'_{j,(na_\tau,\vec{x})} \rightarrow A''_{j,(na_\tau,\vec{x})} = A'_{j,(na_\tau,\vec{x})} - \Delta_j \chi''_{\vec{x}}$ is defined by (see J. D. Bjorken, S. D. Drell, *Relativistic Quantum Fields* (1965), chapter 14)

$$\chi''_{\vec{x}} = - \sum_{\vec{x}'} D_{\vec{x},\vec{x}'}^{(d-1)} \left(\sum_k \Delta_{(\vec{x})}^k A'_{k,(na_\tau,\vec{x}'-\hat{k})} \right), \quad (D.84)$$

where propagator $D_{\vec{x},\vec{x}'}^{(d-1)}$ is the Green's function of the (discrete) Laplace operator in the $(d-1)$ -dim. plane

$$\Delta^{(d-1)} \Delta^{(d-1)} D_{\vec{x},\vec{x}''}^{(d-1)} = \sum_{k=1}^{d-1} \left(D_{\vec{x}+\hat{k},\vec{x}''}^{(d-1)} + D_{\vec{x}-\hat{k},\vec{x}''}^{(d-1)} - 2 D_{\vec{x},\vec{x}''}^{(d-1)} \right) = -\delta_{\vec{x},\vec{x}''}. \quad (D.85)$$

And hence $\sum_j \Delta_{(\vec{x})}^j A''_{j,(na_\tau,\vec{x}-\hat{j})} = \sum_j \Delta_{(\vec{x})}^j A'_{j,(na_\tau,\vec{x}-\hat{j})} - \Delta_{(\vec{x})}^{(d-1)} \Delta_{(\vec{x})}^{(d-1)} \chi''_{\vec{x}} = 0$. The property that $A''_{\tau,(na_\tau,\vec{x})} = A'_{\tau,(na_\tau,\vec{x})} - \Delta_\tau \chi''_{\vec{x}} = A'_{\tau,(na_\tau,\vec{x})} = 0$ follows from field equation (D.83):

$$\Delta_\tau \chi''_{\vec{x}} = - \sum_{\vec{x}'} D_{\vec{x},\vec{x}'}^{(d-1)} \underbrace{\left(\sum_k \Delta_{(\vec{x}')}^k \Delta_\tau A'_{k,(na_\tau,\vec{x}'-\hat{k})} \right)}_0. \quad (D.86)$$

It is important to notice that if the energy had contained more $A_{j,(na_\tau,\vec{x})}$ -terms than those written in (D.81), then equation (D.83) would include more terms and thus the choice of $A_{\tau,(na_\tau,\vec{x})} = 0$ together with $\sum_j \Delta_{(\vec{x})}^j A_{j,(na_\tau,\vec{x}-\hat{j})} = 0$ would not be possible: in order to remove the gauge freedom (which leads to infinities in the partition function) one should retain only one of these conditions but not both of them.

A analogous discussion can be carried out for the model in the “Hamiltonian language” ([53] and [82]). Let us define the following operator ²⁴

$$\hat{\mathcal{G}}_{\vec{x}'} = \sum_{k'=1}^{d-1} \left(\hat{\mathcal{L}}_{k',\vec{x}'} - \hat{\mathcal{L}}_{k',\vec{x}'-\hat{k}'} \right) \equiv \sum_{k'} \Delta_{k'} \hat{\mathcal{L}}_{k',\vec{x}'-\hat{k}'}. \quad (D.87)$$

By considering the relation $\hat{\mathcal{G}}_{\vec{x}'} \hat{\mathcal{H}} = \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} + \frac{a^{d-5}}{2e^2} \hat{\mathcal{G}}_{\vec{x}'} \left\{ \sum_{\vec{x}} \sum_{j,k} [1 - \cos \hat{F}_{jk,\vec{x}}] \right\}$ together with the result of operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ applied on $\hat{F}_{jk,\vec{x}}$

$$\hat{\mathcal{L}}_{k',\vec{x}'} \left\{ \hat{F}_{jk,\vec{x}} \right\} = \delta_{k'j} \delta_{\vec{x}',\vec{x}} + \delta_{k'k} \delta_{\vec{x}',\vec{x}+\hat{j}} - \delta_{k'j} \delta_{\vec{x}',\vec{x}+\hat{k}} - \delta_{k'k} \delta_{\vec{x}',\vec{x}},$$

²⁴We should not forget that we are using $\hat{\mathcal{T}} \equiv \exp \left[-a_\tau \hat{\mathcal{H}}/\eta \right]$, i.e., the continuum τ coordinate becomes the imaginary time in the Hamiltonian “way of writing”. Thus the gradient along the τ direction acts as the (imaginary) time derivative or equivalently, as the momentum density operator in the Hamiltonian operator. In our case and choosing the gauge $A_{\tau,(na_\tau,\vec{x})} = 0$, we should note that term $-(A_{j,((n+1)a_\tau,\vec{x})} - A_{j,(na_\tau,\vec{x})})$ of energy (D.75) becomes for $a_\tau \rightarrow 0$ the electric field in the equivalent Hamiltonian formulation; this can be immediately seen by considering the equation for the electric field for a real time coordinate: $E_j = -\partial_t A_j - \partial_j A_t \stackrel{\text{gauge } A_t=0}{=} -\partial_t A_j$. The same in operator language, operator $\hat{\mathcal{L}}_{j,(na_\tau,\vec{x})}$ corresponding to such a τ gradient (compare equation (D.75) with (D.76)) fulfils the commutation relation expected for the electric field operator in Quantum Field Theory: $[\hat{\mathcal{L}}_{k,(na_\tau,\vec{x}')}, \hat{A}_{j,(na_\tau,\vec{x})}] = -i\eta \delta_{kj} \delta_{\vec{x}',\vec{x}}$. Hence, if one prefers the interpretation of $\hat{\mathcal{L}}_{k',\vec{x}'}$ as electric field operator in the Hamiltonian formulation with imaginary time, operator $\hat{\mathcal{G}}_{\vec{x}'} = \sum_{k'} \Delta_{k'} \hat{\mathcal{L}}_{k',\vec{x}'-\hat{k}'}$ corresponds to the (discrete) divergence of the electric field operator.

one arrives at

$$\begin{aligned}
\hat{\mathcal{G}}_{\vec{x}'} \left\{ \sum_{\vec{x}} \sum_{j,k} \cos \hat{F}_{jk,\vec{x}} \right\} = & -i\eta \left\{ \underbrace{-\sum_{k,k'} \sin \hat{F}_{k'k,\vec{x}'}}_{k \leftrightarrow k'} - \sum_{j,k'} \sin \hat{F}_{jk',\vec{x}'-\hat{j}} + \sum_{k,k'} \sin \hat{F}_{k'k,\vec{x}'-\hat{k}} + \right. \\
& \underbrace{+ \sum_{j,k'} \sin \hat{F}_{jk',\vec{x}'}}_{j \leftrightarrow k'} + \underbrace{+ \sum_{k,k'} \sin \hat{F}_{k'k,\vec{x}'-\hat{k}'}}_{k \leftrightarrow k'} + \underbrace{+ \sum_{j,k'} \sin \hat{F}_{jk',\vec{x}'-\hat{j}-\hat{k}'}}_{j \leftrightarrow k'} - \underbrace{\sum_{k,k'} \sin \hat{F}_{k'k,\vec{x}-\hat{k}-\hat{k}'}}_{k \leftrightarrow k'} - \\
& \left. \underbrace{- \sum_{j,k'} \sin \hat{F}_{jk',\vec{x}'-\hat{k}'}}_{j \leftrightarrow k'} \right\} = 0, \tag{D.88}
\end{aligned}$$

and finally

$$\hat{\mathcal{G}}_{\vec{x}'} \hat{\mathcal{H}} = \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} . \tag{D.89}$$

Therefore, there are a set of eigenvectors of operator $\hat{\mathcal{H}}$ which are also eigenvectors of $\hat{\mathcal{G}}_{\vec{x}'}$. Let us consider one of such eigenvectors: $\psi [A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \psi [A_{k,\vec{x}}] + \sum_{\vec{x},k} (-\Delta_k \zeta_{\vec{x}}) \frac{\partial \psi[\cdot]}{\partial A_{k,\vec{x}}} +$

$$+ \frac{1}{2} \sum_{\vec{x}_1,k_1} \sum_{\vec{x}_2,k_2} (-\Delta_{k_1} \zeta_{\vec{x}_1}) (-\Delta_{k_2} \zeta_{\vec{x}_2}) \frac{\partial^2 \psi[\cdot]}{\partial A_{k_1,\vec{x}_1} \partial A_{k_2,\vec{x}_2}} + \dots$$

$$\psi [A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\sum_{\vec{x}} \sum_k (-\Delta_k \zeta_{\vec{x}}) \frac{\partial}{\partial A_{k,\vec{x}}} \right] \{ \psi [A_{k,\vec{x}}] \} , \tag{D.90}$$

where the brackets means that the operator in the exponential is acting on the content inside the brackets. By noticing that $\sum_{\vec{x},k} (\zeta_{\vec{x}+\hat{k}} - \zeta_{\vec{x}}) \hat{\mathcal{L}}_{k,\vec{x}} = \sum_{\vec{x},k} \zeta_{\vec{x}} (\hat{\mathcal{L}}_{k,\vec{x}-\hat{k}} - \hat{\mathcal{L}}_{k,\vec{x}})$ (we assume that the values taken on by operator $\hat{\mathcal{L}}_{k,\vec{x}}$ approach zero rapidly enough for $\vec{x} \rightarrow \infty$), one obtains

$$\psi [A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \{ \psi [A_{k,\vec{x}}] \} , \tag{D.91}$$

or equivalently

$$\psi [A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \psi [A_{k,\vec{x}}] \exp \left[-\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] . \tag{D.92}$$

Since $\hat{\mathcal{G}}_{\vec{x}'} \hat{\mathcal{H}} = \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'}$ and the eigenvalues of $-i\eta \frac{\hat{\partial}}{\partial A_{k,\vec{x}}}$ are integer (times η), operator $\hat{\mathcal{G}}_{\vec{x}'}$ acting on the eigenvectors $\psi [A_{k,\vec{x}}]$ of the Hamiltonian produces a *real* value: we can therefore introduce the real integer scalar function $\varrho_{\vec{x}'}$ as follows

$$\hat{\mathcal{G}}_{\vec{x}'} \{ \psi [A_{k,\vec{x}}] \} = \varrho_{\vec{x}'} \psi [A_{k,\vec{x}}] , \tag{D.93}$$

and thus according to (D.91)

$$\psi[A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \varrho_{\vec{x}'} \right] \psi[A_{k,\vec{x}}] . \quad (\text{D.94})$$

Gauge vector $A_{k,\vec{x}}$ can be split into a longitudinal $(A_{\parallel})_{k,\vec{x}}$ and a transversal part $(A_{\perp})_{k,\vec{x}}$, where both “longitudinal” and “transversal” should be intended inside the $(d-1)$ -dimensional plane normal to the τ direction, i.e, fields which fulfil $\sum_{j,k=1}^{d-1} \epsilon_{ijk} \Delta_j (A_{\parallel})_{k,\vec{x}} = 0$

and $\sum_{k=1}^{d-1} \Delta_k (A_{\perp})_{k,\vec{x}-\hat{k}} = 0$. The longitudinal part can thus be written as the (discrete) gradient of a scalar function and a general $A_{k,\vec{x}}$ takes the form $A_{k,\vec{x}} = (A_{\perp})_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}$. From equation (D.94) we obtain

$$\psi[A_k] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}} \zeta_{\vec{x}} \varrho_{\vec{x}} \right] \psi[(A_{\perp})_k] . \quad (\text{D.95})$$

According to (D.85), we define propagator $D_{\vec{x},\vec{x}''}^{(d-1)}$ of the (discrete) Laplace operator in the $(d-1)$ -dim. plane, $\sum_{k=1}^{d-1} (D_{\vec{x}+\hat{k},\vec{x}''}^{(d-1)} + D_{\vec{x}-\hat{k},\vec{x}''}^{(d-1)} - 2 D_{\vec{x},\vec{x}''}^{(d-1)}) = -\delta_{\vec{x},\vec{x}''}$, and thus the exponent in equation (D.95) can be expressed as follows

$$\sum_{\vec{x}} \zeta_{\vec{x}} \varrho_{\vec{x}} = \sum_{\vec{x},\vec{x}''} \zeta_{\vec{x}} \underbrace{\delta_{\vec{x},\vec{x}''}}_{\substack{(d-1) \\ \Delta(\vec{x})}} \varrho_{\vec{x}''} \stackrel{\text{sum by parts}}{=} - \sum_{\vec{x},\vec{x}'} \left(\sum_{k=1}^{d-1} \zeta_{\vec{x}+\hat{k}} + \zeta_{\vec{x}-\hat{k}} - 2\zeta_{\vec{x}} \right) D_{\vec{x},\vec{x}''}^{(d-1)} \varrho_{\vec{x}''} ,$$

where considering the form of a general vector field $A_{k,\vec{x}} = (A_{\perp})_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}$, for which relation $\sum_{k=1}^{d-1} \Delta_k (A_{\perp})_{k,\vec{x}-\hat{k}} = 0$ holds, one obtains $\sum_{\vec{x}} \zeta_{\vec{x}} \varrho_{\vec{x}} = \sum_{\vec{x},\vec{x}''} \left(\sum_{k=1}^{d-1} \Delta_k A_{k,\vec{x}-\hat{k}} \right) D_{\vec{x},\vec{x}''}^{(d-1)} \varrho_{\vec{x}''}$ and

$$\psi[A_k] = \exp \left[\frac{i}{\eta} \sum_{\vec{x},\vec{x}''} \left(\sum_{k=1}^{d-1} \Delta_k A_{k,\vec{x}} \right) D_{\vec{x},\vec{x}''}^{(d-1)} \varrho_{\vec{x}''} \right] \psi[(A_{\perp})_k] . \quad (\text{D.96})$$

This last relation and equation (D.93) lead to the result that operator $\hat{\mathcal{G}}_{\vec{x}'}$ applied on $\psi[(A_{\perp})_{k,\vec{x}}]$ gives zero

$$\begin{aligned} \varrho_{\vec{x}'} \psi[A_{k,\vec{x}}] &= \hat{\mathcal{G}}_{\vec{x}'} \{ \psi[A_{k,\vec{x}}] \} = \\ &= -i\eta \sum_{k'=1}^{d-1} \left[\frac{\partial}{\partial A_{k',\vec{x}'}} - \frac{\partial}{\partial A_{k',\vec{x}'-\hat{k}'}} \right] \left\{ \exp \left[\frac{i}{\eta} \sum_{\vec{x},\vec{x}''} \left(\sum_{k=1}^{d-1} A_{k,\vec{x}} - A_{k,\vec{x}-\hat{k}} \right) D_{\vec{x},\vec{x}''}^{(d-1)} \varrho_{\vec{x}''} \right] \psi[(A_{\perp})_k] \right\} = \\ &= \left[\sum_{\vec{x},\vec{x}''} \underbrace{\left(\sum_{k'} D_{\vec{x}',\vec{x}''}^{(d-1)} - D_{\vec{x}'+\hat{k}',\vec{x}''}^{(d-1)} - D_{\vec{x}'-\hat{k}',\vec{x}''}^{(d-1)} + D_{\vec{x}',\vec{x}''}^{(d-1)} \right)}_{\delta_{\vec{x}',\vec{x}''}} \varrho_{\vec{x}''} \right] \psi[A_k] + \end{aligned}$$

$$\begin{aligned}
& + \exp \left[\frac{i}{\eta} \sum_{\vec{x}, \vec{x}''} \left(\sum_k \Delta_k A_{k, \vec{x}} \right) D_{\vec{x}, \vec{x}''}^{(d-1)} \varrho_{\vec{x}''} \right] \hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [(A_\perp)_k] \right\} , \\
\implies & \hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [(A_\perp)_{k, \vec{x}}] \right\} = 0 .
\end{aligned} \tag{D.97}$$

We arrive thus at the following result: after fixing partially the gauge by means of $A_{\tau, (na_\tau, \vec{x})} = 0$, which has led to the Hamiltonian operator at which the present discussion has started, any arbitrary function of the gauge fields which depends only on the $(d-1)$ -dim. transversal component $(A_\perp)_{k, \vec{x}}$ fulfils $\hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [(A_\perp)_{k, \vec{x}}] \right\} = 0$; but operator $\hat{\mathcal{L}}_{k', \vec{x}'}$ corresponds to the gradient of $A_{k', \vec{x}}$ along the τ direction (see the step from equation (D.75) to (D.76)) and this implies that operator $\hat{\mathcal{G}}_{\vec{x}'} = \sum_{k'} \Delta_{k'} \hat{\mathcal{L}}_{k', \vec{x}' - \hat{k}'}$ together with the above operator relation (D.97) is the equivalent operator formulation of the field equation (D.83) (one should not forget that during the previous discussion with operators, this field equation has not been used). On the other side, operator $\hat{\mathcal{G}}_{\vec{x}'}$ commutes with the Hamiltonian: in order to remove the residual gauge freedom which remains after choosing $A_\tau = 0$, the configurations contributing to the partition function should fulfil the condition that they remain unchanged under a gauge transformation from those still allowed; or equivalently and by looking at eq. (D.91) $\psi [A_{k, \vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \left\{ \psi [A_{k, \vec{x}}] \right\}$, these contributing configurations are singlets of symmetry operator $\exp \left[\frac{i}{\eta} \hat{\mathcal{G}}_{\vec{x}'} \right] : \hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [A_{k, \vec{x}}] \right\} = 0$ [28].

As already mentioned, both conditions $A_{\tau, \vec{x}} = 0$ and $\sum_k \Delta_k A_{k, \vec{x} - \hat{k}} = 0$ can only be imposed for the pure gauge model. If $A_{k, \vec{x}}$ is coupled, for example, to a scalar field, then equation (D.83) does not longer hold and one has to choose either $A_{\tau, \vec{x}} = 0$ or $\sum_k \Delta_k A_{k, \vec{x} - \hat{k}} = 0$ as a condition on the gauge field. Or in the operator formulation, conditions $\hat{A}_{\tau, \vec{x}} = 0$ and $\hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [A_{k, \vec{x}}] \right\} = 0$ are not fulfilled; the solution of $\psi [A_{k, \vec{x}}]$ in this case for an easy problem will be considered in the following Note, together with the origin of the name “confinement” for the phase in a compact gauge theory at temperature zero and large coupling constant.

Note. $\hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [A_{k, \vec{x}}] \right\}$ for a gauge field coupled to an external source: an example. The “confining” phase (I).

Let us consider the following easy case: working in the gauge $A_{\tau, (na_\tau, \vec{x})} = 0$, equation (D.82) still holds but instead of equation (D.83) we have the following one

$$-a_\tau^{-2} \sum_{j=1}^{d-1} \Delta_j \Delta_\tau A_{j, (na_\tau, \vec{x} - \hat{j})} = q (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) \quad \text{with } q \text{ an integer} , \tag{D.98}$$

which leads in the operator language to the relation

$$\hat{\mathcal{G}}_{\vec{x}'} \left\{ \psi [A_k] \right\} = q \eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) \left\{ \psi [A_k] \right\} , \tag{D.99}$$

since the gauge field has compact character and therefore operator $\hat{\mathcal{L}}_{k',\vec{x}}$ has as eigenvalues an integer number multiplied by η . The solution of such an operator equation is ([99], p. 428)

$$\psi_{1 \rightarrow 2} [A_k] = \exp \left[-iq \sum_{C_{1 \rightarrow 2}} A_{k,\vec{x}} \right], \quad (D.100)$$

where $C_{1 \rightarrow 2}$ is an arbitrary path connecting site $\vec{x}_1$ to site $\vec{x}_2$ (both sites on the $(d-1)$ -dim. plane normal to the τ direction). Under an infinitesimal gauge transformation *independent* of the τ coordinate, $A_{k,\vec{x}} \rightarrow A'_{k,\vec{x}} = A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}$, equation (D.100) changes into

$$\begin{aligned} \psi_{1 \rightarrow 2} [A'_k] &= \psi_{1 \rightarrow 2} [A_k] \exp \left[iq \underbrace{\sum_{C_{1 \rightarrow 2}} \Delta_k \zeta_{\vec{x}}} \right] = \psi_{1 \rightarrow 2} [A_k] + iq (\zeta_{\vec{x}_2} - \zeta_{\vec{x}_1}) \psi_{1 \rightarrow 2} [A_k], \quad (D.101) \\ &(\zeta_{\vec{x}_1 + \vec{k}_1} - \zeta_{\vec{x}_1}) + (\zeta_{\vec{x}_1 + \vec{k}_1 + \vec{k}_2} - \zeta_{\vec{x}_1 + \vec{k}_1}) \dots (\zeta_{\vec{x}_2} - \zeta_{\vec{x}_1 + \vec{k}_1 + \dots \vec{k}_n}) \end{aligned}$$

and by comparing this result to equation $\psi_{1 \rightarrow 2} [A_k - \Delta_k \zeta] = \psi_{1 \rightarrow 2} [A_k] + \sum_{\vec{x}', k'} (-\Delta_{k'} \zeta_{\vec{x}'}) \frac{\partial \psi_{1 \rightarrow 2} [A_k]}{\partial A_{k', \vec{x}'}}$ for

an infinitesimal $\zeta_{\vec{x}}$, one obtains $\sum_{\vec{x}', k'} (-\Delta_{k'} \zeta_{\vec{x}'}) \left(-i\eta \frac{\partial \psi_{1 \rightarrow 2} [A_k]}{\partial A_{k', \vec{x}'}} \right) = \sum_{\vec{x}'} \zeta_{\vec{x}'} \sum_{k'} \Delta_{k'} \left(-i\eta \frac{\partial \psi_{1 \rightarrow 2} [A_k]}{\partial A_{k', \vec{x}'}} \right) =$
 $= q\eta (\zeta_{\vec{x}_2} - \zeta_{\vec{x}_1}) \psi_{1 \rightarrow 2} [A_k]$ and therefore

$$\sum_{k'} \Delta_{k'} \left(-i\eta \frac{\partial \psi_{1 \rightarrow 2} [A_k]}{\partial A_{k', \vec{x}'}} \right) \equiv \hat{\mathcal{G}}_{\vec{x}'} \{ \psi_{1 \rightarrow 2} [A_k] \} = q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) \psi_{1 \rightarrow 2} [A_k]. \quad (D.102)$$

Equations $\begin{cases} a_\tau^{-2} \Delta_\tau (\Delta_\tau A_{k, (na_\tau, \vec{x})} - \Delta_k A_{\tau, (na_\tau, \vec{x})}) + a^{-2} \sum_j \Delta_j (\sin F_{jk, (na_\tau, \vec{x})}) = 0 \\ a_\tau^{-2} \sum_j \Delta_j (-\Delta_\tau A_{j, (na_\tau, \vec{x}-j)} + \Delta_j A_{\tau, (na_\tau, \vec{x})}) = q (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) \end{cases}$ (before choosing the gauge $A_{\tau, (na_\tau, \vec{x})} = 0$) can be obtained from a model where the gauge field is coupled to a scalar field $\varphi_{(na_\tau, \vec{x})}$ through the covariant derivative term $\frac{a^{d-1} |\psi|^2}{2a_\tau} (\Delta_\tau \varphi_{(na_\tau, \vec{x})} - q A_{\tau, (na_\tau, \vec{x})})^2$ with $a^{d-1} e^2 |\psi|^2 a_\tau^{-1} \Delta_\tau \varphi_{(na_\tau, \vec{x})} = (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1})$. For simplicity, we assume that the other components of the covariant derivative vanish, $\frac{a^{d-3} a_\tau |\psi|^2}{2} (\Delta_k \varphi_{(na_\tau, \vec{x})} - q A_{k, (na_\tau, \vec{x})})^2 = 0$, such that the first equation $a_\tau^{-2} \Delta_\tau (\Delta_\tau A_{k, (na_\tau, \vec{x})} - \Delta_k A_{\tau, (na_\tau, \vec{x})}) + a^{-2} \sum_j \Delta_j (\sin F_{jk, (na_\tau, \vec{x})}) = 0$ holds; in fact, this condition together with the constraint on $\Delta_\tau \varphi_{(na_\tau, \vec{x})}$ means that field $\varphi_{(na_\tau, \vec{x})}$ is not (yet) a degree of freedom, only a variable controled from the exterior. Therefore, we need not to introduce a term in the energy (or in the Hamiltonian) which takes account of this external variable. The more general case, where $\Delta_\tau \varphi_{(na_\tau, \vec{x})}$ can take on arbitrary values, will be considered in the next Note.

For the above model let us study the range of large e^2 , such that the Hamiltonian operator $\frac{a^{3-d} e^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k, \vec{x}}^2 + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \left[1 - \cos(\hat{A}_{j, \vec{x}} + \hat{A}_{k, \vec{x}+j} - \hat{A}_{j, \vec{x}+\vec{k}} - \hat{A}_{k, \vec{x}}) \right]$ can be approximated by

$$\hat{\mathcal{H}}_{\text{large } e^2} = \frac{a^{3-d} e^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k, \vec{x}}^2. \quad (D.103)$$

And now let us calculate the energy of a configuration fulfilling equation (D.98): using the operator “way of writing”, we have to evaluate the expected value of the Hamiltonian operator applied to a function for which equation (D.99) holds.

As in the previous section, let us denote by $|0\rangle$ the eigenvector of the above Hamiltonian, whose eigenvalue is the smallest of all eigenvalues of $\hat{\mathcal{H}}_{\text{large } e^2}$. Since this operator only produces the sum of positive values, $|0\rangle$ must have the following property

$$\hat{\mathcal{L}}_{k, \vec{x}}^2 |0\rangle = 0. \quad (D.104)$$

And state $|w_{1 \rightarrow 2}\rangle$ which fulfils relation (D.99) is obtained [59] by the application of operator

$$\hat{\theta}_{1 \rightarrow 2} = \exp[-i\hat{\varphi}_{\vec{x}_1}] \exp\left[-iq \sum_{C_{1 \rightarrow 2}} \hat{A}_{k,\vec{x}}\right] \exp[+i\hat{\varphi}_{\vec{x}_2}] , \quad (\text{D.105})$$

on $|0\rangle$. The above operator is invariant under a general gauge transformation (general in the sense that it belongs to the already remaining gauge freedom of transformations independent of the τ coordinate)

$$\begin{cases} \varphi_{(na_\tau, \vec{x})} & \rightarrow \varphi'_{(na_\tau, \vec{x})} = \varphi_{(na_\tau, \vec{x})} - \zeta_{(na_\tau, \vec{x})} , \\ A_{k,(na_\tau, \vec{x})} & \rightarrow A'_{k,(na_\tau, \vec{x})} = A_{k,(na_\tau, \vec{x})} - \frac{1}{q} \left(\zeta_{(na_\tau, \vec{x}+\hat{k})} - \zeta_{(na_\tau, \vec{x})} \right) . \end{cases} \quad (\text{D.106})$$

as one can easily prove by looking at relation (D.100). Considering the following relation

$$\hat{\mathcal{L}}_{k', \vec{x}'} \left\{ e^{\mp iq \hat{A}_{k, \vec{x}}} \right\} \equiv \left[\hat{\mathcal{L}}_{k', \vec{x}'}, e^{\mp iq \hat{A}_{k, \vec{x}}} \right] = \mp \eta q \delta_{k', k} \delta_{\vec{x}', \vec{x}} e^{\mp iq \hat{A}_{k, \vec{x}}} , \quad (\text{D.107})$$

it is straightforward to evaluate the energy of function $|w_{1 \rightarrow 2}\rangle = \hat{\theta}_{1 \rightarrow 2} |0\rangle$ in the range of large e^2

$$\hat{\mathcal{H}}_{\text{large } e^2} |w_{1 \rightarrow 2}\rangle = \frac{a^{3-d} e^2}{2} \sum_{\vec{x}'} \sum_{k'} \hat{\mathcal{L}}_{k', \vec{x}'}^2 |w_{1 \rightarrow 2}\rangle \sim q^2 e^2 \mathcal{P} |w_{1 \rightarrow 2}\rangle , \quad (\text{D.108})$$

with $\mathcal{P}$ the length of path $C_{1 \rightarrow 2}$ connecting site $\vec{x}_1$ to site $\vec{x}_2$. Therefore, from all the configurations with the property $-a_\tau^{-2} \sum_j \Delta_j \Delta_\tau A_{j, (na_\tau, \vec{x}-\hat{j})} = q (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1})$, the main contribution at large e^2 to the partition function arises from that configuration where $C_{1 \rightarrow 2}$ runs directly from $\vec{x}_1$ to $\vec{x}_2$ through a straight line, i.e., for the configuration whose energy grows like $|\vec{x}_2 - \vec{x}_1|$. It should be noted that such an energy is actually the *free* energy, since as we have seen in equation (D.11), the largest eigenvalue of transfer matrix $\mathcal{T} = \exp[-a_\tau \hat{\mathcal{H}}/\eta]$ (and therefore the smallest of the eigenvalues of $\hat{\mathcal{H}}$) gives the leading contribution to the free energy.

As a departing comment, one should notice that function (D.100) $\psi_{1 \rightarrow 2}[A_k] = \exp\left[-iq \sum_{C_{1 \rightarrow 2}} A_{k, \vec{x}}\right]$

for the case where the begin $\vec{x}_1$ and the end point $\vec{x}_2$ would coincide (and so $C_{1 \rightarrow 2}$ would become a closed path) would be just the correlation function from which we have calculated the expected value $W(C)$ in the previous Appendix C and which has allowed us to distinguish between the different phases of a gauge invariant model. However, in our present case this would mean a closed path in the $(d-1)$ -dim. sublattice normal to the periodic direction and as we shall see later, in order to investigate the phases of gauge models in a lattice periodic along one direction, the closed path defining $W(C)$ has to stretch along the complete periodic dimension: sites $\vec{x}_1$ and $\vec{x}_2$ will become thus the projection on the $(d-1)$ -dim. sublattice of the repetitions of closed path C along the periodic direction (see figure on page 564).

It is from the calculations just discussed, where the name “confinement” for such a model comes from. As one has seen in the footnote on page 544, the gradient along the τ direction of gauge field $A_{k, \vec{x}}$ acts in the Hamiltonian ²⁵ formulation (always working in the gauge $A_{\tau, \vec{x}} = 0$) as the electric field and

²⁵At the cost of boring the reader, let us stress again that the Hamiltonian formulation is only another form to compute the partition function in a lattice periodic in one direction (called the τ -direction), direction along which the lattice's discreteness gets lost $a_\tau \rightarrow 0$. The partition function is obtained by means of transfer matrix operator $\hat{\mathcal{T}}_{a_\tau}, Z = \text{Tr} \left[\left(\hat{\mathcal{T}}_{a_\tau} \right)^N \right]$, which for its part can be written as the exponential of another operator $\hat{\mathcal{H}}$: $\hat{\mathcal{T}} = \exp[-a_\tau \hat{\mathcal{H}}/\eta]$ with a_τ infinitesimal. Now it depends on the model we are studying whether lattice's length Na_τ along the periodic direction becomes infinite, and thus the periodicity (which has been therefore only introduced to avoid boundary effects) plays no role longer and one only has a continuum theory in 1 dimension plus a lattice theory in the remaining $(d-1)$ dimensions; or whether Na_τ is kept as a finite value ω during the process of taking limit $a_\tau \rightarrow 0$: in this last case, the lattice's periodicity is maintained and the calculation of partition function becomes the solution of a model in a $(d-1)$ -dim. lattice, where the model interactions are described by operator $\hat{\mathcal{H}}$ whose eigenvalues determine the free energy, being η/ω the scale to which every configuration energy has to be compared and which in this work we shall assume to be proportional to the temperature.

thus the equation $\hat{\mathcal{G}}_{\vec{x}'} \{(\cdot)\} = q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) (\cdot)$ corresponds to an electric field created by q positive units of the fundamental charge e located at $\vec{x}_2$ and q negative units at $\vec{x}_1$. The solution of smallest energy for such an equation *at large* e^2 has an energy which grows like the distance between the two point charges, and such a property is called confinement. As we shall see in the next subsection, if the temperature of the system is increased (and we have already seen that for a periodic coordinate in the τ direction, the temperature is inversely correlated to this periodic length), the confining property can be destroyed.

D.2.2 The “deconfinement” transition in the pure compact gauge model at high temperatures

In Appendix C we have learnt that the pure gauge compact model in a d -dimensional space with $d \geq 3$ shows a phase at large e^2 characterized by $W(C) \sim e^{-\text{Area}}$; in fact, for $d = 3$ this phase occurs for any e^2 , not only for large values of e^2 , but this particular question will be considered in the next section. In the dislocation model developed in this work, such a behaviour has been identified with the Stage II, i.e., with the dislocation state where the formation of dense networks of sessile dislocations confines the gliding dislocations, which are retained inside small regions and are not able to scape from them; this leads to a large increase in the resistance to plastic deformation. Now we shall study whether this behaviour, characterized by the phase of large e^2 at the temperature $T = 0$, remains when the temperature is increased. From the comment in the preceding paragraph, the phase at large e^2 in a pure compact gauge theory can be also interpreted in the “Hamiltonian language” as a charge confining phase and therefore, the breaking through heating of the phase characterized at $T = 0$ by $W(C) \sim e^{-\text{Area}}$, will be sometimes called “deconfinement”.

Let us return to the general equation of the partition function $\mathcal{Z} = \text{Tr} [\hat{\mathcal{T}}^N]$, where in order to study the effect of a non vanishing temperature, we consider a model periodic along the direction τ which goes to the continuum. The temperature is related to the finite lattice’s length along such a direction, $\omega \equiv Na_\tau$: for the Quantum Statistical Mechanics by means of relation $\frac{1}{k_B T} = \frac{\omega}{\hbar c}$, for the dislocation model through the ansatz

(2.93) $\frac{\omega}{\eta} = \frac{\lambda_{\text{scale}}}{k_B T}$. Since the calculation of the partition function can be formulated as the trace over a complete set of eigenvectors of a Hamiltonian operator, with such an operator defined through the relation $\hat{\mathcal{T}} = \exp [-a_\tau \hat{\mathcal{H}}/\eta]$, we obtain

$$\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{Na_\tau}{\eta} \hat{\mathcal{H}} \right) \right] \quad \text{with} \quad Na_\tau = \omega. \quad (\text{D.109})$$

Following the argument by Polyakov [82] and Susskind [90], we shall now evaluate this partition function for the case of large e^2 : $\hat{\mathcal{H}} \approx \hat{\mathcal{H}}_{\text{large } e^2} = \frac{a^{3-d} e^2}{2} \sum_{\vec{x}'} \sum_{k'} \hat{\mathcal{L}}_{k', \vec{x}'}^2$. We begin with the case where all eigenvectors of $\hat{\mathcal{H}}$ fulfil condition $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi \} = 0$. Since according to (D.89) operator $\hat{\mathcal{G}}_{\vec{x}'} = \sum_{k'} \hat{\mathcal{L}}_{k', \vec{x}'} - \hat{\mathcal{L}}_{k', \vec{x}' - \hat{k}'}$ commutes with the Hamiltonian, and further,

the gauge field is compact (or equivalently, has angular character), the eigenvalues of operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ acting of the eigenvectors of the Hamiltonian are equal to an *integer* number multiplied by η . Hence partition function (D.109) at large e^2 is given by

$$\mathcal{Z}_{\text{large } e^2} = \sum_{\{n_{k',\vec{x}'}\}} \prod_{\vec{x}} \delta \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, 0 \right) \exp \left[-\frac{\omega}{\eta} \frac{a^{3-d} e^2 \eta^2}{2} \sum_{\vec{x}'} \sum_{k'=1}^{d-1} n_{k',\vec{x}'}^2 \right], \quad (\text{D.110})$$

where $n_{k',\vec{x}'}$ is an *integer* field and the already known notation

$$\sum_{\{n_{k',\vec{x}'}\}} \equiv \prod_{k'=1}^{d-1} \underbrace{\sum_{n_{k',\vec{1}}=-\infty}^{n_{k',\vec{1}}=+\infty} \sum_{n_{k',\vec{2}}=-\infty}^{n_{k',\vec{2}}=+\infty} \cdots \sum_{n_{k',\vec{N}}=-\infty}^{n_{k',\vec{N}}=+\infty}}_{\text{all } N = |\Lambda_\perp| \text{ sites of lattice } \perp a_\tau\text{-direction}} = \prod_{\vec{x}'} \prod_{k'=1}^{d-1} \sum_{n_{k',\vec{x}'}=-\infty}^{n_{k',\vec{x}'}=+\infty},$$

has been used. The constraint $\prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k,\vec{x}-\hat{k}}, 0 \right)$ can be implemented as follows

$$\prod_{\vec{x}} \delta \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, 0 \right) = \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) \exp \left[-i \overbrace{\sum_{\vec{x},k} \theta_{\vec{x}} \Delta_k n_{k,\vec{x}-\hat{k}}}^{-\sum_{\vec{x},k} n_{k,\vec{x}} \Delta_k \theta_{\vec{x}}} \right]. \quad (\text{D.111})$$

The resulting partition, nevertheless, takes a form which we have already met in Appendix C. Let us interrupt briefly the discussion line and consider again the XY model using the method presented in equation (C.481): applying the Poisson summation formula

$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} db f(b) e^{-i2\pi b m}$ to the partition function, one obtains

$$\begin{aligned} \mathcal{Z}_{\text{large } e^2} &= \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) \left\{ \sum_{\{n_{k,\vec{x}}\}} \exp \left[i \sum_{\vec{x},k} n_{k,\vec{x}} \Delta_k \theta_{\vec{x}} \right] \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2} \sum_{\vec{x},k} n_{k,\vec{x}}^2 \right] \right\} = \\ &= \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) \int_{-\infty}^{\infty} \left(\prod_{\vec{x},k} db_{k,\vec{x}} \right) \sum_{\{m_{k,\vec{x}}\}} \exp \left[i \sum_{\vec{x},k} b_{k,\vec{x}} (\Delta_k \theta_{\vec{x}} - 2\pi m_{k,\vec{x}}) \right] \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2} \sum_{\vec{x},k} b_{k,\vec{x}}^2 \right], \\ \mathcal{Z}_{\text{large } e^2} &= \underbrace{\left(\sqrt{\frac{2\pi}{e^2 \omega \eta}} \right)^{(d-1)|\Lambda_\perp|} \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) \sum_{\{m_{k,\vec{x}}\}} \exp \left[-\frac{a^{d-3}}{2e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (\Delta_k \theta_{\vec{x}} - 2\pi m_{k,\vec{x}})^2 \right]}_{\text{D.112}}. \end{aligned}$$

Comparing the underlined term with equation (C.79), it results that the partition function of the d -dimensional pure compact gauge theory for large e^2 and in a lattice periodic along one dimension is, up to a multiplicative constant, equal to the Villain approximation of the $(d-1)$ -dim. XY model with a value of $a^{(d-1)-2} |\psi|_{XY}^2$ equal to $a^{d-3}/e^2 \omega$.

The d -dimensional XY model has a phase transition for $d \geq 2$: for $d = 2$ it is a Kosterlitz-Thouless transition and for $d \geq 3$ is a second order transition. Therefore the d -dim. pure compact gauge theory for large e^2 shows also a phase transition for $d \geq 3$ when the lattice's length along the periodic direction decreases: a Kosterlitz-Thouless

transition in the 3-dimensional model and a second order phase transition for $d = 4$ ²⁶. There is no phase transition for the 2-dim. gauge model; moreover, the compact character of the gauge field plays no role in this dimension, as we already know from the discussion following equation (C.319). For the dislocation model and according to the ansatz $\frac{\omega}{\eta} = \frac{\lambda_{\text{scale}}}{k_B T}$, we can therefore estimate the temperature at which the Stage II (the phase characterized by the behaviour of the compact model at $T = 0$ and large e^2) disappears for a 3-dimensional crystal. Before going further, let us finish the discussion on “confinement”.

Note. The “confining” phase (II).

In equation (D.108) of the previous Note, one has seen that for a field configuration which fulfils $\hat{\mathcal{G}}_{\vec{x}'} \{(\cdot)\} = q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) \{(\cdot)\}$ (where term $q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1})$ arises from the coupling of the gauge field to a scalar field controlled from outside), the energy corresponding to the Hamiltonian formulation, i.e. the free energy, for large e^2 grows like the distance $|\vec{x}_2 - \vec{x}_1|$. The same result can be also obtained from the equivalence of the pure gauge model with the XY model. Let us call $\mathcal{Z}'_{\text{large } e^2} = \text{Tr}' \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}}_{\text{large } e^2} \right) \right]$ to the partition function evaluated for configurations for which $\hat{\mathcal{G}}_{\vec{x}'} \{(\cdot)\} = q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1}) (\cdot)$ holds, whilst we shall denote by $\mathcal{Z}_{\text{large } e^2} = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}}_{\text{large } e^2} \right) \right]$ that where the configurations, on which we are summing, fulfil $\hat{\mathcal{G}}_{\vec{x}'} \{(\cdot)\} = 0$. Hence $\mathcal{V} = -\frac{\eta}{\omega} \ln \left(\frac{\mathcal{Z}'_{\text{large } e^2}}{\mathcal{Z}_{\text{large } e^2}} \right)$ represents the free energy increase in configurations with $\hat{\mathcal{G}}_{\vec{x}'} = q\eta (\delta_{\vec{x}', \vec{x}_2} - \delta_{\vec{x}', \vec{x}_1})$ related to those for which $\hat{\mathcal{G}}_{\vec{x}'} = 0$ holds. This $\mathcal{V}$ takes the form

$$\exp \left(-\frac{\omega}{\eta} \mathcal{V} \right) = \frac{\sum_{\{n_{k,\vec{x}}\}} \prod_{\vec{x}} \delta \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, q\delta_{\vec{x},\vec{x}_2} - q\delta_{\vec{x},\vec{x}_1} \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} n_{k,\vec{x}}^2 \right]}{\sum_{\{n_{k,\vec{x}}\}} \prod_{\vec{x}} \delta \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, 0 \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} n_{k,\vec{x}}^2 \right]}, \quad (\text{D.113})$$

and following the same steps which lead from equation (D.110) to result (D.112), one obtains

$$\exp \left(-\frac{\omega}{\eta} \mathcal{V} \right) = \frac{1}{\mathcal{Z}_{(d-1)-\text{dim. XY}}} \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) e^{iq\theta_{\vec{x}_2}} e^{-iq\theta_{\vec{x}_1}} \sum_{\{m_{k,\vec{x}}\}} \exp \left[-\frac{a^{d-3}}{2e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (\Delta_k \theta_{\vec{x}} - 2\pi m_{k,\vec{x}})^2 \right],$$

$$\exp \left(-\frac{\omega}{\eta} \mathcal{V} \right) = \left\langle e^{iq(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)-\text{dim. XY}}, \quad (\text{D.114})$$

²⁶By looking at equations (C.154) and (C.226), we can obtain an estimation for the critical values of these transtions: $\left(\frac{a}{\omega a e^2 \eta} \right)_{\text{critical}} \approx \frac{2}{\pi} \Rightarrow \frac{\omega_{\text{crit}}}{a} = \frac{\pi}{2a e^2 \eta}$ for the 3-dim. gauge model, and $\left(\frac{a}{\omega e^2 \eta} \right)_{\text{critical}} \approx 0.322 \Rightarrow \frac{\omega_{\text{crit}}}{a} \approx \frac{3.106}{e^2 \eta}$ for the 4-dim. model. Nevertheless, and in order to be exact, both critical values can only be estimations since they have been obtained by means of the Villain approximation. Such an approximation, although works rather well for all ranges in $|\psi|_{XY}^2$, it is only well founded for large $|\psi|_{XY}^2$, range which, in the above equivalence, corresponds to the weak coupling regime, not to the range of large e^2 which we are studying. However, the existence of the phase transitions does not depend on this approximation (only the exact value of the critical points do depend) and the above qualitative results hold.

where as in equation (D.112), the expected value is calculated with respect to a $(d-1)$ -dim. XY model with $\frac{|\psi|_{XY}^2}{\eta} = \frac{1}{e^2\omega\eta}$. Such a correlation has been already calculated for $(d-1)$ equal to 2 and 3 (and 4, but this case is of no interest here): at small $\frac{|\psi|_{XY}^2}{\eta} = \frac{1}{e^2\omega\eta}$, the power expansion method leads to a exponential decay like $|\vec{x}_2 - \vec{x}_1|$ for any dimension (see Tables C.2 and C.2)

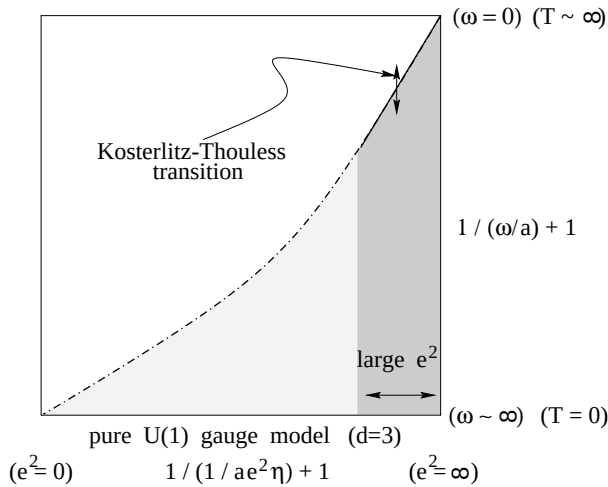
$$\left\langle e^{iq(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim. } XY} = \exp \left[-q|\vec{x}_2 - \vec{x}_1| \ln \left(\frac{2e^2\omega\eta}{a^{d-3}} \right) \right]. \quad (\text{D.115})$$

Applying this result to the definition of $\mathcal{V}$ (D.114), one obtains that for large $e^2\omega\eta$, $\mathcal{V}$ grows like the distance $|\vec{x}_2 - \vec{x}_1|$, as expected from the previous result (D.108) and property which is called “confinement”. The fact that in (D.114) an *integer* factor q appears does not change the result: at small $\frac{1}{e^2\omega\eta}$ the leading contribution to correlation function $\left\langle e^{iq(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{XY}$ is obtained when each bond along the shortest path from $\vec{x}_1$ to $\vec{x}_2$ are covered q times by the bond factor $e^{i\theta_{\vec{x}}}$ arising from the power expansion of the XY term $\exp \left[-\frac{1}{e^2\omega\eta} (1 - \cos(\Delta_k \theta_{\vec{x}})) \right]$. It is noteworthy that if q were not an integer, there would be no factor in the power expansion to combine with $e^{iq(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})}$ and to produce thus a nonvanishing contribution. Nevertheless the integrity of q is ensured by the compact character of the gauge field, which makes operator $\hat{\mathcal{L}}_{k',\vec{x}'}$ have integer values times η .

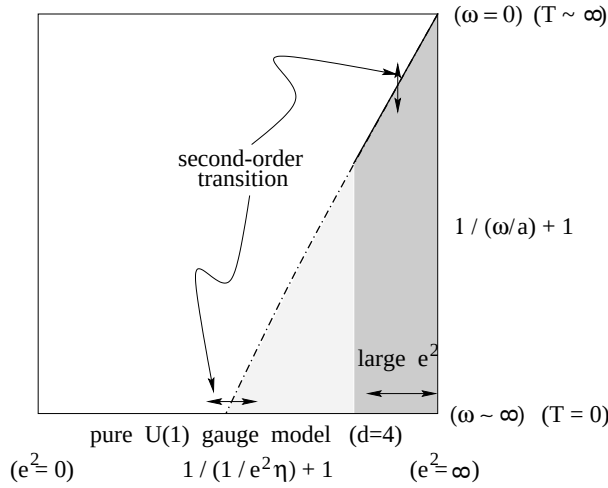
On the other hand, at large $\frac{1}{e^2\omega\eta}$ one can neglect the integer field $m_{k,\vec{x}}$ in (D.114), the integration is then trivial and the correlation function is (approximately) given by

$$\left\langle e^{iq(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim. } XY} = \exp \left[+\frac{q^2 e^2\omega\eta}{2a^{d-3}} 2 D_{\vec{x}_2, \vec{x}_1}^{(d-1)} \right], \quad (\text{D.116})$$

where $D_{\vec{x}_2, \vec{x}_1}^{(d-1)}$ is the propagator of the $(d-1)$ -dim. (discrete) Laplace operator and it given at large distances $|\vec{x}_2 - \vec{x}_1|$ (large if compared to the lattice spacing a) by $D_{\vec{x}_2, \vec{x}_1}^{(d-1)} = -\frac{1}{2\pi} \ln |\vec{x}_2 - \vec{x}_1|$ for $(d-1) = 2$, or by $D_{\vec{x}_2, \vec{x}_1}^{(d-1)} = \frac{1}{4\pi} \frac{1}{|\vec{x}_2 - \vec{x}_1|}$ for $d = 4$. Therefore, $\mathcal{V}$ is proportional to $\ln |\vec{x}_2 - \vec{x}_1|$ ($d = 3$) or to $1/|\vec{x}_2 - \vec{x}_1|$ ($d = 4$) at small $e^2\omega\eta$. Since the temperature is related to length ω , depending on the relation between T and the coupling constant e^2 we can have a phase characterized by the strong growth of $\mathcal{V}$ like $|\vec{x}_2 - \vec{x}_1|$ or another phase with a much weaker form in $\mathcal{V}$. The transition from the first to the second phase will be called the breaking of the “confining” phase.



The pure compact gauge model for $d = 3$ and at $\omega = \infty$ ($T \sim 1/\omega = 0$) shows no transition (see diagram on page 449), and the same phase at large e^2 extends over the range e^2 small. Such a phase corresponds at $\omega < \infty$ to the shaded region in the contiguous figure and is characterized by $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$; in this figure, the diagonal from $(e^2 = 0, \omega = \infty)$ to $(e^2 = \infty, \omega = 0)$ represents $e^2\eta\omega = 1$ and curves under this diagonal satisfy $e^2\eta\omega > 1$. At large e^2 and by decreasing ω , or by increasing T , the phase $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ undergoes a Kosterlitz-Thouless transition to a phase with $\mathcal{V} \sim \ln |\vec{x}_2 - \vec{x}_1|$. The line separating both behaviours, although only proved for large e^2 , can be continued to the region of small e^2 , range which will be studied in more detail in the next section on page 609.



For the 4-dimensional pure gauge model, the situation is a bit different, since at $\omega = \infty$ ($T = 0$) the system already shows a second order phase transition (see diagram on page 465). With decreasing ω (or increasing T), the phase at large e^2 in the shaded region has also a second order transition, just as the (non-periodic) 3-dimensional XY model (to which the pure gauge model in this range is equivalent). As before, we can expect that the separating line, which is a line of second-order phase transitions, continues until the critical point of the model at $\omega = \infty$ is reached. The shaded area in the expected diagram of the figure denotes the phase characterized by $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$, whilst the unshaded region that with $\mathcal{V} \sim 1/|\vec{x}_2 - \vec{x}_1|$.

Now let us consider the Higgs model with the compact gauge coupled to scalar field $\varphi_{(na_\tau, \vec{x})}$, which now is an actual degree of freedom, in a lattice periodic along one of its dimensions. The corresponding operator $\hat{\mathcal{H}}$ for the calculation of the partition function and of correlation functions is given by the sum of a pure gauge part (with the choice $A_{\tau, (na_\tau, \vec{x})} = 0$) $\hat{\mathcal{H}}_{\text{gauge}} = -\frac{a^{3-d}e^2\eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \frac{\hat{\partial}^2}{\partial A_{k, \vec{x}}^2} + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \left[1 - \cos \left(\hat{A}_{j, \vec{x}} + \hat{A}_{k, \vec{x}+\hat{j}} - \hat{A}_{j, \vec{x}+\hat{k}} - \hat{A}_{k, \vec{x}} \right) \right]$ plus the part containing the covariant derivative which couples the gauge field to the scalar field. This last scalar part is obtained by starting from the following energy

$$\begin{aligned} \frac{E_{\text{scalar}}}{\eta} = \sum_n \left[\frac{\beta_\tau}{\eta} \sum_{\vec{x} \in \text{plane}_n} \left[1 - \cos \left(\varphi_{((n+1)a_\tau, \vec{x})} - \varphi_{(na_\tau, \vec{x})} \right) \right] + \right. \\ \left. + \frac{\beta_x}{\eta} \sum_{\vec{x} \in \text{plane}_n} \sum_{k=1}^{d-1} \left[\frac{2}{2} - \frac{\cos \left(\varphi_{(na_\tau, \vec{x}+\hat{k})} - \varphi_{(na_\tau, \vec{x})} - qA_{k, (na_\tau, \vec{x})} \right)}{2} - \right. \right. \\ \left. \left. - \frac{\cos \left(\varphi_{((n+1)a_\tau, \vec{x}+\hat{k})} - \varphi_{((n+1)a_\tau, \vec{x})} - qA_{k, ((n+1)a_\tau, \vec{x})} \right)}{2} \right] \right], \quad (\text{D.117}) \end{aligned}$$

where $\Delta_\tau \varphi_{(na_\tau, \vec{x})}$ is not coupled to the gauge field due to the gauge chosen. After carrying out the same steps as in model (D.27), one will arrive at the result $\beta_\tau \sim a_\tau^{-1}$, which allows us to approximate $1 - \cos \left(\varphi_{((n+1)a_\tau, \vec{x})} - \varphi_{(na_\tau, \vec{x})} \right)$ by $\frac{1}{2} \left(\varphi_{((n+1)a_\tau, \vec{x})} - \varphi_{(na_\tau, \vec{x})} \right)^2$ in the continuum limit $a_\tau \rightarrow 0$. The resulting transfer matrix operator is given, up to a multiplicative factor, by

$$\hat{\mathcal{T}} = \exp \left[-\frac{\beta_x}{2\eta} \sum_{\vec{x}, k} \left(\frac{1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k, \vec{x}} \right)}{2} \right) \right] \times$$

$$\times \exp \left[\frac{\eta}{2\beta_\tau} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \varphi_{\vec{x}}^2} \right] \exp \left[-\frac{\beta_x}{\eta} \sum_{\vec{x},k} \left(\frac{1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right)}{2} \right) \right], \quad (\text{D.118})$$

which, by comparing to $\hat{\mathcal{T}} = \exp \left[-a_\tau \hat{\mathcal{H}}/\eta \right]$ (and $a_\tau \rightarrow 0$), produces $\beta_x, \frac{1}{\beta_\tau} \sim a_\tau$: let us denote the proportionality constant between β_x, β_τ^{-1} and a_τ/a by $a^{d-2}|\psi|^2$ (with a the lattice spacing along the non-periodic dimensions). Thus operator $\hat{\mathcal{H}}$ for the scalar part is equal to $\hat{\mathcal{H}}_{\text{scalar}} = -\frac{a^{1-d}\eta^2}{2|\psi|^2} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \varphi_{\vec{x}}^2} + a^{d-3}|\psi|^2 \sum_{\vec{x}} \sum_{k=1}^{d-1} \left[1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right]$ and for the complete compact Higgs model

$$\begin{aligned} \hat{\mathcal{H}} = & -\frac{a^{3-d}e^2\eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \frac{\hat{\partial}^2}{\partial A_{k,\vec{x}}^2} + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \left[1 - \cos \left(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}} \right) \right] - \\ & -\frac{a^{1-d}\eta^2}{2|\psi|^2} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \varphi_{\vec{x}}^2} + a^{d-3}|\psi|^2 \sum_{\vec{x}} \sum_{k=1}^{d-1} \left[1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right]. \end{aligned} \quad (\text{D.119})$$

As for the pure gauge model, one has to remove the remaining gauge invariance stated by equation

$$\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}} \hat{\mathcal{H}} = \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}} \quad \text{or} \quad \exp \left[\frac{i}{\eta} \hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}} \right] \hat{\mathcal{H}} = \hat{\mathcal{H}} \exp \left[\frac{i}{\eta} \hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}} \right], \quad (\text{D.120})$$

with ²⁷

$$\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}} = -i\eta \sum_{k'=1}^{d-1} \left(\Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} - q \frac{\hat{\partial}}{\partial \varphi_{\vec{x}'}} \right). \quad (\text{D.121})$$

Hence, for the calculation of the partition function by means of the transfer matrix method, the states ψ over which $\hat{\mathcal{H}}$ acts have to satisfy the condition of being singlets

²⁷As one can show after considering equation (D.89) $\sum_{k'=1}^{d-1} \Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} \hat{\mathcal{H}}_{\text{gauge}} = \hat{\mathcal{H}}_{\text{gauge}} \sum_{k'=1}^{d-1} \Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}}$ and result $\left(\frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} - \frac{\hat{\partial}}{\partial A_{k',\vec{x}'-\hat{k}'}} \right) A_{k,\vec{x}} = \delta_{k,k'} \left(\delta_{\vec{x},\vec{x}'} - \delta_{\vec{x},\vec{x}'-\hat{k}'} \right)$, which leads to the following relations:

$$\begin{aligned} \sum_{k'=1}^{d-1} \Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} \left(\sum_{k=1}^{d-1} \left[1 - \cos \left(\Delta_k \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right] \right) &= \sum_{k=1}^{d-1} \left[1 - \cos \left(\Delta_k \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right] \left(\sum_{k'=1}^{d-1} \Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} \right) + \\ &+ \sum_{k'} \left[-q \sin \left(\Delta_{k'} \hat{\varphi}_{\vec{x}'} - q\hat{A}_{k',\vec{x}'} \right) + q \left(\Delta_{k'} \hat{\varphi}_{\vec{x}'-\hat{k}'} - q\hat{A}_{k',\vec{x}'-\hat{k}'} \right) \right], \end{aligned}$$

and

$$\begin{aligned} -q \frac{\hat{\partial}}{\partial \varphi_{\vec{x}'}} \left(\sum_{k=1}^{d-1} \left[1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right] \right) &= \sum_{k=1}^{d-1} \left[1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right] \left(-q \frac{\hat{\partial}}{\partial \varphi_{\vec{x}}} \right) + \\ &+ \sum_k \left[-q \sin \left(\Delta_k \hat{\varphi}_{\vec{x}'-\hat{k}} - q\hat{A}_{k,\vec{x}'-\hat{k}} \right) + q \left(\Delta_k \hat{\varphi}_{\vec{x}'} - q\hat{A}_{k,\vec{x}'} \right) \right]. \end{aligned}$$

under the action of operator $\exp\left[\frac{i}{\eta}\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}}\right]: \exp\left[\frac{i}{\eta}\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}}\right]\left\{\psi[\varphi_{\vec{x}}, A_{k,\vec{x}}]\right\} = \psi[\varphi_{\vec{x}}, A_{k,\vec{x}}]$ or equivalently

$$\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}}\left\{\psi[\varphi_{\vec{x}}, A_{k,\vec{x}}]\right\} = 0, \quad (\text{D.122})$$

condition which selects one unique configuration from the set of equivalent configurations (related among them by gauge transformations) which produce the same contribution to the partition function, just as condition $-\eta \sum_{k'=1}^{d-1} \Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} \left\{\psi[A_{k,\vec{x}}]\right\} = 0$ in the pure gauge model selected out the states which only depended on the transversal part (in the non-periodic sublattice) of the gauge field. On the other hand, if free energy increase $\mathcal{V}$, due to two external sources located at sites $\vec{x}_1$ and $\vec{x}_2$, is calculated, the above condition must be changed into

$$\hat{\mathcal{G}}_{\vec{x}'}^{\text{coupled}}\left\{\psi[\varphi_{\vec{x}}, A_{k,\vec{x}}]\right\} = p\eta(\delta_{\vec{x}',\vec{x}_2} - \delta_{\vec{x}',\vec{x}_1})\psi[\varphi_{\vec{x}}, A_{k,\vec{x}}]. \quad (\text{D.123})$$

As one can expect from the discussion of the Higgs model in a non-periodic lattice (Appendix B), the fact $p = q$ or $p \neq q$ will be play an important role. It is important to note that due to the angular character of both $A_{k,\vec{x}}$ and phase $\varphi_{\vec{x}}$, differential operators $-i\frac{\hat{\partial}}{\partial A_{k,\vec{x}}}$ and $-i\frac{\hat{\partial}}{\partial \varphi_{\vec{x}}}$ have integer values.

Following Susskind [90] and Banks & Rabinovici [19], let us consider the limit of large e^2 and small $|\psi|^2$. In this case, operator $\mathcal{H}$ for the Higgs model is given by

$$\hat{\mathcal{H}} \approx \frac{a^{3-d}e^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k,\vec{x}}^2 + \frac{\gamma}{2} \sum_{\vec{x}} \hat{Q}_{\vec{x}}^2 \quad \text{with} \quad \gamma = \frac{\eta^2}{a^{d-1}|\psi|^2}, \quad (\text{D.124})$$

where $\hat{\mathcal{L}}_{k',\vec{x}'} = -i\eta \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}}$ and (multiplicative) operator field $\hat{Q}_{\vec{x}}$ is *integer* valued since it is the result of applying differential operator $-i\frac{\hat{\partial}}{\partial \varphi_{\vec{x}}}$. The states over which we are summing for the calculation of $\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{Na_\tau}{\eta} \hat{\mathcal{H}} \right) \right]$, fulfil the following equation

$$\hat{\mathcal{G}}_{\vec{x}'}\left\{\psi[Q_{\vec{x}}, A_{k,\vec{x}}]\right\} \equiv \sum_{k'=1}^{d-1} \Delta_{k'} \hat{\mathcal{L}}_{k',\vec{x}'-\hat{k}'} \left\{\psi[Q_{\vec{x}}, A_{k,\vec{x}}]\right\} = q\eta Q_{\vec{x}'} \psi[Q_{\vec{x}}, A_{k,\vec{x}}]. \quad (\text{D.125})$$

Similar to equation (D.110) and since $\eta^{-1}\hat{\mathcal{L}}_{k,\vec{x}}$ has integer eigenvalues, the partition at large e^2 is given by

$$\begin{aligned} \mathcal{Z}_{\text{large } e^2} &= \prod_{\vec{x}} \sum_{Q_{\vec{x}}=-\infty}^{\infty} \prod_{\vec{x},k} \sum_{n_{k,\vec{x}}=-\infty}^{\infty} \prod_{\vec{x}} \delta \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, qQ_{\vec{x}} \right) \times \\ &\times \exp \left[-\frac{\omega}{\eta} \frac{a^{3-d}e^2\eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} n_{k,\vec{x}}^2 - \frac{\omega}{\eta} \frac{\gamma}{2} \sum_{\vec{x}} Q_{\vec{x}}^2 \right]. \end{aligned} \quad (\text{D.126})$$

By using $\prod_{\vec{x}} \delta\left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, qQ_{\vec{x}}\right) = \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) \exp\left[-i \sum_{\vec{x}} \theta_{\vec{x}} \left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}} - qQ_{\vec{x}}\right)\right]$ and applying the Poisson summation formula to replace sums $\sum_{\{Q_{\vec{x}}\}} \equiv \prod_{\vec{x}} \sum_{Q_{\vec{x}}=-\infty}^{\infty}$ and $\sum_{\{n_{k,\vec{x}}\}}$ by integrals over continuous variables $\sum_{\{m_{\vec{x}}\}} \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} d\zeta_{\vec{x}}\right)$ and $\sum_{\{m_{k,\vec{x}}\}} \int_{-\infty}^{\infty} \left(\prod_{\vec{x},k} db_{k,\vec{x}}\right)$, one obtains

$$\begin{aligned} \mathcal{Z}_{\text{large } e^2} &= \sum_{\{m_{\vec{x}}\}} \sum_{\{m_{k,\vec{x}}\}} \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} d\zeta_{\vec{x}}\right) \int_{-\infty}^{\infty} \left(\prod_{\vec{x},k} db_{k,\vec{x}}\right) e^{-i2\pi \sum_{\vec{x}} (\zeta_{\vec{x}} m_{\vec{x}} + \sum_k b_{k,\vec{x}} m_{k,\vec{x}})} \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) \times \\ &\times \exp\left[-\frac{a^{3-d}e^2\omega\eta}{2} \sum_{\vec{x},k} b_{k,\vec{x}}^2 - \frac{\gamma\omega}{2\eta} \sum_{\vec{x}} \zeta_{\vec{x}}^2 + iq \sum_{\vec{x}} \theta_{\vec{x}} \zeta_{\vec{x}} + i \sum_{\vec{x},k} b_{k,\vec{x}} \Delta_k \theta_{\vec{x}}\right] = \left(\sqrt{\frac{4\pi^2}{\gamma a^{3-d}e^2\omega^2}}\right)^{|\Lambda_{\perp}|} \times \\ &\times \int_{-\infty}^{\infty} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) \sum_{\{m_{\vec{x}}\}} \sum_{\{m_{k,\vec{x}}\}} \exp\left[-\frac{a^{d-3}}{2e^2\omega\eta} \sum_{\vec{x},k} (\Delta_k \theta_{\vec{x}} - 2\pi m_{k,\vec{x}})^2 - \frac{\eta}{2\gamma\omega} \sum_{\vec{x}} (q\theta_{\vec{x}} - 2\pi m_{\vec{x}})^2\right] = \\ &\equiv \left(\int_{-\pi}^{\pi} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) \exp\left[-\frac{a^{d-3}}{2e^2\omega\eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{h}{\eta} \sum_{\vec{x}} (1 - \cos q\theta_{\vec{x}})\right]\right). \quad (\text{D.127}) \end{aligned}$$

which, up to a constant factor, is equal to the partition function in the Villain approximation of the $(d-1)$ -dimensional XY model with $|\psi|_{XY}^2 \equiv \frac{1}{e^2\omega}$ under the action of an external magnetic field $h = \frac{\eta^2}{\gamma\omega} = \frac{a^{d-1}|\psi|^2}{\omega}$.

If in the compact $U(1)$ Higgs model at large e^2 and small $|\psi|^2$ one acts from the exterior such that condition (D.125) reads now

$$\sum_{k'=1}^{d-1} \Delta_{k'} \hat{\mathcal{L}}_{k',\vec{x}'-\hat{k}'} \left\{ \psi [Q_{\vec{x}}, A_{k,\vec{x}}] \right\} = [q\eta Q_{\vec{x}'} + p\eta (\delta_{\vec{x}',\vec{x}_2} - \delta_{\vec{x}',\vec{x}_1})] \psi [Q_{\vec{x}}, A_{k,\vec{x}}],$$

and free energy increase $\mathcal{V}$ due to this external action is calculated, one obtains

$$\begin{aligned} \exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) &= \frac{\sum_{\{Q_{\vec{x}}, n_{k,\vec{x}}\}} \prod_{\vec{x}} \delta\left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, qQ_{\vec{x}} + p(\delta_{\vec{x}',\vec{x}_2} - \delta_{\vec{x}',\vec{x}_1})\right) \exp\left[-\frac{a^{3-d}e^2\omega\eta}{2} \sum_{\vec{x},k} n_{k,\vec{x}}^2 - \dots\right]}{\sum_{\{Q_{\vec{x}}\}} \sum_{\{n_{k,\vec{x}}\}} \prod_{\vec{x}} \delta\left(\sum_{k=1}^{d-1} \Delta_k n_{k,\vec{x}-\hat{k}}, qQ_{\vec{x}}\right) \exp\left[-\frac{a^{3-d}e^2\omega\eta}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} n_{k,\vec{x}}^2 - \frac{\omega}{\eta} \frac{\gamma}{2} \sum_{\vec{x}} Q_{\vec{x}}^2\right]} = \\ &= \frac{\int_{-\pi}^{\pi} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) e^{ip(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \exp\left[-\frac{a^{d-3}}{e^2\omega\eta} \sum_{\vec{x},k} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma\omega} \sum_{\vec{x}} (1 - \cos q\theta_{\vec{x}})\right]}{\int_{-\pi}^{\pi} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi}\right) \exp\left[-\frac{a^{d-3}}{e^2\omega\eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma\omega} \sum_{\vec{x}} (1 - \cos q\theta_{\vec{x}})\right]}, \quad (\text{D.128}) \end{aligned}$$

or equivalently $\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) = \left\langle e^{ip(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim. XY under } h \cos(q\theta_{\vec{x}})}$.

Let us study this result in more detail by considering the case $q = p = 1$. For the XY model and since the external magnetic field $h = \eta^2/\gamma\omega$ is nonzero, the system is always magnetized, i.e., $\langle e^{i\theta_{\vec{x}}} \rangle \equiv \mathcal{M} \neq 0$, and shows no phase transition. Hence, correlation function $\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \rangle$ at large distances $|\vec{x}_2 - \vec{x}_1|$ does not decay and becomes equal to $\mathcal{M}^2$: free energy increase $\mathcal{V}$ shows therefore no dependence on the distance and the behaviour $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ for large e^2 , characteristic in the *pure* gauge model, disappears when the gauge field is coupled to a scalar field.

For $q \neq 1$, the denominator in (D.128), the partition function of the model, is invariant under the discrete *global* transformation

$$\theta_{\vec{x}} \rightarrow \theta_{\vec{x}} + n \frac{2\pi}{q} \quad \forall \vec{x} \quad \text{with} \quad n = 1, \dots, q-1. \quad (\text{D.129})$$

If $p = q$, term $e^{ip(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})}$ and so the numerator shows the same symmetry as the denominator, similarly to the case $p = q = 1$: the result of nonexistence of a phase transition or of $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ holds as for $p = q = 1$. A completely different behaviour is shown when $p \neq q$: let us consider the case $p = 1$ and $q = 2$

$$\frac{\int_0^{2\pi} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \exp \left[-\frac{a^{d-3}}{e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma \omega} \sum_{\vec{x}} (1 - \cos 2\theta_{\vec{x}}) \right]}{\int_0^{2\pi} \left(\prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \right) \exp \left[-\frac{a^{d-3}}{e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma \omega} \sum_{\vec{x}} (1 - \cos 2\theta_{\vec{x}}) \right]}.$$

Due to the invariance of $\exp \left[-\frac{a^{d-3}}{e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma \omega} \sum_{\vec{x}} (1 - \cos 2\theta_{\vec{x}}) \right]$ under the global transformation $\theta_{\vec{x}} \rightarrow \theta_{\vec{x}} + \pi \quad \forall \vec{x}$, which changes $e^{-i\theta_{\vec{x}}}$ into $(-1)e^{-i\theta_{\vec{x}}}$, the above equation can be rewritten as the correlation function of the $Z(2)$ Ising model

$$\left\langle e^{-i\theta_{\vec{x}_1}} e^{i\theta_{\vec{x}_2}} \right\rangle_{(d-1)\text{-dim. } XY \text{ under } h \cos(2\theta_{\vec{x}})} = \frac{\prod_{\vec{x}} \sum_{\sigma_{\vec{x}}=\pm 1} \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} e^{-\beta^* \mathcal{E}[\sigma_{\vec{x}}]}}{\prod_{\vec{x}} \sum_{\sigma_{\vec{x}}=\pm 1} e^{-\beta^* \mathcal{E}[\sigma_{\vec{x}}]}}, \quad (\text{D.130})$$

where effective energy $\mathcal{E}[\sigma_{\vec{x}}] \equiv \mathcal{E}(\sigma_{\vec{1}}, \dots, \sigma_{\vec{N}})$, defined as $\exp[-\beta^* \mathcal{E}(\sigma_{\vec{1}}, \dots, \sigma_{\vec{N}})] = \int_{(\sigma_{\vec{1}}-1)\pi/2}^{(\sigma_{\vec{1}}+1)\pi/2} \frac{d\theta_{\vec{1}}}{2\pi} \dots \int_{(\sigma_{\vec{N}}-1)\pi/2}^{(\sigma_{\vec{N}}+1)\pi/2} \frac{d\theta_{\vec{N}}}{2\pi} \exp \left[-\frac{a^{d-3}}{e^2 \omega \eta} \sum_{\vec{x}, k} (1 - \cos \Delta_k \theta_{\vec{x}}) - \frac{\eta}{\gamma \omega} \sum_{\vec{x}} (1 - \cos 2\theta_{\vec{x}}) \right]$, is invariant under the global transformation $\sigma_{\vec{x}'} \rightarrow -\sigma_{\vec{x}'} \quad \forall \vec{x}'$ (or $\theta_{\vec{x}'} \rightarrow \theta_{\vec{x}'} + \pi$); coupling strength β^* is a decreasing function of $e^2 \omega \eta$, and $N = |\Lambda_{\perp}|$ is the number of lattice sites in the $(d-1)$ -dimensional space. According to (D.130) and since the effective energy is originated by interactions between neighbour sites, the different behaviours of the correlation function $\left\langle e^{-i\theta_{\vec{x}_1}} e^{i\theta_{\vec{x}_2}} \right\rangle_{(d-1)\text{-dim. } XY \text{ under } h \cos(2\theta_{\vec{x}})}$ are similar to those of the correlation function $\langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle$ in the $(d-1)$ -dimensional $Z(2)$ Ising model [90] (model which will be denoted by “Ising” in the following equations). And looking at the phase diagram of the

Ising model for $(d - 1) \geq 2$ (see Table C.1), we obtain the following results for $\mathcal{V}$

$$\begin{aligned} \exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) &= \langle e^{-i\theta\vec{x}_1} e^{i\theta\vec{x}_2} \rangle_{(d-1)-\dim.XY} \equiv \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{(d-1)-\dim. \text{ "Ising" }} \xrightarrow{|\vec{x}_2 - \vec{x}_1| \rightarrow \infty} \mathcal{M}^2 \neq 0, \quad \text{large } \beta^* \\ \exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) &= \langle e^{-i\theta\vec{x}_1} e^{i\theta\vec{x}_2} \rangle_{(d-1)-\dim.XY} \equiv \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{(d-1)-\dim. \text{ "Ising" }} \sim e^{-|\vec{x}_2 - \vec{x}_1| \ln(1/\beta^*)}, \quad \text{small } \beta^* \end{aligned} \quad (\text{D.131})$$

or equivalently, if $p \neq q$, $\mathcal{V}$ in the model with gauge coupling to a scalar field keeps its “confining” behaviour for large $e^2\omega\eta$ (small β^*), $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$, as in the pure gauge model. A similar behaviour has been already seen as the d -dimensional compact Higgs model at $T = 0$ was discussed: the phase characteristic at large e^2 in the pure gauge model, i.e., that described by $W(C) \sim e^{-\text{Area}}$, is maintained for the model with the gauge field coupled to a scalar field only if $p \neq q$ (the reader can compare Fig. C.6 and Fig. C.7 for the 3-dimensional model; for the 4-dimensional model the result is similar). All this discussion holds for a small $q \neq 1$: as we shall see later, the case of a large q is a bit different.

The phase at small $e^2\omega\eta$ is characterized in both cases ($p = q$ and $p \neq q$) by a weaker $\mathcal{V}$; a more exact result for $\langle e^{-i\theta\vec{x}_1} e^{i\theta\vec{x}_2} \rangle_{(d-1)-\dim.XY} \equiv \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{(d-1)-\dim. \text{ "Ising" }}$ than that equal to $\mathcal{M}^2 \neq 0$ can be obtained from the connected scalar correlation function (C.47) which one calculated for the Ising model at large β^*

$$\Gamma_{\vec{x}_1\vec{x}_2} = \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle - \mathcal{M}^2 = (\text{constant}) e^{-4\beta^*|\vec{x}_2 - \vec{x}_1|}, \quad (\text{D.132})$$

from which free energy increase $\mathcal{V}$ becomes

$$\begin{aligned} \ln \left[\mathcal{M}^2 \left(1 + \text{const. } e^{-4\beta^*|\vec{x}_2 - \vec{x}_1|} \right) \right] &\stackrel{\text{small } e^2\omega\eta}{\approx} 2 \ln \mathcal{M} + \text{const. } e^{-4\beta^*|\vec{x}_2 - \vec{x}_1|}, \\ \mathcal{V} &\sim \text{constant} + e^{-|\vec{x}_2 - \vec{x}_1|}. \end{aligned} \quad (\text{D.133})$$

If in the Hamiltonian language, with its interpretation of the free energy increase $\mathcal{V}$ as the potential energy between two electric charges placed at $\vec{x}_1$ and $\vec{x}_2$, $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ characterizes a “confining” phase, then the above result (D.133) corresponds to a Yukawa type potential, i.e., to a screened interaction between such charges.

Returning to the gauge model in a periodic lattice along the τ direction, if the relation between the lattice’s length ω along this periodic dimension and the temperature is of the kind $\omega \sim T^{-1}$, from the previous result we see that the phase with $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ gets broken at large temperatures.

D.2.3 The 3-dimensional pure compact gauge model at high temperatures [92]

In the last section we have seen that the behaviour of the pure compact gauge model in a d -dimensional lattice periodic along one of its dimensions when the coupling constant is large is completely similar to the behaviour of the $(d - 1)$ -dimensional XY model. Now we shall see that this similarity occurs not only at large e^2 but also at small e^2 as long as periodic length ω is small. Let us return therefore to the Villain formulation

(the suitable approximation for small e^2) of the 3-dimensional compact gauge model without coupling to any other field (C.271). A site on the d -dim. lattice is denoted by $\vec{r} \equiv (na_\tau, \vec{x})$, where for the moment the lattice is not yet periodic along the τ -direction; a greek index $\mu = 1, \dots, d$ denotes vector components in the d -dim. lattice whilst a latin one $k = 1, \dots, (d-1)$ does in the $(d-1)$ -dim. subspace normal to the τ -direction. The partition function of the model is given by

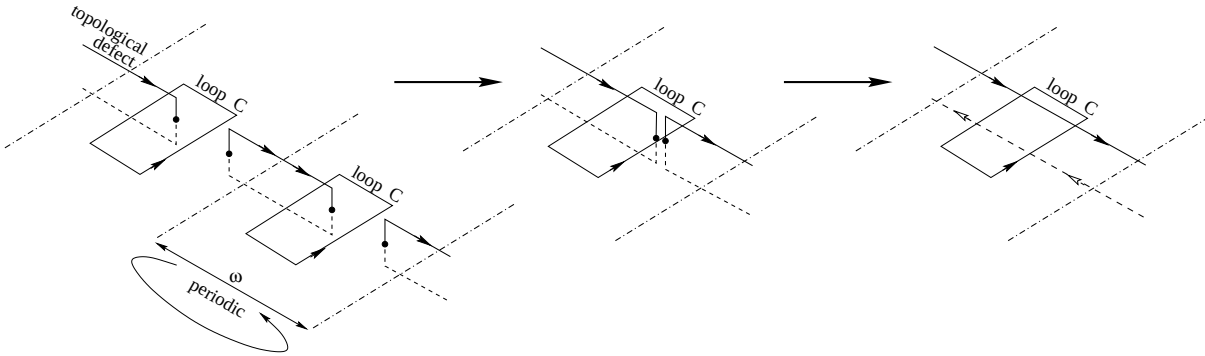
$$\mathcal{Z} = \mathcal{Z}_{\text{non-compact}} \sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{2\pi^2}{ae^2\eta} D(0) \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}} \sum_{\vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'} \right], \quad (\text{D.134})$$

with

$$\mathcal{Z}_{\text{non-compact}} = \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \left(\frac{dA_{\mu, \vec{r}}}{2\pi} \right) \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 \right]. \quad (\text{D.135})$$

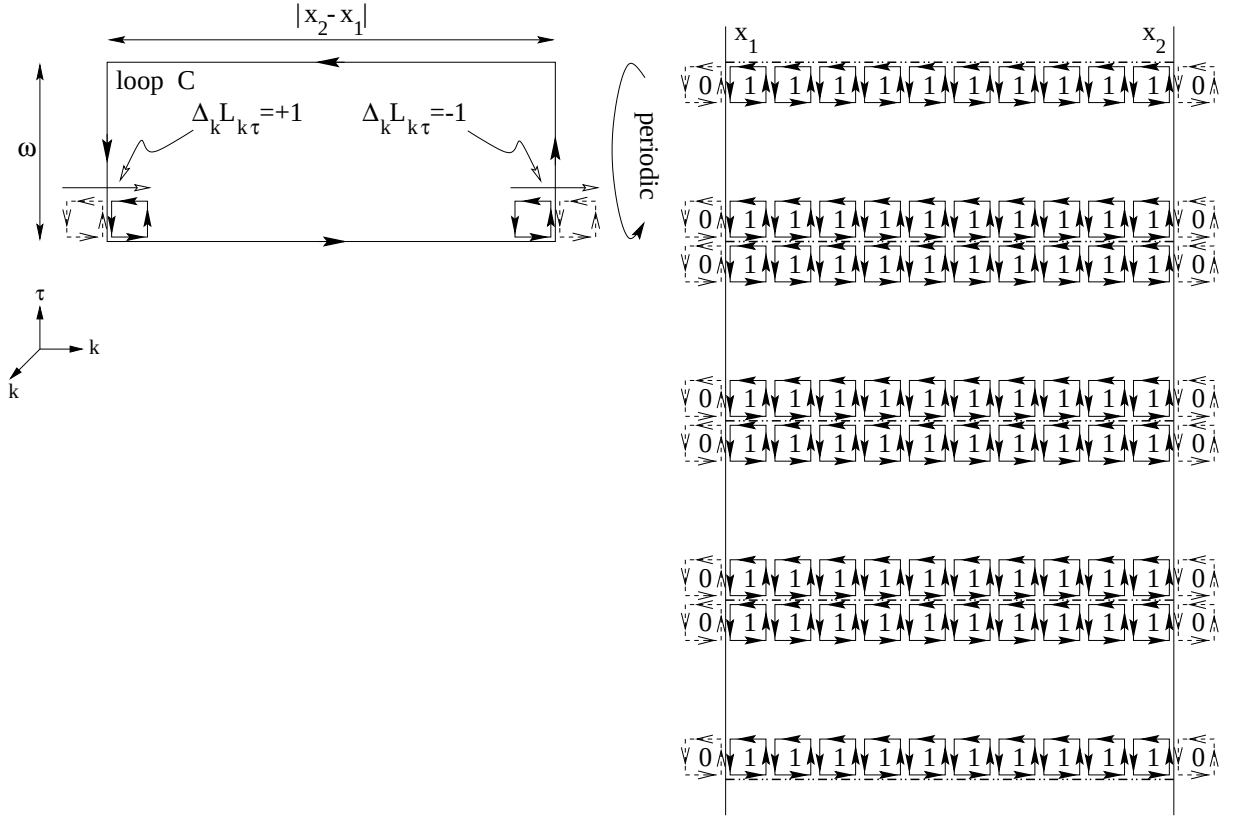
Integer field $G_{\vec{r}} = G_{(na_\tau, \vec{x})}$ carries the compact (or angular) character of gauge field $A_{\mu, (na_\tau, \vec{x})}$; in the frame of an electromagnetic theory, $G_{(na_\tau, \vec{x})}$ describes a magnetic monopole field, whilst in the dislocation language, it characterizes the density of sites at which sessile dislocations end. The propagator $D_{\vec{r}, \vec{r}'}$ in a 3-dimensional (non-periodic) lattice is given by (C.270) $\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}$ with $D(0) \approx 0.253$.

Now let us study in a periodic lattice along direction τ the behaviour of correlation function $W(C)$ for a loop C with two of its sides extending along the *complete* lattice's length ω in the periodic τ -direction. The reason to study such a loop is the very lattice's periodicity: let us suppose that C stretches along the τ -direction but without covering the complete ω . For the behaviour of $W(C)$, the number of topological defects which cross C is decisive. In a non-periodic lattice, a topological defect which crosses C cannot be deformed in order to avoid this crossing; but for a lattice periodic along one dimension this does not longer hold: let us consider the example of the figure, with a topological defect crossing loop C .



Due to the lattice's periodicity, the same defect can be deformed so that it “appears” again outside the loop; one can further deform the topological defect, maintaining always the crossing point inside C and thus keeping the contribution to $W(C)$, until the loop's part just inside C touches its part just outside C , becoming thus two topological lines which do not cross C . Therefore, the contributions to $W(C)$ can be deformed to the identity and are not stable: $W(C) = 1$. However, if one of the sides of C extends over

the complete ω , the just discussed deformation cannot take place and any topological defect which crossed C cannot escape. Of course, one could consider loops C which were located on the $(d-1)$ -dim. sublattice normal to the periodic direction, but this would mean the study of the $(d-1)$ -dim. model in a non-periodic lattice, which has been already considered in Appendix C. From here on, we shall restrict ourselves to loops C whose projection on a plane containing the τ -direction stretches over the whole lattice's length in the periodic dimension. Antisymmetric field $L_{\mu\nu, \vec{r}}$ which defines the contour of path C through relation $\sum_{\mu} \Delta_{\mu} L_{\mu\nu, \vec{r}-\hat{\mu}} = -J_{\nu, \vec{r}}$ (with $J_{\nu, (na_{\tau}, \vec{x})}$ nonzero for those bonds located on C) is given as follows: $L_{\mu\nu, (na_{\tau}, \vec{x})} = \pm 1$ if the square plaquette given by the points $(na_{\tau}, \vec{x}) \rightarrow (na_{\tau}, \vec{x} + \hat{k}) \rightarrow ((n+1)a_{\tau}, \vec{x} + \hat{k})$ is placed inside C and its direction is equal/opposite to the direction of C ; and $L_{\mu\nu, (na_{\tau}, \vec{x})} = 0$ otherwise. A nonvanishing $L_{\mu\nu, (na_{\tau}, \vec{x})}$ -value is represented by a solid square plaquette in the next figure, whilst $L_{\mu\nu, (na_{\tau}, \vec{x})} = 0$ is represented by a dashed plaquette:



$$W(C) = \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu=1}^{d=3} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'} \right] \times \quad (\text{D.136})$$

$$\times \frac{\sum_{\{G_{\vec{r}}\}} \exp \left[-i2\pi \sum_{\vec{r}'} \sum_{\vec{r}, \lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \left(\frac{\Delta_{\vec{r}}}{\vec{r}} \right) L_{\mu\nu, \vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'} - \frac{2\pi^2 D(0)}{ae^2\eta} \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'} \right]}{\sum_{\{G_{\vec{r}}\}} \exp \left[-\frac{2\pi^2 D(0)}{ae^2\eta} \sum_{\vec{r}} G_{\vec{r}}^2 - \frac{\pi}{2ae^2\eta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'} \right]}.$$

It is important to notice that since the closed loop which defines $W(C)$ contains completely lattice's size in the τ -direction and along this direction the lattice is periodic, then the evaluation of $W(C)$ comprises the calculation of the expectation value an infinite set of plaquettes $L_{\mu\nu,(na_\tau,\vec{x})} \neq 0$ extending along the periodic dimension τ . Thus the (discrete) curl of field $L_{\mu\nu,\vec{r}}$ can be written as follows

$$\sum_{\lambda=1}^{d=3} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda L_{\mu\nu,\vec{r}} = \left(\sum_{j,k=1}^{d-1=2} \epsilon_{jk} \Delta_j L_{k\tau,\vec{r}} \right) \times \text{number of sites along the } \tau\text{-direction},$$

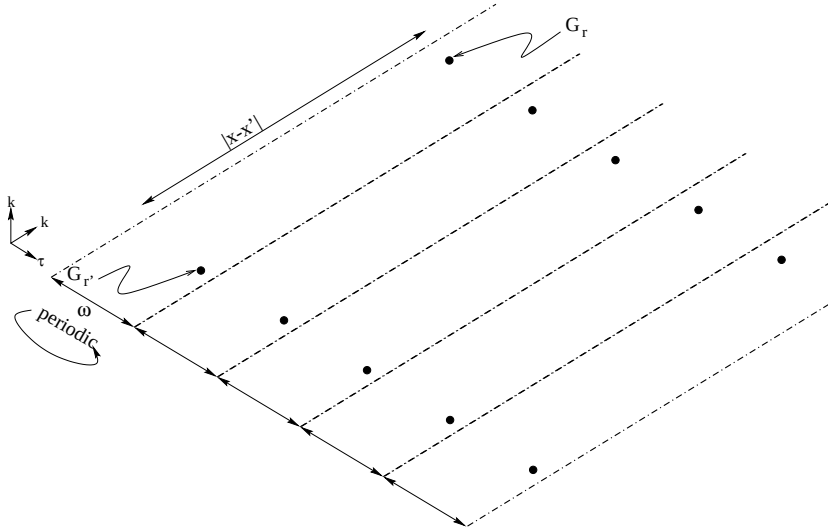
$$\sum_{\lambda=1}^3 \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_\lambda L_{\mu\nu,\vec{r}} = \frac{\omega}{a} \sum_{j,k=1}^2 \epsilon_{jk} \Delta_j L'_{k,\vec{x}}, \quad (\text{D.137})$$

where field $L'_{k,\vec{x}}$ is equal to one for any path contained in the surface enclosed by C and which goes from the line at site $\vec{x}_1$ to the line at site $\vec{x}_2$; $L'_{k,\vec{x}} = 0$ otherwise. Such a field fulfils the following equation ²⁸

$$\sum_{k=1}^{(d-1)} \Delta_k L'_{k,\vec{x}-\hat{k}} = \delta_{\vec{x},\vec{x}_1} - \delta_{\vec{x},\vec{x}_2}. \quad (\text{D.138})$$

This result holds in any lattice periodic along one dimension and for $d \geq 3$ (by writing the corresponding full antisymmetrical ϵ -tensors), and it is just the origin of the similarity between the pure d -dimensional compact gauge model and the $(d-1)$ -dimensional XY model.

Let us focus our attention on term $\sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} G_{\vec{r}'}$:



due to the lattice's periodicity, the interaction is not only between the topological defect located at $\vec{r}$ with that at $\vec{r}'$, but also with all those “images” originated in the periodic repetition of the lattice along the τ -direction.

If $a|\vec{x} - \vec{x}'| \gg \omega$, one can approximate the interaction in d dimensions between $G_{\vec{r}}$ and $G_{\vec{r}'}$ (and their corresponding images) as the interaction in a $(d-1)$ space between the

²⁸As one can easily see from the above figure by taking the d th-dimension as the τ -dimension: let us suppose that loop C is located in the 23-plane with $\tau = \hat{3}$. Then $\sum_{\mu=1}^{d=3} \Delta_\mu L_{\mu\nu,\vec{r}-\hat{\mu}} = \Delta_2 L_{2\tau,(na_\tau,\vec{x})}$, which is equal to $+1$ at the left loop's border or to -1 at the right border.

line at $\vec{x}$, formed by $G_{(na_\tau, \vec{x})}$ and its periodic images, with the line at $\vec{x}'$. The propagator $D_{\vec{x}, \vec{x}'}^{(d-1)}$ for such an interaction can be easily obtained by using an electrostatic picture: for a 3-dimensional space, $D_{\vec{r}, \vec{r}'}^{(d=3)} = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}$ (coordinate vector $\vec{r}$ has been defined dimensionless) is the electrostatic potential created by a point charge located at $\vec{r}$ at site $\vec{r}'$. If many point charges form a continuous charge distribution along one of the space directions, the corresponding electrostatic potential is $-\frac{\lambda}{2\pi} \ln \rho$ where $\rho = |\vec{x} - \vec{x}'|$ is the distance normal to the linear charge distribution (i.e., through a $(d-1)$ -dimensional space) and λ is the linear charge density. Therefore, the propagator of the actual interaction between $G_{\vec{r}}$ and $G_{\vec{r}'}$ in a periodic lattice is equal to $\lambda D_{\vec{x}, \vec{x}'}^{(d=2)}$; in our case, λ is given by the inverse of the spacing along the τ -direction between the images of each topological defect: $\lambda = \frac{a}{\omega}$, where lattice spacing a has been included in order to continue working with dimensionless coordinates. If $\omega \ll a|\vec{x} - \vec{x}'|$, it holds in general for any dimension $d \geq 3$

$$D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}, \quad (\text{D.139})$$

where $\vec{r}$ denotes the site coordinates in the whole lattice and $\vec{x}$ in the subspace normal to the periodic direction τ , and thus

$$\sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} G_{\vec{r}'} = D^{(d=3)}(0) \sum_{\vec{r}} G_{\vec{r}}^2 + \sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} G_{\vec{r}'},$$

with $\sum_{\vec{r}} G_{\vec{r}}^2 = \sum_{\vec{x}} G_{\vec{x}}^2$ and

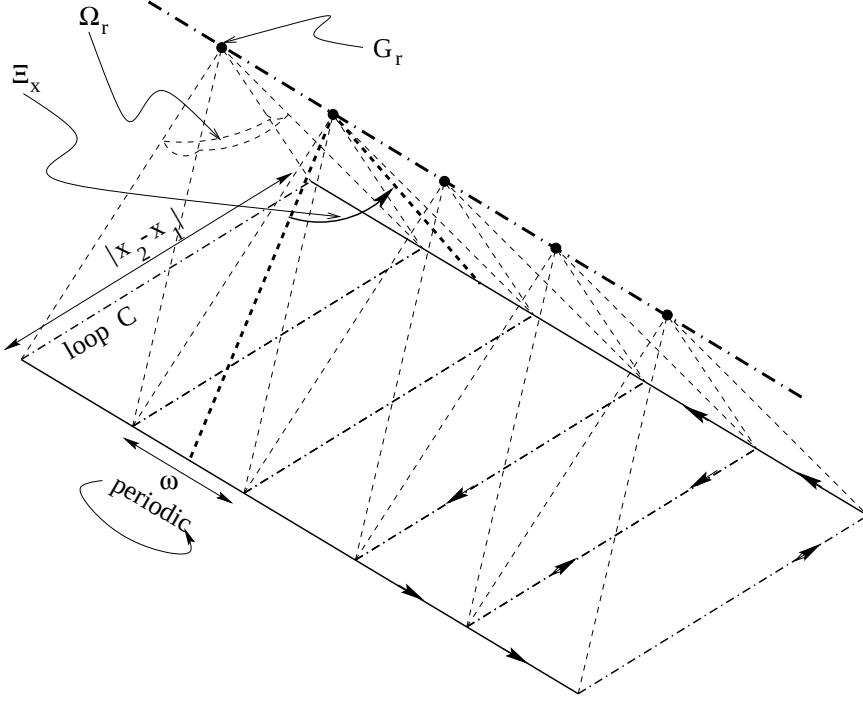
$$\sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} G_{\vec{r}'} \approx \frac{a}{\omega} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} G_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(d=2)} G_{\vec{x}'} = -\frac{a}{\omega} \frac{1}{2\pi} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} G_{\vec{x}} \ln |\vec{x} - \vec{x}'| G_{\vec{x}'}. \quad (\text{D.140})$$

If the relation between ω and the temperature is of the kind $\omega \sim T^{-1}$, then a small ω corresponds to the high temperature range. It is worth noting that approximation $\sum_{\vec{r}, \vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} G_{\vec{r}'} \approx \frac{a}{\omega} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} G_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(d=2)} G_{\vec{x}'}$ holds as long as the separation between the topological defects is small compared with the lattice spacing along the periodic direction: this is achieved either when, independent of ω , the density of topological fields is high, which by regarding selfenergy term $\frac{2\pi^2}{ae^2\eta} D^{(d=3)} \sum_{\vec{x}} G_{\vec{x}}^2$ occurs at large e^2 ; or, if the density of topological defects is small, when ω is small.

The same process applies to the following $W(C)$'s term

$$\sum_{\vec{r}'} \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} G_{\vec{r}'} \equiv \frac{1}{4\pi} \sum_{\vec{r}'} \Omega_{\vec{r}'} G_{\vec{r}'},$$

where $\Omega_{\vec{r}'} = 4\pi \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{(\vec{r})}^\lambda L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}$ is the solid angle under which the whole area



enclosed by C is seen from site $\vec{r}'$. Due to the lattice's periodicity along the τ -direction (assuming $\omega \ll a|\vec{x}_2 - \vec{x}_1|$) and to relations (D.137), (D.138), the term under study can be rewritten as the plane angle $\Xi_{\vec{x}}$ under which the line of continuously distributed $G_{(na\tau, \vec{x})}$ sees distance $|\vec{x}_2 - \vec{x}_1|$. According to (C.121), this leads to the following result

$$\begin{aligned} \frac{1}{4\pi} \sum_{\vec{r}'} \Omega_{\vec{r}'} G_{\vec{r}'} \Big|_{\text{periodic}} &= \sum_{\vec{r}'} \sum_{\vec{r}, \lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{(\vec{r})}^{\lambda} L_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(d=3)} G_{\vec{r}'} \approx \\ &\approx \frac{\omega}{a} \frac{a}{\omega} \sum_{\vec{x}', \vec{x}} \sum_{j,k=1}^{d-1=2} \epsilon_{jk} \Delta_j L'_{k, \vec{x}} D_{\vec{x}, \vec{x}'}^{(d=2)} G_{\vec{x}'} = -\frac{1}{2\pi} \sum_{\vec{x}'} \Xi_{\vec{x}'} G_{\vec{x}'} . \end{aligned} \quad (\text{D.141})$$

Results (D.140) and (D.141) substituted into correlation function (D.136) implies that the 3-dim. pure compact gauge model for small e^2 in a lattice periodic along one dimension is completely similar to the 2-dim. XY model (C.127) with change $\left(\frac{1}{e^2 \omega} \right)_{\text{gauge}} \rightarrow |\psi|_{XY}^2$, as already seen in equation (D.112) for the range of large e^2 . Also $\sum_{\vec{r}, \vec{r}'} \sum_{\mu=1}^{d=3} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} J_{\mu, \vec{r}'}$ can be reformulated as $2 \frac{\omega}{a} D_{\vec{x}_1, \vec{x}_2}^{(d=2)}$: in the interaction between parallel loop elements contained into $\sum_{\vec{r}, \vec{r}'} \sum_{\mu=1}^{d=3} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} J_{\mu, \vec{r}'}$, those arising from the sides along the distance $|\vec{x}_2 - \vec{x}_1|$ get compensated with each other since these sides are covered twice in opposite directions. It only remains the interaction between the two lines running along the τ -direction in opposite directions and separated by the distance normal to these lines, and this produces the following contribution

$$\begin{aligned} \sum_{\vec{r}, \vec{r}'} \sum_{\mu=1}^{d=3} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} J_{\mu, \vec{r}'} &\approx - \sum_{\vec{x}, \vec{x}' \neq \vec{x}} \frac{\omega}{a} (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d=2)} \frac{\omega}{a} (\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2}) = \\ &= -\frac{\omega}{a} \frac{1}{\pi} \ln |\vec{x}_2 - \vec{x}_1| , \end{aligned} \quad (\text{D.142})$$

and therefore

$$W(C)_{(d=3)-\text{dim. period. comp. gauge}} = \left\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d=1=2)-\text{dim. non-per. } XY} \stackrel{\text{eq. (D.114)}}{=} \exp\left(-\frac{\omega}{\eta} \mathcal{V}_{\vec{x}_1, \vec{x}_2}\right), \quad (\text{D.143})$$

where $\left(\frac{1}{e^2\omega}\right)_{\text{gauge}} = |\psi|_{XY}^2$ and $\vec{x}_1, \vec{x}_2$ the sites where C crosses the subspace normal to the τ -dimension. There are two small differences between $\left\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{\text{non-period. } XY}$ and $W(C)_{\text{period. compact gauge}}$. In the former correlation function, topological field $M_{\vec{r}}$ has to satisfy $\sum_{\vec{r} \in \mathbf{R}^2} M_{\vec{r}} = 0$ due to the divergence of discrete propagator $D_{\vec{r}, \vec{r}' = \vec{r}}^{(d=2)}$ (as different from finite $D_{\vec{r}, \vec{r}' = \vec{r}}^{(d)} \equiv D^{(d)}(0)$ for $d \geq 3$), constraint which is easily read off the following relation: $\sum_{\vec{r}, \vec{r}' \in \mathbf{R}^2} M_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=2)} M_{\vec{r}'} = \sum_{\vec{r}, \vec{r}' \neq \vec{r}} M_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(d=2)} M_{\vec{r}'} + D_{\vec{r}, \vec{r}' = \vec{r}}^{(d=2)} \left(\sum_{\vec{r}} M_{\vec{r}}\right)^2$. For the case of $W(C)_{\text{period. compact gauge}}$ such a divergence in the propagator does not occur, since the substitution of $D_{\vec{r}, \vec{r}'}^{(d=3)}$ by $D_{\vec{x}, \vec{x}'}^{(d=2)}$ in the periodic lattice holds only for distances $a|\vec{x} - \vec{x}'| \gg \omega$; however, at small values of ω (or equivalently, at high temperatures if relation $\omega \sim T^{-1}$ holds) the substitution of the propagators is valid also at short distances and thus the constraint on the topological fields (in the 2-dim. plane normal to the periodic direction) is satisfied approximately: $\sum_{\vec{x}} G_{\vec{x}} \approx 0$. As already mentioned, for large e^2 the density of $G_{\vec{x}}$'s is high and the small separation between them already satisfies $a|\vec{x} - \vec{x}'| \gg \omega$ for almost any ω . The second difference is related to the constant term determining the selfenergy of the topological fields in $W(C)_{\text{period. compact gauge}}$

$$\exp\left[-\frac{e^2\omega\eta}{2\pi} \ln|\vec{x}_2 - \vec{x}_1|\right] \sum_{\{G_{\vec{x}}\}} \exp\left[i \sum_{\vec{x}'} \Xi_{\vec{x}'} G_{\vec{x}'} - \overbrace{\frac{2\pi^2 D(0)}{ae^2\eta} \sum_{\vec{x}} G_{\vec{x}}^2}^{\downarrow} - \frac{\pi}{e^2\omega\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} G_{\vec{x}} \ln|\vec{x} - \vec{x}'| G_{\vec{x}'}\right]$$

$$\prod_{\vec{x}} \sum_{G_{\vec{x}} = -\infty}^{\infty} \exp\left[i \sum_{\vec{x}'} \Xi_{\vec{x}'} G_{\vec{x}'} - \underbrace{\frac{2\pi^2 D(0)^{(d=3)}\omega}{ae^2\omega\eta} \sum_{\vec{x}} G_{\vec{x}}^2}_{\uparrow} - \frac{\pi}{e^2\omega\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} G_{\vec{x}} \ln|\vec{x} - \vec{x}'| G_{\vec{x}'}\right] \quad (\text{D.144})$$

whereas in the XY correlation function, with $\left(\frac{1}{e^2\omega}\right)_{\text{gauge}} = |\psi|_{XY}^2$, the term indicated by an arrow is equal to $\frac{\pi^2 |\psi|_{XY}^2}{2\eta} \sum_{\vec{x}} M_{\vec{x}}^2$.

The complete discussion leading to equation (D.143) can be also repeated (with minor changes) for the 4-dim. pure compact gauge model, obtaining for *both* large and small e^2 the equivalence between compact gauge model in a 4-dim. lattice periodic along one direction at small ω and the XY model in a 3-dim. lattice. As a result of the previous discussion, the schematic phase diagram on page 553 holds also for the horizontal strip at small ω and any e^2 , strip where there is no transition different from that studied at large e^2 . In this diagram, only the sector of small e^2 and large ω remains to be considered: in such a region the behaviour of the correlation function introduced in Appendix C (and called the 't-Hooft loop) (C.485) is studied. This will be dealt with

later in equation (D.227), but one can already advance that in such a sector no phase transition will occur at $d = 3, 4$. The reason is easy: at small e^2 , Hamiltonian operator $\frac{a^{3-d}e^2}{2} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k,\vec{x}}^2 + \frac{a^{d-5}}{2e^2} \sum_{\vec{x} \in \Lambda_\perp} \sum_{j,k=1}^{d-1} [1 - \cos(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}})]$ can be approximated by the second term and thus the partition function is given by

$$\mathcal{Z}_{\text{small } e^2} = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}}_{\text{small } e^2} \right) \right] = \int_{-\pi}^{\pi} \left(\prod_{\vec{x} \in (d-1)\text{-dim } \Lambda_\perp} \prod_{k=1}^{d-1} \frac{dA_{k,\vec{x}}}{2\pi} \right) \times \quad (\text{D.145})$$

$$\times \exp \left[-\frac{\omega a^{d-5}}{2e^2 \eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{j,k=1}^{d-1} \left(1 - \cos(A_{j,\vec{x}} + A_{k,\vec{x}+\hat{j}} - A_{j,\vec{x}+\hat{k}} - A_{k,\vec{x}}) \right) \right],$$

which is the partition function of the non-periodic $(d-1)$ -dimensional compact $U(1)$ gauge model with an equivalent coupling constant of $\left(\frac{1}{a^{4-(d-1)}e^2\eta} \right)_{\text{non-period.}} = \frac{\omega}{a} \frac{1}{a^{4-d}e^2\eta}$. This last model at $d-1 = 2$ and $d-1 = 3$ shows no phase transition as we already know from Appendix C: the 2-dim. non-periodic $U(1)$ gauge model, independently of the compact character of the gauge field (see equation (C.320)), as well as the 3-dim. compact (non-periodic) $U(1)$ gauge model (result (C.314)) shows only one phase characterized by the area decay of correlation function $W(C)$ calculated for those models. Hence, in the pure compact $U(1)$ gauge model at $d = 3$ and $d = 4$ when the lattice is periodic along one dimension, except for the phase transition line proven at large e^2 and represented on page 553 another line of transition points does not appear: the diagrams on this page hold for any e^2 and the line separates the phase depicted in the shaded region (large $\omega e^2 \eta$), which was called the “confining” phase and is characterized by $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ for both $d = 3$ and $d = 4$, and a phase outside this region (small $\omega e^2 \eta$) with $W(C) \sim e^{-\ln |\vec{x}_2 - \vec{x}_1|}$ for $d = 3$ or $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|^{-1}}$ for $d = 4$. The transition from one phase to the other occurs when ω decreases, or assuming the relation between ω and T as $\frac{\omega}{\eta} = \frac{\lambda_{\text{scale}}}{k_B T}$, when the temperature is increased ²⁹. This transition is of second-order type for $d = 4$ or a Kosterlitz-Thouless transition for the pure compact gauge model in a 3-dim. lattice periodic along one direction.

The Kosterlitz-Thouless transition has been already qualitatively discussed as the 2-dim. XY model was studied. Let us return to the point at which this discussion was

²⁹A quick form to obtain the behaviour of $W(C)$ for the gauge model at small ω (or high temperatures) is to take the relations calculated in the previous Appendix C for a non-periodic lattice: although in the periodic lattice one has assumed that the loop’s side along the τ -direction fulfils $\omega \ll |\vec{x}_2 - \vec{x}_1| \equiv R$, since the segment ω gets repeated along its direction, the resulting length of the loop along this periodic length still fulfils $\tau \gg R$ and one can use the results for $W(C)$. Hence the results for $W(C)$ in the model at $\omega = \infty$ equal to: $W(C) \sim e^{-\text{Area}_C} = e^{-\tau R}$, $W(C) \sim e^{-\tau \ln R}$, $W(C) \sim e^{\tau/R}$, produces at high temperatures (small ω) the following behaviours: $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|/T}$, $W(C) \sim e^{-T^{-1} \ln |\vec{x}_2 - \vec{x}_1|}$, $W(C) \sim e^{(T|\vec{x}_2 - \vec{x}_1|)^{-1}}$, since the total loop’s length along the τ -direction is proportional to $\omega \sim T^{-1}$. To behaviour $W(C) \sim e^{-\text{Perimeter}_C}$ at $\omega = \infty$ corresponds $W(C) \sim \text{constant}$ at small ω , but this will be proved in the Note on page 630.

stopped in order to see in detail the characteristics of such a transition: then we arrived at the following relation ³⁰

$$\left(\frac{\eta}{|\psi|^2}\right)_{\text{with small topolog. defects}} = \frac{\eta}{|\psi|^2} + 4\pi^3 \exp\left(-\frac{\pi^2|\psi|^2}{\eta}\right) \int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a}\right)^{3-\frac{2\pi|\psi|^2}{\eta}}, \quad (\text{D.146})$$

where in the pure gauge model $\frac{\eta}{|\psi|^2}$ corresponds to $e^2\omega\eta$ and ρ , which was equal the size of a small dislocation pair $M_{\vec{r}} - M_{\vec{r}'}$ in the 2-dim. XY model, corresponds to the distance (measured in the non-periodic $(3-1)$ -dim. subspace) between the line formed by a monopole $G_{(na_{\vec{r}}, \vec{x})}$ (and its periodic images) and a near monopole of opposite charge. As it was discussed after equation (C.153) in the XY model, the effect of small pairs of topological defects is to *reduce* slightly the coupling strength $\frac{|\psi|^2}{\eta}$; in analogy with a dielectricum, where small electric dipoles screen the effect of an exterior electric field, this effect is called the (*Debye*) *screening*. Factor $e^{-\frac{\pi^2|\psi|^2}{\eta}}$ arises from the term containing the selfenergy of dislocation fields $M_{\vec{r}}$'s, $\frac{\pi^2|\psi|^2}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2$, which controls the density of topological defects; the term which contains the interaction between different defects causes the behaviour stated in equation (D.146). As it has been seen in the previous Appendix on phase transitions, by approaching the critical transition point from below, i.e., from the phase where the correlation function of the order parameter shows a weaker decay at large distances than above the critical point, a high density of large topological defects appears, and the interactions among these over large distances determine the behaviour of the model. An useful tool to study this long-range effects is the so-called renormalization equations. Before going into their details, let us obtain an intuitive picture of what these equations are stating. Since the large range is determining in the neighbourhood of a critical point, let us imagine we change from observing the system with a high resolution magnifying glass to watching through a magnifying glass of lower resolution. Hence we lose details in the short-range scale and only keep those characteristics whose range is larger than the loss in magnifying power. Stated mathematically, one integrates out the interactions among the fields describing the model at very short distances, for instance between the lattice spacing a and some lattice spacings sa (with $\ln s \ll 1$); after this process, (dimensionless) distances as $\rho = |\vec{r} - \vec{r}'|$ have to be rescaled $\rho \rightarrow \rho' = \rho/s$ in order to obtain the less magnified picture of the system. If we want now to study the resulting system, the energy which describes it will not in general have the same form as the original energy: the latter contained interactions between fields separated by only one lattice spacing whereas the new energy contains interactions between fields which were separated by several lattice spacings in the original system. Nevertheless, and this is the main idea in the renormalization equations, if the phase transitions which we are considering show *scale invariance* (as a second order phase transition and one of Kosterlitz-Thouless kind do), then in the neighbourhood of

³⁰Where in the 3-dim. periodic pure gauge model the selfenergy term $\frac{\pi^2|\psi|^2}{\eta}$ has to be changed by $4\pi^2 D^{(d=3)}(0) \frac{\omega}{a e^2 \omega \eta}$ as just discussed.

such a transition the only relevant energy terms at large distances are the same as the energy terms which describes the interactions at the very short-range scale. The only change which occurs in such cases from considering the magnified picture to studying the original system's picture can be a variation in the (external) parameters contained in the energy: for instance, let us imagine we start at a close picture of the system at a value $|\psi|^2/\eta$ where the only possible defects are pairs formed by topological defects of opposite charge separated by a couple of lattice spacings. If we decrease the magnification power so that only distances greater than sa can be distinguished, these small pairs disappear and the picture corresponds to the original system but with a larger selfenergy for the defect's pairs [88] which makes less favourable their existence.

It has to be noticed that by taking these coarser pictures, we are not changing any parameter of the model, all pictures represent the same state although the lesser magnified pictures are similar to the look of the original close picture at a different value of such parameters. By studying the evolution of these parameters as we goes to more "defocussed" pictures we can determine the behaviour at large distances of the original model (which only contains interactions among nearest neighbours). Now we can start with the quantitative discussion of the renormalization equations for the Kosterlitz-Thouless transition in the 2-dim. XY model. Let us extend a bit the 2-dim. XY model, by allowing for a adjustable parameter ζ which controls the density of the topological fields through the selfenergy term: the defect's energy part $\frac{\pi^2}{2} \frac{|\psi|^2}{\eta} \sum_{\vec{x} \in \mathbf{R}^2} M_{\vec{x}}^2 + 2\pi^2 \frac{|\psi|^2}{\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} M_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(2)} M_{\vec{x}'}$ is generalized into

$$(-\ln \zeta) \sum_{\vec{x} \in \mathbf{R}^2} M_{\vec{x}}^2 + 2\pi^2 \frac{|\psi|^2}{\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} M_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(2)} M_{\vec{x}'}, \quad \zeta = \zeta_0 e^{-\frac{\pi^2 |\psi|^2}{2\eta}}, \quad (\text{D.147})$$

with ζ_0 is equal to 1 for the XY model without modification. Limit $\zeta \rightarrow 0$ forces $M_{\vec{x}} = 0 \forall \vec{x} \in \mathbf{R}^2$ in the partition function. From the just discussed example, it is clear that at large values of $|\psi|^2/\eta$ for which only small dipoles of topological defects are possible, the process of defocussing results in the limit $\zeta \rightarrow 0$. On the other hand, starting at a small $|\psi|^2/\eta$ where there are many topological defects not bound into small dipoles, the loss in magnifying power results in an larger effective density of these large defects by eliminating the small dipoles to which the isolated topological defects could be linked, or equivalently, pushing ζ toward large values.

Returning to equation (D.146) together with modification (D.147)

$$\left(\frac{\eta}{|\psi|^2} \right)_{\text{with small top. defects}} = \frac{\eta}{|\psi|^2} + 4\pi^3 \zeta^2 \int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi |\psi|^2}{\eta}}, \quad (\text{D.148})$$

let us now carry out quantitatively the above sketched discussion by integrating out distances between lattice spacing a and sa with $\ln s \ll 1$ (and thus $s^k \approx 1 + k \ln s$)

$$\int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi |\psi|^2}{\eta}} = \int_a^{sa} \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi |\psi|^2}{\eta}} + \int_{sa}^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi |\psi|^2}{\eta}}.$$

Hence one obtains

$$\begin{aligned} \left(\frac{\eta}{|\psi|^2} \right)_{\text{screened}} &= \left(\frac{\eta}{|\psi|^2} + 4\pi^3 \zeta^2 \ln s \right) + 4\pi^3 \zeta^2 \underbrace{s^{4 - \frac{2\pi|\psi|^2}{\eta}}}_{1 + \left(4 - \frac{2\pi|\psi|^2}{\eta}\right) \ln s} \int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi|\psi|^2}{\eta}} \equiv \\ &\equiv \left(\frac{\eta}{|\psi|^2} \right)_{\text{defocussed}} + 4\pi^3 \zeta_{\text{defocussed}}^2 \int_a^\infty \frac{d\rho}{a} \left(\frac{\rho}{a} \right)^{3 - \frac{2\pi|\psi|^2}{\eta}}, \end{aligned}$$

where one has rescaled $\rho \rightarrow \rho/s$ in the integral. From this last equation the evolution of the model's parameters $|\psi|^2/\eta$ and ζ during the defocussing process is directly obtained

$$\begin{cases} \frac{d \left(\frac{\eta}{|\psi|^2} \right)}{d \ln s} = 4\pi^3 \zeta^2, \\ \frac{d \zeta^2}{d \ln s} = 2 \left(2 - \frac{\pi|\psi|^2}{\eta} \right) \zeta^2, \end{cases} \quad (\text{D.149})$$

It is important to note that although the second renormalization equation could be already expected from the above discussion on the effect of coarse graining on the density of topological defects, the first equation is a specific result of the 2-dim. XY model which states the screening produced by small defect dipoles on the coupling strength.

Looking at the second equation in (D.149) one obtains that value $\frac{|\psi|^2}{\eta} = \frac{2}{\pi}$ separates the $|\psi|^2/\eta$ -values for which ζ^2 increases with the defocussing process (i.e. $\frac{d\zeta^2}{d \ln s} > 0$) from those values at which ζ^2 goes to zero. Let us study the neighbourhood of $|\psi|^2/\eta = 2/\pi$, where the first renormalization equation can be approximated by

$$\frac{d \left(\frac{|\psi|^2}{\eta} \right)}{d \ln s} = - \left(\frac{|\psi|^2}{\eta} \right)^2 \frac{d \left(\frac{\eta}{|\psi|^2} \right)}{d \ln s} \approx -16\pi^2 \zeta^2. \quad (\text{D.150})$$

The solutions of these equations near to $|\psi|^2/\eta = 2/\pi$ are the hiperbolic trajectories given by

$$\left(2 - \frac{\pi|\psi|^2}{\eta} \right)^2 - 16\pi^2 \zeta^2 = \text{const.}, \quad (\text{D.151})$$

and represented in Fig. D.1. All points on the same hiperbola gives the same physical state whilst the direction points toward the parameter region to which the system goes during the coarse graining process. If we start at any point of one of the hiperbolas under the line $2 - \frac{\pi|\psi|^2}{\eta} = -4\pi\zeta$, the defocussing flows toward the values $\frac{|\psi|^2}{\eta} > \frac{2}{\pi}$ and $\zeta = 0$, i.e, toward the state with no topological defects at all. Therefore, for $\frac{|\psi|^2}{\eta} > \frac{2}{\pi} + 4\zeta$ the topological defects play no relevant role and the correlation function can be calculated

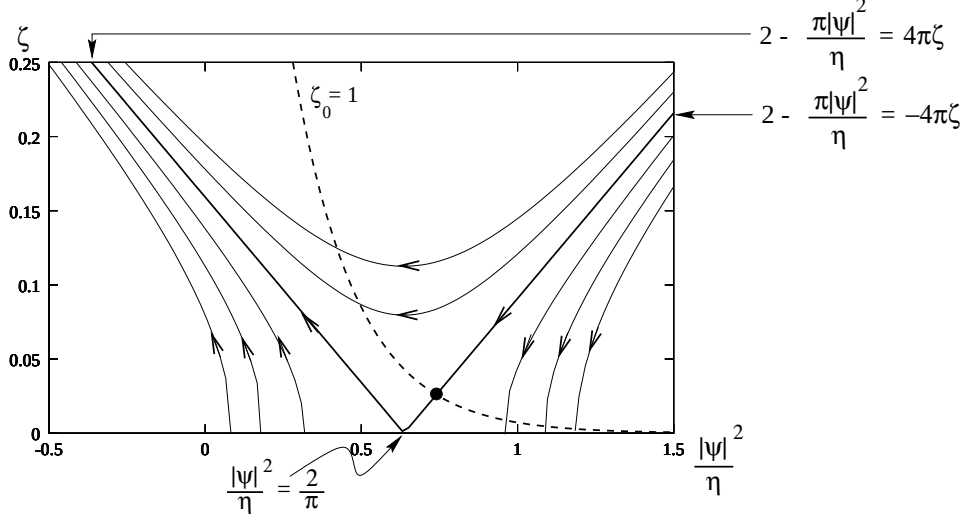


Figure D.1: Trajectories followed by the parameters of the 2-dim. XY model during the process of coarse graining (or defocussing).

without taking into account the compact character of field $\theta_{\vec{x}}$ in the XY model; this was already seen since correlation function $\langle e^{i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2})} \rangle_{\text{large } \frac{|\psi|^2}{\eta}} \approx \exp\left(-\frac{\eta}{2\pi|\psi|^2} \ln |\vec{r}_2 - \vec{r}_1|\right)$ can be directly obtained by approximating energy's term $\frac{|\psi|^2}{\eta} \sum_{\vec{x} \in \mathbf{R}^2} \sum_{\mu=1,2} [1 - \cos \Delta_\mu \theta_{\vec{x}}]$ by $\frac{|\psi|^2}{2\eta} \sum_{\vec{x} \in \mathbf{R}^2} \sum_{\mu=1,2} [\Delta_\mu \theta_{\vec{x}}]^2$ and thus $\theta_{\vec{x}}$ loses its angular character. On the contrary, all those trajectories above and on the left of the line $2 - \frac{\pi|\psi|^2}{\eta} = -4\pi\zeta$ flow toward the region with $\frac{|\psi|^2}{\eta} < \frac{2}{\pi}$ and $\zeta \rightarrow \infty$, and this means toward a large-range behaviour dominated by many isolated topological defects. Therefore for $\frac{|\psi|^2}{\eta} < \frac{2}{\pi} + 4\zeta$ the compact character leads to a behaviour completely different to that for $\frac{|\psi|^2}{\eta} > \frac{2}{\pi} + 4\zeta$. Hence the critical value of $|\psi|^2/\eta$ is given by

$$\left(\frac{|\psi|^2}{\eta}\right)_{\text{crit}} = \frac{2}{\pi} + 4\zeta_c, \quad (\text{D.152})$$

for which we have already obtained an estimation in equation (C.154): $\left(\frac{|\psi|^2}{\eta}\right)_{\text{crit}} \approx \frac{2}{\pi}$;

$4\zeta_c = 4\zeta_0 e^{-\frac{\pi^2}{2} \left(\frac{|\psi|^2}{\eta}\right)_{\text{crit}}}$ is a small correction to that estimation. The XY model in the Villain approximation, which has $\zeta_0 = 1$, lies on the dashed line of Fig. D.1 and its critical value is represented by the solid point. This is the general form how the renormalization equations works: one obtains how the model's parameters change by decreasing the magnification power under which the system is observed (this can be done by firstly

obtaining the effect of topological defects on the parameters as in equation (D.2.3) and then by changing the scale from a to sa or by integrating out directly in the energy the distances between a and sa) and then one studies toward which values these parameters flow during the defocussing process.

After this discussion about the Kosterlitz-Thouless transition, which completes that carried out in the previous Appendix, it is time we return to the pure gauge 3-dim. compact gauge model periodic along one dimension, which is completely similar to the 2-dim. XY model by substituting

$$\begin{cases} M_{\vec{x}} & \longleftrightarrow G_{(na_\tau, \vec{x})} + \text{periodic images along the } \tau\text{-direction,} \\ \frac{|\psi|^2}{\eta} & \longleftrightarrow \frac{1}{e^2 \omega \eta}. \end{cases}$$

The critical value is therefore equal to

$$\left(\frac{a}{\omega} \frac{1}{ae^2 \eta} \right)_{\text{crit}} = \frac{2}{\pi} + 4\zeta_0 e^{-\frac{2\pi^2 D^{(d=3)}(0)}{(ae^2 \eta)_{\text{crit}}}}. \quad (\text{D.153})$$

From the above discussion, and assuming the relation $\omega \sim T^{-1}$, one obtains that for large T/e^2 the topological defects are not relevant for the large-range behaviour since they form tightly bound dipoles. But this is the result we should expect for the dislocation model: remembering that the topological defects $G_{(na_\tau, \vec{x})}$ play the role of the stopping points (due to sessile dislocations) for the gliding dislocations, a tightly bound dipole of a negative and a positive charged $G_{(na_\tau, \vec{x})}$ represents a point where mobile dislocations ends, because they have become sessile, located at a near distance of another point where gliding mobile emerges, because they have been liberated again; the length of the sessile dislocation segment is thus equal to the distance between the two $G_{(na_\tau, \vec{x})}$'s. And it is just in the high temperature stage of a dislocation system where the effective length of sessile segments decreases due to cross-slip among different primary slip systems, the latter being activated by the temperature. On the other hand, at low T/e^2 the high density of start points for the sessile dislocations, points not compensated by the opposite charged element — defining thus long segments of sessile dislocations —, produces the rapid hardening and simultaneously, the disorder characterized by the stronger decay with the distance of the correlation function $W(C)$ in the pure gauge model.

D.2.4 The pure $Z(N)$ gauge model at high temperatures

By looking at the already studied phase diagrams at $\omega = \infty$ (or at vanishing temperature) as for instance Fig. C.6 and in order to have the complete the phase diagram for $T \neq 0$, beside the pure compact $U(1)$ gauge model discussed up to this point we need the behaviour of the pure $Z(N)$ gauge model for a lattice periodic along one direction. Due to the equivalence between the pure compact $U(1)$ gauge model with the XY model in a lattice of a lower dimensionality, we can expect a similar analogy between the pure $Z(N)$ gauge model in a d -dim. lattice periodic along one dimension with the $Z(N)$ Ising model in a $(d-1)$ -dim. nonperiodic lattice, at least at large e^2 (since in this range the behaviour of the compact $U(1)$ and of the $Z(N)$ gauge models is very similar). For the

range of large e^2 this is shown in a very similar form as for the $U(1)$ model, simply by using the Hamiltonian formulation (formulation which already implies that the lattice discreteness along the periodic τ -dimension, is not observable when compared with the other nonperiodic dimensions). Since the final step to obtain this equivalence at e^2 large is the Villain formulation of the non-periodic (i.e., $\omega = \infty$) $Z(N)$ Ising model, let us consider this last model in the following Note.

Note. The $Z(N)$ Ising model in the Villain approximation [37].

The $Z(N)$ Ising model is defined by an energy where nearest neighbour scalar fields $e^{i\varphi_{\vec{r}}}$ are coupled to each other. Phase $\varphi_{\vec{r}}$ instead of having a continuous range $\varphi_{\vec{r}} \in [0, 2\pi)$ as in the XY model takes values from a discrete set: $\varphi_{\vec{r}} = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, (N-1)\frac{2\pi}{N} = \left\{ n\frac{2\pi}{N} : n = 0, 1, \dots, (N-1) \right\}$. The partition function of the $Z(N)$ Ising model in a d -dim. non-periodic lattice is thus given in the Villain approximation by the following equation

$$\mathcal{Z} = \prod_{\vec{r}} \sum_{n_{\vec{r}}=0}^{N-1} \prod_{\vec{r}} \prod_{\mu=1, \dots, d} \sum_{m_{\mu, \vec{r}}=-\infty}^{\infty} \exp \left[-\frac{\beta}{2\eta} \sum_{\vec{r}, \mu} \left(\frac{2\pi}{N} \Delta_{\mu} n_{\vec{r}} - 2\pi m_{\mu, \vec{r}} \right)^2 \right], \quad (\text{D.154})$$

where one has sum $\prod_{\vec{r}} \sum_{n_{\vec{r}}=0}^{N-1} \equiv \sum_{\substack{\{n_{\vec{r}}=N-1\} \\ \{n_{\vec{r}}=0\}}}$ instead of integral $\int_0^{2\pi} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi}$ for the XY model; coupling strength β becomes $a^{d-2}|\psi|_{XY}^2$ in the limit $N \rightarrow \infty$, which is the XY model. The discrete-valued phase $n_{\vec{r}}$ can be substituted by a continuous variable $\frac{N}{2\pi}\theta_{\vec{r}}$ with $\theta_{\vec{r}} \in [0, 2\pi)$ by using the Fourier transform of $\sum_{n=0}^{N-1} \delta\left(\theta - \frac{2\pi}{N}n\right)$

$$\sum_{n=0}^{N-1} \delta\left(\theta - \frac{2\pi}{N}n\right) = \frac{1}{2\pi} \sum_{Q=-\infty}^{\infty} e^{iNQ\theta} \int_0^{2\pi} \sum_{n=0}^{N-1} \delta\left(\theta - \frac{2\pi}{N}n\right) e^{-iNQ\theta} d\theta = \frac{N}{2\pi} \sum_{Q=-\infty}^{\infty} e^{iNQ\theta}. \quad (\text{D.155})$$

Hence, the partition function becomes

$$\begin{aligned} \mathcal{Z} &= \int_0^{2\pi} \prod_{\vec{r}} d\theta_{\vec{r}} \sum_{\substack{\{n_{\vec{r}}=N-1\} \\ \{n_{\vec{r}}=0\}}} \prod_{\vec{r}} \delta\left(\theta_{\vec{r}} - \frac{2\pi}{N}n_{\vec{r}}\right) \sum_{\substack{\{m_{\mu, \vec{r}}=\infty\} \\ \{m_{\mu, \vec{r}}=-\infty\}}} \exp \left[-\frac{\beta}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu, \vec{r}})^2 \right] = \\ &= (N)^{|\Lambda|} \int_0^{2\pi} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\substack{\{Q_{\vec{r}}=\infty\} \\ \{Q_{\vec{r}}=-\infty\}}} \sum_{\substack{\{m_{\mu, \vec{r}}=\infty\} \\ \{m_{\mu, \vec{r}}=-\infty\}}} \exp \left[iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} - \frac{\beta}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu, \vec{r}})^2 \right], \quad (\text{D.156}) \end{aligned}$$

with $|\Lambda|$ the number of lattice sites. Scalar-scalar correlation function $\left\langle e^{i\frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle$ takes the form

$$\begin{aligned} \left\langle e^{i\frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle &= \frac{(N)^{|\Lambda|}}{\mathcal{Z}} \int_0^{2\pi} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\{Q_{\vec{r}}\}} \exp \left[i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2}) + iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} \right] \times \\ &\times \sum_{\{m_{\mu, \vec{r}}\}} \exp \left[-\frac{\beta}{2\eta} \sum_{\vec{r}, \mu} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu, \vec{r}})^2 \right], \quad (\text{D.157}) \end{aligned}$$

Starting at this last equation, the calculation follows the same steps as in the XY model: firstly choosing a condition of the sums $\sum_{\{m_{\mu,\vec{r}}\}}$ permits to extend $\int_{-\pi}^{\pi} \prod_{\vec{r}} d\theta_{\vec{r}} \rightarrow \int_{-\infty}^{\infty} \prod_{\vec{r}} d\theta_{\vec{r}}$ as in the comments previous to equation (C.78)

$$\frac{\int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\{m_{\mu,\vec{r}}\}} \sum_{\{Q_{\vec{r}}\}} \exp \left[i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2}) + iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} - \frac{\beta}{2\eta} \sum_{\vec{r},\mu} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right]}{\int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\{m_{\mu,\vec{r}}\}} \sum_{\{Q_{\vec{r}}\}} \exp \left[iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} - \frac{\beta}{2\eta} \sum_{\vec{r},\mu} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu,\vec{r}})^2 \right]}, \quad (\text{D.158})$$

where $\sum_{\{m_{\mu,\vec{r}}\}}$ denotes the constrained sum, although for the following the tilde will not be written more. Then, one introduces an auxiliary field $b_{\mu,\vec{r}}$

$$\begin{aligned} \left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle &= \frac{(N)^{|\Lambda|}}{\mathcal{Z}} \int_{-\infty}^{\infty} \prod_{\vec{r},\mu} \frac{db_{\mu,\vec{r}}}{\sqrt{2\pi\beta/\eta}} \int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\{Q_{\vec{r}}\}} \exp \left[i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2}) + iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} \right] \times \\ &\times \sum_{\{m_{\mu,\vec{r}}\}} \exp \left[-\frac{\eta}{2\beta} \sum_{\vec{r},\mu} b_{\mu,\vec{r}}^2 + i \sum_{\vec{r},\mu} b_{\mu,\vec{r}} (\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu,\vec{r}}) \right], \end{aligned} \quad (\text{D.159})$$

and carries out all the integrations on $d\theta_{\vec{r}}$ which in the numerator produces the following constraint: $-\sum_{\mu} \Delta_{\mu} b_{\mu,\vec{r}-\hat{\mu}} + N Q_{\vec{r}} + (\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}) = 0$, whose solution for a 2-dim.³¹ (non-periodic) lattice is equal to $b_{\mu,\vec{r}} = \sum_{\nu} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{r}-\hat{\nu}} + N R_{\mu,\vec{r}} + L_{\mu,\vec{r}}$; field $L_{\mu,\vec{r}}$ is defined as in equation (C.95), i. e., equal to +1 for any bond on a path connecting site $\vec{r}_1$ with $\vec{r}_2$, and it therefore fulfils $\sum_{\mu} \Delta_{\mu} L_{\mu,\vec{r}-\hat{\mu}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}$.

Integer-valued field $R_{\mu,\vec{r}}$ is similarly defined by the relation $\sum_{\mu} \Delta_{\mu} R_{\mu,\vec{r}-\hat{\mu}} = Q_{\vec{r}}$ and thus does not vanish on the bonds located on paths connecting the different sites where $Q_{\vec{r}} \neq 0$. In the denominator of the partition function $b_{\mu,\vec{r}}$ is given by the same equation but with $L_{\mu,\vec{r}}$ set equal to zero.

Let us follow the study of this model for a 2-dim. (non-periodic) lattice, by substituting the integrations on $db_{\mu,\vec{r}}$ by integrals over $da_{\vec{r}}$ (in the 3-dim. model, one has to fix a condition on $a_{\lambda,\vec{r}}$ in order to eliminate the over-counting of the same configuration due to invariance of the model's energy under a gauge transformation $a_{\lambda,\vec{r}} \rightarrow a_{\lambda,\vec{r}} - \Delta_{\lambda} \chi_{\vec{r}-\hat{\lambda}}$). By noting that $\sum_{\vec{r},\mu} (N R_{\mu,\vec{r}} + L_{\mu,\vec{r}}) m_{\mu,\vec{r}}$ is an integer, and after integrating these $da_{\vec{r}}$'s, one obtains

$$\left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle = \frac{\sum_{\{M_{\vec{r}}\}}' \sum_{\{Q_{\vec{r}}\}}' \exp \left\{ -\frac{\eta}{2\beta} \sum_{\vec{r},\mu} (N R_{\mu,\vec{r}} + L_{\mu,\vec{r}})^2 \right\} \times}{\sum_{\{M_{\vec{r}}\}}' \sum_{\{Q_{\vec{r}}\}}' \exp \left\{ -\frac{N^2 \eta}{2\beta} \sum_{\vec{r},\mu} R_{\mu,\vec{r}}^2 \right\} \times} \quad (\text{D.160})$$

³¹For a 3-dim. (non-periodic) lattice the constraint's solution is given by $b_{\mu,\vec{r}} = \sum_{\nu,\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda,\vec{r}-\hat{\nu}} + N R_{\mu,\vec{r}} + L_{\mu,\vec{r}}$

with integer-valued fields $R_{\mu,\vec{r}}$ and $L_{\mu,\vec{r}}$ as for the 2-dim. lattice.

$$\times \exp \left\{ -\frac{\beta}{2\eta} \sum_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}} - \frac{i\eta}{\beta} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} (N R_{\mu, \vec{r}} + L_{\mu, \vec{r}}) \right] D_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}'} - \frac{i\eta}{\beta} \sum_{\nu', \mu'} \epsilon_{\nu'\mu'} \Delta_{(\vec{r}')}^{\nu'} (N R_{\mu', \vec{r}'} + L_{\mu', \vec{r}'}) \right] \right\} \\ \frac{\times \exp \left\{ -\frac{\beta}{2\eta} \sum_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}} - \frac{iN\eta}{\beta} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} R_{\mu, \vec{r}} \right] D_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}'} - \frac{iN\eta}{\beta} \sum_{\nu', \mu'} \epsilon_{\nu'\mu'} \Delta_{(\vec{r}')}^{\nu'} R_{\mu', \vec{r}'} \right] \right\}}{}$$

where $M_{\vec{r}} = \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{\nu} m_{\mu, \vec{r}}$ is the point-dislocation density (or the point-vortex density). The sum over

$M_{\vec{r}}, \prod_{\vec{r}} \sum_{M_{\vec{r}}=-\infty}^{\infty} \exp \left[-\frac{2\pi^2\beta}{\eta} \sum_{\vec{r}, \vec{r}'} M_{\vec{r}} D_{\vec{r}, \vec{r}'} M_{\vec{r}'} \right]$, is restricted to the condition $\sum_{\vec{r}} M_{\vec{r}} = 0$ due to the divergence of $D(0)$ in two dimensions³². The same constraint must be imposed on $\prod_{\vec{r}} \sum_{Q_{\vec{r}}}$ too, since

term $\exp \left[-\frac{N^2\eta}{2\beta} \sum_{\vec{r}, \vec{r}'} Q_{\vec{r}} D_{\vec{r}, \vec{r}'} Q_{\vec{r}'} \right]$ appears also after operating in equation (D.160): by using result (C.124)

$$\Delta_{(\vec{r})}^{L, \perp} \Delta_{(\vec{r}')}^{L, \perp} D_{\vec{r}, \vec{r}'} + \Delta_{L, \parallel} \Delta_{L, \parallel}' D_{\vec{r}, \vec{r}'} = \sum_{\mu} L_{\mu, \vec{r}}^2 \delta_{\vec{r}, \vec{r}'} \quad \text{with} \quad \begin{cases} \Delta_{L, \perp} &= \sum_{\mu, \nu} L_{\mu, \vec{r}} \epsilon_{\mu\nu} \Delta_{\nu} \\ \Delta_{L, \parallel} &= \sum_{\mu} L_{\mu, \vec{r}} \Delta_{\mu} \end{cases},$$

and a similar result for $\Delta_{R, \parallel} = \sum_{\mu} R_{\mu, \vec{r}} \Delta_{\mu}$ and $\Delta_{R, \perp} = \sum_{\mu, \nu} R_{\mu, \vec{r}} \epsilon_{\mu\nu} \Delta_{\nu}$, one obtains

$$-\frac{\beta}{2\eta} \sum_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}} - \frac{i\eta}{\beta} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{(\vec{r})}^{\nu} (N R_{\mu, \vec{r}} + L_{\mu, \vec{r}}) \right] D_{\vec{r}, \vec{r}'} \left[2\pi M_{\vec{r}'} - \frac{i\eta}{\beta} \sum_{\nu', \mu'} \epsilon_{\nu'\mu'} \Delta_{(\vec{r}')}^{\nu'} (N R_{\mu', \vec{r}'} + L_{\mu', \vec{r}'}) \right] = \\ = -\frac{2\pi^2\beta}{\eta} \sum_{\vec{r}, \vec{r}'} M_{\vec{r}} D_{\vec{r}, \vec{r}'} M_{\vec{r}'} + \frac{\eta}{2\beta} \sum_{\vec{r}, \mu} \underbrace{(N R_{\mu, \vec{r}} + L_{\mu, \vec{r}})^2}_{\text{underlined}} + i2\pi \sum_{\vec{r}, \vec{r}'} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{\nu} (N R_{\mu, \vec{r}} + L_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} M_{\vec{r}'} - \\ - \frac{\eta}{2\beta} \sum_{\vec{r}, \vec{r}'} \underbrace{\left[\sum_{\mu} N R_{\mu, \vec{r}} + L_{\mu, \vec{r}} \right] \Delta_{(\vec{r})}^{\mu} \Delta_{(\vec{r}')}^{\mu'} D_{\vec{r}, \vec{r}'} \left[\sum_{\mu'} N R_{\mu', \vec{r}'} + L_{\mu', \vec{r}'} \right]}_{\text{underlined}}, \\ + \sum_{\vec{r}, \vec{r}'} [N Q_{\vec{r}} + (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})] D_{\vec{r}, \vec{r}'} [N Q_{\vec{r}'} + (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2})]$$

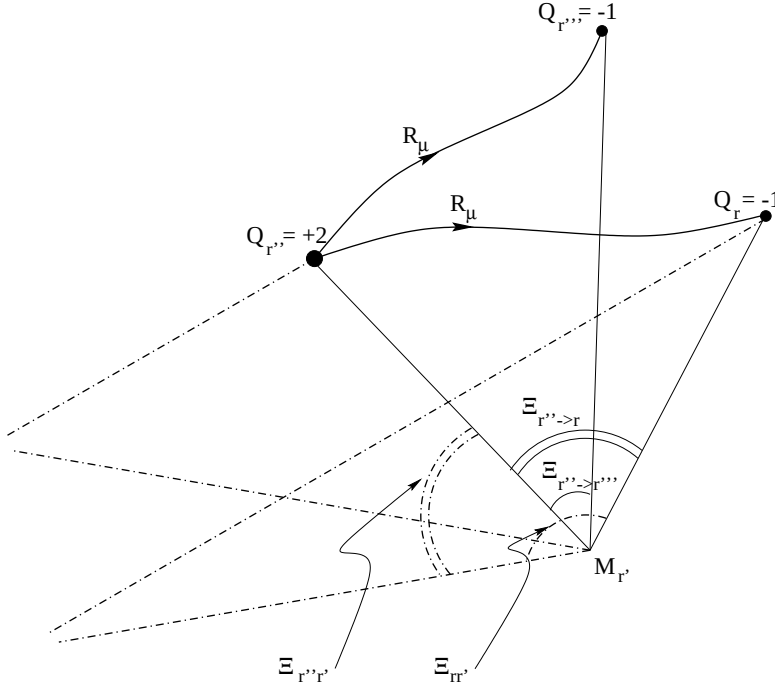
where the underlined term gets compensated by numerator's factor $\exp \left[-\frac{\eta}{2\beta} \sum_{\vec{r}, \mu} (N R_{\mu, \vec{r}} + L_{\mu, \vec{r}})^2 \right]$.

Remembering that $\sum_{\vec{r}'} M_{\vec{r}'} = 0$ together with result (C.121), the third and imaginary summand in the last equation can be rewritten as

$$i2\pi \sum_{\vec{r}, \vec{r}'} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{\nu} (N R_{\mu, \vec{r}} + L_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} M_{\vec{r}'} = iN \sum_{\vec{r}, \vec{r}'} Q_{\vec{r}} \Xi_{\vec{r}, \vec{r}'} M_{\vec{r}'} - i \sum_{\vec{r}'} \Xi_{\vec{r}'} M_{\vec{r}'} , \quad (\text{D.161})$$

³²In the 3-dim. model, such divergence does not occur but fields $M_{\lambda, \vec{r}}$ by definition must fulfil the following relation:
 $\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} = 0$.

where $\Xi_{\vec{r}}$ is the parallax angle of distance $|\vec{r}_1 - \vec{r}_2|$ observed from site $\vec{r}'$, and $\Xi_{\vec{r},\vec{r}'}$ is the parallax angle (observed from site $\vec{r}'$) drawn from a reference direction to site $\vec{r}$.



This last result can be better seen from an example with three non-zero integer-valued $Q_{\vec{r}}$ as in the figure. Field $R_{\mu,\vec{r}}$ runs from the positive $Q_{\vec{r}''}=+2$ to the two single negative $Q_{\vec{r}'''}$ and $Q_{\vec{r}}$: let $\Xi_{\vec{r}'' \rightarrow \vec{r}'''}$ and $\Xi_{\vec{r}'' \rightarrow \vec{r}}$ be the parallax angles observed from site $\vec{r}'$ of the path $\vec{r}'' \rightarrow \vec{r}'''$ and $\vec{r}'' \rightarrow \vec{r}$ respectively. The sum of both parallaxes can be formulated as

$$\begin{aligned} \Xi_{\vec{r}'' \rightarrow \vec{r}} + \Xi_{\vec{r}'' \rightarrow \vec{r}'''} &= \\ &= (\Xi_{\vec{r},\vec{r}'} - \Xi_{\vec{r}''',\vec{r}'}) + \\ &+ (\Xi_{\vec{r}''',\vec{r}'} - \Xi_{\vec{r}'',\vec{r}'}) = \\ &= -[Q_{\vec{r}}\Xi_{\vec{r},\vec{r}'} + Q_{\vec{r}'''}\Xi_{\vec{r}''',\vec{r}'} + \\ &+ Q_{\vec{r}''}\Xi_{\vec{r}'',\vec{r}'}] , \end{aligned} \quad (D.162)$$

where the notation can be read off the figure. Substituting the expression for the Coulomb propagator in a 2-dim. lattice (C.109)³³, the correlation function in the $Z(N)$ Ising model takes in the Villain approximation the following form

$$\begin{aligned} \left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle &= \frac{\sum'_{\{M_{\vec{r}}\}} \sum'_{\{Q_{\vec{r}}\}} \exp \left[-\frac{\pi^2 \beta}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 \right] \times}{\prod_{\vec{x}} \sum_{\vec{P}_{\vec{r}}=-\infty}^{\infty} \delta \left(\sum_{\vec{r}} \vec{P}_{\vec{r}}, 0 \right) \prod_{\vec{x}} \sum_{G_{\vec{r}}=-\infty}^{\infty} \delta \left(\sum_{\vec{r}} G_{\vec{r}}, 0 \right) \exp \left[-\frac{\pi^2 \beta}{2\eta} \sum_{\vec{r}} M_{\vec{r}}^2 \right] \times} \quad (D.163) \\ &\times \exp \left[-\frac{N^2 \eta}{8\beta} \sum_{\vec{r}} \left[Q_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}) \right]^2 + iN \sum_{\vec{r},\vec{r}'} \left[Q_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}) \right] \Xi_{\vec{r},\vec{r}'} M_{\vec{r}'} \right] \times \\ &\times \exp \left[-\frac{N^2 \eta}{8\beta} \sum_{\vec{r}} Q_{\vec{r}}^2 + iN \sum_{\vec{r},\vec{r}'} Q_{\vec{r}} \Xi_{\vec{r},\vec{r}'} M_{\vec{r}'} \right] \times \\ &\times \exp \left[\frac{N^2 \eta}{4\pi\beta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} \left[Q_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}) \right] \ln |\vec{r} - \vec{r}'| \left[\right]_{\vec{r}'} + \frac{\pi\beta}{\eta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right] \\ &\times \exp \left[\frac{N^2 \eta}{4\pi\beta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} Q_{\vec{r}} \ln |\vec{r} - \vec{r}'| Q_{\vec{r}'} + \frac{\pi\beta}{\eta} \sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} \ln |\vec{r} - \vec{r}'| M_{\vec{r}'} \right] \end{aligned}$$

³³And by noting that, since $\sum_{\vec{r}'} M_{\vec{r}'} = 0$, then $\sum_{\vec{r},\vec{r}' \neq \vec{r}} M_{\vec{r}} M_{\vec{r}'} = -\sum_{\vec{r}} M_{\vec{r}}^2$.

It is clear from the last line in the previous equation that the 2-dim. $Z(N)$ Ising model is self-dual by interchanging the roles of $M_{\vec{r}}$ and $Q_{\vec{r}}$ and their respective coupling strengths $\frac{\beta}{\eta}$ and $\frac{N^2}{4\pi^2} \frac{\eta}{\beta}$ in the partition function (the denominator of the above equation)

$$\left\{ \begin{array}{l} M_{\vec{r}} \longleftrightarrow Q_{\vec{r}}, \\ \frac{\beta}{\eta} \longleftrightarrow \frac{N^2}{4\pi^2} \frac{\eta}{\beta}, \end{array} \right. \quad (\text{D.164})$$

or by interchanging $M_{\vec{r}}$ and $\bar{Q}_{\vec{r}} \equiv Q_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})$ in the correlation's numerator. Such a result which has been already proved for the $Z(2)$ model and “guessed” for the general $Z(N)$ model on page 425. The limit $N \rightarrow \infty$ forces all $Q_{\vec{r}} = 0$ due to term $\frac{N^2}{4\pi\beta} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} Q_{\vec{r}} \ln |\vec{r} - \vec{r}'| Q_{\vec{r}'}$, and we recover thus the XY model. It is to be noticed that if the $Z(N)$ Ising model had only two phases, one dual to the other and separated by the critical coupling strength $\left(\frac{\beta}{\eta}\right)_{\text{crit}} = \frac{N^2}{4\pi^2} \frac{1}{(\beta/\eta)_{\text{crit}}} \Rightarrow \left(\frac{\beta}{\eta}\right)_{\text{crit}} = \frac{N}{2\pi}$, then limit $N \rightarrow \infty$ would imply that the XY would have only one phase from $\beta = 0$ as far as $\beta = \beta_{\text{crit}} = \infty$; but since the XY has already two phases, for N large enough it must exist another phase between the two phases which are related to each other by the dual transformation of equation (D.164).

A similar calculation can be carried out for the 3-dim. $Z(N)$ Ising model and the final result can be directly read off equation (D.163) with minor changes: replacing the 2-dimensional propagator $-\frac{1}{2\pi} \ln |\vec{r} - \vec{r}'|$ by the 3-dim. one $\frac{1}{4\pi |\vec{r} - \vec{r}'|}$, substituting parallax angles by solid angles and now topological field has to fulfil $\sum_{\lambda} M_{\lambda, \vec{r}} = 0$ whilst sum $\prod_{\vec{r}} \sum_{Q_{\vec{r}} = -\infty}^{\infty}$ is not longer constrained by $\sum_{\vec{r}} Q_{\vec{r}} = 0$. However, the 3-dim. model is not self-dual since topological field $M_{\lambda, \vec{r}}$ is a vector field whereas $Q_{\vec{r}}$ is a scalar field.

Let us now return to the pure $Z(N)$ gauge model by considering the $Z(2)$ gauge model in a d -dimensional lattice periodic along the τ -direction, similarly to energy (D.70)

$$\begin{aligned} E = \sum_n \left\{ \beta_{\tau} \sum_{\vec{x} \in \text{plane}_n} \sum_{j=1}^{d-1} \left[1 - S_{j, (na_{\tau}, \vec{x})}^{\text{gauge}} S_{\tau, (na_{\tau}, \vec{x} + \hat{j})}^{\text{gauge}} S_{j, ((n+1)a_{\tau}, \vec{x})}^{\text{gauge}} S_{\tau, (na_{\tau}, \vec{x})}^{\text{gauge}} \right] + \right. \\ \left. + \beta_x \sum_{\vec{x} \in \text{plane}_n} \sum_{j < k=1}^{d-1} \left[1 - S_{j, (na_{\tau}, \vec{x})}^{\text{gauge}} S_{k, (na_{\tau}, \vec{x} + \hat{j})}^{\text{gauge}} S_{j, (na_{\tau}, \vec{x} + \hat{k})}^{\text{gauge}} S_{k, (na_{\tau}, \vec{x})}^{\text{gauge}} \right] \right\}, \quad (\text{D.165}) \end{aligned}$$

where n denotes the planes normal to the τ -direction, and fields $S_{k, (na_{\tau}, \vec{x})}^{\text{gauge}} = e^{iA_{k, (na_{\tau}, \vec{x})}}$, $S_{\tau, (na_{\tau}, \vec{x})}^{\text{gauge}} = e^{iA_{\tau, (na_{\tau}, \vec{x})}}$ take the values ± 1 (or $A_{k, (na_{\tau}, \vec{x})} = 0, \pi = A_{\tau, (na_{\tau}, \vec{x})}$). Now the discussion follows as for the $U(1)$ model: one reduces partially the gauge freedom by the choice $A_{\tau, (na_{\tau}, \vec{x})} = 0 \forall n, \vec{x}$ or equivalently

$$S_{\tau, (na_{\tau}, \vec{x})}^{\text{gauge}} = e^{iA_{\tau, (na_{\tau}, \vec{x})}} = 1 \quad \forall n, \vec{x}, \quad (\text{D.166})$$

and thus the energy in the partition function takes the form $\frac{E}{\eta} = \sum_n L(n+1, n)$ with

$$L(n+1, n) = \frac{\beta_{\tau}}{2\eta} \sum_{\vec{x} \in \text{plane}_n} \sum_{j=1}^{d-1} \left(S_{j, ((n+1)a_{\tau}, \vec{x})}^{\text{gauge}} - S_{j, (na_{\tau}, \vec{x})}^{\text{gauge}} \right)^2 + \quad (\text{D.167})$$

$$+ \frac{\beta_x}{\eta} \sum_{\vec{x} \in \text{plane}_n} \sum_{j < k=1}^{d-1} \left[\frac{1 - S_{j, (na_\tau, \vec{x})}^{\text{gauge}} S_{k, (na_\tau, \vec{x} + \hat{j})}^{\text{gauge}} S_{j, (na_\tau, \vec{x} + \hat{k})}^{\text{gauge}} S_{k, (na_\tau, \vec{x})}^{\text{gauge}}}{2} + \right. \\ \left. + \frac{1 - S_{j, ((n+1)a_\tau, \vec{x})}^{\text{gauge}} S_{k, ((n+1)a_\tau, \vec{x} + \hat{j})}^{\text{gauge}} S_{j, ((n+1)a_\tau, \vec{x} + \hat{k})}^{\text{gauge}} S_{k, ((n+1)a_\tau, \vec{x})}^{\text{gauge}}}{2} \right].$$

After obtaining the different elements of the transfer matrix, given by the exponential of $-L((n+1)\text{th plane}, n\text{th plane})$, for states in the $(n+1)$ -plane with no, one, two, ... flipped $S_{k, ((n+1)a_\tau, \vec{x})}^{\text{gauge}}$'s related to the n -plane

$$\mathcal{T}_{0 \text{ flips}} = \exp \left[-\frac{\beta_x}{\eta} \sum_{\vec{x}} \sum_{j < k=1}^{d-1} 1 - S_{j, \vec{x}}^{\text{gauge}} S_{k, \vec{x} + \hat{j}}^{\text{gauge}} S_{j, \vec{x} + \hat{k}}^{\text{gauge}} S_{k, \vec{x}}^{\text{gauge}} \right], \\ \mathcal{T}_{1 \text{ flip}} = e^{-2\beta_\tau/\eta} \exp \left[-\frac{\beta_x}{\eta} \sum_{\vec{x}} \sum_{j < k=1}^{d-1} \left(\frac{1 - S_{j, (na_\tau, \vec{x})}^{\text{gauge}}}{2} \cdots + \frac{1 - S_{j, ((n+1)a_\tau, \vec{x})}^{\text{gauge}}}{2} \cdots \right) \right],$$

and after taking the continuum limit along the periodic τ -direction, limit which relates the transfer matrix to the operator $\hat{\mathcal{H}}$ by means of $\hat{\mathcal{T}} \approx \mathbf{1} - \frac{a_\tau}{\eta} \hat{\mathcal{H}}$,

$$\mathcal{T}_{0 \text{ flips}} \approx 1 - \frac{\beta_x}{\eta} \sum_{\vec{x}} \sum_{j < k=1}^{d-1} \left(1 - S_{j, \vec{x}}^{\text{gauge}} S_{k, \vec{x} + \hat{j}}^{\text{gauge}} S_{j, \vec{x} + \hat{k}}^{\text{gauge}} S_{k, \vec{x}}^{\text{gauge}} \right) \approx 1 - \frac{a_\tau}{\eta} \mathcal{H}_{0 \text{ flips}}, \\ \mathcal{T}_{1 \text{ flip}} \approx e^{-2\beta_\tau/\eta} \left(\right) \approx -\frac{a_\tau}{\eta} \mathcal{H}_{1 \text{ flip}},$$

one obtains in a way similar to equation (D.34) the following operator $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{0 \text{ flips}} + \hat{\mathcal{H}}_{1 \text{ flip}}$

$$\hat{\mathcal{H}}_{0 \text{ flips}} = \frac{\beta_x}{a_\tau} \sum_{\vec{x}} \sum_{j < k=1}^{d-1} \left[1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x} + \hat{j}} (\hat{\sigma}_3)_{j, \vec{x} + \hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right], \\ \hat{\mathcal{H}}_{1 \text{ flip}} = -\eta \frac{e^{-2\beta_\tau/\eta}}{a_\tau} \sum_{\vec{x}} \sum_{j=1}^{d-1} (\hat{\sigma}_1)_{j, \vec{x}}.$$
(D.168)

In the last two equation, $(\hat{\sigma}_1)_{j, \vec{x}}$ and $(\hat{\sigma}_3)_{j, \vec{x}}$ are the corresponding Pauli matrices acting as operators on the bond connecting site $\vec{x}$ with $\vec{x} + \hat{j}$, where $\vec{x}$ denotes as usual a site on the subspace normal to the τ -direction. Since $\hat{\mathcal{T}} \xrightarrow{a_\tau \rightarrow 0} \mathbf{1}$, and thus $\mathcal{T}_{0 \text{ flips}} \rightarrow \mathbf{1}$ and $\mathcal{T}_{1 \text{ flips}} \rightarrow 0$ for $a_\tau \rightarrow 0$, the behaviour of both coupling strengths at the continuum limit is given by

$$\begin{cases} a_\tau \sim \beta_x & \text{and} & \beta_x \xrightarrow{a_\tau \rightarrow 0} 0, \\ a_\tau \sim e^{-2\beta_\tau/\eta} & \text{and} & \beta_\tau, e^{+2\beta_\tau/\eta} \xrightarrow{a_\tau \rightarrow 0} \infty. \end{cases}$$
(D.169)

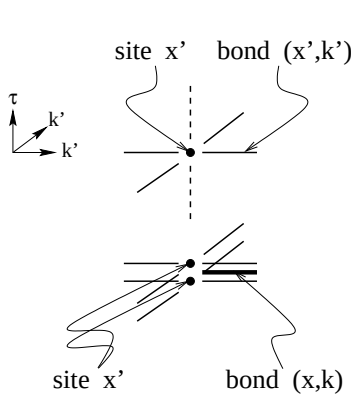
By comparing (D.168) with the operator of the $U(1)$ gauge model, $\hat{\mathcal{H}} = \frac{a^{3-d}e^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \hat{\mathcal{L}}_{k,\vec{x}}^2 + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} [1 - \cos(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}})]$, which is the limit $N \rightarrow \infty$ of the $Z(N)$ gauge model, one obtains the dependence of β_x and β_τ on coupling constant e^2 , $\frac{e^{-2\beta_\tau/\eta}}{a_\tau} \sim e^2$, $\frac{\beta_x}{a_\tau} \sim e^{-2}$, although the exact dependence will be determined later.

However, there is still a remaining infinite overcounting originated by the gauge invariance and which has not been removed by the gauge fixing $A_{\tau,(na_\tau,\vec{x})} = 0 \forall n, \vec{x}$, just as in the $U(1)$ gauge model. And the form of solving the question is the same as in the $U(1)$ model, after noticing [42] that the operator ³⁴

$$\hat{\mathcal{G}}_{\vec{x}'} = \prod_{k'=1}^{d-1} (\hat{\sigma}_1)_{k',\vec{x}'-\hat{k}'} (\hat{\sigma}_1)_{k',\vec{x}'} , \quad (\text{D.170})$$

fulfil the following properties

$$\begin{aligned} \hat{\mathcal{G}}_{\vec{x}'}^{-1} (\hat{\sigma}_1)_{k,\vec{x}} \hat{\mathcal{G}}_{\vec{x}'} &= (\hat{\sigma}_1)_{k,\vec{x}} && \text{for all bonds } (\vec{x}, k), \\ \hat{\mathcal{G}}_{\vec{x}'}^{-1} (\hat{\sigma}_3)_{k,\vec{x}} \hat{\mathcal{G}}_{\vec{x}'} &= -(\hat{\sigma}_3)_{k,\vec{x}} && \text{if bond } (\vec{x}, k) \text{ is attached to site } \vec{x}', \\ \hat{\mathcal{G}}_{\vec{x}'}^{-1} (\hat{\sigma}_3)_{k,\vec{x}} \hat{\mathcal{G}}_{\vec{x}'} &= (\hat{\sigma}_3)_{k,\vec{x}} && \text{if bond } (\vec{x}, k) \text{ is not attached to site } \vec{x}'. \end{aligned} \quad (\text{D.171})$$



These properties can be easily shown: operator $\hat{\mathcal{G}}_{\vec{x}'}$ is the product of all operators $(\hat{\sigma}_1)_{k',\vec{x}'}$ which are located on the bonds (of the subspace normal to the τ -direction) attached to site $\vec{x}'$. Further, $\hat{\mathcal{G}}_{\vec{x}} \hat{\mathcal{G}}_{\vec{x}'} = \mathbf{1}$ and thus $\hat{\mathcal{G}}_{\vec{x}'}^{-1} = \hat{\mathcal{G}}_{\vec{x}}$. Since $(\hat{\sigma}_1)_{k',\vec{x}'}$ commutes with $(\hat{\sigma}_1)_{k,\vec{x}}$, the first relation of (D.171) follows. Operators $(\hat{\sigma}_1)_{k',\vec{x}'}$ and $(\hat{\sigma}_3)_{k,\vec{x}}$ commutes with each other when applied on different bonds, and only in the case where bond $(\vec{x}, k)$ coincides with bond $(\vec{x}', k')$ or bond $(\vec{x}' - \hat{k}', k')$ then these operators do not commute: the example, represented picturically in the figure, reads mathematically

$$\hat{\mathcal{G}}_{\vec{x}'}^{-1} (\hat{\sigma}_3)_{k,\vec{x}} \hat{\mathcal{G}}_{\vec{x}'} = 1 \cdot \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{k'=k, \vec{x}'=\vec{x}} \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)_{k,\vec{x}} \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{k,\vec{x}} = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right)_{k,\vec{x}} = -(\hat{\sigma}_3)_{k,\vec{x}} .$$

Since the “Hamiltonian” (D.168) contains an even power of $(\hat{\sigma}_3)_{k,\vec{x}}$ ’s, it follows

$$\hat{\mathcal{G}}_{\vec{x}'} \hat{\mathcal{H}} = \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} \quad \text{or} \quad \hat{\mathcal{H}} = \hat{\mathcal{G}}_{\vec{x}'}^{-1} \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} . \quad (\text{D.172})$$

³⁴Although one has maintained the same letter $\mathcal{G}$ for the generator of *gauge* transformations as in the $U(1)$ gauge model, operator $\hat{\mathcal{G}}_{\vec{x}'}$ for the $Z(2)$ gauge model actually corresponds to operator $\exp\left[\frac{i}{\eta} \pi \hat{\mathcal{G}}_{\vec{x}'}\right]$ in the $U(1)$ gauge model.

According to (D.171) the effect of $\hat{\mathcal{G}}_{\vec{x}'}$ is to reverse *locally* the state of $(\hat{\sigma}_3)_{k,\vec{x}}$, whose eigenvalues are determined by gauge field $A_{k,\vec{x}}$ through $(\sigma_3)_{k,\vec{x}} = e^{iA_{k,\vec{x}}}$, only on the bonds attached to $\vec{x}'$ whereas the other sites remain untouched, and since operator $\mathcal{H}$ (which determines the configuration's energy in a model with a lattice periodic along the continuum direction) is invariant under the action of $\hat{\mathcal{G}}_{\vec{x}'}$, then operator $\hat{\mathcal{G}}_{\vec{x}'}$ is the generator of the remaining gauge invariance which we want to remove in order to count each different contribution only once. This can be achieved by demanding that the states ψ , on which operator $\hat{\mathcal{H}}$ is acting, are singlets under the action of $\hat{\mathcal{G}}_{\vec{x}'}$ [28]

$$\hat{\mathcal{G}}_{\vec{x}'} \{ \psi \} = \psi . \quad (\text{D.173})$$

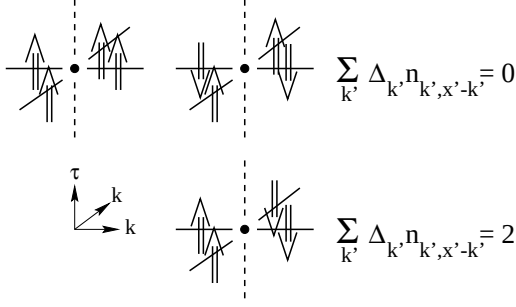
This works for the pure model, without any coupling of the gauge fields to any other field; if now there is a coupling to another field or to an external source, equation (D.173) must be modified to allow that at some lattice sites $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi \}$ does not vanish, similarly to the $U(1)$ gauge model as the coupling to an external (i.e. controled from the exterior, not a degree of freedom) source was implemented by changing constraint $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi [A_k] \} = 0$ by $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi [A_k] \} = q\eta (\delta_{\vec{x}',\vec{x}_2} - \delta_{\vec{x}',\vec{x}_1}) \psi [A_k]$. This is a question which will be treated in the next Note, but before arriving at that point, let us finish the discussion about the equivalence of the pure $Z(2)$ gauge model at large e^2 with the Ising model in one dimension lower.

Remembering that the coupling strengths are related to e^2 by $e^{-2\beta\tau/\eta}/a_\tau \sim e^2$, $\beta_x/a_\tau \sim e^{-2}$, partition function $\mathcal{Z} = \text{Tr} \left[\exp \left(-\frac{Na_\tau}{\eta} \hat{\mathcal{H}} \right) \right] = \text{Tr} \left[\exp \left(-\frac{\omega}{\eta} \hat{\mathcal{H}} \right) \right]$ at large e^2 can be written as follows

$$\begin{aligned} \hat{\mathcal{H}}_{\text{large } e^2} &\approx -\eta \frac{e^{-2\beta\tau/\eta}}{a_\tau} \sum_{\vec{x}} \sum_{j=1}^{d-1} (\hat{\sigma}_1)_{j,\vec{x}} , \\ \mathcal{Z}_{\text{large } e^2} &= \prod_{\vec{x}} \prod_{j=1}^{d-1} \sum_{n_{j,\vec{x}}=0,1} \prod_{\vec{x}'} \delta \left(\sum_{k'} \Delta_{k'} n_{k',\vec{x}'-\hat{k}'} , 0 \bmod 2 \right) \times \\ &\times \exp \left[-\omega \frac{e^{-2\beta\tau/\eta}}{a_\tau} \sum_{\vec{x}} \sum_{j=1}^{d-1} 1 - \cos(\pi n_{j,\vec{x}}) \right] , \end{aligned} \quad (\text{D.174})$$

where a constant factor $\exp \left[-\omega \frac{e^{-2\beta\tau/\eta}}{a_\tau} (d-1) |\Lambda_\perp| \right]$ has been introduced for convenience and it has no effect in the phase transitions since it only produces a constant in the free energy. The above equation has been formulated as a sum over a discrete field $n_{j,\vec{x}}$ because operator $(\hat{\sigma}_1)_{k,\vec{x}}$ only produces as eigenvalues ± 1 : $(\sigma_1)_{j,\vec{x}} = e^{i\pi n_{j,\vec{x}}} = e^{-i\pi n_{j,\vec{x}}} = \cos(\pi n_{j,\vec{x}})$ with $n_{j,\vec{x}} = 0, 1$. The product of Dirac deltas implements constraint (D.173)

$$\hat{\mathcal{G}}_{\vec{x}'} \{ \psi \} = \psi \quad \implies \quad \prod_{k'=1}^{d-1} (\sigma_1)_{k',\vec{x}'-\hat{k}'} (\sigma_1)_{k',\vec{x}'} = 1 = e^{i\pi \sum_{k'} (n_{k',\vec{x}'} - n_{k',\vec{x}'-\hat{k}'})} ,$$



and thus $\sum_{k'} \Delta_{k'} n_{k', \vec{x}' - \hat{k}'} = 0, \pm 2$. An example of several configurations for which $\prod_{k'=1}^{d-1} (\sigma_1)_{k', \vec{x}' - \hat{k}'} (\sigma_1)_{k', \vec{x}'} = 1$ holds is represented in the figure, where $\uparrow, \downarrow$ denotes the eigenvectors of $(\sigma_1)_{k', \vec{x}'}$ with $n_{k', \vec{x}'} = 0, 1$ respectively. Moreover, e^2 large forces $n_{j, \vec{x}} = 0$ almost everywhere (all $(\hat{\sigma}_1)_{k, \vec{x}}$ -eigenvectors equal to $\uparrow$), and therefore one can approximate $1 - \cos(\pi n_{j, \vec{x}}) \approx \frac{\pi^2 n_{j, \vec{x}}^2}{2}$

$$\mathcal{Z}_{\text{large } e^2} = \prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=0,1} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, 0 \bmod 2 \right) \exp \left[-\frac{\omega}{2} \frac{e^{-2\beta_\tau/\eta}}{a_\tau} \sum_{\vec{x}, k} (\pi n_{k, \vec{x}})^2 \right]. \quad (\text{D.175})$$

If instead of $Z(2)$ -fields $(\sigma_1)_{k, \vec{x}} = e^{i\pi n_{k, \vec{x}}}$ (with $n_{k, \vec{x}} = 0, 1$) one would consider the limit $N \rightarrow \infty$ of the $Z(N)$ gauge model with fields $\psi = e^{i\frac{2\pi}{N} n_{k, \vec{x}}}$ with $n_{k, \vec{x}} = 0, \pm 1, \pm 2, \dots, \pm \infty$, one would obtain the pure $U(1)$ gauge model at large e^2 , whose partition function is given in (D.110), $\prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=-\infty}^{\infty} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, 0 \right) \exp \left[-\frac{\omega}{\eta} \frac{a^{3-d} e^2 \eta^2}{2} \sum_{\vec{x}, k} n_{k, \vec{x}}^2 \right]$, and comparing (D.175) with such equation, one obtains

$$\frac{e^{-2\beta_\tau/\eta}}{a_\tau} = \frac{1}{\pi^2 a} a^{4-d} e^2 \eta. \quad (\text{D.176})$$

as the complete dependence of $e^{-2\beta_\tau/\eta}$ on e^2 ³⁵. The final step runs as follows: integer variables $n_{k, \vec{x}}$ are substituted by continuous fields $\theta_{k, \vec{x}} \in [0, 2\pi)$ by using the Fourier

³⁵In a similar way to operator $\hat{\mathcal{H}}$ in the Hamiltonian formulation of the Ising model, another form of writing the Hamiltonian operator for the $Z(2)$ gauge model is $\hat{\mathcal{H}} = -\eta \sum_{\vec{x}, k} (\hat{\sigma}_1)_{k, \vec{x}} + \kappa_e \sum_{\vec{x}} \sum_{j < k} \left[1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x} + \hat{j}} (\hat{\sigma}_3)_{j, \vec{x} + \hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right]$ (if one wants to restore lattice spacing along the

non-periodic directions a , then the previous equation is equal to $\hat{\mathcal{H}}/a$), where κ_e is the proportionality constant relating β_x and $e^{-2\beta_\tau/\eta}$, both of them proportional to a_τ : $\beta_x = \kappa_e e^{-2\beta_\tau/\eta}$. As in the discussion following (D.33) in the Ising model in a lattice with $\omega = Na_\tau = \infty$, the different phases of the model are characterized by a large or a small value of κ_e ; a small κ_e means that the term multiplied by $e^{-2\beta_\tau/\eta}$ dominates over the Hamiltonian's term containing β_x . In fact, a small κ_e corresponds to large e^2 and viceversa. In order to see this relation between κ_e and e^2 , let us consider a closed loop C perpendicular to the τ -direction and then calculate by means of the Hamiltonian formulation in a lattice of $\omega = \infty$ the expectation value of $\prod_{\vec{x} \in \text{closed loop } C} (\hat{\sigma}_3)_{k, \vec{x}}$: this

is $W(C)$ for the non-periodic $Z(2)$ gauge model. At κ_e small the leading part of the Hamiltonian is $-\eta \sum_{\vec{x}, k} (\hat{\sigma}_1)_{k, \vec{x}}$ whilst $\frac{\kappa_e}{\eta} \eta \sum_{\vec{x}} \sum_{j < k} \left[1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x} + \hat{j}} (\hat{\sigma}_3)_{j, \vec{x} + \hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right]$ can be treated as a perturbation of order $\lambda = \kappa_e/\eta$. Since we are here considering an infinite lattice along the τ -dimension, the leading contribution to $W(C)$ comes from the eigenvector of lowest $\hat{\mathcal{H}}$ -eigenvalue, which will be

transform $\sum_{n=0}^{N-1} \delta\left(\theta - \frac{2\pi}{N}n\right) = \frac{N}{2\pi} \sum_{m=-\infty}^{\infty} e^{iNm\theta}$, and then, since we are considering the limit $e^2 \rightarrow \infty$, we can extend the integration range of $\theta_{k,\vec{x}}$ to be $(-\infty, \infty)$: partition function $\mathcal{Z}_{\text{large } e^2}$ becomes thus

$$\int_0^{2\pi} \prod_{\vec{x}} d\theta_{\vec{x}} \prod_{\vec{x},k} \sum_{n_{k,\vec{x}}=0}^{n_{k,\vec{x}}=1} \delta(\theta_{k,\vec{x}} - \pi n_{k,\vec{x}}) \prod_{\vec{x}} \delta\left(\sum_k \Delta_k n_{k,\vec{x}-\hat{k}}, 0 \bmod 2\right) \exp\left[-\frac{a^{3-d}e^2\omega\eta}{2\pi^2} \sum_{\vec{x},k} (\pi n_{k,\vec{x}})^2\right] =$$

$$= \int_{-\infty}^{\infty} \prod_{\vec{x},k} \frac{d\theta_{\vec{x}}}{\pi} \prod_{\vec{x},k} \sum_{m_{k,\vec{x}}=-\infty}^{\infty} \exp\left[i2 \sum_{\vec{x},k} m_{k,\vec{x}} \theta_{k,\vec{x}} - \frac{a^{3-d}e^2\omega\eta}{2\pi^2} \sum_{\vec{x},k} \theta_{k,\vec{x}}^2\right] \prod_{\vec{x}} \delta\left(\sum_k \frac{1}{\pi} \Delta_k \theta_{k,\vec{x}-\hat{k}}, 0 \bmod 2\right).$$

After rewriting the last product of Dirac deltas as $\prod_{\vec{x}} \delta\left(\sum_k \frac{1}{\pi} \Delta_k \theta_{k,\vec{x}-\hat{k}}, 0 \bmod 2\right) =$
 $= \prod_{\vec{x}} \sum_{P_{\vec{x}}=-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{\vec{x}} \frac{d\chi_{\vec{x}}}{2\pi} \exp\left[i \sum_{\vec{x}} \chi_{\vec{x}} \left(\sum_k \frac{1}{\pi} \Delta_k \theta_{k,\vec{x}-\hat{k}} - 2P_{\vec{x}}\right)\right]$, the integration of $\prod_{k,\vec{x}} d\theta_{k,\vec{x}}$ produces

$$\mathcal{Z}_{\text{large } e^2} = \left(\sqrt{\frac{2\pi}{a^{3-d}e^2\omega\eta}}\right)^{(d-1)|\Lambda_{\perp}|} \underbrace{\int_{-\infty}^{\infty} \prod_{\vec{x}} \frac{d\chi_{\vec{x}}}{2\pi} \sum_{\{m_{k,\vec{x}}\}} \sum_{\{P_{\vec{x}}\}} \exp\left[-i2 \sum_{\vec{x}} P_{\vec{x}} \chi_{\vec{x}}\right]}_{\times \exp\left[-\frac{\pi^2 a^{d-3}}{2 e^2 \omega \eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} \left(\frac{1}{\pi} \Delta_k \chi_{k,\vec{x}} - 2m_{k,\vec{x}}\right)^2\right]}, \quad (\text{D.177})$$

denoted by $|0\rangle_{\text{small } \kappa_e}$. Such an eigenstate is equal to the eigenvector of $\hat{\mathcal{H}}_{\kappa_e=0} = -\eta \sum_{\vec{x},k} (\hat{\sigma}_1)_{k,\vec{x}}$, de-

noted by $|\uparrow \dots \uparrow\rangle$, plus contributions of order $\left(\frac{\kappa_e}{\eta}\right)^n$: since the effect of $(\hat{\sigma}_3)_{k,\vec{x}}$ in the basis of $(\hat{\sigma}_1)_{k,\vec{x}}$ -eigenvectors (basis in which the matrix representation of $(\hat{\sigma}_3)_{k,\vec{x}}$ is off-diagonal) is to reverse the spin $\uparrow$ located on the bond connecting $\vec{x}$ and $\vec{x} + \hat{k}$, then the contribution to $|0\rangle_{\text{small } \kappa_e}$ of order $\left(\frac{\kappa_e}{\eta}\right)^n$ contains n plaquettes of reversed spins (denoted schematically $|n_{\downarrow\downarrow\downarrow\downarrow}\rangle$), produced by the action n consecutive times of $\hat{\mathcal{H}}_1 = \eta \sum_{\vec{x}} \sum_{j < k} \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}}\right]$ on $|\uparrow \dots \uparrow\rangle$. Clearly

$\langle \uparrow \dots \uparrow | \prod_{\text{loop } C} (\hat{\sigma}_3)_{k,\vec{x}} | \uparrow \dots \uparrow \rangle = 0$, and the same vanishing result holds if $\langle \uparrow \dots \uparrow |$ is “sandwiched”

with a state $\prod_{\text{loop } C} (\hat{\sigma}_3)_{k,\vec{x}} |n_{\downarrow\downarrow\downarrow\downarrow}\rangle$ in which all the different n reversed plaquettes do not correspond to all the plaquettes tiling the interior of C and thus the reversed spins have not been flipped back to $\uparrow$ by the operators contained in $\prod_{\text{loop } C} (\hat{\sigma}_3)_{k,\vec{x}} = \prod_{\text{plaquettes } \subset \text{Area}_C} \underbrace{\left[(\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}}\right]}_{\text{plaquette}}$. Since

each flipped plaquette is produced by the action of $\hat{\mathcal{H}}_1$ and each $\hat{\mathcal{H}}_1$ is accompanied by a $\lambda = \kappa_e/\eta$, the first non-vanishing contribution is of the order $W(C) =_{\text{small } \kappa_e} \langle 0 | \prod_{\text{loop } C} (\hat{\sigma}_3)_{k,\vec{x}} | 0 \rangle_{\text{small } \kappa_e} \sim \left(\frac{\kappa_e}{\eta}\right)^{\text{Area}_C}$,

where Area_C is the number of unit plaquettes covering the area enclosed by loop C . However, from the usual calculation of $W(C)$ by means of the “high-temperature” expansion, we already know that $W(C) \sim e^{-\text{Area}_C \ln(\eta/\kappa_e)}$ is the characteristic behaviour of $W(C)$ at large e^2 , and this implies that κ_e is a decreasing function of e^2 .

where the underlined part is, as expected, equal to the partition function of the $(d-1)$ -dim. Ising model (see denominator of equation (D.158)) for a value of β equal to $1/e^2\omega$, just as the relation between the periodic $U(1)$ gauge model and the $(d-1)$ -dim. XY model (one should remember that the coupling strength β in a $(d-1)$ -dim. Ising model corresponds to $a^{(d-1)-2}|\psi|_{XY}^2$ in the XY model). Before continuing with the proof of the equivalence between the $Z(N)$ gauge model in d -dim. periodic along one direction and the $(d-1)$ -dim. $Z(N)$ Ising model for the range of small e^2 , the case of the $Z(N)$ gauge field coupled to another field will be discussed in the next Note.

Note. The “confining” phase (III).

As for the $U(1)$ model, we shall firstly consider the coupling of the gauge field to an external source, and then a generalization, by considering the coupling to a scalar field which is a degree of freedom. Although we have already seen that the effect of operator $\hat{\mathcal{G}}_{\vec{x}'}$ is to “flip” field operators $(\hat{\sigma}_3)_{k,\vec{x}} \equiv e^{i\hat{A}_{k,\vec{x}}}$, becoming thus the generator of gauge transformations in the $Z(2)$ model, it is interesting to formulate the result of applying $\hat{\mathcal{G}}_{\vec{x}'}$ on a given state in the same form (D.92) as the gauge-transformed state in the continuous $U(1)$ model. A state in the $Z(2)$ gauge model which gets transformed under a general gauge transformation acting on several different sites, is given by

$$\psi [(\sigma'_3)_{k,\vec{x}}] = \left(\prod_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \right) \psi [(\sigma_3)_{k,\vec{x}}] \left(\prod_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right), \quad (\text{D.178})$$

where $\zeta_{\vec{x}'} = 0, 1$ depending whether gauge variable on site $\vec{x}'$ remains untouched or not. Since $\hat{\mathcal{G}}_{\vec{x}'}$ produces as value only ± 1 , product $\prod_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'}$ can be also written as $\prod_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} = \prod_{\vec{x}'} \exp \left[i \frac{\pi}{2} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right]$ and thus the transformed state (D.178) takes the form

$$\psi [(\sigma'_3)_{k,\vec{x}}] = \exp \left[i \frac{\pi}{2} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \psi [(\sigma_3)_{k,\vec{x}}] \exp \left[-i \frac{\pi}{2} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right], \quad (\text{D.179})$$

or equivalently

$$\psi_{\text{transformed}} = \exp \left[i \frac{\pi}{2} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \{ \psi \}, \quad (\text{D.180})$$

where $\{ \}$ means that all operators on the left are acting only inside the brackets. This last result has the same form as the state resulting from a gauge transformation in the $U(1)$ gauge model:

$$\psi [A_{k,\vec{x}} - \Delta_k \zeta_{\vec{x}}] = \exp \left[\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \psi [A_{k,\vec{x}}] \exp \left[-\frac{i}{\eta} \sum_{\vec{x}'} \zeta_{\vec{x}'} \hat{\mathcal{G}}_{\vec{x}'} \right] \text{ where operator } \hat{\mathcal{G}}_{\vec{x}'} \text{ in the continuous model is given by } \hat{\mathcal{G}}_{\vec{x}'} = -i\eta \sum_{k'=1}^{d-1} \left(\frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} - \frac{\hat{\partial}}{\partial A_{k',\vec{x}'-k'}} \right).$$

As well as constraint $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi [A_{k,\vec{x}}] \} = 0$ for the $U(1)$ model means that the states only depends on the transversal component of the gauge field (i.e., that component which remains invariant under a gauge transformation) and thus together with choice $A_{\tau,\vec{x}} = 0$ all gauge freedom is removed, condition $\hat{\mathcal{G}}_{\vec{x}'} \{ \psi [(\sigma_3)_{k,\vec{x}}] \} = \psi [(\sigma_3)_{k,\vec{x}}]$ for the $Z(2)$ model reduces the set of contributing states to those which are gauge invariant, in order to avoid that infinitely many different states which related to each other by means of a gauge transformation, would produce the same energy (since the energy remains invariant under a gauge transformation); hence this condition, together with $A_{\tau,\vec{x}} = 0$, removes completely the gauge invariance in the pure gauge model. However, if the gauge field is coupled to an external source,

both conditions cannot be required and constraint $\hat{\mathcal{G}}_{\vec{x}'} \{\psi\} = \psi$ must be changed by allowing state ψ to be other than a gauge singlet at some sites. The modification we shall study now is given by

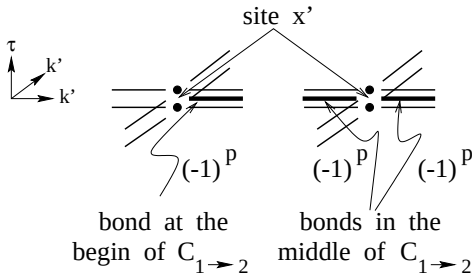
$$\hat{\mathcal{G}}_{\vec{x}'} \{\psi\} = (-1)^{p(\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2})} \psi, \quad (\text{D.181})$$

with $p = 1, 2$ and whose solution is the product of operators $\left[(\hat{\sigma}_3)_{k, \vec{x}}\right]^p$ along a path $C_{1 \rightarrow 2}$ connecting site $\vec{x}_1$ to site $\vec{x}_2$, and such a product acting on the “ground state” $|0\rangle$ ³⁶

$$\psi_{1 \rightarrow 2} = \prod_{(k, \vec{x}) \in C_{1 \rightarrow 2}} (\hat{\sigma}_3)_{k, \vec{x}}^p |0\rangle. \quad (\text{D.182})$$

The fact that this solution, analogue of equation (D.100) for the $U(1)$ model, fulfils (D.181) is shown by using property $\hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} = \mp (\hat{\sigma}_3)_{k, \vec{x}}$ (depending on whether bond-operator $(\hat{\sigma}_3)_{k, \vec{x}}$ is or not attached to site $\vec{x}'$)

$$\begin{aligned} \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}}^p \hat{\mathcal{G}}_{\vec{x}'}^{-1} &= \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \dots \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} = \left(\hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{k, \vec{x}} \hat{\mathcal{G}}_{\vec{x}'}^{-1}\right)^p, \\ \hat{\mathcal{G}}_{\vec{x}'} \{\psi_{1 \rightarrow 2}\} &= \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{\text{begin of } C_{1 \rightarrow 2}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \dots \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{\text{middle of } C_{1 \rightarrow 2}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \dots \hat{\mathcal{G}}_{\vec{x}'} (\hat{\sigma}_3)_{\text{end of } C_{1 \rightarrow 2}} \hat{\mathcal{G}}_{\vec{x}'}^{-1} \hat{\mathcal{G}}_{\vec{x}'} |0\rangle, \end{aligned}$$



and by noting that if site $\vec{x}'$ is located in the middle of path $C_{1 \rightarrow 2}$ then there is *two* bond-operators $(\hat{\sigma}_3)_{k, \vec{x}' - \hat{k}}, (\hat{\sigma}_3)_{k, \vec{x}'}$ attached to $\vec{x}'$ and which produce thus $(-1)^p (-1)^p = 1$. On the other hand, if $\vec{x}'$ is placed just at the begin or at the end of the path, then there is only one factor $(-1)^p$, as represented in the figure. All this leads to $\hat{\mathcal{G}}_{\vec{x}'} \{\psi_{1 \rightarrow 2}\} = (-1)^{p(\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2})} \psi_{1 \rightarrow 2}$.

In the $Z(2)$ gauge model at large e^2 , let us calculate the free energy increase $\mathcal{V}$ due to the change from constraint

$$\prod_{\vec{x}'} \delta \left(\sum_{k'} \Delta_{k'} n_{k', \vec{x}' - \hat{k}', 0 \bmod 2} \right) \text{ representing } \hat{\mathcal{G}}_{\vec{x}'} \{\psi\} = \psi,$$

to $\prod_{\vec{x}'} \delta \left(\sum_{k'} \Delta_{k'} n_{k', \vec{x}' - \hat{k}', p(\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2}) \bmod 2} \right)$, which is equivalent to $\hat{\mathcal{G}}_{\vec{x}'} \{\psi\} = (-1)^{p(\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2})} \psi$, in the partition function at large e^2

$$\exp\left(-\frac{\omega}{\eta} \mathcal{V}\right) = \frac{\prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=0}^{n_{k, \vec{x}}=1} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, p(\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) \bmod 2 \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2\pi^2} \sum_{\vec{x}, k} (\pi n_{k, \vec{x}})^2 \right]}{\prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=0}^{n_{k, \vec{x}}=1} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, 0 \bmod 2 \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2\pi^2} \sum_{\vec{x}, k} (\pi n_{k, \vec{x}})^2 \right]} \quad (\text{D.183})$$

It is important to note that for the $Z(2)$ gauge model, the only value of $p = 1, 2$ which can produce a nontrivial $\mathcal{V}$ is $p = 1$; in general, for the $Z(N)$ gauge model where $\mathcal{V}$ is given by

$$\exp\left(-\frac{\omega}{\eta} \mathcal{V}\right) = \frac{\prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=0}^{N-1} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, p(\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) \bmod N \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2\pi^2} \sum_{\vec{x}, k} \left(\frac{2\pi}{N} n_{k, \vec{x}} \right)^2 \right]}{\prod_{\vec{x}, k} \sum_{n_{k, \vec{x}}=0}^{N-1} \prod_{\vec{x}} \delta \left(\sum_k \Delta_k n_{k, \vec{x} - \hat{k}}, 0 \bmod N \right) \exp \left[-\frac{a^{3-d} e^2 \omega \eta}{2\pi^2} \sum_{\vec{x}, k} \left(\frac{2\pi}{N} n_{k, \vec{x}} \right)^2 \right]},$$

³⁶State $|0\rangle$ is defined as the eigenvector of operator $\hat{\mathcal{H}}$ with the smallest eigenvalue (or equivalently the eigenvalue of transfer matrix $\hat{\mathcal{T}}$ with the largest modulus). Adding a constant term to $\hat{\mathcal{H}}$ one can choose that this smallest eigenvalue is 0: $\hat{\mathcal{H}} |0\rangle = 0$. Since, as it was argued in the Note on page 513, the ground state is unique, then $\hat{\mathcal{G}}_{\vec{x}'} |0\rangle = |0\rangle \quad \forall \vec{x}'$ holds as one can easily show by using the gauge invariance of the Hamiltonian operator $\hat{\mathcal{H}} = \hat{\mathcal{G}}_{\vec{x}'}^{-1} \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'}$:

$$0 = \hat{\mathcal{H}} |0\rangle = \hat{\mathcal{H}} = \hat{\mathcal{G}}_{\vec{x}'}^{-1} \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} |0\rangle \implies \hat{\mathcal{H}} \hat{\mathcal{G}}_{\vec{x}'} |0\rangle = 0 \implies \hat{\mathcal{G}}_{\vec{x}'} |0\rangle = |0\rangle.$$

and integer $p = 1, 2, \dots, N$ plays a relevant role as long as $p \neq N$, otherwise $\mathcal{V} = 0$.

Following the same steps as for $\mathcal{Z}_{\text{large } e^2}$, we arrive at the following result for $\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right)$

$$\frac{\int_{-\infty}^{\infty} \prod_{\vec{x}} \frac{d\chi_{\vec{x}}}{2\pi} e^{ip\chi_{\vec{x}_2}} e^{-ip\chi_{\vec{x}_1}} \sum_{\{m_{k,\vec{x}}\}} \sum_{\{P_{\vec{x}}\}} \exp\left[-i2 \sum_{\vec{x}} P_{\vec{x}} \chi_{\vec{x}} - \frac{a^{d-3}}{2e^2\omega\eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (\Delta_k \chi_{k,\vec{x}} - 2\pi m_{k,\vec{x}})^2\right]}{\int_{-\infty}^{\infty} \prod_{\vec{x}} \frac{d\chi_{\vec{x}}}{2\pi} \sum_{\{m_{k,\vec{x}}\}} \sum_{\{P_{\vec{x}}\}} \exp\left[-i2 \sum_{\vec{x}} P_{\vec{x}} \chi_{\vec{x}} - \frac{a^{d-3}}{2e^2\omega\eta} \sum_{\vec{x}} \sum_{k=1}^{d-1} (\Delta_k \chi_{k,\vec{x}} - 2\pi m_{k,\vec{x}})^2\right]},$$

which according to (D.157) is equal to the scalar-scalar correlation function in a $(d-1)$ -dim. Ising model

$$\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) = \left\langle e^{ip\pi(n_{\vec{x}_1} - n_{\vec{x}_2})} \right\rangle_{(d-1)\text{-dim. Ising}} \equiv \left\langle \sigma_{\vec{x}_1}^p \sigma_{\vec{x}_2}^p \right\rangle_{(d-1)\text{-dim. Ising}}, \quad (\text{D.184})$$

for a coupling strength in the Ising model $\frac{\beta_{\text{Ising}}}{a^{(d-1)-2}} = \frac{1}{e^2\omega}$. Therefore, large $e^2\omega$ implies the disordered phase of the $Z(N=2)$ Ising model, where the scalar “up/down”-symmetry is not spontaneously broken and thus the scalar-scalar correlation for $p = 1 \neq N = 2$ decays exponentially like $\langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{(d-1)\text{-dim. Ising}} = e^{-|\vec{x}_2 - \vec{x}_1| \ln(1/\beta_{\text{Ising}})}$ (see Table C.1)

$$\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right) = \exp\left[-|\vec{x}_2 - \vec{x}_1| \ln\left(a^{4-d} e^2 \eta \frac{\omega}{a}\right)\right], \quad (\text{D.185})$$

with the result $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ characteristic for the large e^2 phase in a pure gauge model, and which has been called “confinement”. Although we have not yet seen the equivalence between the d -dim. periodic $Z(N)$ model and the $(d-1)$ -dim. $Z(N)$ Ising model for the range of small $e^2\omega$, which will be shown by means of the Villain formulation³⁷, we can nevertheless advance the final result and extend relation $\mathcal{V}_{\text{gauge } Z(N)} = -\frac{\eta}{\omega} \ln \left\langle e^{ip\frac{2\pi}{N}(n_{\vec{x}_1} - n_{\vec{x}_2})} \right\rangle_{(d-1)\text{-dim. } Z(N) \text{ Ising}}$ also for small $e^2\omega$; this leads us to the behaviour $\mathcal{V} \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$ at small $e^2\omega$ and $p \neq N$, which arises from the form of the connected scalar-scalar correlation (C.47) in the Ising model at large Ising coupling strength $\beta_{\text{Ising}} = \frac{1}{a^{3-d} e^2 \omega}$

$$\Gamma_{\vec{x}_1 \vec{x}_2} = \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle - \text{const.} = (\text{const.}) e^{-4\beta_{\text{Ising}} |\vec{x}_2 - \vec{x}_1|}.$$

Before leaving this Note it is interesting to notice that, since we are aiming at the description of the phase diagram of the (compact) $U(1)$ Higgs model, the pure $Z(N)$ gauge model arises as limit $|\psi|^2 = \infty$ of the Higgs model. For a periodic lattice, such a limiting model can be obtained for instance by returning to operator (D.119)

$$\begin{aligned} \hat{\mathcal{H}} = & -\frac{a^{3-d} e^2 \eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \frac{\hat{\partial}^2}{\partial A_{k,\vec{x}}^2} + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \left[1 - \cos\left(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}}\right) \right] - \\ & -\frac{a^{1-d} \eta^2}{2|\psi|^2} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \varphi_{\vec{x}}^2} + a^{d-3} |\psi|^2 \sum_{\vec{x}} \sum_{k=1}^{d-1} \left[1 - \cos\left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q \hat{A}_{k,\vec{x}}\right) \right], \end{aligned} \quad (\text{D.186})$$

³⁷It may be noteworthy the fact that, similarly to equation (D.143), the calculation of $\mathcal{V}$ also yields the behaviour of gauge correlation function $W(C)$ for a path C with one side covering the total lattice's length ω along the periodic direction: $W(C)_{d\text{-dim. periodic gauge } Z(N)} = \left\langle e^{ip\frac{2\pi}{N}(n_{\vec{x}_2} - n_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim. } Z(N) \text{ Ising}}$, being $\vec{x}_1$ and $\vec{x}_2$ the sites where C crosses the $(d-1)$ -dim. subspace normal the periodic direction. Such a result, although up to now has been only proved for the $U(1)$ model, holds also for the $Z(N)$ gauge model, as well as for the Higgs model, as we shall see in the Note on page 615.

and instead of working with $\hat{A}_{k,\vec{x}}$ and $\hat{\phi}_{\vec{x}}$, one formulates $\hat{\mathcal{H}}$ by using the following fields [19]³⁸

$$\hat{\mathcal{A}}_{k,\vec{x}} = A_{k,\vec{x}} - \frac{1}{q} (\varphi_{\vec{x}+\hat{k}} - \varphi_{\vec{x}}) \in [0, 2\pi) \quad \text{and} \quad \phi_{\vec{x}} = \varphi_{\vec{x}} \in [0, 2\pi), \quad (\text{D.187})$$

which produces $\frac{\hat{\partial}}{\partial \varphi_{\vec{x}}} = \sum_{\vec{x}',k} \frac{\partial \mathcal{A}_{k,\vec{x}'}}{\partial \varphi_{\vec{x}}} \frac{\hat{\partial}}{\partial \mathcal{A}_{k,\vec{x}'}} + \sum_{\vec{x}'} \frac{\partial \phi_{\vec{x}'}}{\partial \varphi_{\vec{x}}} \frac{\hat{\partial}}{\partial \phi_{\vec{x}'}} = \frac{1}{q} \sum_k \left(\frac{\hat{\partial}}{\partial \mathcal{A}_{k,\vec{x}'-\hat{k}}} - \frac{\hat{\partial}}{\partial \mathcal{A}_{k,\vec{x}}} \right) + \frac{\hat{\partial}}{\partial \phi_{\vec{x}}}$ and $\frac{\hat{\partial}}{\partial A_{k,\vec{x}}} = \frac{\hat{\partial}}{\partial \mathcal{A}_{k,\vec{x}}}$. Hence $\mathcal{H}$ becomes

$$\begin{aligned} \hat{\mathcal{H}} = & -\frac{a^{3-d}e^2\eta^2}{2} \sum_{\vec{x}} \sum_{k=1}^{d-1} \frac{\hat{\partial}^2}{\partial \mathcal{A}_{k,\vec{x}}^2} + \frac{a^{d-5}}{2e^2} \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \left[1 - \cos \left(\hat{\mathcal{A}}_{j,\vec{x}} + \hat{\mathcal{A}}_{k,\vec{x}+\hat{j}} - \hat{\mathcal{A}}_{j,\vec{x}+\hat{k}} - \hat{\mathcal{A}}_{k,\vec{x}} \right) \right] - \\ & - \frac{a^{1-d}\eta^2}{2|\psi|^2} \sum_{\vec{x}} \left(\frac{1}{q} \sum_k \Delta_k \frac{\hat{\partial}}{\partial \mathcal{A}_{k,\vec{x}'-\hat{k}}} + \frac{\hat{\partial}}{\partial \phi_{\vec{x}}} \right)^2 + a^{d-3}|\psi|^2 \sum_{\vec{x}} \sum_{k=1}^{d-1} \left[1 - \cos \left(q \hat{\mathcal{A}}_{k,\vec{x}} \right) \right], \end{aligned} \quad (\text{D.188})$$

where now the condition to remove the remaining gauge invariance from the transfer matrix reads

$$-i\eta \sum_{k'=1}^{d-1} \left(\Delta_{k'} \frac{\hat{\partial}}{\partial A_{k',\vec{x}'}} - q \frac{\hat{\partial}}{\partial \varphi_{\vec{x}'}} \right) \left\{ \psi [\varphi_{\vec{x}}, A_{k,\vec{x}}] \right\} = +iq\eta \frac{\hat{\partial}}{\partial \phi_{\vec{x}'}} \left\{ \psi [\phi_{\vec{x}}, \mathcal{A}_{k,\vec{x}}] \right\} = 0. \quad (\text{D.189})$$

Now it is clear from (D.188) that limit $|\psi|^2 = \infty$ forces $\cos(q\hat{\mathcal{A}}_{k,\vec{x}}) = 1$ and thus compact gauge field $\mathcal{A}_{k,\vec{x}} \in [0, 2\pi)$ is constrained to take the values of the pure $Z(q)$ gauge model: $\mathcal{A}_{k,\vec{x}} = 0, \frac{2\pi}{q}, \dots, (q-1)\frac{2\pi}{q}$, and thus one obtains for $q = 2$ the already discussed operator (D.168)

$$\hat{\mathcal{H}} = -\frac{a^{4-d}e^2\eta^2}{\pi^2 a} \sum_{\vec{x} \in (d-1)\text{-dim. } \Lambda_{\perp}} \sum_{j=1}^{d-1} (\hat{\sigma}_1)_{j,\vec{x}} + \frac{1}{2a^{5-d}e^2} \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j,k=1}^{d-1} \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right] \quad (\text{D.190})$$

Integer parameter q , which in the covariant derivative couples the gauge field to the scalar field, determines therefore the index $N = q$ of the pure $Z(N)$ gauge model, which is limit $|\psi|^2 = \infty$ of the compact $U(1)$ Higgs model.

Now let us return to the behaviour of $\mathcal{V}$ for the Higgs model at very small $|\psi|^2$ and for $p \neq q$ (D.131): $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ at large $e^2\omega\eta$ and $\mathcal{V} \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$ at small $e^2\omega\eta$. And these are the same behaviours as in the pure $Z(N)$ gauge model. Therefore one cannot expect that by relaxing limit $|\psi|^2 = \infty$ in the Higgs model towards a large but finite $|\psi|^2$, the behaviour of $\mathcal{V}$ changes; only the extension of the parameter's range for each phase is modified by decreasing $|\psi|^2$.

Let us now study the pure $Z(N)$ gauge model in a d -dim. lattice which is periodic along one direction and whose periodicity length ω is small: this will be achieved by means of the Villain formulation of such a model, firstly in a non-periodic lattice and later by making the τ -direction periodic. As different from the pure $U(1)$ gauge model, the final result is not exactly the $(d-1)$ -dim. $Z(N)$ Ising, since at small e^2 the behaviour of the periodic $Z(N)$ gauge model for $d \geq 4$ deviate from that which would correspond in the $(d-1)$ -dim. $Z(N)$ scalar model.

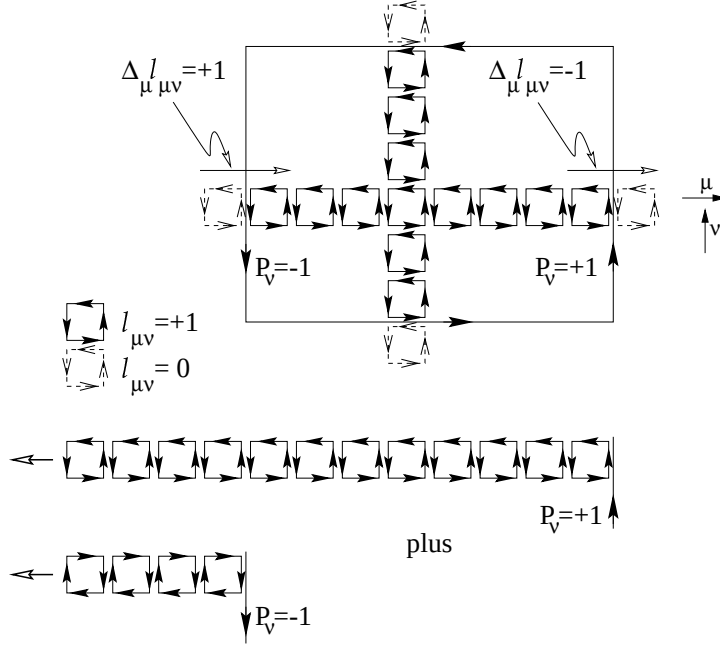
³⁸Where since the new gauge field has the form of a gauge transformation, then the discrete curl of the gauge field remains invariant $\sum_{j,k=1}^{d-1} \left[1 - \cos \left(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}} \right) \right] = \sum_{j,k=1}^{d-1} \left[1 - \cos \left(\hat{\mathcal{A}}_{j,\vec{x}} + \hat{\mathcal{A}}_{k,\vec{x}+\hat{j}} - \hat{\mathcal{A}}_{j,\vec{x}+\hat{k}} - \hat{\mathcal{A}}_{k,\vec{x}} \right) \right]$.

The Villain approximation for the $Z(N)$ gauge model in a d -dim. lattice without periodicity is defined by an energy $\frac{1}{2a^{4-d}e^2} \sum_{\vec{r}} \sum_{\nu, \mu < \nu=1}^d \left(\frac{2\pi}{N} \Delta_\mu n_{\nu, \vec{r}} - \frac{2\pi}{N} \Delta_\nu n_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}} \right)^2$ with $n_{\nu, \vec{r}} = 0, 1, \dots, N-1$ and where integer fields $g_{\mu\nu, \vec{r}} = 0, \pm 1, \pm 2, \dots, \pm \infty$ carry the compact character of the gauge field (or equivalently, they approximate the cosine term in the energy). Therefore, correlation function $W(C)$ is similar to (C.262) but with the implementation of $\sum_{n=0}^{N-1} \delta\left(A - \frac{2\pi}{N}n\right) = \frac{N}{2\pi} \sum_{P=-\infty}^{\infty} e^{iNPA}$ to take into account the discrete set of values of field $n_{\nu, \vec{r}}$

$$W(C) = \frac{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu < \nu} dB_{\mu\nu, \vec{r}} \sum_{\{P_{\mu, \vec{r}}\}} \sum_{\{g_{\mu\nu, \vec{r}}\}} \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} + iN \sum_{\vec{r}, \mu} P_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \times}{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu < \nu} dB_{\mu\nu, \vec{r}} \prod_{\vec{r}, \mu} \sum_{P_{\mu, \vec{r}}=-\infty}^{\infty} \prod_{\vec{r}, \mu < \nu} \sum_{g_{\mu\nu, \vec{r}}=-\infty}^{\infty} \exp \left[+iN \sum_{\vec{r}, \mu} P_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \times} \times \exp \left[-\frac{a^{4-d}e^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}}) \right] \times \exp \left[-\frac{a^{4-d}e^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}}) \right], \quad (D.191)$$

where the extension from integration range $A_{\mu, \vec{r}} \in [0, 2\pi)$ to $A_{\mu, \vec{r}} \in (-\infty, \infty)$ has been done by means of imposing a constraint on sum $\prod_{\vec{r}, \mu < \nu} \sum_{g_{\mu\nu, \vec{r}}=-\infty}^{\infty}$. Integer field $J_{\mu, \vec{r}}$ defines closed contour C , as already seen: $J_{\nu, \vec{r}} = -\sum_{\mu} \Delta_\mu L_{\mu\nu, \vec{r}-\hat{\mu}}$ with integer field $L_{\mu\nu, \vec{r}}$ equal to $+1/-1$ for plaquettes inside C , being $+1$ the value if the plaquette is covered in the same direction as path C .

Integration of $\prod_{\vec{r}, \mu} dA_{\mu, \vec{r}}$ produces $\sum_{\mu} \Delta_\mu B_{\mu\nu, \vec{r}-\hat{\mu}} = J_{\nu, \vec{r}} + NP_{\mu, \vec{r}}$ in the numerator (and the same constraint but with $J_{\nu, \vec{r}} = 0$ in the denominator). Before solving such a constraint one should note that, since with gauge invariant energies only gauge invariant correlation functions can be calculated, integer field $P_{\mu, \vec{r}}$ cannot take any value, but only those for which $\sum_{\mu} \Delta_\mu P_{\mu, \vec{r}-\hat{\mu}} = 0$ holds, as one can see by looking at term $iN \sum_{\vec{r}, \mu} P_{\mu, \vec{r}} A_{\mu, \vec{r}}$: a local change $A_{\mu, \vec{r}} \rightarrow A_{\mu, \vec{r}} - \Delta_\mu \chi_{\vec{r}}$ does not have any influence in the partition function as long as $0 = \sum_{\vec{r}, \mu} P_{\mu, \vec{r}} \Delta_\mu \chi_{\vec{r}} = -\sum_{\vec{r}, \mu} (\Delta_\mu P_{\mu, \vec{r}-\hat{\mu}}) \chi_{\vec{r}}$ (or equivalently, those $P_{\mu, \vec{r}}$ -configurations which do not fulfil $\sum_{\mu} \Delta_\mu P_{\mu, \vec{r}-\hat{\mu}} = 0$ will produce an exactly vanishing contribution since every $A_{\mu, \vec{r}}$ is accompanied by infinite many different $\chi_{\vec{r}}$'s).



Therefore, one can write $P_{\nu, \vec{r}} = -\sum_{\mu} \Delta_{\mu} l_{\mu\nu, \vec{r}-\hat{\mu}}$ with $l_{\mu\nu, \vec{r}}$ an antisymmetric integer field defined analogously to $L_{\mu\nu, \vec{r}}$: $+|P_{\nu}| - |P_{\nu}|$ for plaquettes inside the closed path defined by $P_{\nu, \vec{r}}$ with the same/opposite direction as $P_{\nu, \vec{r}}$. Equivalently, one can define $l_{\mu\nu, \vec{r}}$ as a plaquette integer field, with a value equal to the amount of $P_{\nu, \vec{r}}$ on the site attached to plaquette $(\vec{r}, \mu\nu)$, and covering with this value all plaquettes from the site of $P_{\nu, \vec{r}}$ toward a reference point along the μ -direction (in the figure, far on the left). The solution of the constraint on $B_{\mu\nu, \vec{r}}$ is

$$B_{\mu\nu, \vec{r}} = \sum_{\lambda} \epsilon_{\mu\nu\lambda} \Delta_{\lambda} C_{\vec{r}-\hat{\lambda}} - L_{\mu\nu, \vec{r}} - N l_{\mu\nu, \vec{r}}, \quad (\text{D.192})$$

for a 3-dim. lattice (the 4-dim. case is obtained straightforwardly). From this point on we shall focus our attention only on the 3-dim. case, whose discussion runs parallel to that for the Villain formulation of the pure compact $U(1)$ gauge model, by changing $L_{\mu\nu, \vec{r}} \rightarrow L_{\mu\nu, \vec{r}} + N l_{\mu\nu, \vec{r}}$ ³⁹. By comparing with (C.275) one obtains

$$W(C) = \frac{\prod_{\vec{r}, \mu < \nu} \sum_{l_{\mu\nu, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \sum_{G_{\vec{r}} = -\infty}^{\infty} \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} (L_{\mu\nu, \vec{r}} + N l_{\mu\nu, \vec{r}})^2 \right] \times}{\prod_{\vec{r}, \mu < \nu} \sum_{l_{\mu\nu, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \sum_{G_{\vec{r}} = -\infty}^{\infty} \exp \left[-\frac{N^2 ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} l_{\mu\nu, \vec{r}}^2 \right] \times} \quad (\text{D.193})$$

$$\times \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} \left(G_{\vec{r}} + i \frac{ae^2\eta}{2\pi} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} L_{\mu\nu, \vec{r}} + N \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} l_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(\begin{pmatrix} \\ \end{pmatrix}_{\vec{r}'} \right) \right]$$

$$\times \exp \left[-\frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} \left(G_{\vec{r}} + i \frac{Nae^2\eta}{2\pi} \sum_{\lambda, \mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})\lambda} l_{\mu\nu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(G_{\vec{r}'} + i \frac{Nae^2\eta}{2\pi} \sum_{\lambda', \mu' < \nu'} \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{r}')\lambda'} l_{\mu'\nu', \vec{r}'} \right) \right],$$

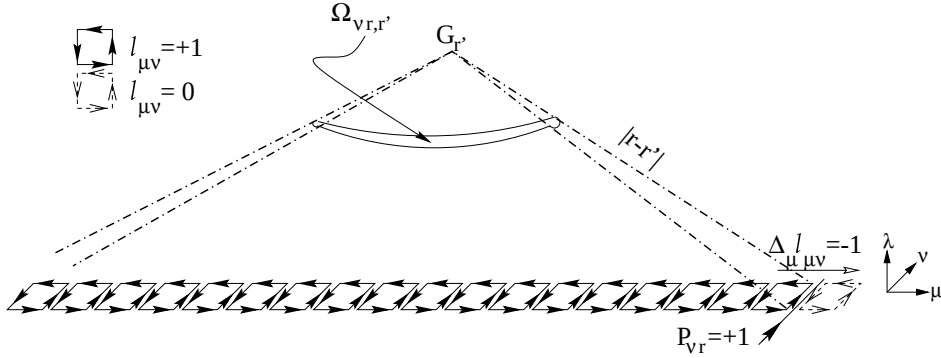
where $G_{\vec{r}} = \sum_{\lambda=1}^{d=3} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$ is the point-monopole density (or the density of points

³⁹Integer field $l_{\mu\nu, \vec{r}}$ has no effect in the product with $i2\pi B_{\mu\nu, \vec{r}} g_{\mu\nu, \vec{r}}$ since $g_{\mu\nu, \vec{r}}$ is also an integer field.

where able-to-glide dislocations become sessile)⁴⁰. The above equation is simplified: firstly, by using the relation between the derivative parallel and perpendicular to the area enclosed by $J_{\mu,\vec{r}}$ and by $P_{\mu,\vec{r}}$ ⁴¹

$$\begin{aligned} & \sum_{\vec{r},\vec{r}'} \left(\sum_{\lambda,\mu<\nu} \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})}^{\lambda} L_{\mu\nu,\vec{r}} + N \epsilon_{\lambda\mu\nu} \Delta_{(\vec{r})}^{\lambda} l_{\mu\nu,\vec{r}} \right) D_{\vec{r},\vec{r}'} \left(\sum_{\lambda',\mu'<\nu'} \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{r}')}^{\lambda'} L_{\mu'\nu',\vec{r}'} + N \epsilon_{\lambda'\mu'\nu'} \Delta_{(\vec{r}')}^{\lambda'} l_{\mu'\nu',\vec{r}'} \right) - \\ & - \sum_{\vec{r},\mu<\nu} (L_{\mu\nu,\vec{r}} + N l_{\mu\nu,\vec{r}})^2 = - \sum_{\vec{r},\vec{r}'} \sum_{\mu,\mu'} \delta_{\mu\mu'} \left[\underbrace{\left(\sum_{\nu} \Delta_{\nu} L_{\mu\nu,\vec{r}-\hat{\nu}} \right)}_{J_{\mu,\vec{r}}} + N \underbrace{\left(\sum_{\nu} \Delta_{\nu} l_{\mu\nu,\vec{r}-\hat{\nu}} \right)}_{P_{\mu,\vec{r}}} \right] D_{\vec{r},\vec{r}'} \left[\right]_{\mu',\vec{r}'} , \end{aligned}$$

and secondly by noting that since $P_{\mu,\vec{r}}$ is a closed path, then $\sum_{\vec{r},\lambda} \sum_{\mu<\nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{(\vec{r})}^{\lambda} l_{\mu\nu,\vec{r}} \right) D_{\vec{r},\vec{r}'} = \frac{|P_{\mu}|}{4\pi} (\Omega_P)_{\vec{r}'}$, where site $\vec{r}'$ observes the $P_{\mu,\vec{r}}$ -closed path under a solid angle equal to $(\Omega_P)_{\vec{r}'}$, just as in equation (C.277):



Since the $P_{\nu,\vec{r}}$ -closed path can bifurcate into several closed paths or into segments which end again in the primitive closed path, it is better to write $\sum_{\vec{r},\lambda} \sum_{\mu<\nu} \epsilon_{\lambda\mu\nu} \left(\Delta_{(\vec{r})}^{\lambda} l_{\mu\nu,\vec{r}} \right) D_{\vec{r},\vec{r}'}$ as $\frac{1}{4\pi} \sum_{\vec{r},\nu} P_{\nu,\vec{r}} \Omega_{\nu,\vec{r},\vec{r}'}$, where $\Omega_{\nu,\vec{r},\vec{r}'}$ is equal to the solid angle observed from site $\vec{x}'$ of the plaquette strip above defined which starts at bond $(\vec{r},\nu)$ of the $P_{\nu,\vec{r}}$ -path. This can be

⁴⁰In the 4-dim. model instead of $G_{\vec{r}}$ one has a vector monopole density $G_{\rho,\vec{r}} = - \sum_{\lambda} \sum_{\mu<\nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu,\vec{r}}$ (with $\epsilon_{1234} = 1$) which fulfils $\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}} = 0$.

⁴¹If one defines the derivative perpendicular to the closed path described by $P_{\mu,\vec{r}}$ as $\Delta_{(\vec{r})}^{\perp} = \sum_{\lambda} \sum_{\mu<\nu} l_{\mu\nu,\vec{r}} \epsilon_{\mu\nu\lambda} \Delta_{(\vec{r})}^{\lambda}$ and the parallel derivatives along the two directions on the surface where such a closed path is resting $\Delta_{(\vec{r})}^{\parallel}(\mu) = \sum_{\nu} l_{\mu\nu,\vec{r}} \Delta_{(\vec{r})}^{\nu}$, then the following holds

$$\Delta_{\perp} \Delta_{\perp} D_{\vec{r},\vec{r}'} + \sum_{\mu,\mu'} \Delta_{\parallel}(\mu) \Delta_{\parallel}(\mu') \delta_{\mu\mu'} D_{\vec{r},\vec{r}'} = \sum_{\mu<\nu} l_{\mu\nu,\vec{r}} l_{\mu\nu,\vec{r}'} \delta_{\vec{r},\vec{r}'} ,$$

and the same for $L_{\mu\nu,\vec{r}}$.

easily seen as follows: since $\Delta_\lambda D_{\vec{r}, \vec{r}'}^{(d=3)} = \frac{1}{4\pi} \frac{r_\lambda - r'_\lambda}{|\vec{r} - \vec{r}'|} \frac{1}{|\vec{r} - \vec{r}'|^2}$, $\sum_{\lambda, \mu < \nu} l_{\mu\nu, \vec{r}} \Delta_\lambda D_{\vec{r}, \vec{r}'}$ is just the solid angle stretched by the plaquette $l_{\mu\nu, \vec{r}}$ as seen from site $\vec{r}'$ (where topological field $G_{\vec{r}'}$ is located; see figure for an example with $|P_\nu| = 1$. Not forgetting the constraint on $P_{\mu, \vec{r}}$ imposed by the gauge invariance of the model, correlation function $W(C)$ takes the following form

$$\begin{aligned} & \frac{\sum_{\{P_{\mu, \vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\mu} \Delta_{\mu} P_{\mu, \vec{r} - \hat{\mu}}, 0\right) \sum_{\{G_{\vec{r}'}\}} \exp\left\{-\frac{iN}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} \left[P_{\nu, \vec{r}} + \frac{1}{N} J_{\nu, \vec{r}}\right] \Omega_{\nu, \vec{r}, \vec{r}'} G_{\vec{r}'}\right\} \times}{\prod_{\vec{r}, \mu} \left[\sum_{P_{\mu, \vec{r}} = -\infty}^{\infty} \delta\left(\sum_{\mu} \Delta_{\mu} P_{\mu, \vec{r} - \hat{\mu}}, 0\right) \right] \prod_{\vec{r}} \sum_{G_{\vec{r}} = -\infty}^{\infty} \exp\left\{-\frac{iN}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} \Omega_{\nu, \vec{r}, \vec{r}'} G_{\vec{r}'}\right\} \times} \\ & \times \exp\left\{-\frac{N^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} \left[P_{\nu, \vec{r}} + \frac{1}{N} J_{\nu, \vec{r}}\right] D_{\vec{r}, \vec{r}'} \left[P_{\nu, \vec{r}'} + \frac{1}{N} J_{\nu, \vec{r}'}\right] - \frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'}\right\} \\ & \times \exp\left\{-\frac{N^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} D_{\vec{r}, \vec{r}'} P_{\nu, \vec{r}'} - \frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'}\right\}, \end{aligned} \quad (\text{D.194})$$

with $\sum_{\vec{r}, \mu} J_{\mu, \vec{r}} \Omega_{\nu, \vec{r}, \vec{r}'} = \Omega_{\vec{r}'}$ and $\Omega_{\vec{r}'}$ the solid angle for path C observed from $\vec{r}'$ ⁴².

For the 4-dim. model the proof runs analogously, although both topological fields $G_{\mu, \vec{r}}$ and the plaquette strip's solid angle $\Omega_{\nu, \vec{r}, \mu, \vec{r}'}$ are tensors of one order higher than in the 3-dim. model; further, the topological field fulfils $\sum_{\mu} \Delta_{\mu} G_{\mu, \vec{r}} = 0$. As a result, the 4-dim. $Z(N)$ gauge model in the Villain approximation becomes self-dual by interchanging

$$\begin{cases} G_{\mu, \vec{r}} & \longleftrightarrow & P_{\mu, \vec{r}}, \\ \frac{1}{e^2 \eta} & \longleftrightarrow & \frac{N^2}{4\pi^2} e^2 \eta, \end{cases} \quad (\text{D.195})$$

very similarly to the self-duality relation (D.164) for the 2-dim. $Z(N)$ Ising model. And as in that model, if the 4-dim. $Z(N)$ had only two phases, something which only occurs for small N , then the phase-transition point would be given by $(e^2 \eta)_{\text{crit}} = \frac{2\pi}{N}$; however, since limit $N \rightarrow \infty$ is the pure $U(1)$ gauge model, which shows also two phases, but the limit of self-dual transition point vanishes, which would indicate the existence of one unique phase, there must exist a third phase for N large enough. The two transition points $(e^2 \eta)_{\text{crit}, 1}$, $(e^2 \eta)_{\text{crit}, 2}$ which form the limits of this third phase are related to each other by duality equation $\frac{1}{(e^2 \eta)_{\text{crit}, 1}} = \frac{N^2}{4\pi^2} (e^2 \eta)_{\text{crit}, 2}$.

⁴²Limit $N \rightarrow \infty$ together with term $\frac{N^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} D_{\vec{r}, \vec{r}'} P_{\nu, \vec{r}'}$ freezes out field $P_{\mu, \vec{r}}$ to zero and one recovers thus the Villain formulation of the compact $U(1)$ gauge model.

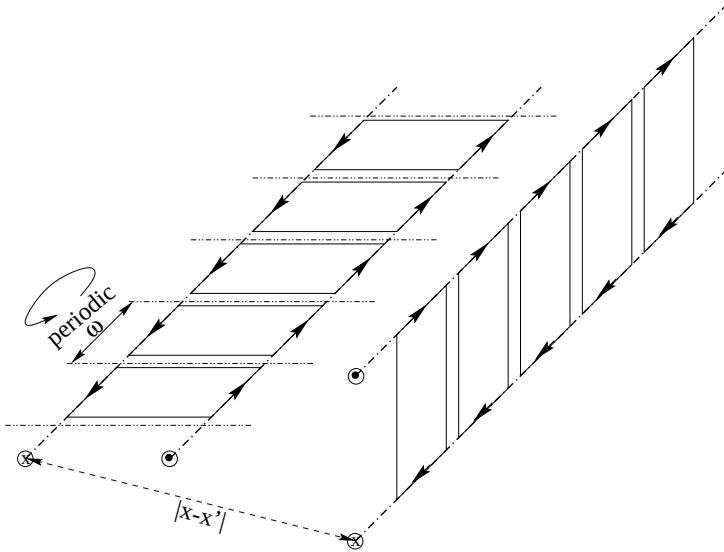
The critical value for the 4-dim. $Z(2)$ gauge model obtained by means of the Villain formulation $(e^2\eta)_{\text{crit}} = \pi$ is an approximation to the exact value $(e^2\eta)_{\text{crit}} = 0.441^{-1}$ evaluated in (C.368).

Up to this point all the above results hold for a d -dim. non-periodic lattice. For the study of such a model at a nonvanishing temperature T , one has to consider the model in a lattice which is periodic along a direction (the τ -direction), assuming that its periodic length ω is related to the temperature by means of $\omega \sim T^{-1}$. Path C which defines $W(C)$ is chosen so that one of its sides runs parallel to the τ -direction and has as length ω . At small ω , or at any ω but with e^2 large, the lattice's periodicity makes topological point defects $G_{\vec{r}}$ form a line denoted by $G_{\vec{x}}$, being $\vec{x}$ the site where such a line crosses the plane normal to the τ -direction. By using relation $D_{\vec{r},\vec{r}'}^{(d=3)} \approx \frac{a}{\omega} D_{\vec{x},\vec{x}'}^{(d=2)}$, which holds as long as $a|\vec{x} - \vec{x}'| \gg \omega$, one obtains

$$\sum_{\vec{r},\vec{r}'} G_{\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} G_{\vec{r}'} = D(0) \sum_{\vec{r}} G_{\vec{r}}^2 + \sum_{\vec{r},\vec{r}' \neq \vec{r}} G_{\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} G_{\vec{r}'} \rightarrow D^{(d=3)}(0) \sum_{\vec{x}} G_{\vec{x}}^2 - \frac{a}{\omega} \frac{1}{2\pi} \sum_{\vec{x},\vec{x}' \neq \vec{x}}' G_{\vec{x}} \ln |\vec{x} - \vec{x}'| G_{\vec{x}'}$$

with $D^{(d=3)}(0) \approx 0.253$ and where sum $\sum_{\vec{x},\vec{x}' \neq \vec{x}}'$ means that the following relation holds

for the topological field: $\sum_{\vec{x}} G_{\vec{x}} \approx 0$, just as discussed for the $U(1)$ gauge model.



For large $P_{\nu,\vec{r}}$ -loops which stretch along the periodic direction as in the figure, $\sum_{\vec{r},\vec{r}' \neq \vec{r}} \sum_{\nu} P_{\nu,\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} P_{\nu,\vec{r}'}$ also becomes the interaction, carried by propagator $\frac{a}{\omega} D_{\vec{x},\vec{x}'}^{(d=2)}$, of scalar fields $\bar{P}_{\vec{x}}$ located on the plane normal to the periodic direction:

$$\begin{aligned} \sum_{\vec{r},\vec{r}'} \sum_{\nu} P_{\nu,\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} P_{\nu,\vec{r}'} &\approx \\ &\approx D^{(d=3)}(0) \sum_{\vec{r}} \left(\frac{\omega}{a} \bar{P}_{\vec{x}} \right)^2 - \\ &- \frac{a}{\omega} \sum_{\vec{x},\vec{x}' \neq \vec{x}}' \frac{\omega}{a} \bar{P}_{\vec{x}} \frac{\ln |\vec{x} - \vec{x}'|}{2\pi} \frac{\omega}{a} \bar{P}_{\vec{x}'} \end{aligned}$$

where one has assumed that the length of the $P_{\nu,\vec{r}}$ -loops along the periodic direction is of the same order as ω : for the range at which we are working, this is satisfied for ω small. In the range of large e^2 , and large ω , energy's term $\frac{N^2 a e^2 \eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\nu} P_{\nu,\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} P_{\nu,\vec{r}'}$

forbids large $P_{\nu,\vec{r}}$ -loops, but in such a range the role of these loops is irrelevant as we have already seen by means of the Hamiltonian formulation: the behaviour of $\mathcal{V}_{\vec{x}_1,\vec{x}_2} \sim |\vec{x}_2 - \vec{x}_1|$ is the same as in the compact $U(1)$ gauge model, where $P_{\nu,\vec{r}}$ does not appear. However,

another range where large $P_{\nu,\vec{r}}$ would be secured for any ω were at e^2 small, but there the density of $G_{\vec{r}}$ would be so low that relation $D_{\vec{r},\vec{r}'}^{(d=3)} \approx \frac{a}{\omega} D_{\vec{x},\vec{x}'}^{(d=2)}$ would not hold for the interactions between the $G_{\vec{r}}$'s if ω is not small enough. This is an indication that at small e^2 , the appearance of large $P_{\nu,\vec{r}}$ -loops may produce a different behaviour on the $Z(N)$ gauge model when compared with the behaviour of the $U(1)$ gauge model, something which will be considered in the next Note.

Remaining in the range of small ω so that relation $\frac{ae^2\eta}{2} \sum_{\vec{r},\vec{r}'} \sum_{\nu} P_{\nu,\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} P_{\nu,\vec{r}'} \approx \frac{e^2\omega^2\eta}{2a} D^{(d=3)}(0) \sum_{\vec{r}} \bar{P}_{\vec{x}}^2 + \frac{e^2\omega\eta}{2} \sum_{\vec{x},\vec{x}' \neq \vec{x}} ' \bar{P}_{\vec{x}} \left(-\frac{\ln|\vec{x}-\vec{x}'|}{2\pi} \right) \bar{P}_{\vec{x}'}$ is satisfied, where sum $\sum_{\vec{x},\vec{x}' \neq \vec{x}} '$ denotes that field $\bar{P}_{\vec{x}}$ has to fulfil neutrality condition $\sum_{\vec{x}} \bar{P}_{\vec{x}} = 0$ (since this is a density of crossing points belonging to closed paths and thus there are so many positive as negative points), also those terms involving solid angles take the form of $\Xi_{\vec{x},\vec{x}'}$ -angles measured in the $(d-1)$ -dim. plane normal to the τ -direction: $\frac{1}{4\pi} \sum_{\vec{r},\nu} P_{\nu,\vec{r}} \Omega_{\nu\vec{r},\vec{r}'} = \sum_{\vec{r}} \sum_{\lambda,\mu < \nu} \epsilon_{\lambda\mu\nu} \frac{\Delta_{(\vec{r})}}{\lambda} l_{\mu\nu,\vec{r}} D_{\vec{r},\vec{r}'}^{(d=3)} \rightarrow \sum_{\vec{x}} \sum_{j,k=1}^{d-1} \frac{\omega}{a} \epsilon_{j k (\vec{x})} \bar{l}_{k,\vec{x}} \frac{a}{\omega} D_{\vec{x},\vec{x}'}^{(d=2)} = \frac{1}{2\pi} \sum_{\vec{x}} \bar{P}_{\vec{x}} \Xi_{\vec{x},\vec{x}'}$, together with $\sum_{k=1}^{d-1} \Delta_k \bar{l}_{k,\vec{x}-\hat{k}} = -\bar{P}_{\vec{x}}$. After substituting these results into (D.194), $W(C)$ for the 3-dim. periodic $Z(N)$ gauge model at small ω takes the following form

$$\begin{aligned}
W(C) = & \frac{\sum_{\{\bar{P}_{\vec{x}}\}} ' \sum_{\{G_{\vec{x}}\}} ' \exp \left[-\frac{N^2 e^2 \omega^2 \eta}{2a} D(0) \sum_{\vec{x}} \left[\bar{P}_{\vec{x}} + \frac{(\delta_{\vec{x},\vec{x}_1} - \delta_{\vec{x},\vec{x}_2})}{N} \right]^2 - \frac{2\pi^2 D(0)}{ae^2\eta} \sum_{\vec{x}} G_{\vec{x}}^2 \right] \times}{\prod_{\vec{x}} \sum_{\bar{P}_{\vec{x}}=-\infty}^{\infty} \delta \left(\sum_{\vec{x}} \bar{P}_{\vec{x}}, 0 \right) \prod_{\vec{x}} \sum_{G_{\vec{x}}=-\infty}^{\infty} \delta \left(\sum_{\vec{x}} G_{\vec{x}}, 0 \right) \exp \left[-\frac{N^2 e^2 \omega^2 \eta}{2a} D(0) \sum_{\vec{x}} \bar{P}_{\vec{x}}^2 - \frac{2\pi^2 D(0)}{ae^2\eta} \sum_{\vec{x}} G_{\vec{x}}^2 \right] \times} \\
& \times \exp \left[-iN \sum_{\vec{x},\vec{x}'} \left[\bar{P}_{\vec{x}} + \frac{(\delta_{\vec{x},\vec{x}_1} - \delta_{\vec{x},\vec{x}_2})}{N} \right] \Xi_{\vec{x},\vec{x}'} G_{\vec{x}'} - \frac{2\pi^2}{e^2\omega\eta} \sum_{\vec{x},\vec{x}' \neq \vec{x}} G_{\vec{x}} D_{\vec{x},\vec{x}'}^{(d=2)} G_{\vec{x}'} \right] \times \\
& \times \exp \left[-iN \sum_{\vec{x},\vec{x}'} \bar{P}_{\vec{x}} \Xi_{\vec{x},\vec{x}'} G_{\vec{x}'} - \frac{2\pi^2}{e^2\omega\eta} \sum_{\vec{x},\vec{x}' \neq \vec{x}} G_{\vec{x}} D_{\vec{x},\vec{x}'}^{(d=2)} G_{\vec{x}'} \right] \times \\
& \times \exp \left[-\frac{N^2 e^2 \omega \eta}{2} \sum_{\vec{x},\vec{x}' \neq \vec{x}} \left[\bar{P}_{\vec{x}} + \frac{(\delta_{\vec{x},\vec{x}_1} - \delta_{\vec{x},\vec{x}_2})}{N} \right] D_{\vec{x},\vec{x}'}^{(d=2)} \left[\bar{P}_{\vec{x}'} + \frac{(\delta_{\vec{x}',\vec{x}_1} - \delta_{\vec{x}',\vec{x}_2})}{N} \right] \right] \\
& \times \exp \left[-\frac{N^2 e^2 \omega \eta}{2} \sum_{\vec{x},\vec{x}' \neq \vec{x}} \bar{P}_{\vec{x}} D_{\vec{x},\vec{x}'}^{(d=2)} \bar{P}_{\vec{x}'} \right],
\end{aligned}$$

with $\vec{x}_1$ and $\vec{x}_2$ the sites where path C crosses the plane normal to the τ -direction. As expected, the above equation has the same form, up to the selfenergy terms, as the Villain formulation for the correlation function of the $Z(N)$ Ising model in a 2-dim.

non-periodic lattice (D.163)

$$W(C)_{d\text{-dim. period. } Z(N) \text{ gauge}}^{\text{small } \omega} = \left\langle e^{i \frac{2\pi}{N} (n_{\vec{x}_2} - n_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim. non-period. } Z(N) \text{ Ising}}, \quad (\text{D.196})$$

with $\beta_{\text{Ising}} = \frac{1}{e^2 \omega}$. The same holds for the 4-dim. $Z(N)$ gauge model, with $\beta_{\text{Ising}} = \frac{a}{e^2 \omega}$. According to relation $\omega \sim T^{-1}$, approximation $\omega \ll a|\vec{x} - \vec{x}'|$ is satisfied at high temperatures.

According to equation (D.196), as well as to result (D.184) $\exp\left(-\frac{\omega}{\eta} \mathcal{V}\right)_{d\text{-dim. per. } Z(2) \text{ gauge}}^{\text{large } e^2} = \left\langle e^{i\pi(n_{\vec{x}_1} - n_{\vec{x}_2})} \right\rangle_{(d-1)\text{-dim. Ising}}$ obtained by using the Hamiltonian version of Statistical Physics, for the $Z(N=2)$ model and $d \geq 3$ the behaviour of $W(C)$ and of $\mathcal{V}$ can be directly read off how the scalar-scalar correlation function of the Ising model $\langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle$ behaves:

$$\begin{aligned} \text{large } e^2 \omega &\Leftrightarrow \text{small } \beta_{\text{Ising}} \implies \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{\text{Ising}} \equiv W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}, \\ \text{small } e^2 \omega &\Leftrightarrow \text{large } \beta_{\text{Ising}} \implies \langle \sigma_{\vec{x}_1} \sigma_{\vec{x}_2} \rangle_{\text{Ising}} \equiv W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}. \end{aligned} \quad (\text{D.197})$$

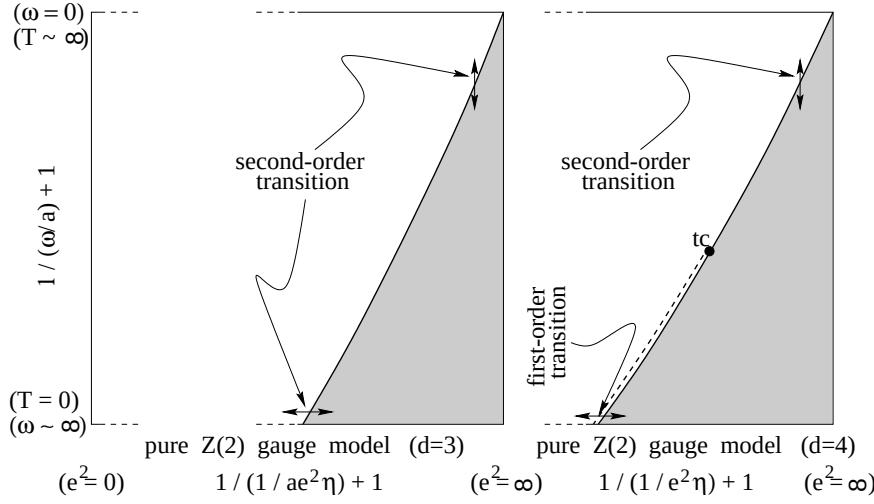
The critical value $(e^2 \omega \eta)_{\text{crit}}$ which separates both behaviours is given by the critical values of the coupling strength of the $(d-1)$ -dim. Ising model coupling⁴³

$$\begin{aligned} \text{for } (d-1) = 2 \quad \left(\frac{\beta_{\text{Ising}}}{\eta} \right)_{\text{crit}} &\equiv \left(\frac{1}{e^2 \omega \eta} \right)_{\text{crit}} = 0.441, \\ \text{for } (d-1) = 3 \quad \left(\frac{\beta_{\text{Ising}}}{\eta} \right)_{\text{crit}} &\equiv \left(\frac{1}{e^2 \omega \eta} \right)_{\text{crit}} = 0.227, \end{aligned} \quad (\text{D.198})$$

where the first value was calculated in (C.366) using the self-duality property of the 2-dim. Ising model, and the second value has been computed numerically (in D. J. Wallace *et al.*, *Phys. Rev. B* **29**, 403 (1986), quoted in [78], page 138).

The transition of the $(d-1)$ -dim. $Z(2)$ Ising model is of second-order kind, both for $d-1=2$ and $d-1=3$, whereas the transition of the non-periodic ($\omega = \infty$) $Z(2)$ gauge model is second-order at $d=3$ but first-order for $d=4$. Hence, the phase diagram of the d -dim. (periodic) $Z(2)$ gauge model in dependence of ω (or of temperature $T \sim 1/\omega$) in the range of large e^2 is similar to the diagram of the 4-dim. (periodic) compact $U(1)$ gauge model in page 554 but with the difference that for $d=4$ the transition at $\omega = \infty$ is of first order.

⁴³The lattice spacing a along the non-periodic dimensions has been set equal to 1; if one prefers to work with it, the coupling strength for the 4-dimensional model has to be written as $\left(\frac{a \beta_{\text{Ising}}}{\eta} \right)_{\text{crit}} \equiv \left(\frac{a}{\omega e^2 \eta} \right)_{\text{crit}} = 0.227$.



However, such a transition is not stable when the lattice becomes periodic with ω small, since then the effective propagator at large distances $D_{\vec{x},\vec{x}'}^{(d=3)}_{\text{period.}} \sim |\vec{x} - \vec{x}'|^{-1}$ is stronger than the $\omega = \infty$ propagator $D_{\vec{r},\vec{r}'}^{(d=4)}_{\text{non-period.}} \sim |\vec{r} - \vec{r}'|^{-2}$: a change at the long range behaviour changes the character of the phase transition.

Therefore there exists a tricritical point (denoted by tc in the above figure) which separates the line of transition of first-order kind (solid-dashed line) from those of second-order type. The shaded area of the above diagrams denotes the phase characterized by $\mathcal{V} \sim |\vec{x}_2 - \vec{x}_1|$ whilst the unshaded regions are those where $\mathcal{V} \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$, with $\mathcal{V} = -\frac{\eta}{\omega} \ln W(C)$ and C a path covering the complete lattice periodic length ω .

Note that in the above phase diagrams, the behaviour of the $Z(2)$ gauge model at small e^2 has not been depicted: from the Hamiltonian operator describing the model

$$\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in (d-1)\text{-dim. } \Lambda_{\perp}} \sum_{j=1}^{d-1} (\hat{\sigma}_1)_{j,\vec{x}} + \kappa_e \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j=1}^{d-1} \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right], \text{ where}$$

κ_e is a decreasing function of e^2 , it becomes clear that the partition of the system at small e^2 (large κ_e) is that of the $(d-1)$ -dim. non-periodic $Z(2)$ gauge model

$$\begin{aligned} \mathcal{Z}_{\text{large } \kappa_e} &= \text{Tr} \left[\exp \left(-\frac{\omega \kappa_e}{\eta} \sum_{\vec{x} \in (d-1)\text{-dim. } \Lambda_{\perp}} \sum_{j=1}^{d-1} \left(1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right) \right) \right] = \\ &= \prod_{\vec{x} \in (d-1)\text{-dim. } \Lambda_{\perp}} \prod_{j=1}^{d-1} \sum_{\sigma_{j,\vec{x}} = \pm 1} \exp \left[-\frac{\omega \kappa_e}{\eta} \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j=1}^{d-1} \left(1 - \sigma_{j,\vec{x}} \sigma_{k,\vec{x}+\hat{j}} \sigma_{j,\vec{x}+\hat{k}} \sigma_{k,\vec{x}} \right) \right], \quad (\text{D.199}) \end{aligned}$$

with coupling strength $\beta_e \equiv \kappa_e \omega$. The $Z(2)$ gauge model in a 2-dimensional lattice non-periodic along any dimension has no phase transition (C.347); on the contrary, in a 3-dimensional it does take place. Hence, the d -dimensional $Z(2)$ gauge model in a lattice of periodicity ω along the τ -direction will not show any phase transition at small e^2 if $d = 3$; if $d = 4$, on the other hand, in the range of small e^2 there will occur a phase-transition line given by $\frac{\kappa_e \omega}{\eta} = \left(\frac{\beta_e}{\eta} \right)_{\text{crit. 3-dim. non-per. } Z(2) \text{ gauge}}$, where $\left(\frac{\beta_e}{\eta} \right)_{\text{crit.}} = 0.761$ is the critical value for the 3-dimensional non-periodic $Z(N=2)$ gauge model (C.364). In the following Note, the range of small e^2 will be considered in more detail in order to cal-

culate a correlation function able to distinguish between the two phases at $d = 4$.

Note. The phase diagram of the d -dimensional $Z(N = 2)$ gauge model in a periodic lattice along one direction at small e^2 and for $d = 3, 4$.

The Hamiltonian formulation has been especially useful when a lattice of periodicity ω along the τ -dimension was studied in the range of small κ_e (large e^2), and thus the equivalence with scalar models in a non-periodic lattice of one dimensionality lower was set. For κ_e small the same technique can be used but applied to the dual Hamiltonian obtained by the pass to the dual lattice (defined as a lattice of the same dimensionality as the original one but displaced a half of the lattice spacing along the diagonal), with the accompanying change of site-, bond- and plaquette-interactions by the corresponding dual interactions which depend on the lattice's dimensionality: for instance, to a site $\vec{x}$ in a 2-dimensional lattice the dual is the plaquette whose center is $\vec{x}$, while in a 3-dimensional lattice, the dual is a cube surrounding $\vec{x}$. The utility of the formulation by means of the dual Hamiltonian is the resulting interchange between the range of large/small parameters in the original model and the small/large value of the parameters describing the model in the dual formulation: thus to study the original model at large κ_e (small e^2) will become equivalent to consider the dual Hamiltonian for the small range of the corresponding coupling strength, range in which the same technique as in the original model at small κ_e follows.

The above strategy has been already applied on the search for the different phases of the Ising model in a periodic lattice (see equations (D.58) and (D.63)); let us apply it now on the 4-dimensional $Z(2)$ gauge model, whose Hamiltonian operator, formulated in the 3-dimensional sublattice $\Lambda_\perp$ normal to

the τ -direction, is ⁴⁴ $\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in 3-\text{dim. } \Lambda_\perp} \sum_{j=1}^3 (\hat{\sigma}_1)_{j,\vec{x}} + \kappa_e \sum_{\vec{x} \in \Lambda_\perp} \sum_{j < k=1}^3 \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right]$.

From the direct calculation of the partition function in a non-periodic lattice we already know that the 4-dim. $Z(2)$ gauge model is self-dual (C.367), property which is of course maintained in a periodic lattice. Now we shall re-obtain this self-duality but in the Hamiltonian version.

Returning to the operator language, the repetition of the same contribution, arising from different configurations related to each other by a gauge transformation, is avoided by means of constraint

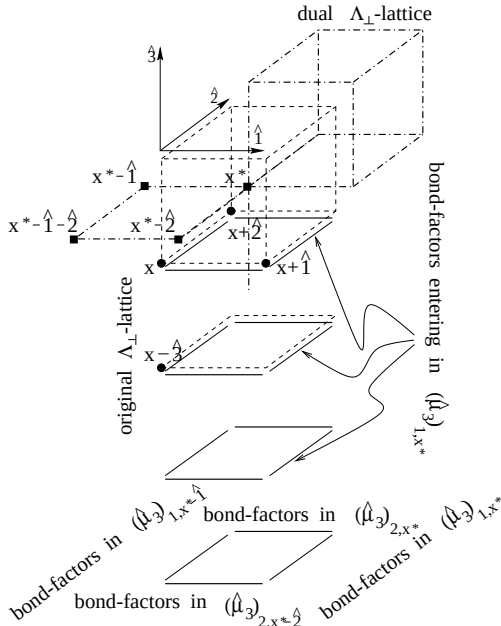
$$(\hat{\sigma}_1)_{1,\vec{x}} (\hat{\sigma}_1)_{-1,\vec{x}} (\hat{\sigma}_1)_{2,\vec{x}} (\hat{\sigma}_1)_{-2,\vec{x}} (\hat{\sigma}_1)_{3,\vec{x}} (\hat{\sigma}_1)_{-3,\vec{x}} = 1 \quad \forall \vec{x} \in \Lambda_\perp, \quad (\text{D.200})$$

imposed on the partition function; one should remember that the above constraint can be relaxed in some points in we are calculating correlation functions. Relation (D.200) allows us to solve $(\hat{\sigma}_1)_{\pm 3,\vec{x}}$ in function of onl $(\hat{\sigma}_1)_{\pm 1,\vec{x}}$'s and $(\hat{\sigma}_1)_{\pm 2,\vec{x}}$'s [42]

$$\begin{aligned} (\hat{\sigma}_1)_{3,\vec{x}} &= \left[(\hat{\sigma}_1)_{1,\vec{x}} (\hat{\sigma}_1)_{-1,\vec{x}} (\hat{\sigma}_1)_{2,\vec{x}} (\hat{\sigma}_1)_{-2,\vec{x}} \right] \underbrace{(\hat{\sigma}_1)_{-3,\vec{x}}}_{(\hat{\sigma}_1)_{3,\vec{x}-\hat{3}}}, \\ (\hat{\sigma}_1)_{3,\vec{x}-\hat{3}} &= \left[(\hat{\sigma}_1)_{1,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{-1,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{2,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{-2,\vec{x}-\hat{3}} \right] (\hat{\sigma}_1)_{3,\vec{x}-2\hat{3}}, \\ (\hat{\sigma}_1)_{3,\vec{x}} &= \left[(\hat{\sigma}_1)_{1,\vec{x}} (\hat{\sigma}_1)_{-1,\vec{x}} (\hat{\sigma}_1)_{2,\vec{x}} (\hat{\sigma}_1)_{-2,\vec{x}} \right] \left[(\hat{\sigma}_1)_{1,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{-1,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{2,\vec{x}-\hat{3}} (\hat{\sigma}_1)_{-2,\vec{x}-\hat{3}} \right] \times \\ &\quad \times \left[(\hat{\sigma}_1)_{1,\vec{x}-2\hat{3}} (\hat{\sigma}_1)_{-1,\vec{x}-2\hat{3}} (\hat{\sigma}_1)_{2,\vec{x}-2\hat{3}} (\hat{\sigma}_1)_{-2,\vec{x}-2\hat{3}} \right] \times \cdots, \end{aligned} \quad (\text{D.201})$$

⁴⁴In this Note the above Hamiltonian's form with $\kappa_e \sum_{\vec{x} \in \Lambda_\perp} \sum_{j < k=1}^{d-1} \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right]$ has been preferred to $\hat{\mathcal{H}} = -\frac{e^2 \eta^2}{\pi^2 a} \sum_{\vec{x} \in 3-\text{dim. } \Lambda_\perp} \sum_{j=1}^3 (\hat{\sigma}_1)_{j,\vec{x}} + \frac{1}{\pi^2 a e^2} \sum_{\vec{x} \in \Lambda_\perp} \sum_{j < k=1}^3 \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right]$ of equation (D.190) in order to stress the the likeness to the Ising model. One form can be obtained from the other by factorizing a $(\kappa_e \eta)^{1/2}$ out.

very similarly to (D.62) in the 2-dim. Ising model. From this result, all $(\hat{\sigma}_1)_{3,\vec{x}}$'s have been eliminated as real field variables in the Hamiltonian and therefore any $(\hat{\sigma}_3)_{3,\vec{x}}$ commutes with $\hat{\mathcal{H}}$ (which now does not contain $(\hat{\sigma}_1)_{3,\vec{x}}$) and with the transfer matrix: the value of $(\hat{\sigma}_3)_{3,\vec{x}}$ remains constant along the τ -dimension and one can fix this constant value by means of a boundary condition: $(\hat{\sigma}_3)_{3,\vec{x}} = 1 \quad \forall \vec{x}$. As a consequence, the variable dual to $(\hat{\sigma}_3)_{3,\vec{x}}$ will lose also the role of an independent field variable; in the original model there are 2 independent degrees of freedom, $(\hat{\sigma}_{1,3})_{1,\vec{x}}$ and $(\hat{\sigma}_{1,3})_{2,\vec{x}}$, as well as 2 is also the number of real degrees of freedom in the dual model.



Now let us define the following operators located on the dual bonds (see figure)

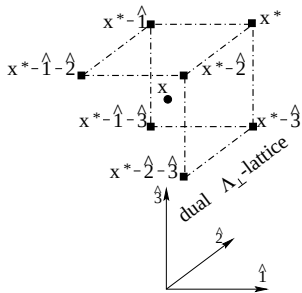
$$\begin{aligned} (\hat{\mu}_3)_{-1,\vec{x}^*} &= (\hat{\mu}_3)_{1,\vec{x}^*-\hat{1}} = \prod_{n' \geq 0} (\hat{\sigma}_1)_{2,\vec{x}-n'\cdot\hat{3}}, \\ (\hat{\mu}_3)_{-2,\vec{x}^*} &= (\hat{\mu}_3)_{2,\vec{x}^*-\hat{2}} = \prod_{n' \geq 0} (\hat{\sigma}_1)_{1,\vec{x}-n'\cdot\hat{3}}, \\ (\hat{\mu}_3)_{1,\vec{x}^*} &= \prod_{n' \geq 0} (\hat{\sigma}_1)_{2,(\vec{x}+\hat{1})-n'\cdot\hat{3}}, \\ (\hat{\mu}_3)_{2,\vec{x}^*} &= \prod_{n' \geq 0} (\hat{\sigma}_1)_{1,(\vec{x}+\hat{2})-n'\cdot\hat{3}}, \end{aligned} \quad (\text{D.202})$$

where $(\hat{\mu}_3)_{3,\vec{x}^*}$ needs not to be considered since, as discussed, it is not a real independent field operator and has been set equal to 1 (in fact one will assume that this choice holds and then show that the final result is consistent with such an assumption). The above definitions lead to the following direct results

$$\begin{aligned} (\hat{\mu}_3)_{-1,\vec{x}^*}(\hat{\mu}_3)_{-1,\vec{x}^*-\hat{3}} &= (\hat{\sigma}_1)_{2,\vec{x}} \left[\prod_{n' \geq 0} (\hat{\sigma}_1)_{2,\vec{x}-n'\cdot\hat{3}} \right]^2 = (\hat{\sigma}_1)_{2,\vec{x}}, \\ (\hat{\mu}_3)_{-2,\vec{x}^*}(\hat{\mu}_3)_{-2,\vec{x}^*-\hat{3}} &= (\hat{\sigma}_1)_{1,\vec{x}} \left[\prod_{n' \geq 0} (\hat{\sigma}_1)_{1,\vec{x}-n'\cdot\hat{3}} \right]^2 = (\hat{\sigma}_1)_{1,\vec{x}}, \end{aligned}$$

as well as, according to equation (D.201), to the following relation for the dual plaquette whose vertices are dual sites $\vec{x}^* - \hat{1} - \hat{2}$, $\vec{x}^* - \hat{2}$, $\vec{x}^*$ and $\vec{x}^* - \hat{1}$

$$\begin{aligned} (\hat{\mu}_3)_{1,(\vec{x}^*-\hat{1}-\hat{2})}(\hat{\mu}_3)_{2,(\vec{x}^*-\hat{1}-\hat{2})+\hat{1}}(\hat{\mu}_3)_{1,(\vec{x}^*-\hat{1}-\hat{2})+\hat{2}}(\hat{\mu}_3)_{2,(\vec{x}^*-\hat{1}-\hat{2})} &= (\hat{\mu}_3)_{-2,\vec{x}^*}(\hat{\mu}_3)_{-2,\vec{x}^*-\hat{1}}(\hat{\mu}_3)_{-1,\vec{x}^*}(\hat{\mu}_3)_{-1,\vec{x}^*-\hat{2}} = \\ &= \prod_{n' \geq 0} \left[(\hat{\sigma}_1)_{1,\vec{x}-n'\cdot\hat{3}} (\hat{\sigma}_1)_{1,(\vec{x}-\hat{1})-n'\cdot\hat{3}} (\hat{\sigma}_1)_{2,\vec{x}-n'\cdot\hat{3}} (\hat{\sigma}_1)_{2,(\vec{x}-\hat{2})-n'\cdot\hat{3}} \right] = (\hat{\sigma}_1)_{3,\vec{x}}. \end{aligned}$$



All this, and remembering choice $(\hat{\mu}_3)_{\pm 3,\vec{x}^*} = 1 \quad \forall \vec{x}^*$, allows us to write bond-operator $(\hat{\sigma}_1)_{k,\vec{x}}$ acting on the bond from site $\vec{x}$ to site $\vec{x} + \hat{k}$ as the product of dual operators $(\hat{\mu}_3)_{i,\vec{x}^*}$ located on the four vertices of the dual plaquette crossed by such a bond $(\vec{x}, k)$

$$\begin{aligned} (\hat{\mu}_3)_{2,\vec{x}^*-\hat{2}-\hat{3}}(\hat{\mu}_3)_{3,\vec{x}^*-\hat{3}}(\hat{\mu}_3)_{2,\vec{x}^*-\hat{2}}(\hat{\mu}_3)_{3,\vec{x}^*-\hat{2}-\hat{3}} &= (\hat{\sigma}_1)_{1,\vec{x}}, \\ (\hat{\mu}_3)_{1,\vec{x}^*-\hat{1}-\hat{3}}(\hat{\mu}_3)_{3,\vec{x}^*-\hat{3}}(\hat{\mu}_3)_{1,\vec{x}^*-\hat{1}}(\hat{\mu}_3)_{3,\vec{x}^*-\hat{1}-\hat{3}} &= (\hat{\sigma}_1)_{2,\vec{x}}, \\ (\hat{\mu}_3)_{1,\vec{x}^*-\hat{1}-\hat{2}}(\hat{\mu}_3)_{2,\vec{x}^*-\hat{2}}(\hat{\mu}_3)_{1,\vec{x}^*-\hat{1}}(\hat{\mu}_3)_{2,\vec{x}^*-\hat{1}-\hat{2}} &= (\hat{\sigma}_1)_{3,\vec{x}}. \end{aligned} \quad (\text{D.203})$$

The other set of bond-operators in the dual lattice are defined as

$$\begin{aligned} (\hat{\mu}_1)_{-2,\vec{x}^*} &= (\hat{\sigma}_3)_{1,\vec{x}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{1}}(\hat{\sigma}_3)_{1,\vec{x}+\hat{3}}(\hat{\sigma}_3)_{3,\vec{x}}, \\ (\hat{\mu}_1)_{2,\vec{x}^*} &= (\hat{\sigma}_3)_{1,\vec{x}+\hat{2}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{2}+\hat{1}}(\hat{\sigma}_3)_{1,\vec{x}+\hat{2}+\hat{3}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{2}}, \\ (\hat{\mu}_1)_{-1,\vec{x}^*} &= (\hat{\sigma}_3)_{2,\vec{x}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{2}}(\hat{\sigma}_3)_{2,\vec{x}+\hat{3}}(\hat{\sigma}_3)_{3,\vec{x}}, \\ (\hat{\mu}_1)_{1,\vec{x}^*} &= (\hat{\sigma}_3)_{2,\vec{x}+\hat{1}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{1}+\hat{2}}(\hat{\sigma}_3)_{2,\vec{x}+\hat{1}+\hat{3}}(\hat{\sigma}_3)_{3,\vec{x}+\hat{1}}, \end{aligned} \quad (\text{D.204})$$

in function of the actual independent fields of the original model: those acting on bonds directed along the 1- and the 2-directions. Although not independent, one can also define $(\hat{\mu}_1)_{\pm 3, \vec{x}^*}$ in an analogous form

$$\begin{aligned} (\hat{\mu}_1)_{-3, \vec{x}^*} &= (\hat{\sigma}_3)_{1, \vec{x}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{1}} (\hat{\sigma}_3)_{1, \vec{x}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{x}} , \\ (\hat{\mu}_1)_{3, \vec{x}^*} &= (\hat{\sigma}_3)_{1, \vec{x}+\hat{3}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{3}+\hat{1}} (\hat{\sigma}_3)_{1, \vec{x}+\hat{3}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{3}} . \end{aligned}$$

From the above definition together with (D.204), it follows immediately

$$(\hat{\mu}_1)_{-1, \vec{x}^*} (\hat{\mu}_1)_{1, \vec{x}^*} (\hat{\mu}_1)_{-2, \vec{x}^*} (\hat{\mu}_1)_{2, \vec{x}^*} (\hat{\mu}_1)_{-3, \vec{x}^*} (\hat{\mu}_1)_{3, \vec{x}^*} = 1 \quad \forall \vec{x}^* , \quad (\text{D.205})$$

since each factor corresponds to one face of a closed cube and thus in the product of the six faces, each bond appears twice. The above relation is just the same constraint as (D.200) but now on the dual variables. This constraint justifies that only $(\hat{\mu}_1)_{1, \vec{x}^*}$ and $(\hat{\mu}_1)_{2, \vec{x}^*}$ have been chosen as independent variables, since from (D.205) operator $(\hat{\mu}_1)_{3, \vec{x}^*}$ can be solved in function of $(\hat{\mu}_1)_{1, \vec{x}^*}$ and $(\hat{\mu}_1)_{2, \vec{x}^*}$. Moreover, operators $(\hat{\mu}_1)_{k, \vec{x}^*}$ and $(\hat{\mu}_3)_{k, \vec{x}^*}$ satisfies the same algebra as the Pauli matrices, as one can easily prove: $(\hat{\mu}_1)_{k, \vec{x}^*}^2 = 1 = (\hat{\mu}_3)_{k, \vec{x}^*}^2 \quad \forall \vec{x}^*$ and $(\hat{\mu}_1)_{k, \vec{x}^*} (\hat{\mu}_3)_{k, \vec{x}^*} = -(\hat{\mu}_3)_{k, \vec{x}^*} (\hat{\mu}_1)_{k, \vec{x}^*}$ but $(\hat{\mu}_1)_{k, \vec{x}^*} (\hat{\mu}_3)_{j, \vec{y}^*} = (\hat{\mu}_3)_{j, \vec{y}^*} (\hat{\mu}_1)_{k, \vec{x}^*}$ if $\vec{y}^* \neq \vec{x}^*$ or $j \neq k$ ⁴⁵.

Now the Hamiltonian describing the 4-dim. model, when written using the dual variables (D.203) and (D.204), takes the following form

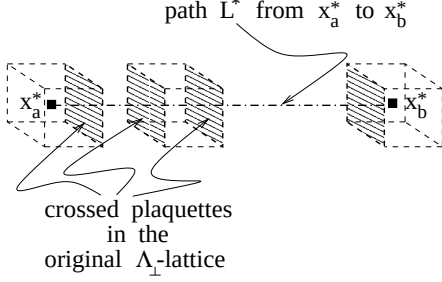
$$\begin{aligned} \hat{\mathcal{H}} &= -\eta \sum_{\vec{x} \in 3\text{-dim. } \Lambda_{\perp}} \sum_{j=1}^3 (\hat{\sigma}_1)_{j, \vec{x}} + \kappa_e \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j < k=1}^3 \left(1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x}+\hat{j}} (\hat{\sigma}_3)_{j, \vec{x}+\hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right) \rightarrow \\ &\rightarrow \frac{\kappa_e}{\eta} \left[\underbrace{\eta}_{(\kappa_e)_{\text{dual}}} \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j < k=1}^3 \left(1 - (\hat{\mu}_3)_{j, \vec{x}^*} (\hat{\mu}_3)_{k, \vec{x}^*+\hat{j}} (\hat{\mu}_3)_{j, \vec{x}^*+\hat{k}} (\hat{\mu}_3)_{k, \vec{x}^*} \right) - \eta \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} \right] , \quad (\text{D.206}) \end{aligned}$$

up to an irrelevant additive constant. This last result proves the self-duality of the 4-dim. $Z(2)$ gauge model by means of the Hamiltonian formulation: note that, as usual in the dual relations, the range of large κ_e corresponds to $(\kappa_e)_{\text{dual}}$ small, and viceversa. Since in the range of small κ_e (large e^2) the 4-dim. $Z(2)$ gauge model shows a phase transition when ω decreases, the same kind of transition will therefore occur for κ_e large. In order to characterize such a transition at small e^2 , and following a similar argument as in the Note on page 422, let us introduce the following correlation function in the original $\Lambda_{\perp}$ -lattice

$$H(\vec{x}_a^*, \vec{x}_b^*) = \frac{Z_{\kappa_e, - \rightarrow + \text{ in plaquettes crossed by } L^*}}{Z_{\kappa_e}} , \quad (\text{D.207})$$

where $\kappa_e \left(1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x}+\hat{j}} (\hat{\sigma}_3)_{j, \vec{x}+\hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right)$ is changed into $\kappa_e \left(1 + (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x}+\hat{j}} (\hat{\sigma}_3)_{j, \vec{x}+\hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right)$ for those plaquettes crossed by the path L^* connecting dual sites $\vec{x}_a^*$ and $\vec{x}_b^*$ (see next figure).

⁴⁵For instance $(\hat{\mu}_1)_{1, \vec{x}^*} (\hat{\mu}_3)_{1, \vec{x}^*} = (\hat{\sigma}_3)_{3, \vec{x}+\hat{1}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{1}+\hat{3}} (\hat{\sigma}_3)_{3, \vec{x}+\hat{1}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{1}} \left[\frac{(\hat{\sigma}_1)_{2, \vec{x}+\hat{1}} \prod_{n' > 0} (\hat{\sigma}_1)_{2, (\vec{x}+\hat{1})-n' \cdot \hat{3}}}{n' > 0} \right] =$
 $= -(\hat{\mu}_3)_{1, \vec{x}^*} (\hat{\mu}_1)_{1, \vec{x}^*}$. Without forgetting boundary condition $(\hat{\mu}_3)_{3, \vec{x}^*} = 1 \quad \forall \vec{x}^*$, one obtains that in $(\hat{\mu}_1)_{k, \vec{x}^*} (\hat{\mu}_3)_{j \neq k, \vec{x}^*}$ it does not appear both $(\hat{\sigma}_1)_{i, \vec{x}}$ and $(\hat{\sigma}_3)_{i, \vec{x}}$ acting on the same bond and therefore $(\hat{\mu}_1)_{k, \vec{x}^*} (\hat{\mu}_3)_{j \neq k, \vec{x}^*} = (\hat{\mu}_3)_{j, \vec{x}^*} (\hat{\mu}_1)_{k, \vec{x}^*}$. If $j = k$ (equal, for example, to $j = k = 1$) in $(\hat{\mu}_1)_{k, \vec{x}^*} (\hat{\mu}_3)_{j, \vec{y}^*}$ but $\vec{x}^* \neq \vec{y}^*$ then it can occur that, either no $(\hat{\sigma}_3)_{2, \vec{x}}$ from $(\hat{\mu}_1)_{1, \vec{x}^*}$ coincides on the same bond with a $(\hat{\sigma}_3)_{2, \vec{y}=\vec{x}}$ from the "ladder" $(\hat{\mu}_3)_{1, \vec{y}^*} = \prod_{n' \geq 0} (\hat{\sigma}_1)_{2, (\vec{y}+\hat{1})-n' \cdot \hat{3}}$, or that the two $(\hat{\sigma}_3)_{2, \vec{x}+\hat{1}}$, $(\hat{\sigma}_3)_{2, \vec{x}+\hat{1}+\hat{3}}$ meet on their respective bonds one $(\hat{\sigma}_1)_{2, \vec{x}}$ as in the following case $(\hat{\mu}_1)_{1, \vec{x}^*} (\hat{\mu}_3)_{1, \vec{x}^*+2 \cdot \hat{3}} =$
 $(\hat{\sigma}_3)_{3, \vec{x}+\hat{1}+\hat{2}} (\hat{\sigma}_3)_{3, \vec{x}+\hat{1}} \frac{(\hat{\sigma}_3)_{2, \vec{x}+\hat{1}+\hat{3}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{1}}}{n' > 0} \left[\frac{(\hat{\sigma}_1)_{2, \vec{x}+\hat{1}+2 \cdot \hat{3}} (\hat{\sigma}_1)_{2, \vec{x}+\hat{1}+\hat{3}} (\hat{\sigma}_1)_{2, \vec{x}+\hat{1}} \prod_{n' > 0} (\hat{\sigma}_1)_{2, (\vec{x}+\hat{1})-n' \cdot \hat{3}}}{n' > 0} \right] = (\hat{\mu}_3)_{1, \vec{x}^*+2 \cdot \hat{3}} (\hat{\mu}_1)_{1, \vec{x}^*} .$



For $Z_{\kappa_e, - \rightarrow +}$ the calculation follows the same steps as those leading to (D.206) except for the definition of dual bond-operators $(\hat{\mu}_1)_{k, \vec{x}^*}$ which change in a sign for those plaquettes crossed by L^*

$$\begin{aligned} (\hat{\mu}_1)_{-2, \vec{x}^*}^{\text{not crossed}} &= (\hat{\sigma}_3)_{1, \vec{x}} (\hat{\sigma}_3)_{3, \vec{x}+\hat{1}} (\hat{\sigma}_3)_{1, \vec{x}+\hat{3}} (\hat{\sigma}_3)_{3, \vec{x}} , \\ (\hat{\mu}_1)_{-1, \vec{x}^*}^{\text{not crossed}} &= (\hat{\sigma}_3)_{2, \vec{x}} (\hat{\sigma}_3)_{3, \vec{x}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{3}} (\hat{\sigma}_3)_{3, \vec{x}} , \\ (\hat{\mu}_1)_{-2, \vec{y}^*}^{\text{crossed}} &= - (\hat{\sigma}_3)_{1, \vec{y}} (\hat{\sigma}_3)_{3, \vec{y}+\hat{1}} (\hat{\sigma}_3)_{1, \vec{y}+\hat{3}} (\hat{\sigma}_3)_{3, \vec{y}} , \\ (\hat{\mu}_1)_{-1, \vec{y}^*}^{\text{crossed}} &= - (\hat{\sigma}_3)_{2, \vec{y}} (\hat{\sigma}_3)_{3, \vec{y}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{y}+\hat{3}} (\hat{\sigma}_3)_{3, \vec{y}} . \end{aligned} \quad (\text{D.208})$$

The calculation of the product of the six plaquettes for the cube surrounding each $\vec{x}^*$ leads to the result that condition (D.205) on $(\hat{\mu}_1)_{k, \vec{x}^*}$ will hold everywhere except for sites $\vec{x}_a^*$ and $\vec{x}_b^*$, since the cube which surrounds each of these two sites contains one plaquette crossed by L^* ; the cube for the other sites $\vec{x}^*$ has none or two faces crossed by L^* (see figure above)

$$(\hat{\mu}_1)_{-1, \vec{x}^*} (\hat{\mu}_1)_{1, \vec{x}^*} (\hat{\mu}_1)_{-2, \vec{x}^*} (\hat{\mu}_1)_{2, \vec{x}^*} (\hat{\mu}_1)_{-3, \vec{x}^*} (\hat{\mu}_1)_{3, \vec{x}^*} = (-1)^{(\delta_{\vec{x}^*, \vec{x}_b} - \delta_{\vec{x}^*, \vec{x}_a})} \mathbf{1} . \quad (\text{D.209})$$

Hence correlation $H(\vec{x}_a^*, \vec{x}_b^*)$ formulated in the dual lattice is given by

$$\begin{aligned} H(\vec{x}_a^*, \vec{x}_b^*) &= \frac{\text{Tr} \left[\prod_{\vec{x}^* \in \Lambda_\perp} \delta \left(\prod_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} (\hat{\mu}_1)_{-j, \vec{x}^*}, (-1)^{(\delta_{\vec{x}^*, \vec{x}_b} - \delta_{\vec{x}^*, \vec{x}_a})} \mathbf{1} \right) \right]}{\text{Tr} \left[\prod_{\vec{x}^* \in \Lambda_\perp} \delta \left(\prod_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} (\hat{\mu}_1)_{-j, \vec{x}^*}, \mathbf{1} \right) \right]} \times \\ &\times \exp \left\{ -\frac{\omega \kappa_e}{\eta^2} \left[-\eta \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} + \eta \frac{\eta}{\kappa_e} \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j < k=1}^3 \left(1 - (\hat{\mu}_3)_{j, \vec{x}^*} (\hat{\mu}_3)_{k, \vec{x}^*+\hat{j}} (\hat{\mu}_3)_{j, \vec{x}^*+\hat{k}} (\hat{\mu}_3)_{k, \vec{x}^*} \right) \right] \right\} \\ &\times \exp \left\{ -\frac{\omega \kappa_e}{\eta} \left[-\eta \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} + \eta \frac{\eta}{\kappa_e} \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j < k=1}^3 \left(1 - (\hat{\mu}_3)_{j, \vec{x}^*} (\hat{\mu}_3)_{k, \vec{x}^*+\hat{j}} (\hat{\mu}_3)_{j, \vec{x}^*+\hat{k}} (\hat{\mu}_3)_{k, \vec{x}^*} \right) \right] \right\} . \end{aligned} \quad (\text{D.210})$$

But, according to the discussion leading to equation (D.183)n, the above result for $H(\vec{x}_a^*, \vec{x}_b^*)$ is just equal to $\exp \left(-\frac{\omega \kappa_e}{\eta^2} \mathcal{V}_{(\kappa_e)_{\text{dual}}}^* \right)$ where $\mathcal{V}_{(\kappa_e)_{\text{dual}}}^*$ is the free energy increase due to change from constraint

$$\prod_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} (\hat{\mu}_1)_{-j, \vec{x}^*} = 1 \text{ to } \prod_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} (\hat{\mu}_1)_{-j, \vec{x}^*} = (-1)^{(\delta_{\vec{x}^*, \vec{x}_b} - \delta_{\vec{x}^*, \vec{x}_a})} \mathbf{1} \text{ and } (\kappa_e)_{\text{dual}} = \eta \frac{\eta}{\kappa_e} . \text{ Finally, in the range of small } e^2 \text{ (large } \kappa_e \text{ or equivalently small } (\kappa_e)_{\text{dual}}), \text{ the leading term in the Hamiltonian is } -\eta \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{j=1}^3 (\hat{\mu}_1)_{j, \vec{x}^*} \text{ and therefore}$$

$$\begin{aligned} H(\vec{x}_a^*, \vec{x}_b^*) &= \frac{\prod_{\vec{x}^*, k} \sum_{n_{k, \vec{x}^*}=0}^1 \prod_{\vec{x}^*} \delta \left(\sum_k \Delta_k n_{k, \vec{x}^*-\hat{k}}, (\delta_{\vec{x}^*, \vec{x}_b} - \delta_{\vec{x}^*, \vec{x}_a}) \bmod 2 \right) \exp \left[-\frac{\kappa_e \omega}{\eta} \sum_{\vec{x}^*, k} \left(1 - \cos(\pi n_{k, \vec{x}^*}) \right) \right]}{\prod_{\vec{x}^* \in \Lambda_\perp} \prod_{k=1}^3 \sum_{n_{k, \vec{x}^*}=0}^1 \prod_{\vec{x}^*} \delta \left(\sum_k \Delta_k n_{k, \vec{x}^*-\hat{k}}, 0 \bmod 2 \right) \exp \left[-\frac{\kappa_e \omega}{\eta} \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{k=1}^3 \left(1 - \cos(\pi n_{k, \vec{x}^*}) \right) \right]} \\ &= \frac{\int_{-\infty}^{\infty} \prod_{\vec{x}^*} \frac{d\chi_{\vec{x}^*}}{2\pi} e^{i\chi_{\vec{x}_b^*}} e^{-i\chi_{\vec{x}_a^*}} \sum_{\{m_{k, \vec{x}^*}\}} \sum_{\{P_{\vec{x}^*}\}} \exp \left[-i2 \sum_{\vec{x}^*} P_{\vec{x}^*} \chi_{\vec{x}^*} - \frac{\eta}{2\kappa_e \omega} \sum_{\vec{x}^*, k} \left(\frac{1}{\pi} \Delta_k \chi_{k, \vec{x}^*} - 2m_{k, \vec{x}^*} \right)^2 \right]}{\int_{-\infty}^{\infty} \prod_{\vec{x}^*} \frac{d\chi_{\vec{x}^*}}{2\pi} \prod_{\vec{x}^* \in \Lambda_\perp} \prod_{k=1}^3 \sum_{m_{k, \vec{x}^*}=-\infty}^{\infty} \prod_{\vec{x}^*} \sum_{P_{\vec{x}^*}=-\infty}^{\infty} \exp \left[-i2 \sum_{\vec{x}^* \in \Lambda_\perp} P_{\vec{x}^*} \chi_{\vec{x}^*} - \frac{\eta}{2\kappa_e \omega} \sum_{\vec{x}^* \in \Lambda_\perp} \sum_{k=1}^3 \left(\frac{1}{\pi} \Delta_k \chi_{k, \vec{x}^*} - 2m_{k, \vec{x}^*} \right)^2 \right]} \end{aligned}$$

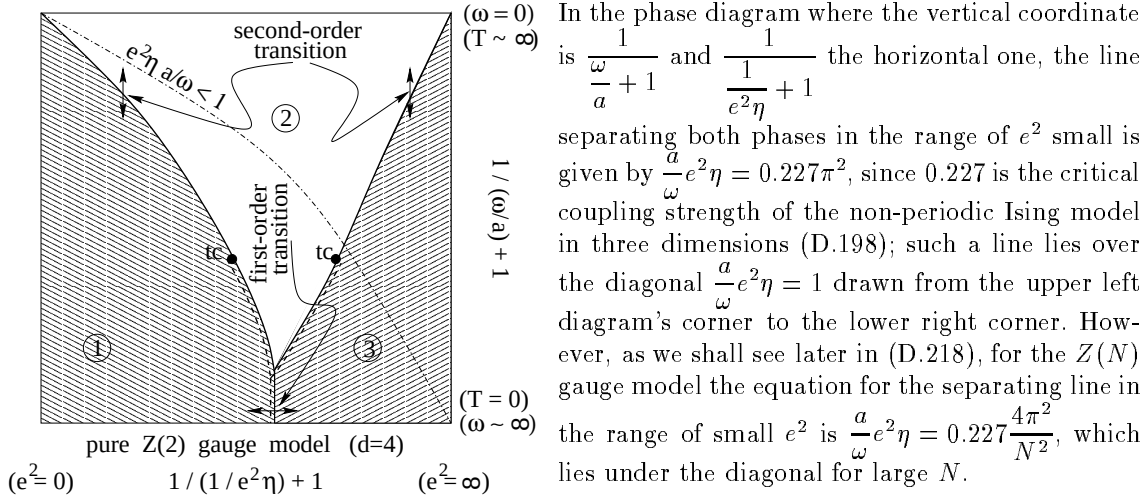
$$H(\vec{x}_a^*, \vec{x}_b^*)_{\text{small } e^2} = \exp \left(-\frac{\omega \kappa_e}{\eta} \mathcal{V}_{(\kappa_e)_{\text{dual}}}^* \right) = \left\langle \sigma_{\vec{x}_a^*} \sigma_{\vec{x}_b^*} \right\rangle_{3\text{-dim. non-period. Ising}}, \quad (\text{D.211})$$

similar to (D.184) but now the coupling strength for the equivalent Ising model is equal to $\frac{\beta_{\text{Ising}}}{\eta} = \frac{\eta}{\pi^2 \kappa_e \omega}$.

Or equivalently if the notation by means of e^2 is preferred ⁴⁶

$$\frac{a\beta_{3\text{-dim. Ising}}}{\eta} = \frac{1}{\pi^2} \frac{a}{\omega} (e^2 \eta)_{4\text{-dim. } Z(2) \text{ gauge}}. \quad (\text{D.212})$$

Therefore, remaining inside the range of small e^2 , a large ω corresponds to β_{Ising} small and correlation $H(\vec{x}_a^*, \vec{x}_b^*)$ decays like the distance separating $\vec{x}_a$ and $\vec{x}_b$; on the contrary, at small ω $H(\vec{x}_a^*, \vec{x}_b^*)$ remains constant even for large $|\vec{x}_b - \vec{x}_a|$. The phase transition which separates both behaviours is of second-order kind since this is the transition character of the 3-dim. Ising model in a lattice non-periodic along the τ -direction. This is not a surprise since the phase transition at large e^2 when ω decreases is also of second-order type and the considered gauge model is self-dual: the change in the behaviour shown by the system in the range of e^2 small can be mapped in the change of phases occurring at large e^2 .



For the 4-dimensional pure $Z(2)$ gauge model in a lattice periodic along the τ -dimension, and by collecting the results of the phase transition at large e^2 seen in the diagram of page 593 together with

⁴⁶Since as discussed $\kappa_e \sim \frac{1}{e^2}$, and by restoring the length dimension of lattice spacing a , Hamiltonian operator (D.190)

$$\frac{\hat{\mathcal{H}}}{\eta} = -\frac{e^2 \eta}{\pi^2 a} \sum_{\vec{x} \in 3\text{-dim. } \Lambda_{\perp}} \sum_{j=1}^3 (\hat{\sigma}_1)_{j,\vec{x}} + \frac{1}{a e^2 \eta} \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{j < k=1}^3 [1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}}]$$

$$\text{dual variables becomes } \frac{\hat{\mathcal{H}}}{\eta} = \frac{e^2 \eta}{\pi^2 a} \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j < k=1}^3 (1 - (\hat{\mu}_3)_{j,\vec{x}^*} (\hat{\mu}_3)_{k,\vec{x}^*+\hat{j}} (\hat{\mu}_3)_{j,\vec{x}^*+\hat{k}} (\hat{\mu}_3)_{k,\vec{x}^*}) - \frac{1}{a e^2 \eta} \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j=1}^3 (\hat{\mu}_1)_{j,\vec{x}^*}.$$

$$\text{Hence } \frac{\hat{\mathcal{H}}_{\text{small } e^2}}{\eta} = -\frac{1}{a e^2 \eta} \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{j=1}^3 (\hat{\mu}_1)_{j,\vec{x}^*} \text{ and repeating the calculation of (D.211)}$$

$$H(\vec{x}_a^*, \vec{x}_b^*) = \frac{\int_{-\infty}^{\infty} \prod_{\vec{x}^*} \frac{d\chi_{\vec{x}^*}}{2\pi} e^{i\chi_{\vec{x}_b^*}} e^{-i\chi_{\vec{x}_a^*}} \sum_{\{m_{k,\vec{x}^*}\}} \sum_{\{P_{\vec{x}^*}\}} \exp \left[-i2 \sum_{\vec{x}^*} P_{\vec{x}^*} \chi_{\vec{x}^*} - \frac{e^2 \eta}{2\pi^2} \frac{a}{\omega} \sum_{\vec{x}^*,k} (\Delta_k \chi_{k,\vec{x}^*} - 2\pi m_{k,\vec{x}^*})^2 \right]}{\int_{-\infty}^{\infty} \prod_{\vec{x}^*} \frac{d\chi_{\vec{x}^*}}{2\pi} \prod_{\vec{x}^* \in \Lambda_{\perp}} \prod_{k=1}^3 \sum_{m_{k,\vec{x}^*}=-\infty}^{\infty} \prod_{\vec{x}^*} \sum_{P_{\vec{x}^*}=-\infty}^{\infty} \exp \left[-i2 \sum_{\vec{x}^* \in \Lambda_{\perp}} P_{\vec{x}^*} \chi_{\vec{x}^*} - \frac{e^2 \eta}{2\pi^2} \frac{a}{\omega} \sum_{\vec{x}^* \in \Lambda_{\perp}} \sum_{k=1}^3 (\Delta_k \chi_{k,\vec{x}^*} - 2\pi m_{k,\vec{x}^*})^2 \right]},$$

being thus $\frac{a\beta_{\text{Ising}}}{\eta} = \frac{1}{\pi^2} e^2 \eta \frac{a}{\omega}$ the coupling strength in the equivalent 3-dim. (non-periodic) Ising model.

the just obtained phase transition at small e^2 , the complete phase diagram takes the form of the above figure. Inside the shaded left region, $H(\vec{x}_a^*, \vec{x}_b^*)$ decays exponentially like the distance, and the same occurs inside the shaded right region for correlation function $W(C)$ (where the exponent's distance is that between the intersection points of path C with sublattice $\Lambda_\perp$). At small and at large e^2 the phase transition with decreasing ω is second-order. The region between the shaded zones is characterized by a constant value of both $W(C)$ and $H(\vec{x}_a^*, \vec{x}_b^*)$ even at large distances. The behaviour of $W(C)$ in the shaded region denoted by ①, as well as the form of $H(\vec{x}_a^*, \vec{x}_b^*)$ inside phase ③, deserves a brief discussion. Let us consider firstly $W(C)$: at $\omega = \infty$, $W(C)$ for large e^2 shows the strongest decay like the area contained into closed path C while at small e^2 the decay in $W(C)$ is like the perimeter of C . When the lattice's periodicity along the τ -direction, ω , becomes relevant because small, then only the intersection of path C with the sublattice normal to the τ -direction, $\Lambda_\perp$, characterizes the $W(C)$'s behaviour: at large e^2 the area decay of $\omega = \infty$ gets transformed into a decay like the distance of the projection of C onto $\Lambda_\perp$; and to the perimeter decay at $\omega = \infty$ corresponds a constant value of $W(C)$ for ω small. In the case of correlation function $H(\vec{x}_a^*, \vec{x}_b^*)$ studied in the lattice of finite ω , dual sites $\vec{x}_a^*$ and $\vec{x}_b^*$ can also be considered as the points where a closed path C^* , whose one side stretches along the periodic direction, cuts the sublattice which is normal to the τ -coordinate and which is dual to $\Lambda_\perp$; therefore $H(\vec{x}_a^*, \vec{x}_b^*)$ at a finite lattice's periodicity ω would correspond to a correlation function $H(C^*)$ at $\omega = \infty$. According to equations (C.372), in the lattice without periodicity along the τ -direction, the following relations holds: $H(C^*) \sim e^{-\text{Area}_{C^*}}$ at small e^2 and $H(C^*) \sim e^{-\text{Perimeter}_{C^*}}$ for e^2 large. And thus at finite and small ω , one can expect $H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}$ at small e^2 and $H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}$ for e^2 large. The characterization of the different phases shown by the 4-dimensional $Z(2)$ gauge model by means of $W(C)$ and $H(\vec{x}_a^*, \vec{x}_b^*)$ is as follows:

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(C^*) \sim e^{-\mathcal{A}_{C^*}} \end{cases}$
②	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*}} \end{cases}$

At $\omega = \infty$ (no periodicity along the τ -direction), the model shows a first order transition at the value $\frac{1}{e^2\eta} = 0.441$, obtained in (C.368) by means of the usual calculation of the partition function. Such a transition is also obtained by means of the Hamiltonian formulation at value $\kappa_e/\eta = 1$, calculated in the same way as for the 2-dim. Ising model (a self-dual model too):

- Either by returning to the Hamiltonian of equation (D.168), $\mathcal{H} = -\eta \frac{e^{-2(\beta_e)_\tau/\eta}}{a_\tau} \sum_{\vec{x} \in \Lambda_\perp} \sum_{j=1}^3 (\hat{\sigma}_1)_{j,\vec{x}} + \frac{\beta_x}{a_\tau} \sum_{\vec{x}} \sum_{j < k=1}^3 \left[1 - (\hat{\sigma}_3)_{j,\vec{x}} (\hat{\sigma}_3)_{k,\vec{x}+\hat{j}} (\hat{\sigma}_3)_{j,\vec{x}+\hat{k}} (\hat{\sigma}_3)_{k,\vec{x}} \right]$, where from the relation between the transfer matrix and $\mathcal{H}$ in the limit $a_\tau \rightarrow 0$ one obtains $\begin{cases} (\beta_e)_x & \sim a_\tau \\ e^{-2(\beta_e)_\tau/\eta} & \sim a_\tau \end{cases}$ and thus κ_e is defined through $\frac{(\beta_e)_x}{\eta} = \frac{\kappa_e}{\eta} e^{-2(\beta_e)_\tau/\eta}$; the selfduality causes that coupling strengths $(\beta_e)_\tau^*$, $(\beta_e)_x^*$ in the dual model are related to those of the original model by means of $\begin{cases} \frac{(\beta_e)_\tau^*}{\eta} = -\frac{1}{2} \ln \tanh \left(\frac{(\beta_e)_x}{\eta} \right) \\ \frac{(\beta_e)_x^*}{\eta} = -\frac{1}{2} \ln \tanh \left(\frac{(\beta_e)_\tau}{\eta} \right) \end{cases}$, duality relations which only works for an infinite large lattice in all directions, i.e., for the case of $\omega = \infty$. Therefore the transition point is determined by the following equation

$$\sinh \left(2 \frac{(\beta_e)_x, \text{transit}}{\eta} \right) \sinh \left(2 \frac{(\beta_e)_\tau, \text{transit}}{\eta} \right) = 1,$$

which for $a_\tau \rightarrow 0$ produces $\frac{2(\beta_e)_{x, \text{tr}}}{\eta} \approx \sinh\left(2\frac{(\beta_e)_{x, \text{tr}}}{\eta}\right) = \frac{1}{\sinh\left(2\frac{(\beta_e)_{\tau, \text{tr}}}{\eta}\right)} \approx \frac{e^{-2(\beta_e)_{\tau, \text{tr}}/\eta}}{2}$, i.e.,

$\left(\frac{\kappa_e}{\eta}\right)_{\text{transit}} = 1$: this is the same transition value as in equations (D.32) and (D.33) of the 2-dimensional Ising model, although there the phase transition was of second-order kind and not first order as here.

- Or by repeating the argument on page 580. Always staying at $\omega = \infty$, let us calculate correlation function $W(C) = \left\langle \prod_{\text{loop } C} (\hat{\sigma}_3)_{k, \vec{x}} \right\rangle$ for both ranges of small and large κ_e . For κ_e small (large

e^2) the leading part of the Hamiltonian is $-\eta \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^3 (\hat{\sigma}_1)_{k, \vec{x}}$ and the rest of operator $\mathcal{H}$,

$\frac{\kappa_e}{\eta} \hat{\mathcal{H}}_1 = \frac{\kappa_e}{\eta} \eta \sum_{\vec{x} \in \Lambda_\perp} \sum_{j < k=1}^3 \left[1 - (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x}+\hat{j}} (\hat{\sigma}_3)_{j, \vec{x}+\hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right]$, acts as a perturbation of order

$\lambda = \kappa_e/\eta$. Denoting by “spin up” $\uparrow_{k, \vec{x}}$ the $(\hat{\sigma}_1)_{k, \vec{x}}$ -eigenstate of eigenvalue +1, the eigenstate of the whole Hamiltonian at κ_e/η small (and for $\omega = \infty$) is equal to $|\uparrow \dots \uparrow\rangle$, plus the perturbative contributions which contain n plaquettes with the spins flipped down, which are of order $\left(\frac{\kappa_e}{\eta}\right)^n$

and which arise from the action n consecutive times of $\hat{\mathcal{H}}_1$ on $|\uparrow \dots \uparrow\rangle$; such a state in which the spins of n plaquettes are pointing down will be denoted schematically by $|n_{\downarrow\downarrow\downarrow\downarrow}\rangle$. Since $\langle \uparrow \dots \uparrow | \prod_{\text{loop } C} (\hat{\sigma}_3)_{k, \vec{x}} | n_{\downarrow\downarrow\downarrow\downarrow} \rangle = 0$ unless all the different n reversed plaquettes correspond to

all the plaquettes tiling the interior of C , then the first non-vanishing contribution to $W(C)$ is of order $\left(\frac{\kappa_e}{\eta}\right)^{\text{Area}_C}$. This perturbative expansion will work as long as $\kappa_e/\eta < 1$. For the range

$\kappa_e/\eta > 1$, $W(C)$ is calculated by means of the dual model, which in our case takes the same form as the original model because of its selfduality. The product of $(\hat{\sigma}_3)_{k, \vec{x}}$ along the bonds on C can be also written as the product of the unit plaquettes tiling the interior of the area enclosed by

C and thus $W(C) = \left\langle \prod_{\text{plaq. inside } C} (\hat{\sigma}_3)_{j, \vec{x}} (\hat{\sigma}_3)_{k, \vec{x}+\hat{j}} (\hat{\sigma}_3)_{j, \vec{x}+\hat{k}} (\hat{\sigma}_3)_{k, \vec{x}} \right\rangle$, which formulated by means

of the dual variables, according to (D.204), is equal to the product of the dual bond-operators $(\hat{\mu}_1)_{i, \vec{x}^\star}$ crossing as “toothpicks” the plaquettes inside C

$$W(C) = \left\langle \prod_{\text{bonds crossing interior } C} (\hat{\mu}_1)_{\epsilon_{ijk}, \vec{x}^\star} \right\rangle.$$

Hence the calculation of $W(C)$ for large κ_e becomes very easy if carried out in the dual model:

in such κ_e -range the leading part of Hamiltonian is $-\eta \sum_{\vec{x}^\star \in \Lambda_\perp} \sum_{k=1}^3 (\hat{\mu}_1)_{k, \vec{x}}$ whilst the perturbation

assumes the form of $\frac{\eta}{\kappa_e} \hat{\mathcal{H}}_1 = \frac{\eta}{\kappa_e} \eta \sum_{\vec{x}^\star \in \Lambda_\perp} \sum_{j < k=1}^3 \left[1 - (\hat{\mu}_3)_{j, \vec{x}^\star} (\hat{\mu}_3)_{k, \vec{x}^\star+\hat{j}} (\hat{\mu}_3)_{j, \vec{x}^\star+\hat{k}} (\hat{\mu}_3)_{k, \vec{x}^\star} \right]$. The

eigenstate with the lowest $\hat{\mathcal{H}}$ -eigenvalue is equal to $|\uparrow^\star \dots \uparrow^\star\rangle$ (with $(\hat{\mu}_1)_{i, \vec{x}^\star} |\uparrow^\star_{i, \vec{x}^\star}\rangle = + |\uparrow^\star_{i, \vec{x}^\star}\rangle$) plus perturbative contributions of states with the (dual) spins of some plaquettes reversed: up to order η/κ_e , this ground state of $\hat{\mathcal{H}}$ is given by

$$|0\rangle_{\text{large } \kappa_e} \approx |\uparrow^\star \dots \uparrow^\star\rangle + \left(\frac{\eta}{\kappa_e}\right) \hat{\mathcal{H}}_1 |\uparrow^\star \dots \uparrow^\star\rangle + \dots =$$

$$= |\uparrow^* \dots \uparrow^*\rangle + \left(\frac{\eta}{\kappa_e}\right) \frac{\eta}{-\eta(3|\Lambda_\perp| - 4) - \eta \cdot (-1) \cdot 4 - (-3|\Lambda_\perp|\eta)} \sum_{\text{plaq. } \subset \Lambda_\perp} |1_{\Psi^* \Psi^* \Psi^* \Psi^*}\rangle + \dots,$$

where $|1_{\Psi^* \Psi^* \Psi^* \Psi^*}\rangle$ denotes the state with one reversed plaquette. Since the effect of operator $\prod_{\text{bonds crossing interior } C} (\hat{\mu}_1)_{i, \vec{x}^*}$ is to count (mod. 2) the number of bonds with reversed $(\hat{\mu}_1)_{i, \vec{x}^*}$ -

eigenvectors, then $\langle 1_{\Psi^* \Psi^* \Psi^* \Psi^*} | \prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}^*} | 1_{\Psi^* \Psi^* \Psi^* \Psi^*} \rangle = -1$ if the reversed plaquette is linked to C (i.e., is crossed by C) since then one of the reversed plaquette's bonds crosses the interior of C ; on the other hand, $\langle 1_{\Psi^* \Psi^* \Psi^* \Psi^*} | \prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}^*} | 1_{\Psi^* \Psi^* \Psi^* \Psi^*} \rangle = +1$ if the reversed plaquette is placed outside C or cuts perpendicularly the interior of C but without being crossed by C (and thus *two* of the reversed plaquette's bonds coincides with two of the bonds contained into $\prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}^*}$)

$$\frac{\langle 0 | \prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}^*} | 0 \rangle_{\text{large } \kappa_e}}{\langle 0 | 0 \rangle_{\text{large } \kappa_e}} \approx \frac{1 + \left(\frac{\eta}{8\kappa_e}\right)^2 [\mathcal{P}_C \cdot (-1) + (3|\Lambda_\perp| - \mathcal{P}_C) \cdot (+1)] + \dots}{1 + \left(\frac{\eta}{8\kappa_e}\right)^2 3|\Lambda_\perp| + \dots},$$

$$W(C)_{\text{large } \kappa_e} \approx \exp \left(-2\mathcal{P}_C \left(\frac{\eta}{8\kappa_e} \right)^2 \right), \quad (\text{D.213})$$

with $\mathcal{P}_C$ the perimeter of closed path C . This perimeter exponent holds as long as the perturbative expansion works ⁴⁷, i.e., for $\kappa_e/\eta > 1$. Thus, the transition at $\omega = \infty$ occurs for

⁴⁷Contributions of higher order in $\frac{\eta}{\kappa_e}$ do not spoil the perimeter decay: let us calculate the next perturbative order, as

$$\text{in the footnote following equation (D.42), by using } \begin{cases} \langle \uparrow^* \dots \uparrow^* | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle = -\frac{3|\Lambda_\perp|}{8\eta} \\ \hat{g}^2 \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle = -\frac{1}{8^2 \eta} \sum_{\text{plaq. } \subset \Lambda_\perp} |1_{\Psi^* \Psi^* \Psi^* \Psi^*}\rangle \end{cases} \text{ as well}$$

$$\text{as } \begin{cases} \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle = \frac{1}{8 \cdot 12} \sum_{\text{plaq. } \subset \Lambda_\perp} |2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{contig.}}\rangle + \frac{1}{8 \cdot 16} \sum_{\text{plaq. } \subset \Lambda_\perp} |2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{separ.}}\rangle \\ \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle = \left[\frac{4 \cdot 3}{8^2 \cdot 12} + \frac{2(3|\Lambda_\perp| - (2 \cdot 2 + 1))}{8^2 \cdot 16} \right] \sum_{\text{plaq. } \subset \Lambda_\perp} |1_{\Psi^* \Psi^* \Psi^* \Psi^*}\rangle + \dots \end{cases} \text{ where state } |2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{contig.}}\rangle$$

denotes the state with two contiguous reversed plaquettes, and $|2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{separ.}}\rangle$ corresponds to the state where the reversed dual plaquettes are not touching each other; further, $\dots$ denotes a state with three flipped dual plaquettes. By noting that $\langle \uparrow^* \dots \uparrow^* | \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle = 0$, the perturbative contributions at large but finite κ_e to the eigenstate of lowest $\hat{\mathcal{H}}$ -eigenvalue $|0\rangle_{\text{large } \kappa_e} \approx | \uparrow^* \dots \uparrow^* \rangle + \left(\frac{\eta}{\kappa_e}\right) \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle + \left(\frac{\eta}{\kappa_e}\right)^3 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle - \left(\frac{\eta}{\kappa_e}\right)^3 \langle \uparrow^* \dots \uparrow^* | \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle \hat{g}^2 \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle + \left(\frac{\eta}{\kappa_e}\right)^2 \hat{g} \hat{\mathcal{H}}_1 \hat{g} \hat{\mathcal{H}}_1 | \uparrow^* \dots \uparrow^* \rangle$ produce as $|0\rangle_{\text{large } \kappa_e}$

$$| \uparrow^* \dots \uparrow^* \rangle + \left(\frac{\eta}{4\kappa_e}\right) \left[1 + 3 \left(\frac{\eta}{8\kappa_e}\right)^2 \right] \sum_{\text{plaq. } \subset \Lambda_\perp} |1_{\Psi^* \Psi^* \Psi^* \Psi^*}\rangle + \left(\frac{\eta}{4\kappa_e}\right)^2 \left[\frac{2}{3} \sum_{\text{plaq. } \subset \Lambda_\perp} |2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{contig.}}\rangle + \frac{1}{2} \sum_{\text{plaq. } \subset \Lambda_\perp} |2_{\Psi^* \Psi^* \Psi^* \Psi^*}^{\text{separ.}}\rangle \right] + \dots,$$

and thus

$$\frac{1}{\langle 0 | 0 \rangle_{\text{large } \kappa_e}} \approx 1 - \left(\frac{\eta}{8\kappa_e}\right)^2 (3|\Lambda_\perp|) \left[1 + 6 \left(\frac{\eta}{8\kappa_e}\right)^2 \right] - \left(\frac{\eta}{8\kappa_e}\right)^4 \left[\frac{16(3|\Lambda_\perp|)}{3} + \frac{(3|\Lambda_\perp|)(3|\Lambda_\perp| - 5)}{2} \right] + \left(\frac{\eta}{8\kappa_e}\right)^4 (3|\Lambda_\perp|)^2,$$

$$\langle 0 | \prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}^*} | 0 \rangle_{\text{large } \kappa_e} \approx 1 - \left(\frac{\eta}{8\kappa_e}\right)^2 (3|\Lambda_\perp| - 2\mathcal{P}_C) \left[1 + 6 \left(\frac{\eta}{8\kappa_e}\right)^2 \right] -$$

$\kappa_e/\eta = 1$, something which could be already guessed because of the self-duality relation (D.206): $\hat{\mathcal{H}}(\text{gauge}, \kappa_e) = \frac{\kappa_e}{\eta} \hat{\mathcal{H}}\left(\text{gauge}, \eta \frac{\eta}{\kappa_e}\right)$. Of course, the same discussion could be carried out for correlation function $H(C^*) = \left\langle \prod_{\text{dual bonds along } C^*} (\hat{\mu}_3)_{k, \vec{x}} \right\rangle$ with C^* a closed loop in the dual lattice; as expected from the behaviour of $W(C)$, one would obtain $H(C^*) \sim e^{-\text{Area}_{C^*}}$ at small $\kappa_e^* \equiv \eta \frac{\eta}{\kappa_e}$ (or equivalently, at large κ_e), $H(C^*) \sim e^{-\mathcal{P}_{C^*}}$ at small κ_e , and the transition point at $\kappa_e/\eta = 1$. The behaviours of $W(C)$ and $H(C^*)$ in a non-periodic 4-dim. lattice ($\omega = \infty$) have been already obtained in (C.372) but by using another method.

The previous results hold for the $Z(2)$ gauge model in a 4-dimensional lattice of periodicity ω along one dimension. Let us now see the phase diagram for the 3-dimensional model, which differs considerably from the diagram on page 598: in the range of small e^2 there is no phase transition for any $\omega \neq \infty$. This property has been already obtained in (D.199) as it was shown that the model in a lattice of periodicity ω in such a range (large e^2 corresponds to κ_e small) is described by Hamiltonian $\hat{\mathcal{H}}_{\text{large } e^2} \approx \kappa_e \sum_{\vec{x} \in 2\text{-dim. } \Lambda_\perp} \left(1 - (\hat{\sigma}_3)_{1, \vec{x}} (\hat{\sigma}_3)_{2, \vec{x}+\hat{1}} (\hat{\sigma}_3)_{1, \vec{x}+\hat{2}} (\hat{\sigma}_3)_{2, \vec{x}}\right)$ and thus by partition function $\mathcal{Z} = \prod_{\vec{x} \in 2\text{-dim. } \Lambda_\perp} \sum_{\sigma_{1, \vec{x}} = \pm 1} \sum_{\sigma_{2, \vec{x}} = \pm 1} \exp \left[-\frac{\omega \kappa_e}{\eta} \sum_{\vec{x} \in \Lambda_\perp} \left(1 - \sigma_{1, \vec{x}} \sigma_{2, \vec{x}+\hat{1}} \sigma_{1, \vec{x}+\hat{2}} \sigma_{2, \vec{x}}\right) \right]$, which is the usual partition function of the 2-dimensional $Z(2)$ gauge model in a non-periodic lattice at a coupling strength of $\beta_e = \kappa_e \omega$. Since this last model shows no transition for any β_e , the 3-dimensional $Z(2)$ gauge model in a periodic lattice displays only one phase for small e^2 . The system does have a transition points at $\omega = \infty$, i.e., in the non-periodic model, which was calculated in (C.364) $\frac{1}{ae^2\eta} = 0.761$ and is of second-order type.

Taking into account this property as well as phase diagram on page 593, the phase diagram for the 3-dim. periodic model becomes as represented in the next figure. The absence of any transition at small e^2 justifies the substitution, at *any* e^2 , of integer field $P_{\mu, \vec{r}} \rightarrow \frac{\omega}{a} \bar{P}_{\vec{r}}$ when the 3-dimensional lattice becomes periodic, being $P_{\mu, \vec{r}}$ the fields which carry the discreteness of the set of values taken on by the spin variables. Another consequence of such a transition absence is the fact that correlation function $W(C)$ already distinguishes the two existing phases. However, one still needs the other correlation function, $H(\vec{r}_a^*, \vec{r}_b^*)$, in order to compare the phase in the unshaded region with phase ① or with phase ② of the 4-dimensional model. Such a correlation function is defined as follows

$$H(\vec{r}_a^*, \vec{r}_b^*) = \frac{\mathcal{Z}_{\kappa_e, - \rightarrow + \text{ in plaquettes crossed by } L^*}}{\mathcal{Z}_{\kappa_e}}, \quad (\text{D.214})$$

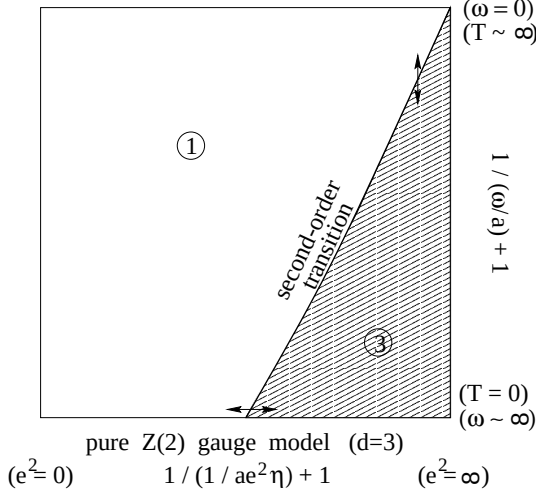
$$- \left(\frac{\eta}{8\kappa_e} \right)^4 \left[\frac{16(3|\Lambda_\perp| - 4\mathcal{P}_C)}{3} + \frac{(3|\Lambda_\perp| - 4\mathcal{P}_C)(3|\Lambda_\perp| - 5)}{2} \right],$$

where one has kept terms only up to order $(\eta/\kappa_e)^4$. Hence $W(C)$ up to such perturbative order is given by

$$\frac{\text{large } \kappa_e \langle 0 | \prod_{\text{bonds across interior } C} (\hat{\mu}_1)_{i, \vec{x}} | 0 \rangle_{\text{large } \kappa_e}}{\text{large } \kappa_e \langle 0 | 0 \rangle_{\text{large } \kappa_e}} \approx 1 - 2\mathcal{P}_C \left(\frac{\eta}{8\kappa_e} \right)^2 \left[1 + 6 \left(\frac{\eta}{8\kappa_e} \right)^2 + \frac{32}{3} \left(\frac{\eta}{8\kappa_e} \right)^2 - 5 \left(\frac{\eta}{8\kappa_e} \right)^2 \right] +$$

$$+ \underbrace{|\Lambda_\perp| \left(\frac{\eta}{8\kappa_e} \right)^4 \left(-\frac{12\mathcal{P}_C}{2} + 6\mathcal{P}_C \right) + |\Lambda_\perp|^2 \left(\frac{\eta}{8\kappa_e} \right)^4 (-9 + 9)}_{= 0}.$$

Note that the effect of the higher orders is to increase the coefficient multiplying the perimeter of C , something which was to expect since when the transition point is reached, the correlation function changes the perimeter decay in a more strong decay like the area contained into C . It has no sense, nevertheless, to calculate the critical exponent for such a transition as it was done in (D.45), since the considered transition is of first order kind.



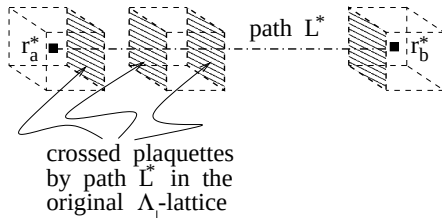
where, as in equation (D.207), in the numerator $\kappa_e \left(1 - (\hat{\sigma}_3)_{1,\vec{x}} (\hat{\sigma}_3)_{2,\vec{x}+\hat{1}} (\hat{\sigma}_3)_{1,\vec{x}+\hat{2}} (\hat{\sigma}_3)_{2,\vec{x}} \right)$ is changed by term $\kappa_e \left(1 + (\hat{\sigma}_3)_{1,\vec{x}} (\hat{\sigma}_3)_{2,\vec{x}+\hat{1}} (\hat{\sigma}_3)_{1,\vec{x}+\hat{2}} (\hat{\sigma}_3)_{2,\vec{x}} \right)$ for those plaquettes crossed by an open path L^* which covers in the dual lattice the distance from site $\vec{r}_a^*$ to site $\vec{r}_b^*$; these two sites are placed in a plane normal to the τ -direction and therefore can also be denoted by $\vec{x}_a^*$ and $\vec{x}_b^*$. For the following, the coupling strength for the scalar and for the gauge model will be denoted, respectively, by β_{Ising} and β_e if we are calculating correlations in an infinite lattice with the same spacing along all directions, or they will be named κ and κ_e if we are working with the Hamiltonian formulation; both β_e and κ_e are decreasing functions of e^2 .

The 3-dimensional $Z(2)$ gauge model is dual to the 3-dimensional $Z(2)$ Ising model, property proved for a non-periodic lattice ($\omega = \infty$) in equation (C.363) and for a lattice of periodicity ω in equation (D.66), and therefore one can expect that correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ will become the scalar-scalar correlation of the Ising model when formulated by using the dual variables. Although the proof of such an equivalence is rather lengthy and will not be carried out here, one can make plausible the duality relation between the mentioned correlation functions as follows:

- In the non-periodic lattice ($\omega = \infty$) we can return to the usual formulation without the operator language, and correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$

$$\frac{\sum_{\{\sigma_{\mu,\vec{r}}=\pm 1\}} \left[\prod_{\substack{\text{plaq. not} \\ \text{crossed by } L^*}} \left(1 + \sigma_{\mu,\vec{r}} \sigma_{\nu,\vec{r}+\hat{\mu}} \sigma_{\mu,\vec{r}+\hat{\nu}} \sigma_{\nu,\vec{x}} \tanh\left(\frac{1}{ae^2\eta}\right) \right) \right] \left[\prod_{\substack{\text{plaq.} \\ \text{crossed by } L^*}} \left(1 - \sigma_{\mu,\vec{r}} \cdots \tanh\left(\frac{1}{ae^2\eta}\right) \right) \right]}{\prod_{\vec{r} \in 3-\text{dim. } \Lambda} \prod_{\mu,\nu=1}^3 \sum_{\sigma_{\mu,\vec{r}}=\pm 1} \left[\prod_{\text{all plaquettes}} \left(1 + \sigma_{\mu,\vec{r}} \sigma_{\nu,\vec{r}+\hat{\mu}} \sigma_{\mu,\vec{r}+\hat{\nu}} \sigma_{\nu,\vec{x}} \tanh\left(\frac{1}{ae^2\eta}\right) \right) \right]}$$

in the range of large e^2 is calculated (as in equation (C.371)) by expanding the above relation in increasingly higher powers of $\tanh(1/ae^2\eta)$. Taking into account summation $\prod_{\vec{r} \in \Lambda} \prod_{\mu,\nu=1}^3 \sum_{\sigma_{\mu,\vec{r}}=\pm 1}$, the non-vanishing contributions to the correlation arise from closed surfaces tiled by plaquette factors $\mp \sigma_{\mu,\vec{r}} \sigma_{\nu,\vec{r}+\hat{\mu}} \sigma_{\mu,\vec{r}+\hat{\nu}} \sigma_{\nu,\vec{x}} \tanh\left(\frac{1}{ae^2\eta}\right)$ from the power expansion; closed surfaces because in this way each bond-variable $\sigma_{\mu,\vec{r}}$ appears twice.



The first contribution is 1; the next non-vanishing contribution corresponds to a closed unit cube formed by 6 plaquette-factors which will produce a contribution of the order $\left(\mp \tanh\left(\frac{1}{ae^2\eta}\right) \right)^6$, where sign $\mp$ of the plaquette-factors will depend on, from the possible $|\Lambda|$ places which the cube can be located, whether the cube's faces are crossed or not by path L^* .

If the cube surrounds the start or the end of path L^* , then the contribution will be negative. On the other hand, if the cube is not crossed in any form by L^* , or if it is completely “pierced” by

L^* , an even number of $-\tanh\left(\frac{1}{ae^2\eta}\right)$ appears and the contribution is positive. Hence $H(\vec{r}_a^*, \vec{r}_b^*)$ at $\omega = \infty$ and for large e^2 is equal to

$$\frac{1 + (|\Lambda| - 2)\left(\tanh\left(\frac{1}{ae^2\eta}\right)\right)^6 + (-2)\left(\tanh\left(\frac{1}{ae^2\eta}\right)\right)^6 + \dots}{1 + |\Lambda|\left(\tanh\left(\frac{1}{ae^2\eta}\right)\right)^6 + \dots} \approx \exp\left[-4\left(\tanh\left(\frac{1}{ae^2\eta}\right)\right)^6\right],$$

$$H(\vec{r}_a^*, \vec{r}_b^*)_{\text{large } e^2}^{\omega=\infty} = \text{constant}. \quad (\text{D.215})$$

For small e^2 , the studied correlation function

$$\frac{\sum_{\{\sigma_{\mu,\vec{r}}=\pm 1\}} \exp\left[-\frac{1}{ae^2\eta} \sum_{\substack{\text{plaq. not} \\ \text{crossed by } L^*}} (1 - \sigma_{\mu,\vec{r}}\sigma_{\nu,\vec{r}+\vec{\mu}}\sigma_{\mu,\vec{r}+\vec{\nu}}\sigma_{\nu,\vec{x}}) - \frac{1}{ae^2\eta} \sum_{\substack{\text{plaq.} \\ \text{crossed by } L^*}} (1 + \sigma_{\mu,\vec{r}}\sigma_{\nu,\vec{r}+\vec{\mu}}\sigma_{\mu,\vec{r}+\vec{\nu}}\sigma_{\nu,\vec{x}})\right]}{\prod_{\vec{r} \in 3\text{-dim. } \Lambda} \prod_{\mu,\nu=1}^3 \sum_{\sigma_{\mu,\vec{r}}=\pm 1} \exp\left[-\frac{1}{ae^2\eta} \sum_{\text{all plaquettes}} (1 - \sigma_{\mu,\vec{r}}\sigma_{\nu,\vec{r}+\vec{\mu}}\sigma_{\mu,\vec{r}+\vec{\nu}}\sigma_{\nu,\vec{x}})\right]}$$

the main contribution to the partition function (the denominator of the above equation) comes from the configuration with all gauge spins “up”, where here “up” is to be understood as the representative of the infinite many configurations obtained from the configuration with $\sigma_{\mu,\vec{r}} = 1 \quad \forall \vec{r}, \mu$ by a gauge transformation; one must sum only over such representatives in order to avoid over-repetition of the same contribution. Although the contribution of all spins “up” to the partition function is 1, in the numerator of $H(\vec{r}_a^*, \vec{r}_b^*)$ such a configuration produces $\exp\left[-\frac{1}{ae^2\eta} 2 \cdot |\vec{r}_b^* - \vec{r}_a^*|\right]$, which corresponds to the plaquettes crossed by L^* . Therefore, the correlation function at $\omega = \infty$ and for the range of small e^2 shows an exponential decay like the distance separating dual sites $\vec{r}_a^*$ and $\vec{r}_b^*$

$$H(\vec{r}_a^*, \vec{r}_b^*)_{\text{small } e^2}^{\omega=\infty} \sim e^{-|\vec{r}_b^* - \vec{r}_a^*| 2/ae^2\eta}. \quad (\text{D.216})$$

Behaviours (D.215) and (D.216) are, nevertheless, those expected for the decay of the scalar-scalar correlation function of the Ising model, $\langle \sigma_{\vec{r}_a^*} \sigma_{\vec{r}_b^*} \rangle$, at large and small coupling strength β_{Ising} respectively. This proves for $\omega = \infty$ the stated property that correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ becomes $\langle \sigma_{\vec{r}_a^*} \sigma_{\vec{r}_b^*} \rangle$ in the dual model, with the respective coupling strengths being related through: large/small gauge coupling strength $\beta_e = \frac{1}{ae^2}$ corresponds to small/large scalar coupling strength β_{Ising} .

- For a 3-dimensional lattice of finite periodicity ω along one dimension, and returning to the Hamiltonian operator formulation, one should proceed as in equation (D.65) in order to prove the equivalence of the correlations under study. However, one can argue more simply: since the model at finite ω shows no phase transition for small e^2 , then the behaviour of $H(\vec{r}_a^*, \vec{r}_b^*)$ at $\omega = \infty$ in that range must be maintained for any $\omega < \infty$. For large e^2 and at $\omega = \infty$, correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ presents no decay with the distance, as has been obtained by means of the power expansion in (D.215). However, for large e^2 a phase transition does exist when ω becomes small: such a transition (see accompanying comments to diagram on page 532) is related to the breaking of the power expansion which has produced $H(\vec{r}_a^*, \vec{r}_b^*) \sim \text{const.}$, passing to the behaviour in which all powers in $\kappa_e \sim \frac{1}{ae^2}$ are relevant; but this is just the change from the validity of equation (D.215) to behaviour (D.216). Therefore, at large e^2 and ω small enough, one has $H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}$. This again the expected behaviour with decreasing ω of scalar-scalar correlation function at large coupling strength, as represented in the diagram on page 538. And hence for the 3-dimensional $Z(2)$ gauge model one obtains

$$\left\{ \begin{array}{l} H(\vec{r}_a^*, \vec{r}_b^*)_{\text{small/large } \beta_e \equiv 1/ae^2}^{\text{non-period. gauge}} \equiv \left\langle \sigma_{\vec{r}_a^*} \sigma_{\vec{r}_b^*} \right\rangle_{\text{large/small } \beta_{\text{Ising}}}^{\text{non-period. Ising}}, \\ H(\vec{r}_a^*, \vec{r}_b^*)_{\text{small } \kappa_e \sim 1/e^2}^{\text{period. gauge}} \equiv \left\langle \sigma_{\vec{r}_a^*} \sigma_{\vec{r}_b^*} \right\rangle_{\text{large } \kappa}^{3\text{-dim. per. Ising}} \equiv \left\langle \sigma_{\vec{x}_a^*} \sigma_{\vec{x}_b^*} \right\rangle_{\beta_{\text{Ising}} = \kappa \omega}^{2\text{-dim. non-period. Ising}}, \end{array} \right. \quad (\text{D.217})$$

where the last equivalence has been already proved as the phase diagram of the Ising model in a lattice of periodicity ω was studied: Hamiltonian $\hat{\mathcal{H}} = -\eta \sum_{\vec{x} \in 2\text{-dim. } \Lambda_\perp} (\hat{\sigma}_1)_{\vec{x}} + \kappa \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^2 (1 - (\hat{\sigma}_3)_{\vec{x}+\hat{k}} (\hat{\sigma}_3)_{\vec{x}})$ for the

3-dim. scalar model produces a partition function $\mathcal{Z} = \prod_{\vec{x} \in \Lambda_\perp} \sum_{\sigma_{\vec{x}} = \pm 1} \exp \left[-\frac{\kappa \omega}{\eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^2 (1 - \sigma_{\vec{x}+\hat{k}} \sigma_{\vec{x}}) \right]$

at large coupling strength. For small e^2 , $H(\vec{r}_a^*, \vec{r}_b^*)_{\text{small } e^2}^{\text{period. gauge}} = H(\vec{r}_a^*, \vec{r}_b^*)_{\text{small } e^2}^{\text{non-period. gauge}} \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}$.

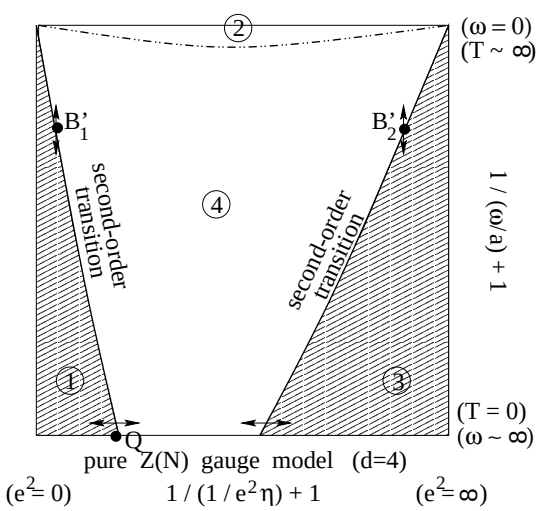
Relations (D.217) prove that the phase in the unshaded region of the above diagram (on page 603) corresponds to the phase denoted by ①, in the 4-dimensional $Z(2)$ gauge model: both for $\omega = \infty$ and e^2 small, as well as for small ω and e^2 , correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ shows a decay like distance $|\vec{x}_b^* - \vec{x}_a^*|$. The phases of the 3-dimensional $Z(2)$ gauge model are described as:

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim \text{const.} \end{cases}$

As a finishing point on the discussion of the $Z(N)$ gauge model in a lattice which is periodic along one direction, let us now consider instead of the lowest value of N possible, $N = 2$, the opposite case: limit $N \rightarrow \infty$, which will result in the occurrence of another different phase. The reason: the difference between, on the one hand, the phase shown in a non-periodic lattice by the scalar as well as by the gauge model with symmetry $Z(N = 2)$ in the range of small coupling strength β_{Ising} , $\beta_e \equiv \frac{1}{a^{4-d}e^2}$, and on the other hand, the phase appearing for the same range of the model's parameters when the symmetry is $U(1)$. Remembering that limit $N \rightarrow \infty$ of the $Z(N)$ symmetry corresponds to the continuous $U(1)$ symmetry, for N large enough (see the comments and the footnote on page 425) the $Z(N)$ scalar as well as the $Z(N)$ gauge model shows, in addition to the two phases already present in the $Z(N = 2)$ model, a third intermediate phase which corresponds to that at small coupling strength in the $U(1)$ model. With increasing N such an intermediate phase gradually displaces the $Z(2)$ -model's phase at small coupling strength until taking completely the place of the latter. These are general results: the specific form of the phase diagram will again differ depending whether the 3-dimensional or the 4-dimensional is under study.

Maintaining the same order as in the previous Note, let us discuss firstly the 4-dimensional $Z(N)$ gauge model. We shall consider only the range of large e^2 , since the characteristics at small e^2 can be easily deduced by the self-duality which relates both ranges. In a lattice of periodicity ω along one dimension, correlation function $W(C)$ in the 4-dim. $Z(N)$ gauge model at large e^2 is equivalent to the scalar-scalar correlation function of 3-dim. $Z(N)$ Ising model in a non-periodic lattice and with $1/e^2\omega$ as coupling strength β_{Ising} ; this is only a slight generalization of result (D.177) and (D.184), or

as in equation (D.196). Since, as just mentioned, the $Z(N)$ (non-periodic) Ising model shows *three* phases when N is large enough, with decreasing ω (and thus with increasing β_{Ising}) $W(C)$ will have the following behaviours⁴⁸: $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ at large ω , $W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}}$ at a smaller ω , and finally $W(C) \sim \text{const.}$ at a very small ω , where $\vec{x}_1, \vec{x}_2 \in \Lambda_\perp$ are the sites where closed path C crosses the subspace normal to the periodic direction. The fact that phase ② given by $W(C) \sim \text{const.}$, characteristic of the model whose symmetry is the discrete $Z(2)$ group, “survives” only at a very small ω could be expected by noting that term $\frac{N^2 e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=4)} P_{\nu, \vec{r}'}$, which controls the carrier $P_{\nu, \vec{r}}$ of the discrete set of values for the spin field in the $Z(N)$ model, becomes $\frac{N^2 e^2 \eta}{2} \sum_{\vec{x}, \vec{x}' \in \Lambda_\perp} \left(\frac{\omega}{a} \bar{P}_{\vec{x}} \right) \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d=3)} \left(\frac{\omega}{a} \bar{P}_{\vec{x}'} \right) = \frac{N^2 \omega e^2 \eta}{2a} \sum_{\vec{x}, \vec{x}'} \bar{P}_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(d=3)} \bar{P}_{\vec{x}'}$ in a periodic lattice of small ω : according to this term, a large N will suppress the $\bar{P}_{\vec{x}}$ -fields, leaving thus only field variables with a continuous range (i.e., a $U(1)$ symmetry), unless ω is very small. Because of the self-duality in the gauge model, the two transitions which occur at large e^2 when ω is decreased are also present in the range of small e^2 , where now the equivalence with the scalar-scalar correlation function of the 3-dim. non-periodic $Z(N)$ Ising model corresponds to correlation function $H(\vec{x}_a^*, \vec{x}_b^*)$:



$H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}$ at large ω ,
 $H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}}$ at a smaller ω ,
 and $H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}$ at a very small ω . The case of $\omega = \infty$ has been already studied in Fig. C.10 and the three phases are characterized by $\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(C^*) \sim e^{-\mathcal{A}_{C^*}} \end{cases}$ at small e^2 , $\begin{cases} W(C) \sim e^{-\mathcal{P}_C + \tau/|\vec{x}_2 - \vec{x}_1|} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*} + \tau^*/|\vec{x}_b^* - \vec{x}_a^*|} \end{cases}$ at e^2 larger, and $\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*}} \end{cases}$ at e^2 large. The resulting phase diagram can be easily drawn as in the left figure. At $\omega = \infty$, the

system presents also two transitions and the intermediate phase, that corresponding to large $\beta_e \equiv e^{-2}$ in the pure $U(1)$ gauge model, tends to occupy the whole range of small e^2 in the limit $N \rightarrow \infty$: transition point Q tends toward $e^2 = 0$ in such a limit. These critical points are both of second order: the right transition point will become the

⁴⁸The phases of the $Z(N)$ Ising model in a non-periodic 3-dimensional lattice are: the already known disordered and ordered phases characterized by $\langle e^{i \frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \rangle \sim e^{-|\vec{r}_2 - \vec{r}_1|}$ and $\langle e^{i \frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \rangle \sim \text{const.}$, shown at a very small and at large β_{Ising} respectively, as well as the intermediate phase $\langle e^{i \frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \rangle \sim e^{|\vec{r}_2 - \vec{r}_1|^{-1}}$ occurring at a not so small β_{Ising} and which is the phase at small β_{XY} in the XY model. For the 2-dimensional model, this intermediate state is characterized by $\langle e^{i \frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \rangle \sim e^{-\ln |\vec{r}_2 - \vec{r}_1|}$, just the same behaviour of the scalar-scalar correlation function in the 2-dim. XY model.

critical point of the $U(1)$ gauge model in a 4-dim. non-periodic lattice, transition which is of second order, and the left transition point Q is dual to the other and therefore also second-order.

Also the transition at large e^2 from phase ③ to phase ④ is of second order, since this is the character of the 3-dim. non-periodic XY model⁴⁹ and therefore, by self-duality, this is also the transition kind across the line separating phase ① and phase ④ in the range of e^2 small. Transition points B'_1 and B'_2 are related to each other by a self-dual equation

$$\frac{1}{(e^2)_{B'_1} \omega \eta} \equiv \frac{\beta_{3\text{-dim. non-per. Ising}}}{\eta} \equiv \frac{q^2 (e^2)_{B'_2} \eta}{4\pi^2 \omega} \implies \frac{1}{(e^2)_{B'_1} \eta} = \frac{q^2}{4\pi^2} (e^2)_{B'_2} \eta, \quad (\text{D.218})$$

and thus the left shaded region eventually disappears at $q \rightarrow \infty$.

In addition to the just discussed phases, there exists a “pocket” of phase ②, characterized by $W(C)$, $H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}$, which will eventually disappear in the limit $N \rightarrow \infty$. The phases of the above diagram are summarized in the following table:

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(C^*) \sim e^{-\mathcal{A}_{C^*}} \end{cases}$
②	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*}} \end{cases}$
④	$\begin{cases} W(C) \sim e^{ \vec{x}_2 - \vec{x}_1 ^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{ \vec{x}_b^* - \vec{x}_a^* ^{-1}} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C + \tau/ \vec{x}_2 - \vec{x}_1 } \\ H(C^*) \sim e^{-\mathcal{P}_{C^*} + \tau^*/ \vec{x}_b^* - \vec{x}_a^* } \end{cases}$

where τ and τ^* are, respectively, the length of C and C^* along the will-become-periodic a_τ -coordinate.

For the 3-dimensional $Z(N)$ gauge model, the phase named ② in the 4-dimensional diagram disappears. The reason is the same as for the 3-dimensional $Z(2)$ gauge model: for small e^2 the leading term in the Hamiltonian produces a partition function which is that of the $Z(N)$ gauge model in a non-periodic 2-dimensional lattice, being the coupling strength β_e of this last model proportional to ω . But the 2-dimensional (non-periodic) $Z(N)$ gauge model, as the $Z(2)$ and the $U(1)$ gauge models, shows no phase transitions and therefore, correlation function $H(\vec{x}_a^*, \vec{x}_b^*)$ in the range of small e^2 displays always the same behaviour. Such a behaviour can be obtained as follows: $H(\vec{x}_a^*, \vec{x}_b^*)$ becomes, when written by means of the dual variables, scalar-scalar correlation function $\left\langle e^{i \frac{2\pi}{N} (n_{r_a^*} - n_{r_b^*})} \right\rangle$

⁴⁹One should remember that of the two phase transitions points of the $Z(N)$ model in a 3-dimensional non-periodic lattice, that point separating the intermediate phase from the phase at small coupling strength $\beta_e = \frac{1}{ae^2}$ corresponds to the only transition point existing in the XY model, and this is a second-order (critical) point. Further, the scalar coupling strength β_{Ising} is proportional to $\frac{1}{\omega}$ in the gauge model when its range of large e^2 is studied, and thus such a second-order phase transition occurs by passing from a large to a small ω .

of the $Z(N)$ Ising model in a 3-dimensional lattice of the same periodicity ω as the original lattice; scalar coupling strength κ for the last model is proportional to N^2 and inversely proportional to gauge coupling strength $\kappa_e \sim 1/e^2$ ⁵⁰. Thus, at small

⁵⁰The result can be argued as for the case of the $Z(2)$ symmetry, firstly by proving it for a non-periodic lattice ($\omega = \infty$) in which correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ shows no decay according to the “high-temperature” expansion in powers of $\frac{1}{ae^2\eta}$; and then going to the model at finite ω by noting that the system has no phase transition in the range of e^2 small, and thus in such a range $H(\vec{x}_a^*, \vec{x}_b^*)$ has the same form as at $\omega = \infty$, and that the phase transitions at large e^2 when ω decreases is related to the breaking of the power expansion on which the behaviour at $\omega = \infty$ bases. Returning to $\omega = \infty$, instead of obtaining the equivalence between $H(\vec{x}_a^*, \vec{x}_b^*)$ and the scalar-scalar correlation function of the 3-dimensional $Z(N)$ Ising model by means of the “high-temperature” expansion (which only works for large e^2), let us prove it by using the Villain formulation, which holds approximately well for any e^2 and further gives us the relation between the gauge coupling strength of the original model and the scalar coupling strength of the dual model. In the $Z(N)$ scalar model, instead of spin field $\sigma_{\vec{r}}$ the scalar variable is written $e^{i\frac{2\pi}{N}n_{\vec{r}}}$, with $n_{\vec{r}} = 0, 1, \dots, (N-1)$ and similarly for the gauge field variable. Returning to scalar-scalar correlation function of the $Z(N)$ Ising model in a non-periodic d -dim. lattice (D.159)

$$\frac{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} db_{\mu, \vec{r}} \int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \sum_{\{Q_{\vec{r}}\}} \sum_{\{m_{\mu, \vec{r}}\}} \exp \left[i \left(\theta_{\vec{r}_1} - \theta_{\vec{r}_2} \right) + iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} - \frac{\eta}{2\beta_{\text{Ising}}} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} \left(\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu, \vec{r}} \right) \right]}{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} db_{\mu, \vec{r}} \int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \prod_{\vec{r}} \sum_{Q_{\vec{r}}=-\infty}^{\infty} \prod_{\vec{r}, \mu} \sum_{m_{\mu, \vec{r}}=-\infty}^{\infty} \exp \left[iN \sum_{\vec{r}} Q_{\vec{r}} \theta_{\vec{r}} - \frac{\eta}{2\beta_{\text{Ising}}} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} \left(\Delta_{\mu} \theta_{\vec{r}} - 2\pi m_{\mu, \vec{r}} \right) \right]}$$

the solution of constraint $-\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{r}-\hat{\mu}} + NQ_{\vec{r}} + (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) = 0$, resulting from the integration of $\prod_{\vec{r}} d\theta_{\vec{r}}$,

for a ($d=3$)-dimensional lattice is $b_{\mu, \vec{r}} = \sum_{\nu, \lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda, \vec{r}-\hat{\nu}} + NR_{\mu, \vec{r}} + L_{\mu, \vec{r}}$, with $\sum_{\mu} \Delta_{\mu} L_{\mu, \vec{r}-\hat{\mu}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}$ and

$\sum_{\mu} \Delta_{\mu} R_{\mu, \vec{r}-\hat{\mu}} = Q_{\vec{r}}$. By defining $M_{\lambda, \vec{r}} = \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu, \vec{r}}$, by substituting $b_{\mu, \vec{r}}$ into $\left\langle e^{i\frac{2\pi}{N}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle$ and finally

carrying out all integrations as in (D.163), one obtains

$$\prod_{\vec{r}} \sum_{Q_{\vec{r}}=-\infty}^{\infty} \prod_{\vec{r}, \lambda} \left[\sum_{M_{\lambda, \vec{r}}=-\infty}^{\infty} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \right] \exp \left[-\frac{iN}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} \Omega_{\lambda, \vec{r}, \vec{r}'} \left[Q_{\vec{r}'} + \frac{1}{N} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) \right] \right] \times \\ \times \exp \left[-\frac{N^2\eta}{2\beta_{\text{Ising}}} \sum_{\vec{r}, \vec{r}'} \left[Q_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) \right] D_{\vec{r}, \vec{r}'} \left[Q_{\vec{r}'} + \frac{1}{N} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) \right] - \frac{2\pi^2\beta_{\text{Ising}}}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'} M_{\lambda, \vec{r}'} \right]$$

in the numerator of the scalar-scalar correlation function; the same but without terms $\frac{1}{N} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})$ in the denominator. Now let us consider $\mathcal{Z}$ for the 3-dim. non-periodic $Z(N)$ gauge model, which is equal to the denominator of equation

$$(D.191) \quad \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu < \nu} dB_{\mu\nu, \vec{r}} \prod_{\vec{r}, \mu} \sum_{P_{\mu, \vec{r}}=-\infty}^{\infty} \prod_{\vec{r}, \mu < \nu} \sum_{g_{\mu\nu, \vec{r}}=-\infty}^{\infty} \exp \left[iN \sum_{\vec{r}, \mu} P_{\mu, \vec{r}} A_{\mu, \vec{r}} - \frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 \right] \times \\ \times \exp \left[i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}} \right) \right]; \text{ the numerator of correlation function } H(\vec{r}_a^*, \vec{r}_b^*) \text{ is equal to } \mathcal{Z} \text{ but}$$

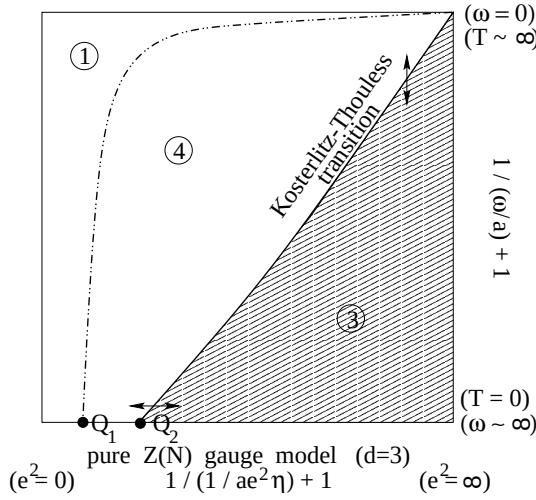
with the change $\cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}) \rightarrow \cos \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{N} \sum_{\rho} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^* \right)$, where integer field $L_{\lambda, \vec{r}}^*$

is the unit tangent vector to path L^* and satisfies thus $\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^* = (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$; the careful reader has

surely noticed that this last relation should be written $\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{r}^*-\hat{\lambda}}^* = (\delta_{\vec{r}^*, \vec{r}_a^*} - \delta_{\vec{r}^*, \vec{r}_b^*})$ since antisymmetric ten-

sor $\epsilon_{\mu\nu\lambda}$ in $\epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}}^*$ produces the coupling of a field in the original lattice with a field placed in the dual lattice, but for the following calculation and in order to avoid an excessive notation, sites $\vec{r}_a^*$ and $\vec{r}_b^*$ will be denoted without the star, keeping in mind where they are located. The above change in the numerator's cosine, nevertheless,

gauge coupling strength (large e^2) the behaviour of $H(\vec{x}_a^*, \vec{x}_b^*)$ is given by the scalar-scalar correlation function of the 2-dimensional non-periodic $Z(N)$ Ising model, being this last correlation function equivalent to $\left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_a^*} - n_{\vec{r}_b^*})} \right\rangle$ in a 3-dim. periodic lattice; the coupling strength of the 2-dim. non-periodic scalar model is proportional to ω . In addition to the following behaviours for large e^2 , $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} & \text{large } \omega \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|} & \text{smaller } \omega \end{cases}$, the just discussed equivalence leads to result $H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}$ for large e^2 and very small ω , behaviour which is just the same as for small e^2 and $\omega = \infty$, proving therefore that the same phase extends between these two ranges (see next diagram). And such a phase corresponds in the 4-dimensional model to state ①, remaining excluded thus phase ② from the considered ranges.



At $\omega = \infty$, the 3-dimensional gauge model displays two transition points as the 4-dimensional model: the existence of three phases can be easily seen by noticing that correlation function $H(\vec{r}_a^*, \vec{r}_b^*) \sim \left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle$ presents three different behaviours. However, and this is the difference with the 4-dimensional gauge model, both transition points Q_1, Q_2 tend to $e^2 = 0$ in the limit $N \rightarrow \infty$: this could be already expected for point Q_1 due to dual relation $\beta_{Z(N) \text{ scalar, dual model}} = \frac{N^2}{4\pi^2} a e^2$ and the fact that $\lim_{N \rightarrow \infty} \beta_{\text{scalar}} \neq 0$.

The fact that transition point $e_{Q_2}^2$, which would correspond to the transition point of the $U(1)$ gauge model, tends nevertheless to zero too is because of the absence of more than one phase in the 3-dimensional compact $U(1)$ gauge model, phase which is characterized by $W(C) \sim e^{-\mathcal{A}_C}$ and $H(\vec{r}_a^*, \vec{r}_b^*) \sim \text{constant}$.

can be also obtained in the Villain formulation by means of substitution $g_{\mu\nu, \vec{r}} \rightarrow g_{\mu\nu, \vec{r}} + \frac{1}{N} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \vec{\lambda}}^*$, or equivalently, $G_{\vec{r}} \rightarrow G_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$; integer field $G_{\vec{r}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$ is the carrier of the topological content of the gauge field. After carrying out all integrations, partition function $\mathcal{Z}$ of the $Z(N)$ gauge model becomes equal to the denominator of equation (D.194): $\sum_{\{G_{\vec{r}}\}} \prod_{\vec{r}, \mu} \left[\sum_{P_{\mu, \vec{r}} = -\infty}^{\infty} \delta \left(\sum_{\mu} \Delta_{\mu} P_{\mu, \vec{r} - \vec{\mu}}, 0 \right) \right] \exp \left[-\frac{iN}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} \Omega_{\nu, \vec{r}, \vec{r}'} G_{\vec{r}'} \right] \times \exp \left[-\frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'} - \frac{N^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\nu} P_{\nu, \vec{r}} D_{\vec{r}, \vec{r}'} P_{\nu, \vec{r}'} \right]$, which is also the denominator of $H(\vec{r}_a^*, \vec{r}_b^*)$. The numerator of this last correlation function is $\mathcal{Z}$ but with the change $G_{\vec{r}} \rightarrow G_{\vec{r}} + \frac{1}{N} (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$, equal thus to the numerator of scalar-scalar correlation function $\left\langle e^{i \frac{2\pi}{N} (n_{\vec{r}_a} - n_{\vec{r}_b})} \right\rangle$ for the 3-dimensional $Z(N)$ Ising model with $\beta_{\text{Ising}} = a e^2 \frac{N^2}{4\pi^2} \equiv \frac{N^2}{4\pi^2} \frac{1}{\beta_e}$.

Returning to ω finite, $W(C)$ also presents three different behaviours at large e^2 , belonging intermediate phase ④ to the same state as the intermediate phase at $\omega = \infty$. The transition in the range of large e^2 from phase ③ to phase ④ is of Kosterlitz-Thouless type, which is the transition character of the 2-dimensional non-periodic XY model (limit $N \rightarrow \infty$ of the $Z(N)$ Ising model). The transition at point Q_2 is, however, of unknown type but one can at least exclude that it is Kosterlitz-Thouless: the reason is that remaining on the separating line which starts at Q_2 , if one leaves this point and approaches the region of small ω the propagator of the interactions changes from $D_{\vec{r}, \vec{r}'}^{(d=3)} = \frac{1}{4\pi|\vec{r} - \vec{r}'|}$, which decreases at large distances, to $\frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d=2)} = -\frac{a}{2\pi\omega} \ln|\vec{x} - \vec{x}'|$, which increases with $|\vec{x} - \vec{x}'|$. This change in the long-range behaviour indicates a change in the phase-transition character. Transition point Q_2 comes from the only transition point existing in the $Z(N=2)$ gauge model at $\omega = \infty$, which was second-order; since by increasing N the number of orientations available to the spin fields are larger, an important change in the spin's orientations at a large scale can be carried out even in a more smooth way as in the $Z(N=2)$ gauge model, and thus the transition at Q_2 can be expected to be second-order⁵¹.

Together with the disappearing of point Q_1 at $N \rightarrow \infty$, phase ① disappears too: if the separating line between phases ① and ④ would remain, then it would exist a phase transition at small e^2 for some ω , something which can not occur for the 3-dimensional gauge model. For $U(1)$ as gauge symmetry group, the system only has two phases: ④ and ③, occupying the latter the whole e^2 -axis at $\omega = \infty$. On the other hand, the different phases of the 3-dimensional $Z(N)$ gauge model for large N are given by:

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) & \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
③	$\begin{cases} W(C) & \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim \text{const.} \end{cases}$	$\begin{cases} W(C) & \sim e^{-\mathcal{A}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim \text{const.} \end{cases}$
④	$\begin{cases} W(C) & \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{-\ln \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{ \vec{r}_b^* - \vec{r}_a^* ^{-1}} \end{cases}$

⁵¹It may be worth noting that in certain aspects, a Kosterlitz-Thouless and a second order phase transition are similar. Both are continuous phase transitions (without “jumps” at the transition point from a phase to the other, something which is characteristic of a first order phase transition) and both display critical behaviour at the transition point, i.e., the aspect of the system is the same independent of the “magnifying” power under which we observe the system. This scale invariance is mathematically expressed by the divergence of the correlation length at the transition point, where correlation length ξ is defined as the characteristic length for the decay of the gauge connected correlation function. According to results (C.479) and (C.480), such a connected correlation looks like $\Gamma_{\text{gauge}}^{\text{D}}(|\vec{r}_2 - \vec{r}_1|) \sim e^{-|\vec{r}_2 - \vec{r}_1|} \sim \Gamma_{\text{gauge}}^{\text{D}}(|\vec{r}_2 - \vec{r}_1|)$ and $\Gamma_{\text{gauge}}^{\text{D}}(|\vec{r}_2 - \vec{r}_1|) \sim \frac{1}{|\vec{r}_2 - \vec{r}_1|}$. The difference between the second order phase transition and that of Kosterlitz-Thouless kind resides in the form in which correlation length diverges when critical point e_{crit}^2 is approached from the phase where the correlation length is finite. If this divergence is of a power-type $\xi \sim \frac{1}{(e^2 - e_{\text{crit}}^2)^\nu}$, with ν a real positive number, the transition is called of second-order kind. If, on the contrary, the correlation length is essentially divergent $\xi \sim \exp\left(\frac{1}{(e^2 - e_{\text{crit}}^2)^\nu}\right)$ then it corresponds to a Kosterlitz-Thouless phase transition.

where τ at $\omega = \infty$ is the length dimension of C along the, then non-periodic, a_τ -coordinate.

D.3 The phase diagram of the compact $U(1)$ Higgs model at high temperatures

Now we shall collect the results for the pure $U(1)$ and $Z(N)$ gauge model to obtain the phase diagram of the compact $U(1)$ Higgs model, whose limits $|\psi|^2 = 0$ and $|\psi|^2 = \infty$ are, respectively, these just discussed pure gauge models. One should remember that integer q in the covariant derivative, which couples the gauge to the scalar field in the Higgs model, determines N (for the $Z(N)$ gauge model) at $|\psi|^2 = \infty$, namely $N = q$. However, the case $e^2 = 0$ will be discussed previously in detail, case which corresponds to the XY model and has been considered inside the Note on page 538 whose results will be here summarized by calculating the scalar-scalar correlation function $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ using again the Hamiltonian formulation; sites $\vec{r}_1$ and $\vec{r}_2$ are assumed to be placed in the sublattice normal to the will-become-periodic τ -direction. By looking at (D.119) and (D.120) one can immediately identify operator $\hat{\mathcal{H}}$ for the pure $U(1)$ scalar model without any coupling

to a gauge field: $\hat{\mathcal{H}} = \frac{a^{1-d}}{2|\psi|^2} \sum_{\vec{x} \in (d-1)\text{-dim. } \Lambda_\perp} \hat{\mathcal{L}}_{\vec{x}}^2 + a^{d-3}|\psi|^2 \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} \left[1 - \cos(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}}) \right]$, where

operator $\hat{\mathcal{L}}_{\vec{x}'} \equiv -i\eta \frac{\partial}{\partial \varphi_{\vec{x}'}}$ has integer values (times η) since $\varphi_{\vec{x}}$ is an angular variable, and further $[\hat{\mathcal{L}}_{\vec{x}'}, \hat{\mathcal{H}}] = 0$. Operator $\hat{\mathcal{L}}_{\vec{x}'}$ satisfies $\hat{\mathcal{L}}_{\vec{x}'} \{e^{\mp i \hat{\varphi}_{\vec{x}}}\} \equiv [\hat{\mathcal{L}}_{\vec{x}'}, e^{\mp i \hat{\varphi}_{\vec{x}}}] = \mp \eta \delta_{\vec{x}', \vec{x}} e^{\mp i \hat{\varphi}_{\vec{x}}}$. Let us denote by $|0\rangle_{|\psi|^2=0}$ the (unique) eigenvector of lowest eigenvalue of the Hamiltonian at $|\psi|^2 = 0$, $\hat{\mathcal{H}}_0 = \frac{1}{2a} \frac{1}{a^{d-2}|\psi|^2} \sum_{\vec{x} \in \Lambda_\perp} \hat{\mathcal{L}}_{\vec{x}}^2$, i.e. $\hat{\mathcal{L}}_{\vec{x}'} |0\rangle_{|\psi|^2=0} = 0$ since $\hat{\mathcal{L}}_{\vec{x}'}$ (as well as $\hat{\mathcal{H}}$) has

only positive integer eigenvalues. Therefore $_{|\psi|^2=0} \langle 0 | e^{i \hat{\varphi}_{\vec{x}}} | 0 \rangle_{|\psi|^2=0} = 0$ as one can easily check

$$\begin{aligned} -\eta \left(_{|\psi|^2=0} \langle 0 | e^{i \hat{\varphi}_{\vec{x}}} | 0 \rangle_{|\psi|^2=0} \right) &= _{|\psi|^2=0} \langle 0 | [\hat{\mathcal{L}}_{\vec{x}}, e^{i \hat{\varphi}_{\vec{x}}}] | 0 \rangle_{|\psi|^2=0} = \underbrace{_{|\psi|^2=0} \langle 0 | \hat{\mathcal{L}}_{\vec{x}} e^{i \hat{\varphi}_{\vec{x}}} | 0 \rangle_{|\psi|^2=0}}_0 - \\ &- \underbrace{_{|\psi|^2=0} \langle 0 | e^{i \hat{\varphi}_{\vec{x}}} \hat{\mathcal{L}}_{\vec{x}} | 0 \rangle_{|\psi|^2=0}}_0 = 0. \end{aligned}$$

At $\omega = \infty$ the only relevant eigenvector is that of lowest $\hat{\mathcal{H}}$ -eigenvalue and which is, for small $|\psi|^2$, equal to $|0\rangle_{|\psi|^2=0}$ plus the perturbative contributions of order $\left(\frac{a^{d-2}|\psi|^2}{\eta} \right)^n$ resulting from the action n consecutive times on $\hat{\mathcal{H}}_0$ -eigenstate $|0\rangle_{|\psi|^2=0}$ of perturbation $\hat{\mathcal{H}}_1 = \frac{\eta}{a} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} \left[1 - \cos(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}}) \right]$. As seen $_{|\psi|^2=0} \langle 0 | e^{i(\hat{\varphi}_{\vec{x}_1} - \hat{\varphi}_{\vec{x}_2})} | 0 \rangle_{|\psi|^2=0} = 0$ and the same holds for $_{|\psi|^2=0} \langle 0 | e^{i(\hat{\varphi}_{\vec{x}_1} - \hat{\varphi}_{\vec{x}_2})} [\hat{\mathcal{H}}_1]^n | 0 \rangle_{|\psi|^2=0} = 0$ unless $[\hat{\mathcal{H}}_1]^n$ contains $e^{-i(\hat{\varphi}_{\vec{x}_1} - \hat{\varphi}_{\vec{x}_2})}$.

This will occur when the expansion of $\left[\sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} \left(1 - \frac{e^{i(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}})} + e^{-i(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}})}}{2} \right) \right]^n$ in powers of bond-factor $e^{\pm i(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}})}$ will produce an unbroken line a bond-factors covering any path from $\vec{x}_1$ to $\vec{x}_2$: the leading nonzero contribution to the scalar-scalar correlation function $\langle e^{i(\varphi_{\vec{x}_1} - \varphi_{\vec{x}_2})} \rangle$ for small $|\psi|^2$ and $\omega = \infty$ will correspond to the shortest path connecting $\vec{x}_1$ and $\vec{x}_2$, and is therefore equal to $\left(\frac{a^{d-2} |\psi|^2}{\eta} \right)^{|\vec{x}_2 - \vec{x}_1|}$. As expected, one obtains the exponential decay like the distance characteristic of the disordered phase in the scalar model for small coupling strength.

Bevor considering the behaviour of the scalar-scalar correlation function at small $|\psi|^2$ when ω is small, let us firstly study, for any finite ω , the range of large $|\psi|^2$ where the Hamiltonian is well approximated by $\hat{\mathcal{H}}_{\text{large } |\psi|^2} = a^{d-3} |\psi|^2 \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} \left[1 - \cos(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}}) \right]$. Therefore, the resulting correlation function is given by

$$\langle e^{i(\varphi_{\vec{x}_1} - \varphi_{\vec{x}_2})} \rangle = \frac{\int_{-\pi}^{\pi} \prod_{\vec{x} \in \Lambda_\perp} \frac{d\varphi_{\vec{x}}}{2\pi} e^{i(\varphi_{\vec{x}_1} - \varphi_{\vec{x}_2})} \exp \left[-\frac{\omega a^{d-3} |\psi|^2}{\eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} [1 - \cos(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}})] \right]}{\int_{-\pi}^{\pi} \prod_{\vec{x} \in (d-1)\text{-dim. } \Lambda_\perp} \frac{d\varphi_{\vec{x}}}{2\pi} \exp \left[-\frac{\omega a^{d-3} |\psi|^2}{\eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} [1 - \cos(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}})] \right]},$$

and the partition function describing the system in the range under study is just the corresponding to the XY model in a $(d-1)$ -dimensional non-periodic lattice for a coupling strength $|\psi|_{(d-1)\text{-dim. non-period. } XY}^2 = |\psi|_{d\text{-dim. period. } XY}^2 \omega^{52}$

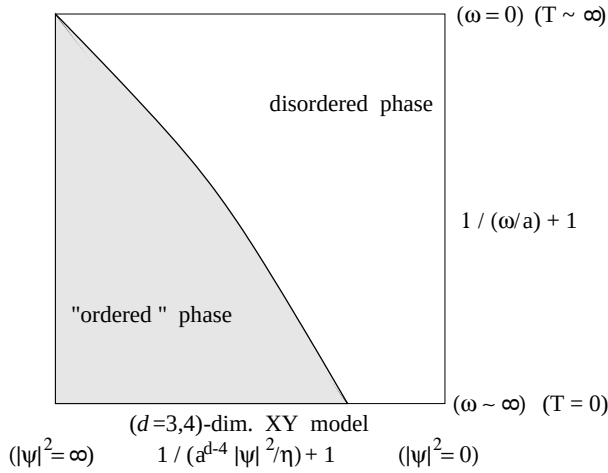
$$\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{\text{large } |\psi|_{\text{period.}}^2}^{(d=3,4)\text{-dim. period. } XY} = \langle e^{i(\varphi_{\vec{x}_1} - \varphi_{\vec{x}_2})} \rangle_{|\psi|^2 \equiv \omega |\psi|_{\text{period.}}^2}^{(d-1=2,3)\text{-dim. non-period. } XY}. \quad (\text{D.219})$$

⁵²Such a relation between coupling strengths of d -dim. periodic XY and the $(d-1)$ -dim. non-periodic XY model can be also obtained by means of the Villain formulation of the scalar-scalar correlation function. For instance, let us sketch how this is achieved for the 3-dimensional case: at $\omega = \infty$ (no periodicity), $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle$ for the 3-dimensional XY model is given by equation (C.218),

$$\frac{\exp \left[\frac{\eta/4\pi a |\psi|^2}{|\vec{r}_2 - \vec{r}_1|} \right] \sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[-\frac{i}{2} \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}} \left(\delta \Omega_{\lambda, \vec{r}}^{(\vec{r}_1)} - \delta \Omega_{\lambda, \vec{r}}^{(\vec{r}_2)} \right) - \frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} M_{\lambda, \vec{r}'} \right]}{\prod_{\vec{r}, \lambda} \sum_{M_{\lambda, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} M_{\lambda, \vec{r}'} \right]},$$

where $M_{\lambda, \vec{r}}$ is the model's topological field, which forms closed loops. $\sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}} \delta \Omega_{\lambda, \vec{r}}^{(\vec{r}_1)}$ denotes the solid parallax of closed loop $M_{\lambda, \vec{r}}$ observed from site $\vec{r}_1$. This solid angle can be rewritten as $\sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}} \Omega_{\lambda, \vec{r}, \vec{r}_1}$,

where $\Omega_{\lambda, \vec{r}, \vec{r}_1}$ is equal to the solid angle observed from site $\vec{r}_1$ for the strip of plaquettes which starts at bond $(\vec{r}, \lambda)$ of the $M_{\lambda, \vec{r}}$ -loop and ends at a certain reference point. This is because of $\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} = 0$, which allows us to define a (plaquette) antisymmetric tensor $m_{\mu\lambda, \vec{r}}$ such that $M_{\lambda, \vec{r}} = -\sum_{\mu} \Delta_{\mu} m_{\mu\lambda, \vec{r} - \hat{\mu}}$, with $m_{\mu\lambda, \vec{r}} = +1$ for those plaquettes inside and with the same direction as the $M_{\lambda, \vec{r}}$ -loop and which form a strip reaching up to the reference point as in the figure of page 587.



According to this result, for large $|\psi|^2$ and small ω the scalar correlation function decays exponentially like distance $|\vec{x}_2 - \vec{x}_1|$, just as for small $|\psi|^2$ and $\omega = \infty$. Therefore, both ranges belong to the same phase and at small $|\psi|^2$ the model displays no phase transition for any finite ω . The phase diagram results as in the figure. The transition at large $|\psi|^2$ when ω decreases is Kosterlitz-Thouless for $d = 3$ and second order for $d = 4$; the transition point at $\omega = \infty$ is of second-order type for both $d = 3$ and $d = 4$.

The absence of a phase transition in the range of small $|\psi|^2$ for the $(d = 3, 4)$ -dimensional XY model when ω decreases was already shown in a more rigorous way on page 538 as such an absence was obtained by means of exact duality relations for the $(d = 3, 4)$ -dimensional $Z(N)$ Ising model, model of which the XY model is limit $N \rightarrow \infty$. In the

Hence $\sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}} \left(\delta \Omega_{\lambda, \vec{r}}^{(\vec{r}_1)} - \delta \Omega_{\lambda, \vec{r}}^{(\vec{r}_2)} \right) = \sum_{\vec{r}, \lambda} \sum_{\vec{r}'} M_{\lambda, \vec{r}} \Omega_{\lambda \vec{r}, \vec{r}'} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2})$. If the 3-dim. lattice becomes periodic along one direction with periodicity ω small enough, periodic repetitions of topological loops stretching along this direction and of sites $\vec{r}_1, \vec{r}_2$ also occur, and the resulting interactions among these periodic images can be formulated as taking place in the 2-dim. subspace normal to the periodic dimension and as being carried by propagator $D_{\vec{r}, \vec{r}'}^{(d=3)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1=2)} = -\frac{a}{\omega} \frac{1}{2\pi} \ln |\vec{x} - \vec{x}'|$. Periodicity ω is assumed to be small enough such that the length of a $M_{\mu, \vec{r}}$ -loop stretching along the periodic dimension can be approximated by ω and thus $M_{\mu, \vec{r}} \rightarrow \frac{\omega}{a} M_{\vec{x}}$, being $M_{\vec{x}}$ the integer field which describes the points where *closed* loop $M_{\mu, \vec{r}}$ crosses the 2-dimensional sublattice $\Lambda_{\perp}$. According to the $M_{\vec{x}}$ -definition, relation $\sum_{\vec{x} \in \Lambda_{\perp}} M_{\vec{x}} = 0$ is trivially satisfied. Further

$$\sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} M_{\lambda, \vec{r}'} = D(0) \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2 + \sum_{\vec{r}, \vec{r}' \neq \vec{r}, \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} M_{\lambda, \vec{r}'} \rightarrow D^{(d=3)}(0) \frac{\omega^2}{a^2} \sum_{\vec{x}} M_{\vec{x}}^2 + \frac{\omega^2}{a^2} \frac{a}{\omega} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} M_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(d=2)} M_{\vec{x}'},$$

$$\text{which substituted into the scalar correlation gives } \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle = \frac{\exp \left[-\frac{\eta \ln |\vec{x}_2 - \vec{x}_1|}{2\pi |\psi|^2 \omega} \right] \sum_{\{M_{\vec{x}}\}} \delta \left(\sum_{\vec{x}} M_{\vec{x}}, 0 \right) \times}{\prod_{\vec{x}} \sum_{M_{\vec{x}} = -\infty}^{\infty} \delta \left(\sum_{\vec{x}} M_{\vec{x}}, 0 \right) \times}$$

$$\times \exp \left[i \sum_{\vec{x}} M_{\vec{x}} \Xi_{\vec{x}, \vec{x}'} (\delta_{\vec{x}_1} - \delta_{\vec{x}_2}) - \frac{2\pi^2 |\psi|^2 \omega^2}{a\eta} D(0) \sum_{\vec{x}} M_{\vec{x}}^2 + \frac{\pi |\psi|^2 \omega}{\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} M_{\vec{x}} \ln |\vec{x} - \vec{x}'| M_{\vec{x}'} \right] \times \exp \left[-\frac{2\pi^2 |\psi|^2 \omega^2}{a\eta} D^{(d=3)}(0) \sum_{\vec{x}} M_{\vec{x}}^2 + \frac{\pi |\psi|^2 \omega}{\eta} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} M_{\vec{x}} \ln |\vec{x} - \vec{x}'| M_{\vec{x}'} \right].$$

But this last result, up to the self-energy term, is the Villain formulation of the scalar-scalar correlation function in the 2-dimensional non-periodic XY model with a coupling strength, as expected, equal to $|\psi|_{2\text{-dim. non-period. XY}}^2 = |\psi|_{3\text{-dim. period. XY}}^2 \omega$. Although the "translation" $M_{\mu, \vec{r}} \rightarrow \frac{\omega}{a} M_{\vec{x}}$ when the lattice becomes periodic is not rigorous, it is justified because it leads to the same result as the more exact proof by means of the Hamiltonian formalism.

above phase diagram, the disordered phase is characterized by $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \sim e^{-|\vec{x}_2 - \vec{x}_1|}$, whilst the “ordered” phase corresponds to the state with a weaker or no decay of the scalar-scalar correlation function: $\begin{cases} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \sim e^{|\vec{r}_2 - \vec{r}_1|^{-1}} & \omega = \infty \\ \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} & \text{small } \omega \end{cases}$ for the 3-dimensional XY model, and $\begin{cases} \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \sim e^{-|\vec{r}_2 - \vec{r}_1|^{-2}} & \omega = \infty \\ \langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} & \text{small } \omega \end{cases}$ for the 4-dimensional case.

The equation of the separating line between the two phase of the XY model is given by $|\psi|_{3,4\text{-dim. per. } XY}^2 \omega = |\psi|_{2,3\text{-dim. non-per. } XY}^2$ when the critical coupling strength of the non-periodic XY model is substituted. Since the transition point of the 2-dim. (non-periodic) XY model, obtained in (D.152) by means of the Villain formulation, is equal to $\left(\frac{|\psi|_{2\text{-dim. non-per.}}^2}{\eta} \right)_{\text{crit}} = \frac{2}{\pi} + 4\zeta_0 e^{-\frac{\pi^2}{2} \left(\frac{|\psi|_{2\text{-dim. non-per.}}^2}{\eta} \right)_{\text{crit}}}$, the critical value for the transition in the periodic 3-dim. XY model decreases for ω small according to

$$\left(\frac{\omega a |\psi|_{3\text{-dim. per. } XY}^2}{\eta} \right)_{\text{crit}} = \frac{2}{\pi} + 4\zeta_0 e^{-2\pi^2 D^{(d=3)}(0) \left(\frac{\omega^2 a |\psi|_{\text{per.}}^2}{a^2 \eta} \right)_{\text{crit}}}, \quad (\text{D.220})$$

where, in addition to change $|\psi|_{\text{non-per.}}^2 \rightarrow \omega |\psi|_{\text{per.}}^2$ when one passes from the non-periodic to the periodic model in one higher dimension, one has to undo substitution $D_{\vec{r}, \vec{r}'}^{(d)} \rightarrow \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ in the term which contains the self-interaction of the topological field and which is proportional to propagator $D(0)$: this is due to the fact that the substitution in the propagators only holds for $a|\vec{x} - \vec{x}'| \gg \omega$ and this is not satisfied for $\vec{x} = \vec{x}'$. The above result means also that, if a relation of the kind $\omega \sim T^{-1}$ holds, the minimal value of $\eta^{-1} |\psi|_{3\text{-dim. period. } XY}^2$ necessary to have $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{3\text{-dim. period. } XY} \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|}$ increases with the temperature.

In the case of the 4-dimensional periodic XY model, and its relation for large $|\psi|^2$ with the 3-dimensional non-periodic XY model, the critical value of this last model which separates the disordered from the ordered phase was estimated in (C.226): $\left(\frac{a |\psi|_{3\text{-dim. non-per.}}^2}{\eta} \right)_{\text{crit}} = \frac{\ln \mu}{2\pi^2 D^{(d=3)}(0)}$. This is obtained by counterbalancing $\ln \mu$, where $\mu \approx 5$ determines through μ^L the number of random walks of length L in 3 dimensions, with the selfenergy per unit length of the topological defects. As above, one has to change $D^{(d=3)}(0) \rightarrow \frac{\omega}{a} D^{(d=4)}(0)$ back: $\left(\frac{a \omega |\psi|_{4\text{-dim. per.}}^2}{\eta} \right)_{\text{crit}} = \frac{\ln \mu}{2\pi^2 \frac{\omega}{a} D^{(d=4)}(0)}$

$$\left(\frac{\omega^2 a^2 |\psi|_{4\text{-dim. per. } XY}^2}{\eta} \right)_{\text{crit}} = \frac{\ln \mu}{2\pi^2 D^{(d=4)}(0)} \approx 0.526. \quad (\text{D.221})$$

The minimal necessary coupling strength $a^2 |\psi|_{4\text{-dim. per. } XY}^2$, such that the system shows the ordered state with $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{4\text{-dim. per. } XY} \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}}$, decreases with ω , whilst under this value the disordered phase characterized by $\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \rangle_{4\text{-dim. per. } XY} \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ occurs.

Let us return to the Higgs model. Before summarizing all the results obtained for the gauge models considered in the previous sections, the behaviour of $W(C)$ for the Higgs model at small $|\psi|^2$ will be calculated in the following Note.

Note. Equivalence between $\mathcal{V}_{\vec{x}_1, \vec{x}_2}$ and $W(C)$ in the compact Higgs model.

In a d -dim. lattice periodic along one direction and for the pure gauge models, both with $U(1)$ as well as with $Z(N)$ symmetry, it has been proved that free energy increase $\mathcal{V}_{\vec{x}_1, \vec{x}_2}$, produced by two opposite charged and external sources at $\vec{x}_1$ and $\vec{x}_2$, is given at large e^2 by the scalar-scalar correlation function in a (non-periodic) $(d-1)$ -dim. scalar model with the corresponding continuous $U(1)$ or discrete $Z(N)$ symmetry: results (D.114) and (D.184), obtained by means of the Hamiltonian formulation. On the other hand, it has been also proved that such a scalar-scalar correlation function determines the behaviour of $W(C)$ for a closed loop C with one of its sides converging the complete periodic length ω , and with $\vec{x}_1$ and $\vec{x}_2$ are the points where C crosses the sublattice normal to the τ -direction; this has been calculated by using the Villain formulation in a lattice of small periodicity ω : results (D.143) and (D.196). For the Higgs model at large e^2 and small $|\psi|^2$ only $\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right)$ has been discussed (D.128)⁵³, but as can be expected, this will result equivalent to the behaviour $W(C)$ for the Higgs model and it is the question which will be studied now.

Let us start with correlation function $\left\langle e^{ip(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{\text{under } h \cos(q\theta_{\vec{x}})}$, which is equivalent to $\exp\left(-\frac{\omega}{\eta}\mathcal{V}\right)$. The numerator of such a correlation function in the Villain formulation is the exponential of the following terms $ip(\theta_{\vec{x}_1} - \theta_{\vec{x}_2}) - \frac{\eta}{2|\psi|_{XY}^2} \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}}^2 + i \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}} (\Delta_{\vec{\tau}} \theta_{\vec{\tau}} - 2\pi m_{\mu, \vec{\tau}}) - \frac{\eta}{2h} \sum_{\vec{\tau}} b_{\vec{\tau}}^2 + i \sum_{\vec{\tau}} b_{\vec{\tau}} (q\theta_{\vec{\tau}} - 2\pi m_{\vec{\tau}})$ with sums over $m_{\mu, \vec{\tau}}, m_{\vec{\tau}}$ and integrals over $b_{\mu, \vec{\tau}}, b_{\vec{\tau}}$ and $\theta_{\vec{\tau}}$. This last integral produces constraint $\sum_{\mu} \Delta_{\mu} b_{\mu, \vec{\tau} - \hat{\mu}} = qb_{\vec{\tau}} + p(\delta_{\vec{\tau}, \vec{x}_1} - \delta_{\vec{\tau}, \vec{x}_2})$ whose solution in a 2-dim. lattice is $b_{\mu, \vec{\tau}} = \sum_{\nu} \epsilon_{\mu\nu} \Delta_{\nu} a_{\vec{\tau} - \hat{\nu}} + pL_{\mu, \vec{\tau}} + qc_{\mu, \vec{\tau}}$, with $\sum_{\mu} \Delta_{\mu} L_{\mu, \vec{\tau} - \hat{\mu}} = \delta_{\vec{\tau}, \vec{x}_1} - \delta_{\vec{\tau}, \vec{x}_2}$ and $\sum_{\mu} \Delta_{\mu} c_{\mu, \vec{\tau} - \hat{\mu}} = b_{\vec{\tau}}$. The Gaussian integration over $c_{\mu, \vec{\tau}}$ is immediately carried out by looking for the terms which include $c_{\mu, \vec{\tau}}$: $-\frac{\eta}{2h} \sum_{\vec{\tau}, \mu} c_{\mu, \vec{\tau}} \left(-\Delta\Delta + \frac{hq^2}{|\psi|_{XY}^2}\right) c_{\mu, \vec{\tau}} - i \sum_{\vec{\tau}, \mu} c_{\mu, \vec{\tau}} \left(-2\pi\Delta_{\mu} m_{\vec{\tau}} + 2\pi q m_{\mu, \vec{\tau}} - i \frac{\eta p}{h} \frac{hq^2}{|\psi|_{XY}^2} L_{\mu, \vec{\tau}} - i \frac{\eta q}{|\psi|_{XY}^2} \sum_{\nu} \epsilon_{\nu\mu} \Delta_{\nu} a_{\vec{\tau} - \hat{\nu}}\right)$; the remaining terms include $-\frac{\eta}{2|\psi|_{XY}^2} \sum_{\vec{\tau}} a_{\vec{\tau}} (-\Delta\Delta) a_{\vec{\tau}} - i \sum_{\vec{\tau}} a_{\vec{\tau}} \left(2\pi M_{\vec{\tau}} - i \frac{\eta p}{|\psi|_{XY}^2} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{\nu} L_{\mu, \vec{\tau}}\right) - \frac{\eta p^2}{2|\psi|_{XY}^2} \sum_{\vec{\tau}, \mu} L_{\mu, \vec{\tau}}^2$, with $M_{\vec{\tau}} = \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta_{\nu} m_{\mu, \vec{\tau}}$. We define $m^2 = \frac{hq^2}{|\psi|_{XY}^2}$. Integration over $\prod_{\vec{\tau}, \mu} dc_{\mu, \vec{\tau}}$ and $\prod_{\vec{\tau}} da_{\vec{\tau}}$ ⁵⁴, followed by regroup-

⁵³By implementing in the Hamiltonian formulation the following constraint on the states $\psi[\varphi_{\vec{x}}, A_{k, \vec{x}}]$ on which operator $\hat{H}$ is acting: $-i\eta \sum_{k'=1}^{d-1} \left(\Delta_{k'} \frac{\hat{\partial}}{\partial A_{k', \vec{x}'}} - q \frac{\hat{\partial}}{\partial \varphi_{\vec{x}'}} \right) \{ \psi[\varphi_{\vec{x}}, A_{k, \vec{x}}] \} = p(\delta_{\vec{x}_2} - \delta_{\vec{x}_1}) \psi[\varphi_{\vec{x}}, A_{k, \vec{x}}]$.

⁵⁴Integrating $dc_{\mu, \vec{\tau}}$ gives $-\frac{\eta p^2}{2|\psi|_{XY}^2} \sum_{\vec{\tau}, \mu} L_{\mu, \vec{\tau}}^2 - \frac{2\pi^2 h}{\eta} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} \left(-\Delta_{\mu} m_{\vec{\tau}} + q m_{\mu, \vec{\tau}} - i \frac{\eta}{2\pi h} \frac{p}{q} \frac{hq^2}{|\psi|_{XY}^2} L_{\mu, \vec{\tau}} \right) D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} \left(\right)_{\lambda, \vec{\tau}'}$
 $\frac{-\eta}{2|\psi|_{XY}^2} \sum_{\vec{\tau}, \vec{\tau}'} a_{\vec{\tau}} \left((-\Delta\Delta) \delta_{\vec{\tau}, \vec{\tau}'} - \frac{hq^2}{|\psi|_{XY}^2} \sum_{\nu} \Delta_{\nu} \Delta'_{\nu} D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} \right) a_{\vec{\tau}'} - i 2\pi \sum_{\vec{\tau}, \vec{\tau}'} a_{\vec{\tau}} \left(\delta_{\vec{\tau}, \vec{\tau}'} - \frac{hq^2}{|\psi|_{XY}^2} D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} \right) \left(M_{\vec{\tau}'} - \frac{i\eta p}{2\pi |\psi|_{XY}^2} \sum_{\nu, \mu} \epsilon_{\nu\mu} \Delta'_{\nu} L_{\mu, \vec{\tau}'} \right)$.
The discrete differential operator in the second line of the previous equation has the same form as (C.402), with $m^2 = \frac{hq^2}{|\psi|_{XY}^2}$, whilst the linear terms in $a_{\vec{\tau}}$ are similar to those in (C.403): the result of the integration over $\prod_{\vec{\tau}} da_{\vec{\tau}}$ becomes

ing of the terms containing $L_{\mu,\vec{\tau}}$, produces

$$\begin{aligned}
& -\frac{\eta p^2}{2|\psi|_{XY}^2} \sum_{\vec{\tau},\vec{\tau}'} (\delta_{\vec{\tau},\vec{\tau}_1} - \delta_{\vec{\tau},\vec{\tau}_2}) D_{\vec{\tau},\vec{\tau}'}^{(m^2)} (\delta_{\vec{\tau}',\vec{\tau}_1} - \delta_{\vec{\tau}',\vec{\tau}_2}) - \frac{2\pi^2 |\psi|_{XY}^2}{\eta} \sum_{\vec{\tau},\vec{\tau}'} M_{\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} M_{\vec{\tau}'} - \frac{2\pi^2 h}{\eta} \sum_{\vec{\tau},\vec{\tau}'} \sum_{\nu=1}^2 V_{\nu,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} V_{\nu,\vec{\tau}'} + \\
& + i2\pi p \sum_{\vec{\tau},\vec{\tau}'} M_{\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \sum_{\nu,\mu} \epsilon_{\nu\mu} \Delta'_{\nu} L_{\mu,\vec{\tau}'} + i2\pi \frac{p}{q} \frac{hq^2}{|\psi|_{XY}^2} \sum_{\vec{\tau},\vec{\tau}'} \sum_{\nu=1}^2 V_{\nu,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \sum_{\mu} \epsilon_{\nu\mu} L_{\mu,\vec{\tau}'} , \tag{D.222}
\end{aligned}$$

with sums over integer fields $M_{\vec{\tau}}$ and $V_{\nu,\vec{\tau}}$; the latter is defined as $V_{\nu,\vec{\tau}} = \sum_{\mu} \epsilon_{\nu\mu} (-\Delta_{\mu} m_{\vec{\tau}} + q m_{\mu,\vec{\tau}})$ and satisfies $\sum_{\nu} \Delta_{\nu} V_{\nu,\vec{\tau}} = q M_{\vec{\tau}}$.

Let us return to the Villain formulation of $W(C)$ for the 3-dim. compact Higgs model, whose numerator contains the exponential of

$$ip \sum_{\vec{\tau},\mu} J_{\mu,\vec{\tau}} A_{\mu,\vec{\tau}} - \frac{1}{2ae^2\eta} \sum_{\vec{\tau},\mu < \nu} (\Delta_{\mu} A_{\nu,\vec{\tau}} - \Delta_{\nu} A_{\mu,\vec{\tau}} - 2\pi g_{\mu\nu,\vec{\tau}})^2 - \frac{a|\psi|^2}{2\eta} \sum_{\vec{\tau},\mu} (\Delta_{\mu} \varphi_{\vec{\tau}} - q A_{\mu,\vec{\tau}} - 2\pi m_{\mu,\vec{\tau}})^2 ,$$

with sums over $g_{\mu\nu,\vec{\tau}}$, $m_{\mu,\vec{\tau}}$ and integrals over $A_{\mu,\vec{\tau}}$, $\varphi_{\vec{\tau}}$. Integrating the latter one obtains (C.414)

$$\begin{aligned}
W(C) &= \frac{1}{\mathcal{Z}_{V \text{ and } G}} \exp \left[-\frac{p^2 ae^2 \eta}{2} \sum_{\vec{\tau},\vec{\tau}'} \sum_{\mu} J_{\mu,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} J_{\mu,\vec{\tau}'} \right] \sum_{\{M_{\lambda,\vec{\tau}}, G_{\vec{\tau}}\}} \prod_{\vec{\tau}} \delta \left(\sum_{\lambda} \Delta_{\lambda} V_{\lambda,\vec{\tau}} - q G_{\vec{\tau}}, 0 \right) \times \\
&\times \exp \left[i2\pi \frac{p}{q} m^2 \sum_{\vec{\tau},\vec{\tau}'} \sum_{\lambda} V_{\lambda,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu,\vec{\tau}'} + i2\pi p \sum_{\vec{\tau},\vec{\tau}'} G_{\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \sum_{\lambda,\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} L_{\mu\nu,\vec{\tau}'} \right] \times \\
&\times \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{\tau},\vec{\tau}',\lambda} V_{\lambda,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} V_{\lambda,\vec{\tau}'} - \frac{2\pi^2}{ae^2\eta} \sum_{\vec{\tau},\vec{\tau}'} G_{\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} G_{\vec{\tau}'} \right] , \tag{D.223}
\end{aligned}$$

with $\vec{\tau}, \vec{\tau}' \in \mathbf{R}^3$, and $G_{\vec{\tau}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu,\vec{\tau}}$ and $V_{\lambda,\vec{\tau}} = \sum_{\mu,\nu} \epsilon_{\lambda\mu\nu} \Delta_{\mu} m_{\nu,\vec{\tau}} + q \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} g_{\mu\nu,\vec{\tau}}$. Now the lattice becomes periodic along one direction, which we have called the τ -direction, and further, closed loop C defined by $\sum_{\nu=1}^3 \Delta_{\nu} L_{\mu\nu,\vec{\tau}-\hat{\nu}} = J_{\mu,\vec{\tau}}$ is located on a plane containing the τ -direction such that C cover the complete lattice's length ω along this direction. If ω is small, or the density and size of the topological fields is high (something which occurs at large e^2 and small $|\psi|^2$), one can assume that these topological fields form continuous distributions along the τ -direction. Since the direction of $J_{\mu,\vec{\tau}}$ (as well as one of the indices of $L_{\mu\nu,\vec{\tau}}$) is the the periodic direction, then the direction of vector $V_{\lambda,\vec{\tau}}$ which plays a role does not correspond to the τ -direction because of term $\sum_{\lambda} V_{\lambda,\vec{\tau}} D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} L_{\mu\nu,\vec{\tau}'}$.

$$\begin{aligned}
& -\frac{\eta p^2}{2|\psi|_{XY}^2} \sum_{\vec{\tau},\mu} L_{\mu,\vec{\tau}}^2 - \frac{2\pi^2 |\psi|_{XY}^2}{\eta} \sum_{\vec{\tau},\vec{\tau}'} \left(M_{\vec{\tau}} - i \frac{\eta p}{2\pi |\psi|_{XY}^2} \sum_{\nu,\mu} \epsilon_{\nu\mu} \Delta_{\nu} L_{\mu,\vec{\tau}} \right) D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \left(M_{\vec{\tau}'} - i \frac{\eta p}{2\pi |\psi|_{XY}^2} \sum_{\nu',\mu'} \epsilon_{\nu'\mu'} \Delta'_{\nu'} L_{\mu',\vec{\tau}'} \right) - \\
& -\frac{2\pi^2 h}{\eta} \sum_{\vec{\tau},\vec{\tau}'} \sum_{\nu} \left(\sum_{\mu} \epsilon_{\nu\mu} (-\Delta_{\mu} m_{\vec{\tau}} + q m_{\mu,\vec{\tau}}) - i \frac{\eta}{2\pi h} \frac{p}{q} \frac{hq^2}{|\psi|_{XY}^2} \sum_{\mu} \epsilon_{\nu\mu} L_{\mu,\vec{\tau}} \right) D_{\vec{\tau},\vec{\tau}'}^{(m^2)} \left(\right)_{\nu,\vec{\tau}'} .
\end{aligned}$$

By taking μ as the periodic direction, one obtains the following “translation rules” from the 3-dim. periodic lattice to the projections onto the 2-dim. sublattice

$$\begin{aligned}
V_{\lambda, \vec{r}}, G_{\vec{r}} &\rightarrow V_{k, \vec{x}}, G_{\vec{x}} && \text{with } k = 1, \dots, (d-1), \vec{x} \in \mathbf{R}^2, \\
J_{\mu, \vec{r}} &\rightarrow \frac{\omega}{a} (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) && , \\
L_{\mu\nu, \vec{r}} &\rightarrow \frac{\omega}{a} L_{k, \vec{x}} && \text{with } \sum_{k=1}^2 \Delta_k L_{k, \vec{x}-\vec{k}} = (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}), \\
D_{\vec{r}, \vec{r}'}^{(m^2)} &\rightarrow \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(m^2)} && \text{with } \vec{x}, \vec{x}' \in \mathbf{R}^2 \text{ and } \vec{x}' \neq \vec{x}.
\end{aligned} \tag{D.224}$$

For the 4-dim. lattice, one obtain similar correspondences. And substitution of (D.224) into (D.223) gives (D.222), proving thus that at small ω or for large e^2 and small $|\psi|^2$ the following relation holds for the Higgs model

$$W(C) \equiv \left\langle e^{ip \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\vec{r}, \mu}} \right\rangle_{d-\text{dim compact Higgs}} = \left\langle e^{ip(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{(d-1)-\text{dim. } XY \text{ under } h \cos(q\theta_{\vec{x}})}, \tag{D.225}$$

with $|\psi|_{XY}^2 = \frac{1}{e^2 \omega}$, $h = \frac{a}{\omega} a^{d-2} |\psi|^2$, and $\vec{x}_1, \vec{x}_2$ the sites where closed loop C crosses the $(d-1)$ -dim. subspace normal to the periodic direction. Equivalently $W(C)_{\text{compact Higgs}} = \exp\left(-\frac{\omega}{\eta} \mathcal{V}_{\vec{x}_1, \vec{x}_2}\right)_{\text{compact Higgs}}$.

It may be interesting to discuss how the different behaviours of $\left\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{2-\text{dim. } XY \text{ under } h \cos(q\theta_{\vec{x}})}$ takes place for the $(d=3)$ -dimensional gauge model and at $p=1$. In the range of small coupling strength in the equivalent scalar model (large e^2), the scalar-scalar correlation function is calculated by means of the many times used “high-temperature” power expansion: but in our case, relations

$$\text{lations } \begin{cases} \int_{-\pi}^{\pi} \frac{d\theta_{\vec{x}}}{2\pi} e^{i\theta_{\vec{x}}} = 0 \\ \int_{-\pi}^{\pi} \frac{d\theta_{\vec{r}}}{2\pi} e^{i\theta_{\vec{r}}} e^{-i\theta_{\vec{x}}} = 1 \end{cases} \text{ does not always lead to a decay of } \left\langle e^{i(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{XY \text{ under } h \cos(q\theta_{\vec{x}})} \text{ like}$$

distance $|\vec{x}_2 - \vec{x}_1|$. The reason is that for $q=1$ the two powers $e^{i\theta_{\vec{x}_1}}$ and $e^{i\theta_{\vec{x}_2}}$ can also be cancelled out not by a chain of bond-factors $\left(\frac{|\psi|_{XY}^2}{2\eta}\right) e^{\pm i(\theta_{\vec{x}+\vec{k}} - \theta_{\vec{x}})}$, which arises from the expansion of

part $\exp\left[-\frac{|\psi|_{XY}^2}{\eta} \sum_{\vec{x} \in \Lambda_{\perp}} \sum_{k=1}^2 [1 - \cos(\Delta_k \theta_{\vec{x}})]\right]$, covering a continuous path from $\vec{x}_1$ to $\vec{x}_2$: the solution

arises from the expansion of the other part of the system’s energy, $\exp\left[-\frac{h}{\eta} \sum_{\vec{x} \in \Lambda_{\perp}} \underbrace{\cos(q\theta_{\vec{x}})}_{q=1}\right]$, directly

in site-powers $\left(\frac{h}{2\eta}\right) e^{\pm i\theta_{\vec{x}}}$. Thus, for $q=1$ and in the range of large e^2 , the leading contribution to

$\left\langle e^{i(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{XY \text{ under } h \cos(q\theta_{\vec{x}})}$ is $\left(\frac{h}{2\eta}\right)^2$, which is a constant. On the other hand, the above argument fails to work if $q > 1$, since in such a case the expansion of the energy’s part containing h produces powers like $\left(\frac{h}{2\eta}\right) e^{\pm i q \theta_{\vec{x}}}$ which are not able to cancel completely the single powers $e^{i\theta_{\vec{x}_1}}$ and thus one

need again $(q-1)$ chains formed by link-powers $\left(\frac{|\psi|_{XY}^2}{2\eta}\right) e^{\pm i(\theta_{\vec{x}+\vec{k}} - \theta_{\vec{x}})}$, producing again an exponential

decay like the distance between sites $\vec{x}_1$ and $\vec{x}_2$, $\left(\frac{|\psi|_{XY}^2}{2\eta}\right)^{(q-1)|\vec{x}_2 - \vec{x}_1|}$. This difference between $q=1$

and $q > 1$ in the range of large e^2 (small $|\psi|_{XY}^2$) can be also understood by looking at equation (D.222):

small $|\psi|_{XY}^2$ means $m^2 = \frac{h q^2}{|\psi|_{XY}^2}$ large and therefore propagator $D_{\vec{x}, \vec{x}'}^{(m^2)}$ only reaches very short distances

and can be approximated by $D_{\vec{x},\vec{x}'}^{(m^2)} \approx \frac{1}{m^2} \delta_{\vec{x},\vec{x}'}$. By substituting this into (D.222), one obtains that the term containing $V_{k,\vec{x}}$ (“footprint” of the presence of the external magnetic field) which differs in the numerator from that in the denominator, $i2\pi \frac{1}{q} \frac{hq^2}{|\psi|_{XY}^2} \sum_{\vec{x},\vec{x}'} \sum_{j=1}^2 V_{k,\vec{x}} D_{\vec{x},\vec{x}'}^{(m^2)} \sum_{k=1}^2 \epsilon_{jk} L_{j,\vec{x}'}$, is equal to

$$\exp \left[i2\pi \frac{1}{q} \frac{m^2}{m^2} \sum_{\vec{x},\vec{x}'} \sum_{j=1}^2 V_{k,\vec{x}} \delta_{\vec{x},\vec{x}'} \sum_{k=1}^2 \epsilon_{jk} L_{j,\vec{x}'} \right] = e^{i \frac{2\pi}{q} \times \text{integer}}. \quad (\text{D.226})$$

Only if $q \neq 1$ the numerator of the scalar-scalar correlation function is different from its denominator: this difference can be maintained as long as integer field $V_{k,\vec{x}}$ “touches” the path defined by $L_{j,\vec{x}'}$, something which can be maintained for $|\vec{x}_2 - \vec{x}_1|$ leading to the above discussed exponential decay.

For the range of small e^2 (large $|\psi|_{XY}^2$), one can approximate correlation $\left\langle e^{i(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{XY \text{ under } h \cos(q\theta_{\vec{x}})}$ by the (exponential of) term $-\frac{\eta}{2|\psi|_{XY}^2} \sum_{\vec{x},\vec{x}'} (\delta_{\vec{x},\vec{x}_1} - \delta_{\vec{x},\vec{x}_2}) D_{\vec{x},\vec{x}'}^{(m^2)} (\delta_{\vec{x}',\vec{x}_1} - \delta_{\vec{x}',\vec{x}_2})$ in equation (D.222),

because topological field $M_{\vec{x}}$ is strongly suppressed by the large $|\psi|_{XY}^2$. And this term produces a constant as value of the correlation function, since the propagator is short-range. However, this last argument no longer works for large q , since in such a case the term containing the external mag-

netic field, $\exp \left[-\frac{h}{\eta} \sum_{\vec{x} \in \Lambda_{\perp}} \cos(q\theta_{\vec{x}}) \right]$, oscillates too rapidly to have an effect and can therefore be neglected: $W(C) \equiv \left\langle e^{i \sum_{\vec{r},\mu} J_{\mu,r} A_{\vec{r},\mu}} \right\rangle_{d-\dim \text{ per. compact Higgs}} \approx \left\langle e^{i(\theta_{\vec{x}_1} - \theta_{\vec{x}_2})} \right\rangle_{(d-1)-\dim. XY \text{ without } h \cos(q\theta_{\vec{x}})}$ and thus

the scalar-scalar correlation function in the range of small e^2 and for large q decays like the exponential of the long-range propagator of the $(d-1)$ -dimensional scalar model, $D_{\vec{x},\vec{x}'}^{(d-1)} = \begin{cases} -\frac{\ln|\vec{x} - \vec{x}'|}{1^{2\pi}} & d = 3 \\ \frac{1}{4\pi|\vec{x} - \vec{x}'|} & d = 4 \end{cases}$.

Only if the coupling strength of the equivalent scalar model is small (large e^2) and the magnetic field is strong (large $|\psi|^2$), then the effect of $h \cos(q\theta_{\vec{x}})$ can be maintained even at large q . All the above discussion can be similarly applied to the $(d=4)$ -dimensional gauge model.

Remark. It is worth noting that there is a further case in the range of small e^2 where the effect of the $h \cos(q\theta_{\vec{x}})$ may become also negligible: for *large* enough ω and simultaneously for small enough $|\psi|^2$ since then $h = \frac{a}{\omega} a^{d-2} |\psi|^2$ is very weak. In such a case a small “pocket” of the phase characterized by $W(C) \sim e^{D_{\vec{x}_1,\vec{x}_2}}$ cannot be ruled out.

$d=3, q=2, \omega$ small

$$\begin{array}{cc} |\psi|^2 = 0 & \begin{array}{c} W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ \hline W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \end{array} \\ \text{small } |\psi|^2 & \hline \end{array}$$

$$\begin{array}{cc} |\psi|^2 = \infty & \begin{array}{c} W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ \hline e^2 = 0 \quad \quad \quad e^2 = \infty \end{array} \end{array}$$

By collecting the diverse results for $\mathcal{V}_{\vec{x}_1,\vec{x}_2}$, one obtains the behaviour of correlation function $W(C) = e^{ip \sum_{\vec{r},\mu} A_{\vec{r},\mu} J_{\mu,\vec{r}}}$ $\equiv \exp \left(-\frac{\omega}{\eta} \mathcal{V}_{\vec{x}_1,\vec{x}_2} \right)$, from here on $p = 1$, which in a d -dimensional lattice periodic along one direction and where closed loop C covers whole lattice's length ω along the periodic dimension, is schematica-

$d=4, q=2, \omega$ small

$$\begin{array}{c} |\psi|^2 = 0 \\ \text{small } |\psi|^2 \end{array} \quad \begin{array}{c} \overline{W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}} \\ \overline{W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}} \end{array}$$

$$\begin{array}{c} |\psi|^2 = \infty \\ e^2 = 0 \end{array} \quad \begin{array}{c} W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \\ W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \end{array} \quad \begin{array}{c} W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ e^2 = \infty \end{array}$$

lly represented in the accompanying figure for the case of $q=2$ and for small ω ; $\vec{x}_1$ and $\vec{x}_2$ are the intersection points of loop C with $\Lambda_\perp$, the $(d-1)$ -dim. subspace normal to the periodic τ -direction. In the upper figure is the case $d=3$ and in the figure below, that for $d=4$. The difference between the two phases at $|\psi|^2 = \infty$ characterized both by $W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$ will be studied in the following.

Further, since at small e^2 the behaviour of $W(C)$ is the same for small $|\psi|^2$ as at large $|\psi|^2$ — although for a not so small ω , at very small $|\psi|^2$ and e^2 the small “pocket” discussed in the remark on the previous page can occur —, we can conclude that if there is no other correlation function able to distinguish between both ranges, there is no phase transition at small e^2 . In fact, whether the critical point of the XY model (limit $e^2 = 0$ of the Higgs model) is connected with one of the transition points at $|\psi|^2 = \infty$ (the $Z(q)$ gauge model) or, on the contrary, remains isolated⁵⁵ will depend on if it is possible at small e^2 to distinguish different phases for large and for small $|\psi|^2$, and this for its part will depend on the value of q : $q \neq 1$ or $q = 1$.

The above diagram can be qualitatively understood in an easy form: as we already know from the spin systems, there are several spin configurations which have the same energy due to the invariance in the model’s energy under a global symmetry, invariance which also occurs for an extreme strong coupling strength. Breaking of this global symmetry, i.e, when all spins are aligned in a given direction implies a higher degree of order or equivalently, a weaker decay of correlation functions with the distance; if, on the other hand, the value of the model’s parameters allows for topological configurations which gradually deform a local alignment inside a certain direction into a different spin alignment inside another local region, the symmetry is restored and the correlation functions displays a stronger decay with the distance. The same holds when the symmetry is a gauge (or local) symmetry but with the modification that it applies to the set of representatives, where among all configurations which are related to each other by means of a gauge transformation and belong thus to the same class, only one representing configuration is chosen; this is actually what is done by including a strong enough gauge-fixing term in the formulation of the partition function with field variables covering a non-compact range. Beginning at $|\psi|^2 = \infty$, each configuration with all $S_{\mu, \vec{r}}^{\text{gauge}} = e^{iA_{\mu, \vec{r}}}$ ’s aligned in one of the q possible orientations $A_{\mu, \vec{r}} = 0, \frac{2\pi}{q}, \dots, (q-1)\frac{2\pi}{q}$ can be chosen as representative for all those configurations which are obtained from this

⁵⁵Isolated for a fixed ω , but if we represent the critical point of XY model as a function of ω , it draws a line.

by means of gauge transformations. At very small e^2 the system must choose one of these representatives as ground state, and then the symmetry is broken and $W(C)$ shows a weak decay. By increasing e^2 , there eventually appear other configurations which are able to connect different regions where there are different representatives: the symmetry is restored and $W(C)$ decays more strongly. This, of course, only occurs for $q > 1$, since for $|\psi|^2 = \infty$ and $q = 1$ there is one unique value for the gauge field, namely $A_{\mu,\vec{r}} = 0$, and the symmetry is already broken: $W(C)$ is trivially equal to 1.

At finite $|\psi|^2$, fluctuations in $S_{\mu,\vec{r}}^{\text{gauge}} = e^{iA_{\mu,\vec{r}}}$ for $q > 1$ make easier the existence of the “connecting configurations” (see the schematic figure on page 444). Moreover, at finite $|\psi|^2$ one has the compact superconductor: we have already seen two cases of such a model, the “ $SO(3)$ ”-superconductor (compact $U(1)$ Higgs model for $q = 2$) and the “ $SU(2)$ ”-superconductor (compact $U(1)$ Higgs model for $q = 1$). In both cases, after imposing the Coulomb gauge (or another gauge fixing term which is strong enough to remove the gauge degeneracy), the gauge configurations are classified according to their vorticity. However not any amount of vorticity is topologically stable in a compact superconductor: for $q = 2$ there are gauge transformations between states whose vorticity differs in two units (see equation (B.160)) and therefore there are only two classes of stable vortices, the class formed by vortices whose winding number is an even integer, and the class of odd integer vorticity; for $q = 1$ all vortices belong to one unique class. The *stable* vortex classes are the alluded representatives. It is then clear that only for the cases where there are more than only one representative, i.e., for $q > 1$, the possibility of “connecting configurations” between different representatives can exist: the strongest decay in $W(C)$ can only appear for $q > 1$. This strongest decay is given by $W(C) \sim e^{-\text{Area}_C}$ at $\omega = \infty$, or by $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ at small ω . On the contrary, for $q = 1$ and at $|\psi|^2 \neq 0$, we shall only obtain the weaker decay in $W(C)$: $W(C) \sim e^{-\text{Perimeter}_C}$ at $\omega = \infty$ or $W(C) \sim \text{const.}$ at small ω . A special case is $|\psi|^2 = 0$ where there are no superconductor or vortices and the behaviour of $W(C)$ is only determined by the topological defects (monopoles) of the compact pure gauge theory. Only in this case the restoration of the

$$\begin{array}{lcl}
 q = 1, \omega \text{ small} & & \\
 W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} & d = 3 & \\
 |\psi|^2 = 0 & \overline{W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|^{-1}} \quad d = 4} & W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\
 & \overline{W(C) \sim \text{const.}} & \\
 \text{small } |\psi|^2 & \overline{\hspace{10em}} &
 \end{array}$$

$$\begin{array}{c}
 W(C) = 1 \\
 |\psi|^2 = \infty \quad \overline{\hspace{10em}} \quad e^2 = 0 \hspace{1em} e^2 = \infty
 \end{array}$$

broken symmetry (always referred to the representatives of the gauge configurations) is even possible for $q = 1$. The phase diagram for $q = 1$ and small ω is depicted schematically in the figure, and as we shall see, the transition point of the XY model in this case is an isolated point, not connected with the also isolated critical value of the pure $U(1)$ gauge model.

The opposite case corresponds to a large q . Since in this case, the pure $Z(q)$ gauge model itself (limit $|\psi|^2 = \infty$) is very near to the pure $U(1)$ gauge model (limit $|\psi|^2 = 0$), one can expect that the behaviour of $W(C)$ in the whole range of $|\psi|^2$ will be determined

by its behaviour at the pure $U(1)$ gauge model. This can be also qualitatively understood by noting that now there are many different vortex classes, and therefore many different representatives, and thus the possibility of the “connecting configurations” is higher; and this implies a decay of $W(C)$ stronger than in the case of small $q > 1$. At small $|\psi|^2 \neq 0$ we have already obtained $W(C) = \left\langle e^{i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\vec{r}, \mu}} \right\rangle_{d\text{-dim per. comp. Higgs}} \equiv \left\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim non-per. } XY \text{ under } h \cos(q\theta_{\vec{x}})}$ which holds for large e^2 ; for its part, the $(d-1)$ -dim. non-periodic XY with a term $h \sum_{\vec{x}} \cos(q\theta_{\vec{x}})$ shows the same phases as the $Z(q)$ Ising model in a $(d-1)$ -dim. non-periodic lattice, being the coupling strength in this last scalar model a decreasing function of $e^2\omega$. A similar equivalence to the $(d-1)$ -dim. non-periodic $Z(q)$ Ising model holds also for large e^2 in limit $|\psi|^2 = \infty$ of the d -dim. compact Higgs model, i.e. in the d -dim. $Z(q)$ gauge model, when the lattice is periodic along one direction. In both cases, small $|\psi|^2 \neq 0$ and $|\psi|^2 = \infty$, scalar coupling strength β_{Ising} of the equivalent $Z(q)$ Ising model decreases with increasing $e^2\omega$, although for $|\psi|^2 = \infty$ in a different way as for $|\psi|^2 \neq 0$. As we already know from the discussion of the phase diagram of the d -dimensional $Z(q)$ gauge model, the existence of three phases in the $Z(q)$ Ising model⁵⁶ is directly related to the three phases shown by the d -dimensional $Z(q)$ gauge model at $|\psi|^2 = \infty$:

- For $d = 3$ (see diagram on page 609) the three phases for large e^2 correspond to the same phases existing at $|\psi|^2 = \infty$, since the model shows no phase transition in the range of small e^2 . A cut of the phase diagram at any ω , therefore, presents three phases.
- For $d = 4$ (see diagram on page 606), the self-duality of the $Z(q)$ gauge model produces a distribution of the phases for small e^2 symmetrical to that for e^2 large. Again the existence of three phases for large e^2 , direct consequence of the three phases of the $Z(q)$ Ising model, makes that a cut of the diagram at any ω displays three phases (one of these phases changes if the cut is made at a very small ω , meeting thus the small “pocket” appearing in the middle range of e^2 and for $\omega \approx 0$).

Nevertheless, such two transition points existing at $|\psi|^2 = \infty$ does not appear at small enough $|\psi|^2 \neq 0$ because the phase corresponding to behaviour $\left\langle e^{i \frac{2\pi}{q}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle \sim \text{const.}$ in the $Z(q)$ Ising model, and which should appear at small e^2 , does not occur for q large enough. The reasons are the following: the effect of term $h \cos(q\theta_{\vec{x}})$, responsible for the equivalence to the $Z(q)$ Ising model instead to the XY model directly, practically

⁵⁶The ordered phase at large scalar coupling strength and characterized by $\left\langle e^{i \frac{2\pi}{q}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle \sim \text{const.}$, the disordered phase $\left\langle e^{i \frac{2\pi}{q}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle \sim e^{-|\vec{r}_2 - \vec{r}_1|}$ at a very small coupling strength, being both phases the only ones present in $Z(2)$ Ising model, plus the intermediate phase appearing at large q and gradually covering a larger part of the range of small coupling strength when $q \rightarrow \infty$, which is described by $\left\langle e^{i \frac{2\pi}{q}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle \sim e^{-\ln |\vec{r}_2 - \vec{r}_1|}$ in the 2-dimensional (non-periodic) $Z(q)$ Ising model, or by $\left\langle e^{i \frac{2\pi}{q}(n_{\vec{r}_1} - n_{\vec{r}_2})} \right\rangle \sim e^{|\vec{r}_2 - \vec{r}_1|^{-1}}$ in the 3-dimensional one.

vanishes at large q since $\cos(q\theta_{\vec{x}})$ oscillates then too quickly to have an effect on the system; h is proportional to $|\psi|^2$; and because relation $\exp\left(-\frac{\omega}{\eta}\mathcal{V}_{\vec{x}_1, \vec{x}_2}\right) \equiv \equiv \left\langle e^{i(\theta_{\vec{x}_2} - \theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim per. } XY \text{ under } h \cos(q\theta_{\vec{x}})}$ has been deduced for the range of large e^2 . Thus the d -dim. $W(C)$'s behaviour at large q and for very small but non vanishing $|\psi|^2$ is approximately given by scalar-scalar correlation function of the $(d-1)$ -dim. XY model *without* magnetic term $h \sum_{\vec{x}} \cos(q\theta_{\vec{x}})$, and this last model only shows two phases, characterized by $\left\langle e^{i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2})} \right\rangle \sim e^{-|\vec{r}_1 - \vec{r}_2|}$ for small scalar coupling strength (large e^2) and, in the range of large coupling strength (small e^2) by $\left\langle e^{i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2})} \right\rangle \sim e^{-\ln|\vec{r}_1 - \vec{r}_2|}$ at $d=3$ or by $\left\langle e^{i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2})} \right\rangle \sim e^{|\vec{r}_1 - \vec{r}_2|^{-1}}$ at $d=4$.

The change in the range of small $|\psi|^2 \neq 0$ and not so large e^2 for the behaviour of $W(C)$, from $W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$ at small $q > 1$ to $\begin{cases} W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} & d=3 \\ W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} & d=4 \end{cases}$ at large q , is further supported by the fact that the range under study belong to the same phase as that of the pure $Z(q)$ gauge model characterized by $\begin{cases} W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} & d=3 \\ W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} & d=4 \end{cases}$ ⁵⁷; and since as already discussed, with decreasing $|\psi|^2$ the decay in $W(C)$ becomes stronger, then $W(C)$ cannot be equal to a constant value for small $|\psi|^2$ if it already

$d=3, q \text{ large}, \omega \text{ small}$

$$\begin{array}{l} |\psi|^2 = 0 \\ \text{small } |\psi|^2 \end{array} \quad \begin{array}{c} \xrightarrow{W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|}} \\ \xrightarrow{W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}} \end{array} \quad \begin{array}{c} W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \end{array}$$

$$\begin{array}{c} |\psi|^2 = \infty \\ e^2 = 0 \end{array} \quad \begin{array}{c} W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \\ \xrightarrow{e^{-\ln|\vec{x}_2 - \vec{x}_1|}} \end{array} \quad \begin{array}{c} W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ \xrightarrow{e^{-|\vec{x}_2 - \vec{x}_1|}} \end{array} \quad \begin{array}{c} W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\ e^2 = \infty \end{array}$$

displays a power decay at $|\psi|^2 = \infty$. The schematic phase diagram for the 3-dim. compact Higgs model for large q is represented in the figure, where the left transition point at $|\psi|^2 = \infty$ tends to zero with increasing q . For the 4-dimensional case, the diagram is similar by only replacing behaviour $W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|}$ with $W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}}$.

As we shall see in the next section, a completely different case is the non-compact Higgs model. In such a case every vorticity is stable and forms an own class. The analogy with a spin system, nevertheless, does not work since the gauge field has lost its angular character and we cannot regard $e^{iA_{\mu, \vec{r}}}$ as a (gauge) spin field. The same stated in another way, the change in the energy from the compact formulation given by $1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})$ to the non-compact form $\frac{1}{2}(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2$ removes the

⁵⁷The phase for $|\psi|^2 \neq 0$ and e^2 small cannot be the same as the short phase present in the pure $Z(q)$ gauge model at very small e^2 and characterized by $W(C) \sim \text{const.}$: the extent of this last phase becomes smaller and smaller with increasing q .

possibility of the “connecting configurations”. These had made possible the symmetry restoration by connecting different local regions where there were configurations belonging to different classes. Hence, in the non-compact Higgs model the strongest decay in $W(C)$ is never present.

In order to distinguish, for a small $q > 1$, between the two phases which show $W(C) \sim \text{const.} + e^{-|\vec{x}_2 - \vec{x}_1|}$ at small e^2 , or between those with the same behaviour in $W(C)$ but in the 4-dimensional $Z(q)$ gauge model at $|\psi|^2$, we have to turn again to correlation function $H(\vec{x}_a^*, \vec{x}_b^*)$:

- For the 3-dimensional non-periodic Higgs model it is defined as

$$H(\vec{r}_a^*, \vec{r}_b^*) = \frac{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - qA_{\mu, \vec{r}})] \right] \times}{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \exp \left[-\frac{a|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_{\mu} \varphi_{\vec{r}} - qA_{\mu, \vec{r}})] \right] \times} \\ \times \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left[1 - \cos \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^* \right) \right] \right] \Bigg/ \exp \left[-\frac{1}{ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})] \right], \quad (\text{D.227})$$

where $L_{\lambda, \vec{r}}^*$ is the unit tangent vector of a path connecting (dual) sites $\vec{r}_a^*$ and $\vec{r}_b^*$ ⁵⁸ and which is normal to the will-become periodic τ -direction; in $\sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^*$ either μ or ν corresponds to this τ -dimension. Since in the Villain formulation of (D.227) topological defects $G_{\vec{r}} = \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$ arise through approximation $\exp \left[1 - \cos \left(\Delta_{\mu} A_{\nu, \vec{r}} \dots - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^* \right) \right] \approx \sum_{g_{\mu\nu, \vec{r}} = -\infty}^{\infty} \exp \left[\frac{1}{2} \left(\Delta_{\mu} A_{\nu, \vec{r}} \dots - 2\pi g_{\mu\nu, \vec{r}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^* \right)^2 \right]$, then $H(\vec{r}_a^*, \vec{r}_b^*)$ is equal to the exponential of (minus) the free energy increase due to substitution $g_{\mu\nu, \vec{r}} \rightarrow g_{\mu\nu, \vec{r}} + \frac{1}{q} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^*$, or equivalently $G_{\vec{r}} \rightarrow G_{\vec{r}} + \frac{1}{q} (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$, i.e., the energy increase produced by the introduction by hand of two opposite charged topological defects at sites $\vec{r}_a^*$ and $\vec{r}_b^*$. According with the above comments on symmetry breaking, we can already guess the behaviour of $H(\vec{r}_a^*, \vec{r}_b^*)$: if this (free) energy increase with distance $|\vec{r}_b^* - \vec{r}_a^*|$, and therefore $H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{-|\vec{r}_b^* - \vec{r}_a^*|}$,

⁵⁸Although due to $\epsilon_{\mu\nu\lambda}$ inside term $\cos \left(\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r} - \hat{\lambda}}^* \right)$, the interaction of vector field $A_{\mu, \vec{r}}$ placed in the original lattice is with vector field $L_{\lambda, \vec{r} - \hat{\lambda}}^*$ actually located in the dual lattice, $\vec{r}_{a,b}^*$ will be written for the following as $\vec{r}_{a,b}$ in order to simplify the notation. Further, since $\vec{r}_{a,b}^* \in \Lambda_{\perp}$, both sites can be also denoted by $\vec{x}_a^*$ and $\vec{x}_b^*$. Star * will be written explicitly only to stress that the points are located on the dual lattice normal to the τ -direction.

it is an indication for a high rigidity and order in the system ⁵⁹ and therefore for symmetry breakdown, which for its part is characterized by a weak decay in $W(C)$. If on the contrary the system has produced many large defects, which are highly correlated among them forming configurations which restore the broken symmetry, the system loses its rigidity and $W(C)$ decays strongly; and these produced monopoles arrange themselves with the two external monopoles in order to reduce their energy and thus $H(\vec{r}_a, \vec{r}_b)$ decays weakly with the distance (or does not decay at all). Summarizing: a strong decay in $W(C)$ implies a weak decay in $H(\vec{r}_a, \vec{r}_b)$, and a strong decay in $H(\vec{r}_a, \vec{r}_b)$ implies a weak decay in $W(C)$.

A special case is $q = 1$, for which $\cos\left(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_\lambda \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^*\right) = \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})$ is satisfied since $L_{\lambda, \vec{r}}^*$ is an integer field, and thus, by looking at equation (D.227), $H(\vec{r}_a, \vec{r}_b) = 1$ at any e^2 or $|\psi|^2$. This property holds independently of the lattice's periodicity.

- For the 4-dimensional non-periodic Higgs model the correlation under study, called the 't-Hooft loop, takes the following form

$$H(C^*) = \frac{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \exp\left[-\frac{a^2|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}})]\right]}{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \exp\left[-\frac{a^2|\psi|^2}{\eta} \sum_{\vec{r}, \mu} [1 - \cos(\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}})]\right]} \times$$

$$\times \exp\left[-\frac{1}{e^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left[1 - \cos\left(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}}^*\right)\right]\right] \Bigg] \Bigg]_{(D.228)}$$

$$\times \exp\left[-\frac{1}{e^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})]\right]$$

where C^* is a closed path with one pair of its sides normal to the will-become periodic τ -direction. Antisymmetric integer field $L_{\rho\sigma, \vec{r}}^*$ is defined as only different from zero and equal to ± 1 for those plaquettes inside C^* , such that $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r}-\hat{\rho}}^* = -J_{\sigma, \vec{r}}^*$,

with $J_{\sigma, \vec{r}}^*$ the unit vector tangent to closed path C^* and therefore $\sum_{\rho} \Delta_{\rho} J_{\rho, \vec{r}}^* = 0$. As

for the 3-dimensional model, substitution $g_{\mu\nu, \vec{r}} \rightarrow g_{\mu\nu, \vec{r}} + \frac{1}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}}^*$ takes

place in the numerator of $H(C^*)$, or equivalently, the following substitution in the

⁵⁹An example of such a behaviour occurs in a superconductor, where the energy of two magnetic monopoles (the topological defects of the gauge field), separated a distance $|\vec{r}_b - \vec{r}_a|$, increases with this distance in the superconducting state. This is due to the fact that the magnetic field generated by the monopoles, because of the Meissner effect, have to be confined inside the vortices, whose energy depends on the vortex length. The existence of vortices and the Meissner effect are the main characteristics of symmetry breaking. On the other hand, the energy of the two monopoles shows a weaker dependence with the distance in the normal conducting state, where there are no vortices as isolated stable objects.

monopole fields $G_{\rho,\vec{r}} \rightarrow G_{\rho,\vec{r}} - \frac{1}{q} J_{\rho,\vec{r}}^*$, where the topological defect corresponding to the compact character of the gauge field is defined by $G_{\rho,\vec{r}} = - \sum_{\lambda} \sum_{\mu < \nu} \epsilon_{\rho\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu,\vec{r}}$ (with $\epsilon_{1234} = 1$) and satisfies therefore $\sum_{\rho} \Delta_{\rho} G_{\rho,\vec{r}} = 0$. Note that since $G_{\rho,\vec{r}}$ and $J_{\rho,\vec{r}}^*$ fulfil the same equation, in the case of $q = 1$ integer field $J_{\rho,\vec{r}}^*$ can be absorbed into $G_{\rho,\vec{r}}$ and thus $H(C^*) = 1$. According to $G_{\rho,\vec{r}} \rightarrow G_{\rho,\vec{r}} - \frac{1}{q} J_{\rho,\vec{r}}^*$, $H(C^*)$ measures the energy increase produced by the introduction by hand of a topological defect which draws a closed loop C^* . If $q = 1$, the charge of this external defect is the same as the charge of the defects which the system itself can produce, and thus the topological charge carried by $J_{\rho,\vec{r}}^*$ is completely screened by the $G_{\rho,\vec{r}}$'s which tend to surround the external defect in order to decrease the total system's energy: mathematically this is denoted by the variable change $G'_{\rho,\vec{r}} = G_{\rho,\vec{r}} - J_{\rho,\vec{r}}^*$, which results in a vanishing energy increase, or equivalently, in $H(C^*) = 1$. On the contrary, if $q \neq 1$ the topological charge of $J_{\rho,\vec{r}}^*$ cannot be completely screened by the $G_{\rho,\vec{r}}$'s and it will depend on the density of $G_{\rho,\vec{r}}$ whether the energy increase shows a strong or a weak dependence on the size of the external topological defect; one can nevertheless guess that the higher the density of $G_{\rho,\vec{r}}$ is (and thus the symmetry gets restored and the system's rigidity becomes low), the easier to screen the defect introduced by hand is and the weaker the decay shown by $H(C^*)$. As for the 3-dim. model, the behaviours of $H(C^*)$ and $W(C)$ are opposite to each other: a strong decay in $H(C^*)$ corresponds to a weak decay in $W(C)$, and viceversa with the roles of $H(C^*)$ and $W(C)$ interchanged.

The above definitions for the correlation function which displays a behaviour complementary to that of $W(C)$ hold for a non-periodic d -dimensional lattice ($\omega = \infty$). In the case of a lattice periodic along one direction, and from the definitions above for $L_{\lambda,\vec{r}}^*$ or $L_{\rho\sigma,\vec{r}}$, the correlation function which has to be calculated is $H(\vec{x}_a^*, \vec{x}_b^*)$ where $\vec{x}_a^*$ and $\vec{x}_b^*$ are sites located in the (dual) $(d-1)$ -dim. sublattice normal to the periodic

$d=3, q=2, \omega$ small

$$\begin{array}{ccccc} & W(C) \sim & & W(C) \sim & \\ & \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} & & e^{-|\vec{x}_2 - \vec{x}_1|} & \\ |\psi|^2 = \infty & \xrightarrow{\hspace{1.5cm}} & & \xrightarrow{\hspace{1.5cm}} & |\psi|^2 = \infty \\ e^2 = 0 & & & & e^2 = \infty \\ & H(\vec{r}_a, \vec{r}_b) \sim & & H(\vec{x}_a^*, \vec{x}_b^*) \sim & \\ & e^{-|\vec{x}_b^* - \vec{x}_a^*|} & & \text{cst.} + e^{-|\vec{x}_b^* - \vec{x}_a^*|} & \end{array}$$

$d=3, q$ large, ω small

$$\begin{array}{ccccc} & W(C) \sim & & W(C) \sim & & W(C) \sim \\ & \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} & & e^{-\ln |\vec{x}_2 - \vec{x}_1|} & & e^{-|\vec{x}_2 - \vec{x}_1|} \\ |\psi|^2 = \infty & \xrightarrow{\hspace{1.5cm}} & & \xrightarrow{\hspace{1.5cm}} & & \xrightarrow{\hspace{1.5cm}} & |\psi|^2 = \infty \\ e^2 = 0 & & & & & & e^2 = \infty \\ & H(\vec{r}_a, \vec{r}_b) \sim & & H(\vec{r}_a, \vec{r}_b) \sim & & H(\vec{x}_a^*, \vec{x}_b^*) \sim & \\ & e^{-|\vec{x}_b^* - \vec{x}_a^*|} & & e^{-\ln |\vec{x}_b^* - \vec{x}_a^*|} & & \text{cst.} + e^{-|\vec{x}_b^* - \vec{x}_a^*|} & \end{array}$$

direction and connected by a path defined by integer field $L_{k,\vec{x}}^*$. The behaviour for such a correlation function in the pure $Z(q)$ gauge model has been already considered: for $d=3$, in the diagram and in the table on pages 603 and 605 respectively for $q=2$, and on pages 609 and 610 for large

d=4, $q = 2$, ω small

$$\begin{array}{c}
W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\
|\psi|^2 = \infty \quad e^2 = 0 \quad \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \end{array} \quad |\psi|^2 = \infty \quad e^2 = \infty \\
H(\vec{r}_a, \vec{r}_b) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{r}_a, \vec{r}_b) \sim \text{cst.} + e^{-|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{cst.} + e^{-|\vec{x}_b^* - \vec{x}_a^*|}
\end{array}$$

d=4, q large, ω small

$$\begin{array}{c}
W(C) \sim \text{cst.} + e^{-|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|} \\
|\psi|^2 = \infty \quad e^2 = 0 \quad \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \end{array} \quad |\psi|^2 = \infty \quad e^2 = \infty \\
H(\vec{r}_a, \vec{r}_b) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{r}_a, \vec{r}_b) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{cst.} + e^{-|\vec{x}_b^* - \vec{x}_a^*|}
\end{array}$$

q ; for $d = 4$, in the diagram and in the table on pages 598 and 599 respectively for $q = 2$, and on pages 606 and 607 for large q . These results are sketched in the left figures for *small* ω . The behaviour of $H(\vec{x}_a^*, \vec{x}_b^*)$ in the pure $U(1)$ gauge model is easily deduced from the $Z(q)$ gauge model in the limit $q \rightarrow \infty$.

It remains to study $H(\vec{x}_a^*, \vec{x}_b^*)$ in the Higgs model to have a complete picture of the phase diagram of such a model. This will be carried out in the following by noting that in the range of e^2 large and small $|\psi|^2$, $H(\vec{x}_a^*, \vec{x}_b^*)$ is given by as the following quotient of partition functions of the 2-dim. non-periodic XY model under the action of an external magnetic field

$$H(\vec{x}_a^*, \vec{x}_b^*) = \frac{\mathcal{Z}_{2\text{-dim. } XY, \text{ with } \cos(\Delta_k \theta_{\vec{x}}) \rightarrow \cos(\Delta_k \theta_{\vec{x}} + \frac{2\pi}{q} \sum_j \epsilon_{kj} L_{j, \vec{x} - \hat{j}}^*)}}{\mathcal{Z}_{2\text{-dim. } XY, |\psi|_{XY}^2 = 1/e^2 \omega}}, \quad (\text{D.229})$$

where the coupling strength $|\psi|_{XY}^2$ in the equivalent XY model is inversely proportional to $e^2 \omega$. This can be proved either by repeating the steps which produced result (D.127) using the Hamiltonian formulation for large e^2 , or by returning to the Villain formulation and equation (D.222) ⁶⁰. Let us follow this second path: the partition function

⁶⁰A “mnemonic rule” to obtain the equivalence between, on the one side, the d -dimensional periodic Higgs model in the range of large e^2 and small $|\psi|^2$, and on the other side, the $(d - 1)$ -dimensional non-periodic XY model under the action of an external magnetic field, is the following: starting at the energy describing the Higgs model

$$\frac{1}{a^{4-d} e^2 \eta} \sum_{\vec{r}} \sum_{\mu < \nu = 1}^d [1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})] + \frac{a^{d-2} |\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu=1}^d [1 - \cos(\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}})]$$

one carries out the following variable change $\begin{cases} \varphi_{\vec{r}} \rightarrow \phi_{\vec{r}} = \varphi_{\vec{r}} \\ A_{\mu, \vec{r}} \rightarrow B_{\mu, \vec{r}} = A_{\vec{r}} - \frac{1}{q} \Delta_\mu \varphi_{\vec{r}} \end{cases}$ which produces as en-

$$\text{ergy } \frac{1}{a^{4-d} e^2 \eta} \sum_{\vec{r}} \sum_{\mu < \nu = 1}^d [1 - \cos(\Delta_\mu B_{\nu, \vec{r}} - \Delta_\nu B_{\mu, \vec{r}})] + \frac{a^{d-2} |\psi|^2}{\eta} \sum_{\vec{r}} \sum_{\mu=1}^d [1 - \cos(q B_{\mu, \vec{r}})]$$

since the pure gauge part is invariant under such a change. From the discussion by using the Hamiltonian formulation, we have learnt that, firstly, there is a dimension reduction in one unity; and further, that in the range of small coupling strengths, both gauge and scalar (i.e., large e^2 and small $|\psi|^2$), the leading part in the Hamiltonian and thus in the energy entering in the partition function is that where the coupling occurs on bonds along the periodic direction. In the considered range one can approximate the partition function of the (periodic) model by

$$\mathcal{Z} = \int_{-\pi}^{\pi} \prod_{\vec{x} \in \Lambda_\perp} \frac{d\theta_{\vec{x}}}{2\pi} \exp \left[-\frac{a^{d-3}}{e^2 \omega \eta} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^{d-1} [1 - \cos(\Delta_k B_{\tau, \vec{x}})] - \frac{a^{d-1} |\psi|^2}{\omega \eta} \sum_{\vec{x} \in \Lambda_\perp} [1 - \cos(q B_{\tau, \vec{x}})] \right], \text{ where}$$

of the non-periodic Higgs model is defined by terms $-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} V_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} V_{\lambda, \vec{r}'} - \frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} G_{\vec{r}'}$ of equation (D.223) and hence $H(\vec{x}_a^*, \vec{x}_b^*)$ for the 3-dimensional periodic Higgs model in the range of large e^2 and small $|\psi|^2$ is given by

$$\frac{\sum_{\{V_{k, \vec{x}}, G_{\vec{x}}\}} \prod_{\vec{x}} \delta \left(\sum_{k=1}^2 \Delta_k V_{k, \vec{x}} - q G_{\vec{x}}, 0 \right) \exp \left[-\frac{2\pi^2 a^2 |\psi|^2}{\omega \eta} \sum_{\vec{x}, \vec{x}', k} (V_{k, \vec{x}} + L_{k, \vec{x}}^*) D_{\vec{x}, \vec{x}'}^{(m^2)} (V_{k, \vec{x}'} + L_{k, \vec{x}'}^*) \right]}{\prod_{\vec{x} \in \Lambda_{\perp}} \prod_{k=1}^2 \sum_{V_{k, \vec{x}}, G_{\vec{x}} = -\infty}^{\infty} \prod_{\vec{x}} \delta \left(\sum_{k=1}^2 \Delta_k V_{k, \vec{x}} - q G_{\vec{x}}, 0 \right) \exp \left[-\frac{2\pi^2 a^2 |\psi|^2}{\omega \eta} \sum_{\vec{x}, \vec{x}' \in \Lambda_{\perp}} \sum_{k=1}^2 V_{k, \vec{x}} D_{\vec{x}, \vec{x}'}^{(m^2)} V_{k, \vec{x}'} \right]} \times \exp \left[-\frac{2\pi^2}{e^2 \omega \eta} \sum_{\vec{x}, \vec{x}'} \left(G_{\vec{x}} + \frac{(\delta_{\vec{x}, \vec{x}_a} - \delta_{\vec{x}, \vec{x}_b})}{q} \right) D_{\vec{x}, \vec{x}'}^{(m^2)} \left(G_{\vec{x}'} + \frac{(\delta_{\vec{x}', \vec{x}_a} - \delta_{\vec{x}', \vec{x}_b})}{q} \right) \right] \times \exp \left[-\frac{2\pi^2}{e^2 \omega \eta} \sum_{\vec{x}, \vec{x}' \in \Lambda_{\perp}} G_{\vec{x}} D_{\vec{x}, \vec{x}'}^{(m^2)} G_{\vec{x}'} \right], \quad (\text{D.230})$$

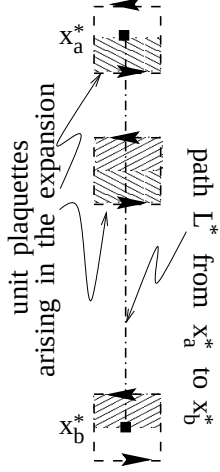
with $\sum_{k=1}^2 \Delta_k L_{k, \vec{x} - \hat{k}}^* = (\delta_{\vec{x}, \vec{x}_a} - \delta_{\vec{x}, \vec{x}_b})$; $G_{\vec{x}}$ is the topological (integer) field carrier of the compact character of the gauge field, and $V_{k, \vec{x}}$ is the vorticity content due to the angular nature of scalar phase $\theta_{\vec{x}}$ (more exactly they are, respectively, the projection of $G_{\vec{r}}$ and $V_{\lambda, \vec{r}}$ onto sublattice $\Lambda_{\perp}$ normal to the periodic direction). By comparing equation (D.230) with result (D.222) one concludes that in the scalar model in presence of an external field, model equivalent to the gauge model under study, integer field $L_{k, \vec{x}}^*$ enters in the partition function through change $m_{k, \vec{x}} \rightarrow m_{k, \vec{x}} - \frac{1}{q} \epsilon_{kj} L_{j, \vec{x} - \hat{j}}^*$, or equivalently, $H(\vec{x}_a^*, \vec{x}_b^*) =$

$$= \frac{\int_{-\pi}^{\pi} \prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \exp \left[-\frac{1}{e^2 \omega \eta} \sum_{\vec{x}, k} \left[1 - \cos \left(\Delta_k \theta_{\vec{x}} + \frac{2\pi}{q} \sum_j \epsilon_{kj} L_{j, \vec{x} - \hat{j}}^* \right) \right] - \frac{a^2 |\psi|^2}{\omega \eta} \sum_{\vec{x}} \cos(q \theta_{\vec{x}}) \right]}{\int_{-\pi}^{\pi} \prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \exp \left[-\frac{1}{e^2 \omega \eta} \sum_{\vec{x}, k} [1 - \cos(\Delta_k \theta_{\vec{x}})] - \frac{a^2 |\psi|^2}{\omega \eta} \sum_{\vec{x}} \cos(q \theta_{\vec{x}}) \right]},$$

which is relation (D.229) for a scalar coupling strength $|\psi|_{XY}^2 \equiv \frac{1}{e^2 \omega}$ and a magnetic field $h \equiv \frac{a^2 |\psi|^2}{\omega}$.

Now, following the same strategy as in the discussion previous and after equation (D.226), the behaviour of $H(\vec{x}_a^*, \vec{x}_b^*) = \frac{\mathcal{Z}_{2\text{-dim. } XY, \text{ with } \cos(\Delta_k \theta_{\vec{x}}) \rightarrow \cos(\Delta_k \theta_{\vec{x}} + \frac{2\pi}{q} \sum_j \epsilon_{kj} L_{j, \vec{x} - \hat{j}}^*)}}{\mathcal{Z}_{2\text{-dim. } XY, |\psi|_{XY}^2 = 1/e^2 \omega}}$

factor a/ω on the coupling strengths arises from the relation between propagators $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$. By renaming $\theta_{\vec{x}} \equiv B_{\tau, \vec{x}}$, the above result is the partition function of the $(d-1)$ -dimensional non-periodic XY model under the action of an external magnetic field $h \equiv \frac{a^{d-1} |\psi|^2}{\omega}$, being $a^{(d-1)-2} |\psi|_{XY}^2 \equiv \frac{a^{d-3}}{e^2 \omega}$ the coupling strength in such scalar model: this is the same result as in equation (D.225).



in the range of small $|\psi|^2$ and large e^2 is easy to obtain. Large e^2 justifies the numerator's expansion in powers of bonds $\left(\frac{1}{2e^2\omega\eta}\right) e^{\pm i(\theta_{\vec{x}+\hat{k}} - \theta_{\vec{x}} + 2\pi/q \sum_j \epsilon_{kj} L_{j,\vec{x}-\hat{j}}^*)}$ forming closed 1×1 plaquettes: as it is clear from the figure, the only form that the contribution of such a plaquette in the numerator differs from that in the denominator is that the “tile” surrounds one of the two points $\vec{x}_a^*$ or $\vec{x}_b^*$; in a “tile” crossed twice by path L^* , the effect of $\pm \frac{2\pi}{q} \sum_j \epsilon_{kj} L_{j,\vec{x}-\hat{j}}^*$ on each of the two crossed plaquette's sides gets cancelled due to the opposite directions of the sides:

$$H(\vec{x}_a^*, \vec{x}_b^*) \approx \frac{1 + (|\Lambda_\perp| - 2) \cdot \left(\frac{1}{2e^2\omega\eta}\right)^4 + 1 \cdot \left(\frac{1}{2e^2\omega\eta}\right)^4 e^{i2\pi/q} + 1 \cdot \left(\frac{1}{2e^2\omega\eta}\right)^4 e^{-i2\pi/q} + \dots}{1 + |\Lambda_\perp| \cdot \left(\frac{1}{2e^2\omega\eta}\right)^4 + \dots}$$

$$H(\vec{x}_a^*, \vec{x}_b^*)_{\text{large } e^2, \text{small } |\psi|^2} \approx 1 - 2 \left(\frac{1}{2e^2\omega\eta}\right)^4 \left(1 - \cos \frac{2\pi}{q}\right) \sim \text{constant}. \quad (\text{D.231})$$

The expansion of $\mathcal{Z}$ in site-powers $\left(\frac{a^2|\psi|^2}{2\omega\eta}\right) e^{\pm i(q\theta_{\vec{x}})}$ alone does not produce a non-vanishing contribution, since one needs again q chains of bond-powers to cancel the $e^{\pm i(q\theta_{\vec{x}})}$'s produced in the expansion in site-powers. For e^2 smaller (large coupling strength $|\psi|_{XY}^2$ in the equivalent scalar model), $\mathcal{Z}_{XY}$ under $h \cos(q\theta_{\vec{x}})$, with $M_{\vec{x}} \rightarrow (M_{\vec{x}} + \frac{1}{q}(\delta_{\vec{x},\vec{x}_a} - \delta_{\vec{x},\vec{x}_b}))$ can be approximated by $\prod_{\vec{x}} \sum_{M_{\vec{x}}=-\infty}^{\infty} \exp \left[-\frac{2\pi^2|\psi|_{XY}^2}{\eta} \sum_{\vec{x},\vec{x}' \in \Lambda_\perp} \left(M_{\vec{x}} + \frac{(\delta_{\vec{x},\vec{x}_a} - \delta_{\vec{x},\vec{x}_b})}{q} \right) D_{\vec{x},\vec{x}'}^{(m^2)} \right]_{\vec{x}'}$ with all $M_{\vec{x}}$'s suppressed by term $\frac{2\pi^2|\psi|_{XY}^2}{\eta} \sum_{\vec{x},\vec{x}'} M_{\vec{x}} D_{\vec{x},\vec{x}'}^{(m^2)} M_{\vec{x}'}$. Hence

$$H(\vec{x}_a^*, \vec{x}_b^*)_{\text{smaller } e^2, \text{small } |\psi|^2} \approx \exp \left[-\frac{2\pi^2}{e^2\omega\eta} \frac{(\delta_{\vec{x},\vec{x}_a} - \delta_{\vec{x},\vec{x}_b})}{q} D_{\vec{x},\vec{x}'}^{(m^2)} \frac{(\delta_{\vec{x}',\vec{x}_a} - \delta_{\vec{x}',\vec{x}_b})}{q} \right], \quad (\text{D.232})$$

which is a constant too due to the short-range propagator. Hence, $H(\vec{x}_a^*, \vec{x}_b^*)$ in the range of small $|\psi|^2$ and for small q does not change its behaviour by decreasing e^2 from a large value to a smaller one; if q becomes too large, the effect of $h \cos(q\theta_{\vec{x}})$ becomes negligible and instead of equation (D.232) one obtains $H(\vec{x}_a^*, \vec{x}_b^*)_{\text{smaller } e^2, \text{small } |\psi|^2} \approx \exp \left[-\frac{2\pi^2}{q^2 e^2 \omega \eta} (\delta_{\vec{x},\vec{x}_a} - \delta_{\vec{x},\vec{x}_b}) D_{\vec{x},\vec{x}'}^{(d-1)} (\delta_{\vec{x}',\vec{x}_a} - \delta_{\vec{x}',\vec{x}_b}) \right]$, with $D_{\vec{x},\vec{x}'}^{(d-1)}$ the long-range propagator. Returning to the case of a small $q > 1$, the phase transition which occurs in the range under study is only revealed by the change in correlation function $W(C)$.

d=3, $q = 2$, ω small

$$|\psi|^2 = 0 \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}}{W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$$\text{small } |\psi|^2 \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}}{W(C) \sim \text{const.} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$$|\psi|^2 = \infty \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}}{W(C) \sim \text{const.} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$e^2 = 0 \quad \quad \quad e^2 = \infty$

by $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \\ W(C) \sim \text{const.} \end{cases}$ to the left corner at very small e^2 . This intermediate state is connected with the phase at small e^2 in the region of small $|\psi|^2$.

d=4, $q = 2$, ω small

$$|\psi|^2 = 0 \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}}{W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$$\text{small } |\psi|^2 \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}}{W(C) \sim \text{const.} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$$|\psi|^2 = \infty \quad \frac{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{cst.} \quad H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{cst.}}{W(C) \sim \text{cst.} \quad W(C) \sim \text{cst.} \quad W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}}$$

$e^2 = 0 \quad \quad \quad e^2 = \infty$

for e^2 small ⁶¹. As in the 3-dimensional model, the behaviour of correlation function

⁶¹Although the proof of such a statement will not be carried out here, one can, nevertheless, gain some insight by studying the range of small e^2 and large $|\psi|^2$: the Hamiltonian of the Higgs model (D.119), $\hat{\mathcal{H}} = -\frac{e^2\eta^2}{2a} \sum_{\vec{x} \in \Lambda_\perp} \sum_{k=1}^3 \frac{\hat{\partial}^2}{\partial A_{k,\vec{x}}^2} + \frac{1}{ae^2} \sum_{\vec{x}} \sum_{j < k=1}^3 \left[1 - \cos \left(\hat{A}_{j,\vec{x}} + \hat{A}_{k,\vec{x}+\hat{j}} - \hat{A}_{j,\vec{x}+\hat{k}} - \hat{A}_{k,\vec{x}} \right) \right] - \frac{\eta^2}{2a^3|\psi|^2} \sum_{\vec{x}} \frac{\hat{\partial}^2}{\partial \varphi_{\vec{x}}^2} + a|\psi|^2 \sum_{\vec{x}} \sum_{k=1}^3 \left[1 - \cos \left(\hat{\varphi}_{\vec{x}+\hat{k}} - \hat{\varphi}_{\vec{x}} - q\hat{A}_{k,\vec{x}} \right) \right]$, for small e^2 and large $|\psi|^2$ produces a partition function equal to that of the compact Higgs model but in a non-periodic 3-dimensional lattice, being the gauge and scalar coupling strengths in this last model equal to $\left(\frac{1}{ae^2} \right)_{\text{3-dim. non-per. Higgs}} \equiv \frac{\omega}{ae^2}$

and $(a|\psi|^2)_{\text{3-dim. non-per. Higgs}} \equiv a|\psi|^2\omega$ respectively. The behaviour of $H(\vec{x}_a^*, \vec{x}_b^*)$ for small e^2 and at $|\psi|^2 = \infty$ is an exponential decay like $|\vec{x}_b^* - \vec{x}_a^*|$; if we now look back at the phase diagram of the 3-dimensional non-periodic Higgs model for $q = 2$ (see Fig. C.6 on page 449), the strong decay in $H(\vec{x}_a^*, \vec{x}_b^*)$, which is equivalent to the weak decay of $W(C)$, gets lost by decreasing either the scalar or the gauge coupling strength of the non-periodic model, i.e., by the decreasing $|\psi|^2\omega$ or ω/e^2 in the 4-dimensional periodic model at small e^2 . Hence eventually a phase transition occurs to a state characterized by the strongest decay in $W(C)$, or equivalently, by the weakest form in $H(\vec{x}_a^*, \vec{x}_b^*)$:

For $q = 2$, the different phases displayed by the compact Higgs model in a 3-dimensional lattice of small periodicity ω along one direction are schematically represented in the diagram left. For q large enough, the pure $Z(q)$ gauge model (limit $|\psi|^2 = \infty$) shows the intermediate phase characterized by $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|} \\ W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} \end{cases}$, relegating the phase characterized

All the above discussion applies also to the 4-dimensional Higgs model, whose phase diagram at small ω for $q = 2$ is represented schematically in the left figure. As for the 3-dimensional model in the range of small $|\psi|^2$, $H(\vec{x}_a^*, \vec{x}_b^*)$ remains constant at large e^2 and behaves like $H(\vec{x}_a^*, \vec{x}_b^*) \sim \begin{cases} \text{const.} & q = 2 \\ e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} & \text{large } q \end{cases}$

$H(\vec{x}_a^*, \vec{x}_b^*)$ allows us to distinguish between the two phases at not large e^2 which show $W(C) \sim \text{constant}$. For large q , the small region at the left lower corner (small e^2 and large $|\psi|^2$) becomes gradually smaller; further, the intermediate phase at $|\psi|^2 = \infty$, as well as the phase for small $|\psi|^2$ and e^2 , gets characterized by $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \\ W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \end{cases}$.

It is to be noted that even for $q = 2$ but for a not so small ω , according to the remark on page 618, in the range of both e^2 and $|\psi|^2$ small (the circle in the above scheme for $d = 4$) a small “pocket” of phase $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \\ W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \end{cases}$, or of phase $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|} \\ W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} \end{cases}$ for $d = 3$, can exist connecting with the same phase occurring at $|\psi|^2 = 0$ and small e^2 .

Before continuing with the complete phase diagram of the Higgs model at any ω , let us discuss briefly the already met equivalence between behaviour $W(C) \sim e^{-\text{Perimeter}_C}$ for a non-periodic lattice and $W(C) \sim \text{const.}$ for a lattice of periodicity ω small.

Note. Equivalence between $W(C) \sim e^{-\text{Perimeter}_C}$ at $\omega = \infty$ and $W(C) \sim \text{const.}$ at small ω , and between $W(C) \sim e^{-\text{Area}_C}$ at $\omega = \infty$ and $W(C) \sim e^{-|\vec{x}_2 - \vec{x}_1|}$ at small ω .

For the compact Higgs model in a non-periodic lattice ($\omega = \infty$), behaviour $W(C) \sim e^{-\mathcal{P}_C}$ occurs in the following cases:

- In the range of small e^2 and large $|\psi|^2$ such that $m^2 = a^2 q^2 e^2 |\psi|^2$ is still large: due to the small e^2 and to the large $|\psi|^2$ the density of topological fields belonging to the compact character of the gauge and the phase fields is strongly suppressed, and due to the large m^2 propagator $D_{\vec{\tau}, \vec{\tau}'}^{(m^2)}$ can be approximated as $D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} \approx \frac{1}{m^2} \delta_{\vec{\tau}, \vec{\tau}'}$. Thus $W(C) = \left\langle e^{i \sum_{(\vec{\tau}, \mu) \in C} J_{\mu, \vec{\tau}} A_{\mu, \vec{\tau}}} \right\rangle$ in

$$(C.414) \text{ is approximately equal to } W(C) \approx \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} J_{\mu, \vec{\tau}} D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} J_{\mu, \vec{\tau}'} \right] D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} \approx \frac{1}{m^2} \delta_{\vec{\tau}, \vec{\tau}'}$$

$$\approx \exp \left[-\frac{\eta}{2q^2 a |\psi|^2} \mathcal{P}_C \right]. \text{ In a periodic lattice of small } \omega, \text{ where } C \text{ covers the complete periodic length, then } \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} J_{\mu, \vec{\tau}} D_{\vec{\tau}, \vec{\tau}'}^{(m^2)} J_{\mu, \vec{\tau}'} \approx \frac{\omega^2}{a^2} \frac{a}{\omega} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) D_{\vec{x}, \vec{x}'}^{(m^2)} (\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2}) \text{ and at large } |\psi|^2 \text{ and small } e^2, \text{ with } e^2 |\psi|^2 \text{ large, one obtains } W(C) \approx \exp \left(-\frac{\omega\eta}{q^2 a^2 |\psi|^2} \right) = \text{constant.}$$

- Although not belonging exactly to the cases under study, let us consider the 4-dimensional pure $U(1)$ gauge model at small e^2 , where the topological defects of the gauge field play no role and

$$\text{thus } W(C) \approx \exp \left[-\frac{e^2\eta}{2} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} J_{\mu, \vec{\tau}} D_{\vec{\tau}, \vec{\tau}'}^{(d=4)} J_{\mu, \vec{\tau}'} \right], \text{ which leads to } W(C) \sim e^{-\mathcal{P}_C + \tau/|\vec{x}_2 - \vec{x}_1|} \text{ with } \tau \text{ the length of path } C \text{ along the will-become-periodic direction. Therefore, for small } \omega \text{ } W(C) \text{ is equal to } \exp \left[-\frac{e^2\omega\eta}{2a} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) \frac{1}{4\pi|\vec{x} - \vec{x}'|} (\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2}) \right] \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \text{ at small } e^2$$

$H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{constant}$. And this transition is just to the phase of small $|\psi|^2$ and larger e^2 in the periodic Higgs model, proving thus the above stated behaviour for the correlation function under study.

which, although not constant, does not decay for large distances $|\vec{x}_2 - \vec{x}_1|$. This conclusion, nevertheless, does not apply to the 3-dimensional pure $U(1)$ gauge model, where the magnetic monopoles (the topological defects of the gauge field) are always present: see Polyakov's result (C.316) and the Note which follows to that equation, and therefore $W(C)$ can never be

approximated by $\exp\left[-\frac{ae^2\eta}{2}\sum_{\vec{r},\vec{r}'}\sum_{\mu}J_{\mu,\vec{r}}D_{\vec{r},\vec{r}'}^{(d=3)}J_{\mu,\vec{r}'}\right]$ ⁶².

- A more interesting case is the compact Higgs model at $q = 1$ at large e^2 and small $|\psi|^2$, where the topological fields and the compact character of gauge field $A_{\mu,\vec{r}}$ are relevant. For a lattice with $\omega = \infty$, the behaviour of $W(C)$ was calculated by the “high temperature” expansion: in order to obtain a nonvanishing contribution when all $\prod_{\vec{r},\mu}\int_{-\pi}^{\pi}\frac{dA_{\mu,\vec{r}}}{2\pi}$ are integrated, each bond-factor $e^{iA_{\mu,\vec{r}}}$ contained in $W(C)$ has to be covered by the corresponding power $e^{-iA_{\mu,\vec{r}}}$, power which arises from the expansion of terms $e^{-\frac{1}{ae^2\eta}\sum_{\vec{r},\mu}<\nu}[1-\cos(\Delta_{\mu}A_{\nu,\vec{r}}-\Delta_{\nu}A_{\mu,\vec{r}})]$ and $e^{-\frac{a|\psi|^2}{\eta}\sum_{\vec{r},\mu}[1-\cos(\Delta_{\mu}\varphi_{\vec{r}}-qA_{\mu,\vec{r}})]}$ $\Big|_{q=1}$ in plaquette-factors $e^{i(\Delta_{\mu}A_{\nu,\vec{r}}-\Delta_{\nu}A_{\mu,\vec{r}})}$ and in bond-factors $e^{i(\Delta_{\mu}\varphi_{\vec{r}}-qA_{\mu,\vec{r}})}\Big|_{q=1}$ respectively. Tiling the interior of path C with plaquette factors produces, in leading e^2 order, contribution $\left(\frac{1}{2ae^2\eta}\right)^{\text{Area}_C}$. For a large loop C , this last contribution is much smaller than that produced by surrounding C with bond-factors, which gives $\left(\frac{a|\psi|^2}{2\eta}\right)^{\mathcal{P}_C}$, something which only holds as long as $q = 1$, since otherwise bond-factors $e^{-iqA_{\mu,\vec{r}}}$ produced in the expansion of $e^{i(\Delta_{\mu}\varphi_{\vec{r}}-qA_{\mu,\vec{r}})}$ would not be able to cancel the $e^{iA_{\mu,\vec{r}}}$'s contained in $W(C)$; in the $q \neq 1$ case, the leading contribution is therefore equal to $\left(\frac{1}{2ae^2\eta}\right)^{\mathcal{A}_C}$.

If now the d -dim. lattice becomes periodic along one direction, $W(C)$ in the range of large e^2 (and small $|\psi|^2$) is equivalent, according to the Susskind's result (D.225), to the behaviour of the scalar-scalar correlation function of the $(d-1)$ -dim. XY model at small $|\psi|_{XY}^2$ but in the presence of an external magnetic field term $h\sum_{\vec{x}}\cos(q\theta_{\vec{x}})\Big|_{q=1}$, model which shows *always* the ordered state: the scalar-scalar correlation function tends to a nonvanishing constant at large distances, and therefore $W(C) \sim \text{const.}$. On the other hand, and as a by-product of this discussion, the behaviour which in an infinite non-periodic lattice is characteristic when the compact nature of the gauge field is relevant (i.e., at large e^2), $W(C) = \exp[-\mathcal{A}_C \ln(2ae^2\eta)]$, corresponds in the case of lattice of periodicity ω to $W(C) \sim e^{-|\vec{x}_2-\vec{x}_1|}$, since the area decay in $W(C)_{\omega=\infty}$ occurs for $q > 1$ and in such a case correlation function $W(C)_{\text{small } \omega} = \left\langle e^{i(\theta_{\vec{x}_2}-\theta_{\vec{x}_1})} \right\rangle_{(d-1)\text{-dim } XY \text{ under } h\cos(q\theta_{\vec{x}})}$ shows exponential decay $e^{-|\vec{x}_2-\vec{x}_1|}$ since the magnetic field term still allows for a symmetry restoring at $q \neq 1$.

Finally and summarizing the two diagrams above on page 629 for the 3-dimensional Higgs model and 629 for the 4-dimensional model, as well as the phase diagrams for d -dimensional pure $Z(q)$ gauge models (on pages 603 and 609 for $d = 3$, and on pages 598 and 606 for $d = 4$), we shall study the complete phase diagram for the d -dimensional compact $U(1)$ Higgs model in dependence on the lattice's periodicity ω along one direction (or depending on the temperature by assuming a relation $\omega \sim 1/T$).

⁶²One should be a bit careful with the 3-dimensional pure $U(1)$ gauge model, since $W(C) \sim e^{-\mathcal{A}_C}$ at any e^2 for the non-periodic lattice. But at small ω , only in the range of large e^2 (range which decreases with decreasing ω) behaviour $W(C) \sim e^{-|\vec{x}_2-\vec{x}_1|}$ occurs.

- The phase diagram for $d = 3$ and $q = 2$ is depicted in Fig. D.2 on page 747, where two sections of the diagram at $\omega = \infty$ and at small ω have also been drawn. The transition point for the XY and the pure $U(1)$ gauge model, points A' and E' of Fig. D.2 respectively, are of Kosterlitz-Thouless type for small ω ; the lines which start at these points are nevertheless lines of second-order phase transitions, as we shall now see. As already seen, in the range of large e^2 and small $|\psi|^2$ $\mathcal{Z}_{d-\text{dim.}, \text{per. comp. Higgs}} = \mathcal{Z}_{(d-1)-\text{dim. } XY \text{ under } h \cos(q\theta_{\vec{x}})}$ is satisfied, and the XY model in a non-periodic lattice and under an external magnetic field's term $h \cos(q\theta_{\vec{x}})$ still maintains, for its part, an unbroken $Z(q)$ symmetry if $q > 1$: thus for $|\psi|^2 \neq 0$ and near to point E' , the phase transition when e^2 is changed is of the same kind as the transition of the $Z(q)$ Ising model, which for the case of $q = 2$ is of second-order kind. In order to discuss whether the transition near to point A' (when e^2 is “turned on”) keeps the same character as at A' , we need another equivalence relation similar to that obtained in (D.225), but without integrating one of the continuous fields: returning to the Villain formulation of the non-periodic 3-dim. Higgs model (C.380), its partition function is equal

$$\text{to the exponential of } -\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}}) - \\ - \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}}), \text{ with sum over } g_{\mu\nu, \vec{r}}, m_{\mu, \vec{r}} \text{ and}$$

integrals over $A_{\mu, \vec{r}}, \varphi_{\vec{r}}, B_{\mu\nu, \vec{r}}, b_{\mu, \vec{r}}$. By using Poisson formula, $\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} db e^{i2\pi mb} f(b) = \sum_{l=-\infty}^{\infty} f(l)$, one simplifies $\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} db_{\mu, \vec{r}} \prod_{\vec{r}, \mu} \sum_{m_{\mu, \vec{r}}=-\infty}^{\infty}$ by $\prod_{\vec{r}, \mu} \sum_{l_{\mu, \vec{r}}=-\infty}^{\infty}$, producing thus

$$\text{in the partition function } -\frac{ae^2\eta}{2} \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}}^2 + i \sum_{\vec{r}, \mu < \nu} B_{\mu\nu, \vec{r}} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi g_{\mu\nu, \vec{r}}) - \\ - \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}}). \text{ Integration of } \prod_{\vec{r}} d\varphi_{\vec{r}} \text{ produces constraint } \\ \sum_{\mu} \Delta_\mu l_{\mu, \vec{r}-\hat{\mu}} = 0, \text{ whose solution is } l_{\mu, \vec{r}} = \sum_{\nu} \Delta_\nu l_{\mu\nu, \vec{r}-\hat{\nu}} \text{ with } l_{\mu\nu, \vec{r}} \text{ an antisymmetric} \\ \text{integer field. Integration of } \prod_{\vec{r}, \mu} dA_{\mu, \vec{r}} \text{ results in } B_{\mu\nu, \vec{r}} = \sum_{\lambda} \epsilon_{\mu\nu\lambda} \Delta_\lambda C_{\vec{r}-\hat{\lambda}} + q L_{\mu\nu, \vec{r}}, \text{ and} \\ \text{thus integration of } \prod_{\vec{r}} dC_{\vec{r}} \text{ gives }^{63} \text{ finally}$$

$$\sum_{\{l_{\mu, \vec{r}}\}} \sum_{\{G_{\vec{r}}\}} \prod_{\vec{r}} \delta\left(\sum_{\mu} \Delta_\mu l_{\mu, \vec{r}-\hat{\mu}}, 0\right) \exp\left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 - \frac{aq^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} l_{\mu, \vec{r}'}\right] \times \\ \times \exp\left[+i2\pi q \sum_{\vec{r}, \vec{r}'} \sum_{\nu} l_{\nu, \vec{r}} \Omega_{\nu\vec{r}, \vec{r}'} G_{\vec{r}'} - \frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \vec{r}'} G_{\vec{r}} D_{\vec{r}, \vec{r}'} G_{\vec{r}'}\right], \quad (\text{D.233})$$

⁶³By considering also result $\sum_{\lambda, \lambda'} \sum_{\mu < \nu, \mu' < \nu'} \epsilon_{\lambda\mu\nu} (\Delta_{\vec{r}} \lambda) l_{\mu\nu, \vec{r}} D_{\vec{r}, \vec{r}'} \epsilon_{\lambda'\mu'\nu'} (\Delta_{\vec{r}'} \lambda') l_{\mu'\nu', \vec{r}'} = \sum_{\vec{r}} \sum_{\mu < \nu} L_{\mu\nu, \vec{r}}^2 - \\ - \sum_{\vec{r}} \sum_{\mu} \left(\sum_{\nu} \Delta_\nu l_{\mu\nu, \vec{r}-\hat{\nu}}\right) D_{\vec{r}, \vec{r}'} \left(\sum_{\nu'} \Delta_{\nu'} l_{\mu\nu', \vec{r}'-\hat{\nu}'}\right).$

with $G_{\vec{r}} = \sum_{\lambda=1}^{d=3} \sum_{\mu < \nu} \epsilon_{\lambda\mu\nu} \Delta_{\lambda} g_{\mu\nu, \vec{r}}$. At $e^2 = 0$ (the XY model) $G_{\vec{r}}$ is frozen to 0 due to the last term of the above equation, remaining only the short-range interaction of term $\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$; but when $e^2 \neq 0$, long range interactions carried by propagator $D_{\vec{r}, \vec{r}'}$ appear, and since the long-distance interactions are those which determine the character of the phase interaction, that displayed at $e^2 = 0$ should change when the gauge field is turned on. On the contrary, passing from the transition point at $|\psi|^2 = \infty$ to a near point along the separating line (line BA at $\omega = \infty$ or line $B'D'$ at small ω) with a large but finite $|\psi|^2$, does not change the type of long-range interactions since term $\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$ only adds a contribution to the very short range. Therefore the character of the transition at the pure $Z(q)$ gauge model is maintained when $|\psi|^2$ becomes finite. By carrying out the corresponding substitutions in propagator $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ and in the integer fields when the lattice becomes periodic with ω small, all the above general results based on the long-range propagator works also at small ω . Thus, when point B' is abandoned, where for $q = 2$ its transition character is second-order, and since the transition type at A' must change when one leaves point A' , line $B'E'$ (except for point E') as well as line $B'A'$ (except for point A') are lines of second-order phase transitions. To complete the description of Fig. D.2, the two transition points at $\omega = \infty$ are those of the (non-periodic) 3-dim. XY model and of the 3-dim. $Z(2)$ gauge model respectively, as in Fig. C.6 (page 449). As already mentioned, a small “pocket” of phase $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|} \\ W(C) \sim e^{-\ln|\vec{x}_2 - \vec{x}_1|} \end{cases}$ cannot be excluded for a ω large enough.

- For $d = 3$ and large q , whose phase diagram is represented in Fig. D.3 on page 748, there are some differences when compared with the case of $q = 2$. Transition point B'_2 , the only point of the pure $Z(q)$ gauge model which survives in the limit $q \rightarrow \infty$, is now Kosterlitz-Thouless since this is the character of the 2-dimensional XY model in a non-periodic lattice; since point E' is also of Kosterlitz-Thouless type, and the above argument has shown that the transition points at $|\psi|^2 = \infty$ maintains their character for large but finite $|\psi|^2$, then the whole line connecting B'_2 and E' is formed by Kosterlitz-Thouless transition points. This is further supported by the fact that at large e^2 and small $|\psi|^2$, equivalence $\mathcal{Z}_{d-\text{dim.}, \text{per. comp. Higgs}} = \mathcal{Z}_{(d-1)-\text{dim. } XY \text{ under } h \cos(q\theta_{\vec{x}})}$ leads to the result that in the considered range the phase transition is of the same kind as that in the (non-periodic) $Z(q)$ Ising model, model which for large q is approximately equal to the XY model; this becomes even clearer since a large q almost decouples the effect of the rapid oscillating magnetic-field term $h \sum_{\vec{x} \in \Lambda_1} \cos(q\theta_{\vec{x}})$. Since the phase transition of the XY model in the lattice's dimensionality under study is of Kosterlitz-Thouless type, the same kind as the pure 3-dim. $U(1)$ gauge model at small ω , the character of the phase transition across the line $E'B'_2$ for points near E' is the same as at E' . For points near to B'_2 , we have already seen after equation (D.233) that the transition

maintains the same character as at B'_2 , where it is Kosterlitz-Thouless. Therefore, whole line $E'B'_2$ is formed by transition points of Kosterlitz-Thouless kind. On the other hand, transition point B_2 at $\omega = \infty$ is expected to be second-order ⁶⁴, and thus line B_2A are formed by points of this character ⁶⁵. Point B'_1 tends toward $e^2 = 0$ with increasing q ; the same occurs for both transition points B_1 and B_2 at $\omega = \infty$ do since the 3-dimensional non-periodic pure $U(1)$ gauge model displays only one phase.

The case of $q = 1$ is, on the other side, quickly discussed: except for the limiting cases of $|\psi|^2 = 0$ or $e^2 = 0$, where the phase transitions of the pure models discussed in the previous sections take place, the rest of the diagram is occupied by one unique phase characterized by $W(C) \sim \text{const.}$ at small ω , or by $W(C) \sim e^{-\text{Perimeter}_C}$ at $\omega = \infty$ (according to the correspondence between these two behaviours discussed in the previous Note).

- In the phase diagram for 4-dimensional Higgs model in a lattice periodic along one direction and for $q = 2$, represented in Fig. D.4 on page 749, the main difference if compared to the 3-dimensional model is the existence in a periodic lattice of three phases at $|\psi|^2 = \infty$, even at $q = 2$. This is due to the self-duality of the model. In the range of large e^2 one phase transition occurs when ω is decreased (point B'_2), since the model in such a range is equivalent to the $Z(2)$ Ising model in a 3-dimensional non-periodic lattice and with a scalar coupling strength inversely proportional to $e^2\omega$. By the self-duality property which relates the range of large e^2 to the range of e^2 small, an analogous phase transition when ω decreases occurs also for small e^2 (point B'_1), since now the partition function of the gauge model is equivalent to that of the 3-dimensional non-periodic $Z(2)$ Ising model, being the coupling strength of the last model proportional this time to e^2/ω (see equation (D.212)). This self-duality is expressed mathematically by the following relation

$$\frac{1}{(e^2)_{B'_1}\omega\eta} \equiv \frac{\beta_{3\text{-dim. non-per. Ising}}}{\eta} \equiv \frac{(e^2)_{B'_2}\eta}{\pi^2\omega} \Rightarrow \frac{1}{(e^2)_{B'_1}\eta} = \frac{1}{\pi^2}(e^2)_{B'_2}\eta, \quad (\text{D.234})$$

result which holds independently of the lattice's periodicity ω and which is case $q = 2$ of equation (D.218).

Both transitions at B'_1 and B'_2 are second-order, since this is the character of the phase transition in the 3-dimensional $Z(2)$ scalar model. From the stability

⁶⁴Point B_2 was denoted by Q_2 in the phase diagram on page 609, page on which it was argued that it should be a critical point of second-order type.

⁶⁵The effect of “turning on” e^2 when point A is left, is not so relevant at large q as it was for $q = 2$. The reason is that term $\frac{q^2 a e^2 \eta}{2} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} l_{\mu, \vec{\tau}} D_{\vec{\tau}, \vec{\tau}'}^{(d)} l_{\mu, \vec{\tau}'}$, or the corresponding substitution with

$D_{\vec{\tau}, \vec{\tau}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ at small ω , forbids field $l_{\mu, \vec{\tau}}$ to form large rings or dipoles for large q , and thus the interaction among the $l_{\mu, \vec{\tau}}$'s still remains confined to short distances. One may still expect the phase transition to maintain its character near A' as that at point A' , since the long-range behaviour does not suffer such a strong change as for $q = 2$. Nevertheless, a very short segment on $A'B'_2$ cannot be excluded, segment where the character of the phase transition may change before becoming again Kosterlitz-Thouless and continuing in this way until point B'_2 is reached.

argument presented for the 3-dimensional gauge model, and which applies to the 4-dimensional case too, the transition kind at B'_1 and B'_2 is maintained for large $|\psi|^2 < \infty$. Also E' is a second-order transition point (this is the character of the transition in the 3-dim. non-periodic XY model); further, the phase transition of the 3-dimensional XY under a external magnetic term $h \sum_{\vec{x} \in \Lambda_\perp} \cos(q\theta_{\vec{x}}) \Big|_{q=2}$, model

equivalent to the Higgs model for large e^2 and small $|\psi|^2$, is the same as the $Z(2)$ Ising model: the transition across line $E'B'_2$ near to E' is therefore second-order. And thus, the whole line B'_2E' is formed by second-order transition points.

On the other hand, the transition at point A' , which is given by equation (D.221) $\left(\frac{a^2 |\psi|_{4\text{-dim. per. } XY}^2}{\eta} \right)_{\text{crit}} \approx 0.526 \frac{a^2}{\omega^2}$, is of second-order type; but it is not stable under the “turning on” of $e^2 \neq 0$, becoming first-order. Since near to B'_1 the transition does keep the second-order character for $|\psi|^2 < \infty$, then there must exist a tricritical transition point near to A' where, by approaching B'_1 , the phase transition passes from first-order to second-order.

As in the 3-dimensional model, a small “pocket” of phase ④ characterized by $\begin{cases} H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \\ W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \end{cases}$ may well exist for a large enough ω , pocket connecting not only with the same phase at $|\psi|^2 = 0$ and small ω but also, and especially, with this phase which already occurs for a wide range of small e^2 and $|\psi|^2$ at $\omega = \infty$.

- For large q the form of the phase diagram of the 4-dimensional Higgs model (Fig. D.5 on page 750) is rather similar to the previous case $q = 2$. Again the self-duality of the 4-dimensional pure $Z(q)$ gauge model (both at $\omega = \infty$ and at small ω) leads to the existence of two transition points B'_1 and B'_2 related to each other by equation (D.218), $\frac{1}{(e^2)_{B'_1} \eta} = \frac{q^2}{4\pi^2} (e^2)_{B'_2} \eta$. Both transition points are, as for $q = 2$, of second-order type and the discussion of the transition’s character across lines B'_2E' and B'_1A' is the same as for $q = 2$. Note that the self-duality relation causes $\lim_{q \rightarrow \infty} (e^2)_{B'_1} = 0$ since the other transition point of the pure $Z(q)$ gauge model, B'_2 , tends toward the only transition point of the pure $U(1)$ gauge model, being this last model limit $q \rightarrow \infty$ of the $Z(q)$ gauge model. A slight difference occurs on line B'_1A' , since the instability for points on such a line when one leaves point A' , i.e., when one turns on the coupling to the gauge field, is less severe for large q than in the case $q = 2$ due to term $\frac{q^2 a e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'}$, just as discussed for the 3-dimensional model. Therefore, the segment near to A' where the transition points are first-order is expected to be very short. The transition points at $\omega = \infty$ have been already discussed in Fig. C.10.

The main difference with the phase diagram of the 4-dimensional model at $q = 2$ lies in the phase limited by transition points B'_1 and B'_2 : for $q = 2$ this phase was always characterized by a constant value for any distance in both correlation functions $W(C)$ and $H(\vec{x}_a^*, \vec{x}_b^*)$. And such a phase extended over the whole central

region of the phase diagram (safe limit $|\psi|^2 = 0$ in the range of small e^2). For large q , nevertheless, the phase enclosed at $|\psi|^2 = \infty$ between B'_1 and B'_2 shows a slight stronger decay in both correlation functions, behaviour which extends again over the greatest part of the diagram. Only at very small ω there exists a “pocket” of $W(C)$, $H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.}$ whose extent decreases with increasing q .

The case of $q = 1$ is, as in the 3-dim. Higgs model, much more simplified. Only at $|\psi|^2 = 0$ or $e^2 = 0$ there appear two phases; the rest of the diagram is occupied by one unique phase characterized by $W(C) \sim \text{const.}$ at small ω , or by $W(C) \sim e^{-\text{Perimeter}_C}$ at $\omega = \infty$. The only difference between the 3-dim. and the 4-dim. Higgs model at $q = 1$ consists of the pure $U(1)$, which at $d = 4$ has already two phases at $\omega = \infty$ whereas the 3-dim. one shows only one (compare Fig. C.6, page 449, and Fig. C.10 on page 481).

D.4 The phase diagram of the non-compact $U(1)$ Higgs model at high temperatures (I)

The non-compact $U(1)$ Higgs model is defined for a gauge field which has no angular character. Instead of term $1 - \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})$ with $A_{\mu, \vec{r}} \in [0, 2\pi)$, the model contains as energy's part for the gauge field the following term: $\frac{1}{2}(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})^2$, where the range of the gauge field is not compact, $A_{\mu, \vec{r}} \in (-\infty, \infty)$. When $\prod_{\vec{r}, \mu} dA_{\mu, \vec{r}}$ is integrated out, it is thus necessary the inclusion of a choice for the gauge field, in order to remove the overcounting of all configurations related to each other only by a gauge transformation, and which therefore produce the same contribution⁶⁶. But the main consequences of the non-angular character of the gauge field is the absence in correlation function $W(C)$ of the strongest decay like the area of C , and the fact that parameter q in covariant derivative $\cos(\Delta_\mu \varphi_{\vec{r}} - q A_{\mu, \vec{r}})$ plays no relevant role in the characteristics of the different model's phases. Before we consider the effect of the lattice periodicity, and thus of the temperature, let us firstly summarize the results obtained for the non-periodic lattice ($\omega = \infty$).

If we return to the Polyakov's argument (see page 344 for its application to the scalar model and page 397 for its application to the gauge model), the partition function for the pure gauge field, in both the compact and the non-compact formulation, can be approximated by $\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{2a^{4-d}e^2\eta} \int d^d \vec{r} F_{\mu\nu}^2(\vec{r}) \right]$, with $F_{\mu\nu}(\vec{r}) = \partial_\mu A_\nu(\vec{r}) - \partial_\nu A_\mu(\vec{r})$ (called the magnetic field if one prefers the analogy with the superconductor). The main contribution to the partition function arises from the configuration which minimizes $\int d^d \vec{r} F_{\mu\nu}^2(\vec{r})$. But it is here where the non-compact differs from the compact model,

⁶⁶In fact, in the Villain formulation of the compact gauge model such a gauge fixing factor was also necessary when, after putting the angular character of $A_{\mu, \vec{r}}$ into integer fields $g_{\mu\nu, \vec{r}}$ and imposing a constraint on sum $\prod_{\vec{r}} \prod_{\mu < \nu} \sum_{g_{\mu\nu, \vec{r}} = -\infty}^{\infty}$, the $A_{\mu, \vec{r}}$ -integration range was extended to $(-\infty, \infty)$.

since in the former the minimum is obtained by solving equation $\partial_\mu F_{\mu\nu} = 0$ whereas in the latter the minimum corresponds to the solution of $\partial_\mu F_{\mu\nu} = 0 \pmod{2\pi}$, where this “mod 2π ” allows for the jumps which the *compact* gauge field can suffer. According to the Polyakov solution for the 3-dimensional gauge model, it is just the configurations which are solutions of $\partial_\mu F_{\mu\nu} = 0 \pmod{2\pi}$ those which lead to equation (C.308): $\nabla^2 \phi_{cl} = 2\pi \Theta_C(y, z) \partial_x \delta(x) + M^2 \sin \phi_{cl}$. And the solution of this last equation substituted into $W(C) = \exp\left(-\frac{a^2 e^2 \eta}{2} \delta(0) \mathcal{A}_C\right) \frac{\exp\left[-\frac{e^2 \eta}{4\pi^2} \int d^3 \vec{r} \left(\frac{1}{2} (\vec{\nabla} \phi_{cl})^2 + \phi 2\pi \Theta_C \partial_x \delta(x) - M^2 \cos \phi_{cl}\right)\right]}{\exp\left[-\frac{e^2 \eta}{4\pi^2} \int d^3 \vec{r} \left(\frac{1}{2} (\vec{\nabla} \phi_{cl})^2 M^2 \cos \phi_{cl}\right)\right]}$ is responsible for the $W(C)$ ’s decay like area $\mathcal{A}_C$ at $\omega = \infty$ even for small e^2 . For the $(d = 3, 4)$ -dimensional model at large $e^2 \eta$, term $\exp\left[\frac{1}{a^{4-d} e^2 \eta} \sum_{\vec{r}, \mu < \nu} \cos(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})\right] = \prod_{\vec{r}, \mu < \nu} \exp\left[\frac{1}{a^{4-d} e^2 \eta} \frac{e^{i(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})} + e^{-i(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})}}{2}\right]$ of the compact formulation, together with the integration range of the gauge field, leads to the area decay of correlation $W(C) = \left\langle e^{i \sum_{(\vec{r}, \mu) \in C} J_{\mu, \vec{r}} A_{\mu, \vec{r}}} \right\rangle$: each bond-factor $e^{i A_{\mu, \vec{r}}}$ on C must be covered by power $e^{-i A_{\mu, \vec{r}}}$ arising from the expansion of the energy’s cosine function in plaquette-powers $\left(\frac{1}{2a^{4-d} e^2 \eta}\right) e^{\pm i(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})}$, remaining no bond-power without covering. And thus the leading contribution with the smallest power in $\left(\frac{1}{2a^{4-d} e^2 \eta}\right)$ corresponds to the configuration where the interior of C is completely tiled by the expansion’s plaquette-powers. On the other hand, the non-compact model produces an immediate Gaussian integration $W(C) = \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp\left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} - \frac{1}{2a^{4-d} e^2 \eta} \sum_{\vec{r}, \mu < \nu} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})^2\right] = \text{const.} \times \exp\left[-\frac{a^{4-d} e^2 \eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}\right]$, and $\sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} J_{\mu, \vec{r}'}$ shows for $d \geq 3$ a dependence on the size of C always weaker than its area.

The extension to the Higgs model, with gauge field coupled to the phase of a scalar field, reads $\partial_\mu F_{\mu\nu} - q^2 e^2 |\psi|^2 A_\nu = 0$ for the non-compact model, and for the compact case $\partial_\mu F_{\mu\nu} - q^2 e^2 |\psi|^2 A_\nu = 0 \pmod{2\pi}$. Term $q^2 e^2 |\psi|^2 A_\nu$ makes the gauge field short range, and this, together with the angular character of the scalar’s phase and the vanishing of the covariant derivative at the spatial infinity, is responsible for the existence of vortices. If such vortices occur as isolated stable objects, one says that the symmetry is broken. From here on, short-range gauge fields *plus* the existence of topologically stable vortices (and no isolated monopoles), on the one hand, and the complete breaking of the gauge symmetry ⁶⁷ on the other hand, will be used as synonyms (see also (B.217)). The ab-

⁶⁷On a plane normal to the vortex line in the (external) physical space, and considering a closed path S_∞^1 encircling at large distances this vortex, the vortex configuration is defined by means of a *nonvanishing* value of scalar field ψ (origin of the symmetry breaking) whose phase drew one (or several) complete rotations in its internal symmetry space as the closed S_∞^1 was covered in the external space. If such a configuration was *topologically stable* then there was no possibility of adjusting this

sence of the compact character in the gauge field makes stable any amount of vorticity (which for its part determines the symmetry breaking) and moreover, it impedes any kind of configuration which could connect states with different winding number. Therefore, in the non-compact Higgs model the strongest area decay in $W(C)$ cannot occur⁶⁸. This can be also seen as follows: with decreasing $|\psi|^2$, the number of vortices increases, since their energy is proportional to $|\psi|^2$; and eventually they will be so near to each other that they condense all together and form uniform distributions of the gauge field where isolated vortices are no longer present. In such configurations the gauge field can spread through long distances and its short range disappears. Therefore the symmetry is no longer broken and one can expect a stronger decay in $W(C)$ when $|\psi|^2$ strongly decreases; but since in the non-compact model, even at $|\psi|^2 = 0$, $W(C)$ does not reach the area law, then $W(C)$ does not show a decay like the area of C at any $|\psi|^2$.

After having discussed the qualitative behaviour of $W(C)$ at $\omega = \infty$ and by looking at the phase diagrams at such a ω -value Fig. C.8 (page 465) and Fig. C.9 (page 477), one can guess immediately the form of $W(C)$ for the d -dim. non-compact Higgs model in a lattice of small periodicity ω ; C is a closed path which stretches over the whole periodic τ -dimension and crosses at sites $\vec{x}_1$ and $\vec{x}_2$ the $(d - 1)$ -dimensional subspace normal to the τ -direction (subspace denoted by $\Lambda_\perp$). Behaviour $W(C) \sim \text{const.}$ corresponds to the phase with $W(C) \sim e^{-\mathcal{P}_C}$ at $\omega = \infty$, as we already know from the Note on page

630. This phase occurs at large $|\psi|^2$ and small e^2 . And $\begin{cases} W(C) \sim e^{-\ln|\vec{x}_2-\vec{x}_1|} & d = 3 \\ W(C) \sim e^{1/|\vec{x}_2-\vec{x}_1|} & d = 4 \end{cases}$

is the phase at small ω corresponding to the phase $\begin{cases} W(C) \sim e^{-\tau \ln|\vec{x}_2-\vec{x}_1|} & d = 3 \\ W(C) \sim e^{-\mathcal{P}_C + \tau/|\vec{x}_2-\vec{x}_1|} & d = 4 \end{cases}$

at $\omega = \infty$, where τ is the length of C along the will-become-periodic τ -direction. This phase occupies the complete e^2 -range for small $|\psi|^2$, as well as the small region of large e^2 for large $|\psi|^2$. Hence, phase ③ of the phase diagrams on pages 747-750, phase which is characteristic of the compact nature of the gauge field and shows $W(C) \sim e^{-|\vec{x}_2-\vec{x}_1|}$, as expected does not appear. And the same occurs to phase ②, which was only displayed at small $q \neq 1$ and was due to the equivalence relation between, on the one hand, the

boundary behaviour at large distances with a gradient of the scalar field which vanished rapidly enough when the spatial infinity was approached. And if such a configuration was to have a finite energy, then a coupling to a gauge field was necessary: this coupling, on the one hand, made the gauge field short range, and on the other hand, forced the integral of the gauge field, along a closed path around the topological configuration, to be proportional to an integer which was called the winding number.

Further, isolated monopoles not-attached to a vortex — i.e., whose magnetic field is not constrained to be confined into a vortex — requires a long range gauge field.

⁶⁸For the sake of completeness, let us mention the behaviour of the compact Higgs model, which is completely different. For $q \geq 2$ there are several classes of stable vortices: as long as those configurations which connect states belonging to different classes are not energetically possible, symmetry breaking exists and $W(C)$ shows a weak decay in dependence on the size of C . Eventually $W(C)$ will decay like the area when the connecting configurations are allowed, configurations originated by the compact character of the gauge field, in the same way as the domain walls in the scalar models are possible by the angular character of the scalar's phase (and they are the responsible of the restoring of the broken symmetry in the spin systems). For $q = 1$, i.e., the “ $SU(2)$ ”-superconductor, there is only one class of vortices and thus there is no connecting configuration between different classes: the symmetry is always broken and $W(C)$ shows only the weak perimeter decay.

original compact model in certain ranges of e^2 and $|\psi|^2$, and the $Z(2)$ Ising model on the other hand. Such an equivalence relation does not take place in the non-compact model for any value of e^2 or of $|\psi|^2$, because such a model is not sensitive to whether q is small or not: limit $|\psi|^2 = \infty$, limit which in the compact model displayed in the clearest way the difference between large and small q , is dual-related for any lattice's periodicity ω to models whose symmetry is $U(1)$ and not $Z(q)$.

The rest of this section is devoted to proving the above argued results for $W(C)$ in the non-compact Higgs model. The strategy for the study of $W(C)$ runs as follows: by immediate calculation in the limit $|\psi|^2 = 0$ (as well as for large e^2 at $|\psi|^2 = \infty$); and by the already mentioned duality relation in the limit $|\psi|^2 = \infty$. Since, as argued, the only phase occurring at $|\psi|^2 = 0$ will result to be the same as that for large e^2 and $|\psi|^2 = \infty$, the model for any finite $|\psi|^2 \neq 0$ will need no further consideration. For the following, limit $|\psi|^2 = \infty$ of the non-compact Higgs model will be named the Peskin model, as in Appendix C.

The numerator of $W(C)$ in the non-compact Higgs model at any $|\psi|^2$ is given by

$$W(C) = \frac{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} db_{\mu, \vec{r}} \exp \left[i \sum_{\vec{r}, \mu} J_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] \sum_{\{m_{\mu, \vec{r}}\}} \times}{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}} \frac{d\varphi_{\vec{r}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} db_{\mu, \vec{r}} \prod_{\vec{r}, \mu} \sum_{m_{\mu, \vec{r}}=-\infty}^{\infty} \times}$$

$$\times \exp \left[-\frac{1}{2a^{4-d}e^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 - \frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}}) \right]$$

$$\times \exp \left[-\frac{1}{2a^{4-d}e^2\eta} \sum_{\vec{r}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}})^2 - \frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - q A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}}) \right].$$

By means of the Poisson formula, one can substitute $\prod_{\vec{r}, \mu} \left(db_{\mu, \vec{r}} \sum_{m_{\mu, \vec{r}}=-\infty}^{\infty} \right) e^{-i2\pi \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} m_{\mu, \vec{r}}}$ by $\prod_{\vec{r}, \mu} \sum_{l_{\mu, \vec{r}}=-\infty}^{\infty}$. The integration of $\prod_{\vec{r}} d\varphi_{\vec{r}}$ produces a constraint on $l_{\mu, \vec{r}}$ given by factor $\prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}-\hat{\mu}}, 0 \right)$, and Gaussian integration of $\prod_{\vec{r}, \mu} dA_{\mu, \vec{r}}$, after having included a gauge choice, gives as final result the following equation

$$W(C)_{\text{non-comp.}} = \frac{\prod_{\vec{r}, \mu} \sum_{l_{\mu, \vec{r}}=-\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 \right] \times}{\prod_{\vec{r}, \mu} \sum_{l_{\mu, \vec{r}}=-\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}-\hat{\mu}}, 0 \right) \exp \left[-\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 \right] \times}$$

$$\times \exp \left[-\frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} \left(l_{\mu, \vec{r}} - \frac{1}{q} J_{\mu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(d)} \left(l_{\mu, \vec{r}'} - \frac{1}{q} J_{\mu, \vec{r}'} \right) \right]$$

$$\times \exp \left[-\frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'} \right], \quad (\text{D.235})$$

where $\sum_{\mu} \Delta_{\mu} J_{\mu, \vec{r}-\hat{\mu}} = 0$, as well as $\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{r}-\hat{\mu}} = 0$, is satisfied. Result (D.235) holds for a d -dim. non-periodic lattice ($\omega = \infty$).

It is noteworthy that limit $|\psi|^2 = 0$ at *any* e^2 produces the same behaviour as limit $|\psi|^2 = \infty$ at *large* e^2 : in both cases integer field $l_{\mu, \vec{r}}$ is suppressed, in the first case by term $\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$, and in the second case by term $\frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'}$ ⁶⁹.

This holds also for the periodic lattice: at small ω , and using relation $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$, the behaviour of $W(C)$ at $|\psi|^2 = 0$, as well as $|\psi|^2 = \infty$ and large e^2 , becomes equal to

$$\begin{aligned} W(C)_{|\psi|^2=0, \text{non-comp.}} &= W(C)_{\text{large } e^2, |\psi|^2=\infty, \text{non-comp.}} = \\ &= \exp \left[-\frac{a^{4-d}e^2\eta}{2} \sum_{\vec{x}, \vec{x}' \neq \vec{x}} \frac{\omega}{a} (\delta_{\vec{x}, \vec{x}_1} - \delta_{\vec{x}, \vec{x}_2}) \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)} \frac{\omega}{a} (\delta_{\vec{x}', \vec{x}_1} - \delta_{\vec{x}', \vec{x}_2}) \right] = \quad (\text{D.236}) \\ &= \begin{cases} \exp \left[-\frac{e^2\omega\eta}{2\pi} \ln |\vec{x}_2 - \vec{x}_1| \right] & d = 3, \\ \exp \left[\frac{e^2\omega\eta}{4\pi a} \frac{1}{|\vec{x}_2 - \vec{x}_1|} \right] & d = 4, \end{cases} \end{aligned}$$

which corresponds, respectively, to $W(C) = \exp \left[-\frac{ae^2\eta}{2\pi} \tau \ln |\vec{x}_2 - \vec{x}_1| \right]$ and to $W(C) = \exp \left[-\frac{e^2\eta}{2} \mathcal{P}_C \left(D(0) + \frac{1}{2a\pi^2} \right) + \frac{e^2\eta}{4\pi} \frac{1}{|\vec{x}_2 - \vec{x}_1|} \right]$ at $\omega = \infty$.

Let us complete the picture of the phase diagram by studying the phases existing at $|\psi|^2 = \infty$, i.e., the Peskin model. This is achieved by means of the duality relation, already proved in Appendix C for the non-periodic lattice, between, on the one hand, the d -dim. Peskin model and, on the other hand, the XY model or the pure compact $U(1)$ gauge model for $d = 3$ or $d = 4$ respectively:

- For $d = 3$, equation (D.235) is equal to $\frac{\mathcal{Z}_{3\text{-dim. } XY, \text{ with } M_{\lambda, \vec{r}} \rightarrow M_{\lambda, \vec{r}} - \frac{1}{q} J_{\lambda, \vec{r}}}}{\mathcal{Z}_{3\text{-dim. } XY}}$ with a scalar coupling strength for the XY model given by $\frac{a|\psi|^2_{XY}}{\eta} \equiv \frac{q^2}{4\pi^2} ae^2\eta$

⁶⁹One cannot simply absorb term $\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$ into the (underlined) selfenergy term which arises from $\frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'} = \frac{a^{4-d}q^2e^2\eta}{2} D^{(d)}(0) \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2 + \frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}' \neq \vec{r}} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'}$ and therefore simply state that the behaviour of the model is the same for any $|\psi|^2$: the reason is the fact that $\frac{\eta}{2a^{d-2}|\psi|^2} \sum_{\vec{r}, \mu} l_{\mu, \vec{r}}^2$ contains another different parameter, $|\psi|^2$, which does control the density of field $l_{\mu, \vec{r}}$ independently of interaction term $\frac{a^{4-d}q^2e^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} l_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d)} l_{\mu, \vec{r}'}$, just as we have seen.

$$W_{\text{non-comp.}}^{|\psi|^2=\infty} = \frac{\int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \exp \left[-\frac{q^2 e^2 a \eta}{4\pi^2} \sum_{\vec{r}, \mu} \left(1 - \cos \left(\Delta_{\mu} \theta_{\vec{r}} + \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right) \right) \right]}{\int_{-\pi}^{\pi} \prod_{\vec{r}} \frac{d\theta_{\vec{r}}}{2\pi} \exp \left[-\frac{q^2 e^2 a \eta}{4\pi^2} \sum_{\vec{r}, \mu} (1 - \cos(\Delta_{\mu} \theta_{\vec{r}})) \right]}.$$

Antisymmetric field $L_{\rho\sigma, \vec{r}}$ is defined, as usual, different from zero and equal to ± 1 for those plaquettes tiling the interior of closed path C : $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r}-\hat{\rho}} = -J_{\sigma, \vec{r}}$, with $J_{\sigma, \vec{r}}$ the unit vector tangent to C . As characteristic for dual relations, the scalar coupling strength of the equivalent model, $\frac{a|\psi|_{XY}^2}{\eta}$, is inversely proportional to the gauge coupling strength, $\frac{1}{ae^2\eta}$, of the original model.

The ordered phase of the scalar model (large $|\psi|_{XY}^2$), where the angular character of the scalar phase can be neglected⁷⁰, corresponds to the range of large e^2 in the Peskin model, where all integer fields $l_{\mu, \vec{r}}$ are suppressed and thus one obtains $W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2=\infty} \approx \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=3)} J_{\mu, \vec{r}'} \right]$. On the contrary, the disordered phase of the equivalent dual 3-dimensional XY model, with its corresponding expansion in bond-powers $\left(\frac{|\psi|_{XY}^2}{2\eta} \right) e^{\pm i(\theta_{\vec{r}+\hat{\mu}} - \theta_{\vec{r}})}$, is related to the phase at small e^2 in the limit $|\psi|^2 = \infty$ of the non-compact Higgs model. Moreover, such a power expansion applied to $\frac{\mathcal{Z}_{3\text{-dim. } XY, \text{ with } M_{\lambda, \vec{r}} \rightarrow M_{\lambda, \vec{r}} - \frac{1}{q} J_{\lambda, \vec{r}}}}{\mathcal{Z}_{3\text{-dim. } XY}}$ leads to the result of a perimeter decay in $W(C)$: the only leading numerator's contributions different from those in the denominator occur when the unit closed plaquette formed by 4 bond-powers is crossed by path C , producing $W(C)_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2=\infty} \approx \approx 1 + \mathcal{P}_C \left(\frac{|\psi|_{XY}^2}{2\eta} \right)^4 \left(-1 + \frac{e^{i\frac{2\pi}{q}} + e^{-i\frac{2\pi}{q}}}{2} \right) \approx \exp \left[-\mathcal{P}_C \left(\frac{aq^2 e^2 \eta}{8\pi^2} \right)^4 \left(1 - \cos \frac{2\pi}{q} \right) \right]$.

Even more easily calculated is the complementary correlation function $H(\vec{r}_a, \vec{r}_b) = \frac{\mathcal{Z}_{\text{non-comp. gauge, with } (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}) \rightarrow (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi/q \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^*)}}{\mathcal{Z}_{\text{non-comp. gauge}}}$, where $L_{\lambda, \vec{r}}^*$ is the unit vector tangent to a path connecting (dual) sites $\vec{r}_a$ and $\vec{r}_b$: it takes the form of scalar-scalar correlation function $\left\langle e^{i(\theta_{\vec{r}_a} - \theta_{\vec{r}_b})} \right\rangle$ when translated to the dual model

⁷¹. Hence the behaviour of $H(\vec{r}_a, \vec{r}_b)$ in the Peskin model is $\begin{cases} e^{-|\vec{r}_b - \vec{r}_a|} & \text{small } e^2 \\ e^{1/|\vec{r}_b - \vec{r}_a|} & \text{large } e^2 \end{cases}$,

⁷⁰And thus scalar-scalar correlation function $\left\langle e^{i(\theta_{\vec{r}_1} - \theta_{\vec{r}_2})} \right\rangle$ for the equivalent (i.e. dual) 3-dimensional XY model can be approximated by $\exp \left[\frac{\eta}{a|\psi|_{XY}^2} D_{\vec{r}_1, \vec{r}_2}^{(d=3)} \right]$.

⁷¹As in equation (D.235), and again by using the Villain formulation for any e^2 and $|\psi|^2$, correlation

corresponding respectively to the behaviour of the scalar-scalar correlation function in the disordered (small $|\psi|_{XY}^2$) and in the ordered phase (large $|\psi|_{XY}^2$) of the 3-dimensional XY model.

- For limit $|\psi|^2 = \infty$ in the 4-dim. periodic non-compact Higgs model, and according to (D.235), $W(C)$ at $\omega = \infty$ is given by $\frac{\mathcal{Z}_{4\text{-dim. pure comp. } U(1), \text{ with } G_{\rho, \vec{\tau}} \rightarrow G_{\rho, \vec{\tau}} - \frac{1}{q} J_{\rho, \vec{\tau}}}}{\mathcal{Z}_{4\text{-dim. pure comp. } U(1)}}$, being the coupling strength in the dual model, the 4-dimensional pure compact $U(1)$ gauge model, proportional to the inverse of the gauge coupling strength of the

$$\text{function } H(\vec{r}_a, \vec{r}_b) = \frac{\int_{-\infty}^{\infty} \prod_{\vec{\tau}, \mu} \frac{dA_{\mu, \vec{\tau}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{\tau}} \frac{d\varphi_{\vec{\tau}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{\tau}, \mu} db_{\mu, \vec{\tau}} \prod_{\vec{\tau}, \mu} \sum_{m_{\mu, \vec{\tau}}=-\infty}^{\infty} \exp\left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}}^2\right] \times}{\int_{-\infty}^{\infty} \prod_{\vec{\tau}, \mu} \frac{dA_{\mu, \vec{\tau}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{\tau}} \frac{d\varphi_{\vec{\tau}}}{2\pi} \int_{-\infty}^{\infty} \prod_{\vec{\tau}, \mu} db_{\mu, \vec{\tau}} \prod_{\vec{\tau}, \mu} \sum_{m_{\mu, \vec{\tau}}=-\infty}^{\infty} \exp\left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}}^2\right] \times} \\ \times \exp\left[i \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}} (\Delta_{\mu} \varphi_{\vec{\tau}} - q A_{\mu, \vec{\tau}} - 2\pi m_{\mu, \vec{\tau}}) - \frac{1}{2ae^2\eta} \sum_{\vec{\tau}, \mu < \nu} \left(\Delta_{\mu} A_{\nu, \vec{\tau}} - \Delta_{\nu} A_{\mu, \vec{\tau}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{\tau}-\hat{\lambda}}^*\right)^2\right], \\ \times \exp\left[i \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}} (\Delta_{\mu} \varphi_{\vec{\tau}} - q A_{\mu, \vec{\tau}} - 2\pi m_{\mu, \vec{\tau}}) - \frac{1}{2ae^2\eta} \sum_{\vec{\tau}, \mu < \nu} (\Delta_{\mu} A_{\nu, \vec{\tau}} - \Delta_{\nu} A_{\mu, \vec{\tau}})^2\right]$$

with $\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{\tau}-\hat{\lambda}}^* = (\delta_{\vec{\tau}, \vec{\tau}_a} - \delta_{\vec{\tau}, \vec{\tau}_b})$, is calculated as follows: $\prod_{\vec{\tau}, \mu} \left(db_{\mu, \vec{\tau}} \sum_{m_{\mu, \vec{\tau}}=-\infty}^{\infty} \right) e^{-i2\pi \sum_{\vec{\tau}, \mu} b_{\mu, \vec{\tau}} m_{\mu, \vec{\tau}}}$ is

substituted by $\prod_{\vec{\tau}, \mu} \sum_{l_{\mu, \vec{\tau}}=-\infty}^{\infty}$ by means the Poisson formula. Integration of $\prod_{\vec{\tau}} d\varphi_{\vec{\tau}}$ produces constraint

$\prod_{\vec{\tau}} \delta\left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{\tau}-\hat{\mu}}, 0\right)$. And Gaussian integration of $\prod_{\vec{\tau}, \mu} dA_{\mu, \vec{\tau}}$, after having included a necessary gauge

choice, gives $H_{\vec{r}_a, \vec{r}_b} = \frac{\exp\left[-\frac{1}{2} \frac{4\pi^2}{q^2 ae^2\eta} \sum_{\vec{\tau}, \lambda} (L_{\lambda, \vec{\tau}}^*)^2\right] \sum_{\{l_{\mu, \vec{\tau}}\}} \prod_{\vec{\tau}} \delta\left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{\tau}-\hat{\mu}}, 0\right) \exp\left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{\tau}, \mu} l_{\mu, \vec{\tau}}^2\right] \times}{\prod_{\vec{\tau}, \mu} \sum_{l_{\mu, \vec{\tau}}=-\infty}^{\infty} \prod_{\vec{\tau}} \delta\left(\sum_{\mu} \Delta_{\mu} l_{\mu, \vec{\tau}-\hat{\mu}}, 0\right) \exp\left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{\tau}, \mu} l_{\mu, \vec{\tau}}^2\right] \times}$

$\times \exp\left[-\frac{1}{2} \frac{q^2 ae^2\eta}{4\pi^2} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} \left(2\pi l_{\mu, \vec{\tau}} - \frac{i4\pi^2}{q^2 ae^2\eta} \sum_{\nu, \lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} L_{\lambda, \vec{\tau}}^*\right) D_{\vec{\tau}, \vec{\tau}'}^{(d=3)} \left(2\pi l_{\mu, \vec{\tau}'} - \frac{i4\pi^2}{q^2 ae^2\eta} \sum_{\nu', \lambda'} \epsilon_{\mu\nu'\lambda'} \Delta_{\nu'} L_{\lambda', \vec{\tau}'}^*\right)\right]$

$$\times \exp\left[-\frac{q^2 ae^2\eta}{2} \sum_{\vec{\tau}, \vec{\tau}'} \sum_{\mu} l_{\mu, \vec{\tau}} D_{\vec{\tau}, \vec{\tau}'}^{(d=3)} l_{\mu, \vec{\tau}'}\right]$$

Limit $|\psi|^2 = \infty$ on the previous result, together with $\sum_{\vec{\tau}, \vec{\tau}', \mu} \sum_{\nu, \lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} L_{\lambda, \vec{\tau}}^* D_{\vec{\tau}, \vec{\tau}'}^{(d=3)} \sum_{\nu', \lambda'} \epsilon_{\mu\nu'\lambda'} \Delta_{\nu'} L_{\lambda', \vec{\tau}'}^* =$

$= \sum_{\vec{\tau}, \lambda} (L_{\lambda, \vec{\tau}}^*)^2 - \sum_{\vec{\tau}, \vec{\tau}'} \left(\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{\tau}-\hat{\lambda}}^*\right) D_{\vec{\tau}, \vec{\tau}'}^{(d=3)} \left(\sum_{\lambda'} \Delta_{\lambda'} L_{\lambda', \vec{\tau}'-\hat{\lambda}'}^*\right)$, proves the equivalence between

$H(\vec{r}_a, \vec{r}_b)$ and the scalar-scalar correlation function $\langle e^{i(\theta_{\vec{r}_a} - \theta_{\vec{r}_b})} \rangle$ calculated in a 3-dimensional XY of

coupling strength $\frac{a|\psi|_{XY}^2}{\eta} \equiv \frac{q^2 ae^2\eta}{4\pi^2}$.

non-compact gauge model: $\frac{1}{\eta e^2_{\text{comp. } U(1)}} \equiv \frac{q^2}{4\pi^2} e^2 \eta$. And hence $W_{\text{non-comp.}}^{|\psi|^2=\infty}$ takes the following form

$$\frac{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dW_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{q^2 e^2 \eta}{4\pi^2} \sum_{\vec{r}, \mu < \nu} \left(1 - \cos \left(\Delta_{\mu} W_{\nu, \vec{r}} - \dots - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}} \right) \right) \right]}{\int_{-\pi}^{\pi} \prod_{\vec{r}, \mu} \frac{dW_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{q^2 e^2 \eta}{4\pi^2} \sum_{\vec{r}} \sum_{\mu < \nu} (1 - \cos (\Delta_{\mu} W_{\nu, \vec{r}} - \Delta_{\nu} W_{\mu, \vec{r}})) \right]}.$$

As for the 3-dimensional Peskin model, the phase for the dual model where the compact character of the gauge variables can be neglected (large $1/\eta e^2_{\text{comp. } U(1)}$) corresponds again in the non-compact Higgs model at $|\psi|^2 = \infty$ to the phase characterized by $W(C)_{\text{non-comp.}}^{\text{large } e^2, |\psi|^2=\infty} \approx \exp \left[-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\mu} J_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(d=4)} J_{\mu, \vec{r}'} \right]$. And the phase for small $1/\eta e^2_{\text{comp. } U(1)}$ in the pure compact $U(1)$ gauge model, where the expansion in plaquette-powers $\left(\frac{1}{2\eta e^2_{\text{comp.}}} \right) e^{\pm i(\Delta_{\mu} W_{\nu, \vec{r}} - \Delta_{\nu} W_{\mu, \vec{r}})}$ must be used, corresponds in the Peskin model for e^2 small to the perimeter decay of $W(C)$, decay produced by the leading contribution in the power-expansion of $\frac{\mathcal{Z}_{\text{comp. } U(1), \text{ with } G_{\rho, \vec{r}} \rightarrow G_{\rho, \vec{r}} - \frac{1}{q} J_{\rho, \vec{r}}}}{\mathcal{Z}_{4\text{-dim. pure comp. } U(1)}}$ when a unit cube formed by 6 plaquettes surrounds, in the 4-dimensional space, each point of path C : $W(C)_{\text{non-comp.}}^{\text{small } e^2, |\psi|^2=\infty} \approx 1 + \mathcal{P}_C \left(\frac{1}{2\eta e^2_{\text{comp.}}} \right)^6 \left(-1 + \frac{e^{i\frac{2\pi}{q}} + e^{-i\frac{2\pi}{q}}}{2} \right) \approx \approx \exp \left[-\mathcal{P}_C \left(\frac{q^2 e^2 \eta}{8\pi^2} \right)^6 \left(1 - \cos \frac{2\pi}{q} \right) \right]$.

Let us consider in the dual lattice a closed path C^* of dimensions $\tau^* \times |\vec{x}_b^* - \vec{x}_a^*|$: this path is defined by an integer plaquette-field $L_{\rho\sigma}^*$ which fulfils $\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r}-\hat{\rho}}^* = -J_{\sigma, \vec{r}}^*$, with $J_{\sigma, \vec{r}}^*$ the unit vector tangent to C^* . Then correlation function

$$H(C^*) = \frac{\mathcal{Z}_{\text{non-comp. gauge, with } (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}}) \rightarrow (\Delta_{\mu} A_{\nu, \vec{r}} - \Delta_{\nu} A_{\mu, \vec{r}} - 2\pi/q \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}}^*)}}{\mathcal{Z}_{\text{non-comp. gauge}}}, \text{ when}$$

formulated by using the dual variables of the pure compact $U(1)$ gauge model, becomes $W(C^*)$ for such a compact model. This is shown just by following the same steps as in the previous footnote ⁷². Therefore the form of $H(C^*)$ for the Peskin

model in a non-periodic lattice is $\begin{cases} e^{-\mathcal{A}_{C^*}} & \text{small } e^2 \\ e^{-\mathcal{P}_{C^*} + \tau^*/|\vec{x}_b^* - \vec{x}_a^*|} & \text{large } e^2 \end{cases}$, which are the be-

⁷²And by using relation $\sum_{\vec{r}, \vec{r}', \mu} \left(\sum_{\nu} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} \Delta_{\nu} L_{\rho\sigma, \vec{r}}^* \right) D_{\vec{r}, \vec{r}'}^{(d=4)} \left(\sum_{\nu'} \sum_{\rho' < \sigma'} \epsilon_{\mu\nu'\rho'\sigma'} \Delta_{\nu'} L_{\rho'\sigma', \vec{r}'}^* \right) = \sum_{\vec{r}} \sum_{\rho < \sigma} (L_{\rho\sigma, \vec{r}}^*)^2 - \sum_{\vec{r}, \vec{r}'} \left(\sum_{\rho} \Delta_{\rho} L_{\rho\sigma, \vec{r}-\hat{\rho}}^* \right) D_{\vec{r}, \vec{r}'}^{(d=4)} \left(\sum_{\rho'} \Delta_{\rho'} L_{\rho'\sigma', \vec{r}'-\hat{\rho}'}^* \right)$.

haviours of $W(C^*)$ in the pure compact $U(1)$ model at small and large gauge coupling strength $\frac{1}{\eta e^2_{\text{comp.}}}$ respectively.

If now the lattice becomes periodic along one direction, the description of the d -dimensional Peskin model is directly obtained from the above duality relations to the $(d = 3)$ -dimensional XY model and to the $(d = 4)$ -dimensional pure compact $U(1)$ gauge model. As usual, path C of correlation function $W(C)$ is assumed to cover completely the lattice's length along the now periodic a_τ -coordinate. In the same way, open path $L_{\lambda, \vec{r}}^*$ of correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ for the 3-dimensional model stretches along a direction normal to the periodic dimension, lying thus in $\Lambda_\perp$, the subspace normal to the τ -direction. And closed path C^* of correlation function $H(C^*)$ for the 4-dimensional model has, as C , one pair of sides which cover the whole τ -direction, denoting $\vec{x}_a^*, \vec{x}_b^*$ the points where C^* crosses $\Lambda_\perp$. Therefore, for the periodic lattice, the correlation function complementary to $W(C)$ will be written as $H(\vec{x}_a^*, \vec{x}_b^*)$. The behaviour of both correlation functions for the Higgs model at $|\psi|^2 = \infty$ in a lattice of periodicity ω is obtained as follows:

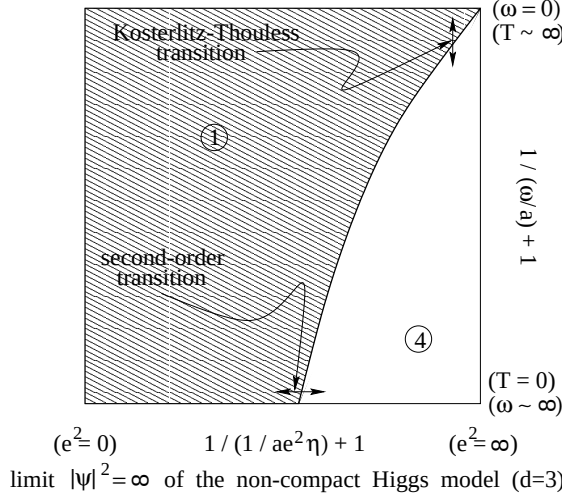
- For $d = 3$, the XY model in a 3-dimensional lattice periodic along one direction (see figure on page 613) shows always the disordered state in the range of small $|\psi|_{XY}^2$ (large e^2 in the Peskin model), whilst for $|\psi|_{XY}^2$ large the model is equivalent to the XY model in a non-periodic 2-dimensional lattice, being its scalar coupling strength equal to $\frac{|\psi|_{2\text{-dim. non-per. } XY}^2}{\eta} \equiv \frac{\omega}{a} \frac{a|\psi|_{3\text{-dim per. } XY}^2}{\eta}$. Therefore, the 3-dimensional Peskin model at ω finite shows in the range of small e^2 always the phase characterized by $\begin{cases} W(C) & \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \end{cases}$, which corresponds at $\omega = \infty$ to $\begin{cases} W(C) & \sim e^{-\mathcal{P}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{-|\vec{r}_b^* - \vec{r}_a^*|} \end{cases}$. And in the range of large e^2 , the system displays a phase transition when ω is decreased at a e^2 -value of $\frac{q^2}{4\pi^2} \left(\frac{\omega}{a} a e^2_{3\text{-dim. per. Peskin}} \eta \right)_{\text{crit}} = \left(\frac{|\psi|_{2\text{-dim. non-per. } XY}^2}{\eta} \right)_{\text{crit}} \approx \frac{2}{\pi}$. Remaining at large e^2 , in the model dual to the 3-dimensional Peskin model in such a e^2 -range, i.e., in the 2-dimensional non-periodic XY model, the transition occurs from the ordered phase of this last model, which produces $\begin{cases} W(C) & \sim e^{-\ln |\vec{x}_2 - \vec{x}_1|} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{-\ln |\vec{x}_b^* - \vec{x}_a^*|} \end{cases}$ at finite ω or $\begin{cases} W(C) & \sim e^{-\tau \ln |\vec{x}_2 - \vec{x}_1|} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{1/|\vec{r}_b^* - \vec{r}_a^*|} \end{cases}$ at $\omega = \infty$, to the disordered phase of the scalar model; for such a disordered phase, the expansion in bond-powers $\left(\frac{|\psi|_{2\text{-dim. non-per. } XY}^2}{2\eta} \right) e^{\pm i(\theta_{\vec{x}+\vec{k}} - \theta_{\vec{x}})}$ gives in the Peskin model as leading terms

$$W(C) \approx \frac{1 + (|\Lambda_\perp| - 2) \left(\frac{|\psi|_{\text{non-per. } XY}^2}{2\eta} \right)^4 + \left(\frac{|\psi|_{\text{non-per. } XY}^2}{2\eta} \right)^4 (e^{i\frac{2\pi}{q}} + e^{-i\frac{2\pi}{q}})}{1 + |\Lambda_\perp| \left(\frac{|\psi|_{2\text{-dim. non-per. } XY}^2}{2\eta} \right)^4} \approx$$

$$\approx 1 + 2 \left(\frac{\omega |\psi|_{3\text{-dim. per. } XY}^2}{2\eta} \right)^4 \left(-1 + \frac{e^{i\frac{2\pi}{q}} + e^{-i\frac{2\pi}{q}}}{2} \right) \approx$$

$$\approx \exp \left[-2 \left(\frac{q^2 \omega e_{3\text{-dim. Peskin}}^2 \eta}{8\pi^2} \right)^4 \left(1 - \cos \frac{2\pi}{q} \right) \right],$$

produced when 4 of the expansion's bond-powers form a closed plaquette which surrounds the two cut points $\vec{x}_1$ and $\vec{x}_2$ where C intersects $\Lambda_\perp$.



By using the same power expansion, it can be proved that with *decreasing* ω the system transits from phase $H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln|\vec{x}_b^* - \vec{x}_a^*|}$ to the disordered phase of the dual scalar model, which in the Peskin model at large e^2 produces

$$H(\vec{x}_a^*, \vec{x}_b^*) \approx \left(\frac{|\psi|_{2\text{-dim. non-per. } XY}^2}{2\eta} \right)^{|\vec{x}_b^* - \vec{x}_a^*|} =$$

$$= \exp \left[-|\vec{x}_b^* - \vec{x}_a^*| \ln \left(\frac{8\pi^2}{q^2 \omega e_{3\text{-dim. Peskin}}^2 \eta} \right) \right]$$

when the expansion's bond-powers cover the shortest path from $\vec{x}_a^*$ to $\vec{x}_b^*$.

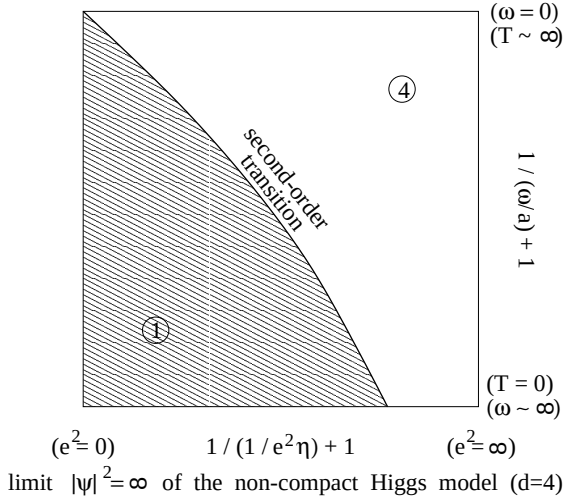
The just discussed transition in the range of large e^2 is of Kosterlitz-Thouless type, since this is the character of the phase transition in the 2-dimensional (non-periodic) XY model. The phase diagram of the Peskin model takes the form of the above figure, with the two phases characterized as in the following table

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) & \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-P_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
④	$\begin{cases} W(C) & \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{-\ln \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{1/ \vec{r}_b^* - \vec{r}_a^* } \end{cases}$

- For the 4-dimensional Peskin model, one only needs consider the phase diagram of the pure compact $U(1)$ gauge model (see diagram on page 554), model which only shows a phase transition with decreasing ω in the range of large $e_{\text{comp.}}^2$, where the model is equivalent to the XY model in a 3-dimensional non-periodic lattice and with a scalar coupling strength given by $\frac{a|\psi|_{3\text{-dim. non-per. } XY}^2}{\eta} \equiv \frac{a}{\omega e_{\text{comp.}}^2 \eta}$. Hence, the Peskin model will display a change in the behaviours of the correlation functions only in the range of small e^2 and for a critical value equal to $\frac{q^2}{4\pi^2} \left(\frac{a}{\omega} e_{4\text{-dim. per. Peskin}}^2 \eta \right)_{\text{crit}} = \left(\frac{a|\psi|_{3\text{-dim. non-per. } XY}^2}{\eta} \right)_{\text{crit}} \approx 0.322$; note the dependence on ω , which is the opposite to that in the 3-dimensional Peskin model. For large e^2

the phase is always the same, characterized by $\begin{cases} W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \end{cases}$ at finite

ω , or by $\begin{cases} W(C) \sim e^{-\mathcal{P}_C + \tau/|\vec{x}_2 - \vec{x}_1|} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*} + \tau^*/|\vec{x}_b^* - \vec{x}_a^*|} \end{cases}$ at $\omega = \infty$: this is because in the model dual to the Peskin model, i.e., in the 4-dim. pure compact $U(1)$ gauge model, for small $e_{\text{comp.}}^2$ one can neglect the compact nature of the gauge field and directly integrate.



Focussing our attention on the range of small e^2 , and since the 4-dimensional pure compact $U(1)$ gauge model in the corresponding range (large $e_{\text{comp.}}^2$) is equivalent to the 3-dimensional non-periodic XY model, when ω increases there occurs a second-order phase transition (the transition type in the non-periodic 3-dimensional XY model), which separates the ordered phase in the scalar model, characterizing the Peskin model with $\begin{cases} W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \end{cases}$, from the phase which requires the power-expansion

in the XY model. Such an expansion in bond-powers $\left(\frac{a|\psi|_{\text{non-per. } XY}^2}{2\eta}\right) e^{\pm i(\theta_{\vec{x}+\vec{k}} - \theta_{\vec{x}})}$ follows exactly the same argumentation as in the 3-dimensional Peskin model and produces similar results⁷³ too: $W(C) \approx 1 + 2\left(\frac{a}{\omega e_{\text{comp.}}^2 \eta}\right)^4 \left(-1 + \frac{e^{i\frac{2\pi}{q}} + e^{-i\frac{2\pi}{q}}}{2}\right) \approx \approx \exp\left[-2\left(\frac{aq^2 e_{\text{Peskin}}^2 \eta}{\omega 8\pi^2}\right)^4 \left(1 - \cos\frac{2\pi}{q}\right)\right]$, and $H(\vec{x}_a^*, \vec{x}_b^*) \approx \left(\frac{a|\psi|_{\text{non-per. } XY}^2}{2\eta}\right)^{|\vec{x}_b^* - \vec{x}_a^*|} = \exp\left[-|\vec{x}_b^* - \vec{x}_a^*| \ln\left(\frac{\omega 8\pi^2}{aq^2 e_{\text{Peskin}}^2 \eta}\right)\right]$. The phase diagram is represented above, with the phases as in the following table:

Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(C^*) \sim e^{-\mathcal{A}_{C^*}} \end{cases}$
④	$\begin{cases} W(C) \sim e^{ \vec{x}_2 - \vec{x}_1 ^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{ \vec{x}_b^* - \vec{x}_a^* ^{-1}} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C + \tau/ \vec{x}_2 - \vec{x}_1 } \\ H(C^*) \sim e^{-\mathcal{P}_{C^*} + \tau^*/ \vec{x}_b^* - \vec{x}_a^* } \end{cases}$

⁷³It is important to note that the results are, nevertheless, not the same since the dependence on ω is the inverse to that in the 3-dimensional model: the equivalence relation to the XY model in a non-periodic lattice is $|\psi|_{3\text{-dim. non-per. } XY}^2 \sim \frac{1}{\omega}$ for the model under study, whilst for the 3-dimensional Peskin model it was $|\psi|_{2\text{-dim. non-per. } XY}^2 \sim \omega$. This is the reason why the phase transition towards the disordered phase of the scalar XY model occurs in the 4-dimensional Peskin model with increasing ω and not with decreasing ω as in the 3-dimensional case.

For limit $|\psi|^2 = 0$ of the non-compact Higgs model, i.e., the pure non-compact $U(1)$ gauge model, which is defined by energy $E = \frac{1}{2a^{4-d}e^2} \sum_{\vec{r}, \mu < \nu} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})^2$, it only remains to calculate the correlation function complementary to $W(C)$, $H(\vec{x}_a^*, \vec{x}_b^*)$; $W(C)$ for such a model has been already obtained in equation (D.236). But this calculation is quickly achieved, since the model in such a range displays only one phase whose characterizing correlation functions are calculated immediately by Gaussian integration, using propagator $D_{\vec{r}, \vec{r}'}^{(d)}$ for the non-periodic lattice, or propagator $\frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ for a d -dimensional lattice of small periodicity ω along one direction. For instance, correlation function $H(\vec{r}_a, \vec{r}_b)$ for the 3-dimensional pure non-compact $U(1)$ model

$$\frac{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} \left(\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\lambda} \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^* \right)^2 \right]}{\int_{-\infty}^{\infty} \prod_{\vec{r}, \mu} \frac{dA_{\mu, \vec{r}}}{2\pi} \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}} \sum_{\mu < \nu} (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})^2 \right]},$$

with $\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^* = (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$, becomes equal to ⁷⁴

$$H(\vec{r}_a, \vec{r}_b) = \exp \left[-\frac{2\pi^2}{q^2 ae^2\eta} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b}) D^{(d=3)}(\vec{r}, \vec{r}') (\delta_{\vec{r}', \vec{r}_a} - \delta_{\vec{r}', \vec{r}_b}) \right],$$

$d=3$, non-compact, ω small

$$|\psi|^2 = 0 \quad \frac{W(C) \sim e^{-\ln |\vec{x}_2 - \vec{x}_1|}}{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln |\vec{x}_b^* - \vec{x}_a^*|}}$$

after having introduced a gauge fixing factor which removes the repetition of the same contribution, caused by the gauge invariance. The above result gives $H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{|\vec{r}_b^* - \vec{r}_a^*|^{-1}}$ in a 3-dimensional non-periodic lattice ($\omega = \infty$), as well as $H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln |\vec{x}_b^* - \vec{x}_a^*|}$ for small ω , according to the relation valid for such a ω -value: $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$. In the figure is

$$|\psi|^2 = \infty \quad \frac{W(C) \sim \text{const.}}{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|}} \quad \bullet \quad \frac{W(C) \sim e^{-\ln |\vec{x}_2 - \vec{x}_1|}}{H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln |\vec{x}_b^* - \vec{x}_a^*|}} \quad e^2 = \infty$$

represented a section at small ω of the complete phase diagram for the 3-dim. non-compact Higgs model. The section is similar for the 4-dim. model, but with the phase covering the whole e^2 -range at small $|\psi|^2$ and reaching the left lower corner characterized

$$\text{by } \begin{cases} W(C) \sim e^{|\vec{x}_2 - \vec{x}_1|^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{|\vec{x}_b^* - \vec{x}_a^*|^{-1}} \end{cases}.$$

Finally, the complete phase diagram for the d -dimensional non-compact Higgs model is depicted in Fig. D.6 (page 751) for $d = 3$ and in Fig. D.7 (page 752) for $d = 4$. For

$$^{74} \sum_{\vec{r}, \vec{r}', \mu} \sum_{\nu, \lambda} \epsilon_{\mu\nu\lambda} \Delta_\nu L_{\lambda, \vec{r}}^* D_{\vec{r}, \vec{r}'} \sum_{\nu', \lambda'} \epsilon_{\mu\nu'\lambda'} \Delta_{(\vec{r}')}^{\nu'} L_{\lambda', \vec{r}'}^* = \sum_{\vec{r}, \lambda} (L_{\lambda, \vec{r}}^*)^2 - \sum_{\vec{r}, \vec{r}'} \left(\sum_{\lambda} \Delta_{\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^* \right) D_{\vec{r}, \vec{r}'} \left(\sum_{\lambda'} \Delta_{\lambda'} L_{\lambda', \vec{r}'-\hat{\lambda}'}^* \right)$$

the 3-dim. case, transition points A' and B' of the diagram are given by

$$\begin{aligned} A' : \quad \left(\frac{\omega}{a} \frac{a|\psi|^2}{\eta} \right)_{\text{crit}} &= \frac{2}{\pi} + 4\zeta_0 e^{-2\pi^2 D(0) \left(\frac{\omega^2}{a^2} \frac{a|\psi|^2}{\eta} \right)_{\text{crit}}}, \\ B' : \quad \left(\frac{\omega}{a} a e^2 \eta \right)_{\text{crit}} &= \frac{4\pi^2}{q^2} \left(\frac{2}{\pi} + 4\zeta_0 e^{-\frac{q^2}{2} D(0) \left(\frac{\omega^2}{a^2} a e^2 \eta \right)_{\text{crit}}} \right), \end{aligned} \quad (\text{D.237})$$

with $D(0) \equiv D^{(d=3)}(0) = 0.253$. The transition at both points is Kosterlitz-Thouless, since this is the transition type of the XY model in a 2-dim. non-periodic lattice. The stability argument used for the compact Higgs model applies also for the non-compact model: point B' ($|\psi|^2 = \infty$) is stable under a decreasing in $|\psi|^2$, whilst transition at A' (the transition point of the XY model) changes its character when e^2 is turned on⁷⁵. Since the transition character at B' must be maintained when one leaves it along the line from B' to A' , most of such a line $B'A'$ is formed by transition points of Kosterlitz-Thouless character. But since by abandoning A' along line $A'B'$ the transition type changes, then a short segment on line $A'B'$ near to point A' should be formed by first-order transition-points, becoming again of Kosterlitz-Thouless character when one continues toward B' .

For the 4-dimensional non-compact Higgs model, transition points A' and B' , which are both second-order (once again because this is the transition type of the 3-dimensional non-periodic XY model), are given by

$$\begin{aligned} A' : \quad \left(\frac{\omega^2}{a^2} \frac{a^2|\psi|^2}{\eta} \right)_{\text{crit}} &\approx 0.526, \\ B' : \quad \left(\frac{a}{\omega} e^2 \eta \right)_{\text{crit}} &\approx 0.322 \frac{4\pi^2}{q^2}. \end{aligned} \quad (\text{D.238})$$

Again the stability argument shows that the transitions across most of line $B'A'$ are second-order, becoming first-order only for a short segment of this line near A' , segment whose length decreases with decreasing ω since the whole line $B'A'$ gets shorter too.

It is worth noting that, according to the equation of transition point B' , for both the 3-dimensional and the 4-dimensional non-compact Higgs model, the region occupied by phase ①, that characterized by $\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-|\vec{x}_b^* - \vec{x}_a^*|} \end{cases}$, decreases in size with increasing q . This is a property shared also by the compact Higgs model. In fact, for large

⁷⁵This is easily seen by returning to the partition function of the d -dimensional non-compact Higgs model, which is the denominator of equation (D.235): limit $|\psi|^2 = \infty$ leaves only the term containing the long-range interactions, carried by $D_{\vec{r}, \vec{r}'}^{(d)}$ at $\omega = \infty$ or by $\frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ at small ω . If $|\psi|^2$ becomes large but finite, only the interaction at a very short distance is modified; however, only long-range interactions are the responsible for the transition character and thus the phase transition type for large $|\psi|^2$ is determined by that at $|\psi|^2 = \infty$. On the contrary, turning on the coupling to the gauge field destabilizes the transition character of the XY model's transition point, since for $e^2 = 0$ only the contact (i.e., very short-range) interactions are present, and by allowing $e^2 \neq 0$ long-range interactions appear.

q the phase diagram of both models are similar; the only difference is the existence in the compact model of phase ③, which with its strongest decay in $W(C)$ is characteristic for the compact gauge field.

D.5 The phase diagram of the non-compact $U(1)$ Higgs model at high temperatures (II)

D.5.1 Connection to the Abrikosov model of Appendix A

In the previous section we have discussed the phase diagram of the Higgs model in a d -dimensional (with $d = 3, 4$) lattice, both non-periodic and periodic along one direction, being the length period in the last case proportional to the inverse of the temperature. Let us return to the non-periodic case for a while. The first point is to note that the non-compact $U(1)$ Higgs model placed in a discrete lattice displays the same two phases as the usual superconductor. This is best seen when one compares the behaviour of correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ for $d = 3$ or of correlation function $H(C^*)$ for $d = 4$, which measures the free energy increase due, respectively, to the inclusion of two external monopoles placed at $\vec{r}_a^*$ and $\vec{r}_b^*$, or to the introduction by hand of a monopole ring along C^* ⁷⁶. Thus in the phase for large $|\psi|^2/\eta$ and small $e^2\eta$, denoted by ①, the just mentioned correlation function shows its strongest decay in each respective model,

$$\begin{cases} H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{-|\vec{r}_b^* - \vec{r}_a^*|} & d = 3 \\ H(C^*) \sim e^{-\text{Area}_{C^*}} & d = 4 \end{cases},$$

indication as we already know of the large rigidity in the system, whose energy $\begin{cases} -\eta \ln H(\vec{r}_a^*, \vec{r}_b^*) \\ -\eta \ln H(C^*) \end{cases}$ increases with distance $|\vec{r}_b^* - \vec{r}_a^*|$ in the 3-dimensional model or with area $\mathcal{A}_{C^*} \equiv \tau^* |\vec{x}_b^* - \vec{x}_a^*|$ in the 4-dimensional model ⁷⁷. In the periodic lattice, the energy increase grows like $|\vec{x}_b^* - \vec{x}_a^*|$, being $\vec{x}_a^*$ and $\vec{x}_b^*$ the points where the projection of the monopole pair or the monopole ring intersects the sublattice normal to the periodic dimension. On the other hand, and arguing from the standpoint of

⁷⁶For instance, for the 3-dimensional Higgs model, correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ is defined as $\frac{\mathcal{Z}_{\text{with } (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}}) \rightarrow (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - 2\pi/q \sum_\lambda \epsilon_{\mu\nu\lambda} L_{\lambda, \vec{r}-\hat{\lambda}}^*)}}{\mathcal{Z}}$, where $L_{\lambda, \vec{r}}^*$ is the unit vector tangent to a path from $\vec{r}_a^*$ to $\vec{r}_b^*$: $\sum_\lambda \Delta_\lambda L_{\lambda, \vec{r}-\hat{\lambda}}^* = \delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b}$. Hence $H(\vec{r}_a^*, \vec{r}_b^*)$ corresponds to $(-\eta \text{ times})$ the exponential of the free energy increase created by the substitution of a magnetic field $F_{\mu\nu, \vec{r}} \equiv (\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}})$, whose differential divergence vanishes $\frac{1}{2} \sum_{\rho, \mu, \nu} \epsilon_{\rho\mu\nu} \Delta_\rho F_{\mu\nu, \vec{r}} = 0$, by a magnetic field which satisfies $\frac{1}{2} \sum_{\rho, \mu, \nu} \epsilon_{\rho\mu\nu} \Delta_\rho F_{\mu\nu, \vec{r}} = -\frac{2\pi}{q} \sum_\lambda \Delta_\lambda L_{\lambda, \vec{r}-\hat{\lambda}}^* = -\frac{2\pi}{q} (\delta_{\vec{r}, \vec{r}_a} - \delta_{\vec{r}, \vec{r}_b})$. For the 4-dimensional model, the definition is similar but with change $\Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} \rightarrow \Delta_\mu A_{\nu, \vec{r}} - \Delta_\nu A_{\mu, \vec{r}} - \frac{2\pi}{q} \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma, \vec{r}-\hat{\rho}}^*$ in the numerator's partition function; integer field $L_{\rho\sigma, \vec{r}}^*$ is non-zero and equal to ± 1 only for those plaquettes tiling the interior of C , and satisfies therefore $\sum_\rho \Delta_\rho L_{\rho\sigma, \vec{r}-\hat{\rho}}^* = -J_{\sigma, \vec{r}}^*$, with $J_{\sigma, \vec{r}}^*$ the unit vector tangent to C^* .

⁷⁷The 2-dimensional Higgs model will be not considered since it only shows one unique phase.

a superconductor, the energy of a monopole pair for $d = 3$ (or of a monopole ring C^* for $d = 4$) grows like the distance separating both monopoles (or like the area contained into C^*) because the magnetic field created by the monopoles is constrained to “live” only inside a vortex (or a extensive vortex-plane, formed by the displacement along the fourth dimension of the 3-dimensional vortex-line); and the energy of a vortex-line (or vortex-plane) is proportional to its length (or area for $d = 4$).

Outside the just discussed superconducting phase, characterized by the strongest decay possible in correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$ (or in $H(C^*)$), a different phase occurs

$$\text{where the behaviour for such correlations, } \begin{cases} -\eta \ln H(\vec{r}_a^*, \vec{r}_b^*) \sim -\frac{1}{|\vec{r}_b^* - \vec{r}_a^*|_{\tau^*}} \\ -\eta \ln H(C^*) \sim \mathcal{P}_{C^*} - \frac{1}{|\vec{x}_b^* - \vec{x}_a^*|} \end{cases}, \text{ results}$$

$$\text{also from the direct integration of } \frac{2\pi^2 a^{4-d} e^2}{q^2} \sum_{\vec{r}, \vec{r}', (\sigma)} \left[\sum_{\lambda} \frac{\Delta_{\lambda}}{(\vec{r})} L_{\lambda(\sigma), \vec{r}-\hat{\lambda}}^* \right] D_{\vec{r}, \vec{r}'} \left[\sum_{\lambda'} \frac{\Delta_{\lambda'}}{(\vec{r}') } L_{\lambda'(\sigma), \vec{r}'-\hat{\lambda}'}^* \right]$$

$$\text{by using long-range propagator } D_{\vec{r}, \vec{r}'} = \begin{cases} \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} & d = 3 \\ \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|^2} & d = 4 \end{cases}. \text{ Such a phase has been}$$

denoted by ④ in the phase diagrams of the $(d = 3, 4)$ -dimensional non-compact Higgs model (see the diagram on page 751 and on page 752), and is the behaviour one should expect in a normal conducting phase where monopole's interaction is carried by the Coulomb propagator $D_{\vec{r}, \vec{r}'}$. For the lattice periodic along one direction, the results are similar but with the substitution $D_{\vec{r}, \vec{r}'}^{(d)} \approx \frac{a}{\omega} D_{\vec{x}, \vec{x}'}^{(d-1)}$ if lattice's periodicity ω is small. All the above discussion proves that the two phases of the non-compact Higgs model are the same as those displayed by the superconductor ⁷⁸.

Let us still stay at the non-periodic model, i.e., at the model at temperature $T = 0$. In Appendix A, we have seen another form of describing the superconductor at $T = 0$: the Landau-Ginzburg theory of superconductivity, or equivalently, the Abrikosov model. Let us briefly summarize the results of such a description: instead of proceeding as in the lattice formulation carried out up to this point, by summing over all possible configurations created by the fields placed at every site of a discrete lattice $\{\psi_{\vec{r}} = |\psi| e^{i\varphi_{\vec{r}}}, A_{\mu, \vec{r}}\}$ with $|\psi|^2$ always different from zero, one defines an effective continuum model for an order parameter $\bar{\psi}(\vec{r}) = \langle \psi \rangle e^{i\bar{\varphi}(\vec{r})}$, being $\langle \psi \rangle$ the expectation value of (microscopic) variable $\psi_{\vec{r}}$ after removing the gauge invariance by means of a gauge fixing. This order parameter is coupled to an (effective) gauge field $\bar{A}_{\mu}(\vec{r})$, which describes the spreading at large distances of the collective effect by all the (microscopic) gauge field $A_{\mu, \vec{r}}$. Every microscopic configuration in the lattice Higgs model gets always defined by its content in topological configurations, called vortices, flux lines or dislocations, plus fluctuations without topological character; the two phases of the non-compact lattice model at small

⁷⁸In fact, the equivalence between, on the one hand, phase ① and phase ④, and on the other hand, the superconducting and the normal conducting phase of a superconductor holds also for the *compact* $U(1)$ Higgs model. But in the latter, the compact nature of the gauge field produces other phases which have no correspondence with the usual superconductor. Such differences would only be manifest if the $U(1)$ symmetry of the superconductor would be embedded in a non-Abelian symmetry group, like $SO(3)$ or $SU(2)$.

$e^2\eta$ (at large $e^2\eta$ the model only shows one phase, the normal conducting one) are characterized by a large amount of such vortices at small $|\psi|^2/\eta$, and by a small density of topological objects at large $|\psi|^2/\eta$, with the behaviour of the partition function in the last case being given mainly by the fluctuations without topological content. As it has been discussed in Appendix B on page 209 (see also [11]), the field variables of the Abrikosov model coincides with those defining each microscopic configuration (over which we are summing to obtain the partition function) only for the superconducting phase, which also in the Abrikosov model is characterized by the existence of topological vortices. Such a phase is described by a non-vanishing order parameter $\langle\psi\rangle$ in the Abrikosov model. In the normal conducting phase, also called the Coulomb phase, the microscopic field variables are still the vortices and their fluctuations, but its description in the Abrikosov model is, nevertheless, given by the disappearance of the topological vortices as isolated objects and the appearance of a long-spreading (effective) gauge field; moreover for the normal conducting phase $\langle\psi\rangle = 0$, although $|\psi|^2$ is still nonzero in the lattice model. Also in the latter phase, the strength of the coupling between the effective fields — resulting from the average over all the microscopic configurations living in the lattice — can differ from the coupling constant defined on the lattice. Summarizing:

Phase	Lattice (non-compact) Higgs model for $e^2\eta$ small	Abrikosov model
① supercond.	low vortex density	vortices as macroscopic objects
	large $ \psi ^2/\eta$	$\langle\psi\rangle \neq 0$
④ Coulomb	high vortex density	long-range gauge field
	small $ \psi ^2/\eta$	$\langle\psi\rangle = 0$

Although the equivalence between the phases of the lattice Higgs model and those of a superconductor has been argued basing on the behaviour of correlation function $H(\vec{r}_a^*, \vec{r}_b^*)$, it would be desirable to obtain an order parameter in the lattice formulation which, at large distances, would vanish in the normal conducting phase but would be nonzero in the superconducting one. Correlation function $W(C)$ is not suited for such an order parameter, since it always decays to zero for a large path C : at least so strong as the perimeter of C for the Coulomb phase (but always weaker as the area), and like the perimeter for the superconducting phase. Neither the correlation function complementary to $W(C)$, i.e., $H(\vec{r}_a^*, \vec{r}_b^*)$ and $H(C^*)$, satisfies the requisite⁷⁹. Let us try it with the following correlation function

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[iq \sum_{C_1 \rightarrow 2} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle, \quad (\text{D.239})$$

where $L_{\mu, \vec{r}}$ is a unit lattice vector tangent to a path from lattice site $\vec{r}_1$ to site $\vec{r}_2$: $\sum_{\mu} \Delta_{\mu} L_{\mu, \vec{r}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}$. Equation (D.239) is a slight modification of $W(C)$ ⁸⁰, and is

⁷⁹ $H(\vec{r}_a^*, \vec{r}_b^*)$ could be useful only for the 3-dim. Higgs model, since $H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{-|\vec{r}_b^* - \vec{r}_a^*|} \xrightarrow{|\vec{r}_b^* - \vec{r}_a^*| \rightarrow \infty} 0$ for phase ① and $H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{1/|\vec{r}_b^* - \vec{r}_a^*|} \xrightarrow{|\vec{r}_b^* - \vec{r}_a^*| \rightarrow \infty} 1$ for the normal conducting phase ④. However, for the 4-dimensional model, $H(C^*)$ decays again to zero in both phases.

⁸⁰ $W(C) = \left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[iq \sum_{C_1 \rightarrow 2} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ when start and end point, $\vec{r}_1$ and $\vec{r}_2$, coincide.

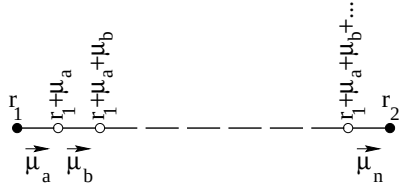
also invariant under a gauge transformation $\begin{cases} \varphi_{\vec{r}} \rightarrow \varphi_{\vec{r}}^{\text{gauged}} = \varphi_{\vec{r}} - q\chi_{\vec{r}} \\ A_{\mu,\vec{r}} \rightarrow A_{\mu,\vec{r}}^{\text{gauged}} = A_{\mu,\vec{r}} - (\chi_{\vec{r}+\hat{\mu}} - \chi_{\vec{r}}) \end{cases}$

$$(\varphi_{\vec{r}_1} - q\chi_{\vec{r}_1}) + q \left(\sum_{\vec{r},\mu} L_{\mu,\vec{r}} A_{\mu,\vec{r}} - \underbrace{\sum_{\vec{r},\mu} L_{\mu,\vec{r}} (\chi_{\vec{r}+\hat{\mu}} - \chi_{\vec{r}})}_{\text{boundary terms}} \right) - (\varphi_{\vec{r}_2} - q\chi_{\vec{r}_2}) = \varphi_{\vec{r}_1} + q \sum_{\vec{r},\mu} L_{\mu,\vec{r}} A_{\mu,\vec{r}} - \varphi_{\vec{r}_2} - \sum_{\vec{r}} \left(\sum_{\mu} \Delta_{\mu} L_{\mu,\vec{r}-\hat{\mu}} \right) \chi_{\vec{r}}$$

$$\left\langle e^{i\varphi_{\vec{r}_1}^{\text{gaug}} \exp \left[iq \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}}^{\text{gaug}} \right]} e^{-i\varphi_{\vec{r}_2}^{\text{gaug}}} \right\rangle = \left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[iq \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle. \quad (\text{D.240})$$

Note that the proof of the gauge invariance bases on relation $\sum_{\mu} \Delta_{\mu} L_{\mu,\vec{r}-\hat{\mu}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}$ only, independently whether $L_{\mu,\vec{r}}$ is an integer field or not. Since in the non-compact Higgs model paramater q plays no relevant role, it will be taken as $q = 1$ from here on.

However, even correlation function (D.239) does not produce the desired result either: at small $e^2\eta$, both for large $|\psi|^2/\eta$ and for small $|\psi|^2/\eta$, it shows a decay⁸¹ like distance $|\vec{r}_2 - \vec{r}_1|$. The proof of such a behaviour for small $|\psi|^2/\eta$ is easily obtained by noting that



$e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] e^{-i\varphi_{\vec{r}_2}}$ can be also written as the product of $e^{\pm i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - A_{\mu,\vec{r}})}$ for all bonds along path $C_{1 \rightarrow 2}$

$$\sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \equiv A_{\mu_a,\vec{r}_1} + A_{\mu_b,\vec{r}_1+\hat{\mu}_a} + A_{\mu_c,\vec{r}_1+\hat{\mu}_a+\hat{\mu}_b} + \dots + A_{\mu_n,\vec{r}_1+\hat{\mu}_a+\hat{\mu}_b+\dots},$$

$$e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \equiv e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_1+\hat{\mu}_a} - A_{\mu_a,\vec{r}_1})} e^{i(\varphi_{\vec{r}_1+\hat{\mu}_a} - \varphi_{\vec{r}_1+\hat{\mu}_a+\hat{\mu}_b} - A_{\mu_b,\vec{r}_1+\hat{\mu}_a})} \times$$

$$\times \dots \times e^{i(\varphi_{\vec{r}_1+\hat{\mu}_a+\hat{\mu}_b+\dots} - \varphi_{\vec{r}_2} - A_{\mu_n,\vec{r}_2-\hat{\mu}_n})}.$$

Therefore, at small $|\psi|^2/\eta$ and for the non-compact Higgs model, where only phase $\varphi_{\vec{r}}$ has angular character, the expansion of $\exp \left[-\frac{a^{d-2}|\psi|^2}{2\eta} \sum_{\vec{r},\mu} [1 - \cos(\Delta_{\mu}\varphi_{\vec{r}} - A_{\mu,\vec{r}})] \right]$ in bond-powers $\left(\frac{a^{d-2}|\psi|^2}{2\eta} \right) e^{\pm i(\varphi_{\vec{r}+\hat{\mu}} - \varphi_{\vec{r}} - A_{\mu,\vec{r}})}$, together with relation $\int_{-\pi}^{\pi} \frac{d\varphi_{\vec{r}}}{2\pi} e^{i\varphi_{\vec{r}}} = 0$, leads to the result that the leading contribution to $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ arises

⁸¹Something which could already be expected since $W(C)$, correlation function which is a special case of $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu,\vec{r}} A_{\mu,\vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ for $\vec{r}_1 = \vec{r}_2$, displays a perimeter decay as weakest decay.

from that expansion's term where only each bond on $C_{1 \rightarrow 2}$ is covered by a bond-power:

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle \approx \left(\frac{a|\psi|^2}{2\eta} \right)^{|\vec{r}_2 - \vec{r}_1|}, \text{ or equivalently,}$$

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-compact}}^{\text{small } e^2, \text{ small } |\psi|^2} \approx \exp \left[-|\vec{r}_2 - \vec{r}_1| \ln \left(\frac{2\eta}{a^{d-2}|\psi|^2} \right) \right]. \quad (\text{D.241})$$

In order to study the range of large $|\psi|^2/\eta$ and small $e^2\eta$, we shall use general result (D.262) for the 3-dimensional non-compact Higgs model

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle = \quad (\text{D.242})$$

$$= \exp \left[-\frac{\eta m^2}{2a|\psi|^2} \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} L_{\mu, \vec{r}'} - \frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'}^{(m^2)} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) \right] \times$$

$$\times \frac{\sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[i 2\pi \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{\nu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} - \frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}{\prod_{\vec{r}, \lambda} \sum_{M_{\lambda, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}$$

proved in the next Note (see page 656). In the above equation, $m^2 = a^2 e^2 |\psi|^2$ and $D_{\vec{r}, \vec{r}'}^{(m^2)}$ is the short-range propagator satisfying $-\sum_{\mu} \Delta_{(\vec{r})}^{\mu} \Delta_{(\vec{r}')}^{\mu} D_{\vec{r} - \hat{\mu}, \vec{r}'}^{(m^2)} + m^2 D_{\vec{r}, \vec{r}'}^{(m^2)} = \delta_{\vec{r}, \vec{r}'}$. For large

$|\psi|^2/\eta$, $D_{\vec{r}, \vec{r}'}^{(m^2)}$ can be approximated by $\frac{1}{m^2} \delta_{\vec{r}, \vec{r}'}$; further, small $e^2\eta$ suppresses most of topological field $M_{\lambda, \vec{r}}$ because of term $\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \approx \frac{2\pi^2}{a e^2 \eta} \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2$.

Hence the correlation under study is approximately given by

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{C_{1 \rightarrow 2}} L_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-compact}}^{\text{small } e^2, \text{ large } |\psi|^2} \approx \text{const.} \times \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 \right]. \quad (\text{D.243})$$

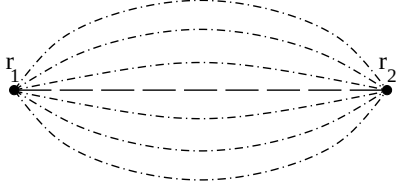
And $\sum_{\vec{r}, \mu} L_{\mu, \vec{r}}^2 = |\vec{r}_2 - \vec{r}_1|$, showing thus the expected decay also for large $|\psi|^2/\eta$. The result for the 4-dimensional model is the same but with $a|\psi|^2$ substituted by $a^2|\psi|^2$.

The correlation function we are looking for is similar to (D.239)

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle, \quad (\text{D.244})$$

but where $\mathcal{L}_{\mu, \vec{r}}^{\text{smear}}$ is no longer an integer field different from zero only on the lattice bonds along a path $C_{1 \rightarrow 2}$ but a field defined on any lattice bond through equation [57]

$$\mathcal{L}_{\mu, \vec{r}}^{\text{smear}} = -\Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_1} + \Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_2}, \quad (\text{D.245})$$



where $D_{\vec{r},\vec{r}'}$ is the propagator of the Laplace operator $\left(-\frac{\Delta}{(\vec{r})}\frac{\Delta}{(\vec{r}')}\right)D_{\vec{r},\vec{r}'} \equiv -\sum_{\rho}\frac{\Delta}{(\vec{r})}\rho\frac{\Delta}{(\vec{r}')}\rho D_{\vec{r}-\hat{\rho},\vec{r}'} = \sum_{\rho}\frac{\Delta}{(\vec{r})}\rho\frac{\Delta}{(\vec{r}')}\rho D_{\vec{r},\vec{r}'} = \delta_{\vec{r},\vec{r}'}$. Field $\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}}$ is schematically represented as a path from $\vec{r}_1$ to $\vec{r}_2$ “smeared” over the whole lattice. It satisfies

$$\sum_{\mu}\Delta_{\mu}\mathcal{L}_{\mu,\vec{r}-\hat{\mu}}^{\text{smeared}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}. \quad (\text{D.246})$$

This last equation shows that $\left\langle e^{i\varphi_{\vec{r}_1}} \exp\left[i\sum_{\vec{r},\mu}\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}}A_{\mu,\vec{r}}\right] e^{-i\varphi_{\vec{r}_2}}\right\rangle$ is gauge invariant, just as in result (D.240). For the 3-dimensional non-compact Higgs model, such a correlation function is given by result (D.262)

$$\begin{aligned} & \exp\left[-\frac{\eta m^2}{2a|\psi|^2}\sum_{\vec{r},\vec{r}',\mu}\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}}D_{\vec{r},\vec{r}'}^{(m^2)}\mathcal{L}_{\mu,\vec{r}'}^{\text{smeared}} - \frac{\eta}{2a|\psi|^2}\sum_{\vec{r},\vec{r}'}(\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2})D_{\vec{r},\vec{r}'}^{(m^2)}(\delta_{\vec{r}',\vec{r}_1} - \delta_{\vec{r}',\vec{r}_2})\right] \times \\ & \times \frac{\sum_{\{M_{\lambda,\vec{r}}\}}\prod_{\vec{r}}\delta\left(\sum_{\lambda}\Delta_{\lambda}M_{\lambda,\vec{r}},0\right)\exp\left[-\frac{2\pi^2a|\psi|^2}{\eta}\sum_{\vec{r},\vec{r}',\lambda}M_{\lambda,\vec{r}}D_{\vec{r},\vec{r}'}^{(m^2)}M_{\lambda,\vec{r}'}\right]}{\prod_{\vec{r},\lambda}\sum_{M_{\lambda,\vec{r}}=-\infty}^{\infty}\prod_{\vec{r}}\delta\left(\sum_{\lambda}\Delta_{\lambda}M_{\lambda,\vec{r}},0\right)} \times \\ & \times \exp\left[i2\pi\sum_{\vec{r},\vec{r}',\lambda}\sum_{\nu,\mu}\epsilon_{\lambda\nu\mu}\left(\Delta_{\nu}L_{\mu,\vec{r}} + m^2\sum_{\vec{y}}D_{\vec{r},\vec{y}}\Delta_{(\vec{y})\nu}\left(L_{\mu,\vec{y}} - \mathcal{L}_{\mu,\vec{y}}^{\text{smeared}}\right)\right)D_{\vec{r},\vec{r}'}^{(m^2)}M_{\lambda,\vec{r}'}\right] \\ & \times \exp\left[-\frac{2\pi^2a|\psi|^2}{\eta}\sum_{\vec{r},\vec{r}'}\sum_{\lambda}M_{\lambda,\vec{r}}D_{\vec{r},\vec{r}'}^{(m^2)}M_{\lambda,\vec{r}'}\right], \quad (\text{D.247}) \end{aligned}$$

where $L_{\mu,\vec{r}}$ is the usual integer field fulfilling $\sum_{\mu}\Delta_{\mu}L_{\mu,\vec{r}-\hat{\mu}} = \delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2}$. This equation can be further simplified into ⁸²

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp\left[i\sum_{\vec{r},\mu}\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}}A_{\mu,\vec{r}}\right] e^{-i\varphi_{\vec{r}_2}}\right\rangle =$$

$$\begin{aligned} & \overset{82}{\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}} = -\frac{\Delta}{(\vec{r})}\mu D_{\vec{r},\vec{r}_1} + \frac{\Delta}{(\vec{r})}\mu D_{\vec{r},\vec{r}_2}} \text{ is the formulation in discrete differential form of the electrostatic field created by} \\ & \text{two static charges of opposite sign placed at } \vec{r}_1 \text{ and } \vec{r}_2, \text{ and thus its curl } \sum_{\lambda,\nu,\mu}\epsilon_{\lambda\nu\mu}\Delta_{\nu}\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}} \text{ always vanishes. Fur-} \\ & \text{ther, } \sum_{\vec{r},\vec{r}',\mu}\mathcal{L}_{\mu,\vec{r}}^{\text{smeared}}D_{\vec{r},\vec{r}'}^{(m^2)}\mathcal{L}_{\mu,\vec{r}'}^{\text{smeared}} = \underbrace{\sum_{\vec{r},\vec{r}'}\left(-\frac{\Delta}{(\vec{r})}\mu D_{\vec{r},\vec{r}_1}\right)D_{\vec{r},\vec{r}'}^{(m^2)}\left(-\frac{\Delta}{(\vec{r}')}\mu D_{\vec{r}',\vec{r}_1} + \frac{\Delta}{(\vec{r}')}\mu D_{\vec{r}',\vec{r}_2}\right)}_{\sum_{\vec{r},\vec{r}'}D_{\vec{r},\vec{r}_1}D_{\vec{r},\vec{r}'}^{(m^2)}\left[\left(-\sum_{\mu}\frac{\Delta}{(\vec{r})}\mu\frac{\Delta}{(\vec{r}')}\mu D_{\vec{r}'-\hat{\mu},\vec{r}_1}\right) - \delta_{\vec{r}',\vec{r}_2}\right]} + \underbrace{\sum_{\vec{r}}\left(-D_{\vec{r},\vec{r}_2}D_{\vec{r},\vec{r}_1}^{(m^2)} + D_{\vec{r},\vec{r}_2}D_{\vec{r},\vec{r}_2}^{(m^2)}\right)}_{m^{-2}D_{\vec{r}_1,\vec{r}_2} - m^{-2}D_{\vec{r}_2,\vec{r}_2}} \\ & = \frac{1}{m^2}\sum_{\vec{r},\vec{r}'}(\delta_{\vec{r},\vec{r}_1} - \delta_{\vec{r},\vec{r}_2})\left(D_{\vec{r},\vec{r}'} - D_{\vec{r},\vec{r}'}^{(m^2)}\right)(\delta_{\vec{r}',\vec{r}_1} - \delta_{\vec{r}',\vec{r}_2}), \text{ after using relation } D_{\vec{r},\vec{r}'} - D_{\vec{r},\vec{r}'}^{(m^2)} = m^2\sum_{\vec{y}}D_{\vec{r},\vec{y}}D_{\vec{y},\vec{r}'}^{(m^2)}. \text{ And} \\ & \sum_{\vec{r}',\vec{y}}m^2\left(\sum_{\vec{r}}D_{\vec{r},\vec{y}}\Delta_{(\vec{y})\nu}\mathcal{L}_{\mu,\vec{y}}^{\text{smeared}}D_{\vec{r},\vec{r}'}^{(m^2)}\right)M_{\lambda,\vec{r}'} = \sum_{\vec{r}',\vec{y}}M_{\lambda,\vec{r}'}\left(D_{\vec{r}',\vec{y}} - D_{\vec{r}',\vec{y}}^{(m^2)}\right)\left(\frac{\Delta}{(\vec{y})}\nu\mathcal{L}_{\mu,\vec{y}}^{\text{smeared}}\right). \end{aligned}$$

$$= \exp \left[-\frac{\eta}{2a|\psi|^2} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) \right] \times \quad (\text{D.248})$$

$$\frac{\sum_{\{M_{\lambda, \vec{r}}\}} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[i 2\pi \sum_{\vec{r}, \vec{r}', \lambda} \left(\sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{\nu} L_{\mu, \vec{r}} \right) D_{\vec{r}, \vec{r}'} M_{\lambda, \vec{r}'} - \frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}{\prod_{\vec{r}, \lambda} \sum_{M_{\lambda, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}.$$

It is noteworthy that result (D.248) is very similar to correlation function $\langle e^{i\varphi_{\vec{r}_1}} e^{-i\varphi_{\vec{r}_2}} \rangle$ of the 3-dimensional XY model: the only difference is the term describing the interaction between the topological fields, interaction which in the case of the XY model is carried by long-range propagator $D_{\vec{r}, \vec{r}'}$, and here by short-range propagator $D_{\vec{r}, \vec{r}'}^{(m^2)}$. Hence

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-comp. Higgs}} \leq \left\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \right\rangle_{XY}, \quad (\text{D.249})$$

where the inequality arises since the disordering effect of $M_{\lambda, \vec{r}}$ is more pronounced in the case of $\sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'}$ than for $\sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'} M_{\lambda, \vec{r}'}$ (which suppresses more

effectively the topological field): therefore $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ decays at least so strongly as the scalar-scalar correlation function of the XY model⁸³. A similar result holds for the 4-dimensional Higgs model.

For large $|\psi|^2/\eta$ and small $e^2\eta$, equation (D.248) can be approximated by its first factor, since term $\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \approx \frac{2\pi^2}{ae^2\eta} \sum_{\vec{r}, \lambda} M_{\lambda, \vec{r}}^2$ freezes $M_{\lambda, \vec{r}}$ to zero almost everywhere, and thus

$$\begin{aligned} \left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-compact}}^{\text{small } e^2, \text{ large } |\psi|^2} &\approx \exp \left[\frac{\eta}{a^{d-2} |\psi|^2} D_{\vec{r}_1, \vec{r}_2} \right] = \quad (\text{D.250}) \\ &= \begin{cases} \exp \left[\frac{\eta}{4\pi a |\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|} \right] & d = 3 \\ \exp \left[\frac{\eta}{4\pi a^2 |\psi|^2} \frac{1}{|\vec{r}_2 - \vec{r}_1|^2} \right] & d = 4 \end{cases} \xrightarrow{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \text{const.} \neq 0. \end{aligned}$$

⁸³Relation $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-comp. Higgs}} \leq \left\langle e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \right\rangle_{XY}$ is a special case of a more general result called the “diamagnetic inequality”, proved in D. Brydges, J. Fröhlich, E. Seiler, *Ann. Phys.* **121**, 227 (1979) by means of techniques somewhat different from those leading to (D.249).

On the other hand, for small $|\psi|^2/\eta$ the scalar-scalar correlation function of the XY model already decays exponentially like $|\vec{r}_2 - \vec{r}_1|$, and inequality (D.249) leads therefore to

$$\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}}^{\text{smear}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-compact}}^{\text{any } e^2, \text{ small } |\psi|^2} \sim e^{-|\vec{r}_2 - \vec{r}_1|} \xrightarrow{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} 0. \quad (\text{D.251})$$

For the 3-dimensional model, the origin of the above decay could have been also obtained by the method presented in (C.231), based on the assumption that at $|\psi|^2/\eta$ small the $M_{\lambda, \vec{r}}$ -loops are large enough to self-cancel their mutual interactions; this holds even in a more justified way for equation (D.248) since $\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'}$ allows for loops even larger than those permitted by the XY -term $\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'} M_{\lambda, \vec{r}'}$.

From results (D.250) and (D.251), one arrives thus at an order parameter for the lattice Higgs model, which displays the same form as the order parameter of the Abrikosov model

$$\langle \psi \rangle^2 \equiv \lim_{|\vec{r}_2 - \vec{r}_1| \rightarrow \infty} \left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle_{\text{non-comp. Higgs}}, \quad (\text{D.252})$$

$$\text{with } \mathcal{L}_{\mu, \vec{r}} = -\Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_1} + \Delta_{(\vec{r})}^{\mu} D_{\vec{r}, \vec{r}_2},$$

vanishing at small $|\psi|^2/\eta$ (the normal conducting phase), but remaining nonzero in the range of large $|\psi|^2/\eta$ and small $e^2\eta$ (the superconducting state).

The general form taken by correlation function $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ for any $\mathcal{L}_{\mu, \vec{r}}$ will be obtained in the following Note for the 3-dimensional non-compact Higgs model.

Note. Calculation of $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ for the 3-dim. non-compact Higgs model.

Correlation function $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$, where field $\mathcal{L}_{\mu, \vec{r}}$ is not necessary an integer field but satisfies $\sum_{\mu} \Delta_{\mu} \mathcal{L}_{\mu, \vec{r} - \hat{\mu}} = (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})$, is calculated as follows: its numerator is given by

$$\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} - \frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} (\Delta_{\mu} \varphi_{\vec{r}} - A_{\mu, \vec{r}} - 2\pi m_{\mu, \vec{r}}) \right] \times$$

$$\times \exp \left[-\frac{1}{2ae^2\eta} \sum_{\vec{r}, \lambda, \nu} A_{\lambda, \vec{r}} \left(-\delta_{\lambda\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} A_{\nu, \vec{r} - \hat{\rho}} + \Delta_{\lambda} \Delta_{\nu} A_{\nu, \vec{r} - \hat{\lambda}} \right) \right] \text{ with integrations over } A_{\mu, \vec{r}}, \varphi_{\vec{r}} \text{ and } b_{\mu, \vec{r}}$$

(with an infinite range) and sum over integer field $m_{\mu, \vec{r}}$, which is the carrier of the complete rotations drawn by phase $\varphi_{\vec{r}}$; the denominator of the correlation under study, i.e., the partition function, is

equal to the just written equation but without term $e^{i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2})} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right]$. In order to carry out the integration of $\prod_{\vec{r}, \mu} dA_{\mu, \vec{r}}$, and because of the gauge invariance, one needs to fix a gauge choice by means of a term like $\left(\sqrt{\frac{1}{2\pi a \epsilon^2 \eta \alpha_1}} \right)^{|\Lambda|} \det(-\Delta \Delta) \exp \left[-\frac{1}{2a \epsilon^2 \eta \alpha_1} \sum_{\vec{r}, \mu} (\Delta_\mu A_{\mu, \vec{r}})^2 \right]$: the term quadratic in $A_{\mu, \vec{r}}$, and which defines the propagator of this gauge field, is equal to

$$-\frac{1}{2\pi a \epsilon^2 \eta} \sum_{\vec{r}} \sum_{\lambda, \nu} A_{\lambda, \vec{r}} \left[-\delta_{\lambda \nu} \sum_{\rho} \Delta_\rho \Delta_\rho A_{\nu, \vec{r} - \hat{\rho}} + \left(1 - \frac{1}{\alpha_1} \right) \Delta_\lambda \Delta_\nu A_{\nu, \vec{r} - \hat{\lambda}} \right].$$

According to results (C.524)-(C.528), the propagator of such a (discrete) differential operator is ⁸⁴

$$\delta_{\lambda \rho} D_{\vec{r}, \vec{r}'} + (1 - \alpha_1) \sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'}, \quad (\text{D.253})$$

where $D_{\vec{r}, \vec{r}'}$ is the propagator of the (discrete) Laplace operator: $\left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} \right) D_{\vec{r}, \vec{r}'} \equiv -\sum_{\rho} \frac{\Delta_\rho}{(\vec{r})} \frac{\Delta_\rho}{(\vec{r})} D_{\vec{r} - \hat{\rho}, \vec{r}'} = \sum_{\rho} \frac{\Delta_\rho}{(\vec{r})} \frac{\Delta_\rho}{(\vec{r}')} D_{\vec{r}, \vec{r}'} = \delta_{\vec{r}, \vec{r}'}$. Further the determinant of the differential operator acting on $A_{\mu, \vec{r}}$ is equal to $\frac{(\sqrt{\alpha_1})^{|\Lambda|}}{[\det(-\Delta \Delta)]^{3/2}}$. Integration of variables $\prod_{\vec{r}, \mu} dA_{\mu, \vec{r}}$ produces thus $\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{3|\Lambda|} \left(\sqrt{2\pi a \epsilon^2 \eta} \right)^{2|\Lambda|} \times \frac{1}{\sqrt{\det(-\Delta \Delta)}} \exp \left[-\frac{a \epsilon^2 \eta}{2} \sum_{\vec{r}, \vec{r}', \lambda, \rho} \left(b_{\lambda, \vec{r}} - \mathcal{L}_{\lambda, \vec{r}} \right) \left(\delta_{\lambda \rho} D_{\vec{r}, \vec{r}'} + (1 - \alpha_1) \sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'} \right) \left(b_{\rho, \vec{r}'} - \mathcal{L}_{\rho, \vec{r}'} \right) \right] \times \exp \left[i(\varphi_{\vec{r}_1} - \varphi_{\vec{r}_2}) + i \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} \Delta_\mu \varphi_{\vec{r}} - \frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} b_{\mu, \vec{r}}^2 - i 2\pi \sum_{\vec{r}, \mu} b_{\mu, \vec{r}} m_{\mu, \vec{r}} \right].$

Now the integration of $\prod_{\vec{r}} d\varphi_{\vec{r}}$ produces constraint $\sum_{\mu} \Delta_\mu b_{\mu, \vec{r} - \hat{\mu}} = (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2})$, whose general solution is $b_{\mu, \vec{r}} = \sum_{\nu, \lambda} \epsilon_{\mu \nu \lambda} \Delta_\nu a_{\lambda, \vec{r} - \hat{\nu}} + L_{\mu, \vec{r}}$, with $L_{\mu, \vec{r}}$ the, often met, integer field equal to ± 1 for lattice bonds on a path connecting $\vec{r}_1$ and $\vec{r}_2$ ⁸⁵. It is important to note that the constraint on $b_{\mu, \vec{r}}$, together with the equation which holds for $\mathcal{L}_{\mu, \vec{r}}$, leads to relation $\sum_{\mu} \Delta_\mu (b_{\mu, \vec{r} - \hat{\mu}} - \mathcal{L}_{\mu, \vec{r} - \hat{\mu}}) = 0$, and this for its part eliminates any dependence on parameter α_1 (coming from the gauge fixing factor) in the correlation function:

$$\frac{(1 - \alpha_1) \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \rho} \left(b_{\lambda, \vec{r}} - \mathcal{L}_{\lambda, \vec{r}} \right) \left(\sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'} \right) \left(b_{\rho, \vec{r}'} - \mathcal{L}_{\rho, \vec{r}'} \right)}{\delta_{\vec{r}, \vec{r}'}} =$$

$$\begin{aligned} & \text{⁸⁴Also shown directly: } \left[\delta_{\nu \lambda} \left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} \right) + \left(1 - \frac{1}{\alpha_1} \right) \frac{\Delta_\lambda}{(\vec{r})} \frac{\Delta_\nu}{(\vec{r})} \right] \left(\delta_{\lambda \rho} D_{\vec{r}, \vec{r}'} + (1 - \alpha_1) \sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'} \right) = \delta_{\nu \rho} \delta_{\vec{r}, \vec{r}'} + \\ & \left(1 - \frac{1}{\alpha_1} \right) \frac{\Delta_\rho}{(\vec{r})} \frac{\Delta_\nu}{(\vec{r})} D_{\vec{r}, \vec{r}'} + (1 - \alpha_1) \sum_{\vec{y}'} \underbrace{\left(-\frac{\Delta}{(\vec{r})} \frac{\Delta}{(\vec{r})} \right) D_{\vec{r}, \vec{y}'} \frac{\Delta_\nu}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'}}_{\delta_{\vec{r}, \vec{y}'}} - \frac{(1 - \alpha_1)^2}{\alpha_1} \sum_{\vec{y}'} \underbrace{\left(-\sum_{\lambda} \frac{\Delta_\lambda}{(\vec{r})} \frac{\Delta_\lambda}{(\vec{y}')} D_{\vec{r}, \vec{y}'} \right)}_{-\delta_{\vec{r}, \vec{y}'}} \left(+ \frac{\Delta_\nu}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'} \right) = \\ & = \delta_{\nu \rho} \delta_{\vec{r}, \vec{r}'} . \text{ To simplify the notation of the sites connected by the propagator, } \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}' - \hat{\lambda}, \vec{r}'} \text{ has been abbreviated by } \frac{\Delta_\lambda}{(\vec{y}')} \frac{\Delta_\rho}{(\vec{y}')} D_{\vec{y}', \vec{r}'} . \end{aligned}$$

⁸⁵One should not confuse $L_{\mu, \vec{r}}$ with $\mathcal{L}_{\mu, \vec{r}}$, where the latter, although it satisfies $\sum_{\mu} \Delta_\mu \mathcal{L}_{\mu, \vec{r} - \hat{\mu}} = \delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}$ too, need not to be an integer field.

$$\begin{aligned}
&= -(1 - \alpha_1) \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \rho} \left(b_{\lambda, \vec{r}} - \mathcal{L}_{\lambda, \vec{r}} \right) \underbrace{\left(\sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} \Delta_{(\vec{y}')}^{\lambda} \Delta_{(\vec{r}')}^{\rho} D_{\vec{y}' - \hat{\lambda}, \vec{r}' - \hat{\rho}} \right)}_{+ \sum_{\vec{y}'} \Delta_{(\vec{r})}^{\lambda} D_{\vec{r} - \hat{\lambda}, \vec{y}'} \Delta_{(\vec{r}')}^{\rho} D_{\vec{y}', \vec{r}' - \hat{\rho}}} \left(b_{\rho, \vec{r}'} - \mathcal{L}_{\rho, \vec{r}'} \right) = \\
&= -(1 - \alpha_1) \sum_{\vec{r}, \vec{r}'} \underbrace{\left[\sum_{\lambda} \Delta_{(\vec{r})}^{\lambda} \left(b_{\lambda, \vec{r}} - \mathcal{L}_{\lambda, \vec{r}} \right) \right]}_0 \sum_{\vec{y}'} D_{\vec{r}, \vec{y}'} D_{\vec{y}', \vec{r}'} \underbrace{\left[\sum_{\rho} \Delta_{(\vec{r}')}^{\rho} \left(b_{\rho, \vec{r}'} - \mathcal{L}_{\rho, \vec{r}'} \right) \right]}_0 = 0. \quad (\text{D.254})
\end{aligned}$$

On the other hand, term $-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} \left(b_{\lambda, \vec{r}} - \mathcal{L}_{\lambda, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(b_{\lambda, \vec{r}'} - \mathcal{L}_{\lambda, \vec{r}'} \right)$ can be rewritten as follows

$$\begin{aligned}
&-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} a_{\lambda, \vec{r}} \underbrace{\left(\sum_{\nu} \Delta_{(\vec{r})}^{\nu} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}, \vec{r}'} \right)}_{\delta_{\vec{r}, \vec{r}'}} a_{\lambda, \vec{r}'} + \frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}'} \left(\sum_{\lambda} \Delta_{(\vec{r})}^{\lambda} a_{\lambda, \vec{r}} \right) D_{\vec{r}, \vec{r}'} \left(\sum_{\nu} \Delta_{(\vec{r}')}^{\nu} a_{\nu, \vec{r}'} \right) - \\
&-\frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'} - ae^2\eta \sum_{\vec{r}, \vec{r}', \lambda} a_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'} \left(\sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{(\vec{r}')}^{\nu} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'} \right)).
\end{aligned}$$

For the change of the remaining integration variable from $b_{\mu, \vec{r}} = \sum_{\nu, \lambda} \epsilon_{\mu\nu\lambda} \Delta_{\nu} a_{\lambda, \vec{r} - \hat{\nu}} + L_{\mu, \vec{r}}$ to $a_{\lambda, \vec{r}}$, and since infinite many $a_{\lambda, \vec{r}}$'s produces the same $b_{\mu, \vec{r}}$, one needs once again a fixing factor in order to remove this repetition. This is achieved by the inclusion of a factor like $\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2 \alpha_2}} \right)^{|\Lambda|} \det(-\Delta\Delta) \times$
 $\times \exp \left[-\frac{\eta}{2a |\psi|^2 \alpha_2} \sum_{\vec{r}, \lambda} (\Delta_{\lambda} a_{\lambda, \vec{r}})^2 \right]$ in the correlation function, which becomes

$$\begin{aligned}
&\exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} (L_{\mu, \vec{r}})^2 - \frac{ae^2\eta}{2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'} \right] \left(\dots \right) \int_{-\infty}^{\infty} \prod_{\vec{r}, \lambda} \frac{da_{\lambda, \vec{r}}}{2\pi} \sum_{\{m_{\mu, \vec{r}}\}} \times \\
&\frac{\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{4|\Lambda|} \left(\sqrt{2\pi ae^2\eta} \right)^{2|\Lambda|} \frac{\sqrt{\det(-\Delta\Delta)}}{(\sqrt{\alpha_2})^{|\Lambda|}} \int_{-\infty}^{\infty} \prod_{\vec{r}, \lambda} \frac{da_{\lambda, \vec{r}}}{2\pi} \prod_{\vec{r}, \mu} \sum_{m_{\mu, \vec{r}} = -\infty}^{\infty} \exp \left[-i2\pi \sum_{\vec{r}, \mu} L_{\mu, \vec{r}} m_{\mu, \vec{r}} \right] \times}{1} \\
&\times \exp \left[\sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}} \left(-i2\pi \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu, \vec{r}} - \frac{\eta}{a |\psi|^2} \left(\sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}, \nu, \mu} D_{\vec{r}, \vec{y}} \epsilon_{\lambda\nu\mu} \Delta_{(\vec{y})}^{\nu} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) \right] \times \\
&\frac{\times \exp \left[-i2\pi \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}} \underbrace{\left(\sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \Delta_{\nu} m_{\mu, \vec{r}} \right)}_{M_{\lambda, \vec{r}}} \right] \times}{M_{\lambda, \vec{r}}} \\
&\times \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \nu < \lambda} (\Delta_{\nu} a_{\lambda, \vec{r}} - \Delta_{\lambda} a_{\nu, \vec{r}})^2 - \frac{\eta}{2a |\psi|^2 \alpha_2} \sum_{\vec{r}, \lambda} (\Delta_{\lambda} a_{\lambda, \vec{r}})^2 - \frac{\eta}{2a |\psi|^2} (a^2 e^2 |\psi|^2) \sum_{\vec{r}, \lambda} a_{\lambda, \vec{r}}^2 \right] \times \\
&\times \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}} \sum_{\lambda, \nu} a_{\lambda, \vec{r}} \left(-\delta_{\lambda\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} a_{\nu, \vec{r} - \hat{\rho}} + m^2 \delta_{\lambda\nu} a_{\nu, \vec{r}} + \left(1 - \frac{1}{\alpha_2} \right) \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r} - \hat{\lambda}} \right) \right] \times
\end{aligned}$$

$$\frac{\times \exp \left[-\frac{\eta}{a|\psi|^2} (-m^2) \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \nu} a_{\lambda, \vec{r}} \left(\sum_{\vec{r}'} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}-\hat{\lambda}, \vec{r}'-\hat{\nu}} \right) a_{\nu, \vec{r}'} \right]}{\times \exp \left[-\frac{\eta}{a|\psi|^2} (+m^2) \sum_{\vec{r}, \vec{r}'} \sum_{\lambda, \nu} a_{\lambda, \vec{r}} \left(\sum_{\vec{r}'} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}-\hat{\lambda}, \vec{r}'} \right) a_{\nu, \vec{r}'} \right]}, \quad (\text{D.255})$$

with $m^2 = a^2 e^2 |\psi|^2$. As already found in many occasions, integer field $M_{\lambda, \vec{r}} \equiv \sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{\nu} m_{\mu, \vec{r}}$ measures the vorticity (or the dislocation density) at site $\vec{r}$. In order to change the sum over $m_{\mu, \vec{r}}$ by a sum over $M_{\lambda, \vec{r}}$, a constraint on $m_{\mu, \vec{r}}$ must be fixed which eliminates overcounting of the same configuration; such a constraint has had to be imposed from the very beginning when the integration range of variable $\varphi_{\vec{r}}$ was extended in the Villain approximation from $[-\pi, \pi)$ to $(-\infty, \infty)$. Term $\exp \left[-i2\pi \sum_{\vec{r}, \mu} L_{\mu, \vec{r}} m_{\mu, \vec{r}} \right]$ has been substituted by 1 since both $m_{\mu, \vec{r}}$ and $L_{\mu, \vec{r}}$ are integer fields.

The integration over the remaining continuous variable $a_{\lambda, \vec{r}}$ is a bit more difficult: the propagator of the differential operator contained in the terms quadratic in $a_{\lambda, \vec{r}}$

$$a_{\lambda, \vec{r}} \left(-\delta_{\lambda \nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} a_{\nu, \vec{r}-\hat{\rho}} + m^2 \delta_{\lambda \nu} a_{\nu, \vec{r}} + \left(1 - \frac{1}{\alpha_2} \right) \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r}-\hat{\lambda}} + m^2 \sum_{\vec{r}'} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}-\hat{\lambda}, \vec{r}'} a_{\nu, \vec{r}'} \right)$$

is more easily calculated by Fourier transforming this differential operator, whose form acting on Fourier component $\tilde{a}_{\lambda}$ is

$$\left(6 - 2 \sum_{\rho} \cos k_{\rho} \right) \delta_{\lambda \nu} + m^2 \delta_{\lambda \nu} + \left(1 - \frac{1}{\alpha_2} + \frac{m^2}{6 - 2 \sum_{\rho} \cos k_{\rho}} \right) e^{-ik_{\lambda}} (e^{ik_{\lambda}} - 1) (e^{ik_{\nu}} - 1), \quad (\text{D.256})$$

where $\frac{1}{6 - 2 \sum_{\rho} \cos k_{\rho}}$ is the Fourier transform of propagator $D_{\vec{r}, \vec{r}'}$. Let us firstly evaluate the determinant of the matrix (D.256), with the abbreviations $\Sigma \equiv 6 - 2 \sum_{\rho} \cos k_{\rho} = \sum_{\rho} e^{-ik_{\rho}} (e^{ik_{\rho}} - 1)^2$ and

$$\left(\dots \right) \equiv 1 - \frac{1}{\alpha_2} + \frac{m^2}{6 - 2 \sum_{\rho} \cos k_{\rho}}$$

$$\begin{vmatrix} m^2 + \Sigma + \left(\dots \right) e^{-ik_x} (e^{ik_x} - 1)^2 & \left(\dots \right) e^{-ik_x} (e^{ik_x} - 1) (e^{ik_y} - 1) & \left(\dots \right) e^{-ik_x} (e^{ik_x} - 1) (e^{ik_z} - 1) \\ \left(\dots \right) e^{-ik_y} (e^{ik_y} - 1) (e^{ik_x} - 1) & m^2 + \Sigma + \left(\dots \right) e^{-ik_y} (e^{ik_y} - 1)^2 & \left(\dots \right) e^{-ik_y} (e^{ik_y} - 1) (e^{ik_z} - 1) \\ \left(\dots \right) e^{-ik_z} (e^{ik_z} - 1) (e^{ik_x} - 1) & \left(\dots \right) e^{-ik_z} (e^{ik_z} - 1) (e^{ik_y} - 1) & m^2 + \Sigma + \left(\dots \right) e^{-ik_z} (e^{ik_z} - 1)^2 \end{vmatrix}$$

which is equal to

$$\frac{1}{\alpha_2} \left(m^2 + 6 - 2 \sum_{\rho} \cos k_{\rho} \right)^2 \left(6 - 2 \sum_{\rho} \cos k_{\rho} \right). \quad (\text{D.257})$$

Hence, the determinant of the differential operator in

$$a_{\lambda, \vec{r}} \left(-\delta_{\lambda\nu} \sum_{\rho} \Delta_{\rho} \Delta_{\rho} a_{\nu, \vec{r}-\hat{\rho}} + m^2 \delta_{\lambda\nu} a_{\nu, \vec{r}} + \left(1 - \frac{1}{\alpha_2}\right) \Delta_{\lambda} \Delta_{\nu} a_{\nu, \vec{r}-\hat{\lambda}} + m^2 \sum_{\vec{r}'} \Delta_{(\vec{r})}^{\lambda} \Delta_{(\vec{r}')}^{\nu} D_{\vec{r}-\hat{\lambda}, \vec{r}'} a_{\nu, \vec{r}'} \right)$$

is given by $\prod_{\vec{k}} \left[\frac{1}{\alpha_2} \left(m^2 + 6 - 2 \sum_{\rho} \cos k_{\rho} \right)^2 \left(6 - 2 \sum_{\rho} \cos k_{\rho} \right) \right]$, or equivalently

$$\left(\frac{1}{\alpha_2} \right)^{|\Lambda|} \left[\det (-\Delta\Delta + m^2) \right]^2 \left[\det (-\Delta\Delta) \right]. \quad (\text{D.258})$$

The propagator is obtained by calculating the inverse of the matrix, whose determinant we have just obtained: for example, the inverse of the (1, 1) and (1, 2) elements are respectively

$$\frac{\Sigma + \left(1 - \alpha_2 - \frac{\alpha_2 m^2}{\Sigma}\right) e^{-ik_x} (e^{ik_x} - 1)^2}{(\Sigma + m^2) \Sigma} \quad \text{and} \quad \frac{\left(1 - \alpha_2 - \frac{\alpha_2 m^2}{\Sigma}\right) (e^{ik_x} - 1) e^{-ik_y} (e^{ik_y} - 1)}{(\Sigma + m^2) \Sigma},$$

and the propagator of the discrete differential equation under study takes the form

$$\delta_{\lambda\rho} D_{\vec{r}, \vec{r}}^{(m^2)} + (1 - \alpha_2) \sum_{\vec{y}'} D_{\vec{r}, \vec{y}'}^{(m^2)} \Delta_{(\vec{y}')}^{\lambda} \Delta_{(\vec{y}')}^{\rho} D_{\vec{y}'-\hat{\lambda}, \vec{r}'} - \alpha_2 m^2 \sum_{\vec{y}, \vec{y}'} D_{\vec{r}, \vec{y}}^{(m^2)} D_{\vec{y}, \vec{y}'} \Delta_{(\vec{y}')}^{\lambda} \Delta_{(\vec{y}')}^{\rho} D_{\vec{y}'-\hat{\lambda}, \vec{r}'} . \quad (\text{D.259})$$

The integration of $\prod_{\vec{r}, \lambda} da_{\lambda, \vec{r}}$ is now immediate and produces as multiplicative factor the follow-

ing one, $\left(\sqrt{\frac{2\pi a |\psi|^2}{\eta}} \right)^{3|\Lambda|} \frac{(\sqrt{\alpha_2})^{|\Lambda|}}{\det (-\Delta\Delta + m^2) \sqrt{\det (-\Delta\Delta)}}$, which multiplies in (D.255) the already existing factor $\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{4|\Lambda|} \left(\sqrt{2\pi a e^2 \eta} \right)^{2|\Lambda|} \frac{\sqrt{\det (-\Delta\Delta)}}{(\sqrt{\alpha_2})^{|\Lambda|}}$. The numerator of correlation function $\left\langle e^{i\varphi \tau_1} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi \tau_2} \right\rangle$ becomes finally equal to the exponential of the following equation (multiplied by $-2\pi^2 a |\psi|^2 / \eta$)

$$\begin{aligned} & \sum_{\vec{r}, \vec{r}', \lambda} \left(M_{\lambda, \vec{r}} - \frac{i\eta}{2\pi a |\psi|^2} \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^{\nu} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \times \\ & \times \left(M_{\lambda, \vec{r}'} - \frac{i\eta}{2\pi a |\psi|^2} \sum_{\nu', \mu'} \epsilon_{\lambda\nu'\mu'} \left(\Delta_{(\vec{r}')}^{\nu'} L_{\mu', \vec{r}'} + m^2 \sum_{\vec{y}'} D_{\vec{r}', \vec{y}'} \Delta_{(\vec{y}')}^{\nu'} (L_{\mu', \vec{y}'} - \mathcal{L}_{\mu', \vec{y}'}) \right) \right) - \\ & -(1 - \alpha_2) \sum_{\vec{r}, \vec{r}', \lambda, \rho} \left(M_{\lambda, \vec{r}} - \frac{i\eta}{2\pi a |\psi|^2} \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^{\nu} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) \times \\ & \times \sum_{\vec{z}'} D_{\vec{r}, \vec{z}'}^{(m^2)} \Delta_{(\vec{z}')}^{\lambda} \Delta_{(\vec{r}')}^{\rho} D_{\vec{z}'-\hat{\lambda}, \vec{r}'-\hat{\rho}} \left(\right)_{\rho, \vec{r}'} + \\ & + \alpha_2 m^2 \sum_{\vec{r}, \vec{r}', \lambda, \rho} \left(M_{\lambda, \vec{r}} - \frac{i\eta}{2\pi a |\psi|^2} \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \left(\Delta_{(\vec{r})}^{\nu} L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^{\nu} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) \times \end{aligned} \quad (\text{D.260})$$

$$\times \sum_{\vec{z}, \vec{z}'} D_{\vec{r}, \vec{z}}^{(m^2)} D_{\vec{z}, \vec{z}'} \Delta_{(\vec{z}')}^\lambda \Delta_{(\vec{r}')}^\rho D_{\vec{z}' - \hat{\lambda}, \vec{r}' - \hat{\rho}} \left(\begin{array}{c} \\ \rho, \vec{r}' \end{array} \right),$$

with sum $\prod_{\vec{r}, \lambda} \sum_{M_{\lambda, \vec{r}} = -\infty}^{\infty}$. All dependence on parameter α_2 disappears after operating, just as in the case of the previous gauge choice (see (D.254)): summation by parts produces the relations

$$\begin{aligned} \sum_{\vec{r}, \vec{r}'} \left[\begin{array}{c} \\ \lambda, \vec{r} \end{array} \right] \underbrace{\sum_{\vec{z}, \vec{z}'} D_{\vec{r}, \vec{z}}^{(m^2)} \Delta_{(\vec{z}')}^\lambda \Delta_{(\vec{r}')}^\rho D_{\vec{z}' - \hat{\lambda}, \vec{r}' - \hat{\rho}}}_{\rho, \vec{r}'} \left[\begin{array}{c} \\ \rho, \vec{r}' \end{array} \right] &= + \sum_{\vec{r}, \vec{r}'} \left(\Delta_{(\vec{r})}^\lambda \left[\begin{array}{c} \\ \lambda, \vec{r} \end{array} \right] \right) \sum_{\vec{z}, \vec{z}'} D_{\vec{r}, \vec{z}}^{(m^2)} D_{\vec{z}, \vec{z}'} \Delta_{(\vec{r}')}^\rho \left(\left[\begin{array}{c} \\ \rho, \vec{r}' \end{array} \right] \right), \\ &+ \sum_{\vec{z}'} \left(\Delta_{(\vec{r})}^\lambda D_{\vec{r} - \hat{\lambda}, \vec{z}'}^{(m^2)} \right) \Delta_{(\vec{r}')}^\rho D_{\vec{z}', \vec{r}' - \hat{\rho}} \end{aligned}$$

$$\begin{aligned} \sum_{\vec{r}, \vec{r}'} \left[\begin{array}{c} \\ \lambda, \vec{r} \end{array} \right] \underbrace{\sum_{\vec{z}, \vec{z}'} D_{\vec{r}, \vec{z}}^{(m^2)} D_{\vec{z}, \vec{z}'} \Delta_{(\vec{z}')}^\lambda \Delta_{(\vec{r}')}^\rho D_{\vec{z}' - \hat{\lambda}, \vec{r}' - \hat{\rho}}}_{\rho, \vec{r}'} \left[\begin{array}{c} \\ \rho, \vec{r}' \end{array} \right] &= + \sum_{\vec{r}, \vec{r}'} \left(\Delta_{(\vec{r})}^\lambda \left[\begin{array}{c} \\ \lambda, \vec{r} \end{array} \right] \right) \sum_{\vec{z}, \vec{z}'} D_{\vec{r}, \vec{z}}^{(m^2)} D_{\vec{z}, \vec{z}'} D_{\vec{z}', \vec{r}'} \Delta_{(\vec{r}')}^\rho \left(\left[\begin{array}{c} \\ \rho, \vec{r}' \end{array} \right] \right), \\ &+ \sum_{\vec{z}, \vec{z}'} \left(\Delta_{(\vec{r})}^\lambda D_{\vec{r} - \hat{\lambda}, \vec{z}}^{(m^2)} \right) D_{\vec{z}, \vec{z}'} \Delta_{(\vec{r}')}^\rho D_{\vec{z}', \vec{r}' - \hat{\rho}} \end{aligned}$$

which substituted into (D.259) leads to the result

$$\sum_{\vec{r}, \vec{r}', \lambda} \left(M_{\lambda, \vec{r}} - \frac{i\eta}{2\pi a |\psi|^2} \sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \left(\Delta_{(\vec{r})}^\nu L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^\nu (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\begin{array}{c} \\ \lambda, \vec{r}' \end{array} \right) + 0,$$

since $\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}} = 0$ and $\sum_{\lambda, \nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{\lambda} \Delta_{\nu} L_{\mu, \vec{r}} = 0 = \sum_{\lambda, \nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{\lambda} \Delta_{\nu} \mathcal{L}_{\mu, \vec{r}}$: all terms containing the parameter of the gauge fixing factor have been eliminated. This result proves thus the invariance of the correlation function under the choice of different gauge choices.

As last question in this Note, let us obtain the final form of the correlation function. In order to achieve it, we need some previous relations: one relating long-range and short-range propagators $D_{\vec{r}, \vec{y}'} - D_{\vec{r}, \vec{y}'}^{(m^2)} = m^2 \sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{y}'}$, easily shown by Fourier transforming the relation, $\frac{1}{6 - 2 \sum_{\rho} \cos k_{\rho}} -$

$-\frac{1}{m^2 + 6 - 2 \sum_{\rho} \cos k_{\rho}} = \frac{m^2}{(6 - 2 \sum_{\rho} \cos k_{\rho})(m^2 + 6 - 2 \sum_{\rho} \cos k_{\rho})}$; and the second set of relations are

of the kind $\sum_{\vec{r}, \vec{r}', \lambda} \left(\sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{(\vec{r})}^\nu L_{\mu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\sum_{\nu', \mu'} \epsilon_{\lambda \nu' \mu'} \Delta_{(\vec{r}')}^{\nu'} L_{\mu', \vec{r}'} \right) = \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} \left(\sum_{\nu} \Delta_{(\vec{r})}^\nu \Delta_{(\vec{r}')}^\nu D_{\vec{r}, \vec{r}'}^{(m^2)} \right) L_{\mu, \vec{r}'} -$
 $-\sum_{\vec{r}, \vec{r}'} \underbrace{\left(\sum_{\mu} \Delta_{(\vec{r})}^\mu L_{\mu, \vec{r} - \hat{\mu}} \right)}_{\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}} D_{\vec{r}, \vec{r}'}^{(m^2)} \underbrace{\left(\sum_{\nu} \Delta_{(\vec{r}')}^\nu L_{\nu, \vec{r}' - \hat{\nu}} \right)}_{\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}},$ and similar for those terms involving $\mathcal{L}_{\mu, \vec{r}}$ ⁸⁶. Hence

$$-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} (L_{\mu, \vec{r}})^2 - \frac{ae^2 \eta}{2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) -$$

⁸⁶These other relations are given by $\sum_{\vec{r}, \vec{r}', \lambda} \left(\sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \Delta_{(\vec{r})}^\nu L_{\mu, \vec{r}} \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\sum_{\vec{y}', \nu', \mu'} \epsilon_{\lambda \nu' \mu'} D_{\vec{r}, \vec{y}'} \Delta_{(\vec{y}')}^{\nu'} (L_{\mu', \vec{y}'} - \mathcal{L}_{\mu', \vec{y}'}) \right) =$

$$\begin{aligned}
& -\frac{2\pi^2 a |\psi|^2}{\eta} \left(\frac{i\eta}{2\pi a |\psi|^2} \right)^2 \sum_{\vec{r}, \vec{r}', \lambda} \left(\sum_{\nu, \mu} \epsilon_{\lambda \nu \mu} \left(\Delta_{(\vec{r})}^\nu L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^\nu (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \Bigg|_{\lambda, \vec{r}'} = \\
& = -\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \mu} (L_{\mu, \vec{r}})^2 - \frac{ae^2 \eta}{2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) + \\
& + \frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} \underbrace{\left(\sum_{\nu} \Delta_{(\vec{r})}^\nu \Delta_{(\vec{r}')}^\nu D_{\vec{r}, \vec{r}'}^{(m^2)} \right)}_{\delta_{\vec{r}, \vec{r}'} - m^2 D_{\vec{r}, \vec{r}'}^{(m^2)}} L_{\mu, \vec{r}'} - \frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'}^{(m^2)} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) + \\
& + \frac{2m^2 \eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) + \frac{m^2 \eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) - \\
& - \frac{m^2 \eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'}^{(m^2)} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) = \\
& = -\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'}^{(m^2)} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) - \frac{ae^2 \eta}{2} \sum_{\vec{r}, \vec{r}', \mu} \mathcal{L}_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \mathcal{L}_{\mu, \vec{r}'} . \quad (D.261)
\end{aligned}$$

And the correlation function is equal to

$$\begin{aligned}
& \left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle = \exp \left[-\frac{\eta}{2a |\psi|^2} \sum_{\vec{r}, \vec{r}'} (\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}) D_{\vec{r}, \vec{r}'}^{(m^2)} (\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}) \right] \times \\
& \times \frac{\exp \left[-\frac{\eta}{2a |\psi|^2} m^2 \sum_{\vec{r}, \vec{r}', \mu} \mathcal{L}_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} \mathcal{L}_{\mu, \vec{r}'} \right] \left(\prod_{\{M_{\lambda, \vec{r}}\}} \sum_{\vec{r}} \prod_{\lambda} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \right) \times}{\left(\sqrt{\frac{\eta}{2\pi a |\psi|^2}} \right)^{|\Lambda|} \frac{\left(\sqrt{2\pi a e^2 \eta} \right)^{2|\Lambda|}}{\sqrt{(-\Delta \Delta + m^2)}} \prod_{\vec{r}, \lambda} \sum_{M_{\lambda, \vec{r}} = -\infty}^{\infty} \prod_{\vec{r}} \delta \left(\sum_{\lambda} \Delta_{\lambda} M_{\lambda, \vec{r}}, 0 \right) \times} \quad (D.262) \\
& = \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} \underbrace{\left(\sum_{\nu} \Delta_{(\vec{r})}^\nu \Delta_{(\vec{r}')}^\nu D_{\vec{r}, \vec{r}'}^{(m^2)} \right)}_{\delta_{\vec{r}, \vec{r}'} - m^2 D_{\vec{r}, \vec{r}'}^{(m^2)}} \sum_{\vec{y}'} D_{\vec{r}', \vec{y}'} (L_{\mu, \vec{y}'} - \mathcal{L}_{\mu, \vec{y}'}) - \sum_{\vec{r}, \vec{r}'} \underbrace{\left(\sum_{\mu} \Delta_{(\vec{r})}^\mu L_{\mu, \vec{r}} - \hat{\mu} \right)}_{\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}} D_{\vec{r}, \vec{r}'}^{(m^2)} \underbrace{\left(\sum_{\vec{y}', \nu} \Delta_{(\vec{r}')}^\nu D_{\vec{r}', \vec{y}'} - \hat{\nu} \right)}_{\delta_{\vec{r}', \vec{r}_1} - \delta_{\vec{r}', \vec{r}_2}} (L_{\mu, \vec{y}'} - \mathcal{L}_{\mu, \vec{y}'}) = \\
& = \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) - \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} \underbrace{\sum_{\vec{r}'} m^2 D_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{y}'} (L_{\mu, \vec{y}'} - \mathcal{L}_{\mu, \vec{y}'})}_{D_{\vec{r}, \vec{y}'} - D_{\vec{r}, \vec{y}'}^{(m^2)}} + 0 = \sum_{\vec{r}, \vec{r}', \mu} L_{\mu, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}). \text{ And} \\
& \sum_{\vec{r}, \vec{r}', \lambda} \left(\sum_{\vec{y}, \nu, \mu} \epsilon_{\lambda \nu \mu} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^\nu (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) D_{\vec{r}, \vec{r}'}^{(m^2)} \left(\sum_{\vec{y}', \nu', \mu'} \epsilon_{\lambda \nu' \mu'} D_{\vec{r}', \vec{y}'} \Delta_{(\vec{y}')}^{\nu'} (L_{\mu', \vec{y}'} - \mathcal{L}_{\mu', \vec{y}'}) \right) = \sum_{\vec{r}, \mu} \sum_{\vec{y}, \vec{y}'} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \times \\
& \times \underbrace{\left(\sum_{\nu} \Delta_{(\vec{y})}^\nu \Delta_{(\vec{r})}^\nu D_{\vec{r}, \vec{y}} \right)}_{\delta_{\vec{y}, \vec{r}}} \underbrace{\sum_{\vec{r}'} D_{\vec{r}, \vec{r}'}^{(m^2)} D_{\vec{r}', \vec{y}'} (L_{\mu, \vec{y}'} - \mathcal{L}_{\mu, \vec{y}'})}_{m^{-2} D_{\vec{r}, \vec{y}'} - m^{-2} D_{\vec{r}, \vec{y}'}^{(m^2)}} - \sum_{\vec{r}, \vec{r}'} \underbrace{\left(\sum_{\vec{y}, \mu} \Delta_{(\vec{y})}^\mu D_{\vec{r} - \hat{\mu}, \vec{y}} (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right)}_{\delta_{\vec{r}, \vec{r}_1} - \delta_{\vec{r}, \vec{r}_2}} D_{\vec{r}, \vec{r}'}^{(m^2)} \sum_{\vec{y}', \nu} D_{\vec{r}', \vec{y}'} \times 0 = \\
& + \sum_{\vec{y}, \mu} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^\mu (L_{\mu, \vec{y} - \hat{\mu}} - \mathcal{L}_{\mu, \vec{y} - \hat{\mu}}) = 0 \\
& = \frac{1}{m^2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}) - \frac{1}{m^2} \sum_{\vec{r}, \vec{r}', \mu} (L_{\mu, \vec{r}} - \mathcal{L}_{\mu, \vec{r}}) D_{\vec{r}, \vec{r}'}^{(m^2)} (L_{\mu, \vec{r}'} - \mathcal{L}_{\mu, \vec{r}'}).
\end{aligned}$$

$$\frac{\exp \left[i2\pi \sum_{\vec{r}, \vec{r}', \lambda} \sum_{\nu, \mu} \epsilon_{\lambda\nu\mu} \left(\Delta_\nu L_{\mu, \vec{r}} + m^2 \sum_{\vec{y}} D_{\vec{r}, \vec{y}} \Delta_{(\vec{y})}^\nu (L_{\mu, \vec{y}} - \mathcal{L}_{\mu, \vec{y}}) \right) D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} - \frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}', \lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}{\times \exp \left[-\frac{2\pi^2 a |\psi|^2}{\eta} \sum_{\vec{r}, \vec{r}'} \sum_{\lambda} M_{\lambda, \vec{r}} D_{\vec{r}, \vec{r}'}^{(m^2)} M_{\lambda, \vec{r}'} \right]}$$

with $m^2 = a^2 e^2 |\psi|^2$ the square of the inverse of the gauge field's penetration length.

Remark. Although the introduction of $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ has the goal of having a correlation function for the non-compact model with a nonvanishing value in the state characterized by the existence of isolated topologically stable vortices, and with zero value for the state without vortices, such a correlation function can be also calculated for a compact-gauge model. Such a calculation will be not carried here: for those phases ① and ④ where the compact nature of the gauge field is not relevant, the behaviour $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$ is the same as in the non-compact model. Further, and though not obtained for that phase ③ distinctive of the angular character in the gauge field, the expected behaviour of the discussed correlation function is easily obtained for this phase: decay towards zero for large distances. The reason: for the correlation between two gauge fields at arbitrary sites, what actually measures the gauge-invariant correlation function of the type here studied (or $W(C)$ too), the existence of a much larger set of allowed gauge transformations, which — without any energy cost since the energy is invariant under such transformations — can deform gauge field configurations into each other, makes possible a more independently assignment in the gauge field's value, makes possible the removing of certain amounts of vorticity and it is just this topologically stable vorticity that characteristic described by a nonvanishing value of $\left\langle e^{i\varphi_{\vec{r}_1}} \exp \left[i \sum_{\vec{r}, \mu} \mathcal{L}_{\mu, \vec{r}} A_{\mu, \vec{r}} \right] e^{-i\varphi_{\vec{r}_2}} \right\rangle$. And in a model with a compact gauge field, where even certain “jumps” are allowed because of the angular nature of the gauge field, such a correlation function is therefore much weaker and thus the decay with the distance in this gauge-invariant correlation function for a certain value of the model's parameters is stronger than that displayed in the non-compact model for the same parameters. If in a certain region of the phase diagram for the *non-compact* model, the correlation function under study here already decays towards zero for large distances $|\vec{r}_2 - \vec{r}_1|$, then a similar behaviour is to be expected in the compact model also.

D.5.2 The effective potential $V_{\text{eff}}(\bar{\rho})$

In order to unify the description of the non-compact Higgs model between that in a discrete lattice and the Landau-Ginzburg model describing superconductivity, we have considered an order parameter defined for the lattice model which shows the expected behaviour of the order parameter in the (continuum) Landau-Ginzburg theory of superconductivity. This last theory, also called the Abrikosov model, results from the coupling of the order parameter of the Landau-Ginzburg model to an effective gauge field. The effect of the temperature has been already studied for the lattice model and now the picture of the non-compact Higgs model will be completed by studying at high temperatures the effective macroscopic model represented by the Abrikosov model: as in the

lattice model, this will be achieved by letting that one of the space dimensions becomes periodic. It must be stressed that we are not aiming at a deduction of the temperature effects on the Abrikosov model by starting at a more basic quantum-field microscopic model: the task here is to begin with an already given effective model and to study the effect of periodicity in one of the space variables on which the fields describing the model depend. More specifically, we shall use the formulation by means of the effective model to calculate the value of lattice's periodicity ω at the transition point between the two phases displayed by the model in the range of small e^2 .

It should be noted that although both models, the lattice non-compact Higgs model and the Abrikosov model, present the same phases, this does not imply that the ranges of the respective model's parameters where the different phases occurs, should coincide. In fact, a naive identification between order parameter $\langle\psi\rangle^2$ in the Abrikosov model and $|\psi|^2$ in the lattice mode would lead to the contradiction that the normal conducting phase is characterized in the first model by a vanishing $\langle\psi\rangle^2$ whilst the same phase occurs in the second model for a small but non-zero $|\psi|^2$. Let us review how the Abrikosov model describes the superconductor, where we shall assume that the penetration depth of the effective gauge field is larger than the correlation length of effective scalar field $\bar{\psi}(\vec{r}) = \langle\psi\rangle e^{i\bar{\varphi}(\vec{r})}$: this means that in the Abrikosov model, the superconducting phase is characterized by the existence of isolated flux lines, just as in the lattice model. One starts at an effective energy given by

$$E[\bar{\psi}(\vec{r}), \bar{A}_\mu(\vec{r})] = \int d^d\vec{r} \left[\frac{1}{2} \sum_{\mu, \nu < \mu} (\partial_\mu \bar{A}_\nu(\vec{r}) - \partial_\nu \bar{A}_\mu(\vec{r}))^2 + \right. \\ \left. + \frac{1}{2} \sum_\mu |\partial_\mu \bar{\psi}(\vec{r}) - ie \bar{A}_\mu \bar{\psi}|^2 + V(\bar{\psi}) \right], \quad (\text{D.263})$$

$$\text{with potential } V(\bar{\psi}) = V(|\bar{\psi}|) = -|\alpha||\bar{\psi}|^2 + \frac{|\beta|}{2}|\bar{\psi}|^4,$$

and looks for solutions which minimizes it; as a matter of notation, in the continuum model coupling strength e is usually not included into the gauge field. Potential's energy

part has a minimum for $|\bar{\psi}| \equiv \langle\psi\rangle = \sqrt{\frac{|\alpha|}{|\beta|}} \neq 0$; note that as long as we work within the

framework of the Landau-Ginzburg model coupled to a gauge field, we are identifying scalar field length's $|\bar{\psi}|$ with expectation value $\langle\psi\rangle$. As next step, one looks for field solu-

tions of type $\bar{\psi}(\vec{r}) \xrightarrow{|\vec{r}| \rightarrow \infty} \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\theta}$, being θ the polar angle in a real 2-dimensional plane

⁸⁷. Such solutions, called the flux lines or vortices, require a gauge field $\bar{A}_{\mu, \vec{r}}$ characterized by an integer nonvanishing circulation $\oint_{\text{path at } |\vec{r}|=\infty} \bar{A}_\mu(\vec{r}) d\ell^\mu \neq 0$. As different from the lattice

⁸⁷Solutions in scalar field $\bar{\psi}(\vec{r}) = \langle\psi\rangle e^{i\bar{\varphi}(\vec{r})}$ which are topologically trivial, such as $\bar{\psi} = \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\bar{\varphi}_0}$ with a constant phase $\bar{\varphi}_0$ over the whole space, are of lesser interest here since then the coupling to a gauge field would play no role. Potential $V(\bar{\psi}) = V(|\bar{\psi}|)$ only fixes a constant $|\bar{\psi}| \equiv \langle\psi\rangle$, letting free the scalar field's phase: thus, there exists a rich set of configurations for which, when a closed path is covered at spatial infinity, the scalar's phase change in a multiple of 2π , i.e., $\bar{\varphi}$ has angular character. These

Higgs model, coupling constant e in the Abrikosov model is taken as a fixed parameter of small value, and the phase transition towards a normal conducting phase gets originated by the action of an external magnetic field; however, an increasing external magnetic can be also modeled if, for a given and fixed external field, the coupling to the superconductor is increased, coupling which takes place in covariant derivative $\partial_\mu \bar{\psi}(\vec{r}) - ie\bar{A}_\mu \bar{\psi}$. As a final step in the discussion using the Abrikosov model, one calculates under the action of a given external field the Gibbs free energy by summing over a spatial distribution of vortices, letting distance d between these (and thus the vortex density) to be adjusted by the condition of minimum in the Gibbs energy. As long as the vortex density is not too high, and according to equation (A.44) $d \approx \lambda \ln \left(\frac{z 2\pi/e}{(H_{\text{ext}} - H_{c1})8\pi\lambda^2} \right)$, distance d between

vortices results to be approximately proportional to penetration length $\lambda = \frac{1}{\sqrt{|\bar{\psi}|^2 e^2}}$ of

the effective gauge field (the logarithmic term shows a much weaker dependence on λ). Analogously to the lattice (non-compact) Higgs model, the transition towards the normal conducting phase, characterized by an increase in the density of flux lines which eventually condense building long-range lines of magnetic field, can be modeled for a fixed external field H_{ext} and a fixed system's rigidity $|\bar{\psi}|^2$ by means of an increase in e^2 .

It is worth noting that the approximation used in the calculation of the Gibbs energy by assuming that the effective gauge field is given by the superposition of the

$$\text{gauge field of each isolated vortex, } \begin{cases} \vec{A}(x, y) \approx \frac{1}{e} \sum_{\text{positions } \vec{\rho}_a} \hat{\theta}_{\vec{\rho}_a} \left[\frac{1}{|\vec{\rho} - \vec{\rho}_a|} - \frac{1}{\lambda} K_1 \left(\frac{|\vec{\rho} - \vec{\rho}_a|}{\lambda} \right) \right] \\ \bar{\psi}(x, y) \approx \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\varphi(x, y)} \end{cases}$$

with scalar phase's satisfying $\partial_j \varphi = - \sum_k \epsilon_{jk} \sum_{\text{positions } \vec{\rho}_a} \frac{(\vec{\rho} - \vec{\rho}_a)_k}{|\vec{\rho} - \vec{\rho}_a|^2}$ and $|\vec{\rho}_a| = \sqrt{x_a^2 + y_a^2}$ and

$$\hat{\theta}_{\vec{\rho}_a} = -\hat{x} \frac{y - y_a}{|\vec{\rho} - \vec{\rho}_a|} + \hat{y} \frac{x - x_a}{|\vec{\rho} - \vec{\rho}_a|} \text{ the azimuthal angle around the } a\text{-vortex, breaks down}$$

configurations would be irrelevant if they would be topologically unstable, such that one configuration could be deformed (transformed) into a configuration of trivial topology. But as already discussed for the non-compact Higgs model, where the $U(1)$ gauge symmetry is not embedded into a larger symmetry, two configurations with different (integer) winding number $\frac{1}{2\pi} \oint_{\text{path at } |\vec{r}|=\infty} \partial_\mu \bar{\varphi}(\vec{r}) dl^\mu$ cannot be transformed

into each other; as a reminder, the same property does not apply to the compact model, since there the angular character also in the gauge field (angular character which arises from the fact that the $U(1)$ gauge symmetry is a subgroup of a larger compact group like $SO(3)$) allows for transformations which do connect configurations of different topological content. However, the topological stability of a configuration of nonzero winding number causes a nonvanishing $|\partial\bar{\psi}(\vec{r})|^2$ for a wide range of $\vec{r}$, and this for its part, originates an infinite energy for the configuration; such a divergence is actually that

of equation (1.7). The extending of pure scalar model, $E[\bar{\psi}(\vec{r})] = \int d^d \vec{r} \left[\frac{1}{2} \sum_\mu |\partial_\mu \bar{\psi}(\vec{r})|^2 + V(\bar{\psi}) \right]$, to

a gauge invariant one was required just to avoid such a divergence, achieved by imposing that covariant derivative at spatial infinity for such a configuration, $\partial_\mu \left(\sqrt{\frac{|\alpha|}{|\beta|}} e^{i\bar{\varphi}} \right) - ie\bar{A}_\mu \sqrt{\frac{|\alpha|}{|\beta|}} e^{i\bar{\varphi}}$, has to vanish. As

a final note, the angular character of scalar field's phase $\bar{\varphi}$ can be also formulated as $\nabla^2 \bar{\varphi} = 0 \pmod{2\pi}$, in the same way as in equation (C.132) of the lattice model.

near the transition point when distance d approaches a value near to the transversal size of the vortex (which of the order of scalar's correlation length ξ). As a consequence, $\bar{\psi}$ becomes small and $|\bar{\psi}| \approx 0$. In a more mechanical way, the transition from the superconducting to the normal conducting phase is described (within the framework of the Landau-Ginzburg model coupled to a gauge field), as the change from a constant nonvanishing $\langle\psi\rangle$, corresponding to the minimum of potential $V(\bar{\psi})$ with $|\bar{\psi}| \equiv \langle\psi\rangle$, towards a state characterized by $\langle\psi\rangle = 0$: such a transition is also called symmetry restoring⁸⁸.

If up to this point we have identified $|\bar{\psi}|$ with expectation value $\langle\psi\rangle$, which is equivalent to the identification of the lowest value of the energy with the free energy, we shall now go a step further in our discussion of the Abrikosov model by introducing the effect of fluctuations in the calculation of the Gibbs free energy. This is necessary if we want to describe in a more exact way the phase transition which occurs in the non-compact Higgs model, since it can occur that through a fluctuation one can connect a state which minimizes the model's energy with another state of (almost) the same low energy (and which was until that moment inaccessible), opening thus new paths through which the phase transition is enabled. Moreover, if we “zoom in” the system under a “magnifying glass” fluctuations can be responsible for new interaction forms among the model's fields, interactions which were not relevant at the original more “defocussed” pictures of the model, but which — through the information of how these interactions vary in strength when the magnifying power is modified — allows us to determine the behaviour of the correlation functions even at large distances.

By definition, fluctuations are perturbations about those solutions which minimizes the system's energy. Thus one starts with a model depending on one (or more) general field $\phi(\vec{r})$ and described by energy functional $E[\phi(\vec{r})]$ and for such a model one calculates

$$\text{correlation functions as } \langle\phi(\vec{r}_1) \cdots \phi(\vec{r}_n)\rangle \stackrel{\text{def}}{=} \frac{\sum_{\text{all config. } \phi(\vec{r})} \phi(\vec{r}_1) \cdots \phi(\vec{r}_n) \exp\left(-\frac{E[\phi(\vec{r})]}{\eta}\right)}{\sum_{\text{all config. } \phi(\vec{r})} \exp\left(-\frac{E[\phi(\vec{r})]}{\eta}\right)} =$$

$$= \eta^n \left(\frac{1}{\mathcal{Z}[J(\vec{r})]} \frac{\delta^n \mathcal{Z}[J(\vec{r})]}{\delta J(\vec{r}_1) \cdots \delta J(\vec{r}_n)} \right) \Big|_{J=0}, \text{ being the partition function given by } \mathcal{Z}[J(\vec{r})] =$$

$$= \sum_{\text{all config. } \phi(\vec{r})} \exp\left(-\frac{E[\phi(\vec{r})]}{\eta} + \frac{1}{\eta} \int d^d \vec{r} J(\vec{r}) \phi(\vec{r})\right). \text{ Although the main contribution to}$$

the partition comes from that configuration $\phi_{\text{cl}}(\vec{r})$ for which the energy is minimum (configuration usually called the saddle point), there are also other configurations whose

⁸⁸The reason for such a name: when $\langle\psi\rangle \neq 0$ one spoke of symmetry breaking (see Note on page 201). It is important to note, first of all, that we have been identifying scalar field's length $|\bar{\psi}|$ with expectation value $\langle\psi\rangle$ and thus symmetry breaking occurs if $\frac{dV(\bar{\psi})}{d|\bar{\psi}|} = 0$ for $|\bar{\psi}| \neq 0$. The generalization is to consider

also fluctuations about those configurations characterized by $|\bar{\psi}| \neq 0$, and then calculate $\langle\psi\rangle$ by means of defining a partition function. The second important point to note is that, after including fluctuations, the symmetry displayed by the Abrikosov model is a gauge (local) one and, as discussed in the Note on page 210, the expectation value of a non gauge-invariant expression, like $\langle\psi\rangle$, vanishes exactly if the gauge invariance is not removed by hand. But the latter is just what occurs in the Abrikosov model by imposing constraint $\partial_\mu \bar{A}_\mu = 0$ on the effective gauge field.

energy does not lie “too far” from that of $\phi_{\text{cl}}(\vec{r})$ and which also contribute to the partition function. The question is to know in which range of the model’s parameter, those fluctuations can change the description which would have been obtained by minimizing $E[\phi(\vec{r})] - \int d^d \vec{r} J(\vec{r}) \phi(\vec{r})$ directly. From the definition of the partition function, it is clear that in the limit $\eta \rightarrow 0$ the only contributing configuration is saddle point $\phi_{\text{cl}}(\vec{r})$, fact which enables us to make a bit more concrete the above “not too far”: general field variable $\phi(\vec{r})$ in the energy of every configuration will be descomposed as follows

$$\phi(\vec{r}) = \phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r}),$$

where $\gamma(\vec{r})$ denotes the fluctuation; $\left(\right)$ denotes some model’s small parameter such that $\sqrt{\eta} \left(\right)$ is dimensionless (this will be dealt with later in this discussion). After expanding $E[\phi(\vec{r})]$ in powers of $\gamma(\vec{r})$ and taking into account the definition of the saddle-point configuration, $\left. \frac{\delta E[\phi]}{\delta \phi(\vec{r})} \right|_{\phi=\phi_{\text{cl}}} = J(\vec{r})$

$$\begin{aligned} & \frac{E[\phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r})]}{\eta} - \frac{1}{\eta} \int d^d \vec{r} J(\vec{r}) (\phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r})) = \\ & = \frac{1}{\eta} \left(E[\phi_{\text{cl}}(\vec{r})] - \int d^d \vec{r} J(\vec{r}) \phi_{\text{cl}}(\vec{r}) \right) + 0 + \\ & + \frac{(\)^2}{2!} \int d^d \vec{z}_1 d^d \vec{z}_2 \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{z}_1) \delta \phi(\vec{z}_2)} \right|_{\phi=\phi_{\text{cl}}} \gamma(\vec{z}_1) \gamma(\vec{z}_2) + \frac{\eta^{1/2} (\)^3}{3!} \int d^d \vec{z}_1 d^d \vec{z}_2 d^d \vec{z}_3 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{z}_1) \delta \phi(\vec{z}_2) \delta \phi(\vec{z}_3)} \right|_{\phi=\phi_{\text{cl}}} \gamma(\vec{z}_1) \gamma(\vec{z}_2) \gamma(\vec{z}_3) + \dots \end{aligned}$$

the partition function and of the Gibbs free energy becomes a perturbative expansion in increasing higher η -powers: a larger η -power indicates a larger content in fluctuations⁸⁹. Anticipating the result of the long Note at the end of this Appendix (see equation (D.373)), the just discussed expansion for the Gibbs free energy takes the following form

$$\begin{aligned} G[\varphi(\vec{r})] = & \text{constant} + \left. E[\phi(\vec{r})] \right|_{\phi=\varphi} + \frac{\eta}{2} \text{Tr} \ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \delta \phi(\vec{r}_1)} \right|_{\phi=\varphi} + \\ & + \frac{\eta^2}{8} \int d^d \vec{z}_1 \dots d^d \vec{z}_4 \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{z}_1) \dots \delta \phi(\vec{z}_4)} \right|_{\phi=\varphi} D_{(\varphi)}(\vec{z}_1, \vec{z}_2) D_{(\varphi)}(\vec{z}_3, \vec{z}_4) - \\ & - \frac{\eta^2}{12} \int d^d \vec{z}_1 \dots d^d \vec{z}_3 d^d \vec{y}_1 \dots d^d \vec{y}_3 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{z}_1) \delta \phi(\vec{z}_2) \delta \phi(\vec{z}_3)} \right|_{\phi=\varphi} \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{y}_1) \delta \phi(\vec{y}_2) \delta \phi(\vec{y}_3)} \right|_{\phi=\varphi} \times \end{aligned}$$

⁸⁹The definition of $\gamma(\vec{r})$ with the square root of η , and not another different power of η , is only a convenience so that the bilinear term in $\gamma(\vec{r})$ of the above expansion contains no power of η , term which can be directly (Gaussian-)integrated in partition function $\mathcal{Z}[J(\vec{r})] = \sum_{\text{all config. } \gamma(\vec{r})} \exp \left[-\frac{E[\phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r})]}{\eta} + \frac{1}{\eta} \int d^d \vec{r} J(\vec{r}) (\phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r})) \right]$. The exponential of other terms containing higher powers of $\gamma(\vec{r})$ than the second, terms which also include some power of η , has to be expanded.

$$\times D_{(\varphi)}(\vec{z}_1, \vec{y}_1) D_{(\varphi)}(\vec{z}_2, \vec{y}_2) D_{(\varphi)}(\vec{z}_3, \vec{y}_3) + \dots, \quad (\text{D.264})$$

with $\varphi(\vec{r})$ the field variable on which Gibbs free energy $G[\varphi(\vec{r})] = -\eta \ln \mathcal{Z}[J(\vec{r})] \Big|_{J=J[\varphi]} + \int d^d \vec{r}' J(\vec{r}') \varphi(\vec{r}')$ depends⁹⁰: $\varphi(\vec{r}') = \eta \frac{\delta \ln \mathcal{Z}[J(\vec{r})]}{\delta J(\vec{r}')}.$ In the above relation, $D_{(\varphi)}(\vec{r}_1, \vec{r}_1')$ gets defined through $\int d^d \vec{r}_1' D_{(\varphi)}(\vec{r}_1, \vec{r}_1') \left(\frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1') \delta \phi(\vec{r}_2')} \Big|_{\phi=\varphi} \right) = \delta(\vec{r}_1, \vec{r}_2')$ and the constant term is equal to $-\eta \ln \left(\sqrt{2\pi\eta} \right) \int d^d \vec{r}.$ Before continuing, and for the sake of completeness, let us obtain the exact form of $\left(\right)$ in $\phi(\vec{r}) = \phi_{\text{cl}}(\vec{r}) + \sqrt{\eta} \left(\right) \gamma(\vec{r})$ for the effective energy of the Landau-Ginzburg theory coupled to a gauge field (D.263): such an energy will be provisionally formulated on a discrete d -dimensional lattice of spacing a in order to identify better the dimension of each parameter

$$\begin{aligned} \frac{E[\bar{\psi}(\vec{r}), \bar{A}_\mu(\vec{r})]}{\eta} &\rightarrow \sum_{\vec{r}} \left[\frac{a^{d-4}}{2e^2\eta} \sum_{\mu, \nu < \mu} \left(\Delta_\mu(ae\bar{A}_{\nu, \vec{r}}) - \Delta_\nu(ae\bar{A}_{\mu, \vec{r}}) \right)^2 + \right. \\ &\quad \left. + \frac{a^{d-2}}{2\eta} \sum_{\mu} \left| \Delta_\mu \bar{\psi}_{\vec{r}} - i(ae\bar{A}_{\mu, \vec{r}}) \bar{\psi}_{\vec{r}} \right|^2 - a^2 |\alpha| \left(\frac{a^{d-2} \bar{\psi}_{\vec{r}} \bar{\psi}_{\vec{r}}^*}{\eta} \right) + \frac{a^{4-d} |\beta| \eta}{2} \left(\frac{a^{d-2} \bar{\psi}_{\vec{r}} \bar{\psi}_{\vec{r}}^*}{\eta} \right)^2 \right], \end{aligned} \quad (\text{D.265})$$

resulting from the continuum formulation by means of relations $\begin{cases} \int d^d \vec{r}' \rightarrow a^d \sum_{\vec{r}'} \\ \partial_\mu \rightarrow a^{-1} \Delta_\mu \end{cases}.$

One can directly read off equation (D.265) that, beside $a^2 |\alpha|$, parameters $a^{4-d} |\beta| \eta$ and $a^{4-d} e^2 \eta$ are dimensionless: these two last parameters correspond, in the Abrikosov model, to $\eta(\cdot)^2$ for the above discussed expansion although they have not to be of the same magnitude⁹¹; they remain, nevertheless, always small. Hence we shall restrict ourselves to the range of small $|\beta|$ and small e^2 so that the perturbative expansion in η -powers holds. These ranges corresponds, according to relation $|\bar{\psi}|^2 \sim \frac{|\alpha|}{|\beta|}$, to strong coupling

strengths $\frac{\beta_\psi}{\eta} \equiv \frac{a^{d-2} |\psi|^2}{\eta}$, $\frac{\beta_e}{\eta} \equiv \frac{1}{a^{4-d} e^2 \eta}$ in the lattice Higgs model (one should not confuse the scalar and gauge coupling strengths in the lattice Higgs model with parameter $|\beta|$ of the potential in the Abrikosov model).

⁹⁰From result (D.364), Helmholtz free energy $-\eta \ln \mathcal{Z}[J(\vec{r})]$ is, up to a constant, equal to $\left(E[\phi_{\text{cl}}(\vec{r})] - \int d^d \vec{r}' J(\vec{r}') \phi_{\text{cl}}(\vec{r}') \right) + \frac{\eta}{2} \text{Tr} \ln \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1') \delta \phi(\vec{r}_1')} \Big|_{\phi=\phi_{\text{cl}}} + (\text{terms with higher powers of } \eta).$ Therefore $\varphi(\vec{r}') = \eta \frac{\delta \ln \mathcal{Z}[J(\vec{r})]}{\delta J(\vec{r}')}.$ is equal to $\phi_{\text{cl}}(\vec{r}') + \eta(\dots)$: only in the limit $\eta \rightarrow 0$, $\varphi(\vec{r})$ coincides with saddle-point configuration $\phi_{\text{cl}}(\vec{r})$, and energy $E[\phi_{\text{cl}}(\vec{r})]$ with Gibbs free energy $G[\varphi(\vec{r}) = \phi_{\text{cl}}(\vec{r})].$

⁹¹And therefore it can occur, for instance, that a contribution of second perturbative order in $a^{4-d} e^2 \eta$ is of similar magnitude as the first perturbative order in $a^{4-d} |\beta| \eta$ [27].

Let us return to the general energy $E[\phi(\vec{r})]$, depending on one or more field variables $\phi(\vec{r})$. As already mentioned, the characterization of the phases in the non-compact Higgs model will be done (within the framework of Abrikosov model) by means of the magnitude of the order parameter: the main aim of our discussion will be to calculate whether the following expectation value

$$\langle \phi(\vec{r}') \rangle = \eta \frac{1}{\mathcal{Z}[J(\vec{r})]} \frac{\delta \mathcal{Z}[J(\vec{r})]}{\delta J(\vec{r}')} \Big|_{J=0} \equiv \lim_{J \rightarrow 0} \varphi(\vec{r}'), \quad (\text{D.266})$$

is nonzero or not. From the definition of the Gibbs free energy, the $J(\vec{r}')$ related to a given $\varphi(\vec{r})$ is given by $J(\vec{r}') = \frac{\delta G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}')}$; thus expectation value $\langle \phi(\vec{r}') \rangle = \lim_{J \rightarrow 0} \varphi(\vec{r}')$ has to satisfy $\frac{\delta G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}')} \Big|_{\varphi(\vec{r})=\langle \phi(\vec{r}') \rangle} = 0$. We shall focus our attention on models which shows

translation invariance, and under such a premise $\langle \phi(\vec{r}') \rangle = \eta \frac{1}{\mathcal{Z}[J(\vec{r})]} \frac{\delta \mathcal{Z}[J(\vec{r})]}{\delta J(\vec{r}')} \Big|_{J=0} =$

$$= \frac{\sum_{\text{all config. } \phi(\vec{r})} \exp\left(-\frac{E[\phi(\vec{r})]}{\eta}\right) \phi(\vec{r}')}{\sum_{\text{all config. } \phi(\vec{r})} \exp\left(-\frac{E[\phi(\vec{r})]}{\eta}\right)}$$
 is actually a constant, independent of $\vec{r}$: $\langle \phi(\vec{r}') \rangle = \langle \phi \rangle$

([31] and [52]). It therefore suffices to consider $G[\varphi(\vec{r})]$ for $\varphi(\vec{r}) = \bar{\rho} = \text{constant}$. The question of interest has been reduced to whether equation $\frac{dG[\varphi(\vec{r}) = \bar{\rho}]}{d\bar{\rho}} = 0$ has as solution $\bar{\rho} \neq 0$ or not. Such a problem is best studied by means of the so-called effective potential $V_{\text{eff}}(\bar{\rho})$, defined through relation ⁹²

$$G[\varphi(\vec{r})] = \int d^d \vec{r} \left(V_{\text{eff}}(\varphi(\vec{r})) + \text{terms containing derivatives } \partial_\mu \varphi(\vec{r}) \right), \quad (\text{D.267})$$

where for $\varphi(\vec{r}) = \bar{\rho} = \text{constant}$ one can extract directly $V_{\text{eff}}(\bar{\rho})$

$$G[\varphi(\vec{r}) = \bar{\rho}] = V_{\text{eff}}(\bar{\rho})|\Lambda|, \quad (\text{D.268})$$

with $V_{\text{eff}}(\bar{\rho})$ no longer a functional but an ordinary function. And therefore one has to study if equation $\frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} = 0$ has as solution $\bar{\rho} \neq 0$. The (infinite) volume of the d -dimensional system, $|\Lambda| = \int d^d \vec{r}$, can be also written as $|\Lambda| = (2\pi)^d \delta^{(d)}(\vec{0})$ according to definition $\delta^{(d)}(\vec{k}) = \int_{-\infty}^{\infty} \frac{d^d \vec{r}}{(2\pi)^d} e^{i\vec{k} \cdot \vec{r}}$.

For the cases we shall later study, energy functional $E[\phi(\vec{r})]$ at which all this discussion has started always contains a part with the derivatives of field $\phi(\vec{r})$ plus a potential

⁹²For the reader knowledgeable in Quantum Field Theory, the effective potential in the latter is defined as $[\varphi(\vec{r})] = \int d^d \vec{r} \left(-V_{\text{eff}}(\varphi(\vec{r})) + \text{terms with } \partial_\mu \varphi(\vec{r}) \right)$, since energy $E[\phi(\vec{r})]$ of Statistical Physics plays the role of the action in Quantum Field Theory, and thus the potential energy part enters with a different sign in the latter.

part without any derivative and of the form $V(\phi(\vec{r})) = -|\alpha||\phi(\vec{r})|^2 + \frac{|\beta|}{2}|\phi(\vec{r})|^4$. According to result (D.264), the effective potential takes the following form

$$V_{\text{eff}}(\bar{\rho}) = \text{const.} + V(\bar{\rho}) + \frac{\eta}{2} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \text{Tr} \ln \frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r}_1)\delta\phi(\vec{r}_1)} \Big|_{\phi(\vec{r})=\bar{\rho}} + \mathcal{O}(\eta^2), \quad (\text{D.269})$$

with $\text{const.} = -\frac{\eta}{2} \ln(2\pi\eta)$. As expected, $V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho})$ in the limit $\eta \rightarrow 0$ where all fluctuations are suppressed; this limit is also called the Landau or the tree approximation. Since the point at which one starts to measure the free energy is irrelevant, the above constant term of the effective potential will be adjusted for convenience depending on the case, for instance, to remove an infinite constant term resulting from the evaluation of the trace.

A more comfortable (and mechanical) form of calculating the *first* perturbative order in $V_{\text{eff}}(\bar{\rho})$ runs as follows [31]: in energy functional $E[\phi(\vec{r})]$ one substitutes $\phi(\vec{r})$ by $\bar{\rho} + \chi(\vec{r})$, $E[\phi(\vec{r}) = \bar{\rho} + \chi(\vec{r})] = \dots + \frac{1}{2} \int d^d \vec{r}_1 d^d \vec{r}_2 \frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r}_1)\delta\phi(\vec{r}_2)} \Big|_{\phi(\vec{r})=\bar{\rho}} \chi(\vec{r}_1)\chi(\vec{r}_2) + \dots$, and

keeps only the term bilinear in $\chi(\vec{r})$. If in the energy defining the model there are several field variables $\phi_a(\vec{r})$ (as it is the case of the Abrikosov model, where there are scalar and gauge fields), we only need “shift” $\phi_a(\vec{r}) = \bar{\rho}_a + \chi_a(\vec{r})$ those fields in whose expectation value we are interested: in the Abrikosov model, these are only the scalar field; the rest of the fields remains unshifted $\phi_a(\vec{r}) = \chi_a(\vec{r})$. And then only the terms containing any two of the $\chi(\vec{r})$ ’s is kept, being the effective potential (up to order η) equal to

$$V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) - \eta \ln \left[\int [d\chi(\vec{r})] \exp \left(- \frac{E_{\text{bilin. in } \chi(\vec{r})} \left[\begin{matrix} (\phi_{\text{scal}})_a = \bar{\rho}_a + \chi_a(\vec{r}) \\ (\phi_{\text{gauge}})_\mu = \chi_\mu(\vec{r}) \end{matrix} \right]}{\eta} \right) \right], \quad (\text{D.270})$$

as one can easily prove by carrying out the functional Gaussian integration over $[d\chi(\vec{r})] \equiv \prod_{\vec{r}} \prod_{a,\mu} d\chi_{a,\mu}(\vec{r})$ ⁹³; this result is determined up to an additive constant which we can neglect by adjusting the origin of energy. Beside the expansion in η -powers of the Gibbs

⁹³The rest of the expansion’s terms in $V_{\text{eff}}(\bar{\rho})$, containing a second (or higher) power of η , as for instance term $\frac{\eta^2}{8} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \int d^d \vec{z}_1 \dots d^d \vec{z}_4 \frac{\delta^4 E[\phi(\vec{r})]}{\delta\phi(\vec{z}_1) \dots \delta\phi(\vec{z}_4)} \Big|_{\phi(\vec{r})=\bar{\rho}} D_{(\varphi)}(\vec{z}_1, \vec{z}_2) \Big|_{\phi(\vec{r})=\bar{\rho}} D_{(\varphi)}(\vec{z}_3, \vec{z}_4) \Big|_{\phi(\vec{r})=\bar{\rho}}$, can be graphically represented by diagrams as explained on page 735: functional derivatives $\frac{\delta^n E[\phi(\vec{r})]}{\delta\phi(\vec{z}_1) \dots \delta\phi(\vec{z}_n)} \Big|_{\phi(\vec{r})=\bar{\rho}}$ as vertices of n legs, and propagator $D_{(\varphi)}(\vec{z}, \vec{z}') \Big|_{\phi(\vec{r})=\bar{\rho}}$ as internal line connecting two legs of the vertices. For the previous example-term, such a dia-

free energy, the latter can be also expanded in powers of its field variable $\varphi(\vec{r})$ ⁹⁴

$$G[\varphi(\vec{r})] = \underbrace{G[\varphi(\vec{r})] \Big|_{\varphi(\vec{r})=0}}_{\text{constant}} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^d \vec{r}_1 \cdots d^d \vec{r}_n \Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | 0) \varphi(\vec{r}_1) \cdots \varphi(\vec{r}_n), \quad (\text{D.271})$$

$$\text{with } \Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | 0) = \frac{\delta^n G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1) \cdots \delta \varphi(\vec{r}_n)} \Big|_{\varphi=0}.$$

For the following it will be useful to work with the Fourier transforms

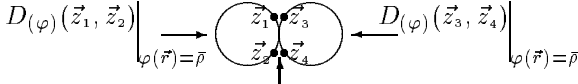
$$\begin{aligned} \tilde{\Gamma}^{(n)}(\vec{k}_1, \dots, \vec{k}_n | 0) (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \dots + \vec{k}_n) = \\ = \int d^d \vec{r}_1 \cdots d^d \vec{r}_n \Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | 0) e^{i(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 + \dots + \vec{k}_n \cdot \vec{r}_n)}. \end{aligned} \quad (\text{D.272})$$

Hence, by comparing equation (D.271) with definition $G[\varphi(\vec{r}) = \bar{\rho}] = V_{\text{eff}}(\bar{\rho}) (2\pi)^d \delta^{(d)}(\vec{0})$, one obtains

$$V_{\text{eff}}(\bar{\rho}) = \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} G[\varphi(\vec{r})] \Big|_{\varphi(\vec{r})=0} + \sum_{n=1}^{\infty} \frac{1}{n!} \bar{\rho}^n \tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0} | 0), \quad (\text{D.273})$$

where every Fourier transform $\tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0} | 0)$ is written as an expansion in powers of η . Thus one obtains immediately

$$\frac{d^n V_{\text{eff}}}{d\bar{\rho}^n} \Big|_{\bar{\rho}=0} = \tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0} | 0), \quad (\text{D.274})$$

gram looks as . As it will be proved in equation

$$\frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{z}_1) \cdots \delta \phi(\vec{z}_4)} \Big|_{\phi(\vec{r})=\bar{\rho}}$$

tion (D.373), it is a property of all diagrams which represents the terms entering in

$V_{\text{eff}}(\bar{\rho}) - V(\bar{\rho}) + \eta \ln \left[\int \left(\prod_{\vec{r}, a, \mu} d\chi_{a, \mu}(\vec{r}) \right) \exp \left(- \frac{E_{\text{bilin. in } \chi(\vec{r})} \left[\begin{smallmatrix} (\phi_{\text{scal}})_a = \bar{\rho}_a + \chi_a(\vec{r}) \\ (\phi_{\text{gaug}})_\mu = \chi_\mu(\vec{r}) \end{smallmatrix} \right]}{\eta} \right) \right]$ that each one cannot be separated into two isolated pieces by cutting only one of its internal lines.

⁹⁴Alternatively, one can write $G[\varphi(\vec{r})] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d \vec{r}_1 \cdots d^d \vec{r}_n \Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | 0) \varphi(\vec{r}_1) \cdots \varphi(\vec{r}_n)$, with

$\Gamma^{(n=0)} = G[\varphi(\vec{r})] \Big|_{\varphi=0} = -\frac{\eta}{2} \ln(2\pi\eta) (2\pi)^d \delta^{(d)}(\vec{0}) + \frac{\eta}{2} \text{Tr} \ln \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \delta \phi(\vec{r}_1)} \Big|_{\phi(\vec{r})=0} + \dots$, being $\Gamma^{(0)}$ independent of field variable $\varphi(\vec{r})$.

or if we are interested in $\left. \frac{\partial^n V_{\text{eff}}}{\partial \bar{\rho}^n} \right|_{\bar{\rho}=v}$, with v constant, one can expand (D.271) as $G[\varphi(\vec{r})] = \underbrace{G[\varphi(\vec{r})]}_{\text{constant}} \Big|_{\varphi(\vec{r})=v} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^d \vec{r}_1 \cdots d^d \vec{r}_n \Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | v) (\varphi(\vec{r}_1) - v) \cdots (\varphi(\vec{r}_n) - v)$, with

$$\Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n | v) = \frac{\delta^n G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1) \cdots \delta \varphi(\vec{r}_n)} \Big|_{\varphi=v}, \text{ obtaining thus}$$

$$\left. \frac{d^n V_{\text{eff}}}{d \bar{\rho}^n} \right|_{\bar{\rho}=v} = \tilde{\Gamma}^{(n)}(\vec{0}, \dots, \vec{0} | v). \quad (\text{D.275})$$

By using $\left. \frac{d^n V_{\text{eff}}}{d \bar{\rho}^n} \right|_{\bar{\rho}=v} = \left. \frac{d^n V_{\text{eff}}}{d \bar{\rho}^n} \right|_{\bar{\rho}=0} + \left. \frac{d^{n+1} V_{\text{eff}}}{d \bar{\rho}^{n+1}} \right|_{\bar{\rho}=0} v + \frac{1}{2!} \left. \frac{d^{n+2} V_{\text{eff}}}{d \bar{\rho}^{n+2}} \right|_{\bar{\rho}=0} v^2 + \dots$, it follows that term $\tilde{\Gamma}^{(n)}(\vec{0}, \dots, \vec{0} | v)$ is a polynomial in function of $\tilde{\Gamma}^{(m)}(\vec{0}, \dots, \vec{0} | 0)$ with $m \geq n$.

It has been already mentioned that each $\tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0} | v)$ owns an expansion in powers of η : for instance, and taking into account (D.381), $\tilde{\Gamma}^{(2)}(\vec{k}_1 = \vec{0}, \vec{k}_2 = \vec{0} | 0)$ up to order η^1 is given by

$$\begin{aligned} \tilde{\Gamma}^{(2)}(\vec{k}_1 = \vec{0}, \vec{k}_2 = \vec{0} | 0) &= \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}_1 = \vec{0}) \delta \tilde{\phi}(\vec{k}_2 = \vec{0})} \Big|_{\vec{q}=0} - \\ &- \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}_1 = \vec{0}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}=0} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}_2 = \vec{0}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{p})} \Big|_{\vec{q}=0} \tilde{D}(\vec{p} | 0) \tilde{D}(-\vec{p} | 0) + \\ &+ \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}=0} \tilde{D}(\vec{p} | 0), \end{aligned} \quad (\text{D.276})$$

with relations $\frac{1}{\tilde{D}(\vec{p} | 0)} = \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}=0}$ and $\frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \cdots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\vec{q}=0} (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) = \int d^d \vec{r}_1 \cdots d^d \vec{r}_n \frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \cdots \delta \phi(\vec{r}_n)} \Big|_{\phi=\varphi=0} e^{i(\vec{q}_1 \cdot \vec{r}_1 + \dots + \vec{q}_n \cdot \vec{r}_n)}$. The above equation, and in general any $\tilde{\Gamma}^{(n)}$, can be reminded in a graphical way as follows:

- Energy's functional derivative $\frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \cdots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\vec{q}=0}$ gets represented by a vertex
- with n “legs”, assigning to each leg a vector $\vec{q}_i$:
- $$\begin{aligned} \text{Diagram with } n \text{ legs} \Big|_{\vec{q}=0} &\equiv \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}=0}, \\ \text{Diagram with } 2 \text{ legs} \Big|_{\vec{q}=0} &\equiv \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}=0}. \end{aligned}$$

- Propagator $\tilde{D}(\vec{p}|0) = \left(\frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\tilde{\phi}=0} \right)^{-1}$ as a line $\bullet \xrightarrow{\vec{p}} \bullet \Big|_{\tilde{\phi}=0}$ which connects two legs of the vertices.
- Integrating over all different $\vec{p}$ -vectors of the internal lines. All external legs of $\tilde{\Gamma}^{(n)}(\vec{0}, \dots, \vec{0}|0)$ carry vector $\vec{k} = \vec{0}$.

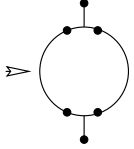
$$\tilde{\Gamma}^{(1)}(\vec{0}|0) = \frac{\delta E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0} + \frac{\eta}{2} \text{ (loop diagram) } \Big|_{\tilde{\phi}=0}, \quad (\text{D.277})$$

$$\tilde{\Gamma}^{(2)}(\vec{0}, \vec{0}|0) = \frac{1}{\tilde{D}(\vec{0}|0)} - \frac{\eta}{2} \text{ (bubble diagram) } \Big|_{\tilde{\phi}=0} + \frac{\eta}{2} \text{ (self-energy diagram) } \Big|_{\tilde{\phi}=0}, \quad (\text{D.278})$$

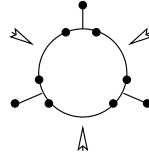
$$\begin{aligned} \tilde{\Gamma}^{(3)}(\vec{0}, \vec{0}, \vec{0}|0) &= \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0} - 3 \frac{\eta}{2} \text{ (triangle diagram) } \Big|_{\tilde{\phi}=0} + 2 \frac{\eta}{2} \text{ (circular diagram) } \Big|_{\tilde{\phi}=0} + \\ &+ \frac{\eta}{2} \text{ (self-energy diagram) } \Big|_{\tilde{\phi}=0}, \end{aligned} \quad (\text{D.279})$$

$$\begin{aligned} \tilde{\Gamma}^{(4)}(\vec{0}, \dots, \vec{0}|0) &= \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \dots \delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0} - 4 \frac{\eta}{2} \text{ (triangle diagram) } \Big|_{\tilde{\phi}=0} - 3 \frac{\eta}{2} \text{ (circular diagram) } \Big|_{\tilde{\phi}=0} + \\ &+ 12 \frac{\eta}{2} \text{ (circular diagram) } \Big|_{\tilde{\phi}=0} - 6 \frac{\eta}{2} \text{ (circular diagram) } \Big|_{\tilde{\phi}=0} + \frac{\eta}{2} \text{ (self-energy diagram) } \Big|_{\tilde{\phi}=0}, \end{aligned} \quad (\text{D.280})$$

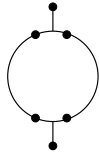
where the sign of each diagram is positive or negative depending whether the number of propagators (or of internal lines connecting the internal legs of the vertices) is, respectively, odd or even. The combinatoric factor multiplying each diagram contained in $\tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0}|0)$ is equal to the product of all combinations of the external legs, which is given by $n!$, divided by a factor arising from the permutations of the external legs attached to the same vertex and from all possible reflections and rotations of the diagram. For some diagrams, the combinatoric factor is easily calculated: for diagram



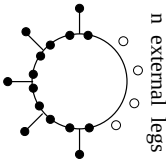
contained in $\tilde{\Gamma}^{(n=2)}$ there are 2 forms of attaching a third external leg



(denoted by the arrows), for $\tilde{\Gamma}^{(n=3)}$'s diagram



fore, if the combinatoric factor of is $1 \times \frac{\eta}{2}$, then the factor for $\tilde{\Gamma}^{(n)}$'s diagram

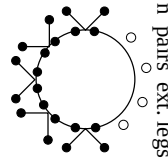


becomes $(n-1)! \times \frac{\eta}{2}$, factor which is equal to the all possible permuta-

tions of the n external legs divided by the n equivalent forms of inserting each leg:

$$\frac{n!}{n} \times \frac{\eta}{2}.$$

In the same way, the combinatoric factor for



, which is contained

in $\tilde{\Gamma}^{(2n)}$, is $\frac{(2n)!}{\underbrace{n! \cdot 2! \cdot \dots \cdot 2!}_{n \text{ times}}} \times \frac{\eta}{2}$, where the $2!$ -factors arise from the permutations of the

two external legs attached to the same vertex. The same representation by means of the above diagrams holds also for $\tilde{\Gamma}^{(n)}(\vec{k}_1 = \vec{0}, \dots, \vec{k}_n = \vec{0}|v)$ but with vertices and propagators taken not at $\varphi = 0$ (and $\tilde{\varphi} = 0$) but at $\varphi = v = \text{const.}$ (and $\tilde{\varphi}(\vec{q}) = v(2\pi)^d \delta^{(d)}(\vec{q})$). It is curious to note that all diagrams in equations (D.277)-(D.280) are formed by only an internal loop, loop number which coincides with the order in η : if we had written the next order in η , all diagrams multiplied by η^2 would have contained two loops, and so on.

Note. Calculation of the effective potential: an example.

For a model described by an energy functional $E[\phi(\vec{r})]$ whose highest non vanishing derivative is the 4th one (all models we shall study satisfy this property), let us calculate effective potential (D.269) up to order η : $V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) + \frac{\eta}{2} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \text{Tr} \ln \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r})} \Big|_{\phi(\vec{r})=\bar{\rho}}$. The “strategy” for obtaining

$\text{Tr} \ln \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r})} \Big|_{\phi(\vec{r})=\bar{\rho}}$ as function of the Fourier transforms of the functional derivat. $\frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_n)} \equiv$

$\equiv \int \frac{d^d \vec{k}_1}{(2\pi)^d} \cdots \frac{d^d \vec{k}_n}{(2\pi)^d} \frac{\delta^n E[\tilde{\phi}(\vec{k})]}{\delta \tilde{\phi}(\vec{k}_1) \cdots \delta \tilde{\phi}(\vec{k}_n)} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \cdots + \vec{k}_n) e^{-i(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 + \cdots + \vec{k}_n \cdot \vec{r}_n)}$ is the following: one expands the logarithm in powers of constant value $\bar{\rho}$, followed by writting the expansion in terms of Fourier transforms $\left. \frac{\delta^n E[\tilde{\phi}(\vec{k})]}{\delta \tilde{\phi}(\vec{k}_1) \cdots \delta \tilde{\phi}(\vec{k}_n)} \right|_{\vec{\phi}=0}$; the next step is to take the trace, $\int \frac{d^d \vec{r}}{\left|_{\vec{r}'=\vec{r}} \right.}$, which will produce a factor $(2\pi)^d \delta^{(d)}(\vec{0})$ cancelling thus the $\frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})}$ in the definition of V_{eff} . Let us start: for the energy functional under consideration, $\left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\bar{\rho}} = \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_0 + \int d^d z_1 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}') \delta \phi(\vec{z}_1)} \right|_0 \bar{\rho} + \frac{1}{2} \int d^d z_1 d^d z'_1 \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}') \delta \phi(\vec{z}_1) \delta \phi(\vec{z}'_1)} \right|_0 \bar{\rho}^2$ holds, and therefore $\ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\phi=\bar{\rho}} = \ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\phi=0} + \ln \left[1 + \bar{\rho} \int d^d r'_1 d^d z_1 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1)} \right|_0 D(\vec{r}'_1 - \vec{r}'|0) + \frac{\bar{\rho}^2}{2} \int d^d r'_1 d^d z_1 d^d z'_1 \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1) \delta \phi(\vec{z}'_1)} \right|_0 D(\vec{r}'_1 - \vec{r}'|0) \right]$, with $D(\vec{r}'_1 - \vec{r}'|0) \equiv \int \frac{d^d \vec{p}_1}{(2\pi)^d} \tilde{D}(\vec{p}_1) e^{i\vec{p}_1 \cdot (\vec{r}'_1 - \vec{r}')}$ the propagator defined by $\int d^d \vec{r}'_1 \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1)} \right|_{\phi=0} D(\vec{r}'_1 - \vec{r}'|0) = \delta^{(d)}(\vec{r} - \vec{r}')$. The first terms in the expansion of the underlined logarithm in powers of $\bar{\rho}$ are

$$\begin{aligned}
& \bar{\rho} \int d^d \vec{r}'_1 d^d z_1 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1)} \right|_{\phi=0} D(\vec{r}'_1 - \vec{r}'|0) + \\
& + \frac{\bar{\rho}^2}{2} \int d^d \vec{r}'_1 d^d z_1 d^d z'_1 \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1) \delta \phi(\vec{z}'_1)} \right|_{\phi=0} D(\vec{r}'_1 - \vec{r}'|0) - \\
& - \frac{\bar{\rho}^2}{2} \int d^d \vec{r}'_1 \cdots d^d z_2 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1)} \right|_{\phi=0} D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_2) \delta \phi(\vec{r}'_2) \delta \phi(\vec{z}_2)} \right|_{\phi=0} D(\vec{r}'_2 - \vec{r}'|0) - \\
& - \frac{\bar{\rho}^3}{2} \int d^d \vec{r}'_1 \cdots d^d z_2 \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1)} \right|_0 D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_2) \delta \phi(\vec{r}'_2) \delta \phi(\vec{z}_2) \delta \phi(\vec{z}'_2)} \right|_0 D(\vec{r}'_2 - \vec{r}'|0) + \\
& + \frac{\bar{\rho}^3}{3} \int d^d \vec{r}'_1 \cdots d^d z_3 \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r})(\vec{r}'_1)(\vec{z}_1)} \right|_0 D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r}_2)(\vec{r}'_2)(\vec{z}_2)} \right|_0 D(\vec{r}'_2 - \vec{r}_3|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r}_3)(\vec{r}'_3)(\vec{z}_3)} \right|_0 D(\vec{r}'_3 - \vec{r}'|0) - \\
& - \frac{\bar{\rho}^4}{8} \int d^d \vec{r}'_1 \cdots d^d z_2 \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}'_1) \delta \phi(\vec{z}_1) \delta \phi(\vec{z}'_1)} \right|_0 D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_2) \delta \phi(\vec{r}'_2) \delta \phi(\vec{z}_2) \delta \phi(\vec{z}'_2)} \right|_0 D(\vec{r}'_2 - \vec{r}'|0) + \\
& + \frac{\bar{\rho}^4}{2} \int d^d \vec{r}'_1 \cdots d^d z_3 \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r})(\vec{r}'_1)(\vec{z}_1)} \right|_0 D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r}_2)(\vec{r}'_2)(\vec{z}_2)} \right|_0 D(\vec{r}'_2 - \vec{r}_3|0) \left. \frac{\delta^4 E[\phi(\vec{r})]}{(\vec{r}_3)(\vec{r}'_3)(\vec{z}_3)(\vec{z}'_3)} \right|_0 D(\vec{r}'_3 - \vec{r}'|0) - \\
& - \frac{\bar{\rho}^4}{4} \int d^d \vec{r}'_1 \cdots d^d z_4 \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r})(\vec{r}'_1)(\vec{z}_1)} \right|_0 D(\vec{r}'_1 - \vec{r}_2|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r}_2)(\vec{r}'_2)(\vec{z}_2)} \right|_0 D(\vec{r}'_2 - \vec{r}_3|0) \left. \frac{\delta^3 E[\phi(\vec{r})]}{(\vec{r}_3)(\vec{r}'_3)(\vec{z}_3)} \right|_0 D(\vec{r}'_3 - \vec{r}_4|0) \times \\
& \quad \times \left. \frac{\delta^3 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_4) \delta \phi(\vec{r}'_4) \delta \phi(\vec{z}_4)} \right|_{\phi=0} D(\vec{r}'_4 - \vec{r}'|0) + \dots,
\end{aligned}$$

where in each integral one has to integrate over all coordinates except for $\vec{r}$ and $\vec{r}'$; in some of the above lines, a self-explanatory abbreviation has been used. After formulating the above equation in Fourier

transforms, one gets the expansion of V_{eff} in $\bar{\rho}$ -powers

$$\begin{aligned}
& \bar{\rho} \int \frac{d^d \vec{p}}{(2\pi)^d} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) + \\
& + \frac{\bar{\rho}^2}{2!} \int \frac{d^d \vec{p}}{(2\pi)^d} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \left[\left. \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) - \left(\left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right)^2 \right] + \\
& + \frac{\bar{\rho}^3}{3!} \int \frac{d^d \vec{p}}{(2\pi)^d} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \left[-3 \left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \left. \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) + \right. \\
& \quad \left. + 2 \left(\left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right)^3 \right] + \\
& + \frac{\bar{\rho}^4}{4!} \int \frac{d^d \vec{p}}{(2\pi)^d} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \left[-3 \left(\left. \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right)^2 + \right. \\
& \quad + 12 \left(\left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right)^2 \left. \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) - \right. \\
& \quad \left. - 6 \left(\left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right)^4 \right]. \tag{D.281}
\end{aligned}$$

The effect of taking the trace, $\text{Tr} \left[\left(\begin{array}{c} \end{array} \right) (\vec{r}, \vec{r}') \right] \equiv \int d^d \vec{r} \left(\begin{array}{c} \end{array} \right) (\vec{r}, \vec{r}') \Big|_{\vec{r}' = \vec{r}}$, is to produce an isolated integral

$\int_{-\infty}^{\infty} d^d \vec{r} \equiv (2\pi)^d \delta^{(d)}(\vec{0})$ which cancels the delta in the denominator of the effective potential's definition.

By looking at relation $\left. \frac{d^n V_{\text{eff}}}{d\bar{\rho}^n} \right|_{\bar{\rho}=0} = \tilde{\Gamma}^{(n)}(\vec{0}, \dots, \vec{0}|0)$ we can immediately read off (D.281) the diagrams

in equations (D.277)-(D.280) whose vertices have up to four legs. By reconstructing the expansion of the logarithm, we obtain finally

$$V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) + \frac{\eta}{2} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \text{Tr} \ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\phi(\vec{r})=0} + \tag{D.282}$$

$$+ \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln \left[1 + \bar{\rho} \left. \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) + \frac{\bar{\rho}^2}{2} \left. \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \right|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0) \right].$$

Constant term $\frac{\eta}{2} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \text{Tr} \ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\phi(\vec{r})=0}$ can be written as

$$\frac{\eta}{2} \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} \text{Tr} \ln \left. \frac{\delta^2 E[\phi(\vec{r})]}{\delta \phi(\vec{r}) \delta \phi(\vec{r}')} \right|_{\phi(\vec{r})=0} = \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln \left[1 + \left(\frac{1}{\tilde{D}(\vec{p}|0)} - 1 \right) \right], \tag{D.283}$$

by following the same strategy, this time applied to $\frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')}\Big|_{\phi=0} = 1 + \left(\frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')}\Big|_{\phi=0} - 1 \right)$,

where $\frac{1}{\tilde{D}(\vec{p}|0)} \equiv \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})}\Big|_{\tilde{\phi}=0}$. Hence one obtains

$$V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) + \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})}\Big|_{\tilde{\phi}}, \quad (\text{D.284})$$

from $(2\pi)^d \delta^{(d)}(\vec{p} + \vec{p}') \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(\vec{p}')}\Big|_{\tilde{\phi}} = \int d^d \vec{r} d^d \vec{r}' e^{i(\vec{p}\cdot\vec{r} + \vec{p}'\cdot\vec{r}')} \frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')}\Big|_{\phi(\vec{r})=\bar{\rho}} = \int d^d \vec{r} d^d \vec{r}' e^{i(\vec{p}\cdot\vec{r} + \vec{p}'\cdot\vec{r}')} \times$
 $\times \left[\frac{\delta^2 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')}\Big|_{\phi=0} + \int d^d \vec{z}_1 \frac{\delta^3 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')\delta\phi(\vec{z}_1)}\Big|_{\phi=0} \bar{\rho} + \frac{1}{2} \int d^d \vec{z}_1 d^d \vec{z}_1' \frac{\delta^4 E[\phi(\vec{r})]}{\delta\phi(\vec{r})\delta\phi(\vec{r}')\delta\phi(\vec{z}_1)\delta\phi(\vec{z}_1')}\Big|_{\phi=0} \bar{\rho}^2 \right]$ and there-
 fore $\frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})}\Big|_{\tilde{\phi}} = \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})}\Big|_{\tilde{\phi}=0} + \bar{\rho} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})\delta\tilde{\phi}(\vec{0})}\Big|_{\tilde{\phi}=0} + \frac{\bar{\rho}^2}{2} \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})\delta\tilde{\phi}(\vec{0})\delta\tilde{\phi}(\vec{0})}\Big|_{\tilde{\phi}=0} \equiv$
 $\equiv \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta\tilde{\phi}(\vec{p})\delta\tilde{\phi}(-\vec{p})}\Big|_{\tilde{\phi}(\vec{q})=\bar{\rho}(2\pi)^d \delta^{(d)}(\vec{q})}$. One can generalize straightforwardly all the above discussion for an
 energy functional with higher non-vanishing derivatives.

Now the formalism is ready to be applied to the question we want to investigate [31]: the Abrikosov model up to perturbative order η in a d -dimensional system ($d = 3, 4$) where one dimension is periodic with periodicity ω . We begin with an energy functional $E[\phi^a(\vec{r})]$ which depends on a set of field variables which we shall denote generally by $\phi^a(\vec{r})$ ⁹⁵ and which includes a potential depending only on the length of the scalar field, whose expectation value $\langle\phi_{\text{scal}}\rangle$ we want to study. One calculates effective potential $V_{\text{eff}}(\bar{\rho})$ by shifting the scalar field (just as explained in equation (D.270)), potential which will assumed to be a function only of $\bar{\rho}^2 \equiv \sum_a \bar{\rho}_a \bar{\rho}_a$ ⁹⁶. Further, we suppose that in a non-periodic system the effective potential has a nonvanishing $\langle\phi_{\text{scal}}\rangle$ as a minimum:

⁹⁵In order to simplify a bit the notation when complex fields are considered, their real and imaginary parts will be taken as different fields: thus term $\sum_{\mu} |\partial_{\mu} \bar{\psi}(\vec{r}) - ie \bar{A}_{\mu} \bar{\psi}|^2$ becomes

$\sum_{\mu} \sum_{a,b=1,2} (\partial_{\mu} \phi^a + e \bar{A}_{\mu} \epsilon_{ab} \phi^b)^2$, with scalar field written as $\bar{\psi} \equiv \phi^{a=1} + i\phi^{a=2}$. This is the same as

to change the $U(1)$ internal symmetry by the equivalent $SO(2)$ one, as discussed in equation (B.35). Gauge field $\bar{A}_{\mu}(\vec{r})$ is also a further general field variable $\phi^a(\vec{r})$ in functional $E[\phi^a(\vec{r})]$, something which should be borne in mind for the following, although for the moment its notation will be kept as $\bar{A}_{\mu}(\vec{r})$ in order to distinguish it from the field variables in whose expectation value we are interested.

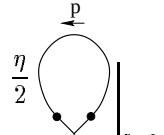
⁹⁶All models we shall consider fulfil such a condition: pure scalar model $E[\phi_{\text{scal}}(\vec{r})] = \int d^d \vec{r} \left[\frac{1}{2} \sum_{\mu=1}^d (\partial_{\mu} \phi_{\text{scal}})^2 - |\alpha| \phi_{\text{scal}}^2 + \frac{|\beta|}{2} \phi_{\text{scal}}^4 \right]$ as well as Abrikosov model $E[\phi_{\text{scal}}^{a=1,2}(\vec{r}), \phi_{\text{gauge}}^{\mu=1,\dots,d}(\vec{r})] = \int d^d \vec{r} \left[\frac{1}{2} \sum_{\mu, \nu < \mu} (\partial_{\mu} \phi_{\text{gauge}}^{\nu} - \partial_{\nu} \phi_{\text{gauge}}^{\mu})^2 + \frac{1}{2} \sum_{\mu} \sum_{a,b=1,2} (\partial_{\mu} \phi_{\text{scal}}^a + e \phi_{\text{gauge}}^{\mu} \epsilon_{ab} \phi_{\text{scal}}^b)^2 - |\alpha| \left(\sum_a \phi_{\text{scal}}^a \phi_{\text{scal}}^a \right) + \frac{|\beta|}{2} \left(\sum_a \phi_{\text{scal}}^a \phi_{\text{scal}}^a \right)^2 \right]$. For instance, the first term of the effective potential which multi-

$\frac{\partial V_{\text{eff}}(\bar{\rho}^2)}{\partial \bar{\rho}_a} \Big|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle \neq 0}^{\omega=\infty} = 0$. Now one of the system's dimensions becomes periodic and

we ask ourselves whether this periodicity removes the symmetry breaking characterized by the non-vanishing expectation value of the scalar field, i.e., if the following equation

$$0 = \frac{\partial V_{\text{eff}}(\bar{\rho}^2)}{\partial \bar{\rho}_a}^{(\omega)} = 2\bar{\rho}_a \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2}^{(\omega)}$$

has as only solution $\bar{\rho} = 0$.

plies an odd power of $\bar{\rho}$ is $\frac{\partial V_{\text{eff}}}{\partial \bar{\rho}_a} \Big|_{\bar{\rho}=0} = \Gamma_a^{(1)}(\vec{k} = \vec{0}|0)$, represented by diagram 

and equal to $\frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\tilde{\phi}=0} \tilde{D}(\vec{p}|0)$ for the pure scalar model, or to

$$\frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \left[\sum_{b,b'} \frac{\delta^3 E[\tilde{\phi}^{a,\mu}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{0}) \delta \tilde{\phi}_{\text{scal}}^b(\vec{p}) \delta \tilde{\phi}_{\text{scal}}^{b'}(-\vec{p})} \Big|_{\tilde{\phi}=0} \tilde{D}_{\text{scal}}^{bb'}(\vec{p}|0) + \sum_{\mu,\nu} \frac{\delta^3 E[\tilde{\phi}^{a,\mu}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{0}) \delta \tilde{\phi}_{\text{gauge}}^\mu(\vec{p}) \delta \tilde{\phi}_{\text{gauge}}^\nu(-\vec{p})} \Big|_{\tilde{\phi}=0} \tilde{D}_{\text{gauge}}^{\mu\nu}(\vec{p}|0) \right]$$

for the gauge model. And in both models the considered diagram vanishes since its vertex

$$\left. \begin{array}{c} \text{p} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \text{-p} \end{array} \right|_{\tilde{\phi}=0} \equiv \frac{\delta^3 E[\tilde{\phi}^{a,\mu}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{0}) \delta \tilde{\phi}_{\text{scal}}^b(\vec{p}) \delta \tilde{\phi}_{\text{scal}}^{b'}(-\vec{p})} \Big|_{\tilde{\phi}=0}, \frac{\delta^3 E[\tilde{\phi}^{a,\mu}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{0}) \delta \tilde{\phi}_{\text{gauge}}^\mu(\vec{p}) \delta \tilde{\phi}_{\text{gauge}}^\nu(-\vec{p})} \Big|_{\tilde{\phi}=0}$$

is zero. The same holds, for example, for $\frac{\partial^3 V_{\text{eff}}}{\partial \bar{\rho}_a \partial \bar{\rho}_b \partial \bar{\rho}_c} \Big|_{\bar{\rho}=0} = \Gamma_{abc}^{(3)}(\vec{0}, \vec{0}, \vec{0}|0)$ since all its diagrams contain the just discussed

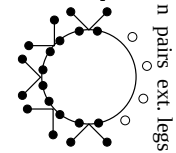
vertex of three legs or contain vertices with more than four legs (and which are thus identically

zero from the very form of $E[\tilde{\phi}^{a,\mu}(\vec{q})]$). Only in the gauge model, $\frac{\delta^3 E[\tilde{\phi}^{a,\mu}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{0}) \delta \tilde{\phi}_{\text{scal}}^b(\vec{p}) \delta \tilde{\phi}_{\text{gauge}}^\nu(-\vec{p})} \Big|_{\tilde{\phi}=0}$

is different from zero, but such a vertex does not appear in the diagrams of effective potential's $\Gamma_a^{(1)}(\vec{k} = \vec{0}|0)$ if we are studying the expectation value of the *scalar* field: there is no propagator which connects a leg of scalar field with a leg of gauge field. As an example, let us calculate $V_{\text{eff}}(\bar{\rho}^2)$ for

the model described by energy functional $E[\phi_{\text{scal}}(\vec{r})] = \int d^d \vec{r} \left[\frac{1}{2} \sum_{\mu=1}^d (\partial_\mu \phi_{\text{scal}})^2 + \alpha \phi_{\text{scal}}^2 + \frac{|\beta|}{2} \phi_{\text{scal}}^4 \right]$.

Since the Fourier transform of 4-legs vertex is equal to $12|\beta|$, the Fourier transform of the propagator is given by $\tilde{D}(\vec{p}|0) = \frac{1}{\vec{p}^2 + 2\alpha}$, and since the only occuring diagrams are those with

n pairs of external legs , whose combinatorial factor is $\frac{(2n)!}{n2^n}$, then we obtain

up to order η the following results: $\tilde{\Gamma}^{(2)}(\vec{0}, \vec{0}|0) = 2\alpha + \frac{\eta}{2} \left(\begin{array}{c} \text{p} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \text{-p} \end{array} \right) \Big|_{\tilde{\phi}=0} \equiv 2\alpha + \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{12|\beta|}{\vec{p}^2 + 2\alpha},$

$\tilde{\Gamma}^{(4)}(\vec{0}, \dots, \vec{0}|0) = 12|\beta| - 3\frac{\eta}{2} \left(\begin{array}{c} \text{p} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \text{-p} \end{array} \right) \Big|_{\tilde{\phi}=0} \equiv 12|\beta| - 3\frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \left(\frac{12|\beta|}{\vec{p}^2 + 2\alpha} \right)^2$, and $\tilde{\Gamma}^{(2n)}(\vec{0}, \dots, \vec{0}|0) =$

Hence a necessary condition for $\langle \phi_{\text{scal}} \rangle = 0$ must be $\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle=0}^{(\omega)} \neq 0$, and on the contrary, $\langle \phi_{\text{scal}} \rangle \neq 0$ implies $\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle}^{(\omega)} = 0$, summarized as follows

$$\left\{ \begin{array}{l} \text{symm. breaking:} \quad \langle \phi_{\text{scal}} \rangle \neq 0 \implies \left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle \neq 0}^{(\omega)} = 0, \\ \text{symm. restoring:} \quad \left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega)} \neq 0 \implies \langle \phi_{\text{scal}} \rangle = 0. \end{array} \right. \quad (\text{D.285})$$

If we separate the effective potential in a part for $\omega = \infty$ (and thus independent of ω) and in a part depending on a finite ω , $V_{\text{eff}}(\bar{\rho}^2)^{(\omega)} = V_{\text{eff}}(\bar{\rho}^2)^{(\omega=\infty)} + V_{\text{eff}}(\bar{\rho}^2)^{(\text{dep.}\omega)}$, the necessary condition for $\langle \phi_{\text{scal}} \rangle = 0$ reads

$$\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega=\infty)} + \left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\text{dep.}\omega)} \neq 0,$$

Finally, and according to (D.285), the critical value of periodicity ω for which the symmetry breaking gets lost, transiting from $\langle \phi_{\text{scal}} \rangle \neq 0$ to $\langle \phi_{\text{scal}} \rangle = 0$, is given by

$$\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\omega=\omega_{\text{crit}}, \bar{\rho}=0}^{(\text{dep.}\omega)} = - \left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega=\infty)}. \quad (\text{D.286})$$

This last relation can be also be understood as follows: since the effective potential only contains even powers of $\bar{\rho}$, plus the fact that V_{eff} must be positive at large $\bar{\rho}^2$ in order to have a minimum, then the location of this minimum at $\bar{\rho}^2 \neq 0$ or at $\bar{\rho}^2 = 0$ depends on the sign, negative or positive, of the coefficient multiplying $\bar{\rho}^2$, i.e., of $\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega)}$.

Thus, the transition from the non-vanishing $\bar{\rho}^2$ to the (so-called symmetric) minimum

$$= (-1)^{n+1} \frac{(2n)!}{n2^n} \left. \text{diagram} \right|_{\bar{\phi}=0} \equiv (-1)^{n+1} \frac{(2n)!}{n2^n} \int \frac{d^d \vec{p}}{(2\pi)^d} \left(\frac{12|\beta|}{\bar{p}^2 + 2\alpha} \right)^n \text{ for } n > 2. \text{ Hence effective}$$

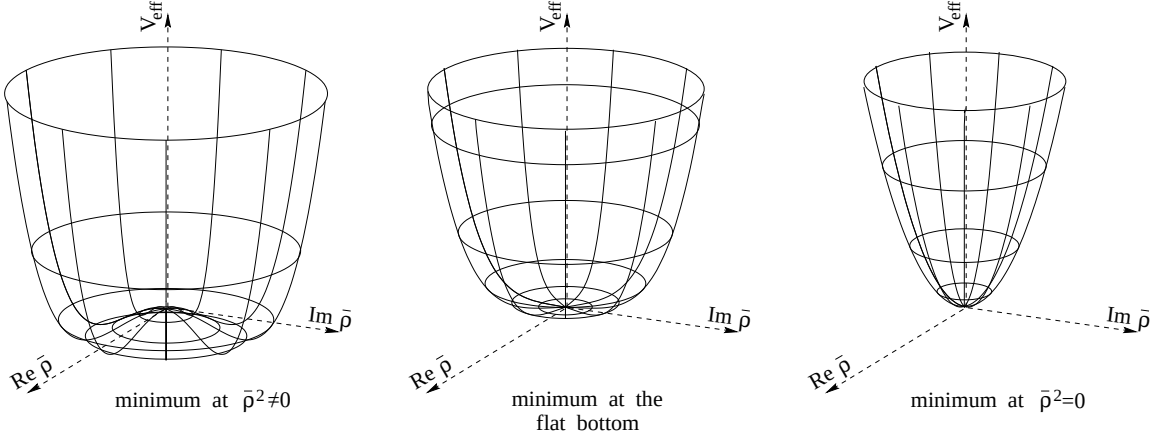
$$\text{potential (D.273) is equal to } \left(V_{\text{eff}}(\bar{\rho}) - \frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} G[\varphi(\vec{r})] \right) \Big|_{\varphi(\vec{r})=0} = \sum_{m=1}^{\infty} \frac{1}{m!} \bar{\rho}^m \tilde{\Gamma}^{(m)}(\vec{0}, \dots, \vec{0}|0) =$$

$$= \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 + \frac{\eta}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int \frac{d^d \vec{p}}{(2\pi)^d} \left(\frac{6|\beta| \bar{\rho}^2}{\bar{p}^2 + 2\alpha} \right)^n = \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 + \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln \left(1 + \frac{6|\beta| \bar{\rho}^2}{\bar{p}^2 + 2\alpha} \right),$$

just as the result already obtained in (D.282); constant term $\frac{1}{(2\pi)^d \delta^{(d)}(\vec{0})} G[\varphi(\vec{r})] \Big|_{\varphi(\vec{r})=0}$ is equal to

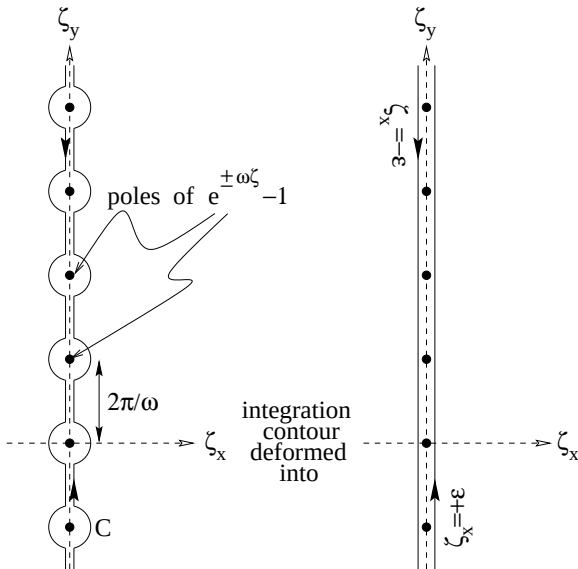
$$\frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln(\bar{p}^2 + 2\alpha).$$

at $\bar{\rho}^2 = 0$ occurs for $\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\omega)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0} = 0$, as one can intuitively see in the intermediate potential of the next figure, characterized by a flat bottom which covers a wide range containing $\bar{\rho}^2 = 0$ as well as some $\bar{\rho}^2 \neq 0$.



The calculation of $V_{\text{eff}}^{(\omega)}$ and of $\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\omega)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}$ will be carried out by means of their

Fourier transforms. When one of the dimensions becomes periodic with periodicity ω , dimension which will be denoted by r_τ , all scalar and gauge fields of the energy functional satisfy $\phi^{a,\mu}(r_\tau + \omega, \vec{x}) = \phi^{a,\mu}(r_\tau, \vec{x})$ and the Fourier integral corresponding to r_τ becomes a Fourier series: $\int_{-\infty}^{\infty} \frac{dp_\tau}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{d-1} \left(\frac{dp_j}{2\pi} \right) \tilde{f}(p_\tau, p_j) \rightarrow \frac{1}{\omega} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{d-1} \left(\frac{dp_j}{2\pi} \right) \tilde{f}(2\pi n/\omega, p_j)$. Such a sum is carried out by means of the following technique [56]: a general sum $\frac{1}{\omega} \sum_{n=-\infty}^{\infty} f(p_\tau = 2\pi n/\omega) \equiv \frac{1}{\omega} \sum_{n=-\infty}^{\infty} f(\zeta = ip_\tau = i2\pi n/\omega)$ for a regular function $f(\text{Re } \zeta, \text{Im } \zeta)$



can be formulated by means of the Cauchy residuum formula as $\frac{1}{\omega} \sum_{n=-\infty}^{\infty} f(\zeta = i2\pi n/\omega) = \frac{1}{4\pi i} \oint_C d\zeta f(\zeta) \coth\left(\frac{\omega\zeta}{2}\right)$, with integration contour C as in the figure and lying on the complex plane. The proof of such a result runs as follows: from relation $\coth\left(\frac{\omega\zeta}{2}\right) = \left(-1 - \frac{2}{e^{-\omega\zeta} - 1}\right)$, and from the fact that $\frac{1}{2\pi i} \oint_C d\zeta f(\zeta) = 0$ since $f(\zeta)$ is a regular function, one obtains

$$\frac{1}{2\pi i} \oint_C d\zeta f(\zeta) \frac{\omega}{2} \coth\left(\frac{\omega\zeta}{2}\right) = -\frac{1}{2\pi i} \oint_C d\zeta f(\zeta) \frac{\omega}{2} \left(1 + \frac{2}{e^{-\omega\zeta} - 1}\right) \stackrel{\zeta = -\omega\zeta}{=}$$

$$= \frac{1}{2\pi i} \oint_C d\zeta f(-\zeta/\omega) \underbrace{\frac{1}{e^\zeta - 1}}_{\text{poles at } \zeta = -i2\pi n} = \sum_{n=-\infty}^{\infty} f(-\zeta/\omega) \Big|_{\zeta = -i2\pi n} = \sum_{n=-\infty}^{\infty} f(i2\pi n/\omega).$$

The same holds for $\coth\left(\frac{\omega\zeta}{2}\right) = \left(1 + \frac{2}{e^{\omega\zeta} - 1}\right)$ with the substitution $\zeta \rightarrow \omega\zeta$. It must be noted that the premise of the regularity of $f(\zeta)$ is too strong here: from the definition of the above integration contour, it suffices if $f(\zeta)$ does not have any singularity along the axis ζ_y (the imaginary part of ζ) to have $\frac{1}{2\pi i} \oint_C d\zeta f(\zeta) = 0$. Finally

$$\begin{aligned} \frac{1}{\omega} \sum_{n=-\infty}^{\infty} f(\zeta = i2\pi n/\omega) &= \frac{1}{2\pi i} \oint_C d\zeta f(\zeta) \frac{1}{2} \coth\left(\frac{\omega\zeta}{2}\right) = \\ &= \frac{1}{2\pi i} \int_{i\infty-\varepsilon}^{-i\infty-\varepsilon} d\zeta f(\zeta) \left(-\frac{1}{2} - \frac{1}{e^{-\omega\zeta} - 1}\right) + \frac{1}{2\pi i} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\zeta f(\zeta) \left(\frac{1}{2} + \frac{1}{e^{\omega\zeta} - 1}\right), \end{aligned}$$

where the last equality arises from the deformation of contour C to a vertical line downward at $\zeta = -i\varepsilon$ plus a vertical line upward at $\zeta = +i\varepsilon$ (with ε infinitesimal). By changing $\zeta \rightarrow -\zeta$ in the first integral of the last line, one arrives at the result

$$\begin{aligned} \frac{1}{\omega} \sum_{n=-\infty}^{\infty} f(\zeta = i2\pi n/\omega) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\zeta \frac{f(\zeta) + f(-\zeta)}{2} + \\ &+ \frac{1}{2\pi i} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\zeta \frac{f(\zeta) + f(-\zeta)}{e^{\omega\zeta} - 1}, \end{aligned} \quad (\text{D.287})$$

for a $f(\zeta)$ without singularities along $\Im\zeta$. The first integral above does not depend on ω and belongs in the effective potential (or in its derivatives) to the part $V_{\text{eff}}^{(\omega=\infty)}$.

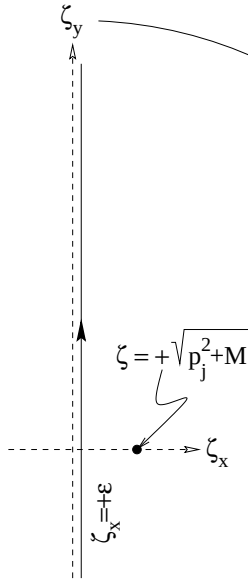
D.5.3 The effective potential for the pure scalar model

Let us gain a bit more insight into result (D.287) by considering the pure scalar model defined by energy $E[\phi_{\text{scal}}(\vec{r})] = \int d^d \vec{r} \left[\frac{1}{2} \sum_{\mu=1}^d (\partial_\mu \phi_{\text{scal}})^2 + V(\phi_{\text{scal}}) \right]$ with potential $V(\phi_{\text{scal}}) = \alpha \phi_{\text{scal}}^2 + \frac{|\beta|}{2} \phi_{\text{scal}}^4$ (α positive or negative). The effective potential is given in general by equation (D.284)

$$V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) + \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{p}}, \quad (\text{D.288})$$

$$\frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{p}} = \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{p}=0} + \bar{\rho} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0})} \Big|_{\vec{p}=0} + \frac{\bar{\rho}^2}{2} \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \Big|_{\vec{p}=0} + \dots$$

which yields $V_{\text{eff}}(\bar{\rho}) = \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 + \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \ln(\bar{p}^2 + 2\alpha + 6|\beta|\bar{\rho}^2)$ for the pure scalar model (as it has been proved in the footnote on page 679). When the τ -direction becomes periodic, one has to calculate the two integrals of equation (D.287) for $f(\zeta) = f(-\zeta) =$



$= \ln \left[\left(\sum_{j=1}^{d-1} p_j^2 + 2\alpha + 6|\beta|\bar{\rho}^2 \right) - \zeta^2 \right]$, where the second integral is carried out by derivat-
ing $\frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\zeta \frac{2 \ln(p_j^2 + M - \zeta^2)}{e^{\omega\zeta} - 1}$ with re-
spect to $p_j^2 \equiv \sum_{j=1}^{d-1} p_j^2$ (for the following, the sum over repeated indices will not be usually written explicitly), obtaining thus $\frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\zeta \frac{2}{p_j^2 + M - \zeta^2} \frac{1}{e^{\omega\zeta} - 1}$. The closed integration contour of the figure, with $\Re \zeta = +\epsilon > 0$, closes at $|\zeta| = \infty$ where $\frac{1}{e^{\omega\zeta} - 1}$ falls off very rapidly, and only encloses pole $\zeta = +\sqrt{p_j^2 + M}$

$$\begin{aligned} \frac{d}{d(p_j^2)} \left[\frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\zeta \frac{2 \ln(p_j^2 + M - \zeta^2)}{e^{\omega\zeta} - 1} \right] &= \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\zeta \frac{2}{p_j^2 + M - \zeta^2} \frac{1}{e^{\omega\zeta} - 1} = \\ &= \frac{1}{\sqrt{p_j^2 + M}} \frac{1}{e^{\omega\sqrt{p_j^2 + M}} - 1}, \end{aligned} \quad (\text{D.289})$$

$$\frac{\eta}{2} \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\zeta \ln(p_j^2 + M - \zeta^2) \frac{2}{e^{\omega\zeta} - 1} = \frac{\eta}{\omega} \ln \left(1 - e^{-\omega\sqrt{p_j^2 + M}} \right). \quad (\text{D.290})$$

The effective potential in a system periodic along one direction is then given by

$$\begin{aligned} V_{\text{eff}}(\bar{\rho})^{(\omega)} &= \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 + \underbrace{\frac{\eta}{2} \int_{-\infty}^{\infty} \frac{dp_{\tau}}{2\pi} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d-1} \frac{dp_j}{2\pi} \right) \ln \left(p_{\tau}^2 + \sum_{j=1}^{d-1} p_j^2 + 2\alpha + 6|\beta|\bar{\rho}^2 \right)}_{\zeta \equiv ip_{\tau} : \text{const.} + \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d-1} \frac{dp_j}{2\pi} \right) \left(\sum_{j=1}^{d-1} p_j^2 + 2\alpha + 6|\beta|\bar{\rho}^2 \right)^{1/2}} \\ &+ \frac{\eta}{\omega^d} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d-1} \frac{dp_j}{2\pi} \right) \ln \left[1 - \exp \left(- \sqrt{p_j^2 + \omega^2 \underbrace{(2\alpha + 6|\beta|\bar{\rho}^2)}_{V''(\bar{\rho})}} \right) \right], \end{aligned} \quad (\text{D.291})$$

where the (infinite) constant in the first integral only shifts the origin of the effective potential and can therefore be neglected. In spite of this neglecting, the remaining integral in the part independent of ω is still divergent: $\lim_{\Lambda_c \rightarrow \infty} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^d} p^{d-2} dp \sqrt{p^2 + V''(\bar{\rho})} \sim \sim \lim_{\Lambda_c \rightarrow \infty} \Lambda_c^d$, with S_{d-1} the surface of the unit-sphere in a $(d-1)$ -dim. space. It is important to notice that the range of $p \sim \Lambda_c \sim \infty$ in the Fourier transforms corresponds

to the range of small $|\vec{r}|$ in the back-transform (since otherwise, exponential e^{ipr} would oscillate too rapidly to produce a contribution to the Fourier integral): the system under study diverges thus in the range of short distances. In order to solve such a problem [99], one has introduced as an intermediate step a large (but finite) cut-off Λ_c . The pass from integral $\int_0^\infty dp$ to $\int_0^{\Lambda_c} dp$ is equivalent to the modification of the short-range behaviour for distances smaller than $1/\Lambda_c$: such a process is called regularization. This modification actually means that the model under consideration is only an approximately partial description of a system which already contains a fundamental scale not explicitly occurring in the model⁹⁷. Now we shall study whether this modification eventually allows us to redefine the V_{eff} 's original parameters as functions of Λ_c such that the derivatives of V_{eff} (and thus $\tilde{\Gamma}^{(n)}$) remain finite in the limit $\Lambda_c \rightarrow \infty$. It is very important to note that during this process we are only redefining the parameters *already present* in the original model, without introducing new parameters. If one is successful in the just discussed construction, then an effective model for large distances $|\vec{r}| \gg 1/\Lambda_c$ (the range of interest for phase transitions) will be obtained without being necessary to know exactly what kinds of interactions occur at the very small scale. If, on the contrary, the elimination of divergences at the range of short distances requires the introduction of new terms in the model's energy (or Lagrangian), with their corresponding new parameters, it does not mean that we are dealing with a "bad" model: only that we were not so lucky to start with a model already containing the interactions occurring within a wide range of distances, and in the moment we leave the distance at which the model was initially guessed and try to go to smaller distances, new interactions appear.

Those models which permit such a process of eliminating divergences at very short distances to get a large-scale model depending on effective terms (of the same form as those of the original model) are called renormalizable models. Further, it will result that even if it is necessary to introduce an arbitrary scale μ_0 (called the renormalization point) to define correctly the effective parameters, the form of the renormalized model does not depend on the choice of μ_0 : more explicitly, the effective parameters defined at

⁹⁷In a lattice model one does not find such a short-distance problem, since the lattice formulation already has a built-in cut-off from the very beginning: lattice spacing a , which is the above mentioned "hidden" fundamental scale. The divergences in the (continuum) effective potential arise from those integrals of closed loops of the propagator; analogously to equation (C.107) which gave the propaga-

tor of the lattice Laplace operator, $D_{\vec{r}, \vec{r}'} = \int_{-\pi/a}^{\pi/a} \frac{d^d \vec{k}}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{\left(4 - 2 \sum_{\mu=1}^d \cos(ak_\mu)\right)/a^2}$, the propagator of

differential operator $\left(-\sum_{\mu=1}^d \partial_\mu \partial_\mu + V''(\bar{\rho})\right) \delta^{(d)}(\vec{r} - \vec{r}')$ in the continuum formulation ($a \rightarrow 0$) yields

$$D(\vec{r}, \vec{r}') = \lim_{\Lambda_c \rightarrow \infty} \int_{-\Lambda_c}^{\Lambda_c} \frac{d^d \vec{k}}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{\vec{k}^2 + \text{const} \left(\vec{k}^2\right)^2 / \Lambda_c^2 + V''(\bar{\rho})}, \text{ with } \Lambda_c = \frac{\pi}{a}.$$

A large but finite cut-off in the continuum formulation is therefore equivalent to a discrete lattice model with propagator equal $\frac{1}{\vec{k}^2 + \text{const} \frac{\vec{k}^4}{\Lambda_c^2} + V''(\bar{\rho})}$ in Fourier transform, being the lattice spacing equal to $\frac{\pi}{\Lambda_c}$ [78].

arbitrary scale μ_0 are related to those defined at another different renormalization point μ'_0 in such a way that, independently of which set of renormalized parameters are used, the form of the renormalized model remains the same⁹⁸. It can be proved that the form in which the *renormalized* model is constructed does not depend on the intermediate steps done to introduce the finite large cut-off Λ_c , although this proof will be not carried out here.

It should stressed once more that the need of introducing Λ_c signals the existence of a fundamental scale, such as for instance the lattice spacing, which determines the limit up to which one can go with only rearranging the parameters of the original model at which we have started: if one would go to distances shorter than this fundamental scale, the interactions occuring at such distances would be described by Lagrangian's terms which were different than those of the original model. If the model is renormalizable, after the renormalization process has been carried out such cut-off gets substituted by an arbitrary scale, the renormalization point (see also the Note on page 291).

Let us restrict ourselves to order η , or more exactly, up to the first order in the perturbative small and dimensionless parameter $k\eta|\beta|$ with $k \sim |\alpha|^{(d-4)/2}$ (see comments after equation (D.265)). We focus our attention on the effective potential's part independent of ω for a large but finite cut-off Λ_c

$$V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} = V(\bar{\rho}) + \frac{\eta}{2} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + V''(\bar{\rho})}$$

$$\text{with } V(\bar{\rho}) = \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 \quad \text{and} \quad V''(\bar{\rho}) = 2\alpha + 6|\beta|\bar{\rho}^2. \quad (\text{D.292})$$

and on its first four derivatives⁹⁹

⁹⁸The word “renormalization” has been already met in this work during the discussion on page 567: there it appeared when two pictures of the 2-dim. *XY* model at a certain $|\psi|^2/\eta$, but taken under two “magnifying glasses” of slightly different magnifying powers, were compared to each other. It resulted that the coarser picture was also the picture which, under the finer magnifying glass, we would observe of the same model but at a different $|\psi|^2/\eta$. The magnifying power plays there the role of renormalization point μ_0 : the smallest length which could still be distinguished. It is to be expected for a renormalizable model, both pictures are described by the original model without being necessary to introduce new interactions (at least as long as we are studying pictures of the model at large distances). The details specific to the 2-dim. *XY* model are of no interest here, but only the main idea that the loss in magnifying power (i.e., the integrating out the interactions occuring at the short distances which are no longer observed) corresponds to model's parameters which depend on the length scale which can be still distinguished after the “defocussing” process. Thus, by starting with a model which only contains interactions among nearest neighbours, one obtains a series of successive more defocussed pictures with its corresponding evolution of the model's parameters towards a picture at large distances: this allows us to understand the long-distance behaviour of the original model.

⁹⁹All the above V_{eff} 's derivatives correspond, of course, to the diagrams of equations (D.277)-(D.280). Since the highest non-vanishing functional derivative of energy (D.292) is the fourth one, one gets that the contributions of order η (of order $\eta|\beta|$) to, for instance, $\frac{d^4 V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^4}$ are represented by

the following diagrams: $-3\frac{\eta}{2} \text{p} \downarrow \text{p} \uparrow \Big|_{\bar{\rho}} + 12\frac{\eta}{2} \text{p} \downarrow \text{p} \uparrow \Big|_{\bar{\rho}} - 6\frac{\eta}{2} \text{p} \downarrow \text{p} \uparrow \Big|_{\bar{\rho}}$, where $\Big|_{\bar{\rho}}$ means values

$$\begin{aligned}
\frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} &= \frac{dV(\bar{\rho})}{d\bar{\rho}} + \frac{\eta}{4}(12|\beta|)\bar{\rho} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''(\bar{\rho})}}, \\
\frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} &= \frac{d^2V(\bar{\rho})}{d\bar{\rho}^2} + \frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''(\bar{\rho})}} - \frac{\eta}{8}(12|\beta|)^2 \bar{\rho}^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}}, \\
\frac{d^3V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^3} &= \frac{d^3V(\bar{\rho})}{d\bar{\rho}^3} - \frac{3\eta}{8}(12|\beta|)^2 \bar{\rho} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}} + \frac{3\eta}{16}(12|\beta|)^3 \bar{\rho}^3 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{5/2}}, \\
\frac{d^4V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^4} &= \frac{d^4V(\bar{\rho})}{d\bar{\rho}^4} - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}} + \\
&\quad + \boxed{\frac{18\eta}{16}(12|\beta|)^3 \bar{\rho}^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{5/2}} - \frac{15\eta}{32}(12|\beta|)^4 \bar{\rho}^4 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{7/2}}},
\end{aligned}$$

where the set of integrals enclosed into the box, denoted by $\boxed{\text{finite}(\bar{\rho})}$ for the following, is finite¹⁰⁰ at $d \leq 4$ in the limit $\Lambda_c \rightarrow \infty$

$$\boxed{\text{finite}(\bar{\rho})} = \begin{cases} \frac{3\eta}{16\pi^2}(12|\beta|)^3 \frac{\bar{\rho}^2}{V''(\bar{\rho})} - \frac{\eta}{32\pi^2}(12|\beta|)^4 \left(\frac{\bar{\rho}^2}{V''(\bar{\rho})} \right)^2 + O\left(\frac{1}{\Lambda_c^2} \right) & d = 4 \\ \frac{3\eta}{16\pi} \frac{(12|\beta|)^3}{\sqrt{V''(\bar{\rho})}} \frac{\bar{\rho}^2}{V''(\bar{\rho})} - \frac{3\eta}{64\pi} \frac{(12|\beta|)^4}{\sqrt{V''(\bar{\rho})}} \left(\frac{\bar{\rho}^2}{V''(\bar{\rho})} \right)^2 + O\left(\frac{1}{\Lambda_c^3} \right) & d = 3 \\ \frac{3\eta}{8\pi} \frac{(12|\beta|)^3}{V''(\bar{\rho})} \frac{\bar{\rho}^2}{V''(\bar{\rho})} - \frac{\eta}{8\pi} \frac{(12|\beta|)^4}{V''(\bar{\rho})} \left(\frac{\bar{\rho}^2}{V''(\bar{\rho})} \right)^2 + O\left(\frac{1}{\Lambda_c^2} \right) & d = 2 \end{cases} \quad (\text{D.293})$$

Note that the derivatives of effective potential $V_{\text{eff}}^{(\omega=\infty)}$ (up to order η) higher than the above ones contain only integrals which are finite for $d \leq 4$ in the limit of an infinite cut-off. The form of removing the divergences in the above integrals will be to include the following terms in the potential: $V(\phi_{\text{scal}}) = \alpha\phi_{\text{scal}}^2 + \frac{|\beta|}{2}\phi_{\text{scal}}^4 \rightarrow V(\phi_{\text{scal}}) - \delta\alpha\phi_{\text{scal}}^2 - \frac{\delta\beta}{2}\phi_{\text{scal}}^4$,

taken at $\tilde{\varphi}(\vec{q}) = \bar{\rho}(2\pi)^d \delta^{(d)}(\vec{q})$. The integration over dp_τ by means of result $\int_0^\infty \frac{dp_\tau}{2\pi} \frac{1}{(p_\tau^2 + p_j^2 + V'')^4} = \frac{5}{6(p_j^2 + V'')} \int_0^\infty \frac{dp_\tau}{2\pi} \frac{1}{(p_\tau^2 + p_j^2 + V'')^3} = \frac{5}{6(p_j^2 + V'')} \frac{3}{4(p_j^2 + V'')} \int_0^\infty \frac{dp_\tau}{2\pi} \frac{1}{(p_\tau^2 + p_j^2 + V'')^2} = \frac{5}{6(p_j^2 + V'')} \frac{3}{4(p_j^2 + V'')} \frac{1}{(p_j^2 + V'')} \int_0^\infty \frac{dp_\tau}{2\pi} \frac{1}{p_\tau^2 + p_j^2 + V''} = \frac{5}{6(p_j^2 + V'')} \frac{3}{4(p_j^2 + V'')} \frac{1}{2(p_j^2 + V'')} \frac{1}{\sqrt{p_j^2 + V''}}$, with $p_j^2 \equiv \sum_{j=1}^{d-1} p_j^2$, also produces the above results for the derivatives of V_{eff} .

$$^{100}\text{For a finite and large } \Lambda_c, \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V'')^{5/2}} = \begin{cases} \frac{1}{6\pi^2 V''} + O\left(\frac{1}{\Lambda_c^2} \right) & d = 4 \\ \frac{1}{6\pi (V'')^{3/2}} + O\left(\frac{1}{\Lambda_c^3} \right) & d = 3 \\ \frac{1}{3\pi (V'')^2} + O\left(\frac{1}{\Lambda_c^2} \right) & d = 2 \end{cases} \quad \text{and}$$

$$\int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V'')^{7/2}} = \begin{cases} \frac{1}{15\pi^2 (V'')^2} + O\left(\frac{1}{\Lambda_c^2} \right) & d = 4 \\ \frac{1}{10\pi (V'')^{5/2}} + O\left(\frac{1}{\Lambda_c^5} \right) & d = 3 \\ \frac{4}{15\pi (V'')^3} + O\left(\frac{1}{\Lambda_c^2} \right) & d = 2 \end{cases} \quad \text{hold.}$$

where the new terms are called the counterterms, have the same form as the terms of potential $V(\bar{\rho})$ and have to cancel the integrals' divergences. It should be noted that $\delta\alpha, \delta\beta \sim O(\eta)$ since the divergent integrals are all multiplied by η . The same effect would have been obtained if in the original parameters α, β one separates a part finite at $\Lambda_c \rightarrow \infty$ plus an infinite part which cancels the divergences in the effective potential's derivatives: $\alpha = \alpha_1 - \delta\alpha$ and $|\beta| = |\beta_1| - \delta\beta$. By reorganizing the above derivatives, it can be easily seen that the just introduced counterterms are able to remove all divergences at order η (more exactly, at order $\eta|\beta|$)

$$\begin{aligned}
\frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} &= \bar{\rho} \left[-2\delta\alpha - 2\delta\beta\bar{\rho}^2 + \frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''(\bar{\rho})}} \right] + \\
&\quad + \bar{\rho} (2\alpha + 2|\beta|\bar{\rho}^2) , \\
\frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} &= \left[-2\delta\alpha - 2\delta\beta\bar{\rho}^2 + \frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''(\bar{\rho})}} \right] + \\
&\quad + \frac{\bar{\rho}^2}{3} \left[-12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}} \right] + \\
&\quad + 2\alpha + 6|\beta|\bar{\rho}^2 , \\
\frac{d^3V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^3} &= \bar{\rho} \left[-12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}} \right] + \\
&\quad + 6|\beta|\bar{\rho} + \frac{3\eta}{16}(12|\beta|)^3 \bar{\rho}^3 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{5/2}} , \\
\frac{d^4V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^4} &= \left[-12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\bar{\rho}))^{3/2}} \right] + \\
&\quad + 12|\beta| + \boxed{\text{finite}(\bar{\rho})} ,
\end{aligned} \tag{D.294}$$

where there are only *two* different bracketed groups which present problems, just the number of counterterms ¹⁰¹. The above integral inside the double brackets diverges

¹⁰¹One should note that in the above integrals, $\eta \frac{1}{(p^2 + V''(\bar{\rho}))^k}$ has not been substituted by $\eta \frac{1}{(p^2 + V''(\bar{\rho}) - 2\delta\alpha + 6\delta\beta\bar{\rho}^2)^k}$ since $\delta\alpha, \delta\beta$ are already of order η and their contribution inside the integrals appears thus only at order η^2 . Moreover, for the $d = 2, 3, 4$ the results of the problematic integrals at a large (but finite) Λ_c are:

$$\begin{aligned}
\int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''}} &= \begin{cases} \frac{\Lambda_c^2}{4\pi^2} + \frac{V''}{8\pi^2} \left[\ln\left(\frac{V''}{4\Lambda_c^2}\right) + 1 \right] + O\left(\frac{1}{\Lambda_c^2}\right) & d = 4 \\ \frac{\Lambda_c}{2\pi} - \frac{1}{2\pi} \sqrt{V''} + O\left(\frac{1}{\Lambda_c}\right) & d = 3 \\ -\frac{1}{4\pi} \ln\left(\frac{V''}{4\Lambda_c^2}\right) + O\left(\frac{1}{\Lambda_c^2}\right) & d = 2 \end{cases} , \\
\int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V'')^{3/2}} &= \begin{cases} -\frac{1}{4\pi^2} \left[\ln\left(\frac{V''}{4\Lambda_c^2}\right) + 2 \right] + O\left(\frac{1}{\Lambda_c^2}\right) & d = 4 \\ \frac{1}{2\pi} \frac{1}{\sqrt{V''}} + O\left(\frac{1}{\Lambda_c}\right) & d = 3 \\ \frac{1}{2\pi} \frac{1}{V''} + O\left(\frac{1}{\Lambda_c^2}\right) & d = 2 \end{cases} .
\end{aligned}$$

like the logarithm of Λ_c at $d = 4$ (and remains finite for $\Lambda_c \rightarrow \infty$ at $d \leq 3$): thus in $\ln\left(\frac{4\Lambda_c^2}{V''}\right) = \ln\left(\frac{4\Lambda_c^2}{\mu^2}\right) - \ln\left(\frac{V''}{\mu^2}\right)$ one introduces an arbitrary scale parameter μ to separate the divergent part, which has to be cancelled by $\delta\beta$:

$$\begin{aligned} -12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_3}{(2\pi)^3} \frac{p^2 dp}{(p^2 + V''(\bar{\rho}))^{3/2}} = \\ = -12\delta\beta - \frac{3\eta}{32\pi^2}(12|\beta|)^2 \ln\left(\frac{4\Lambda_c^2}{\mu^2}\right) + \underbrace{\frac{3\eta}{32\pi^2}(12|\beta|)^2 \ln\left(\frac{V''}{\mu^2}\right)}_{\text{finite at } \Lambda_c \rightarrow \infty}. \end{aligned}$$

In the same way as for $\delta\beta$, from the first bracketed group in (D.294) one gets $\delta\alpha$, being $\delta\alpha$ and $\delta\beta$ equal to

$$\begin{aligned} \delta\beta = \begin{cases} -\frac{3\eta}{32\pi^2} |\beta|(12|\beta|) \ln\left(\frac{4\Lambda_c^2}{\mu^2}\right) + O(\eta^2) & d = 4 \\ 0 \cdot \eta + O(\eta^2) & d = 2, 3 \end{cases} \\ \delta\alpha = \begin{cases} \frac{\eta}{32\pi^2}(12|\beta|) \left[\Lambda_c^2 + |\alpha| \ln\left(\frac{4\Lambda_c^2}{\mu^2}\right) \right] + O(\eta^2) & d = 4 \\ \frac{\eta}{16\pi}(12|\beta|) \Lambda_c + O(\eta^2) & d = 3 \\ \frac{\eta}{32\pi}(12|\beta|) \ln\left(\frac{4\Lambda_c^2}{\mu^2}\right) + O(\eta^2) & d = 2 \end{cases} \quad (\text{D.295}) \end{aligned}$$

up to a finite constant, which can be absorbed into the arbitrary scale μ . For $d = 3$ the logarithmic divergence in $\delta\alpha$, and thus scale μ , appears firstly in the term of order η^2 .

As already mentioned, all integrals contained in $\frac{d^n V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^n}$ are finite at $\Lambda_c = \infty$ for $n \geq 5$, and therefore they do not introduce new divergences. This proves (although the proof is not yet complete, since it only holds up to order $\eta|\beta|$) that the effective potential of model (D.292) can be rendered finite by means of the two counterterms

$$\begin{aligned} V_{\text{eff,ren}}(\bar{\rho})^{(\omega=\infty)} = V(\bar{\rho}) + \frac{\eta}{2} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + V''} - \delta\alpha \bar{\rho}^2 - \frac{\delta\beta}{2} \bar{\rho}^4 \\ \text{with } V(\bar{\rho}) = \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 \end{aligned} \quad (\text{D.296})$$

and is thus renormalizable. The same result holds also for the same scalar model when it is coupled to a gauge field: the reason is that *a*) the propagator of the gauge field (obtained after removing the gauge invariance through a choice of gauge) has a dependence on p^2 similar to that in the propagator of the scalar field, and that *b*) the new introduced interactions produces vertices with up to four legs and the new occurring diagrams are of exactly the same form as in the pure scalar model. Since these the two kinds of renormalizable models are the only ones which we shall consider, it will be understood for the following that we have included the necessary counterterms in the effective potential and the subindex “ren” will be not written explicitly for V_{eff} .

If at $\omega = \infty$ parameter α becomes negative, $\alpha = -|\alpha|$, then the minimum of the effective potential occurs for a non-vanishing $\bar{\rho} = \langle \phi_{\text{scal}} \rangle \neq 0$

$$\left. \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle}^{(\omega=\infty)} = 0, \quad (\text{D.297})$$

where $\langle \phi_{\text{scal}} \rangle$ is equal to $\frac{-\alpha}{|\beta|} = \frac{|\alpha|}{|\beta|}$ (with $V''(-\alpha/|\beta|) > 0$) *plus* the finite correction of order η . In the same way as the parameters of the potential at order η^0 can be defined

through $\left\{ \begin{array}{l} \left. \frac{d^2 V(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=\text{minimum of } V(\bar{\rho})} = -4\alpha = 4|\alpha|, \\ \left. \frac{d^4 V(\bar{\rho})}{d\bar{\rho}^4} \right|_{\bar{\rho}=\text{minimum of } V(\bar{\rho})} = 12|\beta| \end{array} \right.$, the renormalized parameters of the effective potential get defined by

$$\left\{ \begin{array}{l} \left. \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle}^{(\omega=\infty)} = 4|\alpha|_{\text{ren}}, \\ \left. \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle}^{(\omega=\infty)} = 12|\beta|_{\text{ren}}. \end{array} \right. \quad (\text{D.298})$$

From these definitions as well as from equations (D.294) one obtains expectation value $\langle \phi_{\text{scal}} \rangle$ directly as function of the finite renormalized parameters

$$\begin{aligned} 4|\alpha|_{\text{ren}} \equiv \left. \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle}^{(\omega=\infty)} &= 2\alpha - 2\delta\alpha + \underbrace{\frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(\langle \phi \rangle)}} + 2|\beta| \langle \phi \rangle^2 - 2\delta\beta \langle \phi \rangle^2 +}_{\left. \frac{1}{\langle \phi \rangle} \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \right|_{\bar{\rho}=\langle \phi \rangle}^{(\omega=\infty)} = 0} \\ &+ \underbrace{4|\beta| \langle \phi \rangle^2 - 4\delta\beta \langle \phi \rangle^2 - \frac{\eta}{8}(12|\beta|)^2 \langle \phi \rangle^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\langle \phi \rangle))^{3/2}}}_{4|\beta|_{\text{ren}} \langle \phi \rangle^2 \left[1 - \frac{1}{12|\beta|_{\text{ren}}} \boxed{\text{finite}(\bar{\rho})} \right]_{\bar{\rho}=\langle \phi \rangle}} \\ \langle \phi_{\text{scal}} \rangle^2 &= \left(\frac{|\alpha|_{\text{ren}}}{|\beta|_{\text{ren}}} \right) \left[1 + \frac{1}{12|\beta|_{\text{ren}}} \boxed{\text{finite}(\bar{\rho})} \right]_{\bar{\rho}=\langle \phi \rangle} + O(\eta^2), \quad (\text{D.299}) \end{aligned}$$

where inside $\frac{1}{12|\beta|_{\text{ren}}} \boxed{\text{finite}(\bar{\rho})} \Big|_{\bar{\rho}=\langle \phi \rangle}$, which is already of order η , if one only keeps terms up to order $\eta|\beta|$, one can replace original parameters $\alpha, |\beta|$ by the renormalized ones

$\alpha_{\text{ren}}, |\beta|_{\text{ren}}$, being the latter pair equal to ¹⁰²

$$|\beta|_{\text{ren}} = \begin{cases} |\beta| \left[1 + \frac{3\eta}{32\pi^2} (12|\beta|) \left(\ln \left(\frac{V''(\langle\phi\rangle)}{\mu^2} \right) + 5 \right) + O(\eta^2) \right] & d = 4 \\ |\beta| \left[1 - \frac{3\eta}{64\pi} \frac{(12|\beta|)}{\sqrt{V''(\langle\phi\rangle)}} + O(\eta^2) \right] & d = 3 \\ |\beta| \left[1 - \frac{3\eta}{16\pi} \frac{(12|\beta|)}{V''(\langle\phi\rangle)} + O(\eta^2) \right] & d = 2 \end{cases} \quad (\text{D.300})$$

and

$$\alpha_{\text{ren}} = \begin{cases} \alpha \left[1 + \frac{\eta}{32\pi^2} (12|\beta|) \left(\ln \left(\frac{V''(\langle\phi\rangle)}{\mu^2} \right) + 4 \right) + O(\eta^2) \right] & d = 4 \\ \alpha \left[1 + \frac{\eta}{16\pi} \frac{(12|\beta|)}{\sqrt{V''(\langle\phi\rangle)}} + O(\eta^2) \right] & d = 3 \\ \alpha \left[1 + \frac{\eta}{32\pi} \frac{(12|\beta|)}{|\alpha|} \left(\ln \left(\frac{V''(\langle\phi\rangle)}{\mu^2} \right) - \frac{3}{2} \right) + O(\eta^2) \right] & d = 2 \end{cases} \quad (\text{D.301})$$

¹⁰²Using $\eta \frac{\langle\phi_{\text{scal}}\rangle^2}{V''(\langle\phi_{\text{scal}}\rangle)} = \eta \frac{-\alpha/|\beta| + O(\eta)}{-4\alpha + O(\eta)} = \frac{3\eta}{12|\beta|} + O(\eta^2)$ when $\bar{\rho} = \langle\phi_{\text{scal}}\rangle \neq 0$ ($\alpha < 0$) into

(D.293), one obtains $\left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi_{\text{scal}}\rangle} = \begin{cases} \frac{9\eta}{32\pi^2} (12|\beta|)^2 & d = 4 \\ \frac{9\eta}{64\pi} \frac{(12|\beta|)^2}{\sqrt{V''(\langle\phi_{\text{scal}}\rangle)}} & d = 3 \\ 0 & d = 2 \end{cases}$. Hence, from relation $12|\beta|_{\text{ren}} \equiv$

$\equiv \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \Big|_{\bar{\rho}=\langle\phi_{\text{scal}}\rangle}^{(\omega=\infty)} = 12|\beta| + \underbrace{\left[-12\delta\beta - \frac{3\eta}{8} (12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\langle\phi\rangle))^{3/2}} \right]}_{= \text{finite part of the integral}} + \left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi\rangle}$ one

finds $|\beta|_{\text{ren}} = |\beta| \left[1 - \frac{3\eta}{8} (12|\beta|) \left(\text{finite part of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\langle\phi\rangle))^{3/2}} \right) + \frac{1}{12|\beta|} \left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi\rangle} \right]$.

Taking account of this last result and (D.299), and keeping terms only up to order η , i.e., using $\delta\beta \langle\phi\rangle^2 = \delta\beta \frac{-\alpha}{|\beta|} + O(\eta^2)$ as well as $\frac{1}{12|\beta|_{\text{ren}}} \left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi\rangle} = \frac{1}{12|\beta|} \left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi\rangle} + O(\eta^2)$, one obtains from $\frac{1}{\langle\phi_{\text{scal}}\rangle} \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \Big|_{\bar{\rho}=\langle\phi\rangle}^{(\omega=\infty)} = 0$ in (D.294) the following relation for parameter $\alpha_{\text{ren}} = -|\alpha|_{\text{ren}}$:

$0 = 2\alpha + 2|\beta| \langle\phi\rangle^2 + \underbrace{\left[-2\delta\alpha - 2\delta\beta \langle\phi\rangle^2 + \frac{\eta}{4} (12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(\langle\phi\rangle)}} \right]}_{= \text{finite part of the integral}} = 2\alpha + 2|\beta| \frac{|\alpha|_{\text{ren}}}{|\beta|} \left[1 + \frac{\left[\text{finite}(\bar{\rho}) \right]}{12|\beta|_{\text{ren}}} \right]_{\bar{\rho}=\langle\phi\rangle} +$

$+ \frac{3\eta}{8} (12|\beta|) \left(\text{fin. part} \int_0^{\Lambda_c} \left(\frac{p^{d-2} dp}{(p^2 + V''(\langle\phi\rangle))^{3/2}} \right) - \frac{\left[\text{finite}(\bar{\rho}) \right]}{12|\beta|} \right)_{\bar{\rho}=\langle\phi\rangle} + \frac{\eta}{4} (12|\beta|) \left(\text{fin. part} \int_0^{\Lambda_c} \left(\frac{p^{d-2} dp}{\sqrt{p^2 + V''(\langle\phi\rangle)}} \right) + O(\eta^2) \right)$

and therefore $2\alpha_{\text{ren}} = 2\alpha \left[1 + \frac{\eta}{8} \frac{(12|\beta|)}{\alpha} \left(\text{finite part of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(\langle\phi\rangle)}} \right) - \frac{3\eta}{8} (12|\beta|) \times \right.$

$\times \left(\text{finite part of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\langle\phi\rangle))^{3/2}} \right) + O(\eta^2) \Big]$, with $V''(\langle\phi\rangle) = 4|\alpha| + O(\eta)$.

as well as the expectation value of the scalar field as function of the original parameters is given by

$$\langle \phi \rangle^2 = \begin{cases} \left(\frac{|\alpha|}{|\beta|} \right) \left[1 - \frac{2\eta}{32\pi^2} (12|\beta|) \left(\ln \left(\frac{V''(\langle \phi \rangle)}{\mu^2} \right) + 1 \right) + O(\eta^2) \right] & d = 4 \\ \left(\frac{|\alpha|}{|\beta|} \right) \left[1 + \frac{2\eta}{8\pi} \frac{(12|\beta|)}{\sqrt{V''(\langle \phi \rangle)}} + O(\eta^2) \right] & d = 3 \\ \left(\frac{|\alpha|}{|\beta|} \right) \left[1 + \frac{2\eta}{64\pi} \frac{(12|\beta|)}{|\alpha|} \ln \left(\frac{V''(\langle \phi \rangle)}{\mu^2} \right) + O(\eta^2) \right] & d = 2 \end{cases} \quad (\text{D.302})$$

with $V''(\langle \phi \rangle) = -4\alpha + O(\eta) = -4\alpha_{\text{ren}} + O(\eta)$ ($\alpha, \alpha_{\text{ren}} < 0$). This shows how to construct the finite renormalized effective potential up to order η : from relations (D.294)

$$\begin{aligned} \frac{1}{\bar{\rho}} \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} = & \begin{cases} 2\alpha_{\text{ren}} \left(1 - \frac{3\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{12\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + \\ 2\alpha_{\text{ren}} \left(1 - \frac{\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle \phi \rangle)}} \right) + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 + \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle \phi \rangle)}} \right) - \\ 2\alpha_{\text{ren}} \left(1 - \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right) + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right) - \end{cases} \\ & + \frac{\eta(12|\beta|_{\text{ren}})}{32\pi^2} (2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2) \ln \left| \frac{V''(\bar{\rho})}{V''(\langle \phi \rangle)} \right| + O(\eta^2) \quad d = 4 \\ & - \frac{\eta(12|\beta|_{\text{ren}})}{8\pi} \sqrt{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2} + O(\eta^2) \quad d = 3, \quad (\text{D.303}) \\ & - \frac{\eta(12|\beta|_{\text{ren}})}{16\pi} \ln \left| \frac{V''(\bar{\rho})}{V''(\langle \phi \rangle)} \right| + O(\eta^2) \quad d = 2 \end{aligned}$$

we can obtain the effective potential, equal (up to a constant independent of $\bar{\rho}$ which is the value of the effective potential at $\bar{\rho} = 0$) to

$$\begin{aligned} V_{\text{eff}}^{(\omega=\infty)} = & \begin{cases} \alpha_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{3\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 - \frac{12\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + \\ \alpha_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle \phi \rangle)}} \right) + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 + \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle \phi \rangle)}} \right) - \\ \alpha_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right) + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 - \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right) - \end{cases} \\ & + \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)^2}{64\pi^2} \left(\ln \left| \frac{V''(\bar{\rho})}{V''(\langle \phi \rangle)} \right| - \frac{1}{2} \right) + O(\eta^2) \quad d = 4 \\ & - \frac{\eta}{12\pi} (2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)^{3/2} + O(\eta^2) \quad d = 3, \quad (\text{D.304}) \\ & - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)}{16\pi} \left(\ln \left| \frac{V''(\bar{\rho})}{V''(\langle \phi \rangle)} \right| - 1 \right) + O(\eta^2) \quad d = 2 \end{aligned}$$

with $V''(\bar{\rho}) = 2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2 + O(\eta)$. The above potential is now function of finite parameters. It is noteworthy that although the corrections of order η are negative to the

parameter of the square power of $\bar{\rho}$ (such corrections may even introduce new powers of $\bar{\rho}$ as for $d = 3$), this leads nevertheless to no new minima nor to the disappearance of the already existing minimum: all these corrections are multiplied by $\eta|\beta|$, which from the very beginning has been assumed to be very small, and therefore cannot counterbalance a number of order 1.

All the above process has been carried out for the case $\alpha < 0$. If we were interested in the opposite case, when the potential's minimum is $\langle\phi\rangle = 0$ and $\alpha > 0$, the renormalized parameters are defined analogously as in (D.298): since the original parameters fulfil

$$\begin{cases} V''(0) = 2\alpha > 0 \\ V''''(0) = 12|\beta| \end{cases}, \text{ the renormalized ones gets defined by } \begin{cases} \left. \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega=\infty)} = 2\alpha_{\text{ren}} \\ \left. \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \right|_{\bar{\rho}=0}^{(\omega=\infty)} = 12|\beta|_{\text{ren}} \end{cases};$$

$$\left. \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \right|_{\bar{\rho}=0}^{(\omega=\infty)} = 0 \text{ is identically satisfied since } \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \Big|_{\bar{\rho}=0}^{(\omega=\infty)} \sim \bar{\rho}. \text{ The counterterms,}$$

needed to render finite the effective model at very short distances, are just the same as in the case of a potential's nonvanishing minimum: relations (D.295) ¹⁰³. For the sake

¹⁰³It is important to note that the value of $\delta\alpha$ and $\delta\beta$ does not depend on which $\bar{\rho}$ has been taken for the derivatives of the effective potential. If we have obtained $\delta\alpha, \delta\beta$ at a certain value $\bar{\rho}_0$, their values at $\bar{\rho}_1$ become as follows: from equations (D.294) one obtains for the fourth equation

$$\begin{aligned} -12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\bar{\rho}_1^2))^{3/2}} &= \underbrace{-12\delta\beta - \frac{3\eta}{8}(12|\beta|)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\bar{\rho}_0^2))^{3/2}}}_{\text{finite by definition of } \delta\beta \text{ at } \bar{\rho}_0} + \\ &+ \underbrace{\frac{9\eta}{32}(12|\beta|)^3 (\bar{\rho}_1^2 - \bar{\rho}_0^2) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\bar{\rho}_0^2))^{5/2}}}_{\text{finite at } \Lambda_c \rightarrow \infty \text{ for } d \leq 4} + \dots, \end{aligned}$$

and for the second equation of (D.294)

$$\begin{aligned} -2\delta\alpha - 2\delta\beta\bar{\rho}_1^2 + \frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(\bar{\rho}_1^2)}} &= \underbrace{-2\delta\alpha - 2\delta\beta\bar{\rho}_0^2 + \frac{\eta}{4}(12|\beta|) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(\bar{\rho}_0^2)}}}_{\text{finite by definition of } \delta\alpha \text{ at } \bar{\rho}_0} \\ &+ \underbrace{-2\delta\beta (\bar{\rho}_1^2 - \bar{\rho}_0^2) - \frac{\eta}{16}(12|\beta|)^2 (\bar{\rho}_1^2 - \bar{\rho}_0^2) \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\bar{\rho}_0^2))^{3/2}}}_{\text{finite by definition of } \delta\beta \text{ at } \bar{\rho}_0} + \\ &+ \underbrace{\frac{3\eta}{64}(12|\beta|)^3 (\bar{\rho}_1^2 - \bar{\rho}_0^2)^2 \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(\bar{\rho}_0^2))^{5/2}}}_{\text{finite at } \Lambda_c \rightarrow \infty \text{ for } d \leq 4} + \dots, \end{aligned}$$

of completeness, let us write the effective potential for $\alpha > 0$ ¹⁰⁴

$$V_{\text{eff,ren}}^{(\omega=\infty)} = \begin{cases} \alpha_{\text{ren}} \bar{\rho}^2 + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 - \frac{3\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + \\ \alpha_{\text{ren}} \bar{\rho}^2 \left(1 + \frac{\eta}{8\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{2\alpha_{\text{ren}}}} \right) + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 + \frac{3\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{2\alpha_{\text{ren}}}} \right) - \\ \alpha_{\text{ren}} \bar{\rho}^2 + \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 + \frac{3\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{2\alpha_{\text{ren}}} \right) - \\ + \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)^2}{64\pi^2} \left(\ln \left(\frac{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2}{2\alpha_{\text{ren}}} \right) - \frac{1}{2} \right) + O(\eta^2) & d = 4 \\ - \frac{\eta}{12\pi} (2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)^{3/2} + O(\eta^2) & d = 3 \\ - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2)}{16\pi} \left(\ln \left(\frac{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2}{2\alpha_{\text{ren}}} \right) - 1 \right) + O(\eta^2) & d = 2 \end{cases} \quad (\text{D.305})$$

being thus both $\left. \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \right|_{\bar{\rho}=\bar{\rho}_1}^{(\omega=\infty)}$ and $\left. \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=\bar{\rho}_1}^{(\omega=\infty)}$ finite. The renormalized parameters for

$$\langle \phi \rangle = 0 \text{ and } \alpha > 0 \text{ are: } |\beta|_{\text{ren}} = |\beta| \left[1 - \frac{3\eta}{8} (12|\beta|) \left(\text{finite part of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{(p^2 + V''(0))^{3/2}} \right) + O(\eta^2) \right] =$$

$$= \begin{cases} |\beta| \left[1 + \frac{3\eta}{32\pi^2} (12|\beta|) \left(\ln \left(\frac{2\alpha}{\mu^2} \right) + 2 \right) + O(\eta^2) \right] & d = 4 \\ |\beta| \left[1 - \frac{3\eta}{16\pi} \frac{(12|\beta|)}{\sqrt{2\alpha}} + O(\eta^2) \right] & d = 3 \\ |\beta| \left[1 - \frac{3\eta}{16\pi} \frac{(12|\beta|)}{2\alpha} + O(\eta^2) \right] & d = 2 \end{cases} \quad \text{as well as } \alpha_{\text{ren}} = \alpha \left[1 + \frac{\eta}{8} \frac{(12|\beta|)}{\alpha} \times \right.$$

$$\left. \times \left(\text{fin.p.of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + V''(0)}} \right) + O(\eta^2) \right] = \begin{cases} \alpha \left[1 + \frac{\eta}{32\pi^2} (12|\beta|) \left(\ln \left(\frac{2\alpha}{\mu^2} \right) + 1 \right) + O(\eta^2) \right] & d = 4 \\ \alpha \left[1 - \frac{\eta}{8\pi} \frac{(12|\beta|)}{\sqrt{2\alpha}} + O(\eta^2) \right] & d = 3 \\ \alpha \left[1 - \frac{\eta}{16\pi} \frac{(12|\beta|)}{2\alpha} \ln \left(\frac{2\alpha}{\mu^2} \right) + O(\eta^2) \right] & d = 2 \end{cases} .$$

$$^{104} \text{By integrating result } \frac{1}{\bar{\rho}} \frac{dV_{\text{eff}}}{d\bar{\rho}} \Big|_{\bar{\rho}=\bar{\rho}_1}^{(\omega=\infty)} = \begin{cases} 2\alpha_{\text{ren}} + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 - \frac{3\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) + \\ 2\alpha_{\text{ren}} \left(1 + \frac{\eta}{8\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{2\alpha_{\text{ren}}}} \right) + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 + \frac{3\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{2\alpha_{\text{ren}}}} \right) - \\ 2\alpha_{\text{ren}} + 2|\beta|_{\text{ren}} \bar{\rho}^2 \left(1 + \frac{3\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{2\alpha_{\text{ren}}} \right) - \end{cases}$$

$$+ \frac{\eta(12|\beta|_{\text{ren}})}{32\pi^2} (2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2) \ln \left(\frac{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2}{2\alpha_{\text{ren}}} \right) + O(\eta^2) \quad d = 4 \\ - \frac{\eta(12|\beta|_{\text{ren}})}{8\pi} \sqrt{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2} + O(\eta^2) \quad d = 3 \\ - \frac{\eta(12|\beta|_{\text{ren}})}{16\pi} \ln \left(\frac{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}} \bar{\rho}^2}{2\alpha_{\text{ren}}} \right) + O(\eta^2) \quad d = 2$$

and for $\alpha = 0$ ¹⁰⁵

$$V_{\text{eff, ren}}^{(\omega=\infty)} = \begin{cases} \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 + \frac{\eta (12|\beta|_{\text{ren}})^2}{256\pi^2} \bar{\rho}^4 \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) - \frac{25}{6} \right) & d = 4 \\ \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 - \frac{\eta (12|\beta|_{\text{ren}})^{3/2}}{24\sqrt{2}\pi} \bar{\rho}^3 & d = 3 \\ \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 - \frac{\eta (12|\beta|_{\text{ren}})}{32\pi} \bar{\rho}^2 \left(\frac{\bar{\rho}^2}{6\mu_0^2} + \ln\left(\frac{6|\beta|_{\text{ren}}\bar{\rho}^2}{\mu^2}\right) - 1 \right) & d = 2 \end{cases} \quad (\text{D.306})$$

¹⁰⁵A special case is $\alpha = 0$: the potential's minimum still corresponds to $\langle \phi \rangle = 0$ and the counterterm containing $-2\delta\alpha$ cancels completely integral $\frac{\eta}{4}(12|\beta|)\int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2+0}}$, ensuring thus

$2\alpha_{\text{ren}} = \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0}^{(\omega=\infty)} = 2\alpha = 0$. Furtherly, since in the original potential $V''''(\bar{\rho}) = 12|\beta| \quad \forall \bar{\rho}$, one chooses to define renormalized $|\beta|_{\text{ren}}$ at a certain arbitrary renormalization point $\bar{\rho} = \mu_0$ (instead of taking $\bar{\rho} = 0$), $\frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \Big|_{\bar{\rho}=\mu_0}^{(\omega=\infty)} = 12|\beta|_{\text{ren}}$, in order to avoid nonsensical divisions by zero which

would arise from $V''(0) = 0$. For $\alpha = 0$, one gets $\boxed{\text{finite}(\bar{\rho})} = \begin{cases} \frac{\eta}{4\pi^2} (12|\beta|)^2 & d = 4 \\ \frac{3\eta}{16\pi} \frac{(12|\beta|)^2}{\sqrt{V''(\bar{\rho})}} & d = 3 \\ \frac{\eta}{4\pi} \frac{(12|\beta|)^2}{V''(\bar{\rho})} & d = 2 \end{cases}$ and therefore

$$|\beta|_{\text{ren}} = |\beta| \left[1 - \frac{3\eta}{8} (12|\beta|) \left(\text{finite part of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{(p^2 + V''(\mu_0))^{3/2}} \right) + \frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})} \Big|_{\bar{\rho}=\mu_0} + O(\eta^2) \right] =$$

$$= \begin{cases} |\beta| \left[1 + \frac{3\eta(12|\beta|)}{32\pi^2} \left(\ln\left(\frac{6|\beta|\mu_0^2}{\mu^2}\right) + \frac{14}{3} \right) + O(\eta^2) \right] & d = 4 \\ |\beta| \left[1 + O(\eta^2) \right] & d = 3 \\ |\beta| \left[1 + \frac{\eta}{8\pi\mu_0^2} + O(\eta^2) \right] & d = 2 \end{cases}, \text{ yielding } \frac{1}{\bar{\rho}} \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} = 2|\beta|\bar{\rho}^2 + \frac{\eta}{4}(12|\beta|) \times$$

$$\times \left(\text{finite p.of } \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}dp}{\sqrt{p^2 + V''(\bar{\rho})}} \right) = \begin{cases} 2|\beta|_{\text{ren}}\bar{\rho}^2 + \frac{\eta (12|\beta|_{\text{ren}})^2}{64\pi^2} \bar{\rho}^2 \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) - \frac{11}{3} \right) & d = 4 \\ 2|\beta|_{\text{ren}}\bar{\rho}^2 - \frac{\eta (12|\beta|_{\text{ren}})^{3/2}}{8\sqrt{2}\pi} \bar{\rho} & d = 3 \\ 2|\beta|_{\text{ren}}\bar{\rho}^2 \left[1 - \frac{\eta}{8\pi\mu_0^2} \right] - \frac{\eta (12|\beta|_{\text{ren}})}{16\pi} \ln\left(\frac{6|\beta|_{\text{ren}}\bar{\rho}^2}{\mu^2}\right) & d = 2 \end{cases},$$

relation from which effective potential (D.306) follows directly through integration. Following is noteworthy: if we would define the renormalized parameter at another different renormalization point

$$\mu'_0 \text{ (only relevant for } d = 4 \text{ and } d = 2 \text{ and shown here for } d = 4), |\beta'|_{\text{ren}} = \frac{1}{12} \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \Big|_{\bar{\rho}=\mu'_0}^{(\omega=\infty)} =$$

$$= |\beta| \left[1 + \frac{3\eta}{32\pi^2} (12|\beta|) \left(\ln\left(\frac{6|\beta|\mu_0'^2}{\mu^2}\right) + \frac{14}{3} \right) + O(\eta^2) \right] = |\beta|_{\text{ren}} + \frac{\eta (12|\beta|_{\text{ren}})^2}{64\pi^2} \ln\left(\frac{\mu_0'^2}{\mu_0^2}\right) + O(\eta^2), \quad \text{the}$$

same effective potential results [27]

$$\frac{1}{\bar{\rho}^3} \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} = 2|\beta'|_{\text{ren}} + \frac{\eta (12|\beta'|_{\text{ren}})^2}{64\pi^2} \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0'^2}\right) - \frac{11}{3} \right) = 2|\beta|_{\text{ren}} + \frac{\eta (12|\beta|_{\text{ren}})^2}{64\pi^2} \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) - \frac{11}{3} \right) + O(\eta^2),$$

a particular example of the model's independence of the choice of μ_0 . For $d = 2$ the analogous proof goes along the same steps although then it is almost trivial since up to order η following holds:

$$|\beta'|_{\text{ren}} \left[1 - \frac{\eta}{8\pi\mu_0'^2} \right] = |\beta| = |\beta|_{\text{ren}} \left[1 - \frac{\eta}{8\pi\mu_0^2} \right].$$

where, for $d = 4$ (and $d = 2$), μ_0 is the arbitrary renormalization point used to define renormalized parameter $|\beta|_{\text{ren}} = \frac{1}{12} \frac{d^4 V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}}{d\bar{\rho}^4} \Big|_{\bar{\rho}=\mu_0}$ and has not to coincide with the scale μ used to “mutilate” the short-range content. Such a μ_0 , nevertheless, plays no role in the physics: for instance at $d = 4$, and due to the relation between the renormalized parameter at a certain scale $\bar{\rho}$ and the value of the same parameter at $\bar{\rho} = \mu_0$, one gets

$$\frac{1}{12|\beta|_{\text{ren, at } \bar{\rho}}} = \frac{1}{12|\beta|_{\text{ren, at } \bar{\rho}=\mu_0}} - \frac{3}{32\pi^2} \ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right), \quad (\text{D.307})$$

and as a result, the form of the effective potential remains unchanged when written in terms of $|\beta|_{\text{ren, at } \bar{\rho}}$. Equation (D.307) is a particular case of a so-called (renormalized) running coupling constant. It is important to stress that all the above discussion has been possible because we are considering that $\eta|\beta|$ is a perturbative small parameter.

After having discussed how to remove divergences from the short-distance behaviour of the effective potential at $\omega = \infty$, let us return to study of the ω -dependent part (D.291):

$V_{\text{eff}}^{(\text{dep. } \omega)} = \frac{\eta}{\omega^d} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - e^{-\sqrt{p^2 + \omega^2 V''(\bar{\rho})}} \right]$ (with p dimensionless). This part, nevertheless, and its derivatives does not produce new divergences, as one can see, for instance, from $\frac{dV_{\text{eff}}^{(\text{dep. } \omega)}}{d\bar{\rho}^2} = \frac{12\eta|\beta|}{4\omega^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + \omega^2 V''(\bar{\rho})}} \frac{1}{e^{\sqrt{p^2 + \omega^2 V''(\bar{\rho})}} - 1}$: in the

short-distance range $p \sim \frac{\Lambda_c}{\omega} \sim \infty$ there occurs no divergence due to the damping exponential. Therefore the counterterms needed to remove the divergences in the ω -independent part of the effective potential already suffices for the whole potential [56]; and at order $\eta|\beta|$, the parameters of $V''(\bar{\rho})$ in the integral (which is already multiplied by $\eta|\beta|$) can be substituted by the renormalized ones: $V''(\bar{\rho}) = 2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2 + O(\eta)$, although for the following the subindex “ren” will often be avoided in order to shorten the notation. The question under study: for a system which at $\omega = \infty$ has $\alpha = -|\alpha| < 0$ and therefore shows a $\langle\phi\rangle \neq 0$ as minimum of the effective potential ¹⁰⁶, calculate the

¹⁰⁶ From result (D.302), the corrections of order $\eta|\beta|$ (and those of higher orders) in the *pure scalar model* cannot make $\langle\phi\rangle$ vanish due to the very nature of a perturbative parameter: if for $\eta = 0$ we had $\langle\phi\rangle \neq 0$, the same will hold for a very small $\eta \neq 0$, at least as long as $\omega = \infty$ and there is no other parameter which could compete with the (perturbative small) dimensionless $\eta|\beta|_{\text{ren}}$ ($|\alpha|_{\text{ren}}^{(d-4)/2}$). The same holds in the opposite case: if $\alpha = 0$ (and thus $\langle\phi\rangle = 0$) for $\eta = 0$, there is no symmetry breaking caused by corrections of order η . For instance, at $d = 4$ effective potential (D.306) could have a nonvanishing minimum $\bar{\rho} = \langle\phi\rangle \neq 0$: $0 = \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} = \bar{\rho}^3 \left[2|\beta|_{\text{ren}} + \frac{\eta(12|\beta|_{\text{ren}})^2}{64\pi^2} \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) - \frac{11}{3} \right) \right]$ for $\eta(12|\beta|_{\text{ren}}) \ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) = -\frac{32\pi^2}{3} \left(1 - \frac{11}{32}\eta|\beta|_{\text{ren}} \right) \eta|\beta|_{\text{ren}} \approx^{\text{very small}} -\frac{32\pi^2}{3}$, but since we would then equate a quantity of (perturbative) order $\eta|\beta|$ with a quantity of order 1, the result would lie outside the range where perturbation in first order is valid ([27] and [53]). A completely different case occurs if the model, beside $|\beta|$, owns another parameters λ in which we can expand the effective potential for the expectation value of the scalar field: in such a case we would obtain $\frac{\eta|\beta|_{\text{ren}}}{(\eta\lambda)^n} \ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) \sim \text{order } 1$, and although both $\eta|\beta|$ and $\eta\lambda$ are very small, their quotient can be of order 1. This already occurs for $\lambda = e^2$, with e^2

critical value of periodicity ω for which the minimum moves to $\langle\phi\rangle = 0$. According to equation (D.286), we have to solve

$$\frac{12\eta|\beta|_{\text{ren}}}{4\omega_{\text{crit}}^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega_{\text{crit}}^2} V''(0)} \frac{dp}{e^{\sqrt{p^2 + \omega_{\text{crit}}^2} V''(0)} - 1} = - \frac{dV_{\text{eff}}(\bar{\rho}^2)}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0}^{(\omega=\infty)}, \quad (\text{D.308})$$

where $\frac{dV_{\text{eff}}(\omega=\infty)}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0} = \frac{1}{2} \left(\frac{1}{\bar{\rho}} \frac{dV_{\text{eff}}(\omega=\infty)}{d\bar{\rho}} \right) \Big|_{\bar{\rho}=0}$ is obtained from (D.303). In the range of *small* ω_{crit} , more exactly, when $\omega_{\text{crit}}^2 \ll \frac{1}{|V''(0)|}$, equation (D.308) can be solved by using the results of the following Note for integral $I_n(M) = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1}$, which will be obtained as an expansion in powers of $M^2 \equiv \omega^2 V''(0)$.

Note. Calculation of $I_n(M) = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1}$ for $n \geq 1$ ([31] and [56]).

The first step in order to calculate such an integral is to show the following property

$$\begin{aligned} \frac{dI_{n+1}(M)}{dM} &= 2M \frac{dI_{n+1}}{d(M^2)} = \frac{2M}{n!} \int_0^\infty \frac{\partial}{\partial x^2} \left[\frac{1}{\sqrt{x^2 + M^2} (e^{\sqrt{x^2 + M^2}} - 1)} \right] \underbrace{x^n \frac{d(x^2)}{2x}}_{\text{partial integr.}} \\ &\quad \frac{\partial}{\partial M^2} \left[\frac{1}{\sqrt{x^2 + M^2} (e^{\sqrt{x^2 + M^2}} - 1)} \right] \frac{1}{2} (x^2)^{\frac{n-1}{2}} \\ &= 0 - \frac{M}{n!} (n-1) \int_0^\infty \frac{x^{n-2}}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1} = -\frac{M}{n} I_{n-1}(M). \end{aligned} \quad (\text{D.309})$$

Thus one only needs $I_1(M)$ and $I_2(M)$; the latter is calculated directly $\int_0^\infty \frac{x}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1} = \left[\ln(1 - e^{-\sqrt{x^2 + M^2}}) \right]_0^\infty = -\ln(1 - e^{-M})$, whose expansion in M -powers is

$$I_2(M) = -\ln(1 - e^{-M}) = -\ln M - \ln \left(1 - \frac{M}{2!} + \frac{M^2}{3!} - \frac{M^3}{4!} + \frac{M^4}{5!} + \dots \right),$$

$$I_2(M) = -\ln M + \frac{1}{2}M - \frac{1}{12}M^2 - \frac{11}{720}M^4 + \dots \quad (\text{D.310})$$

$I_1(M)$ is a bit more involved: in order to avoid singularities in the integral, one multiplies the integrand by $x^{-\epsilon}$ (with $\epsilon > 0$ but very small), carries out the integration and at the end takes limit $\lim_{\epsilon \rightarrow 0}$

$$I_1(M) = \lim_{\epsilon \rightarrow 0} I_{1,\epsilon}(M), \quad \text{with} \quad I_{1,\epsilon}(M) = \int_0^\infty \frac{x^{-\epsilon}}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1}.$$

Using relation $\sum_{n=1}^\infty \frac{z}{z^2 + n^2} = -\frac{1}{2z} + \frac{\pi}{2} \coth(\pi z)$, or equivalently $\sum_{n=1}^\infty \frac{z}{z^2 + (2\pi n)^2} = -\frac{1}{2z} + \frac{1}{4} \frac{e^z + 1}{e^z - 1}$

$$\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + 2 \sum_{n=1}^\infty \frac{z}{z^2 + (2\pi n)^2},$$

the coupling constant to the gauge field.

which substituted into $I_{1,\epsilon}(M)$ gives

$$I_{1,\epsilon}(M) = -\frac{1}{2} \int_0^\infty \frac{x^{-\epsilon}}{\sqrt{x^2 + M^2}} dx + \int_0^\infty \frac{x^{-\epsilon}}{x^2 + M^2} dx + 2 \sum_{n=1}^\infty \int_0^\infty \frac{x^{-\epsilon}}{x^2 + M^2 + (2\pi n)^2} dx. \quad (\text{D.311})$$

The first integral is calculated with the help of Beta function $B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ and the behaviour of Gamma function $\Gamma(\epsilon) \sim \frac{1}{\epsilon}$ for an infinitesimal ϵ ¹⁰⁷:

$$\begin{aligned} -\frac{1}{2} \int_0^\infty \frac{x^{-\epsilon}}{\sqrt{x^2 + M^2}} dx &= -\frac{M^{-\epsilon}}{2} \int_0^\infty \frac{x^{-\epsilon}}{\sqrt{1+x^2}} dx \stackrel{t=x^2}{=} -\frac{M^{-\epsilon}}{4} \int_0^\infty \frac{t^{\frac{1-\epsilon}{2}-1}}{\sqrt{1+t}} dt = -\frac{M^{-\epsilon}}{4} B\left(\frac{1}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \approx \\ &\approx -\frac{M^{-\epsilon}}{4} B\left(\frac{1}{2}, \frac{\epsilon}{2}\right) = -\frac{1}{8} \left(\frac{M}{2}\right)^{-\epsilon} \underbrace{B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)}_{\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(\epsilon)}} \stackrel{(M/2)^{-\epsilon} \approx 1 - \epsilon \ln(M/2)}{\approx} -\frac{1}{2\epsilon} + \frac{1}{2} \ln\left(\frac{M}{2}\right). \end{aligned} \quad (\text{D.312})$$

By using $\int_0^\infty \frac{t^{p-1}}{1+t} dt = \frac{\pi}{\sin(p\pi)}$ with $0 < p < 1$, the second integral in (D.311) becomes

$$\int_0^\infty \frac{x^{-\epsilon}}{x^2 + M^2} dx = \frac{1}{M^{\epsilon+1}} \int_0^\infty \frac{x^{-\epsilon}}{1+x^2} dx \stackrel{t=x^2}{=} \frac{1}{2M^{\epsilon+1}} \int_0^\infty \frac{t^{\frac{1-\epsilon}{2}-1}}{1+t} dt = \frac{1}{2M^{\epsilon+1}} \frac{\pi}{\cos\left(\frac{\epsilon\pi}{2}\right)} \stackrel{\epsilon \approx 0}{\approx} \frac{\pi}{2M^{\epsilon+1}}. \quad (\text{D.313})$$

Using this last equation, integral $\int_0^\infty \frac{x^{-\epsilon}}{x^2 + M^2 + (2\pi n)^2} dx$ results in $\frac{\pi}{2\left(\sqrt{M^2 + (2\pi n)^2}\right)^{\epsilon+1}}$ and thus the third integral in (D.311) takes the following form

$$\begin{aligned} 2 \sum_{n=1}^\infty \int_0^\infty \frac{x^{-\epsilon}}{x^2 + M^2 + (2\pi n)^2} dx &= \pi \sum_{n=1}^\infty \frac{1}{(2\pi n)^{\epsilon+1}} \left\{ \left[\frac{1}{\left(\sqrt{1 + \left(\frac{M}{2\pi n}\right)^2}\right)^{\epsilon+1}} - 1 \right] + 1 \right\} = \\ &= \pi \sum_{n=1}^\infty \frac{1}{(2\pi n)^{\epsilon+1}} + \pi \sum_{n=1}^\infty \frac{1}{(2\pi n)^{\epsilon+1}} \underbrace{\left[\frac{1}{\left(\sqrt{1 + \left(\frac{M}{2\pi n}\right)^2}\right)^{\epsilon+1}} - 1 \right]}_{-\frac{\epsilon+1}{2} \left(\frac{M}{2\pi n}\right)^2 + \frac{(\epsilon+1)(\epsilon+3)}{8} \left(\frac{M}{2\pi n}\right)^4 + \dots}, \end{aligned}$$

where the second summand has been expanded in powers of M^2 . The first of the above sums is equal to $\frac{\pi}{(2\pi)^{\epsilon+1}} \zeta(1+\epsilon)$, where $\zeta(x)$ is the Riemann zeta function defined as $\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x}$: for an infinitesimal

¹⁰⁷Plus relation $B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right)$ (equation **8.384.4** in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego (1994)).

positive ϵ , $\zeta(\epsilon + 1)$ is given [31] by $\zeta(\epsilon + 1) = (2\pi)^\epsilon \left(\frac{1}{\epsilon} - \ln(2\pi) + \gamma \right)$ with $\gamma = 0.5772 \dots$ the Euler's constant. The second sum contains $\zeta(3 + \epsilon), \zeta(5 + \epsilon), \dots$ which show no singularity for $\epsilon \rightarrow 0$ and thus

$$2 \sum_{n=1}^{\infty} \int_0^{\infty} \frac{x^{-\epsilon}}{x^2 + M^2 + (2\pi n)^2} dx = \frac{1}{2} \left(\frac{1}{\epsilon} - \ln(2\pi) + \gamma \right) + \frac{\pi}{(2\pi)^{\epsilon+1}} \left[-\frac{\epsilon+1}{2} \left(\frac{M}{2\pi} \right)^2 \zeta(3 + \epsilon) + \right. \\ \left. + \frac{(\epsilon+1)(\epsilon+3)}{8} \left(\frac{M}{2\pi} \right)^4 \zeta(5 + \epsilon) + \dots \right]. \quad (\text{D.314})$$

Substituting (D.312), (D.313) and (D.314) into (D.311) and then taking limit $\epsilon \rightarrow 0$, one obtains

$$I_1(M) = \frac{\pi}{2M} + \frac{1}{4} \ln \left(\frac{M^2}{16\pi^2} \right) + \frac{\gamma}{2} - \frac{1}{4} \left(\frac{M}{2\pi} \right)^2 \zeta(3) + \frac{3}{16} \left(\frac{M}{2\pi} \right)^4 \zeta(5) + \dots, \quad (\text{D.315})$$

with $\zeta(3) = 1.202 \dots$ and $\zeta(5) = 1.037 \dots$. Once $I_1(M)$ and $I_2(M)$ are calculated, applying relation $\frac{dI_{n+1}(M)}{dM} = -\frac{M}{n} I_{n-1}(M)$ one can obtain the other $I_n(M)$'s ¹⁰⁸, being the first ones equal to

$$I_2(M) = -\frac{1}{2} \ln M^2 + \frac{1}{2} M - \frac{1}{12} M^2 - \frac{11}{720} M^4 + \dots, \quad (\text{D.316})$$

$$I_3(M) = \frac{\pi^2}{12} - \frac{\pi}{4} M - \frac{M^2}{16} \left(\ln \left(\frac{M^2}{16\pi^2} \right) - 1 + 2\gamma \right) + \frac{\zeta(3)}{128\pi^2} M^4 + \dots, \quad (\text{D.317})$$

$$I_4(M) = \frac{\zeta(3)}{3} + \frac{M^2}{12} (\ln M^2 - 1) - \frac{M^3}{18} + \frac{M^4}{144} + \frac{11}{12960} M^6 \dots, \quad (\text{D.318})$$

$$I_5(M) = \frac{\pi^4}{360} - \frac{\pi^2}{96} M^2 + \frac{\pi}{48} M^3 + \frac{M^4}{256} \left(\ln \left(\frac{M^2}{16\pi^2} \right) - \frac{3}{2} + 2\gamma \right) - \frac{\zeta(3)}{3072\pi^2} M^6 + \dots \quad (\text{D.319})$$

It should be understood that $\ln M^2$ must be substituted by $\ln |M^2|$ for $M^2 < 0$.

The critical ω -value for the transition from an “ordered” state at large ω , and characterized by $\langle \phi \rangle \neq 0$, to a “disordered” phase with $\langle \phi \rangle = 0$ is given by the solution of equation $\frac{12\eta|\beta|_{\text{ren}}}{2\omega_{\text{crit}}^{d-2}} \frac{S_{d-1}}{(2\pi)^{d-1}} (d-2)! I_{d-1}(\omega_{\text{crit}} \sqrt{V''(0)}) = - \left(\frac{1}{\bar{\rho}} \frac{dV_{\text{eff}}(\omega=\infty)}{d\bar{\rho}} \right) \Big|_{\bar{\rho}=0}$. If the model's parameters are such that the transition occurs in the range of small ω , and since the equation to solve already contains $\frac{1}{\omega_{\text{crit}}^{d-2}}$, one gets ω_{crit} by substituting the constant term of the just obtained expansions (D.315)-(D.317) for $I_{d-1}(M)$ (or, in absence of such a constant, by the weakest dependence on M^2). It should be noted that, because of $V''(0) = -2|\alpha|_{\text{ren}} + O(\eta)$, if we take higher orders in the $I_{d-1}(M)$'s expansion (i.e., for the range of not so small ω_{crit}), we obtain results which one cannot rely on since $\sqrt{V''(0)}$ is imaginary (plus correction of order η which may be real). This is the price we have to pay for having worked only with the first perturbative η -order in the effective potential: a more accurate value of ω_{crit} requires to expand V_{eff} up to higher orders in $\eta|\beta|$ [31]. By

¹⁰⁸By noting that $I_n(M=0) = \frac{1}{(n-1)!} \int_0^{\infty} \frac{x^{n-2}}{e^x - 1} dx = \frac{1}{n-1} \zeta(n-1)$ with $n > 2$ natural, and that $\zeta(n)$ is the Riemann zeta function (equation 3.411.1 in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego (1994)); $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202 \dots$, $\zeta(4) = \frac{\pi^4}{90}$.

taking into account (D.303) and for the range of small ω_{crit} , the latter is given in a first approximation by

$$\begin{cases} \frac{1}{24} \frac{12\eta|\beta|_{\text{ren}}}{\omega_{\text{crit}}^2} \approx -2\alpha_{\text{ren}} \left[1 + \frac{\eta(12|\beta|_{\text{ren}})}{32\pi^2} \left(\ln \left| \frac{V''(0)}{V''(\langle\phi\rangle)} \right| - 3 \right) \right] & d = 4, \\ -\frac{1}{8\pi} \frac{12\eta|\beta|_{\text{ren}}}{\omega_{\text{crit}}} \ln |\omega_{\text{crit}}^2 V''(0)| \approx -2\alpha_{\text{ren}} \left[1 - \frac{\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle\phi\rangle)}} \right] & d = 3. \end{cases}$$

The 2-dim. pure scalar model has no solution at first order in η . Since $1/\omega_{\text{crit}}$ is assumed large, and $\eta|\beta|_{\text{ren}}$ is perturbatively small, the above results can be approximated by

$$\begin{cases} \frac{1}{\omega_{\text{crit}}^2} \approx 24 \frac{2|\alpha|_{\text{ren}}}{12\eta|\beta|_{\text{ren}}} & d = 4, \\ \frac{1}{\omega_{\text{crit}}} \approx \text{constant} \times 8\pi \frac{2|\alpha|_{\text{ren}}}{12\eta|\beta|_{\text{ren}}} & d = 3. \end{cases} \quad (\text{D.320})$$

The solution for the 3-dim. model is only reliable if $\omega_{\text{crit}}^2 |\alpha|_{\text{ren}}$ is not too small, i.e., if $|\alpha|_{\text{ren}}$ is large, so that $\frac{\ln |\omega_{\text{crit}}^2 V''(0)|}{\omega_{\text{crit}}}$ is large. Otherwise, the above result for $d = 3$ would

not be reliable since it would yield a small $\frac{1}{\omega_{\text{crit}}}$, contrary to the assumption made to evaluate the integrals which have led us to result (D.320): this only would mean that the transition would not occur at a small ω_{crit} , as assumed, and that we would need to go to lower orders in $1/\omega$ in the effective potential and this for its part would require taking account of higher order in the perturbative expansion on powers of $\eta|\beta|$.

It is to note that the ordered and the disordered phase, occurring in the lattice XY model for large and small $|\psi|_{XY}^2$ respectively, correspond in the here considered (continuum) pure scalar model to the states characterized by a nonvanishing and a zero expectation value of the scalar field, also respectively: thus one can make correspond $|\psi|_{XY}^2$ to $\frac{|\alpha|}{|\beta|}$. And therefore the solution for $d = 4$ in the effective model has a similar

form to equation (D.221), $\left(\frac{\omega^2 |\psi|_{4\text{-dim.per. XY}}^2}{\eta} \right)_{\text{crit}} \approx 0.526$, valid for the XY model in a 4-dimensional lattice periodic along one direction. Also the dependence of ω_{crit} on $|\psi|_{XY}^2$ in the 3-dimensional (lattice) XY model (D.220), $\left(\frac{\omega |\psi|_{3\text{-dim.per. XY}}^2}{\eta} \right)_{\text{crit}} \approx \frac{2}{\pi}$, is somewhat similar to that in (D.320). Hence ω_{crit} for the scalar model at $d = 3, 4$ can be obtained by means of the critical behaviour in the periodic lattice model, and in the continuum formulation, by means of an effective model, effective in the sense that the actual short-distance interactions are irrelevant and we are averaging over them; thus the specific details of each models, such as the value of the constant proportional to $\omega_{\text{crit}}^{d-2} \frac{|\alpha|}{\eta|\beta|}$, vary as a consequence of this averaging over the lattice's small distance scale to obtain the effective model. Only as a reminder, we have from the very beginning assumed that ω is inversely proportional to the temperature, and therefore, the above results apply to the range of high temperatures.

Relations (D.320) can be also obtained by calculating directly the effective potential: since $\int_0^\infty x^{d-2} dx \ln(1 - e^{-\sqrt{x^2+M^2}})$ $\stackrel{\text{partial integr.}}{=} \underbrace{0}_{d \geq 2} - \frac{1}{d-1} \int_0^\infty \frac{x^d}{\sqrt{x^2+M^2}} \frac{dx}{e^{\sqrt{x^2+M^2}} - 1}$, the

$$\omega\text{-dependent part of } V_{\text{eff}} \text{ is equal to } V_{\text{eff}}(\bar{\rho})^{(\text{dep.}\omega)} = -\frac{\eta}{\omega^d} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{d!}{d-1} I_{d+1} \left(\omega \sqrt{V''(\bar{\rho})} \right) =$$

$$= \begin{cases} -\frac{\eta\pi^2}{90\omega^4} + \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)}{24\omega^2} - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)^{3/2}}{12\pi\omega} - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)^2}{64\pi^2} \left(\ln\left(\frac{V''(\bar{\rho})}{16\pi^2/\omega^2}\right) - \frac{3}{2} + 2\gamma \right) + O(\eta^2) \\ -\frac{\eta\zeta(3)}{2\pi\omega^3} - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)}{8\pi\omega} \left(\ln\left(\frac{V''(\bar{\rho})}{1/\omega^2}\right) - 1 \right) + \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)^{3/2}}{12\pi} + O(\eta^2) \\ -\frac{\eta\pi}{12\omega^2} + \frac{\eta\sqrt{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2}}{4\omega} + \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)}{16\pi} \left(\ln\left(\frac{V''(\bar{\rho})}{16\pi^2/\omega^2}\right) - 1 + 2\gamma \right) + O(\eta^2) \end{cases}$$

for respectively $d = 4$, $d = 3$ and $d = 2$. The underlined terms appear also, but with the opposite sign, in the ω -independent part of the effective potential (D.304). The complete effective potential is therefore given by

$$V_{\text{eff}}^{(\omega)} = \begin{cases} \alpha_{\text{ren}}\bar{\rho}^2 \left[1 - \frac{3\eta(12|\beta|_{\text{ren}})}{32\pi^2} + \frac{1}{\omega^2} \frac{\eta(12|\beta|_{\text{ren}})}{48\alpha_{\text{ren}}} \right] & + \\ \alpha_{\text{ren}}\bar{\rho}^2 \left[1 - \frac{\eta}{16\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle\phi\rangle)}} - \frac{1}{\omega} \frac{\eta(12|\beta|_{\text{ren}})}{16\pi\alpha_{\text{ren}}} \left(\ln\left(\frac{V''(\bar{\rho})}{1/\omega^2}\right) - 1 \right) \right] & + \\ \alpha_{\text{ren}}\bar{\rho}^2 \left[1 - \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right] & + \end{cases}$$

$$+ \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 - \frac{12\eta(12|\beta|_{\text{ren}})}{32\pi^2} \right) - \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)^{3/2}}{12\pi\omega} +$$

$$+ \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 + \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\sqrt{V''(\langle\phi\rangle)}} \right) + \quad (D.321)$$

$$+ \frac{|\beta|_{\text{ren}}}{2} \bar{\rho}^4 \left(1 + \frac{3\eta}{64\pi} \frac{(12|\beta|_{\text{ren}})}{\alpha_{\text{ren}}} \right) + \frac{\eta\sqrt{2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2}}{4\omega} -$$

$$+ \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)^2}{64\pi^2} \left(\ln\left(\frac{16\pi^2/\omega^2}{V''(\langle\phi\rangle)}\right) + 1 - 2\gamma \right) + \left(-\frac{\eta\pi^2}{90\omega^4} + \frac{\eta\alpha_{\text{ren}}}{12\omega^2} \right) + O(\eta^2) \quad d = 4$$

$$- \frac{\eta\alpha_{\text{ren}}}{4\pi\omega} \left(\ln\left(\frac{V''(\bar{\rho})}{1/\omega^2}\right) - 1 \right) + \left(-\frac{\eta\zeta(3)}{2\pi\omega^3} \right) + O(\eta^2) \quad d = 3$$

$$- \frac{\eta(2\alpha_{\text{ren}} + 6|\beta|_{\text{ren}}\bar{\rho}^2)}{16\pi} \left(\ln\left(\frac{16\pi^2/\omega^2}{V''(\langle\phi\rangle)}\right) - 2\gamma \right) + \left(-\frac{\eta\pi}{12\omega^2} \right) + O(\eta^2) \quad d = 2$$

where one can read off ω_{crit} (D.320) (for the range of small ω) at the transition from symmetry breaking ($\langle\phi\rangle \neq 0$) towards symmetry restoring ($\langle\phi\rangle = 0$): at $\omega = \omega_{\text{crit}}$ the coefficient multiplying $|\alpha|_{\text{ren}}\bar{\rho}^2$ becomes zero, as just discussed in the figure on page 680.

For practical use, and taking account of relation $\frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\omega)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0} = \frac{1}{2} \frac{d^2 V_{\text{eff}}^{(\omega)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0} =$
 $= \frac{1}{2} \tilde{\Gamma}^{(2)}(\vec{0}, \vec{0}|0)$, the equation for ω_{crit} , $\frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\text{dep.}\omega)}}{d\bar{\rho}^2} \Big|_{\omega=\omega_{\text{crit}}, \bar{\rho}=0} + \frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\omega=\infty)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0} = 0$,

can be graphically remembered (D.278) as

$$\left. \frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\omega)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0} = \alpha - \delta\alpha - \frac{\eta}{4} \left[\text{diagram 1} \right] + \frac{\eta}{4} \left[\text{diagram 2} \right] \Big|_{\tilde{\rho}=\tilde{\rho}=0} = 0, \quad (\text{D.322})$$

where the diagrams form the (Fourier transform of the) so-called self-mass element Σ (D.376). One only needs to integrate over p_τ (Fourier momentum corresponding to the periodic coordinate) according to (D.287); the counterterm $-\delta\alpha\bar{\rho}^2$ of the ω -independent part of V_{eff} will cancel the divergence at very short distances $p \rightarrow \infty$ which can occur in the integral over the diagrams' closed loops.

After the “training” with the pure scalar model, let us turn our attention to the actual model of interest: when a complex scalar field is coupled to a gauge field and the model's $U(1)$ -symmetry becomes thus a local (or gauge) symmetry.

D.5.4 The effective potential for the Abrikosov model

A complex scalar field $\bar{\psi} \equiv \phi_{\text{scal}}^{a=1} + i\phi_{\text{scal}}^{a=2}$ is coupled to a $U(1)$ gauge field such that any coordinate-dependent change of the scalar's orientation in its internal space, space spanned by the real components $\{\phi_{\text{scal}}^{a=1}, \phi_{\text{scal}}^{a=2}\}$, leaves invariant the energy defining the

model, given by $E[\phi_{\text{scal}}^{a=1,2}(\vec{r}), \phi_{\text{gauge}}^{\mu=1,2,\dots,d}(\vec{r})] = \int d^d\vec{r} \left[\frac{1}{2} \left((\partial_\mu \phi_{\text{gauge}}^\nu)^2 - (\partial_\mu \phi_{\text{gauge}}^\nu)(\partial_\nu \phi_{\text{gauge}}^\mu) \right) + \frac{1}{2} (\partial_\mu \phi_{\text{scal}}^a + e\phi_{\text{gauge}}^\mu \epsilon_{ab} \phi_{\text{scal}}^b)^2 + \alpha (\phi_{\text{scal}}^a \phi_{\text{scal}}^a) + \frac{|\beta|}{2} (\phi_{\text{scal}}^a \phi_{\text{scal}}^a)^2 \right]$, summing over any repeated index and with $\alpha = -|\alpha| < 0$ for the case of symmetry breaking¹⁰⁹. The calculation of the effective potential up to order η , being the (constant) expectation value of the scalar field the effective potential's variable, is carried out as explained in equation

(D.270): one has to shift only the scalar field by a constant $\begin{cases} \phi_{\text{scal}}^a(\vec{r}) = \bar{\rho}_a + \chi_{\text{scal}}^a(\vec{r}) \\ \phi_{\text{gauge}}^\mu(\vec{r}) = \chi_{\text{gauge}}^\mu(\vec{r}) \end{cases}$, keep those terms which only contain any two of the $\chi_{\text{scal,gauge}}^{a,\mu}$ and finally calculate integral

$$V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) - \eta \ln \left[\int [d\chi(\vec{r})] \exp \left(- \frac{E_{\text{bilin. in } \chi(\vec{r})} \left[\begin{matrix} \phi_{\text{scal}}^a = \bar{\rho}_a + \chi_{\text{scal}}^a(\vec{r}) \\ \phi_{\text{gauge}}^\mu = \chi_{\text{gauge}}^\mu(\vec{r}) \end{matrix} \right]}{\eta} \right) \right], \text{ with}$$

$$E_{\text{bilin. in } \chi(\vec{r})} \left[\begin{matrix} \phi_{\text{scal}}^a = \bar{\rho}_a + \chi_{\text{scal}}^a(\vec{r}) \\ \phi_{\text{gauge}}^\mu = \chi_{\text{gauge}}^\mu(\vec{r}) \end{matrix} \right] = \int d^d\vec{r} \left[\frac{1}{2} \left((\partial_\mu \chi_{\text{gauge}}^\nu)^2 - (\partial_\mu \chi_{\text{gauge}}^\nu)(\partial_\nu \chi_{\text{gauge}}^\mu) \right) + \frac{1}{2} (\partial_\mu \chi_{\text{scal}}^a)^2 - e\epsilon_{cd}\bar{\rho}_c (\partial_\mu \chi_{\text{scal}}^d) \phi_{\text{gauge}}^\mu + \frac{e^2}{2} \bar{\rho}_a^2 (\phi_{\text{gauge}}^\mu)^2 + \right]$$

¹⁰⁹The energy describing the model is invariant under the local transformation $\begin{cases} \begin{pmatrix} \phi_{\text{scal}}^{a=1} \\ \phi_{\text{scal}}^{a=2} \end{pmatrix} \rightarrow \begin{pmatrix} \phi_{\text{scal}}^{a=1} \\ \phi_{\text{scal}}^{a=2} \end{pmatrix} = \begin{pmatrix} \cos \xi(\vec{r}) & \sin \xi(\vec{r}) \\ -\sin \xi(\vec{r}) & \cos \xi(\vec{r}) \end{pmatrix} \begin{pmatrix} \phi_{\text{scal}}^{a=1} \\ \phi_{\text{scal}}^{a=2} \end{pmatrix} \\ \phi_{\text{gauge}}^\mu \rightarrow \phi_{\text{gauge}}^\mu = \phi_{\text{gauge}}^\mu - \frac{1}{e} \partial_\mu \xi(\vec{r}) \end{cases}$, where the $U(1)$ transformation has been written by means of orthogonal $SO(2)$ matrices.

$$+\alpha(\chi_{\text{scal}}^a\chi_{\text{scal}}^a)+2|\beta|\bar{\rho}_a\bar{\rho}_b\chi_{\text{scal}}^a\chi_{\text{scal}}^b+|\beta|\bar{\rho}_a^2\left(\chi_{\text{scal}}^b\chi_{\text{scal}}^b\right)^2]. \quad (\text{D.323})$$

However, the integration over field $d\chi_{\text{gaug}}^\mu$ presents a problem since the differential operator acting on the term bilinear in ϕ_{gaug}^μ has no well-defined propagator, because of the very form of $\left(\partial_\mu\chi_{\text{gaug}}^\nu\right)^2 - \left(\partial_\mu\chi_{\text{gaug}}^\nu\right)\left(\partial_\nu\chi_{\text{gaug}}^\mu\right)$ to ensure the invariance under a transformation of the kind $\chi_{\text{gaug}}^\mu \rightarrow \chi_{\text{gaug}}'^\mu = \chi_{\text{gaug}}^\mu - \partial_\mu\xi(\vec{r})$: not all but only some of the gauge field's components, the transversal ones¹¹⁰, enters in the considered differential operator

¹¹⁰As already discussed in equation (C.183), the following property holds: $\int d^d\vec{r} \sum_\lambda \bar{A}_\lambda(\vec{r}) \left(-\sum_\nu \partial_\nu \partial_\nu \bar{A}_\lambda(\vec{r}) + \partial_\lambda \sum_\nu \partial_\nu \bar{A}_\nu(\vec{r}) \right) = \int d^d\vec{r} \sum_\lambda \bar{A}_\lambda^T \left(-\sum_\nu \partial_\nu \partial_\nu \right) \bar{A}_\lambda^T$, with $\bar{A}_\lambda^T(\vec{r})$ the transversal component of the gauge field, component which remains invariant under a gauge transformation $\bar{A}_\lambda \rightarrow \bar{A}'_\lambda = \bar{A}_\lambda - \partial_\lambda \xi(\vec{r})$ [25]. Let us remember that the transversal and longitudinal components of the gauge field are defined by

$$\begin{cases} \bar{A}_\lambda^T \equiv \sum_\nu P_{\lambda\nu}^T \bar{A}_\nu = \sum_\nu \int \frac{d^d\vec{k}}{(2\pi)^d} \left(\delta_{\lambda\nu} - \frac{k_\lambda k_\nu}{k^2} \right) e^{i\vec{k}\cdot\vec{r}} \tilde{A}_\nu(\vec{k}) \\ \bar{A}_\lambda^L \equiv \sum_\nu P_{\lambda\nu}^L \bar{A}_\nu = \sum_\nu \int \frac{d^d\vec{k}}{(2\pi)^d} \left(\frac{k_\lambda k_\nu}{k^2} \right) e^{i\vec{k}\cdot\vec{r}} \tilde{A}_\nu(\vec{k}) \end{cases},$$

where $P_{\lambda\nu}^T, P_{\lambda\nu}^L$ are, respectively, the transverse and longitudinal operators, which satisfy $\left(P_{\lambda\nu}^{T(L)}\right)^{-1} = P_{\lambda\nu}^{T(L)}$ and $\sum_\nu P_{\lambda\nu}^{T(L)} P_{\nu\mu}^{T(L)} = P_{\lambda\mu}^{T(L)}$. The independence of longitudinal component

$\bar{A}_\lambda^L(\vec{r})$ in differential operator $\left(-\delta_{\lambda\mu} \sum_\nu \partial_\nu \partial_\nu + \partial_\lambda \partial_\mu \right) \delta^{(d)}(\vec{r} - \vec{r}')$ makes impossible to obtain the corresponding propagator, i.e., the differential operator's inverse: since $\sum_\lambda \bar{A}_\lambda(\vec{r}) \left(-\sum_\nu \partial_\nu \partial_\nu \bar{A}_\lambda(\vec{r}) + \partial_\lambda \sum_\nu \partial_\nu \bar{A}_\nu(\vec{r}) \right) = \sum_{\lambda,\mu} \bar{A}_\lambda P_{\lambda\mu}^T \left(-\sum_\nu \partial_\nu \partial_\nu \right) \bar{A}_\mu + \sum_\lambda \bar{A}_\lambda P_{\lambda\mu}^L \cdot 0 \cdot \bar{A}_\mu$, the

propagator would be $D_{\text{gaug}\mu\nu}(\vec{r}, \vec{r}') = \int \frac{d^d\vec{k}}{(2\pi)^d} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + \int \frac{d^d\vec{k}}{(2\pi)^d} \left(\frac{k_\mu k_\nu}{k^2} \right) \frac{1}{0} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}.$

And the propagator is necessary when the gauge field is integrated out in calculations such as the partition function. In order to see in more detail this question, let us consider the Fourier transform of a differential operator a bit more general than $\left(-\delta_{\lambda\mu} \sum_\nu \partial_\nu \partial_\nu + \partial_\lambda \partial_\mu \right) \delta^{(d)}(\vec{r} - \vec{r}')$ (whose Fourier transform is $\delta_{\lambda\mu} \vec{k}^2 - k_\lambda k_\mu$) [31]: $\delta_{\lambda\mu} \sum_\nu k_\nu k_\nu - k_\lambda k_\mu - d_\lambda(k) d_\mu(-k)$, and study the condition which arbitrary vector field $d_\mu(k)$ has to fulfil such that the just written differential operator has an inverse. This condition is its non vanishing determinant

$$\begin{vmatrix} \vec{k}^2 - k_1^2 - d_1(k) d_1(-k) & -k_1 k_2 - d_1(k) d_2(-k) & -k_1 k_3 - d_1(k) d_3(-k) & \cdots \\ -k_1 k_2 - d_1(-k) d_2(k) & \vec{k}^2 - k_2^2 - d_2(k) d_2(-k) & -k_2 k_3 - d_2(k) d_3(-k) & \cdots \\ -k_1 k_3 - d_1(-k) d_3(k) & -k_2 k_3 - d_2(-k) d_3(k) & \vec{k}^2 - k_3^2 - d_3(k) d_3(-k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} =$$

$$= \left(\vec{k}^2 \right)^{d-2} \left(\sum_\lambda k_\lambda d_\lambda(k) \right) \left(\sum_\mu (-k_\mu) d_\mu(-k) \right) \neq 0.$$

Thus, the generalized differential operator whose Fourier transform is given by $\delta_{\lambda\mu} \sum_\nu k_\nu k_\nu - k_\lambda k_\mu - d_\lambda(k) d_\mu(-k)$ owns a well-defined propagator if $\sum_\lambda k_\lambda d_\lambda(k) \neq 0$. Clearly,

and therefore, the integration on the longitudinal components will produced an infinite multiplicative constant (infinite because any gauge transformation does not modify the “transversal” content of a configuration). This problem is solved as follows: either by identifying and integrating only over the gauge field’s components on which energy’s part $\left(\partial_\mu \chi_{\text{gaug}}^\nu\right)^2 - \left(\partial_\mu \chi_{\text{gaug}}^\nu\right) \left(\partial_\nu \chi_{\text{gaug}}^\mu\right)$ really depends, i.e., by removing explicitly the gauge invariance; or by integrating over all components but, simultaneously, including a factor which compensates the infinite number of times the same contribution occurs as a consequence of all those configurations related to each other by a gauge transformation. This last strategy is achieved by putting a so-called gauge fixing factor into the integral: a special case is to formulate the integral as

$$\frac{1}{\mathcal{N}} \det(-\partial_\nu \partial_\nu) \int [df(\vec{r})] \int [d\chi_{\text{gaug}}^\mu(\vec{r})] \prod_{\vec{r}} \delta(\partial_\mu \chi_{\text{gaug}}^\mu - f(\vec{r})) \times \\ \times G[f(\vec{r})] \exp\left(-\frac{E_{\text{bilin. in } \chi(\vec{r})}[\chi_{\text{scal}}^a, \chi_{\text{gaug}}^\mu]}{\eta}\right), \quad (\text{D.324})$$

just as explained in the Faddeev-Popov “recipe” (C.197) for lattice models; functional $G[f(\vec{r})]$ is usually chosen to be $\exp\left(-\frac{1}{2\alpha_{\text{fix}}\eta} \int d^d \vec{r} (f(\vec{r}))^2\right)$ and $\mathcal{N} = \left(\sqrt{\frac{2\pi\alpha_{\text{fix}}}{\eta}}\right)^{|\Lambda|}$ with $|\Lambda| = \int d^d \vec{r}$ the (infinite) volume of the d -dimensional system. As we saw the Faddeev-Popov method of removing the infinite overcounting of the same contribution, the resulting partition function had no dependence on the parameter of the gauge fixing factor: as an example, the lattice $U(1)$ Higgs model was solved for two different choices of the fixing factor, obtaining the same result: equations (C.410) and (C.537). The proof holds completely analogous for the continuum model, showing thus that the complete effective potential does not depend on α_{fix} . For the effective potential *up to order* η $V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) - \eta \ln [\text{integral (D.324)}]$, neverthelss, we are not integrating the complete gauge invariant energy but only the truncated $E_{\text{bilin. in } \chi(\vec{r})}$ (D.323), where though part

original differential operator $\left(-\delta_{\lambda\mu} \sum_\nu \partial_\nu \partial_\nu + \partial_\lambda \partial_\mu\right) \delta^{(d)}(\vec{r} - \vec{r}')$, which corresponds to $d_\lambda(k) = 0$, does not satisfy such a condition. It is important to stress that the origin of this absence of inverse is the gauge invariance of $\bar{A}_\lambda(\vec{r}) \left(-\sum_\nu \partial_\nu \partial_\nu \bar{A}_\lambda(\vec{r}) + \partial_\lambda \sum_\nu \partial_\nu \bar{A}_\nu(\vec{r})\right)$: for instance, we already know that the inclusion in the energy of a gauge fixing factor such as $\frac{1}{\alpha_{\text{fix}}} \left(\sum_\lambda \partial_\lambda \bar{A}_\lambda\right)^2$ removes the gauge invariance; but now the differential operator whose propagator we want to obtain has become $\left(-\delta_{\lambda\mu} \sum_\nu \partial_\nu \partial_\nu + \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) \partial_\lambda \partial_\mu\right) \delta^{(d)}(\vec{r} - \vec{r}')$, which after writting its Fourier transform $\delta_{\lambda\mu} \sum_\nu k_\nu k_\nu - \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) k_\lambda k_\mu$, corresponds to $d_\lambda(k) = \frac{1}{\sqrt{\alpha_{\text{fix}}}} k_\lambda$. And this last $d_\lambda(k)$ does satisfy $\sum_\lambda k_\lambda d_\lambda(k) \neq 0$, proving thus the existence of a propagator for $-\delta_{\lambda\mu} \sum_\nu \partial_\nu \partial_\nu + \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) \partial_\lambda \partial_\mu$.

$\frac{1}{2} \left((\partial_\mu \chi_{\text{gaug}}^\nu)^2 - (\partial_\mu \chi_{\text{gaug}}^\nu) (\partial_\nu \chi_{\text{gaug}}^\mu) \right)$ is gauge invariant, part $\frac{1}{2} (\partial_\mu \chi_{\text{scal}}^a + e \epsilon_{ab} \bar{\rho}_b \chi_{\text{gaug}}^\mu)^2$ it is *not* (since $\bar{\rho}_b$ is constant, not a field). Hence, the proof of the independence of α_{fix} holds no longer for the effective potential up to order η , and although the gauge fixing factor enables us to calculate the propagator of the gauge field in (D.324), the resulting effective potential will, in general, depend on α_{fix} ¹¹¹. On the other hand, since we know

¹¹¹As mentioned, to include any gauge fixing factor in the integral of partition function $\mathcal{Z}$ is equivalent to factorize out the number of (infinite) configurations which are related to each other through a gauge transformation. In the continuum formulation, the partition function and Helmholtz free energy $F[J(\vec{r})] = -\eta \ln \mathcal{Z}[J(\vec{r})] = -\eta \ln [\det(-\partial_\mu \partial_\mu)] - \eta \ln \int [d\phi_{\text{gaug}}^\mu] [d\phi_{\text{scal}}^a] \times$
 $\times \exp \left(-\frac{E[\phi_{\text{gaug}}^\mu(\vec{r}), \phi_{\text{scal}}^a(\vec{r})]}{\eta} - \frac{1}{2\alpha_{\text{fix}}\eta} \int d^d \vec{r} (\partial_\mu \phi_{\text{gaug}}^\mu)^2 + \frac{1}{\eta} \int d^d \vec{r} J(\vec{r}) \phi_{\text{scal}}^a \right)$ are independent of the parameters on which the gauge fixing factor depends, just as in the lattice model, and the same property is satisfied by Gibbs free energy $G[\varphi(\vec{r})]$, i.e., by the Legendre transformation of the Helmholtz free energy $G[\varphi(\vec{r})] = -\eta \ln \mathcal{Z}[J(\vec{r})] + \int d^d \vec{r} J(\vec{r}) \varphi(\vec{r})$, and by the effective potential which is proportional to $G[\varphi(\vec{r})]$ for $\varphi(\vec{r})$ equal to a constant value $\bar{\rho}$. Nevertheless, and as different from the lattice model where the gauge field was only coupled to the gradient of the scalar field's phase (the length of the scalar field was taken as a constant value $|\psi|^2$), in the continuum model the gauge field is coupled to the gradient of the whole scalar field, causing thus that the calculation of the partition function is, in most of the cases, impossible to be carried out. The only form to compute $\mathcal{Z}[J(\vec{r})]$, and thus to obtain the complete effective potential, is a perturbative expansion around the configuration $\{\phi_{\text{cl scal}}^a, \phi_{\text{cl gaug}}^\mu\}$ which minimizes

the exponential inside the partition function. By writing $\begin{cases} \phi_{\text{scal}}^a(\vec{r}) = \phi_{\text{cl scal}}^a(\vec{r}) + \sqrt{\eta} \left(\gamma_{\text{scal}}^a(\vec{r}) \right), \\ \phi_{\text{gaug}}^\mu(\vec{r}) = \phi_{\text{cl gaug}}^\mu(\vec{r}) + \sqrt{\eta} \left(\gamma_{\text{gaug}}^\mu(\vec{r}) \right), \end{cases}$ one obtains the following expansion in powers of $\gamma_{\text{scal}}^a, \gamma_{\text{gaug}}^\mu$

$$\begin{aligned} \mathcal{Z}[J(\vec{r})] = \text{const.} \times \exp & \left(-\frac{E[\phi_{\text{cl gaug}}^\mu, \phi_{\text{cl scal}}^a]}{\eta} - \frac{\int d^d \vec{r} (\partial_\mu \phi_{\text{cl gaug}}^\mu)^2}{2\alpha_{\text{fix}}\eta} + \frac{\int d^d \vec{r} J(\vec{r}) \phi_{\text{cl scal}}^a}{\eta} \right) \int [d\gamma_{\text{gaug}}^\mu] [d\gamma_{\text{scal}}^a] \times \\ & \times \exp \left[-\frac{()^2}{2!} \int d^d \vec{r}_1 d^d \vec{r}_2 \frac{\delta^2 E[\phi_{\text{scal}}^a(\vec{r}), \phi_{\text{gaug}}^\mu(\vec{r})]}{\delta \phi_{\text{scal,gaug}}^{a,\lambda}(\vec{r}_1) \delta \phi_{\text{scal,gaug}}^{b,\mu}(\vec{r}_2)} \right]_{\phi=\phi_{\text{cl}}} \gamma_{\text{scal,gaug}}^{a,\lambda}(\vec{r}_1) \gamma_{\text{scal,gaug}}^{b,\mu}(\vec{r}_2) - \frac{()^2}{2\alpha_{\text{fix}}} \int d^d \vec{r} (\partial_\mu \gamma_{\text{gaug}}^\mu)^2 - \\ & - \frac{\eta^{1/2}()^3}{3!} \int d^d \vec{r}_1 d^d \vec{r}_2 d^d \vec{r}_3 \frac{\delta^3 E[\phi_{\text{scal}}^a(\vec{r}), \phi_{\text{gaug}}^\mu(\vec{r})]}{\delta \phi_{\text{scal,gaug}}^{a,\lambda}(\vec{r}_1) \delta \phi_{\text{scal,gaug}}^{b,\mu}(\vec{r}_2) \delta \phi_{\text{scal,gaug}}^{c,\nu}(\vec{r}_3)} \right]_{\phi=\phi_{\text{cl}}} \gamma_{\text{scal,gaug}}^{a,\lambda}(\vec{r}_1) \gamma_{\text{scal,gaug}}^{b,\mu}(\vec{r}_2) \gamma_{\text{scal,gaug}}^{c,\nu}(\vec{r}_3) + \dots \end{aligned}$$

where expansion's factors $\eta()$ are proportionnal to $\sqrt{\eta e^2}$ and $\sqrt{\eta|\beta|}$ for the gauge and the scalar fields, respectively. If one breaks the above expansion in the term quadratic in $\gamma_{\text{scal,gaug}}^{a,\mu}$, one obtains a free energy containing only contributions of order η , and from such a free energy comes the effective potential up to order η discussed above. However, in such a process we are dropping interactions between more

than two fields contained in derivatives such as $\frac{\delta^3 E[\phi_{\text{scal}}^a(\vec{r}), \phi_{\text{gaug}}^\mu(\vec{r})]}{\delta \phi_{\text{scal,gaug}}^{a,\lambda}(\vec{r}_1) \delta \phi_{\text{scal,gaug}}^{b,\mu}(\vec{r}_2) \delta \phi_{\text{scal,gaug}}^{c,\nu}(\vec{r}_3)} \Big|_{\phi=\phi_{\text{cl}}}$, and the

original model's energy, which was gauge invariant, contained interactions up to four fields arising from the covariant derivative's term $\frac{1}{2} (\partial_\mu \phi_{\text{scal}}^a + e \phi_{\text{gaug}}^\mu \epsilon_{ab} \phi_{\text{scal}}^b)^2$: therefore, by neglecting such terms, the gauge invariance gets lost. And it was just this invariance which allowed us to factorize out the infinite factor which removes the gauge fixing factor. Hence, the no-dependence on α_{fix} is not satisfied, just as a consequence of having worked with a finite number of terms in a perturbative expansion.

that the whole effective potential does not depend on α_{fix} , its part of order η will be obtained by isolating in $V_{\text{eff}}(\bar{\rho}) = V(\bar{\rho}) - \eta \ln [\text{integral (D.324)}]$ those (spurious) terms of which are proportional to any power of the gauge fixing's parameter and neglecting them; or equivalently, by taking the limit $\alpha_{\text{fix}} \rightarrow 0$.

For the calculation of ω_{crit} in the range of small ω , nevertheless, we need not worry about such a problem, since as we have seen in (D.320), keeping only up to the first perturbative order in the effective potential already forces us to keep only the term with the weakest dependence on ω (usually a constant) in the integrals obtained during the calculation of ω_{crit} . And such a constant term occurs to be independent of the parameter of the gauge fixing factor ¹¹².

After including the gauge fixing operator, the propagator corresponding to part bilinear in χ_{gauge}^ν , $\chi_{\text{gauge}}^\lambda \left(-\partial_\nu \partial_\nu \delta_{\lambda\mu} + \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) \partial_\lambda \partial_\mu + e^2 \bar{\rho}^2 \delta_{\lambda\mu} \right) \chi_{\text{gauge}}^\mu$, is equal to ¹¹³

$$D_{\text{gauge } \mu\nu}(\vec{r}, \vec{r}') = \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 + e^2 \bar{\rho}^2} \left(\delta_{\mu\nu} + \frac{(\alpha_{\text{fix}} - 1) p_\mu p_\nu}{\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')}.$$
 (D.325)

The differential operator on χ_{scal}^a , $\left(-\partial_\nu \partial_\nu \delta_{ab} + (2\alpha + 2|\beta| \bar{\rho}^2) \delta_{ab} + 4|\beta| \bar{\rho}_a \bar{\rho}_b \right) \delta^{(d)}(\vec{r} - \vec{r}')$, has as propagator ¹¹⁴

¹¹²Except for some pathological gauge choices: see comments on page 3332 of [31].

¹¹³The inverse of the differential operator's Fourier transform, $(\vec{k}^2 + e^2 \bar{\rho}^2) \delta_{\lambda\mu} - \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) k_\lambda k_\mu$, is of the form $A \delta_{\mu\nu} + B k_\mu k_\nu$, where A, B are determined from the equation defining the propagator, $\left[(\vec{k}^2 + e^2 \bar{\rho}^2) \delta_{\lambda\mu} - \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) k_\lambda k_\mu \right] (A \delta_{\mu\nu} + B k_\mu k_\nu) = \delta_{\lambda\nu}$, yielding $A \delta_{\mu\nu} + B k_\mu k_\nu = \frac{1}{\vec{k}^2 + e^2 \bar{\rho}^2} \left(\delta_{\mu\nu} + \frac{(\alpha_{\text{fix}} - 1) k_\mu k_\nu}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right)$. Integrating $[d\chi_{\text{gauge}}^\mu]$ in the effective potential produces the $(\text{determinant})^{-1/2}$ of the differential operator under study

$$\det D_{\text{gauge } \lambda\mu}(\vec{r}, \vec{r}') = \frac{1}{\det \left(-\partial_\nu \partial_\nu \delta_{\lambda\mu} + \left(1 - \frac{1}{\alpha_{\text{fix}}}\right) \partial_\lambda \partial_\mu + e^2 \bar{\rho}^2 \delta_{\lambda\mu} \right)} = \prod_{\vec{k}} \left(\frac{1}{\vec{k}^2 + e^2 \bar{\rho}^2} \right)^d \times$$

$$\times \begin{vmatrix} 1 + \frac{(\alpha_{\text{fix}} - 1) k_1^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} & \frac{(\alpha_{\text{fix}} - 1) k_1 k_2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} & \dots \\ \frac{(\alpha_{\text{fix}} - 1) k_1 k_2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} & 1 + \frac{(\alpha_{\text{fix}} - 1) k_2^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} = \prod_{\vec{k}} \left(\frac{1}{\vec{k}^2 + e^2 \bar{\rho}^2} \right)^{d-1} \frac{\alpha_{\text{fix}}}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2},$$

where $\prod_{\vec{k}}$ denotes a product of a continuous range of $\vec{k}$, and its logarithm is equal $\int d^d \vec{k}$.

¹¹⁴As for the gauge propagator, we solve $\left[(\vec{k}^2 + 2\alpha + 2|\beta| \bar{\rho}^2) \delta_{ab} + 4|\beta| \bar{\rho}_a \bar{\rho}_b \right] (A \delta_{bc} + B \bar{\rho}_b \bar{\rho}_c) = \delta_{ab}$, which produces $A \delta_{bc} + B \bar{\rho}_b \bar{\rho}_c = \frac{1}{\vec{k}^2 + 2\alpha + 2|\beta| \bar{\rho}^2} \left(\delta_{bc} - \frac{4|\beta| \bar{\rho}_b \bar{\rho}_c}{\vec{k}^2 + 2\alpha + 6|\beta| \bar{\rho}^2} \right)$. The determinant of the

$$D_{\text{scalar } bc}(\vec{r}, \vec{r}') = \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 + 2\alpha + 2|\beta|\bar{\rho}^2} \left(\delta_{bc} - \frac{4|\beta|\bar{\rho}_b \bar{\rho}_c}{\vec{p}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} \right) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} . \quad (\text{D.326})$$

The Gaussian integral of $[d\chi_{\text{gauge}}^\mu]$ yields as effective potential to first perturbative order in η the following equation

$$\begin{aligned} V_{\text{eff}} = & \underbrace{V(\bar{\rho}^2) - \eta \int d^d \vec{p} \ln \vec{p}^2 + (d-1) \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + e^2 \bar{\rho}^2] + \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2]}_{\text{gauge fixing factor}} + \\ & -\eta \ln \int [d\chi_{\text{scal}}^a] \exp \left[-\frac{1}{2\eta} \int d^d \vec{r} \chi_{\text{scal}}^a \left(-\partial_\nu \partial_\nu \delta_{ab} + (2\alpha + 2|\beta|\bar{\rho}^2) \delta_{ab} + 4|\beta|\bar{\rho}_a \bar{\rho}_b \right) \chi_{\text{scal}}^b - \right. \\ & \left. -e^2 \int d^d \vec{r} d^d \vec{r}' \epsilon_{ac} \epsilon_{bd} \bar{\rho}_a \bar{\rho}_b (\partial_\mu \chi_{\text{scal}}^c(\vec{r})) D_{\text{gauge } \mu\nu}(\vec{r}, \vec{r}') \left(\frac{\partial_\nu}{(\vec{r}')} \chi_{\text{scal}}^d(\vec{r}') \right) \right] , \quad (\text{D.327}) \end{aligned}$$

where the constant term $-\frac{\eta}{2} \ln \left(\frac{2\pi \alpha_{\text{fix}}}{\eta} \right) \int d^d \vec{k}$ is canceled out by the $+\frac{\eta}{2} (\ln \mathcal{N}) |\Lambda|$ of the gauge fixing factor (see (D.324)). The underlined term, $\ln \det (-\partial_\nu \partial_\nu) = \int d^d \vec{p} \ln \vec{p}^2$, is also part of the same gauge fixing factor. Integrating $[d\chi_{\text{scal}}^a]$, one obtains finally ¹¹⁵

$$\begin{aligned} V_{\text{eff}} = & V(\bar{\rho}^2) - \eta \int d^d \vec{p} \ln \vec{p}^2 + (d-1) \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + e^2 \bar{\rho}^2] + \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2] + \\ & + \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + 2\alpha + 6|\beta|\bar{\rho}^2] - \frac{\eta}{2} \int d^d \vec{p} \ln [\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2] + \end{aligned}$$

propagator of scalar field χ_{scal}^a becomes

$$\begin{aligned} \det D_{\text{scalar } ab}(\vec{r}, \vec{r}') &= \frac{1}{\det \left(-\partial_\nu \partial_\nu \delta_{ab} + (2\alpha + 2|\beta|\bar{\rho}^2) \delta_{ab} + 4|\beta|\bar{\rho}_a \bar{\rho}_b \right)} = \prod_{\vec{k}} \left(\frac{1}{\vec{k}^2 + 2\alpha + 2|\beta|\bar{\rho}^2} \right)^2 \times \\ & \times \left| \begin{array}{cc} 1 - \frac{4|\beta|\bar{\rho}_1^2}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} & -\frac{4|\beta|\bar{\rho}_1 \bar{\rho}_2}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} \\ -\frac{4|\beta|\bar{\rho}_1 \bar{\rho}_2}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} & 1 - \frac{4|\beta|\bar{\rho}_2^2}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} \end{array} \right| = \prod_{\vec{k}} \frac{1}{\vec{k}^2 + 2\alpha + 2|\beta|\bar{\rho}^2} \frac{1}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} . \end{aligned}$$

¹¹⁵Operator $\left(-\partial_\nu \partial_\nu \delta_{ab} + (2\alpha + 2|\beta|\bar{\rho}^2) \delta_{ab} + 4|\beta|\bar{\rho}_a \bar{\rho}_b \right) \delta^{(d)}(\vec{r} - \vec{r}') - e^2 \epsilon_{ac} \epsilon_{bd} \bar{\rho}_a \bar{\rho}_b \frac{\partial_\mu}{(\vec{r}')} \frac{\partial_\nu}{(\vec{r}')} D_{\text{gauge } \mu\nu}(\vec{r}, \vec{r}')$ has as determinant

$$\begin{aligned} & \prod_{\vec{k}} \left| \begin{array}{cc} \left(\vec{k}^2 + 2\alpha + 2|\beta|\bar{\rho}^2 \right) + 4|\beta|\bar{\rho}_1^2 - \frac{\alpha_{\text{fix}} e^2 \vec{k}^2 \bar{\rho}_2^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} & \left(4|\beta| + \frac{\alpha_{\text{fix}} e^2 \vec{k}^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) \bar{\rho}_1 \bar{\rho}_2 \\ \left(4|\beta| + \frac{\alpha_{\text{fix}} e^2 \vec{k}^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) \bar{\rho}_1 \bar{\rho}_2 & \left(\vec{k}^2 + 2\alpha + 2|\beta|\bar{\rho}^2 \right) + 4|\beta|\bar{\rho}_2^2 - \frac{\alpha_{\text{fix}} e^2 \vec{k}^2 \bar{\rho}_1^2}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \end{array} \right| = \\ & = \prod_{\vec{k}} \left(\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2 \right) \frac{\left(\vec{k}^2 \right)^2 + \vec{k}^2 (2\alpha + 2|\beta|\bar{\rho}^2) + \alpha_{\text{fix}} e^2 \bar{\rho}^2 (2\alpha + 2|\beta|\bar{\rho}^2)}{\vec{k}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} . \end{aligned}$$

$$+\frac{\eta}{2} \int d^d \vec{p} \ln \left[\underbrace{\left(\vec{p}^2 \right)^2 + \vec{p}^2 \left(2\alpha + 2|\beta|\bar{\rho}^2 \right) + \alpha_{\text{fix}} e^2 \bar{\rho}^2 \left(2\alpha + 2|\beta|\bar{\rho}^2 \right)}_{\left(\vec{p}^2 + R_1^2 \right) \left(\vec{p}^2 + R_2^2 \right)} \right] , \quad (\text{D.328})$$

with $R_{1,2}^2 = \frac{2\alpha + 2|\beta|\bar{\rho}^2}{2} \left(1 \pm \sqrt{1 - \frac{4\alpha_{\text{fix}} e^2 \bar{\rho}^2}{2\alpha + 2|\beta|\bar{\rho}^2}} \right)$; the above underlined terms cancel against each other.

If the d -dimensional system is periodic along coordinate r_τ with periodicity ω , the integration of its corresponding p_τ (with $\vec{p}^2 = p_\tau^2 + \sum_{j=1}^{d-1} p_j^2$) in (D.328) produces, according to relations (D.287) and (D.290), the following effective potential

$$V_{\text{eff}}(\bar{\rho})^{(\omega)} = V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} + V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)}$$

$$\begin{aligned} V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} &= \alpha \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4 + (d-1) \frac{\eta}{2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d-1} \frac{dp_j}{2\pi} \right) \left(\sum_{j=1}^{d-1} p_j^2 + e^2 \bar{\rho}^2 \right)^{1/2} + \\ &+ \frac{\eta}{2} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + 2\alpha + 6|\beta|\bar{\rho}^2} + \\ &+ \frac{\eta}{2} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + R_1^2} + \\ &+ \frac{\eta}{2} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + R_2^2} + \\ &- \eta \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-1} dp , \end{aligned} \quad (\text{D.329})$$

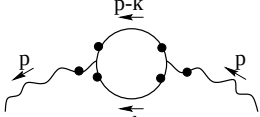
$$\begin{aligned} V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)} &= (d-1) \frac{\eta}{\omega^d} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d-1} \frac{dp_j}{2\pi} \right) \ln \left[1 - \exp \left(- \sqrt{\sum_{j=1}^{d-1} p_j^2 + \omega^2 e^2 \bar{\rho}^2} \right) \right] + \\ &+ \frac{\eta}{\omega^d} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - \exp \left(- \sqrt{p^2 + \omega^2 (2\alpha + 6|\beta|\bar{\rho}^2)} \right) \right] + \\ &+ \frac{\eta}{\omega^d} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - \exp \left(- \sqrt{p^2 + \omega^2 R_1^2} \right) \right] + \\ &+ \frac{\eta}{\omega^d} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - \exp \left(- \sqrt{p^2 + \omega^2 R_2^2} \right) \right] + \\ &- 2 \frac{\eta}{\omega^d} \int_0^{\infty} \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - e^{-p} \right] , \end{aligned} \quad (\text{D.330})$$

plus an (infinite) constant which is irrelevant. As expected, the effective potential depends on parameter α_{fix} (of the gauge fixing factor) through $R_{1,2}^2$. The dependence on p in the integrals of $V_{\text{eff}}^{(\omega=\infty)}$ is the same as for the pure scalar model (D.292), and thus the divergences, which occur up to order η when one integrates the range of $p \approx \infty$, are com-

pletely cancelled by the counterterms corresponding to α and β ¹¹⁶, $-\delta\alpha\bar{\rho}^2 - \frac{\delta\beta}{2}(\bar{\rho}^2)^2$, whose dependence on cut-off Λ_c will become similar to that in equation (D.295).

As for the pure scalar model, we assume that the possible transition will occur in the range of small ω such that $\frac{1}{\omega^2} \gg |\alpha|$ and $\frac{1}{\omega^2} \gg |\beta|\bar{\rho}^2$ hold for those ω -values near to the phase transition point. As different from the pure scalar model, where there appeared only one parameter, $\eta|\beta|$, for which the perturbative expansion was carried out, in the Abrikosov model there are two of such parameters, $\eta|\beta|$ and ηe^2 , and although both are kept perturbative small, depending on the relation between them the phase diagram will display different characteristics. Let us consider the following two cases: $\eta|\beta| \gg \eta e^2$ and $\eta|\beta| \ll \eta e^2$.

¹¹⁶In the model there exists another parameter which might be necessary to renormalize: coupling constant e . Such a renormalization is, nevertheless, required to avoid the divergence produced by an internal closed loop of scalar field's propagator attached to the gauge field's propagator, as

for instance in propagator , where wavy lines represent the propagator of

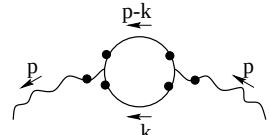
the gauge field, and the other lines, the scalar field's propagator. Since the part in the energy which couples two scalar fields with one gauge field is $\int d^d \vec{r} [-e\epsilon_{cd}\phi_{\text{scal}}^c(\vec{r})(\partial_\mu \phi_{\text{scal}}^d(\vec{r}))\phi_{\text{gaug}}^\mu(\vec{r})] =$

$$= \int \frac{d^d \vec{k}_1}{(2\pi)^d} \frac{d^d \vec{k}_2}{(2\pi)^d} \frac{d^d \vec{k}_3}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left[-ie\epsilon_{cd}(k_2)_\mu \tilde{\phi}_{\text{scal}}^c(\vec{k}_1) \tilde{\phi}_{\text{scal}}^d(\vec{k}_2) \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3) \right], \text{ and therefore}$$

$$\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^{a'}(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3)} = -ie\epsilon_{aa'} \left((k_2)_\mu - (k_1)_\mu \right), \text{ the just mentioned internal loop corresponds to}$$

$$e^2 \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{\vec{k}^2 + 2\alpha + 2|\beta|\bar{\rho}^2} \frac{1}{(\vec{p} - \vec{k})^2 + 2\alpha + 2|\beta|\bar{\rho}^2} \left(2 - \frac{4|\beta|\bar{\rho}^2}{\vec{k}^2 + 2\alpha + 6|\beta|\bar{\rho}^2} - \frac{4|\beta|\bar{\rho}^2}{(\vec{p} - \vec{k})^2 + 2\alpha + 6|\beta|\bar{\rho}^2} \right) \times \\ \times \underbrace{(k_\nu k_\sigma + k_\nu(p_\sigma - k_\sigma) + k_\sigma(p_\nu - k_\nu) + (p_\sigma - k_\sigma)(p_\nu - k_\nu))}_{p_\nu p_\sigma} \equiv e^2 p_\nu p_\sigma \mathcal{I}(\vec{p}) \quad \text{according to defi-}$$

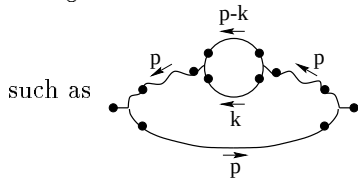
nition (D.326). Hence the propagator of the gauge field



represents

$$\frac{1}{\vec{p}^2 + e^2 \bar{\rho}^2} \left(\delta_{\lambda\nu} + \frac{(\alpha_{\text{fix}} - 1)p_\lambda p_\nu}{\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) \left(e^2 p_\nu p_\sigma \mathcal{I}(\vec{p}) \right) \frac{1}{\vec{p}^2 + e^2 \bar{\rho}^2} \left(\delta_{\sigma\mu} + \frac{(\alpha_{\text{fix}} - 1)p_\sigma p_\mu}{\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) = \frac{\alpha_{\text{fix}}^2 e^2 \mathcal{I}(\vec{p}) p_\lambda p_\mu}{(\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2)^2},$$

where integral $\mathcal{I}$ at $d = 4$ diverges logarithmically in the range of very short distances (for $k \sim \infty$), divergence which has to be removed by renormalizing e . However, such a propagator occurs in diagrams



such as which contributes to $\frac{\partial^2 V_{\text{eff}}}{\partial \phi_{\text{scal}}^c \partial \phi_{\text{scal}}^d}$ but at order η^2 : as it will be shown in

the comments after equation (D.366), the expansion of the Gibbs free energy (and thus of the effective potential) in powers of η corresponds also to an expansion in the number of closed internal loops. Hence, since we have restricted ourselves to the first order in η , coupling constant e does not need to be renormalized.

$\eta|\beta| \gg \eta e^2$ In this case $\frac{1}{\omega^2} \gg e^2 \bar{\rho}^2$ is already ensured since $\frac{1}{\omega^2} \gg |\beta| \bar{\rho}^2$. Therefore, in the ω -dependent part of the effective potential

$$V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)} = -\frac{\eta}{\omega^d} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{d!}{(d-1)} \left[(d-1) I_{d+1} \left(\omega \sqrt{e^2 \bar{\rho}^2} \right) + \right. \\ \left. + I_{d+1} \left(\omega \sqrt{2\alpha + 6|\beta| \bar{\rho}^2} \right) + \right. \\ \left. + I_{d+1} \left(\omega \sqrt{R_1^2} \right) + I_{d+1} \left(\omega \sqrt{R_2^2} \right) - 2I_{d+1}(0) \right], \quad (\text{D.331})$$

all integrals can be calculated by means of approximations (D.317)-(D.319)

$$V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega, d=4)} = -\frac{2\eta\pi^2}{90\omega^4} + \frac{\eta}{24\omega^2} [e^2 \bar{\rho}^2 + (2\alpha + 6|\beta| \bar{\rho}^2) + (2\alpha + 2|\beta| \bar{\rho}^2)] - \\ -\frac{\eta}{12\pi\omega} \left[(e^2 \bar{\rho}^2)^{3/2} + (2\alpha + 6|\beta| \bar{\rho}^2)^{3/2} + R_1^3 + R_2^3 \right] - \\ -\frac{\eta}{64\pi^2} \left(-\frac{3}{2} + 2\gamma \right) \left[(e^2 \bar{\rho}^2)^2 + (2\alpha + 6|\beta| \bar{\rho}^2)^2 + (2\alpha + 2|\beta| \bar{\rho}^2)^2 - \right. \\ \left. - 2\alpha_{\text{fix}} e^2 \bar{\rho}^2 (2\alpha + 2|\beta| \bar{\rho}^2) \right] + \dots \\ V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega, d=3)} = -\frac{2\eta\zeta(3)}{2\pi\omega^3} - \frac{\eta}{8\pi\omega} \left[(2\alpha + 6|\beta| \bar{\rho}^2) \left(\ln \left(\frac{2\alpha + 6|\beta| \bar{\rho}^2}{1/\omega^2} \right) - 1 \right) + \right. \\ \left. + e^2 \bar{\rho}^2 \left(\ln \left(\frac{e^2 \bar{\rho}^2}{1/\omega^2} \right) - 1 \right) + R_1^2 \left(\ln \left(\frac{R_1^2}{1/\omega^2} \right) - 1 \right) + \right. \\ \left. + R_2^2 \left(\ln \left(\frac{R_2^2}{1/\omega^2} \right) - 1 \right) \right] + \dots \\ V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega, d=2)} = -\frac{2\eta\pi}{12\omega^2} + \frac{\eta}{4\omega} \left[\sqrt{e^2 \bar{\rho}^2} + \sqrt{2\alpha + 6|\beta| \bar{\rho}^2} + \sqrt{R_1^2} + \sqrt{R_2^2} \right] + \dots$$

where in the above results, those terms which also appear but with the opposite sign in the ω -independent part of the effective potential have been not written (see comment previous to equation (D.321)).

The calculation of this ω -independent part $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ in the case $e^2 \eta \ll |\beta| \eta$ is easily carried out: in $\frac{d}{d\bar{\rho}} V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ the term originated by the gauge coupling and which contains e^2 can be neglected when compared to those terms with the scalar-scalar coupling and thus the problem reduces itself to a model similar to the pure scalar one, although there occurs a new difficulty which is solved by means of the gauge fixing factor ¹¹⁷. Since the calculation in this

¹¹⁷When one calculates the corrections to order η in the derivatives of effective potential $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$, apart from the range of very small distances $|\vec{p}| \approx \infty$, which produces divergences removed by the counterterms, there is in fact another source of divergences, this time originated from the range of large distances $|\vec{p}| \approx 0$. Thus the terms containing R_1^2 and R_2^2 in the effective potential's derivatives,

$$\frac{p^{d-2} dp}{(\sqrt{p^2 + R_1^2})^n} R_1^2 \stackrel{e^2 \eta \ll |\beta| \eta}{\approx} \stackrel{\approx}{\approx} 2\alpha + 2|\beta| \bar{\rho}^2 \frac{p^{d-2} dp}{(\sqrt{p^2 + 2\alpha + 2|\beta| \bar{\rho}^2})^n}, \text{ when evaluated in the case of } \alpha < 0 \text{ at}$$

the nonvanishing $\bar{\rho} = \langle \phi_{\text{scal}} \rangle = \sqrt{\frac{|\alpha|}{|\beta|}} + \eta(\dots)$ diverges at $p \rightarrow 0$ for $d \geq n+1$. To solve this problem, which only appears when the scalar field is coupled to a gauge invariant model, one can modify slightly the last and $\bar{\rho}$ -independent term of $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ (see equation (D.329)) by introducing an infinitesimal

case follows the same steps as in the pure scalar model, only the results for the potential's renormalized parameters will be written ¹¹⁸

$$|\beta|_{\text{ren}} = \begin{cases} |\beta| \left[1 + \frac{(d-1)\eta e^4}{32\pi^2|\beta|} \left(\ln \left(\frac{e^2 \langle \phi \rangle^2}{\mu^2} \right) + \frac{14}{3} \right) + \frac{3\eta(12|\beta|)}{32\pi^2} \left(\ln \left(\frac{4|\alpha|}{\mu^2} \right) + 5 \right) + O(\eta^2) \right] & d=4 \\ |\beta| \left[1 + 0 \cdot \eta e^4 - \frac{3\eta(12|\beta|)}{64\pi\sqrt{4|\alpha|}} + O(\eta^2) \right] & d=3 \\ |\beta| \left[1 + \frac{(d-1)\eta e^2}{48\pi|\alpha|} - \frac{3\eta(12|\beta|)}{64\pi|\alpha|} + O(\eta^2) \right] & d=2 \end{cases} \quad (\text{D.332})$$

term $\varepsilon \bar{\rho}^2$ in the following way: $-\eta \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-1} dp \rightarrow -2\frac{\eta}{2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \sqrt{p^2 + \varepsilon \bar{\rho}^2}$, and at the end, taking limit $\varepsilon \rightarrow 0$. The minus sign and the factor 2 will compensate at $\varepsilon \rightarrow 0$ the divergent contributions originated by each one of the terms containing $R_{1,2}^2$ (which have the opposite sign). It is important to note that the just discussed modification only acts on the term produced by the gauge fixing factor, as one can immediately see by looking at the underlined term in equation (D.327), term which produces $-\eta \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-1} dp$ after carrying out the integration over the periodic dp_τ .

The inclusion of $\varepsilon \bar{\rho}^2$ is equivalent, simply by comparing such a modification with potential's term $\sqrt{p^2 + 2\alpha + 6|\beta|\bar{\rho}^2}$, to the addition of a term $\frac{\varepsilon}{12}\bar{\rho}^4$ to the effective potential. Following [31] we could, instead of taking limit $\varepsilon \rightarrow 0$ at the end of the calculations, keep ε perturbative small but nonzero: taking into account that $\sqrt{p^2 + R_{1,2}^2} \Big|_{\phi=\langle\phi_{\text{scal}}\rangle} \stackrel{e^2\eta \ll |\beta|\eta}{\approx} \sqrt{p^2 \mp \alpha_{\text{fix}} e^2 \bar{\rho}^2}$, then keeping ε would correspond to set it of the same order as e^2 .

$$\stackrel{118}{\text{According to derivatives}} \left\{ \begin{aligned} \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}} &= 2\bar{\rho} \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} \\ \frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} &= 2 \frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} + 4\bar{\rho}^2 \frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^2} \\ \frac{d^3V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^3} &= 12\bar{\rho} \frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^2} + 8\bar{\rho}^3 \frac{d^3V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^3} \\ \frac{d^4V_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^4} &= 12 \frac{d^2V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^2} + \boxed{48\bar{\rho}^2 \frac{d^3V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^3} + 16(\bar{\rho}^2)^2 \frac{d^4V_{\text{eff}}^{(\omega=\infty)}}{d(\bar{\rho}^2)^4}} \end{aligned} \right.$$

the terms in the box, denoted by $\boxed{\text{finite}(\bar{\rho})}$, are finite for $d \leq 4$ even in the range of very small dis-

$$\text{tances } (p \rightarrow \infty): \frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})} = \left\{ \begin{aligned} & \frac{(d-1)\eta e^4}{12\pi^2|\beta|} + \frac{3\eta(12|\beta|)^2\bar{\rho}^2}{16\pi^2(2\alpha + 6|\beta|\bar{\rho}^2)} - \frac{\eta(12|\beta|)^3\bar{\rho}^4}{32\pi^2(2\alpha + 6|\beta|\bar{\rho}^2)^2} + \\ & \frac{(d-1)\eta e^4}{16\pi|\beta|\sqrt{e^2\bar{\rho}^2}} + \frac{3\eta(12|\beta|)^2\bar{\rho}^2}{16\pi(2\alpha + 6|\beta|\bar{\rho}^2)^{3/2}} - \frac{3\eta(12|\beta|)^3\bar{\rho}^4}{64\pi(2\alpha + 6|\beta|\bar{\rho}^2)^{5/2}} + \\ & \frac{(d-1)\eta e^4}{12\pi|\beta|(e^2\bar{\rho}^2)} + \frac{3\eta(12|\beta|)^2\bar{\rho}^2}{8\pi(2\alpha + 6|\beta|\bar{\rho}^2)^2} - \frac{\eta(12|\beta|)^3\bar{\rho}^4}{8\pi(2\alpha + 6|\beta|\bar{\rho}^2)^3} + \\ & + \frac{\eta(2|\beta|)^2\bar{\rho}^2}{4\pi^2(2\alpha + 2|\beta|\bar{\rho}^2)} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{12\pi^2(2\alpha + 2|\beta|\bar{\rho}^2)^2} & d=4 \\ & + \frac{\eta(2|\beta|)^2\bar{\rho}^2}{4\pi(2\alpha + 2|\beta|\bar{\rho}^2)^{3/2}} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{8\pi(2\alpha + 2|\beta|\bar{\rho}^2)^{5/2}} & d=3 \\ & + \frac{\eta(2|\beta|)^2\bar{\rho}^2}{2\pi(2\alpha + 2|\beta|\bar{\rho}^2)^2} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{3\pi(2\alpha + 2|\beta|\bar{\rho}^2)^3} & d=2 \end{aligned} \right\} \stackrel{\text{up to } \eta^1}{\approx} \frac{1}{12|\beta|_{\text{ren}}} \boxed{\text{finite}(\bar{\rho})}; \quad \text{the underlined}$$

and

$$\alpha_{\text{ren}} = \begin{cases} \alpha \left[1 + \frac{(d-1)\eta e^4}{32\pi^2|\beta|} + \frac{\eta}{32\pi^2}(12|\beta|) \left(\ln\left(\frac{4|\alpha|}{\mu^2}\right) + 4 \right) + O(\eta^2) \right] & d = 4 \\ \alpha \left[1 + \frac{(d-1)\eta e^4}{16\pi|\beta|\sqrt{e^2\langle\phi\rangle^2}} + \frac{\eta}{16\pi} \frac{(12|\beta|)}{\sqrt{4|\alpha|}} + O(\eta^2) \right] & d = 3 \\ \alpha \left[1 + \frac{(d-1)\eta e^2}{16\pi|\alpha|} \left(\ln\left(\frac{e^2\langle\phi\rangle^2}{\mu^2}\right) - 1 \right) + \frac{\eta}{32\pi} \frac{(12|\beta|)}{|\alpha|} \left(\ln\left(\frac{4|\alpha|}{\mu^2}\right) - \frac{3}{2} \right) + O(\eta^2) \right] & d = 2 \end{cases} \quad (\text{D.333})$$

contributions are those which become problematic in the case of $\alpha < 0$ at the nonvanishing $\bar{\rho} = \langle\phi_{\text{scal}}\rangle = \sqrt{\frac{|\alpha|}{|\beta|}} + \eta(\dots)$ and whose removing has been discussed in the previous footnote and therefore will not be considered in the following. Hence for $\alpha = -|\alpha|$

$$\text{one gets } \frac{1}{12|\beta|} \left[\text{finite}(\bar{\rho}) \right]_{\bar{\rho}=\langle\phi_{\text{scal}}\rangle} = \begin{cases} \frac{(d-1)\eta e^4}{12\pi^2|\beta|} + \frac{9\eta(12|\beta|)}{32\pi^2} & d = 4 \\ \frac{(d-1)\eta e^4}{16\pi|\beta|\sqrt{e^2\langle\phi_{\text{scal}}\rangle^2}} + \frac{9\eta(12|\beta|)}{64\pi\sqrt{4|\alpha|}} & d = 3 \\ \frac{(d-1)\eta e^4}{12\pi|\beta|(e^2\langle\phi_{\text{scal}}\rangle^2)} + 0 & d = 2 \end{cases} \quad \text{and, according}$$

to the minimum condition $\left. \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}} \right|_{\bar{\rho}=\langle\phi_{\text{scal}}\rangle}^{(\omega=\infty)} = 0 \Rightarrow \left. \frac{dV_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \right|_{\bar{\rho}=\langle\phi_{\text{scal}}\rangle}^{(\omega=\infty)} = 0$ together with

$$\text{the definition of the renormalized potential's parameters } \begin{cases} |\alpha|_{\text{ren}} = \frac{1}{4} \frac{d^2 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^2} \Big|_{\bar{\rho}=\langle\phi\rangle}^{(\omega=\infty)} \\ |\beta|_{\text{ren}} = \frac{1}{12} \frac{d^4 V_{\text{eff}}(\bar{\rho})}{d\bar{\rho}^4} \Big|_{\bar{\rho}=\langle\phi\rangle}^{(\omega=\infty)} \end{cases},$$

one obtains $|\alpha|_{\text{ren}} = \langle\phi\rangle^2 |\beta|_{\text{ren}} \left[1 - \frac{\text{finite}(\bar{\rho})}{12|\beta|_{\text{ren}}} \right]_{\bar{\rho}=\langle\phi\rangle}$, which is the same relation as (D.299) in the pure scalar model. The renormalized constants are:

$$|\beta|_{\text{ren}} = |\beta| \left[1 - \frac{\eta}{8|\beta|} \left(\text{sum finite p.} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{\text{constant} \times p^{d-2} dp}{(p^2 + \text{terms with } \bar{\rho}^2)^{3/2}} \text{ at } \Lambda_c \rightarrow \infty \right) \right]_{\bar{\rho}=\langle\phi\rangle} + \frac{\text{finite}(\bar{\rho})}{12|\beta|} \Big|_{\bar{\rho}=\langle\phi\rangle}$$

as well as $\alpha_{\text{ren}} = \alpha \left[1 + \frac{\eta}{4\alpha} \left(\text{sum of finite parts} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{\text{constant} \times p^{d-2} dp}{\sqrt{p^2 + \text{terms with } \bar{\rho}^2}} \text{ at } \Lambda_c \rightarrow \infty \right) \right]_{\bar{\rho}=\langle\phi\rangle} -$

$$- \frac{\eta}{8|\beta|} \left(\text{sum of finite parts} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{\text{constant} \times p^{d-2} dp}{(p^2 + \text{terms with } \bar{\rho}^2)^{3/2}} \text{ at } \Lambda_c \rightarrow \infty \right) \Big|_{\bar{\rho}=\langle\phi\rangle} + O(\eta^2) \Big]; \text{ in the}$$

previous result, as already mentioned, the diverging contributions which arise from the gauge-dependent terms at $\bar{\rho} = \sqrt{\frac{|\alpha|}{|\beta|}} + \eta(\dots)$ have been conveniently removed by means of a slight modification on the gauge fixing term.

together with expectation value $\langle \phi_{\text{scal}} \rangle^2 = \left(\frac{|\alpha|_{\text{ren}}}{|\beta|_{\text{ren}}} \right) \left[1 + \frac{\boxed{\text{finite}(\bar{\rho})}}{12|\beta|_{\text{ren}}} \right]_{\bar{\rho}=\langle \phi \rangle}$, being

$$\frac{1}{12|\beta|_{\text{ren}}} \boxed{\text{finite}(\bar{\rho})} \Big|_{\bar{\rho}=\langle \phi_{\text{scal}} \rangle} \stackrel{\text{up to } \eta^1}{\approx} \begin{cases} \frac{(d-1)\eta e^4}{12\pi^2|\beta|_{\text{ren}}} + \frac{9\eta(12|\beta|_{\text{ren}})}{32\pi^2} & d=4 \\ \frac{(d-1)\eta e^4}{16\pi|\beta|_{\text{ren}}\sqrt{e^2\langle \phi_{\text{scal}} \rangle^2}} + \frac{9\eta(12|\beta|_{\text{ren}})}{64\pi\sqrt{4|\alpha|_{\text{ren}}}} & d=3 \\ \frac{(d-1)\eta e^4}{12\pi|\beta|_{\text{ren}}(e^2\langle \phi_{\text{scal}} \rangle^2)} + 0 & d=2 \end{cases},$$

or equivalently

$$\langle \phi_{\text{scal}} \rangle^2 = \begin{cases} \left(\frac{|\alpha|}{|\beta|} \right) \left[1 - \frac{(d-1)\eta e^4}{32\pi^2|\beta|} \left(\ln \left(\frac{e^2\langle \phi \rangle^2}{\mu^2} \right) + 1 \right) - \frac{2\eta}{32\pi^2}(12|\beta|) \left(\ln \left(\frac{4|\alpha|}{\mu^2} \right) + 1 \right) + O(\eta^2) \right] & d=4 \\ \left(\frac{|\alpha|}{|\beta|} \right) \left[1 + \frac{(d-1)\eta e^4}{8\pi|\beta|\sqrt{e^2|\alpha|/|\beta|}} + \frac{2\eta(12|\beta|)}{8\pi\sqrt{4|\alpha|}} + O(\eta^2) \right] & d=3 \\ \left(\frac{|\alpha|}{|\beta|} \right) \left[1 + \frac{(d-1)\eta e^2}{16\pi|\alpha|} \ln \left(\frac{e^2\langle \phi \rangle^2}{\mu^2} \right) + \frac{2\eta(12|\beta|)}{64\pi|\alpha|} \ln \left(\frac{4|\alpha|}{\mu^2} \right) + O(\eta^2) \right] & d=2 \end{cases} \quad (\text{D.334})$$

where μ is the scale at which we arbitrarily start considering the long distance behaviour. It is to note that the above results hold also for *any* relation between ηe^2 and $\eta|\beta|$ as long as we are working in the gauge choice $\alpha_{\text{fix}} = 0$. We are now in a position to calculate the critical ω_{crit} for the symmetry restoring, i.e, at which the model transits from a $\langle \phi_{\text{scal}} \rangle \neq 0$ as a minimum of the effective potential towards a phase characterized by a minimum at $\langle \phi_{\text{scal}} \rangle = 0$.

According to equation (D.286), ω_{crit} gets defined by $\frac{dV_{\text{eff}}(\bar{\rho}^2)^{(\text{dep.}\omega)}}{d\bar{\rho}^2} \Big|_{\omega=\omega_{\text{crit}}, \bar{\rho}=0} =$

$$= -\frac{dV_{\text{eff,ren}}(\bar{\rho}^2)^{(\omega=\infty)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0} \quad \text{with}$$

$$\begin{aligned} \frac{dV_{\text{eff}}^{(\text{dep.}\omega)}}{d\bar{\rho}^2} &= (d-1) \frac{\eta e^2}{2\omega^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega^2 e^2 \bar{\rho}^2}} \frac{dp}{e^{\sqrt{p^2 + \omega^2 e^2 \bar{\rho}^2}} - 1} + \\ &+ \frac{\eta(12|\beta|)}{4\omega^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega^2 (2\alpha + 6|\beta|\bar{\rho}^2)}} \frac{dp}{e^{\sqrt{p^2 + \omega^2 (2\alpha + 6|\beta|\bar{\rho}^2)}} - 1} + \\ &+ \frac{\eta}{2\omega^{d-2}} \frac{dR_1^2}{d\bar{\rho}^2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega^2 R_1^2}} \frac{dp}{e^{\sqrt{p^2 + \omega^2 R_1^2}} - 1} + \\ &+ \frac{\eta}{2\omega^{d-2}} \frac{dR_2^2}{d\bar{\rho}^2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega^2 R_2^2}} \frac{dp}{e^{\sqrt{p^2 + \omega^2 R_2^2}} - 1} \end{aligned} \quad (\text{D.335})$$

and with, up to order η and always for $\eta e^2 \ll \eta|\beta|$, the approximation of

substituting $-\frac{dV_{\text{eff, ren}}(\bar{\rho}^2)^{(\omega=\infty)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0}$ for $-\alpha_{\text{ren}} \equiv |\alpha|_{\text{ren}}$ since $\frac{dV_{\text{eff}}^{(\text{dep.}\omega)}}{d\bar{\rho}^2}$ already is of order η (see comments after equation (D.340)).

At first sight the above result will produce (through the gauge fixing parameter α_{fix} which is already contained into $R_{1,2}^2$) a gauge-dependent ω_{crit} . However, since all this discussion is being carried out under the assumption that the transition occurs in the range of small ω_{crit} , by expanding the two integrals containing $R_{1,2}^2$, $\frac{1}{\sqrt{p^2 + \omega_{\text{crit}}^2 R_{1,2}^2} \left(e^{\sqrt{p^2 + \omega_{\text{crit}}^2 R_{1,2}^2}} - 1 \right)} \approx \frac{1}{p(e^p - 1)} -$

$-\omega_{\text{crit}}^2 R_{1,2}^2 \frac{e^p - 1 + p}{2p^3(e^p - 1)^2} + O(\omega_{\text{crit}}^4)$, one gets that the leading contribution at

small ω of both integrals together becomes equal to $\frac{\eta}{2\omega^{d-2}} \left(\frac{dR_1^2}{d\bar{\rho}^2} + \frac{dR_2^2}{d\bar{\rho}^2} \right) \times \times \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{p(e^p - 1)}$. And since $\frac{dR_1^2}{d\bar{\rho}^2} + \frac{dR_2^2}{d\bar{\rho}^2} = 2|\beta|$, as one can show from $R_{1,2}^2 = \frac{2\alpha + 2|\beta|\bar{\rho}^2}{2} \left(1 \pm \sqrt{1 - \frac{4\alpha_{\text{fix}} e^2 \bar{\rho}^2}{2\alpha + 2|\beta|\bar{\rho}^2}} \right)$, and thus independent of α_{fix} ,

by using (D.315)-(D.317) for $I_{d-1}(M) = \frac{1}{(d-2)!} \int_0^\infty \frac{x^{d-2}}{\sqrt{x^2 + M^2}} \frac{dx}{e^{\sqrt{x^2 + M^2}} - 1}$ one obtains the critical ω -value

$$\begin{cases} \frac{1}{\omega_{\text{crit}}^2} \approx 12 \frac{2|\alpha|_{\text{ren}}}{\left(\frac{12\eta|\beta|_{\text{ren}}}{2} + \frac{12\eta|\beta|_{\text{ren}}}{6} \right) + (d-1)\eta e^2} & d = 4, \\ \frac{1}{\omega_{\text{crit}}^2} \approx \frac{-4\pi|\alpha|_{\text{ren}}/|\beta|_{\text{ren}}}{\eta \left(6 \ln |2\alpha|_{\text{ren}} \omega_{\text{crit}}^2 + \left(\frac{(d-1)\eta e^2}{\eta|\beta|_{\text{ren}}} + 2 \right) \lim_{p \rightarrow 0} \ln p \right)} & d = 3, \end{cases} \quad (\text{D.336})$$

where up to order η one can substitute $|\beta|$ for $|\beta|_{\text{ren}}$. The above result, based in the approximation for integral $I_{d-1}(M)$, only holds if the transition takes place at a small ω_{crit} , where “small” means that we are working in the range where

$$\frac{1}{\omega^2} \gg |\alpha|_{\text{ren}}, |\beta|_{\text{ren}} \bar{\rho}^2, \quad \left(\frac{1}{\omega^2} \gg e^2 \bar{\rho}^2 \right), \quad (\text{D.337})$$

is satisfied; the third condition is already fulfilled since in this paragraph we are assuming $\eta|\beta| \gg \eta e^2$.

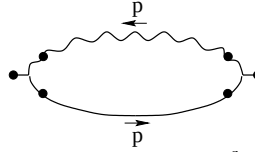
One arrives at the same result (D.336) by solving¹¹⁹ the following (Feynman)

¹¹⁹One has to consider the following diagram's vertices which are connected by the corresponding scalar or gauge propagator: to energy's part $\int d^d \vec{r} [-e\epsilon_{cd} \phi_c(\vec{r}) (\partial_\mu \phi_d(\vec{r})) \bar{A}_\mu(\vec{r})] = \int \frac{d^d \vec{k}_1}{(2\pi)^d} \frac{d^d \vec{k}_2}{(2\pi)^d} \frac{d^d \vec{k}_3}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) [-ie\epsilon_{cd}(k_2)_\mu \tilde{\phi}(\vec{k}_1) \tilde{\phi}(\vec{k}_2) \tilde{\bar{A}}_\mu(\vec{k}_3)]$ corresponds the 3-

diagrams

$$\begin{aligned}
\left. \frac{dV_{\text{eff}}^{(\omega)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0} &= 0 = \left. \frac{dV_{\text{eff}}^{(\text{dep. } \omega)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0} + \left. \frac{dV_{\text{eff, ren}}^{(\omega=\infty)}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0} = \alpha - \delta\alpha - \\
& - \frac{\eta}{4} \left. \left(\text{diagram 1} \right) \right|_{\tilde{\varphi}=\tilde{\rho}=0} - \frac{\eta}{4} \left. \left(\text{diagram 2} \right) \right|_{\tilde{\varphi}=\tilde{\rho}=0} - \\
& - 2 \times \frac{\eta}{4} \left. \left(\text{diagram 3} \right) \right|_{\tilde{\varphi}=\tilde{\rho}=0} + \frac{\eta}{4} \left. \left(\text{diagram 4} \right) \right|_{\tilde{\varphi}=\tilde{\rho}=0} + \frac{\eta}{4} \left. \left(\text{diagram 5} \right) \right|_{\tilde{\varphi}=\tilde{\rho}=0} = 0,
\end{aligned} \tag{D.338}$$

for $\omega = \omega_{\text{crit}}$, just as in equation (D.322) for the pure scalar model. In the above diagrams, the wavy line in the internal loop represents the propagator of the gauge field, and the factor 2 multiplying the first diagram on the third line is due to the two possible positions (up and down) of the gauge field's propagator. As already noted in the previous footnote, each one of the two ver-

tices in diagram  yields a contribution proportional

to e since that proportional to e^2 , which would be produced by derivative

$$\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^b(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3)} = 2e^2 \delta_{ab} \tilde{\phi}_{\text{gaug}}^\mu \text{ from } \int d^d \vec{r} \left[\frac{e^2}{2} \phi_c^2(\vec{r}) \bar{A}_\mu^2(\vec{r}) \right], \text{ vanishes}$$

legs vertex $\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^{a'}(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3)} \equiv \frac{\delta^3 E[\tilde{\phi}_a(\vec{q}), \tilde{A}_\mu(\vec{q})]}{\delta \tilde{\phi}_a(\vec{k}_1) \delta \tilde{\phi}_{a'}(\vec{k}_2) \delta \tilde{A}_\mu(\vec{k}_3)} = -ie \epsilon_{aa'} \left((k_2)_\mu - (k_1)_\mu \right)$. To part

$$\int d^d \vec{r} \left[\frac{e^2}{2} \phi_c^2(\vec{r}) \bar{A}_\mu^2(\vec{r}) \right] = \int \frac{d^d \vec{k}_1}{(2\pi)^d} \dots \frac{d^d \vec{k}_4}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \dots + \vec{k}_4) \left[\frac{e^2}{2} \tilde{\phi}_c(\vec{k}_1) \tilde{\phi}_c(\vec{k}_2) \tilde{A}_\mu(\vec{k}_3) \tilde{A}_\mu(\vec{k}_4) \right], \text{ the}$$

4-legs vertex $\frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^b(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3) \delta \tilde{\phi}_{\text{gaug}}^\nu(\vec{k}_4)} \equiv \frac{\delta^4 E[\tilde{\phi}_a(\vec{q}), \tilde{A}_\mu(\vec{q})]}{\delta \tilde{\phi}_a(\vec{k}_1) \delta \tilde{\phi}_b(\vec{k}_2) \delta \tilde{A}_\mu(\vec{k}_3) \delta \tilde{A}_\nu(\vec{k}_4)} = 2e^2 \delta_{ab} \delta_{\mu\nu}$.

And finally to energy's part $\int d^d \vec{r} \left[\frac{|\beta|}{2} (\phi_c(\vec{r}) \phi_c(\vec{r}))^2 \right] = \int \frac{d^d \vec{k}_1}{(2\pi)^d} \dots \frac{d^d \vec{k}_4}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \dots + \vec{k}_4) \times$
 $\times \left[|\beta| \tilde{\phi}_c(\vec{k}_1) \tilde{\phi}_c(\vec{k}_2) \tilde{\phi}_d(\vec{k}_3) \tilde{\phi}_d(\vec{k}_4) \right]$, the vertex of 4-legs $\frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^b(\vec{k}_2) \delta \tilde{\phi}_{\text{scal}}^c(\vec{k}_3) \delta \tilde{\phi}_{\text{scal}}^d(\vec{k}_4)} =$
 $= 4|\beta| (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc})$. Although the last energy's part could also produce functional deriva-

tive $\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^b(\vec{k}_2) \delta \tilde{\phi}_{\text{scal}}^c(\vec{k}_3)}$, this will not contribute since it is proportional to $\tilde{\phi}$ and in the diagrams

we set at the end $\tilde{\phi} = 0$. The same holds for the contribution to vertex $\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{scal}}^b(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_3)}$,

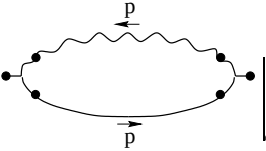
or for $\frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}_{\text{scal}}^a(\vec{k}_1) \delta \tilde{\phi}_{\text{gaug}}^\mu(\vec{k}_2) \delta \tilde{\phi}_{\text{gaug}}^\nu(\vec{k}_3)}$, which comes from part $\int d^d \vec{r} \left[\frac{e^2}{2} \phi_c^2(\vec{r}) \bar{A}_\mu^2(\vec{r}) \right]$.

because it is valued at $\tilde{\phi}_{\text{scal}}^a = \tilde{\phi}_{\text{gaug}}^\mu = 0$. For a similar reason, both diagrams in the second line of (D.338) are equal to zero since their vertices contain $\tilde{\phi}_{\text{scal}}^a$. It should be also noted that since we are studying the expectation value of the scalar field, no external legs in the vertices (i.e., those legs taken at zero Fourier momentum) can correspond to the gauge field. Taking into account the scalar and gauge propagator, (D.325) and (D.326), the above diagrammatic equation yields ¹²⁰

$$|\alpha| - \delta\alpha = \left[8|\beta|\eta \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 - 2|\alpha|} + (d-1)e^2\eta \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2} \right] - \alpha_{\text{fix}} e^2 \eta \left[\int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 - 2|\alpha|} - \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2} \right], \quad (\text{D.339})$$

when evaluated at $\omega = \omega_{\text{crit}}$. And now the integration of the Fourier momentum corresponding to the periodic coordinate by using relation (D.289), $\int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 + M} = \frac{1}{2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \frac{1}{\sqrt{p^2 + M}} \frac{1}{e^{\omega\sqrt{p^2 + M}} - 1}$, gives the following result

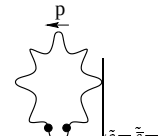
$$|\alpha|_{\text{ren}} = \frac{\eta}{2\omega_{\text{crit}}^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \frac{8|\beta|}{\sqrt{p^2 - 2\omega_{\text{crit}}^2|\alpha|} (e^{\omega\sqrt{p^2 - 2\omega_{\text{crit}}^2|\alpha|}} - 1)} + \frac{\eta}{2\omega_{\text{crit}}^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \frac{(d-1)e^2}{p(e^p - 1)} - \quad (\text{D.340})$$

¹²⁰To diagram  , with both external legs at zero momentum, corresponds

$$\int \frac{d^d \vec{p}}{(2\pi)^d} \left[-ie\epsilon_{aa'} \left((p)_\mu - 0 \right) \right] \frac{1}{\vec{p}^2 + e^2 \bar{\rho}^2} \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} \left(\delta_{\mu\nu} + \frac{(\alpha_{\text{fix}} - 1)p_\mu p_\nu}{\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} \frac{\delta_{a'b'}}{\vec{p}^2 - 2|\alpha|} \left[-ie\epsilon_{ab'} \left((-p)_\nu - 0 \right) \right] =$$

$$= 2\alpha_{\text{fix}} e^2 \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 - 2|\alpha|}. \quad \text{And similarly to } \img alt="A diagram showing a loop with two external legs. The top leg is a wavy line with momentum p pointing left. The bottom leg is a straight line with momentum p pointing right. The loop is represented by a horizontal line with two vertices, each having a small circle attached to it." data-bbox="460 630 550 690"/> $\equiv \int \frac{d^d \vec{p}}{(2\pi)^d} 4|\beta| (2\delta_{a'b'} + \delta_{a'b'} + \delta_{a'b'}) \times$

$$\times \frac{1}{\vec{p}^2 - 2|\alpha| + 2|\beta|\bar{\rho}^2} \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} \left(\delta_{a'b'} - \frac{4|\beta|\bar{\rho}_{a'}\bar{\rho}_{b'}}{\vec{p}^2 - 2|\alpha| + 6|\beta|\bar{\rho}^2} \right) \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} = 32|\beta| \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2 - 2|\alpha|}, \quad \text{and finally to}$$

diagram  the integral $\int \frac{d^d \vec{p}}{(2\pi)^d} 2e^2 \underbrace{\delta_{aa}}_{=2} \delta_{\mu\nu} \frac{1}{\vec{p}^2 + e^2 \bar{\rho}^2} \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} \left(\delta_{\mu\nu} + \frac{(\alpha_{\text{fix}} - 1)p_\mu p_\nu}{\vec{p}^2 + \alpha_{\text{fix}} e^2 \bar{\rho}^2} \right) \bigg|_{\vec{p}=0}^{\vec{p}=\vec{p}=0} =$

$$= (4e^2(d-1) + 4e^2\alpha_{\text{fix}}) \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{\vec{p}^2}.$$$$

$$-\frac{\alpha_{\text{fix}} e^2 \eta}{2\omega_{\text{crit}}^{d-2}} \int_0^\infty \frac{S_{d-1} p^{d-2} dp}{(2\pi)^{d-1}} \left[\frac{1}{\sqrt{p^2 - 2\omega_{\text{crit}}^2 |\alpha|} (e^{\sqrt{p^2 - 2\omega_{\text{crit}}^2 |\alpha|}} - 1)} - \frac{1}{p(e^p - 1)} \right],$$

where counterterm $\delta\alpha$ has removed the divergences contained in $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ and its derivatives. Although $\left. \frac{dV_{\text{eff, ren}}}{d\bar{\rho}^2} \right|_{\bar{\rho}=0}^{(\omega=\infty)}$ is not exactly $\alpha_{\text{ren}} = -|\alpha|_{\text{ren}}$ for there are other contributions of order η , however these last contributions are of order $\frac{(\eta e^2)^2}{\eta|\beta|}$ at $d = 4$ or of order $\sqrt{\frac{(\eta e^2)^3}{\eta|\beta|}}$ at $d = 3$ (see (D.333)) and therefore negligible since we are considering the case $\eta e^2 \ll \eta|\beta|$: in this case we don't need to consider the corrections of order η in $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ [14]. Or equivalently, such corrections do not change the main property of the original effective potential: the existence of a nonvanishing expectation value; and thus one can directly substitute in the original potential the original parameters for the renormalized ones as long as $\eta|\beta| \gg \eta e^2$. A completely different case will occur when we study the case $\eta|\beta| \ll \eta e^2$ (remaining both parameters perturbative small).

Just as after equation (D.335), the apparent dependence on the choice of gauge through α_{fix} disappears when the leading contribution for small ω_{crit} is calculated. The minus inside the last integrand causes this integral to be of order $\frac{1}{\omega_{\text{crit}}^{d-4}}$ due to expansion $\frac{1}{\sqrt{p^2 - 2\omega_{\text{crit}}^2 |\alpha|} (e^{\sqrt{p^2 - 2\omega_{\text{crit}}^2 |\alpha|}} - 1)} \approx \frac{1}{p(e^p - 1)} + \omega_{\text{crit}}^2 |\alpha| \frac{e^p - 1 + p}{p^3(e^p - 1)^2} + O(\omega_{\text{crit}}^4)$ and therefore negligible when compared with the two first terms of (D.340): the resulting ω_{crit} is then gauge-independent and equal to (D.336). Of course the whole discussion could be carried out for lower powers of $1/\omega_{\text{crit}}$ but only provided we also take higher perturbation terms in $\eta|\beta|$ and ηe^2 , since otherwise the use of relations (D.315)-(D.317) would produce nonsensical results.

Let us return to relations (D.336) which defines the critical point for the transition from $\langle \phi_{\text{scal}} \rangle^2 \neq 0$ towards $\langle \phi_{\text{scal}} \rangle^2 = 0$ when ω is decreased, keeping ω small. For the 3-dimensional model and due to the divergence at large distances of $\lim_{p \rightarrow 0} \ln p$ one obtains $\frac{1}{\omega_{\text{crit}}} \approx 0$, result which lies well beyond the assumed small ω_{crit} : the only form such that result (D.336) could hold also for $d = 3$ would be a large $|\alpha|_{\text{ren}}$. Such breaking in the assumption of small ω_{crit} means that for integral $\int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{x^{d-2}}{\sqrt{x^2 + (|\beta|, e^2)\bar{\rho}^2}} \frac{dx}{e^{\sqrt{x^2 + (|\beta|, e^2)\bar{\rho}^2}} - 1}$ we need higher expansion terms in $\eta|\beta|$ and in ηe^2 to obtain a reliable value for ω_{crit} for $d = 3$: in any way, this will result in a much weaker dependence of ω_{crit} on the effective potential's parameters, than that displayed in the 4-dimensional model, where ω_{crit} does already have a result according to the

assumption of its smallness. A similar weaker dependence in the critical value of ω_{crit} for the 3-dimensional compared to that of the 4-dim. model can be also observed in the non-compact lattice models in the range of small ηe^2 when phase diagrams on pages 751 and 752 are compared to each other.

For the $d = 4$ case, critical ω_{crit} increases with e^2 according to relation $\frac{1}{\omega_{\text{crit}}^2} \approx 3 \frac{|\alpha|_{\text{ren}}}{\eta |\beta|_{\text{ren}}} \left[1 - \frac{(d-1)\eta e^2}{8\eta |\beta|_{\text{ren}}} \right]$, as always only valid for a transition which occurs at small ω_{crit} . This is also the qualitative behaviour one expects from the 4-dim. non-compact lattice Higgs model¹²¹: considering different sections of constant *large* $|\psi|^2/\eta$ (parameter which corresponds to $|\alpha|_{\text{ren}}/\eta |\beta|_{\text{ren}}$ in the here discussed effective Abrikosov model) in the phase diagram on page 752, one observes that the line separating the two phases and characterizing thus ω_{crit} shows a negative slope in the region of small ηe^2 (note that a small ω corresponds to the top of the phase diagram's vertical axis). And such a slope becomes steeper with an increasing larger $|\psi|^2/\eta$: this corresponds qualitatively also to the obtained result $\frac{1}{\omega_{\text{crit}}^2} \approx 3 \frac{|\alpha|_{\text{ren}}}{\eta |\beta|_{\text{ren}}} \left[1 - \frac{(d-1)\eta e^2}{8\eta |\beta|_{\text{ren}}} \right]$. As it was also the case for the pure scalar model, the non-compact lattice Higgs model as well as the continuum effective Abrikosov model display, up to specific numerical details, the same behaviour. It should be noted also that also the character of the phase transition, of continuous (or critical) type, coincides in both models: in the non-compact lattice Higgs model, most of the line $A'B'$ in the upper corner of small ω are points of second-order transitions since the tricritical point tc approaches point A' in the considered range.

Also expected from the discussion on the non-compact lattice Higgs model, the 2-dimensional case trivially does not display any transition when one of its coordinates becomes periodic; according to it, any dependence on ω_{crit} in equation (D.340) disappears at $d = 2$.

$\eta |\beta| \ll \eta e^2$ In the previous case, as the coupling e^2 to the gauge field was assumed to be much smaller than the self-coupling $|\beta|$ of the 4th-power of the scalar field, we found that the corrections of order η^1 does not change the form of $\frac{dV_{\text{eff, ren}}(\bar{\rho}^2)}{d\bar{\rho}^2} \Big|_{\bar{\rho}=0}^{(\omega=\infty)}$ compared to its original form at order η^0 (see equation

¹²¹In order to “translate” the results from the effective potential to those of the non-compact lattice Higgs model at small ηe^2 , one has to remember that the lattice model's phase occurring for large $\frac{|\psi|^2}{\eta}$ and small ηe^2 coincides with the effective model's phase with non-vanishing expectation value $\frac{\langle \phi_{\text{scal}} \rangle^2}{\eta}$, characteristic for this last model at $\eta |\beta| \gg \eta e^2$: this was proven by evaluating correlation function (D.252) in the lattice model, correlation which displays the expected behaviour of the expectation value in the effective Abrikosov model. Note that in the range $\eta |\beta| \gg \eta e^2$ we are now working, the corrections of order η do not spoil the symmetry breaking of the original (without corrections) potential as has been shown after equation (D.340). The expectation value of the effective potential's scalar field is approximately, up to order $\frac{(\eta e^2)^{n=2,3}}{\eta |\beta|} \ll 1$, given by $\frac{\alpha_{(\text{ren})}}{\eta |\beta|_{(\text{ren})}}$.

(D.335) and the comments following (D.340)) and thus such corrections did not spoil the “mexican-hat” form of the effective potential at $\omega = \infty$, as in the left figure on page 680. Moreover, when the space became periodic along one direction, in the case of a small ω satisfying $\frac{1}{\omega^2} \gg |\beta|\bar{\rho}^2$ one was able to use the approximations (D.315)-(D.317) in *all* the integrals which occur in the perturbative expansion up to order η^1 of the effective potential: such approximation of small ω were valid also for the integral containing gauge coupling e^2 since $\eta|\beta| \gg \eta e^2$. Thus the critical ω -value was defined as the value at which the “mexican-hat” bottom became flat, just as in the middle figure on page 680. Now, in the case of $e^2\eta \gg |\beta|\eta$, we shall see that for certain relations between ηe^2 and $\eta|\beta|$ (always keeping both couplings perturbatively small) the corrections of order η do modify substantially the original form of the effective potential and further the approximation of small ω previously used does not hold for the integral containing e^2 . All this will cause that the critical (or continuous) behaviour displayed at $e^2\eta \ll |\beta|\eta$ turns to be of first-order ¹²² type [14].

As in the previous case, it will result that the effective potential up to order η is gauge dependent (although if we would consider the complete effective potential at all orders in η , it will become gauge independent). However, when the transition ω -value is calculated in the range of small ω , and as long as we only keep the leading contribution, the resulting $\omega_{\text{transition}}$ will become gauge independent for the same reason as that discussed after equation (D.335) (or after (D.339)). Knowing that, and in order to obtain more manageable equations, we shall from the very beginning choose the gauge $\alpha_{\text{fix}} = 0$: this allows us to profit from results (D.332)-(D.334), which were already obtained in the case $\eta|\beta| \gg \eta e^2$ for *any* α_{fix} , but which also hold at $\alpha_{\text{fix}} = 0$ independent of the relation between ηe^2 and $\eta|\beta|$. As it is clear from defi-

inition $R_{1,2}^2 = \frac{2\alpha + 2|\beta|\bar{\rho}^2}{2} \left(1 \pm \sqrt{1 - \frac{4\alpha_{\text{fix}}e^2\bar{\rho}^2}{2\alpha + 2|\beta|\bar{\rho}^2}} \right)$, limit $\lim_{\eta e^2/\eta|\beta| \rightarrow 0}$ produces the same result as limit $\lim_{\alpha_{\text{fix}} \rightarrow 0}$. On the other hand, the integrals containing $R_{1,2}^2$ which appear in the ω -independent part of the effective potential (and in its derivatives) at order η can show a divergence at $p \rightarrow 0$, divergence which is removed by a part of the gauge fixing factor (see footnote on page 709), and after this removing, these integrals do not yield any contribution to the renormalized parameters α_{ren} and $|\beta|_{\text{ren}}$ of the effective potential (see footnote on page 710).

As a matter of notation, the nonvanishing potential’s minimum $\langle \phi_{\text{scal}} \rangle$ calculated in (D.334) will be denoted by $\langle \phi_{\text{scal}}^{(\text{m})} \rangle$ for the following: this is only

¹²²Only as a reminder: as different from the critical or continuous phase transition where at the transition point the energy of the old and of the new phase are the same without being separated by any barrier and the transition takes place continuously, in the first-order or discontinuous phase transition the old phase disappears by means of a fluctuation large enough that there exists another phase of still lower energy and, once the system has sprung to this more stable state, the new phase develops very fast.

to remind that now, under certain conditions, such a $\langle \phi_{\text{scal}} \rangle$ may become a *metastable* state, i.e, a local minimum of the effective potential but not the deepest one.

Before starting with the discussion on the effects of the corrections of order η on the effective potential at $\omega = \infty$, let us firstly consider the consequence of $\eta e^2 \gg \eta|\beta|$ when the potential's part at finite small ω is calculated. More concrete, how the contribution for $\frac{1}{\omega^2} \gg |\alpha|$, $|\beta|\bar{\rho}^2$ looks in the effective potential evaluated at $\bar{\rho} = 0$ and at $\bar{\rho} = \langle \phi_{\text{scal}}^{(m)} \rangle$: if we had calculated the critical value ω_{crit} by starting at (D.335)

$$\frac{dV_{\text{eff}}^{(\text{dep.}\omega)}}{d\bar{\rho}^2} \Big|_{\eta e^2 \gg \eta|\beta|} \approx \frac{(d-1)\eta e^2}{2\omega^{d-2}} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2}}{\sqrt{p^2 + \omega^2 e^2 \bar{\rho}^2}} \frac{dp}{e\sqrt{p^2 + \omega^2 e^2 \bar{\rho}^2} - 1},$$

and then applying approximation $\int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{x^{d-2}}{\sqrt{x^2 + M^2}} \frac{dx}{e\sqrt{x^2 + M^2} - 1} \Big|_{|M| \ll 1} \approx$

$$\Big|_{|M| \ll 1} \begin{cases} \frac{1}{12} & d = 4 \\ -\frac{1}{2\pi} \ln |M| & d = 3 \end{cases}, \text{ we would have obtained as critical } \omega\text{-value the}$$

$$\text{following one: } \begin{cases} \frac{1}{\omega_{\text{crit}}^2} \approx 12 \frac{2|\alpha|}{(d-1)\eta e^2} & d = 4 \\ \frac{1}{\omega_{\text{crit}}} \approx \frac{-4\pi|\alpha|}{(d-1)\eta e^2 \lim_{p \rightarrow 0} \ln p} & d = 3 \end{cases} \text{ (we are not yet taking}$$

into account the corrections of order η in $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$). However, and always assuming $\frac{1}{\omega^2} \gg |\alpha|$, $|\beta|\bar{\rho}^2$, the above approximation is based on condition $M^2 \equiv \omega^2 e^2 \bar{\rho}^2 \ll 1$ which held for the case $\eta|\beta| \gg \eta e^2$ for any value of $\bar{\rho}$ but which in the case under consideration now, with the opposite relation between ηe^2 and $\eta|\beta|$, does *not* work: especially at $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$ (minimum of $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)}$ without corrections of order η) and for $d = 4$

$$e^2 \bar{\rho}^2 = e^2 \frac{|\alpha|}{|\beta|} = \left(\frac{(\eta e^2)^2}{\eta|\beta|} \right) \frac{1}{8\omega_{\text{crit}}^2} \implies \omega_{\text{crit}}^2 e^2 \bar{\rho}^2 \gg 1, \quad (\text{D.341})$$

for $(\eta e^2)^2 \gg \eta|\beta|$. For $d = 3$ the similar result

$$e^2 \bar{\rho}^2 = e^2 \frac{|\alpha|}{|\beta|} = \left(\frac{(\eta e^2)^3}{\eta|\beta|} \right) \frac{1}{4\pi^2|\alpha|} \left(\lim_{p \rightarrow 0} \ln p \right)^2 \frac{1}{\omega_{\text{crit}}^2}, \quad (\text{D.342})$$

is less conclusive since the divergent $\lim_{p \rightarrow 0} \ln p$ is an indication that we have to go to the correction of order η^2 in $\frac{dV_{\text{eff}}^{(\text{dep.}\omega)}}{d\bar{\rho}^2}$.

Let us therefore return to the general contribution of finite ω to the effective potential, equation (D.330)

$$V_{\text{eff}}(\bar{\rho})^{(\text{dep.}\omega)} = \frac{\eta}{\omega^d} \sum_{M_a^2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - \exp\left(-\sqrt{p^2 + \omega^2 M_a^2}\right) \right],$$

whilst the ω -independent part of the potential will be still considered without corrections of order η , i.e. $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} = -|\alpha|\bar{\rho}^2 + \frac{|\beta|}{2}\bar{\rho}^4$. The above integral $\int_0^\infty p^{d-2} dp \ln \left[1 - \exp \left(-\sqrt{p^2 + \omega^2 M_a^2} \right) \right] = -\frac{d!}{d-1} I_{d+1} \left(\omega \sqrt{M_a^2} \right)$ must be calculated for $M^2 = \epsilon^2 \bar{\rho}^2$ ($d-1$ times), $M^2 = (2\alpha + 6|\beta|\bar{\rho}^2)$, $(2\alpha + 2|\beta|\bar{\rho}^2)$, 0 (once) and $M^2 = 0$ (two further times but with opposite sign, arising from part of the gauge fixing factor). Always fulfilling $\frac{1}{\omega^2} \gg |\alpha|, |\beta|\bar{\rho}^2$, for $\bar{\rho}^2 = 0$ approximation $\omega^2 M^2 \ll 1$ holds for all integrals in $V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)}$ and thus

$$\begin{aligned} V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}=0} &= V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} \Big|_{\bar{\rho}=0} + V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)} \Big|_{\bar{\rho}=0} \approx \\ &\approx \begin{cases} -\frac{\eta}{\omega^4} \frac{\pi^2}{90} ((d-1) + (3-2)) & d=4, \\ -\frac{\eta}{\omega^3} \frac{\zeta(3)}{2\pi} ((d-1) + (3-2)) & d=3. \end{cases} \end{aligned} \quad (\text{D.343})$$

For $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$ and $(\eta e^2)^2 \gg \eta|\beta|$, on the other hand, let us assume (assumption which will be proven later to be self-consistent) that it occurs something similar to that in relations (D.341)-(D.342): $\omega^2 e^2 \bar{\rho}^2 \gg 1$. Then the integral $(d-1) \frac{\eta}{\omega^d} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} p^{d-2} dp \ln \left[1 - \exp \left(-\sqrt{p^2 + \omega^2 e^2 \bar{\rho}^2} \right) \right]$ produces no relevant contribution due to the exponential damping of its integrand and therefore

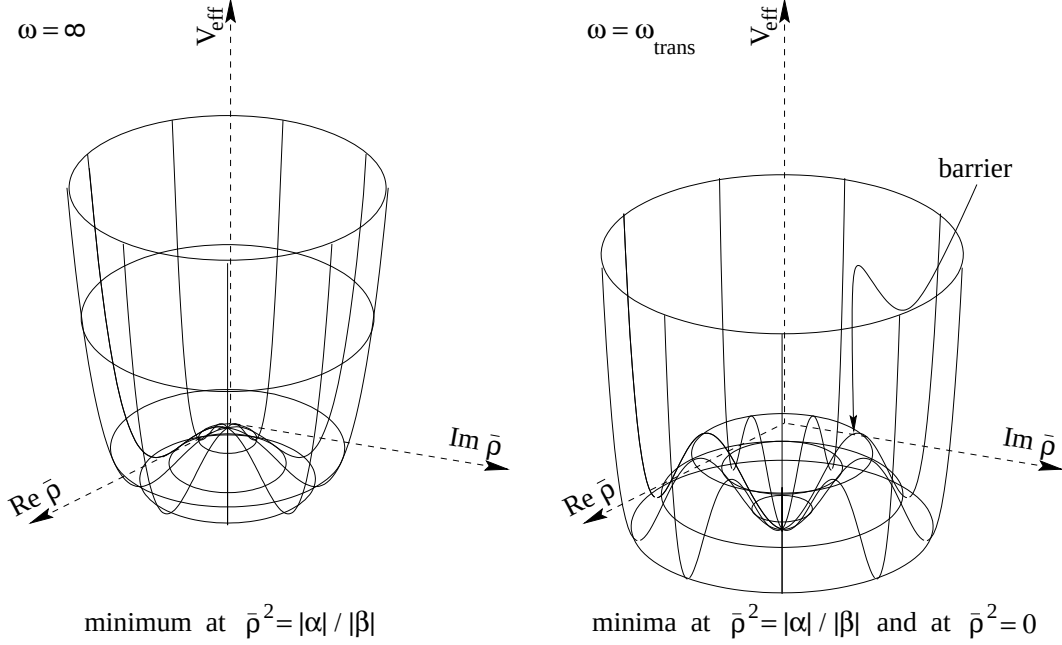
$$\begin{aligned} V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}^2=|\alpha|/|\beta|} &= V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} \Big|_{\bar{\rho}^2=|\alpha|/|\beta|} + V_{\text{eff}}(\bar{\rho})^{(\text{dep. } \omega)} \Big|_{\bar{\rho}^2=|\alpha|/|\beta|} \approx \\ &\approx \begin{cases} -\frac{|\alpha|^2}{2|\beta|} - \frac{\eta}{\omega^4} \frac{\pi^2}{90} \times 1 & d=4, \\ -\frac{|\alpha|^2}{2|\beta|} - \frac{\eta}{\omega^3} \frac{\zeta(3)}{2\pi} \times 1 & d=3. \end{cases} \end{aligned} \quad (\text{D.344})$$

Comparing this last equation and (D.343), and since the largest ω -contribution (at small ω) decreases in a larger amount the value of the effective potential at $\bar{\rho}^2 = 0$ than that at $\bar{\rho}^2 = |\alpha|/|\beta|$, one arrives at the result that for

$$\begin{cases} \frac{1}{\omega_{\text{trans}}^2} \approx \left(\frac{15}{\pi^2} \frac{|\alpha|^2}{\eta|\beta|} \right)^{1/2} & d=4, \\ \frac{1}{\omega_{\text{trans}}^2} \approx \left(\frac{\pi}{2\zeta(3)} \frac{|\alpha|^2}{\eta|\beta|} \right)^{2/3} & d=3, \end{cases} \quad (\text{D.345})$$

$\bar{\rho}^2 = 0$ has the same effective potential value as $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$, and for $\omega < \omega_{\text{trans}}$ it lies even deeper in energy than $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$. Since both $\bar{\rho}^2$ -values are separated

by a barrier (see next figure) the transition from the asymmetric minimum $\langle \phi_{\text{scal}} \rangle^2 = \frac{|\alpha|}{|\beta|}$ to the symmetric one $\langle \phi_{\text{scal}} \rangle^2 = 0$ as soon as ω has decreased under the value ω_{trans} is of first-order type.



It is important to note that only because we are assuming $\eta e^2 \gg \eta|\beta|$ we have obtained the above first order transition; otherwise, the largest ω -contribution to $V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}=0}$ and to $V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}^2=|\alpha|/|\beta|}$ would have been *the same*, i.e., both multiplied by $(d-1) + (3-2)$, and one would have needed to go to the next order in ω , proportional to $\bar{\rho}^2$, next order which gives the leading contribution to $\frac{dV_{\text{eff}}^{(\text{dep. } \omega)}}{d\bar{\rho}^2}$ and is thus responsible for the second-order transition as we have considered in the previous paragraph for $\eta|\beta| \gg \eta e^2$. It only remains to check such an assumption ¹²³ which has led us to (D.345), i.e., $\omega_{\text{trans}}^2 e^2 \bar{\rho}^2 \gg 1$ with $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$: at $d=4$ $\omega_{\text{trans}}^2 e^2 \frac{|\alpha|}{|\beta|} = \sqrt{\frac{\pi^2}{15}} \left(\frac{(\eta e^2)^2}{\eta|\beta|} \right)^{1/2} \gg 1$ for $(\eta e^2)^2 \gg \eta|\beta|$,

¹²³For $d=2$, as expected, there is no transition, not even of first order type, when the 2-dimensional space becomes periodic along one direction. If one had repeated the previous steps for the 2-dimensional model, one would obtain $V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}=0} = -\frac{\eta}{\omega^2} \frac{\pi}{12} \left((d-1) + (3-2) \right)$; and for $\bar{\rho}^2 = \frac{|\alpha|}{|\beta|}$ and supposing $\omega^2 e^2 \bar{\rho}^2 \gg 1$, $V_{\text{eff}}(\bar{\rho})^{(\omega)} \Big|_{\bar{\rho}^2=|\alpha|/|\beta|} = -\frac{|\alpha|^2}{2|\beta|} - \frac{\eta}{\omega^2} \frac{\pi}{12}$. This, nevertheless, would lead to a ω -value at the first-order transition equal to $\frac{1}{\omega_{\text{trans}}^2} = \frac{6|\alpha|^2}{\pi\eta|\beta|}$ which does not fulfil the starting assumption: $\omega^2 e^2 \bar{\rho}^2 \Big|_{\bar{\rho}^2=|\alpha|/|\beta|} = \frac{\pi}{6|\alpha|} \eta e^2$ contains only the (perturbatively small) expansion variable ηe^2 , with no $\eta|\beta|$ to which ηe^2 could be compared and thus producing a large $\omega^2 e^2 \bar{\rho}^2$.

and at $d = 3$ $\omega_{\text{trans}}^2 e^2 \frac{|\alpha|}{|\beta|} = \sqrt[3]{\frac{4(\zeta(3))^2}{\pi^2}} \left(\frac{(\eta e^2)^3}{\eta |\beta|} \right)^{1/3} \gg 1$ for $(\eta e^2)^3 \gg \eta |\beta|$.

When through decreasing ω the system transits from an ordered state with nonvanishing expectation value for the scalar field at $\omega = \infty$ towards a state characterized by a minimum of the effective potential at $\bar{\rho}^2 = 0$; and as just studied, during this process the behaviour of the model for $\eta |\beta| \ll \eta e^2$, differs from the behaviour shown by the model for $\eta |\beta| \gg \eta e^2$. Originally, at $\omega = \infty$, state $\bar{\rho}^2 = 0$ was a maximum of the effective potential, keeping this property also if the corrections of order η were included into $V_{\text{eff}}^{(\omega=\infty)}$ for the case $\eta |\beta| \gg \eta e^2$. The mentioned difference according to the relation between ηe^2 and $\eta |\beta|$ is, however, not the only one: for $\eta |\beta| \ll \eta e^2$ it can occur that, still at $\omega = \infty$, the potential's absolute minimum is already placed at $\bar{\rho}^2 = 0$. In this case therefore, there is no phase transition at all when ω is decreased since from the very beginning there is no symmetry breaking, i.e, the state is described by a vanishing expectation value of the scalar field even at $\omega = \infty$. Let us see how, for a certain relation between ηe^2 and $\eta |\beta| \ll \eta e^2$, the effect of the fluctuation's contribution of order η to the effective potential at $\omega = \infty$ can spoil the existence of a nonvanishing expectation value as unique minimum of the potential. As already mentioned, gauge choice $\alpha_{\text{fix}} = 0$ will be used throughout the following discussion, such that the results for $V_{\text{eff}}^{(\omega=\infty)}$ derived in (D.332)-(D.334) holds also here. One starts with

$$\frac{dV_{\text{eff,ren}}^{(\omega=\infty)}}{d\bar{\rho}^2} \Big|_{\eta e^2 \gg \eta |\beta|} \approx \alpha + |\beta| \bar{\rho}^2 + \frac{\eta(d-1)e^2}{4} \left(\text{finite part of } \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + e^2 \bar{\rho}^2}} \right)$$

where counterterms $\delta\alpha + \delta\beta \bar{\rho}^2$ have removed the divergent part from the above integral (and form those of the higher derivatives too). The nonvanishing minimum of the effective potential will be denoted by $\langle \phi_{\text{scal}}^{(m)} \rangle$, where the superindex $^{(m)}$ reminds us that such a expectation value, although fulfilling $\frac{dV_{\text{eff}}^{(\omega=\infty)}}{d\bar{\rho}^2} \Big|_{\bar{\rho}=\langle \phi_{\text{scal}}^{(m)} \rangle} = 0$, can be not the minimum of *lowest* energy, being separated from it by a barrier. The relation of the original potential's parameters to the renormalized ones up to order η can be read off (D.332)

$$|\beta| \Big|_{\eta e^2 \gg \eta |\beta|} \approx \begin{cases} |\beta|_{\text{ren}} \left[1 - \frac{(d-1)\eta e^4}{32\pi^2 |\beta|_{\text{ren}}} \left(\ln \left(\frac{e^2 \langle \phi_{\text{scal}}^{(m)} \rangle^2}{\mu^2} \right) + \frac{14}{3} \right) + O(\eta^2) \right] & d = 4 \\ |\beta|_{\text{ren}} \left[1 + 0 \cdot \eta e^4 + \frac{3\eta(12|\beta|_{\text{ren}})}{64\pi\sqrt{4|\alpha|_{\text{ren}}}} + O(\eta^2) \right] & d = 3 \\ |\beta|_{\text{ren}} \left[1 - \frac{(d-1)\eta e^2}{48\pi |\alpha|_{\text{ren}}} + O(\eta^2) \right] & d = 2 \end{cases}$$

and off (D.333) with $\alpha = -|\alpha|$

$$\alpha \stackrel{\eta e^2 \gg \eta |\beta|}{\approx} \begin{cases} \alpha_{\text{ren}} \left[1 - \frac{(d-1)\eta e^4}{32\pi^2 |\beta|_{\text{ren}}} + O(\eta^2) \right] & d = 4 \\ \alpha_{\text{ren}} \left[1 - \frac{(d-1)\eta e^4}{16\pi |\beta|_{\text{ren}} \sqrt{e^2 \langle \phi_{\text{scal}}^{(m)} \rangle^2}} + O(\eta^2) \right] & d = 3 \\ \alpha_{\text{ren}} \left[1 - \frac{(d-1)\eta e^2}{16\pi |\alpha|_{\text{ren}}} \left(\ln \left(\frac{e^2 \langle \phi_{\text{scal}}^{(m)} \rangle^2}{\mu^2} \right) - 1 \right) + O(\eta^2) \right] & d = 2 \end{cases}$$

and thus the effective potential at $\omega = \infty$, sum of the original (i.e., without the effect of fluctuations) $V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} = -|\alpha| \bar{\rho}^2 + \frac{|\beta|}{2} \bar{\rho}^4$ plus the corrections of order η , becomes ¹²⁴

$$\frac{1}{|\beta|_{\text{ren}} \langle \phi_{\text{scal}}^{(m)} \rangle^4} V_{\text{eff, ren}}(\bar{\rho})^{(\omega=\infty)} =$$

¹²⁴Up to order η^1 , in those terms already proportional to η one can directly substitute α , $|\beta|$ and $\langle \phi_{\text{scal}}^{(m)} \rangle$ for α_{ren} , $|\beta|_{\text{ren}}$ and $\frac{|\alpha|_{\text{ren}}}{|\beta|_{\text{ren}}}$ respectively. Hence, by integrating $\frac{dV_{\text{eff, ren}}^{(\omega=\infty)}}{d\bar{\rho}^2} = \alpha + |\beta| \bar{\rho}^2 + \frac{\eta(d-1)e^2}{4} \left(\text{finite part of } \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + e^2 \bar{\rho}^2}} \right)$ and substituting the above relations for $|\alpha|$ and $|\beta|$, one obtains for $d = 4$

$$V_{\text{eff, ren}}^{(\omega=\infty)} = -|\alpha|_{\text{ren}} \bar{\rho}^2 \left[1 - \frac{(d-1)\eta e^4}{32\pi^2 |\beta|_{\text{ren}}} \right] + \frac{|\beta|_{\text{ren}}}{2} (\bar{\rho}^2)^2 \left[1 + \frac{(d-1)\eta e^4}{32\pi^2 |\beta|_{\text{ren}}} \left(\ln \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right) - \frac{25}{6} \right) \right],$$

for $d = 3$

$$V_{\text{eff, ren}}^{(\omega=\infty)} = -|\alpha|_{\text{ren}} \bar{\rho}^2 \left[1 - \frac{(d-1)\eta e^2}{16\pi \sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{e^2}{|\beta|_{\text{ren}}}} \right] - \frac{(d-1)\eta e^3}{12\pi} (\bar{\rho}^2)^{3/2} + \frac{|\beta|_{\text{ren}}}{2} (\bar{\rho}^2)^2 \left[1 + \frac{3\eta(12|\beta|_{\text{ren}})}{128\pi \sqrt{|\alpha|_{\text{ren}}}} \right],$$

and for $d = 2$ the following renormalized effective potential

$$V_{\text{eff, ren}}^{(\omega=\infty)} = -|\alpha|_{\text{ren}} \bar{\rho}^2 \left[1 + \frac{(d-1)\eta e^2}{16\pi |\alpha|_{\text{ren}}} \ln \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right) \right] + \frac{|\beta|_{\text{ren}}}{2} (\bar{\rho}^2)^2 \left[1 - \frac{(d-1)\eta e^2}{48\pi |\alpha|_{\text{ren}}} \right].$$

Taking into account $\frac{1}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} = \left(\frac{|\beta|_{\text{ren}}}{|\alpha|_{\text{ren}}} \right) \left[1 - \frac{\boxed{\text{finite}(\bar{\rho})}}{12|\beta|_{\text{ren}}} \right]_{\bar{\rho}^2 = |\alpha|_{\text{ren}}/|\beta|_{\text{ren}}}$ (note that $\boxed{\text{finite}(\bar{\rho})}$ already

is of order η) and $\frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})} \Big|_{\bar{\rho}^2 = |\alpha|_{\text{ren}}/|\beta|_{\text{ren}}} \stackrel{\eta e^2 \gg \eta |\beta|}{\approx} \begin{cases} \frac{(d-1)\eta e^4}{12\pi^2 |\beta|_{\text{ren}}} & d = 4 \\ \frac{(d-1)\eta e^2}{16\pi \sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{e^2}{|\beta|_{\text{ren}}}} & d = 3 \\ \frac{(d-1)\eta e^2}{12\pi |\alpha|_{\text{ren}}} & d = 2 \end{cases}$, result (D.346) follows

by direct substitution.

$$\begin{aligned}
&= \left\{ \begin{aligned} &-\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \left[1 - \frac{11(d-1)}{96\pi^2} \frac{(\eta e^2)^2}{\eta|\beta|_{\text{ren}}} \right] + \\ &-\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \left[1 - \frac{(d-1)}{8\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} \right] - \\ &-\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \left[1 + \frac{(d-1)\eta e^2}{16\pi|\alpha|_{\text{ren}}} \left(\ln \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right) - \frac{4}{3} \right) \right] + \end{aligned} \right. \quad (\text{D.346}) \\
&+ \frac{1}{2} \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right)^2 \left[1 + \frac{(d-1)}{32\pi^2} \frac{(\eta e^2)^2}{\eta|\beta|_{\text{ren}}} \left(\ln \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right) - \frac{25}{6} \right) \right] \quad d = 4 \\
&- \frac{(d-1)}{12\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right)^{3/2} + \frac{1}{2} \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right)^2 \left[1 + \frac{3\eta(12|\beta|_{\text{ren}})}{128\pi\sqrt{|\alpha|_{\text{ren}}}} \right] \quad d = 3 . \\
&+ \frac{1}{2} \left(\frac{\bar{\rho}^2}{\langle \phi_{\text{scal}}^{(m)} \rangle^2} \right)^2 \left[\underline{1 - \frac{(d-1)\eta e^2}{48\pi|\alpha|_{\text{ren}}}} \right] \quad d = 2
\end{aligned}$$

It should be noted that the potential in the $d = 2$ case remains bounded from below (it will not become negative for large $\bar{\rho}$): the underlined term never becomes negative since ηe^2 is always kept perturbatively small.

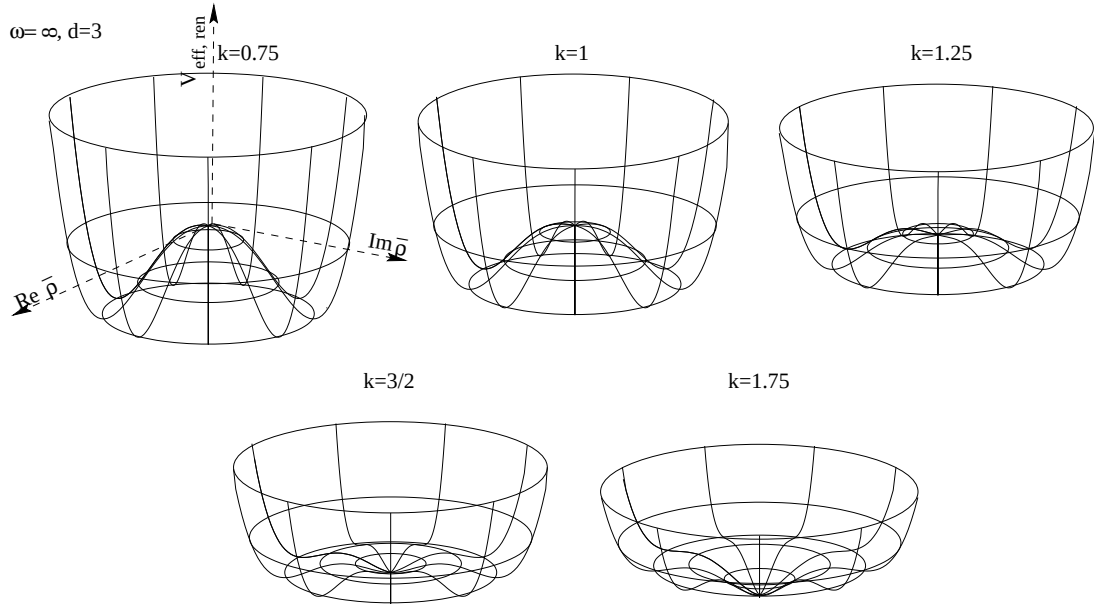
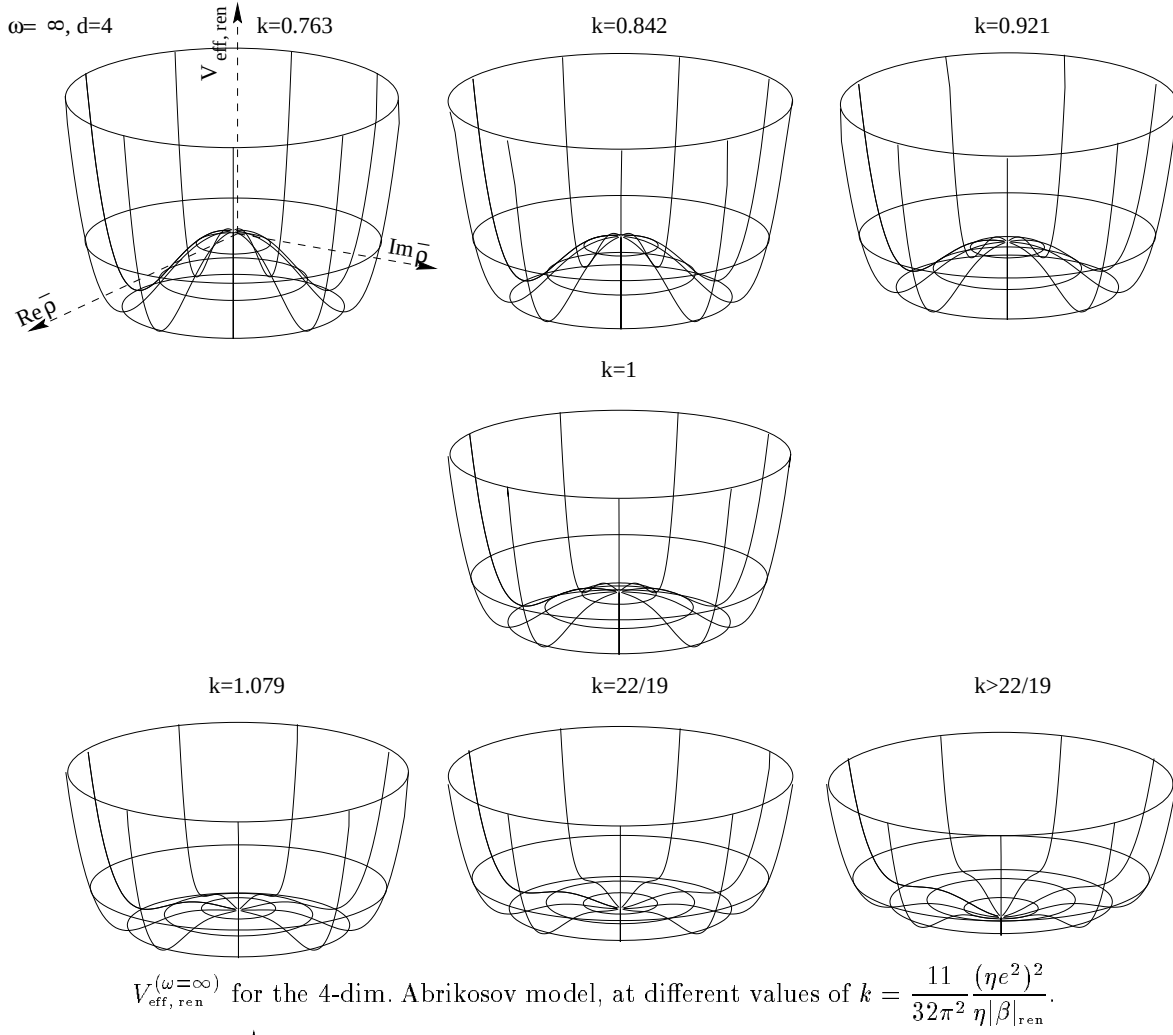
In the original effective potential with no corrections of order η

$$V_{\text{eff}}(\bar{\rho})^{(\omega=\infty)} = -|\alpha|\bar{\rho}^2 + \frac{|\beta|}{2}(\bar{\rho}^2)^2,$$

the symmetry breaking, or equivalently the existence of a nonvanishing minimum, was due to the negative parameter multiplying $\bar{\rho}^2$. On the other hand, observing (D.346) for the cases $d = 3$ and $d = 4$ and since, although ηe^2 and $\eta|\beta|_{\text{ren}}$ are perturbatively small, quotient $\frac{(\eta e^2)^n}{\eta|\beta|_{\text{ren}}}$ can be well large, the correction of order η makes the parameter multiplying $\bar{\rho}^2$ change its sign when following relations hold [58]:

$$\left\{ \begin{aligned} &(\eta e^2)^2 > \eta|\beta|_{\text{ren}} \frac{96\pi^2}{11(d-1)} \quad d = 4 \\ &(\eta e^2)^3 > \eta|\beta|_{\text{ren}} \frac{64\pi^2|\alpha|_{\text{ren}}}{(d-1)^2} \quad d = 3 \end{aligned} \right. \quad (\text{D.347})$$

For the 2-dimensional model, such correction of order η is not able to change the form of the effective potential since it is too small, it only contains parameter ηe^2 without being divided by any power of $\eta|\beta|_{\text{ren}}$. In the three cases, the parameter multiplying $\bar{\rho}^4$ is always positive for large $\bar{\rho}$ in order to ensure the potential is bounded from below.



For $d = 4$ and $d = 3$, the form of the effective potential at $\omega = \infty$ is represented in the figures on page 724. By changing the value of ηe^2 in relation to that of $\eta|\beta|$ (but always keeping both smaller than 1), for a value

$$k = 1 \text{ which corresponds to } \begin{cases} (\eta e^2)^2 = \eta|\beta|_{\text{ren}} \frac{96\pi^2}{11(d-1)} & d = 4 \\ (\eta e^2)^3 = \eta|\beta|_{\text{ren}} \frac{64\pi^2|\alpha|_{\text{ren}}}{(d-1)^2} & d = 3 \end{cases} \quad \text{the poten-}$$

tial begins to have a local minimum placed at $\bar{\rho}^2 = 0$, local minimum which eventually develops to be absolute when

$$V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=\langle\phi_{\text{scal}}^{(m)}\rangle \neq 0} > V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=0}.$$

Hence, for a certain relation between a ηe^2 larger than $\eta|\beta|$, the existence of a nonvanishing expectation value for the scalar field is no longer fulfilled, not even at $\omega = \infty$: the symmetry breaking of the original effective potential without any η -corrections is “restored” once such corrections are taken into account, i.e., when fluctuations are energetically possible and are therefore considered¹²⁵. And further, the phase transition from a nonvanishing towards a zero expectation value when ω is decreased does not take place at all in such a case.

Equivalently, if in the effective potential up to order η and for small ηe^2 one starts at a value of $\eta|\beta| \approx \eta|\beta|_{\text{ren}}$ larger than ηe^2 , allowing thus the symmetry breaking characteristic of the original effective potential without corrections,

$$\text{and then } \eta|\beta| \text{ is decreased such that } \begin{cases} \eta|\beta| < \frac{11(d-1)}{96\pi^2}(\eta e^2)^2 & d = 4 \\ \eta|\beta| < \frac{(d-1)^2}{64\pi^2|\alpha|_{\text{ren}}}(\eta e^2)^3 & d = 3 \end{cases} \quad \text{holds,}$$

the system will finally transit over the barrier (by means of a first-order transition) towards the now deepest symmetric minimum $\bar{\rho}^2 = 0$.

This fact, that by decreasing $\eta|\beta|$ until $\eta|\beta| \ll \eta e^2$ the most stable state becomes described by a zero expectation value for the effective potential's

¹²⁵In order to be more exact, the system can be described by a nonzero expectation value even at a value of ηe^2 for which $V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=\langle\phi_{\text{scal}}^{(m)}\rangle \neq 0} > V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=0}$, value which corresponds to

$$k = \begin{cases} \frac{11(d-1)}{96\pi^2} \frac{(\eta e^2)^2}{\eta|\beta|_{\text{ren}}} = \frac{22}{19} & d = 4 \\ \frac{(d-1)}{8\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} = \frac{3}{2} & d = 3 \end{cases} \quad \text{as long as there exists a barrier separating the actual mini-}$$

um $\bar{\rho}^2 = 0$ from the metastable asymmetric one. But in the moment when the effect of a non-infinite space's periodicity ω is taken into account (or in the moment any other fluctuation, arising from another external source, acts on the system), this will cause that the system transits to the symmetric deeper minimum. Of course, the same holds but with the roles of $\bar{\rho}^2 \neq 0$ and $\bar{\rho}^2 = 0$ interchanged for a value $k \geq 1$ but smaller than the just mentioned value: the zero expectation value is a local minimum and it may occur that the system is originally in this state although any fluctuation or the contributions of a finite ω can push the system to the absolute asymmetric minimum.

scalar field, is not so surprising when one returns to the phase diagrams of the non-compact lattice Higgs model at $\omega = \infty$. Let us remember the “tool” developed in equation (D.252) to compare the phase’s description between the lattice Higgs model and the Abrikosov model, not forgetting that such comparison works for a (perturbative) small ηe^2 . The superconducting phase was described in the non-compact lattice model by a perimeter decay in the correlation function $W(C)$, occurring at large $|\psi|^2/\eta$ and small $e^2\eta$, whilst the normal conducting (or Coulomb) phase was characterized by a stronger decay $W(C) \sim \exp\left[-\oint_C d^d\vec{r} \oint_C d^d\vec{r}' J_\mu(\vec{r})D(\vec{r}-\vec{r}')J_\mu(\vec{r}')\right]$ and took place at small $|\psi|^2/\eta$. On the other hand, each phase corresponds respectively to the symmetry breaking’s nonzero $\langle\phi_{\text{scal}}\rangle^2 \neq 0$ and to a vanishing expectation value in the effective potential formulation of the Abrikosov model; moreover, a diminishing $\eta|\beta|$ will turn the sign of the parameter multiplying $\bar{\rho}^2$, impeding thus the existence of an asymmetric minimum, which combined with the coupling to a gauge field made the latter short-range and such a short-range gauge field was responsible for the perimeter decay of $W(C)$ in the corresponding phase of the lattice Higgs model. Hence, always placed in the range of small ηe^2 , starting at a “large” $\eta|\beta|$ (large compared to ηe^2) and decreasing it one transits from the state characterized by the symmetry breaking (the superconducting phase ① of the lattice Higgs model) towards the state described by a symmetric deepest minimum in the effective potential (the Coulomb phase ④): in the 3-dim. and the 4-dim. non-compact Higgs model such a transition for small ηe^2 occurs on the line between point A and tc (see diagrams on pages 465 and 477) and is of first-order type, just the same type¹²⁶ as in the effective potential up to order η .

If for the effective potential we put the results of the corrections of order η at $\omega = \infty$ for $\eta|\beta| \ll \eta e^2$ plus the effect of a small periodicity ω (space periodicity along one direction) discussed in the figure on page 720, together with the studied second-order phase transition for $\eta|\beta| \gg \eta e^2$, we arrive at the qualitative diagram on page 753 for the ($d = 3, 4$)-dimensional Abrikosov model. As in the phase diagram of the ($d = 3, 4$)-dim. non-compact lattice Higgs model, with decreasing ω the diagram’s space occupied by the superconducting state diminishes, state described in the effective potential of the Abrikosov model by a nonvanishing $\bar{\rho}^2$ as its lowest minimum. In the effective potential, for ηe^2 small compared to $\eta|\beta|$ the loss of superconductivity, to transit toward a zero $\bar{\rho}^2$

¹²⁶This similarity between the two models, nevertheless, should not be taken word for word. In fact, for the non-compact lattice Higgs model the just discussed phase transition from the phase described by the perimeter decay of $W(C)$ towards a stronger decay takes place also for large $|\psi|^2/\eta$ and large ηe^2 , and this time, the transition is second-order. However, the ηe^2 value in that case (the coupling to the gauge field in the lattice model has been written as qe) fulfils $\eta e^2 > 1$, which lies beyond the perturbative expansion carried out for the effective potential of the Abrikosov model. On the contrary, the behaviour displayed by both models at small ηe^2 does coincide largely, as we have just seen.

as lowest potential's minimum, occurs for a small ω -value given by (D.336)

$$\left\{ \begin{array}{l} \frac{1}{\omega_{\text{crit}}^2} \approx 3 \frac{|\alpha|_{\text{ren}}}{\eta|\beta|_{\text{ren}}} \left[1 - \frac{(d-1)\eta e^2}{8\eta|\beta|_{\text{ren}}} \right] \\ \frac{1}{\omega_{\text{crit}}^2} \approx \frac{-4\pi|\alpha|_{\text{ren}}/|\beta|_{\text{ren}}}{\eta(6\ln|2|\alpha|_{\text{ren}}\omega_{\text{crit}}^2 + 2\lim_{p \rightarrow 0} \ln p)} \left[1 - \frac{(d-1)\eta e^2}{2\eta|\beta|_{\text{ren}}} \right] \end{array} \right. \quad \begin{array}{l} d=4, \\ \text{large } |\alpha|_{\text{ren}} \end{array} \quad d=3, \quad (\text{D.348})$$

and such a transition is of continuous (or critical) type, i.e., of second-order or of Kosterlitz-Thouless kind. In the range of ηe^2 larger than $\eta|\beta|$, on the other hand, such a transition is of first-order type since there exists a barrier separating the states described by $\bar{\rho}^2 = 0$ and by $\bar{\rho}^2 \neq 0$. The ω -value at which such a phase transition takes place gets calculated as in (D.344)-(D.343)

$$\left\{ \begin{array}{l} -\frac{\eta}{\omega_{\text{trans}}^4} \frac{\pi^2}{90} \times (d-1) = V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=\langle \phi_{\text{scal}}^{(m)} \rangle} \\ -\frac{\eta}{\omega_{\text{trans}}^3} \frac{\zeta(3)}{2\pi} \times (d-1) = V_{\text{eff, ren}}^{(\omega=\infty)} \Big|_{\bar{\rho}=\langle \phi_{\text{scal}}^{(m)} \rangle} \end{array} \right. \quad \begin{array}{l} d=4, \\ d=3, \end{array}$$

where after incorporating the corrections of order η to $V_{\text{eff, ren}}^{(\omega=\infty)}$, it becomes

$$\left\{ \begin{array}{l} \frac{1}{\omega_{\text{trans}}^2} \approx \left(\frac{15|\beta|_{\text{ren}} \langle \phi_{\text{scal}}^{(m)} \rangle^4}{\pi^2 \eta} \left[1 - \frac{19}{22}k \right] \right)^{1/2} \\ \frac{1}{\omega_{\text{trans}}^2} \approx \left(\frac{\pi}{2\zeta(3)} \frac{|\beta|_{\text{ren}} \langle \phi_{\text{scal}}^{(m)} \rangle^4}{\eta} \left[1 - \frac{2}{3}k \right] \right)^{2/3} \end{array} \right. \quad \begin{array}{l} d=4, \\ d=3, \end{array} \quad (\text{D.349})$$

$$\text{with } k = \begin{cases} \frac{11(d-1)(\eta e^2)^2}{96\pi^2 \eta|\beta|_{\text{ren}}} & d=4 \\ \frac{(d-1)}{8\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} & d=3 \end{cases}. \text{ When } \omega \text{ decreases such a transition does not}$$

take place if $k \geq k_1 = \begin{cases} 22/19 & d=4 \\ 3/2 & d=3 \end{cases}$ since it has already occurred at $\omega = \infty$: for such a k -value, not even at $\omega = \infty$ there is an asymmetric absolute minimum.

In the range $\eta|\beta| \ll \eta e^2$ (D.349) as well as for $\eta|\beta| \gg \eta e^2$ (D.348) (but $\eta|\beta|$ always smaller than 1), a slight increase in ηe^2 pushes towards larger values the ω corresponding to the transition from the asymmetric nonzero to the symmetric vanishing minimum, something which also occurs in the phase diagrams for the lattice non-compact Higgs model¹²⁷. Further, and always for small ηe^2 , since $\eta|\beta| \ll \eta e^2$ corresponds in the lattice

¹²⁷Beside this, for the 3-dim. non-compact Higgs model in the range of large $\eta e^2 > 1$ the transition value of ω does decreases with increasing ηe^2 (see phase diagram on page 751), but such a behaviour cannot be compared to that shown by the effective potential of the Abrikosov model since it occurs for a value of ηe^2 larger than 1 and thus not within the range where the perturbative expansion considered for the effective potential is valid.

Higgs model to a $|\psi|^2/\eta$ near to the range of (small) $|\psi|^2/\eta$ for the transition line at $\omega = \infty$ (because $\eta|\beta| \ll \eta e^2$ can cause the potential's symmetry breaking to be destroyed and a zero symmetric minimum corresponds to the phase occurring at small $|\psi|^2/\eta$ in the lattice Higgs model), the corresponding ω -value for the transition in the non-compact Higgs model lies at a larger value (or at a low position in the phase diagrams on pages 751 and 752) than the ω -value for large $|\psi|^2/\eta$ (also a larger $\eta|\beta| \gg \eta e^2$ in the effective potential, which is able to maintain the symmetry breaking occurring in the effective potential without corrections of order η). Because, in the non-compact Higgs model, the first ω -value lies on the segment between A' and tc (tricritical point tc has not yet reached point A'), the phase transition which takes place is first-order, just as in the effective potential. For the second ω -value, placed in a higher position within the phase diagram of the non-compact lattice Higgs model, the transition point is already second-order since the segment $A' - tc$ has been almost reduced to point A' ; and this again the behaviour displayed by the effective model, for which the value $1/\omega_{\text{trans}}$ corresponding to the first-order transition is smaller than $1/\omega_{\text{crit}}$.

As in the pure scalar model, for the sake of completeness, the final result for the effective potential at $\omega = \infty$ in the case $|\alpha| = 0$ will be given, valid for any relation regarding ηe^2 and $\eta|\beta|$. Also in this case we shall apply the gauge choice $\alpha_{\text{fix}} = 0$. The resulting effective potential up to order η is equal to ¹²⁸

$$V_{\text{eff, ren}}^{(\omega=\infty)} = \begin{cases} \frac{|\beta|_{\text{ren}}}{2}(\bar{\rho}^2)^2 + \\ \frac{|\beta|_{\text{ren}}}{2}(\bar{\rho}^2)^2 - \frac{(d-1)\eta e^3}{12\pi}(\bar{\rho}^2)^{3/2} - \\ \frac{|\beta|_{\text{ren}}}{2}(\bar{\rho}^2)^2 \left[1 - \frac{(d-1)\eta e^2}{48\pi|\beta|_{\text{ren}}\mu_0^2} - \frac{\eta}{6\pi\mu_0^2} \right] - \end{cases} \quad (\text{D.350})$$

¹²⁸Working with $\alpha_{\text{fix}} = 0$ allows us to take profit from the results obtained in the footnote on page 709, although these were obtained for any gauge choice but with $\eta|\beta| \gg \eta e^2$. Thus $\frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})} \Big|_{\alpha=0}$

$$= \begin{cases} \frac{(d-1)\eta e^4}{12\pi^2|\beta|} + \left(\frac{3\eta(12|\beta|)^2\bar{\rho}^2}{16\pi^2(6|\beta|\bar{\rho}^2)} - \frac{\eta(12|\beta|)^3\bar{\rho}^4}{32\pi^2(6|\beta|\bar{\rho}^2)^2} \right) + \left(\frac{\eta(2|\beta|)^2\bar{\rho}^2}{4\pi^2(2|\beta|\bar{\rho}^2)} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{12\pi^2(2|\beta|\bar{\rho}^2)^2} \right) & d = 4 \\ \frac{(d-1)\eta e^4}{16\pi|\beta|\sqrt{e^2\bar{\rho}^2}} + \left(\frac{3\eta(12|\beta|)^2\bar{\rho}^2}{16\pi(6|\beta|\bar{\rho}^2)^{3/2}} - \frac{3\eta(12|\beta|)^3\bar{\rho}^4}{64\pi(6|\beta|\bar{\rho}^2)^{5/2}} \right) + \left(\frac{\eta(2|\beta|)^2\bar{\rho}^2}{4\pi(2|\beta|\bar{\rho}^2)^{3/2}} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{8\pi(2|\beta|\bar{\rho}^2)^{5/2}} \right) & d = 3 \\ \frac{(d-1)\eta e^4}{12\pi|\beta|(e^2\bar{\rho}^2)} + \left(\frac{3\eta(12|\beta|)^2\bar{\rho}^2}{8\pi(6|\beta|\bar{\rho}^2)^2} - \frac{\eta(12|\beta|)^3\bar{\rho}^4}{8\pi(6|\beta|\bar{\rho}^2)^3} \right) + \left(\frac{\eta(2|\beta|)^2\bar{\rho}^2}{2\pi(2|\beta|\bar{\rho}^2)^2} - \frac{\eta(2|\beta|)^3\bar{\rho}^4}{3\pi(2|\beta|\bar{\rho}^2)^3} \right) & d = 2 \end{cases}$$

$$\text{or } \frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})} = \begin{cases} \frac{1}{12\pi^2} \left((d-1)\frac{(\eta e^2)^2}{\eta|\beta|} + 3\eta(12|\beta|) + \frac{\eta}{3}(12|\beta|) \right) & d = 4 \\ \frac{1}{16\pi\sqrt{|\beta|\bar{\rho}^2}} \left((d-1)\sqrt{\frac{(\eta e^2)^3}{\eta|\beta|}} + \sqrt{\frac{3}{2}}\eta(12|\beta|) + \frac{\eta}{3\sqrt{2}}(12|\beta|) \right) & d = 3 \\ \frac{1}{12\pi|\beta|\bar{\rho}^2} \left((d-1)(\eta e^2) + \frac{\eta}{2}(12|\beta|) + \frac{\eta}{6}(12|\beta|) \right) & d = 2 \end{cases} \quad ; \quad \text{since}$$

$\frac{1}{12|\beta|} \boxed{\text{finite}(\bar{\rho})}$ is already of order η , we can substitute $|\beta|$ for $|\beta|_{\text{ren}}$ inside it as long as we are

$$\begin{aligned}
& + \frac{|\beta|_{\text{ren}}(\bar{\rho}^2)^2}{64\pi^2} \left((d-1) \frac{(\eta e^2)^2}{\eta|\beta|_{\text{ren}}} + 3\eta(12|\beta|_{\text{ren}}) + \frac{\eta}{3}(12|\beta|_{\text{ren}}) \right) \left(\ln\left(\frac{\bar{\rho}^2}{\mu_0^2}\right) - \frac{25}{6} \right) \quad d=4 \\
& - \frac{\eta(12|\beta|_{\text{ren}})^{3/2}}{24\sqrt{2}\pi} (\bar{\rho}^2)^{3/2} - \frac{\eta(12|\beta|_{\text{ren}})^{3/2}}{72\sqrt{6}\pi} (\bar{\rho}^2)^{3/2} \quad d=3 \\
& - \frac{(d-1)\eta e^2}{16\pi} \bar{\rho}^2 \left(\ln\left(\frac{e^2 \bar{\rho}^2}{\mu^2}\right) - 1 \right) - \frac{\eta(12|\beta|_{\text{ren}})}{32\pi} \bar{\rho}^2 \left(\ln\left(\frac{6|\beta|_{\text{ren}} \bar{\rho}^2}{\mu^2}\right) - 1 \right) - \\
& \quad - \frac{\eta(12|\beta|_{\text{ren}})}{96\pi} \bar{\rho}^2 \left(\ln\left(\frac{2|\beta|_{\text{ren}} \bar{\rho}^2}{\mu^2}\right) - 1 \right) \quad d=2
\end{aligned}$$

where μ_0 is a renormalization point; μ is another arbitrary scale, which has not obligatory to coincide with μ_0 , at which one has set (arbitrarily too) the starting point for long distances.

As already advanced in the footnote on page 694, and although at $\omega = \infty$ no symmetry breaking occurs for the original effective potential since it has no nonvanishing $\bar{\rho}^2 = 0$ as minimum, now the corrections of order η do produce such an asymmetric minimum, at least for the 4-dim. model. Since actually these corrections arise from summing up the effect of the fluctuations about the model's fields, and therefore they are determined by the interactions contained in the energy describing the model, the appearance of a nonvanishing asymmetric minimum is called a dynamical symmetry breaking. This time such a nonzero minimum is valid within the perturbative expansion we have carried out through the whole discussion on the effective potential [27], [53]:

only expanding up to such an order. As in the pure scalar model (see equation (D.306) and the comments there), and since the minimum of the original potential $V_{\text{eff}}^{(\omega=\infty)} = \frac{|\beta|}{2} \bar{\rho}^4$ vanishes, the renormalized $|\beta|_{\text{ren}}$ is defined at a certain nonzero arbitrary renormalization point $\bar{\rho} = \mu_0$ obtaining thus

$$\begin{aligned}
|\beta|_{\text{ren}} &= |\beta| \left[1 - \frac{\eta}{8|\beta|} \left(\text{sum finite p.} \int_0^{\Lambda_c} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{\text{constant} \times p^{d-2} dp}{(p^2 + \text{terms with } \bar{\rho}^2)^{3/2}} \text{ at } \Lambda_c \rightarrow \infty \right) \right]_{\bar{\rho}=\mu_0} + \frac{\boxed{\text{finite}(\bar{\rho})}}{12|\beta|} \Big|_{\bar{\rho}=\mu_0} \\
&= \begin{cases} |\beta| \left[1 + \frac{(d-1)\eta e^4}{32\pi^2|\beta|} \left(\ln\left(\frac{e^2 \mu_0^2}{\mu^2}\right) + \frac{14}{3} \right) + \frac{3\eta(12|\beta|)}{32\pi^2} \left(\ln\left(\frac{6|\beta| \mu_0^2}{\mu^2}\right) + \frac{14}{3} \right) + \right. \\ \quad \left. + \frac{\eta(12|\beta|)}{96\pi^2} \left(\ln\left(\frac{2|\beta| \mu_0^2}{\mu^2}\right) + \frac{14}{3} \right) + O(\eta^2) \right] & d=4 \\ |\beta| \left[1 + 0 \cdot \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|}} + 0 \cdot \eta(12|\beta|) + O(\eta^2) \right] & d=3 \\ |\beta| \left[1 + \frac{(d-1)\eta e^2}{48\pi|\beta|\mu_0^2} + \frac{\eta}{8\pi\mu_0^2} + \frac{\eta}{24\pi\mu_0^2} + O(\eta^2) \right] & d=2 \end{cases} \quad \text{Finally, from} \\
\frac{dV_{\text{eff,ren}}^{(\omega=\infty)}}{d\bar{\rho}^2} &= |\beta|\bar{\rho}^2 + \frac{\eta}{4} \left(\text{finite part of } (d-1)e^2 \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + e^2 \bar{\rho}^2}} + \frac{12|\beta|}{2} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + 6|\beta|\bar{\rho}^2}} + \right. \\
&\quad \left. + \frac{12|\beta|}{6} \int_0^\infty \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + 2|\beta|\bar{\rho}^2}} \right) \quad \text{and from result} \quad \left(\text{finite p. of } \int_0^{\Lambda_c \rightarrow \infty} \frac{S_{d-1}}{(2\pi)^{d-1}} \frac{p^{d-2} dp}{\sqrt{p^2 + C\bar{\rho}^2}} \right) = \\
&= \begin{cases} \frac{C\bar{\rho}^2}{8\pi^2} \left[\ln\left(\frac{C\bar{\rho}^2}{\mu^2}\right) + 1 \right] + O\left(\frac{1}{\Lambda_c^2}\right) & d=4 \\ -\frac{1}{2\pi} \sqrt{C\bar{\rho}^2} + O\left(\frac{1}{\Lambda_c}\right) & d=3 \\ -\frac{1}{4\pi} \ln\left(\frac{C\bar{\rho}^2}{\mu^2}\right) + O\left(\frac{1}{\Lambda_c^2}\right) & d=2 \end{cases} \quad \text{, direct integration and substitution of } |\beta| \text{ as function of} \\
|\beta|_{\text{ren}} &\text{ leads to equation (D.350).}
\end{aligned}$$

derivative $\frac{dV_{\text{eff, ren}}}{d\bar{\rho}^2} \stackrel{(\omega=\infty)}{}$ has a non-zero root $\bar{\rho}^2 = \langle \phi_{\text{scal}} \rangle^2$ (of course, different from that in the effective potential with $|\alpha| \neq 0$) for

$$\mathcal{C} \ln \left(\frac{\langle \phi_{\text{scal}} \rangle^2}{\mu_0^2} \right) = -32\pi^2 \left(1 - \frac{11}{96\pi^2} \mathcal{C} \right) \quad d = 4 \quad (\text{D.351})$$

with $\mathcal{C} = (d-1) \frac{(\eta e^2)^2}{\eta |\beta|_{\text{ren}}} + 3\eta(12|\beta|_{\text{ren}}) + \frac{\eta}{3}(12|\beta|_{\text{ren}})$

where $\mathcal{C}$, although includes $3\eta(12|\beta|_{\text{ren}}) + \frac{\eta}{3}(12|\beta|_{\text{ren}})$ which is much smaller than 1, also contains $(d-1) \frac{(\eta e^2)^2}{\eta |\beta|_{\text{ren}}}$ which can be well of order 1 if ηe^2 is larger than $\eta |\beta|_{\text{ren}}$ (although both parameters remain of perturbative order). This, nevertheless, does not occur for $d = 2, 3$: for instance, the 3-dimensional model would require to equate $|\beta|_{\text{ren}} \langle \phi_{\text{scal}} \rangle^2$, which is of order η^0 , with $\left(\frac{3(d-1)\eta e^3}{24\pi} + \frac{3\eta(12|\beta|_{\text{ren}})^{3/2}}{48\sqrt{2}\pi} + \frac{3\eta(12|\beta|_{\text{ren}})^{3/2}}{144\sqrt{6}\pi} \right) \sqrt{\langle \phi_{\text{scal}} \rangle^2}$, which is of order η^1 since it does not contain any quotient between the small perturbation parameters; an analogous case holds also for $d = 2$.

From relation (D.351), one can choose the arbitrary renormalization point μ_0 to coincide with the nonvanishing expectation value, being the latter a free parameter not determined by any other potential's terms. One speaks then of the “transmutation” of the renormalization point, whose very appearance is already a footprint of the existence of a fundamental scale in the model (see comments on page 684), into an arbitrarily chosen non-zero expectation value.

This closes the long study in the two last Appendices of the phase diagram for the model described by a scalar field coupled to a compact or a non-compact Abelian gauge field. It has been considered the different phases for a model “living” in a d -dimensional space without any periodicity and also for the case when one direction of such a space becomes periodic, being the periodicity length related to the inverse of the temperature. A large part of the discussion has been carried out putting the model's field into a lattice of spacing a . And in the case of the non-compact gauge field, a different formulation has been also obtained to describe the phases displayed by the model: such a formulation has been the effective potential, which although based on a perturbative expansion and is thus not exact, allows us to get an effective simplified description according to the existence or not of an asymmetric minimum for such a potential. In the following last Note, the general formulation for the Feynman diagrams and their associated correlation functions, on which the effective potential is based, will be presented.

Note. Generating functionals of correlation functions ([25] and [99], ch. 5).

Let us start with the partition function of a model described by the energy $E[\phi_{\vec{r}}]$, which depends locally on a general field $\phi_{\vec{r}}$. Although this discussion applies in general for a continuum model, the discrete notation will be maintained in order to visualize better the calculations; the functional for-

mulation is obtained by changing partial derivative $\frac{\partial}{\partial \phi_{\vec{r}_1}}$ by functional derivative $\frac{\delta}{\delta \phi(\vec{r}_1)}$, sum $\sum_{\vec{r}}$ by d -dim. integral $\int d^d \vec{r}$, and by the change of a N -integration $\prod_{\vec{r}} d\phi_{\vec{r}}$ (with $N = |\Lambda|$ the number of sites in the discretized space) by a functional integration $[d\phi(\vec{r})]$. The partition function is equal to

$$\mathcal{Z}[J_{\vec{r}}] = \int \prod_{\vec{r}} d\phi_{\vec{r}} \exp \left(-\frac{E[\phi_{\vec{r}}]}{\eta} + \frac{1}{\eta} \sum_{\vec{r}} J_{\vec{r}} \phi_{\vec{r}} \right), \quad (\text{D.352})$$

where a general external source $J_{\vec{r}}$ has been also introduced, as for instance an external magnetic field in the Ising model. By means of term $\sum_{\vec{r}} J_{\vec{r}} \phi_{\vec{r}}$, correlation functions can be formulated as derivatives of the

$$\text{partition function: } \langle \phi_{\vec{r}_1} \cdots \phi_{\vec{r}_n} \rangle = \frac{\sum_{\text{all config. } \phi_{\vec{r}}} \phi_{\vec{r}_1} \cdots \phi_{\vec{r}_n} \exp \left(-\frac{E[\phi_{\vec{r}}]}{\eta} \right)}{\sum_{\text{all config. } \phi_{\vec{r}}} \exp \left(-\frac{E[\phi_{\vec{r}}]}{\eta} \right)} = \eta^n \left(\frac{1}{\mathcal{Z}[J_{\vec{r}}]} \frac{\partial^n \mathcal{Z}[J_{\vec{r}}]}{\partial J_{\vec{r}_1} \cdots \partial J_{\vec{r}_n}} \right) \Big|_{J_{\vec{r}}=0},$$

or in functional notation $\eta^n \left(\frac{1}{\mathcal{Z}[J]} \frac{\delta^n \mathcal{Z}[J]}{\delta J(\vec{r}_1) \cdots \delta J(\vec{r}_n)} \right) \Big|_{J(\vec{r})=0}$. The partition is thus equal to

$$\begin{aligned} \mathcal{Z}[J_{\vec{r}}] &= \mathcal{Z}[J_{\vec{r}}=0] \sum_{n=0}^{\infty} \frac{1}{\eta^n n!} \sum_{\vec{r}_1} \cdots \sum_{\vec{r}_n} \langle \phi_{\vec{r}_1} \cdots \phi_{\vec{r}_n} \rangle J_{\vec{r}_1} \cdots J_{\vec{r}_n}, \\ \mathcal{Z}[J_{\vec{r}}] &= \mathcal{Z}[J_{\vec{r}}=0] \sum_{n=0}^{\infty} \frac{1}{\eta^n n!} \left\langle \left(\sum_{\vec{r}} \phi_{\vec{r}} J_{\vec{r}} \right)^n \right\rangle. \end{aligned} \quad (\text{D.353})$$

By defining the connected correlation functions as follows ([78], ch. 4)

$$\begin{aligned} \langle \phi_{\vec{r}_1} \rangle &= \langle \phi_{\vec{r}_1} \rangle_{\text{connected}}, \\ \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle &= \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle_{\text{conn}} + \langle \phi_{\vec{r}_1} \rangle_{\text{conn}} \langle \phi_{\vec{r}_2} \rangle_{\text{conn}}, \\ \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \phi_{\vec{r}_3} \rangle &= \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \phi_{\vec{r}_3} \rangle_{\text{conn}} + \langle \phi_{\vec{r}_1} \rangle_{\text{conn}} \langle \phi_{\vec{r}_2} \phi_{\vec{r}_3} \rangle_{\text{conn}} + \langle \phi_{\vec{r}_2} \rangle_{\text{conn}} \langle \phi_{\vec{r}_1} \phi_{\vec{r}_3} \rangle_{\text{conn}} + \langle \phi_{\vec{r}_3} \rangle_{\text{conn}} \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle_{\text{conn}} + \\ &\quad + \langle \phi_{\vec{r}_1} \rangle_{\text{conn}} \langle \phi_{\vec{r}_2} \rangle_{\text{conn}} \langle \phi_{\vec{r}_3} \rangle_{\text{conn}}, \end{aligned}$$

$$\text{one expands } \ln \left(\frac{\mathcal{Z}[J_{\vec{r}}]}{\mathcal{Z}[0]} \right) = \ln \left[1 + \frac{1}{\eta} \left\langle \sum_{\vec{r}} \phi_{\vec{r}} J_{\vec{r}} \right\rangle + \frac{1}{\eta^2 2!} \left\langle \left(\sum_{\vec{r}} \phi_{\vec{r}} J_{\vec{r}} \right)^2 \right\rangle + \frac{1}{\eta^3 3!} \left\langle \left(\sum_{\vec{r}} \phi_{\vec{r}} J_{\vec{r}} \right)^3 \right\rangle + \dots \right],$$

where after regrouping in powers of $J_{\vec{r}}$'s

$$\begin{aligned} \ln \left(\frac{\mathcal{Z}[J_{\vec{r}}]}{\mathcal{Z}[J_{\vec{r}}=0]} \right) &= \frac{1}{\eta} \left(\sum_{\vec{r}_1} \langle \phi_{\vec{r}_1} \rangle J_{\vec{r}_1} \right) + \frac{1}{\eta^2} \left(\frac{1}{2!} \sum_{\vec{r}_1, \vec{r}_2} \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle J_{\vec{r}_1} J_{\vec{r}_2} - \frac{1}{2} \sum_{\vec{r}_1, \vec{r}_2} \langle \phi_{\vec{r}_1} \rangle \langle \phi_{\vec{r}_2} \rangle J_{\vec{r}_1} J_{\vec{r}_2} \right) + \\ &\frac{1}{\eta^3} \left(\frac{1}{3!} \sum_{\vec{r}_1, \vec{r}_2, \vec{r}_3} \langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \phi_{\vec{r}_3} \rangle J_{\vec{r}_1} J_{\vec{r}_2} J_{\vec{r}_3} - \frac{1}{2} \sum_{\vec{r}_1, \vec{r}_2, \vec{r}_3} \langle \phi_{\vec{r}_1} \rangle \langle \phi_{\vec{r}_2} \phi_{\vec{r}_3} \rangle J_{\vec{r}_1} J_{\vec{r}_2} J_{\vec{r}_3} + \frac{1}{3} \sum_{\vec{r}_1, \vec{r}_2, \vec{r}_3} \langle \phi_{\vec{r}_1} \rangle \langle \phi_{\vec{r}_2} \rangle \langle \phi_{\vec{r}_3} \rangle J_{\vec{r}_1} J_{\vec{r}_2} J_{\vec{r}_3} \right) + \dots \end{aligned}$$

one obtains the following result

$$\ln \mathcal{Z}[J_{\vec{r}}] = \ln \mathcal{Z}[J_{\vec{r}}=0] + \sum_{n=1}^{\infty} \frac{1}{\eta^n n!} \sum_{\vec{r}_1} \cdots \sum_{\vec{r}_n} \langle \phi_{\vec{r}_1} \cdots \phi_{\vec{r}_n} \rangle_{\text{conn}} J_{\vec{r}_1} \cdots J_{\vec{r}_n}. \quad (\text{D.354})$$

Thus, partition function $\mathcal{Z}[J_{\vec{r}}]$ is the generating functional of the correlation functions and $\ln \mathcal{Z}[J_{\vec{r}}]$, which up to a multiplicative constant $-\eta$ is the (Helmholtz) free energy, is the generating functional

of the connected correlation functions through $\langle \phi(\vec{r}_1) \cdots \phi(\vec{r}_n) \rangle_{\text{conn}} = \eta^n \frac{\delta^n \ln \mathcal{Z}[J]}{\delta J(\vec{r}_1) \cdots \delta J(\vec{r}_n)} \Big|_{J(\vec{r})=0}$. Also Gibbs free energy $G[\varphi_{\vec{r}}]$, defined as the Legendre transform of free energy $F[J_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}]$

$$G[\varphi_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}] \Big|_{J=J[\varphi]} + \sum_{\vec{r}'} J_{\vec{r}'} \varphi_{\vec{r}'}, \quad \text{with } \varphi_{\vec{r}} = \eta \frac{\partial \ln \mathcal{Z}[J_{\vec{r}}]}{\partial J_{\vec{r}}} \text{ or } \varphi(\vec{r}) = \eta \frac{\delta \ln \mathcal{Z}[J]}{\delta J(\vec{r})}, \quad (\text{D.355})$$

is the generating functional of another kind of correlation functions, the so-called one-line irreducible diagrams; these will be dealt with in equation (D.373) at the end of this Note. Expectation value $\langle \phi \rangle$ is the limit of $\varphi_{\vec{r}} = \eta \frac{\partial \ln \mathcal{Z}[J_{\vec{r}}]}{\partial J_{\vec{r}}}$ at $J_{\vec{r}} \rightarrow 0$. On the other hand, if we have a given $\varphi_{0\vec{r}}$ and want to know what external source $J_{\vec{r}}$ produces such a $\varphi_{0\vec{r}}$, we have to solve the following equation

$$\begin{aligned} \frac{\partial G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}}} \Big|_{\varphi_{\vec{r}}=\varphi_{0\vec{r}}} &= - \left[\sum_{\vec{r}'} \underbrace{\frac{\partial \eta \ln \mathcal{Z}[J_{\vec{r}}]}{\partial J_{\vec{r}'}}}_{\varphi_{\vec{r}'}} \frac{\partial J_{\vec{r}'}}{\partial \varphi_{\vec{r}}} \right] \Big|_{\varphi_{\vec{r}}=\varphi_{0\vec{r}}} + \left[\sum_{\vec{r}'} \frac{\partial J_{\vec{r}'}}{\partial \varphi_{\vec{r}}} \varphi_{\vec{r}'} \right] \Big|_{\varphi_{\vec{r}}=\varphi_{0\vec{r}}} + J_{\vec{r}}, \\ \frac{\partial G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}}} \Big|_{\varphi_{\vec{r}}=\varphi_{0\vec{r}}} &= J_{\vec{r}}. \end{aligned} \quad (\text{D.356})$$

Returning to equation $\mathcal{Z}[J_{\vec{r}}] = \int \prod_{\vec{r}} d\phi_{\vec{r}} \exp \left(-\frac{E[\phi_{\vec{r}}]}{\eta} + \frac{1}{\eta} \sum_{\vec{r}} J_{\vec{r}} \phi_{\vec{r}} \right)$, all models considered in this work satisfy that energy $E[\phi_{\vec{r}}]$ contains a term like $\frac{a^{d-2}}{2} \sum_{\vec{r}} \sum_{\mu=1}^d (\Delta_{\mu} \phi_{\vec{r}})^2 \leftrightarrow \frac{1}{2} \int d^d \vec{r} \sum_{\mu} (\partial_{\mu} \phi(\vec{r}))^2$, term which can be multiplied by a coupling strength $\beta_{e,\psi}$, as for example in the gauge model where it is equal to $1/e^2$. In order to avoid an overloaded notation, $\beta_{e,\psi}$ will be absorbed into η from here on, $\frac{\eta}{\beta_{e,\psi}} \rightarrow \eta$, since in the final result of the free energy it will not appear. For $\eta \rightarrow 0$ (which includes thus a strong coupling strength), the configuration whose contribution dominates the integrand of $\mathcal{Z}[J_{\vec{r}}]$ is the so-called saddle-point configuration $\phi_{\text{cl}\vec{r}}$, i.e., that configuration for which the integrand's exponent is a minimum

$$\frac{\partial}{\partial \phi_{\vec{r}}} \left(-E[\phi_{\vec{r}}] + \sum_{\vec{r}'} J_{\vec{r}'} \phi_{\vec{r}'} \right) \Big|_{\phi_{\vec{r}}=\phi_{\text{cl}\vec{r}}} = 0, \quad (\text{D.357})$$

or in functional notation $\frac{\delta E[\phi]}{\delta \phi(\vec{r})} \Big|_{\phi(\vec{r})=\phi_{\text{cl}}(\vec{r})} = J(\vec{r})$. For a bit larger η , about $\phi_{\text{cl}\vec{r}}$ there appear fluctuations $\gamma_{\vec{r}}$ which also contribute to the partition function and can be treated as perturbations ¹²⁹

$$\phi_{\vec{r}} = \phi_{\text{cl}\vec{r}} + \sqrt{\eta} \gamma_{\vec{r}}, \quad (\text{D.358})$$

and we can expand the energy of these configurations $E[\phi_{\vec{r}}]$ in powers of $\gamma_{\vec{r}}$

$$E[\phi_{\text{cl}\vec{r}} + \sqrt{\eta} \gamma_{\vec{r}}] - \sum_{\vec{r}} J_{\vec{r}} (\phi_{\text{cl}\vec{r}} + \sqrt{\eta} \gamma_{\vec{r}}) = \left(E[\phi_{\text{cl}\vec{r}}] - \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}} \right) + 0 + \frac{\eta}{2!} \sum_{\vec{z}_1, \vec{z}_2} \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2}} \Big|_{\phi(\vec{r})=\phi_{\text{cl}}(\vec{r})} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} +$$

¹²⁹Taking $\eta \approx 0$ enables us to isolate the effect of taking configurations which do not correspond exactly to the minimum of the energy which defines the partition function. As we shall see now, an expansion in powers of fluctuation $\gamma_{\vec{r}}$ is equivalently to an expansion in η -powers, and thus those terms in the resulting free energy which are of proportional to some power of η are the contributions of the fluctuations.

$$+ \frac{\eta^{3/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \frac{\partial^3 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2} \partial \phi_{\vec{z}_3}} \Big|_{\phi(\vec{r}) = \phi_{\text{cl}}(\vec{r})} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} + \frac{\eta^2}{4!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4} \frac{\partial^4 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2} \partial \phi_{\vec{z}_3} \partial \phi_{\vec{z}_4}} \Big|_{\phi(\vec{r}) = \phi_{\text{cl}}(\vec{r})} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4} + \dots$$

By defining propagator $D_{\text{cl}\vec{z}_1, \vec{z}_2}$ through relation $\sum_{\vec{z}_2} \frac{\partial^2 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}} D_{\text{cl}\vec{z}_2, \vec{z}_3} = \delta_{\vec{z}_1, \vec{z}_3}$, the partition function becomes

$$\mathcal{Z} [J_{\vec{r}}] = \exp \left(-\frac{E [\phi_{\text{cl}\vec{r}}]}{\eta} + \frac{1}{\eta} \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}} \right) (\sqrt{\eta})^{|\Lambda|} \int \prod_{\vec{r}} d\gamma_{\vec{r}} \exp \left[-\frac{1}{2!} \sum_{\vec{z}_1, \vec{z}_2} \gamma_{\vec{z}_1} \left(\frac{\partial^2 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}} \right) \gamma_{\vec{z}_2} - \right. \\ \left. - \frac{\eta^{1/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} - \frac{\eta}{4!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4} - \dots \right],$$

with the following abbreviations: $\partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \equiv \frac{\partial^3 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1} \partial \phi_{\vec{z}_2} \partial \phi_{\vec{z}_3}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}}$, ... The above equation can be now expanded in powers of η , but before doing it let us writing $\mathcal{Z} [J_{\vec{r}}]$ in a more useful form: by using that $\int \prod_{\vec{r}} d\gamma_{\vec{r}} \gamma_{\vec{r}_1}^n \exp (-\mathcal{E} [\gamma_{\vec{r}}]) = \left[\left(\frac{\partial}{\partial j_{\vec{r}_1}} \right)^n \left\{ \int \prod_{\vec{r}} d\gamma_{\vec{r}} \exp \left(-\mathcal{E} [\gamma_{\vec{r}}] + \sum_{\vec{r}} j_{\vec{r}} \gamma_{\vec{r}} \right) \right\} \right] \Big|_{j_{\vec{r}}=0}$, the above equation takes the following form (do not confuse $J_{\vec{r}}$ with the just introduced $j_{\vec{r}}$)

$$\mathcal{Z} [J_{\vec{r}}] = \exp \left(-\frac{E [\phi_{\text{cl}\vec{r}}]}{\eta} + \frac{1}{\eta} \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}} \right) (\sqrt{\eta})^{|\Lambda|} \times \\ \times \left[\exp \left(-\frac{\eta^{1/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \frac{\partial}{\partial j_{\vec{z}_1}} \frac{\partial}{\partial j_{\vec{z}_2}} \frac{\partial}{\partial j_{\vec{z}_3}} - \frac{\eta}{4!} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \frac{\partial}{\partial j_{\vec{z}_1}} \dots \frac{\partial}{\partial j_{\vec{z}_4}} - \dots \right) \right\} \\ \underbrace{\int \prod_{\vec{r}} d\gamma_{\vec{r}} \exp \left(-\frac{1}{2!} \sum_{\vec{r}, \vec{r}'} \gamma_{\vec{r}} \left(\frac{\partial^2 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{r}} \partial \phi_{\vec{r}'}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}} \right) \gamma_{\vec{r}'} + \sum_{\vec{r}} j_{\vec{r}} \gamma_{\vec{r}} \right) \Big\} \Big|_{j_{\vec{r}}=0}}_{\frac{(\sqrt{2\pi})^{|\Lambda|}}{\sqrt{\det \left(\frac{\partial^2 E [\phi_{\vec{r}}]}{\partial \phi_{\vec{r}} \partial \phi_{\vec{r}'}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}}} \right)} \exp \left(\frac{1}{2} \sum_{\vec{r}, \vec{r}'} j_{\vec{r}} D_{\text{cl}\vec{r}, \vec{r}'} j_{\vec{r}'} \right)}$$
(D.359)

equation which can simplified by using Fourier transformations: let f, g be two general functions of N variables $f(x_1, \dots, x_N) = \prod_{k=1}^N \sum_{q_k=-\infty}^{\infty} e^{i \sum_k q_k x_k} \tilde{f}(q_1, \dots, q_N)$ and $g(x_j) = \sum_{\{p_j\}} e^{i \sum_j p_j x_j} \tilde{g}_p$. Then $f\left(\frac{\partial}{\partial x_k}\right)$ acting on function $g(x_j)$ is equal to $f\left(\frac{\partial}{\partial x_k}\right) \{g(x_j)\} = \sum_{\{q_k\}} \sum_{\{p_j\}} \tilde{f}_q \tilde{g}_p e^{i \sum_k q_k \left(\frac{\partial}{\partial x_k}\right)} \left\{ e^{i \sum_j p_j x_j} \right\} = \sum_{\{q_k\}} \sum_{\{p_j\}} \tilde{f}_q \tilde{g}_p e^{i \sum_j p_j (i q_j + x_j)}$; on the other hand one has as well $\left[g\left(\frac{\partial}{\partial y_j}\right) \left\{ f(y_k) e^{\sum_m x_m y_m} \right\} \right] \Big|_{y=0} = \left[\sum_{\{q_k\}} \sum_{\{p_j\}} \tilde{f}_q \tilde{g}_p e^{i \sum_j p_j (i q_j + x_j)} e^{i \sum_k (q_k + x_k) y_k} \right] \Big|_{y=0} = f\left(\frac{\partial}{\partial x_k}\right) \{g(x_j)\}$. Therefore

$$f\left[\frac{\partial}{\partial j_{\vec{r}}}\right] \{g[j_{\vec{r}}]\} = \left[f\left[\frac{\partial}{\partial \gamma_{\vec{r}}}\right] \left\{ g[\gamma_{\vec{r}}] e^{\sum_{\vec{r}} j_{\vec{r}} \gamma_{\vec{r}}} \right\} \right] \Big|_{j_{\vec{r}}=0}, \quad (D.360)$$

which applied into the previous partition function (D.359) with $g[j_{\vec{\tau}}] = \exp\left(\frac{1}{2} \sum_{\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}} D_{\text{cl}\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}'}\right)$ and $f\left[\frac{\partial}{\partial j_{\vec{\tau}}}\right] = \exp\left(-\frac{\eta^{1/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \frac{\partial}{\partial j_{\vec{z}_1}} \frac{\partial}{\partial j_{\vec{z}_2}} \frac{\partial}{\partial j_{\vec{z}_3}} - \frac{\eta}{4!} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \frac{\partial}{\partial j_{\vec{z}_1}} \dots \frac{\partial}{\partial j_{\vec{z}_4}} - \dots\right)$ produces the following result

$$\begin{aligned} \mathcal{Z}[J_{\vec{\tau}}] &= \frac{\left(\sqrt{2\pi\eta}\right)^{|\Lambda|} \exp\left(-\frac{E[\phi_{\text{cl}\vec{\tau}}]}{\eta} + \frac{1}{\eta} \sum_{\vec{\tau}} J_{\vec{\tau}} \phi_{\text{cl}\vec{\tau}}\right)}{\sqrt{\det\left(\frac{\partial^2 E[\phi_{\vec{\tau}}]}{\partial \phi_{\vec{\tau}} \partial \phi_{\vec{\tau}'}}\right)_{\phi_{\vec{\tau}} = \phi_{\text{cl}\vec{\tau}}}}} \left[\exp\left(\frac{1}{2} \sum_{\vec{\tau}, \vec{\tau}'} D_{\text{cl}\vec{\tau}, \vec{\tau}'} \frac{\partial}{\partial \gamma_{\vec{\tau}}} \frac{\partial}{\partial \gamma_{\vec{\tau}'}}\right) \left\{ \right. \\ &\quad \left. \exp\left(-\frac{\eta^{1/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} - \frac{\eta}{4!} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \gamma_{\vec{z}_1} \dots \gamma_{\vec{z}_4} - \dots + \sum_{\vec{\tau}} j_{\vec{\tau}} \gamma_{\vec{\tau}}\right) \right\} \right]_{\gamma_{\vec{\tau}} = 0}, \\ \mathcal{Z}[J_{\vec{\tau}}] &= \left(\right) \left[\exp\left(\frac{1}{2} \sum_{\vec{\tau}, \vec{\tau}'} D_{\text{cl}\vec{\tau}, \vec{\tau}'} \frac{\partial}{\partial \gamma_{\vec{\tau}}} \frac{\partial}{\partial \gamma_{\vec{\tau}'}}\right) \left\{ 1 - \frac{\eta^{1/2}}{3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} - \right. \right. \\ &\quad \left. \left. - \frac{\eta}{4!} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \gamma_{\vec{z}_1} \dots \gamma_{\vec{z}_4} + \frac{1}{2!} \frac{\eta}{3!3!} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \sum_{\vec{y}_1, \vec{y}_2, \vec{y}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \gamma_{\vec{z}_1} \dots \gamma_{\vec{z}_3} \gamma_{\vec{y}_1} \dots \gamma_{\vec{y}_3} + \dots \right\} \right]_{\gamma_{\vec{\tau}} = 0} \end{aligned} \quad (\text{D.361})$$

Since after carrying out the derivations contained in operator $\exp\left(\frac{1}{2} \sum_{\vec{\tau}, \vec{\tau}'} D_{\text{cl}\vec{\tau}, \vec{\tau}'} \frac{\partial}{\partial \gamma_{\vec{\tau}}} \frac{\partial}{\partial \gamma_{\vec{\tau}'}}\right) = 1 + \frac{1}{2} \sum_{\vec{\tau}_1, \vec{\tau}'_1} D_{\text{cl}\vec{\tau}_1, \vec{\tau}'_1} \frac{\partial}{\partial \gamma_{\vec{\tau}_1}} \frac{\partial}{\partial \gamma_{\vec{\tau}'_1}} + \frac{1}{2!} \left(\frac{1}{2}\right)^2 \sum_{\vec{\tau}_1, \vec{\tau}'_1} \sum_{\vec{\tau}_2, \vec{\tau}'_2} D_{\text{cl}\vec{\tau}_1, \vec{\tau}'_1} D_{\text{cl}\vec{\tau}_2, \vec{\tau}'_2} \frac{\partial}{\partial \gamma_{\vec{\tau}_1}} \frac{\partial}{\partial \gamma_{\vec{\tau}'_1}} \frac{\partial}{\partial \gamma_{\vec{\tau}_2}} \frac{\partial}{\partial \gamma_{\vec{\tau}'_2}} + \dots$ one has to set $\gamma_{\vec{\tau}} = 0$, only the terms with an even power of $\gamma_{\vec{\tau}}$'s will remain from the power expansion of the above equation, expansion which thus will be also an expansion in *integer* powers of η (since each $\gamma_{\vec{\tau}}$ contributes with $\sqrt{\eta}$). Apart from the underlined term, $\mathcal{Z}[J_{\vec{\tau}}]$'s expansion up to order η is given by the terms containing a 4th. and a 6th. power of $\gamma_{\vec{\tau}}$'s ¹³⁰

$$1 - \frac{1}{2!} \left(\frac{1}{2}\right)^2 \frac{\eta}{4!} \sum_{\vec{\tau}_1, \vec{\tau}'_1} \sum_{\vec{\tau}_2, \vec{\tau}'_2} \sum_{\vec{z}_1, \dots, \vec{z}_4} D_{\text{cl}\vec{\tau}_1, \vec{\tau}'_1} D_{\text{cl}\vec{\tau}_2, \vec{\tau}'_2} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \left(\frac{\partial}{\partial \gamma_{\vec{\tau}_1}} \frac{\partial}{\partial \gamma_{\vec{\tau}'_1}} \frac{\partial}{\partial \gamma_{\vec{\tau}_2}} \frac{\partial}{\partial \gamma_{\vec{\tau}'_2}} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4} \right) +$$

¹³⁰It is noteworthy that if we would have kept coupling strength $\beta_{e, \psi}$ as the fluctuation was defined, $\phi_{\vec{\tau}} = \phi_{\text{cl}\vec{\tau}} + \sqrt{\frac{\eta}{\beta_{e, \psi}}} \gamma_{\vec{\tau}}$, then the bilinear term in $\gamma_{\vec{\tau}}$ in the expansion of $E[\phi_{\vec{\tau}}]$ would have been equal to

$\frac{1}{2! \beta_{e, \psi}} \sum_{\vec{\tau}, \vec{\tau}'} \frac{\partial^2 E[\phi_{\vec{\tau}}]}{\partial \phi_{\vec{\tau}} \partial \phi_{\vec{\tau}'}} \Big|_{\phi = \phi_{\text{cl}}} \gamma_{\vec{\tau}} \gamma_{\vec{\tau}'}$ and in equation (D.359), instead of obtaining $\exp\left(\frac{1}{2} \sum_{\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}} D_{\text{cl}\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}'}\right)$, we would

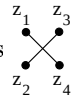
have obtained $\exp\left(\frac{\beta_{e, \psi}}{2} \sum_{\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}} D_{\text{cl}\vec{\tau}, \vec{\tau}'} j_{\vec{\tau}'}\right)$. In this last case, partition function (D.361) would take the follow-

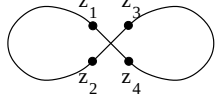
ing form: $\mathcal{Z}[J_{\vec{\tau}}] = \text{constant} \times \left[\exp\left(\frac{\beta_{e, \psi}}{2} \sum_{\vec{\tau}, \vec{\tau}'} D_{\text{cl}\vec{\tau}, \vec{\tau}'} \frac{\partial}{\partial \gamma_{\vec{\tau}}} \frac{\partial}{\partial \gamma_{\vec{\tau}'}}\right) \left\{ 1 - \frac{\eta^{1/2}}{3! \beta_{e, \psi}^{3/2}} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} - \right. \right.$

$$+ \frac{1}{3!} \left(\frac{1}{2} \right)^3 \frac{\eta}{2!(3!)^2} \sum_{\vec{r}_1, \dots, \vec{r}_3} \sum_{\vec{z}_1, \dots, \vec{z}_3} D_{\text{cl}\vec{r}_1, \vec{r}_1'} D_{\text{cl}\vec{r}_2, \vec{r}_2'} D_{\text{cl}\vec{r}_3, \vec{r}_3'} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \cdots \frac{\partial}{\partial \gamma_{\vec{r}_3}} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{y}_3} \right) + \dots$$

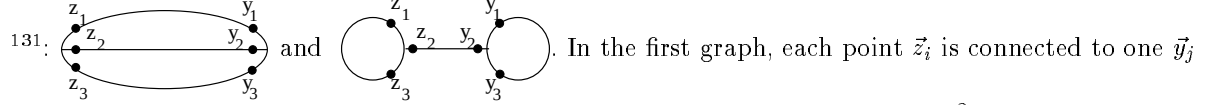
In term $\frac{\partial}{\partial \gamma_{\vec{r}_1}} \frac{\partial}{\partial \gamma_{\vec{r}_1'}} \frac{\partial}{\partial \gamma_{\vec{r}_2}} \frac{\partial}{\partial \gamma_{\vec{r}_2'}} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4}$, derivative $\frac{\partial}{\partial \gamma_{\vec{r}_1}}$ can act on any of the four points $\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4$; next derivative $\frac{\partial}{\partial \gamma_{\vec{r}_1'}}$ can act on one of the three remaining points, and so on: the combinatorial factor is therefore $4!$ and thus

$$\begin{aligned} & - \frac{\eta}{4!8} \sum_{\vec{r}_1, \vec{r}_1'} \sum_{\vec{r}_2, \vec{r}_2'} \sum_{\vec{z}_1, \dots, \vec{z}_4} D_{\text{cl}\vec{r}_1, \vec{r}_1'} D_{\text{cl}\vec{r}_2, \vec{r}_2'} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \frac{\partial}{\partial \gamma_{\vec{r}_1'}} \frac{\partial}{\partial \gamma_{\vec{r}_2}} \frac{\partial}{\partial \gamma_{\vec{r}_2'}} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4} \right) = \\ & = - \frac{\eta}{8} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4} D_{\text{cl}\vec{z}_1, \vec{z}_2} D_{\text{cl}\vec{z}_3, \vec{z}_4} . \end{aligned} \quad (\text{D.362})$$

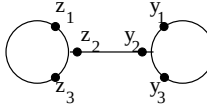
Equation (D.362) can be graphically represented by a vertex with 4 points  to denote $\partial^4 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4}$ together with two lines connecting two of such points, where each line represents propagator $D_{\text{cl}\vec{r}, \vec{r}'}$:



further, the sum (integral) over all coordinates inside the diagram is assumed. On the other hand, term $\frac{\partial}{\partial \gamma_{\vec{r}_1}} \frac{\partial}{\partial \gamma_{\vec{r}_1'}} \frac{\partial}{\partial \gamma_{\vec{r}_2}} \frac{\partial}{\partial \gamma_{\vec{r}_2'}} \frac{\partial}{\partial \gamma_{\vec{r}_3}} \frac{\partial}{\partial \gamma_{\vec{r}_3'}} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{z}_3} \gamma_{\vec{y}_1} \cdots \gamma_{\vec{y}_3}$ contains two different diagrams



and the corresponding combinatorial factor $8(3!)^2$ is obtained as follows: after $\frac{\partial}{\partial \gamma_{\vec{r}_1}}$ has acted on any of the 6 possible points, for $\frac{\partial}{\partial \gamma_{\vec{r}_1'}}$ remains only the 3 points belonging to the vertex where $\frac{\partial}{\partial \gamma_{\vec{r}_1}}$ has not acted; to derivative $\frac{\partial}{\partial \gamma_{\vec{r}_2}}$ there are 4 points available, whereas to $\frac{\partial}{\partial \gamma_{\vec{r}_2'}}$ only 2, and finally $\frac{\partial}{\partial \gamma_{\vec{r}_3}}$ can act

on one of the 2 remaining points. For diagram  only one point of vertex $\partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3}$

$$\left. - \frac{\eta}{4! \beta_{e, \psi}^{4/2}} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{z}_4} + \frac{1}{2!} \frac{\eta}{3!3! \beta_{e, \psi}^{2 \times 3/2}} \sum_{\vec{z}_1, \dots, \vec{y}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{z}_3} \gamma_{\vec{y}_1} \cdots \gamma_{\vec{y}_3} + \dots \right] \Bigg|_{\gamma_{\vec{r}}=0} .$$

And by noticing that each propagator $D_{\text{cl}\vec{r}, \vec{r}'}$ contributes with a power $\beta_{e, \psi}$, and each vertex $\frac{\partial^n E[\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_1}, \dots, \partial \phi_{\vec{z}_n}} \Big|_{\phi=\phi_{\text{cl}}}$ with a power $\beta_{e, \psi}^{-n/2}$, then (as already advanced) the only remaining terms in the expansion of $\mathcal{Z}[J_{\vec{r}}]$ contain no $\beta_{e, \psi}$:

$$\begin{aligned} & 1 - \frac{1}{2!} \left(\frac{\beta_{e, \psi}}{2} \right)^2 \frac{\eta}{4! \beta_{e, \psi}^{4/2}} \sum_{\vec{r}_1, \vec{r}_1'} \sum_{\vec{r}_2, \vec{r}_2'} \sum_{\vec{z}_1, \dots, \vec{z}_4} D_{\text{cl}\vec{r}_1, \vec{r}_1'} D_{\text{cl}\vec{r}_2, \vec{r}_2'} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \frac{\partial}{\partial \gamma_{\vec{r}_1'}} \frac{\partial}{\partial \gamma_{\vec{r}_2}} \frac{\partial}{\partial \gamma_{\vec{r}_2'}} \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \gamma_{\vec{z}_4} \right) + \\ & + \frac{1}{3!} \left(\frac{\beta_{e, \psi}}{2} \right)^3 \frac{\eta}{2!(3!)^2 \beta_{e, \psi}^{2 \times 3/2}} \sum_{\vec{r}_1, \dots, \vec{r}_3} \sum_{\vec{z}_1, \dots, \vec{y}_3} D_{\text{cl}\vec{r}_1, \vec{r}_1'} D_{\text{cl}\vec{r}_2, \vec{r}_2'} D_{\text{cl}\vec{r}_3, \vec{r}_3'} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \cdots \frac{\partial}{\partial \gamma_{\vec{r}_3}} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{y}_3} \right) + \dots \end{aligned}$$

¹³¹All these diagrams are called Feynman diagrams or graphs. These are the names used in Quantum Field Theory, where instead of speaking of energy scale η and of energy $E[\phi_{\vec{r}}]$, Helmholtz free energy $F[J_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}]$ and Gibbs free energy $G[\varphi_{\vec{r}}]$, one speaks of Planck constant $\hbar$ and of action $iS[\phi_{\vec{r}}]$, generating functional $W[J_{\vec{r}}] = -i\hbar \ln \mathcal{Z}[J_{\vec{r}}]$ and effective action $\Gamma[\varphi_{\vec{r}}]$.

is connected to a point of the other vertex, and its combinatorial factor is equal to $2(3!)^3$: after $\frac{\partial}{\partial \gamma_{\vec{r}_1}}$ has chosen on any of the 6 possible points, $\frac{\partial}{\partial \gamma_{\vec{r}'_1}}$ can act either on any of the 2 points of the same vertex chosen by $\frac{\partial}{\partial \gamma_{\vec{r}_1}}$ (and the following factors are $4 \cdot 3 \cdot 2$) or on any of the the 3 points of the other vertex (being $4 \cdot 1 \cdot 2$ the following factors); thus $6 \cdot 2 \cdot 4 \cdot 3 \cdot 2 + 6 \cdot 3 \cdot 4 \cdot 2 = 2(3!)^3$

$$\begin{aligned} & \frac{\eta}{16(3!)^3} \sum_{\vec{r}_1, \dots, \vec{r}'_3} \sum_{\vec{z}_1, \dots, \vec{y}_3} D_{\text{cl}\vec{r}_1, \vec{r}'_1} D_{\text{cl}\vec{r}_2, \vec{r}'_2} D_{\text{cl}\vec{r}_3, \vec{r}'_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \cdots \frac{\partial}{\partial \gamma_{\vec{r}'_3}} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{y}_3} \right) = \\ & = \eta \sum_{\vec{z}_1, \dots, \vec{y}_3} \partial^3 E_{\text{cl}\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}\vec{y}_1, \vec{y}_2, \vec{y}_3} \left(\frac{1}{12} D_{\text{cl}\vec{z}_1, \vec{y}_1} D_{\text{cl}\vec{z}_2, \vec{y}_2} D_{\text{cl}\vec{z}_3, \vec{y}_3} + \frac{1}{8} D_{\text{cl}\vec{z}_1, \vec{z}_3} D_{\text{cl}\vec{z}_2, \vec{y}_2} D_{\text{cl}\vec{y}_1, \vec{y}_3} \right), \quad (\text{D.363}) \end{aligned}$$

which together with (D.362) enables us to write (Helmholtz) free energy $F[J_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}]$ (the previous η is the energy scalar alone, it does not contain coupling strength $\beta_{e,\psi}$) up to order η^2 after substituting into (D.361)

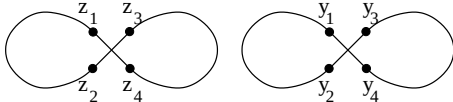
$$\begin{aligned} F[J_{\vec{r}}] = & \text{const} + \left(E[\phi_{\text{cl}\vec{r}}] - \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}} \right) + \frac{\eta}{2} \ln \det \left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}} \partial \phi_{\vec{r}'}} \right)_{\phi_{\vec{r}} = \phi_{\text{cl}\vec{r}}} + \\ & + \frac{\eta^2}{8} \text{diagram 1} - \frac{\eta^2}{12} \text{diagram 2} - \frac{\eta^2}{8} \text{diagram 3} + \dots \quad (\text{D.364}) \end{aligned}$$

with $\text{const} = -\eta \frac{|\Lambda|}{2} \ln(2\pi\eta)$. One should not forget that the above equation depends on $J_{\vec{r}}$ through $\phi_{\text{cl}\vec{r}}$ and all the $\frac{\partial^n E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \cdots \partial \phi_{\vec{r}_n}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}\vec{r}}}$, $\phi_{\text{cl}\vec{r}}$ being determined by means of $\frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}\vec{r}}} = J_{\vec{r}}$. By functional derivation of $F[J_{\vec{r}}]$ one obtains, by definition, the connected correlation functions; and we shall now see that the (Helmholtz) free energy only contains connected diagrams.

It is just by taking the logarithm of $\mathcal{Z}[J_{\vec{r}}]$ when the disconnected diagrams (i.e. diagrams formed by isolated elements) disappear: let us see an example by considering one of the many set of contributions of order η^2 in $\mathcal{Z}[J_{\vec{r}}]$ (D.361) which arises from $+\frac{1}{4!} \left(\frac{1}{2}\right)^4 \frac{\eta^2}{2!(4!)^2} \sum_{\vec{r}_1, \dots, \vec{r}'_4} \sum_{\vec{z}_1, \dots, \vec{y}_4} D_{\text{cl}\vec{r}_1, \vec{r}'_1} D_{\text{cl}\vec{r}_2, \vec{r}'_2} \times$
 $\times D_{\text{cl}\vec{r}_3, \vec{r}'_3} D_{\text{cl}\vec{r}_4, \vec{r}'_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{y}_1, \dots, \vec{y}_4} \left(\frac{\partial}{\partial \gamma_{\vec{r}_1}} \cdots \frac{\partial}{\partial \gamma_{\vec{r}'_4}} \gamma_{\vec{z}_1} \cdots \gamma_{\vec{y}_4} \right)$. Such term contains the following diagrams (with their corresponding combinatorial factors)

$$\begin{aligned} & \sum_{\vec{z}_1, \dots, \vec{y}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{y}_1, \dots, \vec{y}_4} D_{\text{cl}\vec{z}_1, \vec{y}_1} D_{\text{cl}\vec{z}_2, \vec{y}_2} D_{\text{cl}\vec{z}_3, \vec{y}_3} D_{\text{cl}\vec{z}_4, \vec{y}_4} + \\ & \quad \text{diagram 4} \rightarrow 8 \cdot 4 \cdot 6 \cdot 3 \cdot 4 \cdot 2 \cdot 2, \\ & + \sum_{\vec{z}_1, \dots, \vec{y}_4} \partial^4 E_{\text{cl}\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{\text{cl}\vec{y}_1, \dots, \vec{y}_4} D_{\text{cl}\vec{z}_1, \vec{y}_1} D_{\text{cl}\vec{z}_2, \vec{y}_2} D_{\text{cl}\vec{z}_3, \vec{z}_4} D_{\text{cl}\vec{y}_3, \vec{y}_4} + \\ & \quad \text{diagram 5} \rightarrow 8 \cdot 4 \cdot 6 \cdot 2 \cdot 4 \cdot 3 \cdot 2 + \\ & \quad + 8 \cdot 4 \cdot 6 \cdot 3 \cdot 4 \cdot 1 \cdot 2 + \\ & \quad + 8 \cdot 3 \cdot 6 \cdot 4 \cdot 4 \cdot 3 \cdot 2, \end{aligned}$$

$$+ \sum_{\vec{z}_1, \dots, \vec{y}_4} \partial^4 E_{\text{cl}} \vec{z}_1, \dots, \vec{z}_4 \partial^4 E_{\text{cl}} \vec{y}_1, \dots, \vec{y}_4 D_{\text{cl}} \vec{z}_1, \vec{z}_2 D_{\text{cl}} \vec{z}_3, \vec{z}_4 D_{\text{cl}} \vec{y}_1, \vec{y}_2 D_{\text{cl}} \vec{y}_3, \vec{y}_4$$

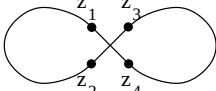

→ 8 · 4 · 6 · 3 · 4 · 2 · 2,

and thus the partition function partially expanded up to order η^2 becomes

$$\mathcal{Z}[J_{\vec{r}}] = \text{const.} \frac{\exp\left(-\frac{E[\phi_{\text{cl}} \vec{r}]}{\eta} + \frac{1}{\eta} \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}} \vec{r}\right)}{\sqrt{\det\left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}} \partial \phi_{\vec{r}'}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}}\right)}} \left[1 - \underbrace{\frac{\eta}{8} \text{ (diagram with } z_1, z_2, z_3, z_4 \text{)}}_{\uparrow} + \right.$$

$$+ \frac{\eta}{12} \text{ (diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{)}} + \frac{\eta}{8} \text{ (diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{)}} + \frac{\eta^2}{128} \text{ (diagram with } z_1, z_2, z_3, z_4 \text{)}} \text{ (diagram with } y_1, y_2, y_3, y_4 \text{)}} +$$

$$+ \frac{\eta^2}{48} \text{ (diagram with } z_1, z_2, z_3, z_4, y_1, y_2, y_3, y_4 \text{)}} + \frac{\eta^2}{16} \text{ (diagram with } z_1, z_2, z_3, z_4, y_1, y_2, y_3, y_4 \text{)}} + \dots \left. \right] \quad (\text{D.365})$$

But by taking the logarithm, the disconnected diagram  gets compensated by the square of the diagram indicated by the arrow and which is contained in the power expansion $\ln(1+z) = z - \underbrace{\frac{z^2}{2}}_{\uparrow} + \dots$

$$- \frac{1}{2} \frac{\eta^2}{64} \left(\text{diagram with } z_1, z_2, z_3, z_4 \text{}} \right)^2 + \frac{\eta^2}{128} \text{ (diagram with } z_1, z_2, z_3, z_4 \text{)}} \text{ (diagram with } y_1, y_2, y_3, y_4 \text{)}} = 0. \quad (\text{D.366})$$

Disconnected diagrams $-\frac{1}{2} \frac{\eta^2}{144} \left(\text{diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{}} \right)^2$ and $-\frac{1}{2} \frac{\eta^2}{64} \left(\text{diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{}} \right)^2$ are also com-

pensated in $\ln \mathcal{Z}[J_{\vec{r}}]$ by some of the diagrams arising from $+\frac{1}{6!} \left(\frac{1}{2}\right)^6 \frac{\eta^2}{4!(3!)^4} \left(\sum_{\vec{r}_1, \dots, \vec{r}'_1} D_{\text{cl}} \vec{r}_1, \vec{r}'_1 \frac{\partial}{\partial \gamma_{\vec{r}_1}} \frac{\partial}{\partial \gamma_{\vec{r}'_1}} \right)^6$

$$\left(\sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{\text{cl}} \vec{z}_1, \vec{z}_2, \vec{z}_3 \gamma_{\vec{z}_1} \gamma_{\vec{z}_2} \gamma_{\vec{z}_3} \right)^4.$$

It is curious to note that, aside of containing only integer η -powers

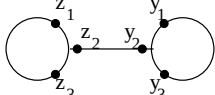
and no disconnected diagrams, $F[J_{\vec{r}}] = \text{cst} + \left(E[\phi_{\text{cl}} \vec{r}] - \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}} \vec{r} \right) + \frac{\eta}{2} \ln \det \left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}} \vec{r}} \right) +$

$$+ \frac{\eta^2}{8} \text{ (diagram with } z_1, z_2, z_3, z_4 \text{)}} - \frac{\eta^2}{12} \text{ (diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{)}} - \frac{\eta^2}{8} \text{ (diagram with } z_1, z_2, z_3, y_1, y_2, y_3 \text{)}} - \frac{\eta^3}{48} \text{ (diagram with } z_1, z_2, z_3, z_4, y_1, y_2, y_3, y_4 \text{)}} -$$

$-\frac{\eta^3}{16}$  + ..., is formed also by the expansion in closed loops of in-

ternal lines, being the power of η the number of loops too. Therefore, such an expansion is also called the loop expansion.

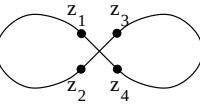
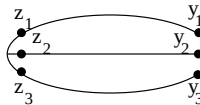
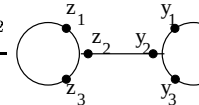
Finally, let us discuss the meaning of Gibbs free energy $G[\varphi_{\vec{r}}]$ as a generating functional. Advancing the final result, the Gibbs free energy will only contain one-line irreducible graphs (also called one-particle irreducible diagrams), i.e., diagrams which cannot be disconnected by cutting only one line.

Thus graph  will not occur in $G[\varphi_{\vec{r}}]$ since it can be separated into two independents parts by cutting the line between $\vec{z}_2$ and $\vec{y}_2$. Let us return to the definition of the Gibbs free energy

$G[\varphi_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}} J_{\vec{r}} \varphi_{\vec{r}}$, where its variable $\varphi_{\vec{r}}$ is defined by $\varphi_{\vec{r}} = -\frac{\partial(-\eta \ln \mathcal{Z}[J_{\vec{r}}])}{\partial J_{\vec{r}}}$ which is equivalent to state that equation $G[\varphi_{\vec{r}}] = -\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}} J_{\vec{r}} \varphi_{\vec{r}}$ remains invariant under changes of $J_{\vec{r}}$ for a fixed $\varphi_{\vec{r}}$

$$\underbrace{-\eta \ln \mathcal{Z}[J_{\vec{r}} + \delta J_{\vec{r}}]}_{-\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}'} \frac{\partial(-\eta \ln \mathcal{Z}[J_{\vec{r}}])}{\partial J_{\vec{r}'}} \delta J_{\vec{r}'}} - \sum_{\vec{r}'} (J_{\vec{r}'} + \delta J_{\vec{r}'}) \varphi_{\vec{r}'} = G[\varphi_{\vec{r}}] \Big|_{\text{fixed } \varphi_{\vec{r}}} = -\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}'} J_{\vec{r}'} \varphi_{\vec{r}'} . \quad (\text{D.367})$$

Looking at free energy (D.364) $F[J_{\vec{r}}] = \text{const} + \left(E[\phi_{\text{cl}\vec{r}}] - \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}} \right) + \frac{\eta}{2} \ln \det \left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi_{\vec{r}} = \phi_{\text{cl}\vec{r}}} \right) +$

$$+ \frac{\eta^2}{8} \text{  } - \frac{\eta^2}{12} \text{  } - \frac{\eta^2}{8} \text{  } + O(\eta^3), \text{ at limit } \eta \rightarrow 0 \text{ the}$$

free energy becomes $F[J_{\vec{r}}] = \text{const} + E[\phi_{\text{cl}\vec{r}}] - \sum_{\vec{r}} J_{\vec{r}} \phi_{\text{cl}\vec{r}}$, and thus $\varphi_{\vec{r}'} = -\frac{\partial F[J_{\vec{r}}]}{\partial J_{\vec{r}'}} = \phi_{\text{cl}\vec{r}'}$ as well as

$G[\varphi_{\vec{r}}] = \text{const} + E[\phi_{\text{cl}\vec{r}}]$. The corrections to this last result due to the fluctuations $\gamma_{\vec{r}}$'s for small $\eta \neq 0$ is given by ¹³²

$$\varphi_{\vec{r}'} = -\frac{\partial F[J_{\vec{r}}]}{\partial J_{\vec{r}'}} = \phi_{\text{cl}\vec{r}'} - \frac{\eta}{2} \frac{\partial}{\partial J_{\vec{r}'}} \left(\underbrace{\ln \det \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi = \phi_{\text{cl}}}}_{\text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi = \phi_{\text{cl}}}} \right) =$$

¹³²Let $\tilde{M}_{ij}$ be a $N \times N$ symmetrical matrix: $\tilde{M} = \tilde{\mathcal{A}}^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix} \tilde{\mathcal{A}}$, $\det \tilde{M} = \prod_{s=1}^N \lambda_s$ and $\ln \det \tilde{M} = \sum_{s=1}^N \ln \lambda_s$.

Since $\exp \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \zeta_N \end{pmatrix} = \begin{pmatrix} e^{\zeta_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\zeta_N} \end{pmatrix}$, one obtains $\ln \tilde{M} = \ln(\tilde{\mathcal{A}}^{-1} \tilde{\mathcal{A}}) + \ln \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix} =$

$= 0 + \begin{pmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ln \lambda_N \end{pmatrix}$ and thus $\text{Tr} \ln \tilde{M} = \sum_{s=1}^N \ln \lambda_s = \ln \det \tilde{M}$. The same applies to operator $\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi = \phi_{\text{cl}}}$

which is a $|\Lambda| \times |\Lambda|$ matrix, with $|\Lambda|$ the number of points in the discretized space.

$$= \phi_{\text{cl}\vec{r}'} - \frac{\eta}{2} \sum_{\vec{r}_1, \text{ with } \vec{r}_2 = \vec{r}_1} \sum_{\vec{r}'_1, \vec{z}} \underbrace{\left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \right)_{\phi = \phi_{\text{cl}}}}_{D_{\text{cl}\vec{r}_1, \vec{r}'_1}}^{-1} \frac{\partial^3 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_2} \partial \phi_{\vec{r}'_1} \partial \phi_{\vec{z}}} \bigg|_{\phi = \phi_{\text{cl}}} \frac{\partial \phi_{\text{cl}\vec{z}}}{\partial J_{\vec{r}'_1}}.$$

It is worth noting that, although $F[J_{\vec{r}}]$ has been calculated up to order η^2 and the same order will be reached for $G[\varphi_{\vec{r}}]$, the above $\varphi_{\vec{r}'}$ has nevertheless reached only order η^1 , and the same applies to its corresponding $J_{\vec{r}'}$. This due to the very form of Gibbs free energy $G[\varphi_{\vec{r}}]$: since this is stationary under a change of $J_{\vec{r}'}$ for fixed $\varphi_{\vec{r}'}$ as seen in (D.367), a correction of order η in the dependence between $J_{\vec{r}'}$ and $\varphi_{\vec{r}'}$ will produce a change in $-\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}} J_{\vec{r}} \varphi_{\vec{r}}$ of

$$\left[-\eta \ln \mathcal{Z}[J_{\vec{r}} + \delta J_{\vec{r}}] + \sum_{\vec{r}} (J_{\vec{r}} + \delta J_{\vec{r}}) \varphi_{\vec{r}} \right] - \left[-\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}} J_{\vec{r}} \varphi_{\vec{r}} \right] =$$

$$\stackrel{\text{eq. (D.367)}}{=} 0 + \frac{1}{2} \sum_{\vec{r}, \vec{r}'} \frac{\partial^2}{\partial J_{\vec{r}} \partial J_{\vec{r}'}} \left[-\eta \ln \mathcal{Z}[J_{\vec{r}}] + \sum_{\vec{r}} J_{\vec{r}} \varphi_{\vec{r}} \right] \underbrace{\delta J_{\vec{r}} \delta J_{\vec{r}'}}_{\uparrow},$$

which is of order η^2 . Due to the same argument, in $\varphi_{\vec{r}'}$'s term *of order* η one can replace propagators $D_{\text{cl}\vec{r}_1, \vec{r}'_1}$ and derivatives calculated at $\phi = \phi_{\text{cl}}$ by propagators $D_{(\varphi)\vec{r}_1, \vec{r}'_1}$ and $\frac{\partial^3 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1} \partial \phi_{\vec{z}}} \bigg|_{\phi = \varphi} \equiv \partial^3 E_{(\varphi)\vec{r}_1, \vec{r}'_1, \vec{z}}$ since these substitutions produce a correction of order η^2 in $\varphi_{\vec{r}'}$ ([99], ch. 5). Propagator $D_{(\varphi)\vec{r}'_1, \vec{r}_1}$ is defined by equation $\sum_{\vec{r}'_1} \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_2} \partial \phi_{\vec{r}'_1}} \bigg|_{\phi = \varphi} D_{(\varphi)\vec{r}'_1, \vec{r}_1} = \delta_{\vec{r}_2, \vec{r}_1}$. Therefore, $\varphi_{\vec{r}'}$ up to order η is equal to ¹³³

$$\varphi_{\vec{r}'} = \phi_{\text{cl}\vec{r}'} - \frac{\eta}{2} \sum_{\vec{z}, \vec{r}_1, \vec{r}'_1} D_{(\varphi)\vec{r}_1, \vec{r}'_1} \partial^3 E_{(\varphi)\vec{r}_1, \vec{r}'_1, \vec{z}} D_{(\varphi)\vec{z}, \vec{r}'} . \quad (\text{D.368})$$

Similar calculations lead to the equation for the $J_{\vec{r}'}$ which generates $\varphi_{\vec{r}'}$

$$J_{\vec{r}'} = \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \bigg|_{\phi = \phi_{\text{cl}}} = \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \bigg|_{\phi = \varphi} + \sum_{\vec{y}} \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'} \partial \phi_{\vec{y}}} \bigg|_{\phi = \varphi} (\phi_{\text{cl}\vec{y}} - \varphi_{\vec{y}}) = \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \bigg|_{\phi = \varphi} +$$

$$+ \frac{\eta}{2} \sum_{\vec{z}, \vec{r}_2, \vec{r}'_2} D_{(\varphi)\vec{r}_2, \vec{r}'_2} \partial^3 E_{(\varphi)\vec{r}_2, \vec{r}'_2, \vec{z}} \underbrace{\left(\sum_{\vec{y}} \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'} \partial \phi_{\vec{y}}} \bigg|_{\phi = \varphi} D_{(\varphi)\vec{z}, \vec{y}} \right)}_{\delta_{\vec{z}, \vec{r}'_1}},$$

$$J_{\vec{r}'} = \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \bigg|_{\phi = \varphi} + \frac{\eta}{2} \sum_{\vec{r}_2, \vec{r}'_2} D_{(\varphi)\vec{r}_2, \vec{r}'_2} \partial^3 E_{(\varphi)\vec{r}_2, \vec{r}'_2, \vec{r}'_1}, \quad (\text{D.369})$$

¹³³From the equation defining $\phi_{\text{cl}\vec{z}}$, $\frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{y}}} \bigg|_{\phi = \phi_{\text{cl}}} = J_{\vec{y}}$, one obtains by $\frac{\partial}{\partial J_{\vec{r}'}}$ -derivation that $\sum_{\vec{z}} \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{y}} \partial \phi_{\vec{z}}} \bigg|_{\phi = \phi_{\text{cl}}} \frac{\partial \phi_{\text{cl}\vec{z}}}{\partial J_{\vec{r}'}} = \delta_{\vec{y}, \vec{z}}$, or equivalently, $\frac{\partial \phi_{\text{cl}\vec{z}}}{\partial J_{\vec{r}'}} = D_{\text{cl}\vec{z}, \vec{r}'}$ which can be approximated by $D_{(\varphi)\vec{z}, \vec{r}'}$ if included in the $\varphi_{\vec{r}'}$ -summand already containing η .

as well as

$$\begin{aligned}
\frac{\eta}{2} \text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\phi_{\text{cl}}} &= \frac{\eta}{2} \text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} + \underbrace{\frac{\eta}{2} \sum_{\vec{r}_1, \vec{r}'_1, \vec{z}} \left(\frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} \right)^{-1}}_{D_{(\varphi) \vec{r}_1, \vec{r}'_1}} \frac{\partial^3 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1} \partial \phi_{\vec{z}}} \Big|_{\phi=\varphi} \underbrace{(\phi_{\text{cl} \vec{z}} - \varphi_{\vec{z}})}_{\text{eq. (D.368)}} = \\
&= \frac{\eta}{2} \text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} + \frac{\eta^2}{4} \underbrace{\sum_{\vec{r}_1, \vec{r}'_1, \vec{z}} \sum_{\vec{r}_2, \vec{r}'_2, \vec{y}} \partial^3 E_{(\varphi) \vec{r}_1, \vec{r}'_1, \vec{z}} \partial^3 E_{(\varphi) \vec{r}_2, \vec{r}'_2, \vec{y}} D_{(\varphi) \vec{r}_1, \vec{r}'_1} D_{(\varphi) \vec{r}_2, \vec{r}'_2} D_{(\varphi) \vec{y}, \vec{z}}}_{\text{Diagram: Two circles connected by a line. Left circle has vertices } \vec{r}_1, \vec{r}'_1 \text{ and } \vec{z}. \text{ Right circle has vertices } \vec{r}_2, \vec{r}'_2 \text{ and } \vec{y}.}, \quad (\text{D.370})
\end{aligned}$$

and

$$E[\phi_{\text{cl} \vec{r}}] = E[\varphi_{\vec{r}}] + \sum_{\vec{r}'} \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \Big|_{\phi=\varphi} (\phi_{\text{cl} \vec{r}'} - \varphi_{\vec{r}'}) + \frac{1}{2} \frac{\eta^2}{4} \text{Diagram: Two circles connected by a line.}, \quad (\text{D.371})$$

By using (D.368) and (D.369) one obtains

$$-\sum_{\vec{r}'} J_{\vec{r}'} \phi_{\text{cl} \vec{r}'} + \sum_{\vec{r}'} J_{\vec{r}'} \varphi_{\vec{r}'} = -\sum_{\vec{r}'} \frac{\partial E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}'}} \Big|_{\phi=\varphi} (\phi_{\text{cl} \vec{r}'} - \varphi_{\vec{r}'}) - \frac{\eta^2}{4} \text{Diagram: Two circles connected by a line.}, \quad (\text{D.372})$$

and substituting this last result, together with relations (D.370) and (D.371), into Gibbs free energy $G[\varphi_{\vec{r}}] = F[J_{\vec{r}}] + \sum_{\vec{r}'} J_{\vec{r}'} \varphi_{\vec{r}'}$ produces

$$\begin{aligned}
G[\varphi_{\vec{r}}] &= \left[\text{const} + E[\varphi_{\vec{r}}] + \frac{\eta}{2} \text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} + \left(\frac{\eta^2}{4} + \frac{\eta^2}{8} - \frac{\eta^2}{4} \right) \text{Diagram: Two circles connected by a line.} \right] + \\
&+ \frac{\eta^2}{8} \text{Diagram: Two circles connected by two lines.} - \frac{\eta^2}{12} \text{Diagram: Two circles connected by three lines.} - \frac{\eta^2}{8} \text{Diagram: Two circles connected by a line.}, \\
G[\varphi_{\vec{r}}] &= \text{const} + E[\varphi_{\vec{r}}] + \frac{\eta}{2} \text{Tr} \ln \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} + \eta^2 \left(\underbrace{\frac{1}{8} \sum_{\vec{z}_1, \dots, \vec{z}_4} \partial^4 E_{(\varphi) \vec{z}_1, \dots, \vec{z}_4} D_{(\varphi) \vec{z}_1, \vec{z}_2} D_{(\varphi) \vec{z}_3, \vec{z}_4}}_{\text{Diagram: Two circles connected by two lines.}} - \right. \\
&\left. - \frac{1}{12} \sum_{\vec{z}_1, \vec{z}_2, \vec{z}_3} \sum_{\vec{y}_1, \vec{y}_2, \vec{y}_3} \partial^3 E_{(\varphi) \vec{z}_1, \vec{z}_2, \vec{z}_3} \partial^3 E_{(\varphi) \vec{y}_1, \vec{y}_2, \vec{y}_3} D_{(\varphi) \vec{z}_1, \vec{y}_1} D_{(\varphi) \vec{z}_2, \vec{y}_2} D_{(\varphi) \vec{z}_3, \vec{y}_3} \right)_{\text{Diagram: Two circles connected by three lines.}}, \quad (\text{D.373})
\end{aligned}$$

where, although terms $\frac{\eta^2}{8} \text{Diagram: Two circles connected by two lines.} - \frac{\eta^2}{12} \text{Diagram: Two circles connected by three lines.} - \frac{\eta^2}{8} \text{Diagram: Two circles connected by a line.}$ in the Helmholtz free energy had been calculated for propagators $D_{\text{cl} \vec{r}, \vec{r}'}$ and vertices $\partial^n E_{\text{cl} \vec{r}_1, \dots, \vec{r}_n}$, since all

these terms are already of order η^2 , one can replace the propagators and vertices by $D_{(\varphi)\vec{r},\vec{r}'}$ and $\partial^n E_{(\varphi)\vec{r}_1,\dots,\vec{r}_n}$ respectively; further, the constant above is $-\eta|\Lambda|\ln(\sqrt{2\pi\eta})$. Result (D.373) shows, as already advanced, that $G[\varphi,\vec{r}]$ only contains one-line irreducible diagrams, and further, just as at the discussion after equation (D.366), the expansion in (integer) η -powers is also an expansion in number of closed loops of internal lines.

The $\varphi,\vec{r}$ -dependence of $G[\varphi,\vec{r}]$ is located in propagator $D_{(\varphi)\vec{r}_1,\vec{r}'_1} \equiv \left(\frac{\partial^2 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_1}\partial\phi_{\vec{r}'_1}} \Big|_{\phi=\varphi} \right)^{-1}$ and in vertex $\partial^n E_{(\varphi)\vec{r}_1,\dots,\vec{r}_n} \equiv \frac{\partial^n E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_1}\dots\partial\phi_{\vec{r}_n}} \Big|_{\phi=\varphi}$. As an example, let us calculate $\frac{\partial^2 G[\varphi,\vec{r}]}{\partial\varphi_{\vec{r}}\partial\varphi_{\vec{r}'}}$ up to order η . By starting at equation (D.369), $\frac{\partial G[\varphi,\vec{r}]}{\partial\varphi_{\vec{r}'}} = J_{\vec{r}'} = \frac{\partial E[\phi,\vec{r}]}{\partial\phi_{\vec{r}'}} \Big|_{\phi=\varphi} + \frac{\eta}{2} \sum_{\vec{r}_1,\vec{r}'_1} D_{(\varphi)\vec{r}_1,\vec{r}'_1} \partial^3 E_{(\varphi)\vec{r}_1,\vec{r}'_1,\vec{r}'}$, and using the following relation¹³⁴

$$\frac{\partial}{\partial\varphi_{\vec{r}}} \left(D_{(\varphi)\vec{r}_1,\vec{r}'_1} \right) = - \sum_{\vec{r}_2,\vec{r}'_2} D_{(\varphi)\vec{r}_1,\vec{r}_2} D_{(\varphi)\vec{r}'_1,\vec{r}'_2} \frac{\partial^3 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}\partial\phi_{\vec{r}}} \Big|_{\phi=\varphi}, \quad (\text{D.374})$$

one obtains

$$\begin{aligned} \frac{\partial^2 G[\varphi,\vec{r}]}{\partial\varphi_{\vec{r}}\partial\varphi_{\vec{r}'}} &= \frac{\partial^2 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}}\partial\phi_{\vec{r}'}} \Big|_{\phi=\varphi} - \underbrace{\frac{\eta}{2} \sum_{\vec{r}_2,\vec{r}'_2} \sum_{\vec{r}_1,\vec{r}'_1} \frac{\partial^3 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}\partial\phi_{\vec{r}}} \Big|_{\phi=\varphi} \frac{\partial^3 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_1}\partial\phi_{\vec{r}'_1}\partial\phi_{\vec{r}}} \Big|_{\phi=\varphi} D_{(\varphi)\vec{r}_1,\vec{r}_2} D_{(\varphi)\vec{r}'_1,\vec{r}'_2}}_{\text{Diagram: Ellipse with vertices } r, r_2, r_1, r'} \\ &+ \underbrace{\frac{\eta}{2} \sum_{\vec{r}_1,\vec{r}'_1} \frac{\partial^4 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_1}\partial\phi_{\vec{r}'_1}\partial\phi_{\vec{r}}\partial\phi_{\vec{r}'}} \Big|_{\phi=\varphi} D_{(\varphi)\vec{r}_1,\vec{r}'_1}}_{\text{Diagram: Loop with vertices } r, r_1, r', r_1}}. \end{aligned} \quad (\text{D.375})$$

This result can be schematically written as

$$\begin{aligned} \frac{\partial^2 G[\varphi,\vec{r}]}{\partial\varphi_{\vec{r}}\partial\varphi_{\vec{r}'}} &= D_{(\varphi)\vec{r},\vec{r}'}^{-1} + \Sigma_{\vec{r},\vec{r}'} , \\ \text{with } \Sigma_{\vec{r},\vec{r}'} &= -\frac{\eta}{2} \text{Diagram: Ellipse with vertices } r, r_2, r_1, r' + \frac{\eta}{2} \text{Diagram: Loop with vertices } r, r_1, r', r_1 + \underbrace{\dots}_{\text{higher orders in } \eta}. \end{aligned} \quad (\text{D.376})$$

¹³⁴Calculated from equation $\sum_{\vec{r}'_2} \frac{\partial^2 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}} \Big|_{\phi=\varphi} D_{(\varphi)\vec{r}_2,\vec{r}'_1} = \delta_{\vec{r}_2,\vec{r}'_1}$ by derivation $\frac{\partial}{\partial\varphi_{\vec{r}}}$: $\sum_{\vec{r}'_2} \frac{\partial^3 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}\partial\phi_{\vec{r}}} \Big|_{\phi=\varphi} D_{(\varphi)\vec{r}'_2,\vec{r}'_1} + \sum_{\vec{r}'_2} \frac{\partial^2 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}} \Big|_{\phi=\varphi} \frac{\partial}{\partial\varphi_{\vec{r}}} \left(D_{(\varphi)\vec{r}'_2,\vec{r}'_1} \right) = 0$. Multiplying by $D_{(\varphi)\vec{r}_1,\vec{r}_2}$ and summing (integrating) on index $\vec{r}_2$ leads to $-\sum_{\vec{r}'_2} \delta_{\vec{r}_1,\vec{r}'_2} \frac{\partial}{\partial\varphi_{\vec{r}}} \left(D_{(\varphi)\vec{r}'_2,\vec{r}'_1} \right) = \sum_{\vec{r}_2,\vec{r}'_2} D_{(\varphi)\vec{r}_1,\vec{r}_2} D_{(\varphi)\vec{r}'_1,\vec{r}'_2} \frac{\partial^3 E[\phi,\vec{r}]}{\partial\phi_{\vec{r}_2}\partial\phi_{\vec{r}'_2}\partial\phi_{\vec{r}}} \Big|_{\phi=\varphi}$.

According to the notation usual in Quantum Field Theory, $\Sigma_{\vec{r},\vec{r}'} \Big|_{\varphi=0}$ is called the self-mass element.

The meaning of such an element is easily seen when $\frac{\partial^2 G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}} \partial \varphi_{\vec{r}'}}$ is compared to $\frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{r}'}}$, being the latter related to correlation function $\langle \phi_{\vec{r}} \phi_{\vec{r}'} \rangle_{\text{conn}} = -\eta \frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{r}'}} \Big|_{J=0}$ with $\phi_{\vec{r}}$ the field(s) entering in the energy $E[\phi_{\vec{r}}]$ which defines the model. From $\frac{\partial \varphi_{\vec{r}}}{\partial \varphi_{\vec{r}'}} = \delta_{\vec{r},\vec{r}'}$ and relations $\varphi_{\vec{r}} = -\frac{\partial F[J_{\vec{r}}]}{\partial J_{\vec{r}}}$, $J_{\vec{r}'} = \frac{\partial G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}'}}$, it follows

$$-\sum_{\vec{z}} \frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{z}}} \frac{\partial^2 G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{z}} \partial \varphi_{\vec{r}'}} = \delta_{\vec{r},\vec{r}'} \quad (\text{D.377})$$

or equivalently

$$\begin{aligned} \frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{r}'}} &= - \left(\frac{1}{D_{(\varphi)}^{-1} + \Sigma} \right)_{\vec{r},\vec{r}'} = - \left(D_{(\varphi)} \cdot \frac{1}{1 + \Sigma \cdot D_{(\varphi)}} \right)_{\vec{r},\vec{r}'} = \\ &= -D_{(\varphi)}_{\vec{r},\vec{r}'} + \sum_{\vec{z},\vec{z}'} D_{(\varphi)}_{\vec{r},\vec{z}} \left(-\frac{\eta}{2} \text{z} \cdot \text{z}' + \frac{\eta}{2} \text{z} \cdot \text{z}' \right) D_{(\varphi)}_{\vec{z}',\vec{r}'} + \dots, \quad (\text{D.378}) \end{aligned}$$

up to order η . This last equation can be obtained by direct calculation too ¹³⁵.

¹³⁵Let us return to $\varphi_{\vec{r}'} = \phi_{\text{cl}\vec{r}'} - \frac{\eta}{2} \sum_{\vec{z},\vec{r}_1,\vec{r}_1'} D_{\text{cl}\vec{r}_1,\vec{r}_1'} \partial^3 E_{(\varphi)}_{\vec{r}_1,\vec{r}_1',\vec{z}'} D_{\text{cl}\vec{z}',\vec{r}'}$ (equation (D.368)), where at order η propagators $D_{(\varphi)}_{\vec{r}_1,\vec{r}_1'}$ and $D_{\text{cl}\vec{r}_1,\vec{r}_1'}$ can be interchanged. From relations $\frac{\partial \phi_{\text{cl}\vec{r}'}}{\partial J_{\vec{r}}} = D_{\text{cl}\vec{r}',\vec{r}}$ and $\eta \frac{\partial}{\partial J_{\vec{r}}} \left(D_{\text{cl}\vec{r}_1,\vec{r}_1'} \right) =$

$$\begin{aligned} &= \eta \sum_{\vec{z}} \frac{\partial}{\partial \phi_{\text{cl}\vec{z}}} \left(D_{\text{cl}\vec{r}_1,\vec{r}_1'} \right) D_{\text{cl}\vec{z},\vec{r}} \stackrel{\text{order } \eta}{\approx} \eta \sum_{\vec{z}} \frac{\partial D_{(\varphi)}_{\vec{r}_1,\vec{r}_1'}}{\partial \varphi_{\vec{z}}} D_{(\varphi)}_{\vec{z},\vec{r}} \stackrel{(D.374)}{=} -\eta \sum_{\vec{z},\vec{r}_2,\vec{r}_2'} D_{(\varphi)}_{\vec{r}_1,\vec{r}_2} D_{(\varphi)}_{\vec{r}_1',\vec{r}_2'} \partial^3 E_{(\varphi)}_{\vec{r}_2,\vec{r}_2',\vec{z}} D_{(\varphi)}_{\vec{z},\vec{r}}, \\ \frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{r}'}} &= -\frac{\partial \varphi_{\vec{r}'}}{\partial J_{\vec{r}}} \text{ up to order } \eta \text{ becomes } \frac{\partial^2 F[J_{\vec{r}}]}{\partial J_{\vec{r}} \partial J_{\vec{r}'}} = -\frac{\partial \phi_{\text{cl}\vec{r}'}}{\partial J_{\vec{r}}} + \sum_{\vec{r}_1,\vec{r}_1',\vec{z},\vec{z}'} \partial^4 E_{(\varphi)}_{\vec{r}_1,\vec{r}_1',\vec{z},\vec{z}'} D_{(\varphi)}_{\vec{r}_1,\vec{r}_1'} D_{(\varphi)}_{\vec{z},\vec{r}} D_{(\varphi)}_{\vec{z}',\vec{r}'} - \\ &- \frac{\eta}{2} \sum_{\vec{z},\vec{r}_2,\vec{r}_2'} \sum_{\vec{z}',\vec{r}_1,\vec{r}_1'} \partial^3 E_{(\varphi)}_{\vec{r}_2,\vec{r}_2',\vec{z}} \partial^3 E_{(\varphi)}_{\vec{r}_1,\vec{r}_1',\vec{z}'} D_{(\varphi)}_{\vec{r}_1,\vec{r}_2} D_{(\varphi)}_{\vec{r}_1',\vec{r}_2'} D_{(\varphi)}_{\vec{z},\vec{r}} D_{(\varphi)}_{\vec{z}',\vec{r}'} - \\ &- \frac{\eta}{2} \sum_{\vec{z},\vec{r}_1,\vec{r}_1'} \sum_{\vec{z},\vec{r}_2,\vec{r}_2'} D_{(\varphi)}_{\vec{r}_1,\vec{r}_1'} \partial^3 E_{(\varphi)}_{\vec{r}_1,\vec{r}_1',\vec{z}'} \partial^3 E_{(\varphi)}_{\vec{r}_2,\vec{r}_2',\vec{z}} D_{(\varphi)}_{\vec{z}',\vec{r}_2} D_{(\varphi)}_{\vec{r}',\vec{r}_2'} D_{(\varphi)}_{\vec{z},\vec{r}} \equiv \end{aligned}$$

$$\equiv -D_{\text{cl}\vec{r},\vec{r}'} + \frac{\eta}{2} \text{z} \cdot \text{z}' - \frac{\eta}{2} \text{z} \cdot \text{z}' - \frac{\eta}{2} \text{z} \cdot \text{z}' \quad .$$

The underlined term gets cancelled by the underlined term in the expansion up to order η of $-D_{\text{cl}\vec{r},\vec{r}'} = -D_{(\varphi)}_{\vec{r},\vec{r}'} +$

$$\begin{aligned} &+ \sum_{\vec{r}_2,\vec{r}_2',\vec{z}} D_{(\varphi)}_{\vec{r},\vec{r}_2} D_{(\varphi)}_{\vec{r}',\vec{r}_2'} \partial^3 E_{(\varphi)}_{\vec{r}_2,\vec{r}_2',\vec{z}} \underbrace{(\phi_{\text{cl}\vec{z}} - \varphi_{\vec{z}})} \\ &\quad \frac{\eta}{2} \sum_{\vec{r}_1,\vec{r}_1',\vec{z}'} D_{(\varphi)}_{\vec{r}_1,\vec{r}_1'} \partial^3 E_{(\varphi)}_{\vec{r}_1,\vec{r}_1',\vec{z}'} D_{(\varphi)}_{\vec{z},\vec{z}'} \end{aligned}$$

The same steps can be carried out for the higher derivatives of $G[\varphi_{\vec{r}}]$. By taking into account result (D.374), $\frac{\partial}{\partial \varphi_{\vec{r}}} \left(D_{(\varphi) \vec{z}_1, \vec{z}'_1} \right) = - \sum_{\vec{z}_3, \vec{z}'_3} D_{(\varphi) \vec{z}_1, \vec{z}_3} \frac{\partial^3 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{z}_3} \partial \phi_{\vec{z}'_3} \partial \phi_{\vec{r}}} \Big|_{\phi=\varphi} D_{(\varphi) \vec{z}'_3, \vec{z}'_1}$ represented graphically as $\frac{\partial}{\partial \varphi_{\vec{r}}} (z_1 \bullet \text{---} \bullet z'_1) = - z_1 \bullet \text{---} z_3 \bullet \text{---} z'_3 \bullet \text{---} z'_1$, and by starting at equation (D.376), $\frac{\partial^2 G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}_1} \partial \varphi_{\vec{r}_2}} = \frac{\partial^2 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}_2}} \Big|_{\phi=\varphi} - \frac{\eta}{2} r_1 \bullet \text{---} \bullet r_2 + \frac{\eta}{2} \text{---} \text{---} \text{---} + \dots$, one obtains up to order η the following relations

$$\begin{aligned} \frac{\partial^3 G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}_1} \partial \varphi_{\vec{r}_2} \partial \varphi_{\vec{r}_3}} &= \frac{\partial^3 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}_2} \partial \phi_{\vec{r}_3}} \Big|_{\phi=\varphi} - \frac{\eta}{2} \left[\begin{array}{c} r_3 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_1 \end{array} + \begin{array}{c} r_1 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_2 \end{array} + \begin{array}{c} r_2 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_3 \end{array} \right] + \\ &+ \frac{\eta}{2} \left[\begin{array}{c} r_3 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_1 \end{array} + \begin{array}{c} r_2 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_3 \end{array} \right] + \frac{\eta}{2} r_1 \bullet \text{---} \bullet r_2 \text{---} \bullet r_3, \end{aligned} \quad (\text{D.379})$$

and

$$\begin{aligned} \frac{\partial^4 G[\varphi_{\vec{r}}]}{\partial \varphi_{\vec{r}_1} \partial \varphi_{\vec{r}_2} \partial \varphi_{\vec{r}_3} \partial \varphi_{\vec{r}_4}} &= \frac{\partial^4 E[\phi_{\vec{r}}]}{\partial \phi_{\vec{r}_1} \partial \phi_{\vec{r}_2} \partial \phi_{\vec{r}_3} \partial \phi_{\vec{r}_4}} \Big|_{\phi=\varphi} - \frac{\eta}{2} \left[\begin{array}{c} r_4 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_1 \end{array} + \begin{array}{c} r_3 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_2 \end{array} + \begin{array}{c} r_2 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_3 \end{array} + \begin{array}{c} r_1 \\ \bullet \\ \text{---} \bullet \text{---} \bullet \\ r_4 \end{array} \right] - \\ &- \frac{\eta}{2} \left[\begin{array}{c} r_3 \quad r_4 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + \begin{array}{c} r_2 \quad r_4 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + \begin{array}{c} r_2 \quad r_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} \right] + \frac{\eta}{2} \left[\begin{array}{c} r_3 \quad r_4 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + \begin{array}{c} r_4 \quad r_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + 10 \text{ permutations} \right] - \\ &- \frac{\eta}{2} \left[\begin{array}{c} r_3 \quad r_4 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + \begin{array}{c} r_4 \quad r_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ r_1 \end{array} + 4 \text{ permutations} \right] + \frac{\eta}{2} r_1 \bullet \text{---} \bullet r_2 \text{---} \bullet r_3 \text{---} \bullet r_4. \end{aligned} \quad (\text{D.380})$$

Let us denote by $\tilde{\Gamma}^{(n)}(\vec{k}_1, \dots, \vec{k}_n | v)$ the Fourier transform of $\frac{\delta^n G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1) \dots \delta \varphi(\vec{r}_n)} \Big|_{\varphi=v}$, with v a *constant* value. As a final point in this Note, and returning to the continuum notation, let us calculate some of these Fourier transforms starting with $\tilde{\Gamma}^{(2)}(\vec{k}, \vec{k}' | v)$, defined by relation $\tilde{\Gamma}^{(2)}(\vec{k}, \vec{k}' | v) (2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}') =$

$$= \int d^d \vec{r} d^d \vec{r}' \frac{\delta^2 G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}) \delta \varphi(\vec{r}')} \Big|_{\varphi=v} e^{i(\vec{k} \cdot \vec{r} + \vec{k}' \cdot \vec{r}')} . \text{ By means of the following definitions }^{136}$$

$$\frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_n)} \Big|_{\phi=\varphi=v} = \int \frac{d^d \vec{q}_1}{(2\pi)^d} \dots \frac{d^d \vec{q}_n}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_v e^{i(\vec{q}_1 \cdot \vec{r}_1 + \dots + \vec{q}_n \cdot \vec{r}_n)} ,$$

$$D_{(\varphi)}(\vec{r}, \vec{r}') \Big|_{\varphi=v} = \int \frac{d^d \vec{p}}{(2\pi)^d} \tilde{D}(\vec{p}|v) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} ,$$

part $-\frac{\eta}{2} \left[\text{diagram of a loop with vertices } \vec{r}_1, \vec{r}_2, \vec{r}'_1, \vec{r}'_2 \right] \Big|_{\varphi=v}$ of $\frac{\delta^2 G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}) \delta \varphi(\vec{r}')} \Big|_{\varphi=v}$ produces

$$-\frac{\eta}{2} \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{d^d \vec{q}_2}{(2\pi)^d} \frac{d^d \vec{q}'_2}{(2\pi)^d} \frac{d^d \vec{q}'}{(2\pi)^d} \frac{d^d \vec{q}_1}{(2\pi)^d} \frac{d^d \vec{q}'_1}{(2\pi)^d} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}) \delta \tilde{\phi}(\vec{q}_2) \delta \tilde{\phi}(\vec{q}'_2)} \Big|_v \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}') \delta \tilde{\phi}(\vec{q}_1) \delta \tilde{\phi}(\vec{q}'_1)} \Big|_v \times$$

$$\times (2\pi)^d \delta^{(d)}(\vec{q} + \vec{q}_2 + \vec{q}'_2) (2\pi)^d \delta^{(d)}(\vec{q}' + \vec{q}_1 + \vec{q}'_1) \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{d^d \vec{k}'}{(2\pi)^d} \tilde{D}(\vec{p}|v) \tilde{D}(\vec{p}'|v) \times$$

$$\times (2\pi)^d \delta^{(d)}(\vec{q} - \vec{k}) (2\pi)^d \delta^{(d)}(\vec{q}' - \vec{k}') (2\pi)^d \delta^{(d)}(\vec{q}_2 - \vec{p}) (2\pi)^d \delta^{(d)}(\vec{q}'_2 - \vec{p}') (2\pi)^d \delta^{(d)}(\vec{q}_1 + \vec{p}) (2\pi)^d \delta^{(d)}(\vec{q}'_1 + \vec{p}')$$

$$\frac{136}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_v \text{ is an abbreviation for } \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}(\vec{q})=(2\pi)^d \delta^{(d)}(\vec{q})v} , \text{ with } (2\pi)^d \delta^{(d)}(\vec{q})v \text{ the Fourier}$$

transform of $\phi(\vec{r}) = v = \text{const.}$. This is shown by starting at the definition for $v = 0$ $\frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_n)} \Big|_{\phi(\vec{r})=0} =$

$$= \int \frac{d^d \vec{q}_1}{(2\pi)^d} \dots \frac{d^d \vec{q}_n}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}(\vec{q})=0} e^{-i(\vec{q}_1 \cdot \vec{r}_1 + \dots + \vec{q}_n \cdot \vec{r}_n)} ; \quad \frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_n)} \Big|_{\phi(\vec{r})=v} =$$

$$= \frac{\delta^n E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \delta \phi(\vec{r}_n)} \Big|_{\phi(\vec{r})=0} + v \int d^d \vec{r}_{n+1} \frac{\delta^{n+1} E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_n) \delta \phi(\vec{r}_{n+1})} \Big|_{\phi=0} + \frac{v^2}{2} \int d^d \vec{r}_{n+1} d^d \vec{r}_{n+2} \frac{\delta^{n+2} E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \dots \delta \phi(\vec{r}_{n+2})} \Big|_{\phi=0} + \dots =$$

$$= \int \frac{d^d \vec{q}_1}{(2\pi)^d} \dots \frac{d^d \vec{q}_n}{(2\pi)^d} e^{-i(\vec{q}_1 \cdot \vec{r}_1 + \dots + \vec{q}_n \cdot \vec{r}_n)} \left[(2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}=0} + v \int \frac{d^d \vec{q}_{n+1}}{(2\pi)^d} \underbrace{\int d^d \vec{r}_{n+1} e^{-i\vec{q}_{n+1} \cdot \vec{r}_{n+1}} \times}_{(2\pi)^d \delta^{(d)}(\vec{q}_{n+1})} \right.$$

$$\times (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n + \vec{q}_{n+1}) \frac{\delta^{n+1} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n) \delta \tilde{\phi}(\vec{q}_{n+1})} \Big|_{\tilde{\phi}=0} + \frac{v^2}{2} (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) \frac{\delta^{n+2} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0}$$

$$\left. + \dots \right] \equiv \int \frac{d^d \vec{q}_1}{(2\pi)^d} \dots \frac{d^d \vec{q}_n}{(2\pi)^d} e^{-i(\vec{q}_1 \cdot \vec{r}_1 + \dots + \vec{q}_n \cdot \vec{r}_n)} (2\pi)^d \delta^{(d)}(\vec{q}_1 + \dots + \vec{q}_n) \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_v , \quad \text{where one reads off}$$

$$\frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_v = \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}=0} + v \frac{\delta^{n+1} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n) \delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0} + \frac{v^2}{2} \frac{\delta^{n+2} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n) \delta \tilde{\phi}(\vec{0}) \delta \tilde{\phi}(\vec{0})} \Big|_{\tilde{\phi}=0} + \dots \quad \text{This}$$

$$\text{last result is equal to } \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}(\vec{q})=(2\pi)^d \delta^{(d)}(\vec{0})v} = \frac{\delta^n E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n)} \Big|_{\tilde{\phi}=0} + \int \frac{d^d \vec{q}_{n+1}}{(2\pi)^d} \frac{\delta^{n+1} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_n) \delta \tilde{\phi}(\vec{q}_{n+1})} \Big|_{\tilde{\phi}=0} \times$$

$$\times (2\pi)^d \delta^{(d)}(\vec{q}_{n+1})v + \frac{1}{2} \int \frac{d^d \vec{q}_{n+1}}{(2\pi)^d} \frac{d^d \vec{q}_{n+2}}{(2\pi)^d} \frac{\delta^{n+2} E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}_1) \dots \delta \tilde{\phi}(\vec{q}_{n+2})} \Big|_{\tilde{\phi}=0} \left((2\pi)^d \delta^{(d)}(\vec{q}_{n+1})v (2\pi)^d \delta^{(d)}(\vec{q}_{n+2})v \right)^2 + \dots$$

whereas part $+\frac{\eta}{2} \left[\text{diagram} \right]_{\rho=v}$ gives

$$+\frac{\eta}{2} \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{d^d \vec{q}'}{(2\pi)^d} \frac{d^d \vec{q}_1}{(2\pi)^d} \frac{d^d \vec{q}_1'}{(2\pi)^d} \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{q}) \delta \tilde{\phi}(\vec{q}') \delta \tilde{\phi}(\vec{q}_1) \delta \tilde{\phi}(\vec{q}_1')} \Big|_{\vec{q}=\vec{v}} (2\pi)^d \delta^{(d)}(\vec{q} + \vec{q}' + \vec{q}_1 + \vec{q}_1') \tilde{D}(\vec{p}|v) \times$$

$$\times (2\pi)^d \delta^{(d)}(\vec{q} - \vec{k})(2\pi)^d \delta^{(d)}(\vec{q}' - \vec{k}')(2\pi)^d \delta^{(d)}(\vec{q}_1 - \vec{p})(2\pi)^d \delta^{(d)}(\vec{q}_1' + \vec{p}).$$

Hence, up to order η , $\tilde{\Gamma}^{(2)}(\vec{k}, \vec{k}'|v)$ is given by

$$(2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}') \tilde{\Gamma}^{(2)}(\vec{k}, \vec{k}'|v) = (2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}') \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{k}')} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} - \quad (\text{D.381})$$

$$-(2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}') \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(\vec{k}' - \vec{p})} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}') \delta \tilde{\phi}(-\vec{p}) \delta \tilde{\phi}(\vec{p} - \vec{k}')} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \tilde{D}(\vec{p}|v) \tilde{D}(\vec{k}' - \vec{p}|v) +$$

$$+(2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}') \frac{\eta}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{k}') \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \tilde{D}(\vec{p}|v),$$

with v a constant value. Or diagrammatically, always remembering constraint $\vec{k} = -\vec{k}'$ from $\delta^{(d)}(\vec{k} + \vec{k}')$

$$\tilde{\Gamma}^{(2)}(\vec{k}, \vec{k}'|v) = \frac{\delta^2 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{k}')} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} - \frac{\eta}{2} \left[\text{diagram} \right]_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} + \frac{\eta}{2} \left[\text{diagram} \right]_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})}, \quad (\text{D.382})$$

with vertices $\left[\text{diagram} \right]_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \equiv \frac{\delta^3 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(\vec{k}' - \vec{p})} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})}$, $\left[\text{diagram} \right]_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \equiv \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}) \delta \tilde{\phi}(\vec{k}') \delta \tilde{\phi}(\vec{p}) \delta \tilde{\phi}(-\vec{p})} \Big|_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})}$
 and propagator $\left[\text{diagram} \right]_{\vec{q}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} \equiv \tilde{D}(\vec{p}|v)$. And always carrying out the integration over all $\vec{p}$ -vectors on the internal lines.

In a similar way, one can calculate $\tilde{\Gamma}^{(1)}(\vec{k}_1|v)$, $\tilde{\Gamma}^{(3)}(\vec{k}_1, \vec{k}_2, \vec{k}_3|v)$ and $\tilde{\Gamma}^{(4)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4|v)$, Fourier transforms to $\frac{\delta G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1)} \Big|_{\vec{r}=\vec{v}}$, $\frac{\delta^3 G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1) \delta \varphi(\vec{r}_2) \delta \varphi(\vec{r}_3)} \Big|_{\vec{r}=\vec{v}}$ and $\frac{\delta^4 G[\varphi(\vec{r})]}{\delta \varphi(\vec{r}_1) \delta \varphi(\vec{r}_2) \delta \varphi(\vec{r}_3) \delta \varphi(\vec{r}_4)} \Big|_{\vec{r}=\vec{v}}$ respectively: for

instance, element $\left[\text{diagram} \right]_{\vec{r}=\vec{v}}$ becomes

$$\int d^d \vec{z}_1 \dots d^d \vec{z}_2 \left(\frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_1) \delta \phi(\vec{r}_2) \delta \phi(\vec{z}_1) \delta \phi(\vec{z}_2)} \Big|_{\vec{r}=\vec{v}} D_{(\varphi)}(\vec{z}_1, \vec{z}_1') D_{(\varphi)}(\vec{z}_2, \vec{z}_2') \frac{\delta^4 E[\phi(\vec{r})]}{\delta \phi(\vec{r}_3) \delta \phi(\vec{r}_4) \delta \phi(\vec{z}_1') \delta \phi(\vec{z}_2')} \Big|_{\vec{r}=\vec{v}} \right) =$$

$$= \int \frac{d^d \vec{k}_1}{(2\pi)^d} \frac{d^d \vec{k}_2}{(2\pi)^d} \frac{d^d \vec{k}_3}{(2\pi)^d} \frac{d^d \vec{k}_4}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) e^{-i(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 + \vec{k}_3 \cdot \vec{r}_3 + \vec{k}_4 \cdot \vec{r}_4)} \left[\text{diagram} \right]_{\vec{r}=\vec{v}}.$$

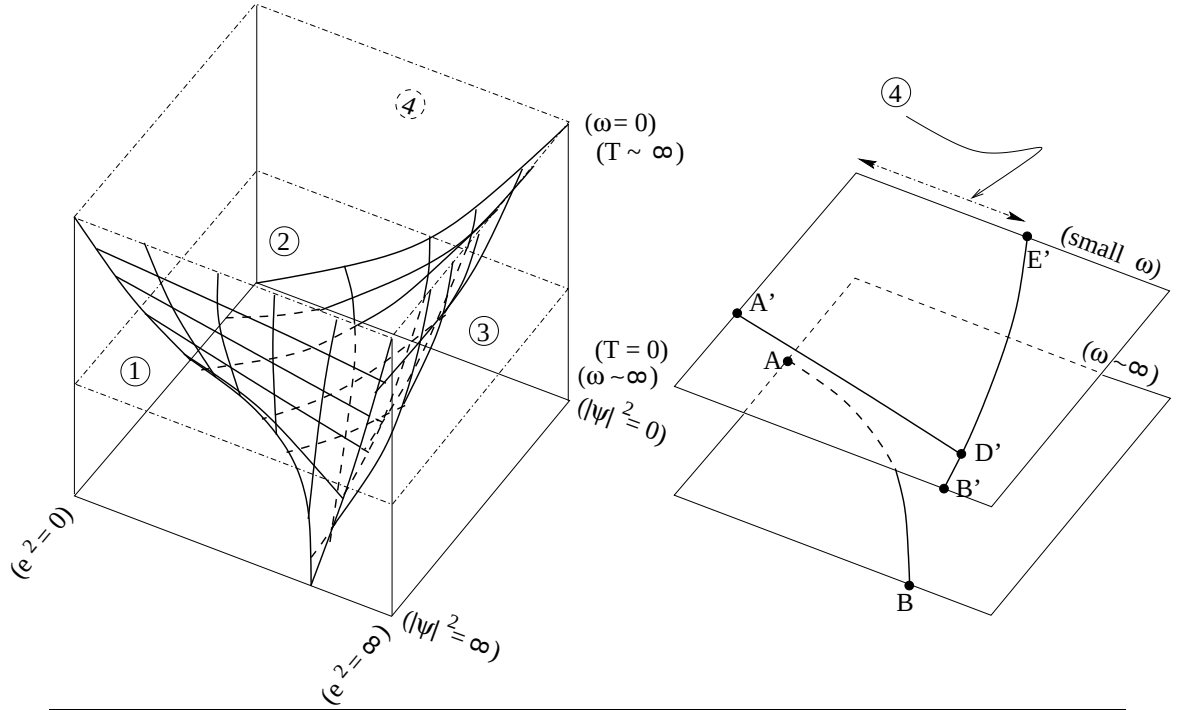
Hence Fourier transform $\tilde{\Gamma}^{(1)}(\vec{k}_1|v)$, according to equation (D.369), takes the following form

$$\tilde{\Gamma}^{(1)}(\vec{k}_1|v) = \frac{\delta E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}_1)} \Big|_{\tilde{\phi}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})} + \frac{\eta}{2} \text{p} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \text{p} \Big|_{\tilde{\phi}(\vec{q})=v(2\pi)^d \delta^{(d)}(\vec{q})}, \quad (\text{D.383})$$

and $\tilde{\Gamma}^{(4)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4|v)$ is given by

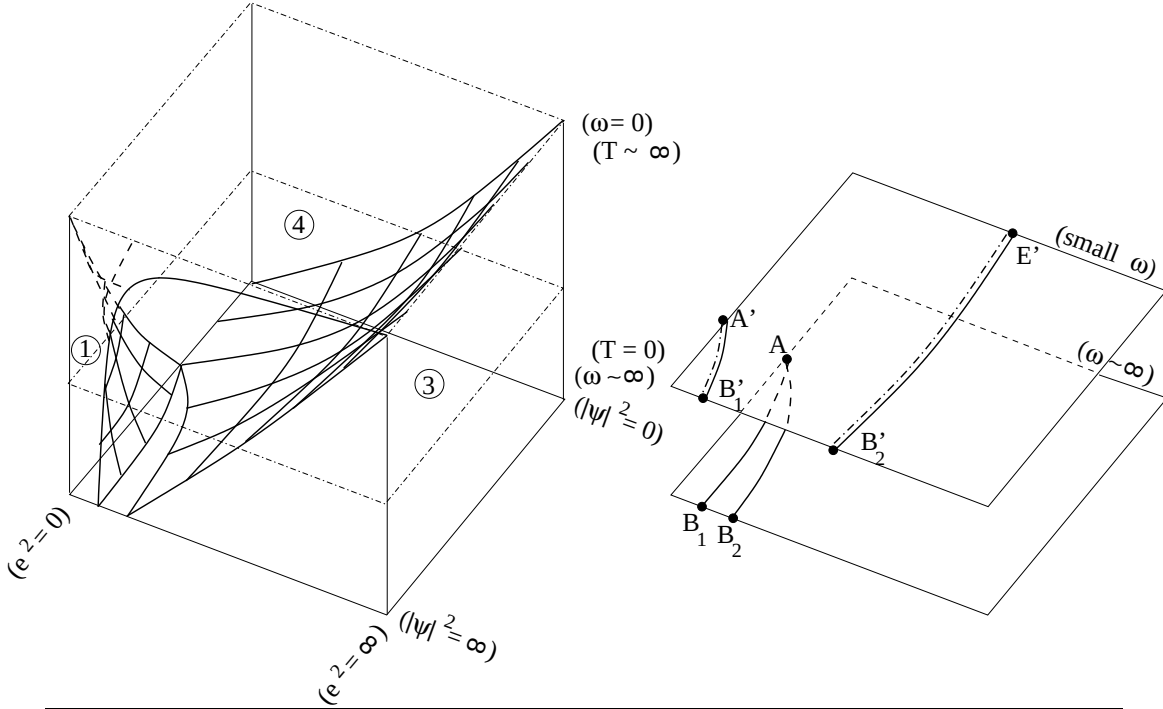
$$\begin{aligned}
\tilde{\Gamma}^{(4)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4|v) = & \frac{\delta^4 E[\tilde{\phi}(\vec{q})]}{\delta \tilde{\phi}(\vec{k}_1) \delta \tilde{\phi}(\vec{k}_2) \delta \tilde{\phi}(\vec{k}_3) \delta \tilde{\phi}(\vec{k}_4)} \Big|_v - \frac{\eta}{2} \left[\text{Diagram 1} + 3 \text{ permutations} \right] - \\
& - \frac{\eta}{2} \left[\text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right] + \\
& + \frac{\eta}{2} \left[\text{Diagram 5} + 11 \text{ permut.} \right] - \frac{\eta}{2} \left[\text{Diagram 6} + 5 \text{ permut.} \right] + \\
& + \frac{\eta}{2} \left[\text{Diagram 7} \right],
\end{aligned}
\tag{D.384}$$

plus constraint $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = \vec{0}$ and without forgetting that v is a constant value.



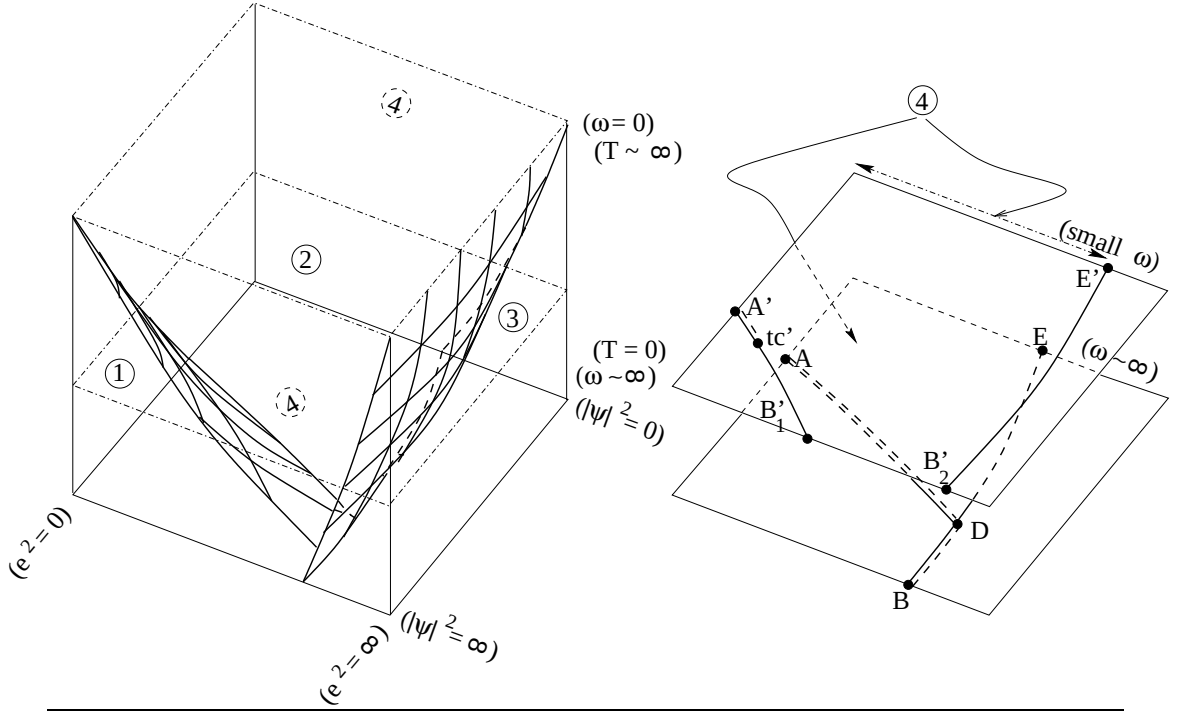
Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
②	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim \text{const.} \end{cases}$
④ (for $ \psi ^2 = 0$)	$\begin{cases} W(C) \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	

Figure D.2: Expected phase diagram for the 3-dimensional compact Higgs model with $q = 2$ showing the dependence on the periodic lattice's length from $\omega = \infty$ (no periodicity) to a very small ω . Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2}{\pi} \frac{a}{\omega}$, point B' by $ae^2\eta = 2.268 \frac{a}{\omega}$ and point E' by $ae^2\eta = \frac{\pi}{2} \frac{a}{\omega}$. Points A and B at $\omega = \infty$ are equal to those in Fig. C.5, page 448: $\frac{a|\psi|^2}{\eta} = 0.322$ and $ae^2\eta = 1.314$ respectively. All lines are lines of second-order phase transitions. A small “pocket” of the phase restricted, at small ω , to $|\psi|^2 = 0$ and small e^2 may exist also for small $|\psi|^2 \neq 0$ for a ω large enough.



Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-P_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim \text{const.} \end{cases}$
④	$\begin{cases} W(C) \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{-\ln \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a^*, \vec{r}_b^*) \sim e^{1/ \vec{r}_b^* - \vec{r}_a^* } \end{cases}$

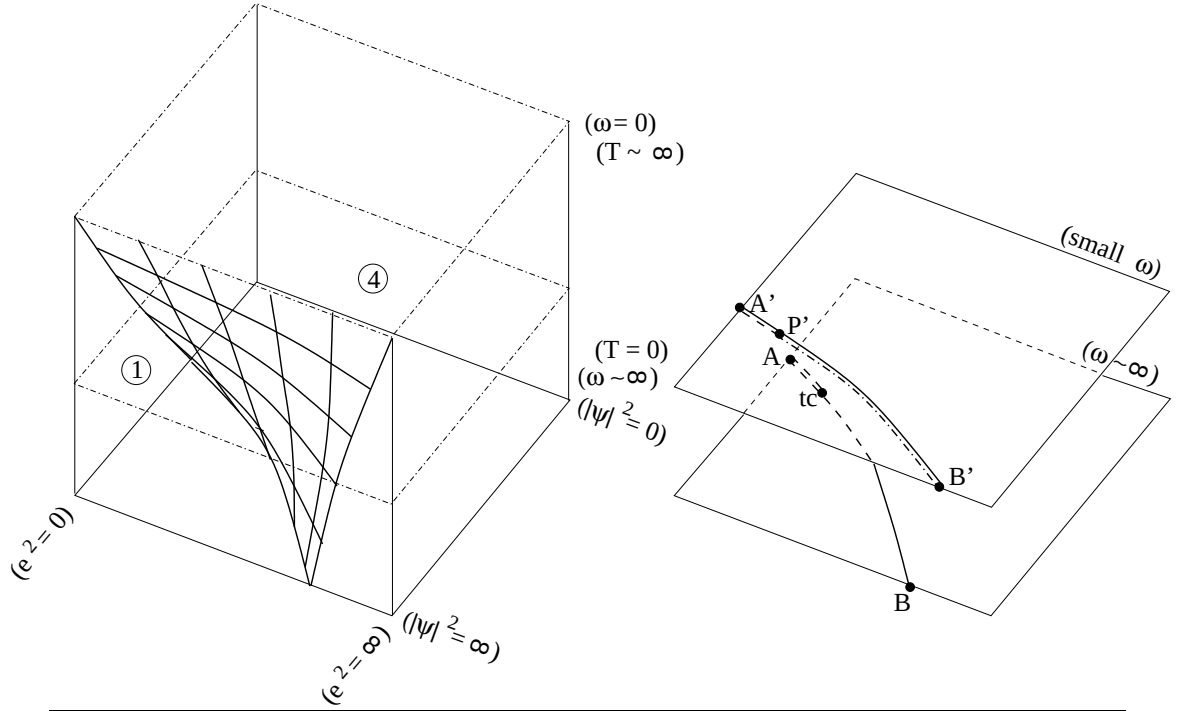
Figure D.3: Expected phase diagram for the 3-dimensional compact Higgs model at large q showing the dependence on the periodic lattice's length from $\omega = \infty$ (no periodicity) to a very small ω . Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2}{\pi} \frac{a}{\omega}$ and point E' by $ae^2\eta = \frac{\pi}{2} \frac{a}{\omega}$. Point B'_2 tends toward the value of E' in the limit $q \rightarrow \infty$. Both point B'_1 and point B_1 decreases with increasing q like $1/q^2$; transition point B_2 tends to $e^2 = 0$ too, since the 3-dimensional $U(1)$ gauge model displays no phase transition points at $\omega = \infty$. Point A is equal to $\frac{a|\psi|^2}{\eta} = 0.322$. Both solid lines at $\omega = \infty$ are expected to be formed by second-order transition points, whilst the dashed-solid lines (small ω) are formed by transition points of Kosterlitz-Thouless type.



Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C} \\ H(C^*) \sim e^{-\mathcal{A}_{C^*}} \end{cases}$
②	$\begin{cases} W(C) \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	
③	$\begin{cases} W(C) \sim e^{- \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim \text{const.} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{A}_C} \\ H(C^*) \sim e^{-\mathcal{P}_{C^*}} \end{cases}$
④	$\begin{cases} W(C) \sim e^{ \vec{x}_2 - \vec{x}_1 ^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) \sim e^{ \vec{x}_b^* - \vec{x}_a^* ^{-1}} \end{cases}$	$\begin{cases} W(C) \sim e^{-\mathcal{P}_C + \tau/ \vec{x}_2 - \vec{x}_1 } \\ H(C^*) \sim e^{-\mathcal{P}_{C^*} + \tau^*/ \vec{x}_b^* - \vec{x}_a^* } \end{cases}$

Figure D.4: Expected phase diagram for the 4-dimensional compact Higgs model with $q = 2$ showing the dependence on the periodic lattice's length from $\omega = \infty$ (no periodicity) to a very small ω . Point A' is given by $\frac{a^2|\psi|^2}{\eta} \approx 0.526 \frac{a^2}{\omega^2}$, point B'_1 by $e^2\eta = 2.240 \frac{\omega}{a}$, point B'_2 by $e^2\eta = 4.405 \frac{a}{\omega}$, and point E' by $e^2\eta = 3.106 \frac{a}{\omega}$. Points A , B and E at $\omega = \infty$ are equal to those

in the right part of Fig. C.10 on page 481: $\frac{a^2|\psi|^2}{\eta} = 0.526$, $e^2\eta = 2.268$ and $e^2\eta = 1.572$ respectively. Solid lines at small ω are lines of second-order phase transitions, and point tc' denotes the tricritical point separating the first-order-transition segment which starts at A' from the rest of the line of second-order transitions. Dashed-solid (and dashed-dashed) line ADB is a line of first-order phase transitions whilst line ED is of second-order type; point D is another tricritical point. For $\omega = \infty$ in the table above, τ and τ^* denote respectively the length of C and C^* along the, in such a case non-periodic, a_τ -coordinate. For ω large enough one can assume the existence of a small “pocket” connecting the phase restricted, at small ω , to $|\psi|^2 = 0$ and small e^2 , with the same phase occurring at $\omega = \infty$ for a wide range of small e^2 and $|\psi|^2$.

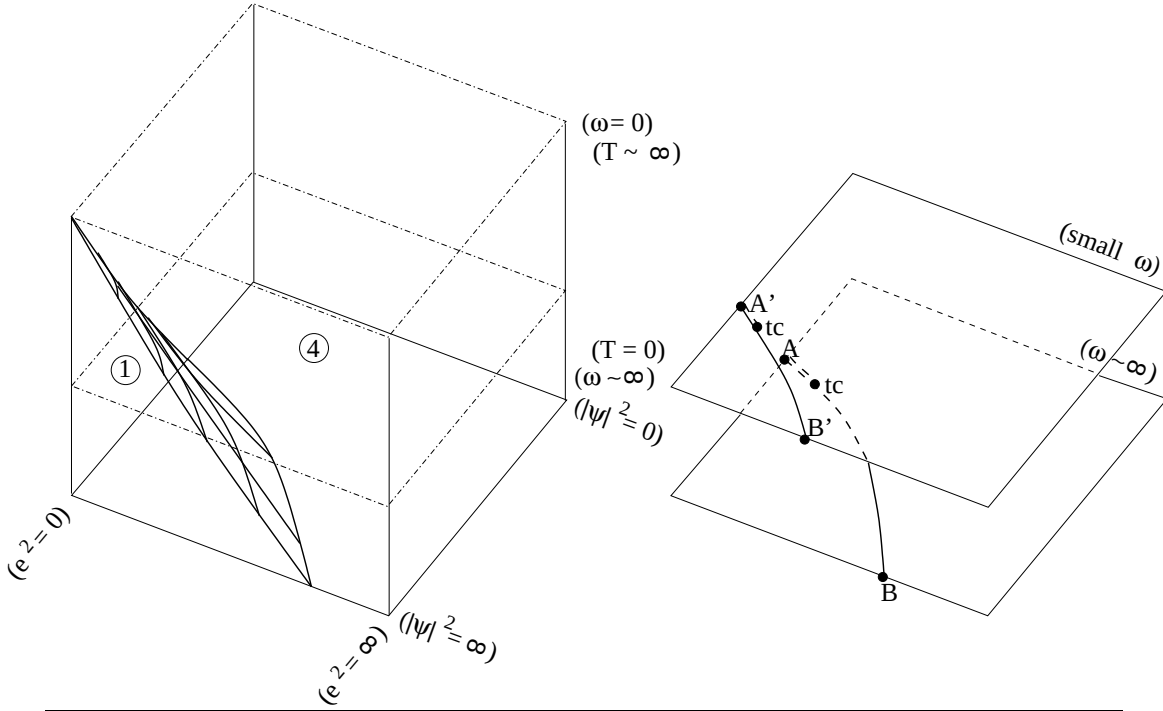


Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) & \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-p_C} \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{- \vec{r}_b^* - \vec{r}_a^* } \end{cases}$
④	$\begin{cases} W(C) & \sim e^{-\ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{-\ln \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-\tau \ln \vec{x}_2 - \vec{x}_1 } \\ H(\vec{r}_a^*, \vec{r}_b^*) & \sim e^{1/ \vec{r}_b^* - \vec{r}_a^* } \end{cases}$

Figure D.6: Expected phase diagram for the 3-dimensional non-compact Higgs model at any q showing the dependence on the periodic lattice's length from $\omega = \infty$ (no periodicity) to a very small ω . Point A' is given by $\frac{a|\psi|^2}{\eta} = \frac{2}{\pi} \frac{a}{\omega}$ and point B' by $ae^2\eta = \frac{8\pi}{q^2} \frac{a}{\omega}$. Points A and B at $\omega = \infty$ are equal to those in the

left part of Fig. C.8 on page 465: $\frac{a|\psi|^2}{\eta} = 0.322$ and $ae^2\eta = \frac{12.71}{q^2}$ respectively.

Dashed-solid line $P'B'$ at small ω is formed by Kosterlitz-Thouless transition points; short segment $A'P'$ is expected to be of first-order type. At $\omega = \infty$, except for the short segment represented by a dashed-dashed line which starts at A and finishes at the tricritical point tc , segment where the phase transitions are of first-order kind, the rest of line AB is of second-order type.



Phase	small ω	$\omega = \infty$
①	$\begin{cases} W(C) & \sim \text{const.} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{- \vec{x}_b^* - \vec{x}_a^* } \end{cases}$	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C} \\ H(C^*) & \sim e^{-\mathcal{A}_{C^*}} \end{cases}$
④	$\begin{cases} W(C) & \sim e^{ \vec{x}_2 - \vec{x}_1 ^{-1}} \\ H(\vec{x}_a^*, \vec{x}_b^*) & \sim e^{ \vec{x}_b^* - \vec{x}_a^* ^{-1}} \end{cases}$	$\begin{cases} W(C) & \sim e^{-\mathcal{P}_C + \tau/ \vec{x}_2 - \vec{x}_1 } \\ H(C^*) & \sim e^{-\mathcal{P}_{C^*} + \tau^*/ \vec{x}_b^* - \vec{x}_a^* } \end{cases}$

Figure D.7: Expected phase diagram for the 4-dimensional non-compact Higgs model at any q showing the dependence on the periodic lattice's length from $\omega = \infty$ (no periodicity) to a very small ω . Point A' is given by $\frac{a^2|\psi|^2}{\eta} = 0.526 \frac{a^2}{\omega^2}$ and point B' by $e^2\eta = \frac{12.71}{q^2} \frac{\omega}{a}$. Points A and B at $\omega = \infty$ are equal to those in the left part of Fig. C.9 on page 477: $\frac{a^2|\psi|^2}{\eta} = 0.526$ and $e^2\eta = \frac{25.11}{q^2}$ respectively. Except for the short segments represented by the solid-dashed line (at small ω) or by a dashed-dashed line (at $\omega = \infty$) which start at A' or A and finish at a tricritical point tc , segments where the phase transitions are of first-order kind, all the other lines are expected to be formed by second-order critical points. Tricritical point tc tends towards A' with decreasing ω .

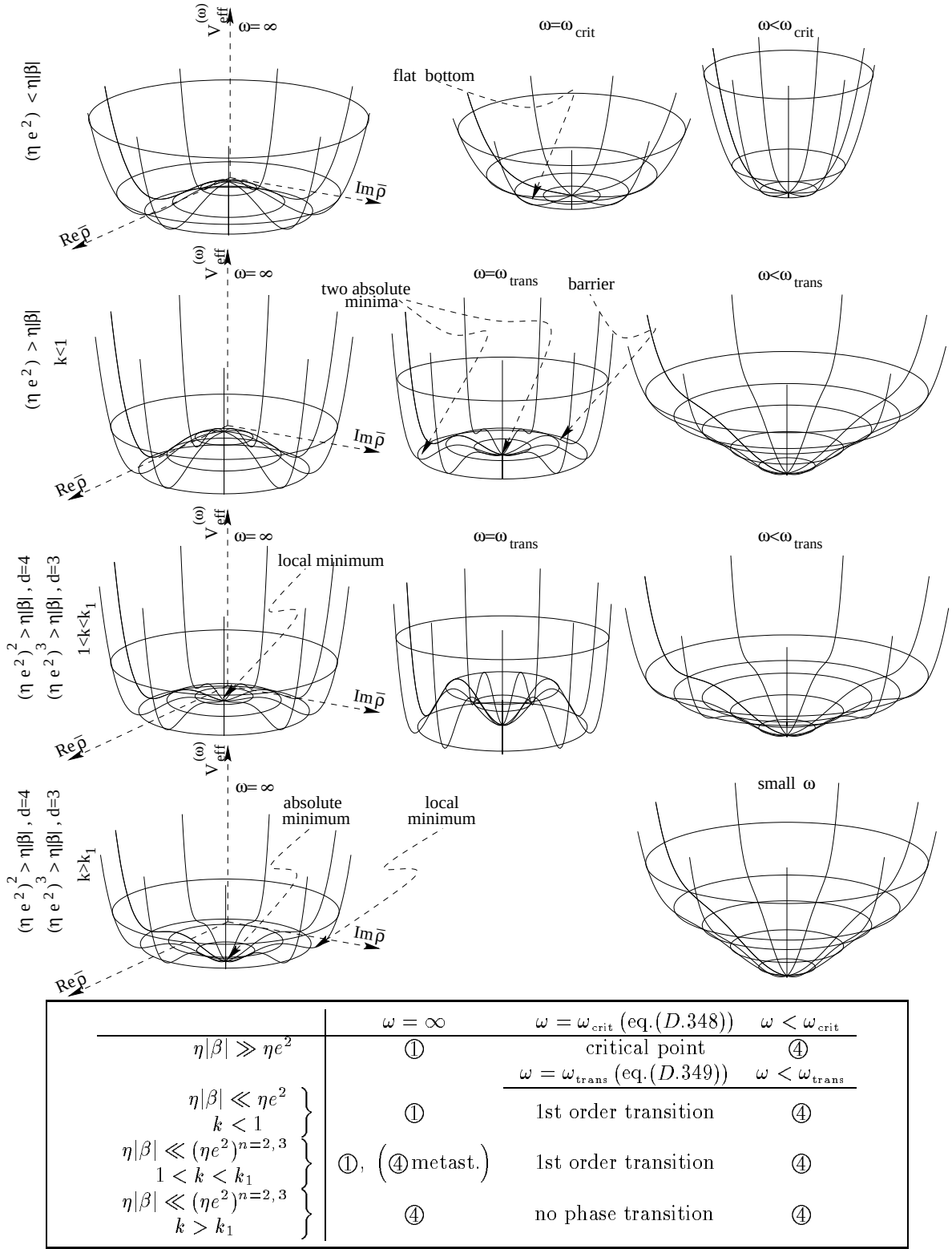


Figure D.8: Expected phases of the effective potential up to order η for the d -dim. Abrikosov model with $d = 3, 4$. Superconducting phase ① gets characterized by symmetry breaking, by a nonvanishing expectation value of the effective potential's scalar field; phase ④ denotes the normal conducting (or Coulomb) state and corresponds to a zero expectation value $\bar{\rho}^2 = 0$ as lowest minimum of the effective potential. Value k_1 is defined by $k = \frac{11(d-1)}{96\pi^2} \frac{(\eta e^2)^2}{\eta|\beta|_{\text{ren}}} = \frac{22}{19}$ for $d = 4$ and by

$$k = \frac{(d-1)}{8\pi\sqrt{|\alpha|_{\text{ren}}}} \sqrt{\frac{(\eta e^2)^3}{\eta|\beta|_{\text{ren}}}} = \frac{3}{2} \text{ for } d = 3.$$

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 Numbers in *italic* refer to footnotes.

Erklärung

Ich versichere, laut der geltenden Promotionsordnung des Fachbereichs Physik der Universität Paderborn, die Dissertation “*The Superconductor as Model for the Hardening Stages in a Dislocated Crystal*” selbständig verfaßt zu haben. Die verwendeten Hilfsmittel habe ich vollständig zitiert und diese Doktorarbeit ist nur bei dem Fachbereich Physik der Universität Paderborn vorgelegt worden.

Logroño (Spanien), den 9. Januar 2002

José-Luis Marqués-López