Spaces of continuous and holomorphic functions with growth conditions

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Dedicated to my father...

...who believed in me, whatever I did.

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1 Weighted (PLB)-spaces of continuous functions

1.1 Introduction to part 1

In the first chapter of this work we investigate the weighted (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ of continuous functions, i.e. for a double sequence $\mathcal{A} := ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) with $a_{n,k+1}(x) \leq a_{n,k}(x) \leq a_{n+1,k}(x)\ \forall\ n,k \in \mathbb{N}, x \in X$, we form the projective limit (with respect to n) of the inductive limits (with respect to k) of the weighted spaces of continuous functions $Ca_{n,k}(X)$ and $C(a_{n,k})_0(X)$, respectively.

Weighted spaces of continuous functions were introduced by Nachbin ([33], [34], [35]). Inductive limits of weighted spaces of continuous and holomorphic functions were studied by Bierstedt, Meise [12] in 1976. In 1982 Bierstedt, Meise, Summers [14] investigated the projective description of weighted inductive limits. They showed that the weighted inductive limit $\mathcal{V}_0C(X)$ is always a topological subspace of its projective hull $C\overline{V}_0(X)$ and that in the (LB)-case $\mathcal{V}_0C(X)$ is complete if and only if $\mathcal{V}_0C(X) = C\overline{V}_0(X)$ holds algebraically (and topologically) if and only if the sequence $\mathcal{V}$ is regularly decreasing. For O-growth conditions Bierstedt, Bonet [6] and Bastin [2] gave a similar result in 1989: For a locally compact and σ-compact space X the (LB)-space $\mathcal{V}C(X)$ equals $C\overline{V}(X)$ topologically (and algebraically) if and only if the sequence $\mathcal{V}$ satisfies condition (D) (compare section 1.4.3).

The more complicated case of weighted (LF)-spaces of continuous functions was investigated by Bierstedt, Bonet [8] in 1994. For weighted (LF)-spaces of continuous functions they used the conditions (Q) and (wQ) of Vogt to obtain results for the projective description.

In the following chapter we investigate the weighted (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ of continuous functions for the first time. We will analyse their topological structures and for o-growth conditions we can characterise when the (PLB)-space $\mathcal{A}_0C(X)$ and the (LF)-space $\mathcal{V}_0C(X)$ are equal algebraically and topologically.

In section 1.2 we give the necessary notations and definitions for the first chapter. In section 1.3 we collect some properties of the spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ which follow from the general theory of Banach spaces and of their inductive and projective limits. After that we recall in section 1.4 several results for general (LF)-spaces: In 1.4.1, we give the definitions of acyclic and weakly acyclic (LF)-spaces in the sense of Palamodov [36], recall the characterisation of Retakh [38] and present the characterisation of Vogt [44] with the conditions (Q) and (wQ) as well as some results of Wengenroth [46]. Next, in 1.4.2 we introduce the conditions (Q) and (wQ) of Vogt in the way Bierstedt, Bonet [8] used them to investigate weighted (LF)-spaces of continuous functions. They reformulated them in terms of the weights, introduced a condition (wQ^*) which is equivalent to (wQ), and constructed many examples of which we present some here, too. At the end of this section in 1.4.3 we collect the main results for the weighted (LB)- and (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ and their projective description which were proved by Bastin [2], Bierstedt, Bonet [8] and Bierstedt, Meise, Summers [14]. Then we present the general theory of projective spectra of (DF)-spaces in section 1.5. These results go back to Palamodov, Retakh, Vogt and Wengenroth ([36], [38], [42], [43], [45]).

After all this we are finally able to investigate the structure of the weighted (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$. In 1.6.1 we show for the weighted (PLB)-spaces $\mathcal{A}_0C(X)$ that condition (wQ) is equivalent to $\mathrm{Proj}^1 A_0 = 0$ and that it is also equivalent to $\mathcal{A}_0C(X)$ ultrabornological (theorem 1.48). Furthermore we prove that the projective spectrum A_0 is of strong P-type if and only if condition (Q) is satisfied (theorem 1.49). In the case of O-growth conditions we show in section 1.6.2 that $\mathrm{Proj}^1 A = 0$ if and only if condition (Q) is satisfied (theorem 1.52) if and only if the projective spectrum A is of strong P-type (remark 1.53).

Next follow the inductive description for weighted Fréchet spaces in section 1.7.1 and the inductive description in the case of weighted (PLB)-spaces in section 1.7.2. In both cases we prove that for a locally compact and σ-compact space X the spaces $C\underline{A}(X)$ and $CA(X)$ resp. $\mathcal{A}C(X)$ are equal algebraically (theorem 1.55 and 1.56). In the case of O-growth conditions it was not possible to give a similar characterisation as in theorem 1.48. But the inductive description allows us to conclude that from $\mathcal{A}C(X)$ barrelled it follows that condtion (wQ) is satisfied (see corollary 1.60 and remark 1.61).

In section 1.8 we compare the weighted (PLB)-spaces of continuous functions with the weighted (LF)-spaces of continuous functions. First we give an example which shows that these spaces are not equal in general. Then we introduce condition (B) of Vogt and prove that $\mathcal{A}C(X) = \mathcal{V}C(X)$ holds algebraically if and only if condition (B) is satisfied (theorem 1.65). To prove the same in the case of o-growth condtions we need the additional condition that $(\mathcal{A}_n)_0C(X)$ is complete for each $n \in \mathbb{N}$ (theorem 1.66). If all $(\mathcal{A}_n)_0C(X)$ are complete we can even prove that $\mathcal{A}_0C(X) = \mathcal{V}_0C(X)$ holds algebraically and topologically if and only if the conditions (B) and (wQ) are satisfied (corollary 1.67).

We finish this chapter in section 1.9 with an example in the case of sequence spaces which illustrates the results given above.

1.2 Notation and definitions

Weighted (PLB)-spaces of continuous functions are defined by a double sequence of weights as the projective limit of the inductive limit of the single weighted Banach spaces of continuous functions. In this work these spaces are investigated for the first time. Taking the limits the other way round one gets the (LF)-spaces of continuous functions which were investigated by Bierstedt, Bonet [8]. From now on let X be a locally compact space and $\mathcal{A} := ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ a double sequence of strictly positive continuous functions on X, called weights, which is decreasing in k and increasing in n, i.e. for each $n, k \in \mathbb{N}$ and $x \in X$

$$a_{n,k+1}(x) \leq a_{n,k}(x) \leq a_{n+1,k}(x)$$

holds. Define the weighted spaces of continuous functions

$$\begin{aligned} Ca_{n,k}(X) &:= \{f \in C(X)\ ;\ ||f||_{n,k} := \sup_{x\in X} a_{n,k}(x)|f(x)| < \infty\}, \\ C(a_{n,k})_0(X) &:= \{f \in C(X)\ ;\ a_{n,k}|f| \text{ vanishes at infinity on } X\} \end{aligned}$$

with continuous inclusions

$$Ca_{n,k}(X) \to Ca_{n,k+1}(X) \text{ and } C(a_{n,k})_0(X) \to C(a_{n,k+1})_0(X)$$

for fixed $n \in \mathbb{N}$.

$C(a_{n,k})_0(X)$ is a closed subspace of $Ca_{n,k}(X)$, and both spaces are complete, hence Banach spaces, where $C(a_{n,k})_0(X)$ carries the induced norm. The unit balls are

$$\begin{aligned} B_{n,k} &:= \{f \in Ca_{n,k}(X); a_{n,k}|f| \leq 1\}, \\ (B_{n,k})_0 &:= \{f \in C(a_{n,k})_0(X); a_{n,k}|f| \leq 1\}. \end{aligned}$$

We form the locally convex inductive limits

$$\begin{aligned} \mathcal{A}_n C(X) &:= \text{ind}_k Ca_{n,k}(X), \\ (\mathcal{A}_n)_0 C(X) &:= \text{ind}_k C(a_{n,k})_0(X) \end{aligned}$$

with $\mathcal{A}_{n+1}C(X) \subset \mathcal{A}_n C(X)$ and $(\mathcal{A}_{n+1})_0 C(X) \subset (\mathcal{A}_n)_0 C(X)$ and the projective limits

$$\begin{aligned} \mathcal{A}C(X) &:= \text{proj}_n \mathcal{A}_n C(X) = \text{proj}_n \text{ind}_k Ca_{n,k}(X), \\ \mathcal{A}_0 C(X) &:= \text{proj}_n (\mathcal{A}_n)_0 C(X) = \text{proj}_n \text{ind}_k C(a_{n,k})_0(X). \end{aligned}$$

1.3 Consequences of known results

In this section we collect known results for the weighted (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ which follow from the general theory of Banach resp. Fréchet spaces and their countable inductive and projective limits. Every Fréchet space F as well as every Banach space E is *webbed* [32], which means that there exists a family $C_{n_1,...,n_k}, n_1, ..., n_k \in \mathbb{N}, k \in \mathbb{N}$, of absolutely convex subsets of F with the following properties:

i) $\cup_{n=1}^{\infty} C_n = F$.

ii) $\cup_{n=1}^{\infty} C_{n_1,...,n_k,n} = C_{n_1,...,n_k}$ for all $n_1, ..., n_k \in \mathbb{N}$ and all $k \in \mathbb{N}$.

iii) For each sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $]0, \infty[$, so that for every sequence $(x_k)_{k\in\mathbb{N}}$ in F with $x_k \in C_{n_1,...,n_k}$ for all $k \in \mathbb{N}$ the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in F.

Remark 1.1. The spaces $C(a_{n,k})(X)$ and $C(a_{n,k})_0(X)$ are webbed for each $k, n \in \mathbb{N}$.

To show that $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ are webbed we need some results of de Wilde [47]:

Theorem 1.2. *i) A countable inductive limit of webbed spaces is webbed.*

ii) A countable projective limit of webbed spaces is webbed.

As a consequence of theorem 1.2 we get:

Corollary 1.3. *The spaces $\mathcal{A}_nC(X)$ and $(\mathcal{A}_n)_0C(X)$ are webbed for each $n \in \mathbb{N}$. The spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ are webbed.*

Fréchet and Banach spaces are ultrabornological (and hence barrelled) [32].

Remark 1.4. The spaces $C(a_{n,k})_0(X)$ and $C(a_{n,k})(X)$ are ultrabornological (and hence barrelled) for each $n, k \in \mathbb{N}$.

Theorem 1.5. *(Meise, Vogt [32]) Let the locally convex space E carry the inductive topology of the system $(j_i : E_i \to E)_{i\in I}$. If all the spaces E_i are barrelled or ultrabornological, then E has the corresponding property, too.*

Corollary 1.6. *The spaces $(\mathcal{A}_n)C(X)$ and $(\mathcal{A}_n)_0C(X)$ are ultrabornological (and hence barrelled) for each $n \in \mathbb{N}$.*

In general the countable projective limit of barrelled resp. ultrabornological spaces need not be barrelled resp. ultrabornological. Conditions under which $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ are barrelled resp. ultrabornological will be discussed later.

1.4 The (LF)-case

Palamodov [36] investigated acyclic and weakly acyclic (LF)-spaces with homological tools. Retakh [38] later called them (LF)-spaces of type (M) and (M_0). In 1992 Vogt investigated these spaces with more functional analytic tools and reformulated the conditions of Retakh. Vogt introduced the conditions (Q) and (wQ) as necessary conditions for acyclic and weakly acyclic (LF)-spaces. In this section we collect some of these results to compare them with the (PLB)-spaces and with the results we will obtain here.

1.4.1 General results for (LF)-spaces

Let E be an (LF)-space. This means here that there is an increasing sequence $(E_k)_{k\in\mathbb{N}}$ of subspaces of E with continuous imbeddings, each E_k is equipped with a Fréchet space topology, and $E := \operatorname{ind}_k E_k$. For each $k \in \mathbb{N}$ there is a fundamental system $(||\ ||_{k,n})_n$ of seminorms in E_k and we assume that

$$||\ ||_{k+1,n} \leq ||\ ||_{k,n} \leq ||\ ||_{k,n+1}$$

holds for each $k, n \in \mathbb{N}$. Identifying $\oplus_k E_k$ with the set $\{x = (x_k)_k \in \prod_{k\in\mathbb{N}} E_k; x_k = 0 \text{ up to finitely many } k\}$ we define the map $q : \oplus_k E_k \to E$ by $q(x) = \sum_{k\in\mathbb{N}} x_k$. $\sigma : \oplus_{k\in\mathbb{N}} E_k \to \oplus_{k\in\mathbb{N}} E_k$, defined by $\sigma(x) = (x_k - x_{k-1})_k,\ x_{-1} = 0$, is an isomorphism onto the kernel of q. Hence we have the canonical exact sequence

$$0 \to \oplus_{k\in\mathbb{N}} E_k \xrightarrow{\sigma} \oplus_{k\in\mathbb{N}} E_k \xrightarrow{q} E \to 0.$$

The inverse $\sigma^{-1} : \ker q \to \oplus_{k\in\mathbb{N}} E_k$ is given by

$$\sigma^{-1}(x) = (\sum_{j=1}^{k} x_j)_k.$$

q is continuous and open, σ is continuous, but not necessarily open onto its range. This means that σ^{-1} need not be continuous.

Definition 1.7. (Palamodov [36]) An inductive spectrum is called *acyclic* if σ^{-1} is continuous; it is called *weakly acyclic* if σ^{-1} is weakly continuous, i.e. continuous with respect to the weak topologies. An (LF)-space E is called (weakly) acyclic if it admits an acyclic resp. a weakly acyclic defining spectrum.

Retakh [38] investigated this behaviour of an inductive spectrum and proved:

Theorem 1.8. *E is (weakly) acyclic if and only if the following condition is fulfilled: There exists a sequence U_μ of absolutely convex neighbourhoods of zero in $E, \mu = 0, 1, 2, ...,$ such that*

i) $U_\mu \subset U_{\mu+1}$ *for all* μ.

ii) *For every μ there exists $K \geq \mu$ such that for all $K \geq k$ the (weak) topology of E_K coincides on U_μ with the (weak) topology of E_k.*

In the notation of Retakh [38] the condition in the acyclic case is called (M), in the weakly acyclic case (M_0). Vogt showed that the following conditions (Q) and (wQ) are necessary for acyclicity resp. weak acyclicity.

Proposition 1.9. *i) If E is acyclic, then condition (Q) holds:*
$\forall\, n\ \exists\, m \geq n, k\ \forall\, \mu \geq m, l, \varepsilon > 0\ \exists\, L, S > 0\ \forall\, x \in E_n$:

$$||x||_{m,l} \leq \varepsilon||x||_{n,k} + S||x||_{\mu,L}.$$

ii) If E is weakly acyclic, then condition (wQ) holds:
$\forall\, n \in \mathbb{N}\ \exists\, m \geq n, k \in \mathbb{N}\ \forall\, \mu \geq m, l \in \mathbb{N}\ \exists\, L \in \mathbb{N}, S > 0\ \forall\, x \in E_n$*:*

$$||x||_{m,l} \leq S(||x||_{n,k} + ||x||_{\mu,L}).$$

These conditions can be evaluated in concrete cases and are more suitable for applications than the characterisations of Retakh.

Definition 1.10. An inductive limit $E = \mathrm{ind}_n E_n$ is called *regular* if every bounded subset of E is contained and bounded in some step E_n.

It is well-known that every complete (LF)-space is regular, but whether the converse holds is an open problem (raised by Grothendieck), even for (LB)-spaces.

Definition 1.11. Let $(E,t) = \operatorname{ind}_n(E_n, t_n)$ be an (LF)-space. The inductive limit (E,t) is called *sequentially retractive* if every convergent sequence in (E,t) is contained and convergent in some step (E_n, t_n). It is called *boundedly retractive* if for any bounded subset B of (E,t) there is $n \in \mathbb{N}$ such that B is contained and bounded in (E_n, t_n) and that the topologies t and t_n coincide on B. The inductive limit (E,t) is called *(sequentially) compactly regular* if every (sequentially) compact subset of the inductive limit is (sequentially) compact in some step.

Palamodov [36] showed that every acyclic (LF)-space is complete, regular and sequentially retractive. In 1996 Wengenroth [46] showed that for a general (LF)-space the conditions (M) and (Q) are equivalent to the properties of being sequentially retractive, boundedly retractive, compactly regular and sequentially compactly regular.

1.4.2 Conditions (Q) and (wQ)

Vogt [44] introduced the conditions (Q) and (wQ) for general acyclic and weakly acyclic (LF)-spaces, but we do not need these general conditions in the sequel. In the case of weighted (PLB)-spaces one can reformulate these conditions in terms of the weights, as follows:

Definition 1.12. i) A sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (Q) if and only if
$\forall\, n\ \exists\, m \geq n, k\ \forall\, \mu \geq m, l, \varepsilon > 0\ \exists\, L, S > 0\ \forall\, x \in X$:

$$\frac{1}{a_{m,l}(x)} \leq \max(\varepsilon \frac{1}{a_{n,k}(x)}, S \frac{1}{a_{\mu,L}(x)}).$$

ii) A sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (wQ) if and only if
$\forall\, n \in \mathbb{N}\ \exists\, m \geq n, k \in \mathbb{N}\ \forall\, \mu \geq m, l \in \mathbb{N}\ \exists\, L \in \mathbb{N}, S > 0\ \forall\, x \in X$:

$$\frac{1}{a_{m,l}(x)} \leq S \max(\frac{1}{a_{n,k}(x)}, \frac{1}{a_{\mu,L}(x)}).$$

Note that condition (wQ) is always satisfied in the (LB)-case.

Bierstedt and Bonet introduced a condition similar to (wQ) which they called condition (wQ^*).

Definition 1.13. (Bierstedt, Bonet [8]) A sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (wQ^*) if
$\exists\ (k(\nu))_{\nu\in\mathbb{N}}\ \forall\ n\ \exists\ m \geq n\ \forall\ \mu \geq m, l\ \exists\ L, S > 0\ \forall\ x \in X$:

$$\frac{1}{a_{m,l}(x)} \leq S \max\left(\min_{1\leq\nu\leq n} \frac{1}{a_{\nu,k(\nu)}(x)}, \frac{1}{a_{\mu,L}(x)}\right).$$

Lemma 1.14. *(Bierstedt, Bonet [8]) Condition (wQ) is equivalent to condition (wQ^*).*

At the end of this section we will present some examples of sequences of weights which satisfy the conditions introduced above.

Example 1.15. (Bierstedt, Bonet [8]) Let

$$\begin{array}{lll} v : X \to \mathbb{R}, & 0 < v(x) \leq 1 & \forall\ x \in X, \\ w : X \to \mathbb{R}, & 0 \leq w(x) & \forall\ x \in X, \end{array}$$

be continuous functions, $r, \rho > 0$ or $+\infty$, and let $(r_n)_{n\in\mathbb{N}}, (\rho_k)_{k\in\mathbb{N}}$ be strictly increasing sequences of positive numbers with $r_n \to r$ and $\rho_k \to \rho$. For each $n, k \in \mathbb{N}$ we put

$$v_{n,k}(x) := v(x)^{r_n} w(x)^{\rho_k}\ \ \forall\, x \in X$$

and $\mathcal{V} := ((v_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$. If $\rho = \infty$, then the sequence $\mathcal{V} = ((v_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (Q) and therefore (wQ).

Next we give an example of a sequence $\mathcal{V} = ((v_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ which satisfies condition (wQ), but not (Q). First we have to define regularly decreasing sequences in the sense of Bierstedt, Meise, Summers [14]:

Definition 1.16. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of weights on X. $\mathcal{V}$ is called *regularly decreasing* if, for given $n \in \mathbb{N}$, there exists $m \geq n$, such that for every $\varepsilon > 0$ and every $k \geq m$, it is possible to find $\delta(k, \varepsilon) > 0$ with

$$v_k(x) \geq \delta(k, \varepsilon) v_n(x) \quad \text{whenever} \quad v_m(x) \geq \varepsilon v_n(x).$$

In other words, $\mathcal{V}$ is regularly decreasing if and only if for given $n \in \mathbb{N}$, there exists $m \geq n$ such that, on each subset of X on which the quotient $\frac{v_m}{v_n}$ is bounded away from zero, also all quotients $\frac{v_k}{v_n}, k \geq m$, are bounded away from zero.

Example 1.17. (Bierstedt, Meise, Summers [15]) Let $X := \mathbb{N} \times \mathbb{N}$. The sequence

$$v_n(i,j) = \begin{cases} \frac{1}{j^i} & i \leq n \\ \frac{1}{j^n} & i \geq n+1 \end{cases}, (i,j) \in \mathbb{N} \times \mathbb{N}$$

is regularly decreasing, and the sequence

$$v_n(i,j) = \begin{cases} \frac{1}{j^n} & i \leq n-1 \\ \frac{1}{i^n} & i \geq n \end{cases}, (i,j) \in \mathbb{N} \times \mathbb{N}$$

is not regularly decreasing.

Now to the example of a sequence $\mathcal{V} = ((v_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ which satisfies condition (wQ), but not (Q).

Example 1.18. (Bierstedt, Bonet [8]) Let $\mathcal{W} = (w_n)_{n\in\mathbb{N}}$ be a decreasing sequence of weights on a locally compact space X which is not regularly decreasing. For

$$v_{n,k} = 2^k w_n, \; n,k \in \mathbb{N},$$

$\mathcal{V} = ((v_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (wQ), but not (Q).

Remark 1.19. (Bierstedt, Bonet [8])

$(Q) \;\Leftrightarrow\;$ (wQ) plus the ″countably regularly decreasing″ condition (cRD) :
$\forall\, n \;\exists\, m,k \;\forall\, \mu, l, \varepsilon > 0 \;\exists\, L, \delta > 0 :$
$v_{m,l}(x) \geq \varepsilon v_{n,k}(x) \Rightarrow v_{\mu,L}(x) \geq \delta v_{n,k}(x).$

Hence in the (LB)-case condition (Q) is equivalent to the regularly decreasing condition.

1.4.3 The inductive limits $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$

In this section we will give a survey on inductive limits of weighted Banach and Fréchet spaces and their projective description. In the case of (LF)-spaces $\mathcal{V}_0C(X)$ Bierstedt, Meise, Summers [15] showed that the topology of the weighted inductive limit $\mathcal{V}_0C(X)$ can always be described by an associated system $\overline{V}$ of weights on X.

In the beginning of this section we will restrict our attention to the (LB)-case; the (LF)-case will be treated later on.

For a decreasing sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) define

$$Cv_n(X) := \{f \in C(X); ||f||_n = \sup_{x\in X} v_n(x)|f(x)| < \infty\},$$

$$C(v_n)_0(X) := \{f \in C(X); v_n f \text{ vanishes at } \infty \text{ on } X\},$$

and the weighted inductive limits of spaces of continuous functions

$$\mathcal{V}C(X) := \text{ind}_n Cv_n(X) \text{ and } \mathcal{V}_0 C(X) := \text{ind}_n C(v_n)_0(X).$$

The associated system $\overline{V}$ of weights was introduced by

$$\overline{V} := \{\overline{v} \in C(X); \forall\, n\ \exists\ \alpha_n > 0, \overline{v} \leq \inf_n \alpha_n v_n \text{ on } X\}.$$

The corresponding weighted spaces for $\overline{V}$ (the projective hulls) are

$$C\overline{V}(X) := \{f \in C(X); \forall\, \overline{v} \in \overline{V} : \sup_{x\in X} \overline{v}(x)|f(x)| < \infty\}$$

and

$$C\overline{V}_0(X) := \{f \in C(X); \forall\, \overline{v} \in \overline{V} : \overline{v}|f| \text{ vanishes at } \infty \text{ on } X\}.$$

$C\overline{V}(X)$ and $C\overline{V}_0(X)$ are complete, and $C(\overline{V})_0(X)$ is a closed subspace of $C\overline{V}(X)$. For the case of o-growth conditions Bierstedt, Meise, Summers [14] showed in 1982:

Theorem 1.20. *In the (LB)-case of $\mathcal{V}_0 C(X)$, the following conditions are equivalent:*

i) $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ *is regularly decrasing, i.e. for given $n \in \mathbb{N}$, there exists $m \geq n$, such that for every $\varepsilon > 0$ and every $k \geq m$, it is possible to find $\delta(k,\varepsilon) > 0$ with*

$$v_k(x) \geq \delta(k,\varepsilon) v_n(x) \quad \text{whenever} \quad v_m(x) \geq \varepsilon v_n(x),$$

ii) $\mathcal{V}_0 C(X)$ *is complete,*

iii) $\mathcal{V}_0 C(X) = C\overline{V}_0(X)$ *holds algebraically (and then also topologically).*

Before we can formulate a result for O-growth conditions we have to introduce condition (D), which was first used by Bierstedt, Meise [13] as a sufficient condition for distinguishedness of echelon spaces. This property generalizes the quasinormable and the reflexive case of echelon spaces. It was inspired by a condition of Grothendieck [23], see also [15].

Definition 1.21. The decreasing sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition (D) if there exists an increasing sequence $J = (X_m)_{m\in\mathbb{N}}$ of subsets X_m of X such that
(N, J) for each $m \in \mathbb{N}$ there is $n_m \geq m$ with $\inf_{x\in X_m} \frac{v_k(x)}{v_{n_m}(x)} > 0, k = n_m + 1, n_m + 2, ...$, while
(M, J) for each $n \in \mathbb{N}$ and each subset Y of X with $Y \cap (X \setminus X_m) \neq \emptyset$ for all $m \in \mathbb{N}$ there is $n' = n'(n, Y) > n$ such that $\inf_{y\in Y} \frac{v_{n'}(y)}{v_n(y)} = 0$.

In the following result we need the assumption that the space X is not only locally compact, but also σ-compact. A characterisation of σ-compactness of a locally compact space X was given by Bastin [1]. The condition $(M, \mathcal{K})$ was introduced by Bierstedt, Meise [13]. Condition $(M_1, \mathcal{K})$ and the continuous domination property were defined by Bastin [1].

Definition 1.22. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous weights on X.

i) $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition $(M, \mathcal{K})$ if, for every non relatively compact subset Y of X, $\forall\, n \in \mathbb{N}\ \exists\, \tilde{n} \in \mathbb{N}$ such that
$$\inf_{x\in Y} \frac{v_{\tilde{n}}(x)}{v_n(x)} = 0.$$

ii) $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition $(M_1, \mathcal{K})$ if, for every non relatively compact subset Y of X, there is $n \in \mathbb{N}$ such that
$$\inf_{x\in Y} \frac{v_n(x)}{v_1(x)} = 0.$$

iii) the family $\overline{V}$ satisfies the continuous domination property if every $\overline{v} \in \overline{V}$ is dominated by a continuous element of $\overline{V}$.

Proposition 1.23. *(Bastin [1]) Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous weights on X. The following conditions are equivalent:*

i) *V satisfies condition* $(M, \mathcal{K})$ *and the continuous domination property,*

ii) *V satisfies condition* $(M_1, \mathcal{K})$ *and the continuous domination property,*

iii) *the space* X *is locally compact and* σ*-compact.*

Theorem 1.24. *(Bierstedt, Bonet [6], Bastin [2]) Let* X *be* σ*-compact. In the* (LB)*-case of* $\mathcal{V}C(X)$ *the following conditions are equivalent:*

i) *The sequence* $\mathcal{V}$ *satisfies condition* (D)*,*

ii) $\mathcal{V}C(X) = C\overline{V}(X)$ *holds algebraically and topologically.*

It follows a collection of results for projective description in the case of (LF)-spaces. The (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ were defined by Bierstedt, Bonet and investigated in [8]. The notation and the main results of this article are given below.

For every $n \in \mathbb{N}$ let $V_n = (v_{n,k})_{k\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on X. Let $\mathcal{V}$ denote the sequence $(V_n)_{n\in\mathbb{N}}$ and let us assume that

$$v_{n+1,k}(x) \le v_{n,k}(x) \le v_{n,k+1}(x)$$

holds for all $n, k \in \mathbb{N}$ and for all $x \in X$. Define

$$\begin{aligned} CV_n(X) &:= \{f \in C(X); \forall\, k \in \mathbb{N} : ||f||_{n,k} := \sup_{x\in X} v_{n,k}(x)|f(x)| < \infty\}, \\ C(V_n)_0(X) &:= \{f \in C(X); \forall\, k \in \mathbb{N}\ \forall\, \varepsilon > 0\ \exists\, K \subset X \text{ compact} : \\ &\quad\ v_{n,k}(x)|f(x)| \le \varepsilon\ \forall\, x \in X \setminus K\}. \end{aligned}$$

For each $n \in \mathbb{N}$ we obtain that $CV_n(X)$ (resp. $C(V_n)_0(X)$) is continuously included in $CV_{n+1}(X)$ (resp. $C(V_{n+1})_0(X)$). The weighted (LF)-spaces of continuous functions are defined by

$$\mathcal{V}C(X) := \text{ind}_n CV_n(X) \text{ and } \mathcal{V}_0C(X) := \text{ind}_n C(V_n)_0(X).$$

Remark 1.25. As an (LF)-space $\mathcal{V}C(X)$ (resp. $\mathcal{V}_0C(X)$) is webbed and ultrabornological.

This holds because $CV_n(X)$ (resp. $C(V_n)_0(X)$) is webbed and ultrabornological as a Fréchet space for each $n \in \mathbb{N}$ and because a countable inductive limit of webbed or ultrabornological spaces is webbed or ultrabornological (see theorem 1.2 and theorem 1.5).

In order to describe $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ algebraically and topologically Bierstedt, Bonet [8] introduced the system $\overline{V}$ of weights associated with $\mathcal{V}$,

$$\overline{V} := \{\overline{v} \in C(X); \overline{v} \geq 0 \text{ and } \forall\ n \in \mathbb{N}\ \exists\ \alpha_n > 0, k(n) \in \mathbb{N} : \overline{v} \leq \alpha_n v_{n,k(n)}\}.$$

The projective hulls of the weighted inductive limits are defined as follows:

$$\begin{aligned} C\overline{V}(X) &:= \{f \in C(X); \forall\ \overline{v} \in \overline{V} : p_{\overline{v}}(f) := \sup_{x \in X} \overline{v}(x)|f(x)| < \infty\}, \\ C\overline{V}_0(X) &:= \{f \in C(X); \forall\ \overline{v} \in \overline{V}\ \forall\ \varepsilon > 0\ \exists\ K \subset X \text{ compact} : \\ &\quad\ \overline{v}(x)|f(x)| \leq \varepsilon\ \forall\ x \in X \setminus K\}. \end{aligned}$$

One has $\mathcal{V}C(X) \subset C\overline{V}(X)$ and $\mathcal{V}_0C(X) \subset C\overline{V}_0(X)$ with continuous inclusions, and $C\overline{V}_0(X)$ and $C\overline{V}(X)$ are complete locally convex spaces.

The main results of [8] are that $\mathcal{V}C(X) = C\overline{V}(X)$ holds algebraically if and only if the sequence $\mathcal{V}$ satisfies condition (wQ), and that the (LF)-space $\mathcal{V}C(X)$ is also complete if and only if the sequence $\mathcal{V}$ satisfies condition (wQ). In the case of o-growth conditions it was proved that $\mathcal{V}_0C(X) = C\overline{V}_0(X)$ is equivalent to $\mathcal{V}$ satisfying condition (Q), and that this is equivalent to $\mathcal{V}_0C(X)$ complete.

1.5 Known results of projective spectra of (DF)-spaces

Before we can investigate the structure of the (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$, we need some general results about projective spectra of (DF)-spaces. All the following results go back to the work of Palamodov [36], Retakh [38], Vogt [42],[43], and Wengenroth [45].

Definition 1.26. A *projective spectrum* $\mathcal{X}$ is a sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces (over the same field of real or complex numbers) and linear maps $\iota_m^n : X_m \to X_n$ for $n \leq m$, satisfying

$$\iota_m^n \circ \iota_k^m = \iota_k^n \text{ for } n \leq m \leq k \text{ and } \iota_n^n = \mathrm{id}_{X_n}.$$

Write $\mathcal{X} = (X_n, \iota_{n+1}^n)_{n\in\mathbb{N}}$.

Definition 1.27. For $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbb{N}}$ set:

$$\begin{aligned} \mathrm{Proj}^0\mathcal{X} &:= \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n; \iota^n_m(x_m) = x_n \ \forall\ n \le m\}, \\ \mathrm{Proj}^1\mathcal{X} &:= \prod_{n\in\mathbb{N}} X_n / B(\mathcal{X}), \end{aligned}$$

where

$$B(\mathcal{X}) = \{(a_n)_n \in \prod_{n\in\mathbb{N}} X_n; \exists\ (b_n)_n \in \prod_{n\in\mathbb{N}} X_n \text{ such that } a_n = \iota^n_{n+1} b_{n+1} - b_n\}.$$

There is a natural exact sequence of linear spaces:

$$(*) \quad 0 \longrightarrow \mathrm{Proj}^0\mathcal{X} \hookrightarrow \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma} \prod_{n\in\mathbb{N}} X_n \xrightarrow{q} \mathrm{Proj}^1\mathcal{X} \longrightarrow 0,$$

where $\sigma : (x_n)_{n\in\mathbb{N}} \to (\iota^n_{n+1} x_{n+1} - x_n)_{n\in\mathbb{N}}$ and q is the quotient map.

In the case of a projective spectrum $\mathcal{X} = (X_n, \iota^n_{n+1})$ of (LB)-spaces every X_n has the form $X_n = \cup_k X_{n,k}$ where $X_{n,k}$ is a Banach space with a norm $||\cdot||_{n,k}$, and X_n carries the locally convex inductive limit topology of the $X_{n,k}$. ι^n_{n+1} is assumed to be continuous. We put $B_{n,k} := \{x \in X_{n,k};\ ||x||_{n,k} \le 1\}$ and assume that $\cup_{k\in\mathbb{N}} B_{n,k} = X_n$ and that $(B_{n,k})_{k\in\mathbb{N}}$ is a fundamental sequence of bounded sets in X_n. Let $X = \mathrm{proj}_n X_n$. $\iota^n : X \to X_n$ denotes the canonical projection onto the n-th component. $\mathcal{X}$ is called *reduced* if $X_n = \overline{\iota^n X}$ for all $n \in \mathbb{N}$. $\mathcal{X}$ is called a (DFS)*-spectrum* if for every k and m there exists M such that the inclusion $X_{k,m} \hookrightarrow X_{k,M}$ is compact. For a locally convex space X we denote by $\mathcal{U}_0(X)$ the filter basis of absolutely convex neighbourhoods of 0. Palamodov and Retakh investigated under which conditions $\mathrm{Proj}^1\mathcal{X} = 0$, i.e. the map σ in the exact sequence $(*)$ is surjective. First Palamodov [36] presented a sufficient condition:

Theorem 1.28. *Let $\mathcal{X}$ be a projective spectrum and assume that each X_n is endowed with a complete metrizable group topology such that the spectral maps are continuous and*

$$\forall\ n \in \mathbb{N}, U \in \mathcal{U}_0(X_n)\ \exists\ m \ge n\ \forall\ \mu \ge m : \quad \iota^n_m X_m \subset \iota^n_\mu X_\mu + U.$$

Then $\mathrm{Proj}^1\mathcal{X} = 0$.

The next result was given by Frerick and Wengenroth [22] and independently by Braun and Vogt [19]. $\mathcal{BD}(X)$ denotes the set of Banach discs in a locally convex space X.

Theorem 1.29. *Let $\mathcal{X} = (X_n, \iota_m^n)$ be a projective spectrum consisting of locally convex spaces and continuous linear maps such that*

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ \mu \geq m\ \exists\ B \in \mathcal{BD}(X_n)\ \forall\ M \in \mathcal{BD}(X_m)$$

$$\exists\ K \in \mathcal{BD}(X_{mu}); \quad \iota_m^n(M) \subset \iota_\mu^n(K) + B.$$

Then $\mathrm{Proj}^1 \mathcal{X} = 0$*.*

Palamodov gave a characterisation in the case of a projective spectrum of Fréchet spaces by showing that then the condition in theorem 1.28 is also necessary. Compare also Wengenroth ([48], 3.2.8).

Theorem 1.30. *For a projective spectrum $\mathcal{X}$ consisting of Fréchet spaces and continuous linear maps the following conditions are equivalent:*

i) $\mathrm{Proj}^1 \mathcal{X} = 0$*,*

ii) $\forall\ n \in \mathbb{N}, U \in \mathcal{U}_0(x_n)\ \exists\ m \geq n\ \forall\ \mu \geq m: \quad \iota_m^n X_m \subset \iota_\mu^n X_\mu + U.$

In the case of projective spectra of (LB)-spaces Retakh gave a necessary and sufficient condition for $\mathrm{Proj}^1 \mathcal{X} = 0$. Compare also [48], 3.2.9.

Theorem 1.31. *For a projective spectrum $\mathcal{X}$ of (LB)-spaces,* $\mathrm{Proj}^1 \mathcal{X} = 0$ *holds if and only if there is a sequence of Banach discs $B_n \subset X_n$ such that*

i) *$\iota_m^n B_m \subset B_n$ for all $n \leq m$,*

ii) *for every n there is $m \geq n$ such that for each $\mu \geq m$ $\iota_m^n(X_m) \subset \iota_\mu^n X_\mu + B_n$ holds.*

Vogt (see [42], theorem 4.4, proposition 4.5, theorem 5.7) reformulated these results and introduced condition (P).

Theorem 1.32. *For a projective spectrum $\mathcal{X}$ of (LB)-spaces,* $\mathrm{Proj}^1 \mathcal{X} = 0$ *holds if and only if the following holds:*
(P) $\exists\ (k(\nu))_{\nu \in \mathbb{N}}\ \forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ \mu \geq m\ \exists\ S$*:*

$$\iota_m^n X_m \subset \iota_m^n X_\mu + S \bigcap_{\nu=1}^{n} (\iota_n^\nu)^{-1} B_{\nu,k(\nu)}.$$

Proposition 1.33. *The following is necessary for* (P)*:*

$$\exists\ (k(\nu))_{\nu\in\mathbb{N}}\ \forall\ n\in\mathbb{N}\ \exists\ m\geq n\ \forall\ l,\mu\ \exists\ L,S:$$
$$\iota_m^n B_{m,l}\subset S(\iota_\mu^n B_{\mu,L}+\bigcap_{\nu=1}^{n}(\iota_n^\nu)^{-1}B_{\nu,k(\nu)}).$$

In [42], 5.7 Vogt showed the connection between the projective spectrum $\mathcal{X}$ and the topological properties ultrabornological resp. barrelled for its projective limit.

Theorem 1.34. *For a projective spectrum* $\mathcal{X}$ *of* (LB)*-spaces the following holds: If* $\mathrm{Proj}^1\mathcal{X}=0$*, then* $X=\mathrm{proj}_n X_n$ *is ultrabornological (and hence barrelled).*

The following properties were defined in [43]:

Definition 1.35. For a projective spectrum $\mathcal{X}$ let

(P_1) $\quad \exists\ k\ \forall\ n\ \exists\ m\ \forall\mu,l\ \exists\ L,S:\iota_m^\mu B_{m,l}\subset S(\iota_n^\mu B_{n,k}+B_{\mu,L}),$

(P_2) $\quad \forall\ n\ \exists\ k,m\ \forall\ \mu,l\ \exists\ L,S:\iota_m^\mu B_{m,l}\subset S(\iota_n^\mu B_{n,k}+B_{\mu,L}).$

Of course (P_1) is stronger than (P_2). A weak variant of condition (P_2) was defined by Wengenroth [45].

Definition 1.36. Let $\mathcal{X}=(X_n,\iota_m^n)$ be a projective spectrum and $\mathcal{B}(X_n)$ the family of all absolutely convex bounded sets. Then
(P_3) $\forall\ n\in\mathbb{N}\ \exists\ m\geq n\ \forall\ \mu\geq m\ \exists\ B\in\mathcal{B}(X_n)\ \forall\ M\in\mathcal{B}(X_k)\ \exists\ K\in\mathcal{B}(X_\mu),S>0:$

$$\iota_m^n(M)\subset S(\iota_\mu^n(K)+B).$$

Vogt turned the conditions (P_1) and (P_2) into inequalities by means of dualisation. The following notation was used: Let $j_n^m:X_n'\to X_m'$ be the transpose of ι_m^n for $n\leq m$. For $y\in X_n'$ set

$$||y||_{n,k}^*=\sup\{|y(x)|\ :\ ||x||_{n,k}\leq 1\}$$

with

$$\begin{aligned}||y||_{n,k}^* &\leq ||y||_{n,k+1}^*,\\ ||y||_{n,k}^* &\geq ||j_n^{k+1}y||_{n+1,k}^*.\end{aligned}$$

For a reduced spectrum $\mathcal{X}$ we identify X_n' with $X_n^* := j^n X_n' \subset X'$, where j^n is the transpose of ι^n. Then $X_n^* \subset X_{n+1}^*$ holds for each $n \in \mathbb{N}$ and we obtain an imbedding spectrum of Fréchet spaces. By X^* we denote the dual space X' equipped with the inductive topology. X_b' denotes X' equipped with the strong topology. The map $\mathrm{id} : X^* \to X_b'$ is continuous.

Definition 1.37.

(P_1^*) $\exists\, k\, \forall\, n\, \exists\, m\, \forall\, \mu, l\, \exists\, L, S > 0\, \forall\, y \in X_n^* : ||j_n^m y||_{m,l}^* \leq S(||j_n^\mu y||_{\mu,L}^* + ||y||_{n,k}^*)$,

(P_2^*) $\forall\, n\, \exists\, k, m\, \forall\, \mu, l\, \exists\, L, S > 0\, \forall\, y \in X_n' : ||j_n^m y||_{m,l}^* \leq S(||j_n^\mu y||_{\mu,L}^* + ||y||_{n,k}^*)$.

Condition (P_1) and (P_1^*) are not equivalent in general. The same holds for (P_2) and (P_2^*). An example for this was given by Dierolf, Frerick, Mangino and Wengenroth [20]. They constructed a projective spectrum of (LB)-spaces of "Moscatelli type" which satisfies the conditions (P_1^*) and (P_2^*), but neither (P_1) nor (P_2). On the other hand, with duality theory and the bipolar theorem it follows:

Remark 1.38. (Vogt [43]) For (DFS)-spectra the conditions (P_1^*) and (P_2^*) are equivalent to (P_1) and (P_2), respectively.

Theorem 1.39. *(Vogt [43]) Let $\mathcal{X}$ be a (DFS)-spectrum. The following implications hold:*

i) $(P_1) \Rightarrow \mathrm{Proj}^1 \mathcal{X} = 0 \Rightarrow (P_2)$,

ii) $(P_1^*) \Rightarrow \mathrm{Proj}^1 \mathcal{X} = 0 \Rightarrow (P_2^*)$.

Now assume that $\mathcal{X}$ is reduced and consider the inductive spectrum $\mathcal{X}^*$. For a reduced spectrum Vogt (see [42] corollary 5.10, theorem 5.11) showed:

Corollary 1.40. *If X is barrelled and $\mathcal{X}$ is reduced, then*
(P_2^*) $\forall\ n\ \exists\ k, m\ \forall\ \mu, l\ \exists\ L, S > 0\ \forall\ y \in X_n'$:

$$||j_n^m y||_{m,l}^* \leq S(||j_n^\mu y||_{\mu,L}^* + ||y||_{n,k}^*)$$

holds.

Theorem 1.41. *If $\mathcal{X}$ is reduced, the following implications hold:*

$$\mathrm{Proj}^1 \mathcal{X} = 0 \Rightarrow X^* \text{ is regular} \Rightarrow (P_2^*).$$

$(P) \Leftrightarrow \mathrm{Proj}^1 \mathcal{X} = 0$				
$\Downarrow$				
X ultrabornological	$\Rightarrow$	X barrelled	$\Leftrightarrow$	Every bounded set in X' is contained and bounded in some X_n^*.
		$\Downarrow$		
		X' sequentially complete (quasi complete)	$\Rightarrow$	
$\Downarrow$				$\Downarrow$
X bornological	$\Rightarrow$	X quasibarrelled	$\Rightarrow$	Every bounded set in X_b' is contained and bounded in some X_n^*.
$\Downarrow$		$\Downarrow$		
X_b' complete		X_b' sequentially complete (quasi complete)	$\Rightarrow$	
				$\Downarrow$
				X^* regular
				$\Downarrow$
				(P_2^*)

For a reduced spectrum $\mathcal{X}$ the implications between the properties discussed in this section are collected in the previous scheme. All these implications were shown by Vogt in [42] and [43].

Asking under which conditions the map

$$\sigma : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n, \sigma((x_n)_{n\in\mathbb{N}}) = (\iota^n_{n+1}x_{n+1} - x_n)_{n\in\mathbb{N}}$$

in the exact sequence $(*)$ is not only surjective, but also every bounded set in $\prod_{n\in\mathbb{N}} X_n$ is contained in the image under σ of a bounded set in $\prod_{n\in\mathbb{N}} X_n$ ("σ lifts bounded sets"), we are led to the following definition.

Definition 1.42. (Bonet, Dierolf, Wengenroth [18]) Let $\mathcal{B}(X_n)$ be the system of bounded sets of $X_n, n \in \mathbb{N}$. A projective spectrum $\mathcal{X} = (X_n, \iota^n_m)$ consisting of locally convex spaces is said to be of *strong P-type* if
(P_s) $\forall\ n \in \mathbb{N}\ \exists\ B_n \in \mathcal{B}(X_n), m \geq n\ \forall\ \mu \geq m, M \in \mathcal{B}(X_m)\ \exists\ K \in \mathcal{B}(X_\mu)$:

$$\iota^n_m(M) \subset \iota^n_\mu(K) + B_n.$$

A similar condition was used by Wengenroth in the following theorem:

Theorem 1.43. *(Wengenroth [48]) Let $\mathcal{X} = (X_n, \iota^n_m)$ be a locally convex projective spectrum of locally complete spaces. Then the condition*

$$(P_{\tilde{s}}) \qquad \forall\ n\ \exists\ m \geq n\ \forall\ \mu \geq m\ \exists\ B \in \mathcal{B}(X_n)\ \forall\ M \in \mathcal{B}(X_m)\ \exists\ K \in \mathcal{B}(X_\mu)$$

$$\iota^n_m(M) \subset \iota^n_\mu(K) + B$$

implies that σ lifts bounded sets, that there are $\tilde{B}_n \in \mathcal{B}(X_n)$ with $\iota^n_m(\tilde{B}_n) \subset \tilde{B}_n$ for $M \geq n$ and that $\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ M \in \mathcal{B}(X_m)\ \exists\ D \in \mathcal{B}(\mathrm{Proj}\mathcal{X})$

$$\iota^n_m(M) \subset \iota^n(D) + \tilde{B}_n.$$

Corollary 1.44. *(Wengenroth [48]) Let $\mathcal{X} = (X_n, \iota^n_m)$ be a locally convex projective spectrum of regular (LB)-spaces. Then σ lifts bounded sets if and only if the condition $(P_{\tilde{s}})$ of 1.43 holds.*

1.6 New results on the weighted (PLB)-spaces $\mathcal{A}_0C(X)$ and $\mathcal{A}C(X)$

1.6.1 Structure of $\mathcal{A}_0C(X)$

Using the general results of Vogt and Wengenroth we now investigate the structure of the weighted (PLB)-space $\mathcal{A}_0C(X)$. In the case of the (LB)-spaces $(\mathcal{A}_n)_0C(X)$ the linear maps

$$\iota_m^n : (\mathcal{A}_m)_0C(X) \to (\mathcal{A}_n)_0C(X)$$

for $n \leq m$ can be chosen as $\iota_m^n = \mathrm{id}_{(\mathcal{A}_m)_0C(X)}$ since $(\mathcal{A}_m)_0C(X) \subset (\mathcal{A}_n)_0C(X)$. The projective spectrum is defined by

$$A_0 := ((\mathcal{A}_n)_0C(X), \mathrm{id}_{(\mathcal{A}_{n+1})_0C(X)})_{n\in\mathbb{N}},$$

and its projective limit $\mathcal{A}_0C(X)$ is

$$\mathcal{A}_0C(X) := \mathrm{proj}_n(\mathcal{A}_n)_0C(X) = \mathrm{proj}_n\mathrm{ind}_kC(a_{n,k})_0(X).$$

Remark 1.45. The projective spectrum A_0 is reduced.

Proof: Let $C_c(\mathbb{R}^n)$ be the space of all continuous functions with compact support and let $f \in C(a_{n,k})_0(X)$. For each $\varepsilon > 0$ there exists a compact set $K \subset X$ with

$$a_{n,k}(x)|f(x)| < \varepsilon \;\forall\; x \in X \setminus K.$$

One can choose a function $\varphi \in C_c(\mathbb{R}_n)$ with $\varphi \equiv 1$ on K and look at $f\varphi \in C_c(\mathbb{R}^n)$. It follows that $C_c(\mathbb{R}^n)$ is dense in $C(a_{n,k})_0(X)$ for each $n, k \in \mathbb{N}$ and hence in each $(\mathcal{A}_n)_0C(X) = \mathrm{ind}_kC(a_{n,k})_0(X)$. Because $C_c(\mathbb{R}^n)$ is dense in each step it is dense in the projective limit $\mathcal{A}_0C(X)$.

Remark 1.46. In general $\mathcal{A}_0C(X)$ is not a (DFS)-spectrum, because the inclusions are not compact.

To see under which conditions $\mathrm{Proj}^1A_0 = 0$, i.e. the map σ in the canonical exact sequence

$$0 \longrightarrow \mathrm{Proj}^0A_0 \hookrightarrow \prod_{n\in\mathbb{N}}(\mathcal{A}_n)_0C(X) \xrightarrow{\sigma} \prod_{n\in\mathbb{N}}(\mathcal{A}_n)_0C(X) \xrightarrow{q} \mathrm{Proj}^1A_0 \longrightarrow 0$$

is surjective, we use condition (wQ) on $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$. Before we can prove the main theorem of this chapter, which will show the connection between condition (wQ), $\mathrm{Proj}^1A_0 = 0$ and $\mathcal{A}_0C(X)$ ultrabornological resp. barrelled, we need the following obvious result:

Lemma 1.47. *Let X be locally compact and $Z_1, Z_2 \subset X$ zero sets of continuous functions h_1, h_2 on X. If $Z_1 \cap Z_2 = \emptyset$, then $h = \frac{|h_1|}{|h_1|+|h_2|}$ defines a continuous function on X with values in $[0,1]$ such that $h_{|Z_1} \equiv 0$ and $h_{|Z_2} \equiv 1$.*

Theorem 1.48. *The following conditions are equivalent:*

i) *The sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (wQ),*

ii) $\mathrm{Proj}^1 A_0 = 0$,

iii) *the (PLB)-space $\mathcal{A}_0C(X)$ is ultrabornological,*

iv) *the (PLB)-space $\mathcal{A}_0C(X)$ is barrelled.*

Proof. We will prove the theorem in the following way

$$i) \overset{1.)}{\Longrightarrow} ii) \overset{2.)}{\Longrightarrow} iii) \overset{3.)}{\Longrightarrow} iv) \overset{4.)}{\Longrightarrow} i).$$

For 1.) we apply theorem 1.32. In the case of the weighted (PLB)-spaces $\mathcal{A}_0C(X)$ condition (P) looks as follows:

$$\exists\ (k(\nu))_{\nu\in\mathbb{N}}\ \forall\ n\in\mathbb{N}\ \exists\ m\geq n\ \forall\ \mu\geq m\ \exists\ S>0:$$

$$(\mathcal{A}_m)_0C(X)\subset(\mathcal{A}_\mu)_0C(X)+S\cap_{\nu=1}^{n}(B_{\nu,k(\nu)})_0.$$

Let $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfy condition (wQ), and therefore (wQ^*). Using condition (wQ^*) one can find a sequence $(k(\nu))_{\nu\in\mathbb{N}}$ which satisfies (P). For given n select m according to (wQ^*), and for given $\mu \geq m$ and $f \in (\mathcal{A}_m)_0C(X)$ one can find l such that $f \in C(a_{m,l})_0(X)$. For μ, l select $L, S > 0$ as in (wQ^*). Now define the sets

$$\begin{aligned}
K &:= \{x\in X; a_{m,l}(x)|f(x)|>\frac{1}{2S}\},\\
X_1 &:= \{x\in X; a_{\mu,L}(x)<2Sa_{m,l}(x)\},\\
X_2 &:= \{x\in X; a_{m,l}(x)|f(x)|<\frac{1}{S}\}\cap\{x\in X; a_{\mu,L}(x)>Sa_{m,l}(x)\}.
\end{aligned}$$

The set K is relatively compact and open in X. We claim that

$$X = (X_1\cup K)\cup X_2$$

holds. To show this, let $x \in X$ be given. The following cases are possible:

$$\text{I)}\quad a_{m,l}(x)|f(x)| > \frac{1}{2S} \quad \text{or} \quad \text{II)}\quad a_{m,l}(x)|f(x)| \leq \frac{1}{2S} < \frac{1}{S}.$$

In the first case $x \in K$. If the second case is true one has to evaluate if

$$\text{a)}\quad a_{\mu,L}(x) < 2Sa_{m,l}(x) \quad \text{or} \quad \text{b)}\quad a_{\mu,L}(x) \geq 2Sa_{m,l}(x) > Sa_{m,l}(x).$$

From a) it follows that $x \in X_1$. If b) is true, then $x \in X_2$ holds.

Now define $Z_1 := X \setminus (X_1 \cup K)$ and $Z_2 := X \setminus X_2$. For Z_1 and Z_2 we obtain

$$Z_1 \cap Z_2 = X \setminus (X_1 \cup K) \cap (X \setminus X_2) = X \setminus (X_1 \cup K \cup X_2) = \emptyset.$$

To apply lemma 1.47 one has to show that Z_1 and Z_2 are zero sets of continuous functions h_1, h_2 on X. The set Z_1 can be written as $Z_1 = (X \setminus X_1) \cap (X \setminus K)$. First we show the existence of suitable continuous functions g_1, g_2 such that $X \setminus X_1$ and $X \setminus K$ are zero sets. Define the continuous functions g_1 and g_2 on X by

$$g_1(x) := \begin{cases} 0 & \text{for } a_{\mu,L}(x) - 2Sa_{m,l}(x) > 0 \\ a_{\mu,L}(x) - 2Sa_{m,l}(x) & \text{elsewhere,} \end{cases}$$

$$g_2(x) := \begin{cases} 0 & \text{for } a_{m,l}(x)|f(x)| - \frac{1}{2S} < 0 \\ a_{m,l}(x)|f(x)| - \frac{1}{2S} & \text{elsewhere.} \end{cases}$$

$X \setminus X_1$ is the zero set of g_1, $X \setminus K$ is the zero set of g_2 and finally Z_1 is the zero set of the function $h_1 := \max(g_1, g_2)$. The case of Z_2 can be treated in a similar way.

By lemma 1.47 there exists a function $h \in C(X, [0,1])$ such that $h_{|Z_1} \equiv 0$ and $h_{|Z_2} \equiv 1$. Write the given function $f \in C(a_{m,l})_0(X)$ as $f = hf + (1 - h)f =: f_1 + f_2$. We have to show that $f_1 \in C(a_{\mu,L})_0(X)$ and therefore $f_1 \in (\mathcal{A}_\mu)_0 C(X)$ and $f_2 \in S(B_{\nu,k(\nu)})_0$ for $\nu = 1, ..., n$. If $x \notin X_1 \cup K$ then $f_1(x) = h(x)f(x) = 0$. If $x \in X_1 \cup K$ it follows that $|f_1(x)| \leq |f(x)|$ and for $x \in X_1$

$$a_{\mu,L}(x)|f_1(x)| \leq a_{\mu,L}(x)|f(x)| \leq 2Sa_{m,l}(x)|f(x)|$$

holds. Since $\overline{K}$ is compact,

$$\sup_{x \in X} a_{\mu,L}(x)|f_1(x)| < \infty.$$

Collecting these facts we obtain that for $x \in X_1 \cup K$ there exists $\tilde{S} > 0$ such that $a_{\mu,L}|f_1| \leq \tilde{S} a_{m,l}|f|$. From $f \in C(a_{m,l})_0(X)$ it follows that $f_1 \in C(a_{\mu,L})_0(X)$. If $x \notin X_2$

$$f_2(x) = f(x)(1 - h(x)) = 0.$$

On the other hand, if $x \in X_2$ we have $|f_2| \leq |f|$ and by the definition of X_2

$$(**) \qquad \frac{1}{a_{m,l}(x)} > \frac{S}{a_{\mu,L}(x)}.$$

With condition (wQ^*) and $(**)$ we obtain that

$$a_{\nu,k(\nu)}(x) \leq S a_{m,l}(x)$$

holds for $\nu = 1, ..., n$, and then

$$a_{\nu,k(\nu)}(x)|f_2(x)| \leq a_{\nu,k(\nu)}(x)|f(x)| \leq S a_{m,l}(x)|f(x)|$$

holds for $\nu = 1, ..., n$. It follows that $f_2 \in C(a_{\nu,k(\nu)})_0(X)$ for $\nu = 1, ..., n$. From $x \in X_2$ we also can conclude that $a_{m,l}(x)|f(x)| < \frac{1}{S}$, which is equivalent to $S a_{m,l}(x)|f(x)| < 1$. Then

$$a_{\nu,k(\nu)}(x)|f_2(x)| \leq S a_{m,l}(x)|f(x)| < 1$$

holds for $\nu = 1, ..., n$ and therefore $f_2 \in (B_{\nu,k(\nu)})_0$ for $\nu = 1, ..., n$. Finally condition (P) and therefore $\mathrm{Proj}^1 A_0 = 0$ follows.

2.) follows immediately from theorem 1.34 and 3.) holds in general. Since the projective spectrum A_0 is reduced we can apply corollary 1.40 to show 4.):

$$(P_2^*) \qquad \forall\, n\ \exists\, m \geq n, k\ \forall\, \mu \geq m, l\ \exists\, L, S\ \forall\, \varphi \in (\mathcal{A}_n)_0 C(X)' :$$

$$||\varphi||^*_{m,l} \leq S(||\varphi||^*_{\mu,L} + ||\varphi||^*_{n,k}),$$

where $||\varphi||^*_{m,l} := \sup\{| < \varphi, f > |; f \in (B_{m,l})_0\}$. Since $(B_{m,l})_0 \subset (B_{n,l})_0$ for $m \geq n$, the sup is finite. We show that condition (wQ) follows for the sequence $\mathcal{A}$. The quantifiers are the same. We fix $x \in X$. The measure $\delta_x : (\mathcal{A}_n)_0 C(X) \to \mathbb{C}, \delta_x(f) := f(x)$, is continuous because the topology of this weighted (LB)-space is finer than the compact-open topology. Now

$$||\delta_x||^*_{m,l} = \sup\{|f(x)|; f \in (B_{m,l})_0\} \leq \frac{1}{a_{m,l}(x)}$$

clearly holds since $|f| \leq \frac{1}{a_{m,l}}$ on X for each $f \in (B_{m,l})_0$. On the other hand, select $\varphi \in C(X)$ with compact support, $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$. Clearly $f_0 = \frac{\varphi}{a_{m,l}}$ belongs to $(B_{m,l})_0$ and

$$\frac{1}{a_{m,l}(x)} = f_0(x) \leq \sup\{|f(x)|; f \in (B_{m,l})_0\} = ||\delta_x||^*_{m,l}.$$

Thus, we have proved that $||\delta_x||^*_{m,l} = \frac{1}{a_{m,l}(x)}$. Therefore

$$\frac{1}{a_{m,l}(x)} \leq S(\frac{1}{a_{\mu,L}(x)} + \frac{1}{a_{n,k}(x)}) \leq 2S \max(\frac{1}{a_{\mu,L}(x)}, \frac{1}{a_{n,k}(x)})$$

which is condition (wQ). □

For the next theorem we need the assumption that the space $(\mathcal{A}_n)_0 C(X)$ is complete. This completeness was characterised by Bierstedt, Bonet, Summers [14] (see theorem 1.20).

Theorem 1.49. *Let $(\mathcal{A}_n)_0 C(X)$ be complete for each $n \in \mathbb{N}$. The projective spectrum A_0 is of strong P-type if and only if the sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (Q).*

To prove this theorem we need some technical tools. First we present a partition of a continuous functions. The idea of this goes back to Ernst, Schnettler [21].

Proposition 1.50. *Let $u, v \in C(X)$ be strictly positive functions. If $f \in C(X)$ satisfies $|f| \leq \max(u, v)$, there exist $g_1, g_2 \in C(X)$ with $|g_1| \leq u$ and $|g_2| \leq v$ on X such that $|f| = g_1 + g_2$.*

Proof. Define

$$\begin{aligned} A &:= \{x \in X;\ |f(x)| \leq u(x)\}, \\ B &:= \{x \in X;\ |f(x)| > u(x)\}, \end{aligned}$$

and $g_1, g_2 \in C(X)$ by $g_1(x) := \min(|f(x)|, u(x))$ and $g_2(x) := |f(x)| - g_1(x)$. Of course, $|f| = g_1 + g_2$ holds. If $x \in A$ then $g_1(x) = |f(x)| \leq u(x)$ and $g_2(x) = 0$. For $x \in B$ we obtain $g_1(x) = u(x)$ and $g_2(x) = |f(x)| - u(x)$. With the definition of the set B and the assumption that $|f| \leq \max(u, v)$ it follows that $v(x) \geq |f(x)| \geq |f(x)| - u(x) = g_2(x)$. We have $0 \leq g_1 \leq u$ and $0 \leq g_2 \leq v$ on X. □

Lemma 1.51. *Let A_0 be of strong P-type. Condition (P_s) (see definition 1.42) can be written as follows:* $\forall\; n\; \exists\; m,k\; \forall\; \mu, l, \varepsilon > 0\; \exists\; L, S > 0$:

$$(B_{m,l})_0 \subset S(B_{\mu,L})_0 + \varepsilon(B_{n,k})_0.$$

Then $\forall\, n\; \exists\, m,k\, \forall\, \mu, l, \varepsilon > 0\, \exists\, L, S > 0\; \forall\; \varphi \in (\mathcal{A}_n)_0 C(X)^*$:

$$||\varphi||^*_{m,l} \leq S||\varphi||^*_{\mu,L} + \varepsilon||\varphi||^*_{n,k}$$

holds.

Proof. Choose $\varphi \in (\mathcal{A}_n)_0 C(X)^*$ and let $||\varphi||^*_{m,l} := \sup\{|\varphi(f)|; f \in (B_{m,l})_0\} \leq 1$. Write f as $f = Sf_1 + \varepsilon f_2$ with $f_1 \in (B_{\mu,L})_0$ and $f_2 \in (B_{n,k})_0$.

$$\begin{aligned} |\varphi(f)| &\leq S|\varphi(f_1)| + \varepsilon|\varphi(f_2)| \\ &\leq S||\varphi||^*_{\mu,L} + \varepsilon||\varphi||^*_{n,k} \end{aligned}$$

holds for all $f \in (B_{m,l})_0$. Then the same inequality holds for the sup, and one gets

$$||\varphi||^*_{m,l} \leq S||\varphi||^*_{\mu,L} + \varepsilon||\varphi||^*_{n,k}.$$

□

Proof of 1.49. Let the projective spectrum A_0 be of strong P-type. We apply lemma 1.51 and get: $\forall\, n\, \exists\, m,k\, \forall\, \mu, l, \tilde{\varepsilon} > 0\, \exists\; L, \tilde{S} > 0$ such that

$$||\varphi||^*_{m,l} \leq \tilde{S}||\varphi||^*_{\mu,L} + \tilde{\varepsilon}||\varphi||^*_{n,k}.$$

Let n, m, k, μ, l be as before and put $\varepsilon = 2\tilde{\varepsilon}$. Let L be as above and put $S = 2\tilde{S}$. Fix $x_0 \in X$ and define $\delta_{x_0} : (\mathcal{A}_n)_0 C(X) \to \mathbb{C}$ as in the proof of theorem 1.48; recall that from 4.) in the proof of theorem 1.48

$$||\delta_{x_0}||^*_{m,l} = \sup\{|f(x_0)|; f \in (B_{m,l})_0\} = \frac{1}{a_{m,l}(x_0)}.$$

It follows that

$$\begin{aligned} \frac{1}{a_{m,l}(x_0)} = ||\delta_{x_0}||^*_{m,l} &\leq \tilde{S}||\delta_{x_0}||^*_{\mu,L} + \tilde{\varepsilon}||\delta_{x_0}||^*_{n,k} \\ &= \tilde{S}\frac{1}{a_{\mu,L}(x_0)} + \tilde{\varepsilon}\frac{1}{a_{n,k}(x_0)}. \end{aligned}$$

We claim that

$$\frac{1}{a_{m,l}(x_0)} \leq \max(S\frac{1}{a_{\mu,L}(x_0)}, \varepsilon\frac{1}{a_{n,k}(x_0)}),$$

which is exactly condition (Q). If not, then

$$\begin{aligned}\frac{1}{a_{m,l}(x_0)} &= \frac{1}{2}\frac{1}{a_{m,l}(x_0)} + \frac{1}{2}\frac{1}{a_{m,l}(x_0)} \\ &> \frac{1}{2}S\frac{1}{a_{\mu,L}(x_0)} + \frac{1}{2}\varepsilon\frac{1}{a_{n,k}(x_0)},\end{aligned}$$

which implies

$$\frac{1}{a_{m,l}(x_0)} > \tilde{S}\frac{1}{a_{\mu,L}(x_0)} + \tilde{\varepsilon}\frac{1}{a_{n,k}(x_0)},$$

a contradiction to the inequality proved above.

Now let condition (Q) be given. Take $f \in B_{m,l}$; then

$$|f| \leq \frac{1}{a_{m,l}} \leq \max(\frac{S}{a_{\mu,l}}, \frac{\varepsilon}{a_{n,k}})$$

holds on X. By proposition 1.50 there exist $f_1, f_2 \in C(X)$ with $|f_1| \leq \frac{S}{a_{\mu,L}}, |f_2| \leq \frac{\varepsilon}{a_{n,k}}$ and $|f| = f_1 + f_2$. It follows that $f_1 \in SB_{\mu,L}$ and $f_2 \in \varepsilon B_{n,k}$.

1.6.2 Structure of $\mathcal{A}C(X)$

In this chapter we investigate the structure of the space $\mathcal{A}C(X)$ which is defined by

$$\mathcal{A}C(X) := \mathrm{proj}_n \mathcal{A}_n C(X) = \mathrm{proj}_n \mathrm{ind}_k C a_{n,k}(X).$$

Similar to the case of o-growth conditions the linear maps

$$\iota_m^n : \mathcal{A}_m C(X) \to \mathcal{A}_n C(X)$$

for $n \leq m$ can be chosen as $\iota_m^n = \mathrm{id}_{\mathcal{A}_m C(X)}$ since $\mathcal{A}_m C(X) \subset \mathcal{A}_n C(X)$. The projective spectrum is defined by

$$A := (\mathcal{A}_n C(X), \mathrm{id}_{\mathcal{A}_{n+1} C(X)})_{n\in\mathbb{N}}.$$

Theorem 1.52. $\mathrm{Proj}^1 A = 0$ *if and only if* $\mathcal{A} = ((a_{n,k})_{\in\mathbb{N}})_{n\in\mathbb{N}}$ *satisfies condition* (Q).

Proof. Without loss of generality we can assume that for each $n \in \mathbb{N}$ the system $(B_{n,k})_{k\in\mathbb{N}}$ is a fundamental system of bounded sets in $\mathcal{A}_nC(X)$. First let $\mathrm{Proj}^1 A = 0$. With theorem 1.31 it follows that for each $n \in \mathbb{N}$ there exists a bounded absolutely convex B_n in $\mathcal{A}_nC(X)$ such that $B_{n+1} \subset B_n$ for each $n \in \mathbb{N}$ and

$$(**) \qquad \forall\, n\ \exists\, m \geq n\ \forall\, \mu \geq m:\ \mathcal{A}_mC(X) \subset \mathcal{A}_\mu C(X) + B_n.$$

Since B_n is bounded in $\mathcal{A}_nC(X)$ and since $\mathcal{A}_nC(X)$ is a regular inductive limit (see [14]), we can select $k(n) \in \mathbb{N}$ such that $B_n \subset B_{n,k(n)}$. Now we have to show that $\mathcal{A} = ((a_{n,k})_{\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (Q). For given n, select m as in $(**)$ and $k = k(n)$ with $B_n \subset B_{n,k(n)}$. Fix $\mu \geq m, l, \varepsilon > 0$; w.l.o.g. $\varepsilon \leq 1$. For the function $f := \frac{1}{\varepsilon a_{m,l}}$, $f \in \mathcal{A}_mC(X)$ holds, and by $(**)$ one can write $f = g + h$ with $g \in \mathcal{A}_\mu C(X)$ and $h \in B_n$. For $g \in \mathcal{A}_\mu C(X)$ there exist $L \in \mathbb{N}$ and $S > 0$ with $a_{\mu,L}|g| \leq S$ on X. For $h \in B_n$ it follows that $h \in B_{n,k(n)}$ and then $a_{n,k}|h| \leq 1$, and therefore $|h| \leq \frac{1}{a_{n,k}}$ holds on X. It follows that

$$\begin{aligned} \frac{1}{\varepsilon a_{m,l}} &= |f| \leq |g| + |h| \leq \frac{S}{a_{\mu,L}} + \frac{1}{a_{n,k}} \\ \Rightarrow \quad \frac{1}{a_{m,l}} &\leq \frac{S\varepsilon}{2a_{\mu,L}} + \frac{\varepsilon}{2a_{n,k}} \leq \max(\frac{S}{a_{\mu,L}}, \frac{\varepsilon}{a_{n,k}}) \end{aligned}$$

holds on X, which is exactly (Q).

In the other direction we show that condition (Q) even implies that the map σ in the exact sequence lifts bounded sets. Then $\mathrm{Proj}^1 A = 0$ follows immediately from theorem 1.29. Let condition (Q) be satisfied, i.e. $\forall\, n\ \exists\, m, k\ \forall\, \mu, l, \varepsilon > 0\ \exists\, L, S > 0:$

$$\frac{1}{a_{m,l}} \leq \max(\frac{S}{a_{\mu,L}}, \frac{\varepsilon}{a_{n,k}})$$

holds on X. Take $f \in B_{m,l}$; then

$$|f| \leq \frac{1}{a_{m,l}} \leq \max(\frac{S}{a_{\mu,l}}, \frac{\varepsilon}{a_{n,k}})$$

holds on X. By proposition 1.50 there exist $f_1, f_2 \in C(X)$ with $|f_1| \leq \frac{S}{a_{\mu,L}}, |f_2| \leq \frac{\varepsilon}{a_{n,k}}$ and $|f| = f_1 + f_2$. It follows that $f_1 \in SB_{\mu,L}$ and $f_2 \in \varepsilon B_{n,k}$, which is condition (P_s) in the case of the space $\mathcal{A}C(X)$ (compare lemma 1.51). ☐

Remark 1.53. We have even shown that $\mathcal{A}C(X)$ has $\mathrm{Proj}^1 A = 0$ if and only if the projective spectrum A is of strong P-type.

Remark 1.54. With theorem 1.52 above and theorem 1.34, a general result for projective spectra of (LB)-spaces, we get the following inclusions for the (PLB)-space $\mathcal{A}C(X)$:

$$(Q) \Rightarrow \mathrm{Proj}^1 A = 0 \Rightarrow \mathcal{A}C(X) \text{ is ultrabornological (and hence barrelled).}$$

But in general the (PLB)-space $\mathcal{A}C(X)$ is not reduced, i.e. we cannot use the general theory of (DF)-spaces to conclude that from $\mathcal{A}C(X)$ barrelled or ultrabornological it follows that the sequence $\mathcal{A}$ satisfies condition (wQ) or (Q). It is unknown if then condition (Q) must be satisfied. The fact that from $\mathcal{A}C(X)$ barrelled it follows that the sequence $\mathcal{A}$ satisfies at least condition (wQ) is indeed right. But to show this, we need the inductive description which will be introduced in the next section.

1.7 Inductive description

From now on through the whole section 1.7 let the space X be locally compact and σ-compact.

1.7.1 Inductive description for Fréchet spaces

First we investigate inductive description for an increasing sequence $\mathcal{A} = (a_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights). For

$$Ca_n(X) := \{f \in C(X); ||f||_n := \sup_{x\in X} a_n(x)|f(x)| < \infty\},$$

the space $CA(X) = \mathrm{proj}_n Ca_n(x)$ is a Fréchet space. Define

$$\underline{A} := \{\underline{a} \in C(X); \underline{a} \geq 0, \forall\ n\ \exists\ \alpha_n > 0 : \underline{a} \geq \alpha_n a_n\},$$

$$C\underline{a}(X) := \{f \in C(X); ||f||_{\underline{a}} := \sup_{x\in x} \underline{a}(x)|f(x)| < \infty\}$$

and the corresponding space

$$C\underline{A}(X) := \mathrm{ind}_{\underline{a}\in\underline{A}} C\underline{a}(X).$$

Theorem 1.55.
$$C\underline{A}(X) = CA(X)$$
holds algebraically, and the canonical mapping $C\underline{A}(X) \to CA(X)$ is continuous.

Proof. Let $f \in C\underline{A}(X)$. There exists $\underline{a} \in \underline{A}, \underline{a} \geq \alpha_n a_n$ for each $n \in \mathbb{N}$, with $f \in C\underline{a}(X)$. It follows that

$$||f||_n = \sup_{x\in X} a_n(x)|f(x)| \leq \frac{1}{\alpha_n} \sup_{x\in X} \underline{a}(x)|f(x)| = \frac{1}{\alpha_n}||f||_{\underline{a}},$$

which means that $f \in Ca_n(X)$ for each $n \in \mathbb{N}$, hence $f \in CA(X)$.

Now let $g \in CA(X)$. Then for each $n \in \mathbb{N}$ there exists $\alpha_n \in \mathbb{R}_+$ with $a_n|g| \leq \alpha_n$ for all $n \in \mathbb{N}$. With the inequality above we obtain $|g| \leq \frac{\alpha_n}{a_n}$ on X for each $n \in \mathbb{N}$. It follows that $|g| \leq \inf_{n\in\mathbb{N}} \frac{\alpha_n}{a_n}$ on X. Define the sequence $(w_n)_{n\in\mathbb{N}}$ by $w_n := \frac{1}{a_n}$. In the notation of section 1.4.3 (but V replaced by W) we obtain $\overline{w} := \inf_{n\in\mathbb{N}} \frac{\alpha_n}{a_n} \in \overline{W}$. With a result of Bierstedt, Meise, Summers ([14], 0.2) it follows that there exists $\tilde{w} \in \overline{W}$ with $\tilde{w}(x) > 0$ for all $x \in X$ and $\overline{w} = \inf_{n\in\mathbb{N}} \frac{\alpha_n}{a_n} \leq \tilde{w}$. Define $\underline{b} := \frac{1}{\tilde{w}}$. Then $|g| \leq \tilde{w} = \frac{1}{\underline{b}}$, and therefore $|g|\underline{b} \leq 1$ holds. We still have to show that $\underline{b} \in \underline{A}$ holds. From $\tilde{w} \in \overline{W}$ it follows that $\forall\ n \in \mathbb{N}\ \exists\ \beta_n > 0$ such that $\frac{1}{\underline{b}} = \tilde{w} \leq \inf_n \beta_n w_n = \inf \beta_n \frac{1}{a_n}$. Then $\underline{b} \geq \sup_n \frac{1}{\beta_n} a_n$, and this means that $\underline{b} \in \underline{A}$ and $g \in C\underline{A}(X)$. □

1.7.2 Inductive description in the (PLB)-case

Now we investigate the (PLB)-case. Again we take a double sequence $\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ of strictly positive weights with

$$a_{n,k+1}(x) \leq a_{n,k}(x) \leq a_{n+1,k}(x) \quad \forall n, k \in \mathbb{N}, \forall x \in X,$$

and define

$$\underline{A} := \{\underline{a} \in C(X); \underline{a} \geq 0,\ \forall\ n \in \mathbb{N}\ \exists\ \alpha_n, k(n) :\ \underline{a} \geq \sup_n \alpha_n a_{n,k(n)}\}.$$

Again we have $v_{n,k} = \frac{1}{a_{n,k}}$ where $v_{n,k} \in \overline{V}$ is a weight as in [8] with

$$\overline{V} = \{\overline{v} \in C(X); \overline{v} > 0,\ \forall\ n \in \mathbb{N}\ \exists\ \alpha_n, k(n) :\ \overline{v} \leq \inf_n \alpha_n v_{n,k(n)}\}.$$

Then $\underline{A} = \{\frac{1}{\overline{v}}; \overline{v} \in \overline{V}\}$. Define

$$C\underline{a}(X) := \{f \in C(X); ||f||_{\underline{a}} := \sup_{x \in X} \underline{a}(x)|f(x)| < \infty\}$$

and

$$C\underline{A}(X) := \mathrm{ind}_{\underline{a} \in \underline{A}} C\underline{a}(X).$$

Since X is σ-compact, one can restrict the attention to the positive elements $\overline{v} \in \overline{V}$ (see [14]) and hence to the positive elements $\underline{a} \in \underline{A}$. Each $C\underline{a}(X)$ is a Banach space, $\underline{a} \in \underline{A}$, and hence $C\underline{A}(X)$ is ultrabornological.

Theorem 1.56.

$$C\underline{A}(X) = \mathcal{A}C(X)$$

holds algebraically, and the canonical mapping $C\underline{A}(X) \to \mathcal{A}C(X)$ is continuous.

Proof. Let $f \in C\underline{A}(X)$, i.e. there exists $\underline{a} \in \underline{A}$ such that $f \in C\underline{a}(X)$. For each $n \in \mathbb{N}$ there are $\alpha_n > 0$ and $k(n) \in \mathbb{N}$ such that $\underline{a} \geq \alpha_n a_{n,k(n)}$. It follows that

$$||f||_{n,k(n)} = \sup_{x \in X} a_{n,k(n)}(x)|f(x)| \leq \frac{1}{\alpha_n} \sup_{x \in X} \underline{a}(x)|f(x)| \leq \frac{1}{\alpha_n}||f||_{\underline{a}} < \infty,$$

which means $f \in Ca_{n,k(n)}(X)$ for each $n \in \mathbb{N}$, hence $f \in \mathcal{A}_n C(X)$ for each $n \in \mathbb{N}$ and therefore $f \in \mathcal{A}C(X)$. The above inequality shows that $C\underline{a}(X)$ is continuously injected in $Ca_{n,k(n)}(X)$ for each $n \in \mathbb{N}$. It follows that the mapping $C\underline{A}(X) \to \mathcal{A}_n C(X)$ is continuous for each $n \in \mathbb{N}$, which proves the last assertion of the theorem.

Now let $f \in \mathcal{A}C(X)$, i.e. for each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $f \in Ca_{n,k(n)}(X)$, i.e. for each $n \in \mathbb{N}$ there exist $k(n) \in \mathbb{N}, b_n > 0$ such that $a_{n,k}|f| \leq b_n \Rightarrow |f| \leq b_n \frac{1}{a_{n,k}}$ for all $n \in \mathbb{N}$. Define $\overline{w} := \inf_n b_n \frac{1}{a_{n,k(n)}}$. One has $\overline{w} \in \overline{V}$, and with [14] there exists $\overline{\overline{w}} \in \overline{V}$ with $\overline{\overline{w}}(x) > 0$ for all $x \in X$ and $\overline{w} \leq \overline{\overline{w}}$. Define $\underline{a} := \frac{1}{\overline{\overline{w}}} \in C(X)$. Then $\underline{a}|f| \leq 1$ because $|f| \leq \overline{w} \leq \overline{\overline{w}}$. It follows that $\underline{a} \in \underline{A}$ and $f \in C\underline{A}(X)$. ∎

Corollary 1.57. *$C\underline{A}(X)$ and $\mathcal{A}C(X)$ have the same bounded sets, and the inductive limit $C\underline{A}(X)$ is regular.*

Proof. After theorem 1.56 it suffices to fix a bounded set $B \subset \mathcal{A}C(X)$ and to show that there exists $\underline{a} \in \underline{A}$ such that B is contained and bounded in $C\underline{a}(X)$. B is bounded in $\mathcal{A}_nC(X)$ for each $n \in \mathbb{N}$. Since $\mathcal{A}_nC(X)$ is a regular inductive limit, there exist $k(n), m_n > 0$ such that $B \subset m_nB_{n,k(n)}$. Then for each $f \in B$

$$a_{n,k(n)}|f| \leq m_n$$

holds. Define $\overline{w} := \inf_n m_n \frac{1}{a_{n,k(n)}}$. With the same arguments as above we obtain $\underline{a} \in \underline{A}, \underline{a} > 0$, such that $\underline{a}|f| \leq 1$ for each $f \in B$, which means that B is contained in $C\underline{a}(X)$ and bounded there. □

Theorem 1.58. *Let the sequence $\mathcal{A}$ satisfy condition (Q). Then*

$$C\underline{A}(X) = \mathcal{A}C(X)$$

holds topologically.

Proof. If condition (Q) is satisfied, with theorem 1.52 it follows that $\mathrm{Proj}^1 A = 0$. Hence $\mathcal{A}C(X)$ is ultrabornological (theorem 1.34) and webbed (corollary 1.3). The space $C\underline{A}(X)$ is ultrabornological and webbed, since $C\underline{A}(X)$ is the ultrabornological space associated with $\mathcal{A}C(X)$ by corollary 1.57 and [26], 13.3.3. Define $\mathrm{id} : C\underline{A}(X) \to \mathcal{A}C(X)$, which is a continuous embedding. With de Wilde's [47] closed graph theorem it follows that $\mathrm{id}^{-1} : \mathcal{A}C(X) \to C\underline{A}(X)$ is continuous, too. So we have a topological isomorphism between the spaces $\mathcal{A}C(X)$ and $C\underline{A}(X)$. □

Theorem 1.59. *Let $(\mathcal{A}_n)_0C(X)$ be complete for each $n \in \mathbb{N}$. If $\mathcal{A}C(X)$ is barrelled, then $\mathcal{A}_0C(X)$ is barrelled.*

Proof. Let T_0 be a barrel in $\mathcal{A}_0C(X)$, i.e. T_0 is absolutely convex, closed and absorbant. Define

$$T := \{f \in \mathcal{A}C(X);\ \varphi f \in T_0\ \forall\ \varphi \in C_c(X), 0 \leq \varphi \leq 1\}.$$

We show that T is a barrel in $\mathcal{A}C(X)$: Choose $f, g \in T, \lambda, \mu \in I\!K = \mathbb{R}$ or $\mathbb{C}$ such that $|\lambda| + |\mu| \leq 1$. Since

$$\varphi(\lambda f + \mu g) = \lambda(\varphi f) + \mu(\varphi g) \in T_0,$$

T is absolutely convex.

Now let $(f_i)_{i\in\mathbb{N}} \subset T$ with $f_i \to f \in \mathcal{A}C(X)$ in $\mathcal{A}C(X)$. For $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$ we claim that $\varphi f_i \to \varphi f$ in $\mathcal{A}_0C(X)$. This is equivalent to $\varphi f_i \to \varphi f$ in $(\mathcal{A}_n)_0C(X)$ for each $n \in \mathbb{N}$. $(\mathcal{A}_n)_0C(X)$ carries the topology induced by its projective hull, say, $C(\overline{A}_n)_0(X)$ for $\overline{A}_n =$ the "$\overline{V}$" associated with $(v_{n,k})_{k\in\mathbb{N}}$. But for each $\overline{v} \in \overline{A}_n$:

$$p_{\overline{v}}(\varphi(f_i - f)) = \sup_{x\in X} \overline{v}(x)\varphi(x)|f_i(x) - f(x)| \leq p_{\overline{v}}(f_i - f) \to 0$$

since $f_i \to f$ in $C\overline{A}_n(X)$ follows from $f_i \to f$ in $\mathcal{A}_nC(X)$. Since the barrel T_0 is closed, $\varphi f_i \in T_0$ for each $i \in \mathbb{N}$ now implies $\varphi f \in T_0$, and hence $f \in T$. We have proved that T is closed.

For $f \in \mathcal{A}C(X)$ define

$$B_f := \{\varphi f; \varphi \in C_c(X), 0 \leq \varphi \leq 1\}.$$

B_f is bounded in $\mathcal{A}_0C(X)$ because for each $n \in \mathbb{N}$ there exist $k(n) \in \mathbb{N}, b_n > 0$ such that $a_{n,k(n)}|f| \leq b_n$ on X and then

$$a_{n,k(n)}|\varphi f| \leq a_{n,k(n)}|f| \leq b_n \text{ on } X,$$

which means that B_f is bounded in $C(a_{n,k(n)})_0(X)$, hence in $(\mathcal{A}_n)_0C(X)$ and then finally in $\mathcal{A}_0C(X)$. After the assumption that $(\mathcal{A}_n)_0C(X)$ is complete (hence locally complete), there exists a Banach disc B with $B_f \subset B$. Since by [37], 3.2.7, a barrel absorbs Banach discs, there is $\beta > 0$ such that $B_f \subset \beta T_0$. Then $\frac{1}{\beta}\varphi f \in T_0 \ \forall \ \varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$, hence $\frac{1}{\beta}f \in T$ or $f \in \beta T$. It follows that T is absorbant. This finishes the proof that T is a barrel in $\mathcal{A}C(X)$.

Since $\mathcal{A}C(X)$ is barrelled, T is a 0-neighbourhood, i.e. there exists $W \in \mathcal{U}_0(\mathcal{A}C(X))$ with $W \subset T$. After our hypothesis $\mathcal{A}_n = (a_{n,k})_{k\in\mathbb{N}}$ is regularly decreasing for each $n \in \mathbb{N}$, hence $\mathcal{A}_n = (a_{n,k})_{k\in\mathbb{N}}$ satisfies condition (D), and thus $\mathcal{A}_nC(X) = C\overline{A}_n(X)$ holds topologically for each $n \in \mathbb{N}$. Now the 0-neighbourhood W in $\mathcal{A}C(X)$ can be taken of the form

$$W = \{f \in \mathcal{A}C(X);\ \sup_{x\in X} \overline{a}_n(x)|f(x)| \leq 1\}$$

with $\overline{a}_n \in \overline{A}_n$ for some $n \in \mathbb{N}$. Define

$$V := \{g \in \mathcal{A}_0C(X); \sup_{x\in X} \overline{a}_n(x)|g(x)| \leq 1\}.$$

$\{g \in (\mathcal{A}_n)_0C(X);\ \sup_{x\in X} \overline{a}_n(x)|f(x)| \leq 1\}$ is a 0-neighbourhood in the space $(\mathcal{A}_n)_0C(X) \subset C(\overline{A}_n)_0(X)$, hence V is a 0-neighbourhood in $\mathcal{A}_0C(X)$, and if we show $V \subset T_0$, then T_0 is a 0-neighbourhood, too, which proves that $\mathcal{A}_0C(X)$ is barrelled. Let $g \in V$. Then $g \in W \subset T$ and hence $\varphi g \in T_0$ for each $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$. For each compact set $K \subset X$ let $\varphi_K \in C_c(X)$ satisfy $\varphi_K(x) = 1$ for each $x \in K$ and $0 \leq \varphi_K \leq 1$ on X. Fix $n \in \mathbb{N}$ and consider $(\varphi_K g)_K$ in $(\mathcal{A}_n)_0C(X)$. By using that $(\mathcal{A}_n)_0C(X)$ carries the topology induced by $C(\overline{A}_n)_0(X)$, one easily sees that $\varphi_K g \to g$ in $(\mathcal{A}_n)_0C(X)$. Hence one has $\varphi_K g \to g$ in $\mathcal{A}_0C(X)$, too, which yields $g \in T_0$ since T_0 is closed in $\mathcal{A}_0C(X)$. □

Even without the hypothesis that $(\mathcal{A}_n)_0C(X)$ is complete for each $n \in \mathbb{N}$, a modification of the proof of theorem 1.59 serves to show.

Corollary 1.60. *If $\mathcal{A}C(X)$ is barrelled, then $\mathcal{A}_0C(X)$ is quasibarrelled.*

Proof. Let T_0 be a bonivorous barrel in $\mathcal{A}_0C(X)$. Again define

$$T := \{f \in \mathcal{A}C(X);\ \varphi f \in T_0\ \forall\ \varphi \in C_c(X), 0 \leq \varphi \leq 1\}$$

and show that T is a barrel in $\mathcal{A}C(X)$. That T is absolutely convex and closed follows exactly in the same way as in the proof of theorem 1.59. But to show that T is absorbant we now proceed as follows: For $f \in \mathcal{A}C(X)$ again define

$$B_f := \{\varphi f; \varphi \in C_c(X), 0 \leq \varphi \leq 1\}.$$

As in the proof of 1.59, it is clear that B_f is bounded in $\mathcal{A}_0C(X)$. Since here T_0 is bornivorous, it follows that B_f is absorbed by T_0. From this point on the rest of the proof follows along the lines of the end of the proof of 1.59. □

Remark 1.61. Let the following conditions be satisfied:

i) The sequence $\mathcal{A}$ satisfies condition (Q),

ii) $\mathrm{Proj}^1 A = 0$,

iii) $\mathcal{A}C(X)$ is barrelled,

iv) $\mathcal{A}_0C(X)$ is quasibarrelled,

v) condition (P_2^*) is satisfied,

vi) the sequence $\mathcal{A}$ satisfies condition (wQ).

Then the implications

$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow v) \Rightarrow vi)$$

hold.

It is an open question if $iv) \Rightarrow ii)$ holds.

Proof. In theorem 1.52 we proved that condition (Q) for the sequence $\mathcal{A}$ is equivalent to $\mathrm{Proj}^1 A = 0$. Hence it follows that $\mathcal{A}C(X)$ is barrelled (1.34). In this section (corollary 1.60) we proved that $\mathcal{A}C(X)$ barrelled implies that $\mathcal{A}_0C(X)$ is quasibarrelled. Vogt proved in the general case of reduced spectra of (LB)-spaces that the space X is barrelled if and only if it is quasibarrelled ([43], 3.1). With corollary 1.40 it follows that condition (P_2^*) is satisfied, which is equivalent to condition (wQ) (1.48).

□

1.8 Comparison of the (PLB)- and the (LF)-space

Now we want to describe under which conditions the (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ are equal to the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$, respectively. This cannot be true in general as the following example shows for the case of $\mathcal{A}C(X)$ and $\mathcal{V}C(X)$.

Example 1.62. First we define a sequence of weights on $X := \mathbb{N} \times \mathbb{N}$ by

$$a_{n,k}(i,j) := \begin{cases} j^{-k} & \text{if } i \leq n \\ 0 & \text{otherwise} \end{cases}, \quad n, k \in \mathbb{N}.$$

$\mathcal{A} = ((a_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ is decreasing in k and increasing in n. Now define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $f(i,j) := j^{i+1/2}$ for each $(i,j) \in \mathbb{N} \times \mathbb{N}$. Fix n, and select $k(n) = n+1$. If $i \leq n$, we have

$$|f(i,j)| = j^{i+1/2} \leq j^{n+1}$$

for each j, which means that $f \in Ca_{n,k(n)}(X)$, hence $f \in \mathcal{A}_nC(X)$. Then we get $f \in \mathcal{A}C(X)$.

Suppose that $f \in \mathcal{V}C(X)$.We find k such that for each n there is $C_n > 0$ with $|f(i,j)| \le C_n j^k$ for $i = 1, ..., n$. Select $n := k$ and $i := k$ to conclude

$$j^{n+1/2} = j^{i+1/2} = |f(i,j)| \le C_n j^k = C_n j^n$$

for each j. This implies $j^{1/2} \le C_n$ for each j, which is a contradiction.

Lemma 1.63. *$\mathcal{V}C(X) \subset \mathcal{A}C(X)$ and $\mathcal{V}_0C(X) \subset \mathcal{A}_0C(X)$ holds in general with continuous inclusions.*

Proof. One can write $\mathcal{V}C(X) = \text{ind}_k\text{proj}_n Ca_{n,k}(X)$. Let $f \in \mathcal{V}C(X)$. Thus there exists $k \in \mathbb{N}$ such that $||f||_{n,k} < +\infty$ for all $n \in \mathbb{N}$. This implies that for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $||f||_{n,k} < +\infty$, hence $f \in \mathcal{A}_nC(X)$ for each $n \in \mathbb{N}$ and therefore $f \in \mathcal{A}C(X)$. A similar argument gives $\mathcal{V}_0C(X) \subset \mathcal{A}_0C(X)$. The argument for the continuous inclusions is as follows:

$$\text{proj}_n Ca_{n,k}(x) \to Ca_{n,k}(X)$$

is continuous for each $n, k \in \mathbb{N}$. Then

$$\text{proj}_n Ca_{n,k}(x) \to \text{ind}_k Ca_{n,k}(X) = \mathcal{A}_nC(X)$$

is continuous for each $n \in \mathbb{N}$, and thus

$$\mathcal{V}C(X) = \text{ind}_k\text{proj}_n Ca_{n,k}(X) \to \mathcal{A}_nC(X)$$

must be continuous for each $n \in \mathbb{N}$, whence the continuity of

$$\mathcal{V}C(X) \to \mathcal{A}C(X).$$

□

Next we introduce a condition which was used by Vogt [41] as a characterisation for Fréchet spaces between which all continuous linear mappings are bounded.

Definition 1.64. Let $\mathcal{A} = ((a_{m,l})_{l\in\mathbb{N}})_{n\in\mathbb{N}}$ be a sequence of weights. $\mathcal{A}$ satisfies condition (B) if $\forall\ k(n)\ \exists\ l \in \mathbb{N}\ \forall\ m \in \mathbb{N}\ \exists\ \tilde{n} \in \mathbb{N}, c > 0$:

$$a_{m,l} \le c \max_{1\le n\le \tilde{n}} a_{n,k(n)}.$$

Theorem 1.65. *$\mathcal{A}C(X) = \mathcal{V}C(X)$ holds algebraically if and only if the sequence $\mathcal{A}$ satisfies condition (B).*

Proof. Let $\mathcal{A}$ satisfy condition (B). $\mathcal{V}C(X) \subset \mathcal{A}C(X)$ holds by 1.63. To show the other inclusion, choose $f \in \mathcal{A}C(X)$. After the definition of $\mathcal{A}C(X)$ it follows that: $\forall\, n \in \mathbb{N}\ \exists\, k(n) \in \mathbb{N}, b_n > 0$:

$$a_{n,k(n)}(x)|f(x)| \leq b_n \ \forall\, x \in X.$$

For given $(k(n))_{n\in\mathbb{N}}$ we can apply condition (B) to find l such that for each m there exist $\tilde{n}, c > 0$ with

$$a_{m,l} \leq c \max_{1\leq n\leq \tilde{n}} a_{n,k(n)}.$$

We claim that $f \in CV_l(X)$, i.e. $f \in Ca_{m,l}C(X)$ for each $m \in \mathbb{N}$. Indeed, for given m, one can select $\tilde{n} := \tilde{n}(m)$ and $c := c_m > 0$ as in condition (B) and then for each $x \in X$:

$$a_{m,l}(x)|f(x)| \leq c \max_{1\leq n\leq \tilde{n}} a_{n,k(n)}(x)|f(x)| \leq c \max_{1\leq n\leq \tilde{n}} b_n < \infty,$$

and $\sup_{x\in X} a_{m,l}(x)|f(x)| < \infty\ \forall\, m \in \mathbb{N}$, hence $f \in CV_l(X) \subset \mathcal{V}C(X)$.

In the other direction, for given $(k(n))_{n\in\mathbb{N}}$ consider the space

$$\begin{aligned} F &:= \cap_{n\in\mathbb{N}} Ca_{n,k(n)}(X) \\ &= \{f \in C(X);\ \sup_{x\in X} a_{n,k(n)}(x)|f(x)| =: ||f||_n < \infty\ \forall\, n \in \mathbb{N}\}. \end{aligned}$$

Clearly $F \subset \mathcal{A}C(X)$. Observe that the norms $||\cdot||_n$ do not satisfy

$$||\cdot||_n \leq ||\cdot||_{n+1}$$

in general. The space F with the norms $p_n(f) := \max_{1\leq\nu\leq n} ||f(x)||_\nu$ is a Fréchet space because F is continuously injected in $\mathcal{A}C(X)$, which has a topology finer than the compact-open topology. By the assumption $\mathcal{A}C(X) = \mathcal{V}C(X)$ we get $F \subset \mathcal{V}C(X) = \mathrm{ind}_k CV_k(X)$, which is an (LF)-space. Morever the inclusion has closed graph because $\mathcal{V}C(X)$ is continuously included in $\mathcal{A}C(X)$. By Grothendieck‘s factorisation theorem there is l such that $F \subset CV_l(X)$, and the inclusion is continuous. This implies $\forall\, m\ \exists\, \tilde{n}, c > 0$ such that, for each $g \in F$,

$$(***) \quad \sup_{x\in X} a_{m,l}(x)|g(x)| \leq c \max_{1\leq n\leq \tilde{n}} \sup_{x\in X} a_{n,k(n)}(x)|g(x)|.$$

Suppose $\exists\, x_0 \in X$ with

$$a_{m,l}(x_0) > \alpha > c \max_{1\le n\le \tilde{n}} a_{n,k(n)}(x_0).$$

By continuity one can find a compact neighbourhood W_0 of x_0 with

$$c \max_{1\le n\le \tilde{n}} a_{n,k(n)}(x) \le \alpha \quad \forall x \in W_0.$$

Select $\varphi \in C(X,[0,1])$ with $\mathrm{supp}\varphi \subset W_0$ and $\varphi(x_0) = 1$. Clearly $\varphi \in F$. We apply $(***)$ to conclude

$$\begin{aligned} a_{m,l}(x_0) &= a_{m,l}(x_0)|\varphi(x_0)| \le \sup_{x\in X} a_{m,l}(x)|\varphi(x)| \\ &\le c \max_{1\le n\le \tilde{n}} \sup_{x\in X} a_{n,k(n)}(x)|\varphi(x)| \le c \max_{1\le n\le \tilde{n}} \sup_{x\in W_0} a_{n,k(n)}(x) \\ &\le \alpha. \end{aligned}$$

This is a contradiction. Hence condition (B) holds. □

Theorem 1.66. *If the sequence $\mathcal{A}$ satisfies condition (B), then the space $\mathcal{A}_0C(X)$ equals $\mathcal{V}_0C(X)$ algebraically. If each $(\mathcal{A}_n)_0(X)$ is complete, which is equivalent to $\mathcal{A}_n = (a_{n,k})_{k\in\mathbb{N}}$ regularly decreasing for each $n \in \mathbb{N}$, then the converse is also true.*

Proof. First we show that from condition (B) it follows that $\mathcal{A}_0C(X) = \mathcal{V}_0C(X)$ holds algebraically. $\mathcal{V}_0C(X) \subset \mathcal{A}_0C(X)$ holds in general (see 1.63). Now let the sequence $\mathcal{A}$ satisfy condition (B). Choose $f \in \mathcal{A}_0C(X)$. Then after the definition of $\mathcal{A}_0C(X)$ it follows that: $\forall\, n \in \mathbb{N}\ \exists\, k(n) \in \mathbb{N}\ \forall\, \varepsilon > 0\ \exists\, K(\varepsilon) \subset X$ compact:

$$(+) \qquad a_{n,k(n)}(x)|f(x)| \le \varepsilon \quad \forall\, x \in X \setminus K(\varepsilon).$$

For given $(k(n))_{n\in\mathbb{N}}$ we can apply condition (B) to find l such that for each m there exist $\tilde{n} = \tilde{n}(m), c = c_m > 0$:

$$a_{m,l} \le c \max_{1\le n\le \tilde{n}} a_{n,k(n)}.$$

We claim that $f \in C(V_l)_0(X)$, i.e. $f \in C(a_{m,l})_0(X)$ for each $m \in \mathbb{N}$. Let $\varepsilon > 0$ be given. For fixed $n \in \mathbb{N}, 1 \le n \le \tilde{n}$, by $(+)$ there exists $k(n)$ such that for $\tilde{\varepsilon} := \frac{\varepsilon}{c}$ there exists a compact set $K_n \subset X$ with

$$a_{n,k(n)}(x)|f(x)| \le \tilde{\varepsilon}\ \forall\, x \in X \setminus K_n.$$

Then

$$a_{m,l}(x)|f(x)| \leq c \max_{1\leq n\leq \tilde{n}} a_{n,k(n)}(x)|f(x)| \leq c\tilde{\varepsilon} = \varepsilon$$

for all $x \in X \setminus K$ with $K := \cup_{\nu=1}^{\tilde{n}} K_n$, which proves our claim.

Now let all $(\mathcal{A}_n)_0 C(X)$ be complete and let $\mathcal{A}_0 C(X) = \mathcal{V}_0 C(X)$ hold algebraically. We have to show condition (B). Similarly to the proof of 1.65, for given $(k(n))_{n\in\mathbb{N}}$ we define $F_0 := \cap_n C(a_{n,k(n)})_0(X)$ and use the same arguments to conclude that condition (B) is satisfied. □

Corollary 1.67. *If all $(\mathcal{A}_n)_0 C(X)$ are complete, then $\mathcal{A}_0 C(X) = \mathcal{V}_0 C(X)$ holds algebraically and topologically if and only if the sequence $\mathcal{A}$ satisfies the conditions (B) and (wQ).*

Proof. When $\mathcal{A}_0 C(X) = \mathcal{V}_0 C(X)$ holds topologically, then the space $\mathcal{A}_0 C(X)$ is ultrabornological as an (LF)-space. With theorem 1.48 it follows that the sequence $\mathcal{A}$ satisfies condition (wQ), and condition (B) follows from 1.66.

Now let $\mathcal{A}$ satisfy condition (wQ). With theorem 1.48 it follows that $\mathcal{A}_0 C(X)$ is ultrabornological. As an (LF)-space $\mathcal{V}_0 C(X)$ is webbed. Define id : $\mathcal{V}_0 C(X) \to \mathcal{A}_0 C(X)$, which is a continuous embedding. With de Wilde's [47] closed graph theorem it follows that $\text{id}^{-1} : \mathcal{A}_0 C(X) \to \mathcal{V}_0 C(X)$ is continuous. So we have a topological isomorphism between $\mathcal{A}_0 C(X)$ and $\mathcal{V}_0 C(X)$. □

1.9 An example in the case of sequence spaces

Before we can illustrate the previous results with an example in the case of sequence spaces we have to introduce Köthe sequence spaces and some of their properties. For the definitions and notations see Bierstedt, Meise, Summers [15]. Some further results which are needed here were given by Bierstedt, Bonet [5], [4].

Definition 1.68. Let $A = (a_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive functions on some index set I; A is called a Köthe matrix. In the following example we chose $I = \mathbb{N}$ and omit it from our notation.

We define the Köthe echelon spaces of order $p, 1 \leq p \leq \infty$ or $p = 0$, as follows:

$$\begin{aligned}
\lambda_p(A) &= \{x \in I\!K^{\mathbb{N}}; ||x||_n := (\sum_{i=1}^{\infty} |x_i a_n(i)|^p)^{1/p} < \infty \ \forall\ n \in \mathbb{N}\}, 1 \le p \le \infty, \\
\lambda_\infty(A) &= \{x \in I\!K^{\mathbb{N}}; ||x||_n := \sup_{i\in\mathbb{N}} |x_i| a_n(i) < \infty \ \forall\ n \in \mathbb{N}\}, \\
\lambda_0(A) &= \{x \in \lambda_\infty(A); \lim_{i\to\infty} x_i a_n(i) = 0 \ \forall\ n \in \mathbb{N}\}
\end{aligned}$$

with

$$\lambda_p(A) = \mathrm{proj}_n l_p(a_n(i)),\ 1 \le p \le \infty,$$

and

$$\lambda_0(A) = \mathrm{proj}_n c_0(a_n(i))$$

algebraically and topologically.

For every Köthe matrix A, the spaces $\lambda_p(A)$, $1 \le p \le \infty$ and $p = 0$, are Fréchet spaces.

Taking $V = (v_n(i))_{n\in\mathbb{N}}$ to denote the associated decreasing sequence of functions $v_n(i) = \frac{1}{a_n(i)}$, we put

$$k_p(V) = \mathrm{ind}_i l_p(v_n(i)),\ 1 \le p \le \infty,$$

and

$$k_0(V) = \mathrm{ind}_i c_0(v_n(i)).$$

These are Köthes‘s co-echelon spaces. Define

$$\overline{V} = \lambda_\infty(A)_+ = \{\overline{v} \in (\mathbb{R}_+)^{\mathbb{N}}; \sup_{i\in\mathbb{N}} \frac{\overline{v}_i}{v_n(i)} < \infty \ \text{ for each } n \in \mathbb{N}\}$$

and

$$K_p(\overline{V}) = \lambda_p(\overline{V}) = \mathrm{proj}_{\overline{v}\in\overline{V}} l_p(\overline{v})$$

for $1 \le p \le \infty$ as well as

$$K_0(\overline{V}) = \mathrm{proj}_{\overline{v}\in\overline{V}} c_0(\overline{v}).$$

Remark 1.69. (Bierstedt, Meise, Summers [15], 1.5) $k_p(V)$ is continuously embedded in $K_p(\overline{V})$.

Let E, F be locally complete locally convex spaces. $L_b(E, F)$ denotes the space of all continuous linear mappings from E to F, endowed with the strong topology. For $\overline{V}$ and a locally convex space E, it is clear how $K_\infty(\overline{V}, E)$ is defined.

Proposition 1.70. *[5] Let E denote a locally complete locally convex space. Then there is a canonical topological isomorphism*

$$K_\infty(\overline{V}, E) = L_b(\lambda_1(A), E)$$

of $K_\infty(E)$ onto the space of all continuous linear mappings from $\lambda_1(A)$ into E, endowed with the topology of uniform convergence on the bounded subsets of $\lambda_1(A)$. In particular, $K_\infty(\overline{V}) = (\lambda_1(A))'_b$ (see [15],2.7).

Now we come to the main example:

Example 1.71. Let $B = (b_k(i))_{k\in\mathbb{N}}, i \in \mathbb{N}$ and $C = (c_n(j))_{n\in\mathbb{N}}, j \in \mathbb{N}$ be Köthe matrices. Consider the space $L_b(\lambda_1(B), \lambda_\infty(C))$. We can write $\lambda_1(B) = \text{proj}_k l_1(b_k(i))$ and $\lambda_\infty(C) = \text{proj}_n l_\infty(c_n(j))$. Suppose that $\lambda_1(B)$ is distinguished, which holds if and only if $(b_k(i))_{k\in\mathbb{N}}$ satisfies condition (D) (see [4], 6.). Bierstedt, Meise, Summers (see [15], 2.8) proved that $\lambda_1(B)$ distinguished is equivalent to $(\lambda_1(B))'_b \cong k_\infty(W)$. With this result and taking $W = (w_k(i))_{k\in\mathbb{N}}, w_k(i) := b_k(i)^{-1}$, we obtain that

$$\lambda_1(B)'_b \cong k_\infty(W) = \text{ind}_k l_\infty(w_k(i)).$$

Now we have $L_b(\lambda_1(B), \lambda_\infty(C)) = \text{proj}_k L_b(\lambda_1(B), l_\infty(c_n(j)))$. Since $\lambda_1(B)$ is distinguished, $L_b(\lambda_1(B), l_\infty(c_n(j)))$ is ultrabornological (see [5]) and by Bierstedt, Bonet ([4], 6. and 7.)

$$\begin{aligned} L_b(\lambda_1(B), l_\infty(c_n(j))) &\cong \text{ind}_k L_b(l_1(b_k(i)), l_\infty(c_n(j))) \\ &\cong \text{ind}_k l_\infty(w_k(i), l_\infty(c_n(j))) \\ &= \text{ind}_k l_\infty(w_k(i) \otimes c_n(j)). \end{aligned}$$

holds. This implies that

$$L_b(\lambda_1(B), \lambda_\infty(C)) = \text{proj}_n \text{ind}_k l_\infty(w_k(i) \otimes c_n(j))$$

holds algebraically and topologically. If $\lambda_1(A)$ is distinguished, the space $L_b(\lambda_1(B), \lambda_\infty(C))$ is of the form $\mathcal{A}C(X)$ with $X = \mathbb{N} \times \mathbb{N}$ and $a_{n,k}(i,j) := \frac{1}{b_k(i)} \otimes c_n(j)$. An easy argument shows that the isomorphism above induces a linear isomorphism between the space $LB(\lambda_1(B), \lambda_\infty(C))$ of bounded linear maps and $\mathcal{V}C(X) = \text{ind}_k \text{proj}_n C a_{n,k}(X)$.

Corollary 1.72. *If* $\mathcal{A} = ((a_{n,k})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ *is defined by* $a_{n,k}(i,j) = \frac{1}{b_k(i)} \otimes c_n(j)$, $i, j \in \mathbb{N}$*, and satisfies condition* (Q)*, then the space* $L_b(\lambda_1(B), \lambda_\infty(C))$ *is barrelled.*

Proof. For a general (PLB)-space $\mathcal{A}C(X)$ we have proved in theorem 1.52 that condition (Q) is satisfied if and only if $\mathrm{Proj}^1 A = 0$. With the general theory of (DF)-spaces (see theorem 1.34) it follows that $L_b(\lambda_1(B), \lambda_\infty(C))$ is barrelled. □

Corollary 1.73. *If* $L_b(\lambda_1(B), \lambda_\infty(C))$ *is barrelled, then* $\mathcal{A} = ((a_{n,k})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ *defined by* $a_{n,k}(i,j) = \frac{1}{b_k(i)} \otimes c_n(j), i, j \in \mathbb{N}$*, satisfies condition* (wQ)*.*

Proof. With corollary 1.60 from $\mathcal{A}C(X)$ barrelled it follows that the space $\mathcal{A}_0C(X)$ is quasibarrelled (corollary 1.60). Again with the general theory of projective spectra of (LB)-spaces (see [43]) this implies that $\mathcal{A}_0C(X)$ is barrelled, and condition (wQ) follows with theorem 1.48. □

2 Weighted spaces of holomorphic functions on the half-plane

2.1 Introduction to part 2

The second chapter of this work deals with weighted Banach spaces of holomorphic functions on the upper half-plane G. Let $v : G \to \mathbb{R}_+$ be a strictly positive, continuous function (weight). The space $Hv_0(G)$ is defined as follows:

$$Hv_0(G) := \{f \in H(G); v|f| \text{ vanishes at infinity on } G\}.$$

This chapter is motivated by a question of Bierstedt [3]. In a survey about weighted inductive limits of spaces of holomorphic functions he asked if the space $Hv_0(G)$ has the approximation property under some conditions of Holtmanns [25]. The problem remains open in general, but we give a positive answer for weights with two additional conditions. Actually we can then even show the existence of a basis.

In section 2.2 we give the necessary notation and an overview about results for weighted spaces of holomorphic functions on certain domains. The main result (theorem 2.13) is given in section 2.3. Next, in section 2.4, we present some preparations before we give the proof of theorem 2.13 in section 2.5. In the last section we give some examples for the weights.

An article similar to this part, except for section 2.2, is accepted for puplication in Bull. Soc. R. Sci. Liège.

2.2 Notation and known results

Let $G \subset \mathbb{C}$ or $\mathbb{C}^N, N \in \mathbb{N}$, and $v : G \to \mathbb{R}_+$ be a weight on G, i.e. a strictly positive, continuous function. Define

$$Hv(G) := \{f \in H(G);\ ||f||_v := \sup_{z \in G} v(z)|f(z)| < \infty\},$$

$$Hv_0(G) := \{f \in H(G);\ vf \text{ vanishes at infinity on } G\}.$$

$Hv_0(G)$ is a closed subspace of $Hv(G)$, and both spaces are complete, hence Banach spaces, where $Hv_0(G)$ carries the induced norm.

The unit balls of these spaces are denoted as follows:

$$B \ \ := \{f \in Hv(G); \sup_{z \in G} v(z)|f(z)| \leq 1\},$$

$$B_0 := \{f \in Hv_0(G); \sup_{z \in G} v(z)|f(z)| \leq 1\}.$$

In 1993 Bierstedt, Bonet, Galbis [10] investigated weighted spaces of holomorphic functions for radial weights v on balanced domains G and proved the metric approximation property of $Hv_0(G)$ if $Hv_0(G)$ contains the polynomials. For starshaped domains and admissible weights Kaballo and Vogt [27] had already proved the approximation property by a different method. More recently Stanev [40] studied weighted spaces of holomorphic functions on the upper half-plane. He gave a characterisation when these spaces are not trivial. In her thesis Holtmanns [25] investigated biduals of weighted spaces of holomorphic functions on the upper-half plane G. She introduced natural conditions on the weight v such that $Hv_0(G)''$ and $Hv(G)$ are isometrically isomorphic. We now present some more details of the results mentioned so far.

Definition 2.1. A Banach space X has the *approximation property* (a.p.) if for any compact subset $M \subset X$ and any $\varepsilon > 0$ there is a linear finite rank operator $L : X \to X$ with $||Lx - x|| \leq \varepsilon$ for every $x \in M$. If there is $\lambda \geq 1$ such that in addition L can always be chosen with $||L|| \leq \lambda$, then X has the *bounded approximation property* (b.a.p.). If λ can be chosen to be 1, one says that X has the *metric approximation property* (m.a.p.).

Definition 2.2. Let G be a *starshaped* bounded open set around zero in $\mathbb{C}^N$, which means that $\overline{G} \subset G_\rho := \{z \in \mathbb{C}^N; \rho z \in G\}$ for $0 < \rho < 1$. A weight $v : G \to \mathbb{R}_+$ with $\lim_{z \to \partial G} v(z) = 0$ is called *admissible* if $v(z) \leq v(\rho z)$ holds for all $z \in G$ and $0 \leq \rho \leq 1$.

Kaballo, Vogt [27] presented the following result in 1980:

Theorem 2.3. *Let $G \subset \mathbb{C}$ be a starshaped bounded open set around zero and $v : G \to \mathbb{R}_+$ be an admissible weight on G. Then $Hv_0(G)$ has the approximation property.*

The theorem was proved by use of the operator $T_\rho : Hv_0(G) \to H(G_\rho)$, $(T_\rho f)(z) = f(\rho z)$. The space $H(G_\rho)$ has the approximation property, and showing that for $\rho \to 1$ the operator T_ρ tends to the identity uniformly on the compact subsets of $Hv_0(G)$, it follows that $Hv_0(G)$ has the approximation property. This proof also shows that $Hv_0(G)$ has the bounded approximation property if $A(\overline{G}) := \{f \in C(\overline{G}); f_{|G} \text{ holomorph}\}$ has. An obvious example for this situation is the unit disc in $\mathbb{C}$, but $A(\overline{G})$ also has the bounded approximation property for bounded balanced domains $G \subset \mathbb{C}^N$.

Definition 2.4. Let G be a balanced open subset of $\mathbb{C}^N$. A weight $v : G \to \mathbb{R}$ is called *radial* if $v(\lambda z) = v(z)$ for all $z \in G$ and all $\lambda \in \mathbb{C}, |\lambda| = 1$.

Theorem 2.5. *(Bierstedt, Bonet, Galbis [10]) Let G be a balanced open subset of $\mathbb{C}^N$, $v : G \to \mathbb{R}_+$ be a radial weight and let $Hv_0(G)$ contain all the polynomials. Then $Hv_0(G)$ has the bounded approximation property, and the polynomials are dense in $Hv_0(G)$.*

In the proof of theorem 2.5 the authors used the Cesàro means of the partial sums of the Taylor series about 0 to construct linear operators of finite rank from $Hv(G)$ into $Hv_0(G)$.

To answer the question of Bierstedt, it is not possible to use the same arguments and ideas as in the case of radial weights on balanced domains.

Now let G be the upper half-plane, $G = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$. Stanev [40] presented conditions under which weighted spaces of holomorphic functions on the upper half-plane are not trivial. His notation is different from the usual one. He considered functions $p : \mathbb{R}_+ \to \mathbb{R}_+$ with $\inf_{t \in [\frac{1}{c}, c]} p(t) > 0$ for all $c > 1$ and the norm

$$||f||_p := \sup_{z \in G} p(\mathrm{Im} z)|f(z)|.$$

Theorem 2.6. *(Stanev [40]) Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a function as above and put $v(z) = v_p(z) := p(\mathrm{Im} z), z \in G$.*

i) *$Hv(G) \neq \{0\}$ if and only if there exist $a, b \in \mathbb{R}$ such that*

$$(-1) \ln p(t) \geq at + b$$

for all $t > 0$.

ii) *$Hv_0(G) \neq \{0\}$ if and only if the following two conditions on the function p are satisfied:*

(a) *there exist $a, b \in \mathbb{R}$ such that $(-1) \ln p(t) \geq at + b$ for all $t > 0$,*

(b) $\lim_{t \to 0_+} p(t) = 0$.

Next we present a result of Holtmanns for weighted spaces of holomorphic functions on the upper half-plane G and their biduals. In her proof she used a general result of Bierstedt, Summers which we will give first.

Proposition 2.7. *(Bierstedt, Summers [16]) If B_0 is dense in B in the compact open topology , then $Hv(G)$ is isometrically isomorphic to the bidual $Hv_0(G)''$.*

Theorem 2.8. *(Holtmanns [25]) Let G be the upper half-plane and let v be a continuous weight on G such that:*

(i) *$v > 0$ on G,*

(ii) $\lim_{\mathrm{Im} z \to 0} v(z) = 0$,

(iii) *there exists $0 < r_0 < 1$ with $v(z) \leq v(z + ir)$ for all $z \in G$ and $0 < r \leq r_0$.*

Then $Hv_0(G)''$ and $Hv(G)$ are isometrically isomorphic.

For $f \in Hv_0(G)$ Holtmanns introduced auxiliary functions

$$f_n(z) := f(z + \frac{1}{n}) \sqrt[n]{\frac{1}{z+i}}, \; z \in G, n \in \mathbb{N}$$

(as in the proof of the classical Phragmen-Lindelöf theorem) to prove the condition of proposition 2.7 in her case. The operators $f \to f_n$ on $Hv_0(G)$ will be important for the proof of our main result, too.

2.3 Commuting b.a.p. and the main result

While we tried to solve the problem of Bierstedt, it turned out that two additional conditions on the weights were needed. With these conditions and a result of Lusky [30] it was even possible to show the existence of a basis.

Definition 2.9. Let X be a Banach space. A sequence $(e_j)_{j\in\mathbb{N}}$ is called *Schauder basis* of X if for each $x \in X$ there is a uniquely determined sequence $(\xi_j(x))_{j\in\mathbb{N}}$ in $I\!K$, for which $x = \sum_{j=1}^{\infty} \xi_j(x) e_j$ is true.

Definition 2.10. Let X be a Banach space. A sequence of bounded linear operators $V_n : X \to X$ of finite rank is called *commuting approximating sequence* (c.a.s.) if $\lim_{n\to\infty} V_n x = x$ for each $x \in X$ and $V_n V_m = V_{\min(n,m)}$ whenever $n \neq m$. If there exists such a sequence $(V_n)_{n\in\mathbb{N}}$ on X, then X is said to have the *commuting bounded approximation property* (CBAP). If $V_n V_m = V_{\min(n,m)}$ holds, in addition, even for $n = m$ then X is said to have a *finite dimensional Schauder decomposition* (FDD).

Clearly, by the Banach-Steinhaus theorem (CBAP) implies the bounded approximation property. It is known that there are Banach spaces with (CBAP) which do not have (FDD).

Definition 2.11. Let X be a given Banach space. For a fixed p with $1 \leq p \leq \infty$ we say that a sequence of continuous linear operators $V_n : X \to X$ *factors uniformly through l_p^m's* with respect to λ if there are suitable integers $m_n \in \mathbb{N}$ and continuous linear operators

$$T_n : X \to l_p^{m_n}, \ S_n : l_p^{m_n} \to X,$$

with

$$V_n = S_n T_n, \ \sup_n ||T_n|| \leq \lambda \text{ and } \sup_n ||S_n|| \leq \lambda.$$

In 1996 Lusky [30] presented the following result which we will use in the case $p = \infty$ to show that $Hv_0(G)$ has a basis.

Theorem 2.12. *Let the Banach space X have a commuting approximating sequence $(V_n)_{n\in\mathbb{N}}$ such that $V_n - V_{n-1}$ factors uniformly through l_p^m's for some $1 \leq p \leq \infty$. Then X has a basis.*

From now on G is the upper half-plane. For our main result we need the following conditions on the weight v. Let $v : G \to \mathbb{R}$ be continuous such that

(i) $v > 0$ on G,

(ii) $\lim_{\mathrm{Im}z \to 0} v(z) = 0$,

(iii) there exists $0 < r_0 < 1$ with $v(z) \leq v(z + ir)$ for all $z \in G$ and $0 < r \leq r_0$,

(iv) for each $\varepsilon > 0$ there exists $b = b(\varepsilon) > 0$ such that $v(z) \geq b$ for all $z \in G$ with $\mathrm{Im}z \geq \varepsilon$,

(v) v is bounded.

The first three conditions were introduced by Holtmanns [25]. She did not require conditions (iv) and (v) for her work, but these conditions seem to be necessary for our result. The following is the main result of the second part of this work.

Theorem 2.13. *Let G be the upper half-plane and v a weight on G which satisfies conditions (i)-(v) above. Then $Hv_0(G)$ has a basis.*

With theorem 2.12 above the proof of theorem 2.13 is reduced to showing that $Hv_0(G)$ has a commuting approximating sequence $\{V_n\}_{n=1}^{\infty}$ such that $V_n - V_{n-1}$ factors uniformly through l_∞^m's.

2.4 Preparations

In the sequel some technical tools are given which are needed for the proof. In her thesis [25] Holtmanns defined linear operators Θ_n as follows:

Definition 2.14. For $f \in Hv_0(G)$ let

$$\Theta_n : Hv_0(G) \to Hv_0(G), \ n \in \mathbb{N}, \ \Theta_n f := f_n$$

$$\text{with } f_n(z) := f(z + \frac{i}{n}) \sqrt[n]{\frac{1}{z+i}} \text{ for } z \in G.$$

The main branch of the n-th root is well-defined since $z \to \frac{1}{z+i}$ maps G into the set $\{z \in \mathbb{C} \,;\, \mathrm{Im}z < 0 \text{ and } |z| < 1\}$. The functions f_n are holomorphic on G since $z + i \neq 0$ for all $z \in G$.

Lemma 2.15. *(Holtmanns [25]) Θ_n is well-defined and continuous as an operator from $Hv_0(G)$ into $Hv_0(G)$. $\Theta_n f$ converges to f in the compact-open topology, $f \in Hv_0(G)$, since $|\sqrt[n]{\frac{1}{z+i}}| \to 1$ for $n \to \infty$.*

Lemma 2.16. *Let $f \in Hv_0(G)$ and Θ_n be as defined before. For each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and a compact set $K \subset G$ with $v(z)|\Theta_n f(z) - f(z)| \leq \varepsilon$ for all $z \in G \setminus K$ and for any fixed $n \in \mathbb{N}, n \geq n_0$.*

Proof. Let $\varepsilon > 0$ be given. Set $\tilde{\varepsilon} = \frac{1}{4}\varepsilon$. $f \in Hv_0(G)$ means that there exist $L > 0$ and $0 < l < \frac{1}{2}$ with

$$v(z)|f(z)| \leq \tilde{\varepsilon} \ \forall \ z \in G \setminus [-L, L] \times i[l, L].$$

Set $K := [-L, L] \times i[\frac{l}{2}, L]$. For all $z \in G \setminus K$ the following inequality holds for $n \in \mathbb{N}$ large enough such that condition (iii) can be applied:

$$\begin{aligned} v(z)|\Theta_n f(z) - f(z)| &\leq v(z)(|f_n(z) - f(z + \tfrac{i}{n})| + |f(z + \tfrac{i}{n}) - f(z)|) \\ &\leq v(z)|f(z + \tfrac{i}{n}) \sqrt[n]{\tfrac{1}{z+i}} - f(z + \tfrac{i}{n})| \\ &\quad + v(z)|f(z + \tfrac{i}{n})| + v(z)|f(z)| \\ &\leq v(z + \tfrac{i}{n})|f(z + \tfrac{i}{n})||\sqrt[n]{\tfrac{1}{z+i}} - 1| \\ &\quad + v(z + \tfrac{i}{n})|f(z + \tfrac{i}{n})| + v(z)|f(z)|. \end{aligned}$$

Let us now show that $v(z + \frac{i}{n})|f(z + \frac{i}{n})| \leq \tilde{\varepsilon}$ for $n \in \mathbb{N}$ large enough. Two cases are possible:
Case 1: $|\text{Re}z| > L$ or $\text{Im}z > L$. Then $z \notin K \Rightarrow z + \frac{i}{n} \notin K \Rightarrow v(z + \frac{i}{n})|f(z + \frac{i}{n})| \leq \tilde{\varepsilon}$.
Case 2: $\text{Im}z < \frac{l}{2}$ and $|\text{Re}z| \leq L$. Then there exists $n_0 \in \mathbb{N}$ with $\frac{1}{n} < \frac{l}{2}$ for all $n \in \mathbb{N}, n \geq n_0$. $z + \frac{i}{n} = x + i(y + \frac{1}{n})$ with $y + \frac{1}{n} < \frac{l}{2} + \frac{1}{n} \leq \frac{l}{2} + \frac{l}{2} = l$
$\Rightarrow \text{Im}(z + \frac{i}{n}) < l \Rightarrow v(z + \frac{i}{n})|f(z + \frac{i}{n})| \leq \tilde{\varepsilon}$.
On the other hand, $\sup_{z \in G} |\sqrt[n]{\frac{1}{z+i}}| = \sup_{z \in G} \sqrt[n]{\frac{1}{|z+i|}} = 1 \ \forall n \in \mathbb{N}$ since $|z + i| \geq |\text{Im}z| + 1 \geq 1 \ \forall z \in G$, and hence $|1 - \sqrt[n]{\frac{1}{z+i}}| \leq 2$.

Using these two estimates in the right hand side of the above inequality yields

$$v(z)|\Theta_n f(z) - f(z)| \leq 2\tilde{\varepsilon} + \tilde{\varepsilon} + \tilde{\varepsilon} \leq \varepsilon$$

for each $z \in G \setminus K$. □

Corollary 2.17. *With lemma 2.15 and lemma 2.16 it follows that for $f \in Hv_0(G)$ and for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $||\Theta_n f - f||_v \leq \varepsilon$ for any fixed $n \in \mathbb{N}, n \geq n_0$.*

Next we define the space $A_0(G)$, extend the operator Θ_n to $\overline{G}$ and show that f_n maps $Hv_0(G)$ to $A_0(G)$ and that there exists a restriction mapping back to $Hv_0(G)$.

Definition 2.18. Define

$$\begin{aligned} A_0(G) := \quad & \{f \in C(\overline{G}); f_{|G} \in H(G),\ \forall\ \eta > 0\ \exists\ N \in \mathbb{R}_+ : \\ & |f(z)| < \eta\ \forall z \in G, |z| \geq N\}, \end{aligned}$$

endowed with the sup-norm and extend $\Theta_n f$ continuously to $\overline{G}$ by taking $(\Theta_n f)(x) = f(x + \frac{1}{n}) \sqrt[n]{\frac{1}{x+i}}$ for $x \in \mathbb{R}$.

Lemma 2.19. *For each $f \in Hv_0(G)$ and each $n \in \mathbb{N}$ we have $\Theta_n f \in A_0(G)$, i.e. there exists a linear mapping*

$$R_n : Hv_0(G) \to A_0(G), \quad R_n f = f_n\ \forall\ n \in \mathbb{N}.$$

Proof. Let $f \in Hv_0(G)$ and $n \in \mathbb{N}$ be fixed. Set $\varepsilon = \frac{1}{n}$. With condition (iv) for the weight v there exists $b = b(\frac{1}{n}) > 0$ with $v(z) \geq b$ for all $z \in G$ with $\mathrm{Im} z \geq \varepsilon$. Then for each $z \in G$, also $v(z + \frac{i}{n}) \geq b$ holds. Now fix $\eta > 0$. $f \in Hv_0(G)$ means that for $\tilde{\eta} := \eta \cdot b$ there exists $N > 0$ such that

$$|f(z + \frac{i}{n})| v(z + \frac{i}{n}) \leq \tilde{\eta}$$

for all $z \in G$ with $|z| \geq N$. Then for f_n and such a $z \in G$ the following estimate holds:

$$\begin{aligned} |f_n(z)| &= |f(z + \tfrac{i}{n})| |\sqrt[n]{\tfrac{1}{z+i}}| \\ &= |f(z + \tfrac{i}{n})| v(z + \tfrac{i}{n}) \tfrac{1}{v(z+\frac{i}{n})} |\sqrt[n]{\tfrac{1}{z+i}}| \\ &\leq \tilde{\eta} \cdot \tfrac{1}{b} = \eta, \end{aligned}$$

hence $f_n \in A_0(G)$. ∎

Lemma 2.20. *The restriction mapping*

$$R : A_0(G) \to Hv_0(G), \quad f \to f_{|G},$$

is well-defined and continuous.

Proof. Fix $f \in A_0(G)$. By condition (v), v is bounded, i.e. there exists $M > 0$ with $v(z) \leq M$ for all $z \in G$. Let $\eta > 0$ be arbitrary, but fixed. Set $\eta' := \frac{\eta}{M}$. For η' there exists $N > 0$ such that $|f(z)| < \eta'$ for all $z \in G$ with $|z| \geq N$. Then $v(z)|f(z)| < M\frac{\eta}{M} = \eta$ for all $z \in G$ with $|z| \geq N$. Define $L := N + 1$. By condition (ii) we can extend v continuously to $\overline{G}$ by putting $\tilde{v}(z) := v(z)$ for $z \in G$ and $\tilde{v}(z) := 0$ elsewhere. $\tilde{v}$ is uniformly continuous on $K := [-L, L] \times i[\delta, L]$ for each $\delta > 0$. f is bounded on K which means that there exists $S > 0$ such that $|f(z)| \leq S$ for all $z \in K$. For $\varepsilon := \frac{\eta}{S} > 0$ there exists $\delta > 0 : z, z' \in K, |z - z'| < \delta \Rightarrow |\tilde{v}(z) - \tilde{v}(z')| < \varepsilon$. We would like to show that $v(z)|f(z)| < \eta$ for all $z \notin K$. The desired inequality holds if $|z| \geq N + 1$. Let $z = x + iy \notin K$ and consider $0 < y < \delta$ and $|z| \leq N + 1$. We get $|x - z| = |x - x - iy| = |y| < \delta$ and $\tilde{v}(z) = \tilde{v}(z) - \tilde{v}(x) < \varepsilon = \frac{\eta}{S}$, hence $v(z)|f(z)| < \frac{\eta}{S}S = \eta$ for all $z \notin K$. □

Lemma 2.21. *The sequence $(R_n)_{n\in\mathbb{N}}$ of linear mappings $R_n : Hv_0(G) \to A_0(G)$ is uniformly bounded.*

Proof. For $n \geq n_0$ large enough so that condition (iii) can be applied, we get

$$\begin{aligned} ||R_n f||_v &= ||f_n||_v = \sup_{z\in G} |f_n(z)|v(z) = \sup_{z\in G} |f(z + \frac{i}{n}) \sqrt[n]{\frac{1}{z+i}}|v(z) \\ &\leq \sup_{z\in G} |f(z + \frac{i}{n})|v(z + \frac{i}{n})| \sqrt[n]{\frac{1}{z+i}}| \\ &\leq ||f||_v. \end{aligned}$$

□

In the next step we define the disc algebra $A(D)$, the space $A_0(D)$, repeat some properties of these spaces and show the existence of an isometric isomorphism between $A_0(D)$ and $A_0(G)$.

Definition 2.22. Let D be the open unit disc, $D := \{z \in \mathbb{C}; |z| < 1\}$. Define the *disc algebra*

$$A(D) := \{f \in C(\overline{D});\ f_{|D} \text{ is holomorphic}\},$$

and the space

$$A_0(D) := \{f \in A(D);\ f(1) = 0\}.$$

Because the polynomials are dense in the disc algebra one can write $A_0(D)$ as

$$A_0(D) = \overline{\text{span}}\{z^j - 1;\ j = 1, 2, ...\}.$$

Bockarev [17] showed in 1974:

Proposition 2.23. *The disc algebra $A(D)$ has a Schauder basis and therefore the bounded approximation property.*

Proposition 2.24. *$A_0(D)$ has the bounded approximation property.*

Proof. By proposition 2.23, $A(D)$ has the bounded approximation property. $p : A(D) \to A_0(D), p(f) = f - f(1), f \in A(D)$, is a bounded projection onto $A_0(D)$. Because of this, $A_0(D)$ is complemented in the disc algebra and inherits the bounded approximation property from $A(D)$.

□

Proposition 2.25. *There exists an isometric isomorphism T between $A_0(G)$ and $A_0(D)$.*

Proof. Compare [39], p. 81. Define $\alpha : G \to D, \alpha(z) := \frac{z-i}{z+i}$ for $z \in G$. α is a linear fractional transformation of the upper half-plane G onto the unit disc D. The inverse mapping of α is $\beta : D \to G, \beta(w) := i\frac{1+w}{1-w}, w \in D$. For each $c \geq 0$, α maps the half plane $\mathrm{Im}z > c$ onto the disc $\{w;\ \ |w - \frac{c}{1+c}| < \frac{1}{1+c}\}$, and α maps the line $\mathrm{Im}z = c$ onto the circle $\{w;\ \ |w - \frac{c}{1+c}| = \frac{1}{1+c}\}$ with the point 1 deleted, also $\beta(1) = \infty$ and $\alpha(\infty) = 1$. Now we can define

$$T : A_0(G) \to A_0(D) \ \text{ as } \ Tf := f \circ \alpha, f \in A_0(G),$$

which is an isometric isomorphism from $A_0(G)$ onto $A_0(D)$. □

From now on we are following a method of Lusky (see [29]) to construct a suitable commuting approximating sequence $(V_n)_{n\in\mathbb{N}}$, $V_n : Hv_0(G) \to Hv_0(G)$ such that $V_n - V_{n-1}$ factors uniformly through l_∞^m's.

Definition 2.26. Let $\mathcal{H}(D) := \{f : \overline{D} \to \mathbb{C}; f \text{ continuous}, f_{|D} \text{ harmonic}\}$ endowed with the sup-norm and let $f \in \mathcal{H}(D)$ have the Fourier series $f(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} \alpha_k r^{|k|} e^{ik\varphi}$.
Define $\tilde{V}_n : \mathcal{H}(D) \to \mathcal{H}(D)$ as

$$(\tilde{V}_n f)(re^{i\varphi}) := \sum_{|k|\leq 2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{2^n<|k|\leq 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi},$$

$\tilde{V}_n$ is the convolution with the de la Vallée Poussin kernel which is defined as

$$\mathcal{V}_n(z) := 2\mathcal{F}_{2^{n+1}}(z) - \mathcal{F}_{2^n}(z),$$

where $\mathcal{F}_n(z)$ is the Fejér kernel

$$\mathcal{F}_{2^n}(z) := \sum_{k=-n}^{n} (1 - \frac{|k|}{2^n}) e^{ik\varphi}.$$

and $V_n : A_0(D) \to A_0(D)$ as

$$V_n f := \tilde{V}_n f - (\tilde{V}_n f)(1) \cdot z^{2^n}, f \in A_0(D).$$

Lemma 2.27. *For the Fourier series $f = \sum \alpha_k r^{|k|} e^{ik\varphi}$ we define the Cesàro means $\sigma_n : \mathcal{H}(D) \to \mathcal{H}(D)$ by $\sigma_n(f) := \sum_{|k| \le 2^n} \frac{2^n - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi}$, cf. [24]. Then*

$$2\sigma_{n+1}(f) - \sigma_n(f) = \tilde{V}_n(f)$$

holds for each $n \in \mathbb{N}$.

Proof. By calculating we obtain

$$\begin{aligned}
& 2\sigma_{n+1}(f) - \sigma_n(f) \\
= \; & 2 \sum_{|k| \le 2^{n+1}} \frac{2^{n+1} - |k|}{2^{n+1}} \alpha_k r^{|k|} e^{ik\varphi} - \sum_{|k| \le 2^n} \frac{2^n - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} \\
= \; & \sum_{2^n < |k| \le 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{|k| \le 2^n} \left(\frac{2^{n+1} - |k|}{2^n} - \frac{2^n - |k|}{2^n} \right) \alpha_k r^{|k|} e^{ik\varphi} \\
= \; & \sum_{2^n < |k| \le 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{|k| \le 2^n} \left(\frac{2^{n+1} - 2^n}{2^n} \right) \alpha_k r^{|k|} e^{ik\varphi} \\
= \; & \sum_{2^n < |k| \le 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{|k| \le 2^n} \alpha_k r^{|k|} e^{ik\varphi} \\
= \; & \tilde{V}_n(f).
\end{aligned}$$

□

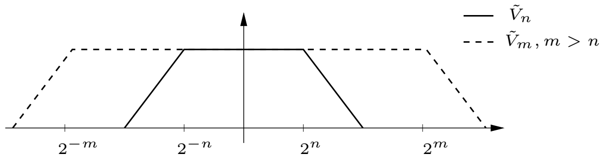
Lemma 2.28. *For $f \in A_0(D)$ and V_n defined as before, the following holds:*

(i) $\lim_{n \to \infty} V_n f = f$ *for each* $f \in A_0(D)$,

(ii) $\dim V_n A_0(D) < \infty$,

(iii) $V_n V_m = V_{\min(n,m)}$, *if* $n \neq m$.

Proof. (i) and (ii) follow immediately from the definition of V_n, respectively of $\tilde{V}_n$ and lemma 2.27 because the Cesàro means are convergent to f in $A(\overline{D})$. To show (iii), we first prove $\tilde{V}_n\tilde{V}_m = \tilde{V}_{\min(n,m)}$, for $n \neq m$. For $m > n$, $\tilde{V}_n\tilde{V}_m = \tilde{V}_n$ follows directly from the definition. $\tilde{V}_n z^k = 0$ if $k \geq 2^{n+1}$, and $\tilde{V}_m z^k = \tilde{V}_n z^k = z^k$ if $k \leq 2^n < 2^m$.



For $n > m$ one can use the same arguments to get $\tilde{V}_n\tilde{V}_m = \tilde{V}_m$. To show the desired equation for V_nV_m, set $W_n(f) = -(\tilde{V}_n f)(1)z^{2^n}$. For $m > n$ we obtain:

$$\begin{aligned}
V_nV_m(f) &= (\tilde{V}_n + W_n)(\tilde{V}_m + W_m)(f) \\
&= (\tilde{V}_n\tilde{V}_m + \tilde{V}_nW_m + W_n\tilde{V}_m + W_nW_m)(f) \\
&= \tilde{V}_n(f) - \tilde{V}_n((\tilde{V}_m f)(1)z^{2^m}) - \tilde{V}_n(\tilde{V}_m f)(1)z^{2^n} - W_n((\tilde{V}_m f)(1)z^{2^m}) \\
&= \tilde{V}_n(f) - (\tilde{V}_m f)(1)\tilde{V}_n(z^{2^m}) - (\tilde{V}_n f)(1)z^{2^n} \\
&\quad +(\tilde{V}_m f)(1)(\tilde{V}_n(z^{2^m}))(1)z^{2^n} \\
&= \tilde{V}_n(f) + W_n(f) \\
&= V_n(f).
\end{aligned}$$

In the case $m < n$ one uses the same arguments and obtains $V_nV_m = V_m$. □

Lemma 2.29. *For trigonometric polynomials $\sum_k \alpha_k r^{|k|} e^{ik\varphi}$ define*

$$P(\sum_k \alpha_k r^{|k|} e^{ik\varphi}) := \sum_{k\geq 0} \alpha_k r^{|k|} e^{ik\varphi},$$

with generally unbounded P. Then

$$P(\tilde{V}_n - \tilde{V}_{n-1})(f) = e^{i2^n\varphi}\sigma_n(e^{-i2^n\varphi}f) - \frac{1}{2}e^{i2^{n-1}\varphi}\sigma_{n-1}(e^{-i2^{n-1}\varphi}f).$$

Hence $P(\tilde{V}_n - \tilde{V}_{n-1})$ is a continuous linear operator and the same then holds for $P(V_n - V_{n-1})$.

Proof. By some calculations we get

$$\begin{aligned}
& P(\tilde{V}_n - \tilde{V}_{n-1})(f) \\
= & \sum_{k=0}^{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=0}^{2^{n-1}} \alpha_k r^k e^{ik\varphi} \\
& - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n - k}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n - k}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \left(1 - \frac{2^n - k}{2^{n-1}}\right) \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \frac{k - 2^{n-1}}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi}
\end{aligned}$$

and

$$\begin{aligned}
& e^{i2^n\varphi} \sigma_n(e^{-i2^n\varphi} f) - \frac{1}{2} e^{i2^{n-1}\varphi} \sigma_{n-1}(e^{-i2^{n-1}}\varphi) \\
= & \sum_{|k-2^n| \le 2^n} \frac{2^n - |k - 2^n|}{2^n} \alpha_k r^k e^{ik\varphi} - \frac{1}{2} \sum_{|k-2^{n-1}| \le 2^{n-1}} \frac{2^{n-1} - |k - 2^{n-1}|}{2^{n-1}} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{0 \le k \le 2^{n+1}} \frac{2^n - |k - 2^n|}{2^n} \alpha_k r^k e^{ik\varphi} - \frac{1}{2} \sum_{0 \le k \le 2^n} \frac{2^{n-1} - |k - 2^{n-1}|}{2^{n-1}} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=0}^{2^n} \frac{2^n - 2^n + k}{2^n} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^n - k + 2^n}{2^n} \alpha_k r^k e^{ik\varphi} \\
& - \sum_{k=0}^{2^{n-1}} \frac{2^{n-1} - 2^{n-1} + k}{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^{n-1} - k + 2^{n-1}}{2^n} \alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=0}^{2^n} \frac{k}{2^n} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{k=0}^{2^{n-1}} \frac{k}{2^n}\alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n-k}{2^n}\alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \left(\frac{k}{2^n} - \frac{2^n-k}{2^n}\right)\alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1}-k}{2^n}\alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \frac{k-2^n+k}{2^n}\alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1}-k}{2^n}\alpha_k r^k e^{ik\varphi} \\
= & \sum_{k=2^{n-1}+1}^{2^n} \frac{k-2^{n-1}}{2^{n-1}}\alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1}-k}{2^n}\alpha_k r^k e^{ik\varphi}.
\end{aligned}
$$

□

Proposition 2.30. *$V_n - V_{n-1}$ factors uniformly through l_∞^m's on $A_0(D)$.*

Proof. By the definition of the Cesàro means, $||\sigma_n|| = 1$ holds for all $n \in \mathbb{N}$; again cf. [24]. With lemma 2.27 we obtain $||\tilde{V}_n|| \leq 3$ for all $n \in \mathbb{N}$. Hence $(V_n)_n$ is uniformly bounded. $C(\partial D)$ is a $\mathcal{L}_\infty$-space, and it is well-known that $\mathcal{H}(D)$ is isometrically isomorphic to $C(\partial D)$. Hence $\mathcal{H}(D)$ is a $\mathcal{L}_\infty$-space. There exists $\lambda > 0$ such that for each $n \in \mathbb{N}$ there is $F \subset \mathcal{H}(D)$ with $\tilde{V}_{n+1}\mathcal{H}(D) \subset F$ and there is an isomorphism $\Phi : F \to l_\infty^M$ with $M = \dim F < \infty$ and $||\Phi||\cdot||\Phi^{-1}|| \leq \lambda$. Note that $A_0(D) \subset \mathcal{H}(D)$. Define $T_n : A_0(D) \to l_\infty^M$ by

$$T_n f := \Phi(V_{n+1} - V_{n-2})f,$$

and $S_n : l_\infty^M \to A_0(D)$ by

$$S_n g := P(V_n - V_{n-1})\Phi^{-1}g - (P(V_n - V_{n-1})\Phi^{-1}g)(1).$$

We have $\sup_n ||S_n|| < \infty$, $\sup_n ||T_n|| < \infty$ and

$$
\begin{aligned}
S_nT_n(f) &= S_n\Phi(V_{n+1} - V_{n-2})f \\
&= P(V_n - V_{n-1})(V_{n+1} - V_{n-2})f \\
&\quad -(P(V_n - V_{n-1})(V_{n+1} - V_{n-2})f)(1) \\
&= P(V_n - V_{n-1})f - (P(V_n - V_{n-1})f)(1) = (V_n - V_{n-1})f
\end{aligned}
$$

where the last but one equality holds because of

$$
\begin{aligned}
(V_n - V_{n-1})(V_{n+1} - V_{n-2}) &= V_nV_{n+1} - V_nV_{n-2} - V_{n-1}V_{n+1} + V_{n-1}V_{n-2} \\
&= V_n - V_{n-2} - V_{n-1} + V_{n-2} = V_n - V_{n-1}.
\end{aligned}
$$

□

2.5 Proof of theorem 2.13

Before we give the proof of theorm 2.13 we will collect the results of section 2.4 in the following diagram. We have

$$Hv_0(G) \xrightarrow{R_n} A_0(G) \xrightarrow{T} A_0(D) \xrightarrow{V_n} A_0(D) \xrightarrow{T^{-1}} A_0(G) \xrightarrow{R} Hv_0(G).$$

With the linear mapping

$$R_n : Hv_0(G) \to A_0(G), R_n f = f_n \;\forall\; n \in \mathbb{N},$$

the isometric isomorphism

$$T : A_0(G) \to A_0(D), Tf := f \circ \beta, f \in A_0(G),$$

the commuting approximating sequence $(V_n)_n$

$$V_n : A_0(D) \to A_0(D), V_n f := \tilde{V}_n f - (\tilde{V}_n f)(1) \cdot z^{2^n}, f \in A_0(D)$$

and the restriction mapping

$$R : A_0(G) \to Hv_0(G), f \to f_{|G}.$$

Now we come to the proof of theorem 2.13: For a suitable sequence $(m_n)_{n\in\mathbb{N}}$ of indices we can assume without loss of generality:

$$(*) \quad R_{m_n} R T^{-1}(r^{|k|}e^{ik\varphi} - 1) = T^{-1}(r^{|k|}e^{ik\varphi} - 1) \;\forall\; |k| \leq 2^{n+1}.$$

If $(*)$ is not true, replace R_{m_n} by

$$\begin{aligned} \tilde{R}_{m_n} &:= R_{m_n}(\mathrm{id} - P_n) + R^{-1}P_n \\ &= (R^{-1} - R_{m_n})P_n + R_{m_n}, \end{aligned}$$

with $E_n := \mathrm{span}\{RT^{-1}(r^{|k|}e^{ik\varphi} - 1);\;\; |k| \leq 2^{n+1}\}$, $E_n \subset Hv_0(G)$ and $P_n : Hv_0(G) \to E_n$ a bounded projection. Then

$$\tilde{R}_{m_n} R T^{-1} = (R^{-1} - R_{m_n}) R T^{-1} + R_{m_n} R T^{-1} = T^{-1}$$

holds on E_n, but we have to show that $\tilde{R}_{m_n}$ is uniformly bounded. By corollary 2.17, one can choose $m_1 < m_2 < ...$ with

$$||R_{m_n} R T^{-1}(r^{|k|}e^{ik\varphi} - 1) - T^{-1}(r^{|k|}e^{ik\varphi} - 1)|| \leq \frac{1}{n 2^{n+2} ||P_n|| w}$$

for all $|k| \leq 2^{n+1}$, where $w := ||R^{-1}{}_{|E_n}||$. By the definition of $\tilde{R}_{m_n}$ we obtain

$$||\tilde{R}_{m_n} - R_{m_n}|| = ||(R^{-1} - R_{m_n})P_n||.$$

Let $x \in E_n$ with $||x||_v = 1$. One can write x as

$$x := \sum_{|k| \leq 2^{n+1}} \alpha_k RT^{-1}(r^{|k|}e^{ik\varphi} - 1).$$

With $U := (R^{-1} - R_{m_n})P_n$ one gets

$$||Ux||_v \leq \sum_{|k| \leq 2^{n+1}} |\alpha_k| \cdot ||URT^{-1}(r^{|k|}e^{ik\varphi} - 1)||_v.$$

Define $F_n := \text{span}\{(r^{|k|}e^{ik\varphi} - 1); \;\; |k| \leq 2^{n+1}\}$. Then $F_n \subset A_0(D)$, $RT^{-1}F_n = E_n$ and $||(RT^{-1}{}_{|F_n})^{-1}|| \leq w \; ||T||$ holds. Set $W := (RT^{-1}{}_{|F_n})^{-1}$ and note and

$$Wx = \sum_{|k| \leq 2^{n+1}} |\alpha_k|(r^{|k|}e^{ik\varphi} - 1).$$

Here the Fourier coefficients can be estimated as follows:

$$|\alpha_k| \leq ||Wx|| \leq ||W|| \cdot ||x||_v = ||W|| \leq w \; ||T||.$$

Putting the estimates together we obtain

$$\begin{aligned} ||\tilde{R}_{m_n} - R_{m_n}|| &= \sup\{||Ux||_v; ||x||_v = 1\} \\ &\leq \sum_{|k| \leq 2^{n+1}} |\alpha_k| \cdot ||URT^{-1}(r^{|k|}e^{ik\varphi} - 1)||_v \\ &\leq 2^{n+2}w||T|| \cdot ||T^{-1}(r^{|k|}e^{ik\varphi} - 1) - R_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi} - 1)|| \\ &\leq \frac{||T||}{n||P_n||}. \end{aligned}$$

Now define $\hat{V}_n : Hv_0(G) \to Hv_0(G)$ by

$$\hat{V}_n := RT^{-1}V_nTR_{m_n}.$$

Comparing the definition of $\hat{V}_n$ with the diagram at the beginning of this section we obtain that $\hat{V}_n$ is welldefined. We claim that $\hat{V}_n$ is a commuting approximating sequence with $\hat{V}_n\hat{V}_m = \hat{V}_{\min(n,m)}$ for $n \neq m$, $\dim \hat{V}_n Hv_0(G) < \infty$ and $\lim_{n\to\infty} \hat{V}_n f = f$ for $f \in Hv_0(G)$. Let $n > m$; then we have:

$$\begin{aligned} \hat{V}_n\hat{V}_m &= RT^{-1}V_nTR_{m_n}RT^{-1}V_mTR_{m_m} \\ &= RT^{-1}V_nTT^{-1}V_mTR_{m_m} \\ &= RT^{-1}V_nV_mTR_{m_m} \\ &= RT^{-1}V_mTR_{m_m} \\ &= \hat{V}_m. \end{aligned}$$

This holds because of $(*)$ and because TT^{-1} is the identity on $A_0(D)$. If $n < m$ we obtain $\hat{V}_n\hat{V}_m = \hat{V}_n$ by the same arguments. In proposition 2.30 we showed that there exist k_n, $T_n : A_0(D) \to l_\infty^{k_n}$ and $S_n : l_\infty^{k_n} \to A_0(D)$ with $\sup_n ||S_n|| < \infty$, $\sup_n ||T_n|| < \infty$ and $S_nT_n = V_n - V_{n-1}$. Set

$$\begin{aligned} \hat{T}_n : Hv_0(G) \to l_\infty^{k_n}, \quad \hat{T}_n &:= T_nTR_{m_n}, \\ \hat{S}_n : l_\infty^{k_n} \to Hv_0(G), \quad \hat{S}_n &:= RT^{-1}S_n. \end{aligned}$$

With $(*)$ and the definition of V_n it follows that

$$(**) \quad V_nTR_{m_j} = V_nTR_{m_n}$$

holds for all $j \geq n$ since $V_nTR_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi} - 1) = V_nTT^{-1}(r^{|k|}e^{ik\varphi} - 1) = V_n(r^{|k|}e^{ik\varphi} - 1)$ for each $|k| \leq 2^{n+1}$. Note that $\sup_n ||\hat{S}_n|| < \infty$, $\sup_n ||\hat{T}_n|| < \infty$ and by $(**)$

$$\begin{aligned} \hat{S}_n\hat{T}_n &= \hat{S}_n(T_nTR_{m_n}) \\ &= RT^{-1}S_nT_nTR_{m_n} \\ &= RT^{-1}(V_n - V_{n-1})TR_{m_n} \\ &= (RT^{-1}V_n - RT^{-1}V_{n-1})TR_{m_n} \\ &= RT^{-1}V_nTR_{m_n} - RT^{-1}V_{n-1}TR_{m_{n-1}} \\ &= \hat{V}_n - \hat{V}_{n-1}. \end{aligned}$$

We have constructed a commuting approximating sequence $\hat{V}_n$ such that $\hat{V}_n - \hat{V}_{n-1}$ factors uniformly through l_∞^m's. With theorem 2.12 it follows that $Hv_0(G)$ has a basis.

2.6 Examples

Example 2.31. (Stanev [40])

i) Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $p(t) = 1$ for all $t \in \mathbb{R}_+$ and define $v(z) = v_p(t) := p(\mathrm{Im}z)$ for $z \in G$. In this case $Hv(G) = H^\infty(G)$ and $Hv_0(G) = \{0\}$, because condition ii) of theorem 2.6 is not satisfied.

ii) Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $p(t) = \exp(t^2)$ for all $t \in \mathbb{R}$ and define $v(z) = v_p(t) := p(\mathrm{Im}z)$ for $z \in G$. From theorem 2.6 it follows that $Hv(G) = \{0\}$ and $Hv_0(G) = \{0\}$.

Example 2.32. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := (\mathrm{Im}z)^r$ for $\mathrm{Im}z \leq 1$ and $v(z) := 1$ elsewhere, $r > 0$. v satisfies the conditions (i) - (v). Hence $Hv_0(G)$ has a basis.

Example 2.33. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := \exp(-1/(\mathrm{Im}z)^2)$. It is easy to see that v satisfies conditions (i)-(v). Hence $Hv_0(G)$ has a basis.

Example 2.34. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := \mathrm{Im}z$. v satisfies conditions (i)-(iv), but v is not bounded. But $Hv_0(G)$ has the bounded approximation property.

Proof: The idea of this construction goes back to Stanev [40]. Let the weight w on the unit disc D be defined by $w(\delta) := (1 - |\delta|^2)$. w is radial and $\lim_{|\delta|\to 1} w(\delta) = 0$. Hence $Hw_0(D)$ has the bounded approximation property [10]. For $f \in Hw_0(D)$ we define the operator $\tilde{T} : Hw_0(D) \to Hv_0(G)$, $\tilde{T}f(z) = (f \circ \tilde{\beta})(z) \cdot (\frac{4}{(1-iz)^2}), z \in G$ with $\tilde{\beta}(z) = \frac{1+iz}{1-iz}$ for $z \in G$. $\tilde{\beta}$ maps the upper half-plane G onto the unit disc D. The operator $\tilde{T}$ is a topological isomorphism from $Hw_0(D)$ onto $Hv_0(G)$ [40].

For $z = x + iy \in G$ set $\tilde{\beta}(z) = \frac{1+iz}{1-iz} = \delta$ and calculate $1 - |\delta|^2$:

$$\begin{aligned}
1 - |\delta|^2 &= 1 - |\tfrac{1+iz}{1-iz}|^2 = \tfrac{|1-iz|^2 - |1+iz|^2}{|1-iz|^2} = \tfrac{(1-iz)(1+\overline{z}) - (1+iz)(1-\overline{z})}{|1-iz|^2} \\
&= \tfrac{1+i\overline{z}-iz+|z|^2-(1-i\overline{z}+iz+|z|^2)}{|1-iz|^2} = \tfrac{2i\overline{z}-2iz}{|1-iz|^2} \\
&= \tfrac{2ix+2y-2ix+2y}{|1-iz|^2} = \tfrac{4y}{|1-iz|^2} \\
&= \tfrac{4\mathrm{Im}z}{|1-iz|^2}
\end{aligned}$$

For $f \in Hw_0(D)$ the following holds:

$$
\begin{aligned}
f \in Hw(D) \quad & \Leftrightarrow (1-|\delta|^2)^p|f(\delta)| < \infty \ \forall \delta \in D \\
& \Leftrightarrow (\tfrac{4\mathrm{Im}z}{|1-iz|^2})^p|f(\tfrac{1+iz}{1-iz})| < \infty \ \forall z \in G \\
& \Leftrightarrow v(z)|\tilde{T}f(z)| < \infty \forall \ z \in G \\
& \Leftrightarrow \tilde{T}f \in Hv(G) \text{ and} \\
f \in Hw_0(D) \quad & \Leftrightarrow f \in Hw(D), \ \lim_{|\delta|\to 1^-}(1-|\delta|^2)^p|f(\delta)| = 0 \\
& \Leftrightarrow \tilde{T}f \in Hv(G), \ \lim_{\mathrm{Im}z\to 0}(\tfrac{4\mathrm{Im}z}{(1-iz)^2})^p f(\tfrac{1+iz}{1-iz}) = 0 \\
& \Leftrightarrow \tilde{T}f \in Hv(G), \ \lim_{\mathrm{Im}z\to 0} v(z)T_f(z) = 0 \\
& \Leftrightarrow \tilde{T}f \in Hv_0(G).
\end{aligned}
$$

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