Mean behaviour of uniformly summable $\mathcal{Q}$-multiplicative functions

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Dedicated to my family

Contents

Introduction

The theory of additive and multiplicative functions has made great progress during the past years; in particular, we want to mention the mean-value theorems by H. Delange [9], E. Wirsing [83], [84] and G. Halász [22], as well as, the first elementary proof of the theorem of Halász by H. Daboussi and K.-H. Indlekofer [8].

H. Delange [9] characterized those multiplicative functions f which satisfy $|f| \leq 1$ and for which a non-zero mean-value

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} f(n)$$

exists. It turned out to be more difficult though to characterize those multiplicative functions f with $|f| \leq 1$, for which a mean-value $M(f)$ exists and is zero. It was done in essentially two steps; for real-valued functions by E. Wirsing [83], [84], this included the proof of an old conjecture, variously ascribed to Erdős and Wintner, to the effect that a mean-value $M(f)$ always exists whenever f assumes only the values ± 1; and for complex-valued functions by G. Halász [22] using an analytic method. The first elementary proof of the theorem of Halász was given by H. Daboussi and K.-H. Indlekofer [8].

The Erdős-Wintner conjecture includes the prime number theorem because the assertion $M(\mu) = 0$, where μ denotes the Möbius function, is equivalent to the prime number theorem

$$\sharp\{p \,|\, p \text{ prime}, \ p \leq x\} =: \pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty \ ,$$

as was shown by E. Landau [56]. The elementary proofs by Wirsing and Daboussi-Indlekofer provide, among other things, an elementary proof of the prime number theorem.

While these theorems use the hypothesis $|f| \leq 1$, many results not based on this condition are also known now. In this context, let us mention the papers by H. Daboussi [6], P.D.T.A. Elliott [13], K.-H. Indlekofer [25], J. Knopfmacher [54] and W. Schwarz [75].

All these results provide valuable methods for the investigation of additive and multiplicative functions, as well as, for prime number theory. However, the actual calculation of values of additive and multiplicative functions requires knowledge of the prime factor decomposition of a number, while usually its q-adic or $\mathcal{Q}$-adic representation ("Cantor expansion") is given.

Let $\{q_r\}_{r\geq 1}$ be a sequence of natural numbers with $q_r \geq 2$, and let $\mathcal{Q}_0 = 1$, $\mathcal{Q}_r = q_r \mathcal{Q}_{r-1}$ for $r \geq 1$. Each nonnegative integer n has a unique $\mathcal{Q}$-adic representation ("Cantor expansion")

$$n = \sum_{r\geq 0} \varepsilon_r(n) \mathcal{Q}_r$$

if the following condition is satisfied

$$0 \leq \varepsilon_r(n) < q_{r+1}\ , \quad r \geq 0.$$

In the case $q_r \equiv q \geq 2$ we will use standard notation of q-adic representation.

This motivates the investigation of functions that are additive or multiplicative with respect to these representations. Such functions are called q-, $\mathcal{Q}$-additive, or q-, $\mathcal{Q}$-multiplicative, respectively. Mean value theorems hold for this setting, too, and the methods and results bear certain analogies to the classical case; however, there are also some peculiarities. The case $|f| \leq 1$ for q-multiplicative functions has been treated by H. Delange [10] to great extent, and its generalization to $\mathcal{Q}$-adic representations for the case $|f| \leq 1$ by J.

Coquet in his thesis [4]. The results for q-additive functions — which can be derived in some cases from the theory of q-multiplicative functions — provide interesting statistical tests for the randomness of data (see E. Manstavičius [63]).

In this thesis, we prove, both for the q-adic case and general $\mathcal{Q}$-adic representations, new theorems about the average of multiplicative functions without the assumption $|f| \leq 1$; it turns out that the class of *uniformly summable functions* is the appropriate generalization. In this context, we also investigate α-almost-periodic q-multiplicative functions.

To make the analogy to "classical" additive and multiplicative functions apparent, it is appropriate to summarize results related to these first.

We proceed as follows: Chapter 1 presents some well-known facts about additive functions. G. H. Hardy and S. Ramanujan proved that ω and Ω have the normal order $\log\log n$. P. Turán found a new proof of Hardy and Ramanujan's result using an inequality which is analogues to Tschebycheff's inequality. This gave M. Kac the idea of thinking about the role of independence in the application of probability theory to number theory. The generalization of this inequality is the famous Turán-Kubilius inequality. The important theorems of P. Erdős [14], M. Kac [15], [17] and A. Wintner [16] are introduced.

Since the main difficulties arise from the fact that the asymptotic density gives only a finitely additive measure (or content or pseudo-measure) on the family of subsets of $\mathbb{N}$, where it is defined, one constructs a sequence of finite, purely probabilistic models which approximate the number theoretical phenomena, and then use arithmetical arguments for "taking the limit". J. Kubilius [55] constructed such finite probability spaces on which independent random variables could be defined to mimic the behaviour of truncated additive functions. K.-H. Indlekofer [46] presents an integration theory on $\mathbb{N}$ using the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ which can be generalized to arbitrary sets.

In Chapter 2, we describe the mean behaviour of complex-valued multiplicative functions f such that $|f(n)| \leq 1$ for every positive integer n. These functions f which satisfy

$|f(n)| \leq 1$ for all $n \in \mathbb{N}$ and for which a non-zero mean-value exists were characterized by H. Delange [9] in 1961, but his method could not be modified to consider the case $M(f) = 0$. In 1967, E. Wirsing [84] proved his celebrated mean-value theorem which asserts, in particular, that any real-valued multiplicative functions f of modulus ≤ 1, has a mean-value. This solved a famous conjecture of Erdős and Wintner. His proof was done by elementary methods (and thus, he gave another elementary proof of the prime number theorem), but he could not handle the complex-valued case in its full generality. Only by an analytic method, found by G. Halász [22] in 1968 the asymptotic behaviour of $\sum_{n \leq x} f(n)$ could be fully determined for all complex-valued multiplicative functions f of modulus smaller than or equal to one.

The first elementary proof of the theorem of Halász was given by H. Daboussi and K.-H. Indlekofer [8] in 1992. More general, K.-H. Indlekofer, I. Kátai and R. Wagner [44] in 2001, compare the asymptotic behavior of $\sum_{n \leq x} f(n)$ and $\sum_{n \leq x} g(n)$ for multiplicative functions f and g, respectively, where $|f| \leq g$. They obtain generalizations of Wirsing's result and extend the theorem of Halász.

In Chapter 3, we introduce the space $\mathcal{L}^*$ of uniformly summable functions; this notion was introduced by K.-H. Indlekofer. Let $\alpha \in \mathbb{R}$ with $\alpha \geq 1$, and let

$$\mathcal{L}^\alpha := \{f | f : \mathbb{N} \to \mathbb{C}, \|f\|_\alpha < \infty\}$$

be the vector-space of arithmetical functions f with bounded semi-norm

$$\|f\|_\alpha := \left(\limsup_{N \to \infty} \frac{1}{N} \sum_{n < N} |f(n)|^\alpha \right)^{\frac{1}{\alpha}}.$$

An arithmetical function $f \in \mathcal{L}^1$ is said to be uniformly summable if

$$\lim_{K \to \infty} \sup_{N \geq 1} \frac{1}{N} \sum_{\substack{n < N \\ |f(n)| \geq K}} |f(n)| = 0,$$

and the space of all uniformly summable functions is denoted by $\mathcal{L}^*$.

Let $\beta > \alpha > 1$, then

$$\mathcal{L}^\beta \subsetneqq \mathcal{L}^\alpha \subsetneqq \mathcal{L}^* \subsetneqq \mathcal{L}^1.$$

The idea of uniform summability turned out to provide the appropriate tools for describing the mean behaviour of a large class of multiplicative functions. As typical results, we mention the theorem by K.-H. Indlekofer, which generalizes results of Daboussi, Delange, Halász and Wirsing. In addition, the spaces $\mathcal{B}^{\alpha}$, $\mathcal{D}^{\alpha}$ and $\mathcal{A}^{\alpha}$ of α-even, α-limit-periodic and α-almost-periodic arithmetical functions are considered. Finally, a complete characterization of α-almost-periodic multiplicative functions given by K.-H. Indlekofer is presented without proof.

The main topic of Chapter 4 is the investigation of q-additive, q-multiplicative functions, and $\mathcal{Q}$-additive, and $\mathcal{Q}$-multiplicative functions, respectively. Observing that q-additive functions are sums of "almost independent random variables", we give a new proof of the Turán-Kubilius inequality for q-additive functions which is much shorter than the proof given by M. Peter and J. Spilker [78] in 2001, and which extends this proof to $\mathcal{Q}$-additive functions. In the case of the q-adic scale, necessary and sufficient conditions for the existence of an asymptotic distribution for a real-valued q-additive function and the mean behaviour of q-multiplicative functions of modulus ≤ 1 have been given by H. Delange [10] in 1972. J. Coquet [4] considered in 1975 the same kind of problems in the cases of $\mathcal{Q}$-adic scales and obtained mainly sufficient conditions. Their main results are formulated.

Chapter 5 and 6 contain our main results. The aim of Chapter 5 is to study the behaviour of the means $\frac{1}{N}\sum_{n<N} f(n)$ and $\frac{1}{N}\sum_{n<N} |f(n)|^{\alpha}$ as $N \to \infty$, $\alpha > 0$, where f is uniformly summable and q-multiplicative, and we give a complete characterization of these means. To our surprise, we find that for q-multiplicative functions the space $\mathcal{L}^{\alpha}$ for every $\alpha > 0$ coincides with the space $\mathcal{L}^{*}$. Furthermore, applying our main results, we investigate finitely distributed q-additive functions and find characterizations for q-multiplicative functions belonging to the space $\mathcal{D}^{1}$ of limit-periodic functions and the space $\mathcal{A}^1$ of almost-periodic functions by their respective spectrum $\sigma(f)$.

In Chapter 6, we extend the results of Chapter 5 to uniformly summable $\mathcal{Q}$-multiplicative functions. In the case of a bounded sequence $\{q_r\}_{r\geq 1}$ we have similar theorems as in the q-adic case. In the case of an unbounded sequence $\{q_r\}_{r\geq 1}$ the situation is quite different. Unavoidable for unbounded sequences $\{q_r\}_{r\geq 1}$ is the existence of a so-called first digit phenomenon.

We investigate the mean behaviour of uniformly summable $\mathcal{Q}$-multiplicative functions that belong to $\mathcal{L}^2$ and for which the first digit condition

$$\max_{1\leq j\leq q_r-1} \frac{1}{j+1} \sum_{a=0}^{j} |f(a\mathcal{Q}_{r-1}) - 1|^2 \to 0 \qquad \text{as } r \to \infty$$

holds.

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Chapter 1

Additive functions

In this chapter, we present some well-known facts about additive functions. Furthermore, the Turán-Kubilius inequality and Erdős' characterization of finitely distributed functions is discussed.

1.1 Definition and introduction

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{P}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, non-negative integers, prime, real, and complex numbers, respectively. An "arithmetical function" is a map $f : \mathbb{N} \to \mathbb{C}$, defined on the set $\mathbb{N}$ of natural number. The set $\mathbb{C}^{\mathbb{N}}$ of arithmetical functions becomes a $\mathbb{C}$-vector-space $(\mathbb{C}^{\mathbb{N}}, +, \cdot)$ by defining addition and scalar multiplication as follows:

$$(f+g) : n \mapsto f(n) + g(n), \quad \lambda \cdot f : n \mapsto \lambda \cdot f(n).$$

Definition 1.1.1. *An arithmetical function g is* **additive** *if*

$$g(m \cdot n) = g(m) + g(n) \tag{1.1}$$

whenever m and n are coprime. If (1.1) holds for all m, n, then f is called **completely additive**. *An additive function g is called* **strongly additive** *if the values of g at prime-*

powers are restricted by the condition

$$g(p^k) = g(p), \qquad \text{if } k = 1, 2, \dots \ .$$

Because of the canonical representation

$$n = \prod_{p \in \mathbb{P}} p^{\alpha_p(n)} \qquad \text{with } p^{\alpha_p(n)} \| n$$

of the integers $n \in \mathbb{N}$ we have

$$g\left(\prod_{p \in \mathbb{P}} p^{\alpha_p(n)}\right) = \sum_{p \in \mathbb{P}} g\left(p^{\alpha_p(n)}\right).$$

G. H. Hardy and S. Ramanujan [23] considered the arithmetical functions ω and Ω, where $\omega(n)$ and $\Omega(n)$ denote the number of different prime divisors and of all prime divisors - i.e. counted with multiplicity - of an integer n, respectively. They proved that ω and Ω have the normal order "$\log\log n$". Here we say, roughly, that an arithmetical function f has the normal order F, if $f(n)$ is approximately $F(n)$ for almost all values of n.[1] More precisely, this means that

$$(1 - \varepsilon)F(n) < f(n) < (1 + \varepsilon)F(n)$$

for every positive ε and almost all values of n.

In 1934, P. Turán [80] gave a simple proof of Hardy and Ramanujan's result. It depended upon the readily obtained estimation

$$\sum_{n \le x} (\omega(n) - \log\log x)^2 \le c \cdot x \cdot \log\log x.$$

for some constant c.

This inequality - reminding us of Tschebycheff's inequality[2] - had a special effect, namely

[1] A property E is said to hold for almost all n if $\lim_{x\to\infty} x^{-1} \#\{n \le x : E \text{ does not hold for } n\} = 0$.

[2] At that time P. Turán knew no probability (see chapter 12 of [12]). The first widely accepted axiom system for probability theory, due to A. N. Kolmogorov, had only appeared in 1933.

gave M. Kac the idea of thinking about the role of independence in the application of probability theory to number theory. Making essential use of the notation of independent random variables, the central limit theorem, and sieve methods, M. Kac together with P. Erdős proved in 1939 [15], and 1940 [17] the following result:

Proposition 1.1.2 (Erdős - Kac). *For a real-valued strongly additive function f, let*

$$A(x) := \sum_{p \le x} \frac{f(p)}{p} \tag{1.2}$$

and

$$B(x) := \left(\sum_{p \le x} \frac{f^2(p)}{p} \right)^{1/2}. \tag{1.3}$$

Then, if $|f(p)| \le 1$, and if $B(x) \to \infty$ as $x \to \infty$, the frequencies

$$F_x(z) := \frac{1}{x} \#\{n \le x : \frac{f(n) - A(x)}{B(x)} \le z\}$$

converge weakly to the limit law

$$G(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} dw$$

as $x \to \infty$ (which we will denote by $F_x(z) \Rightarrow G(z)$).

Proof. see [12], Theorem 12.3.

Thus, for $f(n) = \omega(n)$, P. Erdős and M. Kac obtained a much more general result than G. H. Hardy and S. Ramanujan. In this case,

$$A(x) = \sum_{p \le x} \frac{1}{p} = \log\log x + O(1),$$

and

$$B(x) = \left(\sum_{p \le x} \frac{1}{p} \right)^{1/2} = (1 + o(1))(\log\log x)^{1/2} \ ,$$

so that

$$\frac{1}{x} \#\{n \le x : \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \le z\} \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} dw.$$

A second effect of the above mentioned paper of P. Turán was that P. Erdős, adopting Turán's method of proof, showed in 1938 [14] that, whenever the three series

$$\sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p}, \quad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{f(p)}{p}, \quad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{f^2(p)}{p}$$

converge, then the real-valued strongly additive function f possesses a limiting distribution F, i.e.

$$\frac{1}{x}\#\{n \le x : f(n) \le z\} \Rightarrow F(z)$$

with some suitable distribution function F. It turned out that the convergence of these three series was in fact necessary (see Erdős and Wintner [16]).

All these results can be described as effects of the fusion of (intrinsic) ideas of probability theory and asymptotic estimations. In this context, divisibility by a prime p is an event A_p, and all the $\{A_p\}$ are statistically independent of one another, where the underlying "measure" is given by the **asymptotic density**

$$\begin{aligned} \delta(A_p) &:= \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : n \in A_p\} \\ &= \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x \\ p|n}} 1 \quad \left(= \frac{1}{p}\right) \end{aligned} \tag{1.4}$$

If the limit

$$M(f) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$$

exists, then we say that the function f possesses an (arithmetical) **mean-value** $M(f)$. Then, for strongly additive functions f, we get

$$f = \sum_p f(p)\varepsilon_p$$

where ε_p denotes the characteristic function of A_p, and $M(\varepsilon_p) := \dfrac{1}{p}$.

The main difficulties concerning the immediate application of probabilistic tools arise from the fact that the arithmetical mean-value (1.4) defines only a finitely additive measure (or content or pseudo-measure) on the family of subsets of $\mathbb{N}$ having an asymptotic density; thus, one constructs a sequence of finite, purely probabilistic models, which approximate the number theoretical phenomena, and one then uses arithmetical arguments for ”taking the limit”. This theory, starting with the above mentioned results of P. Erdős, M. Kac, and A. Wintner, was developed by J. Kubilius [55]. He constructed such finite probability spaces on which independent random variables could be defined to mimic the behaviour of truncated additive functions

$$\sum_{p\leq r} f(p)\varepsilon_p.$$

This approach is effective if the ratio $\dfrac{\log r}{\log x}$ essentially tends to zero as x runs to infinity. Then J. Kubilius was able to give necessary and sufficient conditions in order that the frequencies

$$\frac{1}{x}\#\{n \leq x : f(n) - A(x) \leq zB(x)\}$$

weakly converge as $x \to \infty$, assuming that f belongs to a certain class of additive functions. This opened the door for the investigation of the *renormalisation* of additive functions, i.e. the question to determine when a given additive function f may be renormalized by functions $\alpha(x)$ and $\beta(x)$, such that the frequencies

$$\frac{1}{x}\#\{n \leq x : \frac{f(n) - \alpha(x)}{\beta(x)} \leq z\}$$

possess a weak limit as $x \to \infty$ (see Elliott [12], Kubilius [55], Levin and Timofeev [58]).

In [46], K.-H. Indlekofer presented an integration theory on $\mathbb{N}$ (which can be generalized to arbitrary sets) which is based on the following characterization of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$:

If $\mathcal{A}$ is an algebra in $\mathbb{N}$, then

$$\overline{\mathcal{A}} := \{\overline{A} \subset \beta\mathbb{N} \,:\, \overline{A} = cl_{\beta\mathbb{N}}A,\; A \in \mathcal{A}\}$$

is an algebra in $\beta\mathbb{N}$.

If an algebra $\mathcal{A}$ in $\mathbb{N}$, and a finitely additive measure δ on $\mathcal{A}$ are given, then the function $\bar{\delta}$ on $\overline{\mathcal{A}}$ defined by $\overline{\delta}(\overline{A}) = \delta(A)$, $\overline{A} \in \overline{\mathcal{A}}$, is a premeasure on $\overline{\mathcal{A}}$. By a suitable closure of the set of step functions he obtains spaces of number theoretical functions which contain e.g. the Möbius function. Furthermore, the “construction” of these spaces yields new, elementary proofs of the famous results of E. Wirsing, G. Halász and H. Delange for multiplicative functions f, $|f| \leq 1$. We will introduce this in Chapter 3.

1.2 The Turán-Kubilius inequality

If g is strongly additive, then

$$\frac{1}{x}\sum_{n\leq x} g(n) = \frac{1}{x}\sum_{n\leq x}\sum_{p|n} g(p) = \frac{1}{x}\sum_{p\leq x} g(p)\cdot [x/p],$$

and so $g(n)$ is approximate to $\sum\limits_{p\leq x}\dfrac{g(p)}{p}$. The so-called Turán-Kubilius inequality gives an estimation for the difference of the values of the function minus the “expectation”:

$$\left| g(n) - \sum_{p\leq x}\frac{g(p)}{p}\right|$$

in mean square.

Let g be a complex-valued additive arithmetic function, then we set

$$g(n) = \sum_{p^k \| n} g(p^k).$$

For real number $x > 0$, we set

$$A(x) = \sum_{p^k\leq x}\frac{g(p^k)}{p^k},$$

$$E(x) = \sum_{p^k\leq x}\frac{g(p^k)}{p^k}\cdot\left(1 - \frac{1}{p}\right)$$

and

$$D^2(x) = \sum_{p^k \le x} \frac{|g(p^k)|^2}{p^k}.$$

In its general form, the Turán-Kubilius inequality appears as follows.

Proposition 1.2.1 (Turán-Kubilius inequality). *There exist constants* c_1, c_2 *with the property that for every* $x \ge 2$*, and for any additive function* g *the inequalities*

$$\frac{1}{x} \sum_{n \le x} |g(n) - A(x)|^2 \le c_1 \cdot D^2(x)$$

and

$$\frac{1}{x} \sum_{n \le x} |g(n) - E(x)|^2 \le c_2 \cdot D^2(x)$$

hold. In fact, it is possible to take $c_1 = 30$*,* $c_2 = 20$*.*

Proof. see [77], Theorem 4.1.

The Turán-Kubilius inequality has often been applied to the study of additive and multiplicative functions. We formulate an analogous inequality in Chapter 4 for the q-additive and $\mathcal{Q}$-additive functions, and we use this inequality in Chapter 5 and 6.

1.3 Finitely distributed functions

In [18], P. Erdős introduced the notion of finitely distributed functions on $\mathbb{N}$:

A function g is said to be **finitely distributed** if there are positive constants c_1 and c_2, and an unbounded sequence of real numbers $x_1 < x_2 < \ldots$ such that for each x_j at least k positive integers $a_1 < a_2 < \ldots < a_k \le x_j$ may be found, with $k \ge c_1 x_j$, and

$$|g(a_m) - g(a_n)| \le c_2 \qquad 1 \le m \le n \le k.$$

For additive functions he proved the following characterization

Proposition 1.3.1 (Erdős [18]). *An additive function g is finitely distributed if and only if there is a constant c and a function h such that*

$$g(n) = c \log n + h(n),$$

where the series

$$\sum_{|h(p)|>1} \frac{1}{p}\,, \qquad \sum_{|h(p)|\leq 1} \frac{h^2(p)}{p}$$

both converge.

It follows from Proposition (2.2.2) that

$$\lim_{x\to\infty} \frac{1}{x} \left| \sum_{n\leq x} \exp(itg(n)) \right|$$

always exists. It will become clear that g is finitely distributed if and only if there is a set of real t-values of positive Lebesgue measure for which the value of this limit is not zero.

Chapter 2

Multiplicative functions of modulus ≤ 1

In this chapter, we describe the mean behaviour of complex-valued multiplicative functions f such that $|f(n)| \leq 1$ for every positive integer n.

2.1 Definition

Definition 2.1.1. *An arithmetical function f is* **multiplicative** *if $f \neq 0$, and if for all pairs m, n of positive integers the condition $\gcd(m,n) = 1$ implies*

$$f(m \cdot n) = f(m) \cdot f(n). \tag{2.1}$$

If (2.1) holds for all m, n, then f is called **completely multiplicative**.

Every multiplicative function f satisfies $f(1) = 1$, since $f(n \cdot 1) = f(n) \cdot f(1)$, and an integer n may be chosen for which $f(n) \neq 0$. If f_1 and f_2 are multiplicative, then the point-wise product $f_1 \cdot f_2$ also is multiplicative, the same is true for the convolution-product $f_1 * f_2$; if f is multiplicative and $f(n) \neq 0$ for every n, then $1/f$ is multiplicative. A multiplicative function f is determined by its values at the prime-powers:

$$f\left(\prod_{p \in \mathbb{P}} p^{\alpha_p(n)}\right) = \prod_{p \in \mathbb{P}} f\left(p^{\alpha_p(n)}\right).$$

In this formula, according to the fundamental theorem of arithmetic, an integer n is written uniquely as

$$n = \prod_{p \in \mathbb{P}} p^{\alpha_p(n)}$$

as a product of prime powers where $\alpha_p(n) = \max\{\alpha : p^\alpha | n\}$.
An important multiplicative function is the **Möbius function**, defined by

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$

The **Euler totient function** given by

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

is another well-known multiplicative function which enumerates the number of coprime residue classes (mod n).

2.2 Mean-value theorem for multiplicative functions of modulus ≤ 1

The problem of establishing the existence of mean-values was considered by A. Wintner in his book on Erathostenian averages [82], he asserted that if a multiplicative function f may have only values ± 1, then the mean-value $M(f)$ always exists. But, the sketch of his proof could not be substantiated, and the problem remained open as the Erdős-Wintner conjecture.
These functions f which satisfy $|f(n)| \leq 1$ for all $n \in \mathbb{N}$ and for which a *non-zero* mean-value exists were characterized by H. Delange [9] in 1961, he proved

Proposition 2.2.1 (Delange [9]). *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function satisfying $|f| \leq 1$. Then the following conditions are equivalent:*

(A) The mean-value $M(f) = \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n \leq x} f(n)$ exists, and it is non-zero.

(B) *(i) The series* $S_1(f) = \sum_p \frac{f(p)-1}{p}$ *is convergent,*

(ii) $\sum_{0\leq k<\infty} \frac{f(p^k)}{p^k} \neq 0$ *for all primes* p*.*

The assumption $|f| \leq 1$ implies that $\left| \sum_{0\leq k<\infty} \frac{f(p^k)}{p^k} \right| \geq \frac{1}{2}$ for every prime $p \geq 3$. Therefore, as did H. Delange, the validity of *(B(ii))* has to be assumed only for $p = 2$, and it may be substituted by Delange's condition

$$f(2^k) \neq -1 \text{ for some } k \geq 1.$$

But his method could not be modified to consider the case $M(f) = 0$.

To get an impression of the remaining case, we note that if $f = \mu$ is the Möbius function, then the validity of the assertion

$$\frac{1}{x}\sum_{n\leq x} \mu(n) \to 0 \qquad (x \to 0)$$

essentially is (see E. Landau [56]) as difficult as to obtain as the proof of the prime number theorem.

In his paper [84] of 1967, E. Wirsing proved his celebrated mean-value theorem which asserts, in particular, that any real-valued multiplicative functions f of modulus ≤ 1 has a mean-value. This solved the afore-mentioned conjecture of Erdős and Wintner.

In this paper, E. Wirsing adopts a more general formulation: he compares the behaviour of $\sum_{n\leq x} f(n)$ with that of $\sum_{n\leq x} f^*(n)$, where f^* is a nonnegative multiplicative function and $|f| \leq f^*$. His proof was done by elementary methods (and thus, he gave another elementary proof of the prime number theorem), but he could not handle the complex-valued case in full generality. Only by an analytic method, found by G. Halász in 1968, and published in his paper [22], the asymptotic behaviour of $\sum_{n\leq x} f(n)$ could be fully determined for all complex-valued multiplicative functions f of modulus smaller than or equal to one. His main result is given by the following

Proposition 2.2.2 (Halász [22]). *Let f be a complex-valued multiplicative arithmetical function satisfying $|f| \leq 1$.*

(1) If there is a real number a for which the series

$$\sum_p \frac{(1 - \operatorname{Re} f(p)p^{-ia})}{p} \tag{2.2}$$

is convergent, then the asymptotic relation

$$\sum_{n \leq x} f(n) = \frac{x^{1+ia}}{1+ia} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m=1}^{\infty} p^{-m(1+ia)} f(p^m)\right) + o(x)$$

holds.

(2) If the series (2.2) is divergent for every real number a, then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) = 0.$$

(3) In both cases, there are constants D, α, and a slowly oscillating function L of modulus $|L| = 1$ such that the asymptotic formula

$$\sum_{n \leq x} f(n) = D x^{1+i\alpha} L(\log x) + o(x)$$

holds.

The function L, and the constants α, D may explicitly be given (see for example Halász [22]).

In 1986, A. Hildebrand [24] gave a new elementary proof of Wirsing's theorem based on a large sieve inequality which is simpler than Wirsing's proof but does not work in the complex-valued case.

In [8], H. Daboussi and K.-H. Indlekofer succeeded in finding an elementary proof of Halász's theorem, and thus, a new elementary proof of Wirsing's result (see also Indlekofer [26] for a simplified and shorter proof).

Remark. K.-H. Indlekofer, I. Kátai and R. Wagner [44] compare the asymptotic behavior of $\sum_{n\leq x} f(n)$ and $\sum_{n\leq x} g(n)$ for multiplicative functions f and g, respectively, where $|f| \leq g$. Their results extend relevant theorems by E. Wirsing and G. Halász. They established the following theorem which generalizes Wirsing's result and extends the theorem of Halász.

Proposition 2.2.3 (Indlekofer, Kátai, Wagner [44]). *Let g be a multiplicative function and*

$$\sum_{p\leq x} \frac{g(p)}{p} \cdot \log p \sim \tau \cdot \log x, \qquad x \to \infty,$$

hold with a constant $\tau > 0$. Furthermore, let $g(p) = O(1)$ for all primes p, and let

$$\sum_{p} \sum_{k\geq 2} \frac{g(p^k)}{p^k} < \infty.$$

Besides this, if $\tau \leq 1$, then let

$$\sum_{p} \sum_{k\geq 2, p^k\leq x} g(p^k) = O\left(\frac{x}{\log x}\right),$$

and let f be a complex-valued function which satisfies $|f(n)| \leq g(n)$ for every positive integer n. If there exists a real number a_0 such that the series

$$\sum_{p} \frac{(g(p) - \mathrm{Re}\, f(p)p^{-ia})}{p} \tag{2.3}$$

converges for $a = a_0$, then

$$\begin{aligned} \sum_{n\leq x} f(n) &= \frac{x^{ia_0}}{1+ia_0} \prod_{p\leq x} \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{m(1+ia_0)}}\right) \left(1 + \sum_{m=1}^{\infty} \frac{g(p^m)}{p^m}\right)^{-1} \sum_{n\leq x} g(n) \\ &\quad + o\left(\sum_{n\leq x} g(n)\right). \end{aligned} \tag{2.4}$$

as $x \to \infty$. If the series (2.3) diverges for all $a \in \mathbb{R}$, then

$$\sum_{n\leq x} f(n) = o\left(\sum_{n\leq x} g(n)\right) \qquad (x \to \infty). \tag{2.5}$$

In both cases there are constants c, a_0 and a slowly oscillating function $\tilde{L}$ with $|\tilde{L}(u)| = 1$ such that

$$\sum_{n \leq x} f(n) = \Big(c x^{i a_0} \tilde{L}(\log x) + o(1) \Big) \sum_{n \leq x} g(n), \quad as\ x \to \infty. \tag{2.6}$$

Chapter 3

Uniformly summable functions

In this chapter, the spaces of uniformly summable, α-even, α-limit-periodic and α-almost-periodic arithmetical functions are considered. In addition, the mean behaviour of uniformly summable multiplicative functions and a complete characterization of α-almost-periodic multiplicative functions given by K.-H Indlekofer are presented.

3.1 Definition

Let $\alpha \in \mathbb{R}$ with $\alpha \geq 1$, and let

$$\mathcal{L}^{\alpha} := \{f | f : \mathbb{N} \to \mathbb{C}, \|f\|_{\alpha} < \infty\}$$

be the vector space of arithmetical functions f with bounded semi-norm

$$\|f\|_{\alpha} := \left(\limsup_{N \to \infty} \frac{1}{N} \sum_{n < N} |f(n)|^{\alpha} \right)^{\frac{1}{\alpha}}.$$

A characterization of multiplicative functions $f \in \mathcal{L}^{\alpha}$ $(\alpha > 1)$ which possess a nonzero mean-value $M(f)$ was independently given by P.D.T.A. Elliott [13], and using a different method, by H. Daboussi [6].

In 1980, K.-H. Indlekofer [25] introduced the following

Definition 3.1.1. *An arithmetical function $f \in \mathcal{L}^1$ is said to be* **uniformly summable** *if*

$$\lim_{K\to\infty} \sup_{N\geq 1} \frac{1}{N} \sum_{\substack{n<N \\ |f(n)|\geq K}} |f(n)| = 0,$$

and the space of all uniformly summable functions is denoted by $\mathcal{L}^*$.

It is easy to show that, if $\beta > \alpha > 1$,

$$\mathcal{L}^\beta \underset{\neq}{\subseteq} \mathcal{L}^\alpha \underset{\neq}{\subseteq} \mathcal{L}^* \underset{\neq}{\subseteq} \mathcal{L}^1.$$

3.2 Mean behaviour of uniformly summable multiplicative functions

The idea of uniform summability turned out to provide an appropriate tool for the description of the mean behaviour of a large class of multiplicative functions. As typical results, we mention the theorem by K.-H. Indlekofer which generalizes results of H. Daboussi, H. Delange, G. Halász, and E. Wirsing.

Proposition 3.2.1 (Indlekofer [25]). *(A generalization of Delange's result)*
Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative, and let $\alpha \geq 1$. Then, the following two assertions hold.

(i) If $f \in \mathcal{L}^* \cap \mathcal{L}^\alpha$, and if the mean-value

$$M(f) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n)$$

of f exists and is non-zero, then the series

$$\sum_p \frac{f(p)-1}{p}, \sum_{\substack{p \\ |f(p)|\leq \frac{3}{2}}} \frac{|f(p)-1|^2}{p}, \sum_{\substack{p \\ |f(p)-1|\geq \frac{1}{2}}} \frac{|f(p)|^\lambda}{p}, \sum_p \sum_{k\geq 2} \frac{|f(p^k)|^\lambda}{p^k} \tag{3.1}$$

converge for all λ with $1 \leq \lambda \leq \alpha$, and, for each prime p,

$$1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^k} \neq 0. \tag{3.2}$$

(ii) *If the series (3.1) converge, then* $f \in \mathcal{L}^* \cap \mathcal{L}^\alpha$*, and the mean-values* $M(f)$, $M(|f|^\lambda)$ *exist for all* λ *with* $1 \leq \lambda \leq \alpha$*. If, in addition (3.2) holds, then* $M(f) \neq 0$*.*

Note that the membership of $\mathcal{L}^\alpha \cap \mathcal{L}^*$ and the existence of a non-zero mean value together are equivalent to a set of explicit conditions on prime powers. Further, observe that these conditions imply the existence of the mean values $M(|f|^\lambda)$ for all $1 \leq \lambda \leq \alpha$.

Proposition 3.2.2 (Indlekofer [28]). *(A generalization of Wirsing's result)*
Let $f \in \mathcal{L}^*$ *be a real-valued multiplicative function. Then, the existence of the mean value* $M(|f|)$ *implies the existence of* $M(f)$*.*

Note that Theorem 3.2.2 is an appropriate generalization of Wirsing's result, for if f is multiplicative and $|f| \leq 1$, the mean value of $M(|f|)$ always exists.
In this connection it is interesting to mention the following characterization of non-negative multiplicative functions of $\mathcal{L}^*$.

Proposition 3.2.3 (Indlekofer [29]). *Let* $\varepsilon \geq 0$*, and let* $f \in \mathcal{L}^{1+\varepsilon} \cap \mathcal{L}^*$ *be a non-negative multiplicative function. If* $||f||_1 > 0$*, then* $f^{1+\varepsilon} \in \mathcal{L}^\star$*, and there exist positive constants* c_1, c_2 *such that, as* $x \to \infty$*,*

$$M(f^{1+\varepsilon}) = \exp\left(\sum_{p \leq x} \frac{f^{1+\varepsilon}(p) - 1}{p}\right)(c_1 + o(1))$$

$$= \exp\left(\sum_{p \leq x} \frac{f(p) - 1}{p}\right)(c_2 + o(1))$$

from which we deduce that the existence of $M(f^{1+\varepsilon})$ *implies the existence of* $M(f)$*.*

A complete characterization of the asymptotic behaviour of the sums $\sum_{n \leq x} f(n)$ as $x \to \infty$ for complex-valued multiplicative functions $f \in \mathcal{L}^*$ was given by K.-H. Indlekofer in [30]. He proves the following

Proposition 3.2.4 (Indlekofer [30]). *(A generalization of Halász's result)* *Let $f \in \mathcal{L}^*$ be multiplicative, and let $\|f\|_1 > 0$. If we define*

$$\varrho(p) = \begin{cases} \frac{f(p)}{|f(p)|} & \text{if } f(p) \neq 0 \\ 1 & \text{otherwise} \end{cases},$$

then the following two assertions hold.

(i) If there exists a constant $a_0 \in \mathbb{R}$ such that the series

$$\sum_p \frac{(1 - \operatorname{Re} \varrho(p) p^{-ia})}{p} \tag{3.3}$$

converges for $a = a_0$, then there exists a constant $c_0 \in \mathbb{C}$ such that, if $x \to \infty$,

$$\frac{1}{x} \sum_{n \leq x} f(n) = x^{ia_0} \exp\left(\sum_{p \leq x} \frac{f(p)p^{-ia_0} - 1}{p} \right) (c_0 + o(1)),$$

where

$$c_0 = \frac{1}{1 + ia_0} \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{k(1+ia)}}\right) \exp\left\{\frac{1 - f(p)p^{-ia_0}}{p}\right\}.$$

If

$$A^*(x) := \sum_{p \leq x} \frac{\operatorname{Im} f(p) p^{-ia_0}}{p},$$

then

$$\lim_{x \to \infty} \sup_{x \leq y \leq x^2} |A^*(y) - A^*(x)| = 0.$$

(ii) If the series (3.3) diverges for all $a \in \mathbb{R}$, then the mean-value $M(f)$ of f exists and equals zero.

This result generalizes the theorem of Halász [22] on multiplicative functions $|f| \leq 1$. We will extend above proposition to uniformly summable q-multiplicative functions in Chapter 5.

3.3 Mean behaviour of α-almost-periodic multiplicative functions

Definition 3.3.1. *Let r be a positive integer. An arithmetic function f is called*

r-**periodic**, *if $f(n+r) = f(n)$ for every positive integer n,*

r-**even**, *if $f(n) = f(\gcd(n,r))$ for every positive integer n.*

f is termed periodic (resp. even) if there is an r for which f is r-periodic (resp. r-even). Obviously, an r-even function is r-periodic.

Standard examples of r-periodic functions are the exponential functions e_β, where $\beta = \frac{a}{r}$, for $a \in \mathbb{Z}$, $r \in \mathbb{N}$, and where $e_\beta(n) = \exp(2\pi i \cdot \beta \cdot n)$.

The **Ramanujan sum** c_r is a special exponential sum:

$$\begin{aligned} c_r(n) &= \sum_{\substack{1 \le a \le r \\ \gcd(a,r)=1}} \exp\left(2\pi i \cdot \frac{a}{r} \cdot n\right) \\ &= \sum_{t \mid \gcd(r,n)} t \cdot \mu\left(\frac{r}{t}\right). \end{aligned}$$

The vector space $\mathcal{B}_r$ of r-even functions can be generated by the Ramanujan sums c_d, where $d|r$, i.e.,

$$\mathcal{B}_r = \mathrm{Lin}_{\mathbb{C}}[c_d : d|r],$$

and each element of the vector space $\mathcal{D}_r$ of r-periodic functions can be written as a linear combination of exponential functions, i.e.,

$$\mathcal{D}_r = \mathrm{Lin}_{\mathbb{C}}[e_{a/r} : 1 \le a \le r].$$

The vector space of all even and all periodic functions is denoted by $\mathcal{B} := \bigcup_{r=1}^{\infty} \mathcal{B}_r$ and $\mathcal{D} := \bigcup_{r=1}^{\infty} \mathcal{D}_r$, respectively. Finally, we define the vector space

$$\mathcal{A} = \mathrm{Lin}_{\mathbb{C}}[e_\beta : \beta \in \mathbb{R}/\mathbb{Z}]$$

of complex linear combinations of the functions e_β.

Using the semi-norm $\|f\|_\alpha$, the spaces

$\mathcal{B}^\alpha = \|.\|_\alpha$ - closure of $\mathcal{B}$ (α-**almost-even** functions)

$\mathcal{D}^\alpha = \|.\|_\alpha$ - closure of $\mathcal{D}$ (α-**limit-periodic** functions)

$\mathcal{A}^\alpha = \|.\|_\alpha$ - closure of $\mathcal{A}$ (α-**almost-periodic** functions)

may be constructed.

The obvious inclusion relations $\mathcal{B} \subset \mathcal{D} \subset \mathcal{A}$ imply

$$\mathcal{B}^\alpha \subset \mathcal{D}^\alpha \subset \mathcal{A}^\alpha,$$

where $\alpha \geq 1$.

For $\gamma < \alpha$, Hölder's inequality gives

$$\sum_{n\leq N} |f(n)|^\gamma \leq \left\{ \sum_{n\leq N} |f(n)|^\alpha \right\}^{\gamma/\alpha} \cdot \left\{ \sum_{n\leq N} 1 \right\}^{\alpha/(\alpha-\gamma)},$$

therefore,

$$\|f\|_\gamma \leq \|f\|_\alpha \qquad \text{if } \gamma \leq \alpha,$$

and so

$$\mathcal{B}^\alpha \subset \mathcal{B}^\gamma,\ \mathcal{D}^\alpha \subset \mathcal{D}^\gamma,\ \text{and } \mathcal{A}^\alpha \subset \mathcal{A}^\gamma,\ \text{if } \gamma \leq \alpha.$$

Furthermore, we have the inclusions

$$\mathcal{B}^1 \subsetneqq \mathcal{D}^1 \subsetneqq \mathcal{A}^1 \subsetneqq \mathcal{L}^*.$$

For every function $f \in \mathcal{A}^1$, the mean-value $M(f)$, and for every $\beta \in \mathbb{R}$, the **Fourier coefficient**

$$\hat{f}(\beta) := \lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} f(n) e_{-\beta}(n)$$

exist (see, for example, W. Schwarz and J. Spilker [77] Chap. IV and VI).

For $f \in \mathcal{L}^1$, the **Fourier-Bohr spectrum** $\sigma(f)$ is defined by

$$\sigma(f) = \left\{ \beta \in \mathbb{R}/\mathbb{Z} : \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\le N} f(n) e_{-\beta}(n) \right| > 0 \right\}.$$

If $f \in \mathcal{A}^1$, then $\beta \in \sigma(f)$ if and only if $\hat{f}(\beta) \neq 0$.

In his paper [27], K.-H. Indlekofer gave a complete characterization of α-almost-periodic multiplicative functions. He proved the following results.

Proposition 3.3.2 (Indlekofer [27]). *Let $f \in \mathcal{A}^1$ be multiplicative. Then, $M(|f|) = 0$ if and only if $\sigma(f) = \emptyset$.*

Proposition 3.3.3 (Indlekofer [27]). *Let $f \in \mathcal{A}^\alpha$ be multiplicative. Then, f is α-limit-periodic.*

Proposition 3.3.4 (Indlekofer [27]). *Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative. Then, the following assertions are equivalent.*

(i) $f \in \mathcal{A}^\alpha$*, and* $\|f\|_1 > 0$.

(ii) $f \in \mathcal{A}^\alpha$*, and the spectrum* $\sigma(f)$ *of* f *is non-empty.*

(iii) $f \in \mathcal{L}^\alpha \cap \mathcal{L}^*$*, and there exists a Dirichlet-character* χ *such that the mean-value* $M(f\chi)$ *of* $f\chi$ *exists and is different from zero.*

(iv) There exists a Dirichlet-character χ *such that the series*

$$\sum_p \frac{f(p)\chi(p) - 1}{p}, \quad \sum_{\substack{p \\ |f(p)| \le 3/2}} \frac{|f(p)\chi(p) - 1|^2}{p}, \tag{3.4}$$

and

$$\sum_{\substack{p \\ ||f(p)|-1| > 1/2}} \frac{|f(p)|^\lambda}{p}, \quad \sum_p \sum_{k\ge 2} \frac{|f(p^k)|^\lambda}{p^k} \tag{3.5}$$

converge for all λ *with* $1 \le \lambda \le \alpha$.

Remark. The equivalence of (ii) and (iv) was proved by H. Daboussi [7]. The equivalence of (ii), (iii) and (iv) was shown by K.-H. Indlekofer in [30], Corollary 7.

In Chapter 5, we give a complete characterization of α-almost-periodic q-multiplicative functions.

Chapter 4

$\mathcal{Q}$-additive and $\mathcal{Q}$-multiplicative functions

In this chapter, we start our investigation of q-additive and q-multiplicative functions and $\mathcal{Q}$-additive and $\mathcal{Q}$-multiplicative functions, respectively. We give a new proof of the Turán-Kubilius inequality for q-additive functions, and we extend this proof to $\mathcal{Q}$-additive functions.

4.1 Definition

Let $\{q_r\}_{r\geq 1}$ with $q_r \geq 2$ be a sequence of natural numbers, and let $\mathcal{Q}_0 = 1$, $\mathcal{Q}_r = q_r\mathcal{Q}_{r-1}$ when $r \geq 1$. For each nonnegative integer n has a unique representation

$$n = \sum_{r\geq 0} \varepsilon_r(n)\mathcal{Q}_r \tag{4.1}$$

if the following condition is satisfied

$$0 \leq \varepsilon_r(n) < q_{r+1}\ , \quad r \geq 0.$$

If $\varepsilon_k(n) \neq 0$ and $\varepsilon_{k+j}(n) = 0$ for all $j \geq 1$, then $\varepsilon_k(n)$ and k will be called the first digit and the order of n, respectively.

The function $g : \mathbb{N}_0 \to \mathbb{C}$, satisfying the relation

$$g(n) = \sum_{r \geq 0} g(\varepsilon_r(n)\mathcal{Q}_r)$$

for each $n \in \mathbb{N}$ having the form (4.1) with $g(0) = 0$, will be called $\mathcal{Q}$**-additive** function. Similarly, we say that $f : \mathbb{N}_0 \to \mathbb{C}$ is a $\mathcal{Q}$**-multiplicative** function if $f(0) = 1$ and

$$f(n) = \prod_{r \geq 0} f(\varepsilon_r(n)\mathcal{Q}_r)$$

for each $n \in \mathbb{N}$ having the form (4.1).

In the case $q_r \equiv q \geq 2$, we will use standard notation of q-adic representation.
We consider the so-called q**-additive** functions $g : \mathbb{N}_0 \to \mathbb{C}$ which are defined by

$$g(n) = \sum_{r \geq 0} g(\varepsilon_r(n)q^r) \qquad \text{and} \;\; f(0) = 0,$$

and the q**-multiplicative** functions $f : \mathbb{N}_0 \to \mathbb{C}$ which are defined by

$$f(n) = \prod_{r \geq 0} f(\varepsilon_r(n)q^r) \qquad \text{and} \;\; f(0) = 1.$$

These functions were first introduced by A. O. Gelfond [21]. The sum of digits $\sum_{r \geq 0} \varepsilon_r(n)$ of n is a typical and mostly investigated example of q-additive functions (see for example Delange [11]; Coquet [5]). Exponentiating a q-additive function gives a q-multiplicative function.

We recall that a real-valued function $g(n)$ has an asymptotic distribution if there is a distribution function G such that for all continuity points y of G, the probability measures defined by $\mathcal{N}_x(y) := x^{-1}\sharp\{n \leq x; g(n) \leq y\}$ tend to $G(y)$ as x tend to infinity.

4.2 Generalized Turán-Kubilius inequalities for $\mathcal{Q}$-additive functions

Observing that q-additive functions are sums of "almost independent random variables", we prove the following inequality which is interesting in itself.

Theorem 4.2.1. *Let F be an arbitrary nonnegative-valued increasing function satisfying the inequality*

$$F(2x) \leq \rho F(x)$$

for some constant $\rho > 0$. Let $g : \mathbb{N}_0 \to \mathbb{C}$ be q-additive and let $cq^{R-1} \leq N < (c+1)q^{R-1}$ with $R \in \mathbb{N}$ and for $c \in \mathbb{N}$ with $0 < c < q$.
We set

$$E_R(g) = \sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} g(aq^r),$$

and

$$E_{R,c}(g) = E_R(g) + \frac{1}{c} \sum_{a=1}^{c} g(aq^{R-1}).$$

Then, the following assertions hold.

(i) For some constant $M > 0$, we have

$$\begin{aligned} &\frac{1}{N} \sum_{n<N} F(|g(n) - E_{R,c}(g)|) \\ \leq\ & M \cdot \left\{ F\left(\left(\sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} |g(aq^r)|^2 + \frac{1}{c} \sum_{a=1}^{c} |g(aq^{R-1})|^2 \right)^{1/2} \right) \right. \\ & \left. + \sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} F(|g(aq^r)|) + \frac{1}{c} \sum_{a=1}^{c} F(|g(aq^{R-1})|) \right\}. \end{aligned} \tag{4.2}$$

(ii) If, in addition, F fulfills

$$F(x+y) \gg F(x) + F(y),$$

then, for some constant $M' > 0$, we have

$$\begin{aligned} &\frac{1}{N}\sum_{n<N} F(|g(n) - E_{R,c}(g)|) \\ \leq\ & M' \cdot F\left(\left(\sum_{r=0}^{R-2}\frac{1}{q}\sum_{a=0}^{q-1}|g(aq^r)|^2 + \frac{1}{c}\sum_{a=1}^{c}|g(aq^{R-1})|^2\right)^{1/2}\right). \end{aligned} \tag{4.3}$$

To prove Theorem 4.2.1, we use the Burkholder's inequality (see Burkholder [3], Ruzsa [68] and Indlekofer [35])

$$\mathrm{E}F\left(\left|\sum_{j=1}^{n}\xi_j\right|\right) \ll F\left(\left(\sum \mathrm{E}(|\xi_j|^2)\right)^{1/2}\right) + \sum \mathrm{E}F(|\xi_j|)\,, \tag{4.4}$$

where the ξ_j's are independent variables of zero mean, E denotes an expectation, and F is an arbitrary nonnegative-valued increasing function satisfying the inequality

$$F(2x) \leq \rho F(x)$$

for some constant ρ; the value of the implied constant depends on this ρ.

Proof of Theorem 4.2.1.

(i) Each nonnegative integer $n < N$ has a unique representation

$$n = \sum_{r=0}^{R-1}\varepsilon_r(n)q^r,$$

which $0 \leq \varepsilon_r(n) < q$. We obtain

$$g(n) = \sum_{r=0}^{R-1} g(\varepsilon_r(n)q^r).$$

Let $\eta_r(n) = g(\varepsilon_r(n)q^r)$, then $\eta_0, \ldots, \eta_{R-1}$ are independent random variables in the Laplace space $\{0, 1, \cdots, (c+1)q^{R-1}\}$, and $g = \sum_{r=0}^{R-1}\eta_r$.

We define the function g^*

$$g^*(aq^r) = \begin{cases} g(aq^r) & \text{for} \quad r < R-1,\ 0 \leq a < q\ , \\ & \qquad \text{or } r = R-1, 0 \leq a \leq c\ ; \\ 0 & \text{for} \quad r > R-1,\ 0 \leq a < q\ , \\ & \qquad \text{or } r = R-1, c < a < q\ . \end{cases}$$

Then, $\xi_r = \eta_r - E(\eta_r)$ are independent random variables of zero mean, and

$$g^* - E(g^*) = \sum_{r=0}^{R-1} \xi_r \ ,$$

where $E(g^*) = E_{R,c}(g)$.

Applying the Burkholder's inequality (4.4), we obtain

$$\frac{1}{(c+1)q^{R-1}} \sum_{n<(c+1)q^{R-1}} F(|g(n) - E_{R,c}(g)|)$$

$$\ll \underbrace{F\left(\left(\sum_{r=0}^{R-2}\frac{1}{q}\sum_{a=0}^{q-1}\left|g(aq^r) - \frac{1}{q}\sum_{b=0}^{q-1} g(bq^r)\right|^2 + \frac{1}{c}\sum_{a=1}^{c}\left|g(aq^{R-1}) - \frac{1}{c}\sum_{b=1}^{c} g(bq^{R-1})\right|^2\right)^{1/2}\right)}_{I_1}$$

$$+\underbrace{\sum_{r=0}^{R-2}\frac{1}{q}\sum_{a=0}^{q-1} F\left(\left|g(aq^r) - \frac{1}{q}\sum_{b=0}^{q-1} g(bq^r)\right|\right) + \frac{1}{c}\sum_{a=1}^{c} F\left(\left|g(aq^{R-1}) - \frac{1}{c}\sum_{b=1}^{c} g(bq^{R-1})\right|\right)}_{I_2} \ .$$

For the estimation of the first term (I_1), we observe

$$\begin{aligned}
\frac{1}{q}\sum_{a=0}^{q-1}\left|g(aq^r) - \frac{1}{q}\sum_{b=0}^{q-1} g(bq^r)\right|^2 &\leq \frac{1}{q}\left(4\sum_{a=0}^{q-1}|g(aq^r)|^2 + \frac{4}{q}\left|\sum_{b=0}^{q-1} g(bq^r)\right|^2\right) \\
&\leq \frac{1}{q}\left(4\sum_{a=0}^{q-1}|g(aq^r)|^2 + 4\sum_{b=0}^{q-1}|g(bq^r)|^2\right) \\
&= 8\left(\frac{1}{q}\sum_{a=0}^{q-1}|g(aq^r)|^2\right) \ .
\end{aligned}$$

In the same way, we get

$$\frac{1}{c}\sum_{a=1}^{c}\left|g(aq^{R-1}) - \frac{1}{c}\sum_{b=1}^{c} g(bq^{R-1})\right|^2 \leq 8\left(\frac{1}{c}\sum_{a=1}^{c}\left|g(aq^{R-1})\right|^2\right) \ .$$

For the second term (I_2), we have

$$\begin{aligned}\sum_{a=0}^{q-1} F\left(\left|g(aq^r) - \frac{1}{q}\sum_{b=0}^{q-1} g(bq^r)\right|\right) &\le \sum_{a=0}^{q-1} F\left(2 \max_{0\le a<q} |g(aq^r)|\right) \\ &\le \rho \sum_{a=0}^{q-1} F\left(\max_{0\le a<q} |g(aq^r)|\right) \\ &\le q\rho \sum_{a=0}^{q-1} F(|g(aq^r)|) \ .\end{aligned}$$

In the same way, we get

$$\sum_{a=1}^{c} F\left(\left|g(aq^{R-1}) - \frac{1}{c}\sum_{b=1}^{c} g(bq^{R-1})\right|\right) \le c\rho \sum_{a=1}^{c} F\left(\left|g(aq^{R-1})\right|\right) \ .$$

Since

$$\begin{aligned}& \frac{1}{N}\sum_{n<N} F(|g(n) - E_{R,c}(g)|) \\ \le\ & \frac{1}{N}\sum_{n<(c+1)q^{R-1}} F(|g^*(n) - E_{R,c}(g)|) \\ \le\ & \frac{c+1}{c}\cdot\frac{1}{(c+1)q^{R-1}}\sum_{n<(c+1)q^{R-1}} F(|g^*(n) - E_{R,c}(g)|)\end{aligned}$$

the inequality (4.2) follows.

(ii) Since

$$\begin{aligned}\sum_{a=0}^{q-1} F(|g(aq^r)|) &\ll F\left(\sum_{a=0}^{q-1} |g(aq^r)|\right) \\ &\le F\left(\sqrt{q}\left(\sum_{a=0}^{q-1} |g(aq^r)|^2\right)^{1/2}\right) \\ &\ll F\left(\left(\sum_{a=0}^{q-1} |g(aq^r)|^2\right)^{1/2}\right) \ ,\end{aligned}$$

we then have

$$\begin{aligned}
&\sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} F(|g(aq^r)|) + \frac{1}{c} \sum_{a=1}^{c} F(|g(aq^{R-1})|) \\
\ll\ & \sum_{r=0}^{R-2} F\left(\left(\frac{1}{q} \sum_{a=0}^{q-1} |g(aq^r)|^2\right)^{1/2}\right) + F\left(\left(\frac{1}{c} \sum_{a=1}^{c} |g(aq^{R-1})|^2\right)^{1/2}\right) \\
\ll\ & F\left(\left(\sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} |g(aq^r)|^2 + \frac{1}{c} \sum_{a=1}^{c} |g(aq^{R-1})|^2\right)^{1/2}\right) .
\end{aligned}$$

The inequality (4.3) follows.

For $F(x) = x^p$ with $p \geq 1$, we obtain a recent result by M. Peter and J. Spilker [78]

$$\begin{aligned}
&\frac{1}{N} \sum_{n<N} |g(n) - E_{R,c}(g)|^p \\
\leq\ & M'' \cdot \left(\sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} |g(aq^r)|^2 + \frac{1}{c} \sum_{a=1}^{c} |g(aq^{R-1})|^2\right)^{p/2}
\end{aligned}$$

for some constant $M'' > 0$.

If $p = 2$, we obtain an analog of the Turán-Kubilius inequality from Theorem 4.2.1.

Corollary 4.2.2. *Let $g : \mathbb{N}_0 \to \mathbb{C}$ be q-additive, $cq^{R-1} \leq N < (c+1)q^{R-1}$ with $R \in \mathbb{N}$ and some $c \in \mathbb{N}$ with $0 < c < q$.*

We set

$$E_R(g) = \sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} g(aq^r),$$

and

$$E_{R,c}(g) = E_R(g) + \frac{1}{c} \sum_{a=1}^{c} g(aq^{R-1}).$$

Then,

$$\frac{1}{N} \sum_{n<N} |g(n) - E_{R,c}(g)|^2 \leq 2 \left(\sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} |g(aq^r)|^2 + \frac{1}{c} \sum_{a=1}^{c} |g(aq^{R-1})|^2\right) . \qquad (4.5)$$

Analogously, we get

Theorem 4.2.3. *Let F be an arbitrary nonnegative-valued increasing function satisfying the inequality*

$$F(2x) \leq \rho F(x)$$

with some constant $\rho > 0$. Let $g : \mathbb{N}_0 \to \mathbb{C}$ be $\mathcal{Q}$-additive, let $c\mathcal{Q}_{R-1} \leq N < (c+1)\mathcal{Q}_{R-1}$ with $R \in \mathbb{N}$ and some $c \in \mathbb{N}$ with $0 < c < q_R$.
We set

$$E_R(g) = \sum_{r=1}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} g(a\mathcal{Q}_{r-1}),$$

and

$$E_{R,c}(g) = E_R(g) + \frac{1}{c} \sum_{a=1}^{c} g(a\mathcal{Q}_{R-1}).$$

Then, for some constant $M_1 > 0$,

$$\begin{aligned} & \frac{1}{N} \sum_{n<N} F(|g(n) - E_{R,c}(g)|) \\ \leq\ & M_1 \cdot \Bigg\{ F\left(\left(\sum_{r=1}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} |g(a\mathcal{Q}_{r-1})|^2 + \frac{1}{c} \sum_{a=1}^{c} |g(a\mathcal{Q}_{R-1})|^2 \right)^{1/2} \right) \\ & + \sum_{r=1}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} F\left(\left| g(a\mathcal{Q}_{r-1}) - \frac{1}{q_r} \sum_{b=0}^{q_r-1} g(b\mathcal{Q}_{r-1}) \right| \right) \\ & + \frac{1}{c} \sum_{a=1}^{c} F\left(\left| g(a\mathcal{Q}_{R-1}) - \frac{1}{c} \sum_{b=1}^{c} g(b\mathcal{Q}_{R-1}) \right| \right) \Bigg\}. \end{aligned} \tag{4.6}$$

Proof. Each nonnegative integer $n < N$ has a unique representation

$$n = \sum_{r=0}^{R-1} \varepsilon_r(n)\mathcal{Q}_r,$$

which $0 \leq \varepsilon_r(n) < q_{r+1}$. We obtain

$$g(n) = \sum_{r=0}^{R-1} g(\varepsilon_r(n)\mathcal{Q}_r).$$

Let $\eta_r(n) = g(\varepsilon_r(n)\mathcal{Q}_r)$, then $\eta_0, \ldots, \eta_{R-1}$ are independent random variables in the Laplace space $\{0, 1, \cdots, (c+1)\mathcal{Q}_{R-1}\}$, and $g = \sum_{r=0}^{R-1} \eta_r$.

We define the function g^*

$$g^*(a\mathcal{Q}_r) = \begin{cases} g(a\mathcal{Q}_r) & \text{for} \quad r < R-1,\ 0 \le a < q_{r+1}\ , \\ & \quad\ \text{or } r = R-1, 0 \le a \le c\ ; \\ 0 & \text{for} \quad r > R-1,\ 0 \le a < q_{r+1}\ , \\ & \quad\ \text{or } r = R-1, c < a < q_R\ . \end{cases}$$

Then $\xi_r = \eta_r - E(\eta_r)$ are independent random variables of zero mean, and

$$g^* - E(g^*) = \sum_{r=0}^{R-1} \xi_r\ ,$$

where $E(g^*) = E_{R,c}(g)$.

Applying the Burkholder's inequality (4.4), we obtain

$$\begin{aligned} &\frac{1}{(c+1)\mathcal{Q}_{R-1}} \sum_{n<(c+1)\mathcal{Q}_{R-1}} F(|g(n) - E_{R,c}(g)|) \\ \ll\ & F\left(\left(\sum_{r=1}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} \left| g(a\mathcal{Q}_{r-1}) - \frac{1}{q_r} \sum_{b=0}^{q_r-1} g(b\mathcal{Q}_{r-1}) \right|^2 \right.\right. \\ & \left.\left. + \frac{1}{c} \sum_{a=1}^{c} \left| g(a\mathcal{Q}_{R-1}) - \frac{1}{c} \sum_{b=1}^{c} g(b\mathcal{Q}_{R-1}) \right|^2 \right)^{1/2} \right) \\ & + \sum_{r=1}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} F\left(\left| g(a\mathcal{Q}_{r-1}) - \frac{1}{q_r} \sum_{b=0}^{q_r-1} g(b\mathcal{Q}_{r-1}) \right| \right) \\ & + \frac{1}{c} \sum_{a=1}^{c} F\left(\left| g(a\mathcal{Q}_{R-1}) - \frac{1}{c} \sum_{b=1}^{c} g(b\mathcal{Q}_{R-1}) \right| \right)\ . \end{aligned}$$

We estimate the first summand as in Theorem 4.2.1.

$$\begin{aligned}\frac{1}{q_r}\sum_{a=0}^{q_r-1}\left|g(a\mathcal{Q}_{r-1})-\frac{1}{q_r}\sum_{b=0}^{q_r-1}g(b\mathcal{Q}_{r-1})\right|^2 &\le \frac{1}{q_r}\left(4\sum_{a=0}^{q_r-1}|g(a\mathcal{Q}_{r-1})|^2+\frac{4}{q_r}\left|\sum_{b=0}^{q_r-1}g(b\mathcal{Q}_{r-1})\right|^2\right)\\ &\le \frac{1}{q_r}\left(4\sum_{a=0}^{q_r-1}|g(a\mathcal{Q}_{r-1})|^2+4\sum_{b=0}^{q_r-1}|g(b\mathcal{Q}_{r-1})|^2\right)\\ &= 8\left(\frac{1}{q_r}\sum_{a=0}^{q_r-1}|g(a\mathcal{Q}_{r-1})|^2\right) .\end{aligned}$$

In the same way, we get

$$\frac{1}{c}\sum_{a=1}^{c}\left|g(a\mathcal{Q}_{R-1})-\frac{1}{c}\sum_{b=1}^{c}g(b\mathcal{Q}_{R-1})\right|^2 \le 8\left(\frac{1}{c}\sum_{a=1}^{c}|g(a\mathcal{Q}_{R-1})|^2\right) .$$

Since

$$\begin{aligned}&\frac{1}{N}\sum_{n<N}F(|g(n)-E_{R,c}(g)|)\\ \le\ &\frac{1}{N}\sum_{n<(c+1)\mathcal{Q}_{R-1}}F(|g^*(n)-E_{R,c}(g)|)\\ \le\ &\frac{c+1}{c}\cdot\frac{1}{(c+1)\mathcal{Q}_{R-1}}\sum_{n<(c+1)\mathcal{Q}_{R-1}}F(|g^*(n)-E_{R,c}(g)|),\end{aligned}$$

the inequality (4.6) follows.

Corollary 4.2.4. *Let $g:\mathbb{N}_0\to\mathbb{C}$ be $\mathcal{Q}$-additive, $c\mathcal{Q}_{R-1}\le N<(c+1)\mathcal{Q}_{R-1}$ with $R\in\mathbb{N}$ and some $c\in\mathbb{N}$ with $0<c<q_R$.*

We set

$$E_R(g)=\sum_{r=1}^{R-1}\frac{1}{q_r}\sum_{a=0}^{q_r-1}g(a\mathcal{Q}_{r-1}),$$

and

$$E_{R,c}(g)=E_R(g)+\frac{1}{c}\sum_{a=1}^{c}g(a\mathcal{Q}_{R-1}).$$

Then,

$$\frac{1}{N}\sum_{n<N}|g(n)-E_{R,c}(g)|^2 \le 2\left(\sum_{r=1}^{R-1}\frac{1}{q_r}\sum_{a=0}^{q_r-1}|g(a\mathcal{Q}_{r-1})|^2+\frac{1}{c}\sum_{a=1}^{c}|g(a\mathcal{Q}_{R-1})|^2\right). \quad (4.7)$$

4.3 Limit distributions of $\mathcal{Q}$-additive functions

In the case of the q-adic scale, necessary and sufficient conditions for the existence of an asymptotic distribution for a real-valued q-additive function have been given by H. Delange [10] in 1972, he proved the following theorem

Proposition 4.3.1 (Delange [10]). *Let g be a real-valued q-additive function. Then g has a limit distribution if and only if the series*

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1} g(aq^r),$$

and

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1} g^2(aq^r)$$

converges.
The limit distribution has as characteristic function the infinite product

$$\prod_{r=0}^{\infty}\frac{1}{q}\left(1+\sum_{a=1}^{q-1}\exp(itg(aq^r))\right),$$

which converges for all real t.

This is similar to the theorem of Erdős - Wintner [16] for the ordinary additive functions.

J. Coquet [4] considered in 1975 the same kind of problems in the cases of the $\mathcal{Q}$-adic scales and obtained mainly sufficient conditions, he proved the following theorems

- If $\{q_r\}_{r\geq 1}$ is bounded:

Proposition 4.3.2 (Coquet [4]). *Let g be a real-valued $\mathcal{Q}$-additive function.*

Then, g has a limit distribution if and only if the series

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} g(a\mathcal{Q}_{r-1}) \right),$$

and

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} g^2(a\mathcal{Q}_{r-1}) \right)$$

converge.

The limit distribution has as characteristic function the infinite product

$$\prod_{r=1}^{\infty} \frac{1}{q_r} \left(1 + \sum_{a=1}^{q_r-1} \exp(itg(a\mathcal{Q}_{r-1})) \right)$$

which converges for all real t.

- If $\{q_r\}_{r\geq 1}$ is unbounded:

Proposition 4.3.3 (Coquet [4]). *Let g be a real-valued $\mathcal{Q}$-additive function.*

We set

$$g^*(a\mathcal{Q}_{r-1}) = \begin{cases} g(a\mathcal{Q}_{r-1}) & \text{if } |g(a\mathcal{Q}_{r-1})| \leq 1, \\ 1 & \text{if } |g(a\mathcal{Q}_{r-1})| > 1, \end{cases}$$

and

$$\beta_r^* = \sup_{1\leq j\leq q_{r-1}} \left(\frac{1}{j+1} \sum_{a=0}^{j} g^*(a\mathcal{Q}_{r-1}) \right)^2 .$$

If $\beta_r^* \to 0$, and the series

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} g^*(a\mathcal{Q}_{r-1}) \right),$$

and

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} g^*(a\mathcal{Q}_{r-1})^2 \right)$$

are convergent, then g *has a limit distribution, its characteristic function is*

$$\prod_{r=1}^{\infty} \frac{1}{q_r} \left(1 + \sum_{a=1}^{q_r-1} \exp(itg(a\mathcal{Q}_{r-1})) \right).$$

4.4 Mean-value theorem for $\mathcal{Q}$-multiplicative functions

In the same paper, H. Delange [10] asserts that for every q-multiplicative function f with $|f| \leq 1$, where $N_x = \lfloor \frac{\log x}{\log q} \rfloor$,

$$m(x) := \frac{1}{x} \sum_{n<x} f(n) = \prod_{r=0}^{N_x-1} \frac{1}{q} \left(\sum_{a=0}^{q-1} f(aq^r) \right) + o(1),$$

as $x \to \infty$.

From this, he deduced that $\lim_{x\to\infty} |m(x)|$ always exists and equals

$$\prod_{r=0}^{\infty} \left| \frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right|$$

which is nonzero if and only if

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} \operatorname{Re}\left(1 - f(aq^r)\right) \tag{4.8}$$

converges, and

$$\sum_{a=0}^{q-1} f(aq^r) \neq 0 \qquad (\text{for all } r \in \mathbb{N}_0). \tag{4.9}$$

Furthermore, he proved that $\lim_{x\to\infty} m(x)$ exists, and is nonzero if and only if (4.9) holds and the series

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (1 - f(aq^r)) \tag{4.10}$$

is convergent.

As an analogue of the Delange result, J. Coquet [4] proved the following mean-value theorems for $\mathcal{Q}$-multiplicative functions modulus ≤ 1.

- If $\{q_r\}_{r\geq 1}$ is bounded:

Proposition 4.4.1 (Coquet [4]). *Let f be a $\mathcal{Q}$-multiplicative function with $|f| \leq 1$.*

(i) If the mean-value of f exists, and is nonzero, then the series

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} (1 - f(a\mathcal{Q}_{r-1})) \right) \tag{4.11}$$

converges, and

$$1 + \sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1}) \neq 0$$

for all $r \in \mathbb{N}$.

(ii) If the series (4.11) *converges, then the mean-value of f is equal to*

$$\prod_{r=1}^{\infty} \left\{ \frac{1}{q_r} \left(1 + \sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1}) \right) \right\},$$

which converges.

- If $\{q_r\}_{r\geq 1}$ is unbounded:

Proposition 4.4.2 (Coquet [4]). *Let f be a $\mathcal{Q}$-multiplicative function with $|f| \leq 1$.*

(i) If $\max_{1\leq j\leq q_r-1} \left\{ \frac{1}{j+1} \sum_{a=0}^{j} (1 - \operatorname{Re} f(a\mathcal{Q}_{r-1})) \right\} \to 0$, as $r \to \infty$, and the series

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \left(\sum_{a=0}^{q_r-1} (1 - f(a\mathcal{Q}_{r-1})) \right) \tag{4.12}$$

converges, then the mean-value of f is equal to

$$\prod_{r=1}^{\infty}\left\{\frac{1}{q_r}\left(1+\sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1})\right)\right\},$$

which converges.

(ii) *If the mean-value of f exists, and it is nonzero and*

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}(1-\operatorname{Re} f(a\mathcal{Q}_{r-1}))<\infty,$$

then the series (4.12) *converges and*

$$1+\sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1})\neq 0$$

for all $r\in\mathbb{N}$.

It appears to be essential to have information on the difference

$$\frac{1}{x}\sum_{0\le n<x} f(n)-\prod_{0<r\le r(x)}\frac{1}{q_r}\sum_{0\le a<q_r} f(a\mathcal{Q}_{r-1})\ ,$$

where $f(\cdot)$ is any $\mathcal{Q}$-multiplicative function of modulus ≤ 1, and more precisely, to get a characterization of

$$\lim_{x\to\infty}\left(\frac{1}{x}\sum_{0\le n<x} f(n)-\prod_{0<r\le r(x)}\frac{1}{q_r}\sum_{0\le a<q_r} f(a\mathcal{Q}_{r-1})\right)=0. \tag{4.13}$$

In fact, if the sequence $\{q_r\}_{r\ge1}$ is bounded, the relation 4.13 is true always. But if $\{q_r\}_{r\ge1}$ is unbounded, the situation is quite different. For example, in [1], G. Barat constructed a $\mathcal{Q}$-multiplicative function h with values 1 or -1 such that

$$\lim_{x\to\infty}\prod_{0<r\le r(x)}\frac{1}{q_r}\sum_{0\le a<q_r} h(a\mathcal{Q}_{r-1})$$

exists, and it is a positive number while

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n<x} h(n)$$

is less than or equal to zero.

This difference is due to the existence of a *first digit phenomenon* which is unavoidable for unbounded sequences $\{q_r\}_{r\geq 1}$ (see E. Manstavičius [63]).

Chapter 5

Mean behaviour of uniformly summable q-multiplicative functions and its applications

The aim of this chapter is to study the behaviour of the means $\dfrac{1}{N}\displaystyle\sum_{n<N} f(n)$ and $\dfrac{1}{N}\displaystyle\sum_{n<N} |f(n)|^{\alpha}$ as $N \to \infty$, for $\alpha > 0$, where f is uniformly summable and q-multiplicative. To our surprise, we find that for q-multiplicative functions the space $\mathcal{L}^{\alpha}$ for every $\alpha > 0$ coincides with the space $\mathcal{L}^{*}$. Furthermore, applying our main results, we investigate finitely distributed q-additive functions and find characterizations for q-multiplicative functions belonging to the space $\mathcal{D}^{1}$ of limit-periodic functions and the space $\mathcal{A}^{1}$ of almost-periodic functions by their respective spectrum $\sigma(f)$.

5.1 Main results

Here we recall that an arithmetical function $f \in \mathcal{L}^{1}$ is said to be **uniformly summable** if

$$\lim_{K\to\infty} \sup_{N\geq 1} \frac{1}{N} \sum_{\substack{n<N \\ |f(n)|\geq K}} |f(n)| = 0,$$

and the space of all uniformly summable functions is denoted by $\mathfrak{L}^*$. f is q-multiplicative if $f(0) = 1$ and

$$f(aq^r + b) = f(aq^r) \cdot f(b)$$

for every pair of integer (a, b) satisfying

$$0 \leq a < q \text{ and } 0 \leq b < q^r .$$

Definition 5.1.1. *Let f be q-multiplicative function, we define*

$$\widetilde{\Pi_{R,\alpha}} := \prod_{r<R}(1 + \widetilde{u_{r,\alpha}}),$$

and

$$\Pi_R := \prod_{r<R}(1 + u_r)$$

with $\widetilde{u_{r,\alpha}} := \dfrac{1}{q}\sum_{a=1}^{q-1}(|f(aq^r)|^\alpha - 1)$ *and* $u_r := \dfrac{1}{q}\sum_{a=1}^{q-1}(f(aq^r) - 1)$.

The following theorem describes a complete characterization of q-multiplicative uniformly summable functions.

Theorem 5.1.2. *Let f be a q-multiplicative function. Then, the following assertions are equivalent.*

(i) $f \in \mathfrak{L}^*$, *and* $\|f\|_1 > 0$.

(ii) *Let $\alpha > 0$. The series*

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1)^2 \tag{5.1}$$

is convergent, and for some constants $c_1(\alpha), c_2(\alpha) \in \mathbb{R}$, for all R and for some sequence $\{R_i\}$, $R_i \to \infty$, the inequalities

$$\sum_{r<R}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1) \leq c_1(\alpha) < \infty, \tag{5.2}$$

and

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) \geq c_2(\alpha) > -\infty \tag{5.3}$$

hold.

(iii) $f \in \mathfrak{L}^\alpha$*, and* $\|f\|_\alpha > 0$ *for all* $\alpha > 0$.

The mean behaviour of such functions is given in

Theorem 5.1.3. *Let* $f \in \mathfrak{L}^*$ *be a* q*-multiplicative function, and let* $\|f\|_1 > 0$*. Further, let* $q^{R-1} \leq N < q^R$ *with* $R \in \mathbb{N}$*. Then, as* $N \to \infty$*,*

$$\frac{1}{N} \sum_{n<N} f(n) = \Pi_R + o(1)$$

and, for every $\alpha > 0$*,*

$$\frac{1}{N} \sum_{n<N} |f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}} + o(1).$$

An immediate consequence is the following

Corollary 5.1.4. *Let* f *be* q*-multiplicative. Then, the following assertions hold.*

(i) Let $f \in \mathfrak{L}^*$*. If the mean-value* $M(f)$ *of* f *exists, and if it is different from zero, then the series*

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (f(aq^r) - 1), \tag{5.4}$$

and

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 \tag{5.5}$$

converge, and

$$\sum_{a=0}^{q-1} f(aq^r) \neq 0 \quad for\ each\ r \in \mathbb{N}_0.$$

(ii) *If the series (5.4) and (5.5) converge, then $f \in \mathfrak{L}^*$, and the mean-value $M(f)$ of f exists,*

$$M(f) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right),$$

and $\|f - f_R\|_1 \to 0$ as $R \to \infty$, where

$$f_R(n) = \prod_{r \le R} f(\varepsilon_r(n) q^r) \qquad 0 \le \varepsilon_r(n) < q.$$

(iii) *Let $f \in \mathfrak{L}^*$. If the mean-value $M(f)$ of f exists, and if it is different from zero, then the mean-value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha > 0$ (and is different from zero).*

The case of mean-value zero is contained in

Corollary 5.1.5. *Let $f \in \mathfrak{L}^*$ be q-multiplicative. Then, the mean-value $M(f)$ of f is zero if and only if $\Pi_R = o(1)$ as $R \to \infty$.*

5.2 Preliminary results

To prove our main theorem, we need to show the following lemmata

Lemma 5.2.1. *Let $f \in \mathfrak{L}^*$ be q-multiplicative and let $\|f\|_1 > 0$. Then,*

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1)^2 < \infty$$

for all $\alpha > 0$.

Proof. Because of $\|f\|_1 > 0$, we can find a sequence $\{x_i\}$ such that

$$\sum_{\substack{n < x_i \\ \varepsilon < |f(n)|^\alpha < K}} 1 >> x_i,$$

as $i \to \infty$ for some suitable $\varepsilon, K > 0$.

We define an q-additive function g by

$$g(aq^r) = \begin{cases} \log(|f(aq^r)|^\alpha) & \text{if } f(aq^r) \neq 0 \\ 1 & \text{if } f(aq^r) = 0. \end{cases}$$

Then,

$$\sum_{\substack{n<x_i \\ -c_1<g(n)<c_2}} 1 \asymp x_i$$

with $c_1 = \log 1/\varepsilon$ and $c_2 = \log K$.

For real numbers t, we define the functions

$$H(x,t) = \sum_{n<x} \exp(itg(n)),$$

for any $x > 0$.

Delange [10] proved that the limit $l(t) := \lim_{x\to\infty} \frac{1}{x}|H(x,t)|$ always exists, and $l(t) \neq 0$ holds if and only if

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (1 - \cos(tg(aq^r)))$$

converges.

Further, we define the function D by

$$D(\nu) = \begin{cases} \left(\frac{\sin \pi\nu}{\pi\nu}\right)^2 & \text{if } \nu \neq 0, \\ 1 & \text{if } \nu = 0. \end{cases}$$

Then, for each real number y, we have

$$\int_{-\infty}^{\infty} e^{2\pi i\nu y} D(\nu) d\nu = \begin{cases} 1 - |y| & \text{if } |y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Interchanging summation and integration shows that for positive λ

$$\int_{-\infty}^{\infty} \lambda |H(x,t)|^2 D(\lambda t) dt = \sum_{\substack{n_1,n_2 \leq x \\ |g(n_1)-g(n_2)| \leq \lambda}} \left(1 - \frac{1}{\lambda}|g(n_1) - g(n_2)|\right).$$

We divide by x_i, let $x_i \to \infty$, and apply Lebesgue's theorem for dominated convergence. If λ is sufficiently large, then

$$\int_{-\infty}^{\infty} \lambda l(t)^2 D(\lambda t) dt > 0.$$

More exactly, if $g(n)$ satisfies the condition given in the definition of finitely distributed functions, and if $\lambda \geq 2c_2$, then the value of this integral is at least as large as $c_1^2/2$.

It follows that there is a set E, of positive Lebesgue measure, on which $l(t) > 0$.

Now,

$$\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) < \infty$$

for every $0 \leq a \leq q-1$ and for all $t \in E$. It means

$$\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) \leq c$$

for all $t \in E^*$, where E^* is some subset of E and $m(E^*) > 0$. This is equivalent to

$$\sum_{r=0}^{\infty}\sin^2\left(\frac{t}{2}g(aq^r)\right) \leq c < \infty$$

for all $t \in E^*$.

In view of the inequality

$$\sin^2(x \pm y) \leq 2\sin^2 x + 2\sin^2 y$$

and applying Steinhaus's lemma[1] we can find a $T > 0$ such that for all $0 \leq a \leq q-1$, and for $|t| \leq T$, we get

$$\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) \leq 4c < \infty \tag{5.6}$$

Integrating (5.6) from 0 to T and multiplying with $1/T$, we have

$$\sum_{r=0}^{\infty}h(Tg(aq^r)) \leq 4c < \infty \tag{5.7}$$

where $h(u) = 1 - \dfrac{\sin u}{u}$ for $u \neq 0$ and $h(0) = 0$.

Since $h(u) \geq 0$ for all real numbers u, and $h(u) \geq 1/2$ for $u \geq 2$, we conclude that $|g(aq^r)| \geq 2/T$ for only finitely many r.

[1](see [12], Lemma (1.1) The differences generated by a set of real numbers of positive measure, cover an open interval about the origin.)

Thus, there is an $M_a > 0$ such that $|g(aq^r)| \leq M_a$ for all $r \geq 0$, and there is $m_a > 0$ such that $h(u) \geq m_a u^2$ for $|u| \leq TM_a$.

Hence,

$$\sum_{r=0}^{\infty}(g(aq^r))^2 \leq \frac{2q\log 2}{m_a T^2},$$

and the series $\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(g(aq^r))^2$ converges.

Since $(\log|x|)^2 \asymp (|x|-1)^2$ if $||x|-1| \leq 1/2$, the proof of Lemma 5.2.1 is finished.

Lemma 5.2.2. *Let f be q-multiplicative and $R \in \mathbb{N}$. Then,*

$$\sum_{n=0}^{q^R-1}|f(n)|^\alpha = q^R\widetilde{\Pi_{R,\alpha}}$$

for every $\alpha > 0$, and

$$\sum_{n=0}^{q^R-1}f(n) = q^R\Pi_R.$$

Proof. Induction over R yields the following formulas

$$\sum_{n=0}^{q^{R+1}-1}|f(n)|^\alpha = \sum_{a=0}^{q-1}\left(\sum_{l=0}^{q^R-1}|f(aq^R+l)|^\alpha\right),$$

and

$$\sum_{n=0}^{q^{R+1}-1}f(n) = \sum_{a=0}^{q-1}\left(\sum_{l=0}^{q^R-1}f(aq^R+l)\right),$$

which prove Lemma 5.2.2.

Lemma 5.2.3. *Let $f \in \mathfrak{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then,*

$$\widetilde{\Pi_{R,\alpha}} = (c(\alpha,|f|)+o(1))\exp\left(\sum_{r<R}\widetilde{u_{r,\alpha}}\right)$$

for all $\alpha > 0$ with some constant $c(\alpha,|f|) \in \mathbb{R}$.

Proof. It is easy to see that, because of the convergence of the series in Lemma 5.2.1, we have

$$\begin{aligned}\widetilde{\Pi_{R,\alpha}} &= \prod_{r<R}(1+\widetilde{u_{r,\alpha}}) \\ &= \exp\left(\sum_{r<R}\log(1+\widetilde{u_{r,\alpha}})\right) \\ &= \exp\left(\sum_{r<R}\widetilde{u_{r,\alpha}} + O\left(\sum_{r<R}(\widetilde{u_{r,\alpha}})^2\right)\right) \\ &= (c(\alpha,|f|)+o(1))\exp\left(\sum_{r<R}\widetilde{u_{r,\alpha}}\right)\end{aligned}$$

for all $\alpha > 0$, and some constant $c(\alpha, |f|) \in \mathbb{R}$.

Lemma 5.2.4. *Let $f \in \mathcal{L}^*$ be q-multiplicative with $\|f\|_1 > 0$, and let $\alpha > 0$. Then, there are some constants $c_1(\alpha), c_2(\alpha) \in \mathbb{R}$ such that*

$$\sum_{r<R}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1) \leq c_1(\alpha) < \infty \tag{5.8}$$

for all R and

$$\sum_{r<R_i}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1) \geq c_2(\alpha) > -\infty \tag{5.9}$$

for some sequence $\{R_i\}$, $R_i \to \infty$.

Proof. Since $f \in \mathcal{L}^1$ and $\|f\|_1 > 0$, by Lemma 5.2.3, we get the inequalities (5.8) and (5.9) for $\alpha = 1$. Now, let $\alpha > 0$, and let

$$||f(aq^r)| - 1| \leq \frac{1}{2},$$

then

$$\begin{aligned} |f(aq^r)|^\alpha - 1 &= (|f(aq^r)| - 1 + 1)^\alpha - 1 \\ &= \alpha(|f(aq^r)| - 1) + O((|f(aq^r)| - 1)^2), \end{aligned}$$

which implies the inequalities (5.8) and (5.9) for all $\alpha > 0$.

Remark. Let $f \in \mathcal{L}^*$ be q-multiplicative with $\|f\|_1 > 0$, and let $\alpha > 0$.
If

$$\sum_{n<x} |f(n)|^\alpha \asymp x$$

then

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) = O(1)$$

as $R \to \infty$.

The next lemma shows a general method for getting upper estimations.

Lemma 5.2.5. *Let f be q-multiplicative and let $q^{R-1} \leq N < q^R$ with $R \in \mathbb{N}$. Then, for every $h \in \mathbb{N}$, we have*

$$\begin{aligned} \left|\sum_{n<N} f(n)\right| \leq & \sum_{r=1}^{h} \left| q^{R-r} \Pi_{R-r} \prod_{t=1}^{r-1} f(\varepsilon_{R-t}(N) q^{R-t}) \sum_{a=0}^{\varepsilon_{R-r}(N)-1} f(aq^{R-r}) \right| \\ & + \left(\prod_{r=R-h}^{R-1} |f(\varepsilon_r(N) q^r)| \right) \cdot O(q^{R-h}), \end{aligned}$$

where the O-constant depends only on f.

Proof. Let $N = cq^{R-1} + b$, where $1 \leq c < q$, and $b = \sum_{r<R-1} \varepsilon_r(N) q^r < q^{R-1}$, where

$0 \leq \varepsilon_r(N) \leq q-1$. Then,

$$
\begin{aligned}
&\sum_{n<N} f(n) \\
= \; & \sum_{a=0}^{c-1} \left(\sum_{l=0}^{q^{R-1}-1} f(aq^{R-1}+l) \right) + \sum_{l=0}^{b-1} f(cq^{R-1}+l) \\
= \; & \sum_{a=0}^{c-1} f(aq^{R-1}) \sum_{l=0}^{q^{R-1}-1} f(l) + f(cq^{R-1}) \sum_{l=0}^{b-1} f(l) \\
= \; & \quad q^{R-1}\Pi_{R-1} \sum_{a=0}^{c-1} f(aq^{R-1}) \\
& +q^{R-2}\Pi_{R-2} f(cq^{R-1}) \sum_{a=0}^{\varepsilon_{R-2}(N)-1} f(aq^{R-2}) \\
& +q^{R-3}\Pi_{R-3} f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \sum_{a=0}^{\varepsilon_{R-3}(N)-1} f(aq^{R-3}) \\
& + \\
& \vdots \\
& +q^{R-h}\Pi_{R-h} f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \cdots f(\varepsilon_{R-h+1}(N)q^{R-h+1}) \sum_{a=0}^{\varepsilon_{R-h}(N)-1} f(aq^{R-h}) \\
& +f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \cdots f(\varepsilon_{R-h+1}(N)q^{R-h+1}) f(\varepsilon_{R-h}(N)q^{R-h}) \sum_{l=0}^{b_h-1} f(l),
\end{aligned}
$$

where $b_h < q^{R-h}$, and $\left|\sum_{l=0}^{b_h-1} f(l)\right| \leq \sum_{l=0}^{q^{R-h}-1} |f(l)| = O(q^{R-h})$.

In the following lemmata 5.2.6, 5.2.7 and 5.2.8, we collect some more properties of q-multiplicative functions $f \in \mathcal{L}^*$ with $\|f\|_1 > 0$.

Lemma 5.2.6. *Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then, the series*

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2$$

is convergent if and only if

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\operatorname{Re} f(aq^r) - 1) \geq c_3 > -\infty$$

for some constant $c_3 \in \mathbb{R}$ and some sequence $\{R_i\}$ with $R_i \uparrow \infty$.

Proof. We have

$$\begin{aligned} \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 = & \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2 \\ & +2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) \\ & -2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\operatorname{Re} f(aq^r) - 1) \\ =: & \sum\nolimits_1 + 2 \sum\nolimits_2 - 2 \sum\nolimits_3 . \end{aligned}$$

By Lemma 5.2.1, $\sum_1$ is convergent and, by Lemma 5.2.4, $\sum_2$ is bounded from above for some sequence $\{R_i\}$ for $R_i \to \infty$. Thus, Lemma 5.2.6 holds.

Lemma 5.2.7. *Let $f \in \mathcal{L}^*$ be q-multiplicative, and let $\|f\|_1 > 0$. If*

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\operatorname{Re} f(aq^r) - 1) \geq c_3 > -\infty$$

for some constant c_3, and for some sequence $\{R_i\}$ with $R_i \uparrow \infty$, then

$$\begin{aligned} \Pi_R := & \prod_{r<R} (1 + u_r) \\ = & (c(f) + o(1)) \exp \left(\sum_{r<R} u_r \right) \end{aligned}$$

for some constant $c(f) \neq 0$.

Proof. If $\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re}\, f(aq^r) - 1) \geq c_3 > -\infty$ for some constant c_3, and for some sequence $\{R_i\}$ with $R_i \uparrow \infty$, then by Lemma 5.2.6, we have

$$\sum_{r=0}^{\infty} |u_r|^2 \leq \sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 < \infty,$$

and we obtain

$$\begin{aligned} \Pi_R := & \prod_{r<R} (1 + u_r) \\ = & \exp\left(\sum_{r<R} u_r + O\left(\sum_{r<R} |u_r|^2\right)\right) \\ = & (c(f) + o(1)) \exp\left(\sum_{r<R} u_r\right) \end{aligned}$$

with some constant $c(f) \neq 0$.

Lemma 5.2.8. *Let $f \in \mathcal{L}^*$ be q-multiplicative, and let $\|f\|_1 > 0$. If*

$$\lim_{R\to\infty} \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re}\, f(aq^r) - 1) = -\infty,$$

then $\Pi_R \to 0$ as $R \to \infty$.

Proof. Obviously,

$$|\Pi_R| = \exp\left(\sum_{r<R} \log|1 + u_r|\right),$$

and

$$\begin{aligned} \log|1 + u_r| &= \frac{1}{2} \log((1 + \mathrm{Re}\, u_r)^2 + (\mathrm{Im} u_r)^2) \\ &= \frac{1}{2} \log(1 + 2\mathrm{Re}\, u_r + |u_r|^2) \\ &\leq \mathrm{Re}\, u_r + \frac{1}{2}|u_r|^2 . \end{aligned}$$

Since

$$\begin{aligned} |u_r|^2 &\leq \frac{q-1}{q} \cdot \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 \\ &= \frac{q-1}{q} \left\{ \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2 + \frac{2}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) - 2\mathrm{Re}\, u_r \right\}, \end{aligned}$$

we observe

$$\mathrm{Re}\, u_r + \frac{1}{2} \left(\frac{q-1}{q} \cdot (-2\mathrm{Re}\, u_r) \right) = \frac{1}{q} \mathrm{Re}\, u_r$$

which implies

$$|\Pi_R| \ll \exp \left(\sum_{r<R} \frac{1}{q} \cdot \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re}\, f(aq^r) - 1) \right),$$

and the assertion of Lemma 5.2.8 follows.

Remark. Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then, by Lemma 5.2.7 and Lemma 5.2.8, $\Pi_R = o(1)$ if and only if

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re}\, f(aq^r) - 1) \to -\infty$$

as $R \to \infty$.

Using the Turán-Kubilius inequality (Corollary 4.2.2), we prove

Lemma 5.2.9. *Let $f \in \mathcal{L}^*$ be q-multiplicative, $\|f\|_1 > 0$ and $q^{R-1} \leq N < q^R$ where $R \in \mathbb{N}$. Further, let*

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 < \infty.$$

Then, for any $h \in \mathbb{N}$,

$$\left| \frac{1}{N} \sum_{n<N} f(n) - \Pi_R \right| \leq \tilde{c} q^{-h} + o(1)$$

as $N \to \infty$, with some constant $\tilde{c} \in \mathbb{R}$ which only depends on f.

Proof. We set

$$f_R(n) = \prod_{r=0}^{R} f(e_r(n) q^r).$$

Then, for any $h \in \mathbb{N}$, we have

$$\begin{aligned}\left|\frac{1}{N}\sum_{n<N} f(n) - \Pi_R\right| \leq\ & \frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| \\ & + \frac{1}{N}\left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h+1}\right| + |\Pi_{R-h+1} - \Pi_R| \\ =:\ & \sum\nolimits_1 + \sum\nolimits_2 + \triangle\end{aligned}$$

Ad $\sum_1$:

We choose $r_0 \in \mathbb{N}$ such that $|f(aq^r) - 1| \leq \frac{1}{10}$ for all $r,\ a \in \mathbb{N}$ with $r > r_0$, $0 \leq a < q$, and we define the function g_R

$$g_R(n) := \begin{cases} \sum\limits_{r>R} \log f(e_r(n)q^r) & \text{for } R \geq r_0\ , \\ 0 & \text{for } R < r_0\ . \end{cases}$$

Then, the functions g_R are q-additive.

Now,

$$\begin{aligned} & \frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| \\ = \ & \frac{1}{N}\sum_{n<N} |f_{R-h}(n)||\exp(g_{R-h}(n)) - 1| \\ \leq \ & \frac{1}{N}\sum_{n<N} |g_{R-h}(n)|(|f(n)| + |f_{R-h}(n)|) \\ \leq \ & \left(\frac{1}{N}\sum_{n<N} |g_{R-h}(n)|^2\right)^{1/2} \left(\left(\frac{2}{N}\sum_{n<N}|f(n)|^2\right)^{1/2} + \left(\frac{2}{N}\sum_{n<N}|f_{R-h}(n)|^2\right)^{1/2}\right). \end{aligned}$$

Applying the Turán-Kubilius inequality for q-additive functions (Corollary 4.2.2), we obtain

$$
\begin{aligned}
&\frac{1}{N}\sum_{n<N}|g_{R-h}(n)|^2 \\
\leq\ & \frac{2}{N}\sum_{n<N}\left|g_{R-h}(n)-\sum_{R-h<r<R}\frac{1}{q}\sum_{a=0}^{q-1}g_{R-h}(aq^r)\right|^2+\frac{2}{N}\sum_{n<N}\left|\sum_{R-h<r<R}\frac{1}{q}\sum_{a=0}^{q-1}g_{R-h}(aq^r)\right|^2 \\
\leq\ & 4\left(\sum_{R-h<r<R-1}\frac{1}{q}\sum_{a=0}^{q-1}|\log f(aq^r)|^2+\frac{1}{c}\sum_{a=1}^{c}|\log f(aq^{R-1})|^2\right)+ \\
&+2\left|\sum_{R-h<r<R}\frac{1}{q}\sum_{a=0}^{q-1}\log f(aq^r)\right|^2,
\end{aligned}
$$

where $cq^{R-1}\leq N<(c+1)q^{R-1}$ with some integer c with $0<c<q$.

Now, since h is fixed, and $\log f(aq^r)\to 0$ for $r\to\infty$, such that

$$\lim_{R\to\infty}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n<N}|g_{R-h}(n)|^2=0.$$

Using Lemmata 5.2.1, 5.2.3 and 5.2.4 for $\alpha=2$ show f, $f_{R-h}\in\mathcal{L}^2$, and thus,

$$\frac{1}{N}\sum_{n<N}|f(n)-f_{R-h}(n)|=o(1).$$

Ad $\sum_2$:

For all $0\leq a<q$ with $0\leq n<q^{R-h+1}$, we have

$$f_{R-h}(aq^{R-h+1}+n)=f(n)\ ,$$

and for all $l\in\mathbb{N}$, we get

$$\sum_{n=0}^{lq^{R-h+1}-1}f_{R-h}(n)=l\sum_{n=0}^{q^{R-h+1}-1}f(n)=lq^{R-h+1}\Pi_{R-h+1}\ .$$

Further, for $N=lq^{R-h+1}$, we obtain

$$\frac{1}{N}\sum_{n<N}f_{R-h}(n)-\Pi_{R-h+1}=0$$

and for $lq^{R-h+1} < N < (l+1)q^{R-h+1}$ and $l \geq 1$, we conclude

$$\begin{aligned}
& \left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h+1}\right| \\
= & \left|-(N-lq^{R-h+1})\Pi_{R-h+1} + \sum_{n=lq^{R-h+1}}^{N-1} f_{R-h}(n)\right| \\
= & \left|-(N-lq^{R-h+1})\Pi_{R-h+1} + f_{R-h}(lq^{R-h+1}) \sum_{n=0}^{N-lq^{R-h+1}-1} f(n)\right| \\
\leq & \; c(N-lq^{R-h+1}) \\
< & \; cq^{R-h+1}
\end{aligned}$$

with some constant c depending only on f.

Ad $\triangle$: Obviously (cf. proof of Lemma 5.2.8)

$$\begin{aligned}
|\Pi_R - \Pi_{R-h+1}| &= |\Pi_{R-h+1}| \left| \left(\prod_{r=R-h+1}^{R-1} \frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right) - 1 \right| \\
&\leq c \sum_{r=R-h+1}^{R-1} \left| \frac{1}{q} \sum_{a=0}^{q-1} (f(aq^r) - 1) \right| .
\end{aligned}$$

Since h is fixed, and $f(aq^r)$ tends to 1 as r runs to infinity, we have $|\Pi_R - \Pi_{R-h}| = o(1)$ as $R \to \infty$.

5.3 Proof of main results

Proof of Theorem 5.1.2. The implication (i) $\Rightarrow$ (ii) is proved as follows.

If $f \in \mathcal{L}^*$ and $\|f\|_1 > 0$, we conclude, by Lemma 5.2.1, that the series (5.1) is convergent. Lemma 5.2.4 shows the inequalities (5.2) and (5.3) for all $\alpha > 0$.

Proof of (ii) $\Rightarrow$ (iii).

By Lemma 5.2.2 and the convergence of (5.1), we show, as in the proof of Lemma 5.2.3,

$$\frac{1}{q^R}\sum_{n=0}^{q^R-1}|f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}} = (c(\alpha,|f|)+o(1))\exp(\sum_{r<R}\widetilde{u_{r,\alpha}})$$

for all $\alpha > 0$ with some constant $c(\alpha,|f|) \in \mathbb{R}$. Observing, if $q^{R-1} \le N < q^R$,

$$\frac{1}{N}\sum_{n<N}|f(n)|^\alpha \ll \frac{1}{q^R}\sum_{n<q^R}|f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}},$$

and the inequality (5.2) gives $f \in \mathcal{L}^\alpha$, and (5.3) implies $\|f\|_\alpha > 0$.

The implication (iii) $\Rightarrow$ (i) is obvious.

Proof of Theorem 5.1.3. First, we assume that $\Pi_R = o(1)$. Then, by Lemma 5.2.5 $\frac{1}{N}\sum_{n<N} f(n) = o(1)$. Now, let $\Pi_R \neq o(1)$. Then, by Lemma 5.2.6 and Lemma 5.2.9, we have $\frac{1}{N}\sum_{n<N} f(n) = \Pi_R + o(1)$.

Furthermore $\widetilde{\Pi_{R,\alpha}} \neq o(1)$, because of $0 < \|f\|_1 \le \|f\|_\alpha$ for all $\alpha > 0$. Then, by Lemma 5.2.1 and Lemma 5.2.9 the second assertion of Theorem 5.1.3 follows.

Proof of Corollary 5.1.4. (i) Let $f \in \mathcal{L}^*$ be q-multiplicative. If the mean-value $M(f)$ of f exists and is nonzero, then obviously $\|f\|_1 > 0$.

We know that (see the proof of Lemma 5.2.8)

$$|\Pi_R| \ll \exp\left(\sum_{r<R}\frac{1}{q^2}\sum_{a=0}^{q-1}(\mathrm{Re}\, f(aq^r)-1)\right).$$

Further, $\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(\mathrm{Re}\, f(aq^r)-1) > c_3 > -\infty$ for some constant $c_3 \in \mathbb{R}$, since the mean-value $M(f)$ of f exists, and it is different from zero.

By Lemma 5.2.6, the series (5.5) converges, and Lemma 5.2.7 gives

$$\begin{aligned}\Pi_R := & \prod_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \\ = & (c(f)+o(1)) \exp\left(\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (f(aq^r)-1) \right),\end{aligned}$$

with some constant $c(f) \neq 0$.

Since the mean-value $M(f)$ of f exists, and it is nonzero, the series (5.4) converges, and $\sum_{a=0}^{q-1} f(aq^r) \neq 0$ for each $r \in \mathbb{N}_0$.

(ii) If the series (5.4) and (5.5) converge then the infinite product $\lim_{R\to\infty} \Pi_R$ exists, and it is zero if and only if a factor equals zero. Thus, $0 < \widetilde{\Pi_{R,1}}$ for all R and

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|-1) > c_4 > -\infty$$

for some constant $c_4 \in \mathbb{R}$.

Now,

$$\begin{aligned}\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r)-1|^2 = & \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|-1)^2 \\ & +2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|-1) \\ & -2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\operatorname{Re} f(aq^r)-1),\end{aligned}$$

holds, and the convergence of the series (5.4) and (5.5) shows that the series

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|-1)^2,$$

and

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|-1)$$

converge. Then, by Theorem 5.1.2, we have $f \in \mathcal{L}^\alpha$ and $\|f\|_\alpha > 0$.

Furthermore, by Lemma 5.2.6 and Lemma 5.2.9, we know that the mean-value $M(f)$ of

f exists, and $M(f) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right)$.

A small modification of the proof for the estimation of $\sum_1$ in Lemma 5.2.9 yields, because of the convergence of the series (5.4) and (5.5), that $\|f - f_R\|_1 \to 0$ as $R \to \infty$.

(iii) Using Theorem 5.1.2 and the same arguments as above, we conclude that the series

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1),$$

and

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1)^2$$

converge, and thus the mean-value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha > 0$ (and it is different from zero).

Proof of Corollary 5.1.5. Obvious.

5.4 Application to q-additive functions

Let us now turn to q-additive functions. Here we recall that a function $f : \mathbb{N} \to \mathbb{C}$ is q-additive if $f(0) = 0$ and

$$f(aq^r + b) = f(aq^r) + f(b)$$

for every pair of integer (a, b) satisfying

$$0 \leq a < q \text{ and } 0 \leq b < q^r .$$

The main results are as follows.

Theorem 5.4.1. *Let g be q-additive. Then, the following assertions hold.*

(i) If g is finitely distributed, then the series $\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (g(aq^r))^2$ converges.

(ii) If, for some $\alpha(x)$,

$$\frac{1}{x}\sharp\{n \le x : g(n) - \alpha(x) \le y\} \Rightarrow G(y)$$

where G is a distribution function, then g is finitely distributed.

(iii) Let $\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(g(aq^r))^2$ converge, and let $\alpha(x) := \sum_{r<N_x}\frac{1}{q}\sum_{a=0}^{q-1} g(aq^r)$, $N_x := \lfloor\frac{\log x}{\log q}\rfloor$. Then,

$$\frac{1}{x}\sharp\{n \le x : g(n) - \alpha(x) \le y\} \Rightarrow G(y),$$

where G is some distribution function.

Assertion (iii) of Theorem 5.4.1 has already been proved by J. Coquet (see [4], Theorem II. 4).

Proof.

(i) The assertion is an immediate consequence of the proof of Lemma 5.2.1.

(ii) We choose the number γ sufficiently large, and such that $\pm\gamma$ are continuity points of the limiting distribution of $g(n) - \alpha(x)$. Then,

$$S := \frac{1}{x}\sharp\{n \le x : g(n) - \alpha(x) \le \gamma\} > \frac{1}{2}.$$

Moreover, let m and n be any two elements of S, then

$$|g(m) - g(n)| \le |g(m) - \alpha(x)| + |\alpha(x) - g(n)| \le 2\gamma,$$

from which it is clear that $g(n)$ is finitely distributed.

(iii) Let

$$\varphi_x(t) := \frac{1}{x}\sum_{n<x} e^{itg(n)}.$$

Then, we shall prove that

$$\varphi_x(t)e^{-it\alpha(x)} \to \varphi(t) \qquad (x \to \infty)$$

for all $t \in \mathbb{R}$, where $\varphi(t)$ is continuous at $t = 0$.
By Theorem 5.1.3, we have $\frac{1}{x}\sum_{n<x} e^{itg(n)} = \prod_{r<N_x}\left(1 + \frac{1}{q}\sum_{a=0}^{q-1}\left(e^{itg(aq^r)} - 1\right)\right) + o(1)$.

Let $u_r(t) = \frac{1}{q}\sum_{a=0}^{q-1}\left(e^{itg(aq^r)} - 1\right)$ and $v_r(t) = \frac{it}{q}\sum_{a=0}^{q-1} g(aq^r)$.

For $|t| \le T$, we obtain

$$|u_r(t)| \le \frac{T}{q}\sum_{a=0}^{q-1}|g(aq^r)|,$$

$$|u_r(t)|^2 \le \frac{T^2(q-1)}{q^2}\sum_{a=0}^{q-1}(g(aq^r))^2$$

and

$$|u_r(t) - v_r(t)| \le \frac{T^2}{2q}\sum_{a=0}^{q-1}(g(aq^r))^2.$$

Hence, the infinite product $\prod_{r=0}^{\infty}(1 + u_r(t))e^{-v_r(t)}$ is uniformly convergent for $t \in [-T, T]$ and defines the characteristic function of a distribution function G.

5.5 Characterization of almost-periodic q-multiplicative functions

The aim of this section is to find corresponding characterizations for q-multiplicative functions belonging to $\mathcal{D}^1$ and $\mathcal{A}^1$, respectively. Here, we recall that f is called *α-almost-periodic*, if for every $\varepsilon > 0$, there is a linear combination h of exponential functions[2] e_β, $\beta \in \mathbb{R}$, such that $\|f - h\|_\alpha \le \varepsilon$. The linear space of α-almost-periodic functions is denoted by $\mathcal{A}^\alpha$. If h can always be chosen to be periodic then f is called *α-limit-periodic*. The linear space of α-limit-periodic functions is denoted by $\mathcal{D}^\alpha$. We have the inclusions

$$\mathcal{D}^1 \subsetneqq \mathcal{A}^1 \subsetneqq \mathcal{L}^*.$$

[2] $e_\beta : \mathbb{N} \to \mathbb{C}$ with $e_\beta(n) := \exp(2\pi i\beta n)$ is a q-multiplicative function.

A first step in this direction was done by J. Spilker [79] who proved the following

Proposition 5.5.1 (Spilker [79], Theorem 4). *Let f be q-multiplicative and the following two series*

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(f(aq^r)-1) \tag{5.10}$$

and

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}|f(aq^r)-1|^2 \tag{5.11}$$

converge. Then,

(i) $f \in \mathcal{D}^{\alpha},\ \alpha \geq 1.$

(ii) $M(f) = \prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)\right).$

(iii) $\hat{f}(\beta) = \begin{cases} \prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)e_{-\frac{c}{b}}(aq^r)\right) & \text{if } \beta = \frac{c}{b}, \\ 0 & \text{if } \beta \text{ irrational.}\end{cases}$

Remark. Assertion (iii) of Proposition 5.5.1 is not correct as it stands. Choose, for example, $f = 1$ and $\beta = \frac{1}{p}$ where p is a prime which does not divide q. Then, $\hat{f}(\beta) = 0$, and for all $r \in \mathbb{N}_0$,

$$\sum_{a=0}^{q-1} f(aq^r)e_{-\frac{1}{p}}(aq^r) = \frac{1-e(q^{r+1}/p)}{1-e(q^r/p)} \neq 0,$$

i.e. the infinite product $\prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)e_{-\frac{1}{p}}(aq^r)\right)$ does not converge in this case.

We shall characterize the q-multiplicative functions $f \in \mathcal{D}^1$ and $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ by their respective spectrum $\sigma(f)$. First we show that the spectrum is empty only in the trivial case. We prove

Theorem 5.5.2. *Let $f \in \mathcal{A}^1$ be q-multiplicative. Then $M(|f|) = 0$ if and only if $\sigma(f) = \emptyset$.*

In the special case that the mean value exists and is different from zero, using Corollary 5.1.4, we obtain

Theorem 5.5.3. *For every q-multiplicative function f, the following assertions are equivalent:*

(a) $f \in \mathcal{D}^1$ *and the mean-value* $M(f)$ *is nonzero.*

(b) The series (5.10) and (5.11) are both convergent and $\sum_{a=1}^{q-1} f(aq^r) \neq 0$ *for each* $r \in \mathbb{N}_0$.

(c) $f \in \mathcal{L}^*$ *and the mean-value* $M(f)$ *exists and is nonzero.*

(d) $f \in \mathcal{D}^\alpha$ *for all* $\alpha \geq 1$ *and the mean-value* $M(f)$ *is nonzero.*

(e) $f \in \mathcal{A}^1$ *and the mean-value* $M(f)$ *is nonzero.*

(f) $f \in \mathcal{A}^\alpha$ *for all* $\alpha \geq 1$ *and the mean-value* $M(f)$ *is nonzero.*

(g) $f \in \mathcal{L}^\alpha$ *for all* $\alpha \geq 1$ *and the mean-value* $M(f)$ *exists and is nonzero.*

We use the following well-known result to prove Theorem 5.5.2 and Theorem 5.5.3.

Lemma 5.5.4. *(see [77] Chap. VI.8. Proposition 8.2) For* $\alpha \geq 1$ *and every arithmetical function* f , $f \in \mathcal{A}^\alpha$ *if and only if* $f \in \mathcal{A}^1$ *and* $|f| \in \mathcal{A}^\alpha$.

Proof of Theorem 5.5.3. The implications "(a)⇒(e)⇒(c)" are obvious and "(c)⇒(b)⇒(a)" hold by Corollary 5.1.4, (i) and (ii). Using Lemma 5.5.4 together with Corollary 5.1.4 for $|f|^\alpha$, $\alpha \geq 1$, gives "(c)⇒(d)", whereas the implications "(d)⇒(f)⇒(g)⇒(c)" are again obvious. This proves Theorem 5.5.3.

Proof of Theorem 5.5.2. If $M(|f|) = 0$ then obviously $\sigma(f) = \emptyset$. Assume that $M(|f|) \neq 0$. Then, by Theorem 5.5.3, $|f| \in \mathcal{A}^2$ and $M(|f|^2) \neq 0$, and Lemma 5.5.4 implies $f \in \mathcal{A}^2$. By Parseval's equality $M(|f|^2) = \sum_{\beta \in \sigma(f)} |M(f \cdot e_{-\beta})|^2$, and $\sigma(f) = \emptyset$

implies $M(|f|) = M(|f|^2) = 0$. This contradiction proves Theorem 5.5.2.

Concerning the description of the spectrum $\sigma(f)$ for q-multiplicative functions $f \in \mathcal{D}^1$ or $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ we establish

Theorem 5.5.5. *Let $f \in \mathcal{D}^1$ be q-multiplicative with non-empty spectrum $\sigma(f)$.*

(a) If $M(f) \neq 0$ then

$$\sigma(f) \subset \{\beta |\, \beta = \frac{c}{b} \; mod\; 1,\; \frac{c}{b} \in \mathbb{Q};\; p\; prime,\; p|b \Rightarrow p|q; \\ \sum_{a=0}^{q-1} f(aq^r)e_{-\beta}(aq^r) \neq 0 \; for\; all\; r \in \mathbb{N}_0\}.$$

(b) If $M(f) = 0$ then there exists some $\beta_0 \in \mathbb{Q}/\mathbb{Z}$ such that

$$\sigma(f) \subset \{\beta |\, \beta = \beta_0 + \frac{c}{b} \; mod\; 1,\; \frac{c}{b} \in \mathbb{Q};\; p\; prime,\; p|b \Rightarrow p|q; \\ \sum_{a=0}^{q-1} f(aq^r)e_{-\beta}(aq^r) \neq 0 \; for\; all\; r \in \mathbb{N}_0\}.$$

Corollary 5.5.6. *Let $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ be q-multiplicative with non-empty spectrum $\sigma(f)$. Then there exists some $\beta_0 \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that*

$$\sigma(f) \subset \{\beta |\, \beta = \beta_0 + \frac{c}{b} \; mod\; 1,\; \frac{c}{b} \in \mathbb{Q};\; p\; prime,\; p|b \Rightarrow p|q; \\ \sum_{a=0}^{q-1} f(aq^r)e_{-\beta}(aq^r) \neq 0 \; for\; all\; r \in \mathbb{N}_0\}.$$

Proof of Theorem 5.5.5 and Corollary 5.5.6. Let $f \in \mathcal{D}^1$ be q-multiplicative and let the mean-value $M(f)$ be nonzero. Then the series (5.10) and (5.11) both converge for f. Let $\beta \in \sigma(f)$. Then $\beta \in \mathbb{R}/\mathbb{Z}$ and the mean-value $M(f \cdot e_{-\beta})$ is nonzero. Putting $g = f \cdot e_{-\beta}$ implies that

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} |g(aq^r) - 1|^2 \tag{5.12}$$

is convergent. We show that this happens if and only if $\beta = \frac{c}{b}$ is a rational number and each prime divisor of b divides q. We consider three cases.

- **Case 1:** Let β be irrational. The function $e_{-\beta}$ is q-multiplicative and its absolute value is equal to 1. By Delange's result [10] for q-multiplicative functions f of absolute value less or equal to 1 whose mean-value $M(f)$ exists, the series

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |e_{-\beta}(aq^r) - 1|^2 \tag{5.13}$$

converges if and only if the representation $M(e_{-\beta}) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=0}^{q-1} (e_{-\beta}(aq^r)) \right)$ holds. Since $M(e_\beta) = 0$ and $\frac{1}{q} \sum_{a=0}^{q-1} (e_{-\beta}(aq^r)) \neq 0$ for all $r \in \mathbb{N}_0$ the series (5.13) diverges.

- **Case 2:** Let $\beta = \frac{c}{b}$ be rational and assume there is a prime p which divides b, but does not divide q. Then for all r the numbers $\frac{c}{b} q^r$ are not integers. This implies:

$$\left| \exp\left(-\frac{c}{b} q^r \right) - 1 \right| \geq \left| 1 - \exp\left(-\frac{1}{b} \right) \right|,$$

and the series

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |e_{-\frac{c}{b}}(aq^r) - 1|^2 \tag{5.14}$$

diverges.

- **Case 3:** Let $\beta = \frac{c}{b}$ be rational, and assume that for each prime divisor of b divides q, too. Then for all $a = 0, 1, \cdots, q-1$ and all $r \geq r_0$, we have $e_{-\beta}(aq^r) = 1$.

Now

$$|1 - e_{-\beta}(aq^r)|^2 \ll |1 - g(aq^r)|^2 + |1 - f(aq^r)|^2.$$

Since the series (5.11) and (5.12) converge, cases 1 and 2 can not occur. Therefore, the mean-value $M(f \cdot e_{-\beta})$ is zero for the cases 1 and 2. In case 3 the series

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (g(aq^r) - 1)^2$$

and

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (g(aq^r) - 1)$$

converge. Then

$$M(g) = \prod_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} g(aq^r) \tag{5.15}$$

and the mean-value $M(g)$ is nonzero if and only if each factor of (5.15) is nonzero. This proves (a).

For the proof of (b) and Corollary 5.5.6, let the mean-value of f be zero, and let $\beta_0 \in \mathbb{R}/\mathbb{Z}$ such that the mean-value of $f \cdot e_{-\beta_0}$ is nonzero . Then $f \cdot e_{-\beta_0} \in \mathcal{D}^1$. Since $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ if and only if β_0 is irrational, (a) yields (b) and Corollary 5.5.6.

Example. Let $f = e_\beta$ where $\beta \in (R \setminus \mathbb{Q})/\mathbb{Z}$. Then, obviously, the mean-value $M(f)$ equals zero and $\sigma(f) = \{\beta\}$.

Chapter 6

Mean behaviour of uniformly summable $\mathcal{Q}$-multiplicative functions

In this chapter, we extend the results of Chapter 5 to uniformly summable $\mathcal{Q}$-multiplicative functions. In the case of a bounded sequence $\{q_r\}_{r\geq 1}$, we have similar theorems as in the q-adic case. In the case of an unbounded sequence $\{q_r\}_{r\geq 1}$, the situation is quite different. Unavoidable for unbounded sequences $\{q_r\}_{r\geq 1}$ is the existence of a so-called first digit phenomenon. We investigate the mean behaviour of uniformly summable $\mathcal{Q}$-multiplicative functions that belong to $\mathcal{L}^2$ and for which the first digit condition

$$\max_{1\leq j\leq q_r-1} \frac{1}{j+1}\sum_{a=0}^{j} |f(a\mathcal{Q}_{r-1})-1|^2 \to 0 \qquad \text{as } r\to\infty$$

holds.

6.1 Main results

Let $\{q_r\}_{r\geq 1}$ with $q_r \geq 2$, be a sequence of natural numbers, and let $\mathcal{Q}_0 = 1$, $\mathcal{Q}_r = q_r\mathcal{Q}_{r-1}$ when $r \geq 1$. Here, we recall that f is $\mathcal{Q}$-multiplicative if $f(0) = 1$, and

$$f(a\mathcal{Q}_r + b) = f(a\mathcal{Q}_r)\cdot f(b)$$

for every pair of integer (a, b) satisfying

$$0 \leq a < q_{r+1} \text{ and } 0 \leq b < \mathcal{Q}_r .$$

Definition 6.1.1. *Let f be $\mathcal{Q}$-multiplicative function, we define*

$$\widetilde{\Pi_R} := \prod_{r<R}(1 + \widetilde{u_r}),$$

and

$$\Pi_R := \prod_{r<R}(1 + u_r)$$

where $\widetilde{u_r} := \dfrac{1}{q_r}\displaystyle\sum_{a=0}^{q_r-1}(|f(a\mathcal{Q}_{r-1})| - 1)$ *and* $u_r := \dfrac{1}{q_r}\displaystyle\sum_{a=0}^{q_r-1}(f(a\mathcal{Q}_{r-1}) - 1)$.

Theorem 6.1.2. *Let $f \in \mathcal{L}^2$ be $\mathcal{Q}$-multiplicative function. If*

$$\max_{1\leq j\leq q_r-1} \frac{1}{j+1}\sum_{a=0}^{j}|f(a\mathcal{Q}_{r-1}) - 1|^2 \to 0, \tag{6.1}$$

then, for $\mathcal{Q}_{R-2} \leq N < \mathcal{Q}_{R-1}$, $N \to \infty$

(a) $\dfrac{1}{N}\displaystyle\sum_{n<N} f(n) = \Pi_R + o(1)$,

(b) $\dfrac{1}{N}\displaystyle\sum_{n<N} |f(n)| = \widetilde{\Pi}_R + o(1)$.

Theorem 6.1.3. *Let $f \in \mathcal{L}^2$ be $\mathcal{Q}$-multiplicative function. If the conditions*

(i) $\displaystyle\max_{1\leq j<q_r} \frac{1}{j+1}\sum_{a=0}^{j}|f(a\mathcal{Q}_{r-1}) - 1|^2 \to 0$,

(ii) $\displaystyle\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}|f(a\mathcal{Q}_{r-1}) - 1|^2 < \infty$,

(iii) $\displaystyle\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}(f(a\mathcal{Q}_{r-1}) - 1)$ *converges,*

(iv) $1+\sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1}) \neq 0.$

hold, then, the mean-value $M(f)$ of f exists and is different from zero.

Theorem 6.1.4. *Let $f \in \mathcal{L}^2$ be $\mathcal{Q}$-multiplicative function. Suppose the mean-value $M(f)$ of f exists and is different from zero,*

$$\max_{1\leq j<q_r} \frac{1}{j+1}\sum_{a=0}^{j} |f(a\mathcal{Q}_{r-1})-1|^2 \to 0,$$

and

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1})-1|^2 < \infty.$$

Then,

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1} (f(a\mathcal{Q}_{r-1})-1)$$

converges, and

$$1+\sum_{a=1}^{q_r-1} f(a\mathcal{Q}_{r-1}) \neq 0.$$

6.2 Preliminary results

To prove our main theorem, we need to show the following lemmata

Lemma 6.2.1. *Let $z_1, \ldots, z_k \in \mathbb{C}$ are complex numbers, then*

$$|z_1\cdots z_k - 1| \leq \prod_{j=1}^{k}\max(|z_j|,1)\sum_{j=1}^{k}|z_j-1|.$$

Proof.

$$\begin{aligned} |z_1\cdots z_k-1| &\leq |z_k||z_1\cdots z_{k-1}-1| + |z_k-1| \\ &\leq \max(|z_k|,1)(|z_1\cdots z_{k-1}-1|+|z_k-1|) \\ &\leq \prod_{j=1}^{k}\max(|z_j|,1)\sum_{j+1}^{k}|z_j-1|. \end{aligned}$$

Lemma 6.2.2. *Let f be $\mathcal{Q}$-multiplicative and $R \in \mathbb{N}$. Then,*

$$\sum_{n=0}^{\mathcal{Q}_{R-1}-1} f(n) = \mathcal{Q}_{R-1}\Pi_R,$$

and

$$\sum_{n=0}^{\mathcal{Q}_{R-1}-1} |f(n)| = \mathcal{Q}_{R-1}\widetilde{\Pi_R}.$$

Proof. Induction over R yields the following formulas

$$\sum_{n=0}^{\mathcal{Q}_R-1} f(n) = \sum_{a=0}^{q_R-1}\left(\sum_{l=0}^{\mathcal{Q}_{R-1}-1} f(a\mathcal{Q}_{R-1}+l)\right),$$

and

$$\sum_{n=0}^{\mathcal{Q}_R-1} |f(n)| = \sum_{a=0}^{q_R-1}\left(\sum_{l=0}^{\mathcal{Q}_{R-1}-1} |f(a\mathcal{Q}_{R-1}+l)|\right)$$

for all $R \geq 0$, which proves Lemma 6.2.2.

Lemma 6.2.3. *Let f be $\mathcal{Q}$-multiplicative function and*

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}(|f(a\mathcal{Q}_{r-1})|-1)^2 < \infty\ ,$$

then

$$\widetilde{\Pi_R} = (c_1(f)+o(1))\exp\left(\sum_{r\leq R}\widetilde{u_r}\right)$$

for some constant $c_1(f) \in \mathbb{R}$.

Proof. Since

$$\begin{aligned}\sum_{r=1}^{\infty}(\widetilde{u_r})^2 &= \sum_{r=1}^{\infty}\frac{1}{q_r^2}\left(\sum_{a=0}^{q_r-1}(|f(a\mathcal{Q}_{r-1})|-1)\right)^2 \\ &\leq \sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}(|f(a\mathcal{Q}_{r-1})|-1)^2 < \infty\ ,\end{aligned}$$

is easy to see that

$$\begin{aligned}
\widetilde{\Pi_R} &= \prod_{r<R}(1+\widetilde{u_r}) \\
&= \exp\left(\sum_{r<R}\log(1+\widetilde{u_r})\right) \\
&= \exp\left(\sum_{r<R}\widetilde{u_r} + O\left(\sum_{r<R}(\widetilde{u_r})^2\right)\right) \\
&= (c(|f|)+o(1))\exp\left(\sum_{r<R}\widetilde{u_r}\right).
\end{aligned}$$

As a consequence we get

Corollary 6.2.4. *Let f be $\mathcal{Q}$-multiplicative function and*

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}(|f(a\mathcal{Q}_{r-1})|-1)^2 < \infty\ ,$$

then, $f \in \mathcal{L}^1$, and $\|f\|_1 \neq 0$ if and only if for some constants $c_1,\ c_2 \in \mathbb{R}$

$$\sum_{r<R}\widetilde{u_r} \leq c_1 < \infty,$$

as $R \to \infty$, and

$$\sum_{r<R_i}\widetilde{u_r} \geq c_2 > -\infty$$

for some sequence $\{R_i\}$, $R_i \to \infty$.

Lemma 6.2.5. *Let f be $\mathcal{Q}$-multiplicative function and*

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}|f(a\mathcal{Q}_{r-1})-1|^2 < \infty\ ,$$

then, $|\Pi_R| \asymp 1$ if and only if

$$\sum_{r\leq R}(2\mathrm{Re}\,u_r + |u_r|^2) = O(1),$$

as $R \to \infty$.

Proof. We get

$$\begin{aligned}|\Pi_R| &= \prod_{r<R} |1+u_r| \\ &= \exp\left(\log \prod_{r<R} |1+u_r|\right) \\ &= \exp\left(\sum_{r<R} \log|1+u_r|\right).\end{aligned}$$

Since

$$\begin{aligned}\log|1+u_r| &= \frac{1}{2}\log(1+2\mathrm{Re}\,u_r+|u_r|^2) \\ &\asymp 2\mathrm{Re}\,u_r+|u_r|^2\ ,\end{aligned}$$

we obtain

$$\begin{aligned}|\Pi_R| &= \exp\left(\sum_{r<R} \log|1+u_r|\right) \\ &\asymp \exp\left(\sum_{r<R}(2\mathrm{Re}\,u_r+|u_r|^2)\right)\end{aligned}$$

Lemma 6.2.5 yields the following

Corollary 6.2.6. *Let f be $\mathcal{Q}$-multiplicative function and*

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1})-1|^2 < \infty\ ,$$

then,

(i) $|\Pi_R| \to c \neq 0$ *if and only if the series* $\sum_{r\le R}(2\mathrm{Re}\,u_r+|u_r|^2)$ *converges,*

(ii) $|\Pi_R| \to 0$ *if and only if the series* $\sum_{r\le R}(2\mathrm{Re}\,u_r+|u_r|^2)$ *diverges,*

as $R\to\infty$*.*

Example 6.2.7. *Let*

$$u_r = -\frac{1}{r} + i\frac{\sqrt{2}}{\sqrt{r}},$$

where $u_r = \dfrac{1}{r+1}\sum_{a=1}^{r} f(ar!) - 1$ *with* $f(ar!) = 1 - \dfrac{r+1}{r^2} + i\dfrac{(r+1)\sqrt{2}}{r\sqrt{r}}$.

It is easy to see that, the series $\sum_{r=1}^{\infty} \mathrm{Re}\, u_r$ diverges, but the series

$$\sum_{r=1}^{\infty}(2\mathrm{Re}\, u_r + |u_r|^2) = \sum_{r=1}^{\infty}\frac{1}{r^2}$$

converges.

Lemma 6.2.8. *Let f be $\mathcal{Q}$-multiplicative function. If*

$$\max_{1\le j<q_r}\frac{1}{j+1}\sum_{a=0}^{j}|f(a\mathcal{Q}_{r-1}) - 1|^2 \to 0,$$

and

$$\sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}|f(a\mathcal{Q}_{r-1}) - 1|^2 < \infty\ ,$$

then,

$$|\Pi_R| = (c_2(f) + o(1))\exp\left(\sum_{r\le R} 2\mathrm{Re}\, u_r\right)$$

for some constant $c_2(f) \in \mathbb{R}$.

Proof. Since

$$\begin{aligned}\sum_{r=1}^{\infty}|u_r|^2 &= \sum_{r=1}^{\infty}\left|\frac{1}{q_r}\sum_{a=0}^{q_r-1}(f(a\mathcal{Q}_{r-1}) - 1)\right|^2\\ &\le \sum_{r=1}^{\infty}\frac{1}{q_r}\sum_{a=0}^{q_r-1}|f(a\mathcal{Q}_{r-1}) - 1|^2 < \infty\ ,\end{aligned}$$

and applying the proof of Lemma 6.2.5, we obtain

$$\begin{aligned} |\Pi_R| &= \exp\left(\sum_{r<R} \log|1+u_r|\right) \\ &\asymp \exp\left(\sum_{r<R} (2\mathrm{Re}\, u_r + |u_r|^2)\right) \\ &= (c_2(f)+o(1)) \exp\left(\sum_{r\le R} 2\mathrm{Re}\, u_r\right). \end{aligned}$$

Lemma 6.2.9. *Let f be $\mathcal{Q}$-multiplicative function. If*

$$\max_{1\le j<q_r} \frac{1}{j+1} \sum_{a=0}^{j} |f(a\mathcal{Q}_{r-1})-1|^2 \to 0,$$

and

$$\sum_{r=1}^{\infty} \frac{1}{q_r} \sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1})-1|^2 < \infty\ ,$$

then,

$$\Pi_R = (c_3(f)+o(1)) \exp\left(\sum_{r\le R} u_r\right)$$

for some constant $c_3(f) \in \mathbb{R}$.

Proof. By the proof of Lemma 6.2.8, we know that, $\sum_{r=1}^{\infty} |u_r|^2 < \infty$.

Hence,

$$\begin{aligned} \Pi_R := & \prod_{r<R} (1+u_r) \\ = & \exp\left(\sum_{r<R} u_r + O\left(\sum_{r<R} |u_r|^2\right)\right) \\ = & (c_3(f)+o(1)) \exp\left(\sum_{r<R} u_r\right). \end{aligned}$$

6.3 Proof of main results

Proof of Theorem 6.1.2. We set

$$f_T(n) = \prod_{r=1}^{T} f(\varepsilon_r(n)\mathcal{Q}_{r-1})$$

and let $M > \mathcal{Q}_{T-1}$, then

$$\frac{1}{M}\sum_{n<M} |f_T(n)|^2 = \frac{1}{M}\left(\sum_{m<\mathcal{Q}_T} |f(m)|^2 \cdot \sharp\left\{n \,\middle|\, \sum_{r=1}^{T} \varepsilon_r(n)\mathcal{Q}_{r-1} = m \wedge n < M\right\}\right).$$

Put $c\mathcal{Q}_{R_1-1} \leq M < (c+1)\mathcal{Q}_{R_1-1}$ with $0 < c < q_{R_1}$, and to estimate the above term, we obtain

$$\frac{1}{M}\sum_{n<M} |f_T(n)|^2 \leq \frac{K\mathcal{Q}_T \cdot (c+1) q_{R_1-1} q_{R_1-2} \cdots q_{T-1}}{c\mathcal{Q}_{R_1-1}} \leq 2K,$$

where K is constant. Thus f_T is also a $\mathcal{L}^2$ function.

Then, for each $h \in \mathbb{N}$,

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n<N} f(n) - \Pi_R\right| \leq\ & \frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| \\
& + \frac{1}{N}\left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h}\right| + |\Pi_{R-h} - \Pi_R| \\
=\ & \sum\nolimits_1 + \sum\nolimits_2 + \Delta\ .
\end{aligned}$$

Ad $\sum_1$:

We define the $\mathcal{Q}$-additive function g

$$\begin{aligned}
g(n) &= \sum_{r=1}^{\infty} g(\varepsilon_r(n)\mathcal{Q}_{r-1}) \\
&= \sum_{R-h<r<R} g(\varepsilon_r(n)\mathcal{Q}_{r-1}) \\
&= \sum_{R-h<r<R} |f(\varepsilon_r(n)\mathcal{Q}_{r-1}) - 1|
\end{aligned}$$

where

$$g(\varepsilon_r(n)\mathcal{Q}_{r-1}) := \begin{cases} |f(\varepsilon_r(n)\mathcal{Q}_{r-1}) - 1|) & \text{if } R-h < r \leq R\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Applying Lemma 6.2.1, we obtain

$$\begin{aligned} & \frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| \\ = \ & \frac{1}{N}\sum_{n<N} |f_{R-h}(n)| \left| \prod_{R-h<r\leq R} f(\varepsilon_r(n)\mathcal{Q}_{r-1}) - 1 \right| \\ \leq \ & \frac{1}{N}\sum_{n<N} |f_{R-h}(n)| \prod_{R-h<r\leq R} \max(|f(\varepsilon_r(n)\mathcal{Q}_{r-1})|, 1) \sum_{R-h<r\leq R} |f(\varepsilon_r(n)\mathcal{Q}_{r-1}) - 1| \\ \leq \ & \frac{1}{N}\sum_{n<N} |f_{R-h}(n)| \prod_{R-h<r\leq R} \max(|f(\varepsilon_r(n)\mathcal{Q}_{r-1})|, 1) g(n) \\ \leq \ & \left(\frac{1}{N}\sum_{n<N} (g(n))^2\right)^{1/2} \left(\frac{1}{N}\sum_{n<N}\left(|f_{R-h}(n)| \prod_{R-h<r\leq R} \max(|f(\varepsilon_r(n)\mathcal{Q}_{r-1})|, 1)\right)^2\right)^{1/2}. \end{aligned}$$

Using the Turán-Kubilius inequality for $\mathcal{Q}$-additive functions (Corollary 4.2.4), we have

$$\begin{aligned} & \frac{1}{N}\sum_{n<N} (g(n))^2 \\ \leq \ & \frac{2}{N}\sum_{n<N}\left(g(n) - \sum_{R-h<r<R} \frac{1}{q_r}\sum_{a=0}^{q_r-1} g(a\mathcal{Q}_{r-1})\right)^2 + \frac{2}{N}\sum_{n<N}\left(\sum_{R-h<r<R} \frac{1}{q_r}\sum_{a=0}^{q_r-1} g(a\mathcal{Q}_{r-1})\right)^2 \\ \leq \ & 4\left(\sum_{R-h<r<R-1} \frac{1}{q_r}\sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1}) - 1|^2 + \frac{1}{c}\sum_{a=1}^{c} |f(a\mathcal{Q}_{R-2} - 1)|^2\right) \\ & +2\left(\sum_{R-h<r<R} \frac{1}{q_r}\sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1}) - 1|\right)^2, \end{aligned}$$

where $c\mathcal{Q}_{R-2} \leq N < (c+1)\mathcal{Q}_{R-2}$, $0 < c < q_{R-1}$.

Applying the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned} 2\left(\sum_{R-h<r<R} \frac{1}{q_r} \sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1}) - 1|\right)^2 &\leq 2h \sum_{R-h<r<R} \frac{1}{(q_r)^2} \left(\sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1}) - 1|\right)^2 \\ &\leq 2h \sum_{R-h<r<R} \frac{r}{(q_r)^2} \sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1}) - 1|^2. \end{aligned}$$

Since h is fixed, and concerning condition (6.1), we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} (g(n))^2 = 0.$$

Now, we define the $\mathcal{Q}$-multiplicative function $\tilde{f}$ with

$$\tilde{f}(a\mathcal{Q}_{r-1}) := \begin{cases} |f(a\mathcal{Q}_{r-1})| & \text{if } 1 \leq r \leq R-h \\ \max(|f(a\mathcal{Q}_{r-1})|, 1) & \text{if } R-h < r \leq R \end{cases},$$

then,

$$|f_{R-h}(n)| \prod_{R-h<r\leq R} \max(|f(\varepsilon_r(n)\mathcal{Q}_{r-1})|, 1) = \tilde{f}(n),$$

and

$$\begin{aligned} \frac{1}{N} \sum_{n<N} (\tilde{f}(n))^2 &\leq \frac{1}{c\mathcal{Q}_{R-2}} \sum_{n<(c+1)\mathcal{Q}_{R-2}} (\tilde{f}(n))^2 \\ &= \frac{1}{c\mathcal{Q}_{R-2}} \prod_{1\leq r\leq R-h} (q_r) \left(1 + \frac{1}{q_r} \sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1})|^2 - 1\right) \\ &\quad \cdot \prod_{R-h<r\leq R-2} (q_r) \left(1 + \frac{1}{q_r} \sum_{a=0}^{q_r-1} |\tilde{f}(a\mathcal{Q}_{r-1})|^2 - 1\right) \\ &\quad \cdot (c+1) \left(1 + \frac{1}{c+1} \sum_{a=1}^{c} |\tilde{f}(a\mathcal{Q}_{R-2})|^2 - 1\right) \\ &= \prod\nolimits_1 \cdot \prod\nolimits_2 \cdot \prod\nolimits_3, \end{aligned}$$

where $c\mathcal{Q}_{R-2} \leq N < (c+1)\mathcal{Q}_{R-2}$ with $0 < c < q_{R-1}$.

The first product $\prod_1$ equals

$$\begin{aligned}\prod\nolimits_1 &= \frac{1}{c\mathcal{Q}_{R-2}} \prod_{1\le r\le R-h} (q_r) \left(1 + \frac{1}{q_r}\sum_{a=0}^{q_r-1} |f(a\mathcal{Q}_{r-1})|^2 - 1\right) \\ &= \frac{1}{c\mathcal{Q}_{R-2}} \sum_{n<(c+1)\mathcal{Q}_{R-2}} |f_{R-h}(n)|^2.\end{aligned}$$

Since $f_{R-h} \in \mathcal{L}^2$, such that the product $\prod_1$ is bounded.

The equivalent of condition (6.1) is

$$\max_{1\le j<q_r} \frac{1}{j+1} \sum_{\substack{a=0\\ |f(a\mathcal{Q}_{r-1})|>1}}^{j} |f(a\mathcal{Q}_{r-1})|^2 - 1 = O(1),$$

for $r \to \infty$, therefor the products $\prod_2$ and $\prod_3$ are bounded.

Thus, $\dfrac{1}{N}\displaystyle\sum_{n<N} (\tilde{f}(n))^2 \le 4K\tilde{c}^{h-1}$ is bounded, where K is constant only depends on f_{R-h}, and $\tilde{c}$ is constant only depends on $\tilde{f}$.

Ad $\sum_2$:

For all $a \ge 0$, $0 \le n < \mathcal{Q}_{R-h}$,

$$f_{R-h}(a\mathcal{Q}_{R-h} + n) = f(n)$$

holds, and for $1 \le l \le q_{R-h}$, applying Lemma 6.2.2, we have

$$\begin{aligned}\sum_{n=0}^{l\mathcal{Q}_{R-h}-1} f_{R-h}(n) &= \sum_{a=0}^{l-1}\sum_{n'=0}^{\mathcal{Q}_{R-h}-1} f_{R-h}(a\mathcal{Q}_{R-h} + n') \\ &= \sum_{a=0}^{l-1}\sum_{n'=0}^{\mathcal{Q}_{R-h}-1} f(n') \\ &= l\sum_{n'=0}^{\mathcal{Q}_{R-h}-1} f(n') \\ &= l\mathcal{Q}_{R-h}\Pi_{R-h}.\end{aligned}$$

Further, for $N = l\mathcal{Q}_{R-h}$, we obtain

$$\frac{1}{N}\sum_{n<N} f_{R-h}(n) - \Pi_{R-h} = 0,$$

and for $l\mathcal{Q}_{R-h} < N < (l+1)\mathcal{Q}_{R-h}$, we conclude

$$\begin{aligned}
&\left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h}\right| \\
= &\left|-(N - l\mathcal{Q}_{R-h})\Pi_{R-h} + \sum_{n=l\mathcal{Q}_{R-h}}^{N-1} f_{R-h}(n)\right| \\
= &\left|-(N - l\mathcal{Q}_{R-h})\Pi_{R-h} + f_{R-h}(l\mathcal{Q}_{R-h+1}) \sum_{n=0}^{N-l\mathcal{Q}_{R-h}-1} f(n)\right| \\
\leq\ & c(N - l\mathcal{Q}_{R-h}) \\
<\ & c\mathcal{Q}_{R-h},
\end{aligned}$$

with some constant c only depends on f.

Ad Δ:

We get

$$\begin{aligned}
|\Pi_R - \Pi_{R-h}| &= |\Pi_{R-h}| \left| \left(\prod_{r=R-h}^{R-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} f(a\mathcal{Q}_{r-1}) \right) - 1 \right| \\
&\leq c \sum_{r=R-h}^{R-1} \left| \frac{1}{q_r} \sum_{a=0}^{q_r-1} (f(a\mathcal{Q}_{r-1}) - 1) \right| \\
&\leq c \sum_{r=R-h}^{R-1} \frac{\sqrt{q_r - 1}}{q_r} \left(\sum_{a=0}^{q_r-1} |(f(a\mathcal{Q}_{r-1}) - 1)|^2 \right)^{1/2} \\
&\leq c\sqrt{h} \left(\sum_{r=R-h}^{R-1} \frac{q_r - 1}{q_r^2} \sum_{a=0}^{q_r-1} |(f(a\mathcal{Q}_{r-1}) - 1)|^2 \right)^{1/2}.
\end{aligned}$$

Since condition (6.1) holds, and h is fixed, we have $|\Pi_R - \Pi_{R-h}| = o(1)$, as $R \to \infty$.

Altogether, we obtain

$$\left| \frac{1}{N} \sum_{n<N} f(n) - \Pi_R \right| \leq \tilde{c} \frac{1}{q_{R-1} q_{R-2} \cdots q_{R-h+1}} + o(1).$$

The proof of assertion (b) is analogous.

Proof of Theorem 6.1.3. By Lemma 6.2.9 and Theorem 6.1.2, the assertion of Theorem 6.1.3 follows.

Proof of Theorem 6.1.4. Follows by Lemma 6.2.9.

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