Report on the Ph.D. Thesis:
Foliated rho-invariants
by Indrava Roy.

Introduction. Invariants of rho-type were introduced for the first time by Atiyah-Patodi-Singer and then extended to the more general setting of Galois Γ-coverings by Cheeger and Gromov. They are numeric *secondary* invariants of Dirac operators which detect more subtle properties than the corresponding primary invariant, which is the Fredholm index. The geometrically most interesting cases are those defined by the signature operator and by the spin Dirac operator associated to a metric of positive scalar curvature. In the first case the rho-invariant is typically a number which is not homotopy invariant (contrary to the index); similarly, in the second case the rho-invariant is typically different from zero, contrary to the index which is always zero because of the hypothesis of positive scalar curvature. Nevertheless, a fashinating result , established by Navin Keswani and reproved by Piazza and Schick, asserts that under additional hypothesis on the group Γ, namely torsion free and satisfying the maximal Baum-Connes hypothesis, these two rho-invariants behave exactly as an index, thus they are homotopy invariant for the signature operator and equal to zero for the spin Dirac operator associated to a metric of positive scalar curvature. Benameur and Piazza extended this result to a special case of foliations, the so-called foliated flat bundles. Their result for the signature operator is established under a special assumption on the foliated homotopy equivalence. The main goal of this Thesis is to extend this analysis to general (measured) foliations and, more importantly, to remove the additional hypothesis on the foliated homotopy equivalence.

Comments. First of all I would like to say that the goal of this thesis is an ambitious one. I have thought rather hard to this extension of my result with Benameur but I realized rather quickly that it required new ideas and new techniques. This is precisely what is done in this thesis. Before explaining in which way this work of Indrava Roy stands as a very original contribution to the field, I would like to comment on the structure of the thesis and on all the preliminary work leading to the main result contained in it.

The thesis begins with a careful description of all the operator-algebra results that are necessary in the theory. This can be regarded as somewhat standard; on the other hand some of the results explained in Chapther 2 as well as in Chapter 3 are rather often folklore and I am rather impressed by the degree of precision and rigor that Indrava Roy has displayed in preparing these two chapters. This is especially true for the analogue of the Atiyah's theorem on Γ-coverings, which is very carefully explained in this thesis. I am not aware of a published proof of this theorem, even though all the experts know it and would be able to prove it if pressed to do so. The exposition here follows the one in my long paper with Benameur but with all the difficulties and intricacies that are required by dealing with a general measured foliations.

Chapter 4 deals with the stability properties of the foliated rho-invariant. Here, once again, the exposition follows the steps of Cheeger-Gromov and Benameur-Piazza but with all the complications of the foliated case. I was again very impressed by the rigor and the precision of the exposition.

Chapter 5 is the core of the thesis and the chapter where the most original ideas appear. As it has been already explained, the foliated homotopy invariance of the foliated rho-invariant defined by Benameur-Piazza was established by these two authors under restrictive hypothesis on the foliated homotopy equivalence. One consequence of these hypothesis was that it was indeed possible to work with two distinct Hilbert modules, the ones associated to the two homotopy equivalent foliated flat bundles, but that were defined, on the other hand, on *the same C^*-algebra*. This allowed to adapt the original proof of Keswani in a way that did require some work but that was on the other hand not too far from the original argument. In the general case, the one treated by Indrava Roy (in collaboration with his advisor Moulay Benameur), the two Hilbert modules are defined on *different* C^*-algebras and these are "only" Morita equivalent. Here rests the main difficulty of the general case. This crucial difficulty is tackled in this thesis by using and extending the theory of Hilbert-Poincaré complexes, due to Higson and Roe. In the case at hand the complexes are only countably generated; more importantly, as a consequence of this more general theory it is possible to treat homotopy-equivalent Hilbert-Poincaré complexes on Morita equivalent C^*-algebras. In particular, an auxiliary complex is defined and this complex interpolates between the two complexes on the two homotopy equivalent foliations allowing for the construction of an explicit path connecting the two signature-index classes. These precise results are then used in last Chapter of the thesis, Chapter 6, in order to prove the foliated homotopy invariance of the foliated rho-invariant. It is a pity that one step in the argument is not included in the thesis because too tedious and too technical. I have not any doubt that this step will work out but I would have preferred to see it in the thesis.

Conclusions. The thesis of Indrava Roy gives fundamental contributions to a difficult but very active field of research. On the positive side the thesis is very well written, with rigorous and clear arguments. Moreover, it tackles a very difficult problem by developing new techniques, techniques that might very well be useful in future investigations and that have their own intrinsic interest. On the negative side, one key step in the proof of the foliated homotopy invariance is missing. All things considered I believe that this thesis is largely sufficient for granting to Indrava Roy the degree of Doctor in Philosophy. I should like to add, as a final comment, that the results explained in the first five Chapters of the thesis, especially those appearing in Chapter 5, are in my opinion already sufficient to grant the degree of Doctor in Philosophy.

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Referee report on the PhD thesis of Indrava Roy, Foliated rho-invariants.

This thesis contains results which are analogs in foliations with transverse invariant measures, of $L2$-index theorems and $L2$-eta and rho invariants for covering spaces. For instance, $L2$-index theorem of Atiyah and Singer for covering spaces relates a type 1 index theorem on a compact manifold, to a type II index theorem on a covering space, resulting in an integrality result for the type II index in this situation. Here Roy establishes an analog for foliations, of the index for the Dirac operator acting on the leaves, and the index of lift of the Dirac operator to the monodromy groupoid. Both of these are type II indices, and it doesn't appear possible to extract an integrality statement in this generality. Also strictly speaking, Atiyah and Singer established the $L2$ index theorem for pseudodifferential operators and their lifts, whereas here the theorem is established for Dirac type operators.Also, use is made of the maximal version of the Baum-Connes map to precisely state the result.

Next, Roy establishes various properties of the foliated version of the rho invariants such as the diffeomorphism invariance, after giving a fairly complete overview of the literature. Roy then studies Hilbert-Poincare complexes for foliations, following the abstract construction of Higson and Roe, which in turn greatly generalized constructions in the literature due to Mischenkho, Novikov, Kaminker-Miller etc.
He uses this theory to study the homotopy invariance of the index of the leafwise Signature operator. Although this is known in the literature (Hilsum-Skandalis), Roy appears to have a refinement given by an explicit path connecting the representative Signature operators. Finally, Roy studies the homotopy invariance properties of the foliated rho-invariants.
The proof follows that of Keswani (recently also elaborated by Higson and Roe) and is also related to work of Piazza and Schick, done for the rho invariants and the $L2$ analogs.. This part isn't finalized, and is work in progress to be treated by the author elsewhere.

In summary, Indrava Roy's PhD thesis involves extremely sophisticated mathematics in a research area that is very active world-over, and it is clear that he has mastered these concepts and is able to use them effectively to prove extremely interesting results.
Based on my comments, I recommend awarding Indrava Roy a Ph.D.

Minor corrections, (suggested references to be included)

(Singer proved an equivalent theorem at the same time, see:-
Singer, I. M. Some remarks on operator theory and index theory. K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 128-138. Lecture Notes in Math., Vol. 575, Springer, Berlin, 1977)

Chang proves homotopy invariance of $L2$-rho invariants which is topological and analogous to a proof by Weinberger whio studies the ordinary rho-invariant.
(Chang, Stanley, On conjectures of Mathai and Borel. Geom. Dedicata 106 (2004), 161-167.)

Review article by Higson and Roe on the rho invariants