On some geometric and group-theoretic properties of Newton stratifications and Ekedahl-Oort stratifications

Dissertation

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Zusammenfassung

Die vorliegende Arbeit leistet einen Beitrag zu verschiedenen Gebieten der arithmetischen algebraischen Geometrie. Motivation für viele der behandelten Fragestellungen ist der Vergleich der Newton-Stratifizierung und der Ekedahl-Oort-Stratifizierung auf der speziellen Faser von glatten Modellen für Shimuravarietäten vom Hodge-Typ $\mathrm{Sh}_K(\mathbf{G}, X)$ an Primstellen p von guter Reduktion. Hier bezeichnet $\mathbf{G}$ eine zusammenhängende reduktive Gruppe über $\mathbb{Q}$ und $K \subseteq \mathbf{G}(\mathbb{A}^f)$ eine geeignete Untergruppe der adelwertigen Punkte von $\mathbf{G}$. Das Studium von glatten Modellen $\mathscr{S}_K(\mathbf{G}, X)$ solcher Shimuravarietäten spielt eine wichtige Rolle in der Behandlung von bekannten Fällen der lokalen und globalen Langlands-Korrespondenz (siehe etwa Harris-Taylor [HT] und Scholze [Sch]).

Per Konstruktion parametrisieren diese Modelle $\mathscr{S}_K(\mathbf{G}, X)$ eine Familie $\mathcal{A}$ von abelschen Varietäten mit gewissen Zusatzstrukturen, im Fall von PEL-Shimuravarietäten erlauben sie sogar eine explizite Realisierung als Modulräume solcher abelscher Varietäten. Dies führt auf die Definition der Newton-Stratifizierung und Ekedahl-Oort-Stratifizierung auf ihrer speziellen Faser durch Betrachtung der p-divisiblen Gruppen $\mathcal{A}_{\hat{x}}[p^\infty]$ in geometrischen Punkten $\hat{x}$, in Analogie zu den klassischen Stratifizierungen von Modulräumen abelscher Varietäten in positiver Charakteristik (ohne Zusatzstrukturen, siehe Oort [Oo1], [Oo2]). Die moderne Formulierung der Stratifizierungen nutzt die gruppentheoretischen Konzepte der σ-Konjugationsklassen und G-Zips. Dies macht es möglich, viele Fragestellungen, die sich aus dem Vergleich von Newton- und Ekedahl-Oort-Stratifizierung ergeben, in gruppentheoretische Fragestellungen zu übersetzen: Das Grundobjekt ist hier ein reduktives Gruppenschema G über $\mathbb{Z}_p$ mit generischer Faser $\mathbf{G}_{\mathbb{Q}_p}$, die Stratifizierungen korrespondieren zu Zerlegungen von $G(L)$ in Äquivalenzklassen unter σ-Konjugationen, wobei L die Vervollständigung der maximal unverzweigten Erweiterung von $\mathbb{Q}_p$ und σ den Frobeniusautomorphismus von L über $\mathbb{Q}_p$ bezeichnet. Diese Technik zum Studium der speziellen Faser eines glatten Modells $\mathscr{S}_K(\mathbf{G}, X)$ ist wohlbekannt im Fall von PEL-Shimuravarietäten, wir erläutern sie für Shimuravarietäten vom Hodge-Typ in Kapitel II der Arbeit, basierend auf jüngeren Resultaten von Kisin [Ki1] und Zhang [Zh1].

Zum Studium der gruppentheoretischen Fragestellungen, die sich durch dieses Prinzip ergeben, ist es nicht notwendig - und in der Tat hinderlich - sich auf solche Gruppenschemata G zu beschränken, die sich im Kontext von Shimuravarietäten ergeben; sämtliche Konzepte sind sinnvoll in der allgemeineren Situation eines reduktiven Gruppenschemas über dem Bewertungsring eines absolut unverzweigten lokalen Körpers, wobei auch Körper von positiver Charakteristik erlaubt sind. Dies ist der Blickwinkel, den wir in Kapitel I einnehmen werden. Tatsächlich lassen sich einige Fragen, etwa ob zwei vorgegebene Strata einen nichtleeren Schnitt besitzen, unabhängig von der Charakteristik in rein kombinatorische Aussagen übersetzen. Im Falle positiver Charakteristik gibt es hier interessante Bezüge zur Theorie der Schleifengruppen und affinen Deligne-Lusztig-Varietäten. Wir bestimmen in Kapitel I auf diese Weise das Schnittverhalten der Stratifizierungen in einigen konkreten Fällen, und entwickeln auch ein neues hinreichendes kombinatorisches Kriterium für die Existenz sogenannter "fundamentaler" Ekedahl-Oort-Strata, welche unter anderem die Eigenschaft haben vollständig in einem Newton-Stratum enthalten zu sein.

Als eine Hauptanwendung der entwickelten Methoden und gruppentheoretischen Resultate beschreiben wir den "μ-ordinären Ort" in der speziellen Faser eines glatten Modells $\mathscr{S}_K(\mathbf{G}, X)$ für eine Shimuravarietät vom Hodge-Typ: Auf einer offenen und dichten Teilmenge der speziellen Faser ist der Isomorphietyp der zu geometrischen Punkten assoziierten p-divisiblen Gruppen $\mathcal{A}_{\hat{x}}[p^\infty]$ mit Zusatzstruktur konstant, weiter ist er eindeutig bestimmt durch den zugehörigen Isogenietyp und auch durch den Isomorphietyp der p-Torsionsgruppen $\mathcal{A}_{\hat{x}}[p]$. Die verallgemeinert bekannte Resultate von Wedhorn [We1] und Moonen [Mo2] über PEL-Shimuravarietäten.

Conclusion

The aims of the work presented here are twofold: On the one hand, we are interested in the relation of the Newton stratification and Ekedahl-Oort stratification for special fibers of smooth models of certain types of Shimura varieties. These Shimura varieties play an important role in the modern approaches towards the global and local Langlands conjecture (see for example Harris-Taylor [HT] and Scholze [Sch]). Since the very definitions of the stratifications involve reductive groups over p-adic fields, many aspects in their comparison translate into statements about such groups. On the other hand, these group-theoretic questions are interesting in their own right, and can also be studied in situations which do not arise from Shimura varieties, as well as for groups over local fields in positive characteristic. They turn out to be closely related to the study of affine Deligne-Lusztig varieties and the structure of affine Weyl groups (see [GHKR], [He2]). Results obtained in the group-theoretic context may then also be translated back to Shimura varieties.

Let us roughly describe the relationship between the geometric stratifications and the group-theoretic objects, for more details we refer to the introductions of Chapter I and Chapter II. Classically, the Newton stratification and Ekedahl-Oort stratification are defined for a moduli space $\mathscr{M}$ of polarized abelian varieties in positive characteristic p via the classification of p-divisible groups up to isogeny, and of BT1-groups up to isomorphism (see Oort, [Oo1], [Oo2]). Such moduli spaces of abelian varieties arise as special fibers of smooth models of Shimura varieties associated to the general symplectic group $\mathbf{GSp}_{\mathbb{Q}}$.

More generally, consider a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ of PEL-type or of Hodge type. Here $\mathbf{G}$ is a connected reductive group over $\mathbb{Q}$, and $K \subseteq \mathbf{G}(\mathbb{A}_f)$ is an open and compact subroup of the form $K = K_pK^p$ for $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ and $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$. If K_p is hyperspecial and $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ is sufficiently small, then it is known that the Shimura variety has a smooth model $\mathscr{S}_K(\mathbf{G}, X)$ defined over the valuation ring of a finite unramified extension of $\mathbb{Z}_p$, which naturally parametrizes a family $\mathcal{A}$ of abelian varieties with certain additional structures (see Kottwitz [Ko2] and Kisin [Ki1]) - we will make this more precise in Chapter II. This leads to analogous definitions of stratifications for the special fiber of this model: Each $x \in \mathscr{S}_K(\mathbf{G}, X)(\overline{\mathbb{F}}_p)$ defines an abelian variety $\mathcal{A}_x$ with additional structure over $\overline{\mathbb{F}}_p$. The classifications of the associated p-divisible groups $\mathcal{A}_x[p^\infty]$ with extra structure up to isogeny, respectively of their p-torsion groups $\mathcal{A}_x[p]$ up to isomorphism, then yield the Newton stratification and Ekedahl-Oort stratification on $\mathscr{S}_K(\mathbf{G}, X) \otimes \overline{\mathbb{F}}_p$ (see Chapter II).

As K_p is hyperspecial, there exists a reductive model G over $\mathbb{Z}_p$ for $\mathbf{G}_{\mathbb{Q}_p}$ such that $K_p = G(\mathbb{Z}_p)$. Let L be the completion of the maximal unramified extension of $\mathbb{Q}_p$, let $\mathcal{O}$ be its ring of integers. Let σ be the Frobenius automorphism of L over $\mathbb{Q}_p$.

Then the subgroup $G(\mathcal{O})$, and also $G(L)$ itself, act on $G(L)$ by σ-twisted conjugation. Denote by $[g]$ the orbit of $g \in G(L)$ under its σ-twisted conjugation by $G(L)$, these sets form a decomposition of $G(L)$ into equivalence classes. There is another decomposition into equivalence classes $[[g]]$, which are certain unions of σ-twisted $G(\mathcal{O})$-conjugacy classes. These classes were introduced by Viehmann in [Vi1], see Chapter I for the exact definition. Now the Shimura datum $(\mathbf{G}, X)$ determines a cocharacter μ of G and hence a double coset $G(\mathcal{O})\mu(p)G(\mathcal{O}) \subseteq G(L)$, and there is a map (of sets)

$$\gamma\colon \mathscr{S}_K(\mathbf{G}, X)(\overline{\mathbb{F}}_p) \longrightarrow G(\mathcal{O})\mu(p)G(\mathcal{O})/_\sim \tag{$*$}$$

into the set of σ-twisted $G(\mathcal{O})$-conjugacy classes in $G(\mathcal{O})\mu(p)G(\mathcal{O})$ such that $\gamma(x) = \gamma(x')$ if and only if $\mathcal{A}_x[p^\infty]$ and $\mathcal{A}_{x'}[p^\infty]$ are isomorphic, and further $\gamma(x)$ and $\gamma(x')$

1. lie in the same $[\cdot]$-class if and only if $\mathcal{A}_x[p^\infty]$ and $\mathcal{A}_{x'}[p^\infty]$ are isogeneous,
2. lie in the same $[[\cdot]]$-class if and only if $\mathcal{A}_x[p]$ and $\mathcal{A}_{x'}[p]$ are isomorphic,

all with respect to the additional structures. The existence of a map with these properties is well-known for Shimura varieties of PEL-type, and does not come as a surprise in the more general case of Hodge-type Shimura varieties which we will explain in the second chapter (see also the introduction to Chapter II). In the PEL-case it is known that the fibers of γ, which are then called "leaves", are also locally closed in $\mathscr{S}_K(\mathbf{G}, X) \otimes \overline{\mathbb{F}}_p$ (see Mantovan [Ma], Prop. 1). We expect this is to be true for Shimura varieties of Hodge type as well.

The existence of γ allows to study aspects of the stratifications of $\mathscr{S}_K(\mathbf{G}, X)\otimes\overline{\mathbb{F}}_p$ on the right hand side of $(*)$, i.e. by studying the decompositions of $G(\mathcal{O})\mu(p)G(\mathcal{O})$ into $[\cdot]$-classes and $[[\cdot]]$-classes. As the definition of these classes is a purely group-theoretical one, they can - and should - be studied in the more general context of reductive groups over the rings of integers in a totally unramified local field, including fields of positive characteristics. We will take on this point of view in Chapter I, but often with a view towards applications for Shimura varieties. It turns out that many problems regarding the stratifications can be translated into purely combinatorial questions, which are independent of the characteristic. For example, in the group-theoretical context the question of nonemptyness of intersections between strata is closely related to the question of nonemptyness of affine Deligne-Lusztig varieties (see [GHKR] and [GHN]), and the results obtained on this question may also be applied to cases which arise from Shimura varieties. Another interesting question, which asks for the existence of "fundamental" Ekedahl-Oort strata (see Chapter I), may be treated via a combinatorial study of the extended affine Weyl group of G, we give a new criterion for the existence of such fundamental strata in combinatorial terms. All results of the first chapter can also be applied in positive characteristic, for instance to the theory of loop groups.

The results obtained on the group-theoretic side of $(*)$ translate back to statements on Shimura varieties via γ. Our main application of this principle is the description of the μ-ordinary locus for Shimura varieties of Hodge type: It is known for moduli spaces of abelian varieties that there is a unique Newton stratum - the ordinary locus - which is open and dense. The corresponding object for $\mathscr{S}_K(\mathbf{G}, X)\otimes\overline{\mathbb{F}}_p$, the *$\mu$-ordinary locus*, has long been known to be open and dense in the PEL-type (see Wedhorn, [We1]). In the Hodge-type case, as of now the Ekedahl-Oort stratification is better understood than the Newton stratification, for instance we know that there is an open and dense Ekedahl-Oort statum by a result of Zhang [Zh1]. We show in Chapter I that this stratum coincides with the μ-ordinary locus in the group-theoretic context, taking inverse images under γ we thus prove the density of the μ-ordinary locus for a general Shimura variety of Hodge type (in fact we prove more, see Chapter II for the precise statement).

We mention another application: A special case of PEL-type Shimura varieties are "Hilbert-Blumenthal varieties", in this case $\mathscr{S}_K(\mathbf{G}, X)$ is a moduli space over $\mathbb{Z}_{(p)}$ of abelian varieties equipped with an action of the ring of integers of a finite, totally real extension $\mathbb{E}|\mathbb{Q}$ in which p is inert. For these varieties we calculate the precise intersection pattern between Newton strata and Ekedahl-Oort strata on the group-theoretic side. Since the map γ is known to be *surjective* by a result of Viemann and Wedhorn in the case of PEL-type Shimura varieties (see [VW]), this result also gives the intersection properties of the stratifications of $\mathscr{S}_K(\mathbf{G}, X) \otimes \overline{\mathbb{F}}_p$.

Though the study of the Shimura varieties $\mathscr{S}_K(\mathbf{G}, X)$ via $(*)$ has its obvious limitations, for example concerning the schematic properties of the strata and their intersections, we expect that many interesting results will be obtained in the future by methods similar to the ones presented here.

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Contents

Chapter I

Group-theoretic Newton strata and Ekedahl-Oort strata

I.1 Introduction

I.1.1 Stratifications of double cosets

In this first chapter we focus our attention on the group-theoretic manifestations of Newton stratifications and Ekedahl-Oort stratifications. Our main object will be a reductive group G over the valuation ring $\mathcal{O}_F$ of some absolutely unramified local field F. We do not suppose that G arises from a Shimura datum, and we allow valuation rings of equal characteristics, which will provide much flexibility in the arguments.

Throughout the chapter we will fix a maximal Torus T of G and a Borel subgroup B containing T, which are both defined over $\mathcal{O}_F$. We can do so since it is known that G is quasisplit over $\mathcal{O}_F$. (We warn the reader that many definitions, as for example those of standard representatives and standard parabolic subgroups, will depend on the choice of the pair (T, B). However, the eventual results are independent of this choice.) Let (W, S) be the Weyl group of T, endowed with the simple reflections given by B.

Let $\mathcal{O}$ be an absolutely unramified complete discrete valuation ring over $\mathcal{O}_F$ whose residue field k is algebraically closed. Let L be its field of fractions and let σ be the Frobenius isomorphism of $\mathcal{O}$ over $\mathcal{O}_F$ and of L over F. Write $K := G(\mathcal{O}) \subseteq G(L)$, and let $K_1 \subseteq K$ be the subgroup of elements which reduce trivially to $G(k)$. Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$, i.e. of subsets of the form $[g] := \{hg\sigma(h)^{-1} \mid h \in G(L)\}$, and denote by $\mathrm{EO}(G)$ the set of "Ekedahl-Oort classes" of the form $[[g]] := \{hg'\sigma(h)^{-1} \mid g' \in K_1 g K_1,\ h \in K\}$. These classes form decompositions of $G(L)$, we will usually denote σ-conjugacy classes by b and Ekedahl-Oort classes by ξ. The sets $B(G)$ and $\mathrm{EO}(G)$ are endowed with partial order relations $\preceq$ respectively (see Sections I.2.5, I.2.6), in situations which arise from Shimura varieties these partial orders will describe the behaviour of strata under closures. We have a third decomposition of $G(L)$ into K-double cosets (the Cartan decomposition, see Section I.2.4), indexed by the cocharacters μ of T which are dominant with respect to our Borel subgroup B: Let ϵ be a uniformizing element of $\mathcal{O}$, then the elements $\epsilon^\mu := \mu(\epsilon)$ for dominant μ form a set of representatives for the K-double cosets in $G(L)$. By definition, each Ekedahl-Oort class is contained in some K-double coset.

For a dominant cocharacter μ let $B(G, \mu) \subseteq B(G)$ be the subset of classes which meet the double coset $K\epsilon^\mu K$. This set can be described concretely by the classifying maps of σ-conjugacy classes (see Section I.3.1.1). It is known that $B(G, \mu)$ is finite and that it contains a unique minimal and a unique maximal element with respect to the $\preceq$-order on $B(G)$. The "Newton stratification" of $K\epsilon^\mu K$ is then given by its intersections with $b \in B(G, \mu)$. Secondly, let $\mathrm{EO}(G, \mu)$ be the set of Ekedahl-Oort classes which are contained in $K\epsilon^\mu K$. These classes then form the "Ekedahl-Oort stratification" of $K\epsilon^\mu K$. There is an explicit description of $\mathrm{EO}(G, \mu)$ in terms of shortest coset representatives in W, here it is known as well that this is a finite set and that it has a unique maximal and a unique minimal element with respect to the $\preceq$-order on $\mathrm{EO}(G)$ (again, see Section I.3.1.1).

In the situation described in the conclusion, the stratifications of $\mathscr{S}_K(\mathbf{G}, X) \otimes \overline{\mathbb{F}}_p$ are exactly the inverse images of these group-theoretic strata under the map γ from $(*)$, and the partial orders are known or conjectured to describe their closure relations (see Chapter II). On the other hand, if L is of positive characteristic then $K\epsilon^\mu K$ carries the topological structure of an ind-scheme. In this case the group-theoretic strata are locally closed subsets of $K\epsilon^\mu K$ and again the partial orders describe their closures, see for example [Vi1], [Vi2].

I.1.2 Main results

The question of how the two stratifications of $K\epsilon^\mu K$ for some dominant μ compare to each other leads to the problem of describing the intersections $\xi \cap b \cap K\epsilon^\mu K$, or equivalently $\xi \cap b$, for $\xi \in \mathrm{EO}(G, \mu)$ and $b \in B(G, \mu)$.

A major tool in the study of these intersections is the extended affine Weyl group $\widetilde{W} := N_G(T)(L)/T(\mathcal{O}) = X_*(T) \rtimes W$ of G. The group $\widetilde{W}$ provides representatives for $B(G)$, more precisely, every $b \in B(G)$ contains *b-short elements* $\omega_b \in \widetilde{W}$, which have (in addition to lying in b) many favourable properties (see Section I.2.5 for the precise definition of b-short elements). In the case that G is split over $\mathcal{O}_F$ these properties determine a b-short element uniquely. Also, every $\xi \in \mathrm{EO}(G)$ has representatives in $\widetilde{W}$, and in particular a *standard representative* $\tilde{w}_\xi \in \widetilde{W}$ (see Definition I.2.18). The choice of B determines an Iwahori subgroup I of $G(L)$, and it is known that $b \in B(G)$ intersects ξ nontrivially if and only if it intersects $I\tilde{w}_\xi I$ nontrivially (see Section I.3.1.2). Since the σ-conjugacy classes that meet a double coset $I\tilde{w}I$ ($\tilde{w} \in \widetilde{W}$) are determined by the structure of $\widetilde{W}$, this property is in fact independent of the characteristic of L (see Section I.3.1.3). This relates the nonemptyness of intersections $\xi \cap b$ to the nonemptyness of affine Deligne-Lusztig varieties in the affine flag manifold, which has been extensively studied. We report on some known results in Sections I.3.1.3 and I.3.5.1.

It is natural to attempt to study the intersections $\xi \cap b$ by using the representatives in $\widetilde{W}$. For $\xi \in \mathrm{EO}(G, \mu)$ let $\Lambda(\xi)$ be the set of representatives for ξ in $\widetilde{W}$. It turns out that all elements of $\Lambda(\xi)$ lie in the same σ-conjugacy class, so we have a map $\mathrm{Rep}\colon \mathrm{EO}(G, \mu) \to B(G, \mu)$ which sends ξ to this common σ-conjugacy class. In general this map contains little information on the relation of the stratifications, in particular if μ is not minuscule. If μ is minuscule, then the results of Viehmann in [Vi1] and the structure of $\Lambda(\xi)$ imply that Rep: $\mathrm{EO}(G,\mu) \to B(G,\mu)$ is *surjective*, and that there are the following criteria for nontrivial intersections:

Criterion I.A. *Let μ be dominant and minuscule, let $\xi \in \mathrm{EO}(G,\mu)$.*

(i) (necessary criterion) If $b \in B(G,\mu)$ such that $\xi \cap b \neq \emptyset$, then there is some $\xi' \in \mathrm{EO}(G,\mu)$ with $\xi' \preceq \xi$ and $\mathrm{Rep}(\xi') = b$.

(ii) (sufficient criterion) $\{\mathrm{Rep}(\xi') \mid \xi' \preceq \xi\}$ contains a unique maximal element b' with respect to the $\preceq$-order on $B(G,\mu)$, for this element we have $\xi \cap b' \neq \emptyset$.

The criteria are very similar to the ones in ([Vi1], Thm. 1.5., Cor. 1.6.), and they rely heavily on ([Vi1], Prop. 5.5.). We remark that (i) is far from being sufficient, however in simple cases, for example if $G = \mathrm{GU}(2,3)$, it turns out to be helpful (see Section I.3.6.2). Apart from giving these two criteria the map Rep remains rather mysterious (see Remark I.3.16 and Example I.4.13).

It is an interesting aspect in the comparison of the two stratifications of $K\epsilon^\mu K$ to decide whether every Newton stratum contains an Ekedahl-Oort stratum, that is, whether every $b \in B(G,\mu)$ contains some $\xi \in \mathrm{EO}(G,\mu)$. In this case, any two elements of ξ are σ-conjugate by some element of $G(L)$. Of special interest is the case that this is even true for the subgroup K, in other words, that ξ is a single K-σ-conjugacy class. We will call a $\xi \in \mathrm{EO}(G,\mu)$ with this property a *K-fundamental* class. In the situation of $(*)$ in the conclusion, being K-fundamental corresponds to the property that for some $x \in \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p)$ the isomorphism class of $A_x[p^\infty]$ (with additional structures) is determined by the isomorphism class of its p-torsion. In this sense the question was first raised, and answered positively (but without additional structure), by Oort in [Oo4].

A very promising strategy to find a K-fundamental class involves the extended affine Weylgroup: $\tilde{w} \in \widetilde{W}$ will be called *I-fundamental* if every element of $I\tilde{w}I \subseteq G(L)$ is σ-conjugate to $\tilde{w}$ by an element of I. To show that $\xi \in \mathrm{EO}(G,\mu)$ is K-fundamental it suffices to show that its set of $\widetilde{W}$-representatives $\Lambda(\xi)$ contains an I-fundamental element. There are (at least) three types of elements in $\widetilde{W}$ found in the literature which are known to be I-fundamental (see Section I.3.3 for the precise definitions): Straight elements in the sense of [He1], P-fundamental elements in the sense of [Vi1] and [VW], and (J,y,σ)-fundamental elements (see [GHN]). Among these the property of straight element is particularly easy to verify.

Let $b \in B(G,\mu)$. It is known ([He1], 3.5.) that there exists a straight element $\tilde{w} \in \widetilde{W}$ which lies in b, consequently the Ekedahl-Oort class $[[\tilde{w}]]$ is K-fundamental and contained in b. This leaves the question whether there exists a straight $\tilde{w}$ such that $[[\tilde{w}]] \in \mathrm{EO}(G,\mu)$, i.e. such that $\tilde{w} \in W\epsilon^\mu W$. In this direction our main result is the following:

Theorem I.B. *Let μ be dominant, let $b \in B(G,\mu)$. Then $W\epsilon^\mu W \cap b$ contains a straight element if and only if $W\epsilon^\mu W$ contains a b-short element which is σ-balanced. In this case there exists a $\xi \in \mathrm{EO}(G,\mu)$ such that $\xi \subseteq b$ is K-fundamental and $\tilde{w}_\xi$ is straight.*

Here the concept of a *σ-balanced* b-short element appears, which puts another condition on b-short elements. The precise definition is a bit technical, we refer the reader to Section I.3.4. Roughly speaking, ω_b is σ-balanced if the translation parts of its σ-twisted products do not become "too antidominant". The theorem follows from combinatorial calculations involving the Weyl group and the root system of G. Along the way we show that the three concepts mentioned above are in fact equivalent:

Proposition I.C. *An element $\tilde{w} \in \widetilde{W}$ is straight if and only if it is P-fundamental, if and only if it is (J, y, σ)-fundamental.*

The existence of σ-balanced short elements in $W\epsilon^\mu W$ remains an open question. There is some hope to find them for each $b \in B(G,\mu)$ in the case that μ is a minuscule cocharacter (which includes all cases that arise from Shimura varieties). In this case it is known that $W\epsilon^\mu W$ always contains *some* b-short element ω_b. Since this element is unique if G splits over $\mathcal{O}_F$ and there always exists a straight element in b, Theorem I.B gives another proof for the existence of K-fundamental Ekedahl-Oort classes for split groups and minuscule cocharacters (see [VW], Prop. 9.9.). In general, in many examples the condition of being σ-balanced translates into a combinatorial problem.

The unique maximal element of $B(G,\mu)$ plays a distinguished role in the context of Shimura varieties. In accordance with the second chapter we denote it by $b_{\mu-\mathrm{ord}}$ and call it the μ-ordinary element of $B(G,\mu)$. The corresponding stratum $K\epsilon^\mu K \cap b_{\mu-\mathrm{ord}}$ is then the group-theoretic equivalent of the μ-ordinary locus in a Shimura variety. On the other hand, let $\xi_{\max}$ be the unique maximal element of $\mathrm{EO}(G,\mu)$. For pairs (G,μ) which correspond to a PEL-type Shimura variety results of Moonen (formulated for p-divisible groups with additional structures, see [Mo2], 1.3.7., 3.2.7.) show that the μ-ordinary locus coincides with $\xi_{\max}$, and that this is a K-fundamental class in the sense explained above. We show that this holds in full generality:

Theorem I.D. *Let μ be dominant (not necessarily minuscule), let $b_{\mu-\mathrm{ord}} \in B(G,\mu)$ and $\xi_{\max}$ be the unique maximal elements with respect to $\preceq$ respectively. Then $K\epsilon^\mu K \cap b_{\mu-\mathrm{ord}} = \xi_{\max}$, and $\xi_{\max}$ is a K-fundamental Ekedahl-Oort class.*

This theorem will be a major ingredience in the proof of the density of the μ-ordinary locus for Shimura varieties of Hodge type in Chapter II.

Finally we discuss a special example which arises from a well-studied class of Shimura varieties of PEL-type, the so called "Hilbert-Blumenthal" varieties (see for example [GO], [AC]). Here G is given on R-valued points as

$$G(R) = \{g \in \mathrm{GL}_2(\mathcal{O}_{F'} \otimes_{\mathcal{O}_F} R) \mid \det(g) \in R^\times\}, \tag{HB}$$

where $F'|F$ is a finite unramified extension and $\mathcal{O}_{F'}$ denotes the valuation ring of F', and μ is a minuscule cocharacter. This case has been studied previously by Goren and Oort in ([GO]) from the point of view of Shimura varieties, here the authors already determined the generic Newton stratum in each Ekedahl-Oort stratum. Our discussion of this example will be based on reduction techniques for scalar restrictions of reductive groups and normal subgroups with abelian quotients, which seem interesting in their own right.

To explain the results for the Hilbert-Blumenthal example, we briefly describe the sets $B(G,\mu)$ and $\mathrm{EO}(G,\mu)$ in this case. See Section I.4.2.2 for details. Let $d := [F' : F]$, then

- $B(G,\mu)$ contains $\left[\frac{d}{2}\right] + 1$ elements, it is totally ordered with respect to $\preceq$. We write $M := \{\frac{d}{2}, \left[\frac{d+1}{2}\right], \left[\frac{d+1}{2}\right] + 1, \ldots, d\}$, ordered in the obvious way, and identify $B(G,\mu) = \{b_m \mid m \in M\}$ (here m corresponds to the first slope of the Newton polygon of b_m).

- $\mathrm{EO}(G,\mu)$ is as a partially ordered set isomorphic to the power set of $\mathbb{Z}/d\mathbb{Z}$, ordered by inclusion.

In analogy to the definitions in [GO], to each $\xi \in \mathrm{EO}(G,\mu)$ we associate a number $m(\xi) \in M$ (see Definition I.4.7), which determines the "generic" σ-conjugacy class in ξ. We now have the following description of the intersection behaviour between the stratifications:

Theorem I.E. *Let G be given as in (HB), let μ be dominant and minuscule.*

(i) Let $\xi \in \mathrm{EO}(G,\mu)$, let $\tilde{w}_\xi \in \widetilde{W}$ be its standard representative. Then ξ is of one of the following types:

(I) $\tilde{w}_\xi$ is a straight element and $\xi \subseteq b_{m(\xi)}$ is K-fundamental,

(II) $\tilde{w}_\xi$ is not straight, and $\xi \cap b \neq \emptyset$ for all $b \preceq b_{m(\xi)}$.

(ii) If $\xi \in \mathrm{EO}(G,\mu)$ and $b_m \in B(G,\mu)$ with $\xi \cap b_m \neq \emptyset$, then there is a $\xi' \preceq \xi$ of type (I) with $\xi' \subseteq b_m$. In particular, every $b \in B(G,\mu)$ contains a K-fundamental Ekedahl-Oort class.

(iii) If $\xi \in \mathrm{EO}(G,\mu)$ is of type (I), then for every $m' < m(\xi)$ there exists a $\xi' \preceq \xi$ of type (II) with $m(\xi') = m'$.

The theorem shows that in the Hilbert-Blumenthal case for every $b \in B(G,\mu)$ the following holds:

If ξ is a minimal element in $\{\xi' \in \mathrm{EO}(G,\mu) \mid \xi' \cap b \neq \emptyset\}$,
then ξ is a K-fundamental class and $\tilde{w}_\xi$ is straight.

This is in analogy to the case that G is split over $\mathcal{O}_F$, where this property holds as well (see [VW], Rem. 9.21.), and it is also true in the $\mathrm{GU}(2,3)$-example of Section I.3.6.2. It might be interesting to investigate whether a similar statement always holds true if μ is minuscule.

This chapter is structured as follows:

In Section I.2 we fix our notations and introduce the main objects of the theory, we explain some of their properties and interrelations, and present techniques which will be used later. We devote some special care to the extended affine Weyl group, since it will be of great importance and is not treated uniformly in the literature. In this section we present no proofs.

Section I.3 deals with different general aspects of the comparison between the stratifications. In I.3.1 we formulate the intersection problem and explain the structures of $B(G,\mu)$ and $\mathrm{EO}(G,\mu)$. We include some known results and methods for further reference. In I.3.2 we study the structure of the sets $\Lambda(\xi)$ and the map Rep, and show Criterion I.A. Most of the results presented here can also be found elsewhere, or are easy consequences of known results. In I.3.3 we discuss fundamental classes and the different approaches to I-fundamental elements, we introduce σ-balanced elements in I.3.4 and prove Theorem I.B and Proposition I.C. In I.3.5 we discuss the behaviour of the extremal elements in $B(G,\mu)$ and $\mathrm{EO}(G,\mu)$. We include the basic case for completeness, the main result here is Theorem I.D which we prove in I.3.5.2.

In Section I.4 we discuss the situation that G is a normal subgroup of a scalar restricition of some group G_0 such that the quotient is abelian, and the special case of the Hilbert-Blumenthal example. We explain our reduction techniques in I.4.1, and prove Theorem I.E in I.4.2.

I.2 Preparations and main objects

I.2.1 Notations and conventions

We fix a prime number $p > 0$. For any perfect field κ of characteristic p we denote by $W(\kappa)$ the ring of Witt vectors over κ. Let k be an algebraically closed field of characteristic p. Let q be a positive power of p, let $\mathbb{F}_q$ be the finite field with q elements. Let F be either $\mathbb{Q}_q = \mathrm{Frac}(W(\mathbb{F}_q))$ or the field $\mathbb{F}_q((t))$ of Laurent series. We will refer to the former case as the case of mixed characteristics, and to the latter as the equicharacteristic case. Let L be either $\mathrm{Frac}(W(k))$ or $k((t))$ respectively. We denote by $\mathcal{O} \subseteq L$ and $\mathcal{O}_F \subseteq F$ the corresponding valuation rings. Let ϵ be a uniformizing element of $\mathcal{O}_F$ (and hence of $\mathcal{O}$), e.g. $\epsilon = p$ in the case of mixed characteristics or $\epsilon = t$ in the equicharacteristic case. We denote by σ the Frobenius isomorphism $x \mapsto x^q$ of k over $\mathbb{F}_q$, and also the corresponding unique lifts to isomorphisms on $\mathcal{O}$ and on L.

Let G be a reductive group scheme over $\mathcal{O}_F$ in the sense of ([SGA3], Exp. XIX, 2.7.), i.e. a smooth, affine group scheme over $\mathcal{O}_F$ whose geometric fibers are connected, reductive groups. G is then quasisplit over $\mathcal{O}_F$ (see e.g. [VW], A4), and splits over a finite étale extension of $\mathcal{O}_F$ ([SGA3], Exp. XXII, 2.3.), in particular, G is split over $\mathcal{O}$. We fix a Borel pair (T, B) of G which is defined over $\mathcal{O}_F$. Let $X_*(T), X^*(T)$ be the groups of cocharacters and characters of T (over $\mathcal{O}$). Let $\mathfrak{g}$ be the Lie algebra of G, let $\Phi \subseteq X^*(T)$ be set of roots of G with respect to T, and let Δ, Φ^+, Φ^- be the root basis and the sets of positive and negative roots determined by B. We call a cocharacter $\mu \in X_*(T)$ dominant (with respect to B), if $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Phi^+$, and denote by $X_*(T)_{\mathrm{dom}} \subseteq X_*(T)$ the set of dominant cocharacters. Let $W := N_G(T)(\mathcal{O})/T(\mathcal{O})$ the Weyl group of G and $\widetilde{W} := N_G(T)(L)/T(\mathcal{O})$ the extended affine Weyl group of G. The choice of B endows W with a set of simple reflections S, which is in $1-1$-correspondence with the root basis Δ.

The Frobenius σ induces actions on $G(\mathcal{O}), G(L), X_*(T), X^*(T)$ and Φ, which we will denote by σ as well, and the canonical pairing $\langle \cdot, \cdot \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ is σ-equivariant. Since B is defined over $\mathcal{O}_F$, we have $\sigma(\Phi^+) = \Phi^+$, $\sigma(\Delta) = \Delta$ and $\sigma(X_*(T)_{\mathrm{dom}}) = X_*(T)_{\mathrm{dom}}$, and σ operates on (W, S) as a Coxeter automorphism.

Notation I.2.1. We will use the following notations: For any algebraic or geometric object we denote by $(\cdot)_R$ the base change to some ring R, respectively to $\operatorname{Spec} R$. Let $Y \subseteq G(L)$ be any subset, then for $g \in G(L)$ we write ${}^gY := \{gyg^{-1} \mid y \in Y\}$ and for a subgroup $H \subseteq G(L)$ we write $H \cdot_\sigma Y := \{hy\sigma(h)^{-1} \mid y \in Y, h \in H\}$. We embed W and $\widetilde{W}$ in $G(L)$ as follows: $\widetilde{W}$ is a semidirect product $\widetilde{W} = X_*(T) \rtimes W$. For each $w \in W$ we choose a representative in $N_G(T)(\mathcal{O})$ and view $X_*(T)$ as a subgroup of $G(L)$ by sending $\lambda \in X_*(T)$ to $\epsilon^\lambda := \lambda(\epsilon)$. We will denote the longest element of W by w_0, if $W_J \subseteq W$ is the subgroup generated by some subset $J \subseteq S$ then $w_{0,J}$ will denote the longest element of W_J.

I.2.2 The quasi-Coxeter structure of $\widetilde{W}$

As explained in ([Ti], §1) and ([HR], p. 7/8), the extended affine Weyl group has a geometric representation on the apartment for T in the Bruhat-Tits building of G (over L):

Let $V := X_*(T)_{\mathbb{R}}$, this is a finite dimensional $\mathbb{R}$-vector space. $\widetilde{W} = X_*(T) \rtimes W$ acts faithfully on this vector space by affine linear transformations via $(\epsilon^\lambda w)(v) := w(v) + \lambda$, which are equivariant with respect to the actions of σ on $\widetilde{W}$ and V. If $\tilde{w} = \epsilon^\lambda w \in \widetilde{W}$, where $\lambda \in X_*(T)$ and $w \in W$, then we will refer to λ as the translation part of $\tilde{w}$.

Via the pairing $\langle \cdot, \cdot \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ the roots $\alpha \in \Phi$ define linear forms $\langle \alpha, \cdot \rangle$ on V, for each $\alpha \in \Phi$ and $k \in \mathbb{Z}$ the set $H_{\alpha,k} := \{v \in V \mid \langle \alpha, v \rangle = k\}$ is an affine hyperplane in V. The connected components in the complement of the union of the $H_{\alpha,k}$ form the alcoves in V. As the set of these hyperplanes is stable unter the $\widetilde{W}$-action, $\widetilde{W}$ acts on the set of alcoves in V. We denote by

$$C := \{v \in V \mid \langle \alpha, v \rangle > 0,\ \alpha \in \Phi^+\}$$

the dominant Weyl chamber in V. Let $\mathfrak{a}_0$ be the unique alcove contained in C whose closure $\overline{\mathfrak{a}_0}$ contains the origin 0. The reflections S_a along the walls of $\mathfrak{a}_0$ then generate the *affine Weyl group* $W_a \subseteq \widetilde{W}$ of G. This is a finitely generated Coxeter group, which acts simply transitively on the set of alcoves in V, and we have $(W, S) \subset (W_a, S_a)$. The group W_a can be identified with the extended affine Weylgroup of the simply connected cover of the derived group of G, and $\widetilde{W}$ is a semidirect product $\widetilde{W} = W_a \rtimes \Omega$, where

$$\Omega := \{\tilde{w} \in \widetilde{W} \mid \tilde{w}(\mathfrak{a}_0) = \mathfrak{a}_0\}$$

is subgroup in $\widetilde{W}$ of elements preserving $\mathfrak{a}_0$. ([HR], Lemma 14).

The length $l(\tilde{w})$ of an element $\tilde{w} \in \widetilde{W}$ is defined as the number of hyperplanes H_a seperating $\mathfrak{a}_0$ and $\tilde{w}(\mathfrak{a}_0)$. The subgroup Ω consists thus exactly of the elements of length 0 in $\widetilde{W}$. We may calculate $l(\tilde{w})$ by the following formula ([IM], Prop. 1.23):

$$l(\epsilon^\lambda w) = \sum_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^+} |\langle \alpha, \lambda \rangle| + \sum_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} |\langle \alpha, \lambda \rangle - 1| \tag{1}$$

The restriction of l to W_a (resp. W) is the usual length function of W_a (resp. W) as a Coxeter group ([IM], Prop. 1.10). For $w \in W$ the formula simplifies to

$$l(w) = |\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}|. \tag{2}$$

Thus $\widetilde{W} = W_a \rtimes \Omega$ carries the structure of a quasi-Coxeter group, i.e. the semi-direct product of a Coxeter group and an abelian group. If $\tilde{w} \in \widetilde{W}$ then $\tilde{w} = w_a \omega$ for uniquely determined $w_a \in W_a, \omega \in \Omega$ and $l(\tilde{w}) = l(w_a)$. The Bruhat order on W_a is extended to $\widetilde{W}$: If $\tilde{w}_1 = w_1\omega_1, \tilde{w}_2 = w_2\omega_2 \in \widetilde{W} = W_a \rtimes \Omega$, then $\tilde{w}_1 \leq \tilde{w}_2$ if and only $\omega_1 = \omega_2$ and $w_1 \leq w_2$ in W_a. Any inner automorphism given by an element $\omega \in \Omega$ preserves this partial order, as well as the length of elements in $\widetilde{W}$.

Since $\sigma(\mathfrak{a}_0) = \mathfrak{a}_0$, the Frobenius action on $\widetilde{W}$ restricts to actions on (W_a, S_a) and on Ω and preserves the Bruhat order on $\widetilde{W}$ and the length of elements.

Remark I.2.2. The action of $\widetilde{W}$ on V is not canonical. Another common definition is $(\epsilon^\lambda w)(v) := w(v) - \lambda$. Using this definition one obtains the same results as above, but with different groups W_a' and Ω', and with a different Bruhat order and length formula: The relation to the groups defined above is given by $W_a' = w_0 W_a w_0$ and $\Omega' = w_0 \Omega w_0$. We will address the consequences of this ambiguity at the points where they turn up.

I.2.3 Parabolic subgroups and Levi subgroups

By a parabolic subgroup of G over $\mathcal{O}$ we will mean a parabolic subgroup $P \subseteq G_{\mathcal{O}}$ (not necessarily defined over $\mathcal{O}_F$). We call a parabolic subgroup P of G over $\mathcal{O}$ a *semistandard parabolic* if it contains T, and a *standard parabolic* if it contains B. Of course these notions depend on the choice of our Borel pair (T, B). The unique Levi M subgroup containing T of a standard parabolic subgroup will be called a standard Levi subgroup of G (over $\mathcal{O}$).

We will use the following properties of standard parabolics and standard Levi subgroups:

(1) Let $J \subseteq S$ be a subset of simple reflections, let $\Phi_J \subseteq \Phi$ be the roots generated by $\{\alpha_s \mid s \in J\} \subseteq \Delta$. Then there is a unique standard parabolic subgroup P_J such that Φ_J is the root system with respect to T of its standard Levi subgroup, which we denote by M_J. Moreover, $B \cap M_J$ is a Borel subgroup of M_J such that the corresponding positive roots of Φ_J are given as $\Phi_J^+ = \Phi^+ \cap \Phi_J$ (see [SGA3], Exp. XXII, 5.10.1. and Exp. XXVI, 1.4., 6.1.).

(2) Conversely, if $P \subseteq G$ is a standard parabolic subgroup over $\mathcal{O}$, then there is a unique subset $J \subseteq S$ such that $P = P_J$ ([SGA3], Exp. XXVI, 7.7.).

(3) Let $\mu \in X_*(T)$ be dominant, then $M := \mathrm{Cent}_G(\mu)$ is a standard Levi subgroup of G, and we have $M = M_{S_\mu}$, where $S_\mu := \{s \in S \mid s(\mu) = \mu\}$ ([SGA3], Exp. XXVI, 1.4., 6.10.).

For any subset $J \subseteq S$ let $W_J \subseteq W$ be the subgroup generated by J. Then (W_J, J) is again a Coxeter group, it is exactly the Weyl group of the standard Levi subgroup M_J with respect to the Borel pair $(T, B \cap M_J)$. The corresponding notions of length and Bruhat order in W and W_J are compatible in the sense that if l_J denotes the length function on W_J, then $l_J(w) = l(w)$ for all $w \in W_J$. There is a natural set $W^J := \{w \in W \mid l(ws) > l(w) \text{ for all } s \in J\}$ of right coset representatives for W_J. Each $w \in W^J$ is the unique element of shortest length in wW_J, and $l(ww_J) = l(w) + l(w_J)$ for all $w_J \in W_J$ (see [Hu], §1.10.). Symmetrically, ${}^J W := \{w \in W \mid l(sw) > l(w) \text{ for all } s \in J\}$ is a set of left coset representatives with analogous properties (in fact ${}^J W = (W^J)^{-1}$).

A similar result holds for double cosets: For any two subsets $L, J \subseteq S$ the elements in ${}^L W^J := {}^L W \cap W^J$ form a full set of representatives for the double cosets in $W_L \backslash W / W_J$ and each $w \in {}^L W^J$ is the unique element of minimal length in $W_L w W_J$ ([DDPW], (4.3.2)). See Section I.3.2.1 for further properties of ${}^L W^J$.

Recall that $\Phi_J \subseteq \Phi$ is the root system generated by $\{\alpha_s \mid s \in J\}$, and that the positive roots with respect to $B \cap M_J$ are given by $\Phi_J^+ = \Phi_J \cap \Phi^+$. As for example explained in ([Hu], §1.6.), if $w(\alpha_s) \in \Phi^-$ for some $w \in W, s \in S$ then $l(ws) = l(w) - 1$. This implies that for $w \in W$ we have:

- If $w \in W_J$, then $w(\Phi^+ \setminus \Phi_J^+) = \Phi^+ \setminus \Phi_J^+$.
- $w \in W^J$ if and only if $w(\Phi_J^+) \subseteq \Phi^+$.
- $w \in {}^J W$ if and only if $\Phi_J^+ \subseteq w(\Phi^+)$. Equivalently, w lies in ${}^J W$ if and only if $w(\alpha) \in \Phi_J$ implies that $w(\alpha) \in \Phi_J^+$, for any $\alpha \in \Phi^+$.

Shortest coset representatives also exist for the finitely generated Coxeter group W_a, and also for the quasi-Coxeter group $\widetilde{W}$. For example, let $w_a \in W_a$ and $\omega \in \Omega$, let $\tilde{w} = w_a\omega$. Then for every $J \subseteq S_a$ there exists a unique element w_a^J of shortest length in $w_a(W_a)_J$ and a unique element $\tilde{w}^J$ of shortest length in $\tilde{w}(W_a)_J$. These elements are related as follows: For any $J \subseteq S_a$ we have $\tilde{w}^J = w_a^{J'}\omega$, where $J' := {}^\omega J \subseteq S_a$. Again we have the sets of shortest right coset representatives W_a^J and $\widetilde{W}^J$ for $(W_a)_J$, and analogously for left coset representatives and double coset representatives.

Now suppose that $J \subseteq S$ such that $\sigma(J) = J$. Then M_J and thus also the Borel pair $(T, B \cap M_J)$ are defined over $\mathcal{O}_F$. The extended affine Weyl group of M_J is then given by $\widetilde{W}_J = X_*(T) \rtimes W_J \subseteq \widetilde{W}$, and the description from the last subsection applies to this group as well: The action of $\widetilde{W}_J$ on V is just the restriction of the action of $\widetilde{W}$. We have a lenghth function l_J on $\widetilde{W}_J$ given by a formula analogous to (1), and the corresponding subgroup Ω_J of elements of length 0. However, l_J is *not* equal the restriction of l to $\widetilde{W}_J$ in general. For example, if $J = \emptyset$ then $M_J = T$ and $\widetilde{W}_J = X_*(T) = \Omega_J$.

I.2.4 Some group theoretic results

Notation I.2.3. We write $K := G(\mathcal{O}) \subseteq G(L)$. For any $g \in K$ we denote by $\bar{g}$ its image under the natural projection $K \to G(k)$. We set

$$K_1 := \{g \in K \mid \bar{g} = 1\}, \qquad I := \{g \in K \mid \bar{g} \in B^{\mathrm{opp}}(k)\},$$

where B^{opp} is the Borel subgroup opposite of B with respect to T.

By definition, $K_1 \subseteq I \subseteq K$ are σ-invariant subgroups. With respect to the action of $G(L)$ on the Bruhat-Tits building of G, the group $K \subseteq G(L)$ is hyperspecial, namely the stabilizer of $0 \in V$, and I is the subgroup of elements fixing $\mathfrak{a}_0$ pointwise, i.e. an Iwahori subgroup of $G(L)$.

Remark I.2.4. The symmetric definition of the $G(L)$-action (compare I.2.2) would lead to the description $I = \{g \in K \mid \bar{g} \in B(k)\}$, which is also common in the literature.

Decompositions of $G(L)$ Recall that we have the following decompositons (see [Ti], §3):

$$G(L) = \mathop{\dot{\bigcup}}_{\tilde{w} \in \widetilde{W}} I\tilde{w}I \qquad \text{(Bruhat-Tits decomposition)} \tag{3}$$

$$G(L) = \mathop{\dot{\bigcup}}_{\mu \in X_*(T)_{\mathrm{dom}}} K\epsilon^\mu K \qquad \text{(Cartan decomposition)} \tag{4}$$

In the equicharacteristic case, $G(L) = G(k((t)))$ is the group of k-valued points of the loop group LG of $G_{\mathbb{F}_q}$, which is a group ind-scheme (see [Fa], [PR]), so $G(L)$ carries a topology in this case. With respect to this topology the above double cosets are locally closed, and the following closure relations hold:

$$\overline{I\tilde{w}I} = \bigcup_{\tilde{w}' \leq \tilde{w}} I\tilde{w}'I, \qquad \tilde{w} \in \widetilde{W}$$
$$\overline{K\epsilon^{\mu}K} = \bigcup_{\mu' \preceq \mu} K\epsilon^{\mu'}K, \qquad \mu \in X_*(T)_{\mathrm{dom}},$$

where for two dominant cocharacters one writes $\mu' \preceq \mu$ if and only if $\mu - \mu'$ is a sum of positive coroots (compare (5)). These relations are for example shown in ([Ri], Prop. 2.8.).

Change of characteristics We will sometimes use results of combinatorial nature which were obtained in the equicharacteristic case (for example by exploiting the above closure relations) also in the case of mixed characteristics. We may do so by applying the following principles to both $\mathcal{O}_F = \mathbb{Z}_q = W(\mathbb{F}_q)$ and $\mathcal{O}_F = \mathbb{F}_q[[t]]$:

(1) Consider a triple (G, T, B) over $\mathcal{O}_F$ as in Section I.2.1, then $X_*(T), W, \widetilde{W}$ and the σ-actions on these objects depend only on the reductions $G_{\mathbb{F}_q}, T_{\mathbb{F}_q}, B_{\mathbb{F}_q}$ to the special fiber of $\mathrm{Spec}(\mathcal{O}_F)$ (see for example [SGA3], Exp. X, 4.6. for $X_*(T)$ and Exp. XXII, 3.4. for W).

(2) Let G_0 be a reductive group over $\mathbb{F}_q$ with Borel subgroup B_0 and maximal torus $T_0 \subseteq B_0$. Then by ([SGA3], Exp. XXIV, 1.21.), up to isomorphism there is a unique reductive group G over $\mathcal{O}_F$ such that $G_{\mathbb{F}_q}$ is isomorphic to G_0, and the Borel pair (T_0, B_0) of G_0 lifts to a Borel pair (T, B) of G over $\mathcal{O}_F$ by ([SGA3], Exp. XXII, 5.8.3.).

So for example, if (G, T, B) is defined over $\mathbb{Z}_q$, then there is a triple (G', T', B') over $\mathbb{F}_q[[t]]$ which gives rise to the same $\widetilde{W}$ as a σ-module.

Root subgroups and Decompositions of I For every root $\alpha \in \Phi$ let $U_\alpha : \mathbb{G}_{a,\mathcal{O}} \to G_{\mathcal{O}}$ be the associated root group. For any $\alpha \in \Phi$ and $\lambda \in X_*(T)$ we then have the relation

$$\epsilon^{\lambda} U_\alpha(x) \epsilon^{-\lambda} = U_\alpha(\epsilon^{\langle \alpha, \lambda \rangle} x) \quad \text{for all } x \in L.$$

Let $P \subseteq G$ be a semistandard parabolic subgroup. Let M be the unique Levi subgroup of P containing T, let N be the unipotent radical of P and let $\overline{N}$ be the unipotent radical of the parabolic opposite to P (with respect to T), denote by $\mathfrak{m}, \mathfrak{n}, \overline{\mathfrak{n}} \subseteq \mathfrak{g}$ the corresponding Lie algebras. Let $\Phi_M, \Phi_N, \Phi_{\overline{N}} \subseteq \Phi$ be the set of roots such that $\mathfrak{g}^\alpha$ lies in $\mathfrak{m}, \mathfrak{n}, \overline{\mathfrak{n}}$ respectively. Then by the result of ([Ti], §3.1.), we have the decompositions

$$\overline{N}(L) \cap I = \prod_{\alpha \in \Phi_{\overline{N}}} (U_\alpha(L) \cap I)$$
$$N(L) \cap I = \prod_{\alpha \in \Phi_N} (U_\alpha(L) \cap I)$$
$$M(L) \cap I = T(\mathcal{O}) \cdot \prod_{\alpha \in \Phi_M} (U_\alpha(L) \cap I)$$

and

$$I = (\overline{N}(L) \cap I) \cdot (M(L) \cap I) \cdot (N(L) \cap I) \qquad \text{(Iwahori decomposition)}$$

Lang's Theorem The well-known result of Lang on connected algebraic groups over k asserts that the map $G(k) \to G(k),\ g \mapsto g^{-1}\sigma(g)$ is surjective. The fact that $\mathcal{O} = \varprojlim_{n\to\infty} \mathcal{O}/(\epsilon^n)$ allows to show similar results for subgroups of $K = G(\mathcal{O})$. The following variant follows from ([Vi1], Lemma 2.1.):

Proposition I.2.5. *Let $H \subseteq G$ be a connected subgroup over $\mathcal{O}$. Set $H_n := \ker[H(\mathcal{O}) \to H(\mathcal{O}/(\epsilon^n))] \subseteq H(\mathcal{O})$ for $n \geq 0$. Let $g \in G(L)$ such that ${}^{g^{-1}}H_n \subseteq \sigma(H_n)$ for all n. Then the map $H(\mathcal{O}) \to H(\mathcal{O}),\ h \mapsto \sigma^{-1}(g^{-1}h^{-1}g)h$ is surjective.*

We will often use this result in the following form:

Corollary I.2.6. *Let $H \subseteq G$ be a connected subgroup over $\mathcal{O}$, let $g \in K$ such that ${}^{g^{-1}}H \subseteq \sigma(H)$. Then the map $H(\mathcal{O}) \to H(\mathcal{O}),\ h \mapsto \sigma^{-1}(g^{-1}h^{-1}g)h$ is surjective.*

Proof. Since $g \in K = G(\mathcal{O})$, the relation ${}^{g^{-1}}H \subseteq \sigma(H)$ implies that ${}^{g^{-1}}H_n \subseteq \sigma(H_n)$ for all n. □

I.2.5 σ-conjugacy classes

Definition I.2.7.

(a) For an element $g \in G(L)$ we write $[g] := G(L) \cdot_\sigma g$ for its $G(L)$-σ-conjugacy class.

(b) Let $B(G) := \{[g] \mid g \in G(L)\}$.

The structure of $B(G)$ has been studied by Kottwitz in [Ko1], [Ko2] and by Rapoport and Richartz in [RR]. (To be precise, these papers only consider the case of mixed characteristics, but everything also works in the equicharacteristic case.) It is described by two maps:

(I) Define the set of Newton points for G as

$$\mathcal{N}(G) := \big(\mathrm{Hom}_L(\mathbb{D}_L, G_L)/\mathrm{Int}\, G(L)\big)^{\langle\sigma\rangle} = \big((X_*(T)_{\mathbb{Q}})/W\big)^{\langle\sigma\rangle}.$$

Here $\mathbb{D}$ is the diagonizable pro-algebraic group with character group $\mathbb{Q}$, and the second equality follows from the fact that for each $\nu \in \mathrm{Hom}_L(\mathbb{D}_L, G_L)$ some multiple $k \cdot \nu$ factors through $\mathbb{G}_{m,L}$ and is thus $G(L)$-conjugate to an element of $X_*(T)$. In ([Ko1], §4), to each $b \in B(G)$ Kottwitz assigns an element $\nu_b \in \mathcal{N}(G)$, this gives the *Newton map* of G.

$$\nu_G \colon B(G) \longrightarrow \mathcal{N}(G), \quad b \longmapsto \nu_b.$$

We naturally identify $\mathcal{N}(G) = \big((X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q})/W\big)^{\sigma}$ with a subset of the closed dominant Weyl chamber $\overline{C} \subseteq V$ and call $\nu_b \in \overline{C}$ the *dominant Newton vector* of b.
A σ-conjugacy class $b \in B(G)$ is called *basic*, if (any representative of) $\nu_b : \mathbb{D}_L \to G_L$ factors through the center of G, we denote by $B(G)_{\mathrm{bas}} \subseteq B(G)$ the subset of basic elements.

(II) Secondly, let

$$\pi_1(G) := X_*(T)/\langle \alpha^\vee \in \Phi^\vee \rangle$$

be the algebraic fundamental group of G, and let $\pi_1(G)_{\langle\sigma\rangle}$ be its group of σ-coinvariants. There is a natural surjective homomorphism $\eta_G : G(L) \to \pi_1(G)$ which is functorial in G (called ω_G in [Ko2], §7.4., see also [RR], §1). Passing to σ-conjugacy classes, and taking σ-coinvariants on the right hand side one obtains the *Kottwitz map*

$$\kappa_G : B(G) \longrightarrow \pi_1(G)_{\langle\sigma\rangle}$$

By ([Ko1], 5.6.), the map κ_G induces a bijection between the set of basic elements in $B(G)_{\mathrm{bas}}$ and $\pi_1(G)_{\langle\sigma\rangle}$. Further, if $\mu \in X_*(T)$ is dominant such that $g \in K\epsilon^\mu K$ (see (4)), then $\kappa_G(g)$ is equal to the image of μ under the natural projection map $X_*(T) \to \pi_1(G)_{\langle\sigma\rangle}$.

Proposition I.2.8.

(i) (Kottwitz, [Ko3], 4.13.) The map

$$(\nu_G, \kappa_G) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_{\langle\sigma\rangle}$$

is injective.

(ii) (Rapoport-Richartz, [RR], 1.15.(iii)) If $b, b' \in B(G)$ such that $\nu_G(b) = \nu_G(b')$, then $\kappa_G(b) - \kappa_G(b') \in (\pi_1(G)_{\langle\sigma\rangle})_{\mathrm{tors}}$.

In particular, if $\pi_1(G)_{\langle\sigma\rangle}$ is torsion free, then $\nu_G : B(G) \to \mathcal{N}(G)$ is injective.

Of course, the same classification applies for σ-conjugacy classes of standard Levi subgroups M_J of G, where then $\mathcal{N}(M_J)$ and $\pi_1(M_J)$ are given as in (I) and (II) above with respect to W_J and $\Phi_J^\vee$. To every σ-conjugacy class in $G(L)$ we associate a standard Levi subgroup of G:

Definition I.2.9. For $b \in B(G)$ we define its *type* as $J_b := \{s \in S \mid s(\nu_b) = \nu_b\} \subseteq S$. We set $M_b := M_{J_b}$ (see Section I.2.3).

In other words, $M_b := \mathrm{Cent}_G(r \cdot \nu_b) \subseteq G$, where r is some integer such that $r \cdot \nu_b$ is a dominant cocharacter of T. As ν_b is σ-invariant, we always have $\sigma(J_b) = J_b$, and the group M_b is defined over $\mathcal{O}_F$.

Example I.2.10. Let us consider the classical example of the general linear group: Here $F = \mathbb{Q}_p$ and $G = \mathrm{GL}_{n,\mathbb{Z}_p}$. Let T be the torus of diagonal matrices, and let B be the Borel subgroup of upper triangular matrices in GL_n. We then for example have $X^*(T) \cong X_*(T) \cong \mathbb{Z}^n$, $W = S_n$, $\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le n\} \subseteq X^*(T)$, and the σ-action is trivial, the Newton points and the fundamental group are given as

$$\mathcal{N}(G) \simeq \{x \in \mathbb{Q}^n \mid x_1 \ge \cdots \ge x_n\}, \qquad \pi_1(G) = \pi_1(G)_{\langle\sigma\rangle} \simeq \mathbb{Z}.$$

Since $\pi_1(G)_{\langle\sigma\rangle}$ is torsion free, the Newton map $\nu_G\colon B(G) \to \mathcal{N}(G)$ is injective. It has the following realization in this case: The assignment

$$g \longmapsto (L^n, g\sigma)$$

for $g \in \mathrm{GL}_n(L)$ induces a bijection between $B(G)$ and the set of isomorphism classes of isocrystals of height n over k. By the classification of Dieudonné-Manin (see for example [Dem], Ch.IV), these isomorphism classes are described by their *Newton polygons*: By definition, a Newton polygon of length n is a concave, continuous, piecewise linear function Λ on the interval $[0, n]$ with $\Lambda(0) = 0$ such that all break points have integral coordinates (we warn the reader that many different conventions regarding Newton polygons can be found in the literature). A Newton polygon Λ is determined by its slopes in the interval $(i-1, i)$ for $i = 1, \ldots, n$, which are rational numbers. Writing these in the form $(\nu_1^{d_1}, \ldots, \nu_m^{d_m})$ with $\nu_1 > \cdots > \nu_m$ (here $\nu_i^{d_i}$ denotes the d_i-fold repetition of $\nu_i \in \mathbb{Q}$), Λ corresponds to the isomorphism class of the isocrystal $\bigoplus_{i=1}^m N_i$, where N_i is isoclinic of slope ν_i and height d_i.

Now if $b = [g] \in B(G)$ and $\Lambda = (\nu_1^{d_1}, \ldots, \nu_m^{d_m})$ is the Newton polygon of the isomorphism class of $(L^n, g\sigma)$, then

$$\nu_G(b) = (\nu_1^{d_1}, \ldots, \nu_m^{d_m}), \qquad \kappa_G(b) = \sum_{i=1}^m d_i \nu_i,$$

and we have $M_b = \mathrm{GL}_{d_1} \times \cdots \mathrm{GL}_{d_m}$

Order relation on $B(G)$ The set $B(G)$ carries a partial order given as follows: For $\nu, \nu' \in \mathcal{N}(G) \subseteq \overline{C}$ write

$$\nu' \preceq \nu \;:\Longleftrightarrow \nu - \nu' = \sum_{\alpha^\vee \in (\Phi^\vee)^+} n_{\alpha^\vee} \alpha^\vee \text{ for } n_{\alpha^\vee} \in \mathbb{R}^{\geq 0}. \tag{5}$$

This defines a partial order on $\mathcal{N}(G)$, which was introduced by Rapoport and Richartz in ([RR], §2). The order relation on $B(G)$ is obtained by setting

$$b' \preceq b \;:\Longleftrightarrow \nu_G(b') \preceq \nu_G(b). \tag{6}$$

By ([RR], Thm. 3.6.), this order describes the behaviour of a map $S \to B(G)$ associated to an isocrystal with G-structure over S under specializations. In the above example for $G = \mathrm{GL}_n$ we have $b' \preceq b$ if and only if the Newton polygon of b' lies below that of b and both have the same end point, ([RR], Thm. 3.6.) is then nothing but the "Grothendieck specialization theorem", which asserts that the Newton polygons of a family of p-divisible groups go down (with our conventions on Newton polygons) under specialization. For Shimura varieties of PEL-type it is known that $\preceq$ describes the closures of Newton strata (see Section II.5.2).

Representatives in $\widetilde{W}$ Let $\tilde{w} \in \widetilde{W}$, then the σ-conjugacy class $[\tilde{w}]$ does not depend on the representative for $\tilde{w}$ in G by Proposition I.2.5 applied to $H = T$. On the other hand, we will see below in I.2.15 that every σ-conjugacy class in $G(L)$ contains an element of $\widetilde{W}$. As explained in ([He2], 1.3.), in this case there is a more explicit description of the Newton map and the Kottwitz map.

Definition I.2.11. Let $\tilde{w} \in \widetilde{W}$, then we set

$$\nu_{\tilde{w}} := \frac{1}{r}\tilde{w}\sigma(\tilde{w})\cdots\sigma^{r-1}(\tilde{w}) \in X_*(T)_{\mathbb{Q}} \subseteq V, \tag{7}$$

where $r \in \mathbb{N}$ such that σ^r acts trivially on $\widetilde{W}$ and $\tilde{w}\sigma(\tilde{w})\cdots\sigma^{r-1}(\tilde{w}) \in X_*(T)$.

Note that this is well-defined: W is finite, and some power of σ acts trivially on $\widetilde{W}$ (since G splits over a finite unramified extension of $\mathcal{O}_F$), so an $r \in \mathbb{N}$ as in the definition exists. By the usual argument, the definition is independent of the choice of r. We remark that the vector $\nu_{\tilde{w}}$ is not dominant in general.

Now for $b = [\tilde{w}]$, $\tilde{w} \in \widetilde{W}$ the following hold:

(1) Let $\tilde{w} = \epsilon^\lambda w$ for $\lambda \in X_*(T), w \in W$, then $\kappa_G(b)$ is equal to the projection of λ to $\pi_1(G)_{\langle\sigma\rangle}$.

(2) The dominant Newton vector $\nu_b \in \overline{C}$ of b is equal to the unique dominant element in $W \cdot \nu_{\tilde{w}}$. This follows from the characterization of ν_b in ([Ko1], 4.3).

In particular, if $\omega \in \Omega$ then $b = [\omega]$ is basic in G: For r as in Definition I.2.11 we have $l(r \cdot \nu_\omega) = l(\omega\sigma(\omega)\cdots\sigma^{r-1}(\omega))r \cdot l(\omega) = 0$, so the length formula (1) implies that $\langle\alpha, r \cdot \nu_\omega\rangle = 0$ for all $\alpha \in \Phi$, which means that $r \cdot \nu_\omega$ is central in G.

Definition I.2.12. Let $b \in B(G)$, let J_b be its type. An element $\omega_b \in \widetilde{W}$ is called *b-short*, or a *short representative* of b, if $b = [\omega_b]$, $\omega_b \in \Omega_{J_b}$ and $\nu_b = \nu_{\omega_b}$.

Remark I.2.13. In view of property (2) above one can reformulate this Definiton as follows: Let $b \in B(G)$, of type J_b, then an element $\omega_b \in \Omega_{J_b}$ is b-short if and only if $\omega_b \in b$ and ν_{ω_b} is dominant. By the classification of σ-conjugacy classes and the definition of ν_{ω_b}, these conditions are satisfied if and only if $\kappa_G([\omega_b]) = \kappa_G(b)$ and $\omega_b\sigma(\omega_b)\cdots\sigma^{r-1}(\omega_b) = r \cdot \nu_b$ for some $r \in \mathbb{N}$.

This notion of b-short elements, which generalizes the concept of standard representatives in the case that G is split (see [GHKR], Def. 7.2.3), was introduced by Viehmann in ([Vi1], 5.1.), who also showed their existence:

Lemma I.2.14 (Viehmann, [Vi1], 5.3.)**.** *For every $b \in B(G)$ there exists a b-short element ω_b. If G is split over $\mathcal{O}_F$, then ω_b is uniquely determined.*

By the above considerations, for any b-short element ω_b the σ-conjugacy class $[\omega_b]_{M_b} \in B(M_b)$ is basic, so this result can be seen as a "discrete refinement" of ([Ko1], 6.2.) for our types of groups. If G is not split over $\mathcal{O}_F$ then b-short elements are not unique in general, see for example Section I.4.2.2.

Corollary I.2.15. *The map $\widetilde{W} \to B(G),\ \tilde{w} \mapsto [\tilde{w}]$ is surjective.*

Note that by the properties (1) and (2) above this map can be defined only in terms of the structure of $\widetilde{W}$ and its σ-action, it is therefore independent of the characteristic of L in the sense of the last subsection.

I.2.6 Ekedahl-Oort classes

Definition I.2.16.

(i) For $g \in G(L)$ we define its *Ekedahl-Oort class* as

$$[[g]] := K \cdot_\sigma (K_1 g K_1).$$

(ii) Let $\mathrm{EO}(G) := \{[[g]] \mid g \in G(L)\}$.

Since K_1 is a normal subgroup of K, the sets of the form $[[g]]$ for $g \in G(L)$ form a decomposition of $G(L)$ into equivalence classes, which refines the Cartan decomposition: If $g \in K\epsilon^\mu K$ then $[[g]] \subseteq K\epsilon^\mu K$.

Standard representatives The $[[\cdot]]$-classes in $G(L)$ were first defined by Viehmann in [Vi1], who also answered the question of a natural set of representatives, using the alternative description of $\widetilde{W}$ (see Remark I.2.2). Following our conventions, we make the following definitions:
For $\mu \in X_*(T)$ be dominant let $S_\mu := \{s \in S \mid s(\mu) = \mu\}$ and

$$x_\mu := w_{0,S_\mu} w_0, \quad \tau_\mu = \epsilon^\mu x_\mu, \quad J_\mu := \sigma(S_\mu). \tag{8}$$

Then τ_μ is the unique element of shortest length in $W\epsilon^\mu W$, in other words $\tau_\mu \in {}^S\widetilde{W}^S$, and $l(\tau_\mu w) = l(\tau_\mu) + l(w)$ for all $w \in W$.

Proposition I.2.17 (Viehmann, [Vi1], Thm. 1.1.). *Let $\mu \in X_*(T)$ be dominant.*

(i) The elements $\tau_\mu w$ for $w \in W^{J_\mu}$ form a full set of representatives for the $[[\cdot]]$-classes in $K\epsilon^\mu K$ in other words,

$$K\epsilon^\mu K = \mathop{\dot{\bigcup}}_{w \in W^{J_\mu}} [[\tau_\mu w]].$$

(ii) $I\tau_\mu w I \subseteq [[\tau_\mu w]]$ for each $w \in W^{J_\mu}$.

Proof. Using a symmetric approximation strategy, the proof is exactly the same as in ([Vi1], §3). Note however that our τ_μ differs from the one in [Vi1] (where it is given by $w_0 w_{0,S_\mu} \epsilon^\mu$), by our definitions we have $\tau_\mu M_{S_\mu^{\mathrm{opp}}} \tau_\mu^{-1} = M_{S_\mu}$. We will also review parts of the argument in Section I.3.2. $\square$

Definition I.2.18. Let $\xi \in \mathrm{EO}(G)$, then we write $\tilde{w}_\xi$ for the unique representative for ξ of the form $\tau_\mu w$ for $w \in W^{J_\mu}$ given by Proposition I.2.17, and call it the *standard representative* of ξ.

The dual parametrization Let $\mu \in X_*(T)_{\mathrm{dom}}$, let $J_\mu \subseteq S$ and $x_\mu \in W$ as above. Set $L_\mu := S_\mu^{\mathrm{opp}}$, then $L_\mu = x_\mu^{-1} S_\mu x_\mu$. It is shown in ([PWZ1], Prop. 9.13., Prop. 9.14.) that there is a unique bijection

$$\varphi \colon {}^{L_\mu}W \longrightarrow W^{J_\mu}$$

with the property that for each $w \in {}^{L_\mu}W$ there is some $y \in W_{L_\mu}$ with $\varphi(w) = yw\sigma(x_\mu y^{-1} x_\mu^{-1})$, and that $l(\varphi(w)) = l(w)$ for each $w \in {}^{L_\mu}W$. It follows that for every $w \in {}^{L_\mu}W$ one has the identity $[[\tau_\mu w]] = [[\epsilon^\mu x_\mu w]] = [[\epsilon^\mu x_\mu \varphi(w)]] = [[\tau_\mu \varphi(w)]]$, by σ-conjugation with $x_\mu y x_\mu^{-1}$ for $y \in W_{L_\mu}$ as above. So the elements $\tau_\mu w$ for $w \in {}^{L_\mu}W$ also give a set of representatives for the EO-classes in $K\epsilon^\mu K$, we will sometimes refer to them as the *dual standard representatives* of $\mathrm{EO}(G)$. This set of representatives could also be obtained by a variant of the proof of Proposition I.2.17(i) above.

The closure relation It is shown in ([Vi1], Lemma 4.1.) that in the equicharacteristic case the $[[\cdot]]$-classes are the k-valued points of locally closed, irreducible subschemes of the loop group of $G_{\mathbb{F}_q}$ (in the sense of [Vi1], §2.4.). The closure of an EO-class ξ in $G(L)$ is a union of EO-classes, so the relation $\xi' \preceq \xi :\Leftrightarrow \xi' \subseteq \overline{\xi}$ defines a partial order on $\mathrm{EO}(G)$, and thus by transfer also on the set $\{\tilde{w}_\xi \mid \xi \in \mathrm{EO}(G)\}$. By ([Vi1], Thm. 1.4.) the order relation for standard representatives is given as follows:

$$\tilde{w}_{\xi'} \preceq \tilde{w}_\xi \iff z\tilde{w}_{\xi'}\sigma(z)^{-1} \leq \tilde{w}_\xi \text{ for some } z \in W. \tag{9}$$

(But again note that the Bruhat order considered in [Vi1] differs from ours, and that in the proof one has to argue symmetrically.)
On the other hand, the relation given by (9) is purely group theoretical and also meaningful in the case of mixed characteristics, and ([Vi1], Thm. 1.4.) shows that it is a partial order on the set of standard representatives by the principles explained in Section I.2.4. For mixed characteristics we will also write $\xi \preceq \xi'$ if $\tilde{w}_\xi \preceq \tilde{w}_{\xi'}$. For EO-classes which arise from Ekedahl-Oort strata of a Shimura variety this relation then again describes the closure of a stratum (see Section II.6.2).

For further reference we note the following properties of $\preceq$:

Remark I.2.19. Let $\tilde{w} = \tilde{w}_\xi$ and $\tilde{w}' = \tilde{w}_{\xi'}$ be standard representatives of classes $\xi, \xi' \in \mathrm{EO}(G)$.

(1) If $\tilde{w}' \leq \tilde{w}$, then $\tilde{w}' \preceq \tilde{w}$.

(2) If $\tilde{w}' \preceq \tilde{w}$, then $l(\tilde{w}') \leq l(\tilde{w})$. This can be derived from (9) and the fact that $\tilde{w}'$ is of minimal length in its W-σ-conjugacy class (see Proposition I.3.11).

(3) If $\tilde{w}' \preceq \tilde{w}$ and $l(\tilde{w}') = l(\tilde{w})$, then $\tilde{w}' = \tilde{w}$: By (9), in this case $\tilde{w} = z\tilde{w}'\sigma(z)^{-1}$ for some $z \in W$, so $\tilde{w}' \in \xi$, which implies $\xi' = \xi$ and hence $\tilde{w}' = \tilde{w}$ by I.2.17(i).

I.3 Relations between the decompositions

Let $\mu \in X_*(T)_{\mathrm{dom}}$. As motivated in the introduction to this chapter, we will be interested in the comparison of the σ-conjugacy classes and the EO-classes which appear in $K\epsilon^\mu K$. Of special relevance is the case that μ is minuscule, which will always be the case if the pair (G, μ) arises from a Shimura datum as in Chapter II.

I.3.1 The intersection problem

Definition I.3.1. For a dominant cocharacter $\mu \in X_*(T)_{\mathrm{dom}}$ we denote

(a) $B(G,\mu) := \{[g] \mid g \in K\epsilon^\mu K\}$,

(b) $\mathrm{EO}(G,\mu) := \{[[g]] \mid g \in K\epsilon^\mu K\}$.

The study of the intersections between the "group theoretical stratifications" of a K-double coset $K\epsilon^\mu K$ can then be put as follows:

Problem I.3.2. Let $\mu \in X_*(T)_{\mathrm{dom}}$. Given $\xi \in \mathrm{EO}(G,\mu)$ and $b \in B(G,\mu)$, find a description of $\xi \cap b$.

This may be considered from different points of view. A first satisfying answer would be given by a criterion to decide whether $\xi \cap b$ is empty or nonempty. In the equicharacteristic case one may also ask for the topological properties of the intersection, for example its dimension, but we did not consider this question. Another interesting aspect is to decide whether for every $b \in B(G, \mu)$ there exists some $\xi \subseteq b$ such that $\xi = K \cdot_\sigma \tilde{w}_\xi$, we will consider this question in Section I.3.3.

In this subsection we will collect information on the sets $B(G, \mu)$ and $\mathrm{EO}(G, \mu)$ and report on some results that will be used in the sequel.

I.3.1.1 The sets $B(G, \mu)$ and $\mathrm{EO}(G, \mu)$

The set $B(G, \mu)$ is well-known, it is exactly the set of σ-conjugacy classes which meet $K\epsilon^\mu K$. Results of Kottwitz-Rapoport ([KR]), Lucarelli ([Lu]) and Gashi ([Ga]) show that it has the following combinatorial description:

Let $\mu^\natural$ be the image of $\mu \in X_*(T)$ under the projection map to $\pi_1(G)_{\langle\sigma\rangle}$. Let $n \in \mathbb{N}$ such that σ^n acts trivially on $X_*(T)$, and set

$$\bar{\mu} := \frac{1}{n} \sum_{i=0}^{n-1} \sigma^i(\mu) \in \overline{C},$$

then we have

$$B(G, \mu) = \{b \in B(G) \mid \kappa_G(b) = \mu^\natural,\ \nu_G(b) \preceq \bar{\mu}\}, \tag{10}$$

where $\preceq$ is the partial order on $\overline{C}$ explained in Section I.2.5. Note that $\bar{\mu}$ is exactly the vector ν_μ given by the formula (7) of Definition I.2.11, and that this vector is dominant, so we have in fact $\bar{\mu} = \nu_G([\epsilon^\mu])$. Since in addition $\mu^\natural = \kappa_G([\epsilon^\mu])$, we also have the characterization

$$B(G, \mu) = \{b \in B(G) \mid \kappa_G(b) = \kappa_G([\epsilon^\mu]),\ b \preceq [\epsilon^\mu]\}$$

with respect to the partial order on $B(G)$ given by (6).

We note a few important properties of $B(G, \mu)$:

(1) $B(G, \mu)$ is a finite set ([RR], 2.4.).

(2) $B(G, \mu)$ contains a unique maximal element with respect to $\preceq$, namely the σ-conjugacy class $[\epsilon^\mu]$.

(3) $B(G, \mu)$ contains a unique basic element, namely the one which corresponds to $\mu^\natural$ under the bijection $B(G)_{\mathrm{bas}} \simeq \pi_1(G)_{\langle\sigma\rangle}$ given by κ_G. This is also the unique minimal element in $B(G, \mu)$ with respect to $\preceq$.

The description of $\mathrm{EO}(G, \mu)$ is given by Proposition I.2.17(i): We have a bijection $W^{J_\mu} \simeq \mathrm{EO}(G, \mu)$, where $J_\mu = \sigma(S_\mu)$, by sending an element $w \in W^{J_\mu}$ to the standard representative $\tau_\mu w$ (see Definition I.2.18). The restriction of the $\preceq$-relation on $\mathrm{EO}(G)$ given by (9) to the subset $\mathrm{EO}(G, \mu)$ transfers to W^{J_μ} via this bijection, by setting $w' \preceq w :\Leftrightarrow [[\tau_\mu w']] \preceq [[\tau_\mu w]]$. This order can be described as follows:

$$w' \preceq w \iff (x_\mu^{-1} \sigma^{-1}(z) x_\mu) w' z^{-1} \leq w \quad \text{for some } z \in W_{J_\mu} \tag{11}$$

(This characterization follows from (9) by an argument completely analogous to that in the proof of ([Vi1], Cor. 4.7.): If $z \tau_\mu w' \sigma(z)^{-1} \leq \tau_\mu w$ for some $z \in W$, then

$z\tau_\mu w'\sigma(z)^{-1} = \tau_\mu v$ for some $v \leq w$, so $\tau_\mu^{-1} z \tau_\mu \in W$, which implies that $z \in W_{S_\mu}$ and a fortiori that $v = (x_\mu^{-1} z x_\mu) w \sigma(z)^{-1}$.)

It follows from Remark I.2.19 that the order $\preceq$ on W^{J_μ} resp. on $\mathrm{EO}(G,\mu)$ has the following properties.

(1) If $w' \leq w$ then $w' \preceq w$. If $w' \preceq w$, then $l(w') \leq l(w)$ with equality if and only if $w' = w$.

(2) W^{J_μ} has a unique maximal element with respect to $\leq$ and thus also with respect to $\preceq$, namely the shortest right coset representative $w_0 w_{0,J_\mu}$ of $w_0 W^{J_\mu}$. The element $1 \in W^{J_\mu}$ is the unique minimal element of W^{J_μ} with respect to $\preceq$. Consequently, the classes $[[\tau_\mu w_0 w_{0,J_\mu}]], [[\tau_\mu]] \in \mathrm{EO}(G,\mu)$ are the unique maximal and minimal elements with respect to $\preceq$.

Remark I.3.3. Instead of considering W^{J_μ} one can also work with the dual representatives $\tau_\mu w, w \in {}^{L_\mu}W$ for the classes in $\mathrm{EO}(G,\mu)$. The partial order obtained on ${}^{L_\mu}W$ is then given as follows:

$$w' \preceq w \iff z w' \sigma(x_\mu z^{-1} x_\mu^{-1}) \leq w \text{ for some } z \in W_{L_\mu}. \tag{12}$$

This follows from ([PWZ1], Thm. 11.5.) (Note that the condition is the same as in (11), if $z \in W_{J_\mu}$ then $x_\mu^{-1}\sigma^{-1}(z)x_\mu \in W_{L_\mu}$.) With respect to this order, property (1) above holds for ${}^{L_\mu}W$ as well. The unique maximal element of ${}^{L_\mu}W$ with respect to $\preceq$ is then $w_{0,L_\mu} w_0 = w_0 w_{0,S_\mu} = x_\mu^{-1}$ (as we have $w_{0,L_\mu} = w_0 w_{0,S_\mu} w_0$).

I.3.1.2 The connection to affine Deligne-Lusztig sets

Let $\xi \in \mathrm{EO}(G,\mu)$ for some $\mu \in X_*(T)_{\mathrm{dom}}$. By Proposition I.2.17(ii), the question whether $\xi \cap b$ is nonempty is equivalent to the corresponding question for the I-double coset of its standard representative: We have $I\tilde{w}_\xi I \subseteq \xi = K \cdot_\sigma (K_1 \tilde{w}_\xi K_1) \subseteq K \cdot_\sigma (I\tilde{w}_\xi I)$, therefore

$$\xi \cap b \neq \emptyset \quad \iff \quad I\tilde{w}_\xi I \cap b \neq \emptyset \tag{13}$$

for any $b \in B(G)$ (of course the intersection is empty if $b \neq B(G,\mu)$, since then $b \cap K\epsilon^\mu K = \emptyset$). Recall that for $\tilde{w} \in \widetilde{W}$ and $g \in G(L)$ the *affine Deligne-Lusztig set* attached to $\tilde{w}$ and g is defined as

$$X_{\tilde{w}}(g) := \{hI \in G(L)/I \mid h^{-1} g \sigma(h) \in I\tilde{w}I\}. \tag{14}$$

This set only depends on the σ-conjugacy class $[g]$ up to isomorphism, if $g' = h' g \sigma(h')^{-1}$ then multiplication by h' on the left gives $X_{\tilde{w}}(g) \xrightarrow{\sim} X_{\tilde{w}}(g')$. As $X_{\tilde{w}}(g)$ is nonempty if and only if $[g] \cap I\tilde{w}I \neq \emptyset$, by (13) for any $g \in G(L)$ we have $\xi \cap [g] \neq \emptyset \ \Leftrightarrow\ X_{\tilde{w}_\xi}(g) \neq \emptyset$. So the intersection pattern between EO-classes and σ-conjugacy classes would follow from the knowledge of the intersections between I-double cosets and σ-conjugacy classes, or equivalently, the emptyness and nonemptyness of affine Deligne-Lusztig sets. In this direction many partial results are known, see for example ([Be], [GHN], [GH]).

I.3.1.3 Some known techniques and results

The question whether σ-comjugacy classes and I-double cosets intersect is answered by the following algorithm. For $\tilde{w}, \tilde{w}' \in \widetilde{W}$ write $\tilde{w} \rightharpoonup \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0, \ldots, \tilde{w}_k = \tilde{w}'$ such that $l(\tilde{w}_0) \geq \cdots \geq l(\tilde{w}_k)$ and for each $i = 0, \ldots, k-1$ the following holds: There is an $s \in S_a$ such that either $\tilde{w}_{i+1} = s\tilde{w}_i\sigma(s)$ or $\tilde{w}_{i+1} = s\tilde{w}_i$ and $l(s\tilde{w}_i\sigma(s)) = l(\tilde{w}_i) - 2$.

Proposition I.3.4 ([GHN], Thm. 4.3.2.)**.** *Let $\tilde{w} \in \widetilde{W}$ Let $D_{\tilde{w}}$ be the set of elements $\tilde{y} \in \widetilde{W}$ such that $\tilde{w} \rightharpoonup \tilde{y}$ and $\tilde{y}$ is an element of minimal length in its $\widetilde{W}$-σ-conjugacy class. Then the set of σ-conjugacy classes which intersect $I\tilde{w}I$ nontrivially is exactly $\{[\tilde{y}] \mid \tilde{y} \in D_{\tilde{w}}\}$.*

Since $\tilde{w} \rightharpoonup \tilde{w}'$ implies that $l(\tilde{w}') \leq l(\tilde{w})$, and since S_a is a finite set, the set $D_{\tilde{w}}$ can be computed explicitly. As the σ-conjugacy class of an element $\tilde{y} \in \widetilde{W}$ can be calculated inside the extended affine Weyl group (see Section I.2.5), this criterion shows that the emptyness and nonemptyness of intersections between I-double cosets and σ-conjugacy classes depends on nothing but the structure of $\widetilde{W}$ and the σ-action on $\widetilde{W}$. In particular, the question of nonemptyness for these intersections is independent of the characteristic of L in the sense discussed in Section I.2.4.

The following necessary criterion for the nonemptyness of $I\tilde{w}I \cap b$ will play an important role in several arguments in this section:

Proposition I.3.5 ([Vi1], Prop. 5.5.)**.** *Let $\tilde{w} \in \widetilde{W}$. Let $b \in B(G)$, and let $J_b \subseteq S$ be its type in the sense of Definition I.2.9. If $I\tilde{w}I \cap b \neq \emptyset$, then there is a b-short element $\omega_b \in \Omega_{J_b}$ and a $y \in {}^{J_b}W$ such that $y^{-1}\omega_b\sigma(y) \leq \tilde{w}$.*

This result is stated and proven in [Vi1] in the equicharacteristic case, but the notion of a b-short element also only depends on the σ-action on $\widetilde{W}$, so by Proposition I.3.4 it holds for mixed characteristics as well.

In the equicharacteristic case it is shown in ([Vi1], Lemma 4.1.) that each $[[\cdot]]$-class ξ is irreducible, so by the Grothendieck specialization theorem ([RR], Thm. 3.6.) the set $\{b \in B(G) \mid \xi \cap b \neq \emptyset\}$ contains a unique maximal element with respect to $\preceq$. Again, as a consequence of Proposition I.3.4 and the equivalence (13) this also holds true for mixed characteristics.

Definition I.3.6. For $\xi \in \mathrm{EO}(G)$ we call the maximal element of $\{b \in B(G) \mid \xi \cap b \neq \emptyset\}$ the *generic* σ-conjugacy class in ξ.

I.3.2 Representatives of Ekedahl-Oort classes

In this subsection we explain a structure result (Proposition I.3.11) for the representatives in $\widetilde{W}$ of an EO-class ξ, which can be understood as a "discrete version" of Proposition I.2.17(i). In particular it shows a minimality property for standard representatives which will be very useful in the sequel. The result can also be derived from ([He1], §2, §3), we will give a short inductive proof here which shows exactly what we need.

I.3.2.1 Some results on Coxeter groups

In this paragraph we consider an arbitrary finitely generated Coxeter group (W, S). We start with a few well-known results on double coset representatives.

Proposition I.3.7. *Let $J, L \subseteq S$.*

(i) (Kilmoyer, [DDPW], 4.17.) Let $u \in {}^L W^J$. Then

$$^{u^{-1}}W_L \cap W_J = W_{^{u^{-1}}L \cap J} \quad \text{and} \quad W_L \cap {}^u W_J = W_{L \cap {}^u J}.$$

Further, the map $W_{^{u^{-1}}L \cap J} \to W_{L \cap {}^u J}$, $w \mapsto uwu^{-1}$ is length preserving.

(ii) (Howlett, [DDPW], 4.18.) Let $u \in {}^L W^J$. Then every $w \in W_L u W_J$ can be written uniquely as $w = w_L u w_J$ for $w_L \in W_L$ and $w_J \in {}^{(^{u^{-1}}L \cap J)}W_J$, and also as $w = w'_L u w'_J$ for unique $w'_L \in W_L^{L \cap {}^u J}$ and $w'_J \in W_J$. These decompositions satisfy $l(w) = l(w_L) + l(u) + l(w_J)$ and $l(w) = l(w'_L) + l(u) + l(w'_J)$.

Corollary I.3.8. *Let $L, J \subseteq S$, let $u \in {}^L W^J$. Then for $w \in W_L u W_J$ we have*

$$w \in W^J \quad \Longleftrightarrow \quad w = w_L u \text{ for some } w_L \in W_L^{L \cap {}^u J}.$$

Proof. This is an immediate consequence of the length formula in Proposition I.3.7(ii). □

Now consider subsets $L, J \subseteq S$, and let $\delta \colon W_L \to W_J$ be an isomorphism of Coxeter groups. In particular, δ then induces a bijection $L \xrightarrow{\sim} J$. Following He (see [He1], §3.1.), we study the partial δ-conjugation action of W_L: We write $w \longrightarrow_\delta w'$, if there is a series $w = w_0, w_1, \ldots, w_k = w'$ such that $w_i = s_i w_{i-1} \delta(s_i)$ for some $s_i \in L$ and $l(w_i) \leq l(w_{i-1})$ for each $i = 1, \ldots, k$.

Lemma I.3.9. *Let $u \in {}^L W^J$. For every $w \in W_L u W_J$ there is a $v \in W_L$ such that $w \longrightarrow_\delta vu$, and a $v' \in W_J$ such that $w \longrightarrow_\delta uv'$.*

Proof. It suffices to show the first claim, since it implies the second one by considering $u^{-1} \in {}^J W^L$, $w^{-1} \in W_J u^{-1} W_L$ and $\delta^{-1} \colon W_J \to W_L$.
Write $J_L := {}^{u^{-1}}L \cap J$. Let $w = w_L u w_J$ with $w_L \in W_L, w_J \in {}^{J_L}W_J$ be the unique decomposition of w from Proposition I.3.7(ii). We proceed by induction on the length of w_J, the case $l(w_J) = 0$ being trivial. If $l(w_J) \geq 1$, there is $s \in J$ such that $l(w_J s) = l(w_J) - 1$. We have $w_J s = x w'_J$ for some $x \in W_{J_L}, w'_J \in {}^{J_L}W_J$. Now

$$l(ws) = l(w_L u w_J s) \leq l(w_L u) + l(w_J s) = l(w_L) + l(u) + l(w_J) - 1 = l(w) - 1,$$

hence $w \longrightarrow_\delta \delta^{-1}(s) w s$, further

$$\delta^{-1}(s) w s = (\delta^{-1}(s) w_L u x u^{-1}) u w'_J$$

is the unique decomposition of $\delta^{-1}(s) w s$ as in I.3.7(ii) (since $uxu^{-1} \in W_L$). As $l(w'_J) = l(w_J s) - l(x) < l(w_J)$, the claim follows by induction. □

Lemma I.3.10. *Let $w \in W$. There are $\hat{L} \subseteq L, \hat{u} \in {}^{\hat{L}}W^J$ and $\hat{v} \in W_{\hat{L}}$ such that*

(1) ${}^{\hat{u}}\delta(\hat{L}) = \hat{L}$,

(2) $w \longrightarrow_\delta \hat{v}\hat{u}$.

Proof. The proof is a slight refinement of the usual inductive Bédard argument, see for example ([PWZ1], Thm. 9.11.), where we additionally keep track of the length of w. There is a unique $u \in {}^L W^J$ with $w \in W_L u W_J$. By Lemma I.3.9 we find $v \in W_L$ such that $w \longrightarrow_\delta vu$.

Now we proceed by induction on the cardinality of L. If $L = {}^u J$, then $L = {}^u\delta(L)$ and hence the triple (L, u, v) satisfies the properties (1) and (2).

If $L \neq {}^u J$, consider $J_1 := L \cap {}^u J$, $L_1 := \delta^{-1}({}^{u^{-1}}L \cap J)$ and $\delta_1 := \mathrm{int}(u) \circ \delta \colon W_{L_1} \to W_{J_1}$. Applying the induction hypothesis to the coxeter group W_L, the isomorphism δ_1 (note that $L_1, J_1 \subseteq L$) and the element $v \in W_L$, we find $\hat{L} \subseteq L_1$, $\hat{u}_1 \in {}^{\hat{L}}W_L^{J_1}$ and $\hat{v} \in W_{\hat{L}}$ satisfying

(1)' ${}^{\hat{u}_1}\delta_1(\hat{L}) = \hat{L}$,

(2)' $v \longmapsto_{\delta_1} \hat{v}\hat{u}_1$.

Let $\hat{u} := \hat{u}_1 u$, then we have $\hat{u} \in {}^{\hat{L}}W$ since $u \in {}^L W$, and we have $\hat{u} \in W^J$ by Corollary I.3.8, so $\hat{u} \in {}^{\hat{L}}W^J$. Further ${}^{\hat{u}}\delta(\hat{L}) = {}^{\hat{u}_1}\delta_1(\hat{L}) = \hat{L}$ by construction. Since by Proposition I.3.7(i) the conjugation map by u gives a bijection ${}^{u^{-1}}L \cap J \xrightarrow{\sim} L \cap {}^u J$, property (2)' implies that

$$vu \longrightarrow_\delta \hat{v}\hat{u}_1 u = \hat{v}\hat{u}.$$

The triple $(\hat{u}, \hat{L}, \hat{v})$ hence satisfies properties (1) and (2) with respect to w and δ. $\square$

I.3.2.2 Structure of Weyl representatives for EO-classes

Now we give a description of the $\widetilde{W}$-representatives of EO-classes. For such a class $\xi \subseteq G(L)$ let us write

$$\Lambda(\xi) := \{\tilde{w} \in \widetilde{W} \mid \tilde{w} \in \xi\}.$$

Consider the σ-conjugation action of W on $\widetilde{W}$: In analogy to the last paragraph we write $\tilde{w} \longrightarrow_\sigma \tilde{w}'$ for $\tilde{w}, \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0, \ldots, \tilde{w}_k = \tilde{w}'$ such that $\tilde{w}_i = s_i \tilde{w}_{i-1}\sigma(s_i)$ for some $s_i \in S$ and $l(\tilde{w}_i) \leq l(\tilde{w}_{i-1})$ for each $i = 1, \ldots, k$.

Proposition I.3.11. *Let $\xi \subseteq G(L)$ be an EO-class, let $\tilde{w}_\xi = \tau_\mu w$ for $\mu \in X_*(T)$ dominant and $w \in W^{J_\mu}$. Set $L_\mu := S_\mu^{\mathrm{opp}} = w_0 S_\mu w_0$ and*

$$\hat{L}_\xi := \bigcap_{n \geq 0} \varphi^n(L_\mu) \subseteq L_\mu \text{ for } \varphi := \mathrm{int}(w) \circ \sigma \circ \mathrm{int}(x_\mu).$$

(i) $\Lambda(\xi)$ consists of the W-σ-conjugacy classes of the elements $\tau_\mu \hat{v} w$ for $\hat{v} \in W_{\hat{L}_\xi}$, and for every $\tilde{w} \in \Lambda(\xi)$ there is $\hat{v} \in W_{\hat{L}_\xi}$ such that $\tilde{w} \longrightarrow_\sigma \tau_\mu \hat{v} w$.

(ii) All elements in $\Lambda(\xi)$ are K-σ-conjugate, and $\tilde{w}_\xi$ is an element of minimal length in $\Lambda(\xi)$.

Remark I.3.12. $\hat{L}_\xi$ is the maximal subset $\hat{L} \subseteq L_\mu$ with the property that ${}^w\sigma({}^{x_\mu}\hat{L}) = \hat{L}$. Since $\sigma({}^{x_\mu}\hat{L}_\xi) \subseteq J_\mu$, we have $w \in {}^{\hat{L}_\xi}W^{J_\mu}$.

The proof of the proposition uses the following consequence of Lang's Theorem:

Lemma I.3.13. *Let $\mu \in X_*(T)$ be dominant. Let $\hat{L} \subseteq S_\mu^{\mathrm{opp}}$ and $u \in {}^{\hat{L}}W^{J_\mu}$ such that ${}^u\sigma({}^{x_\mu}\hat{L}) = \hat{L}$. Then every element of the form $\tau_\mu \hat{m} u$ for $\hat{m} \in M_{\hat{L}}(\mathcal{O})$ is σ-conjugate to $\tau_\mu u$ by an element of $M_{S_\mu}(\mathcal{O})$.*

Proof. Recall that $x_\mu = w_{0,S_\mu} w_0$ and $\tau_\mu = \epsilon^\mu x_\mu$. The condition on $\hat{L}$ implies that $M_{\hat{L}} = u\sigma(x_\mu M_{\hat{L}} x_\mu^{-1})u^{-1}$. Therefore Lang's Theorem in the form of Corollary I.2.6 applied to $\hat{m} \in M_{\hat{L}}(\mathcal{O})$ and $g := u\sigma(x_\mu)$ shows that there is an m' in $M_{\hat{L}}(\mathcal{O})$ such that

$$\hat{m} = \sigma^{-1}(g^{-1}(m')^{-1}g)m' = x_\mu^{-1}\sigma^{-1}(u^{-1}(m')^{-1}u)x_\mu m'.$$

Let $\tilde{m} := \sigma^{-1}(u^{-1}m'u)$. Since $\sigma^{-1}({}^{u^{-1}}\hat{L}) = {}^{x_\mu}\hat{L} \subseteq {}^{x_\mu}(S_\mu^{\mathrm{opp}}) = S_\mu$, we have $\tilde{m} \in M_{S_\mu}(\mathcal{O})$, so $\tilde{m}$ commutes with ϵ^μ and we find that

$$\tilde{m}\tau_\mu\hat{m}u\sigma(\tilde{m})^{-1} = \tau_\mu u.$$

□

Proof of I.3.11. We show (i) and (ii) simultanously. Let $\hat{v} \in \hat{L}_\xi$, then by Lemma I.3.13 and Remark I.3.12 the element $\tau_\mu\hat{v}w$ is K-σ-conjugate to $\tau_\mu w$, in particular all its W-σ-conjugates lie in $[[\tau_\mu w]] = \xi$. On the other hand, let $\tilde{w} \in \Lambda(\xi)$. Then by the Cartan decomposition (4) we must have $\tilde{w} \in W\epsilon^\mu W = W\tau_\mu W$. We apply the results of the preceeding subsection to $\tilde{w}$:

Let ω be the unique element in Ω such that $\tau_\mu \in W_a \cdot \omega$. Write $S' := {}^\omega S \subset S_a$ and $W' := W_{S'} \subset W_a$, and let $\tau \in W_a$ such that $\tau_\mu = \tau\omega$. Then we have $\tilde{w} = w_a\omega$ for $w_a \in W\tau W'$, and $\tau \in {}^S W_a^{S'}$. Let

$$\delta_1 := \mathrm{int}(\omega) \circ \sigma \colon S \xrightarrow{\sim} S',$$

then by Lemma I.3.9 there is a $v' \in W'$ such that $w_a \longrightarrow_{\delta_1} \tau v'$. This implies that we have $\tilde{w} \longrightarrow_\sigma \tau_\mu v$ for $v := \omega^{-1}v'\omega \in W$.

Now we apply Proposition I.3.10 to $v \in W$ and the isomorphism

$$\delta_2 := \sigma \circ \mathrm{int}(x_\mu) \colon L_\mu \xrightarrow{\sim} J_\mu :$$

There are $\hat{L} \subseteq L_\mu$, $\hat{u} \in {}^{\hat{L}}W^{J_\mu}$ and $\hat{v}$ in $W_{\hat{L}}$ such that $v \longrightarrow_{\delta_2} \hat{v}\hat{u}$ and ${}^{\hat{u}}\delta_2(\hat{L}) = \hat{L}$. Since we have $l(\tau_\mu w') = l(\tau_\mu) + l(w')$ for any $w' \in W$, and every element of W_{S_μ} commutes with ϵ^μ, the first property implies that $\tau_\mu v \longrightarrow_\sigma \tau_\mu\hat{v}\hat{u}$. As $\hat{u} \in {}^{\hat{L}}W^{J_\mu}$, Lemma I.3.13 shows that the element $\tau_\mu\hat{v}\hat{u}$ is further K-σ-conjugate to $\tau_\mu\hat{u}$.

So $\tilde{w}$ is K-σ-conjugate to $\tau_\mu\hat{u}$, in particular $\tilde{w} \in [[\tau_\mu\hat{u}]]$. By the classification of $[[\cdot]]$-classes in Proposition I.2.17 we must therefore have $\hat{u} = w$. Going through the argument again we have shown that there are $\hat{L} \subseteq L_\mu$ with ${}^w\sigma({}^{x_\mu}\hat{L}) = \hat{L}$ and $\hat{v} \in W_{\hat{L}}$ such that $w \in {}^{\hat{L}}W^{J_\mu}$ and $\tilde{w} \longrightarrow_\sigma \tau_\mu\hat{v}w$, and that $\tilde{w}$ is K-σ-conjugate to $\tau_\mu w$. These properties finally imply that $\hat{L} \subseteq \hat{L}_\xi$ and that $l(\tilde{w}) \geq l(\tau_\mu\hat{v}w) = l(\tau_\mu) + l(\hat{v}w) \geq l(\tau_\mu) + l(w) = l(\tau_\mu w)$. □

Corollary I.3.14. *Let $\xi \in \mathrm{EO}(G)$, let $\tilde{w} \in \Lambda(\xi)$ be an element of minimal length. Then we have $K \cdot (I\tilde{w}I) = K \cdot (I\tilde{w}_\xi I)$.*

Proof. This is a variant of ([He2], Lemma 3.1.). By Proposition I.3.11(i), there is a $\hat{v} \in W_{\hat{L}_\xi}$ such that $\tilde{w} \longrightarrow_\sigma \hat{v}\tilde{w}_\xi$. Since by assumption $l(\tilde{w}) = l(\tilde{w}_\xi)$, we must have $\hat{v} = 1$, hence $\tilde{w} \longrightarrow_\sigma \tilde{w}_\xi$, and consequently there is a sequence $\tilde{w} = \tilde{w}_0, \dots, \tilde{w}_k = \tilde{w}_\xi$ such that in each step we have $\tilde{w}_i = s_i\tilde{w}_{i-1}\sigma(s_i)$ for some $s_i \in S$ and $l(\tilde{w}_i) = l(\tilde{w}_{i-1})$. Now the arguments in the proof of Lemma 3.1. in [He2] show that $K \cdot_\sigma (I\tilde{w}_iI) = K \cdot_\sigma (I\tilde{w}_{i-1}I)$ for each $i = 1, \dots, k$. □

I.3.2.3 The map Rep

As a consequence of Proposition I.3.11, the map $\widetilde{W} \to B(G)$, $\tilde{w} \mapsto [\tilde{w}]$ factors via $\mathrm{EO}(G)$, giving a map

$$\mathrm{Rep}\colon\ \mathrm{EO}(G) \longrightarrow B(G), \quad [[\tilde{w}]] \longmapsto [\tilde{w}], \tag{15}$$

which sends $\xi \in \mathrm{EO}(G)$ to the common σ-conjugacy class of its representatives in $\widetilde{W}$. Combining the Proposition I.3.11 with Viehmann's result from Proposition I.3.5, one may conclude the following corollary, which gives another necessary criterion for the nonemptyness of intersections between EO-classes and σ-conjugacy classes:

Corollary I.3.15.

(i) Let $\xi \in \mathrm{EO}(G)$. If $b \in B(G)$ with $\xi \cap b \neq \emptyset$, then there is some $\xi' \preceq \xi$ such that $\mathrm{Rep}(\xi') = b$.

(ii) (compare [Vi1], Cor. 5.6.) For $\xi \in \mathrm{EO}(G)$ the generic σ-conjugacy class in ξ is the unique maximal element in $\{\mathrm{Rep}(\xi') \mid \xi' \preceq \xi\}$.

Proof. Since all involved properties are independent of the chararcteristic of L (see for example Sections I.2.4, I.2.5, I.3.1.3), we may argue in the equicharacteristic case.
(i) If $\xi \cap b \neq \emptyset$, then $I\tilde{w}_\xi I \cap b \neq \emptyset$ by (13), so by Proposition I.3.5 there is some $\tilde{w}' \in b$ with $\tilde{w}' \leq \tilde{w}_\xi$, which means that $\tilde{w}' \in \overline{I\tilde{w}_\xi I}$. Let $\xi' := [[\tilde{w}']]$, then $\mathrm{Rep}(\xi') = b$ by Proposition I.3.11. Since $I\tilde{w}_\xi I \subseteq \xi$, we have $\tilde{w}' \in \overline{I\tilde{w}_\xi I} \subseteq \overline{\xi}$ and therefore $\xi' \cap \overline{\xi} \neq \emptyset$, which implies that $\xi' \subseteq \overline{\xi}$.
(ii) follows from (i), since the generic σ-conjugacy class in ξ is equal to the generic σ-conjugacy class in $\overline{\xi}$. ☐

For each $\mu \in X_*(T)_{\mathrm{dom}}$ the restriction of (15) gives a map

$$\mathrm{Rep}\colon\ \mathrm{EO}(G,\mu) \longrightarrow B(G,\mu). \tag{16}$$

Remark I.3.16. It is not clear whether this map allows a geometric interpretation:

(1) In general, the map $\mathrm{Rep}\colon \mathrm{EO}(G,\mu) \to B(G,\mu)$ is not injective and not order preserving (see Example I.3.6.2 below).

(2) In general, $\mathrm{Rep}\colon \mathrm{EO}(G,\mu) \to B(G,\mu)$ is not surjective: For example, consider $G = \mathrm{GL}_{2,\mathcal{O}_F}$, let $T \subseteq B \subseteq G$ be the diagonal matrices and upper triangular matrices. Let $\mu := (3,0) \in \mathbb{Z}^2 \cong X_*(T)$, i.e. $\epsilon^\mu = \begin{pmatrix} \epsilon^3 & 0 \\ 0 & 1 \end{pmatrix}$. Then under the identification between $B(G)$ and Newton polygons (see Example I.2.10), we have

$$B(G,\mu) = \{(3,0),\ (2,1),\ ((3/2)^2)\},$$

on the other hand we have $J_\mu = S_\mu = \emptyset$, so $|\,\mathrm{EO}(G,\mu)| = |W| = 2$. We will revisit this example in Section I.3.6.1.

(3) In general, $\mathrm{Rep}(\xi)$ is not the generic σ-conjugacy class in ξ. In fact, we will see in Section I.4.2.2 that there may exist $b_1, b_2 \in B(G)$, different from $\mathrm{Rep}(\xi)$ such that $b_1 \preceq \mathrm{Rep}(\xi) \preceq b_2$ and $\xi \cap b_1 \neq \emptyset \neq \xi \cap b_2$.

I.3.2.4 The minuscule case

Recall that a cocharacter μ is called *minuscule* if and only if $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$. The study of Problem I.3.2 for minuscule $\mu \in X_*(T)_{\mathrm{dom}}$ is of special relevance, as a μ which arises from a Shimura datum will always be minuscule (see Section II.3).

If $\mu \in X_*(T)$ is dominant and minuscule, then the element $\tau_\mu = \epsilon^\mu x_\mu$ lies in Ω: By definition, $\langle \alpha, \mu \rangle \in \{0, 1\}$ for each $\alpha \in \Phi^+$, and $x_\mu = w_0 w_{0,J_\mu}$ sends $\alpha \in \Phi^+$ to Φ^- if and only if $\langle \alpha, \mu \rangle = 1$, so $l(\tau_\mu) = 0$ by (1). This has the following consequences:

(a) If $\tilde{w} \in W\epsilon^\mu W$ and $\tilde{w}' \leq \tilde{w}$, then $\tilde{w}' \in W\epsilon^\mu W$.
Indeed, writing $\tilde{w} = w_1 \tau_\mu w_2$ for $w_1, w_2 \in W$ such that $l(w_1) + l(w_2) = l(\tilde{w})$, we must have $\tilde{w}' = w_1' \tau_\mu w_2'$ for $w_1' \leq w_1$ and $w_2' \leq w_2$.

(b) Let $\xi \in \mathrm{EO}(G, \mu)$. If $\xi' \preceq \xi$, then $\xi' \in \mathrm{EO}(G, \mu)$.
This follows from (a) and the description of $\preceq$ for standard representatives in (9).

(c) The map $\mathrm{Rep}\colon \mathrm{EO}(G, \mu) \to B(G, \mu)$ is *surjective*:
For every $b \in B(G, \mu)$ there is $\xi \in \mathrm{EO}(G, \mu)$ with $\xi \cap b \neq \emptyset$. By Corollary I.3.15(i), there is $\xi' \preceq \xi$ with $\mathrm{Rep}(\xi') = b$. By (b), we have $\xi' \in \mathrm{EO}(G, \mu)$.

(d) Corollary I.3.15 takes on the form of Criterion I.A of the introduction to this chapter:
Let $\xi \in \mathrm{EO}(G, \mu)$. If $b \in B(G, \mu)$ such that $\xi \cap b \neq \emptyset$, then there is $\xi' \in \mathrm{EO}(G, \mu)$ with $\xi' \preceq \xi$ and $\mathrm{Rep}(\xi') = b$. The generic σ-conjugacy class in ξ is the maximal element in $\{\mathrm{Rep}(\xi') \mid \xi' \in \mathrm{EO}(G, \mu), \xi' \preceq \xi\}$.

I.3.3 Fundamental classes

We now consider the following aspects of Problem I.3.2:

Question I.3.17. Let $b \in B(G, \mu)$.

(1) Is there a $\xi \in \mathrm{EO}(G, \mu)$ such that $\xi \subseteq b$?

(2) Is there a $\xi \in \mathrm{EO}(G, \mu)$ such that $\xi \subseteq b$ and $\xi = K \cdot_\sigma \tilde{w}_\xi$?

We have seen in the Remark I.3.16 that one may not expect to give a positive answer to these questions in general. On the other hand, these properties are known to be true if μ is minuscule and G is split, and one may hope that they also hold for general G if μ is minuscule (cf. the remarks at the end of Section I.3.4).

If the pair (G, μ) arises from a Shimura variety as in Chapter II, then (1) corresponds to the question whether every Newton stratum contains an Ekedahl-Oort stratum, and (2) asks whether there exists such a stratum which in addition equals a single leaf in the variety.

Definition I.3.18.

(a) We say that $\xi \in \mathrm{EO}(G)$ is *fundamental* if $\xi \subseteq \mathrm{Rep}(\xi) = [\tilde{w}_\xi]$.

(b) We say that $\xi \in \mathrm{EO}(G)$ is *K-fundamental* if $\xi = K \cdot_\sigma \tilde{w}_\xi$.

(c) We say that $\tilde{w} \in \widetilde{W}$ is *I-fundamental* if $I\tilde{w}I = I \cdot_\sigma \tilde{w}$.

With these notions, the questions (1) and (2) ask for the existence of a fundamental resp. K-fundamental class in $K\epsilon^\mu K \cap b$. Of course, if a class ξ is K-fundamental then it is fundamental. There is also the following sufficient criterion:

Criterion I.3.19. *If $\xi \in \mathrm{EO}(G)$ contains an element of $\widetilde{W}$ which is of minimal length in its $\widetilde{W}$-σ-conjugacy class, then ξ is fundamental.*

This follows from the fact that by ([He2], Thm. 3.3.), if an element $\tilde{w} \in \widetilde{W}$ is of minimal length in its $\widetilde{W}$-σ-conjugacy class, then $I\tilde{w}I \subseteq [\tilde{w}]$.

The existence of K-fundamental classes is usually derived from the existence of I-fundamental elements. If $\xi \in \mathrm{EO}(G)$ contains an I-fundamental element of $\widetilde{W}$ then it is K-fundamental: Let $\tilde{w} \in \Lambda(\xi)$ be this element, then

$$\xi = [[\tilde{w}]] = K \cdot_\sigma (K_1 \tilde{w} K_1) \subseteq K \cdot_\sigma (I\tilde{w}I) = K \cdot_\sigma (I \cdot_\sigma \tilde{w}).$$

I-fundamental elements have been discussed from different points of view in the literature, which will turn out to be essentially equivalent.

Straight elements Let $\tilde{w} \in \widetilde{W}$, let $b = [\tilde{w}]$, then the conjugacy class b determines a lower bound for the length of $\tilde{w}$: Let ν_b be the dominant Newton vector of b. Recall that in Definition I.2.11 we defined $\nu_{\tilde{w}} := \frac{1}{r}\tilde{w}\sigma(\tilde{w})\cdots\sigma^{r-1}(\tilde{w})$, for a certain $r \in \mathbb{N}$, and that ν_b is the dominant element in $W \cdot \nu_{\tilde{w}}$. Since $l(\sigma^i(\tilde{w})) = l(\tilde{w})$ for all i and the length function on $\widetilde{W}$ is subadditive, we have $rl(\tilde{w}) \geq l(\tilde{w}\sigma(\tilde{w})\cdots\sigma^{r-1}(\tilde{w})) = l(r \cdot \nu_{\tilde{w}})$. Therefore, denoting as usual by 2ρ the sum of all positive roots in Φ,

$$l(\tilde{w}) \geq \frac{1}{r} l(r \cdot \nu_{\tilde{w}}) = \frac{1}{r} l(r \cdot \nu_b) = \frac{1}{r}\langle 2\rho, r \cdot \nu_b\rangle = \langle 2\rho, \nu_b\rangle.$$

Definition I.3.20. $\tilde{w} \in \widetilde{W}$ is called *straight* if equality in the upper equation holds, that is, if $l(\tilde{w}) = \langle 2\rho, \nu_b\rangle$, where 2ρ is the sum over all elements in Φ^+. In this case we will call $\tilde{w}$ a *straight representative* of $b = [\tilde{w}]$.

It is clear that $\tilde{w}$ is straight if and only if $l(\tilde{w}\sigma(\tilde{w})\cdots\sigma^{k-1}(\tilde{w})) = kl(\tilde{w})$ for all $k \in \mathbb{N}$.

Example I.3.21.

(1) Every $\omega \in \Omega$ is straight: σ acts on Ω, so $\omega\sigma(\omega)\cdots\sigma^{k-1}(\omega) \in \Omega$ and hence $l(\omega\sigma(\omega)\cdots\sigma^{k-1}(\omega)) = 0 = kl(\omega)$ for every $k \in \mathbb{N}$.

(2) For every $\mu \in X_*(T)_{\mathrm{dom}}$ the element ϵ^μ is straight: σ acts on $X_*(T)_{\mathrm{dom}}$, so the length formula (1) shows that

$$l(\epsilon^\mu\sigma(\epsilon^\mu)\cdots\sigma^{k-1}(\epsilon^\mu)) = l(\epsilon^{\mu+\sigma(\mu)+\cdots+\sigma^{k-1}(\mu)}) = \sum_{i=0}^{k-1} l(\epsilon^{\sigma^i(\mu)}) = kl(\epsilon^\mu)$$

for all $k \in \mathbb{N}$. On the other hand, if μ is not dominant and G is not split over $\mathcal{O}_F$, then the element ϵ^μ is not straight in general.

By ([He2], Thm. 3.5.) straight representatives always exist, more precisely, for every $b \in B(G)$ there exists a *unique* $\widetilde{W}$-conjugacy class in $\widetilde{W}$ which contains straight representatives of b. Further we have:

Proposition I.3.22 ([He2], Prop. 4.5.)**.** *If $\tilde{w} \in \widetilde{W}$ is straight, then $\tilde{w}$ is I-fundamental.*

P-fundamental elements In [GHKR] the authors introduced the concept of a fundamental P-alcove in the case that G is a split group, which has since been generalized in two ways:

Let $P \subseteq G$ be a semistandard parabolic subgroup over $\mathcal{O}$. Recall that this means that P contains T, but not necessarily B. Let M be its unique Levi subgroup which contains T, N its unipotent radical and $\overline{N}$ the unipotent radical of the parabolic opposite to P (with respect to T). Set

$$I_N := N(L) \cap I, \quad I_M := M(L) \cap I, \quad I_{\overline{N}} := \overline{N}(L) \cap I.$$

Then an element $\tilde{w} \in \widetilde{W}$ is called *P-fundamental* (see [VW], Def. 9.7. resp. [Vi1], Def. 6.1.), if one has

$$\sigma({}^{\tilde{w}}I_N) \subseteq I_N, \quad \sigma({}^{\tilde{w}}I_{\overline{N}}) \supseteq I_{\overline{N}}, \quad \sigma({}^{\tilde{w}}I_M) = I_M.$$

If $\tilde{w}$ is P-fundamental for some semistandard parabolic P, then it is a straight element. This can be proved by a generalization of the argument in ([GHKR], 13.1.3.), as is for example done in ([Ha], §7.3.). By Proposition I.3.22, $\tilde{w}$ is then I-fundamental, which is also shown in ([Vi1] 6.3.). So the concept of a P-fundamental element is a priori more restrictive than that of a straight element.

In [GHN] the authors give the definition (J, w, σ)-alcoves in the case of a tamely ramified group by means of the action of $\widetilde{W}$ on the apartment $V = X_*(T)_{\mathbb{R}}$, which allows to prove variants of a Hodge-Newton decomposition for affine Deligne-Lusztig varieties (see [GHN], Thm. 3.3.1.). We will only be interested in the corresponding fundamental elements: Recall that we denote by $\mathfrak{a}_0 \subseteq V$ the unique alcove in the dominant Weyl chamber whose closure contains $0 \in V$.

Definition I.3.23 (cf. [GHN], §3)**.**

(a) Let $\mathfrak{a} \subseteq V$ be any alcove and $\alpha \in \Phi$, then we define $k(\alpha, \mathfrak{a}) \in \mathbb{Z}$ as the unique integer k such that $\langle \alpha, \mathfrak{a} \rangle = (k-1, k)$.

(b) Let $J \subseteq S$ such that $\sigma(J) = J$, let $y \in W$. We call $\tilde{w} \in \widetilde{W}$ a *(J, y, σ)-fundamental* element if the following conditions hold:

  (I) $y^{-1}\tilde{w}\sigma(y) \in \widetilde{W}_J$,

  (II) $k(\alpha, \tilde{w}\mathfrak{a}_0) \geq k(\alpha, \mathfrak{a}_0)$ for all $\alpha \in y(\Phi^+ \setminus \Phi_J^+)$,

  (III) $k(\alpha, \tilde{w}\mathfrak{a}_0) = k(\alpha, \mathfrak{a}_0)$ for all $\alpha \in y(\Phi_J^+)$ or equivalently, for all $\alpha \in y(\Phi_J)$.

Remark I.3.24. We can put part (a) of the definition equivalently as follows: Let $\tilde{w} \in \widetilde{W}$ such that $\mathfrak{a} = \tilde{w}\mathfrak{a}_0$, let $U_\alpha \colon \mathbb{G}_{\mathcal{O}} \to G_{\mathcal{O}}$ be the root group of $\alpha \in \Phi$. Then $k(\alpha, \mathfrak{a})$ is determined by the relation

$$U_\alpha(L) \cap {}^{\tilde{w}}I = U_\alpha(\epsilon^{k(\alpha,\mathfrak{a})}\mathcal{O}). \tag{17}$$

For example, we have $k(\alpha, \mathfrak{a}_0) = 1$ for $\alpha \in \Phi^+$ and $k(\alpha, \mathfrak{a}_0) = 0$ for $\alpha \in \Phi^-$.

Lemma I.3.25. *Let $J \subseteq S$ with $\sigma(J) = J$, let $y \in W$ let $\tilde{w} \in \widetilde{W}$ be a (J, y, σ)-fundamental element. Let P_J be the standard parabolic subgroup associated to J, let $P := {}^{\sigma(y)}P_J$. Then $\tilde{w}$ is P-fundamental in the sense explained above. In particular, (J, y, σ)-fundamental elements are a special case of P-fundamental elements.*

Proof. Let M_J be the standard Levi subgroup of P_J, N_J its unipotent radical, let $\overline{N}_J$ be the unipotent radical of the opposite parabolic to P_J. Then $M = {}^{\sigma(y)}M_J$, $N = {}^{\sigma(y)}N_J$, $\overline{N} = {}^{\sigma(y)}\overline{N}_J$ are the Levi subgroup and unipotent radicals associated to P repectively, so their intersections with I are given as

$$I_N = {}^{\sigma(y)}N_J(L) \cap I, \quad I_M = {}^{\sigma(y)}M_J(L) \cap I, \quad I_{\overline{N}} = {}^{\sigma(y)}\overline{N}_J(L) \cap I.$$

Consider the root subgroups U_α for $\alpha \in \Phi$. By Section I.2.4 we have the product decompositions

$$\begin{aligned} {}^yN_J(L) \cap I &= \prod_{\alpha \in y(\Phi^+ \setminus \Phi_J^+)} (U_\alpha(L) \cap I), \\ {}^y\overline{N}_J(L) \cap I &= \prod_{\alpha \in y(\Phi^- \setminus \Phi_J^-)} (U_\alpha(L) \cap I), \\ {}^yM_J(L) \cap I &= T(\mathcal{O}) \cdot \prod_{\alpha \in y(\Phi_J)} (U_\alpha(L) \cap I), \end{aligned}$$

The analogous decompositions also hold for the Iwahori subgroup ${}^{\tilde{w}}I \subseteq G(L)$, so for example ${}^yN(L) \cap {}^{\tilde{w}}I = \prod_{\alpha \in y(\Phi^+ \setminus \Phi_J^+)}(U_\alpha(L) \cap {}^{\tilde{w}}I)$. Now in view of Remark I.3.24 the conditions (II) and (III) of Definition I.3.23(ii) are equivalent to

$$\begin{aligned} U_\alpha(L) \cap {}^{\tilde{w}}I &\subseteq U_\alpha(L) \cap I, \quad \alpha \in y(\Phi^+ \setminus \Phi_J^+), \\ U_\alpha(L) \cap {}^{\tilde{w}}I &= U_\alpha(L) \cap I, \quad \alpha \in y(\Phi_J), \end{aligned}$$

and they imply that $U_\alpha(L) \cap {}^{\tilde{w}}I \supseteq U_\alpha(L) \cap I$ for all $\alpha \in y(\Phi^- \setminus \Phi_J^-)$. By condition (I), the element $y^{-1}\tilde{w}\sigma(y) \in \widetilde{W}_J$ normalizes the groups $N_J, \overline{N}_J$ and M_J, so the relations above imply that

$$\sigma({}^{\tilde{w}}I_N) = \sigma({}^{\tilde{w}\sigma(y)}N_J(L) \cap {}^{\tilde{w}}I) = \sigma({}^yN_J(L) \cap {}^{\tilde{w}}I) \subseteq \sigma({}^yN_J(L) \cap I) = I_N,$$

and in the same manner we see that $\sigma({}^{\tilde{w}}I_{\overline{N}}) \supseteq I_{\overline{N}}$ and $\sigma({}^{\tilde{w}}I_M) = I_M$. ☐

As a consequence, every (J, y, σ)-fundamental element is straight, in particular it is I-fundamental, which was already shown in ([GHN], Prop. 3.4.3.). We sill see in the next subsection that conversely every straight element is (J, y, σ)-fundamental for some J and y.

I.3.4 Balanced short elements

Let $\mu \in X_*(T)_{\mathrm{dom}}$, let $b \in B(G, \mu)$. We have seen that the existence of an I-fundamental element $\tilde{w} \in W\epsilon^\mu W \cap b$ would be sufficient to give a positive answer to Question I.3.17(2) of the last subsection, and that an element $\tilde{w}$ is I-fundamental if it is straight. We will now study the precise conditions for the existence of a straight element in $W\epsilon^\mu W \cap b$ and give a criterion in terms of b-short representatives.

Let $J \subseteq S$ be any subset. We can give a precise description of elements in Ω_J: Let $\lambda \in X_*(T)$, then we say that λ is *J-dominant* if $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Phi_J^+$, and that λ is *J-minuscule* if $|\langle \alpha, \lambda \rangle| \leq 1$ for all $\alpha \in \Phi_J$. For a J-dominant cocharacter $\lambda \in X_*(T)$ set $J_\lambda := \{s \in J \mid s(\lambda) = \lambda\}$ (this is not to be confused with the subset

$J_\mu = \sigma(S_\mu)$ from Definition I.2.17!). Then it follows from the length formula (1), applied to the group $\widetilde{W}_J = X_*(T) \rtimes W_J$, that for an element $\epsilon^\lambda w \in \widetilde{W}_J$ we have

$$\epsilon^\lambda w \in \Omega_J \iff \lambda \text{ is } J\text{-dominant and } J\text{-minuscule, and } w = w_{0,J_\lambda} w_{0,J}. \tag{18}$$

Hence the elements of Ω_J are in $1-1$-correspondence to the set of J-dominant and J-minuscule cocharacters.

If $\omega \in \widetilde{W}$ is a b-short element for some $b \in B(G)$, then $\omega \in \Omega_J$ is given as in (18) for $J = J_b$ (where J_b is the type of b as defined in I.2.9). In this case we have $\sigma(J) = J$, and the vector ν_ω of Definition I.2.11 is equal to the dominant Newton vector ν_b of b, which means that $\nu_b = \frac{1}{r} \cdot (\omega\sigma(\omega)\cdots\sigma^{r-1}(\omega))$ for some $r \in \mathbb{N}$.

Definition I.3.26. Let $J \subseteq S$ such that $\sigma(J) = J$. Let $\omega \in \Omega_J$ be b-short for some $b \in B(G)$ of type $J_b = J$. Write $\omega = \epsilon^\lambda w$ for $w \in W_J$. Consider $\delta := w \circ \sigma \in \widetilde{W} \rtimes \langle\sigma\rangle$ as a map on $X_*(T)$.
We say that ω is *σ-balanced*, if $\langle \alpha, \lambda + \delta(\lambda) + \cdots \delta^i(\lambda)\rangle \geq -1$ for all $i \geq 0$ and all $\alpha \in \Phi^+ \setminus \Phi_J^+$.

Remark I.3.27.

(1) The element $\delta = w \circ \sigma$ can also be considered as a bijection of Φ, it restricts to a bijection of $\Phi^+ \setminus \Phi_J^+$ as $w \in W_J$ and $\sigma(J) = J$. For $\alpha \in \Phi^+ \setminus \Phi_J^+$ and $\lambda \in X_*(T)$ we have $\langle \alpha, \delta(\lambda)\rangle = \langle \delta^{-1}(\alpha), \lambda\rangle$.

(2) Let $\omega = \epsilon^\lambda w \in \Omega_J$ and δ as in the above definition. Then for all $i \geq 0$ the translation part of

$$\omega\sigma(\omega)\cdots\sigma^i(\omega) = \epsilon^\lambda w \sigma(\epsilon^\lambda w)\cdots\sigma^i(\epsilon^\lambda w)$$

is exactly $\lambda + \delta(\lambda) + \cdots + \delta^i(\lambda) \in X_*(T)$. As the element $\omega\sigma(\omega)\cdots\sigma^i(\omega)$ lies again in Ω_J, by (18) this cocharacter is J-dominant for all $i \geq 0$. Thus ω is σ-balanced if and only if $\langle \alpha, \lambda + \delta(\lambda) + \cdots \delta^i(\lambda)\rangle \geq -1$ for all $i \geq 0$ and all $\alpha \in \Phi^+$, which is a condition only for $\alpha \in \Phi^+ \setminus \Phi_J^+$.
Further, there is some $n \in \mathbb{N}$ such that $\delta^n = \mathrm{id}$ and

$$n \cdot \nu_b = \omega\sigma(\omega)\cdots\sigma^{n-1}(\omega) = \lambda + \delta(\lambda) + \cdots + \delta^{n-1}(\lambda) \in X_*(T).$$

Since $\langle \alpha, n \cdot \nu_b\rangle > 0$ for all $\alpha \in \Phi^+ \setminus \Phi_J^+$ by definition of J, in this situation the condition of the definition needs only to be checked for $i \leq n-1$.

The starting point for our criterion is the following observation.

Lemma I.3.28. *Let $\mu \in X_*(T)_{\mathrm{dom}}$, let $b \in B(G,\mu)$. If $\tilde{w} \in W\epsilon^\mu W$ is a straight representative of b, then there exist a b-short element $\omega_b \in W\epsilon^\mu W$ and a $y \in W^{J_b}$ such that $\tilde{w} = y\omega_b\sigma(y)^{-1}$. Further, in this case the standard representative of $[[\tilde{w}]]$ is a straight representative of b.*

Proof. Since $\tilde{w} \in b$, in particular $b \cap I\tilde{w}I \neq \emptyset$. By Proposition I.3.5, there are a b-short element ω_b and an element $y \in W^{J_b}$ such that $y\omega_b\sigma(y)^{-1} \leq \tilde{w}$ (take y as the inverse of the element in ${}^{J_b}W$ given by I.3.5). Since $\tilde{w}$ is of minimal length in $\widetilde{W} \cap b$, this implies that $y\omega_b\sigma(y)^{-1} = \tilde{w}$, so in particular $\omega_b \in W\epsilon^\mu W$.

Let $\xi = [[\tilde{w}]]$. Then by Proposition I.3.11 we have $\tilde{w}_\xi \in b$ and $l(\tilde{w}_\xi) \leq l(\tilde{w})$. Again, by the minimality of $l(\tilde{w})$ this implies that $l(\tilde{w}_\xi) = l(\tilde{w})$, so $\tilde{w}_\xi$ is straight. $\square$

Let us investigate the properties which ensure that conversely a b-short element is conjugate to a straight element as in Lemma I.3.28:

Proposition I.3.29. *Let $b \in B(G)$, write $J := J_b \subseteq S$ for ist type. Let $\omega \in \Omega_J$ be a b-short element, $\omega = \epsilon^\lambda w$ for $w \in W_J$, and let $y \in W^J$. Then $y\omega\sigma(y)^{-1}$ is straight if and only if for all $\alpha \in \Phi^+ \setminus \Phi_J^+$ the following three conditions are satisfied:*

$$\langle \alpha, \lambda \rangle < 0 \implies \langle \alpha, \lambda \rangle = -1 \tag{C0}$$

$$\langle \alpha, \lambda \rangle = -1 \implies y(\alpha) \in \Phi^- \text{ and } \sigma(y)w^{-1}(\alpha) \in \Phi^+ \tag{C1}$$

$$y(\alpha) \in \Phi^+ \text{ and } \sigma(y)w^{-1}(\alpha) \in \Phi^- \implies \langle \alpha, \lambda \rangle > 0 \tag{C2}$$

Proof. Recall the length formula (1). Since $\omega \in \Omega_J$, we have by definition $l_J(\omega) = 0$. The element $w \in W_J$ gives a bijection of $\Phi^+ \setminus \Phi_J^+$, so we have

$$l(\omega) = l_J(\omega) + \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} |\langle \alpha, \lambda \rangle| = \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} |\langle \alpha, \lambda \rangle|. \tag{19}$$

As explained in Remark I.3.27, we have

$$\nu_b = \nu_\omega = \frac{1}{r} \cdot \omega\sigma(\omega) \cdots \sigma^{r-1}(\omega) = \frac{1}{r} \cdot (\lambda + \delta(\lambda) + \cdots + \delta^{r-1}(\lambda))$$

for some $r \in \mathbb{N}$, where $\delta := w \circ \sigma$. Using the facts that $\langle \alpha, \nu_b \rangle = 0$ for all $\alpha \in \Phi_J$ by definition of $J = J_b$, and that δ induces a bijection on $\Phi^+ \setminus \Phi_J^+$, we find that the length of any straight representative of b is given by

$$\begin{aligned} \langle 2\rho, \nu_b \rangle &= \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} \langle \alpha, \nu_b \rangle = \frac{1}{r} \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} \left(\sum_{i=0}^{r-1} \langle \alpha, \delta^i(\lambda) \rangle \right) \\ &= \frac{1}{r} \sum_{i=0}^{r-1} \left(\sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} \langle \alpha, \delta^i(\lambda) \rangle \right) = \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} \langle \alpha, \lambda \rangle. \end{aligned} \tag{20}$$

Consider the "length improvement" $D := l(\omega) - l(y\omega\sigma(y)^{-1})$. The element $y\omega\sigma(y)^{-1}$ is straight if and only if $D = l(\omega) - \langle 2\rho, \nu_b \rangle$, and by (19) and (20) we have

$$l(\omega) - \langle 2\rho, \nu_b \rangle = \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} (|\langle \alpha, \lambda \rangle| - \langle \alpha, \lambda \rangle) = \sum_{\substack{\alpha \in \Phi^+ \setminus \Phi_J^+, \\ \langle \alpha, \lambda \rangle < 0}} 2|\langle \alpha, \lambda \rangle|. \tag{21}$$

We may describe D using the length formula: We have $y\omega\sigma(y)^{-1} = \epsilon^{y(\lambda)} y w \sigma(y)^{-1}$, therefore

$$\begin{aligned} l(y\omega\sigma(y)^{-1}) &= \sum_{\substack{\alpha \in \Phi^+, \\ \sigma(y)w^{-1}y^{-1}(\alpha) \in \Phi^+}} |\langle \alpha, y(\lambda) \rangle| + \sum_{\substack{\alpha \in \Phi^+, \\ \sigma(y)w^{-1}y^{-1}(\alpha) \in \Phi^-}} |\langle \alpha, y(\lambda) \rangle - 1| \\ &= \sum_{\substack{\beta \in y^{-1}(\Phi^+), \\ \sigma(y)w^{-1}(\beta) \in \Phi^+}} |\langle \beta, \lambda \rangle| + \sum_{\substack{\beta \in y^{-1}(\Phi^+), \\ \sigma(y)w^{-1}(\beta) \in \Phi^-}} |\langle \beta, \lambda \rangle - 1|. \end{aligned} \tag{22}$$

For $\beta \in y^{-1}(\Phi^+)$ let β^+ be the element of $\{\beta, -\beta\}$ which lies in Φ^+, so that $\Phi^+ = \{\beta^+ \mid \beta \in y^{-1}(\Phi^+)\}$. We distinguish 3 cases for β:

(1) $\beta \in \Phi_J$.
In this case, since $y^{-1} \in {}^J W$ we must have $\beta \in \Phi_J^+$, and since $\sigma(y) \in W^J$ we have $\sigma(y)w^{-1}(\beta) \in \Phi^-$ if and only $w^{-1}(\beta) \in \Phi_J^-$ (cf. Section I.2.3). So the contribution of β to (22) is equal to the contribution of β to $l_J(\omega)$ in (19), which is zero.

(2) $\beta \in \Phi \setminus \Phi_J$ and $\sigma(y)w^{-1}(\beta) \in \Phi^+$.
In this case, the contribution of β to (22) is equal to the contribution of β^+ to $l(\omega)$ in (19).

(3) $\beta \in \Phi \setminus \Phi_J$ and $\sigma(y)w^{-1}(\beta) \in \Phi^-$.
This happens if and only if the element β^+ lies in either of the following sets:

$$\begin{aligned}\Psi^{\pm} &:= \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y(\alpha) \in \Phi^+,\ \sigma(y)w^{-1}(\alpha) \in \Phi^-\},\\ \Psi^{\mp} &:= \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y(\alpha) \in \Phi^-,\ \sigma(y)w^{-1}(\alpha) \in \Phi^+\}.\end{aligned}$$

If $\beta \in \Phi^+ \setminus \Phi_J^+$, then $\beta^+ = \beta \in \Psi^{\pm}$ and the contribution of β to (22) is equal to $|\langle \beta^+, \lambda\rangle - 1|$. If $\beta \in \Phi^- \setminus \Phi_J^-$, then $\beta^+ = -\beta \in \Psi^{\mp}$ and the contribution of β to (22) is equal to $|\langle \beta^+, \lambda\rangle + 1|$.

Combining (1)-(3), and renaming β^+ by α, we see that

$$l(\omega) - l(y\omega\sigma(y)^{-1}) = \sum_{\alpha \in \Psi^{\pm}} (|\langle \alpha, \lambda\rangle| - |\langle \alpha, \lambda\rangle - 1|) + \sum_{\alpha \in \Psi^{\mp}} (|\langle \alpha, \lambda\rangle| - |\langle \alpha, \lambda\rangle + 1|).$$

Thus if we define

$$\Psi^{\pm}_{>0} := \{\alpha \in \Psi^{\pm} \mid \langle \alpha, \lambda\rangle > 0\}, \qquad \Psi^{\pm}_{\leq 0} := \{\alpha \in \Psi^{\pm} \mid \langle \alpha, \lambda\rangle \leq 0\},$$

and $\Psi^{\mp}_{\geq 0}, \Psi^{\mp}_{<0}$ in the same manner, we have

$$D = |\Psi^{\pm}_{>0}| - |\Psi^{\pm}_{\leq 0}| - |\Psi^{\mp}_{\geq 0}| + |\Psi^{\mp}_{<0}|. \tag{23}$$

Note that $|\Psi^{\pm}| = |\Psi^{\mp}|$: In fact, define

$$\widetilde{\Psi} := \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y(\alpha) \in \Phi^-, \sigma(y)w^{-1}(\alpha) \in \Phi^-\},$$

then

$$\begin{aligned}\Psi^{\pm} \ \dot\cup\ \widetilde{\Psi} &= \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid \sigma(y)w^{-1}(\alpha) \in \Phi^-\},\\ \Psi^{\mp} \ \dot\cup\ \widetilde{\Psi} &= \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y(\alpha) \in \Phi^-\},\end{aligned}$$

and as $y, \sigma(y) \in W^J$ and $w \in W_J$, this implies that

$$|\Psi^{\pm}| + |\widetilde{\Psi}| = l(\sigma(y)) = l(y) = |\Psi^{\mp}| + |\widetilde{\Psi}|.$$

So we have $|\Psi^{\pm}_{>0}| + |\Psi^{\pm}_{\leq 0}| = |\Psi^{\mp}_{\geq 0}| + |\Psi^{\mp}_{<0}|$, and using (23) and (21) we obtain

$$\begin{aligned}D = 2 \cdot (|\Psi^{\mp}_{<0}| - |\Psi^{\pm}_{\leq 0}|) &\leq 2|\Psi^{\mp}_{<0}| \leq 2|\{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid \langle \alpha, \lambda\rangle < 0\}|\\ &\leq l(\omega) - \langle 2\rho, \nu_b\rangle.\end{aligned}$$

Now $y\omega\sigma(y)^{-1}$ is straight if and only if equality holds everywhere in this chain. This is the case for the first inequality if and only if $\Psi^{\pm}_{\leq 0} = \emptyset$, for the second one if and only if $\langle \alpha, \lambda\rangle < 0$ implies that $\alpha \in \Psi^{\mp}$, and (by (21)) for the third one if and only if $\langle \alpha, \lambda\rangle < 0$ implies that $\langle \alpha, \lambda\rangle = -1$, which are exactly the conditions (C0)$-$(C2). $\square$

The following probably well-known lemma will be used in the proof of the next proposition.

Lemma I.3.30. *Let $\eta \in X_*(T)$ be dominant, let $S_\eta = \{s \in S \mid s(\eta) = \eta\}$.*

(i) The assignment $W^{S_\eta} \to W \cdot \eta$, $x \mapsto x(\eta)$ is a bijection. If $\lambda = x(\eta)$ for $x \in W^{S_\eta}$, then for $\alpha \in \Phi^+$ we have

$$\langle \alpha, \lambda \rangle < 0 \iff x^{-1}(\alpha) \in \Phi^-. \tag{*}$$

(ii) Let $J \subseteq S$ be any subset. Then the bijection of (i) restricts to a bijection

$$^J W^{S_\eta} \to \{J\text{-dominant elements in } W \cdot \eta\} \quad , x \mapsto x(\eta).$$

Proof. (i) Clearly, every $\lambda \in W \cdot \eta$ can be written uniquely as $\lambda = x(\eta)$ for $x \in W^{S_\eta}$. In this case

$$\langle \alpha, \lambda \rangle = \langle x^{-1}(\alpha), \eta \rangle \quad \text{for all } \alpha \in \Phi,$$

so the "$\Longrightarrow$"-part of the equivalence is clear since η is dominant. Conversely, if $x^{-1}(\alpha) \in \Phi^-$ for some $\alpha \in \Phi^+$, then $x^{-1}(\alpha) \in \Phi^- \setminus \Phi^-_{S_\eta}$ since $x \in W^{S_\eta}$, which implies that $\langle \alpha, \lambda \rangle < 0$.

(ii) Let $\lambda = x(\eta)$ for $x \in W^{S_\eta}$. If x also lies in $^J W$ then $x^{-1}(\alpha) \in \Phi^+$ for each $\alpha \in \Phi^+_J$ (see Section I.2.3), so the equivalence (*) implies that λ is J-dominant. On the other hand, if λ is J-dominant, then by (*) we must have $x^{-1}(\alpha) \in \Phi^+$ for each $\alpha \in \Phi^+_J$, which implies that $x \in {^J W}$. □

Proposition I.3.31. *Let $b \in B(G)$, write $J := J_b$. Let $\omega \in \Omega_J$ be a b-short element.*

(i) If there exists $y \in W^J$ such that $y\omega\sigma(y)^{-1}$ is straight, then ω is σ-balanced, and further in this case the element $y\omega\sigma(y)^{-1}$ is (J, y, σ)-fundamental.

(ii) Let ω be σ-balanced. Set

$$\Xi := \{\alpha \in \Phi^+ \setminus \Phi^+_J \mid \langle \alpha, \lambda + \cdots + \delta^i(\lambda) \rangle = -1 \text{ for some } i \geq 0\}.$$

Then there is a $y \in W^J$ such that $\Xi = \{\alpha \in \Phi^+ \setminus \Phi^+_J \mid y(\alpha) \in \Phi^-\}$, and for each such y the element $y\omega\sigma(y)^{-1}$ is straight.

Proof. Let $\omega = \epsilon^\lambda w$ for $\lambda \in X_*(T)$ and $w \in W_J$. We reformulate the conditions (C1) and (C2) of Proposition I.3.29 in terms of the map $\delta = w \circ \sigma$ of Definition I.3.26, considered as a map on $\Phi^+ \setminus \Phi^+_J$:
We have $\sigma(y)w^{-1}(\alpha) \in \Phi^+$ if and only if $\sigma^{-1}(\sigma(y)w^{-1}(\alpha)) = y((w \circ \sigma)^{-1}(\alpha)) = y(\delta^{-1}(\alpha)) \in \Phi^+$, therefore, for each $\alpha \in \Phi^+ \setminus \Phi^+_J$, the conditions of I.3.29 are equivalent to

$$\langle \alpha, \lambda \rangle < 0 \implies \langle \alpha, \lambda \rangle = -1 \tag{C0}$$

$$\langle \alpha, \lambda \rangle = -1 \implies y(\alpha) \in \Phi^- \text{ and } y(\delta^{-1}(\alpha)) \in \Phi^+ \tag{C1'}$$

$$y(\alpha) \in \Phi^+ \text{ and } y(\delta^{-1}(\alpha)) \in \Phi^- \implies \langle \alpha, \lambda \rangle = 1 \tag{C2'}$$

Let us prove (i). Suppose that $y \in W^J$ such that $\tilde{w} := y\omega\sigma(y)^{-1}$ is straight. This implies that ω is σ-balanced: By Proposition I.3.29, the properties (C0), (C1') and (C2') above hold for each $\alpha \in \Phi^+ \setminus \Phi^+_J$. Fix an element $\alpha \in \Phi^+ \setminus \Phi^+_J$. By (C0), we have $\langle \alpha, \delta^i(\lambda) \rangle = \langle \delta^{-i}(\alpha), \lambda \rangle \geq -1$ for all $i \in \mathbb{Z}$, so in order to show the condition of Definition I.3.26 for α it suffices to check that the following holds:

If $\langle \alpha, \delta^k(\lambda) \rangle = \langle \alpha, \delta^l(\lambda) \rangle = -1$ for some $k < l$,
then there is some $k < m < l$ such that $\langle \alpha, \delta^m(\lambda) \rangle > 0$.

But for such $k < l$ we have $\langle \delta^{-k}\alpha, \lambda \rangle = \langle \delta^{-l}(\alpha), \lambda \rangle = -1$, therefore $y(\delta^{-k}(\alpha)) \in \Phi^-$, $y(\delta^{-k-1}(\alpha)) \in \Phi^+$ and $y(\delta^{-l}(\alpha)) \in \Phi^-$ by (C1'), so (C2') implies that $\langle \delta^{-m}(\alpha), \lambda \rangle > 0$ for some $k < m < l$.

Now we show that $\tilde{w} = y\omega\sigma(y)^{-1}$ is indeed (J, y, σ)-fundamental, using the original properties (C0)-(C2) of Proposition I.3.29: Recall that $k(\alpha, \mathfrak{a}_0) = 1$ for $\alpha \in \Phi^+$ and $k(\alpha, \mathfrak{a}_0) = 0$ for $\alpha \in \Phi^-$ (see Remark I.3.24). For any $\alpha \in \Phi$ and any alcove $\mathfrak{a} \subseteq V$ we have $k(\alpha, \omega\mathfrak{a}) = k(\alpha, w\mathfrak{a}) + \langle \alpha, \lambda \rangle$. So if $\beta \in y(\Phi^+)$, say $\beta = y(\alpha)$, then

$$\begin{aligned} k(\beta, \tilde{w}\mathfrak{a}_0) &= k(\alpha, y^{-1}\tilde{w}\mathfrak{a}_0) = k(\alpha, \omega\sigma(y)^{-1}\mathfrak{a}_0) \\ &= k(\alpha, w\sigma(y)^{-1}\mathfrak{a}_0) + \langle \alpha, \lambda \rangle \\ &= k(\sigma(y)w^{-1}(\alpha), \mathfrak{a}_0) + \langle \alpha, \lambda \rangle. \end{aligned} \tag{24}$$

Let us check on the conditions of Definition I.3.23(b):

(I) clearly holds, as $y^{-1}\tilde{w}\sigma(y) = \omega \in \Omega_J \subseteq \widetilde{W}_J$.

(II) Let $\beta \in y(\Phi^+ \setminus \Phi_J^+)$, let $\alpha = y^{-1}(\beta) \in \Phi^+ \setminus \Phi_J^+$.
Suppose that $\beta \in \Phi^+$. In this case we have $y(\alpha) \in \Phi^+$, so the properties (C1) and (C2) for α imply that $\langle \alpha, \lambda \rangle \geq 0$, and that $\langle \alpha, \lambda \rangle > 0$ if $\sigma(y)w^{-1}(\alpha) \in \Phi^-$. From (24) it follows that in any case $k(\beta, \tilde{w}\mathfrak{a}_0) \geq 1 = k(\beta, \mathfrak{a}_0)$.
Now suppose that $\beta \in \Phi^-$, i.e. $y(\alpha) \in \Phi^-$. By the properties (C0) and (C1) for α we then have $\langle \alpha, \lambda \rangle \geq -1$ and $\sigma(y)w^{-1}(\alpha) \in \Phi^+$ if $\langle \alpha, \lambda \rangle = -1$, so (24) shows that $k(\beta, \tilde{w}\mathfrak{a}_0) \geq 0 = k(\beta, \mathfrak{a}_0)$.

(III) Let $\beta \in y(\Phi_J^+)$, let $\alpha = y^{-1}(\beta) \in \Phi_J^+$.
As $y \in W^J$, we then have $\beta \in \Phi^+$, so $k(\beta, \mathfrak{a}_0) = 1$. Since $\sigma(y) \in W^J$ and $w^{-1}(\alpha)$ lies in Φ_J, we have $k(\sigma(y)w^{-1}(\alpha), \mathfrak{a}_0) = k(w^{-1}(\alpha), \mathfrak{a}_0)$. Since $l_J(\omega) = 0$, by the length formula we have $\langle \alpha, \lambda \rangle = 1$ if $w^{-1}(\alpha) \in \Phi_J^-$ and $\langle \alpha, \lambda \rangle = 0$ otherwise, so in any case $k(\beta, \tilde{w}\mathfrak{a}_0) = 1 = k(\beta, \mathfrak{a}_0)$ by (24).

Now we prove (ii). Suppose that $y \in W^J$ with $\Xi = \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y(\alpha) \in \Phi^-\}$. Then y satisfies the conditions (C0), (C1') and (C2') and therefore the element $y\omega\sigma(y)^{-1}$ is straight by Proposition I.3.29:

(C0) is clear by the definition of a σ-balanced element.

(C1') Let $\alpha \in \Phi^+ \setminus \Phi_J^+$. If $\langle \alpha, \lambda \rangle = -1$, then $\alpha \in \Xi$ and hence $y(\alpha) \in \Phi^-$. If $\delta^{-1}(\alpha)$ were in Ξ, then we would have

$$-1 = \langle \delta^{-1}(\alpha), \lambda + \cdots + \delta^i(\lambda) \rangle = \langle \alpha, \delta(\lambda) + \cdots + \delta^{i+1}(\lambda) \rangle$$

for some $i \geq 0$. But then $\langle \alpha, \lambda + \cdots + \delta^{i+1}(\lambda) \rangle = -2$, a contradiction to the fact that ω is σ-balanced. Hence we must have $\delta^{-1}(\alpha) \neq \Xi$ and thus $y(\delta^{-1}(\alpha)) \in \Phi^+$.

(C2') If $y(\alpha) \in \Phi^+$ and $y(\delta^{-1}(\alpha)) \in \Phi^-$, then $\delta^{-1}(\alpha) \in \Xi$ and $\alpha \notin \Xi$, so the same argument as above shows that $\langle \alpha, \lambda \rangle$ must be > 0.

To prove the existence of such a $y \in W^J$, note that, since ω is σ-balanced, the set Ξ is a disjoint union $\Xi = \bigcup_{j\geq 0} \Xi_j$ where

$$\Xi_j := \{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid \langle \alpha, \lambda\rangle = \cdots = \langle \alpha,\ \delta^{j-1}(\lambda)\rangle = 0, \langle \alpha, \delta^j(\lambda) = -1\},$$

$\alpha \in \Xi$ lies in Ξ_j if and only if j is minimal with the property that $\langle \alpha, \lambda + \cdots + \delta^j(\lambda)\rangle = -1$. Further, let $n \in \mathbb{N}$ such that $\delta^n = \mathrm{id}$, then $\Xi = \bigcup_{j=0}^{n-1} \Xi_j$, compare Remark I.3.27(ii).

We set $L_{-1} := S, y_{-1} := 1$, and proceed by induction to show that for $i \geq 0$ there exist $L_i \subseteq S$, $y_i \in {}^{L_i}W^J$ and $\mu_i \in X_*(T)$ such that:

(1) $\mu_i = y_i(\delta^i(\lambda))$ is L_{i-1}-dominant,

(2) $L_i = \{s \in L_{i-1} \mid s(\mu_i) = \mu_i\} \subseteq L_{i-1}$,

(3) $y_i = u_i y_{i-1}$ for some $u_i \in {}^{L_i}W_{L_{i-1}}$,

(4) $\{\alpha \in \Phi^+ \setminus \Phi_J^+ \mid y_i(\alpha) \in \Phi^-\} = \bigcup_{j=0}^{i} \Xi_j$.

The element y_n has then the demanded property.

Initial step: $i = 0$. Let $\mu_0 \in X_*(T)_{\mathrm{dom}}$ be the unique element such that $\lambda \in W \cdot \mu_0$, let $L_0 := S_{\mu_0}$. Let $x \in W^{L_0}$ be the unique element such that $\lambda = x(\mu_0)$, then $x \in {}^JW^{L_0}$ by Lemma I.3.30. Set $y_0 := x^{-1}$, then (1)$-$(3) hold by construction, and (4) holds by Lemma I.3.30 (we have $y(\alpha) \in \Phi^+$ for $\alpha \in \Phi_J^+$, since $y \in W^J$).

Induction step: Suppose that elements with the properties (1)$-$(4) are constructed for all $0 \leq j \leq i-1$. Then repeated application of (2)+(3) implies that for each $0 \leq j \leq i-1$ we have $y_{i-1} = uy_j$ for some $u \in W_{L_j}$ and therefore

$$\mu_j = u(\mu_j) = uy_j(\delta^j(\lambda)) = y_{i-1}(\delta^j(\lambda)) \quad \text{for all } 0 \leq j \leq i-1. \tag{*}$$

Set $\lambda_i := y_{i-1}(\delta^i(\lambda))$ and $J_i := L_{i-1} \cap {}^{y_{i-1}}J$. Then λ_i is J_i-dominant: Let $\alpha \in \Phi_{J_i}^+$, then $\alpha \in \Phi_{L_j}^+$ for all $k \leq i-1$ by (2), so

$$\begin{aligned}\langle \alpha, \lambda_i\rangle &\overset{(2)}{=} \langle \alpha, \lambda_i\rangle + \sum_{j=0}^{i-1} \langle \alpha, \mu_j\rangle \overset{(*)}{=} \sum_{j=0}^{i} \langle \alpha, y_{i-1}(\delta^j(\lambda))\rangle \\ &= \langle y_{i-1}^{-1}(\alpha), \sum_{j=0}^{i} \delta^j(\lambda)\rangle \geq 0,\end{aligned}$$

where the last inequality holds because $y_{i-1}^{-1}(\alpha) \in \Phi_J^+$ and $\sum_{j=0}^{i} \delta^j(\lambda)$ is J-dominant, see Remark I.3.27(ii).
Now define μ_i as the unique L_{i-1}-dominant element in $W_{L_{i-1}} \cdot \lambda_i$ and define $L_i \subseteq L_{i-1}$ as in (2), then by Lemma I.3.30 there is an $x \in {}^{J_i}W_{L_{i-1}}^{L_i}$ such that $\lambda_i = x(\mu_i)$. Set $u_i := x^{-1}$ and $y_i := u_i y_{i-1}$, then $y_i \in {}^{L_i}W^J$ by Corollary I.3.8 (since $u_i \in {}^{L_i}W_{L_{i-1}}^{J_i}$) and $y_i(\delta^i(\lambda)) = x^{-1}(\lambda_i) = \mu_i$. Hence the triple (L_i, y_i, μ_i) satisfies (1)$-$(3). As for (4), let $\alpha \in \Phi^+ \setminus \Phi_J^+$ such that $y_i(\alpha) = u_i y_{i-1}(\alpha) \in \Phi^-$. Since $y_{i-1} \in {}^{L_{i-1}}W$ and $u_i \in {}^{L_i}W_{L_{i-1}}$, we are then in exactly one of the following two cases:

- $y_{i-1}(\alpha) \in \Phi^- \setminus \Phi^-_{L_{i-1}}$.
  This is equivalent to $y_{i-1}(\alpha) \in \Phi^-$, which is the case if and only if $\alpha \in \bigcup_{j=0}^{i-1} \Xi_j$ by (4) for y_{i-1}.
- $y_{i-1}(\alpha) \in \Phi^+_{L_{i-1}}$ and $u_i y_{i-1}(\alpha) \in \Phi^-_{L_{i-1}}$.
  This is equivalent $\alpha \in \Xi_i$: Note that we have $\langle \alpha, \delta^j(\lambda)\rangle = \langle y_{i-1}(\alpha), \mu_j\rangle$ for each $0 \leq j \leq i-1$ by (*), so $y_{i-1}(\alpha) \in \Phi_{L_{i-1}}$ if and only if $\langle \alpha, \delta^j(\lambda)\rangle = 0$ for all $0 \leq j \leq i-1$, and in this case automatically $y_{i-1}(\alpha) \in \Phi^+_{L_{i-1}}$. Now if $y_{i-1}(\alpha) \in \Phi^+_{L_{i-1}}$, then Lemma I.3.30 shows that $u_i y_{i-1}(\alpha) = x^{-1} y_{i-1}(\alpha) \in \Phi^-_{L_{i-1}}$ if and only if

$$-1 = \langle y_{i-1}(\alpha), x(\mu_i)\rangle = \langle y_{i-1}(\alpha), \lambda_i\rangle = \langle \alpha, \delta^i(\lambda)\rangle.$$

This shows property (4) for y_i, which concludes the induction step and thus the proof of (ii).

□

We sum up our considerations in the following theorem (cf. Theorem I.B and Proposition I.C from the introduction):

Theorem I.3.32. *Let $\mu \in X_*(T)_{\mathrm{dom}}$, let $b \in B(G, \mu)$.*

(i) For $\tilde{w} \in W\epsilon^\mu W$ the following are equivalent:

(1) $\tilde{w}$ is a straight representative for b.

(2) $\tilde{w} \in b$ and $\tilde{w}$ is P-fundamental for some semistandard parabolic P.

(3) $\tilde{w} \in b$ and $\tilde{w}$ is (J, y, σ)-fundamental for some $J \subseteq S$ and $y \in W$.

(ii) $W\epsilon^\mu W$ contains a straight representative for b if and only if it contains a b-short element which is σ-balanced. Further in this case there exists a $\xi \in \mathrm{EO}(G, \mu)$ such that $\xi \subseteq b$ is K-fundamental and $\tilde{w}_\xi$ is straight.

Proof. We have dicussed the implications (3) ⇒ (2) ⇒ (1) in the previous subsection. The implication (1) ⇒ (3) follows from Lemma I.3.28 and Proposition I.3.31(i), with $J = J_b$ the type of b. The equivalence in (ii) follows from I.3.28 and Proposition I.3.31(i)+(ii), the last statement is a consequence of Lemma I.3.28 and the considerations in the last subsection. □

Corollary I.3.33. *Let $b \in B(G)$, then there exists a σ-balanced b-short element in $\widetilde{W}$. If all b-short elements lie in the same σ-orbit then all of them are σ-balanced.*

Proof. By ([He2], Thm. 3.5.) there exists a straight representative for b in $\widetilde{W}$, so by Theorem I.3.32(ii) there exists a σ-balanced element. On the other hand it is clear that a b-short element ω_b is σ-balanced if and only if $\sigma(\omega_b)$ is σ-balanced. □

If G is split over $\mathcal{O}_F$ then for every $b \in B(G)$ there is a unique b-short element in $\widetilde{W}$ (see I.2.14) so the last corollary asserts that in this case all b-short elements are σ-balanced. Thus the appearance of "unbalanced" b-short elements, as for example in the Hilbert-Blumenthal case discussed in Section I.4.2.2 below, is a phenomenon which only occurs in case of a nontrivial σ-action on $\widetilde{W}$. Note that we gave an indirect proof of Corollary I.3.33, using the existence of straight elements. It would be favourable to find a direct proof using Definition I.3.26.

We conclude with a few comments on the minuscule case:

(1) Suppose that G is split over $\mathcal{O}_F$. Let $\mu \in X_*(T)_{\mathrm{dom}}$ be minuscule, let $b \in B(G,\mu)$. Then the unique b-short element lies in $W\epsilon^\mu W$: This follows from Proposition I.3.5 and I.3.2.4(a). As observed above, this b-short element is automatically σ-balanced. Therefore by Theorem I.3.32(ii) in this case $W\epsilon^\mu W$ always contains a straight (thus in particular I-fundamental) representative for b, which was already observed in ([VW], Prop. 9.9.).

(2) It follows from the description of Ω_J for $J \subseteq S$ in (18) that Theorem I.3.32 also implies the following criterion for the existence of straight elements.

Criterion I.3.34. *Let $\mu \in X_*(T)_{\mathrm{dom}}$ be* minuscule*, let $b \in B(G,\mu)$, let $J := J_b$ be the type of b. The following are equivalent:*

(i) There exists a straight representative of b in $W\epsilon^\mu W$.

(ii) There exists a J-dominant element $\lambda \in W \cdot \mu$ such that $\epsilon^\lambda w_{0,J_\lambda} w_0$ is a σ-balanced b-short element.

The condition of (ii) can be made very explicit and comes down to combinatorial questions in many situations, for example if G is a scalar restriction of GL_n as considered in Section I.4. Concrete examples suggest that these combinatorial questions might be solvable, but unfortunately we were not able to find a proof of criterion (ii) for an interesting family of groups. See also Remark I.4.14.

(3) In ([VW], Thm. 9.18.) the authors show that for a group which arises from a PEL-datum as described in Section II.3 the following holds: If $\mu \in X_*(T)_{\mathrm{dom}}$ is minuscule and $b \in B(G,\mu)$, then there is a possibly different minuscule cocharacter $\mu' \in X_*(T)_{\mathrm{dom}}$ such that $W\epsilon^{\mu'}W$ contains a P-fundamental (hence also a straight) element. It might be interesting to try and show this property in a more general situation, we did not consider this question.

I.3.5 The basic and the μ-ordinary case

Throughout this subsection we fix an element $\mu \in X_*(T)_{\mathrm{dom}}$. As explained in Section I.3.1.1, the set $B(G,\mu)$ contains the unique maximal element $[\epsilon^\mu]$ with respect to $\preceq$. We also denote this element by $b_{\mu-\mathrm{ord}}$ and call it the *μ-ordinary* element of $B(G,\mu)$. This notion is consistent with the corresponding definition for Shimura varieties in Chapter II in the case that μ arises from a Shimura datum. Also, $B(G,\mu)$ contains a unique basic element, which is the unique minimal element in $B(G,\mu)$ with respect to $\preceq$. We denote this element by b_{bas}. In this subsection we discuss Problem I.3.2 for these special σ-conjugacy classes.

I.3.5.1 The basic case

Suppose that $b = b_{\mathrm{bas}} \in B(G,\mu)$ is the basic element. In the case that G is semisimple the intersections between I-double cosets and b have then been investigated in [GHN]. Here the authors give combinatorial criteria in terms of $\widetilde{W}$ and σ for the emptyness and nonemptyness of the intersection ([GHN], Prop. 3.6.4., Thm. 4.4.7.), and also some variants under additional conditions. We note a consequence for EO-classes, which is particularly easy to check:

Criterion I.3.35. *Suppose that G is semisimple, and that μ is not central in G. Let $\xi \in \mathrm{EO}(G,\mu)$, let $w \in {}^{S_\mu^{\mathrm{opp}}}W$ be the unique element such that $\tau_\mu w \in \xi$ (see Section I.2.6).*

(1) If the Dynkin diagram of G is σ-connected and $x_\mu w \in \bigcup_{J\subsetneq S,\sigma(J)=J} W_J$, then $\xi \cap b = \emptyset$.

(2) If $\tau_\mu w$ lies in the shrunken Weyl chambers and $x_\mu w \in W \setminus \bigcup_{J\subsetneq S,\sigma(J)=J} W_J$, then $\xi \cap b \neq \emptyset$.

Here the first condition in (1) can also be put as follows: Let $G_1 \times \cdots \times G_l$ be the decomposition of G_L^{ad} into simple factors. Then there is no strict subset $\{i_1,\dots,i_k\}$ such that $G_{i_1} \times \cdots \times G_{i_k}$ is σ-stable. In (2), an element $\tilde{w} \in \widetilde{W}$ is said to lie in the shrunken Weyl chambers if $k(\alpha, \tilde{w}\mathfrak{a}_0) \neq k(\alpha, \mathfrak{a}_0)$ for every $\alpha \in \Phi$, where $k(\alpha,\cdot)$ is as in Definition I.3.23.

Criterion I.3.35 is derived from the results of [GHN] as follows: We have $l(\tau_\mu w) = l(\tilde{w}_\xi)$, therefore $\xi \cap b \neq \emptyset \iff I\tau_\mu w I \cap b \neq \emptyset$ by Corollary I.3.14 and (13). The element τ_μ lies in ${}^S\widetilde{W}^S$ and w lies in ${}^{S_\mu^{\mathrm{opp}}}W$, where $S_\mu^{\mathrm{opp}} = S \cap \tau_\mu^{-1} S \tau_\mu$, which implies that $\tau_\mu w \in {}^S\widetilde{W}$ by a variant of Proposition I.3.7 for the quasi-Coxeter group $\widetilde{W}$. It follows that the element $\eta_\delta(\tau_\mu w)$ defined in ([GHN], §3.6.) is given by $\sigma^{-1}(x_\mu w)$. The conditions on $x_\mu w$ in (1), (2) are unchanged by the σ-action. So (1) follows from ([GHN], Prop. 3.6.4.), where the additional condition on Newton vectors in this proposition is satisfied because μ is not central, and (2) follows from (loc. cit. Prop. 4.4.9.).

Unfortunately, the criterion (2) is of little use in the case of special interest where μ is minuscule: In this case the only element of the form $\epsilon^\mu w$ for $w \in W$ lying in the shrunken Weyl chambers is $\epsilon^\mu w_{0,S_\mu}$, for all other elements of $\epsilon^\mu w$ one has to apply the criterion from ([GHN], Thm. 4.4.7.).

Now let G be arbitrary again, and consider the unique minimal element $\xi_{\min} = [[\tau_\mu]] \in \mathrm{EO}(G,\mu)$ with respect to the closure relation on $\mathrm{EO}(G,\mu)$. It has the following properties:

Remark I.3.36.

(1) If μ is minuscule, then $\xi_{\min} \subseteq b_{\mathrm{bas}}$ is K-fundamental.

(2) In general, $\xi_{\min} \cap b_{\mathrm{bas}}$ may be empty.

Property (1) was shown in ([VW], Prop. 9.17.). With our preparations the proof is very simple: If μ is dominant and minuscule then $\tau_\mu \in \Omega$, so τ_μ is straight (see Example I.3.21) and therefore I-fundamental (Proposition I.3.22).

We give an example of a case where $\xi_{\min} \cap b_{\mathrm{bas}} = \emptyset$: Let $G = \mathrm{SL}_{3,\mathcal{O}_F}$, let (T,B) be the torus of diagonal matrices and Borel subgroup of upper triangular matrices. In this case, Beazley has determined the intersections between σ-conjugacy classes and I-double cosets in [Be] (but note that the author uses a different convention on Newton polygons and that the Iwahori group considered there is the group $I^{\mathrm{opp}} = w_0 I w_0$ according to our notation). Let $\mu = (2,1,-3) \in X_*(T) \cong \{\mu \in \mathbb{Z}^3 \mid \mu_1 + \mu_2 + \mu_3 = 0\}$. Then we have $S_\mu = \emptyset$ and hence $\tau_\mu = \epsilon^{(2,1,-3)} w_0$. In this case the basic element in $B(G,\mu)$ is also the unique basic element in $B(G)$ (as SL_3 is simply connected), which corresponds to the Newton polygon $(0,0,0)$. Using our notation, the results of [Be] show that $I\tau_\mu I$ intersects exactly the σ-conjugacy classes corresponding to the Newton polygons ν with $(1,-1/2,-1/2) \preceq \nu \preceq (1,1,-2)$, in particular $I\tau_\mu I$ does not intersect b_{bas}.

Remark I.3.37. In certain cases the structure of $K\epsilon^\mu K \cap b_{\mathrm{bas}}$ determines the relation between σ-conjugacy classes and Ekedahl-Oort classes: In [GH] the authors provide a list of cases with semisimple G and minuscule μ for which this "basic locus" is equal to a union of EO-classes whose standard representatives are σ-Coxeter elements (see [GH] §5.1.). In these cases, all Ekedahl-Oort stata which do not lie in b_{bas} can be shown to be K-fundamental ([GH], Thm. 5.2.1. resp. §6.1.).

I.3.5.2 The μ-ordinary case

The aim or this paragraph is to prove the following proposition, which gives a complete answer to Problem I.3.2 for the maximal elements $b_{\mu-\mathrm{ord}}$ and $\xi_{\max}$ and shows Theorem I.D from the introduction.

Proposition I.3.38. *Let $\mu \in X_*(T)_{\mathrm{dom}}$, then for $g \in G(L)$ the following are equivalent:*

(i) $g \in K\epsilon\mu K \cap [\epsilon^\mu]$,

(ii) $g \in K \cdot_\sigma \epsilon^\mu$,

(iii) $g \in [[\epsilon^\mu]]$.

In other words, we have the equalities $[\epsilon^\mu] \cap K\epsilon^\mu K = [[\epsilon^\mu]] = K \cdot_\sigma \epsilon^\mu$.

Remark I.3.39. Proposition I.3.38 should be understood as a generalization of ([Mo2], Thm. 1.3.7. resp. Thm. 3.2.7.): If G arises from a PEL-type Shimura datum, then μ is minuscule, and the element $\epsilon^\mu \in G(L)$ takes the place of the p-divisible group $\underline{X}^{\mathrm{ord}}$ defined in [Mo2].

Corollary I.3.40 (cf. Theorem I.D)**.** *Let $\mu \in X_*(T)_{\mathrm{ord}}$, let $\xi_{\max} \in \mathrm{EO}(G,\mu)$ be the unique maximal element with respect to $\preceq$. Then $\xi_{\max} \subseteq b_{\mu-\mathrm{ord}}$ is K-fundamental, and $\xi_{\max} = K \cdot \epsilon^\mu = K\epsilon^\mu K \cap b_{\mu-\mathrm{ord}}$. Consequently we have*

$$\begin{aligned} \xi \cap b_{\mu-\mathrm{ord}} &= \emptyset \qquad && \text{for all } \xi \in \mathrm{EO}(G,\mu) \setminus \{\xi_{\max}\}, \\ \xi_{\max} \cap b &= \emptyset && \text{for all } b \in B(G,\mu) \setminus b_{\mu-\mathrm{ord}}. \end{aligned}$$

Proof. We have the equality $b_{\mu-\mathrm{ord}} = [\epsilon^\mu]$. On the other hand, $\xi_{\max} = [[\tau_\mu w_{\max}]]$ for $w_{\max} = w_0 w_{0,J_\mu}$, see Section I.3.1.1. Since $\tau_\mu = \epsilon^\mu w_{0,S_\mu} w_0$, we have

$$[[\tau_\mu w_{\max}]] = [[\epsilon^\mu w_{0,S_\mu} w_{0,J_\mu}]] = [[w_{0,S_\mu}\epsilon^\mu \sigma(w_{0,S_\mu})^{-1}]] = [[\epsilon^\mu]],$$

and everything follows from Proposition I.3.38. □

We will prove Proposition I.3.38 throughout the rest of the paragraph. Of course, the implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$ are trivial. The implication $(iii) \Rightarrow (ii)$ follows from the fact that $\epsilon^\mu \in \widetilde{W}$ is straight (see Example I.3.21) and thus I-fundamental.

To show the remaining implication we will use the Hodge-Newton decomposition for affine Deligne-Lusztig sets in the affine Grassmannian, which was first formulated for unramified groups by Kottwitz and later generalized by Mantovan and Viehmann. We will only need the former mentioned version. Let us recall the setup (cf. [Ko4], §4.1.):

Let $\lambda \in X_*(T)$ be a cocharacter (which is usually taken to be dominant) and let $g_0 \in G(L)$, then the affine Deligne-Lusztig set in the affine Grassmannian associated to these elements is defined as

$$X_\lambda^G(g_0) := \{g \in G(L)/K \mid g^{-1} g_0 \sigma(g) \in K\epsilon^\lambda K\}.$$

Let $J \subseteq S$ such that $\sigma(J) = J$, let $M_J \subseteq G$ be the corresponding standard Levi subgroup of G. Then we have the Newton map and Kottwitz map for M_J

$$\nu_{M_J} \colon B(M_J) \longrightarrow (X_*(T)_{\mathbb{Q}}/W_J)^{\langle\sigma\rangle}, \qquad \kappa_{M_J} \colon B(M_J) \longrightarrow \pi_1(M_J)_{\langle\sigma\rangle}.$$

For every $\lambda \in X_*(T)$ and $m_0 \in M_J(L)$ there is the analogous Deligne-Lusztig set

$$X_\lambda^{M_J}(m_0) = \{m \in M_J(L)/M_J(\mathcal{O}) \mid m^{-1} m_0 \sigma(m) \in M_J(\mathcal{O})\epsilon^\mu M_J(\mathcal{O})\},$$

and a natural map $X_\lambda^{M_J}(m_0) \to X_\lambda^G(m_0)$, which is clearly injective.

Recall that $V = X_*(T)_{\mathbb{R}}$ carries compatible actions of W and of σ. Let $V_J \subseteq V$ be the subspace of elements which are invariant under the action of W_J and σ, and let

$$V_J^+ := \{v \in V_J \mid \langle\alpha, v\rangle > 0 \text{ for all } \alpha \in \Phi^+ \setminus \Phi_J^+\}.$$

We fix an $n \in \mathbb{N}$ such that σ^n acts as the identity on V. The composition of the two maps

$$v \longmapsto \frac{1}{n}\sum_{i=0}^{n-1} \sigma^i(v), \qquad v \longmapsto \frac{1}{|W_J|}\sum_{w \in W_J} w(v)$$

gives a projection map $V \to V_J$. The restiction of this map to $X_*(T)$ factors via $\pi_1(M_J)_{\langle\sigma\rangle}$, let $p_J \colon \pi_1(M_J)_{\langle\sigma\rangle} \to V_J$ be the resulting map and define

$$\pi_1(M_J)^+_{\langle\sigma\rangle} := \{x \in \pi_1(M_J)_{\langle\sigma\rangle} \mid p_J(x) \in V_J^+\}.$$

Proposition I.3.41 ([Ko4], Thm. 4.1.)**.** *Let $\lambda \in X_*(T)$ be G-dominant, let $m_0 \in M(L)$ such that $[m_0]_{M_J}$ is basic in $B(M_J)$. If $\kappa_{M_J}([m_0]_{M_J})$ equals the image of λ in $\pi_1(M_J)_{\langle\sigma\rangle}$ and $\kappa_{M_J}([m_0]_{M_J}) \in \pi_1(M_J)^+_{\langle\sigma\rangle}$, then the natural map $X_\lambda^{M_J}(m_0) \hookrightarrow X_\lambda^G(b_0)$ is an isomorphism.*

Now we show the implication $(i) \Rightarrow (ii)$ of Proposition I.3.38: Consider an element $g \in [\epsilon^\mu] \cap K\epsilon^\mu K$. Then $g = h^{-1}\epsilon^\mu\sigma(h)$ for some $h \in G(L)$, we need to show that we may replace h by some element of K. By definition, h lies in the affine Deligne-Lusztig set $X_\mu^G(\epsilon^\mu)$.

Recall that the dominant Newton vector of the σ-conjugacy class $[\epsilon^\mu]$ is given as

$$\bar{\mu} = \frac{1}{n}\sum_{i=0}^{n-1} \sigma^i(\mu) \in X_*(T)_{\mathbb{Q}}, \tag{25}$$

where as before $n \in \mathbb{N}$ is chosen such that σ^n acts as the identity (cf. Section I.3.1.1). Let $J := \{s \in S \mid s(\bar{\mu}) = \bar{\mu}\}$, then the corresponding Levi subgroup is $M_J = \mathrm{Cent}_G(\bar{\mu}) := \mathrm{Cent}_G(n \cdot \bar{\mu}) \subseteq G$. ($J$ is the type of $[\epsilon^\mu]$ in the sense of Definition I.2.9.)

We claim that ϵ^μ is central in M_J: Indeed, we have

$$\mathrm{Z}(M_J) = \bigcap_{\alpha \in \Phi_J} \ker(\alpha) \subseteq T$$

(see [SGA3], Exp. XXII, Cor. 4.1.6.). Let $\alpha \in \Phi_J^+$. By definition, the cocharacter $n \cdot \bar{\mu}$ maps to the center of M_J, so we find that

$$0 = \langle \alpha, n \cdot \bar{\mu} \rangle = \sum_{i=0}^{n-1} \langle \alpha, \sigma^i(\mu) \rangle.$$

Since σ acts on the set of dominant cocharacters, every summand in the upper equation is nonnegative, so they are all equal to zero. In particular, $\langle \alpha, \mu \rangle = 0$ (for every $\alpha \in \Phi_J^+$), which implies that μ is central in M_J as well.

Next, note that the pair (μ, ϵ^μ) satisfies the conditions of the Hodge-Newton decomposition in Proposition I.3.41: ϵ^μ lies in $M_J(L)$, and the J-dominant Newton vector of $[\epsilon^\mu]_{M_J}$ is also given by the formula (25), so it is equal to $\bar{\mu}$, which means that $[\epsilon^\mu]_{M_J}$ is basic in $B(M_J)$. Clearly μ and $[\epsilon^\mu]_{M_J}$ map to the same element in $\pi_1(M_J)_{\langle\sigma\rangle}$. Further, the image of this element in V_J under p_J is the projection of $\mu \in V$ to V_J, which is again equal to $\bar{\mu}$, and this lies in V_J^+ by definition of J.

We have therefore the Hodge-Newton decomposition $X_\mu^{M_J}(\epsilon^\mu) \cong X_\mu^G(\epsilon^\mu)$, so there is an element $m \in M_J(L)$ such that

$$mK = hK \quad \text{and} \quad m^{-1}\epsilon^\mu\sigma(m) \in M_J(\mathcal{O})\epsilon^\mu M_J(\mathcal{O}).$$

Since ϵ^μ commutes with every element of $M_J(L)$, the last equation implies that $m^{-1}\sigma(m) \in M_J(\mathcal{O})$. As M_J is a connected reductive group over $\mathcal{O}$, Lang's Theorem in the form of Corollary I.2.6 implies that there is an $m' \in M_J(\mathcal{O})$ such that $(m')^{-1}\sigma(m') = m^{-1}\sigma(m)$. Let $c \in K$ such that $h = mc$, then altogether we have

$$\begin{aligned} g = h^{-1}\epsilon^\mu\sigma(h) &= c^{-1}(m^{-1}\epsilon^\mu\sigma(m))\sigma(c) \\ &= c^{-1}(m^{-1}\sigma(m)\epsilon^\mu)\sigma(c) \\ &= c^{-1}((m')^{-1}\sigma(m')\epsilon^\mu)\sigma(c) \\ &= c^{-1}(m')^{-1}\epsilon^\mu\sigma(m')\sigma(c) \ \in K \cdot_\sigma \epsilon^\mu, \end{aligned}$$

which was to be shown. This concludes the proof of Proposition I.3.38. □

I.3.6 Two examples

In simple examples the preceeding results allow to determine the intersection behaviour of σ-conjugacy classes and Ekedahl-Oort classes effectively. The GL_2-case is well-known, the example of $\mathrm{GU}(2,3)$ seems to be new.

I.3.6.1 The case $G = \mathrm{GL}_2$

Let $G = \mathrm{GL}_{2,\mathcal{O}_F}$. In this simple case we determine the intersection between σ-conjugacy classes and EO-classes, and also between σ-conjugacy classes and I-double cosets. This easy example will play a crucial role in Section I.4.2.

We choose the usual Borel pair of diagonal matrices and upper triangular matrices $T \subseteq B \subseteq G$. Then we have the identifications $X_*(T) \cong \mathbb{Z}^2$ and $W = \{1, s\} \cong \mathbb{Z}/2\mathbb{Z}$, and a cocharacter $\mu = (a, b) \in \mathbb{Z}$ is dominant if and only if $a \geq b$. As GL_2 is split over $\mathcal{O}_F$, the σ-operation is trivial.

The σ-conjugacy classes in $\mathrm{GL}_2(L)$ are completely determined by their Newton polygons (see Example I.2.10), we identify

$$B(G) = \{\nu = (\nu_1, \nu_2) \mid \nu_1, \nu_2 \in \mathbb{Z}, \nu_1 > \nu_2\} \ \cup \ \{\nu = (\nu_1, \nu_2) \mid \nu_1 = \nu_2 \in \frac{1}{2}\mathbb{Z}\},$$

where the second set corresponds to the set of basic elements. We write b_ν for the class corresponding to $\nu = (\nu_1, \nu_2)$. Then we have $b_\nu \preceq b_{\nu'}$ if and only $\nu \preceq \nu'$, if and only $\nu_1 + \nu_2 = \nu_1' + \nu_2'$ and $\nu_1 \leq \nu_1'$.

Let $\mu = (a, b) \in X_*(T)$ be dominant, then

$$B(G, \mu) = \{\nu \in B(G) \mid a \geq \nu_1 > \nu_2 \geq b, \nu_1 + \nu_2 = a + b\} \ \cup \ \{\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\},$$

we have $b_{\mu-\mathrm{ord}} = b_\mu$, and b_{bas} corresponds to $(\frac{a+b}{2}, \frac{a+b}{2})$.

σ-conjugacy classes and EO-classes Let $\mu = (a, b) \in X_*(T)$ be dominant. Then we are in one of the following cases:

(1) $a = b$.
This is the trivial case where μ is central in G. In this case $J_\mu = S_\mu = W$, $\tau_\mu = \epsilon^\mu$ and therefore $\mathrm{EO}(G, \mu) = \{[[\epsilon^\mu]]\}$, and $B(G, \mu)$ consist only of the element $[\epsilon^\mu] = b_\mu = b_{(a,a)}$. We have

$$K\epsilon^\mu K = [[\epsilon^\mu]] = b_\mu \cap K\epsilon^\mu K = K \cdot_\sigma \epsilon^\mu.$$

(2) $a > b$.
In this case $J_\mu = S_\mu = \emptyset$, $\tau_\mu = \epsilon^\mu s$, and we have $\mathrm{EO}(G, \mu) = \{[[\tau_\mu]], [[\epsilon^\mu]]\}$, so $K\epsilon^\mu K = [[\tau_\mu]] \cup [[\epsilon^\mu]]$. Corollary I.3.40 shows that

$$[[\epsilon^\mu]] = K\epsilon^\mu K \cap b_\mu = K \cdot_\sigma \epsilon^\mu, \qquad [[\tau_\mu]] \cap b_\nu \neq \emptyset \Longleftrightarrow \nu \in B(G, \mu) \setminus \{\mu\}.$$

σ-conjugacy classes and Iwahori double cosets Let us give an explicit description of the product $\widetilde{W} = W_a \rtimes \Omega$: The affine Weyl group is generated by $s \in W$ and the affine reflection $s_a = \epsilon^{(1,-1)}s$, any element in W_a is of the form $ss_as\cdots$ or $s_ass_a\cdots$. The group Ω is isomorphic to $\mathbb{Z}$, and the element $\tau := \epsilon^{(1,0)}s$ is a generating element. We have the relations

$$s_a s = \epsilon^{(1,-1)}, \quad \tau^2 = \epsilon^{(1,1)}, \quad \tau s = \epsilon^{(1,0)}, \quad \tau s \tau^{-1} = s_a, \quad \tau s_a \tau^{-1} = s. \qquad (26)$$

We now reobtain the intersection properties of I-double cosets and σ-conjugacy classes (see [Reu1], §2.3.). They are given as follows:

Example I.3.42. Let $\tilde{w} \in \widetilde{W}$, write $\tilde{w} = w_a\tau^d$ for $w_a \in W_a$ and $d \in \mathbb{Z}$.

(1) If $l(w_a) = 0$, then $\tilde{w} = \tau^d \in \Omega$ is I-fundamental (see Example I.3.21), and we have

$$I\tilde{w}I = I \cdot_\sigma \tilde{w} \subseteq b_\nu, \text{ where } \nu = \left(\frac{d}{2}, \frac{d}{2}\right).$$

(2) If $l(w_a) > 0$ and $d + l(w_a)$ is even, then $\tilde{w}$ is I-fundamental and

$$I\tilde{w}I = I \cdot_\sigma \tilde{w} \subseteq b_\nu, \text{ where } \nu = \left(\frac{d + l(w_a)}{2}, \frac{d - l(w_a)}{2}\right).$$

(3) If $l(w_a) > 0$ and $d + l(w_a)$ is odd, then

$$I\tilde{w}I \cap b_\nu \neq \emptyset \iff \nu \preceq \left(\frac{d+l(w_a)-1}{2}, \frac{d-l(w_a)+1}{2}\right).$$

To see this, note that all the properties involved remain unchanged under conjugation by τ (which is the same as σ-conjugation by τ in this case), so by (26) we may assume that $w_a = s_a s s_a \cdots$. Under this assumption we calculate the element $\tilde{w}$ using the relations (26):

$$\begin{aligned}
&l(w_a) = 0 && &&\Longrightarrow \tilde{w} \in \Omega, \quad \tilde{w}^2 = \tau^{2d} = \epsilon^{(d,d)},\\
&l(w_a) = 2m > 0, && d = 2n &&\Longrightarrow \tilde{w} = (s_a s)^m \tau^{2n} = \epsilon^{(n+m,n-m)},\\
&l(w_a) = 2m+1, && d = 2n+1 &&\Longrightarrow \tilde{w} = (s_a s)^m s_a \tau \tau^{2n} = \epsilon^{(n+m+1,n-m)},\\
&l(w_a) = 2m > 0, && d = 2n+1 &&\Longrightarrow \tilde{w} = (s_a s)^m \tau^{2n} \tau = \epsilon^{(n+m+1,n-m)} \cdot s,\\
&l(w_a) = 2m+1, && d = 2n &&\Longrightarrow \tilde{w} = (s_a s)^m s_a \tau^{2n} = \epsilon^{(n+m+1,n-m-1)} \cdot s.
\end{aligned}$$

So case (1) is clear, and the other cases may be derived from the above results on EO-classes, using Proposition I.2.17(ii) in the form of (13): If $l(w_a) > 0$. then $\tilde{w}$ is the standard representative for its EO-class, and therefore $[[\tilde{w}]] \cap b \neq \emptyset \Leftrightarrow I\tilde{w}I \cap b \neq \emptyset$ for any $b \in B(G)$, further we know that the elements of the form ϵ^μ for dominant μ are I-fundamental by Example I.3.21.

I.3.6.2 The case $G = \mathrm{GU}(2,3)$

Now we consider the example of a unitary group. These unitary groups arise from a PEL-Datum as in Section II.3, where $B = \mathbb{E}$ is an imaginary quadratic extension of $\mathbb{Q}$ such that p is inert in $\mathbb{E}$, and where $*$ is the nontrivial automorphism of $\mathbb{E}$ over $\mathbb{Q}$. The group $G_{\mathcal{O}}$ is then isomorpic to $\mathrm{GL}_{n,\mathcal{O}} \times \mathbb{G}_{m,\mathcal{O}}$, and σ acts trivially on the $\mathbb{G}_{m,\mathcal{O}}$-factor, so it can often be neglected in arguments. In this case we have

$$X_*(T) \cong \mathbb{Z}^n \times \mathbb{Z}, \qquad (W,S) \cong (S_n, \{s_{12}, \ldots, s_{n-1n}\}),$$

where s_{ii+1} is the transposition $(i\ i+1)$, and the σ-action is given as

$$\sigma((x_1, \ldots, x_n; c)) = (c - x_n, \ldots, c - x_1; c), \qquad \sigma(w) = w_0 w w_0$$

(here the natural Borel pair of diagonal and upper triangular matrices is not σ-invariant). Consequently,

$$\begin{aligned}
\mathcal{N}(G) \cong &\{(x,c) \in \mathbb{Q}^n \times \mathbb{Q} \mid x_1 \geq \cdots \geq x_n,\ x_i + x_{n+1-i} = c\},\\
\pi_1(G)_{\langle\sigma\rangle} = &\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}, & n \text{ even},\\ \{0\} \times \mathbb{Z}, & n \text{ odd}.\end{cases}
\end{aligned}$$

We now turn to the special case that $n = 5$, and that μ is the minuscule cocharacter given as $(1^2, 0^3; 1) \in \mathbb{Z}^5 \times \mathbb{Z}$. In this case we can describe the intersections of σ-conjugacy classes and Ekedahl-Oort classes completely:

We have $\pi_1(G)_{\langle\sigma\rangle} = 0$, so an element $b \in B(G, \mu)$ can be identified with its dominant Newton vector. Further, all Newton vectors will be of the form $(\nu; 1)$ for

$\nu \in \mathbb{Q}^5$, we can omit the second entry. With these conventions the set $B(G,\mu)$ consists of the elements

$$\nu_{\mathrm{bas}} = \big((1/2)^5\big), \qquad \nu_1 = \big((3/4)^2,\ (1/2),\ (1/4)^2\big),$$
$$\nu_2 = \big(1,\ (1/2)^3,\ 0\big), \qquad \nu_{\mu-\mathrm{ord}} = \big(1^2,\ 1/2,\ 0^2\big),$$

it is totally ordered: We have $b_{\mathrm{bas}} \prec b_1 \prec b_2 \prec b_{\mu-\mathrm{ord}}$ for the corresponding σ-conjugacy classes.

We have $S_\mu = \{s_{12}, s_{34}, s_{45}\}$, therefore $J_\mu = \sigma(S_\mu) = \{s_{12}, s_{23}, s_{45}\}$. The well-known Coxeter structure of the symmetric group $W = S_5$ shows that

$$W^{J_\mu} = \{\pi \in S_5 \mid \pi(1) < \pi(2) < \pi(3),\ \pi(4) < \pi(5)\} = \{\pi_{ij} \mid 1 \le i < j \le 5\},$$

where π_{ij} is determined by the property that $\pi(4) = i$ and $\pi(5) = j$, so for example $\pi_{45} = \mathrm{id}$. We have $l(\pi_{ij}) = 9 - (i+j)$ and $\pi_{kl} \le \pi_{ij} \Leftrightarrow k \ge i$ and $l \ge j$, the Bruhat order on W^{J_μ} is thus given as follows (to be read from left to right):

$$\begin{tikzcd}
 & & & \pi_{15} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \pi_{14} \arrow[dl, dash] \arrow[dr, dash] & & \pi_{25} \arrow[dl, dash] \arrow[dr, dash] & & \\
\pi_{12} \arrow[r, dash] & \pi_{13} \arrow[dr, dash] & & \pi_{24} \arrow[dl, dash] \arrow[dr, dash] & & \pi_{35} \arrow[dl, dash] \arrow[r, dash] & \pi_{45} \\
 & & \pi_{23} & & \pi_{34} & &
\end{tikzcd}$$

Here elements in the same column have equal length. Every relation with respect to the Bruhat order is a relation with respect to the $\preceq$-order, further we know that two elements of equal length cannot be related under $\preceq$ if they are not equal (see Section I.3.1.1).

Set $\xi_{ij} := [[\tau_\mu \pi_{ij}]]$ for $1 \le i < j \le 5$. To calculate the map $\mathrm{Rep}\colon \mathrm{EO}(G,\mu) \to B(G,\mu)$ in this case, notice that $\sigma^2 = \mathrm{id}$. Furthermore $\sigma(\mu) = (1^3, 0^2; 1)$, and for each $\pi_{ij} \in W^{J_\mu}$ we have the relation

$$\tau_\mu \pi_{ij} \sigma(\tau_\mu \pi_{ij}) = \epsilon^\lambda (x_\mu \pi_{ij} w_0)^2, \quad \text{where } \lambda = \mu + (x_\mu \pi_{ij})(\sigma(\mu)),$$

as explained in Section I.2.5 some power of this element will determine the dominant Newton vector of $\tau_\mu \pi_{ij}$. Easy calculations now show that the map is given as follows:

ξ	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{34}	ξ_{35}	ξ_{45}
$\mathrm{Rep}(\xi)$	$\nu_{\mu-\mathrm{ord}}$	ν_2	ν_2	ν_{bas}	ν_{bas}	ν_1	ν_{bas}	ν_{bas}	ν_{bas}	ν_{bas}

We can now describe intersections of the ξ_{ij} with the σ-conjugacy classes in $B(G,\mu)$.

(1) It follows from Corollary I.3.14(i) in the form of Section I.3.2.4 that the classes $\xi_{45}, \xi_{35}, \xi_{34}, \xi_{25}$ and ξ_{15} are contained in b_{bas}. The element $\tau_\mu = \tau_\mu \pi_{45}$ is straight.

(2) We have $\langle 2\rho, \nu_1 \rangle = 3 = l(\tau_\mu \pi_{24})$. Therefore $\tau_\mu \pi_{24}$ is straight and thus ξ_{24} is contained in b_1.

(3) The variant of Corollary I.3.14(ii) from Section I.3.2.4 shows that the generic σ-conjugacy class in ξ_{23} is equal to b_1. Therefore ξ_{23} intersects b_{bas} and b_1 nontrivially.

(4) We have $\langle 2\rho, \nu_2 \rangle = 4 = l(\tau_\mu \pi_{14})$. Therefore $\tau_\mu \pi_{14}$ is straight and ξ_{14} is contained in b_2. Furthermore we have $\tau_\mu \pi_{13} \in b_2$, on the other hand $l(\tau_\mu \pi_{13}) = 5$, thus the length of every element in the $\widetilde{W}$-σ-conjugacy class of $\tau_\mu \pi_{13}$ will be odd and ≥ 4. So $\tau_\mu \pi_{13}$ is an element of minimal length in its σ-conjugacy class and therefore ξ_{13} is contained in b_2 by Criterion I.3.19.

(5) We have $\langle 2\rho, \nu_{\mu-\mathrm{ord}} \rangle = 6$, so $\tau_\mu \pi_{12}$ is straight and ξ_{12} is contained in $b_{\mu-\mathrm{ord}}$. Of course we already knew this from Section I.3.5.2.

In particular, we see that every σ-conjugacy class contains a K-fundamental EO-class whose standard representative is straight.

Remark I.3.43. We did not give a description of the complete $\preceq$ - order, since the knowledge of the Bruhat order suffices to determine the intersections. In fact, an explicit computation using formula (11) shows that in addition to the Bruhat relations we have $\pi_{34} \preceq \pi_{15}$.

I.4 Subgroups of scalar restrictions

In this section we consider the case that the group G sits inside an exact sequence

$$1 \longrightarrow G \longrightarrow \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} G_{0,\mathcal{O}_{F'}} \longrightarrow D \longrightarrow 1, \tag{27}$$

where $F'|F$ is a finite unramified extension, G_0 is a reductive group over $\mathcal{O}_F$, and D is commutative. In this case it is possible to relate the intersection problem I.3.2 for G to a similar problem for G_0, see Propositions I.4.3 and I.4.4 below. This is useful if the group G_0 better understood, for example if it is split. Situations like this arise in the context of PEL-Shimura varieties (see [VW], §2). We will treat the particularly simple "Hilbert-Blumenthal" case (where $G_0 = \mathrm{GL}_2$) using this method.

I.4.1 Two reduction steps

In the situation of (27) the question of intersections between σ-conjugacy classes and I-double cosets in $G(L)$ is often equivalent to a corresponding problem in $G_0(L)$. We will explain this in two steps: In the case that $G = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} G_{0,\mathcal{O}_{F'}}$ we reduce completely to G_0. Then we deal with the case that $F' = F$, i.e. that $G \subseteq G_0$ is normal and the quotient is commutative. In this case the reduction step works up to two conditions, which can be checked in our application.

I.4.1.1 Scalar restrictions and the "Norm map"

In this subsection we suppose that $G = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} G_{0,\mathcal{O}_{F'}}$ for some reductive group G_0 over $\mathcal{O}_F$, where $\mathcal{O}_{F'}$ is the valuation ring of an unramified extension $F'|F$ of finite degree d. Over $\mathcal{O}$, the group G then decomposes:

$$G_{\mathcal{O}} = \prod_{\gamma \in \Gamma} G_{0,\mathcal{O}},$$

where $\Gamma := \mathrm{Hom}_F(F', L) = \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_{F'}, \mathcal{O})$. The Frobenius σ acts simultanously on Γ and on the factors of the product. The set Γ is of order $d = [F' : F]$ and its σ-action is transitive. We identify $\Gamma \simeq \mathbb{Z}/d\mathbb{Z}$ (not canonically) such that for $g = (g^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}} \in \prod_{i\in\mathbb{Z}/d\mathbb{Z}} G_0(L)$ we have $\sigma(g) = (\sigma(g^{(i+1)}))_{i\in\mathbb{Z}/d\mathbb{Z}}$.

We consider the σ^d-conjugacy classes $B(G_0, \sigma^d)$ in $G_0(L)$. Since σ^d is the Frobenius automorphism of L over F', the set $B(G, \sigma^d)$ is classified by the maps

$$\nu_{G_0,\sigma^d} \colon B(G_0, \sigma^d) \to \mathcal{N}(G_0, \sigma^d), \qquad \kappa_{G_0,\sigma^d} \colon B(G_0, \sigma^d) \to \pi_1(G_0)_{\langle\sigma^d\rangle},$$

defined as in Section I.2.5 with σ replaced by σ^d. We have a "Norm map" (compare [AC] §2, where this map is used to study twisted orbital integrals):

$$\begin{aligned} \widetilde{N} \colon \prod_{i\in\mathbb{Z}/d\mathbb{Z}} G_0(L) &\longrightarrow G_0(L). \\ g = (g^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}} &\longmapsto \mathrm{pr}^{(0)}(g\sigma(g)\cdots\sigma^{d-1}(g)) \\ &\qquad = g^{(0)}\sigma(g^{(1)})\cdots\sigma^{d-1}(g^{(d-1)}). \end{aligned} \tag{28}$$

For $g, h \in G(L) = \prod_{\mathbb{Z}/d\mathbb{Z}} G_0(L)$ we have $\widetilde{N}(hg\sigma(h)^{-1}) = h^{(0)}\widetilde{N}(g)\sigma^d(h^{(0)})^{-1}$, thus $\widetilde{N}$ induces a map

$$N \colon B(G) \longrightarrow B(G_0, \sigma^d), \quad [g] \longmapsto [\widetilde{N}(g)]_{\sigma^d}. \tag{29}$$

Let (T_0, B_0) be a Borel pair for G_0 over $\mathcal{O}_F$, let (W_0, S_0) and I_0 be the associated Weyl and Iwahori group. Let (T, B) be the corresponding Borel pair for G obtained by scalar restriction, then as above we have decompositions

$$T_{\mathcal{O}} = \prod_{\gamma\in\Gamma} T_{0,\mathcal{O}}, \quad B_{\mathcal{O}} = \prod_{\gamma\in\Gamma} B_{0,\mathcal{O}},$$

and analogously for the data $X_*(T), (W, S), \widetilde{W}$ and I.

We study the map N via the classification of σ-conjugacy classes: By our identifications we have

$$\mathcal{N}(G) = (\prod_{i\in\mathbb{Z}/d\mathbb{Z}} (X_*(T_0)_{\mathbb{Q}}/W_0))^{\langle\sigma\rangle}, \qquad \pi_1(G) = \prod_{i\in\mathbb{Z}/d\mathbb{Z}} \pi_1(G_0),$$

where σ acts on the products by $\sigma((v^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}}) = (\sigma(v^{(i+1)}))_{i\in\mathbb{Z}/d\mathbb{Z}}$.

Lemma I.4.1. *We have the commutative diagrams*

$$\begin{tikzcd} B(G) \arrow[r, "N"] \arrow[d, "\nu_G"'] & B(G_0, \sigma^r) \arrow[d, "\nu_{G_0,\sigma^d}"] \\ \mathcal{N}(G) \arrow[r, hook, "d\cdot\mathrm{pr}^{(0)}"] & \mathcal{N}(G_0, \sigma^d), \end{tikzcd} \qquad \begin{tikzcd} B(G) \arrow[r, "N"] \arrow[d, "\kappa_G"'] & B(G_0, \sigma^r) \arrow[d, "\kappa_{G_0,\sigma^d}"] \\ \pi_1(G)_{\langle\sigma\rangle} \arrow[r, "\simeq"] & \pi_1(G_0)_{\langle\sigma^d\rangle}, \end{tikzcd}$$

where the isomorphism in the second diagram is induced by the map

$$\varphi \colon \prod_{i\in\mathbb{Z}/d\mathbb{Z}} \pi_1(G_0) \longrightarrow \pi_1(G_0), \quad x \longmapsto \sum_{i=0}^{d-1} \sigma^i(x^{(i)}).$$

The map $N \colon B(G) \to B(G_0, \sigma^d)$ is a bijection.

Proof. The lower horizontal map in the first diagram is well-defined and injective: If an element $v = (v^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}} \in X_*(T)_\mathbb{Q}/W = \prod_{i\in\mathbb{Z}/d\mathbb{Z}}(X_*(T_0)_\mathbb{Q}/W_0)$ is σ-invariant, then $v^{(i)} = \sigma^{-i}(v^{(0)})$ for each $i \in \mathbb{Z}/d\mathbb{Z}$, and in particular $v^{(0)}$ is invariant under σ^d.

To show the commutativity of the first diagram, we may work with representatives in $\widetilde{W}$: Let $b \in B(G)$, then by Corollary I.2.15 we have $b = [\tilde{w}]$ for some $\tilde{w} \in \widetilde{W}$. Choose $r \in \mathbb{N}$ such that σ^r acts trivially on $\widetilde{W}$ and $\nu_{\tilde{w}} = \frac{1}{r} \cdot (\tilde{w}\sigma(\tilde{w}) \cdots \sigma^{r-1}(\tilde{w}))$ (see Definition I.2.11). If we replace $\tilde{w}$ by $x\tilde{w}\sigma(x)^{-1}$ for some $x \in W$, then $\nu_{\tilde{w}}$ gets replaced by $x(\nu_{\tilde{w}})$, so we may further assume that $\nu_{\tilde{w}} = \nu_G(b)$, i.e. that

$$r \cdot \nu_G(b) = \tilde{w}\sigma(\tilde{w}) \cdots \sigma^{r-1}(\tilde{w}). \tag{30}$$

Now by definition

$$N(b) = [\widetilde{N}(\tilde{w})]_{\sigma^d} = [\mathrm{pr}^{(0)}(\tilde{w}\sigma(\tilde{w}) \cdots \sigma^{d-1}(\tilde{w}))]_{\sigma^d},$$

where the element $\widetilde{N}(\tilde{w})$ lies in $\widetilde{W_0}$, and $(\sigma^d)^r$ acts trivially on $\widetilde{W_0}$. Since (30) implies that

$$rd \cdot \nu_G(b) = \big(\tilde{w} \cdots \sigma^{d-1}(\tilde{w})\big)\sigma^d\big(\tilde{w} \cdots \sigma^{d-1}(\tilde{w})\big) \cdots \sigma^{(r-1)d}\big(\tilde{w} \cdots \sigma^{d-1}(\tilde{w})\big)$$

and $\mathrm{pr}^{(0)}$ commutes with σ^d, we see that

$$r(d \cdot \mathrm{pr}^{(0)}(\nu_G(b))) = \widetilde{N}(\tilde{w})\sigma^d(\widetilde{N}(\tilde{w})) \cdots \sigma^{(r-1)d}(\widetilde{N}(\tilde{w})).$$

The vector $d \cdot \mathrm{pr}^{(0)}(\nu_G(b))$ is dominant, so it is the dominant Newton vector of $N(b)$, which shows the commutativity of the first diagram.

A standard argument shows that the lower horizontal map in the second diagram is an isomorphism: Let $x, z \in \prod_{i\in\mathbb{Z}/d\mathbb{Z}} \pi_1(G_0)$, then $\varphi(x+(z-\sigma(z))) = \varphi(x)+(z^{(0)} - \sigma^d(z^{(0)}))$. So φ factors via $\pi_1(G)_{\langle\sigma\rangle}$, the induced map is clearly surjective. To see injectivity, note that every element of $(\prod_{i\in\mathbb{Z}/d\mathbb{Z}} \pi_1(G_0))_{\langle\sigma\rangle}$ has a representative x such that $x^{(i)} = 0$ if $i \neq 0 \pmod d$. If two such elements x, y are mapped to the same image in $\pi_1(G_0)_{\langle\sigma^d\rangle}$, then there is $z_0 \in \pi_1(G_0)$ such that $y = x + (z_0 - \sigma^d(z_0))$. Define $z \in \prod_{i\in\mathbb{Z}/d\mathbb{Z}} \pi_1(G_0)$ by $z^{(i)} = \sigma^{d-i}(z_0)$ for $i = 1, \dots, d$, then $y = x+(z-\sigma(z))$, so x and y represent the same element in $\pi_1(G')_{\langle\sigma\rangle}$.

Now the commutativity of the second diagram follows by an argument similar to the one above (but easier), using that $X_*(T) \to \pi(G)_{\langle\sigma\rangle}$ is surjective.

The two diagrams and the classification of σ-conjugacy classes (see Proposition I.2.8) show that N is injective. But clearly it is also surjective, since $\widetilde{N}\colon G(L) \to G_0(L)$ is surjective. ☐

Remark I.4.2. The map N is an explicit realization of the Shapiro bijection $B(G) \simeq B(G_0, \sigma^d)$ for the group $G_{0,\mathcal{O}_{F'}}$ (see [Ko1], §1).

Proposition I.4.3. *Let $\tilde{w} \in \widetilde{W}$, $\tilde{w} = (\tilde{w}^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}}$, let $b \in B(G)$. Then*

$$I\tilde{w}I \cap b \neq \emptyset \quad \Longleftrightarrow \quad I_0\tilde{w}^{(0)}I_0\sigma(\tilde{w}^{(1)})I_0 \cdots I_0\sigma^{d-1}(\tilde{w}^{(d-1)})I_0 \cap N(b) \neq \emptyset.$$

Proof. We have $\widetilde{N}(I\tilde{w}I) = I_0\tilde{w}^{(0)}I_0\sigma(\tilde{w}^{(1)})I_0\cdots I_0\sigma^{d-1}(\tilde{w}^{(d-1)})I_0$. If $g \in I\tilde{w}I \cap b \subseteq G(L)$, then by definition $\widetilde{N}(g) \in N(b)$ and $\widetilde{N}(g) \in \widetilde{N}(I\tilde{w}I)$, so the "$\Longrightarrow$"-implication is clear.

Conversely, suppose that $I_0\tilde{w}^{(0)}I_0\sigma(\tilde{w}^{(1)})I_0\cdots I_0\sigma^{d-1}(\tilde{w}^{(d-1)})I_0 \cap N(b) \neq \emptyset$, then there is $g \in I\tilde{w}I$ such that $\widetilde{N}(g) \in N(b)$, and by Lemma I.4.1 this implies that $[g] = b'$ in $B(G)$. □

I.4.1.2 Subgroups with abelian quotients

Now suppose that $G \subseteq G'$ is a normal subgroup, where G' is connected and reductive over $\mathcal{O}_F$, such that the quotient G'/G is commutative. In this situation, by ([SGA3], Exp. XXII, 6.3.1.) we have $(G')^{\mathrm{der}} \subseteq G$. Let (T', B') be a Borel pair of G' defined over $\mathcal{O}_F$, then

$$T := T' \cap G, \qquad\qquad B := B' \cap G$$

are a maximal torus and a Borel subgroup of G (see loc.cit. Exp. XXII, 6.2.8., 6.3.4.), defined over $\mathcal{O}_F$. Further, since by (loc.cit. Exp. XXII 6.2.1.) the natural homomorphism $\mathrm{rad}(G') \to G'/(G')^{\mathrm{der}}$ is surjective, the same holds for $\mathrm{rad}(G') \to G'/G$, and consequently we have a surjective homomorphism $\mathrm{rad}(G') \times G \to G'$. In particular this implies that $\mathrm{Z}(G) = \mathrm{Z}(G') \cap G$, and that the natural map $G^{\mathrm{ad}} \to (G')^{\mathrm{ad}}$ is an isomorphism.

We consider $X_*(T), (W, S), \widetilde{W}$ and I with respect to (T, B), and the corresponding objects for G' with respect to (T', B'). The equality $G^{\mathrm{ad}} = (G')^{\mathrm{ad}}$ implies that the natural map gives an isomorphism $(W, S) \cong (W', S')$, and that we have a canonical bijection between Φ and (Φ'). Further we have natural embeddings $\widetilde{W} \hookrightarrow \widetilde{W}'$ and $I \hookrightarrow I'$, and all these maps are σ-equivariant.

Let $B(G) \to B(G')$, $b \mapsto b'$ be the natural map given by $[g] \mapsto [g']'$. For any $\tilde{w} \in \widetilde{W} \subseteq \widetilde{W}'$ and $b \in B(G)$ we have the trivial implication $I\tilde{w}I \cap b \neq \emptyset \implies I'\tilde{w}I' \cap b' \neq \emptyset$.

Proposition I.4.4. *Suppose that the map* $\mathrm{rad}(G')(\mathcal{O}) \times T(\mathcal{O}) \to T'(\mathcal{O})$ *is surjective. Let* $\tilde{w} \in \widetilde{W} \subseteq \widetilde{W}'$*. If* $b \in B(G)$ *such that* $\kappa_G(b) = \kappa_G([\tilde{w}])$*, then*

$$I\tilde{w}I \cap b \neq \emptyset \iff I'\tilde{w}I' \cap b' \neq \emptyset.$$

Proof. We only have to show the "$\Longleftarrow$"-implication. Suppose that $g' \in I'\tilde{w}I' \cap b'$. By assumption, we have $T'(\mathcal{O}) = T(\mathcal{O}) \cdot \mathrm{rad}(G')(\mathcal{O})$. As for any root $\alpha \in \Phi = \Phi'$ the root subgroups U_α of G and G' coincide, the Iwasawa decomposition (see Section I.2.4) for I and I' implies that $I' = I \cdot \mathrm{rad}(G')(\mathcal{O})$, since further $\mathrm{rad}(G') \subseteq \mathrm{Z}(G')$ we therefore have

$$I'\tilde{w}I' = I\tilde{w}I \cdot \mathrm{rad}(G')(\mathcal{O}).$$

So we may write $g' = z'g$ for $g \in I\tilde{w}I$ and $z' \in \mathrm{rad}(G')(\mathcal{O})$. Since $\mathrm{rad}(G')$ is a torus defined over $\mathcal{O}_F$, by Lang's Theorem in the form of Corollary I.2.6 there is $y' \in \mathrm{rad}(G')(\mathcal{O})$ such that $z' = (y')^{-1}\sigma(y')$, hence $g = y'g'\sigma(y')^{-1}$ (as y' is central in G).

Thus we have found $g \in I\tilde{w}I$ such that $[g]' = [g']' = b'$ in $B(G')$. Now the natural map $\mathcal{N}(G) \to \mathcal{N}(G')$ is *injective*, which follows from the σ-equivariant embedding $X_*(T) \hookrightarrow X_*(T')$ and the equality $W \cong W'$, and both $\nu_G([g])$ and $\nu_G(b)$ are mapped to $\nu_{G'}(b')$, so we have $\nu_G([g]) = \nu_G(b)$. But since $g \in I\tilde{w}I$, by assumption we also have $\kappa_G([g]) = \kappa_G([\tilde{w}]) = \kappa_G(b)$, so the classification of $B(G)$ shows that $[g] = b$, i.e. $g \in I\tilde{w}I \cap b$. $\square$

Remark I.4.5. Let us comment on the two conditions in the last proposition:

(1) Suppose that the map $\pi_1(G)_{\langle\sigma\rangle} \to \pi_1(G')_{\langle\sigma\rangle}$ is injective. Since $\mathcal{N}(G) \to \mathcal{N}(G')$ is always injective, it follows that in this case the map $B(G) \to B(G')$ is injective, so that the condition on $\kappa_G(b)$ may be dropped. This is not the case in general, as the following example shows:

   Let $E|F$ be unramified with $[E : F] = 2$ and $T' = \operatorname{Res}_{\mathcal{O}_E/\mathcal{O}_F} \mathbb{G}_{m,\mathcal{O}_F}$. Let $c\colon T' \to \mathbb{G}_{m,\mathcal{O}_F}$ be the homomorphism given by the field norm $a \mapsto a\sigma(a)$ of E over F, and let $T := \ker(c)$. Then T is a torus of dimension 1, and over $\mathcal{O}$ the exact sequence $1 \to T \to T' \to \mathbb{G}_m \to 1$ identifies with

$$\begin{array}{ccccccccc} 1 \longrightarrow \mathbb{G}_{m,\mathcal{O}} & \longrightarrow & \mathbb{G}_{m,\mathcal{O}} \times \mathbb{G}_{m,\mathcal{O}} & \longrightarrow & \mathbb{G}_{m,\mathcal{O}} \longrightarrow 1 \\ x & \longmapsto & (x, x^{-1}) & & \\ & & (y,z) & \longmapsto & yz \end{array}$$

   The map $\mathbb{Z} \cong X_*(T) \to X_*(T') \cong \mathbb{Z}^2$ is thus given by $\lambda \mapsto (\lambda, -\lambda)$, the Frobenius σ acts on the right hand side by exchanging the entries and on the left hand side by $\lambda \mapsto -\lambda$. So we have $\pi_1(T)_{\langle\sigma\rangle} = \mathbb{Z}/2\mathbb{Z}$, but the map $\pi_1(T)_{\langle\sigma\rangle} \to \pi_1(T')_{\langle\sigma\rangle}$ is the zero map.

(2) The surjectivity of $\operatorname{rad}(G')(\mathcal{O}) \times T(\mathcal{O}) \to T'(\mathcal{O})$ is used in the proof of I.4.3 to conclude that for any $b \in B(G)$ and $\tilde{w} \in \widetilde{W}$ one has

$$I'\tilde{w}I' \cap b' \neq \emptyset \implies I\tilde{w}I \cap b' \neq \emptyset.$$

   This property does not seem to be automatic, although we do not know of an example where it fails.

   The map $\operatorname{rad}(G') \times T \to T'$ is always surjective as a morphism of group schemes, but it is in general not surjective on $\mathcal{O}$-valued points: For example, let $G' = \mathrm{GL}_{n,\mathcal{O}_F}$ and $G = \mathrm{SL}_{n,\mathcal{O}_F}$, and let T' be the subgroup of diagonal matrices. In this case $T \subseteq T'$ is the subtorus of elements of determinant 1, $\operatorname{rad}(G')$ is the one-dimensional torus of scalar matrices, and the map on $\mathcal{O}$-valued points identifies with $\mathcal{O}^\times \times T(\mathcal{O}) \to T'(\mathcal{O}),\ (a,t) \mapsto a \cdot t$, which is not surjective if p divides n.

I.4.2 The Hilbert-Blumenthal case

We now restrict our attention to a particular simple case of the situation of the last subsection: Let $F'|F$ be an unramified extension of finite degree d. Define G over $\mathcal{O}_F$ by the cartesian diagam

$$\begin{tikzcd} G \arrow[r] \arrow[d, hook] & \mathbb{G}_{m,\mathcal{O}_F} \arrow[d, hook, "\text{nat}"] \\ \operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}} \arrow[r, "\operatorname{Res}(\det)"] & \operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathbb{G}_{m,\mathcal{O}_{F'}}, \end{tikzcd} \tag{31}$$

where the right hand embedding $\operatorname{nat}\mathbb{G}_{m,\mathcal{O}_F} \hookrightarrow \operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathbb{G}_{m,\mathcal{O}_{F'}}$ is given by $R^\times \to (R \otimes_{\mathcal{O}_F} \mathcal{O}_{F'})^\times,\ r \mapsto r \otimes 1$. In other words, for any $\mathcal{O}_F$-algebra R by definition

$$G(R) = \{g \in \mathrm{GL}_2(\mathcal{O}_{F'} \otimes_{\mathcal{O}_F} R) \mid \det(g) \in R^\times\}.$$

Groups of this type arise from a PEL-datum as in Section II.3, where $B = \mathbb{E}$ is a totally real extension of $\mathbb{Q}$ such that p is inert in $\mathbb{E}$, where $*$ is the identity on $\mathbb{E}$, and where V is a vector space of dimension two over $\mathbb{E}$.

Let us check that G is a reductive group over $\mathcal{O}_F$: We have an exact sequence $1 \to \operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{SL}_{2,\mathcal{O}_{F'}} \to G \to \mathbb{G}_{m,\mathcal{O}_F} \to 1$, where kernel and cokernel have geometrically connected fibers over $\mathcal{O}_F$, thus the same is true for G. So it follows from ([SGA3], Exp. XXII, 6.3.3.) that G is reductive. We are then in the situation of (27): By definition G is normal in $\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}}$, and the quotient is commutative.

I.4.2.1 Reduction to GL_2

In $\mathrm{GL}_{2,\mathcal{O}_F}$ we consider the Borel pair (T_0, B_0) of diagonal matrices and upper triangular matrices. Following I.4.1.1 and I.4.1.2, we set accordingly

$$T = (\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} T_{0,\mathcal{O}_{F'}}) \cap G, \qquad B = (\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} B_{0,\mathcal{O}_{F'}}) \cap G.$$

Over $\mathcal{O}$ everything decomposes, we then have the identification

$$G(\mathcal{O}) = \{(g^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}} \in \prod_{\mathbb{Z}/d\mathbb{Z}} \mathrm{GL}_2(\mathcal{O}) \mid \det(g^{(i)}) = \text{const}\},$$

where $\sigma((g_i)_{i\in\mathbb{Z}/d\mathbb{Z}}) = (\sigma(g^{(i+1)}))_{i\in\mathbb{Z}/d\mathbb{Z}}$. Analogous decompositions hold for $G(L)$, and also for B and T. We identify $X_*(T_0)$ with $\mathbb{Z}^2$, then

$$X_*(T) = \{(\mu^i)_{i\in\mathbb{Z}/d\mathbb{Z}} \in \prod_{\mathbb{Z}/d\mathbb{Z}} \mathbb{Z}^2 \mid \mu_1^{(i)} + \mu_2^{(i)} = \text{const}\}.$$

For the Weyl group of G with respect to T we have $W \cong \prod_{\mathbb{Z}/d\mathbb{Z}} W_0$, where $W_0 = S_2 = \{1, s\}$, and $\widetilde{W} \subseteq \widetilde{W}_0$ is the subgroup of those elements whose translation parts lie in $X_*(T)$. A cocharacter $\mu = (\mu^{(i)})_{i\in\mathbb{Z}/d\mathbb{Z}} \in X_*(T)$ is dominant if and only if $\mu_1^{(i)} \geq \mu_2^{(i)}$ for all $i \in \mathbb{Z}/d\mathbb{Z}$. Since $G_0 = \mathrm{GL}_{2,\mathcal{O}_F}$ is split, the σ-action on $X_*(T)$, W and $\widetilde{W}$ is simply given by cyclic shift on the entries of $(\cdot)_{i\in\mathbb{Z}/d\mathbb{Z}}$.

The map $\mathcal{N}(G) \to \mathcal{N}(\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}})$ is a bijection: $X_*(T)_{\mathbb{Q}}/W$ is the image of the diagonal embedding $X_*(T_0)_{\mathbb{Q}}/W_0 \hookrightarrow \prod_{\mathbb{Z}/d\mathbb{Z}}(X_*(T_0)_{\mathbb{Q}}/W_0)$, and this image is exactly the set of σ-invariant elements in $\prod_{\mathbb{Z}/d\mathbb{Z}}(X_*(T_0)_{\mathbb{Q}}/W_0)$. We identify

$$\mathcal{N}(G) \cong \mathcal{N}(\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}}) \cong X_*(T_0)_{\mathbb{Q}}/W_0 \cong \{x \in \mathbb{Q}^2 \mid x_1 \geq x_2\}.$$

Similarly, the map $\pi_1(G) \to \pi_1(\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}})$ identifies with the diagonal embedding $\mathbb{Z} \hookrightarrow \prod_{\mathbb{Z}/d\mathbb{Z}} \mathbb{Z}$, where σ acts trivially on the left hand side and by cyclic shift on the right hand side. So we have $\pi_1(G)_{\langle\sigma\rangle} \cong \mathbb{Z}$, and $\pi_1(G)_{\langle\sigma\rangle} \to \pi_1(\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}})_{\langle\sigma\rangle}$ is injective. By the classification of σ-conjugacy classes, the natural map $B(G) \to B(\operatorname{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}})$ is injective as well. Precomposing it with the norm map from Section I.4.1.1 (for the group $G_0 = \mathrm{GL}_{n,\mathcal{O}_F}$) yields an injective map $N\colon B(G) \to B(\mathrm{GL}_2, \sigma^d)$, which is induced by the restriction of the map $\widetilde{N}\colon \prod_{\mathbb{Z}/d\mathbb{Z}} \mathrm{GL}_2(L) \to \mathrm{GL}_2(L)$ of (28) to $G(L)$. Now we have the following reduction lemma for G:

Lemma I.4.6. *Let $\tilde{w} \in \widetilde{W}$, let $b \in B(G)$, then*

$$I\tilde{w}I \cap b \neq \emptyset \iff I_0\tilde{w}^{(0)}I_0\tilde{w}^{(1)}I_0 \cdots I_0\tilde{w}^{(d-1)}I_0 \cap N(b) \neq \emptyset.$$

Proof. First, note that the statement is independent of the residue characteristic $\mathrm{char}(k) = p$: The product on the right hand side is a disjoint union of I_0-double cosets which is determined by the combinatorial structure of $\widetilde{W}_0$. So by Proposition I.3.4 the emptyness or nonemptyness of the intersections on both sides of the equivalence depends only on the structure of $\widetilde{W}$ and $\widetilde{W}_0$, the σ-action on $\widetilde{W}$, and the maps $\widetilde{W} \to B(G)$ and $\widetilde{W}_0 \to B(\mathrm{GL}_2, \sigma^d)$, and all these are independent of p.

So we may assume without loss of generality that $p \neq 2$. By Proposition I.4.3 the intersection on the right hand side is nonempty if and only if $(\prod_{i \in \mathbb{Z}/d\mathbb{Z}} I_0 \tilde{w}^{(i)} I_0) \cap b \neq \emptyset$ in $\prod_{\mathbb{Z}/d\mathbb{Z}} \mathrm{GL}_2(L)$. Now we apply Proposition I.4.4 to $G \subseteq \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}}$: We have seen above that the corresponding map of fundamental groups is injective, so the condition $\kappa_G([\tilde{w}]) = \kappa_G(b)$ can be neglected (Remark I.4.5(1)). As for the second condition, we have

$$\mathrm{rad}(\mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}})(\mathcal{O}) = \prod_{\mathbb{Z}/d\mathbb{Z}} \mathrm{rad}(\mathrm{GL}_2)(\mathcal{O}),$$

$$T(\mathcal{O}) = \{t \in \prod_{i \in \mathbb{Z}/d\mathbb{Z}} (\mathcal{O}^\times)^2 \mid t_1^{(i)} t_2^{(i)} = \mathrm{const}\} \subseteq \prod_{\mathbb{Z}/d\mathbb{Z}} (\mathcal{O}^\times)^2 = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} T_{0,\mathcal{O}_{F'}}(\mathcal{O}),$$

and $\mathrm{rad}(\mathrm{GL}_2)(\mathcal{O})$ consists of the scalar matrices associated to $\mathcal{O}^\times$. Since the map $\mathcal{O}^\times \to \mathcal{O}^\times,\ a \mapsto a^2$ is surjective for $p \geq 3$ (which is not the case if $p = 2$), it follows that $\prod_{\mathbb{Z}/d\mathbb{Z}} (\mathcal{O}^\times)^2 = (\prod_{\mathbb{Z}/d\mathbb{Z}} \mathrm{rad}(\mathrm{GL}_2)(\mathcal{O})) \cdot T(\mathcal{O})$. So Proposition I.4.4 applies and shows that

$$\left(\prod_{i \in \mathbb{Z}/d\mathbb{Z}} I_0 \tilde{w}^{(i)} I_0 \right) \cap b \neq \emptyset \quad \Longleftrightarrow \quad I \tilde{w} I \cap b \neq \emptyset.$$

□

I.4.2.2 Study of the minuscule case

We still use the notation of the last subsection. We now consider the dominant cocharacter $\mu \in X_*(T)$ with $\mu^{(i)} = (1,0) \in X_*(T_0)$ for all $i \in \mathbb{Z}/d\mathbb{Z}$. This is, up to an element of $X_*(\mathrm{rad}(G))$ which only changes the situation up to a shift of Newton vectors, the unique dominant and minuscule element in $X_*(T)$. The main result of this section gives a precise description of the intersections between EO-classes and σ-conjugacy classes in $K\epsilon^\mu K$, using Lemma I.4.6 and the detailed study of the GL_2-case in Section I.3.6.1.

We determine the sets $\mathrm{EO}(G,\mu)$ and $B(G,\mu)$ in this case:

(I) Recall that $W = \prod_{\mathbb{Z}/d\mathbb{Z}} W_0 = \prod_{\mathbb{Z}/d\mathbb{Z}} \{1, s\}$. We have $S_\mu = J_\mu = \{1\} \subseteq W$, so $W^{J_\mu} = W$ and

$$\tau_\mu = \epsilon^\mu w_0 = (\tau_\mu^{(i)})_{i \in \mathbb{Z}/d\mathbb{Z}}, \quad \tau_\mu^{(i)} = \epsilon^{(1,0)} s \ \text{ for all } i \in \mathbb{Z}/d\mathbb{Z}.$$

The set $\mathrm{EO}(G,\mu)$ thus consists of the classes $[[\tau_\mu w]]$, $w \in W$, the order $\preceq$ coincides with the Bruhat order on W. As a partially ordered set, $\mathrm{EO}(G,\mu)$ is isomorphic to the power set of $\mathbb{Z}/d\mathbb{Z}$ oredered by inclusion.

(II) As $\pi_1(G)_{\langle\sigma\rangle} \cong \mathbb{Z}$ is torsion free, an element of $B(G)$ is determined by its dominant Newton point (see Proposition I.2.8). Recall the identification

$$\mathcal{N}(G) \cong \mathcal{N}(\mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}}) \cong \{x \in \mathbb{Q}^2 \mid x_1 \geq x_2\}.$$

The partial order $\preceq$ on $\mathcal{N}(G)$ is given as $x \preceq x' \;\Leftrightarrow\; x_1 \leq x_1',\; x_1+x_2 = x_1'+x_2'$. The cocharacter μ is σ-invariant, so we have $\bar{\mu} = \mu = (1,0) \in \mathcal{N}(G)$.
Note that the classification of σ^d-conjugacy classes in $\mathrm{GL}_2(L)$ is exactly the same as for σ-conjugacy classes (since σ^d is the Frobenius of $\mathcal{O}$ over $\mathcal{O}_{F'}$). We use the convention of Section I.3.6.1 and identify elements of $B(\mathrm{GL}_2, \sigma^d)$ with their Newton polygons. By the first diagram in Lemma I.4.1, for any $b \in B(G)$ the element $d \cdot \nu_G(b) \in \mathcal{N}(\mathrm{GL}_{2,\mathcal{O}_F}, \sigma^d)$ is the Newton polygon of a σ^d-conjugacy class of $\mathrm{GL}_2(L)$ and has therefore integral break points. So the possible Newton vectors of elements in $B(G,\mu) = \{b \in B(G) \mid \nu_G(b) \preceq (1,0)\}$ are

$$\nu_m := \left(\frac{m}{d}, \frac{d-m}{d}\right),\; [d/2]+1 \leq m \leq d; \quad \nu_{\mathrm{bas}} := \left(\frac{1}{2}, \frac{1}{2}\right).$$

It is easy to see that all these lie in the image of ν_G (and it will also follow from Corollary I.4.11 and the discussion of σ-balanced elements at the end of the section). We write

$$B(G,\mu) = \{b_m \mid [d/2]+1 \leq m \leq d\} \;\cup\; \{b_{d/2}\},$$

where $\nu_G(b_m) = \nu_m$ and $b_{d/2} = b_{\mathrm{bas}}$ is the basic element in $B(G,\mu)$ with $\nu_G(b_{d/2}) = \nu_{\mathrm{bas}}$. With this notation we have $b_m \preceq b_{m'}$ if and only if $m \leq m'$, in particular $\preceq$ is a total order on $B(G,\mu)$.

To desribe the intersection pattern for $\xi \in \mathrm{EO}(G,\mu)$ and $b \in B(G,\mu)$ we use the following notation: By an *interval* in $\mathbb{Z}/d\mathbb{Z}$ we mean a subset of the form

$$[i, i+k-1] := \{i, i+1, \ldots, i+k-1\} \subseteq \mathbb{Z}/d\mathbb{Z}, \quad 1 \leq k \leq d.$$

We then call k the *length* of the interval. We use the "parity function"

$$\mathrm{par}\colon \mathbb{Z} \longrightarrow \{0,1\}, \quad \mathrm{par}(a) = \begin{cases} 0, & a \text{ even} \\ 1, & a \text{ odd.} \end{cases}$$

Definition I.4.7. Let $w \in W = \prod_{\mathbb{Z}/d\mathbb{Z}} W_0$.

(a) A *gap* γ of w is an interval $\gamma = [i, i+k-1] \subseteq \mathbb{Z}/d\mathbb{Z}$ such that $w^{(i)} = w^{(i+1)} = \cdots = w^{(i+k-1)} = 1$ and $w^{(i-1)} = w^{(i+k)} = s$.

(b) Let $\{\gamma_1, \ldots, \gamma_t\}$ be the gaps of w, let $k_j \in \{1, \ldots, d-1\}$ be the length of γ_j. We define

$$\begin{aligned} m(1) &:= \frac{d}{2} \\ m(w) &:= d - \sum_{j=1}^{t} \left[\frac{k_j+1}{2}\right] = d - \sum_{j=1}^{t} \frac{k_j + \mathrm{par}(k_j)}{2}, \qquad w \neq 1. \end{aligned} \tag{32}$$

(c) For $\xi = [[\tau_\mu w]] \in \mathrm{EO}(G,\mu)$ we set $m(\xi) := m(w)$.

Example I.4.8. Let $d = 7$, write $w \in W = \prod_{\mathbb{Z}/7\mathbb{Z}} W_0$ in the form $w = (w^{(0)}, \ldots, w^{(6)})$. Then the element $w = (1, s, 1, s, 1, s, 1)$ has two gaps of length 1 and one gap of length 2, and we have $m(w) = 7 - 3 = 4$.

Note that by definition the elements $w = 1$ and $w = w_0$ have no gaps. In the situation of (b) for $w \neq 1$ we have $l(w) = d - \sum_{j=1}^{t} k_j$, so we may write uniformly

$$m(w) = \frac{d + l(w)}{2} - \sum_{j=1}^{t} \frac{\mathrm{par}(k_j)}{2} \qquad \text{for all } w \in W, \tag{33}$$

the sum being zero for $w = 1$.

Remark I.4.9. Our definition compares to the ones of Goren and Oort as follows: In [GO] to each Ekedahl-Oort stratum the authors associate a "type" $\tau \subseteq \mathbb{Z}/d\mathbb{Z}$ (this is not to be confused with τ_μ or with the element $\tau \in \widetilde{W}_0$ which will appear in the proof of I.4.10), and a number $\lambda(\tau)$ (loc. cit. Def. 5.4.8.). Going through the definitions, one sees that if the stratum is associated to $\xi = [[\tau_\mu w]] \in \mathrm{EO}(G, \mu)$ for $w = (w^{(i)})_{i \in \mathbb{Z}/d\mathbb{Z}}$, then the type of ξ is given as $\tau = \{i \in \mathbb{Z}/d\mathbb{Z} \mid w^{(i)} = 1\}$, and $\lambda(\tau)$ is equal to $d - m(w)$.

Proposition I.4.10. *Let G be given as in (31), let (T, B) be as in Section I.4.2.1. Let $\mu = (1, 0)^d \in X_*(T)$. Let $\xi \in \mathrm{EO}(G, \mu)$, write $\tilde{w}_\xi = \tau_\mu w$ for $w \in W = \prod_{\mathbb{Z}/d\mathbb{Z}} W_0$. If $w \neq 1$, let $\{\gamma_1, \ldots, \gamma_t\}$ be the gaps of w, let $\{k_1, \ldots, k_t\}$ be their lengths, let $m(\xi) = m(w)$ be as in (32). Then exactly one of the following cases applies:*

- *(I) $w = 1$ or $w \neq 1$ and all k_j $(j = 1, \ldots, t)$ are even. Then $\tilde{w}_\xi$ is straight, ξ is K-fundamental (in the sense of Definition I.3.18) and*

$$\xi = K \cdot_\sigma \tau_\mu w \subseteq b_{m(w)}.$$

- *(II) $w \neq 1$ and there is some $j \in \{1, \ldots, t\}$ such that k_j is odd. Then for $b \in B(G, \mu)$ we have*

$$\xi \cap b \neq \emptyset \iff b \preceq b_{m(w)}.$$

  Further in this case $\tilde{w}_\xi$ is not straight.

Proof. Since μ is minuscule, the statement for $w = 1$ is clear from Remark I.3.36. From now on, we consider the case that $w \neq 1$. As a first step, we reduce to the case that w is of a special form: We write $w = (w^{(0)}, \ldots, w^{(d-1)})$. As μ is σ-invariant, we have an action of σ on $\mathrm{EO}(G, \mu)$ by $\sigma([[\tau_\mu w]]) = [[\tau_\mu \sigma(w)]]$. Since all the properties involved remain unchanged under this action we may assume without loss of generality that

$$w = (s^{n_1}, 1^{k_1}, \ldots, s^{n_t}, 1^{k_t}) \tag{34}$$

(here the exponents mean repetition of an entry), where $n_1, \ldots, n_t \geq 1$ and $\sum_{j=1}^{t} n_j + k_j = d$. In case (II) we may further assume that k_t is odd.

Next we reduce to $\mathrm{GL}_2(L)$: By (13) and Lemma I.4.6, for any σ-conjugacy class $b \in B(G, \mu)$ we have

$$\begin{aligned} [[\tau_\mu w]] \cap b \neq \emptyset &\iff I \tau_\mu w I \cap b \neq \emptyset \\ &\iff I_0 (\tau_\mu w)^{(0)} I_0 \cdots I_0 (\tau_\mu w)^{(d-1)} I_0 \ \cap N(b) \neq \emptyset. \end{aligned} \tag{35}$$

Let us investigate the right hand side of this equivalence. We have

$$N(b_m) = (m, d - m) \quad \text{for all} \ \ d/2 \leq m \leq d$$

under the identification of $B(\mathrm{GL}_2, \sigma^d)$ with Newton polygons. We keep up the notation of Section I.3.6.1: Let $\tau := \epsilon^{(1,0)}s$, let $s_a := \epsilon^{(1,-1)}s$ be the simple affine reflexion in W_a. So $(\tau_\mu w)^{(i)} = \tau w^{(i)}$ for all $i \in \mathbb{Z}/d\mathbb{Z}$. Since ${}^\tau I_0 = I_0$, denoting $\varphi := \mathrm{int}(\tau)\colon W_a \to W_a$ we may calculate

$$I_0(\tau_\mu w)^{(0)} I_0 \cdots I_0(\tau_\mu w)^{(d-1)} I_0 = I_0\varphi(w^{(0)})I_0 \cdots I_0\varphi^d(w^{(d-1)})I_0 \cdot \tau^d,$$

and

$$\varphi^{i+1}(w^{(i)}) = \begin{cases} 1, & \text{if } w^{(i)} = 1, \\ s_a, & \text{if } w^{(i)} = s \text{ and } i \text{ is even}, \\ s, & \text{if } w^{(i)} = s \text{ and } i \text{ is odd}. \end{cases} \tag{36}$$

From our normalization of w in (34) it follows that

$$I_0(\tau_\mu w)^{(0)} I_0 \cdots I_0(\tau_\mu w)^{(d-1)} I_0 = I_0 x_1 I_0 \cdots I_0 x_t I_0 \cdot \tau^d, \tag{37}$$

where the $x_j \in W_a$ are elements of length $l(x_j) = n_j$, further by the properties of I_0-double cosets we have

$$I_0 x_1 I_0 \cdots I_0 x_t I_0 = \coprod_{x \in X} I_0 x I_0 \tag{38}$$

for some subset $X \subseteq \{y \in W_a \mid y \leq x_1 \cdots x_t\}$.

Suppose that k_1 is even. Then by (36) the last letter of x_1 and the first letter of x_2 are not equal and hence $l(x_1x_2) = l(x_1) + l(x_2)$, which implies that $I_0x_1I_0x_2I_0 = I_0x_1x_2I_0$. On the other hand, suppose that k_1 is odd, then the last letter $s' \in \{s, s_a\}$ of x_1 is equal to the first letter of x_2. In this case $I_0x_1I_0x_2I_0 = I_0x_1s'x_2I_0 \cup \coprod_{x\in X'} I_0xI_0$ for some subset $X' \subseteq \{x \in W_a \mid x < x_1sx_2\}$, furthermore the last letter of x_1sx_2 is equal to the last letter of x_2. Iterating this argument, we see that in any case the set X in (38) contains a *unique element* $x_{\max}$ *of maximal length*, that

$$l(x_{\max}) = \sum_{j=1}^{t} n_j - \sum_{j=1}^{t-1} \mathrm{par}(k_j) = d - \sum_{j=1}^{t-1}(k_j + \mathrm{par}(k_j)) - k_t,$$

and that $X = \{x_{\max}\}$ if all $k_1, \ldots, k_{t-1}$ are even.

Now we consider the cases (I) and (II):

(I) If all $k_1, \ldots, k_t$ are even, then by (37), (38) and the above considerations we have

$$I_0(\tau_\mu w)^{(0)} I_0 \cdots I_0(\tau_\mu w)^{(d-1)} I_0 = I_0 x_{\max}\tau^d I_0,$$

where $l(x_{\max}) = d - \sum_{j=1}^{t} k_j$. Since this implies that $d + l(x_{\max})$ is even, by Example I.3.42(2) we have $I_0x_{\max}\tau^dI_0 \subseteq b_\nu$ in $\mathrm{GL}_2(L)$ for

$$\nu = \left(\frac{2d - \sum_{j=1}^{t} k_j}{2}, \frac{\sum_{j=1}^{t} k_j}{2} \right) = (m(w), d - m(w)),$$

which implies that $[[\tau_\mu w]] \subseteq b_{m(w)}$ by (35). Further in this case we have

$$\langle 2\rho, \nu_{m(w)} \rangle = 2m(w) - d = \sum_{j=1}^{t} n_j = l(w) = l(\tau_\mu w),$$

so $\tau_\mu w$ is straight and hence $[[\tau_\mu w]]$ is K-fundamental.

(II) Now suppose that some k_j is odd. Then k_t is odd by our normalization of w. By (37), (38) and the considerations above, in this case we find

$$I_0(\tau_\mu w)^{(0)} I_0 \cdots I_0(\tau_\mu w)^{(d-1)} I_0 \supseteq I_0 x_{\max} \tau^d I_0,$$

where $l(x_{\max}) = d - \sum_{j=1}^{t-1}(k_j + \mathrm{par}(k_j)) - k_t$. As k_t is odd, we then have $l(x_{\max}) - 1 = d - \sum_{j=1}^{t}(k_j + \mathrm{par}(k_j))$, and further $d + l(x_{\max})$ is odd. So by Example I.3.42(3), $I_0 x_{\max} \tau^d I_0$ meets exactly the σ^d-conjugacy classes in $\mathrm{GL}_2(L)$ corresponding to

$$\nu \preceq \left(\frac{d + l(x_{\max}) - 1}{2}, \frac{d - l(x_{\max}) + 1}{2} \right) = (m(w), d - m(w)).$$

By (35), this implies that $[[\tau_\mu w]] \cap b \neq \emptyset$ for $b \preceq b_{m(w)}$. On the other hand, for all elements $x \in X$ in the decomposition (38) we have $l(x) \leq l(x_{\max})$, so from Example I.3.42(2)+(3) it follows that $[[\tau_\mu w]] \cap b = \emptyset$ if $b \succ b_{m(w)}$.
If $m(w) > \frac{d}{2}$ then we have just seen that ξ intersects at least two σ-conjugacy classes, so $\tilde{w}_\xi$ cannot be straight. If $m(w) = \frac{d}{2}$ then we have $\tilde{w}_\xi \in \xi \subseteq b_{d/2}$. But $\langle 2\rho, \nu_{d/2} \rangle = 0$ and $l(\tilde{w}_\xi) = l(w) > 0$ by assumption, thus $\tilde{w}_\xi$ is not straight.

□

Corollary I.4.11. *Let (G, μ) be as in Proposition I.4.10.*

(i) For all $\xi \in \mathrm{EO}(G, \mu)$, the generic σ-conjugacy class in ξ is given by $b_{m(\xi)}$.

(ii) Let $\xi \in \mathrm{EO}(G, \mu)$. If $b \in B(G, \mu)$ such that $\xi \cap b \neq \emptyset$, then there is a $\xi' \in \mathrm{EO}(G, \mu)$ which is of type (I) in the sense of Proposition I.4.10 such that $\xi' \preceq \xi$ and $\xi' \subseteq b$.
In particular, every $b \in B(G, \mu)$ contains some K-fundamental class in $\mathrm{EO}(G, \mu)$.

(iii) Let $\xi \in \mathrm{EO}(G, \mu)$ be of type (I) in I.4.10, let $m' < m(\xi)$. Then there is a $\xi' \preceq \xi$ of type (II) with $m(\xi') = m'$.

Together with Proposition I.4.10 this proves Theorem I.E from the introduction.

Proof of I.4.11: (i) was already obtained in ([GO], Thm. 5.4.11.), it also follows directly from Propositon I.4.10. For (ii) and (ii), let $\xi \in \mathrm{EO}(G, \mu)$, let $\tilde{w}_\xi = \tau_\mu w$ for $w \in W$, then $m(\xi) = m(w)$ by definition. From the fromula (33) for $m(w)$ we see that ξ is of type (I) in Proposition I.4.10 if and only if $m(w) = \frac{d+l(w)}{2}$ and of type (II) if and only if $m(w) < \frac{d+l(w)}{2}$. Therefore (ii)+(iii) follow from the following lemma:

Lemma I.4.12. *Let $w \in W$.*

(i) If $m(w) < \frac{d+l(w)}{2}$, then there is a $w' < w$ such that $l(w') = l(w) - 1$ and $m(w') = m(w)$.

(ii) If $w \neq 1$ and $m(w) = \frac{d+l(w)}{2}$, then there is $w' < w$ with $l(w') = l(w) - 1$ and either $m(w') = m(w) - 1$ or $w' = 1$ (in this case $m(w') = m(w) - \frac{1}{2}$).

Proof. To see this, it is enough to consider the case that $w \neq 1$. Just as in the proof of I.4.10, after applying some power of σ we may then suppose that $w = (s^{n_1}, 1^{k_1}, \dots, s^{n_t}, 1^{k_t})$ for $n_1, \dots, n_t \geq 1$, $\sum_{j=1}^t n_j + k_j = d$. By (33), we are in the situation of (2) if and only if all $k_1, \dots, k_t$ are even, otherwise we are in the situation of (1), and in this case we may further assume w.l.o.g. that k_t is odd. Set $w' := (s^{n_1}, 1^{k_1}, \dots, s^{n_t - 1}, 1^{k_t+1})$ if $n_t > 1$ and $w' = (s^{n_1}, 1^{k_1}, \dots, s^{n_{t-1}}, 1^{k_{t-1}+k_t+1})$ if $n_t = 1$, then $w' < w$ with $l(w') = l(w) - 1$, and in both cases formula (33) shows that $m(w')$ has the demanded property. $\square$

We conclude this chapter by explaining some of the phenomena and concepts, which were only mentioned before, in the situation of the Hilbert-Blumenthal example:

The map Rep In the situation of Proposition I.4.10 we can give the following description of the map Rep: $\mathrm{EO}(G, \mu) \to B(G, \mu)$ from Section I.3.2.3:

Let $\xi \in \mathrm{EO}(G, \mu)$, let $\tilde{w}_\xi = [[\tau_\mu w]]$ for $w \in W$. Then $\mathrm{Rep}(\xi)$ is determined by its image under the injective map $N\colon B(G) \to B(\mathrm{GL}_2, \sigma^d)$, and by the definitions made in Section I.4.1.1, $N(\mathrm{Rep}(\xi))$ is the σ^d-conjugacy class of $\widetilde{N}(\tau_\mu w) = \mathrm{pr}^{(0)}(\tau_\mu w \sigma(\tau_\mu w) \cdots \sigma^{d-1}(\tau_\mu w))$. This can be made more explicit by reviewing the proof of I.4.10: If $w = 1$, then $\mathrm{Rep}(\xi) = b_{d/2} = b_{\mathrm{bas}}$. If $w \neq 1$, then we may suppose that w is normalized as in (34), in this case $\widetilde{N}(\tau_\mu w)$ is the σ^d-conjugacy class of $x_1 \cdots x_t \tau^d \in \widetilde{W}$, where $x_1, \dots, x_t \in W_a$ are the elements which appear in formula (37), they can be calculated via (36). If $(m, d - m)$ is the Newton polygon which corresponds to this conjugacy class, then $\mathrm{Rep}(\xi) = b_m$.

We now give an example which reflects the seemingly erratic behavior of the map Rep in the case that ξ is not fundamental:

Example I.4.13. Let (G, μ) be as before, where $d = 10$. In this case the elemens of $B(G, \mu)$ are the b_m for $m = 5, \dots, 10$. Consider $\xi \in \mathrm{EO}(G, \mu)$, $\xi = [[\tau_\mu w]]$ for $w \in W = \{1, s\}^{10}$.

1. Let $w = (s^7, 1, s, 1)$.
   Here in the notation of (34) we have $t = 2$ and $n_1 = 7, k_1 = 1, n_2 = 1, k_2 = 1$. So $m(w) = 10 - (1 + 1) = 8$, and one sees easily that $x_1 = s_a s s_a s s_a s s_a$ and $x_2 = s_a$. As in Section I.3.6.1 we calculate
   $$[x_1 x_2 \tau^{10}] = [(s_a s)^3 \tau^{10}] = [\epsilon^{(8,2)}],$$
   as this corresponds to the Newton polygon $(8, 2)$, we have $\mathrm{Rep}(\xi) = b_8$.

2. Let $w = (s^5, 1, s^3, 1)$.
   Here we have $n_1 = 5, k_1 = 1, n_2 = 3, k_2 = 1$, therefore $m(w) = 10 - (1+1) = 8$ and $x_1 = s_a s s_a s s_a, x_2 = s_a s s_a$. In this case,
   $$[x_1 x_2 \tau^{10}] = [s_a s \tau^{10}] = [\epsilon^{(6,4)}]$$
   corresponds to $(6, 4)$, so $\mathrm{Rep}(\xi) = b_6$.

3. Let $w = (s^5, 1^2, s^2, 1)$.
   Here $n_1 = 5, k_1 = 2, n_2 = 2, k_2 = 1$, so $m(w) = 10 - (1 + 1) = 8$ and $x_1 = s_a s s_a s s_a, x_2 = s s_a$. In this case we have
   $$[x_1 x_2 \tau^{10}] = [(s_a s)^3 s_a \tau^{10}] = [\epsilon^{(9,1)} s],$$
   which corresponds to the basic Newton polygon $(5, 5)$, so $\mathrm{Rep}(\xi) = b_5$.

In all three cases, by Proposition I.4.10 we have $\{b \in B(G,\mu) \mid \xi \cap b_m \neq 0\} = \{b_5, b_6, b_7, b_8\}$. In (1), $\mathrm{Rep}(\xi)$ gives the maximal element occuring in this set, while in (3) it is the minimal element, and in (2) neither the one nor the other.

σ-balanced short elements Recall the notion of σ-balanced short elements in Section I.3.4. In the present example we can make many of the observations from this Section very explicit. We have seen in Corollary I.4.11 that for every $b \in B(G,\mu)$ there is an EO-class $\xi \in \mathrm{EO}(G,\mu)$ such that $\tilde{w}_\xi \in W\epsilon^\mu W$ is a straight representative for b. Theorem I.3.32 and Criterion I.3.34 therefore predict the existence of many σ-balanced short elements in $W\epsilon^\mu W$. These elements have a very simple description:

We exclude the trivial case $b = b_{\mathrm{bas}}$. We have $\Phi^+ = \coprod_{\mathbb{Z}/d\mathbb{Z}} \Phi_0^+$, where Φ_0^+ consist of the unique positive root for GL_2, and where σ acts via cyclic shift. Let $b = b_m$ for $d/2 < m \leq d$. Then the type of b is $J_b = \emptyset$, therefore $M_b = T$ and we have $\widetilde{W}_{J_b} = \Omega_{J_b} = X_*(T)$. So b-short elements will be of the form ϵ^λ for $\lambda \in X_*(T)$. For any $\lambda = (\lambda^{(i)})_{i \in \mathbb{Z}/d\mathbb{Z}} \in X_*(T)$ the dominant Newton vector of the σ-conjugacy class $[\epsilon^\lambda]$ is just the dominant representative of $\frac{1}{d}\sum_{i=0}^{d-1} \lambda^{(i)} \in \mathbb{Q}^2$. On the other hand we have $\epsilon^\lambda \in W\epsilon^\mu W$ if and only if λ lies in $W \cdot \mu \subseteq X_*(T)$, which is the case if and only if $\lambda^{(i)} \in \{(1,0),(0,1)\}$ for each $i \in \mathbb{Z}/d\mathbb{Z}$.

For $b = b_m$ and $\lambda \in W \cdot \mu$ we now have

$$\epsilon^\lambda \text{ for } \lambda = (\lambda^{(i)})_{\mathbb{Z}/d\mathbb{Z}} \text{ is } b\text{-short} \iff |\{i \in \mathbb{Z}/d\mathbb{Z} \mid \lambda^{(i)} = (1,0)\}| = m.$$

Further by definition such an element is σ-balanced if and only if for each $i \in \mathbb{Z}/d\mathbb{Z}$ the following holds:

$$\lambda^{(i)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \lambda^{(i+1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{39}$$

Since $m > d/2$, is easy to see that this condition can always be satisfied, and that in most cases it can be satisfied in many ways, but that it is not always satisfied.

For example, let $d = 6$, let $b = b_4$. Then the cocharacters

$$\lambda = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}^2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}),$$
$$\lambda' = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}^3, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

give rise to two σ-balanced b-short elements which are not in the same σ-orbit. Further Proposition I.3.31(ii) asserts that σ-conjugation with $y = (1,1,s,1,1,s)$, respectively with $y' = (1,1,1,s,1,s)$, will produce straight elements. A calculation shows that these elements turn out to be the standard representatives $\tau_\mu w$ and $\tau_\mu w'$ for

$$w = (s,1,1,s,1,1) \qquad w' = (s,s,1,1,1,1),$$

they are not in the same σ-orbit either. This phenomenon was already observed in ([VW], Ex. 9.14.). On the other hand, the cocharacter

$$\lambda'' = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}^4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}^2)$$

will give an element which is b-short, but is not σ-balanced.

Remark I.4.14. Observe that the combinatoric considerations above would also have easily shown the existence of straight representatives in $W\epsilon^{\mu}W$ for each $b \in B(G, \mu)$ by Criterion I.3.34, even without knowing Proposition I.4.10. Similar arguments show that this criterion is also satisfied if $G = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{2,\mathcal{O}_{F'}}$ and $\mu \in X_*(T)$ is an arbitraty dominant and minuscule cocharacter, and also in the case that $G = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{3,\mathcal{O}_{F'}}$ and μ is minuscule and σ-invariant. We suspect that it might be possible to show Criterion I.3.34 for all groups of type $\mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F} \mathrm{GL}_{n,\mathcal{O}_{F'}}$, however, the combinatorics tend to get more complicated if the type J_b of b is nontrivial.

Chapter II

Stratifications for Shimura varieties of Hodge type

II.1 Introduction

In this chapter we explain the Newton stratification and Ekedahl-Oort stratification and the existence of a map γ with the properties mentioned in the conclusion for canonical integral models of Shimura varieties of Hodge type. As a main application the results from the first chapter allow to prove that the μ-ordinary locus for Hodge type Shimura varieties is open and dense, and to give a precise description of its geometric points, which generalizes results that were known in the case of PEL-Shimura varieties.

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ and let X be a $\mathbf{G}(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ such that the pair $(\mathbf{G}, X)$ is a Shimura datum (see Section II.3 for details). Then for any sufficiently small open and compact subgroup $K \subseteq \mathbf{G}(\mathbb{A}_f)$ the associated Shimura variety

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f)/K$$

is a smooth, quasi-projective complex variety and admits a canonical model over the reflex field E of the Shimura datum $(\mathbf{G}, X)$.

Let p be a prime number. Suppose that K_p is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$, and let $\mathrm{Sh}_{K_p}(\mathbf{G}, X) := \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(\mathbf{G}, X)$, where the limit is taken over open compact subgroups $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$. In [La] Langlands suggested that $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ should have an integral canonical model $\mathscr{S}_{K_p}(\mathbf{G}, X)$ over the local ring $o_{E,(v)}$ at any place v of E lying above p, this conjecture was later refined by Milne in [Mi1], see Definition II.3.2 for the precise notion of an integral canonical model. In particular, if $\mathscr{S}_{K^p}(\mathbf{G}, X)$ exists, then for any open compact $K = K_pK^p \subseteq \mathbf{G}(\mathbb{A}_f)$, where K^p is sufficiently small, the quotient $\mathscr{S}_K(\mathbf{G}, X) := \mathscr{S}_{K_p}(\mathbf{G}, X)/K^p$ is a smooth model for the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ over $O_{E,(v)}$.

We will consider the case that the Shimura datum is of Hodge type. In this case the existence of integral canonical models for $\mathrm{Sh}_K(\mathbf{G}, X)$ was shown by Kisin in [Ki1], with some restrictions for $p = 2$. To explain the Newton stratification and Ekedahl-Oort stratification we need to briefly explain the construction of these models: There exists a symplectic vector space (V, ψ) such that there is an embedding

$(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V,\psi), S^{\pm})$ of Shimura data (by the very definition of being of Hodge type). As K_p is hyperspecial, there is a connected reductive group scheme G over $\mathbb{Z}_p$ with generic fiber $\mathbf{G}_{\mathbb{Q}_p}$ such that $K_p = G(\mathbb{Z}_p) \subseteq \mathbf{G}(\mathbb{Q}_p)$. The starting point of the construction is the choice of a lattice $\Lambda \subseteq V$ and a finite set of tensors s over $\Lambda_{\mathbb{Z}_{(p)}}$ such that $\mathbf{G}$ is the stabilizer of $s_{\mathbb{Q}}$ in $\mathrm{GL}(V)$ and $G \subseteq \mathrm{GL}(\Lambda_{\mathbb{Z}_p})$ is the stabilizer of $s_{\mathbb{Z}_p}$. The model $\mathscr{S}_K(\mathbf{G}, X)$ is then defined as the normalization of the closure of $\mathrm{Sh}_K(\mathbf{G}, X)$ in a suitable moduli space of abelian schemes over $o_{E,(v)}$, thus $\mathscr{S}_K(\mathbf{G}, X)$ by construction naturally comes along with an abelian scheme $\mathcal{A}$ on it. The tensors s can be shown to give rise to tensors s_{dR} over $H^1_{\mathrm{dR}}(\mathcal{A} \otimes E/\mathrm{Sh}_K(\mathbf{G}, X))$. The key step in the proof that the schemes $\mathscr{S}_K(\mathbf{G}, X)$ indeed give an integral canonical model for $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ is now to show that $\mathscr{S}_K(\mathbf{G}, X)$ is *smooth*. The technique used in [Ki1] to achieve this shows at the same time that the s_{dR} extend to tensors s°_{dR} over $H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_K(\mathbf{G}, X))$, in a way such that for any closed or geometric point x of $\mathscr{S}_K(\mathbf{G}, X) \otimes \kappa(v)$ one gets induced tensors over the contravariant Dieudonné module $\mathbb{D}(\mathcal{A}_x[p^\infty])$ which are Frobenius invariant and define a subgroup isomorphic to $G_{W(k(x))}$.

We make this more precise: Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. For simplicity, throughout the rest of the introduction we restrict ourselves to $\overline{\mathbb{F}}_p$-valued points. (In fact, to introduce the stratifications we also need points over more general algebraically closed fields and points with finite residue fields. However, since all strata will be locally closed subsets, the $\overline{\mathbb{F}}_p$-valued points contain all topological informations on the stratifications, once defined.) In analogy to the notations in Chapter I set $\mathcal{O} := W(\overline{\mathbb{F}}_p)$ and $L := \mathrm{Frac}(\mathcal{O})$, let σ be the Frobenius automorphism of $\overline{\mathbb{F}}_p$, respectively of $\mathcal{O}$ and L. We fix a σ-invariant Borel pair (B, T) of G over $\mathcal{O}$ (which exists since G is quasisplit). Our Shimura datum defines in the usual way a conjugacy class $[\nu]$ of cocharacters for $\mathbf{G}$ (see Definition II.3.1), and hence for G. We define μ as the unique dominant cocharacter in $X_*(T)$ with respect to B such that $\sigma^{-1}(\mu)^{-1}$ lies in $[\nu]$. Let Λ^* be the dual $\mathbb{Z}$-module of Λ. The tensors s can also be viewed as tensors over $\Lambda^*_{\mathbb{Z}_{(p)}}$ in a canonical way. We let G act on $\Lambda^*_{\mathbb{Z}_p}$ via the contragredient representation

$$\mathrm{GL}(\Lambda) \longrightarrow \mathrm{GL}(\Lambda^*),\ g \longmapsto g^{\vee} := (g^{-1})^*.$$

Now let x be an $\overline{\mathbb{F}}_p$-valued point of $\mathscr{S}_K(\mathbf{G}, X)$, and let $(\mathbb{D}_x, F, V) := \mathbb{D}(\mathcal{A}_x[p^\infty])$ be the associated contravariant Dieudonné module, then the following hold (see Corollary II.4.8, Lemma II.5.4):

Lemma II.A.

(1) The tensors s°_{dR} induce F-invariant tensors $s_{\mathrm{cris},x}$ on $\mathbb{D}_x$, and there is an isomorphism of $\mathcal{O}$-modules $\Lambda^*_{\mathcal{O}} \simeq \mathbb{D}_x$ which identifies $s_{\mathcal{O}}$ with $s_{\mathrm{cris},x}$.

(2) If we identify $\mathbb{D}_x$ with $\Lambda^*_{\mathcal{O}}$ using an isomorphism as in (1), then $F = g^{\vee}(1 \otimes \sigma)$ for some $g \in G(\mathcal{O})\mu(p)G(\mathcal{O})$, and this element is independent of the choice of the isomorphism up to σ-conjugation by an element of $G(\mathcal{O})$.

This is an immediate consequence of the results in [Ki1], though it is not explicitly stated there. See also ([Ki2], §1). The dual lattice appears here due to the fact that we use contravariant Dieudonné theory, this is also the reason for our slightly nonstandard definition of μ.

This lemma implies that, writing $C(G,\mu)$ for the set of $G(\mathcal{O})$-σ-conjugacy classes of the double coset $G(\mathcal{O})\mu(p)G(\mathcal{O}) \subseteq G(L)$, we have a well-defined map

$$\gamma\colon \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p) \longrightarrow C(G,\mu) \tag{$*$}$$

by sending x to the $G(\mathcal{O})$-conjugacy class of the element g, and it is easy to see that $\gamma(x) = \gamma(x')$ if and only if there is an isomorphism $\mathbb{D}_x \simeq \mathbb{D}_{x'}$ compatible with $s_{\mathrm{cris},x}$ and $s_{\mathrm{cris},x'}$ (Lemma II.5.5).

The Newton stratification is easily described in this context: Let $B(G)$ be the set of $G(L)$-σ-conjugacy classes in $G(L)$, then there is a canonical map $\tilde{\theta}\colon C(G,\mu) \to B(G)$. The image of $\tilde{\theta}$ is exactly the subset $B(G,\mu) \subseteq B(G)$ which already showed up in Chapter I (see Section I.3.1.1). Recall that it is endowed with a partial order $\preceq$, and contains a unique maximal element $b_{\mu-\mathrm{ord}} \in B(G,\mu)$ with respect to this order. The Newton strata are now given as the fibers $\mathcal{N}^b = \theta^{-1}(\{b\})$ of the composite map

$$\theta\colon \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p) \xrightarrow{\gamma} C(G,\mu) \xrightarrow{\tilde{\theta}} B(G).$$

Equivalently, two points $x, x' \in \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p)$ lie in the same Newton stratum if and only if there is an isomorphism of isocrystals $\mathbb{D}_x \otimes_{\mathcal{O}} L \simeq \mathbb{D}_{x'} \otimes_{\mathcal{O}} L$ which respects the tensors $s_{\mathrm{cris},x}$ and $s_{\mathrm{cris},x'}$. These strata $\mathcal{N}^b$ are in fact already defined over $\kappa(v)$, and a result of Vasiu ([Va1], 5.3.1.) shows that they are locally closed subsets of $\mathscr{S}_K(\mathbf{G},X) \otimes \kappa(v)$. If the Shimura variety is of PEL-type, then much more is known (see Section II.5.2 for a more detailed discussion).

The Ekedahl-Oort stratification on $\mathscr{S}_K(\mathbf{G},X) \otimes \overline{\mathbb{F}}_p$ has been defined by Zhang in [Zh1]. Just as in the case of a PEL-type Shimura variety its definition relies on the theory of $G_{\mathbb{F}_p}$-zips developed in [PWZ1], [PWZ2]. The precise construction is quite technical, we only state the main results here and refer to Section II.5.3 for details: Let (W,S) be the Weyl group of G with respect to (B,T), and let $J \subseteq S$ be the type of the cocharacter $\sigma(\mu)$. Then the Ekedahl-Oort strata $\mathcal{S}^w \subseteq \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p)$ are parametrized by the set ${}^JW$ of shortest left coset representatives for W_J in W. The set ${}^JW$ carries a partial order $\preceq$, which refines the Bruhat order. By the results of [Zh1], each stratum $\mathcal{S}^w \subseteq \mathscr{S}_K(\mathbf{G},X) \otimes \overline{\mathbb{F}}_p$ is locally closed, and the closure of $\mathcal{S}^w$ is precisely the union of the strata $\mathcal{S}^{w'}$ with $w' \preceq w$. Furthermore two points x, x' lie in the same stratum if and only if there is an isomorphism of Dieudonné spaces $\overline{\mathbb{D}_x} \simeq \overline{\mathbb{D}_{x'}}$ respecting the tensors on both sides.

The map γ defined in $(*)$ has the properties 1. and 2. mentioned in the conclusion: Recall the set $\mathrm{EO}(G,\mu)$ of $[[\cdot]]$-classes contained in $G(\mathcal{O})\mu(p)G(\mathcal{O})$, as defined in Chapter I (using a different notation, see Sections I.2.6 and I.3.1.1). By definition, every $[[\cdot]]$-class is a union of $G(\mathcal{O})$-σ-conjugacy classes, we may therefore view it as a subset of $C(G,\mu)$. In the same way every $b \in B(G,\mu)$ defines a subset of $C(G,\mu)$. Now by definition we have $\theta = \tilde{\theta} \circ \gamma$, and the fiber of $\tilde{\theta}$ over $b \in B(G,\mu)$ is exactly $b \cap C(G,\mu)$. On the other hand, the $\overline{\mathbb{F}}_p$-valued points of the strata $\mathcal{S}^w$ are by definition the fibers of a map $\zeta\colon \mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p) \to {}^JW$. We show in Proposition II.6.6 that there is a surjective map

$$\tilde{\zeta}\colon C(G,\mu) \longrightarrow {}^JW$$

such that $\zeta = \tilde{\zeta} \circ \gamma$ and such that the fibers of $\tilde{\zeta}$ are exactly the classes $[[\cdot]] \in \mathrm{EO}(G,\mu)$. Further we show that the resulting bijection between $\mathrm{EO}(G,\mu)$ and ${}^JW$ identifies ${}^JW$ with a set of standard representatives as considered in Chapter I and is compatible with the $\preceq$-orders on both sides (Lemma II.6.8). Consequently we have a commutative diagram

$$\begin{tikzcd}
 & & B(G,\mu) \\
\mathscr{S}_K(\mathbf{G},X)(\overline{\mathbb{F}}_p) \arrow[urr, "\theta"] \arrow[r, "\gamma"] \arrow[drr, "\zeta"'] & C(G,\mu) \arrow[ur, "\tilde{\theta}"'] \arrow[dr, "\tilde{\zeta}"] & \\
 & & {}^J W
\end{tikzcd}$$

where $\gamma(x)$ and $\gamma(x')$ are mapped to the same element in $B(G,\mu)$ if and only if they lie in the same $[\cdot]$-class in $C(G,\mu)$, and to the same element in ${}^JW$ if and only if they lie in the same $[[\cdot]]$-class in $C(G,\mu)$. We may now apply the results of the first chapter to $\mathscr{S}_K(\mathbf{G},X)\otimes\overline{\mathbb{F}}_p$.

We define the *μ-ordinary locus* $\mathcal{N}^{\mu-\mathrm{ord}}$ in $\mathscr{S}_K(\mathbf{G},X)\otimes\kappa(v)$ as the Newton stratum associated to the maximal element $b_{\mu-\mathrm{ord}}\in B(G,\mu)$. This is the natural generalization of the ordinary locus in a moduli space of abelian varieties $\mathscr{M}$, which is by definition the set of those points $x\in\mathscr{M}$ such that the p-divisible group associated to the abelian variety at a geometric point $\hat{x}$ lying over x is isogenous to a product of étale and multiplicative groups. It has been known for a long time that this ordinary locus is a dense subset of $\mathscr{M}$, see for example Koblitz' proof in [Kob]. For Shimura varieties of PEL-type, the μ-ordinary locus was shown to be open and dense by Wedhorn in [We1] by a deformation theoretic argument. Moonen gave another proof in [Mo2], where he showed that the μ-ordinary Newton stratum coincides with the unique open Ekedahl-Oort stratum, which was by then known to be dense (see [We2]). We follow this second approach: The set ${}^JW$ has a unique maximal element $w_{\max}$ with respect to $\preceq$, by the result of Zhang we know that the corresponding Ekedahl-Oort stratum $\mathcal{S}^{w_{\max}}$ is open and dense. Now Theorem I.D from the introduction of the first chapter implies the following:

Theorem II.B. *The strata $\mathcal{N}^{\mu-\mathrm{ord}}$ and $\mathcal{S}^{w_{\max}}$ are equal as subsets of $\mathscr{S}_K(\mathbf{G},X)\otimes\overline{\mathbb{F}}_p$. Furthermore, for any two $\overline{\mathbb{F}}_p$-valued points x,x' in this set there is an isomorphism of Dieudonné modules $\mathbb{D}_x\simeq\mathbb{D}_{x'}$ which identifies $s_{\mathrm{cris},x}$ with $s_{\mathrm{cris},x'}$.*

As a consequence, we obtain the annonced result on the μ-ordinary locus, which proves a conjecture of Rapoport from 1996.

Theorem II.C. *The μ-ordinary locus is open and dense in $\mathscr{S}_K(\mathbf{G},X)\otimes\kappa(v)$.*

We remark that these questions have also been studied by Vasiu, using a different language, in [Va3].

As another application, the commutative diagram described above allows to translate the combinatorial study of the Hilbert-Blumenthal case in Theorem I.E into geometric properties of the special fiber of the corresponding PEL-Shimura variety, which is a "Hilbert moduli space" of abelian varieties endowed with the action of (a localization of) the ring of integers of a totally real extension of $\mathbb{Q}$, see Proposition II.6.10.

This Chapter is structured as follows: In Section II.2 we fix our main notations and conventions on tensors and σ-linear algebra. In Section II.3 we introduce Shimura varieties of Hodge type and of PEL-type, and explain the notion of a canonical integral model. We review the construction of integral models in the Hodge type case and of the tensors s_{dR} and s_{cris} in Section II.4, we strictly follow the treatment in [Ki1] and do not claim any originality here. In Section II.5 we discuss the Newton and Ekedahl-Oort stratification. In Section II.6, which is the technical heart of this chapter, we finally establish the existence of the map $\tilde{\zeta}$ and its properties, and explain the Theorems II.B and II.C.

II.2 Notations and conventions

II.2.1. For a perfect field k of positive characteristic p we write $W(k)$ for the Witt ring over k, and $L(k)$ for its quotient field. In this chapter σ will generally denote the Frobenius automorphism $a \mapsto a^p$ of k, and also its lift to $W(k)$ and $L(k)$.

Let k be a perfect field of characteristic p. Let R be either k or $W(k)$, and let $R_0 \subseteq R$ be the subring of elements which are fixed by σ (i.e., either $R_0 = \mathbb{F}_p$ or $R_0 = \mathbb{Z}_p$). For any R-module M let $M^{(\sigma)} := M \otimes_{R,\sigma} R$, and for a homomorphism $\beta\colon M \to N$ of R-modules write $\beta^{(\sigma)} := \beta \otimes 1\colon M^{(\sigma)} \to N^{(\sigma)}$. If $f\colon M \to N$ is a σ-linear map of R-modules then

$$M^{(\sigma)} \longrightarrow N, \quad m \otimes a \longmapsto af(m)$$

is R-linear, and if f is σ^{-1}-linear then

$$M \longrightarrow N^{(\sigma)}, \quad m \longmapsto f(m) \otimes 1$$

is R-linear. In both cases we call the resulting homomorphism the *linearization* of f and denote it by f^{lin}.

Now let M_0 be an R_0-module, and let $M = M_0 \otimes_{R_0} R$. Then σ and σ^{-1} act on M via $1 \otimes \sigma$ and $1 \otimes \sigma^{-1}$ respectively. Further, there is a canonical isomorphism

$$M = M_0 \otimes_{R_0} R \xrightarrow{\sim} M \otimes_{R_0} R \otimes_{R,\sigma} R = M^{(\sigma)}, \quad m \otimes a \mapsto m \otimes 1 \otimes a.$$

We will often use this isomorphism to identify M with $M^{(\sigma)}$. For example, if $f\colon M \to N$ is σ-linear, we also write $f^{\mathrm{lin}}\colon M \cong M^{(\sigma)} \to N$, with this notation we then have that $f = f^{\mathrm{lin}} \circ (1 \otimes \sigma)$.

If M_0 is a finitely generated free R_0-module, then σ also acts on $\mathrm{GL}(M)$ and on the group of cocharacters $\mathrm{Hom}_R(\mathbb{G}_{m,R}, \mathrm{GL}(M))$. For $g \in \mathrm{GL}(M)$ we have $\sigma(g) = (1 \otimes \sigma) \circ g \circ (1 \otimes \sigma^{-1})$, and for a cocharacter $\lambda\colon \mathbb{G}_{m,R} \to \mathrm{GL}(M)$ we find that $\sigma(\lambda)(a) = \sigma(\lambda(a))$ for all $a \in R$.

II.2.2. Let R be any ring. If M is a finitely generated free module over R, we denote by $M^{\otimes}$ the direct sum of all R-modules that arise from M by applying the operations of taking duals, tensor products, symmetric powers and exterior powers a finite number of times. An element of $M^{\otimes}$ will be called a *tensor* over M. We have an obvious notion of base change for tensors. Let M^* be the dual R-module of M. Since there is a canonical identificaton of $M^{\otimes}$ with $(M^*)^{\otimes}$ we can view tensors over M as tensors over M^* as well.

Let M and M' be finitely generated free R-modules and let $s = (s_i)_{i \in I}$ and $s' = (s'_i)_{i \in I}$ be families of tensors over M and M' respectively. Every isomorphism

$f\colon M \to M'$ induces an isomorphism $(f^{-1})^*\colon (M)^* \to (M')^*$ and thus also an isomorphism $f^{\otimes}\colon M^{\otimes} \to (M')^{\otimes}$. We will write

$$f\colon (M, s) \longrightarrow (M', s')$$

if and only if $f^{\otimes}$ takes s_i to s_i' for all $i \in I$. We say that a family of tensors $(s_i)_{i\in I}$ over M *defines* the subgroup $G \subseteq \mathrm{GL}(M)$ if

$$G(R') = \{g \in \mathrm{GL}(M_{R'}) \mid g^{\otimes}((s_i)_{R'}) = (s_i)_{R'} \text{ for all } i \in I\}$$

for every R-algebra R'. We have the contragredient representation

$$(\cdot)^{\vee}\colon\ \mathrm{GL}(M) \longrightarrow \mathrm{GL}(M^*), \quad g \longmapsto g^{\vee} := (g^{-1})^*, \tag{40}$$

which is in fact an isomorphism of group schemes over R. Let $(s_i)_{i\in I}$ be a family of tensors over M, defining a subgroup $G \subseteq \mathrm{GL}(M)$. Then these tensors $(s_i)_{i\in I}$, when we consider them as tensors over M^*, define the subgroup $\{g^{\vee} \mid g \in G\} \subseteq \mathrm{GL}(M^*)$.

II.3 Shimura varieties of Hodge type

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ and let X be a $\mathbf{G}(\mathbb{R})$-conjugacy class of algebraic morphisms $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ such that $(\mathbf{G}, X)$ is a Shimura datum, i.e. such that the following conditions are satisfied for each $h \in X$:

(SV1) $\mathrm{ad}\, h$ defines a Hodge structure of type $\{(-1, 1), (0, 0), (1, -1)\}$ on the Lie algebra of $\mathbf{G}_{\mathbb{R}}$.

(SV2) $\mathrm{ad}\, h(i)$ is a Cartan involution of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$.

(SV3) $\mathbf{G}^{\mathrm{ad}}$ has no simple factor defined over $\mathbb{Q}$ on which h projects trivially.

We suppose that the Shimura datum is of Hodge type, by definition this means that there is an embedding

$$(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm}) \tag{41}$$

into a symplectic Shimura datum, which we fix once and for all. Here (V, ψ) is a symplectic vector space over $\mathbb{Q}$ and $S^{\pm}$ is the associated Siegel double space, that is, the set of homomorphisms $h\colon \mathbb{S} \to \mathrm{GSp}(V_{\mathbb{R}}, \psi_{\mathbb{R}})$ which induce a Hodge structure of type $(-1, 0), (0, -1)$ and give rise to a (positive or negative) definite symmetric form $(v, w) \mapsto \psi(v, h(i)w)$ on $V_{\mathbb{R}}$. We will often simply write $\mathrm{GSp}(V)$ for $\mathrm{GSp}(V, \psi)$ with the symplectic pairing implied.

We will say that $(\mathbf{G}, X)$ is of *PEL-type* if it arises from a PEL-datum $(B, *, V, \psi, h)$, where

- $(B, *)$ is a finite dimensional semisimple $\mathbb{Q}$-algebra with positive involution,
- V is a finite dimensional $\mathbb{Q}$-vector space on which B acts on the left,
- ψ is a symplectic form on V with $\psi(bv, w) = \psi(v, b^*w)$ for all $v, w \in V, b \in B$,

such that $\mathbf{G}$ is given by

$$\mathbf{G}(R) = \{g \in \mathrm{GSp}(V_R, \psi_R) \mid g(bv) = bg(v) \text{ for all } b \in B_R, v \in V_R\}$$

for every $\mathbb{Q}$-algebra R, and where $h\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ is such that its composition with the natural embedding $\mathbf{G} \subseteq \mathrm{GSp}(V, \psi)$ lies in the Siegel double space of (V, ψ). In this case X is the $\mathbf{G}(\mathbb{R})$-conjugacy class of h and $(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$ is the obvious embedding.

The datum $(\mathbf{G}, X)$ defines conjugacy classes of cocharacters for $\mathbf{G}$ as follows: Every element $h \in X$ defines a Hodge decomposition $V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$ via the embedding $X \hookrightarrow S^{\pm}$.

Definition II.3.1. (i) We define ν_h to be the cocharacter of $\mathbf{G}_{\mathbb{C}}$ such that $\nu_h(z)$ acts on $V^{(-1,0)}$ through multiplication by z and on $V^{(0,-1)}$ as the identity.

(ii) We denote by $[\nu]$ the unique $\mathbf{G}(\mathbb{C})$-conjugacy class which contains all the cocharacters ν_h, and by $[\nu^{-1}]$ the conjugacy class which contains the ν_h^{-1}.

It is an immediate consequence of the condition (SV1) on a Shimura datum that all the cocharacters ν_h and ν_h^{-1} are *minuscule.* The *reflex field* E of $(\mathbf{G}, X)$ is defined as the field of definition of $[\nu]$ (or equivalently of $[\nu^{-1}]$), this is known to be a finite extension of $\mathbb{Q}$.

We fix a prime number p such that $\mathbf{G}$ is of good reduction at p. Let $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ be a hyperspecial subgroup. Consider subgroups of the type $K = K_p K^p \subseteq \mathbf{G}(\mathbb{A}_f)$, where $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ is open and compact. If K^p is sufficiently small, then the double quotient

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/K$$

(where $\mathbf{G}(\mathbb{Q})$ acts diagonally and K acts on the right factor) has a natural structure as a smooth, quasi-projective variety over $\mathbb{C}$, and further this variety has a canonical model over the reflex field E. In the sequel we will always view $\mathrm{Sh}_K(\mathbf{G}, X)$ as an algebraic variety over E.

The projective limit

$$\mathrm{Sh}_{K_p}(\mathbf{G}, X) := \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}(\mathbf{G}, X),$$

taken over the set of open and compact subgoups of $\mathbf{G}(\mathbb{A}_f^p)$, carries a contiuous right action of $\mathbf{G}(\mathbb{A}_f^p)$ in the sense of Deligne (see [Mi1], 2.1.):
Elements $g \in \mathbf{G}(\mathbb{A}_f^p)$ act by isomorphisms $\mathrm{Sh}_{K_p(gK^pg^{-1})}(\mathbf{G}, X) \to \mathrm{Sh}_{K_pK^p}$ in a way that every $g \in K^p$ gives the identity map on $\mathrm{Sh}_{K_pK^p}(\mathbf{G}, X)$ and that for every normal subgroup $K'^p \subseteq K^p$ the natural covering map induces an isomorphism $\mathrm{Sh}_{K_pK'^p}(\mathbf{G}, X)/(K^p/K'^p) \simeq \mathrm{Sh}_{K_pK^p}(\mathbf{G}, X)$. In particular, we have an equality $\mathrm{Sh}_{K_pK^p}(\mathbf{G}, X) = \mathrm{Sh}_{K_p}(\mathbf{G}, X)/K^p$ for every open and compact $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$.

We fix a place v of E over p. The existence of the hyperspecial subgroup K_p implies that E is unramified at p ([Mi2], 4.7.). Let o_E be the ring of integers in E, and let $o_{E,(v)}$ be its localization at v.

Definition II.3.2 (cf. [Mi1], §2)**.** An *integral canonical model* of $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ over $o_{E,(v)}$ is a projective system $\mathscr{S}_{K_p}(\mathbf{G}, X) = \varprojlim_{K^p} \mathscr{S}_{K_pK^p}(\mathbf{G}, X)$ of schemes over $o_{E,(v)}$, indexed by the set of open and compact subgroups of $\mathbf{G}(\mathbb{A}_f^p)$, together with a continuous right action of $\mathbf{G}(\mathbb{A}_f^p)$ such that:

(i) If K^p is sufficiently small, then $\mathscr{S}_{K_pK^p}(\mathbf{G}, X)$ is smooth over $o_{E,(v)}$ and the map $\mathscr{S}_{K_pK'^p}(\mathbf{G}, X) \to \mathscr{S}_{K_pK^p}(\mathbf{G}, X)$ is étale for every $K'^p \subseteq K^p$.

(ii) $\mathscr{S}_{K_p}(\mathbf{G}, X) \otimes_{o_{E,(v)}} E$ is $\mathbf{G}(\mathbb{A}_f^p)$-equivariantly isomorphic to $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$.

(iii) Let Y be a regular, formally smooth $o_{E,(v)}$-scheme. Then every morphism $Y \otimes_{o_{E,(v)}} E \to \mathscr{S}_{K_p}(\mathbf{G}, X) \otimes_{o_{E,(v)}} E$ extends to a morphism $Y \to \mathscr{S}_{K_p}(\mathbf{G}, X)$.

Note that in the situation of (iii) the extension $Y \to \mathscr{S}_{K_p}(\mathbf{G}, X)$ is automatically unique, since Y is reduced and $Y \otimes_{o_{E,(v)}} E$ is dense in Y. Hence a model in the sense of Definition II.3.2 is unique up to canonical isomorphism which justifies the name "canonical model". In [Mi1] Milne conjectured that an integral canonical model of $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ always exists (for a general Shimura datum, not necessarily of Hodge type), see also the treatment in ([Mo1], §3).

Example II.3.3. Consider a Shimura datum $(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^\pm)$ of PEL-type, given by a PEL-datum $(B, *, V, \psi, h)$. Then p is a prime of good reduction for $\mathbf{G}$ if and only if $B_{\mathbb{Q}_p}$ is unramified. In this case it was shown in ([Ko2], §5 and §6) that a canonical integral model exists (we suppose that $\mathbf{G}$ is connected, so $\mathbf{G}^{\mathrm{ad}}$ has no factor of Dynkin type D), the schemes $\mathscr{S}_{K_pK^p}(\mathbf{G}, X)$ then have an explicit description as a moduli space of abelian schemes with additional structures over $o_{E,(v)}$.

II.4 The integral canonical models

In this section we briefly describe the construction of the canonical integral model for $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$, following Kisin's proof in [Ki1], and introduce the objects which will be fundamental for the study of the closed fiber which follows. In the case $p = 2$ two restrictions arise in order for the construction to work.

II.4.1 Construction of the integral models

Let G be a reductive model of $\mathbf{G}$ over $\mathbb{Z}_p$ such that $K_p = G(\mathbb{Z}_p)$. If $p = 2$, we assume that $\mathbf{G}^{\mathrm{ad}}$ has no factor of Dynkin type B. Then there is a lattice $\Lambda \subseteq V$ and a finite set of tensors $s := (s_i) \subset \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ such that $\mathbf{G}$ and G are identified with the subgroups defined by $s_{\mathbb{Q}} \subseteq V^{\otimes}$ and $s_{\mathbb{Z}_p} \subset \Lambda^{\otimes}_{\mathbb{Z}_p}$ respectively via our chosen embedding $\mathbf{G} \hookrightarrow \mathrm{GSp}(V)$ ([Ki1], 2.3.1., 2.3.2.). Possibly passing to a homothetic lattice, we may and will further assume that the symplectic pairing ψ on V restricts to a pairing $\Lambda \times \Lambda \to \mathbb{Z}$. Note however that Λ will not be self-dual with respect to ψ in general.

Let $\tilde{K}_p$ be the stabilizer of $\Lambda_{\mathbb{Z}_p}$ in $\mathrm{GSp}(V)(\mathbb{Q}_p)$. Then $K_p = \tilde{K}_p \cap \mathbf{G}(\mathbb{Q}_p)$. Let $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ be an open and compact subgroup such that $K := K_pK^p$ leaves $\Lambda_{\hat{\mathbb{Z}}}$ stable (which is the case for all sufficiently small K^p). Then it can be shown that there is an open and compact subgroup $\tilde{K}^p \subseteq \mathrm{GSp}(V)(\mathbb{A}_f^p)$ which contains K^p, such that $\tilde{K} := \tilde{K}_p\tilde{K}^p$ also leaves $\Lambda_{\hat{\mathbb{Z}}}$ stable and such that the natural map $\mathrm{Sh}_K(\mathbf{G}, X) \to \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^\pm) \otimes_{\mathbb{Q}} E$ is a closed embedding ([Ki1], 2.1.2., 2.3.2.), we call a subgroup $\tilde{K}^p$ with these properties *admissible* for K^p. Further, if $K'^p \subseteq K^p$

then there is an open and compact subgroup $\tilde{K}'^p \subseteq \tilde{K}^p$ which is admissible for K'^p and we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_{K'}(\mathbf{G}, X) \arrow[r, hook] \arrow[d] & \mathrm{Sh}_{\tilde{K}'}(\mathrm{GSp}(V), S^{\pm}) \otimes_{\mathbb{Q}} E \arrow[d] \\ \mathrm{Sh}_{K}(\mathbf{G}, X) \arrow[r, hook] & \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm}) \otimes_{\mathbb{Q}} E \end{tikzcd} \tag{42}$$

where the horizontal arrows are closed embeddings.

Construction II.4.1. We denote by Λ' the dual lattice of Λ with respect to ψ. Let $|\Lambda'/\Lambda| = d$, and let $\dim(V) = 2n$. Let $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ be an open and compact subgroup, and let $\tilde{K}^p$ be admissible for K^p. With respect to Λ, consider the moduli space $\mathscr{M}_{n,d,\tilde{K}^p}$ over $\mathbb{Z}_{(p)}$ which parametrizes abelian schemes with a polarization of degree d and a mod-$\tilde{K}^p$ level structure up to isomorphism (see [Ki1], 2.3.3.). By the classical result of Mumford, $\mathscr{M}_{n,d,\tilde{K}^p}$ is representable by a quasi-projective scheme over $\mathbb{Z}_{(p)}$ if $\tilde{K}^p$ is sufficiently small.

Let again $\tilde{K} = \tilde{K}_p\tilde{K}^p$. Due to the moduli interpretation of Shimura varieties of Siegel type, there is an embedding

$$\mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm}) \hookrightarrow \mathscr{M}_{n,d,\tilde{K}^p}$$

of $\mathbb{Z}_{(p)}$-schemes. We give a description of this map on $\mathbb{C}$-valued points, cf. ([Va1], 4.1.): Let

$$[h, g] \in \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm})(\mathbb{C}) = \mathrm{GSp}(V)(\mathbb{Q})\backslash S^{\pm} \times \mathrm{GSp}(V)(\mathbb{A}^f)/\tilde{K}.$$

Let $V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$ be the Hodge decomposition induced by h. There is a unique $\mathbb{Z}$-lattice $\Lambda_g \subset V$ such that $(\Lambda_g)_{\hat{\mathbb{Z}}} = g(\Lambda_{\hat{\mathbb{Z}}})$ and a unique $\mathbb{Q}^{\times}$-multiple $\psi_{h,g}$ of ψ such that $g(\Lambda'_{\hat{\mathbb{Z}}})$ is the dual lattice of $g(\Lambda_{\hat{\mathbb{Z}}})$ with respect to $\psi_{h,g}$ and such that the form $(v, w) \mapsto \psi_{h,g}(v, h(i)w)$ is positive definite on $V_{\mathbb{R}}$. Then $[h, g]$ is mapped to the isomorphism class of (A, λ, η), where $A := V^{(-1,0)}/\Lambda_g$, endowed with the polarization λ induced by $\psi_{h,g}$, is the polarized complex abelian variety associated to $(V, \psi_{h,g}, \Lambda_g, h)$ via Riemann's theorem (see [Del1], 4.7.), and η is the right $\tilde{K}^p$-coset of

$$\Lambda_{\hat{\mathbb{Z}}^p} \xrightarrow{g^p} g^p(\Lambda_{\hat{\mathbb{Z}}^p}) = (\Lambda_g)_{\hat{\mathbb{Z}}^p} \cong H_1(A, \mathbb{Z})_{\hat{\mathbb{Z}}^p} \cong \prod_{l \neq p} T_l(A).$$

Recall that v denotes a place of E over p, and $o_{E,(v)}$ the localization of o_E at v.

Definition II.4.2. Let $K = K_pK^p$ with $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ open and compact such that there is $\tilde{K}^p$ admissible for K^p for which $\mathscr{M}_{n,d,\tilde{K}^p}$ exists as a scheme. Let $\tilde{K} = \tilde{K}_p\tilde{K}^p$, and define $\mathscr{S}_K(\mathbf{G}, X)$ as the normalization of the closure of $\mathrm{Sh}_K(\mathbf{G}, X)$ in $\mathscr{M}_{n,d,\tilde{K}^p} \otimes_{\mathbb{Z}_{(p)}} o_{E,(v)}$ with respect to the embedding

$$\mathrm{Sh}_K(\mathbf{G}, X) \hookrightarrow \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm}) \otimes_{\mathbb{Q}} E \hookrightarrow \mathscr{M}_{n,d,\tilde{K}^p} \otimes_{\mathbb{Z}_{(p)}} o_{E,(v)}.$$

Remark II.4.3. This definition is indeed independent of the choice of $\tilde{K}^p$: Let $\tilde{K}'^p \subseteq \tilde{K}^p$ be an open and compact subgroup which contains K^p (it is then automatically admissible for K^p), then the natural map $\mathscr{M}_{n,d,\tilde{K}'^p} \to \mathscr{M}_{n,d,\tilde{K}^p}$ is finite and there is a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_K(\mathbf{G}, X) \arrow[r, hook] \arrow[dr, hook] & \mathscr{M}_{n,d,\tilde{K}'^p} \otimes_{\mathbb{Z}_{(p)}} o_{E,(v)} \arrow[d] \\ & \mathscr{M}_{n,d,\tilde{K}^p} \otimes_{\mathbb{Z}_{(p)}} o_{E,(v)} . \end{tikzcd}$$

Let Z be a component of $\mathrm{Sh}_K(\mathbf{G}, X)$, and denote by $\overline{Z}'$ and $\overline{Z}$ the closures in the $o_{E,(v)}$-schemes on the right hand side of the diagram respectively. The induced map $\overline{Z}' \to \overline{Z}$ is finite and dominant, and is an isomorphism at the generic points. Hence the corresponding map of the respective normalizations is an isomorphism.

By definition, for every K^p the choice of an admissible $\tilde{K}^p$ gives a natural map $\mathscr{S}_K(\mathbf{G}, X) \to \mathscr{M}_{n,d,\tilde{K}} \otimes_{\mathbb{Z}_{(p)}} o_{E,(v)}$, this defines an abelian scheme over $\mathscr{S}_K(\mathbf{G}, X)$ which is independent of the choice of $\tilde{K}^p$ up to isomorphism by the preceeding remark. If $K' = K_p K'^p$ and $K = K_p K^p$ for $K'^p \subseteq K^p$ then we have a natural map $\mathscr{S}_{K'}(\mathbf{G}, X) \to \mathscr{S}_K(\mathbf{G}, X)$ which is obtained by the choice of suitable admissible subgroups $\tilde{K}'^p \subseteq \tilde{K}^p$ in the diagram (42).

Theorem II.4.4 (Kisin, [Ki1] Theorem 2.3.8.). *If $p = 2$, assume that $\mathbf{G}^{\mathrm{ad}}$ has no factor of Dynkin type B, and that, for each K^p, the dual of each abelian variety associated to a point on the special fiber of $\mathscr{S}_K(\mathbf{G}, X)$ has a connected p-divisible group.*
Then the following hold:

(i) $\mathscr{S}_K(\mathbf{G}, X)$ is a smooth $o_{E,(v)}$-scheme for each K^p.

(ii) The projective limit $\mathscr{S}_{K_p}(\mathbf{G}, X) := \varprojlim_{K^p} \mathscr{S}_K(\mathbf{G}, X)$ is an integral canonical model of $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ over $o_{E,(v)}$ in the sense of Definition II.3.2.

In particular, $\mathscr{S}_{K^p}(\mathbf{G}, X)$ and hence also $\mathscr{S}_K(\mathbf{G}, X) = \mathscr{S}_{K_p}(\mathbf{G}, X)/K^p$ (for K^p sufficiently small) do not depend on the choice of the embedding $(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$, nor on the choices made during the construcion.

II.4.2 Tensors on the de Rham cohomology

Although in general it is not known whether the integral models $\mathscr{S}_K(\mathbf{G}, X)$ allow an interpretation as moduli spaces of abelian schemes with additional structures, each model is by construction naturally endowed with an abelian scheme on it, and the tensors $s \subseteq \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ from the last subsection induce tensors on de Rham cohomology of this abelian scheme. In this subsection we desribe the construction of these tensors and their relation to the tensors s, still following [Ki1].

We will systematically consider the tensors $s \subset \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ chosen in the last subsection as tensors over $\Lambda^*_{\mathbb{Z}_{(p)}}$ and use the contragredient representation

$$(\cdot)^{\vee} \colon \mathrm{GL}(\Lambda) \xrightarrow{\sim} \mathrm{GL}(\Lambda^*),$$

as discussed in Section II.2.2.

Notation II.4.5. In the sequel we will work with a fixed model $\mathscr{S}_K := \mathscr{S}_K(\mathbf{G}, X)$ associated to some sufficiently small subgroup $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ as in Definition II.4.2. We fix an open compact $\tilde{K}^p \subseteq \mathrm{GSp}(V)(\mathbb{A}_f^p)$ which is admissible for K^p in the sense of the last subsection. Note that all the constructions below are in fact *independent* of the choice of $\tilde{K}^p$. In the case $p = 2$ we assume that the assumptions of Theorem II.4.4 hold, so that $\mathscr{S}_K$ is smooth. Let $\pi\colon \mathcal{A} \longrightarrow \mathscr{S}_K$ be the abelian scheme defined by the natural map $\mathscr{S}_K \to \mathscr{M}_{n,d,\tilde{K}^p} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$. Let

$$\mathcal{V}^\circ := H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_K) \quad \text{and} \quad \mathcal{V} := H^1_{\mathrm{dR}}(\mathcal{A} \otimes E/\operatorname{Sh}_K(\mathbf{G}, X)).$$

Then $\mathcal{V}^\circ$ and $\mathcal{V}$ are locally free modules over $\mathscr{S}_K$ and $\mathscr{S}_K \otimes E = \operatorname{Sh}_K(\mathbf{G}, X)$ respectively, and $\mathcal{V} = \mathcal{V}^\circ \otimes E$. Let ∇ denote the Gauß-Manin connection on $\mathcal{V}^\circ$ resp. $\mathcal{V}$. It is known that the Hodge spectral sequence $E_1^{p,q} = R^q\pi_*(\Omega^p_{\mathcal{A}/\mathscr{S}_K}) \implies H^{p+q}_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_K)$ degenerates at E_1 ([BBM], 2.5.2.), giving rise to a filtration

$$\mathcal{V}^\circ = H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_K) \supset \pi_*\Omega^1_{\mathcal{A}/\mathscr{S}_K} =: \operatorname{Fil}^1 \mathcal{V}^\circ,$$

the *Hodge filtration* on $\mathcal{V}^\circ$.

The tensors s_{dR} Let $E'|E$ be any field extension which admits an embedding into $\mathbb{C}$, and let $\xi \in \mathscr{S}_K(E')$. Let $\overline{E'}$ be an algebraic closure of E', choose an embedding $\overline{E'} \hookrightarrow \mathbb{C}$. We denote by $\bar{\xi}$ and $\xi_{\mathbb{C}}$ the $\overline{E'}$-valued and $\mathbb{C}$-valued points corresponding to ξ. From the embedding $\operatorname{Sh}_K(\mathbf{G}, X) \hookrightarrow \mathscr{M}_{n,d,\tilde{K}^p}$ used in Construction II.4.1 we have a natural isomorphism $V \simeq H_1(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Q})$. The dual of this isomorphism maps $s_{\mathbb{Q}} \subset (V^*)^{\otimes}$ to a set of tensors over $H^1(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Q})$, and using the comparison isomorphisms

$$H^1(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Q})_{\mathbb{C}} \cong H^1_{\mathrm{dR}}(\mathcal{A}_{\xi_{\mathbb{C}}}/\mathbb{C}), \quad H^1(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Q})_{\mathbb{Q}_l} \cong H^1_{\acute{e}t}(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Q}_l) \cong H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Q}_l)$$

we obtain tensors $s_{\mathrm{dR},\xi}$ on the algebraic de Rham cohomology of $\mathcal{A}_{\xi_{\mathbb{C}}}$ and $s_{\acute{e}t,l,\xi}$ on the l-adic étale cohomology of $\mathcal{A}_{\bar{\xi}}$ for every prime number l. By a result of Deligne ([Del2], 2.11.), the family $(s_{\mathrm{dR},\xi}, (s_{\acute{e}t,l,\xi}))$ is an absolute Hodge cycle (see loc.cit. §2 for the definition of Hodge cycles and absolute Hodge cycles).

Proposition II.4.6 ([Ki1], 2.2.1., 2.2.2.)**.**

(i) For every $\xi \in \mathscr{S}_K(E')$ as above the tensors $s_{\mathrm{dR},\xi}$ are defined over $H^1_{\mathrm{dR}}(\mathcal{A}_\xi/E')$ and the tensors $s_{\acute{e}t,l,\xi}$ are $\mathrm{Gal}(\overline{E'}|E')$-invariant for each l.

(ii) There exist global sections $s_{\mathrm{dR}} \subset \mathcal{V}^{\otimes}$ defined over E, which are horizontal with respect to the Gauß-Manin connection ∇, such that the pullback of s_{dR} to any $\xi \in \mathscr{S}_K(E')$ as above equals the tensors $s_{\mathrm{dR},\xi} \subset (H^1_{\mathrm{dR}}(\mathcal{A}_\xi/E'))^{\otimes}$.

The tensors s°_{dR} The extension of the tensors s_{dR} to sections of $(\mathcal{V}^\circ)^{\otimes}$ is closely related to the following pointwise construction: Let k be a perfect field of finite trancendence degree over $\mathbb{F}_p$. Let $W(k)$ be the Witt ring over k and let $L(k) := \operatorname{Frac}(W(k))$. Consider a triple $(\tilde{x}, \xi, x)$, where $\tilde{x}$ is a $W(k)$-valued point of $\mathscr{S}_K$ and $\xi \in \mathscr{S}_K(L(k))$, $x \in \mathscr{S}_K(k)$ are the corresponding induced points.

Let $\mathbb{D}_x$ be the contravariant Dieudonné module of the p-divisible group of $\mathcal{A}_x$. Recall that $\mathbb{D}_x$ is a free $W(k)$-module together with a σ-linear map F and a σ^{-1}-linear map V such that $FV = p = VF$. We have canonical isomorphisms

$$H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}}/W(k)) \cong H^1_{\mathrm{cris}}(\mathcal{A}_x/W(k)) \cong \mathbb{D}_x.$$

By our assumption on k, the field $L(k)$ can be embedded into $\mathbb{C}$. The choice of an embedding $\overline{L(k)} \hookrightarrow \mathbb{C}$ hence yields an absolute Hodge cycle $(s_{\mathrm{dR},\xi}, (s_{\acute{e}t,l,\xi}))$ as above. Now there is also an isomorphism

$$\Lambda_{\mathbb{Z}_p} \xrightarrow{\sim} H_1(\mathcal{A}_{\xi_{\mathbb{C}}}, \mathbb{Z})_{\mathbb{Z}_p} \cong T_p(\mathcal{A}_{\xi_{\mathbb{C}}}) \cong T_p(\mathcal{A}_{\bar{\xi}})$$

where, using the notations of Construction II.4.1, if the $\mathbb{C}$-valued point $\xi_{\mathbb{C}}$ corresponds to the element $[h, g] \in \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm})(\mathbb{C})$ then the first arrow is given by g_p. Dualizing this isomorphism, and paying respect to the $\mathrm{Gal}(\overline{L(k)}|L(k))$-operation on the right hand side, yields

$$\Lambda^*_{\mathbb{Z}_p} \simeq T_p(\mathcal{A}_{\bar{\xi}})^*(-1) \cong H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Z}_p),$$

which sends the tensors $s_{\mathbb{Z}_p} \subset (\Lambda^*_{\mathbb{Z}_p})^{\otimes}$ to tensors $s^{\circ}_{\acute{e}t,\xi}$ over $H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Z}_p)$. Since all the isomorphisms involved are compatible, the base change of $s^{\circ}_{\acute{e}t,\xi}$ to tensors over $H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Q}_p)$ is exactly the p-adic component $s_{\acute{e}t,p,\xi}$ of the absolute Hodge cycle defined above. So Proposition II.4.6(i) implies that the $s^{\circ}_{\acute{e}t,\xi}$ are invariant under the action of $\mathrm{Gal}(\overline{L(k)}|L(k))$. Now it follows from Kisin's theory of crystalline representations and $\mathfrak{S}$-modules that the images of these tensors under the p-adic comparison isomorphism

$$H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_x/W(k)) \otimes_{W(k)} B_{\mathrm{cris}} \cong \mathbb{D}_x \otimes_{W(k)} B_{\mathrm{cris}} \tag{43}$$

are F-invariant and are already defined over $\mathbb{D}_x$ ([Ki1], 1.3.6.(1), 1.4.3.(1)). Using the identification $\mathbb{D}_x \cong H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}}/W(k))$ we thus obtain tensors $s^{\circ}_{\mathrm{dR},\tilde{x}}$ over $\mathcal{V}^{\circ}_{\tilde{x}}$.

Proposition II.4.7.

(i) The tensors s_{dR} of Proposition II.4.6 extend (uniquely) to global sections $s^{\circ}_{\mathrm{dR}} \subset (\mathcal{V}^{\circ})^{\otimes}$ which are horizontal with respect to ∇.

(ii) Let $(\tilde{x}, \xi, x)$ be a triple as considered above. Then the tensors $s^{\circ}_{\mathrm{dR},\tilde{x}} \subset (\mathcal{V}^{\circ}_{\tilde{x}})^{\otimes}$ which we obtained via the p-adic comparison isomorphism in the above construction are equal to the pullback of s°_{dR} to $\tilde{x}$.

(iii) In the situation of (ii), assume in addition that k is finite or algebraically closed. Then there is a $W(k)$-linear isomorphism

$$(\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim} (\mathcal{V}^{\circ}_{\tilde{x}}, s^{\circ}_{\mathrm{dR},\tilde{x}})$$

Further, if β is any such isomorphism, then there is a cocharacter λ of $G_{W(k)}$ such that the filtration $\Lambda^*_{W(k)} \supset \beta^{-1}(\mathrm{Fil}^1 \mathcal{V}^{\circ}_{\tilde{x}})$ is induced by $(\cdot)^{\vee} \circ \lambda$, where $(\cdot)^{\vee}$ is the contragredient representation from (40).

Proof. (i) The existence of s°_{dR} is shown in the proof of ([Ki1], 2.3.9.). These extensions are automatically unique, since $\mathscr{S}_K$ is in particular an integral scheme and $\mathcal{V}^{\circ}$ is locally free. By the same reasoning it follows that the s°_{dR} are horizontal with respect to ∇, as they are so over $\mathscr{S}_K \otimes E$.

(ii) If x is a closed point of $\mathscr{S}_K$, then this is immediately clear from the definition of s°_{dR} in ([Ki1], 2.3.9.). In general, as the equality of tensors in $(\mathcal{V}^{\circ}_{\tilde{x}})^{\otimes}$ may be tested over ξ, the statement amounts to the fact that the p-adic comparison isomorphism $H^1_{\acute{e}t}(\mathcal{A}_{\bar{\xi}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^1_{\mathrm{dR}}(\mathcal{A}_{\xi}/L(k)) \otimes_{L(k)} B_{\mathrm{dR}}$ maps the p-adic étale component of the absolute Hodge cycle $(s_{\mathrm{dR},\xi}, (s_{\acute{e}t,l,\xi}))$ to its de Rham component. If $\mathcal{A}_\xi$ can be defined over a number field, this is a theorem of Blasius and Wintenberger ([Bl], 0.3.), and Vasiu ([Va1], 5.2.16.) observed that their result can also be extended to our more general situation.

(iii) Let $\widetilde{\mathbb{D}}$ be the contravariant crystal of the p-divisible group $\mathcal{A}_{\tilde{x}}[p^\infty]$ over $W(k)$. Then we have the natural identification

$$\widetilde{\mathbb{D}}(W(k)) \cong \mathbb{D}_x \cong \mathcal{V}^\circ_{\tilde{x}}$$

which is compatible with the Hodge filtrations on both sides, and by (ii) the tensors $s^\circ_{\mathrm{dR},\tilde{x}}$ get identified with the images of $s^\circ_{\acute{e}t,\xi}$ under the p-adic comparison isomorphism in (43). So the first statement of (iii) follows directly from ([Ki1], 1.4.3. (2)+(3)), applied to the p-divisible group $\mathcal{A}_{\tilde{x}}[p^\infty]$ and the tensors $s^\circ_{\acute{e}t,\xi}$. Likewise, the proof of (4) in loc.cit. (which proves more than what is claimed) shows that the filtration $\Lambda^*_{W(k)} \supset \beta^{-1}(\mathrm{Fil}^1\, \mathcal{V}^\circ_{\tilde{x}})$ is induced by a cocharacter of the subgroup of $\mathrm{GL}(\Lambda^*_{W(k)})$ which is defined by the tensors $s_{W(k)} \subseteq (\Lambda^*_{W(k)})^\otimes$. As this subgroup is exactly the image of $G_{W(k)}$ under $(\cdot)^\vee$, the last claim follows.

□

We remark that the existence of the tensors s°_{dR} is closely related to Theorem II.4.4: In fact, the proof of the smoothness of $\mathscr{S}_K$ in ([Ki1], 2.3.5.) uses a variant of Proposition II.4.7(iii) as a main ingredience, and in turn the arguments given in that proof allow the construction of s°_{dR} in ([Ki1], 2.3.9.).

Corollary II.4.8. *Let $x \in \mathscr{S}_K(k)$, where k is either a finite extension of $\mathbb{F}_p$ or algebraically closed of finite transcendence degree over $\mathbb{F}_p$, and let $\mathbb{D}_x$ be the contravariant Dieudonne module of the p-divisible group $\mathcal{A}_x[p^\infty]$. Let $\tilde{x} \in \mathscr{S}_K(W(k))$ be a lift of x.*
Then the images $s_{\mathrm{cris},x} \subset (\mathbb{D}_x)^\otimes$ of $s^\circ_{\mathrm{dR},\tilde{x}}$ via the identification $H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}}/W(k)) \cong \mathbb{D}_x$ are independent of the choice of $\tilde{x}$. Further, the tensors $s_{\mathrm{cris},x}$ are F-invariant, and there is a $W(k)$-linear isomorphism $(\Lambda^*_{W(k)}, s_{W(k)}) \simeq (\mathbb{D}_x, s_{\mathrm{cris},x})$.

Proof. This follows immediately from (i) and (ii) of the last proposition. □

Remark II.4.9. In the special case of a Shimura variety given by a PEL-datum $(B, *, V, \psi, h)$ it is known that one may choose a lattice $\Lambda \subset V$ such that $\Lambda_{\mathbb{Z}_p}$ is selfdual with respect to ψ and a $*$-invariant $\mathbb{Z}_{(p)}$-order o_B of B such that $o_B \otimes \mathbb{Z}_p$ is a maximal order of $(B_{\mathbb{Q}_p}, *)$ which acts on $\Lambda_{\mathbb{Z}_p}$. In this case the tensors $s \subset \Lambda^\otimes_{\mathbb{Z}_p}$ then encode the action of o_B on $\Lambda_{\mathbb{Z}_p}$, and Corollary II.4.8 is an analogon to the results in ([VW], §2) on p-divisible groups with PEL-structure (but note that the authors use covariant Dieudonné theory in that article).

II.5 Stratifications of the special fiber

Let $\kappa(v)$ be the residue class field of $\mathcal{O}_{E,(v)}$, and let $\overline{\mathbb{F}}_p$ be a fixed algebraic closure of $\mathbb{F}_p$. In this section we describe the Newton stratification on the special fiber $\mathscr{S}_K \otimes \kappa(v)$ of $\mathscr{S}_K$ and the Ekedahl-Oort stratification on $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$. These stratifications arise by considering the isocristals resp. Dieudonné spaces associated to $\mathcal{A}_x$ for points x as in Corollary II.4.8, while paying respect to the tensor structure. The stratifications will be parametrized by combinatorial data which is derived from the Shimura datum $(\mathbf{G}, X)$ (and the choice of the hyperspecial subgroup K_p).

To describe the stratifications we use the group theoretic language for the reductive group scheme G over $\mathbb{Z}_p$ (compare Section I.2.1): G is quasisplit and split over a finite étale extension of $\mathbb{Z}_p$. We fix a Borel pair (T, B) defined over $\mathbb{Z}_p$. Let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ be the root datum associated to (G, T), and let W be the associated Weyl group. The choice of B determines a set $\Phi^+ \subset \Phi$ of positive roots and a set $S \subset W$ of simple reflections which give (W, S) the structure of a finite Coxeter group. A cocharacter $\lambda \in X_*(T)$ will be called dominant, if $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Phi^+$ (where $\langle \cdot, \cdot \rangle$ is the natural pairing between $X^*(T)$ and $X_*(T)$). The group W naturally acts on $X_*(T)$, and the dominant cocharacters form a full set of representatives for the orbits $W \backslash X_*(T)$.

For any local, strictly henselian $\mathbb{Z}_p$-algebra R we have a realization of this data with respect to G_R. In particular we then have $W \cong N_G(T)(R)/T(R)$ and the inclusion $X_*(T) \cong \mathrm{Hom}_R(\mathbb{G}_{m,R}, T_R) \subseteq \mathrm{Hom}_R(\mathbb{G}_{m,R}, G_R)$ induces a bijection between the quotient $W \backslash X_*(T)$ and the set of conjugacy classes of cocharacters for G_R. If $R \to R'$ is a homomorphism of local and strictly henselian $\mathbb{Z}_p$-algebras, then base change to R' yields a bijection between the sets of conjugacy classes of cocharacters for G_R and $G_{R'}$.
Putting $R = \mathbb{C}$, the conjugacy class $[\nu^{-1}]$ from Definition II.3.1 determines an element of $W \backslash X_*(T)$. On the other hand, putting $R = W(k)$ for some algebraically closed field k of characteristic p, we obtain an action of the Frobenius σ on $X_*(T)$, W and Φ (which is independent of the choice of k). This action leaves S, Φ^+ and the set of dominant cocharacters stable, since B and T are defined over $\mathbb{Z}_p$.

Definition II.5.1. We define $\mu \in X_*(T)$ as the unique dominant element such that $\sigma^{-1}(\mu) \in [\nu^{-1}]$.

As remarked in Section II.3, all cocharactes in $[\nu^{-1}]$ are minuscule, so the same is true for μ.

II.5.1 The Dieudonné module at a geometric point

We will use the notations and conventions on σ-linear algebra of Section II.2.1, and especially apply them to the case that $M_0 = \Lambda_{\mathbb{Z}_p}$ or $M_0 = \Lambda_{\mathbb{F}_p}$. Recall that we use the contragredient representation $(\cdot)^\vee \colon \mathrm{GL}(\Lambda) \to \mathrm{GL}(\Lambda^*)$ to let $G(R)$ act on Λ^*_R.

Lemma II.5.2. *Let k be a perfect field of finite transcendence degree over $\mathbb{F}_p$, let $\bar{k}$ be an algebraic closure of k. Let $\tilde{x} \in \mathscr{S}_K(W(k))$, and suppose that there exists an isomorphism*

$$\beta \colon (\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim} (\mathcal{V}^\circ_{\tilde{x}}, s^\circ_{\mathrm{dR},\tilde{x}}).$$

If λ is a cocharacter of $G_{W(k)}$ such that $(\cdot)^\vee \circ \lambda$ induces on $\Lambda^*_{W(k)}$ the filtration $\Lambda^*_{W(k)} \supset \beta^{-1}(\mathrm{Fil}^1 \mathcal{V}^\circ_{\tilde{x}})$, then $\lambda_{W(\bar{k})} \in [\nu^{-1}]$.

Proof. We may assume that $k = \bar{k}$. Let $\xi \in \mathscr{S}_K(L(k))$ be the generic point of $\tilde{x}$, let $\overline{L(k)}$ be an algebraic closure of $L(k)$. We choose an embedding $\overline{L(k)} \hookrightarrow \mathbb{C}$, it suffices then to show that $\lambda_{\mathbb{C}} \in [\nu^{-1}]$.
Let $A := \mathcal{A}_{\xi_{\mathbb{C}}}$. There is a pair $(h, g) \in X \times \mathbf{G}(\mathbb{A}_f)$ such that

$$\xi_{\mathbb{C}} = [h, g] \in \mathrm{Sh}_K(\mathbf{G}, X) = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f)/K.$$

If $V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$ is the Hodge decomposition given by h then, using the notation from Construction II.4.1, we have $A \simeq V^{(-1,0)}/\Lambda_g$, and in turn there is an isomorphism $H_1(A,\mathbb{C}) \simeq (\Lambda_g)_{\mathbb{C}} = V_{\mathbb{C}}$. It follows from the construction of the Riemann correspondence for complex abelian varieties that the dual isomorphism

$$\alpha_{\mathbb{C}} \colon V_{\mathbb{C}}^* \xrightarrow{\sim} H^1(A,\mathbb{C}) \cong H^1_{\mathrm{dR}}(A/\mathbb{C}) = \mathcal{V}_{\xi_{\mathbb{C}}}$$

identifies the Hodge decomposition $H^1_{\mathrm{dR}}(A/\mathbb{C}) = H^{(1,0)} \oplus H^{(0,1)}$ with the decomposition $V_{\mathbb{C}}^* = (V^{(-1,0)})^* \oplus (V^{(0,-1)})^*$ (see e.g. [Mi3], 6.10., 7.5.), and a direct computation shows that the cocharacter $(\cdot)^\vee \circ \nu_h^{-1}$ (with ν_h as in Definition II.3.1) acts on $(V^{(-1,0)})^*$ with weight 1 and on $(V^{(0,-1)})^*$ with weight 0, in other words,

$$(\nu_h(z)^{-1})^\vee|_{(V^{(-1,0)})^*} = z, \qquad (\nu_h(z)^{-1})^\vee|_{(V^{(0,-1)})^*} = 1.$$

This means that $(\cdot)^\vee \circ \nu_h^{-1}$ induces on $V_{\mathbb{C}}^*$ the filtration

$$V_{\mathbb{C}}^* \supset (V^{(-1,0)})^* = \alpha_{\mathbb{C}}^{-1}(H^{(1,0)}) = \alpha_{\mathbb{C}}^{-1}(\mathrm{Fil}^1 \mathcal{V}_{\xi_{\mathbb{C}}}).$$

Further the isomorphism $\alpha_{\mathbb{C}}$ identifies $s_{\mathbb{C}}$ with $s_{\mathrm{dR},\xi_{\mathbb{C}}}$ by construction of s_{dR} (see Section II.4.2). On the other hand, by assumption $\beta_{\mathbb{C}}$ also identifies $s_{\mathbb{C}}$ with $s_{\mathrm{dR},\xi_{\mathbb{C}}}$, and the filtration $V_{\mathbb{C}}^* \supset \beta_{\mathbb{C}}^{-1}(\mathrm{Fil}^1 \mathcal{V}_{\xi_{\mathbb{C}}})$ is induced by $(\cdot)^\vee \circ \lambda_{\mathbb{C}}$. Now the isomorphism $\alpha_{\mathbb{C}}^{-1} \circ \beta_{\mathbb{C}} \colon V_{\mathbb{C}}^* \to V_{\mathbb{C}}^*$ fixes the tensors $s_{\mathbb{C}}$, which means that $\alpha_{\mathbb{C}}^{-1} \circ \beta_{\mathbb{C}} = g_{\mathbb{C}}^\vee$ for some $g_{\mathbb{C}} \in \mathbf{G}(\mathbb{C})$. As we have $g_{\mathbb{C}}^\vee(\beta_{\mathbb{C}}^{-1}(\mathrm{Fil}^1 \mathcal{V}_{\xi_{\mathbb{C}}})) = \alpha_{\mathbb{C}}^{-1}(\mathrm{Fil}^1 \mathcal{V}_{\xi_{\mathbb{C}}})$, after conjugating $\lambda_{\mathbb{C}}$ with $g_{\mathbb{C}}$ we may therefore assume that $(\cdot)^\vee \circ \lambda_{\mathbb{C}}$ and $(\cdot)^\vee \circ \nu_h^{-1}$ both induce the same filtration on $V_{\mathbb{C}}^*$.

Let P be the stabilizer of this filtration in $\mathbf{G}_{\mathbb{C}}$ via $(\cdot)^\vee$, that is, the subgroup of all $\tilde{g} \in \mathbf{G}_{\mathbb{C}}$ such that $\tilde{g}^\vee$ leaves the filtration stable. Then $P \subseteq \mathbf{G}_{\mathbb{C}}$ is a parabolic subgroup, and both $\lambda_{\mathbb{C}}$ and ν_h^{-1} factor through P. Since all maximal tori of P are conjugate over $\mathbb{C}$, after conjugation by an element of $P(\mathbb{C})$ we may further assume that both cocharacters factor via the same (automatically split) maximal torus of P. But this implies that $\lambda_{\mathbb{C}}$ and ν_h^{-1} also induce (via $(\cdot)^\vee$) the same grading on $V_{\mathbb{C}}^*$, and hence that they are equal. $\square$

The following construction is fundamental for the description of $\mathscr{S}_K \otimes \kappa(v)$ in terms of the group G:

Construction II.5.3. Let k be algebraically closed of finite transcendence degree over $\mathbb{F}_p$, let $x \in \mathscr{S}_K(k)$. By Corollary II.4.8 the tensors s°_{dR} induce F-invariant tensors $s_{\mathrm{cris},x} \subset \mathbb{D}_x^\otimes$, and we find an isomorphism $\beta \colon (\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim} (\mathbb{D}_x, s_{\mathrm{cris},x})$. To any such β we attach an element $g_\beta \in G(L(k))$ as follows:

Transporting F via β, we obtain an injective σ-linear map $F_\beta := \beta^{-1} \circ F \circ \beta$ on $\Lambda^*_{W(k)} = (\Lambda^*_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)$. We can write it uniquely as $F_\beta = F_\beta^{\mathrm{lin}} \circ (1 \otimes \sigma)$, where $F_\beta^{\mathrm{lin}} \colon \Lambda^*_{W(k)} \to \Lambda^*_{W(k)}$ is linear and injective. On $\Lambda^*_{L(k)}$ this map is an automorphism which fixes the tensors $s_{L(k)}$. We define $g_\beta \in G(L(k))$ as the unique element such that $F_\beta^{\mathrm{lin}} = g_\beta^\vee$, i.e. such that $F_\beta = g_\beta^\vee \circ (1 \otimes \sigma)$.

We can also summarize the construction as follows: For each $\beta \colon (\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim}$

$(\mathbb{D}_x, s_{\mathrm{cris},x})$ the associated $g_\beta \in G(L(k))$ is the unique element such that the diagram

$$\begin{tikzcd}
\mathbb{D}_x^{(\sigma)} \arrow[r, "F^{\mathrm{lin}}"] & \mathbb{D}_x \\
\Lambda^*_{W(k)} \arrow[u, "\beta^{(\sigma)}"] \arrow[r, "g_\beta^\vee"] & \Lambda^*_{W(k)} \arrow[u, "\beta"']
\end{tikzcd}$$

commutes (where we make the usual identification $(\Lambda^*_{W(k)})^{(\sigma)} \cong \Lambda^*_{W(k)}$, see Section II.2.1).

Lemma II.5.4. *In the situation of Construction II.5.3 the following hold:*

(i) Let $x \in \mathscr{S}_K(k)$, let $\beta\colon (\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim} (\mathbb{D}_x, s_{\mathrm{cris},x})$ be an isomorphism. Then the isomorphisms between $(\Lambda^*_{W(k)}, s_{W(k)})$ and $(\mathbb{D}_x, s_{\mathrm{cris},x})$, are exactly the ones of the form $\beta' = \beta \circ h^\vee$ for $h \in G(W(k))$, further in this case h is uniquely determined and we have $g_{\beta'} = h^{-1} g_\beta \sigma(h)$.

(ii) For every $x \in \mathscr{S}_K(k)$ and every isomorphism $\beta\colon (\Lambda^*_{W(k)}, s_{W(k)}) \xrightarrow{\sim} (\mathbb{D}_x, s_{\mathrm{cris},x})$ we have that $g_\beta \in G(W(k))\mu(p)G(W(k))$.

Proof. (i) This is a standard argument: The automorphisms of $\Lambda^*_{W(k)}$ which fix $s_{W(k)}$ are exactly the elements $h^\vee$ for $h \in G(W(k))$, so the first statements are clear. If $\beta' = \beta \circ h^\vee$, then

$$\begin{aligned}
g_{\beta'}^\vee \circ (1 \otimes \sigma) = F_{\beta'} &= \beta'^{-1} \circ F \circ \beta' \\
&= (h^\vee)^{-1} \circ \big(\beta^{-1} \circ F \circ \beta\big) \circ h^\vee \\
&= (h^\vee)^{-1} \circ \big(g_\beta^\vee \circ (1 \otimes \sigma)\big) \circ h^\vee \\
&= \big((h^\vee)^{-1} \circ g_\beta^\vee \circ \sigma(h^\vee)\big) \circ (1 \otimes \sigma),
\end{aligned}$$

which shows the last claim, since $\sigma(h^\vee) = \sigma(h)^\vee$.

(ii) By the Cartan decomposition for $G(L(k))$ (cf. Section I.2.4) we know that g_β lies in a double coset $G(W(k))\eta(p)G(W(k))$ for a unique dominant cocharacter $\eta \in X_*(T)$. By (i) we may further w.l.o.g. replace β by $\beta \circ h^\vee$ for a suitable $h \in G(W(k))$ to achieve that $g_\beta \in G(W(k))\eta(p)$. By definition of μ in II.5.1, in order to show that $\eta = \mu$ it suffices to check that the base change of $\sigma^{-1}(\eta)$ to k lies in $[\nu^{-1}]$.

Let $\tilde{x} \in \mathscr{S}_K(W(k))$ be a lift of x, and identify $\mathbb{D}_x \cong \mathcal{V}^\circ_{\tilde{x}}$. Consider the pullback

$$\Lambda^*_{W(k)} \supset \beta^{-1}(\mathrm{Fil}^1 \mathcal{V}^\circ_{\tilde{x}})$$

of the Hodge filtration on $\mathcal{V}^\circ_{\tilde{x}}$. By Proposition II.4.7(iii) there is a cocharacter λ of $G_{W(k)}$ such that $(\cdot)^\vee \circ \lambda$ induces this filtration, and by Lemma II.5.2 we know that $\lambda \in [\nu^{-1}]$. Reducing the whole situation modulo p we obtain the contravariant Dieudonne space $(\overline{\mathbb{D}_x}, \overline{F}, \overline{V})$ associated to the p-torsion $\mathcal{A}_x[p]$ and the isomorphism

$$\bar{\beta}\colon \Lambda^*_k \xrightarrow{\sim} \overline{\mathbb{D}_x} \cong \overline{\mathcal{V}^\circ_{\tilde{x}}} = \mathcal{V}^\circ_x = H^1_{\mathrm{dR}}(\mathcal{A}_x/k).$$

By a result of Oda ([Od], 5.11.), we have the equality $\mathrm{Fil}^1 \mathcal{V}_x^\circ = \ker(\overline{F})$, which implies that

$$\overline{\beta^{-1}(\mathrm{Fil}^1 \mathcal{V}_{\tilde{x}}^\circ)} = \bar{\beta}^{-1}(\ker(\overline{F})) = \ker(\bar{\beta}^{-1} \circ \overline{F} \circ \bar{\beta}) = \ker(\overline{F_\beta}).$$

So the filtration $\Lambda_k^* \supseteq \ker(\overline{F_\beta})$ is induced via $(\cdot)^\vee$ by the reduction $\bar{\lambda}$ of λ.

Since $g_\beta = g_0\eta(p)$ for some $g_0 \in G(W(k))$, we have

$$F_\beta = (g_0^\vee \circ \eta(p)^\vee) \circ (1 \otimes \sigma) = (1 \otimes \sigma) \circ (\sigma^{-1}(g_0)^\vee \circ \sigma^{-1}(\eta)(p)^\vee).$$

Let $\Lambda_{W(k)}^* = \bigoplus_{m \in \mathbb{Z}} \Lambda_m^*$ be the grading which is induced by the cocharacter $(\cdot)^\vee \circ \sigma^{-1}(\eta)$ on $\Lambda_{W(k)}^*$. The inclusions $p \cdot \Lambda_{W(k)}^* \subseteq \mathrm{im}(F_\beta) \subseteq \Lambda_{W(k)}^*$ show that we must have $\Lambda_m^* = (0)$ for $m \neq 0, 1$, thus $\Lambda_{W(k)}^* = \Lambda_0^* \oplus \Lambda_1^*$. Reducing the above description of F_β modulo p we find that

$$\ker(\overline{F_\beta}) = \ker(\overline{\sigma^{-1}(\eta)(p)^\vee}) = \overline{\Lambda_1^*}.$$

This implies that the two cocharacters $(\cdot)^\vee \circ \overline{\sigma^{-1}(\eta)}$ and $(\cdot)^\vee \circ \bar{\lambda}$ induce the same filtration on Λ_k^*. Now it follows by the same argument as in the proof of Lemma II.5.2 that $\overline{\sigma^{-1}(\eta)}$ and $\bar{\lambda}$ are $G(k)$-conjugate, which concludes the proof.

□

Lemma II.5.5. *Let k be algebraically closed of finite transcendence degree over $\mathbb{F}_p$, let $x, x' \in \mathscr{S}_K(k)$. Let $g_\beta, g_{\beta'} \in G(W(k))$ be the elements associated to isomorphisms β and β' between $(\Lambda_{W(k)}^*, s_{W(k)})$ and $(\mathbb{D}_x, s_{\mathrm{cris},x})$ resp. $(\mathbb{D}_{x'}, s_{\mathrm{cris},x'})$ by Construction II.5.3.*
Then there is an isomorphism of Dieudonné modules $\mathbb{D}_x \simeq \mathbb{D}_{x'}$ which identifies $s_{\mathrm{cris},x}$ with $s_{\mathrm{cris},x'}$ if and only if $g_{\beta'} = hg_\beta\sigma(h)^{-1}$ for some $h \in G(W(k))$.

Proof. By Construction II.5.3 the existence of an isomorphism $\mathbb{D}_x \simeq \mathbb{D}_{x'}$ which respects the tensors on both sides is equivalent to the existence of an automorphism δ of $(\Lambda_{W(k)}^*, s_{W(k)})$ such that

$$g_{\beta'}^\vee \circ (1 \otimes \sigma) \circ \delta = \delta \circ g_\beta^\vee \circ (1 \otimes \sigma) \qquad (*)$$

Every such automorphism must be of the form $\delta = h^\vee$ for a unique $h \in G(W(k))$, and an easy calculation analogous to that in the proof of II.5.4(i) shows that the property (*) is equivalent to $g_{\beta'} = hg_\beta\sigma(h)^{-1}$. □

II.5.2 The Newton stratification

In order to define the Newton stratification on $\mathscr{S}_K \otimes \kappa(v)$, recall the following facts on σ-conjugacy classes.

Let k be algebraically closed of characteristic p. Let $[g]$ the σ-conjugacy class of an element $g \in G(L(k))$, and let $B(G)$ be the set of all σ-conjugacy classes in $G(L(k))$ (compare I.2.5). This definition is in fact independent of k in the following sense: If k' is any algebraically closed field of characteristic p, then every inclusion $k \subseteq k'$ induces a bijection between the σ-conjugacy classes of $G(L(k))$ and those of $G(L(k'))$ (see [RR], 1.3.).

We have the Newton map $\nu_G\colon B(G) \to \mathcal{N}(G)$ and the Kottwitz map $\kappa_G\colon B(G) \to \pi_1(G)_{\langle\sigma\rangle}$. Let $\mu \in X_*(T)$ be as defined in II.5.1. Recall from Section I.3.1.1 that the subset $B(G,\mu) \subseteq B(G)$ of σ-conjugacy classes which meet the double coset $G(W(k))\mu(p)G(W(k))$ is given (also independently of k in the above sense) as

$$B(G,\mu) := \{b \in B(G) \mid \kappa_G(b) = \mu^\natural,\ \nu_G(b) \preceq \bar{\mu}\},$$

where $\mu^\natural$ is the projection of μ to $\pi_1(G)_{\langle\sigma\rangle}$, $\bar{\mu} = \frac{1}{n}\sum_{i=0}^{n-1}\sigma^i(\mu)$ for some $n \in \mathbb{N}$ such that σ^n acts trivially on $X_*(T)$, and where $\preceq$ is the partial order on $\mathcal{N}(G)$ which generalizes the "lying above" order for Newton polygons. (Note that in the notations of the first chapter we have $G(W(k)) = K$ and $\mu(p) = p^\mu$.)

For $b', b \in B(G,\mu)$ we have $b' \preceq b$ if and only if $\nu_G(b') \preceq \nu_G(b)$. As explained in Section I.3.1.1, $B(G,\mu)$ is a finite set and contains a unique maximal element with respect to $\preceq$, which we denote by $b_{\mu-\mathrm{ord}}$. Further $B(G,\mu)$ contains a unique basic element, which we denote by b_{bas}, this is also the minimal element with respect to $\preceq$.

Let us now define the Newton stratification on $\mathscr{S}_K \otimes \kappa(v)$: Consider a point $x \in \mathscr{S}_K \otimes \kappa(v)$, let $k(x)$ be the residue class field of $\mathscr{S}_K$ in x. Let k be some algebraic closure of $k(x)$, let $\hat{x} \in \mathscr{S}_K(k)$ be the associated geometric point, and choose an isomorphism $\beta\colon (\Lambda^*_{W(k)}, s_{W(k)}) \simeq (\mathbb{D}_{\hat{x}}, s_{\mathrm{cris},\hat{x}})$. Let $g_\beta \in G(L(k))$ be the corresponding element which is given by Construction II.5.3, i.e. by the property $\beta^{-1} \circ F \circ \beta = g_\beta^\vee \circ (1 \otimes \sigma)$, where F is the σ-linear map on $\mathbb{D}_{\hat{x}}$. Lemma II.5.4 shows that $g_\beta \in G(W(k))\mu(p)G(W(k))$ and that its σ-conjugacy class $[g_\beta]$ is independent of the choice of β. Further, this σ-conjugacy class only depends on x and not on the choice of $k(x)$ in the sense explained above. Thus we obtain a well-defined map

$$\mathrm{Newt}\colon \mathscr{S}_K \otimes \kappa(v) \longrightarrow B(G,\mu), \tag{44}$$

which assigns to $x \in \mathscr{S}_K \otimes \kappa(v)$ the σ-conjugacy class of an element g_β associated by Construction II.5.3 to some geometric point of $\mathscr{S}_K \otimes \kappa(v)$ lying above x.

Definition II.5.6.

(i) For an element $b \in B(G,\mu)$ set $\mathcal{N}^b := \mathrm{Newt}^{-1}(\{b\}) \subseteq \mathscr{S}_K \otimes \kappa(v)$. We call $\mathcal{N}^b$ the *Newton stratum* of b.

(ii) We set $\mathcal{N}^{\mu-\mathrm{ord}} := \mathcal{N}^{b_{\mu-\mathrm{ord}}}$, and call $\mathcal{N}^{\mu-\mathrm{ord}}$ the *μ-ordinary locus* in $\mathscr{S}_K \otimes \kappa(v)$.

(iii) We set $\mathcal{N}^{\mathrm{bas}} := \mathcal{N}^{b_{\mathrm{bas}}}$, and call it the *basic locus* in $\mathscr{S}_K \otimes \kappa(v)$.

Remark II.5.7. The same argument as used in Lemma II.5.5 shows that two points $x_1, x_2 \in \mathscr{S}_K \otimes \kappa(v)$ lie in the same Newton stratum if and only if the following holds: If k is any algebraically closed field of finite transcendence degree over $\mathbb{F}_p$ such that $k(x_1)$ and $k(x_2)$ both embed into k (such a field always exists), with associated points $\hat{x}_1, \hat{x}_2 \in \mathscr{S}_K(k)$, then there is an isomorphism of isocrystals $(\mathbb{D}_{\hat{x}_1})_{\mathbb{Q}} \simeq (\mathbb{D}_{\hat{x}_2})_{\mathbb{Q}}$ which identifies the tensors $s_{\mathrm{cris},\hat{x}_1}$ with $s_{\mathrm{cris},\hat{x}_2}$.

A priori, the $\mathcal{N}^b$ are just subsets of $\mathscr{S}_K \otimes \kappa(v)$. In the case of a PEL-type Shimura datum, at each geometric point $\hat{x}$ of $\mathscr{S}_K \otimes \kappa(v)$ the tensors $s_{\mathrm{cris},\hat{x}}$ describe the additonal structure on $\mathbb{D}_{\hat{x}}$ (cf. Remark II.4.9), hence the Newton strata from Definition II.5.6 agree with those considered in the PEL-case. In this case the isocrystal over $\mathscr{S}_K \otimes \kappa(v)$ associated to the p-divisible group $(\mathcal{A} \otimes \kappa(v))[p^\infty]$, with induced additional structure, is an isocrystal with G-strucure in the sense of Rapoport-Richartz ([RR], §3). The "Grothendieck specialization theorem" for isocrystals with G-structure ([RR], Thm. 3.6.) then shows that the behaviour of Newt under specializations on $\mathscr{S}_K \otimes \kappa(v)$ is ruled by the order relation on $B(G,\mu)$:

$$\mathrm{Newt}(x_2) \preceq \mathrm{Newt}(x_1) \quad \text{if } x_1 \text{ specializes to } x_2. \tag{45}$$

It follows from this property that each $\mathcal{N}^b$ is a locally closed subset of $\mathscr{S}_K \otimes \kappa(v)$, and for the closures one has the relation $\overline{\mathcal{N}^b} \subseteq \bigcup_{b' \preceq b} \mathcal{N}^{b'}$. The recent results of Hamacher show that this last inclusion is even an equality, i.e. that

$$\overline{\mathcal{N}^b} = \bigcup_{b' \preceq b} \mathcal{N}^{b'} \tag{46}$$

for Shimura varieties of PEL-type, and also give a precise formula for the dimension of $\mathcal{N}^b$ ([Ha], Thm. 1.1.).

It is natural to ask whether the specialization property (45) also holds for a general Shimura variety of Hodge type. To our knowledge, this result has not yet been established, though we expect it to be true. There is, however, the following result of Vasiu. Since $\mathcal{A} \otimes \kappa(v)$ is in particular a polarized abelian scheme over $\mathscr{S}_K \otimes \kappa(v)$, we have the classical stratification of $\mathscr{S}_K \otimes \kappa(v)$ by Newton polygons, as defined by Oort. For a symmetric Newton polygon $\Delta \in B(\mathrm{GSp}(V))$, denote the corresponding stratum in $\mathscr{S}_K \otimes \kappa(v)$ by $\mathcal{N}^{\Delta}_{\mathrm{NP}}$. Then by Remark II.5.7 every $\mathcal{N}^b$ lies in a unique stratum $\mathcal{N}^{\Delta(b)}_{\mathrm{NP}}$, this defines a map $B(G,\mu) \to B(\mathrm{GSp}(V))$, $b \mapsto \Delta(b)$, which should be thought of as "forgetting the tensor structure".

Proposition II.5.8 ([Va2], 5.3.1.(ii))**.** *For every $b \in B(G,\mu)$ the stratum $\mathcal{N}^b$ is an open and closed subset of $\mathcal{N}^{\Delta(b)}_{\mathrm{NP}}$.*

As a consequence, since the strata $\mathcal{N}^{\Delta}_{\mathrm{NP}}$ are locally closed subsets of $\mathscr{S}_K \otimes \kappa(v)$, the same holds true for the $\mathcal{N}^b$, which justifies the term "Newton strata" in Definition II.5.6 also for Shimura varieties of Hodge type. We endow each $\mathcal{N}^b$ with the structure of a reduced subscheme of $\mathscr{S}_K \otimes \kappa(v)$.

II.5.3 The Ekedahl-Oort stratification

Recall that $\overline{\mathbb{F}}_p$ denotes a fixed algebraic closure of $\mathbb{F}_p$. We now describe the Ekedahl-Oort stratification on $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$ which has been constructed and studied by Zhang in [Zh1]. However, we give a slightly different definition, as we feel that one should work with Λ^* rather than with Λ, further we make the definition independent of the choice of a cocharacter. The main results of [Zh1] remain true with the obvious changes.

The definition of the Ekedahl-Oort stratification is based on the theory of $G_{\mathbb{F}_p}$-Zips which has been developed in [PWZ1], [PWZ2], and which we will apply to a cocharacter in the conjugacy class $[\nu^{-1}]$:

If X is a scheme or a sheaf over an $\mathbb{F}_p$-scheme S, denote by $X^{(\sigma)}$ its pullback by the absolute Frobenius $x \mapsto x^p$ on S, and likewise for morphisms of objects over S. If κ is any algebraic extension of $\mathbb{F}_p$ then, since $G_\kappa = G \otimes \kappa$ is defined over $\mathbb{F}_p$, we have $G_\kappa^{(\sigma)} \cong G_\kappa$ canonically (compare Section II.2.1). In particular, for every subgroup $H \subseteq G_\kappa$ the pullback $H^{(\sigma)}$ is again a subgroup of G_κ.

We view the conjugacy class $[\nu^{-1}]$ as an element of $W\backslash X_*(T)$, as explained at the beginning of this section. The *type* $J \subseteq S$ of this conjugacy class is then given as follows: Let $\chi_{\mathrm{dom}} \in X_*(T)$ be the unique dominant element lying in $[\nu^{-1}]$, then $J := \{s \in S \mid s(\chi_{\mathrm{dom}}) = \chi_{\mathrm{dom}}\}$. Let ${}^JW$ be the set of shortest left coset representatives for the subgroup W_J generated by J (see Section I.2.3). Let w_0 and $w_{0,J}$ be the longest elements of W and W_J respectively and let $x_J := w_0 w_{0,J}$. Then by ([PWZ1], 6.3., see also [PWZ2], §3.5.) the relation given by

$$w' \preceq w \;:\Longleftrightarrow\; zw'\sigma(x_J z x_J^{-1}) \leq w \;\text{ for some } z \in W_J \tag{47}$$

is a partial order $\preceq$ on ${}^JW$. (We will see in Section II.6.2 that this formulation of $\preceq$ is consistent with the order relation on $[[\cdot]]$-classes considered in the first chapter.) This partial order induces a topology on ${}^JW$: A subset $U \subset {}^JW$ is open if and only if for any $w' \in U$ and any $w \in {}^JW$ with $w' \preceq w$ one also has $w \in U$.

Let κ be algebraic over $\mathbb{F}_p$, let χ be a cocharacter of G_κ such that $\chi_{\overline{\mathbb{F}}_p} \in [\nu^{-1}]$. Using the identification $G_\kappa^{(\sigma)} \cong G_\kappa$ we can then view $\chi^{(\sigma)}$ as a cocharacter of G_κ as well. Let $P_{\chi,+}, P_{\chi,-} \subseteq G_\kappa$ be the parabolic subgroups which are characterized by the property that $\mathrm{Lie}(P_\pm)$ is the sum of the non-negative (resp. non-positive) weight spaces with respect to the adjoint operation of χ on $\mathrm{Lie}(G_\kappa)$. Denote by $U_{\chi,+}$ and $U_{\chi,-}$ the corresponding unipotent radicals and by M_χ the common Levi subgroup of $P_{\chi,+}$ and $P_{\chi,-}$.

Definition II.5.9 ([PWZ2], 3.1.)**.** Let S be a scheme over κ. A *$G_{\mathbb{F}_p}$-zip of type χ* over S is a quadruple $\underline{I} = (I, I_+, I_-, \iota)$, where

(1) I is a right G_κ-torsor for the fpqc-topology on S,

(2) $I_+ \subseteq I$ and $I_- \subseteq I$ are subsheaves such that I_+ is a $P_{\chi,+}$-torsor and I_- is a $P_{\chi,-}^{(\sigma)}$-torsor,

(3) $\iota\colon I_+^{(\sigma)}/U_{\chi,+}^{(\sigma)} \xrightarrow{\sim} I_-/U_{\chi,-}^{(\sigma)}$ is an isomorphism of $M_\chi^{(\sigma)}$-torsors.

A morphism $\underline{I} \longrightarrow \underline{I}'$ of $G_{\mathbb{F}_p}$-zips over S is a G_κ-equivariant map $I \to I'$ which maps I_+ to I'_+ and I_- to I'_- and is compatible with ι and ι'.

With the natural notion of pullback the $G_{\mathbb{F}_p}$-zips of type χ form a stack $G_{\mathbb{F}_p}\text{−}\mathtt{Zip}_\kappa^\chi$ over $(\mathbf{Sch}/\kappa)$ ([PWZ2], 3.2.).

Proposition II.5.10 ([PWZ2], 3.12., 3.20., 3.21.)**.** *$G_{\mathbb{F}_p}\text{−}\mathtt{Zip}_\kappa^\chi$ is a smooth algebraic stack of dimension 0 over κ, and there is a homeomorphism of topological spaces*

$$(G_{\mathbb{F}_p}\text{−}\mathtt{Zip}_\kappa^\chi)(\overline{\mathbb{F}}_p) \simeq {}^JW$$

where ${}^JW$ is endowed with the topology given by the partial order $\preceq$ from (47).

So in particular, there is a bijection between the set of isomorphism classes of $G_{\mathbb{F}_p}$-zips of type χ over $\overline{\mathbb{F}}_p$ and the set ${}^JW$. We will give a precise description of this bijection in II.6.5.

Construction II.5.11. Let $\overline{\mathcal{V}^\circ} := H^1_{\mathrm{dR}}(\mathcal{A}\otimes\kappa(v)/\mathscr{S}_K\otimes\kappa(v))$, this is just the reduction mod p of $\mathcal{V}^\circ$. Let $\mathcal{C} := \overline{\mathrm{Fil}^1\,\mathcal{V}^\circ} = \mathrm{Fil}^1\,\overline{\mathcal{V}^\circ}$ be the Hodge filtration, which is a locally direct summand of $\overline{\mathcal{V}^\circ}$. As explained in ([MW], §7), the conjugate Hodge spectral sequence also gives rise to a locally direct summand $\mathcal{D} := R^1\pi_*(\mathscr{H}^0(\Omega^\bullet_{\mathcal{A}\otimes\kappa(v)/\mathscr{S}_K\otimes\kappa(v)}))$ of $\overline{\mathcal{V}^\circ}$, and the (inverse) Cartier homomorphism provides isomorphisms

$$\phi_0 \colon (\overline{\mathcal{V}^\circ}/\mathcal{C})^{(\sigma)} \xrightarrow{\sim} \mathcal{D}, \qquad \phi_1 \colon \mathcal{C}^{(\sigma)} \xrightarrow{\sim} \overline{\mathcal{V}^\circ}/\mathcal{D}.$$

We now fix a cocharacter χ and a *finite* extension κ of $\kappa(v)$ such that χ is defined over κ and such that $\chi_{\overline{\mathbb{F}}_p} \in [\nu^{-1}]$. Recall that G_κ and thus also χ and $\chi^{(\sigma)}$ act on Λ^*_κ via the contragredient representation $(\cdot)^\vee$. Let

$$\Lambda^*_\kappa = \mathrm{Fil}^0_\chi \supset \mathrm{Fil}^1_\chi \supset (0), \qquad (0) \subset \mathrm{Fil}_0^{\chi^{(\sigma)}} \subset \mathrm{Fil}_1^{\chi^{(\sigma)}} = \Lambda^*_\kappa$$

be the descending resp. ascending filtration given in this way by χ and by $\chi^{(\sigma)}$. Then $P_{\chi,+}$ is nothing but the stabilizer of $\mathrm{Fil}^\bullet_\chi$ in G_κ, that is,

$$P_{\chi,+} = \{g \in G_\kappa \mid g^\vee(\mathrm{Fil}^1_\chi) = \mathrm{Fil}^1_\chi\},$$

and in the same fashion $P^{(\sigma)}_{\chi,-}$ is the stabilizer of $\mathrm{Fil}_\bullet^{\chi^{(\sigma)}}$. We denote by $\bar{s}_{\mathrm{dR}}$ the reduction of the tensors s°_{dR} to $\overline{\mathcal{V}^\circ}$, and by $\bar{s}$ the base change of $s \subset (\Lambda^*_{\mathbb{Z}_p})^\otimes$ to Λ^*_κ. Define:

$$\begin{aligned}
I :=& \mathbf{Isom}_{\mathscr{S}_K\otimes\kappa}\big((\Lambda^*_\kappa, \bar{s}) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa},\, (\overline{\mathcal{V}^\circ}, \bar{s}_{\mathrm{dR}}) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa}\big),\\
I_+ :=& \mathbf{Isom}_{\mathscr{S}_K\otimes\kappa}\big((\Lambda^*_\kappa, \bar{s}, \mathrm{Fil}^\bullet_\chi) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa},\, (\overline{\mathcal{V}^\circ}, \bar{s}_{\mathrm{dR}}, \overline{\mathcal{V}^\circ} \supset \mathcal{C}) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa}\big),\\
I_- :=& \mathbf{Isom}_{\mathscr{S}_K\otimes\kappa}\big((\Lambda^*_\kappa, \bar{s}, \mathrm{Fil}_\bullet^{\chi^{(\sigma)}}) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa},\, (\overline{\mathcal{V}^\circ}, \bar{s}_{\mathrm{dR}}, \mathcal{D} \subset \overline{\mathcal{V}^\circ}) \otimes \mathcal{O}_{\mathscr{S}_K\otimes\kappa}\big).
\end{aligned}$$

We have a natural right action of G_κ on I given by $\beta \cdot g := \beta \circ g^\vee$, and I_+ and I_- inherit actions of $P_{\chi,+}$ and $P^{(\sigma)}_{\chi,-}$.

A crucial step in the definition of the Ekedahl-Oort stratification is now the following result of Zhang:

Proposition II.5.12 ([Zh1], 2.4.1.)**.** *The Cartier isomorphisms induce an isomorphism $\iota \colon I_+^{(\sigma)}/U^{(\sigma)}_{\chi,+} \xrightarrow{\sim} I_-/U^{(\sigma)}_{\chi,-}$ such that the tuple $\underline{I} = (I, I_+, I_-, \iota)$ is a $G_{\mathbb{F}_p}$-zip of type χ over $\mathscr{S}_K \otimes \kappa$.*

Proof. We give a few comments on the proof, as we have defined I, I_+ and I_- in a different way than it is done in [Zh1]. Let us show for our definitions that for every closed point x of $\mathscr{S}_K \otimes \kappa$ the fibers I_x and $(I_+)_x$ are trivial torsors for G_κ and $P_{\chi,+}$ respectively: Let $k(x)$ be the residue class field of $\mathscr{S}_K$ at x, which is a finite extension of $\kappa(v)$. Let $\tilde{x} \in \mathscr{S}_K(W(k(x)))$ be a lift of x. By Proposition II.4.7(iii) and Lemma II.5.2 we find an isomorphism $\bar{\beta} \colon (\Lambda^*_{k(x)}, s_{k(x)}) \xrightarrow{\sim} (\overline{\mathcal{V}^\circ}_x, \bar{s}_{\mathrm{dR},x})$ and a cocharacter $\bar{\lambda}$ of $G_{k(x)}$ which induces the filtration $\Lambda^*_{k(x)} \supset \bar{\beta}^{-1}(\mathcal{C}_x)$ via $(\cdot)^\vee$, and further we have $\bar{\lambda}_{\overline{\mathbb{F}}_p} \in [\nu^{-1}]$. By definition, $\bar{\beta}$ then lies in $I_x(k(x))$, which shows that I_x is trivial. Moreover, $\bar{\lambda}$ is conjugate to χ over some finite extension of $k(x)$. This implies that $(I_+)_x$ is a $P_{\chi,+}$-torsor, this torsor must be trivial since $P_{\chi,+}$ is a connected group and $k(x)$ is finite.

Now all the arguments in the sections 2.2. - 2.4. and in the proof of 2.4.1. of [Zh1] carry over to our definition of I, I_+ and I_- with the necessary adjustments. □

By Proposition II.5.12 for every scheme S over $\mathscr{S}_K \otimes \kappa$ one obtains a $G_{\mathbb{F}_p}$-zip over S by pulling back the $G_{\mathbb{F}_p}$-zip $\underline{I}$, in other words, $\underline{I}$ defines a morphism of algebraic stacks

$$\zeta\colon \mathscr{S}_K \otimes \kappa \to G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi}_{\kappa}. \tag{48}$$

Theorem II.5.13 ([Zh1], 3.1.2.)**.** *The morphism ζ is smooth. In particular it induces a continuous and open map of topological spaces*

$$\zeta(\overline{\mathbb{F}}_p)\colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \longrightarrow (G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi}_{\kappa})(\overline{\mathbb{F}}_p) \simeq {}^J W.$$

Proof. Again, the proof of ([Zh1], 3.1.2.) goes through with the obvious changes. $\square$

Remark II.5.14. Though the definition of ζ depends on the choice of a cocharacter χ, the resulting map $\zeta(\overline{\mathbb{F}}_p)\colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \to {}^J W$ is in fact independent of χ. This is a consequence of the following two observations:

(1) Let κ' be a finite extension of κ, let $\chi' = \chi_{\kappa'}$, and let $\underline{I}'$ be the $G_{\mathbb{F}_p}$-zip of type χ' over $\mathscr{S}_K \otimes \kappa'$ given by Construction II.5.11. Then we have $G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi'}_{\kappa'} = (G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi}_{\kappa}) \otimes \kappa'$ and the equality $\underline{I}' = \underline{I} \otimes \kappa'$, which means that χ and χ' induce the same map $\zeta(\overline{\mathbb{F}}_p)$.

(2) Let χ' be a cocharacter of G_κ which is conjugate to χ over κ, say $\chi' = \mathrm{int}(g) \circ \chi$ for some $g \in G(\kappa)$. Let $P_{\chi',\pm} \subseteq G_\kappa$ be the associated parabolic subgroups, with common Levi subgroup $M_{\chi'}$, and again denote by $\underline{I}'$ the $G_{\mathbb{F}_p}$-zip associated to χ' (over κ). Applying the Propositions II.5.12 and II.5.10 to κ and χ' one obtains a map

$$\zeta'(\overline{\mathbb{F}}_p)\colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \to (G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi'}_{\kappa'})(\overline{\mathbb{F}}_p) \simeq {}^J W.$$

As $P_{\chi',\pm} = g(P_{\chi,\pm})g^{-1}$ and consequently $U_{\chi',\pm} = gU_{\chi,\pm}g^{-1}, M_{\chi'} = gM_\chi g^{-1}$, the element g defines an isomorphism of algebraic stacks

$$\begin{aligned}\Xi\colon G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi}_{\kappa} &\xrightarrow{\sim} G_{\mathbb{F}_p}\text{–}\mathtt{Zip}^{\chi'}_{\kappa}\\ \big(I, I_+, I_-, \iota\big) &\longmapsto \big(I,\ (I_+)\cdot g^{-1},\ (I_-)\cdot \sigma(g)^{-1},\ r_{\sigma(g)-1} \circ \iota \circ r_{\sigma(g)}\big),\end{aligned}$$

where $r_{\sigma(g)}$ and $r_{\sigma(g)^{-1}}$ are the obvious isomorphisms given by multiplication with $\sigma(g)$ resp. $\sigma(g)^{-1}$ on the right.
It is easy to see that $\Xi(\underline{I}) = \underline{I}'$. Further, going through the classification of $G_{\mathbb{F}_p}$-zips in [PWZ1], [PWZ2] (see also II.6.5), a straightforward but tedious computation shows that Ξ is compatible with the homeomorphisms from Proposition II.5.10, which implies that $\zeta'(\overline{\mathbb{F}}_p) = \zeta(\overline{\mathbb{F}}_p)$.

Due to Theorem II.5.13 and the definition of the topology on ${}^J W$, the inverse images of elements $w \in {}^J W$ under $\zeta(\overline{\mathbb{F}}_p)$ are the $\overline{\mathbb{F}}_p$-valued points of locally closed subsets $\mathcal{S}^w \subseteq \mathscr{S}_K \otimes \overline{\mathbb{F}}_p$.

Definition II.5.15. For $w \in {}^J W$ we call $\mathcal{S}^w \subseteq \mathscr{S}_K \otimes \overline{\mathbb{F}}_p$ the *Ekedahl-Oort stratum* associated to w. We endow the strata $\mathcal{S}^w$ with the reduced subscheme structure.

Note that this is really a stratification of $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$, in general the strata $\mathcal{S}^w$ are not defined over $\kappa(v)$. It is shown in ([Zh1], 3.2.8. and 3.2.9.) that two points $x, x' \in \mathscr{S}_K(\overline{\mathbb{F}}_p)$ lie in the same Ekedahl-Oort stratum if and only if there is an isomorphism $\underline{\mathbb{D}_x} \simeq \underline{\mathbb{D}_{x'}}$ which identifies $\bar{s}_{\mathrm{cris},x}$ with $\bar{s}_{\mathrm{cris},x'}$. Thus Definition II.5.15

generalizes the definitions of Ekedahl-Oort strata in the Siegel case and in the PEL-case. Recently, Zhang has also annonced a result which shows that the stratification is independent from the choice of the embedding $(\mathbf{G}, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$ in the very beginning of the construction (see [Zh2]).

As the Ekedahl-Oort stratification is defined by taking inverse images under the continuous and open map $\zeta(\overline{\mathbb{F}}_p)$, it inherits many topological properties of $(G_{\mathbb{F}_p}\text{−}\mathtt{Zip}^{\chi}_{\kappa})(\overline{\mathbb{F}}_p) \simeq {}^J W$:

II.5.16.

(1) Each $\mathcal{S}^w$ is either empty or equidimensional of dimension $l(w)$ (see [Zh1], 3.1.6.).

(2) The $\mathcal{S}^w$ form a stratification of $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$ in the strict sense: For every $w \in {}^J W$ we have
$$\overline{\mathcal{S}^w} = \bigcup_{w' \preceq w} \mathcal{S}^w.$$
This follows from the fact that taking inverse images under ζ commutes with taking closures (ζ being an open map), and the structure of the topological space ${}^J W$.

(3) The set ${}^J W$ contains a unique maximal element with respect to $\preceq$, namely $w_{\max} := w_{0,J} w_0$, and a unique minimal element $w_{\min} := 1$. By (2), the associated Ekedahl-Oort stratum $\mathcal{S}^{w_{\max}}$ is the unique open stratum and is dense in $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$, and $\mathcal{S}^{w_{\min}}$ is closed and contained in the closure of each stratum $\mathcal{S}^w$.

(4) In the PEL-case it is known that all Ekedahl-Oort strata are nonempty ([VW], Thm. 10.1.). In the Hodge type case this is an open question. In view of properties (2) and (3) the nonemptyness of all strata would follow from the fact that $\mathcal{S}^{w_{\min}}$ is nonempty.

II.6 Comparing the stratifications

We will now restrict our attention to the geometric fiber $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$, where the Newton stratification and the Ekedahl-Oort stratification are both defined. The question as to how the stratifications are related to each other can be studied by looking at their $\overline{\mathbb{F}}_p$-valued points, since all strata are locally closed subvarieties of $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$. We establish the commutative diagram from the introduction of this chapter, which allows to deduce information on the stratifications from the results of Chapter I. As a major appliciation we prove our theorems on the μ-ordinary locus.

Throughout this section we will mainly take on the group theoretical point of view as in Chapter I, we therefore resume many of the notations from Section I.2.1:

Notation II.6.1. We still denote by $\overline{\mathbb{F}}_p$ a fixed algebraic closure of $\mathbb{F}_p$. Let $\mathcal{O} := W(\overline{\mathbb{F}}_p)$ and $L := L(\overline{\mathbb{F}}_p)$, write $K := G(\mathcal{O}) \subseteq G(L)$. The projection $\mathcal{O} \to \mathcal{O}/(p) = \overline{\mathbb{F}}_p$ induces a surjective homomorphism $K \to G(\overline{\mathbb{F}}_p),\ g \mapsto \bar{g}$. The Frobenius σ of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$ acts on K and on $G(\overline{\mathbb{F}}_p)$, and these operations are compatible in the sense that $\sigma(\bar{g}) = \overline{\sigma(g)}$. Let $K_1 := \{g \in K \mid \bar{g} = 1\}$, this is a normal subgroup of K. For every $w \in W$ we choose a representative $w \in N_G(T)(\mathcal{O})$, and denote its reduction to $N_G(T)(\overline{\mathbb{F}}_p)$ by w as well. All constructions and definitions involving these elements will only depend on $w \in W$ and will be independent of the particular choice of representative. We set $p^\lambda := \lambda(p) \in G(L)$ for $\lambda \in X_*(T)$.

By abuse of notation we will frequently identify geometric objects over $\overline{\mathbb{F}}_p$ with their $\overline{\mathbb{F}}_p$-valued points. For example we denote the $\overline{\mathbb{F}}_p$-valued points of $\mathcal{N}^b$ resp. of $\mathcal{S}^w$ by the same symbols. With these notations we have the decompositions

$$\mathscr{S}_K(\overline{\mathbb{F}}_p) = \overset{\circ}{\bigcup_{b \in B(G,\mu)}} \mathcal{N}^b, \qquad \mathscr{S}_K(\overline{\mathbb{F}}_p) = \overset{\circ}{\bigcup_{w \in {}^J W}} \mathcal{S}^w.$$

II.6.1 The factorization result

Recall that we write $[g]$ for the $G(L)$-σ-conjugacy class of $g \in G(L)$. As in Definition I.2.16, we set

$$[[g]] := K \cdot_\sigma (K_1 g K_1) = \{hg'\sigma(h)^{-1} \mid g' \in K_1 g K_1,\, h \in K\}, \qquad g \in G(L),$$

further we make the following definitions:

Definition II.6.2.

(i) Write $\langle g \rangle := \{hg\sigma(h)^{-1} \mid h \in K\}$ for the K-σ-conjugacy class of an element $g \in G(L)$.

(ii) Set $C(G,\mu) := \{\langle g \rangle \mid g \in Kp^\mu K\}$.

(iii) For any $g' \in Kp^\mu K$ we identify $[[g']]$ and $[g']$ with a subset of $C(G,\mu)$: We write $\langle g \rangle \in [[g']]$ if and only $\langle g \rangle \subseteq [[g']]$ and $\langle g \rangle \in [g']$ if and only $\langle g' \rangle \subseteq [g']$.

It follows from Construction II.5.3 and Lemma II.5.4 that there is a well-defined map

$$\gamma \colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \longrightarrow C(G,\mu) \tag{49}$$

which is given as follows: If $x \in \mathscr{S}_K(\overline{\mathbb{F}}_p)$ and $\beta \colon (\Lambda^*_{\mathcal{O}}, s_{\mathcal{O}}) \simeq (\mathbb{D}_x, s_{\mathrm{cris},x})$ is an isomorphism as in Corollary II.4.8, then

$$\gamma(x) := \langle g_\beta \rangle, \text{ where } g_\beta \in G(L) \text{ with } \beta^{-1} \circ F \circ \beta =: F_\beta = g_\beta^\vee \circ (1 \otimes \sigma).$$

Here as usual $(\cdot)^\vee$ is the contragredient representation.

Remark II.6.3. In the case of a PEL-type Shimura variety the map γ is known to be *surjective* ([VW], Thm. 11.2.). We do not know whether the surjectivity of γ holds in general.

Consider the natural map

$$\tilde{\theta} \colon C(G,\mu) \longrightarrow B(G,\mu), \quad \langle g \rangle \longmapsto [g],$$

and let $\theta := \tilde{\theta} \circ \gamma \colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \to B(G,\mu)$. Then by definition of the Newton stratification in II.5.6 we have $\mathcal{N}^b = \theta^{-1}(\{b\})$ for each $b \in B(G,\mu)$, and further we have the following description of the fibers of $\tilde{\theta}$:

$$\tilde{\theta}^{-1}(\{b\}) = \{\langle g \rangle \mid g \in K\mu(p)K \cap b\} = C(G,\mu) \cap b, \quad b \in B(G,\mu).$$

On the other hand, recall that for every $w \in {}^J W$ the associated Ekedahl-Oort stratum is by definition given as $\mathcal{S}^w = \zeta^{-1}(\{w\})$, where we simply write ζ for the map

$\zeta(\overline{\mathbb{F}}_p)\colon \mathscr{S}_K(\overline{\mathbb{F}}_p) \to {}^JW$ from Theorem II.5.13. We will now explain that one also has a factorization $\tilde{\zeta}\colon C(G,\mu) \to {}^JW$ such that $\zeta = \tilde{\zeta} \circ \gamma$, and that the inverse images $\tilde{\zeta}^{-1}(\{w\})$ for $w \in {}^JW$ are exactly given by the $[[\cdot]]$-classes in $Kp^\mu K$.

II.6.4. We have seen in Remark II.5.14 that the map ζ does not depend on the choice of the cocharacter χ, nor on the choice of κ in Construction II.5.11. Therefore we may and will further assume that χ is the unique dominant cocharacter contained in $[\nu^{-1}]$, in other words that $\chi = \sigma^{-1}(\mu)$. We then have $J = \{s \in S \mid s(\chi) = \chi\}$.

We extend the subgroups of $G_{\overline{\mathbb{F}}_p}$ associated to χ in the last Section to groups over $\mathcal{O}$: Let $M \subseteq G$ be the centralizer of χ, let $P_+ \subseteq G$ be the unique parabolic subgroup which contains M and B, let P_- be the parabolic subgroup opposite to P_+ with respect to T. (In the notations of Section I.2.3 we would have $M = M_J$ and $P_+ = P_J$.) Let $U_\pm$ be the unipotent radical of $P_\pm$ respectively. The fibers of these groups over $\overline{\mathbb{F}}_p$ are then exactly the groups $P_{\chi,\pm}, U_{\chi,\pm}, M_\chi \subseteq G_{\overline{\mathbb{F}}_p}$ defined in Section II.5.3. To simplify notations, we will also write $P_\pm, U_\pm$ and M for these base changes.

II.6.5. Let us review in detail the homeomorphism $(G_{\mathbb{F}_p}\text{-}\mathtt{Zip}^\chi)(\overline{\mathbb{F}}_p) \simeq {}^JW$ from Proposition II.5.10 in this situation. The composition of the relative Frobenius morphism $G_{\overline{\mathbb{F}}_p} \to G^{(\sigma)}_{\overline{\mathbb{F}}_p}$ with the identification $G^{(\sigma)}_{\overline{\mathbb{F}}_p} \cong G_{\overline{\mathbb{F}}_p}$ (see Section II.5.3) is an isogeny of the algebraic group $G_{\overline{\mathbb{F}}_p}$, such that the induced map on $\overline{\mathbb{F}}_p$-valued points coincides with the σ-action on $G(\overline{\mathbb{F}}_p)$. By abuse of notation we denote this isogeny also by σ. It induces a morphism

$$\sigma\colon P_+/U_+ \cong M \longrightarrow M^{(\sigma)} \cong P_-^{(\sigma)}/U_-^{(\sigma)},$$

such that the tuple $(G_{\overline{\mathbb{F}}_p}, P_+, P_-^{(\sigma)}, \sigma)$ is an algebraic zip datum in the sense of ([PWZ1], 3.1.). The associated zip group is defined as

$$E_\chi := \left\{(mu_+, \sigma(m)u_-) \mid u_+ \in U_+, u_- \in U_-^{(\sigma)}, m \in M\right\} \subseteq P_+ \times P_-^{(\sigma)}.$$

It acts on $G_{\overline{\mathbb{F}}_p}$ on the left via $(p_+,p_-)\cdot g = p_+ g p_-^{-1}$. The homeomorphism is now obtained in two steps.

(I) The isomorphism classes of $G_{\mathbb{F}_p}$-zips of type χ over $\overline{\mathbb{F}}_p$ are identified with the orbits of the E_χ-action on $G_{\overline{\mathbb{F}}_p}$ by the following construction: Every $a \in G_{\overline{\mathbb{F}}_p}$ defines a zip $\underline{I}_a$ over $\overline{\mathbb{F}}_p$ by setting

$$\underline{I}_a := (G_{\overline{\mathbb{F}}_p},\, P_+,\, a\cdot P_-^{(\sigma)},\, \iota_a),$$

where ι_a is given by multiplication with a on the left, more precisely,

$$\iota_a\colon P_+^{(\sigma)}/U_+^{(\sigma)} \cong M^{(\sigma)} \xrightarrow{a\cdot} a\cdot M^{(\sigma)} \cong a\cdot P_-^{(\sigma)}/U_-^{(\sigma)}.$$

A zip of this form is called a *standard* $G_{\mathbb{F}_p}$-zip over $\overline{\mathbb{F}}_p$. By ([PWZ2], 3.5.), every $G_{\mathbb{F}_p}$-zip $\underline{I}$ of type χ over $\overline{\mathbb{F}}_p$ is isomorphic to a standard zip:

If $i_+ \in I_+$ and $i_- \in I_-$ such that $\iota(i_+^{(\sigma)}U_+^{(\sigma)}) = i_-U_-^{(\sigma)}$,
and if $a \in G_{\overline{\mathbb{F}}_p}$ such that $i_- = i_+\cdot a$, then $\underline{I} \simeq \underline{I}_a$.

Further, by ([PWZ2], 3.10) the assignment $\underline{I}_a \mapsto E_\chi \cdot a$ is well-defined, and $\underline{I}_a \simeq \underline{I}_{a'}$ if and only if $E_\chi \cdot a = E_\chi \cdot a'$. This construction can be made functorial and induces an isomorphism of stacks $(G_{\mathbb{F}_p}\text{−}\mathtt{Zip}^\chi) \otimes \overline{\mathbb{F}}_p \simeq \left[E_\chi \backslash G_{\overline{\mathbb{F}}_p}\right]$ ([PWZ2], 3.11.).

(II) Let B_- be the Borel subgroup of G which is opposite to B with respect to T, and let $y := w_{0,J}w_0 \in W$. Then the triple (B_-, T, y) is a frame for the zip datum $(G_{\overline{\mathbb{F}}_p}, P_+, P_-^{(\sigma)}, \sigma)$ in the sense of ([PWZ1], 3.6.): We have ${}^yT = T$, ${}^y(B_-) = {}^{w_{0,J}w_0}(B_-) = {}^{w_{0,J}}B$, and (every representative of) $w_{0,J}$ lies in M, so

$$\begin{aligned} B_- \subseteq P_-^{(\sigma)}, \qquad\qquad & {}^y(B_-) \subseteq P_+, \\ \sigma({}^y(B_-) \cap M) = \sigma({}^{w_{0,J}}(B \cap M)) &= \sigma(B_- \cap M) = B_- \cap \sigma(M), \\ \sigma({}^yT \cap M) = \sigma(T) &= T = T \cap \sigma(M) \end{aligned}$$

Here again we suppressed the $(\cdot)_{\overline{\mathbb{F}}_p}$-notation for the base changes of B_- and T. Let (W, S_-) be the Weyl group with respect to the pair (B_-, T). We need to compare it to (W, S), as for example explained in ([PWZ1], §2.3.): There is a unique isomorphism $\delta\colon (W, S) \to (W, S_-)$ of coxeter groups which is induced from an inner automorphism $\mathrm{int}(g)$, where $g \in G(\overline{\mathbb{F}}_p)$ such that $gBg^{-1} = B_-$ and $gTg^{-1} = T$. Since in our case we may choose $g = w_0$, we see that $\delta(w) = w_0ww_0$ for $w \in W$. The results of ([PWZ1], §6, §7), which are formulated for the Weyl group (W, S_-), thus show that the assignment

$${}^JW \longrightarrow E_\chi \backslash G_{\overline{\mathbb{F}}_p}, \quad w \longmapsto O^w := E_\chi \cdot (y\delta(w)) = E_\chi \cdot (w_{0,J}ww_0)$$

is bijective, and that $O^{w'}$ is contained in the closure of O^w if and only if $w' \preceq w$ (we defined $\preceq$ in section II.5.3).

Together these constructions describe the homeomorphism $(G_{\mathbb{F}_p}\text{−}\mathtt{Zip}^\chi)(\overline{\mathbb{F}}_p) \simeq {}^JW$ from II.5.10: It maps the isomorphism class of $\underline{I}_a$ to the unique $w \in {}^JW$ such that $a \in O^w$.

Proposition II.6.6. *Define* $\tilde{\zeta}\colon C(G, \mu) \to {}^JW$ *as the composition*

$$\begin{aligned} C(G, \mu) &\longrightarrow E_\chi \backslash G_{\overline{\mathbb{F}}_p} \quad \simeq \quad {}^JW. \\ \langle h_1 p^\mu h_2 \rangle &\longmapsto E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1) \end{aligned}$$

The following hold:

(i) The map $\tilde{\zeta}$ *is well-defined and surjective.*

(ii) We have the identity $\zeta = \tilde{\zeta} \circ \gamma$.

(iii) For $g, g' \in Kp^\mu K$ *we have* $\tilde{\zeta}(\langle g \rangle) = \tilde{\zeta}(\langle g' \rangle) \iff [[g]] = [[g']]$.

Proof. We will use the root groups $U_\alpha\colon \mathbb{G}_{a,\mathcal{O}} \to G_{\mathcal{O}}$, $(\alpha \in \Phi)$, cf. Section I.2.4. Recall that for every $\alpha \in \Phi$ and $\lambda \in X_*(T)$ we have the relation

$$p^\lambda U_\alpha(x) p^{-\lambda} = U_\alpha(p^{\langle \alpha, \lambda \rangle} x) \quad \text{for all } x \in L. \tag{*}$$

In particular, this implies that if $\langle \alpha, \lambda \rangle > 0$ then $p^\lambda U_\alpha(\mathcal{O}) p^{-\lambda} \subseteq K_1$.

(i) It is clear that $\tilde{\zeta}$ is surjective, once we know that it is well-defined. To prove this it is enough to show that the orbit $E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1)$ is independent of the choice of h_1 and h_2 (see also [HL], where a similar result is proved in Lemma 4.1.). So let $h'_1, h'_2 \in K$ $(i = 1, 2)$ such that $h_1 p^\mu h_2 = h'_1 p^\mu h'_2$. We define

$$c_1 := h_1^{-1} h'_1, \quad \tilde{c}_2 := h_2 (h'_2)^{-1}, \quad c_2 := \sigma^{-1}(\tilde{c}_2).$$

Then $\bar{c}_2(\sigma^{-1}(\bar{h}'_2)\bar{h}'_1)\bar{c}_1^{-1} = \sigma^{-1}(\bar{h}_2)\bar{h}_1$, so it suffices to show that $(\bar{c}_2, \bar{c}_1) \in E_\chi$. Since $c_1 = p^\mu \tilde{c}_2 p^{-\mu}$ and $\chi = \sigma^{-1}(\mu)$, we have $c_2 \in K_\chi := K \cap p^{-\chi} K p^\chi$. This is the stabilizer of two points in the Bruhat-Tits building of G_L, so we may apply the structure theory for these groups to K_χ (again cf. Section I.2.4: For all $\alpha \in \Phi$ define $U_\alpha^\chi := U_\alpha(L) \cap K_\chi$. From (*) we see that $U_\alpha^\chi = U_\alpha(p^a \mathcal{O})$ with $a = \max\{0, -\langle \alpha, \chi \rangle\}$. Note that $U_\alpha^\chi \subseteq K_1$ if $\langle \alpha, \chi \rangle < 0$. Now it follows from ([Ti], 3.1.) that we may write $c_2 = u_- u_+ m$, where

$$u_+ \in \prod_{\langle \alpha, \chi \rangle > 0} U_\alpha^\chi \subseteq U_+(\mathcal{O}), \quad u_- \in \prod_{\langle \alpha, \chi \rangle < 0} U_\alpha^\chi \subseteq K_1, \quad m \in M(\mathcal{O})$$

(here we use that $N_G T(L) \cap K_\chi \subseteq M(\mathcal{O})$ which can be easily checked). Thus we have $\bar{c}_2 \in P_+(\overline{\mathbb{F}}_p)$, with Levi component $\bar{m}$. Using (*) and the equation $\sigma^{-1}(c_1) = p^\chi c_2 p^{-\chi}$ we now see that $c_1 = u'_+ u'_- \sigma(m)$ with $u'_- \in \sigma(U_-(\mathcal{O}))$ and

$$u'_+ \in \sigma\Big(\prod_{\langle \alpha, \chi \rangle > 0} p^\chi U_\alpha^\chi p^{-\chi} \Big) \subseteq K_1.$$

Hence $\bar{c}_1 \in P_-^{(\sigma)}(\overline{\mathbb{F}}_p)$ and has Levi component $\overline{\sigma(m)} = \sigma(\bar{m})$, which shows that $(\bar{c}_2, \bar{c}_1) \in E_\chi$.

(iii) Now we investigate the fibers of $\tilde{\zeta}$ (cf. the proof of Theorem 1.1.(1) in [Vi1]). Let $\langle g \rangle, \langle g' \rangle \in C(G, \mu)$. Since everything only depends on the K-σ-conjugacy classes we may suppose that $g = hp^\mu$ and $g' = h'p^\mu$ for $h, h' \in K$. By definition of $\tilde{\zeta}$ we then have to show that

$$E_\chi \cdot \bar{h} = E_\chi \cdot \bar{h}' \iff [[hp^\mu]] = [[h'p^\mu]].$$

The implication "$\Longleftarrow$" follows directly from the proof of (i). Conversely, let $(p_+, p_-) \in E_\chi$ such that $p_+ \bar{h} p_-^{-1} = \bar{h}'$. We may choose

$$m \in M(\mathcal{O}), \quad u_+ \in U_+(\mathcal{O}), \quad u_- \in \sigma(U_-(\mathcal{O}))$$

such that $p_+ = \bar{u}_+ \bar{m}$ and $p_- = \bar{u}_- \sigma(\bar{m})$. By (*) we have $p^{-\mu} u_-^{-1} p^\mu \in K_1$ and $p^\mu \sigma(u_+) p^{-\mu} \in K_1$. Thus, using the fact that $\sigma(m)^{-1}$ commutes with p^μ, we find that

$$\begin{aligned}
[[h'p^\mu]] &= [[u_+ m h \sigma(m)^{-1} u_-^{-1} p^\mu]] = [[u_+ m h \sigma(m)^{-1} p^\mu]] \\
&= [[m h \sigma(m)^{-1} p^\mu \sigma(u_+)]] = [[m h \sigma(m)^{-1} p^\mu]] \\
&= [[m h p^\mu \sigma(m)^{-1}]] = [[hp^\mu]].
\end{aligned}$$

(ii) Finally we check that $\zeta = \tilde{\zeta} \circ \gamma$. Let $\underline{I}$ be the $G_{\mathbb{F}_p}$-zip associated to χ in Construction II.5.11. Consider a point $x \in \mathscr{S}_K(\overline{\mathbb{F}}_p)$. By definition, $\zeta(x)$ is then the $w \in {}^J W$ which corresponds to the isomorphism class of the pullback $\underline{I}_x$ to $\overline{\mathbb{F}}_p$. On the other hand, let $\beta\colon (\Lambda^*_{\mathcal{O}}, s_{\mathcal{O}}) \to (\mathbb{D}_x, s_{\mathrm{cris},x})$ be an isomorphism, and let $g_\beta \in Kp^\mu K$ such that $F_\beta = g_\beta^\vee \circ (1 \otimes \sigma)$, then $\gamma(x) = \langle g_\beta \rangle$. We may suppose that $g_\beta = hp^\mu$ for some $h \in K$ by passing to a different β (compare Lemma II.5.4). In view of the classification of $G_{\mathbb{F}_p}$-zips over $\overline{\mathbb{F}}_p$ (see II.6.5), to prove that $\tilde{\zeta}(\gamma(x)) = \zeta(x)$ we have to show that $\underline{I}_x$ is isomorphic to the standard zip $\underline{I}_{\bar{h}}$.

Let $\Lambda^*_{\overline{\mathbb{F}}_p} = \Lambda^*_{\chi,0} \oplus \Lambda^*_{\chi,1}$ be the weight decomposition of with respect to the action of χ on $\Lambda^*_{\overline{\mathbb{F}}_p}$ (using, as always, the representation $(\cdot)^\vee$), and likewise for μ. Keeping the notations of II.5.11, by definition we have $\underline{I}_x = (I_x, I_{+,x}, I_{-,x}, \iota_x)$, where

$$\begin{aligned}
I_x &:= \mathbf{Isom}_{\overline{\mathbb{F}}_p}\big((\Lambda^*_{\overline{\mathbb{F}}_p}, \bar{s}),\, (\overline{\mathcal{V}^\circ}_x, \bar{s}_{\mathrm{dR},x})\big),\\
I_{+,x} &:= \mathbf{Isom}_{\overline{\mathbb{F}}_p}\big((\Lambda^*_{\overline{\mathbb{F}}_p}, \bar{s}, \Lambda^*_{\overline{\mathbb{F}}_p} \supset \Lambda^*_{\chi,1}),\, (\overline{\mathcal{V}^\circ}_x, \bar{s}_{\mathrm{dR},x}, \overline{\mathcal{V}^\circ}_x \supset \mathcal{C}_x)\big),\\
I_{-,x} &:= \mathbf{Isom}_{\overline{\mathbb{F}}_p}\big((\Lambda^*_{\overline{\mathbb{F}}_p}, \bar{s}, \Lambda^*_{\mu,0} \subset \Lambda^*_{\overline{\mathbb{F}}_p}),\, (\overline{\mathcal{V}^\circ}_x, \bar{s}_{\mathrm{dR},x}, \mathcal{D}_x \subset \overline{\mathcal{V}^\circ}_x)\big),
\end{aligned}$$

the isomorphism $\iota_x\colon I^{(\sigma)}_{+,x}/U^{(\sigma)}_+ \to I_{-,x}/U^{(\sigma)}_-$ is given as follows: Choose an element $\eta_- \in I_{-,x}$. Then for each $\eta_+ \in I_{x,+}$ the image $\iota_x(\eta_+^{(\sigma)} U_+^{(\sigma)})$ is the $U_-^{(\sigma)}$-coset of the isomorphism

$$\begin{aligned}
\Lambda^*_{\overline{\mathbb{F}}_p} = \Lambda^*_{\mu,0} \oplus \Lambda^*_{\mu,1} &\xrightarrow{\eta_+^{(\sigma)}} \eta_+^{(\sigma)}(\Lambda^*_{\mu,0}) \oplus \mathcal{C}^{(\sigma)} \simeq (\overline{\mathcal{V}^\circ}^{(\sigma)}/\mathcal{C}^{(\sigma)}) \oplus \mathcal{C}^{(\sigma)}\\
&\xrightarrow{\phi_0 \oplus \phi_1} \mathcal{D} \oplus (\overline{\mathcal{V}^\circ}/\mathcal{D}) \simeq \mathcal{D} \oplus \eta_-(\Lambda^*_{\mu,1}) = \overline{\mathcal{V}^\circ} \qquad (50)
\end{aligned}$$

(here ϕ_0 and ϕ_1 are the Cartier isomorphisms and we have omitted the $(\cdot)_x$-subscripts), and this is in fact independent of the choice of η_-.
Let $(\overline{\mathbb{D}_x}, \overline{F}, \overline{V})$ be the reduction mod p of $\mathbb{D}_x$, this is the contravariant Dieudonné space of $\mathcal{A}_x[p]$ (compare the proof of Lemma II.5.4). We use the canonical isomorphism $(\overline{\mathcal{V}^\circ}_x, \bar{s}_{\mathrm{dR},x}) \cong (\overline{\mathbb{D}_x}, s_{\mathrm{cris},x})$. By the result of Oda ([Od], 5.11.) we know that $\mathcal{C}_x$ and $\mathcal{D}_x$ correspond to the subspaces $\ker(\overline{F}) = \mathrm{im}(\overline{V})$ and $\ker(\overline{V}) = \mathrm{im}(\overline{F})$ of $\overline{\mathbb{D}_x}$ respectively, and the isomorphisms ϕ_0 and ϕ_1 get identified with the maps

$$\big(\overline{\mathbb{D}_x}/\ker(\overline{F})\big)^{(\sigma)} \xrightarrow{\overline{F}^{\mathrm{lin}}} \mathrm{im}(\overline{F}), \quad \mathrm{im}(\overline{V})^{(\sigma)} \xrightarrow{(\overline{V}^{-1})^{\mathrm{lin}}} \overline{\mathbb{D}_x}/\ker(\overline{V}).$$

Recall our conventions from II.2.1, and the identities $F = \beta \circ (g_\beta^\vee \circ (1 \otimes \sigma)) \circ \beta^{-1}$ and $F^{\mathrm{lin}} = \beta \circ g_\beta^\vee \circ (\beta^{(\sigma)})^{-1}$ from Construction II.5.3. In particular we have $F \circ \beta = F^{\mathrm{lin}} \circ \beta^{(\sigma)} \circ (1 \otimes \sigma)$. As $V^{\mathrm{lin}}\colon \mathbb{D}_x \to \mathbb{D}_x^{(\sigma)}$ is given by $p \cdot (F^{\mathrm{lin}})^{-1}$, we further have the identity $(\beta^{(\sigma)})^{-1} \circ V^{\mathrm{lin}} \circ \beta = p \cdot (g_\beta^\vee)^{-1}$. Denote by pr_i the projections on the factors of the decomposition $\Lambda^*_{\overline{\mathbb{F}}_p} = \Lambda^*_{\mu,0} \oplus \Lambda^*_{\mu,1}$. The reduction mod p of the map $\mu(p)^\vee\colon \Lambda^*_{\mathcal{O}} \to \Lambda^*_{\mathcal{O}}$ given via $(\cdot)^\vee$ by the element $p^\mu = \mu(p) \in G(L)$ is exactly pr_0, and the reduction of the map $p \cdot (\mu(p)^\vee)^{-1}$ is

pr_1, so we have the commutative diagrams

$$\begin{tikzcd} \overline{\mathbb{D}_x}^{(\sigma)} \arrow[r, "\overline{F}^{\mathrm{lin}}"] & \overline{\mathbb{D}_x} \\ \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}^{(\sigma)}"] \arrow[r, "\bar{h}^\vee \circ \mathrm{pr}_0"] & \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}"'] \end{tikzcd}, \qquad \begin{tikzcd} \overline{\mathbb{D}_x} \arrow[r, "\overline{V}^{\mathrm{lin}}"] & \overline{\mathbb{D}_x}^{(\sigma)} \\ \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}"] \arrow[r, "\mathrm{pr}_1 \circ (\bar{h}^\vee)^{-1}"] & \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}^{(\sigma)}"'] \end{tikzcd}.$$

In particular, the first diagram implies that

$$\begin{aligned} \bar{\beta}^{-1}(\ker(\overline{F})) &= (1 \otimes \sigma^{-1})((\bar{\beta}^{(\sigma)})^{-1}(\ker(\overline{F}^{\mathrm{lin}}))) = (1 \otimes \sigma^{-1})(\ker(\mathrm{pr}_0)) \\ &= (1 \otimes \sigma^{-1})(\Lambda^*_{\mu,1}) = \Lambda^*_{\chi,1}, \\ \bar{\beta}^{-1}(\mathrm{im}(\overline{F})) &= \bar{\beta}^{-1}(\mathrm{im}(\overline{F}^{\mathrm{lin}})) = \bar{h}^\vee(\Lambda^*_{\mu,0}). \end{aligned}$$

This shows that $\bar{\beta} \in I_{x,+}$ and $\bar{\beta} \cdot \bar{h} = \bar{\beta} \circ \bar{h}^\vee \in I_{x,-}$. Putting things together and using the splittings $\mathcal{C}_0 := \bar{\beta}^{(\sigma)}(\Lambda^*_{\mu,0})$ and $\mathcal{D}_1 := \bar{\beta}(\bar{h}^\vee(\Lambda^*_{\mu,1}))$, from the diagrams above we obtain a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{V}^\circ}^{(\sigma)} \simeq \mathcal{C}_0 \oplus \mathcal{C}^{(\sigma)} \arrow[r, "\phi_0 \oplus \phi_1"] & \mathcal{D} \oplus \mathcal{D}_1 \simeq \overline{\mathcal{V}^\circ} \\ \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}^{(\sigma)}"] \arrow[r, "\bar{h}^\vee"] & \Lambda^*_{\overline{\mathbb{F}}_p} \arrow[u, "\bar{\beta}"'] \end{tikzcd},$$

so (50) shows that $\iota_x(\bar{\beta}^{(\sigma)} U_+^{(\sigma)}) = (\bar{\beta} \circ \bar{h}^\vee) U_-^{(\sigma)}$, choosing $\eta_- = \bar{\beta} \cdot \bar{h}$. Thus we have indeed $\underline{I}_x \sim \underline{I}_{\bar{h}}$ (see II.6.5(1)), which concludes the proof.

□

Corollary II.6.7. *Let $w \in {}^J W$. Then $x \in \mathcal{S}^w$ if and only if $\gamma(x) \in [[p^\mu \sigma(w_{0,J} w w_0)]]$.*

Proof. By Proposition II.6.6(ii), $x \in \mathcal{S}^w = \zeta^{-1}(\{w\})$ if and only if $\tilde{\zeta}(\gamma(x)) = w$. Since by definition of $\tilde{\zeta}$ and the description of the bijection $E_\chi \backslash G_{\overline{\mathbb{F}}_p} \simeq {}^J W$ in II.6.5(II) we have $\tilde{\zeta}(\langle p^\mu \sigma(w_{0,J} w w_0) \rangle) = w$, by II.6.6(iii) this is further equivalent to $\gamma(x) \in [[p^\mu \sigma(w_{0,J} w w_0)]]$. □

II.6.2 Group theoretic criteria and applications

Let us summarize the results of the last subsection: The Newton strata $\mathcal{N}^b$ and Ekedahl-Oort strata $\mathcal{S}^w$ are given as the fibers of the maps θ and ζ respectively. We have the commutative diagram

$$\begin{tikzcd} & & B(G,\mu) \\ \mathscr{S}_K(\overline{\mathbb{F}}_p) \arrow[urr, "\theta"] \arrow[r, "\gamma"] \arrow[drr, "\zeta"'] & C(G,\mu) \arrow[ur, "\tilde{\theta}"'] \arrow[dr, "\tilde{\zeta}"] & \\ & & {}^J W \end{tikzcd},$$

for $b \in B(G,\mu)$ we have $\tilde{\theta}^{-1}(\{b\}) = C(G,\mu) \cap b$, and for $w \in {}^JW$ we have $\tilde{\zeta}^{-1}(\{w\}) = [[p^\mu \sigma(w_{0,J} w w_0)]]$ by Corollary II.6.7.

This last result shows in particular that the $[[\cdot]]$-classes in $Kp^\mu K$ are parametrized by elements of ${}^JW$, sending w to $[[p^\mu\sigma(w_{0,J}ww_0)]]$. In order to apply the results from Chapter I, we need to compare this to the parametrization by standard representatives considered there. Let us explain the relation: As in Section I.2.6, set

$$S_\mu := \{s \in S \mid s(\mu) = \mu\}, \quad x_\mu := w_{0,S_\mu} w_0, \quad \tau_\mu := p^\mu x_\mu.$$

Let $L_\mu := S_\mu^{\mathrm{opp}} = w_0 S_\mu w_0$, consider the "dual parametrization" of $[[\cdot]]$-classes in $Kp^\mu K$ given by the elements $\tau_\mu w$ for $w \in {}^{L_\mu}W$ and the order relation $\preceq$ on ${}^{L_\mu}W$ induced by the partial order for $[[\cdot]]$-classes, as explained in Remark I.3.3. The map

$$\phi \colon W \longrightarrow W, \quad w \longmapsto w_0 \sigma(w) w_0 \tag{51}$$

is an automorphism of W which preserves the length of elements and the Bruhat order, and the following holds:

Lemma II.6.8.

(i) ϕ restricts to a bijection ${}^JW \to {}^{L_\mu}W$ which respects the $\preceq$ order relations on both sides.

(ii) For each $w \in {}^JW$ we have $x \in \mathcal{S}^w$ if and only if $\gamma(x) \in [[\tau_\mu \phi(w)]]$.

Proof. In the notations of I.2.6 we have $J = S_\chi$. On the other hand we have $\mu = \sigma(\chi)$, therefore ϕ induces a bijection $J \to L_\mu$ and an isomorphism $W_J \to W_{L_\mu}$. This immediately shows that $\phi\colon {}^JW \to {}^{L_\mu}W$ is a bijection. The second statement of (i) follows from the explicit description of the respective order relations in (12) and (47), and the fact that ϕ preserves the Bruhat order. Statement (ii) is just a reformulation of Corollary II.6.7, since

$$p^\mu \sigma(w_{0,J} w w_0) = p^\mu w_{0,S_\mu} w_0 (w_0 \sigma(w) w_0) = \tau_\mu \phi(w)$$

for each $w \in {}^JW$. $\square$

II.6.9. This has now the following consequences for the comparison of the two stratifications:

(a) For $b \in B(G,\mu)$ and $w \in {}^JW$ there is the following necessary criterion for the corresponding strata to intersect:

$$\text{If } \mathcal{N}^b \cap \mathcal{S}^w \neq \emptyset, \text{ then } (C(G,\mu) \cap b) \cap [[\tau_\mu\phi(w)]] = b \cap [[\tau_\mu \phi(w)]] \neq \emptyset.$$

If the map γ is surjective, then this criterion is also sufficient.

(b) Let $b \in B(G,\mu), w \in {}^JW$. If $[[\tau_\mu\phi(w)]] \subseteq C(G,\mu) \cap b$ (resp. $\supseteq$, resp. $=$), then $\mathcal{S}^w \subseteq \mathcal{N}^b$ (resp. $\supseteq$, resp. $=$).

(c) Let $w \in {}^JW$ such that $[[\tau_\mu\phi(w)]]$ is a K-fundamental class in the sense of Chapter I. Then for any two points $x, x' \in \mathcal{S}^w$ there is an isomorphism $\mathbb{D}_x \simeq \mathbb{D}_{x'}$ which identifies the tensors $s_{\mathrm{cris},x}$ and $s_{\mathrm{cris},x'}$.

Here (a) and (b) are immediate, for (c) recall that being K-fundamental means that $[[\tau_\mu\phi(w)]] = \langle \tau_\mu \phi(w) \rangle$ and apply Lemma II.5.5.

Applications:

Consider the minimal Ekedahl-Oort stratum $\mathcal{S}^{w_{\min}}$ given by the minimal element $w_{\min} = 1 \in {}^J W$: Recall that as a consequence of the properties of a Shimura datum the cocharacter μ is minuscule. Therefore, as remarked in Section I.3.5.1, the EO-class $[[\tau_\mu \phi(w_{\min})]] = [[\tau_\mu]]$ is contained in the basic σ-conjugacy class b_{bas}. Hence by (a) we have $\mathcal{S}^{w_{\min}} \subseteq \mathcal{N}^{\text{bas}}$, just as in the PEL-case.

Consider the PEL-datum $(\mathbb{E}, * = \text{id}, V, \psi, h)$, where $\mathbb{E}|\mathbb{Q}$ is a totally real extension of degree d, $\dim_{\mathbb{E}}(V) = 2$, and where ψ is an $\mathbb{E}$-linear symplectic form on V. In this case the group $\mathbf{G}$ is given as

$$\mathbf{G}(R) = \{g \in \mathrm{GL}_2(\mathbb{E} \otimes_{\mathbb{Q}} R) \mid \det(g) \in R^\times\},$$

there is a unique conjugacy class X of homomorphisms $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ such that $(\mathbf{G}, X)$ is a Shimura datum of PEL-type. Let p be a prime which is inert in $\mathbb{E}$, then p is a prime of good reduction for $\mathbf{G}$ and the group G over $\mathbb{Z}_p$ is of the form that we studied in Section I.4.2. The associated Kottwitz models $\mathscr{S}_K$ are then *Hilbert modular varieties* of pricipally polarized d-dimensional abelian schemes up to prime-to-p isogeny, endowed with an action of $o_{\mathbb{E},(p)}$ and a K^p-level structure. Since the map γ is surjective for PEL-type Shimura varieties, by (a) and (b) we have:

Proposition II.6.10. *In the situation above the intersections of Newton strata and Ekedahl-Oort strata in $\mathscr{S}_K$ are given as explained in Theorem I.E, in particular:*

(i) For every Newton stratum $\mathcal{N}^b$ in $\mathscr{S}_K$ there exists a (generally non unique) Ekedahl-Oort stratum contained in $\mathcal{N}^b$ which is equal to a single leaf.

(ii) There exist an Ekedahl-Oort stratum which has nonempty intersection with every Newton stratum outside the μ-ordinary locus.

In concrete terms, the statement of (i) means that every isogeny class of polarized p-divisible groups with $o_{\mathbb{E},(p)}$-action contains a "minimal element" X in the following sense: For any other p-divisible group X' with these structures, if $X'[p]$ is $o_{\mathbb{E},(p)}$-equivariantly isomorphic to $X[p]$, then X' and X are $o_{\mathbb{E},(p)}$-equivariantly isomorphic.

As another application we can now prove our main results on the μ-ordinary locus in the Hodge-type case:

Theorem II.6.11 (cf. Theorem II.B)**.** *The μ-ordinary locus $\mathcal{N}^{\mu-\text{ord}}$ in $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$ is equal to the unique open Ekedahl-Oort stratum $\mathcal{S}^{w_{\max}}$, in particular it is open and dense. Further, for any two $\overline{\mathbb{F}}_p$-valued points x, x' in the μ-ordinary locus there is an isomorphism $(\mathbb{D}_x, s_{\text{cris},x}) \simeq (\mathbb{D}_{x'}, s_{\text{cris},x'})$.*

Proof. We have $w_{\max} = w_{0,J} w_0$, so $\phi(w_{\max}) = w_0 w_{0,\sigma(J)}$ is the maximal element in ${}^{L_\mu}W$ by Lemma II.6.8. So by Corollary I.3.40 we have $[[\tau_\mu \phi(w_{\max})]] = C(G, \mu) \cap b_{\mu-\text{ord}} = \langle p^\mu \rangle$. Now the equality of the strata holds by observation (b) above, and the last statement follows by (c). As we know that $\mathcal{S}^{w_{\max}}$ is open and dense in $\mathscr{S}_K \otimes \overline{\mathbb{F}}_p$ (see II.5.16(3)), the same holds for the μ-ordinary locus. $\square$

Corollary II.6.12 (cf. Theorem II.C)**.** *The μ-ordinary locus in $\mathscr{S}_K \otimes \kappa(v)$ is open and dense.*

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