## Abstract

# On the Hardness of Computing Local Optima

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In this thesis, we investigate the *complexity of computing locally optimal solutions* of problems arising in the fields of *game theory* and *optimization*. For our investigation, we use the framework of $\mathcal{PLS}$ (short for "Polynomial-time Local Search"), as introduced by Johnson, Papadimtriou, and Yannakakis [56].

Before presenting our results, we first revisit the framework of $\mathcal{PLS}$ and present the necessary notation in Chapter 2. In Chapter 3, we survey the research on the complexity of computing locally optimal solutions. There, we concentrate on well-known and successful local search heuristics for various problems, where our focus is on worst case complexity along with the existence of *sequences of improving steps of exponential length.* We mostly concentrate on surveying research on congestion games, which sparked the interconnection between local search and game theory.

In game theory, congestion games are a widely accepted model to investigate the behavior and performance of large-scale distributed networks with autonomous participants. The class of *restricted network congestion games* is a subclass of congestion games where for each player there exists a set of edges which he is not allowed to use. Rosenthal's potential function guarantees the existence of a Nash equilibrium, as local minima of the potential function coincide with Nash equilibria; moreover, Rosenthal's potential function is polynomial-time computable. This allows to formulate the problem of computing a Nash equilibrium in a given restricted network congestion game as a local search problem. In Chapter 5, we show that computing a Nash equilibrium in a restricted network congestion game with *two players* is $\mathcal{PLS}$*-complete*, using a tight reduction from MaxCut. The result holds for directed networks and for undirected networks.

From the field of optimization, we investigate the complexity of computing locally optimal solutions of the Maximum Constraint Assignment (in short MCA) problem and of weighted standard set problems. In a nutshell, the MCA problem which we study in Chapter 6 is a local search version of weighted Generalized Maximum Satisfiability on constraints (functions mapping assignments to positive integers) over variables with higher valence. The parameters in $(p, q, r)$-MCA$_{k\text{-par}}$ simultaneously limit the maximum length $p$ of each constraint, the maximum appearance $q$ of each variable and its valence $r$;

additionally, the set of constraints is $k$-partite. We focus on hardness results and *show $\mathcal{PLS}$-completeness of* $(3,2,3)$-MCA$_{\text{3-par}}$ and $(2,3,6)$-MCA$_{\text{2-par}}$, using tight reductions from Circuit/Flip. Our results are *optimal* in the sense that $(2,2,r)$-MCA is solvable in polynomial time for every $r \in \mathbb{N}$. We also pay special attention to the case of *binary variables* and show that $(6,2,2)$-MCA is tight $\mathcal{PLS}$-complete. For our results, we extend and refine a technique from Krentel [67].

Finally, in Chapter 7 we study the complexity of computing locally optimal solutions of *weighted standard set problems* such as SetPacking, SetCover, and many more, as pooled in problems [SP1]–[SP10] in the book of Garey and Johnson [40, page 221ff.]. We show that for most of these problems, computing a locally optimal solution is *already $\mathcal{PLS}$-complete* for a *simple natural neighborhood of size one*. For the local search versions of weighted SetPacking and SetCover, we derive tight bounds for a simple neighborhood of size two. To the best of our knowledge, these are one of the very few $\mathcal{PLS}$ results on local search for weighted standard set problems.

The investigations in this thesis are mainly led by showing hardness results for the local search problems outlined above. Ideally, we would like to *demarcate the tractability* of computing locally optimal solutions for these problems. Moreover, we are interested in *commonalities* between the reductions we present, as well as *potential sources of intractability* in the problems we show hardness for. We discuss these superior questions in Chapter 8 and point out potential directions for further research, by stating various open problems.