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A Dynamical Interpretation of
Patterson-Sullivan Distributions

Dissertation

vorgelegt von

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Zusammenfassung

Wir untersuchen auf kompakten reell-hyperbolischen Räumen den Zusammenhang zwischen gewissen Distributionen auf dem Phasenraum, sog. Patterson-Sullivan Distributionen, und dynamischen Zetafunktionen, die mithilfe von Daten des geodätischen Flusses definiert werden und die bekannte logarithmische Ableitung der Selbergschen Zetafunktion verallgemeinern. Für den Flächenfall wurde dies zuerst in [AZ07] getan, wir konzentrieren uns auf den höher-dimensionalen Fall.

Summary

Given a compact real hyperbolic space we study the connection between certain phase space distributions, so called Patterson-Sullivan distributions, and dynamical zeta functions. These zeta functions generalize logarithmic derivatives of classical Selberg zeta functions which are defined by closed geodesics which is data from the geodesic flow on phase space. Patterson-Sullivan distributions are asymptotically equivalent to Wigner distributions which play a key role in quantum ergodicity but they are also invariant under the geodesic flow.

The surface case was studied before in [AZ07] and thus the emphasis in this work lies on the higher dimensional case.

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CHAPTER 1

Statement of results

This work is mainly based on the articles [AZ07] by N. Anantharaman and S. Zelditch and [Zel89] by the latter author. We will shortly summarize the results from [AZ07] and [Zel89], which we want to examine.

Let X_Γ be a compact hyperbolic surface which can be written as

$$X_\Gamma := \Gamma \backslash G / K.$$

Here $G = \mathrm{PSL}_2(\mathbb{R})$, $K = \mathrm{PSO}(2)$ and $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a cocompact, discrete and torsionfree subgroup of G . We also consider the right-regular representation π_R of G on $L^2(\Gamma \backslash G)$ defined by

$$g \cdot f(x) = f(xg).$$

Since Γ is cocompact, this unitary representation decomposes into a discrete sum

$$(1.1) \quad L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} m_\pi V_\pi,$$

where $m_\pi \in \mathbb{N}$. We call the irreducible component V_π spherical, if it possesses a K -fixed vector. These K -fixed vectors are unique up to constants, smooth and give an orthogonal basis of $L^2(X_\Gamma)$ of Laplace eigenfunctions. We fix such a normalized basis $\{\varphi_n\}_n$ with eigenvalues $-(\rho_0^2 + \lambda_n^2)$. It can be shown that one can either assume $\lambda_n \in i\mathbb{R}$ or $\lambda_n \in \mathbb{R}^+$. In the former case, we say that λ_n is in the complementary series, else we say that λ_n is in the principal series. There are only finitely many λ_n in the complementary series. Associated to $\{\varphi_n\}_n$ there are some specific distributions on phase space. They are called Wigner distributions and are given by

$$W_n(\sigma) := \langle \mathrm{Op}(\sigma) \varphi_n, \varphi_n \rangle_{L^2(X_\Gamma)}$$

for $\sigma \in C^\infty(S^*X_\Gamma)$. Here one needs a calculus for pseudodifferential operators, i.e. an assignment

$$C^\infty(S^*X_\Gamma) \rightarrow B(L^2(X_\Gamma)), \quad a \mapsto \mathrm{Op}(a).$$

This is the data from the quantum side we need.

On the side of classical dynamics, we consider the geodesic flow on T^*X_Γ . Since the geodesic flow preserves the metric on T^*X_Γ , it can be considered as a mapping on phase space $S^*X_\Gamma \cong SX_\Gamma$, which can also be identified with $\Gamma \backslash G$. Under this identification the geodesic flow is given by right translation with $\exp tH_0$, where $\mathfrak{a} = \mathbb{R}H_0 = \mathrm{Lie}(A)$ comes from an Iwasawa decomposition of $G = ANK$. Periodic orbits of the geodesic flow are called closed geodesics and it can be shown that there is a bijection between closed geodesics and nontrivial conjugacy classes $[\gamma]$ in Γ . The smallest possible period is called the length of $[\gamma]$ and denoted by L_γ . Finally, a closed geodesic $[\gamma]$ is called prime, if there is no $\gamma_0 \in \Gamma$ and $n \in \mathbb{N}$, $n > 1$, such that

$$(1.2) \quad \gamma = \gamma_0^n.$$

With this data from classical mechanics, in [AZ07] a dynamical zeta function

$$(1.3) \quad \mathcal{Z}(k; \sigma) := \sum_{[\gamma] \neq 1} \frac{e^{-kL_\gamma}}{1 - e^{-L_\gamma}} \cdot \int_{\gamma_0} \sigma$$

for $k \in \mathbb{C}$, $\operatorname{Re}(k) > 1$, is defined. Here $[\gamma_0]$ is the unique prime closed geodesic belonging to $[\gamma]$ and σ is a test function on phase space, that is $\sigma \in C^\infty(\Gamma \backslash G)$.

For $\sigma \equiv 1$ constant, $\mathcal{Z}(\cdot; \sigma)$ just equals the logarithmic derivative of the dynamical Selberg zeta function Z_S

$$\frac{d}{dk} \ln Z_S(k) = \sum_{[\gamma] \neq 1} \frac{e^{-kL_\gamma}}{1 - e^{-L_\gamma}} \cdot L_{\gamma_0}.$$

In this context Anantharaman and Zelditch show the following theorem, see [AZ07, Th.1.3]:

THEOREM 1.0.1. *If σ is a real analytic function on SX_Γ , then $\mathcal{Z}(\cdot; \sigma)$ admits a meromorphic continuation to \mathbb{C} . The poles in the critical strip $0 < \operatorname{Re}(k) < 1$ appear at $s = \frac{1}{2} + i\lambda$, where $-(\frac{1}{4} + \lambda^2)$ is an eigenvalue of the Laplacian. The residue at $k = \frac{1}{2} + i\lambda$ is given by*

$$\sum \langle \sigma, \widehat{\text{PS}}_\varphi \rangle,$$

where this finite sum runs over all (normalized) Patterson-Sullivan distributions $\widehat{\text{PS}}_\varphi$ associated with an eigenfunction φ for the eigenvalue $-(\frac{1}{4} + \lambda^2)$.

Here, (normalized) Patterson-Sullivan distributions $\widehat{\text{PS}}_\varphi$ are certain other kind of phase space distributions which one can associate to the eigenfunctions $\{\varphi_n\}_n$, but in contrast to Wigner distributions they are invariant under the geodesic flow. They were first defined in [AZ07] using the same calculus which was used to construct Wigner distributions. See [HHS12] for an extension of this calculus and a definition of Patterson-Sullivan distributions by this calculus for arbitrary compact locally symmetric spaces. Patterson-Sullivan distributions also play a role in quantum unique ergodicity, since they are asymptotically equivalent to Wigner distributions, see [AZ07], [HHS12].

Note that if the spectrum is simple, i.e. if $m_\pi = 1$ for all spherical components V_π , this theorem yields a definition of Patterson-Sullivan distributions, which only uses knowledge of the length spectrum $\{L_\gamma : [\gamma] \text{ conjugacy class in } \Gamma\}$. See for example the survey article [Sar11] for more information on simple spectra of the Laplacian.

[AZ07] presents two kinds of proofs for Theorem 1.0.1. The first one uses thermodynamic formalism, the second one is of representation theoretical nature and uses a generalized Selberg trace formula, which can be found in [Zel89]. We will pursue the second proof as this seems to be more feasible in generalizing 1.0.1 to higher dimensional spaces. It has the disadvantage as it so far only works for test functions σ , which have only finitely many nontrivial components in the decomposition (1.1). On the other hand this approach makes it possible also to determine poles/residues of $\mathcal{Z}(\sigma)$ outside the strip $0 < \operatorname{Re}(k) < 1$. Furthermore, a generalized Selberg trace formula could be of independent interest, as it connects periodic orbit measures to Wigner distributions.

We will now come to the content of our work. $X_\Gamma := \Gamma \backslash G/K$ will be a compact locally symmetric space of (real) rank one. Here G is a real semisimple, noncompact Lie group with finite center, $K \subset G$ a maximal compact subgroup and $\Gamma \subset G$ a discrete, cocompact and torsionfree subgroup. We also fix an Iwasawa decomposition $G = ANK$ and set $M = Z_K(A)$ for the centralizer of A in K . In the course of this work, we will specialize X_Γ to be a compact real hyperbolic locally symmetric

space, this means $G = SO(1, n)$. We restrict to real rank one symmetric spaces, as this ensures the absence of non trivial elliptic elements in the uniform lattice Γ . The major reason for specializing further to real hyperbolic spaces is that in this case M acts on N with a one dimensional slice and the differential equation which will occur during this work can be solved on this simple slice.

The representation theoretic proof of Theorem 1.0.1 relies mainly on three results, which we will now explain. The first one is an observation using the fact that Patterson-Sullivan distributions are invariant under the geodesic flow. Then one uses the result that the representation of A on irreducible components V_π of $L^2(\Gamma \backslash G)$, which we obtain by restricting the right-regular representation of G , is particularly simple. More precisely, if $G = PSL_2(\mathbb{R})$ and V_π is a spherical component of (1.1), the representation of A on V_π has exactly two invariant subspaces, one of which is generated by the K -fixed vector in V_π . If V_π is not spherical, then it has already a cyclic vector, see [AZ07, Prop. 2.2.]. This yields that in the proof instead of considering all possible $\sigma \in C^\infty(SX_\Gamma)$ one can restrict to σ coming from three basic series. For X_Γ a real hyperbolic space we can generalize the result on the action of A on the spherical spectrum. It will turn out that if X_Γ is of dimension at least 3, the action of A on any spherical component V_π is already irreducible, the K -fixed vector being cyclic, see Theorem 8.1.3. Here we use the fact that the set of M -invariants in the universal enveloping algebra $U(\mathfrak{n})$ is generated by the (euclidean) Laplacian on \mathfrak{n} . If V_π is not spherical, we state a result, but we will not use it, see Proposition 8.1.2.

The procedure for defining $\mathcal{Z}(\sigma)$ is computing the (geometric) trace of a suitable trace class operator $\sigma \cdot \pi_R(f)$, which depends on σ and a suitable function $f \in C^\infty(G)$. Here $\pi_R(f)$ is just the Fourier transform of f with respect to the representation π_R and σ is viewed as a multiplication operator. The trace can be computed in two ways, the first one is the so-called geometric, the latter one is called the spectral trace. We start by computing the geometric trace and thus, the next ingredient is a generalization of Selberg's trace formula depending on σ coming from the three series. We first mention that the computation of the trace formula heavily depends on the rank one assumption. Namely, if X_Γ is a locally symmetric compact space of (real) rank one, then all nontrivial elements in Γ are hyperbolic which means they are conjugate to some element $ma \in MA$. We state a trace formula for general compact locally symmetric spaces of rank one in Theorem 5.2.4.

We further specialize σ to be a function in $C(SX_\Gamma)$, which is only allowed to have finitely many nontrivial components in the *spherical* spectrum and no components in the nonspherical spectrum. Hence in the case of a real hyperbolic space, Lemma 8.1.1 allows us now to reduce to the case where σ equals some eigenfunction φ on X_Γ . The obstacle that occurs in computing a satisfactory trace formula now, is a factor we call the weight function $I_\gamma(\sigma)$. It is a real valued function on N and depends on the element γ in Γ and the function σ .¹ Basically, for $n \in N$, $I_\gamma(\sigma)(n)$ is the integral of the n -translation of σ over the prime closed geodesic belonging to $[\gamma]$. For $\varphi = \sigma \equiv 1$ this weight function is constant and just equals L_{γ_0} but for non constant eigenfunctions the evaluation of the weight function is more complicated. Since φ is a Laplace eigenfunction by assumption, we obtain a differential equation using an expression for the Casimir Ω from Chapter 2.1. But this differential equation is a priori an equation on N and thus it is not clear how $I_\gamma(\varphi)$, which is as mentioned above a real valued function on N , depends in higher dimensions on its value at the neutral element, which is just $\int_{\gamma_0} \varphi$ from the

¹In [AZ07] this weight function is called orbital integral but this seems to not quite compatible with the terminology of the classical Selberg trace formula.

definition of \mathcal{Z} . We circumvent this problem by decomposing $I_\gamma(\varphi)$ into a sum

$$I_\gamma(\varphi) = \sum_{\pi \in \widehat{M}} d_\pi \chi_\pi * I_\gamma(\varphi),$$

see Theorem 5.3.1. Here \widehat{M} consists of all irreducible representations (π, V_π) of M , d_π is the dimension of V_π , $\chi_\pi = \text{Tr}(\pi)$ the character of π and $*$ denotes convolution. Then $d_\pi \chi_\pi * I_\gamma(\varphi)$ is the projection of $I_\gamma(\varphi)$ in the space of M -finite functions of type $\tilde{\pi}$, $\tilde{\pi}$ the contragredient representation to π . Furthermore, each $\chi_\pi * I_\gamma(\varphi)$ is also a Casimir eigenfunction with the same eigenvalue as φ . The observation which helps us now is the fact that in real hyperbolic spaces, the subgroup M acts transitively on spheres in N if the dimension of the real hyperbolic space is at least 3, i.e. slices for this action of M on N are one dimensional. For any slice S we can now restrict the equation for each $\chi_\pi * I_\gamma(\varphi)$ to S and the results of Chapter 3 and 4 allow us to determine $\chi_\pi * I_\gamma(\varphi)|_S$ as a product of a hypergeometric function with a monomial and a scalar which is connected to $I_\gamma(\varphi)$ at the neutral element, see equations (5.18) and (5.19).

The resulting trace formula is now sufficient to define a zeta function

$$(1.4) \quad \mathcal{Z}(k; \sigma) := \sum_{1 \neq [\gamma] \in \text{CT}} \sum_{\pi \in \widehat{M}} c(\gamma, \sigma, \pi, k) e^{(-k + \rho_0)L_\gamma},$$

which converges at least on the half plane $\{k \in \mathbb{C} : \text{Re}(k) > 2\rho_0\}$ and generalizes the one from [AZ07]. Here ρ_0 is a number depending only on the dimension and σ is as above and the coefficients $c(\gamma, \sigma, \pi, k)$ depends on k , the test function σ , π in \widehat{M} and the period length L_{γ_0} of the prime geodesic $[\gamma_0]$.

The meromorphic continuation now follows from the computation of the spectral trace by using the basis of eigenfunctions $\{\varphi_n\}_n$. The preliminary result of the trace, for an eigenfunction φ , is given in Proposition 6.1.3. It uses a calculus for pseudodifferential operators from [Sch10], see also [HS], which is adapted to the rank one setting. Proposition 6.1.3 is valid for any locally symmetric compact space of rank one, but it only involves Wigner distributions. To connect it with Patterson-Sullivan distributions we have to perform computations similar to the ones on the geometric side. In particular, we again encounter a differential equation, which we can only solve when the underlying space is real hyperbolic. The final result for the spectral trace can be found in Theorem 6.2.18.

The meromorphic continuation now follows by standard arguments, we only mention that we have to make some explicit computations for functions on hyperbolic space. These calculations also seem to be possible in other rank one spaces by the classification results but we have not tried to do so.

The main result is as follows, see Proposition 8.1.2.

THEOREM 1.0.2. *Let X_Γ be a compact, locally symmetric real hyperbolic space and σ a function in $C^\infty(SX_\Gamma)$ with only finitely many nontrivial components in the spherical spectrum and no components in the non-spherical spectrum. The associated zeta function $\mathcal{Z}(\sigma)$ defines a meromorphic function on \mathbb{C} . In the strip $\rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}$ the poles of $\mathcal{Z}(\sigma)$ are at $k = \rho_0 + i\lambda$, where $-(\rho_0^2 + \lambda^2)$ is an eigenvalue of the Laplacian, and $k = \rho_0$. Their residues are determined by Patterson-Sullivan distributions. If λ comes from the principal series, then the residue at $k = \rho_0 + i\lambda$ is given (up to a non-zero constant) by normalized Patterson-Sullivan distributions $\widehat{\text{PS}}_\varphi$*

$$(1.5) \quad \sum \langle \sigma, \widehat{\text{PS}}_\varphi \rangle,$$

where the sum runs over all eigenfunctions φ with eigenvalue $-(\lambda^2 + \rho_0^2)$.

REMARK 1.0.3. Let $\sigma = \varphi$ be a Laplace eigenfunction. There are two statements in Theorem 1.3 from [AZ07] which we cannot verify in Theorem 1.0.2 in the case of a compact surface.

The first one is about the form of the zeta function $\mathcal{Z}(\sigma)$ from (1.3). It seems to come from an incomplete integral substitution in [AZ07] which is used to get from equation (9.8) to (9.9). Our definition (1.4) differs in the surface case by a constant which depends on $k \in \mathbb{C}$, the geodesic $[\gamma]$ and the eigenvalues from the K -fixed vectors in the spherical components of σ , see Section 8.2 for the definition. This constant is furthermore holomorphic in k on $\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\}$ and approaches 1, as L_γ goes to infinity. After normalizing $\mathcal{Z}(\sigma)$ this constant is

$$\left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1/2} \cdot {}_2F_1 \left(k - \frac{1}{4} - \frac{ir}{2}, k - \frac{1}{4} + \frac{ir}{2}, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right),$$

where $-\frac{1}{2}(r^2 + \frac{1}{4})$ is the eigenvalue of φ , see Section 8.2. We can only deduce from (1.4) that (1.3) has a meromorphic continuation to the half plane $\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\}$ with the same poles and residues as (1.4). In particular, in the strip $0 < \operatorname{Re}(k) < 1$ the residues of (1.4) are given by (normalized) Patterson-Sullivan distributions.

The second difference is about the location and the residues of the poles of the continuation of $\mathcal{Z}(\sigma)$ in the strip $0 < \operatorname{Re}(k) < 1$. Here the problem seems to be that the constant $\mu_0(s)$ in [AZ07] which relates Wigner- to Patterson-Sullivan distributions could have poles at values $s = 1/2 + ir$, if $-(1/4 + r^2)$ is an eigenvalue from the complementary series, which are not considered in [AZ07]. We can only recover the result on the poles/residues from Theorem 1.3 in [AZ07] in the case where the pole $k = \frac{1}{2} + i\lambda$ corresponds to an eigenvalue $-(\frac{1}{4} + \lambda^2)$ from the principal series.

In the complementary series case, the poles/residue are more complicated. Especially, it makes a difference if $-\rho_0^2$ is an eigenvalue or not.

Preliminaries

In this chapter we collect some facts about the geometry of semisimple Lie groups G and locally symmetric spaces. In Section 2.1 we compute a formula for the Casimir operator Ω which becomes useful if we apply Ω to functions in $C(A \backslash G / K)$ for a fixed Iwasawa decomposition $G = ANK$. In Section 2.2 we discuss the geometry and dynamics of locally symmetric spaces $\Gamma \backslash G / K$. In Section 2.3 we discuss a model for real hyperbolic spaces G / K which makes it possible to do some concrete computations in Section 2.5 where we compute the spherical transform $\mathcal{S}(f)$ of a certain bi- K -invariant function f .

2.1. Some computations on the Casimir element

In [Zel89, p.40] a decomposition of the Casimir operator in $SL_2(\mathbb{R})$ is given, which we want to generalize to arbitrary semisimple Lie algebras. The result is formula (2.3), see also the example at the end of the section.

In what follows let G be a connected, noncompact and semisimple Lie group with finite center and \mathfrak{g} be its Lie algebra with fixed Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and Cartan involution θ . Let K be the analytic subgroup of G corresponding to \mathfrak{k} . Then K is compact. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $A = \exp \mathfrak{a}$.

Further, let \mathfrak{m} and M be the centralizers of \mathfrak{a} in \mathfrak{k} resp. K . Then we have the *Iwasawa decomposition*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$$

which gives the decompositions

$$G = NAK = ANK = KAN,$$

where \mathfrak{n} is a nilpotent Lie subalgebra and $N = \exp \mathfrak{n}$. By slight abuse of notation, we will call all these decompositions *Iwasawa decomposition of G* . If we fix K , the Iwasawa decomposition is unique up to conjugation in K , that is, if

$$G = KAN = KA_1N_1,$$

then there is an element $k \in K$ such that

$$(2.1) \quad A_1 = kAk^{-1} \text{ and } N_1 = kNk^{-1},$$

also $\mathfrak{a}^k := \text{Ad}(k)\mathfrak{a} = \mathfrak{a}_1$ and $\mathfrak{n}^k := \text{Ad}(k)\mathfrak{n} = \mathfrak{n}_1$, [GV88, (2.2.12)].

Let $B(\cdot, \cdot)$ be the Killing form of \mathfrak{g} , then

$$B_\theta(\cdot, \cdot) := -B(\cdot, \theta \cdot)$$

defines an inner product on \mathfrak{g} . We also have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \text{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

for $\alpha \neq 0$ is called the root space of \mathfrak{g} with respect to α and

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* - \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}.$$

We temporarily assume that the rank $n = \dim_{\mathbb{R}} \mathfrak{a}$ of \mathfrak{g} is arbitrary. We need some lemmata:

LEMMA 2.1.1. *For each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$ we have $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$.*

PROOF. [HN12, 12.3.2] □

LEMMA 2.1.2. *If $\alpha, \beta \in \mathfrak{a}^*$ with $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*

PROOF. [HN12, 12.3.4] □

Let X_1, \dots, X_m be any basis of orthonormal elements of \mathfrak{n} with respect to $B_\theta(\cdot, \cdot)$ and set

$$Z_i = -\theta X_i$$

for $i = 1, \dots, m$. Then

$$B(X_i, Z_j) = -B(X_i, \theta X_j) = \delta_{ij}.$$

Further, let H_1, \dots, H_n and M_1, \dots, M_k be any orthonormal bases of \mathfrak{a} resp. \mathfrak{m} with respect to $B_\theta(\cdot, \cdot)$.¹

Then

$$H_1, \dots, H_n, M_1, \dots, M_k, X_1, \dots, X_m, Z_1, \dots, Z_m$$

is a basis of \mathfrak{g} . We denote the dual basis with respect to $B_\theta(\cdot, \cdot)$ with

$$H^1, \dots, H^n, M^1, \dots, M^k, X^1, \dots, X^m, Z^1, \dots, Z^m.$$

Since

$$\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m},$$

while $\mathfrak{m} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$, we see by Lemma 2.1.2 that

$$H^i = H_i$$

and

$$M^i = -M_i.$$

Furthermore, by Lemma 2.1.2,

$$X^i = Z_i$$

and

$$Z^i = X_i.$$

There is an object of special interest, the Casimir operator Ω , which is an element of $U(\mathfrak{g})$, the universal enveloping algebra of \mathfrak{g} . More precisely, Ω lies in $Z(\mathfrak{g})$, that means it commutes with every element in $U(\mathfrak{g})$. If Q_j is any basis of \mathfrak{g} and Q^j the dual basis with $B_\theta(Q_i, Q^j) = \delta_{ij}$, then Ω is defined by

$$(2.2) \quad \Omega := \sum_j Q_j Q^j.$$

This is independent of the choice of Q_j , [GV88, (2.6.58)]. Consequently, we can write the Casimir operator as

$$\Omega = \sum_{i=1}^n H_i^2 - \sum_{i=1}^k M_i^2 + \sum_{i=1}^m (X_i Z_i + Z_i X_i).$$

¹Note that $B(\cdot, \theta \cdot) = -B(\cdot, \cdot)$ on \mathfrak{a} while $B(\cdot, \theta \cdot) = B(\cdot, \cdot)$ on \mathfrak{m} .

We work on the last sum:

$$\begin{aligned}
\sum_{i=1}^m (X_i Z_i + Z_i X_i) &= -\sum_{i=1}^m (X_i \theta X_i + \theta X_i X_i + X_i^2 - X_i^2) \\
&= \sum_{i=1}^m X_i (X_i - \theta X_i) - \sum_{i=1}^m (X_i + \theta X_i) X_i \\
&= \sum_{i=1}^m X_i (2X_i - (X_i + \theta X_i)) - \sum_{i=1}^m (X_i + \theta X_i) X_i = (*).
\end{aligned}$$

Set

$$W_i := X_i + \theta X_i,$$

then

$$W_i \in \mathfrak{k} = \mathfrak{m} \oplus \mathfrak{m}^{\perp \mathfrak{t}} = \mathfrak{m} \oplus (1 + \theta)\mathfrak{n}.$$

Thus,

$$(*) = 2 \sum_{i=1}^m X_i^2 - \sum_{i=1}^m (X_i W_i + W_i X_i).$$

Now,

$$[W_i, X_i] = [X_i + \theta X_i, X_i] = [\theta X_i, X_i] \in \mathfrak{a}.$$

More precisely, a direct computation shows that

$$[\theta X_i, X_i] = H_\alpha,$$

if $X_i \in \mathfrak{g}_\alpha$, see [GV88, (4.2.1)], where $H_\alpha \in \mathfrak{a}$ is defined by

$$\alpha(H) = B(H_\alpha, H)$$

for all $H \in \mathfrak{a}$. Therefore,

$$(*) = 2 \sum_{i=1}^m X_i^2 - 2 \sum_{i=1}^m X_i W_i - 2H_\rho,$$

and

$$(2.3) \quad \Omega = \sum_{i=1}^n H_i^2 - \sum_{i=1}^k M_i^2 + 2 \sum_{i=1}^m X_i^2 - 2 \sum_{i=1}^m X_i W_i - 2H_\rho,$$

if

$$2\rho = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})^+} \dim(\mathfrak{g}_\alpha) \alpha.$$

This decomposition (2.3) is the central result of this section, as it will suffice for our purpose, see Chapter 4.

REMARK 2.1.3. The homogeneous space $X = G/K$ is a symmetric space, in particular, it is a Riemannian space (X, g) with metric g . Using this metric one can define a special differential operator Δ called *Laplace-* or *Laplace-Beltrami-* operator, see [Hel01, Ch. II §2.4]. The Laplace operator equals $-\Omega$ on $C^\infty(X)$, see for example [Sch10, p.97]. This means, if we let elements X in \mathfrak{g} act on functions $f \in C^\infty(G/K)$ by

$$X \cdot f(g) := \left. \frac{d}{dt} \right|_{t=0} f(g \exp tX)$$

as left invariant differential operators and extend this definition to $U(\mathfrak{g})$, then

$$\Delta f = -\Omega f.$$

That is, we have shown in this section, that

$$\Delta = -\sum_{i=1}^n H_i^2 + 2H_\rho - 2 \sum_{i=1}^m X_i^2,$$

see also [AJ99, (4.4.2)],

Now we apply the computation to the case of $SL_2(\mathbb{R})$.

EXAMPLE 2.1.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ with the standard basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the Cartan-Killing form B is defined by the matrix

$$8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

in particular

$$B(H, H) = 8 = -B(W, W).$$

Let $X = \frac{1}{2}V + \frac{1}{2}W = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We want to express Ω in terms of H , X and W . We see that $Z = -2\theta X$ satisfies

$$B(X, Z) = 8$$

since $\theta V = -V$ and $\theta W = W$. Thus,

$$8\Omega = H^2 + 4X^2 - 4XW - 2H$$

is (eight times) the operator from above, see also [Zel89, p.40].

REMARK 2.1.5. This example shows a difference between [AZ07] and our work. For the formula (2.3) we have worked with an orthonormal basis of \mathfrak{n} with respect to B_θ , while in [AZ07] the vector X from Example 2.1.4 is used to identify N with \mathbb{R} , $\exp tX \rightarrow t$. But $B_\theta(X, X) = 4$ not 1. The difference comes from the fact that in [AZ07] the invariant form $\tilde{B}(X, Y) = 2 \cdot \text{tr}(XY)$ is implicitly used. This is off by the factor 2 from our definition $B(X, Y) = 4 \cdot \text{tr}(XY)$.

2.2. Geometry and dynamics of (locally) symmetric spaces

We summarize some notions and facts on the geometry of the (locally) symmetric spaces X and X_Γ . We will mainly cite from the articles [Gan77b] and [Gan77a].

Let G be a semisimple, noncompact connected Lie group of real rank one with finite center, maximal compact subgroup K and Lie algebra \mathfrak{g} . We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $X = G/K$ be the associated (Riemannian) symmetric space and $\Gamma \subset G$ a discrete co-compact and torsionfree subgroup. For short we call Γ a *uniform lattice* in G . Then Γ acts freely (by isometries) on X and the locally symmetric space $X_\Gamma := \Gamma \backslash X$ is a compact (Riemannian) manifold with simply connected covering X , in particular the fundamental group $\pi_1(X)$ is isomorphic to Γ . Further for any $Z \in \mathfrak{g}$ we put

$$|Z| := -B(Z, \theta Z),$$

where B denotes the Killing form on \mathfrak{g} and θ the Cartan involution. We fix an Iwasawa decomposition $G = ANK$ and set M resp. M' to be the centralizer resp. normalizer of A in K . Then M is normal in M' and the quotient group M'/M is called the *Weyl group* W . Since the rank of G is one, W is isomorphic to the group with two elements and we denote the nontrivial element in W by w . On G we fix a Haar measure dg such that

$$\int_G f(g)dg = \int_{ANK} f(ank)dkdnda$$

for integrable function f on G , where da and dn are defined by the euclidean structure on A and N coming from the inner product $B_\theta(\cdot, \cdot)$. The Haar measure dk is assumed to give K unit mass. Then we fix also a G -invariant measure dx on $\Gamma \backslash G$ such that

$$\int_G f(g)dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma x)dx$$

for all $f \in C_c(G)$.

We call $x \in G$ *elliptic*, if it is conjugated to some element in K , which implies in particular that every elliptic element is semisimple, i.e. $\text{Ad}(x) \in \text{End}(\mathfrak{g})$ is semisimple. If $x \in G$ is semisimple but not elliptic, we call it *hyperbolic*.² It is known that $\gamma \in \Gamma$ is elliptic iff it is of finite order, which is again equivalent to γ having a fixedpoint in X . Since Γ is torsionfree, every nontrivial element γ in Γ is hyperbolic. Finally, we note:

PROPOSITION 2.2.1. *Every $\gamma \in \Gamma - \{e\}$ is conjugated to some element $a_\mathfrak{p}a_\mathfrak{p}$ in the Cartan subgroup $A_\mathfrak{p}A$. Here $A_\mathfrak{p}$ is a subgroup contained in K . Even more, by possibly conjugating $a_\mathfrak{p}a_\mathfrak{p}$ with the nontrivial Weyl group element w one can assume that $\gamma \in \Gamma - \{e\}$ is conjugated to some $\delta_\gamma = m_\gamma a_\gamma \in MA^+$.*

PROOF. See [Wil91, Cor.11.5]. □

Note that a_γ is uniquely determined in A^+ by γ while m_γ is determined up to conjugacy in M , see [Wal76, Lem.6.6]. In particular, a_γ and $a_\mathfrak{p}$ generate the same (cyclic) subgroup of A and $|\log a_\gamma| = |\log a_\mathfrak{p}|$, see the proof of [Wil91, Prop.11.4]. By G_γ and $\Gamma_\gamma = \Gamma_\gamma \cap \Gamma$ we denote the centralizer of γ in G and Γ .

PROPOSITION 2.2.2. *Γ_γ is co-compact in G_γ which is reductive and unimodular. Furthermore, the centralizer Γ_γ of any γ in $\Gamma - \{e\}$ is always an infinite cyclic group.*

PROOF. See [Gan77a, p. 407] for the claim on G_γ and [Gan77a, Lem. 4.1.] for the claim on Γ_γ . □

Then we call γ *primitive* if Γ_γ is generated by γ . Since Γ is torsionfree, every $\gamma \in \Gamma - \{e\}$ is the unique positive power of a primitive element γ_0 .

A *geodesic loop* in X_Γ is the image of closed path in X under the orbit map $X \rightarrow X_\Gamma$. We call it a *closed geodesic* if it is a geodesic when viewed as a subset of X . It is well-known that there is a one-to-one correspondence between closed geodesics and nontrivial conjugacy classes in Γ , [Gan77a, p. 404]. We call a closed geodesic a *prime geodesic*, if the corresponding conjugacy class contains a primitive element. By CT we denote the set of conjugacy classes in Γ . For $[\gamma] \in CT$ let $l_\gamma := |\log a_\gamma|$, then

$$(2.4) \quad l_\gamma = \inf_{x \in G} |x^{-1}\gamma x|,$$

see [Gan77a, p. 413]. Here for $g \in G$, $|g| := |X|$, if $g = k \exp X$ with $k \in K$ and $X \in \mathfrak{p}$. We fix a Haar measure dx_γ on G_γ analogous to the Haar measure on G , following the Iwasawa decomposition of $G_\gamma = A_\gamma K_\gamma N_\gamma$ such that K_γ has unit measure. We also have a Haar measure $d\dot{x}_\gamma$ on the quotient $\Gamma_\gamma \backslash G_\gamma$. and we set $|\Gamma_\gamma \backslash G_\gamma| := \int_{\Gamma_\gamma \backslash G_\gamma} d\dot{x}_\gamma$. We can make this more precise, if we consider the centralizer of $\alpha_\gamma \gamma \alpha_\gamma^{-1} = \delta_\gamma$. Then $G_{\delta_\gamma} = A(G_{\delta_\gamma} \cap K) = A(G_{\delta_\gamma} \cap M)$, see [Gan77a, p. 414]. As before we fix Haar measures $dx_{\delta_\gamma} = dadk_{\delta_\gamma}$ on G_δ following the Iwasawa decomposition of G_{δ_γ} such that $K_{\delta_\gamma} := (K \cap G_{\delta_\gamma}) = (M \cap G_{\delta_\gamma}) =: M_{\delta_\gamma}$ has unit

²In all other cases, we call x parabolic. Since $\Gamma \backslash G$ is compact, Γ does not contain any parabolic elements.

measure with respect to dk_{δ_γ} . We set $|\Gamma_\gamma \backslash G_\gamma| := \int_{\Gamma_\gamma \backslash G_\gamma} d\dot{x}_\gamma$ etc.. The number l_γ can now be related to $\Gamma_\gamma \backslash G_\gamma$.

PROPOSITION 2.2.3. *For $\gamma \in \Gamma$ let $\delta_\gamma = \alpha_\gamma \gamma \alpha_\gamma^{-1} \in MA^+$ and $H_\gamma := \alpha_\gamma \Gamma_\gamma \alpha_\gamma^{-1}$. Furthermore, let K_{δ_γ} and G_{δ_γ} be the centralizer of δ_γ in K resp. G . If we identify A with $G_{\delta_\gamma}/M_{\delta_\gamma}$ then*

$$|\Gamma_\gamma \backslash G_\gamma| = |H_\gamma \backslash G_{\delta_\gamma}| = |H_\gamma \backslash G_{\delta_\gamma}/K_{\delta_\gamma}| = |A/\langle a_{\gamma_0} \rangle| = |\log a_{\gamma_0}| = l_{\gamma_0} = l_\gamma j^{-1}.$$

Here $|\cdot|$ always denotes the volume with respect to the invariant measure on quotient space, i.e. $|\Gamma_\gamma \backslash G_\gamma| = \int_{\Gamma_\gamma \backslash G_\gamma} d\dot{x}_\gamma$, $|H_\gamma \backslash G_{\delta_\gamma}| = \int_{H_\gamma \backslash G_{\delta_\gamma}} d\dot{x}_{\delta_\gamma}$ etc.

PROOF. See also [Gan77a, (4.5) pp. 414]. Assume that γ is conjugated to $\delta_\gamma = m_\gamma a_\gamma \in MA^+$, i.e. $\alpha_\gamma \gamma \alpha_\gamma^{-1} = \delta_\gamma$ for some $\alpha_\gamma \in G$. The centralizer G_{δ_γ} of δ_γ in G equals now $M_{m_\gamma} A$, where M_{m_γ} is the centralizer of m_γ in M , see [Wil91, p.185]. Let $\gamma = \gamma_0^j$ with γ_0 primitive and $j \in \mathbb{N}$. If γ_0 is conjugated to $m_{\gamma_0} a_{\gamma_0} \in MA^+$, then it follows by the uniqueness of a_γ that $a_\gamma = a_{\gamma_0}^j$. Then

$$H_\gamma = \alpha_\gamma \Gamma_\gamma \alpha_\gamma^{-1} = \alpha_\gamma \langle \gamma_0 \rangle \alpha_\gamma^{-1} \subset G_{\delta_\gamma} = M_{m_\gamma} A \subset MA.$$

If $\delta' = m'a' \in H_\gamma$, where $a' \in A$ and $m' \in M$, it follows that

$$m' \in M_{m_\gamma} = G_{\delta_\gamma} \cap M = M_{\delta_\gamma} = K_{\delta_\gamma}.$$

Therefore, the action of H_γ on $G_{\delta_\gamma}/K_{\delta_\gamma}$ equals the action of $\{a' : \delta' = m'a' \in H_\gamma\}$ by left translation on A , where we identified

$$G_{\delta_\gamma}/K_{\delta_\gamma} = M_{\delta_\gamma} A/M_{\delta_\gamma}$$

with A . Now $\{a' : \delta' = m'a' \in H_\gamma\} = \langle \tilde{a} \rangle$, where $\tilde{a} \in A$ such that $\alpha_\gamma \gamma_0 \alpha_\gamma^{-1} = \tilde{a} m \in AM$. Since γ_0 is also conjugated to $a_{\gamma_0} m_{\gamma_0} \in A^+ M$, we deduce by possibly replacing α_γ with $\alpha_\gamma w$, that $\tilde{a} \in \{a_{\gamma_0}, a_{\gamma_0}^{-1}\}$, i.e. $\langle \tilde{a} \rangle = \langle a_{\gamma_0} \rangle$. If we fix a Haar measure on $G_{\delta_\gamma} = K_{\delta_\gamma} A$ as we did for G_γ with normalized measure on K_{δ_γ} , then we get by unimodularity

$$|\Gamma_\gamma \backslash G_\gamma| = |H_\gamma \backslash G_{\delta_\gamma}| = |H_\gamma \backslash G_{\delta_\gamma}/K_{\delta_\gamma}| = |A/\langle a_{\gamma_0} \rangle| = |\log a_{\gamma_0}| = l_{\gamma_0} = l_\gamma j^{-1}. \quad \square$$

One can show the following.

PROPOSITION 2.2.4. *The set*

$$\{l_\gamma : 1 \neq [\gamma] \in CT\}$$

is bounded away from zero and has no upper bound.

PROOF. See [Gan77a, Th. 4.4]. □

We denote its infimum by l_{\inf} . The geodesic flow G_t on the tangent bundle TX is given by

$$G_t(V) := \gamma'_V(t),$$

see [Hil05, Ex. 1.1] where $V \in T_x X$ and $\gamma_V : \mathbb{R} \rightarrow X$ is the (unique) geodesic with $\gamma'_V(0) = V$ and $\gamma_V(0) = x$.³ G_t preserves the Riemannian metric on TX , in particular G_t maps the spherical bundle SX into itself.

LEMMA 2.2.5. *The bundle SX can be identified with G/M . Also in the same vein*

$$SX_\Gamma \cong \Gamma \backslash G/M.$$

³More precisely, $\gamma_V(t) = g \exp(tX)K$ for $V = d(xK \mapsto gxK)_o X \in T_g(G/K)$, see [Hil05, p.14].

PROOF. See [Hil05, p.15]. We have the identification

$$TX = T(G/K) \cong G \times_K (\mathfrak{g}/\mathfrak{k}) = G \times_K \mathfrak{p}.$$

The bundle SX gets identified with G/M , since K acts transitively on the spheres in \mathfrak{p} , i.e. the unit sphere in \mathfrak{p} can be written as K/M . Thus $SX \cong G/K \times K/M \cong G/M$. For SX_Γ we project from X to $\Gamma \backslash X$. \square

The geodesic flow on the spherical bundle $SX \cong G/M$ is then given by right-translation with A , i.e. by

$$(gM, \exp(tH_1)) \mapsto G_t(gM) = g \exp(-tH_1)M,$$

where H_1 is the unique unit vector in \mathfrak{a}^+ , see [BO95, p.89]. It also projects to a flow on $\Gamma \backslash G/M$ via

$$(\Gamma gM, \exp(tH_1)) \mapsto G_t(\Gamma gM) = \Gamma g \exp(-tH_1)M.$$

We already remarked that a closed geodesic corresponds to a closed orbit of the geodesic flow. We want to make this more precise. Let $\gamma \neq e$ be conjugated to $m_\gamma a_\gamma$ via α_γ . Then the corresponding orbit is given by

$$c_\gamma := \{\Gamma \alpha_{\gamma^{-1}} \exp(-tH_1)M : t \in \mathbb{R}\} \subset \Gamma \backslash G/M,$$

see [BO95, p.90]. Note that indeed $G_t(\Gamma \alpha_{\gamma^{-1}}M) = \Gamma \alpha_{\gamma^{-1}}M$ for $t = l_\gamma$ and that γ is primitive iff $t = l_\gamma$ is the smallest $t > 0$ with $G_t(\Gamma \alpha_{\gamma^{-1}}M) = \Gamma \alpha_{\gamma^{-1}}M$.

Assume now that a continuous, left- Γ - and right- K -invariant function σ is given. By the the same arguments as in the proof of 2.2.3 we get

$$\begin{aligned} \int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}x)dx &= \int_{H_\gamma \backslash G_{\delta_\gamma}/K_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}x)dx \\ &= \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_{\gamma^{-1}}x)dx \\ &=: \int_{c_{\gamma_0}} \sigma, \end{aligned}$$

where

$$(2.5) \quad c_{\gamma_0} = \{\Gamma \alpha_{\gamma^{-1}} \exp(-tH_1)M : 0 \leq t \leq l_{\gamma_0}\}$$

is the prime, closed orbit in SX_Γ belonging to γ . Furthermore, we note an easy observation:

PROPOSITION 2.2.6. *The mapping $G \rightarrow \mathbb{C}$, $g \mapsto \int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}xg)dx$ is invariant under left translation by elements of G_{δ_γ} . That is, for all $z \in G_{\delta_\gamma}$*

$$\int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}xzg)dx = \int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}xg)dx.$$

PROOF. This follows, because H_γ , which is discrete, and $G_{\delta_\gamma} = M_{m_\gamma}A$ are unimodular. By [Wil91, p. 5] the measure dx on $H_\gamma \backslash G_{\delta_\gamma}$ is invariant under left translation by elements in G_{δ_γ} , that is $\int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}xzg)dx = \int_{H_\gamma \backslash G_{\delta_\gamma}} \sigma(\alpha_{\gamma^{-1}}xg)dx$ for all $z \in G_{\delta_\gamma}$ and continuous functions σ . \square

2.3. Real hyperbolic spaces

We collect some facts concerning real hyperbolic symmetric spaces. We follow [Koo84] and [Qui06].

The real hyperbolic symmetric spaces X form one of the three main series for real symmetric spaces of noncompact type of rank one. If X is a real hyperbolic space of dimension l , it can be described as a quotient $X = \tilde{G}/\tilde{K}$, where $\tilde{G} = O(1, l)$

and $\tilde{K} \subset G$ is a maximal compact subgroup in \tilde{G} isomorphic to $O(l)$. Here $O(1, l)$ is the group of real $(l+1) \times (l+1)$ matrices which preserve the quadratic form

$$[x, y] = x_0y_0 - x_1y_1 - \dots - x_ly_l$$

on \mathbb{R}^{l+1} .

Furthermore $\tilde{K} = \{\pm 1\} \times SO(l)$. Real hyperbolic spaces can also be written as $X = G/K$, where $G = SO_o(1, l)$ is the connected component of identity in $SO(1, l)$ and $SO(1, l) \subset O(1, l)$ is the subgroup of elements with determinant 1. Further, K equals $\{1\} \times SO(l)$. This group G has the advantage of being semisimple while \tilde{G} is only reductive.

We denote the Lie algebras of G and K by \mathfrak{g} resp. \mathfrak{k} . Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with Cartan involution θ defined by

$$\theta(X) := JXJ,$$

where

$$J := \begin{pmatrix} -1 & 0 \\ 0 & I_l \end{pmatrix},$$

I_l the identity matrix in $SO(l)$. The Iwasawa decomposition reads $G = NAK$, where $A = \exp \mathfrak{a} \cong \mathbb{R}$, $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian and $N = \exp \mathfrak{n} \cong \mathbb{R}^{l-1}$, $\mathfrak{n} \subset \mathfrak{g}$ is an abelian Lie subalgebra. We will make the identification of A resp. \mathfrak{a} with \mathbb{R} and of N resp. \mathfrak{n} with \mathbb{R}^{l-1} more precise below, see (2.6) and (2.8). The root space decomposition is simply

$$\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha.$$

Here α is the unique positive root of the pair $(\mathfrak{g}, \mathfrak{a})$ with $\mathfrak{n} = \mathfrak{g}_\alpha$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

It follows that $\rho = \frac{l-1}{2}\alpha$. We can also decompose \mathfrak{g}_0 into $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} \subset \mathfrak{k}$ is the Lie algebra of the centraliser M of A in K . In our setting M is just $\{1\} \times SO(l-1) \times \{1\}$. The subgroup A is given by the set of all matrices of the form

$$a_t := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{l-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

$t \in \mathbb{R}$ and $I_{l-1} \in SO(l-1)$ the identity matrix. The Lie algebra \mathfrak{a} of A is spanned by

$$H_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{l-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where 0_{l-1} is the null matrix in $\mathfrak{so}(l-1)$. One computes $\alpha(H_0) = 1$, $B(H_0, H_0) = 2(l-1)\alpha(H_0)^2 = 2(l-1)$ and $\rho_0 := \rho(H_0) = \frac{l-1}{2}$, see [GV88, p.135]. Here

$$B(X, Y) = (l-1)\text{Tr}(XY) = \text{Tr}(\text{ad}X\text{ad}Y)$$

for $X, Y \in \mathfrak{g}$ denotes the Cartan-Killing form. We identify A resp. \mathfrak{a} with \mathbb{R} via

$$(2.6) \quad \mathbb{R} \rightarrow \mathfrak{a} \rightarrow A, \quad t \mapsto tH_0 \mapsto \exp tH_0.$$

The subgroup N is given by

$$n_u := \begin{pmatrix} 1 + \frac{1}{2}|u|^2 & u^T & -\frac{1}{2}|u|^2 \\ u & I_{l-1} & -u \\ \frac{1}{2}|u|^2 & u^T & 1 - \frac{1}{2}|u|^2 \end{pmatrix},$$

$u \in \mathbb{R}^{l-1}$. Its Lie algebra \mathfrak{n} consists of all matrices X_u , $u \in \mathbb{R}^{l-1}$, of the form

$$X_u := \begin{pmatrix} 0 & u^T & 0 \\ u & 0_{l-1} & -u \\ 0 & u^T & 0 \end{pmatrix}$$

with 0_{l-1} the zero matrix in $\mathbb{R}^{(l-1) \times (l-1)}$. It follows that

$$(2.7) \quad B_\theta(X_{e_1}, X_{e_1}) = -B(X_{e_1}, \theta X_{e_1}) = 4(l-1),$$

where $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^{l-1}$. Similar to (2.6) the identification here is

$$(2.8) \quad \mathbb{R}^{l-1} \rightarrow \mathfrak{n} \rightarrow N, \quad u \mapsto X_u \mapsto \exp X_u$$

Conjugation by A resp. M on N can now be described as

$$a_t n_u a_{-t} = n_{e^t u} \text{ resp. } m n_u m^{-1} = n_{m \cdot u}.$$

Furthermore,

$$n_u n_{u'} = n_{u+u'},$$

while

$$\text{Ad}(a_t)X_u = e^t X_u = X_{e^t u}$$

and

$$(2.9) \quad \text{Ad}(m)X_u = X_{m \cdot u}$$

for $a_t \in A$, $m \in M \cong SO(l-1)$ and $X_u \in \mathfrak{n} \cong \mathbb{R}^{l-1}$. We further recall that the Weyl group $W = N_K(A)/M$, where $N_K(A)$ is the normalizer of A in K , is isomorphic to the symmetric group of two elements.

The space $X = G/K$ can be described as the image of the open set

$$\{x \in \mathbb{R}^{l+1} : [x, x] > 0\}$$

under the mapping from \mathbb{R}^{l+1} to the unit ball $B(\mathbb{R}^l)$ in \mathbb{R}^l given by

$$x \mapsto y, \quad y_i = x_i x_0^{-1}.$$

The group G acts then on $B(\mathbb{R}^l)$ by fractional linear transformation, that is if $g \in G$ is given by

$$(2.10) \quad g = \begin{pmatrix} a & b^T \\ c & d \end{pmatrix}$$

with $a \in \mathbb{R}$, $b, c \in \mathbb{R}^l$ and $d \in \mathbb{R}^{l \times l}$, then

$$(2.11) \quad g \cdot y = (dy + c)(\langle b, y \rangle + a)^{-1}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^l . One can show that, see [vDH97, p.111],

$$(2.12) \quad 1 - |g \cdot y|^2 = \frac{1 - |y|^2}{|\langle b, y \rangle + a|^2},$$

where $|\cdot|$ means the euclidean norm derived from $\langle \cdot, \cdot \rangle$ on \mathbb{R}^l . Hence,

$$(2.13) \quad 1 - |g \cdot 0|^2 = |a|^{-2}.$$

LEMMA 2.3.1. *The mapping*

$$g \mapsto 1 - |g \cdot 0|^2$$

from $SO_o(1, l) \rightarrow \mathbb{R}$ is bi- K -invariant.

PROOF. Since K equals the stabilizer of 0 in G , the right- K -invariance is clear. For the left- K -invariance we note that the product of $g = \begin{pmatrix} a & b^t \\ c & d \end{pmatrix}$ as in (2.10) and $k = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$, $f \in SO(l)$, is of the form

$$k \cdot g = \begin{pmatrix} a & b^t \\ f \cdot c & f \cdot d \end{pmatrix}$$

it follows from (2.13) that for $k \in K$

$$1 - |kg \cdot 0|^2 = 1 - |g \cdot 0|^2.$$

□

For later purpose let us make two computations. For $n_u \in N$ and $a_t \in A$ resp. $m \in M$ we have by (2.13)

$$(2.14) \quad 1 - |a_t n_u \cdot 0|^2 = \left(\cosh t \left(1 + \frac{1}{2}|u|^2 \right) + \frac{1}{2}|u|^2 \sinh t \right)^{-2}$$

$$(2.15) \quad = \left(\cosh t + \frac{|u|^2}{2} e^t \right)^{-2}.$$

For the next computation we identify M with $SO(l-1)$ and m with the $(l-1) \times (l-1)$ -matrix $(m)_{i,j=1}^{l-1}$ in $SO(l-1)$. Let $(m)_{1,1}$ be the first entry on the diagonal. Then by (2.13)

$$\begin{aligned} 1 - |n_{-u} a_t m n_u \cdot 0|^2 &= 1 - |n_{-u} a_t n_{m \cdot u} \cdot 0|^2 \\ &= \left(\left(1 + \frac{|u|^2}{2} \right) \left(\cosh t + \frac{|u|^2}{2} e^{-t} \right) - \langle u, m \cdot u \rangle + \frac{|u|^2}{2} \left(\sinh t - \frac{|u|^2}{2} e^{-t} \right) \right)^{-2} \\ &= \left(-\langle u, m \cdot u \rangle + (1 + |u|^2) \cosh t \right)^{-2}. \end{aligned}$$

In particular, for $u = r \cdot e_1 = (r, 0, \dots, 0)^T \in \mathbb{R}^{l-1}$, $r \in \mathbb{R}$.

$$(2.16) \quad 1 - |n_{-r e_1} a_t m n_{r e_1} \cdot 0|^2 = \left(-(m)_{1,1} r^2 + (1 + r^2) \cosh t \right)^{-2}.$$

Finally, let $H : SO_o(1, l) \rightarrow \mathfrak{a}$ be the Iwasawa projection to $SO_o(1, l) = KAN$ defined by

$$H(k a_t n) = H(k \exp t H_0 n) := t H_0,$$

where $H_0 \in \mathfrak{a}$ with $\alpha(H_0) = 1$. From [Koo84, (5.9)] we cite that for $g = (g)_{i,j=1}^{l+1}$ in $SO_o(1, l)$

$$(2.17) \quad H(g) = \ln |g_{1,1} + g_{0,l+1}|.$$

Then let

$$w := \begin{pmatrix} I_{l-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

be a representative for the nontrivial Weyl group element in $K = \{1\} \times SO(l)$. From (2.17) we conclude that

$$H(n_u w) = t H_0$$

with $t = \ln(1 + |u|^2)$. Also

$$(2.18) \quad H(n_{-u} w) = H(n_u w) = \ln(1 + |u|^2) H_0.$$

2.4. Spherical Transforms

In this section we recall some facts about spherical transforms on semisimple Lie groups. We start by the general definition of a spherical function and spherical transform. In the next section we explicitly calculate the spherical transform of a function f_k depending on some complex parameter k when G/K is a real hyperbolic space.

Let $G = NAK$ be a semisimple, noncompact Lie group, $A : G \rightarrow \mathfrak{a}$ the corresponding projection such that

$$g = n(g) \exp A(g)k(g)$$

and

$$(2.19) \quad \langle gK, kM \rangle := A(k^{-1}g)$$

is the so-called horocycle bracket where $X = G/K$ and boundary $B = K/M$.

DEFINITION 2.4.1. [**Hel01**]

For $f \in C_c^\infty(G/K)$ we define its Fourier transform by

$$(2.20) \quad \mathcal{F}(f, \lambda, b) := \int_{G/K} f(gK) e^{(-i\lambda + \rho)\langle gK, b \rangle} d(gK),$$

where $\langle \cdot, \cdot \rangle$ is the horocycle bracket.

DEFINITION 2.4.2. We call a function in $C^\infty(G//K)$ (*elementary spherical*), if it is an eigenfunction of $U(\mathfrak{g}_{\mathbb{C}})^K$, the subset of K -invariants in $U(\mathfrak{g}_{\mathbb{C}})$.

The following theorem gives the integral formula for spherical functions. It describes spherical functions as integral transforms of the wave $e^{(i\lambda + \rho)A(\cdot)}$.

THEOREM 2.4.3. *Every spherical function is given by*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda + \rho)A(kg)} dk,$$

where $\lambda \in \mathfrak{a}^*$ and $\varphi_\lambda = \varphi_{s\lambda}$ for $s \in W$.

PROOF. [**Hel01**, p.418] □

Note that $A(\cdot)$ is bi- M -invariant, hence the domain of integration can be changed to K/M if dk_M corresponds to the (normalized) measures dk and dm .

We state some useful formulae for spherical functions.

REMARK 2.4.4. If the real rank of G is one and $H : G \rightarrow \mathfrak{a}$, $H(kna) := \log a$, where $\exp \log a = a$, corresponds to the Iwasawa decomposition $G = KAN$, then $H(g^{-1}k) = -A(k^{-1}g)$ and

$$\begin{aligned} \varphi_\lambda(g) &= \int_K e^{(i\lambda + \rho)A(kg)} dk \\ &= \int_K e^{(i\lambda + \rho)A(k^{-1}g)} dk \text{ by unimodularity} \\ &= \int_K e^{(-i\lambda - \rho)H(g^{-1}k)} dk \\ &= \int_K e^{(i\lambda - \rho)H(g^{-1}k)} dk \text{ by Weyl group invariance} \\ &= \int_K e^{(i\lambda - \rho)H(gk)} dk, \text{ see [Sch84, p.89].} \end{aligned}$$

With this preparation we can define the spherical transform on bi- K -invariant functions.

DEFINITION 2.4.5. See [Hel01, p.457]

Let $f \in C^\infty(G//K)$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We define the spherical transform of f by

$$(2.21) \quad \mathcal{S}(f, \lambda) := \int_G f(g) \varphi_{-\lambda}(g) dg,$$

whenever the integral is finite and where $\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)A(kg)} dk$ is an elementary spherical function.

From Remark 2.4.4 one can deduce that the spherical transform can be factorized into a product of the euclidean Fourier transform on A

$$\hat{f}(\lambda) := \int_A f(a) e^{-i\lambda \log a} da$$

and the Abel transform, see Remark 5.4.2 for the definition of the Abel transform F_f .

PROPOSITION 2.4.6. *The spherical transform can be factorized as*

$$\begin{aligned} \mathcal{S}(f, \lambda) &= \widehat{F}_f(\lambda) \\ &= \int_A \int_N f(an) e^{(-i\lambda+\rho) \log a} dn da. \end{aligned}$$

PROOF. [Hel01, Ch.II §5 (37)] or [GV88, Prop.3.3.1.] \square

THEOREM 2.4.7. (*Harish-Chandra's c-function*)

Let G be semisimple, noncompact of real rank one. For $\lambda \in \mathfrak{a}^*$ with $\operatorname{Re}(i\lambda) \in \mathfrak{a}_+^*$ the integral

$$c(\lambda) = \int_{\overline{N}} e^{-(i\lambda+\rho)H(\overline{n})} d\overline{n}$$

converges absolutely and admits a meromorphic extension in the parameter λ to $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$. Here the measure $d\overline{n}$ on $\overline{N} := \theta N$ is normalized such that $\int_{\overline{N}} e^{-2\rho(H(\overline{n}))} d\overline{n} = 1$.

PROOF. [Hel01, p.436] \square

2.5. The spherical transform of f_k

In this section we specialize to $G = SO_o(1, l)$. We want to compute the spherical transform of the bi- K -invariant function, see Lemma 2.3.1,

$$f_k : G \rightarrow \mathbb{R}, g \mapsto (1 - |g \cdot 0|^2)^{k/2}$$

depending on $k \in \mathbb{C}$. We will use f_k to obtain the zeta function $\mathcal{R}(k; \varphi_n)$, see Section 5.4. The spherical transform of f_k will in turn produce the meromorphic continuation of $\mathcal{R}(k; \varphi_n)$ to \mathbb{C} via the spectral trace of the operator $\varphi_n \cdot \pi_R(f_k)$ in Chapter 7.

The spherical transform of f_k is given by Proposition 2.4.6 by

$$\mathcal{S}(f_k, \mu) = \int_A \int_N f_k(an) e^{(-i\mu+\rho) \log a} dn da.$$

Now we identify A with \mathbb{R} by $a_t = \exp tH_0 \mapsto t$, $\alpha(H_0) = 1$, and N with \mathbb{R}^{l-1} , $\exp X_u \mapsto u$, see (2.8). From (2.14ff) it follows that

$$(2.22) \quad f_k(a_t \exp X_u) = \left(\cosh t + \frac{|u|^2}{2} e^t \right)^{-k}.$$

The spherical transform of f_k is hence given by

$$\begin{aligned}
\mathcal{S}(f_k, \mu) &= \int_{-\infty}^{\infty} \int_N f_k(\exp tH_0n) e^{(-i\mu+\rho_0)t} dn dt \\
&\stackrel{(2.22)}{=} \int_{-\infty}^{\infty} \int_N \left(\cosh t + \frac{1}{2}|u|^2 e^t \right)^{-k} e^{(-i\mu+\rho_0)t} du dt \\
&\stackrel{t \rightarrow -t}{=} \int_{-\infty}^{\infty} \int_N \left(\cosh t + \frac{1}{2}|u|^2 e^{-t} \right)^{-k} e^{(i\mu-\rho_0)t} du dt \\
&= \omega_{l-1} \cdot \int_{-\infty}^{\infty} \int_0^{\infty} s^{l-2} \left(\cosh t + \frac{1}{2}s^2 e^{-t} \right)^{-k} e^{(i\mu-\rho_0)t} ds dt \\
&= \omega_{l-1} \cdot \int_{-\infty}^{\infty} (\cosh t)^{-k} e^{(i\mu-\rho_0)t} \int_0^{\infty} s^{l-2} (1+s^2(1+e^{2t})^{-1})^{-k} ds dt,
\end{aligned}$$

where we used polar coordinates, namely we identify $N \cong \mathbb{R}^{l-1}$, $u \mapsto X_u$, see (2.8),

$$\begin{aligned}
\int_N f(n) dn &= \int_{\mathbb{R}^{l-1}} f(\exp X_u) du \\
&= \int_0^{\infty} \int_{\partial B_s(0)} f dS ds \\
&= \omega_{l-1} \int_0^{\infty} s^{l-2} f(\exp sX_{e_1}) ds
\end{aligned}$$

for radial functions f , i.e. $f(n) = \int_M f(m \cdot n) dm$ for all $n \in N$. Here $B_1(0) = \{x \in \mathbb{R}^{l-1} : |x| \leq 1\}$ and $\omega_{l-1} := |\partial B_1(0)|$ with respect to the Lebesgue measure in \mathbb{R}^{l-2} for $l \geq 3$. For $l = 2$ we set $\omega_1 := 2$. Next we substitute $s \mapsto (1+e^{2t})^{1/2}s$ to get

$$\begin{aligned}
&\omega_{l-1} \cdot \int_{-\infty}^{\infty} (\cosh t)^{-k} (1+e^{2t})^{\frac{l-1}{2}} e^{(i\mu-\rho_0)t} dt \int_0^{\infty} s^{l-2} (1+s^2)^{-k} ds \\
&= \omega_{l-1} \cdot 2^{\frac{l-1}{2}} \int_A (\cosh t)^{-k+\frac{l-1}{2}} e^{\frac{l-1}{2}t} \cdot e^{(i\mu-\rho_0)t} dt \int_0^{\infty} s^{l-2} (1+s^2)^{-k} ds.
\end{aligned}$$

Now we remember that $\rho_0 = \frac{l-1}{2}$ to find that we have to compute the integral

$$\begin{aligned}
2^{\frac{l-1}{2}} \int_{-\infty}^{\infty} (\cosh t)^{-k+\frac{l-1}{2}} e^{i\mu t} dt &= 2^{\frac{l-1}{2}} \int_0^{\infty} 2^{k-\frac{l-1}{2}} (v+v^{-1})^{-k+\frac{l-1}{2}} v^{i\mu-1} dv \\
&= 2^k \int_0^{\infty} \frac{v^{i\mu-1}}{\left(\frac{v^2+1}{v}\right)^{k-\frac{l-1}{2}}} dv,
\end{aligned}$$

by the substitution $e^t = v$. Then we substitute $v \mapsto v^{1/2}$ to obtain

$$(2.23) \quad 2^{k-1} \int_0^{\infty} v^{\frac{k+i\mu-\rho_0}{2}-1} (v+1)^{-k+\rho_0} dv.$$

Next we use the integral formula for the Beta function

$$(2.24) \quad B(x, y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

to obtain that (2.23) equals

$$2^{k-1} B\left(\frac{k+i\mu-\rho_0}{2}, \frac{k-i\mu-\rho_0}{2}\right).$$

We recall the following facts on the Gamma function:

REMARK 2.5.1. The Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ converges absolutely for $\operatorname{Re}(z) > 0$ and defines there an entire function. It can be continued analytically to $\mathbb{C} - \{0, -1, -2, \dots\}$. The poles in $-n$, $n \in \mathbb{N}_0$, are of first order with residues $\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n}$. The Gamma function has no zeros, [FB06, Ch. IV 1].

Furthermore, for x and y real we have

$$(2.25) \quad \lim_{|y| \rightarrow \infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi},$$

see [GR65, 8.328].

That is, $B(x, y)$ is defined for

$$\operatorname{Re}(x), \operatorname{Re}(y) \neq 0, -1, -2, \dots,$$

in our case

$$k \neq \rho_0 \pm i\mu, \rho_0 - 2 \pm i\mu, \dots$$

To sum up:

PROPOSITION 2.5.2. *The spherical transform of f_k is given by*

$$(2.26) \quad \begin{aligned} \mathcal{S}(f_k, \mu) &= \omega_{l-1} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} ds \cdot 2^{k-1} B\left(\frac{k + i\mu - \rho_0}{2}, \frac{k - i\mu - \rho_0}{2}\right) \\ &= \omega_{l-1} \cdot 2^{k-1} \frac{\Gamma(k - \rho_0) \Gamma(\rho_0)}{2\Gamma(k)} B\left(\frac{k + i\mu - \rho_0}{2}, \frac{k - i\mu - \rho_0}{2}\right) \\ &= \omega_{l-1} \cdot 2^{k-2} B(k - \rho_0, \rho_0) B\left(\frac{k + i\mu - \rho_0}{2}, \frac{k - i\mu - \rho_0}{2}\right). \end{aligned}$$

As we will later see in Section 5.4 this function f_k leads to an operator of trace class and its trace in turn will produce the (auxiliary) zeta function $\mathcal{R}(\varphi)$, see (5.29).

Polar decomposition and radial parts

In this chapter G will always come from a real hyperbolic symmetric space, which is at least 3 dimensional. That is, $G = SO_o(1, l)$ with $l \geq 3$, see Chapter 2.3. We fix an Iwasawa decomposition $G = ANK$ resp. $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$. Then $M = Z_K(A) \cong SO(l-1)$. For $l = 2$, M is hence trivial and we exclude this case.

In Section 3.1 we discuss the action of M on N which is essentially described by the natural action of $SO(l-1)$ on \mathbb{R}^{l-1} . In the next Section 3.2 we develop a theory of polar decomposition for differential operators D on G for a certain decomposition of G which is similar to the polar decomposition $G = KAK$. This also leads to the definition of radial parts of differential operators for M -invariant functions. For the succeeding chapters, in particular Chapter 4, Section 3.3 is crucial where we apply the theory of polar decomposition from Section 3.1 to the Casimir operator Ω . While Section 3.2 suggests formula (3.4) for Ω , see Theorem 3.2.4, the calculations of Section 3.3 which lead to (3.4) only depend on Section 2.1 and 3.1.

3.1. The action of M on \mathfrak{n}

We assume that the dimension of $\mathfrak{n} = \text{Lie}(N)$ is $l-1$ and we consider the adjoint action of M on \mathfrak{n} . It is content of Kostant's double transitivity theorem, see [Wal73, Th. 8.11.3], that the generic orbits are spheres, i.e. if $0 \neq X \in \mathfrak{n}$, then

$$M \cdot X = \{X' \in \mathfrak{n} : |X'| = |X|\}.$$

It follows that the tangent space $\mathfrak{m} \cdot X$ to the M -orbit through X is $l-2$ dimensional and thus

$$(\mathfrak{m} \cdot X)^{\perp_{\mathfrak{n}}} := \{X' \in \mathfrak{n} : B_{\theta}(X', \mathfrak{m} \cdot X) = 0\}$$

is 1-dimensional. Now it is a general fact that for any compact group H and real, finite dimensional representation of H on some vector space V , the space $(H \cdot v)^{\perp}$ meets every H -orbit in V , see [dad85, Lem. 1]. We call a linear subspace of \mathfrak{n} a *section*, if it intersects every M -orbit. A *slice* is then a subset of a section which intersects every regular orbit exactly once.

LEMMA 3.1.1. *Let $0 \neq X \in \mathfrak{n}$. Then $(\mathfrak{m} \cdot X)^{\perp_{\mathfrak{n}}}$ meets every orbit orthogonally, i.e.*

$$B_{\theta}(Z \cdot X, X') = 0$$

for all $Z \in \mathfrak{m}$ and $X' \in (\mathfrak{m} \cdot X)^{\perp_{\mathfrak{n}}}$.¹

PROOF. The claim will follow as soon as we show $(\mathfrak{m} \cdot X)^{\perp_{\mathfrak{n}}} = \mathbb{R}X$. For this we compute for arbitrary $Z \in \mathfrak{m}$

$$\begin{aligned} B_{\theta}(Z \cdot X, X) &= -B([Z, X], \theta X) \\ &= -B(Z, [X, \theta X]) \\ &= 0, \end{aligned}$$

since $[X, \theta X] \in \mathfrak{a}$ which is orthogonal to \mathfrak{k} with respect to B_{θ} . □

¹The Lemma would still be true if the dimension of $(\mathfrak{m} \cdot X)^{\perp_{\mathfrak{n}}}$ is two. In particular for any semisimple \mathfrak{g} of real rank one, M acts polarly on \mathfrak{n} .

In other words M acts *polarly* on \mathfrak{n} , that is, there is a section which intersects every M -orbit orthogonally. See also [Mic08, Chap.VI 30] for more information on polar actions.

We note the following:

LEMMA 3.1.2. *Let \mathfrak{s}_1 and \mathfrak{s}_2 be 1-dimensional sections for M acting on N . If*

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{a}_1 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_1^{\perp \mathfrak{n}_1} \\ &= \mathfrak{k} \oplus \mathfrak{a}_2 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_2^{\perp \mathfrak{n}_2}, \end{aligned}$$

with $\mathfrak{s}_1 \subset \mathfrak{n}_1$, $\mathfrak{s}_2 \subset \mathfrak{n}_2$, then there is some $k \in K$ such that

$$\mathfrak{a}_1^k = \mathfrak{a}_2 \text{ and } \mathfrak{s}_1^k = \mathfrak{s}_2.$$

In particular, if \mathfrak{k} and $\mathfrak{a}_1 = \mathfrak{a}_2$ are fixed, then there is some $m \in M$ such that $\mathfrak{s}_1^m = \mathfrak{s}_2$.

PROOF. This follows since by (2.1) in Chapter 2.1 there is some $k \in K$ such that

$$\mathfrak{a}_1^k = \mathfrak{a}_2 \text{ and } \mathfrak{s}_1^k + \left(\mathfrak{s}_1^{\perp \mathfrak{n}_1} \right)^k = \mathfrak{s}_2 + \mathfrak{s}_2^{\perp \mathfrak{n}_2}.$$

But \mathfrak{s}_1^k is a one dimensional subspace of \mathfrak{n}_2 , hence a section for $M_2 := Z_K(A_2)$ acting on \mathfrak{n}_2 and consequently there is some $m \in M_2$ with

$$\left(\mathfrak{s}_1^k \right)^m = \mathfrak{s}_2.$$

Still of course,

$$\left(\mathfrak{a}_1^k \right)^m = \mathfrak{a}_2^m = \mathfrak{a}_2.$$

□

We fix $X_1 \neq 0$ of length 1 with respect to $B_\theta(\cdot, \cdot)$ and

$$\mathfrak{s} := (\mathfrak{m} \cdot X_1)^{\perp \mathfrak{n}} = \mathbb{R}X_1.$$

We denote by

$$\mathfrak{s}' := \mathbb{R}X_1 \setminus \{0\}$$

the set of *regular elements*, i.e. the set of elements with maximal orbit dimension. The first aim of this section is to find a decomposition of \mathfrak{g} resp. G according to the section \mathfrak{s} . Later we will study differential equations of invariant functions on \mathfrak{s} .

Let

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s}) := \{Z \in \mathfrak{m} : [Z, X'] = 0 \text{ for all } X' \in \mathfrak{s}\}.$$

be the centralizer of \mathfrak{s} in \mathfrak{m} and let

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp \mathfrak{m}} := \{W \in \mathfrak{m} : B(Z, W) = 0 \text{ for all } Z \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})\}.$$

be its orthogonal complement with respect to the Killing form. Finally,

$$\mathfrak{s}^{\perp \mathfrak{n}} := \{Y \in \mathfrak{n} : B_\theta(X', Y) = 0 \text{ for all } X' \in \mathfrak{s}\}.$$

We start by a lemma showing that $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp \mathfrak{m}}$ is isomorphic to $\mathfrak{s}^{\perp \mathfrak{n}}$.

LEMMA 3.1.3. *The map $\text{ad}X' : \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp \mathfrak{m}} \rightarrow \mathfrak{s}^{\perp \mathfrak{n}}$, $Z \mapsto [Z, X']$ is an isomorphism for all $X' \in \mathfrak{s}'$.*

PROOF. Let $X' \in \mathfrak{s}'$. Since the action is polar, this implies

$$Z_M(X') = Z_M(\mathfrak{s}),$$

[Mic08, 30.23]. Also by polarity of the action we get the orthogonal decomposition

$$\mathfrak{n} = [\mathfrak{m}, X'] \oplus \mathfrak{s},$$

i.e. $[\mathfrak{m}, X'] = \mathfrak{s}^{\perp \mathfrak{n}}$. Furthermore,

$$[\mathfrak{m}, X'] = [\mathfrak{z}_{\mathfrak{m}}(X') \oplus \mathfrak{z}_{\mathfrak{m}}(X')^{\perp \mathfrak{m}}, X'] = [\mathfrak{z}_{\mathfrak{m}}(X')^{\perp \mathfrak{m}}, X'],$$

where $\mathfrak{z}_m(X')$ is the Lie algebra of $Z_M(X') = Z_M(\mathfrak{s})$. That is, $\text{ad}X'$ maps $\mathfrak{z}_m(\mathfrak{s})^{\perp m}$ onto $\mathfrak{s}^{\perp n}$ and both spaces have the same dimension, since

$$M \cdot X' \cong M/Z_M(X').$$

□

The proof shows that lemma is also true for any subgroup $M' \subset M$ acting polarly on \mathfrak{n} , if we replace \mathfrak{s} with a section for M' acting on \mathfrak{n} .

Lemma 3.1.3 allows us now to write \mathfrak{g} as a direct sum involving \mathfrak{s} and $\mathfrak{z}(\mathfrak{s})$. This decomposition is analogous to the one derived from K acting on \mathfrak{p} for a semisimple Lie algebra \mathfrak{g} , see for example [GV88, Chap.4].

PROPOSITION 3.1.4. *Let $\mathfrak{g} = \mathfrak{so}(1, l) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and \mathfrak{s} any section for $M = \exp \mathfrak{m}$ acting on \mathfrak{n} . We have the following direct decompositions for any $X' \in \mathfrak{s}'$:*

$$\begin{aligned} \mathfrak{g} &= (\mathfrak{z}_m(\mathfrak{s})^{\perp m})^{\exp -X'} \oplus \mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{k} \\ &= (\mathfrak{z}_m(\mathfrak{s})^{\perp m} \oplus (1 + \theta)\mathfrak{s})^{\exp -X'} \oplus \mathfrak{s} \oplus \mathfrak{k}. \end{aligned}$$

PROOF. Let $\mathfrak{s} = \mathbb{R}X_1$ for some $0 \neq X_1 \in \mathfrak{n}$. Since the sum $\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ is direct, it suffices to show that $(\mathfrak{z}_m(\mathfrak{s})^{\perp m})^{\exp -X'}$ is not contained in $\mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{k}$. The first claim then follows by dimension counting, see Lemma 3.1.3.

So let $Z^{\exp -X'} \in (\mathfrak{z}_m(\mathfrak{s})^{\perp m})^{\exp -X'} \cap \mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{k}$. Then

$$\begin{aligned} Z^{\exp -X'} &= \text{Ad}(\exp -X')Z \\ &= \sum_{l=0}^{\infty} \frac{(\text{ad} - X')^l}{l!} Z \\ &= Z + [Z, X'] - \frac{[X', [Z, X']]}{2} + \dots \\ &= Z + [Z, X'], \end{aligned}$$

as \mathfrak{n} is abelian and $[Z, X'] \in \mathfrak{n}$. That is, $Z^{\exp -X'} \in \mathfrak{m} \oplus \mathfrak{s}^{\perp n}$ for all $Z \in \mathfrak{m}$. So if we assume

$$Z^{\exp -X'} = Z + [Z, X'] = H + rX_1 + W \in \mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{k},$$

we see that

$$[Z, X'] = rX_1 \in \mathfrak{s}^{\perp n} \cap \mathfrak{s} = 0$$

But by Lemma 3.1.3 this is true for $Z \in \mathfrak{z}_m(\mathfrak{s})^{\perp m}$ precisely iff $Z = 0$. For the second claim, we note that $\dim \mathfrak{s} = 1$, i.e. $\mathfrak{s} = \mathbb{R} \cdot X'$ for all $X' \in \mathfrak{s}'$. Hence,

$$((1 + \theta)\mathfrak{s})^{\exp -X'} \oplus \mathfrak{k} \supset \mathfrak{a}$$

for all $X' \in \mathfrak{s}'$. □

The next lemma is an easy consequence of the fact that spheres are the generic orbits of M in \mathfrak{n} .

LEMMA 3.1.5. *Let $Y \in \mathfrak{s}^{\perp n}$, then there is an analytic function f on \mathfrak{s}' and a (unique) $Z = Z_Y \in \mathfrak{z}_m(\mathfrak{s})^{\perp m}$ such that*

$$Y = f(\exp X')Z^{\exp -X'} - f(\exp X')Z$$

for all $X' \in \mathfrak{s}'$

PROOF. If the dimension of \mathfrak{n} is one, then $\mathfrak{s} = \mathfrak{n}$, i.e. $\mathfrak{s}^{\perp n}$ is trivial and we set

$$f \equiv 0.$$

If $\dim \mathfrak{n} > 1$ we define

$$f(\exp X') = \frac{1}{r}$$

for $X' = rX_1 \in \mathfrak{s}'$. By Lemma 3.1.3 there is a unique $Z = Z_Y \in \mathfrak{z}_m(\mathfrak{s})^{\perp m}$ such that $[Z_Y, X_1] = Y$. Now

$$Z^{\exp -X'} = Z + [Z, X'] = Z + r[Z, X_1]$$

and thus

$$f(\exp X')Z_Y^{\exp -X'} - f(\exp X')Z_Y = [Z, X_1] = Y.$$

□

3.2. Polar decompositions and radial parts for $G = M(A \exp \mathfrak{s})K$

We denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} and by $S(\mathfrak{g})$ the symmetric algebra symmetric algebra. There is a linear bijection

$$\lambda : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

called *symmetrization*, see [War72a, p.161].

From the decomposition $G = M(A \exp \mathfrak{s})K$ we want to obtain a theory of radial parts for differential operators on functions, which are left- M - and right- K -invariant. That is, we construct for given $D \in U(\mathfrak{g})$ some differential operator $\delta(D)$ on $A \exp \mathfrak{s}'$ such that

$$D\phi = \delta(D)\phi$$

on $A \exp \mathfrak{s}'$ for all $\phi \in C^\infty(M \backslash G/K)$. This theory is developed in analogy with the theory for $G = KAK$, see for example [War72b, Chap.9] or [GV88, Chap.4].

PROPOSITION 3.2.1. *The mapping $\psi : M \times A \exp \mathfrak{s} \times K \rightarrow G, (m, ax, k) \mapsto maxk$ is surjective and a submersion on $M \times A \exp \mathfrak{s}' \times K$.*

PROOF. For $Z \in \mathfrak{m}, R \in \mathfrak{a} \oplus \mathfrak{s}, W \in \mathfrak{k}$, the differential $(d\psi)_{m,ax,k}$ computes to

$$\begin{aligned} (d\psi)_{m,ax,k}(Z, R, W) &= (d\psi)_{m,ax,k}(Z, 0, 0) + (d\psi)_{m,ax,k}(0, R, 0) + (d\psi)_{m,ax,k}(0, 0, W) \\ &= \frac{d}{dt}\Big|_{t=0} \psi(m \exp tZ, ax, k) + \frac{d}{dt}\Big|_{t=0} \psi(m, ax \exp tR, k) \\ &\quad + \frac{d}{dt}\Big|_{t=0} \psi(m, ax, k \exp tW) \\ &= Z^{(axk)^{-1}} + R^{k^{-1}} + W \\ &= Z^{(xk)^{-1}} + R^{k^{-1}} + W, \end{aligned}$$

since $\frac{d}{dt}\Big|_{t=0} \psi(g, \exp tL, h) = L^{h^{-1}}$ for all $L \in \mathfrak{g}, g, h \in G$ and since M centralizes A . Thus,

$$(\text{Ad}(k) \circ (d\psi)_{m,ax,k})(Z, R, W) = Z^{x^{-1}} + R + W^k$$

and because $\text{Ad}(k)$ is an isomorphism of \mathfrak{g} , the surjectivity of $d\psi_{m,ax,k}$ follows from the decomposition

$$\mathfrak{g} = \mathfrak{m}^{\exp -X'} + \mathfrak{a} + \mathfrak{s} + \mathfrak{k},$$

which is valid for all $X' \in \mathfrak{s}'$. □

The decomposition of \mathfrak{g} and the *Poincaré-Birkhoff-Witt*-Theorem together yield a decomposition of the universal enveloping algebra which we view as the algebra of differential operators on $C^\infty(G)$ with constant coefficients.

LEMMA 3.2.2. *We have the following decompositions*

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k}) \oplus ((\mathfrak{s}^{\perp n})U(\mathfrak{g})) \\ &= U(\mathfrak{a} \oplus \mathfrak{s}) \oplus ((\mathfrak{s}^{\perp n})U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}). \end{aligned}$$

with projections

$$\pi_1 : U(\mathfrak{g}) \rightarrow U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})$$

and

$$\pi_2 : U(\mathfrak{g}) \rightarrow U(\mathfrak{a} \oplus \mathfrak{s}).$$

Furthermore,

$$\pi_1 \equiv \pi_2 \pmod{U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})\mathfrak{k}}.$$

PROOF. We modify the proof of [GV88, Lemma 2.6.6.]. For the second decomposition we use the direct decomposition of \mathfrak{g}

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{s}^{\perp n} \oplus \mathfrak{k}.$$

It follows by the Poincaré-Birkhoff-Witt-Theorem, see [HN12, Th.7.1.9], that the mapping

$$U(\mathfrak{a} \oplus \mathfrak{s}) \otimes U(\mathfrak{s}^{\perp n}) \otimes U(\mathfrak{k}) \rightarrow U(\mathfrak{g}), \quad a \otimes b \otimes c \mapsto abc$$

defines a linear isomorphism. Because \mathfrak{n} is abelian, \mathfrak{s} and $\mathfrak{s}^{\perp n}$ form indeed subalgebras of \mathfrak{g} . Furthermore,

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{s}^{\perp n})U(\mathfrak{k}) \\ &= U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{s}^{\perp n}) \oplus U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{s}^{\perp n})U(\mathfrak{k})\mathfrak{k} \\ &= U(\mathfrak{a} \oplus \mathfrak{s}) \oplus \mathfrak{s}^{\perp n}U(\mathfrak{s}^{\perp n})U(\mathfrak{a} \oplus \mathfrak{s}) \oplus U(\mathfrak{g})\mathfrak{k}, \end{aligned}$$

since if \mathfrak{g} is the direct sum of two subalgebras \mathfrak{g}_1 and \mathfrak{g}_2 , then

$$U(\mathfrak{g}) = U(\mathfrak{g}_1) \oplus \mathfrak{g}_2U(\mathfrak{g}) = U(\mathfrak{g}_1) \oplus U(\mathfrak{g})\mathfrak{g}_2,$$

see [Var84, Cor. 3.2.7.]. If we apply this to

$$\mathfrak{s}^{\perp n}U(\mathfrak{g}) \subset U(\mathfrak{g}),$$

we get

$$\mathfrak{s}^{\perp n}U(\mathfrak{g}) \subset \mathfrak{s}^{\perp n}U(\mathfrak{s}^{\perp n})U(\mathfrak{a} \oplus \mathfrak{s}) \oplus U(\mathfrak{g})\mathfrak{k},$$

i.e.

$$\mathfrak{s}^{\perp n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathfrak{s}^{\perp n}U(\mathfrak{s}^{\perp n})U(\mathfrak{a} \oplus \mathfrak{s}) \oplus U(\mathfrak{g})\mathfrak{k}.$$

For the first decomposition we proceed as before by using

$$U(\mathfrak{g}) \cong U(\mathfrak{a} \oplus \mathfrak{s}) \otimes U(\mathfrak{k}) \otimes U(\mathfrak{s}^{\perp n})$$

from which we derive this time

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k}) \oplus \mathfrak{s}^{\perp n}U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})U(\mathfrak{s}^{\perp n}) \\ &= U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k}) \oplus \mathfrak{s}^{\perp n}U(\mathfrak{g}). \end{aligned}$$

The last claim

$$\pi_1(D) \equiv \pi_2(D) \pmod{U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})\mathfrak{k}}$$

follows since

$$U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k}) = U(\mathfrak{a} \oplus \mathfrak{s}) \oplus U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})\mathfrak{k}$$

and since $U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k}) \subset U(\mathfrak{g})$. \square

For the definition of a radial part we need to express any $D \in U(\mathfrak{g})$ in polar coordinates adapted to the decomposition of $G = M(A \exp \mathfrak{s})K$. The following theorem is the major ingredient for this.

PROPOSITION 3.2.3. *Let $\mathfrak{g} = \mathfrak{so}(1, l) = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$, $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and \mathfrak{s} a section for $M = \exp \mathfrak{m}$ acting on \mathfrak{n} . The map*

$$\begin{aligned} \Gamma_{a \exp X'} &: \lambda(S(\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp \mathfrak{m}})) \otimes U(\mathfrak{a} \oplus \mathfrak{s}) \otimes U(\mathfrak{k}) \rightarrow U(\mathfrak{g}), \\ &\xi \otimes u \otimes \xi' \mapsto \xi^{\exp(-X')}u\xi' \end{aligned}$$

defines a linear isomorphism for all X' in \mathfrak{s}' .

PROOF. The bijectivity of $\Gamma_{a \exp X'}$ follows from the decomposition

$$\mathfrak{g} = (\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp_{\mathfrak{m}}})^{\exp -X'} \oplus (\mathfrak{s} \oplus \mathfrak{a}) \oplus \mathfrak{k},$$

see Proposition 3.1.4, which is valid for all $X' \in \mathfrak{s}'$, and the *Poincaré-Birkhoff-Witt*-Theorem. \square

We denote the inverse of $\Gamma_{a \exp X'}$ by $\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}$ and interpret it as the local expression of D in polar coordinates $\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}}$ of D around the point $a \cdot \exp X'$ for the decomposition of the dense, open subset in G

$$G' = M(A \exp \mathfrak{s}')K.$$

Then we define \mathcal{F} to be the algebra with unit of functions generated by the function f from Lemma 3.1.5 and let \mathcal{F}^+ be the linear span of monomials of positive degree in this generator f .

THEOREM 3.2.4. *Let $\mathfrak{g} = \mathfrak{so}(1, l) = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ with universal enveloping $U(\mathfrak{g})$, $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and \mathfrak{s} a section for $M = \exp \mathfrak{m}$ acting on \mathfrak{n} . Let D be in $U(\mathfrak{g})$. There are $\Delta_i \in U(\mathfrak{m}) \otimes U(\mathfrak{s} \oplus \mathfrak{a}) \otimes U(\mathfrak{k})$ and $\Delta_0 \in U(\mathfrak{a} \oplus \mathfrak{s}) \otimes U(\mathfrak{k})$, $\varphi_j \in \mathcal{F}^+$ analytic on \mathfrak{s}' such that*

$$\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D) = \Delta_0 + \sum_j \varphi_j(\exp X') \Delta_j$$

for all $X' \in \mathfrak{s}'$.

PROOF. By Proposition 3.2.3 it is equivalent to prove that for any $D \in U(\mathfrak{g})$ there are $D_0 \in U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})$, $\xi_j \in \lambda(S(\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp_{\mathfrak{m}}}))$, $u_j \in U(\mathfrak{a} \oplus \mathfrak{s})$ and $\xi'_j \in U(\mathfrak{k})$ such that

$$D = D_0 + \sum_j \varphi_j \xi_j^{\exp -X'} u_j \xi'_j,$$

where $\varphi_j \in \mathcal{F}^+$.

For $D \in U(\mathfrak{g})$ we set $\Delta_0 := \pi_1(D)$. Next if $D \in U(\mathfrak{s} \oplus \mathfrak{a})U(\mathfrak{k})$, then the theorem is clear, since

$$D = \pi_1(D) + \Gamma_{a \exp X'}(0 \otimes 0 \otimes 0) = \Gamma_{a \exp X'}(\pi_1(D))$$

by the definition of $\Gamma_{a \exp X'}$.

Now we continue with an induction on the degree of $D \in U(\mathfrak{g})$ and note that the claim is true for constants. If the degree is one and D is an element of $\mathfrak{s}^{\perp_{\mathfrak{n}}}$, then there is by Lemma 3.1.5 a unique $Z \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp_{\mathfrak{m}}}$ such that

$$\begin{aligned} D &= f(\exp X') \Gamma_{a \exp X'}(Z \otimes 1 \otimes 1) - f(\exp X') \Gamma_{a \exp X'}(1 \otimes 1 \otimes Z) \\ &= f(\exp X') \Gamma_{a \exp X'}(Z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes Z) \end{aligned}$$

proving the claim for $D \in \mathfrak{s}^{\perp_{\mathfrak{n}}}$.

Since $U(\mathfrak{g}) = U(\mathfrak{s} \oplus \mathfrak{a}) \oplus (\mathfrak{s}^{\perp_{\mathfrak{n}}})U(\mathfrak{g})$ by Lemma 3.2.2, we only have to consider $D \in (\mathfrak{s}^{\perp_{\mathfrak{n}}})U(\mathfrak{g})$ of degree $m+1$ assuming the theorem is true for degree m . Without loss we can assume that $D = YD_1$, where Y is an element of $\mathfrak{s}^{\perp_{\mathfrak{n}}}$ and D_1 of $U(\mathfrak{g})$ of degree m . Again by Lemma 3.1.5

$$Y = f(\exp X')Z^{\exp -X'} - f(\exp X')Z$$

for some $Z \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp_{\mathfrak{m}}}$ and we find for $X \in \mathfrak{s}'$ that

$$\begin{aligned} YD_1 &= f(\exp X')Z^{\exp -X'}D_1 - f(\exp X')ZD_1 \\ &= f(\exp X')Z^{\exp -X'}D_1 - f(\exp X')D_1Z - f(\exp X')[Z, D_1]. \end{aligned}$$

But D_1 and $[Z, D_1]$ are of degree $\leq m$, so we can apply the induction hypothesis

$$\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D_1) = \Delta_0 + \sum_j \varphi_j(\exp X') \Delta_j, \quad \overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}([Z, D_1]) = \tilde{\Delta}_0 + \sum_j \tilde{\varphi}_j(\exp X') \tilde{\Delta}_j$$

resp.

$$D_1 = \Gamma_{a \exp X'} \left(\Delta_0 + \sum_j \varphi_j(\exp X') \Delta_j(D_1) \right) = \Gamma_{a \exp X'}(\Delta_0) + \sum_j \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j)$$

and

$$\begin{aligned} [Z, D_1] &= \Gamma_{a \exp X'} \left(\Delta_0 + \sum_j \varphi_j(\exp X') \Delta_j([Z, D_1]) \right) = \Gamma_{a \exp X'}(\tilde{\Delta}_0) + \sum_l \tilde{\varphi}_l(\exp X') \Gamma_{a \exp X'}(\tilde{\Delta}_l) \\ &= \tilde{\Delta}_0 + \sum_l \varphi_j(\exp X') \Gamma_{a \exp X'}(\tilde{\Delta}_l) \end{aligned}$$

with $D_j, \varphi_j, \tilde{D}_l, \tilde{\varphi}_l$ as claimed in the theorem. That is,

$$\begin{aligned} Z^{\exp -X'} D_1 &= Z^{\exp -X'} \Gamma_{a \exp X'}(\Delta_0) + \sum_j \varphi_j(\exp X') Z^{\exp -X'} \Gamma_{a \exp X'}(\Delta_j) \\ &= Z^{\exp -X'} \Delta_0 + \sum_j \varphi_j(\exp X') Z^{\exp -X'} \Gamma_{a \exp -X'}(\Delta_j), \\ D_1 Z &= \Gamma_{a \exp X'}(\Delta_0) Z + \sum_j \varphi_j(\exp X') \Gamma_{a \exp -X'}(\Delta_j) Z \\ &= \Delta_0 Z + \sum_j \varphi_j(\exp X') \Gamma_{a \exp -X'}(\Delta_j) Z, \end{aligned}$$

It follows that

$$\begin{aligned} Y D_1 &= f(\exp X') Z^{\exp -X'} \Delta_0 + \sum_j f(\exp X') \varphi_j(\exp X') Z^{\exp -X'} \Gamma_{a \exp X'}(\Delta_j) - f(\exp X') \Delta_0 Z \\ &\quad - \sum_j f(\exp X') \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j) Z - f(\exp X') \tilde{\Delta}_0 - \sum_l f(\exp X') \varphi_l(\exp X') \Gamma(\tilde{\Delta}_l) \\ &= -f(\exp X') \Delta_0 Z - f(\exp X') \tilde{\Delta}_j + f(\exp X') Z^{\exp -X'} \Delta_0 + \\ &\quad \sum_j f(\exp X') \varphi_j(\exp X') \left(Z^{\exp -X'} \Gamma_{a \exp X'}(\Delta_j) + \Gamma_{a \exp X'}(\Delta_j) Z \right) \\ &\quad + \sum_l f(\exp X') \varphi_l(\exp X') \Gamma(\tilde{\Delta}_l). \end{aligned}$$

Then we are done, since $\pi_1(Y D_1) = 0$ which implies

$$0 = -f(\exp X') \Delta_0 Z - f(\exp X') \tilde{\Delta}_0.$$

□

It follows that for functions $\phi \in C^\infty(G')$ and $D \in U(\mathfrak{g})$

$$(D\phi)(a \exp X') = \left(\Gamma_{a \exp X'} \left(\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D) \right) \phi \right) (a \exp X')$$

valid for all $a \in A, X' \in \mathfrak{s}'$.

For certain $D \in U(\mathfrak{g})$, the expression of $\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D)$ is simpler.

COROLLARY 3.2.5. *With the notation of Theorem 3.2.4 we have*

$$\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(\pi_1(D)) = \Delta_0.$$

Further, if

$$\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D) = \tilde{\Delta}_0 + \sum_{j=1}^n \varphi_j(\exp X') \Delta_j = \Delta_0 + \sum_{l=1}^m \tilde{\varphi}_l(X') \tilde{\Delta}_l$$

for all $X' \in \mathfrak{s}'$, where $\Delta_j, \tilde{\Delta}_l, \varphi_j, \tilde{\varphi}_l$ as in Theorem 3.2.4 and $n \leq m$, then $\Delta_0 = \tilde{\Delta}_0$ and after a possible permutation of j ,

$$\Delta_1 = \tilde{\Delta}_1, \dots, \Delta_k = \tilde{\Delta}_k, \varphi_1 = \tilde{\varphi}_1, \dots, \varphi_k = \tilde{\varphi}_k.$$

and $\tilde{\Delta}_{k+1} = \dots = \tilde{\Delta}_m = 0$.

PROOF. The first claim, $\mathring{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(\pi_1(D)) = \Delta_0$, follows by the proof of Theorem 3.2.4. For the second claim we note that $\varphi_j(\exp tX') \rightarrow 0, t \rightarrow \infty$, for all $X' \in \mathfrak{s}'$ and $\varphi_j \in \mathcal{F}^+$. So if we assume

$$\Delta_0 + \sum_j \varphi_j(\exp X')\Delta_j = \Delta'_0 + \sum_k \varphi'_k(\exp X')\Delta'_k \text{ resp.}$$

$$\Delta_0 - \Delta'_0 = \sum_l \varphi''_l(\exp X')D''_l,$$

for some $\Delta_0, \Delta'_0 \in U(\mathfrak{s} \oplus \mathfrak{a})U(\mathfrak{k})$, $\Delta_j, \Delta'_k, D''_l \in U(\mathfrak{m})U(\Sigma \oplus \mathfrak{a})U(\mathfrak{k})$, $\varphi_j, \varphi'_k, \varphi''_l \in \mathcal{F}^+$ and all $X' \in \mathfrak{s}'$, then replacing $\exp X'$ by $\exp tX'$ and letting $t \rightarrow \infty$ gives the claim.

The last claim, follows since $\Delta_j, \tilde{\Delta}_l$ are independent of $X' \in \mathfrak{s}'$ and since $\varphi_j \cdot \tilde{\varphi}_k \in \mathcal{F}^+$. \square

We state now a lemma which tells us more about the nature of the projection π_2 from $U(\mathfrak{g})$ onto $U(\mathfrak{s} \oplus \mathfrak{a})$.

LEMMA 3.2.6. *The mapping π_2 is a homomorphism from $U(\mathfrak{g})^K$ into $U(\mathfrak{s} \oplus \mathfrak{a})$.*

PROOF. Let $D, D' \in U(\mathfrak{g})^K$. Then

$$DD' - \pi_2(D)\pi_2(D') = \pi_2(D)(D' - \pi_2(D')) + (D - \pi_2(D))D'.$$

By definition $D - \pi_2(D), D' - \pi_2(D')$ are elements of $(\mathfrak{s}^{\perp n})U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$ and since D' is in $U(\mathfrak{g})^K$, it follows that

$$WD' = D'W$$

for $W \in \mathfrak{k}$. Hence,

$$(D - \pi_2(D))D' \in (\mathfrak{s}^{\perp n})U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}.$$

Since $[\mathfrak{s} \oplus \mathfrak{a}, \mathfrak{s}^{\perp n}] \subset \mathfrak{s}^{\perp n}$ it also follows that

$$\pi_2(D)(D' - \pi_2(D')) \in (\mathfrak{s}^{\perp n})U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}.$$

Thus,

$$\pi_2(DD') = \pi_2(D)\pi_2(D').$$

\square

Now we explain how $\Gamma_{a \exp X'}(D)$ transforms if we apply it to invariant functions $\phi \in C^\infty(MA \backslash G/K)$. This will lead us to the definition of the radial part of D . We therefore compute for $Z \in \mathfrak{z}_m(\mathfrak{s})^{\perp m}, u \in U(\mathfrak{a} \oplus \mathfrak{s}), \xi \in U(\mathfrak{k}), a \exp X' \in A \exp \mathfrak{s}'$ and $\phi \in C^\infty(G)$

$$\begin{aligned} \Gamma_{a \exp X'}(Z \otimes u \otimes \xi)\phi(a \exp X') &= \left(Z^{\exp - X'} u \xi \right) \phi(a \exp X') \\ &= \frac{d}{dt} \Big|_{t=0} u \xi \phi(a \exp X' \exp \text{Ad}(\exp - X')tZ) \\ (3.1) \quad &= \frac{d}{dt} \Big|_{t=0} u \xi \phi(a \exp X' \exp - X' \exp tZ \exp X') \\ &= \frac{d}{dt} \Big|_{t=0} u \xi \phi(a \exp tZ \exp X') \\ &= \frac{d}{dt} \Big|_{t=0} u \xi \phi(\exp tZ a \exp X'), \end{aligned}$$

as $\exp tZ \in M$. In particular it is clear that

$$(3.2) \quad \Gamma_{a \exp X'}(\xi \otimes u \otimes \xi')\phi(a \exp X') = 0$$

if $\phi \in C^\infty(M \backslash G/K)$, $\xi \in U(\mathfrak{m})$, $\xi' \in U(\mathfrak{k})$ and ξ or ξ' is not a constant. If $u = H$ is in \mathfrak{a} and ϕ is left- A -invariant, then a similar computation gives

$$\begin{aligned} Z^{\exp -X'} H \xi \phi(a \exp X') &= \frac{d}{dt_1} \Big|_{t_1=0} \frac{d}{dt_2} \Big|_{t_2=0} \xi \phi(\exp t_1 Z \exp X' \exp t_2 H) \\ &= \frac{d}{dt_1} \Big|_{t_1=0} \frac{d}{dt_2} \Big|_{t_2=0} \xi \phi(\exp t_1 Z \exp e^{-t_2 \alpha(H)} X'). \end{aligned}$$

Let $c : U(\mathfrak{k}) \rightarrow \mathbb{C}$ be the trivial one dimensional representation of $U(\mathfrak{k})$. The map $Z \mapsto c(Z)1$ can be regarded as the projection $U(\mathfrak{k}) \rightarrow \mathbb{C} \cdot 1$ which belongs to the decomposition

$$U(\mathfrak{k}) = \mathfrak{k} \oplus \ker(c),$$

see also [GV88, p.129]. Then for all $D \in U(\mathfrak{g})$ and $\phi \in C^\infty(M \backslash G/K)$, $x = a \exp X' \in A \exp \mathfrak{s}'$,

$$\begin{aligned} D\phi(x) &= \Gamma_{a \exp X'} \left(\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(D) \right) \phi(x) \\ &= \Gamma_{a \exp X'} \left(\Delta_0 + \sum_j \varphi_j(x) \Delta_j \right) \phi(x) \\ &= \Gamma_{a \exp X'}(\Delta_0) \phi(x) + \sum_j \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j) \phi(x) \\ &= \Gamma_{a \exp X'}(\pi_1(D)) \phi(x) + \sum_j \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j) \phi(x) \text{ by the definition of } \Delta_0 \\ &= \pi_1(D) \phi(x) + \sum_j \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j) \phi(x) \text{ by the definition of } \Gamma_{s \exp X'} \\ &= \pi_2(D) \phi(x) + \sum_j \varphi_j(\exp X') \Gamma_{a \exp X'}(\Delta_j) \phi(x), \\ &\quad \text{as } \phi \text{ is right-}K\text{-invariant and } \pi_1 \equiv \pi_2 \pmod{U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})\mathfrak{k}} \\ &= \pi_2(D) \phi(x) + \sum_j \varphi_j(x) c(\xi) c(\xi') u_j \phi(x) \end{aligned}$$

by (3.1) and (3.2), where $\Delta_0, \Delta_j, \varphi_j$ are as in Theorem 3.2.4, i.e. $\varphi_j \in \mathcal{F}^+$ and $u_j \in U(\mathfrak{a} \oplus \mathfrak{s})$. Thus we may define a differential operator $\delta(D)$ on $A \exp \mathfrak{s}'$ by

$$(3.3) \quad \delta(D) := \pi_2(D) + \sum_j \psi_j u_j,$$

where $\psi_j = c(\xi_j) c(\xi'_j) \varphi_j$ and

$$D\phi = \delta(D)\phi$$

on $A \exp \mathfrak{s}'$ for all $\phi \in C^\infty(M \backslash G/K)$. We call $\delta(D)$ the *radial part* of D .

3.3. Example: Polar decomposition of Ω

In this section we apply the theory developed above to the Casimir operator Ω . We start with the computation of the polar decomposition $\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'}(\Omega)$ of Ω .

We recall that we fixed some $X_1 \in \mathfrak{n}$ of unit length. We want to complete X_1 to an orthonormal basis of \mathfrak{n} . Therefore, we remark:

LEMMA 3.3.1. *We can complete X_1 to an orthonormal basis $\{X_1, Y_2, \dots, Y_{l-1}\}$ of \mathfrak{n} such that for all j*

$$[Z_{Y_j}, X_1] = Y_j$$

always implies

$$[X_1, \theta Y_j] = 2Z_{Y_j}$$

and

$$[Y_j, Z_{Y_j}] = X_1.$$

Here $Z_{Y_j} \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})_{\mathfrak{m}}^{\perp}$ is uniquely determined by the relation $[Z_{Y_j}, X_1] = Y_j$, see Lemma 3.1.3.

PROOF. In [Juh01, Chap.2 (2.9)] we find a special orthonormal basis $\{L_j\}_j$ of \mathfrak{n} which satisfies the requirements of this lemma. Because M acts transitively on the sphere in \mathfrak{n} , we can find some $m \in M$ such that $X_1 = m \cdot L_1$. We set $Y_j := m \cdot L_j$ which implies $Z_{Y_j} = m \cdot Z_{L_j}$. But then

$$[X_1, \theta Y_j] = m \cdot [L_1, \theta L_j] = m \cdot 2Z_{L_j} = 2Z_{Y_j}$$

and

$$[Y_j, Z_{Y_j}] = [m \cdot L_j, m \cdot Z_{L_j}] = m \cdot L_1 = X_1.$$

□

We take the orthonormal basis $\{X_1, Y_j\}_j$ of $\mathfrak{s}^{\perp \mathfrak{n}}$ from Lemma 3.3.1. Then we know from Chapter 2.1, equation (2.3), that

$$\begin{aligned} \Omega &= H_1^2 - \sum_{i=1}^k M_i^2 + 2X_1^2 + 2 \sum_{j=2}^{l-1} Y_j^2 - 2X_1 W_1 - 2 \sum_{j=2}^{l-1} Y_j W_j - 2H_{\rho} \\ &= H_1^2 - 2H_{\rho} - \sum_{i=1}^k M_i^2 + 2(X_1^2 - X_1 W) + 2 \sum_{j=2}^{l-1} (Y_j^2 - Y_j W_j), \end{aligned}$$

where $M_i \in \mathfrak{m}$, $H_1 \in \mathfrak{a}$, $H_{\rho} \in \mathfrak{a}$ and $W = W_1, W_j \in \mathfrak{m}^{\perp \mathfrak{t}}$, see Chapter 2.1 for the details.

Let $X' = rX_1 \in \mathfrak{s}'$, then there exists by Lemma 3.1.5 for any $Y_j \in \mathfrak{n}$ some $Z_{Y_j} \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s})^{\perp \mathfrak{m}}$ such that

$$Y_j = f(\exp X') Z_{Y_j}^{\exp -X'} - f(\exp X') Z_{Y_j},$$

where $Z_{Y_j} \cdot X_1 = Y_j$. Hence,

$$\begin{aligned} \sum_j Y_j^2 &= f(\exp X')^2 \sum_j (Z_{Y_j}^{\exp -X'} - Z_{Y_j})(Z_{Y_j}^{\exp -X'} - Z_{Y_j}) \\ &= f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X} - Z_{Y_j}^{\exp -X'} Z_{Y_j} - Z_{Y_j} Z_{Y_j}^{\exp -X} + Z_{Y_j} Z_{Y_j} \\ &= f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} - [Z_{Y_j}, Z_{Y_j}^{\exp -X'}] + Z_{Y_j} Z_{Y_j} \\ &= f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} - [Z_{Y_j}, Z_{Y_j} + [Z_{Y_j}, X']] + Z_{Y_j} Z_{Y_j} \\ &= f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} - [Z_{Y_j}, f(\exp X')^{-1} [Z_{Y_j}, X_1]] + Z_{Y_j} Z_{Y_j} \\ &= f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} - f(\exp X')^{-1} [Z_{Y_j}, Y_j] + Z_{Y_j} Z_{Y_j} \\ &\stackrel{\text{Lem. 3.3.1}}{=} f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} + f(\exp X')^{-1} X_1 + Z_{Y_j} Z_{Y_j}. \end{aligned}$$

For Ω we obtain

$$\begin{aligned} \Omega = \Gamma_{a \exp X'} \left(\overset{\circ}{\perp}_{\mathfrak{a} \oplus \mathfrak{s}, a \exp X'} (\Omega) \right) &= H_1^2 - \sum_i M_i^2 - 2H_\rho + 2X_1^2 + 2(l-2)f(\exp X')X_1 - 2X_1W \\ &\quad + 2f(\exp X')^2 \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'} - 2Z_{Y_j}^{\exp -X'} Z_{Y_j} + Z_{Y_j} Z_{Y_j} \\ (3.4) \quad &\quad - f(\exp X')^{-1} (Z_{Y_j}^{\exp -X'} - Z_{Y_j}) W_j. \end{aligned}$$

From this we can easily derive the radial part of Ω by dropping all terms involving elements from \mathfrak{m} . Note that $(Z_{Y_i} Z_{Y_i})^{\exp -X'}$ operates from the left, i.e. it vanishes if the function is left- M -invariant. Thus,

$$(3.5) \quad \delta(\Omega) = H_1^2 - 2H_\rho + 2X_1^2 + 2(n-2)f \cdot X_1,$$

where $f(\exp X') = \frac{1}{r}$ for $X' = rX_1 \in \mathfrak{s}'$.

3.4. Restrictions of bi- M -invariant functions

Here we want to settle the question what happens if we restrict functions which are bi- M -invariant to the section \mathfrak{s} .

LEMMA 3.4.1. *Let \mathfrak{s} be a section for M acting on \mathfrak{n} , F a fundamental domain for $W = N_M(\mathfrak{s})/Z_M(\mathfrak{s})$ acting on \mathfrak{s} . Then one can choose a slice $F' \subset F$ consisting of regular elements which intersects every regular orbit. The mapping $\phi : M/Z_M(\mathfrak{s}) \times F' \rightarrow \mathfrak{n}'$, where \mathfrak{n}' is the subset of regular points in \mathfrak{n} (always relative to the action of M), is a diffeomorphism.*

PROOF. By definition the map is bijective. For $m_0 \in M$ and $X_0 \in F'$ the differential computes to

$$d\phi_{(m_0, X_0)}(d\tau(m_0)Z, T) = \text{Ad}(m_0)([Z, X_0] + T),$$

where Z is an element of the orthogonal complement of $\mathfrak{z}_\mathfrak{m}(\mathfrak{s})$ in \mathfrak{m} , $T \in \mathfrak{s}$ and $\tau(x)$ is the mapping $mZ_M(\mathfrak{s}) \mapsto xmZ_M(\mathfrak{s})$ from $M/Z_M(\mathfrak{s})$ onto itself.

Now

$$\langle Z \cdot X_0, T \rangle = 0,$$

since the action is polar. Hence, the differential vanishes iff $[Z, X_0] = T = 0$. But $[Z, X_0] = 0$ implies that

$$Z \in \mathfrak{z}_\mathfrak{m}(H_0) = \mathfrak{z}_\mathfrak{m}(\mathfrak{s}),$$

since X_0 is regular. Thus, $Z = 0$. \square

COROLLARY 3.4.2. *The restriction map from $C^\infty(\mathfrak{n}')^M \rightarrow C^\infty(\mathfrak{s}')^W$ is an isomorphism.*

REMARK 3.4.3. Due to the structure of (non exceptional) rank one symmetric spaces most of the theory of this chapter is also applicable to any semisimple, rank one group G coming from the 3 non exceptional series and a subgroup M' of M acting polarly on \mathfrak{n} . Examples for M' are centralizer M_m , $m \in M$. The only case when such centralizers do not act polarly on \mathfrak{n} , is when $G = SU(1, n)$ and M_m is a maximal torus in $SU(1, n)$.

Some differential equations of hypergeometric type

In this chapter we consider a special differential equation coming from the Casimir operator. We make use of the theory of radial parts for the decomposition $SO_o(1, l) = G = M(A \exp \mathfrak{s})K$ from Chapter 3 and consider only functions satisfying a certain equivariance property. We show then that the differential equation resembles a hypergeometric equation and we determine the space of solutions thereof.

4.1. Differential equations for M -equivariant functions on a slice

Let (π, V_π) be an irreducible representation of M on V_π and $F \in C^\infty(X \times_M \text{End}(V_\pi))$, where

$$C^\infty(X \times_M \text{End}(V_\pi)) := \{F \in C^\infty(X, \text{End}(V_\pi)) : F(m \cdot x) = \pi(m)F(x) \text{ f.a. } m \in M \text{ and } x \in X\}.$$

Here we define the action of M on $X = G/K$ by $m \cdot x = mx$. We can also identify X with AN , $xK = anK \mapsto an \cdot o \mapsto an$, $o = K \in G/K$, via the Iwasawa decomposition. The action of M on X is then given by

$$m \cdot x = mx = manK = amnK = a(mnm^{-1})K$$

for $x = an = anK$. We can also define an action of M on G via left translation

$$m * g := mg.$$

Let $\text{pr} : G \rightarrow X = G/K = AN$ be the canonical projection associated to the Iwasawa decomposition $G = ANK$ which maps $g = ank$ to $anK = an$. Then we have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{m*} & G \\ \text{pr} \downarrow & & \downarrow \text{pr} \\ X = G/K = AN & \xrightarrow{m \cdot} & X = G/K = AN \end{array}$$

because

$$\text{pr}(m * g) = \text{pr}(m * ank) = \text{pr}(mank) = \text{pr}(amnm^{-1}mk) = amnm^{-1}K = m \cdot anK = m \cdot \text{pr}(g).$$

Furthermore, we fix a slice $S \subset N$ for the action of M on N . We can assume that $S = \exp \mathbb{R}^+ X_1$, where $X_1 \in \mathfrak{n} \cong \mathbb{R}^{l-1}$ is of unit length with respect to $B_\theta(\cdot, \cdot)$. We set $\mathfrak{s} := \mathbb{R}X_1$. We fix an orthonormal basis $\{X_1, Y_2, \dots, Y_{l-1}\}$ of \mathfrak{n} according to Lemma 3.3.1. Then we compute for a differential operator $Z \in \mathfrak{m}$, $X' = sX_1$ and for any $F : G \rightarrow \text{End}(V_\pi)$ smooth with $F(mg) = \pi(m)F(g)$ for all $m \in M$, $g \in G$

$$\begin{aligned} Z^{\exp -X'} F(\exp sX_1) &= \frac{d}{dt} \Big|_{t=0} F(\exp sX_1 \exp tZ^{\exp -sX_1}) \\ &= \frac{d}{dt} \Big|_{t=0} F(\exp tZ \exp sX_1) \\ &= \frac{d}{dt} \Big|_{t=0} \pi(\exp tZ) F(\exp sX_1) \\ (4.1) \quad &=: \pi(Z)F(\exp sX_1). \end{aligned}$$

We assume in addition that F solves the differential equation

$$(4.2) \quad \Omega F + \mu F = 0$$

for some $\mu \in \mathbb{C}$. Then we use the expression for Ω from equation (3.4)

$$\begin{aligned} \Omega = & H_1^2 - 2H_\rho + 2X_1^2 + 2(l-2)f(\exp X')X_1 + 2f(\exp X')^2 \sum_j \left((Z_{Y_j} Z_{Y_j})^{\exp -X'} \right) \\ & - 2X_1 W - \sum_i M_i^2 - \sum_j 2Z_{Y_j}^{\exp -X'} + Z_{Y_j} Z_{Y_j} - f(\exp X')^{-1} (Z_{Y_j}^{\exp -X'} - Z_{Y_j}) W_j, \end{aligned}$$

which is valid for any $X' = sX_1$, $s \neq 0$. Here $f(\exp X') = f(\exp sX_1) = \frac{1}{s}$, $W, W_j \in \mathfrak{m}^{\perp \mathfrak{t}}$ and $Z_{Y_j} \in \mathfrak{z}_{\mathfrak{m}(\mathfrak{g})}^{\perp \mathfrak{m}}$ as in Lemma 3.3.1. Hence, Ω can be written modulo $U(\mathfrak{g})\mathfrak{k}$ for any $X' = sX_1$, $s \neq 0$, in the form

$$(4.3) \quad H_1^2 - 2H_\rho + 2X_1^2 + 2\frac{l-2}{s}X_1 + \frac{2}{s^2} \sum_j (Z_{Y_j} Z_{Y_j})^{\exp -X'}.$$

We call this the polar coordinate form of Ω . Let \bar{F} be the restriction of F to $S \cdot o \subset X$, where $o = K$ and \tilde{F} the lift of F to a function $\tilde{F} : G \rightarrow \text{End}(V_\pi)$. Then it follows from (4.3) that for $s > 0$

$$\begin{aligned} (\Omega F)(\exp sX_1 \cdot o) &= \left[\left(H_1 - 2H_\rho + 2X_1^2 + \frac{2(l-2)}{s}X_1 \right) \bar{F} \right] (\exp sX_1 \cdot o) \\ &+ \left[\left(\frac{2}{s^2} \text{Ad}(\exp -sX_1) \sum_j Z_{Y_j}^2 \right) \tilde{F} \right] (\exp sX_1). \end{aligned}$$

It follows from (4.1) that

$$\left(\text{Ad}(\exp -sX_1) \sum_j Z_{Y_j}^2 \right) \tilde{F}(\exp sX_1) = \sum_j \pi(Z_{Y_j}^2) \tilde{F}(\exp sX_1).$$

Hence, the restriction \bar{F} of F to $S \cdot o$ satisfies

$$(4.4) \quad \left(H_1^2 - 2H_\rho + 2X_1^2 + 2\frac{l-2}{s}X_1 + \frac{2}{s^2} \sum_j \pi(Z_{Y_j}^2) + \mu \right) \bar{F} = 0,$$

$s \in \mathbb{R}^+$. Let

$$Z_M(S) := \{m \in M : m \cdot \exp sX_1 = \exp sX_1 \text{ for all } s \in \mathbb{R}\}.$$

Since $\text{pr} : G \rightarrow X = AN$ is the identity when restricted to $S \subset N$ and by the commutative diagram from above, it follows that

$$\begin{aligned} Z_M(S) &= \{m \in M : m \cdot \exp sX_1 = \exp sX_1 \text{ for all } s \in \mathbb{R}\} \\ &= \{m \in M : \exp sX_1 m = m \exp sX_1 \text{ for all } s \in \mathbb{R}\}. \end{aligned}$$

That is, $Z_M(S)$ also equals the centralizer of $\exp X_1$ in M .

LEMMA 4.1.1. *For any $(\pi, V_\pi) \in \widehat{M}$ we have that $\dim(V_\pi^{Z_M(S)})$ is either 0 or 1.*

PROOF. This follows since $(M, Z_M(S))$, $M \cong SO(l-1)$, $Z_M(S) \cong SO(l-2)$, is a Gelfand pair. Hence for any irreducible π , the space of vectors fixed under $Z_M(S)$ is at most one dimensional. \square

LEMMA 4.1.2. *Assume $V_\pi^{Z_M(S)}$ is spanned by v . The restriction \bar{F} of F to $S \cdot o$ maps V_π to $V_\pi^{Z_M(S)}$*

PROOF. This is a consequence of $F(m \cdot x) = \pi(m)F(x)$ for any $m \in M, x \in X$. Then for any $v \in V_\pi$

$$\begin{aligned} \pi(m)F(\exp sX_1)v &= F(m \cdot \exp sX_1)v \\ &= F(\exp sX_1)v \end{aligned}$$

for all $s \in \mathbb{R}$ and $m \in Z_M(S)$. This means that $F(\exp sX_1)v$ is $Z_M(S)$ -invariant. \square

LEMMA 4.1.3. *$\sum_j Z_{Y_j}^2$ is $Z_M(S)$ -invariant and $\sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}}$ is negative semidefinite for any $(\pi, V_\pi) \in \widehat{M}$.*

By Lemma 4.1.1 we can hence identify the operator $\sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}} \in \text{End}(V_\pi^{Z_M(S)})$ with a nonnegative number.

PROOF. For the first claim we use the orthogonal decomposition

$$\mathfrak{m} = \mathfrak{z}_\mathfrak{m}(S) \oplus \mathfrak{z}_\mathfrak{m}(S)^{\perp \mathfrak{m}}.$$

Recall the relations from Lemma 3.3.1 on X_1, Y_j, Z_{Y_j} . We compute for $Y_k \neq Y_l$ from $\{Y_j\}_j$

$$\begin{aligned} 0 = -B_\theta(Y_k, Y_l) &= B([Z_{Y_k}, X_1], \theta Y_l) \\ &= B(Z_{Y_k}, [X_1, \theta Y_l]) \\ &\stackrel{\text{Lem. 3.3.1}}{=} 2B(Z_{Y_k}, Z_{Y_l}). \end{aligned}$$

By the same computation $B(Z_{Y_j}, Z_{Y_j}) = 1/2$. That is, $\{Z_{Y_j}\}_j$ is an orthogonal set and $|Z_{Y_j}|^2 = 1/2$. Since the dimension of $\mathfrak{z}_\mathfrak{m}(S)^{\perp \mathfrak{m}}$ equals the dimension of $\mathfrak{s}^{\perp \mathfrak{n}}$, see Lemma 3.1.3, it follows that the set $\{\sqrt{2}Z_{Y_j}\}_j$ forms an orthonormal basis for $\mathfrak{z}_\mathfrak{m}(S)^{\perp \mathfrak{m}}$. If $\{K_i\}_i$ is an orthonormal basis of $\mathfrak{z}_\mathfrak{m}(S)$, then $2\sum_j Z_{Y_j}^2$ can be written as the difference

$$2\sum_j Z_{Y_j}^2 = \Omega_\mathfrak{m} - \sum_i K_i^2$$

of two $Z_M(S)$ -invariant operators, where $\Omega_\mathfrak{m}$ is the Casimir of $U(\mathfrak{m})$.

The second claim is clear, if $V_\pi^{Z_M(S)} = 0$. Otherwise we fix some representation (π, V_π) with $V_\pi^{Z_M(S)} \neq 0$. Since $\sum_j Z_{Y_j}^2$ is $Z_M(S)$ -invariant, this implies that $\sum_j \pi(Z_{Y_j}^2)$ maps $V_\pi^{Z_M(S)}$ into itself. Further, π defines also a representation of \mathfrak{m} by differentiating. Since π is unitary, i.e. there is some preserved inner product on V_π , the mapping $\pi(Z)$ is skew-symmetric for any Z in \mathfrak{m} . Thus,

$$\text{spec}(\pi(Z)) \subset i\mathbb{R} \text{ resp. } \text{spec}(\pi(Z^2)) \subset \mathbb{R}^-$$

for any Z in \mathfrak{m} . \square

Thus, we see that $\sum_j \pi(Z_{Y_j}^2)$, just as $F(\exp sX_1)$ for $s \in \mathbb{R}$, maps $V_\pi^{Z_M(S)}$ into itself. Next, we define the action of X_1 resp. for $H \in \mathfrak{a}$ on $C^\infty(X \times_M V_\pi)$. While the action of X_1 on $\exp \mathbb{R}X_1$ is standard, i.e. by translation, we define an action of A on N by $a \cdot n = a^{-1}na$. This really defines an action, since A is commutative. Indeed,

$$a_1 a_2 \cdot n = a_2 a_1 \cdot n = a_1^{-1} a_2^{-1} n a_2 a_1 = a_1 \cdot (a_2 \cdot n).$$

Furthermore,

$$a^{-1} \exp(X_1) a = \exp(-tH) \exp(X_1) \exp(tH) = \exp(e^{-t\alpha(H)} X_1) \in \exp \mathbb{R}X_1.$$

Thus, we see that the action of A on N restricts to an action on $\exp \mathbb{R}X_1$. The reason for defining the action of A in this non-standard way is that we consider functions on G which are right- K - and left- A -invariant, i.e. we will consider functions on N using the Iwasawa decomposition $G = ANK$. In this way the action of A on N is compatible with the action of vectorfields $V \in \mathfrak{g}$ on $C^\infty(G)$, i.e. for $n \in N$, $H \in \mathfrak{a}$ and $f \in C^\infty(A \backslash G/K)$

$$Hf(n) = \frac{d}{dt} \Big|_{t=0} f(n \exp tH) = \frac{d}{dt} \Big|_{t=0} f(\exp -tHn \exp tH) = \frac{d}{dt} \Big|_{t=0} f(\exp tH \cdot n).$$

Next we determine how the vector fields occurring in (4.4) act on functions of $C^\infty(X, \text{End}(V_\pi))$.

LEMMA 4.1.4. *Identifying $\exp \mathbb{R}X_1$ with \mathbb{R} , the vectorfield X_1 corresponds to $\frac{d}{ds}$ while H_ρ is $-\alpha(H_\rho)s \frac{d}{ds}$ and H_1^2 acts as $\alpha(H_1)^2 \left(s^2 \frac{d^2}{ds^2} + s \frac{d}{ds} \right)$.*

PROOF. Let $F \in C^\infty(X \times_M V_\pi)$. We have for any $H \in \mathfrak{a}$ and $s \in \mathbb{R}$

$$\begin{aligned} (HF)(\exp sX_1) &= \frac{d}{dt} \Big|_{t=0} F(\exp -tH \exp sX_1 \exp tH) \\ &= \frac{d}{dt} \Big|_{t=0} F(\exp sX \exp -tH) \\ &= \frac{d}{dt} \Big|_{t=0} F(\exp(\text{Ad}(\exp -tH)sX_1)) \\ &= \frac{d}{dt} \Big|_{t=0} F(\exp(\exp -t\text{ad}(H)sX_1)) \\ &= \frac{d}{dt} \Big|_{t=0} F(\exp e^{-\alpha(H)t} sX_1) \\ &\stackrel{\text{Identification}}{=} \frac{d}{dt} \Big|_{t=0} F(e^{-\alpha(H)t} s) \\ &= -\alpha(H)sF'(s). \end{aligned}$$

Similarly,

$$\begin{aligned} (H^2F)(\exp sX_1) &= \frac{d^2}{dt^2} \Big|_{t=0} F(\exp -tH \exp sX_1 \exp tH) \\ &= \frac{d^2}{dt^2} \Big|_{t=0} F(\exp e^{-t\alpha(H)} sX_1) \\ &\stackrel{\text{Identification}}{=} \frac{d^2}{dt^2} \Big|_{t=0} F(e^{-t\alpha(H)} s) \\ &= \frac{d}{dt} \Big|_{t=0} -\alpha(H)sF'(e^{-t\alpha(H)} s) \\ &= \alpha(H)^2 (sF'(s) + s^2F''(s)), \end{aligned}$$

where $F(s) := F(\exp sX_1)$. □

We apply equation (4.4) to v and get

$$\left(H_1^2 - 2H_\rho + 2X_1^2 + 2\frac{l-2}{s}X_1 + \frac{2}{s^2} \sum_j \pi(Z_{Y_j}^2) \Big|_{V_\pi^{Z_M(s)}} + \mu \right) F(\exp sX_1)v = 0.$$

i.e.

$$(4.5) \quad \left((\alpha(H_1)^2 s^2 + 2) \frac{d^2}{ds^2} + \left((\alpha(H_1)^2 + 2\alpha(H_\rho))s + 2\frac{l-2}{s} \right) \frac{d}{ds} + \frac{2}{s^2} \sum_j \pi(Z_{Y_j}^2) \Big|_{V_\pi^{Z_M(s)}} + \mu \right) F(\exp sX_1)v = 0.$$

Let $V_\pi^{Z_M(S)} \neq \{0\}$. As $F(\exp sX_1)|_{V_\pi^{Z_M(S)}}$ and $\sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}}$ are elements of $\text{End}(V_\pi^{Z_M(S)})$ which we identify with \mathbb{C} we deduce that

$$(4.6) \quad \left((\alpha(H_1)^2 s^2 + 2) \frac{d^2}{ds^2} + \left((\alpha(H_1)^2 + 2\alpha(H_\rho)) s + 2 \frac{l-2}{s} \right) \frac{d}{ds} + \frac{2}{s^2} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}} + \mu \right) F(s) = 0$$

with the convention $F(\exp sX_1)v =: F(s)v$. We view (4.6) as an ordinary differential equation for a scalar valued function.

LEMMA 4.1.5. *Let $F \in C^\infty(G \times_M V_\pi)$. Then*

$$\text{Tr}(F(s)) = \langle F(s)v, v \rangle_{V_\pi},$$

where $v \in V_\pi$ spans $V_\pi^{Z_M(S)}$.

PROOF. Let v, v_2, \dots, v_{d_π} be an orthonormal basis of V_π w.r.t. the inner product $\langle \cdot, \cdot \rangle_{V_\pi}$. Then

$$\text{Tr}(F(\exp sX_1)) = \langle F(\exp sX_1)v, v \rangle_{V_\pi} + \sum_{i=2}^{d_\pi} \langle F(\exp sX_1)v_i, v_i \rangle_{V_\pi}$$

By Lemma 4.1.2, $F(\exp sX_1)v_i \in V_\pi^{Z_M(S)} = \mathbb{C}v$, hence $\langle F(\exp sX_1)v_i, v_i \rangle_{V_\pi} = 0$ for all i . □

Let M act on $C^\infty(X)$ by the left regular representation, i.e. $m \cdot f(x) = f(m^{-1}x)$ for $m \in M$ and $x \in X$. We call $f \in C^\infty(X)$ M -finite of type $\pi \in \widehat{M}$, if the left regular representation of M restricted to $\text{span}\{M \cdot f\}$ decomposes into finitely many copies of π . We set for $\pi \in \widehat{M}$

$$C^\infty(X)_\pi := \{f \in C^\infty(X) : f \text{ } M\text{-finite of type } \pi\}.$$

For $\pi \in \widehat{M}$ we denote its contragradient representation by $\check{\pi}$.

LEMMA 4.1.6. *For any M -finite function $f \in C^\infty(X)$ of type $\check{\pi} \in \widehat{M}$ we define*

$$f^\pi(x) := d_\pi \int_M f(m \cdot x) \pi(m^{-1}) dm,$$

where $d_\pi = \dim(V_\pi)$. Then

- a) $f^\pi \in C^\infty(X \times_M V_\pi)$,
- b) $f(x) = \text{Tr}(f^\pi(x))$ for all $x \in X$,
- c) $f(\exp sX_1) = \langle F(\exp sX_1)v, v \rangle_{V_\pi}$, in particular $f|_S = 0$, if $V_\pi^{Z_M(S)} = \{0\}$.
- d) There exist finitely many slices S_i for M acting on N such that the restriction to N of $f|_N$ vanishes iff all its restrictions $f|_{S_i}$ vanish.¹

PROOF. Claim a) follows from the computation

$$f^\pi(m' \cdot x) = d_\pi \int_M f(mm' \cdot x) \pi(m^{-1}) dm = d_\pi \int_M f(m \cdot x) \pi(m' m^{-1}) dm = \pi(m) f^\pi(x).$$

From [Hel01, Ch. IV Lem. 1.7] it follows that the mapping from $C(X)$ to $C(X)_{\check{\pi}}$ given by convolution with $d_\pi \chi_\pi := d_\pi \text{Tr}(\pi)$

$$C(X) \rightarrow C(X)_{\check{\pi}}, f \mapsto d_\pi \chi_\pi * f,$$

¹Later in Section 5.3 we will see that this is equivalent to the existence of certain $p_i \in \frac{1}{2}\mathbb{N}_0$ such that $(X_i^{2p_i} f)(e) = 0$ for all i , where $S_i = \exp \mathbb{R}^+ X_i$, X_i of unit length.

where $\chi_\pi * f(x) := \int_M f(m \cdot x) \chi_\pi(m^{-1}) dm$, is a continuous projection. If f is already M -finite of type $\tilde{\pi}$, then

$$\begin{aligned} f(x) &= d_\pi \chi_\pi * f(x) \\ &= d_\pi \int_M f(m \cdot x) \chi_\pi(m^{-1}) dm \\ &= d_\pi \int_M f(m \cdot x) \text{Tr}(\pi(m^{-1})) dm \\ &= \text{Tr}(f^\pi(x)). \end{aligned}$$

Claim c) is a direct consequence of b) and Lemma 4.1.5. For d) let $n \in N$. Then there is some $m \in M$ and $s \geq 0$ such that $n = m \cdot s$. This implies

$$f(n) = f(m \cdot s) = m^{-1} \cdot f(s).$$

We fix a basis f_1, f_2, \dots, f_d of $\text{span}\{M \cdot f\}$. We can assume that $f_1 = f$ and $f_j = m_j \cdot f$ for some $m_j \in M$, $j = 2, \dots, d$. We set $S_j := m_j^{-1} \cdot S$, then $f_j(s) = m_j \cdot f(s) = f(m_j^{-1} s)$, i.e. $f_j|_S = f_{S_j}$. Since $\{M \cdot f\} \subset \text{span}\{f_1, \dots, f_d\}$ the claim follows. \square

We go back to equation (4.6). The application we have in mind is the following. Let $f \in C^\infty(X)_{\tilde{\pi}}$, then $f^\pi \in C^\infty(X \times_M V_\pi)$ and with the convention $f^\pi(\exp sX_1)v =: f^\pi(s)v$, $s \geq 0$, it follows that

$$f^\pi(s) = \langle f^\pi(\exp sX_1)v, v \rangle_{V_\pi} = \text{Tr}(f^\pi(\exp sX_1)) = f(\exp sX_1).$$

Thus, if f satisfies $\Omega f = \mu f$, then by dominated convergence $\Omega f^\pi = \mu f^\pi$ and the restriction $f|_S$ satisfies equation (4.6).

We note that if $\sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(s)}} = 0$, in particular for $\pi = \mathbf{1}$ the trivial representation, this can be transformed to a well-known hypergeometric differential operator, i.e. $F : \mathbb{R} \rightarrow \mathbb{R}$ is given by some hypergeometric function as we will see later. More precisely, we will solve equation (4.6) in the next section.

Let us summarize the results of this section.

THEOREM 4.1.7. *Let $G = SO_o(1, l)$ and $(\pi, V_\pi) \in \widehat{M}$. Furthermore, let $F \in C^\infty(X \times_M V_\pi)$ with $\Omega F = \mu F$ and $\exp \mathbb{R}X_1$ a section for M acting on N . We can assume that $V_\pi^{Z_M(s)} = \mathbb{C}v$, where $v \in V_\pi$ is of unit length or 0.*

- a) *If we define $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(s)v := F(\exp sX_1)v$ for $v \neq 0$ or $F(s) := 0$ for all $s \in \mathbb{R}$ otherwise, then $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies equation (4.6) on \mathbb{R}^+ .*
- b) *For $s \in \mathbb{R}$:*

$$\begin{aligned} F(s) &= \langle F(s)v, v \rangle_{V_\pi} \\ &= \langle F(\exp sX_1)v, v \rangle_{V_\pi} \\ &= \text{Tr}(F(\exp sX_1)). \end{aligned}$$

If $F(x) = f^\pi(x) = d_\pi \int_M f(m \cdot x) \pi(m^{-1}) dm$ for some $f \in C^\infty(X)$, then also

$$\begin{aligned} F(s) &= d_\pi \int_M f(m \cdot s) \chi_\pi(m^{-1}) dm \\ &= d_\pi (\chi_\pi * f)(\exp sX_1) \\ &= d_\pi^2 (\chi_\pi * (\chi_\pi * f))(\exp sX_1). \end{aligned}$$

4.2. Differential equations of hypergeometric type

We continue with working on equation (4.6) and make the substitution $s^2 = u$. The differentials transform according to

$$s \frac{d}{ds} = 2u \frac{d}{du} \text{ resp. } \frac{1}{s} \frac{d}{ds} = 2 \frac{d}{du} \text{ and } \frac{d^2}{ds^2} = 4u \frac{d^2}{du^2} + 2 \frac{d}{du}.$$

Thus, we obtain the new equation

$$\left((4\alpha(H_1)^2 u^2 + 8u) \frac{d^2}{du^2} + ((4\alpha(H_1)^2 + 4\alpha(H_\rho)) u + 4(l-1)) \frac{d}{du} + \frac{2}{u} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(s)} + \mu} \right) F(w) = 0.$$

Then we divide by the positive scalar $4\alpha(H_1)^2$ to obtain

$$\left(u(u+k_1) \frac{d^2}{du^2} + ((1+k_2)u+k_3) \frac{d}{du} + \frac{k_4}{u} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(s)} + k_5} \right) F(w) = 0,$$

where we set $k_1 = \frac{2}{\alpha(H_1)^2}$, $k_2 = \frac{\alpha(H_\rho)}{\alpha(H_1)^2}$, $k_3 = \frac{l-1}{\alpha(H_1)^2}$, $k_4 = \frac{1}{2\alpha(H_1)^2}$ and $k_5 = \frac{\mu}{4\alpha(H_1)^2}$.

Then obviously k_1, k_4 and $k_3 \in \mathbb{R}^+$ but also k_2 is positive, since H_ρ lies in the positive Weyl chamber \mathfrak{a}^+ of \mathfrak{a} . The (regular) singularities of this ordinary differential equation are 0, $-k_1$ and ∞ .

Now we substitute $u \mapsto w = -\frac{u}{k_1}$, i.e. $u = -k_1 w$,

$$\frac{d}{du} = -\frac{1}{k_1} \frac{d}{dw} \text{ and } \frac{d^2}{du^2} = \frac{1}{k_1^2} \frac{d^2}{dw^2}.$$

We get the new equation

$$\left((-k_1 w)((-k_1 w) + k_1) \frac{1}{k_1^2} \frac{d^2}{dw^2} + ((1+k_2)(-k_1 w) + k_3) \frac{1}{(-k_1)} \frac{d}{dw} - \frac{k_4}{k_1 w} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(s)} + k_5} \right) F(w) = 0.$$

After multiplying with (-1) we arrive at

$$(4.7) \quad \left(w(1-w) \frac{d^2}{dw^2} + \left(\frac{k_3}{k_1} - (1+k_2)w \right) \frac{d}{dw} + \frac{k_4}{k_1 w} \sum_i \pi(Z_{Y_i}^2)|_{V_\pi^{Z_M(s)} - k_5} \right) F(w) = 0$$

We call this is an ordinary differential equation of hypergeometric type

$$(4.8) \quad \left(x(1-x) \frac{d^2 y}{dx^2} + [c - (a+b+1)x] \frac{dy}{dx} - aby + \frac{d}{x} \right) F(x) = 0$$

with (regular) singularities at 0, 1 and ∞ . Here $c = \frac{k_3}{k_1} = \frac{l-1}{2}$, $a+b = k_2 = -\frac{\alpha(H_\rho)}{\alpha(H_1)^2}$, $ab = k_5$ and $d = \frac{k_4}{k_1} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(s)}} \leq 0$.

We recall that the dimension of N is $l-1$. Then $\rho = \frac{l-1}{2}\alpha$ and the Cartan-Killing on \mathfrak{a} form is given by, see [GV88, (4.2.10)]

$$B(H, H) = 2(l-1)\alpha(H)^2.$$

We also defined $H_0 \in \mathfrak{a}^+$ such that H_0 satisfies

$$(4.9) \quad \alpha(H_0) = 1.$$

It follows that

$$B(H_0, H_0) = 2(l-1)$$

and

$$H_\alpha = \frac{1}{2(l-1)}H_0.$$

Furthermore, we can assume that H_1 with $B(H_1, H_1) = 1$ lies in the positive Weyl chamber \mathfrak{a}^+ . That is,

$$H_1 = \frac{1}{\sqrt{2(l-1)}}H_0$$

and

$$H_\rho = \frac{1}{4}H_0.$$

For all these facts see [GV88, p.135].

Now

$$a + b = k_2 = \frac{\alpha(H_\rho)}{\alpha(H_1)^2} = \frac{\frac{1}{4}\alpha(H_0)}{\frac{1}{2(l-1)}\alpha(H_0)^2} = \frac{(l-1)}{2} = \rho_0$$

and

$$a \cdot b = k_4 = \frac{\mu}{4\alpha(H_1)^2} = \frac{\mu(l-1)}{2\alpha(H_0)^2} = -\frac{\mu(l-1)}{2} = \mu\rho_0.$$

Let us assume that the eigenvalue is given by

$$\mu = \frac{1}{4} \left(\rho_0 + \frac{r^2}{\rho_0} \right)$$

for some $r \in \mathbb{C}$. Then $a = \frac{1}{2}(\rho_0 + ir)$ and $b = \frac{1}{2}(\rho_0 - ir)$ solves the equations for $a + b$ and $a \cdot b$, where $\rho_0 = \rho(H_0) = \frac{l-1}{2}$.

4.3. Solutions to $\left(x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby + \frac{d}{x}\right) f(x) = 0$

Motivated by the chapter in [Yos97, III,2,3] on solutions to the hypergeometric equation we define $D = x\frac{d}{dx}$. Then it can be shown that the differential operator

$$E(a, b, c) := \left(x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby \right)$$

can be factorized as

$$E(a, b, c) = \left((c+D)(1+D)\frac{1}{x} - (a+D)(b+D) \right),$$

see [Yos97, p.61]. Thus we define

$$\begin{aligned} F(a, b, c, d) &:= x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby + \frac{d}{x} \\ &= E(a, b, c) + \frac{d}{x} \\ &= ((c+D)(1+D) + d)\frac{1}{x} - (a+D)(b+D). \end{aligned}$$

Now it is easy to show that

$$(4.10) \quad D(x^p u(x)) = x^p(p+D)u(x), \text{ i.e. } Dx^p = x^p(p+D),$$

if we view x^p as an operator $u \mapsto x^p u$ for functions $u = u(x)$. Hence also

$$D^2 x^p = x^p(p+D)^2.$$

We now claim that

$$x^{p_1} {}_2F_1(a+p_1, b+p_1, 1+p_1-p_2; x)$$

and

$$x^{p_2} {}_2F_1(a+p_2, b+p_2, 1+p_2-p_1; x)$$

both solve

$$F(a, b, c, d)f(x) = 0,$$

where in turn $p_{1,2}$ solve the indicial equation

$$(4.11) \quad p^2 + (c-1)p + d = 0.$$

Therefore we compute what happens if we apply $F(a, b, c, d)$ to functions of the form $x^p g(x)$. Again we omit the function $g(x)$ viewing x^p as the operator $g \mapsto x^p g$ to keep formulas simple. We compute

$$\begin{aligned} F(a, b, c, d)x^p &= \left[(c+D)(1+D) + d \right] \frac{1}{x} - (a+D)(b+D) \Big] x^p \\ &= [c+d + (c+1)D + D^2] x^{p-1} - [ab + (a+b)D + D^2] x^p \\ &= x^{p-1} [c+d + (c+1)(p-1+D) + (p-1+D)^2] - x^p [ab + (a+b)(p+D) + (p+D)^2] \\ &= x^p [x^{-1} (c+d + (c+1)(p-1+D) + (p-1+D)^2)] - x^p [(a+p+D)(b+p+D)] \\ &= x^p [x^{-1} (p^2 + (c-1)p + d + (c-1+2p)D + D^2)] - x^p [(a+p+D)(b+p+D)] \\ &=: (*). \end{aligned}$$

To get this into the desired form we must therefore have

$$p^2 + (c-1)p + d = 0,$$

i.e.

$$p_{1,2} = \frac{1-c}{2} \pm \sqrt{\left(\frac{1-c}{2}\right)^2 - d} \text{ and } p_1 - p_2 = 2\sqrt{\left(\frac{1-c}{2}\right)^2 - d}.$$

It follows that

$$c-1+2p_{1,2} = \pm 2\sqrt{\left(\frac{1-c}{2}\right)^2 - d} = \pm(p_1 - p_2)$$

and

$$\begin{aligned} &x^{-1} (p_{1,2}^2 + (c-1)p_{1,2} + d + (c-1+2p_{1,2})D + D^2) \\ &= x^{-1} (\pm(p_1 - p_2)D + D^2) \\ &= x^{-1} (-1 + 1 \pm (p_1 - p_2)D + D^2) \\ &= x^{-1} (1 \pm (p_1 - p_2) - (2 \pm (p_1 - p_2)) + (2 \pm (p_1 - p_2))D + 1 - 2D + D^2) \\ &= x^{-1} (1 \pm (p_1 - p_2) + (2 \pm (p_1 - p_2))(-1 + D) + (-1 + D)^2) \\ &\stackrel{(4.10)}{=} (1 \pm (p_1 - p_2) + (2 \pm (p_1 - p_2))D + D^2) x^{-1} \\ &= (1 + D)(1 \pm (p_1 - p_2) + D) \frac{1}{x}, \end{aligned}$$

that is,

$$\begin{aligned} (*) &= x^{p_{1,2}} \left[(1+D)(1 \pm (p_1 - p_2) + D) \frac{1}{x} - (a+p+D)(b+p+D) \right] \\ &= x^{p_{1,2}} E(a+p_{1,2}, b+p_{1,2}, 1 \pm (p_1 - p_2)). \end{aligned}$$

Thus we have shown

$$F(a, b, c, d)[x^{p_{1,2}} g(x)] = x^{p_{1,2}} E(a+p_{1,2}, b+p_{1,2}, 1 \pm (p_1 - p_2)) g(x),$$

hence

$$\begin{aligned} & F(a, b, c, d) \left[x^{p_1, 2} {}_2F_1(a + p_{1/2}, b + p_{1/2}, 1 \pm (p_1 - p_2)) \right] \\ &= x^{p_1, 2} E(a + p_{1, 2}, b + p_{1, 2}, 1 \pm (p_1 - p_2)) {}_2F_1(a + p_{1/2}, b + p_{1/2}, 1 \pm (p_1 - p_2)) \\ &= 0 \end{aligned}$$

as we claimed.

Another point is which solutions are smooth at the origin. We are only interested in solutions which are smooth everywhere, in particular at the origin. While $x^{p_1} {}_2F_1(a + p_1, b + p_2, 1 + p_1 - p_2)$ is always defined as $p_1 \geq 0$, the second solution $x^{p_2} {}_2F_1(a + p_1, b + p_2, 1 - p_1 + p_2)$ is only defined even for $x \neq 0$ if $p_2 - p_1 \neq -1, -2, -3, \dots$. Finally, $x^{p_1, 2} {}_2F_1(a + p_{1/2}, b + p_{1/2}, 1 \pm (p_1 - p_2))$ is smooth at the origin iff $p_{1, 2} \in \mathbb{N}_0$.

It remains to consider when the two solutions are linearly independent, i.e. span the space of solution to $F(a, b, c, d)f = 0$. We will come to this in the next section.

4.3.1. Application of Frobenius method.

Since the second solution

$$x^{p_2} {}_2F_1(a + p_1, b + p_2, 1 - p_1 + p_2)$$

is not defined for special p_2 we shortly explain another method for finding two linearly independent solutions of (4.6) which has the advantage of covering all cases but is less explicit. It is called the *Frobenius method* and we follow Chapter 6 in [Mil]. Let us consider the second order differential equation

$$(4.12) \quad \frac{d^2 y}{dz^2} + P(z) \frac{dy}{dz} + Q(z)y = 0$$

on the punctured disc $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, where P has a pole of at most order 1 at 0 and Q of order 2. Let

$$zP(z) = \sum_{r=0}^{\infty} a_r z^r$$

and

$$z^2 Q(z) = \sum_{s=0}^{\infty} b_s z^s$$

be the Taylor series in D . Then we make the ansatz for a formal solution

$$y(p, z) = z^p \sum_{t=0}^{\infty} c_t z^t.$$

The coefficients can now be determined by putting $y(p, z)$ into (4.12). If

$$(4.13) \quad f(p) := p(p-1) + pa_0 + b_0$$

denotes the indicial equation, one can show that

$$(4.14) \quad c_t = \frac{\sum_{k=1}^t ((t-k+p)a_k + b_k) c_{t-k}}{f(p+t)}.$$

Let now r and s , $s < r$, be the roots of (4.13). Since $r + \mathbb{N}$ does not contain another root the succeeding coefficients can be calculated recursively starting with $c_0 = 1$. Thus we obtain a first solution to (4.12) $y(r, z) = z^r \sum_{t=0}^{\infty} c_t z^t$. If $r - s$ is not an integer we also find a second solution $y(s, z)$ which is linearly independent from the first. If on the other hand $r - s \in \mathbb{N}$, then one can construct another solution independent from $y(r, z)$ but having a logarithmic singularity at $z = 0$.

Going back to equation (4.6) we divide by $\alpha(H_1)^2s^2 + 2$ to find

$$(4.15) \quad \left(\frac{d^2}{ds^2} + P(s)\frac{d}{ds} + Q(s) \right) F(s) = 0,$$

where

$$P(s) := \frac{(\alpha(H_1)^2 + 2\alpha(H_\rho))s + 2\frac{l-2}{s}}{\alpha(H_1)^2s^2 + 2}$$

has a pole of order 1 at $s = 0$ and

$$Q(s) := \frac{\frac{2}{s^2} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi Z_M(s)} + \langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle}{\alpha(H_1)^2s^2 + 2}$$

has a pole of order 2, i.e. the method of Frobenius is applicable. We find that $a_0 = l - 2$, $b_0 = 4d$ in accordance with (4.11). We already know by Lemma 4.1.3 that $d \leq 0$ and we distinguish the two cases $d = 0$ and $d < 0$.

If $d < 0$, then $p_1 > 0$, while $p_2 < 0$. The first solution

$$x^{p_1} {}_2F_1(a + p_1, b + p_1, 1 + p_1 - p_2; x)$$

is always defined and smooth at the origin for $p_1 \in \mathbb{N}_0$, the second solution is only defined even for $x \neq 0$ if $p_2 - p_1 \neq -1, -2, -3, \dots$, as we have seen above. By the Frobenius method one can construct another solution which has a singularity at the origin, i.e. there is only one solution which is smooth at $x = 0$.

If $d = 0$, then $p_1 = 0$ and $p_2 = 1 - c$, i.e. $F(a, b, c, 0)$ reduces to $E(a, b, c)$. The two solutions simplify to

$$x^{p_1} {}_2F_1(a + p_1, b + p_1, 1 + p_1 - p_2; x) = {}_2F_1(a, b, c; x)$$

and

$$x^{p_2} {}_2F_1(a + p_2, b + p_2, 1 + p_2 - p_1; x) = x^{1-c} {}_2F_1(a + 1 - c, b + 1 - c, 2 - c; x).$$

In our case where $G = SO_o(1, l)$ this implies $c = \rho_0 = \frac{l-1}{2}$, i.e. $c = \frac{1}{2}, 1, \frac{3}{2}, \dots$. Thus, except for $c = \frac{1}{2}$ or $c = 1$ there is at most one solution which is smooth at the origin, namely ${}_2F_1(a, b, c; x)$.

Let us discuss the two cases $c = \frac{1}{2}$ and $c = 1$, i.e. the cases $G = SO_o(1, 2)$ and $G = SO_o(1, 3)$. We start with $c = \frac{1}{2}$. For $c = \frac{1}{2}$, $x^{1-c} = x^{1/2}$. Since we made the substitution $s^2 = -u$ to get from equation (4.6) to the hypergeometric type equation (4.8) there is a second smooth solution to (4.6) for $d = 0$ and $c = \frac{1}{2}$, i.e. $G = SO_o(1, 2)$. This solution is - up to constants -

$$is \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\frac{s^2}{4}\right),$$

which is an odd function.

For $c = 1$, i.e. $G = SO_o(1, 3)$, the solutions ${}_2F_1(a, b, c; x)$ and $x^{1-c} {}_2F_1(a + 1 - c, b + 1 - c, 2 - c; x)$ coincide. Then

$$\frac{d}{dc} \Big|_{c=1} x^{1-c} {}_2F_1(a + 1 - c, b + 1 - c, 2 - c; x)$$

is another solution of (4.8) having a singularity at $x = 0$, see [Yos97, p.64].

In any case there is for $d = 0$ - up to constants - only one even, everywhere smooth solution to (4.6).

We make the resubstitution $s^2 = u = -k_1w = -k_1x$, where $k_1 = \frac{2}{\alpha(H_1)^2} = 4(l-1)$. It follows that $x = -\frac{s^2}{4(l-1)}$ and $x^{p_1,2} = \left(\frac{-s^2}{4(l-1)}\right)^{p_1,2}$. Finally let us state the results of this chapter in summarized form:

THEOREM 4.3.1. For $l \geq 3$ the equation (4.6) has up to constants at most one solution which is smooth at the origin $s = 0$. This solution is up to a constant given by

$$\left(\frac{-s^2}{4(l-1)}\right)^p {}_2F_1\left(a+p, b+p, 1+2\sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d}; \frac{-s^2}{4(l-1)}\right).$$

Here $a = \frac{1}{2}(\rho_0 + ir)$, $b = \frac{1}{2}(\rho_0 - ir)$ depend only on the eigenvalue μ given by $\mu = \frac{1}{4}(\rho_0 + \frac{r^2}{\rho_0})$, while $d \leq 0$ depends only on the K -type $\pi \in \widehat{K}$ and the section S . Here

$$p = \frac{1-c}{2} + \sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d} \geq 0$$

solves the indicial equation (4.11). For the solution to be smooth at the origin it is necessary and sufficient that $2p \in \mathbb{N}_0$. This implies also $1 + 2\sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d} \in \mathbb{Q}$. Further, if $d = 0$, then $p = 0$.

For $l = 2$ the space of solution to (4.6) is 2-dimensional and spanned by ${}_2F_1\left(a, b, \frac{1}{2}; \frac{-s^2}{4}\right)$ and ${}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\frac{s^2}{4}\right)$.

REMARK 4.3.2. Let $S = \exp \mathbb{R}^+ X_1$ be a slice. If

$$F(\exp sX_1) = (-1)^p \left(\frac{s^2}{4(l-1)}\right)^p {}_2F_1\left(a+p, b+p, 1+2\sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d}; \frac{-s^2}{4(l-1)}\right),$$

then it follows that

$$F(\exp 2\sqrt{l-1}sX_1) = (-1)^p s^{2p} \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d}; -s^2\right).$$

In particular, if X_1 is proportional to X_{e_1} , see Section 2.3 for the definition of X_{e_1} , this implies $2\sqrt{l-1}X_1 = X_{e_1}$, see (2.7), and

$$F(\exp sX_{e_1}) = (-1)^p s^{2p} \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\left(\frac{1-\rho_0}{2}\right)^2-d}; -s^2\right).$$

The zeta function on $\{\operatorname{Re}(k) > 2\rho_0\}$

In this chapter we want to compute the trace of a certain convolution operator $\sigma \cdot \pi_R(f)$ in order to obtain the (logarithmic derivative of an) auxiliary zeta function $\mathcal{R}(\sigma) = \mathcal{R}(\cdot; \sigma)$ in a half plane of \mathbb{C} . We show that this operator is of trace class and has a kernel. Then we compute its trace by integrating the kernel over the diagonal. For $\sigma \equiv 1$ this procedure yields Selberg's trace formula which is used to derive the classical dynamical zeta function. We generalize this approach to nontrivial eigenfunctions $\sigma = \varphi$ of the Laplacian. From the auxiliary zeta function $\mathcal{R}(\varphi) = \mathcal{R}(\cdot; \varphi)$ we derive the zeta function $\mathcal{Z}(\varphi) = \mathcal{Z}(\cdot; \varphi)$ as a superposition of shifted $\mathcal{R}(\varphi)$. This idea was first used in [AZ07] for $G = SL_2(\mathbb{R})$.

The theory of this chapter is developed for real hyperbolic spaces X as discussed in Section 2.3. In particular, $X = G/K$ with $G = SO_o(1, l)$ and $K \cong SO(l)$, where $l \in \mathbb{N}$, $l \geq 2$. We identify X with AN by using the Iwasawa decomposition $G = ANK$, $A \cong \mathbb{R}$, $N \cong \mathbb{R}^{l-1}$ and $M = Z_K(A) \cong SO(l-1)$. Finally, Γ denotes a uniform lattice, that is, $\Gamma \subset G$ is a discrete, torsion-free and cocompact subgroup. Every nontrivial $\gamma \in \Gamma$ can be conjugate to some $a_\gamma m_\gamma \in A^+M$, see Proposition 2.2.1.

5.1. A generalized trace formula

We let $G = SO_o(1, l)$. Recall the identification of \mathbb{R} with A , $t \mapsto \exp tH_0$ and of \mathbb{R}^{l-1} with N via $u \mapsto \exp X_u$ from Section 2.3. Let da and dn be the left-invariant Haar measures on A and N obtained from this identification.¹ Further we fix a Haar measure dk on K by requiring K to have unit measure, then it follows from [Hel01, Ch.I Prop. 5.1] and [Hel01, Ch. I Cor. 5.3] that there is a Haar measure dg on G such that

$$\int_G f(g)dg = \int_{ANK} f(ank)dadndk$$

for all integrable functions f . We fix a Haar measures on X , $\Gamma \backslash G$ and on $X_\Gamma = \Gamma \backslash G/K$ such that for all integrable functions on G

$$\int_G f(g)dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma g)d(\Gamma g)$$

and

$$\int_G f(g)dg = \int_X \int_K f(gk)dkd(gK)$$

resp. for all integrable functions on $\Gamma \backslash G$

$$\int_{\Gamma \backslash G} f(\Gamma g)d(\Gamma g) = \int_{X_\Gamma} \int_K f(\Gamma gk)dkd(\Gamma gK).$$

Let $C_c^\infty(G//K)$ be the set of smooth functions with compact support that are bi- K -invariant. Denote the right-regular representation of G on $L^2(\Gamma \backslash G)$ by π_R

¹For the integral definition of Harish-Chandra's c -function one normally requires $\int_{\theta N} \exp(-2\rho(H(\theta n)))d\theta n = 1$, see [Hel01, Ch. IV Th. 6.14]. We discuss this in Remark 6.2.16.

and let $\varphi, \sigma \in C^\infty(\Gamma \backslash G)$ and $f \in C_c^\infty(G)$. The Fourier transform of f is defined as an operator on $L^2(\Gamma \backslash G)$ by

$$(5.1) \quad \pi_R(f)\varphi(x) := \int_G f(g) (\pi_R(g)\varphi)(x) dg = \int_G f(g)\varphi(xg) dg = (*\tilde{\varphi})(x^{-1}),$$

where $\tilde{\varphi}(x) := \varphi(x^{-1})$ and convolution is defined by $(\varphi * f)(x) := \int_G \varphi(h)f(h^{-1}x) dh$. Note that $\pi_R(f)\varphi$ is trivially square integrable as $\Gamma \backslash G$ is compact.

We now combine $\pi_R(f)$ with the multiplication operator on $L^2(X_\Gamma)$ which sends $f \mapsto \sigma \cdot f$. By the compact support of f and an application of Fubini's theorem

$$(5.2) \quad \begin{aligned} [\sigma \cdot \pi_R(f)]\varphi(\Gamma x) &= \sigma(\Gamma x) \int_G f(g)\varphi(\Gamma xg) dg \\ &= \sigma(\Gamma x) \int_G \varphi(\Gamma g)f(x^{-1}g) dg \\ &= \sigma(\Gamma x) \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \varphi(\Gamma \gamma g)f(x^{-1}\gamma g) d(\Gamma g) \\ &= \sigma(\Gamma x) \int_{\Gamma \backslash G} \varphi(\Gamma g) \sum_{\gamma \in \Gamma} f(x^{-1}\gamma g) d(\Gamma g). \end{aligned}$$

So, if we define $K_\sigma(x, g) := \sigma(x) \sum_{\gamma \in \Gamma} f(x^{-1}\gamma g)$, then $K_\sigma(\gamma x, g) = K_\sigma(x, \gamma g) = K_\sigma(x, g)$ defines a smooth function $K_\sigma : \Gamma \backslash G \times \Gamma \backslash G \rightarrow \mathbb{C}$ satisfying

$$[\sigma \cdot \pi_R(f)]\varphi(x) = \int_{\Gamma \backslash G} \varphi(\Gamma g) K_\sigma(x, \Gamma g) d(\Gamma g).$$

From [GW75] we adopt the following definition. We call $f \in C^\infty(G//K)$ *admissible*, if

- the operator $\pi_R(f)$ is of trace class on $L^2(\Gamma \backslash G)$ and
- the series $\sum_{\gamma \in \Gamma} f(x\gamma y^{-1})$ converges to a continuous function of (x, y) .

PROPOSITION 5.1.1. *For $\sigma \in C^\infty(\Gamma \backslash G/K)$ and $f \in C^\infty(G//K)$ admissible the operator $\sigma \cdot \pi_R(f)$ is an integral operator with kernel $K_\sigma(\cdot, \cdot)$ as from above. It maps $L^2(X_\Gamma)$ into itself and is a trace class operator on $L^2(X_\Gamma)$.*

PROOF. Standard theory implies that $\pi_R(f)$ is of trace class on $L^2(\Gamma \backslash G)$, see [Wal76, p.172], and hence also on $L^2(X_\Gamma)$ as soon as we check that $\pi_R(f)$ resp. $\sigma \cdot \pi_R(f)$ leave $L^2(X_\Gamma)$ invariant. Note that $\pi_R(f)$ being of trace class implies that $\sigma \cdot \pi_R(f)$ is of trace class as well, since σ is bounded. So we only have to check invariance. Since by assumption σ is right- K -invariant and f is also bi- K -invariant, we get for $k \in K$

$$\begin{aligned} [\sigma \cdot \pi_R(f)]\varphi(xk) &= \sigma(xk)[\pi_R(f)\varphi](xk) \\ &= \sigma(x) \int_G f(g)\varphi(xkg) dg \\ &= \sigma(x) \int_G f(k^{-1}g)\varphi(xg) dg \\ &= \sigma(x) \int_G f(g)\varphi(xg) dg = [\sigma \cdot \pi_R(f)]\varphi(x), \end{aligned}$$

where we applied the transformation $g \mapsto k^{-1}g$ and used the unimodularity of G . \square

From now on we assume that $\sigma \in C^\infty(\Gamma \backslash G/K)$ and $f \in C^\infty(G//K)$ is admissible. We can now compute the trace by

$$\begin{aligned}
\mathrm{Tr}(\sigma \cdot \pi_R(f)) &= \int_{X_\Gamma} \sigma(\Gamma y) \sum_{\gamma \in \Gamma} f(y^{-1}\gamma y) d(\Gamma y) \\
&= f(e) \int_{F_\Gamma} \sigma(x) dx + \sum_{1 \neq \gamma \in \Gamma} \int_{F_\Gamma} \sigma(x) f(x^{-1}\gamma x) dx,
\end{aligned}$$

where $F_\Gamma \subset X$ is some fundamental domain for Γ in X , i.e. F_Γ is measurable and up to a possible set of measure zero it contains exactly one element of every orbit Γx , $x \in X$. In particular, any fundamental domain F_Γ satisfies $\int_{\Gamma \backslash X} f(x) dx = \int_{F_\Gamma} f \circ \mathrm{pr}(g) dg$, $\mathrm{pr} : X \rightarrow \Gamma \backslash X$ canonical projection, for any integrable function $f : \Gamma \backslash X \rightarrow \mathbb{C}$.

We recall notations from Chapter 2.2, in particular that $C\Gamma$ denotes the set of conjugacy classes $[\gamma]$ in Γ and Γ_γ is the centralizer of $\gamma \in \Gamma$. Then we continue our computation with

$$\begin{aligned}
\sum_{1 \neq \gamma \in \Gamma} \int_{F_\Gamma} \sigma(x) f(x^{-1}\gamma x) dx &= \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\gamma' \in [\gamma]} \int_{F_\Gamma} \sigma(x) f(x^{-1}\gamma' x) dx \\
&= \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\Gamma_\gamma \gamma' \in \Gamma_\gamma \backslash \Gamma} \int_{F_\Gamma} \sigma(\gamma' x) f(x^{-1}\gamma'^{-1}\gamma\gamma' x) dx
\end{aligned}$$

In Proposition 2.2.2 it was mentioned that for any nontrivial $\gamma \in \Gamma$ its centralizer Γ_γ is cyclic and infinite as Γ is torsion-free. Furthermore, there is a unique primitive element $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^{n_\gamma}$ for some $n_\gamma \in \mathbb{N}$. It follows $\Gamma_\gamma = \langle \gamma_0 \rangle$ and

$$\sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\Gamma_\gamma \gamma' \in \Gamma_\gamma \backslash \Gamma} \int_{F_\Gamma} \sigma(\gamma' x) f(x^{-1}\gamma'^{-1}\gamma\gamma' x) dx = \sum_{1 \neq [\gamma] \in C\Gamma} \int_{F_{\gamma_0}} \sigma(x) f(x^{-1}\gamma x) dx,$$

where

$$F_{\gamma_0} := \bigsqcup_{\Gamma_\gamma \gamma' \in \Gamma_\gamma \backslash \Gamma} \gamma' F_\Gamma \subset X$$

is a fundamental domain for $\Gamma_\gamma = \langle \gamma_0 \rangle$. Here \bigsqcup means the disjoint union. Then $\langle \gamma \rangle \backslash \Gamma_\gamma = \langle \gamma \rangle \backslash \langle \gamma_0 \rangle \cong \mathbb{Z}/n_\gamma\mathbb{Z}$, i.e. $|\langle \gamma \rangle \backslash \Gamma_\gamma| = n_\gamma$ and

$$\int_{F_{\gamma_0}} \sigma(x) f(x^{-1}\gamma x) dx = \frac{1}{n_\gamma} \sum_{\langle \gamma \rangle \backslash h \in \langle \gamma \rangle \backslash \Gamma_\gamma} \int_{F_{\gamma_0}} \sigma(hx) f(x^{-1}h^{-1}\gamma hx) dx$$

as $f(x^{-1}h^{-1}\gamma hx) = f(x^{-1}\gamma x)$ for $h \in \Gamma_\gamma$. Applying now the transformation $x \mapsto hx$ and using the unimodularity of X , we get

$$\begin{aligned}
&\sum_{1 \neq [\gamma] \in C\Gamma} \frac{1}{n_\gamma} \sum_{\langle \gamma \rangle \backslash h \in \langle \gamma \rangle \backslash \Gamma_\gamma} \int_{F_{\gamma_0}} \sigma(hx) f(x^{-1}h^{-1}\gamma hx) dx \\
&= \sum_{1 \neq [\gamma] \in C\Gamma} \frac{1}{n_\gamma} \sum_{\langle \gamma \rangle \backslash h \in \langle \gamma \rangle \backslash \Gamma_\gamma} \int_{hF_{\gamma_0}} \sigma(x) f(x^{-1}\gamma x) dx
\end{aligned}$$

Now we change to a fundamental domain $F_\gamma \subset X$ for $\langle \gamma \rangle$ to obtain

$$(5.3) \quad \sum_{1 \neq [\gamma] \in C\Gamma} \frac{1}{n_\gamma} \sum_{\langle \gamma \rangle \backslash h \in \langle \gamma \rangle \backslash \Gamma_\gamma} \int_{hF_{\gamma_0}} \sigma(x) f(x^{-1}\gamma x) dx = \sum_{1 \neq [\gamma] \in C\Gamma} \frac{1}{n_\gamma} \int_{F_\gamma} \sigma(x) f(x^{-1}\gamma x) dx$$

But γ is conjugated to $m_\gamma a_\gamma \in MA^+$, i.e.

$$(5.4) \quad \alpha_\gamma \gamma \alpha_\gamma^{-1} = a_\gamma m_\gamma$$

for some $\alpha_\gamma \in G$, and a fundamental domain for $a_\gamma m_\gamma$ can be stated explicitly.

LEMMA 5.1.2. (See [BO94, Lemma 3.1])

Let $ma_t \in MA^+$, where $A = \exp \mathbb{R}H_0$, $a_t = \exp tH_0$, and identify X with AN . A fundamental domain for the cyclic subgroup generated by $a_t m$ is given by

$$F_{a_t m} := \{a_s n : n \in N, 0 \leq s \leq t\}.$$

PROOF. Note that $a_t m$ acts on $x = a_s n$ by $a_t m \cdot x = a_{t+s} m n m^{-1}$. Further,

$$\langle a_t m \rangle = \{a_{zt} m^z : z \in \mathbb{Z}\}.$$

We fix an arbitrary $a_h n \in X = AN$ with $h = c \cdot t + s$, where $c \in \mathbb{Z}$ and $0 \leq s \leq t$. That is,

$$\langle a_t m \rangle \cdot a_h n = \{a_{(z+c)t+s} m^z n m^{-z} : z \in \mathbb{Z}\},$$

in particular $a_s m^{-c} n m^c$ is contained in the orbit of $a_h n$ and $F_{a_t m}$.

Finally, $h \equiv s \pmod{\mathbb{Z}}$, if $a_h n$ is contained in the orbit of $a_s n'$. This shows that $F_{a_t m}$ contains at most one point from any orbit (up to a set of measure zero). \square

LEMMA 5.1.3. Let G be a group acting on a manifold M , $H \subset G$ a subgroup and F_H a fundamental domain for H . Then gF_H is a fundamental domain for gHg^{-1} for all $g \in G$.

PROOF. The proof is simple. Fix $g \in G$ and $m \in M$. If $y \in Hm \cap F_H$, then for some $h \in H$

$$gy = ghm = ghg^{-1}gx \in gHg^{-1}(gx).$$

Since left translation by g is a bijection on G , the claim follows. \square

Thus, a fundamental domain for the cyclic subgroup generated by $\gamma = \alpha_\gamma^{-1} a_\gamma m_\gamma \alpha_\gamma$, see (5.4), is given by

$$F_\gamma := \{\alpha_\gamma^{-1} x : x \in F_{a_\gamma m_\gamma}\} =: \alpha_\gamma^{-1} F_{a_\gamma m_\gamma}.$$

Next we rewrite the integral from the right side of (5.3) as an integral over a subset of AN . Note that the identification $X = AN$ implies $dx = dadn$, see [Hel01, Ch.I 5 Cor. 5.3]

$$\begin{aligned} \sum_{1 \neq [\gamma] \in \text{CT}} \frac{1}{n_\gamma} \int_{F_\gamma} \sigma(x) f(x^{-1} \gamma x) dx &= \sum_{1 \neq [\gamma] \in \text{CT}} \frac{1}{n_\gamma} \int_{A/\langle a_\gamma \rangle} \int_N \sigma(\alpha_\gamma^{-1} an) f(n^{-1} a^{-1} a_\gamma m_\gamma an) dn da \\ &= \sum_{1 \neq [\gamma] \in \text{CT}} \int_N \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} an) f(n^{-1} a^{-1} a_\gamma m_\gamma an) dadn \\ &= \sum_{1 \neq [\gamma] \in \text{CT}} \int_N \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} an) da f(n^{-1} a_\gamma m_\gamma n) dn \end{aligned}$$

In the last but one equation we used again Lemma 5.1.3 for a fundamental domain of $\langle a_\gamma m_\gamma \rangle$ and the fact $a_\gamma = a_{\gamma_0}^{n_\gamma}$, see the proof of Proposition 2.2.3. We call $I_\gamma(\sigma)$ defined by

$$(5.5) \quad I_\gamma(\sigma)(x) := \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} ax) da$$

for $x \in X = AN$ the weight function of the (weighted) orbital integral

$$(5.6) \quad \mathcal{O}_\gamma(f) := \int_N f(n^{-1} a_\gamma m_\gamma n) I_\gamma(\sigma)(n) dn.$$

Thus,

$$\begin{aligned} \sum_{1 \neq [\gamma] \in \text{CT}} \int_N \int_{A / \langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} a n) d a f(n^{-1} a_\gamma m_\gamma n) d n &= \sum_{1 \neq [\gamma] \in \text{CT}} \int_N f(n^{-1} a_\gamma m_\gamma n) I_\gamma(\sigma)(n) d n \\ &= \sum_{1 \neq [\gamma] \in \text{CT}} \mathcal{O}_\gamma(f). \end{aligned}$$

5.2. The weight $I_\gamma(\sigma)$

In this section we want to examine the weight $I_\gamma(\sigma)$ more closely. The next theorem will us show how we can decompose $I_\gamma(\sigma)$. We make use of the following fact which can be found in Helgason's book [Hel01].

THEOREM 5.2.1. ([Hel01, Ch. V Cor. 3.4])

Let X be a manifold with countable base and H a compact, connected Lie transformation group, then for any $f \in C^\infty(X)$

$$f = \sum_{\delta \in \widehat{H}} d_\delta \overline{\chi}_\delta * f = \sum_{\delta \in \widehat{H}} d_\delta \chi_\delta * f$$

with absolute convergence, where $(\phi * f)(x) := \int_H \phi(h) f(h^{-1}x) dh$. Here d_δ is the dimension of δ and $\chi_\delta := \text{Tr}(\delta)$ its character.

For the second equality we use that $\overline{\chi}_\delta = \chi_{\delta^*}$, if δ^* is the dual of δ . Absolute convergence of a series $\sum_{a \in A} v_a$, $\{v_a\}_{a \in A} \subset C^\infty(X)$ means that $\sum_{a \in A} \nu(v_a)$ is absolute convergent for every continuous seminorm ν on $C^\infty(X)$.

We apply this to M acting on $X = AN$ by conjugation to obtain a decomposition of the weight $I_\gamma(\sigma)$. A useful guiding for the theory presented in the remainder of this section is the theory of generalized spherical functions (Eisenstein integrals) for a rank one symmetric space which can be found in [Hel94, Ch.III §2 and §11]. Roughly speaking we replace K acting on \mathfrak{p} with section \mathfrak{a} by M acting on \mathfrak{n} with section \mathfrak{s} .

For a unitary representation $(\pi, V_\pi) \in \widehat{M}$ of dimension d_π and $f \in C^\infty(X)$ we define its π -projection by

$$(5.7) \quad f^\pi(x) := (f)^\pi(x) := d_\pi \int_M f(m \cdot x) \pi(m^{-1}) dm$$

for $x \in X = AN$, where $m \cdot x := mx$. We recall the function space for $(\pi, V_\pi) \in \widehat{M}$ $C^\infty(X \times_M (V_\pi)) := \{F \in C^\infty(X, \text{End}(V_\pi)) : F(m \cdot x) = \pi(m)F(x) \text{ for all } x \in X, m \in M\}$.

Then for $f \in C^\infty(X)$, $x \in X$ and $m' \in M$

$$\begin{aligned} f^\pi(m' \cdot x) &= d_\pi \int_M f(m \cdot (m' \cdot x)) \pi(m^{-1}) dm \\ &= d_\pi \int_M f(mm' \cdot x) \pi(m^{-1}) dm \\ (5.8) \quad &= d_\pi \int_M f(m \cdot x) \pi(m' m^{-1}) dm = \pi(m') f^\pi(x) \end{aligned}$$

by unimodularity of M . Hence, $f^\pi \in C^\infty(X \times_M V_\pi)$.

REMARK 5.2.2. We recall the definition of the action of M on G via left translation

$$m * g := mg.$$

In Section 4.1 we showed that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{m^*} & G \\ \text{pr} \downarrow & & \downarrow \text{pr} \\ X = G/K = AN & \xrightarrow{m} & X = G/K = AN \end{array}$$

We can also extend the definition of the π -projection to functions $f \in C^\infty(G)$ for $(\pi, V_\pi) \in \widehat{M}$, $\dim(V_\pi) = d_\pi$,

$$(5.9) \quad f^\pi(g) := d_\pi \int_M f(m * g) \pi(m^{-1}) dm.$$

It follows again that f^π is in

$C^\infty(G, \operatorname{End}_M(V_\pi)) := \{F \in C^\infty(G, \operatorname{End}(V_\pi)) : F(m * g) = \pi(m)F(g) \text{ for all } m \in M, g \in G\}$.

PROPOSITION 5.2.3. *For any $f \in C^\infty(X)$*

$$f = \sum_{\pi \in \widehat{M}} d_\pi (\chi_\pi * f)(x) = \sum_{\pi \in \widehat{M}} \operatorname{Tr}(f^\pi).$$

PROOF. By the preceding Theorem 5.2.1 we have for $x \in X$

$$\begin{aligned} f(x) &= \sum_{\pi \in \widehat{M}} d_\pi (\chi_\pi * f)(x) \\ &= \sum_{\pi \in \widehat{M}} d_\pi (\operatorname{Tr}(\pi) * f)(x) \\ &= \sum_{\pi \in \widehat{M}} d_\pi \int_M \chi_\pi(m) f(m^{-1}x) dm \\ &= \sum_{\pi \in \widehat{M}} d_\pi \int_M \operatorname{Tr}(\pi(m^{-1})) f(m \cdot x) dm \\ &= \sum_{\pi \in \widehat{M}} \operatorname{Tr} \left(d_\pi \int_M \pi(m^{-1}) f(m \cdot x) dm \right) = \sum_{\pi \in \widehat{M}} \operatorname{Tr}(f^\pi) \end{aligned}$$

by unimodularity of M and the linearity of the trace. \square

Let us summarize the preliminary result of this chapter.

THEOREM 5.2.4. *Let $G = SO_o(1, l) = ANK$, $M = Z_K(A)$ and $\Gamma \subset G$ a uniform lattice. Denote by π_R the right regular representation of G on $L^2(\Gamma \backslash G)$ and fix $\sigma \in C^\infty(X_\Gamma)$ and $f \in C^\infty(G/K)$ admissible. Then the operator $\sigma \cdot \pi_R(f)$, given by (5.2), maps $L^2(X_\Gamma)$ into itself and is of trace class. Its trace can be computed to*

$$\begin{aligned} \operatorname{Tr}(\sigma \cdot \pi_R(f)) &= f(e) \int_{F_\Gamma} \sigma(x) dx + \sum_{1 \neq [\gamma] \in C_\Gamma} \int_N f(n^{-1} a_\gamma m_\gamma n) \sum_{\pi \in \widehat{M}} \operatorname{Tr}(I_\gamma(\sigma)^\pi(n)) dn \\ &= f(e) \int_{F_\Gamma} \sigma(x) dx + \sum_{1 \neq [\gamma] \in C_\Gamma} \int_N f(n^{-1} a_\gamma m_\gamma n) \sum_{\pi \in \widehat{M}} d_\pi (\chi_\pi * I_\gamma(\sigma))(n) dn, \end{aligned}$$

where $1 \neq \gamma$ is conjugate to $a_\gamma m_\gamma$, $F_\Gamma \subset X$ a fundamental domain for Γ and $I_\gamma(\sigma)^\pi$ is defined by (5.7).

REMARK 5.2.5. Let us examine $I_\gamma(\sigma)^\pi$ more closely. We slightly generalize Lemma 4.1.2 in Chapter 4.

LEMMA 5.2.6. *If a $\exp sX_1 \in A \exp \mathbb{R}X$ and $F \in C(X, \times_M V_\pi)$, then $F(a \exp sX_1)$ maps V_π into $V_\pi^{Z_M(S)}$.*

PROOF. The same proof as before applies. We use $F(m \cdot x) = \pi(m)F(x)$ for any $m \in M$, $x \in X$. But then for $v \in V_\pi$

$$\begin{aligned} \pi(m)F(a \exp sX_1)v &= F(m \cdot a \exp sX_1)v = F(am \cdot \exp sX_1)v \\ &= F(a \exp sX_1)v \end{aligned}$$

for all $a \exp sX_1 \in A \exp \mathbb{R}X_1$, $m \in Z_M(S)$. \square

We recall that r_g, l_g denote right- resp. left-translation on G . Let $\gamma \neq 1$ be in Γ with primitive γ_0 . We assume that γ_0 is conjugated to $a_{\gamma_0}m_{\gamma_0} \in A^+M$, where $a_{\gamma_0} = \exp L_{\gamma_0}H_0$. This implies $L_{\gamma_0} = \left(\sqrt{2(l-1)}\right)^{-1} l_{\gamma_0}$, where l_{γ_0} is defined by $a_{\gamma_0} = \exp l_{\gamma_0}H_1$. By definition (5.7)

$$\begin{aligned} (\chi_\pi * I_\gamma(\sigma))(\exp sX_1) &= \int_M I_\gamma(\sigma)(m \cdot \exp sX_1) \chi_\pi(m^{-1}) dm \\ &= \int_M \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} m a \exp sX_1) da \chi_\pi(m^{-1}) dm \\ &= \int_M \int_0^{L_{\gamma_0}} \sigma(\alpha_\gamma^{-1} \exp tH_0 m \exp sX_1) da \chi_\pi(m^{-1}) dm \\ &= \sqrt{2(l-1)} \int_M \int_0^{l_{\gamma_0}} \sigma(\alpha_\gamma^{-1} \exp tH_1 m \exp sX_1) da \chi_\pi(m^{-1}) dm \\ &= \sqrt{2(l-1)} \int_M \int_0^{l_{\gamma_0}} \sigma \circ l_{\alpha_\gamma^{-1}} \circ r_{m \cdot \exp sX_1} da \chi_\pi(m^{-1}) dm \\ &= \sqrt{2(l-1)} \int_M \left(\int_{c_{\gamma_0}} \sigma \circ r_{m \cdot \exp sX_1} \right) \chi_\pi(m^{-1}) dm \\ (5.10) \quad &= \sqrt{2(l-1)} \left(\chi_\pi * \left(x \mapsto \int_{c_{\gamma_0}} \sigma \circ r_x \right) \right) (\exp sX_1) \end{aligned}$$

where we used the definition of the π -projection from (5.7) and for functions f on $\Gamma \backslash X$ the integral over the (prime) closed geodesic belonging to primitive γ_0

$$c_{\gamma_0} = \{\Gamma \alpha_{\gamma_0^{-1}} \exp(-tH_1)M : 0 \leq t \leq l_{\gamma_0}\},$$

see (2.5), is given by

$$\int_{c_{\gamma_0}} f := \int_0^{l_{\gamma_0}} f \circ l_{\alpha_\gamma^{-1}}(\exp tH_1) dt.$$

Finally, we remark that

$$\begin{aligned} \sqrt{2(l-1)} \int_{c_{\gamma_0}} \sigma &= \int_{A/\langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} x) dx \\ (5.11) \quad &= I_\gamma(\sigma)(e) \\ &= \int_M I_\gamma(\sigma)(m) dm, \end{aligned}$$

since for any nontrivial π

$$(\chi_\pi * I_\gamma(\sigma))(e) = 0,$$

where $e \in G$ is the neutral element, as $I_\gamma(\sigma)$ is right- K -invariant and $\int_M \chi_\pi(m) dm = 0$.

5.3. The case of $\sigma = \varphi$ an automorphic eigenfunction

Now we specify σ to be an automorphic eigenfunction of the Laplacian Δ_Γ on the compact manifold X_Γ . So let φ be some fixed Laplace eigenfunction, i.e.

$$\Delta_\Gamma \varphi = \Omega \varphi = \mu \varphi$$

for some eigenvalue μ . More specifically, there are $r \in \mathfrak{a}_\mathbb{C}^* \cong \mathbb{C}$ such that

$$\mu = -\frac{1}{4} \left(\rho_0 + \frac{r^2}{\rho_0} \right) = -\frac{1}{4\rho_0} (\rho_0^2 + r^2).$$

By the ellipticity of Δ_Γ we know that φ is smooth and we can apply the theory of the last section to $\sigma = \varphi$. Then we consider $I_\gamma(\varphi)$ defined in (5.5), so that

$$I_\gamma(\varphi)(x) = \int_{A/ < a_{\gamma_0} >} \varphi(\alpha_\gamma^{-1} a x) da.$$

Since Ω lies in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ it commutes with the right-translation r_g and the left-translation l_g

$$\Omega(\varphi \circ l_g) = (\Omega \varphi) \circ l_g = \mu \varphi \circ l_g$$

for all g in G , in particular for $g = \alpha_\gamma^{-1} a$, $a \in A$.

If we apply now Ω to $I_\gamma(\varphi)$ we can use, as $A/ < a_{\gamma_0} >$ is compact, the theorem of dominated convergence and exchange differentiation with integration to see that $I_\gamma(\varphi)$ is also a Ω -eigenfunction, i.e.

$$\Omega I_\gamma(\varphi) = \mu I_\gamma(\varphi).$$

Since also the action of M and Ω commute, even for any $(\pi, V_\pi) \in \widehat{M}$, by the same reasoning

$$(5.12) \quad \Omega I_\gamma(\varphi)^\pi = \mu I_\gamma(\varphi)^\pi,$$

where, according to (5.7)

$$(5.13) \quad I_\gamma(\varphi)^\pi(x) = d_\pi \int_M I_\gamma(\varphi)(m \cdot x) \pi(m^{-1}) dm.$$

Equations of the form (5.12) have been discussed in Chapter 4. We fix a slice $S \subset N$ for M acting on N and assume that $S = \exp \mathbb{R}^+ X_1$, where $X_1 \in \mathfrak{n}$ is of unit length. We recall that if F is an element of $C^\infty(X \times_M V_\pi)$, then $F|_S$ maps V_π onto $V_\pi^{Z_M(S)}$, see Lemma 4.1.2, where $V_\pi^{Z_M(S)} = \mathbb{C} \cdot v$ for some $v \in V_\pi$.

In Section 4.1 we have shown that the equation

$$\Omega F - \mu F = 0$$

leads to an equation on S

$$(5.14) \quad \left((\alpha(H_1)^2 t^2 + 2) \frac{d^2}{dt^2} + \left((\alpha(H_1)^2 + 2\alpha(H_\rho)) t + 2 \frac{l-2}{t} \right) \frac{d}{dt} + \frac{2}{t^2} \sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}} - \mu \right) F(t) = 0,$$

for the scalar valued function $F(t)$ defined by $F(t)v := F(\exp t X_1)v$, see Theorem 4.1.7. Here α is the unique positive root, $H_1, H_\rho \in \mathfrak{a}^+$ the unique unit vector resp. the defining vector for ρ , $\{Z_{Y_j}\}_j$ the orthogonal set of $\mathfrak{z}_\mathfrak{m}(\mathfrak{s})^{\perp \mathfrak{m}}$ which corresponds to the orthonormal basis $\{Y_j\}_j$ of $\{X_1\}^{\perp \mathfrak{n}}$ via $[Z_{Y_j}, X_1] = Y_j$. By Lemma 4.1.3 we know that $\sum_j \pi(Z_{Y_j}^2)|_{V_\pi^{Z_M(S)}} \in \operatorname{End}(V_\pi^{Z_M(S)})$ can be identified with a scalar ≤ 0 .

We note that $F(t) = d_\pi(\chi_\pi * f)(\exp tX_1)$, if $F = f^\pi$ for some $f \in C^\infty(X)$, see Theorem 4.1.7.

The results of Theorem 4.3.1 now imply that there are constants a, b, d, p , where $a = \rho_0 + ir$ and $b = \rho_0 - ir$ only depend on the eigenvalue, $2p \in \mathbb{N}_0$ and $d \leq 0$ depend only on π and S , such that for some other constant C and for $l \geq 3$ any solution F to (5.14), which is smooth at $t = 0$, can be expressed as

(5.15)

$$\tilde{F}(t) := F(2\sqrt{l-1}t) = (-1)^p C \cdot t^{2p} \cdot {}_2F_1\left(a+l, b+l, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right)$$

for $t > 0$, see Remark 4.3.2. By the Leibniz rule

$$\begin{aligned} \tilde{F}^{(2p)}(t) &= (-1)^p C \left(t^{2p} \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right) \right)^{(2p)} \\ &= (-1)^p C \sum_{i=0}^{2p} \binom{2p}{i} \frac{1}{i!} t^{2p-i} {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right)^{(2p-i)} \\ &= (-1)^p C \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right) \\ &\quad + (-1)^p C \sum_{i=0}^{2p-1} \binom{2p}{i} \frac{1}{i!} t^{2p-i} {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right)^{(2p-i)} \\ &= (-1)^p C \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right) \\ &\quad + (-1)^p C \cdot t \sum_{i=0}^{2p-1} \binom{2p}{i} \frac{1}{i!} t^{2p-i-1} {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -t^2\right)^{(2p-i)} \end{aligned}$$

Hence,

$$\tilde{F}^{(2p)}(0) = (4(l-1))^p F(0)^{(2p)} = (-1)^p C \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; 0\right) = (-1)^p C$$

because ${}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; 0\right) = 1$ by the definition of hypergeometric functions.

For $l = 2$ the space of solutions to (5.14) is spanned by ${}_2F_1\left(a, b, \frac{1}{2}; -\frac{t^2}{4}\right)$ and $it \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\frac{t^2}{4}\right)$, where $a = \rho_0 + ir$ and $b = \rho_0 - ir$ as before. The general solution in this case has thus the form

(5.16)

$$F(2t) := F(\exp 2tX_1) = F(0) \cdot {}_2F_1\left(a, b, \frac{1}{2}; -t^2\right) + 2F'(0)it \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -t^2\right).$$

Now we consider $F := I_\gamma(\varphi)^\pi : \mathbb{R}^+ \rightarrow \mathbb{C}$ defined by $I_\gamma(\varphi)^\pi(s)v = I_\gamma(\varphi)^\pi(\exp sX_1)v$, then also

$$I_\gamma(\varphi)^\pi(s) = d_\pi(\chi_\pi * I_\gamma(\varphi))(\exp sX_1),$$

see Theorem 4.1.7. Under the map $\exp tX_1 \mapsto t$, X_1 is identified with $\frac{d}{dt}$, thus

(5.17)

$$(I_\gamma(\varphi)^\pi)^{(2p)}(0) = d_\pi \frac{d^{2p}}{dt^{2p}}|_{t=0} (\chi_\pi * I_\gamma(\varphi))(\exp tX_1) = d_\pi \left(X_1^{2p} \chi_\pi * I_\gamma(\varphi) \right)(e),$$

where $e \in N$ is the neutral element. For $I_\gamma(\varphi)^\pi : \mathbb{R}^+ \rightarrow \mathbb{C}$, we get from (5.15) for $l \geq 3$

$$\begin{aligned}
d_\pi(\chi_\pi * I_\gamma(\varphi))(\exp 2\sqrt{l-1}sX_1) &= I_\gamma(\varphi)^\pi(2\sqrt{l-1}s) \\
&\stackrel{(5.15)}{=} (-1)^p \widetilde{I_\gamma(\varphi)}^{(2p)}(0) \cdot \\
&\quad \cdot s^{2p} \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right) \\
&= (-1)^p (4(l-1))^p (I_\gamma(\varphi)^\pi)^{(2p)}(0) \cdot \\
&\quad \cdot s^{2p} \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right) \\
&\stackrel{(5.17)}{=} (-1)^p (4(l-1))^p d_\pi \frac{d^{2p}}{dt^{2p}} \Big|_{t=0} (\chi_\pi * I_\gamma(\varphi))(\exp tX_1) s^{2p} \\
&\quad \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right) \\
(5.18) \quad &= (-1)^p (4(l-1))^p d_\pi \left(X_1^{2p} \chi_\pi * I_\gamma(\varphi)\right)(e) \cdot s^{2p} \\
&\quad \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right).
\end{aligned}$$

For $l = 2$ we see from (5.16) that the weight of the orbital integral is given by

$$\begin{aligned}
I_\gamma(\varphi)(2s) &= I_\gamma(\varphi)(\exp 2sX_1) \\
(5.19) \quad &= \left(\int_{c_{\gamma_0}} \varphi\right) \cdot {}_2F_1\left(a, b, \frac{1}{2}; -s^2\right) \\
&\quad + 2i \left(\int_{c_{\gamma_0}} X_1 \varphi\right) \cdot s \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -s^2\right).
\end{aligned}$$

From Lemma 4.1.6, Theorem 4.3.1 and Proposition 5.2.3 we get:

THEOREM 5.3.1. *Let $X = G/K$ be a real hyperbolic space of dimension l , $\Gamma \subset G$ a uniform lattice, $1 \neq \gamma \in \Gamma$ and φ be an automorphic eigenfunction (on X_Γ) with eigenvalue μ . The weight $I_\gamma(\varphi)$ can be decomposed as*

$$I_\gamma(\varphi) = \sum_{\pi \in \widehat{M}} d_\pi(\chi_\pi * I_\gamma(\varphi)).$$

Here $d_\pi(\chi_\pi * I_\gamma(\varphi)) = \operatorname{Tr}(I_\gamma(\varphi)^\pi) \in C^\infty(X)_\pi$ is a Casimir eigenfunction. On each slice $S = \exp \mathbb{R}^+ X_1$, $|X_1| = 1$, for M acting on N , $d_\pi(\chi_\pi * I_\gamma(\varphi))$ satisfies equation (5.14) and is given by (5.18) for $l \geq 3$ resp. (5.19) for $l = 2$.

REMARK 5.3.2. We digress to examine $X_1^{2p} I_\gamma^\pi(\varphi)$ more closely. Using the definition of $I_\gamma(\varphi)^\pi$, see (5.13), and the computations made at the end of the last

section, see Remark 5.2.5, as $\exp tX_1 \in S$ for all $t > 0$,

$$\begin{aligned}
X_1^{2p}(\chi_\pi * I_\gamma(\varphi))(e) &= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_M I_\gamma(\varphi)(m \cdot \exp tX_1) \chi_\pi(m^{-1}) dm \\
&= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_M \int_{A/\langle a_{\gamma_0} \rangle} \varphi(\alpha_\gamma^{-1} am \exp tX_1) da \chi_\pi(m^{-1}) dm \\
&\stackrel{(5.10)}{=} \sqrt{2(l-1)} \frac{d^{2p}}{dt^{2p}}|_{t=0} \left(\chi_\pi * \left(x \mapsto \int_{c_{\gamma_0}} \varphi \circ r_x \right) \right) (\exp tX_1).
\end{aligned}$$

On the other hand,

$$(\chi_\pi * (\varphi \circ l_{\alpha_\gamma^{-1}}))(x) = d_\pi \int_M \varphi \circ l_{\alpha_\gamma^{-1}}(m \cdot x) \chi_\pi(m^{-1}) dm$$

and

$$\begin{aligned}
X_1^{2p}(\chi_\pi * I_\gamma(\varphi))(e) &= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_M I_\gamma(\varphi)(m \cdot \exp tX_1) \chi_\pi(m^{-1}) dm \\
&= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_M \int_{A/\langle a_{\gamma_0} \rangle} \varphi(\alpha_\gamma^{-1} am \exp tX_1) da \chi_\pi(m^{-1}) dm \\
&= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_M \int_{A/\langle a_{\gamma_0} \rangle} \varphi \circ l_{\alpha_\gamma^{-1}}(am \exp tX_1) da \chi_\pi(m^{-1}) dm \\
&= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_{A/\langle a_{\gamma_0} \rangle} \int_M \varphi \circ l_{\alpha_\gamma^{-1}}(m \cdot a \exp tX_1) \chi_\pi(m^{-1}) dmda \\
&= \frac{d^{2p}}{dt^{2p}}|_{t=0} \int_{A/\langle a_{\gamma_0} \rangle} \left(\chi_\pi * (\varphi \circ l_{\alpha_\gamma^{-1}}) \right) (a \exp tX_1) da \\
&= \int_{A/\langle a_{\gamma_0} \rangle} \frac{d^{2p}}{dt^{2p}}|_{t=0} \left(\chi_\pi * (\varphi \circ l_{\alpha_\gamma^{-1}}) \right) (a \exp tX_1) da \\
(5.20) \quad &= \int_{A/\langle a_{\gamma_0} \rangle} \left(X_1^{2p}(\chi_\pi * (\varphi \circ l_{\alpha_\gamma^{-1}})) \right) (a) da
\end{aligned}$$

For $\pi = \mathbf{1}$ trivial, (5.20) simplifies to

$$\begin{aligned}
&\int_{A/\langle a_{\gamma_0} \rangle} \left(X_{e_1}^{2p} \left(x \mapsto \int_M \varphi \circ l_{\alpha_\gamma^{-1}}(m \cdot x) dm \right) \right) (a) da \\
&= \int_{A/\langle a_{\gamma_0} \rangle} \left(X_{e_1}^{2p} \left(\left[x \mapsto \int_M \varphi(m \cdot x) dm \right] \circ l_{\alpha_\gamma^{-1}} \right) \right) (a) da \\
&= \int_{A/\langle a_{\gamma_0} \rangle} \left(\left(X_{e_1}^{2p} \left[x \mapsto \int_M \varphi(m \cdot x) dm \right] \right) \circ l_{\alpha_\gamma^{-1}} \right) (a) da \\
&= \sqrt{2(l-1)} \int_{c_{\gamma_0}} \left(X_{e_1}^{2p} \left[x \mapsto \int_M \varphi(m \cdot x) dm \right] \right).
\end{aligned}$$

5.4. An auxiliary zeta function $\mathcal{R}(\varphi)$

We choose now f from a special family of bi- K -invariant functions f_k . We consider on G the function

$$(5.21) \quad f_k : g \mapsto (1 - |g \cdot 0|^2)^{k/2}$$

for $k \in \mathbb{C}$, where we recall the action of G on $B_1(\mathbb{R}^n)$ given by (2.11) in Chapter 2.3. While this function is bi- K -invariant on G , see Lemma 2.3.1, in particular it is a function on $X = G/K$, it does not have compact support and we have to show that $\pi_R(f_k)$ is indeed of trace class for suitable $k \in \mathbb{C}$.

Recall that we fixed $H_0, H_1 \in \mathfrak{a}^+$ where $\alpha(H_0) = 1$ and $|H_1| = 1$. It follows that $|H_0|^2 = 2(l-1)$, see [GV88, (4.2.10)] and thus $H_0 = \sqrt{2(l-1)}H_1$. Let now $\gamma \in \Gamma$ be conjugated to $a_\gamma m_\gamma \in MA^+$. By definition of l_γ , $a_\gamma = \exp l_\gamma H_1 = \exp \frac{l_\gamma}{\sqrt{2(l-1)}} H_0$. Let us set

$$(5.22) \quad L_\gamma := \frac{l_\gamma}{\sqrt{2(l-1)}}.$$

Then by definition $a_\gamma = \exp L_\gamma H_0$ and we call L_γ the length of the closed geodesic $[\gamma]$. Further, we call the set

$$\{L_\gamma : 1 \neq [\gamma] \in C\Gamma\}$$

the length spectrum of Γ . It follows from Proposition 2.2.4 that $\{L_\gamma : 1 \neq [\gamma] \in C\Gamma\}$ has an infimum $L_{\inf} > 0$.

REMARK 5.4.1. Let $[\gamma] \in C\Gamma$. The definition of the length of $[\gamma]$ from (5.22) differs from l_γ in [Gan77a], see also Section 2.2, by the factor $\left(\sqrt{2(l-1)}\right)^{-1}$. That is $l_\gamma = \sqrt{2(l-1)}L_\gamma$. If $G = SO_o(1, 2)$, then our definition of L_γ agrees with the one which is used in [AZ07]. In this case $\sqrt{2}L_\gamma = l_\gamma$.

The computations made at the end of Chapter 2.3, especially equation (2.16ff), give

$$(5.23) \quad f_k \left((\exp -sX_{e_1}) a_\gamma m_\gamma^{m^{-1}} (\exp sX_{e_1}) \right) = \left(-(m_\gamma^{m^{-1}})_{1,1} s^2 + (1 + s^2) \cosh L_\gamma \right)^{-k},$$

where $a_\gamma = \exp l_\gamma H_1 = \exp L_\gamma H_0 = a_{L_\gamma}$ and

$$(5.24) \quad (m_\gamma^{m^{-1}})_{1,1} := (m^{-1} m_\gamma m)_{1,1}$$

is the first entry in the first row of $m_\gamma^{m^{-1}}$ as we identified $m^{-1} m_\gamma m$ with the $(l-1) \times (l-1)$ -matrix $(m^{-1} m_\gamma m)_{i,j=1}^{l-1}$ in $SO(l-1)$. In other words, $(m_\gamma^{m^{-1}})_{1,1}$ is just the matrix coefficient belonging to the defining representation of M and the first basis vector evaluated at $m_\gamma^{m^{-1}}$. Also, see equation (2.22),

$$(5.25) \quad f_k(\exp tH_0 \exp X_u) = f_k(a_t \exp X_u) = \left(\cosh t + \frac{|u|^2}{2} e^t \right)^{-k}.$$

for X_u in \mathfrak{n} , i.e. $u \in \mathbb{R}^{l-1}$.

In [Gan77b] on page 8 we find the following useful remark for determining whether a function $f \in C^\infty(G//K)$ is admissible.

REMARK 5.4.2. For a function $f \in C^\infty(G//K)$ we consider its Abel transform. Here, we use the formula for the Abel transform given in [Wil91, (9.37)]. We define

$$F_f(a_t) := F_f(t) := e^{\rho_0 t} \int_N f(a_t n) dn,$$

whenever this integral is finite.

The transform F_f , if defined, is a smooth function on A resp. \mathbb{R} . As a function on \mathbb{R} it is even by the bi- K -invariance which translates to a Weyl group invariance on A . If F_f also satisfies for some $\varepsilon > 0$

$$(5.26) \quad \sup_{t \in \mathbb{R}} (\exp(\rho_0 + \varepsilon)|t|) |F_f(t)| < \infty,$$

then $\pi_R(f)$ is admissible.

PROPOSITION 5.4.3. *Let $G = SO_o(1, l)$, π_R the right-regular representation of G on $\Gamma \backslash G$, where Γ is a uniform lattice, and let f_k as defined in (5.21). The function f_k is admissible for $\operatorname{Re}(k) > 2\rho_0$. In particular, the operator $\pi_R(f_k)$ is of trace class for $\operatorname{Re}(k) > 2\rho_0$.*

PROOF. We make use of the criterion from [Gan77b] recalled in Remark 5.4.2. Thus we have to compute the Abel transform F_{f_k} and check (5.26).

$$\begin{aligned}
F_{f_k}(t) &= e^{\rho_0 t} \int_N f_k(a_t n) dn \\
&\stackrel{(5.25)}{=} e^{\rho_0 t} \int_{\mathbb{R}^{l-1}} \left(\cosh t + \frac{|u|^2}{2} e^t \right)^{-k} du \\
&= e^{\rho_0 t} \cosh^{-k} t \int_{\mathbb{R}^{l-1}} \left(1 + \frac{|u|^2 e^t}{2 \cosh t} \right)^{-k} du \\
&= 2^{\rho_0} \cosh^{-k+\rho_0} t \int_{\mathbb{R}^{l-1}} (1 + |u|^2)^{-k} du,
\end{aligned}$$

where we applied for the last equation the coordinate transform $u \mapsto e^{-t/2} \sqrt{2} \cosh^{1/2} t \cdot u$. Thus $e^{\rho_0 |t|} |F_{f_k}(t)|$ behaves as $e^{(2\rho_0 - \operatorname{Re}(k))|t|}$ for large $|t|$. \square

Let us then assume that the automorphic eigenfunction φ on X_Γ with eigenvalue $\mu = -\frac{1}{4}(\rho_0 + \frac{r^2}{\rho_0})$, $r \in \mathfrak{a}_\mathbb{C}^* \cong \mathbb{C}$, is not trivial, so $\int_{\Gamma \backslash G} \varphi(x) dx = 0$. Then the first summand in the formula for $\varphi \cdot \pi_R(f_k)$ in Theorem 5.2.4 vanishes and the trace computes for $l \geq 3$ by (5.23) to

$$\begin{aligned}
\operatorname{Tr}(\varphi \cdot \pi_R(f_k)) &= \sum_{1 \neq [\gamma] \in C\Gamma} \int_N f_k(n^{-1} a_\gamma m_\gamma n) \sum_{\pi \in \widehat{M}} d_\pi (\chi_\pi * I_\gamma(\varphi))(n) dn \\
&= \omega_{l-1} \sum_{1 \neq [\gamma] \in \Gamma} \sum_{\pi \in \widehat{M}} d_\pi \int_M \int_0^\infty s^{l-2} f_k(\exp -s X_{e_1} m^{-1} m_\gamma m \exp s X_1) \\
&\quad \cdot (\chi_\pi * I_\gamma(\varphi))(m \cdot \exp s X_{e_1}) ds dm,
\end{aligned}$$

where we used polar coordinates to obtain the second equality. We recall that if we identify $N \cong \mathbb{R}^{l-1}$, $u \mapsto X_u$, see (2.8), then

$$\begin{aligned}
\int_N f(n) dn &= \int_{\mathbb{R}^{l-1}} f(\exp X_u) du \\
&= \int_0^\infty \int_{\partial B_s(0)} f dS ds \\
&= \omega_{l-1} \int_0^\infty s^{l-2} f(\exp s X_{e_1}) ds
\end{aligned}$$

for radial functions f , i.e. $f(n) = \int_M f(m \cdot n) dm$ for all $n \in N$. Here $B_1(0) = \{x \in \mathbb{R}^{l-1} : |x| \leq 1\}$ and $\omega_{l-1} := |\partial B_1(0)|$ with respect to the Lebesgue measure in \mathbb{R}^{l-2} for $l \geq 3$. For $l = 2$ we set $\omega_1 := 2$.

Now we plug in equation (5.23) to get

$$\begin{aligned}
& \operatorname{Tr}(\varphi \cdot \pi_R(f_k)) \\
&= \omega_{l-1} \sum_{1 \neq [\gamma] \in \Gamma} \sum_{\pi \in \widehat{M}} d_\pi \int_M \int_0^\infty s^{l-2} f_k(\exp -sX_{e_1} m^{-1} m_\gamma m \exp sX_1) \\
&\quad \cdot (\chi_\pi * I_\gamma(\varphi))(m \cdot \exp sX_{e_1}) ds dm \\
&\stackrel{(5.23)}{=} \omega_{l-1} \sum_{1 \neq [\gamma] \in \Gamma} \sum_{\pi \in \widehat{M}} \int_M \int_0^\infty s^{l-2} \left(-(m_\gamma^{m^{-1}})_{1,1} s^2 + (1+s^2) \cosh L_\gamma \right)^{-k} \\
&\quad \cdot (\chi_\pi * I_\gamma(\varphi))(m \cdot \exp sX_{e_1}) ds dm \\
&= \omega_{l-1} \sum_{1 \neq [\gamma] \in \Gamma} \sum_{\pi \in \widehat{M}} d_\pi \cosh^{-k} L_\gamma \int_M \int_0^\infty s^{l-2} \left(s^2 \left(\frac{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}{\cosh L_\gamma} \right) + 1 \right)^{-k} \\
&\quad \cdot (\chi_\pi * I_\gamma(\varphi))(m \cdot \exp sX_{e_1}) ds dm \\
&= \omega_{l-1} \sum_{1 \neq [\gamma] \in \Gamma} \cosh^{-k + \frac{l-1}{2}} L_\gamma \sum_{\pi \in \widehat{M}} d_\pi \int_M \frac{1}{(\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{\frac{l-1}{2}}} \\
&\quad \cdot \int_0^\infty s^{l-2} (s^2 + 1)^{-k} (\chi_\pi * I_\gamma(\varphi)) \left(m \cdot \exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}} sX_{e_1} \right) ds dm,
\end{aligned}$$

where we applied the transformation $s \mapsto \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}} s$ to obtain the last equality.

We define for the automorphic eigenfunction $1 \neq \varphi$, $\gamma \neq 1$ and $\pi \in \widehat{M}$

$$\begin{aligned}
c(\varphi, \gamma, \pi, k) &:= \omega_{l-1} d_\pi \int_M \frac{1}{(\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{\frac{l-1}{2}}} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} \\
(5.27) \quad &\quad \cdot (\chi_\pi * I_\gamma(\varphi)) \left(m \cdot \exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}} sX_{e_1} \right) ds dm,
\end{aligned}$$

$L_\gamma = \sqrt{2(l-1)}^{-1} l_\gamma > 0$. Thus we can write for $\operatorname{Re}(k) > 2\rho_0$ the trace of $\varphi \cdot \pi_R(f_k)$ for any nontrivial automorphic eigenfunction φ as

$$(5.28) \quad \operatorname{Tr}(\varphi \cdot \pi_R(f_k)) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) (\cosh L_\gamma)^{-k + \rho_0}.$$

For this special choice of f_k we call this trace (5.28) the *auxiliary zeta function* for $k \in \mathbb{C}$, $\operatorname{Re}(k) > 2\rho_0$,

$$(5.29) \quad \mathcal{R}(k; \varphi) := \operatorname{Tr}(\varphi \cdot \pi_R(f_k)) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) (\cosh L_\gamma)^{-k + \rho_0},$$

where $\rho_0 = \rho(H_0) = \frac{l-1}{2}$.

We will often omit the argument of the (auxiliary) zeta function and write $\mathcal{R}(\varphi)$ (resp. $\mathcal{Z}(\varphi)$, see the next section) instead of $\mathcal{R}(k; \varphi)$ (resp. $\mathcal{Z}(k; \varphi)$).

REMARK 5.4.4. In (2.7) we have shown that $|X_{e_1}|^2 = 4(l-1)$, i.e. $\frac{1}{2\sqrt{l-1}} X_{e_1}$ has unit length. On the slice

$$S_m := m \cdot \exp \mathbb{R}^+ X_{e_1} = \exp \mathbb{R}^+ m \cdot X_{e_1} = \exp \mathbb{R}^+ m \cdot \frac{1}{2\sqrt{l-1}} X_{e_1}$$

the function $d_\pi(\chi_\pi * I_\gamma(\varphi))$ satisfies for $l \geq 3$, see (5.18),

$$\begin{aligned} d_\pi(\chi_\pi * I_\gamma(\varphi))(\exp s(m \cdot X_{e_1})) &= d_\pi(\chi_\pi * I_\gamma(\varphi))\left(\exp 2\sqrt{l-1}s\left(\frac{m \cdot X_{e_1}}{2\sqrt{l-1}}\right)\right) \\ &= (-1)^p (4(l-1))^p d_\pi\left(\left(\frac{m \cdot X_{e_1}}{2\sqrt{l-1}}\right)^{2p} \chi_\pi * I_\gamma(\varphi)\right)(e) \cdot s^{2p} \\ &\quad \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right) \\ &= (-1)^p d_\pi\left((m \cdot X_{e_1})^{2p} \chi_\pi * I_\gamma(\varphi)\right)(e) \cdot s^{2p} \\ &\quad \cdot {}_2F_1\left(a+p, b+p, 1+2\sqrt{\frac{(1-\rho_0)^2}{4}-d}; -s^2\right) \end{aligned}$$

for some constants a, b, d, p depending on the eigenvalue μ, π and the slice S_m . For $l = 2$, see (5.19)

$$\begin{aligned} I_\gamma(\varphi)(\exp sX_{e_1}) &= I_\gamma(\varphi)\left(\exp 2s\frac{X_{e_1}}{2}\right) \\ &= \left(\int_{c_{\gamma_0}} \varphi\right) \cdot {}_2F_1\left(a, b, \frac{1}{2}; -s^2\right) \\ &\quad + 2i \left(\int_{c_{\gamma_0}} X_1 \varphi\right) \cdot s \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -s^2\right). \end{aligned}$$

REMARK 5.4.5. For the case of $G = SO_o(1, 2)$ we note that $N = \{\exp uX_{e_1} : u \in \mathbb{R}\}$. It follows thus from (5.6), as M is trivial,

$$\begin{aligned} \mathcal{O}_\gamma(f_k) &= \int_N f_k(n^{-1}a_\gamma n) I_\gamma(\varphi_j)(n) dn \\ &= \int_{-\infty}^{\infty} f_k(\exp(-uX_{e_1})a_\gamma \exp(uX_{e_1})) I_\gamma(\varphi_j)(\exp uX) du \\ &= \left(\int_{c_{\gamma_0}} \varphi_j\right) \int_{-\infty}^{\infty} f_k(\exp(-uX_{e_1})a_\gamma \exp(uX_{e_1})) {}_2F_1\left(a, b, \frac{1}{2}; -u^2\right) du \\ &\quad + 2i \left(\int_{c_{\gamma_0}} X_{e_1} \varphi_j\right) \int_{-\infty}^{\infty} f_k(\exp -uX a_\gamma \exp uX) \cdot u \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -u^2\right) du. \end{aligned}$$

The restriction $f|_N$ for any $f \in C(G//K)$ is even, see Corollary 3.4.2. Hence

$$\int_{-\infty}^{\infty} u f(\exp(-uX_{e_1})a_\gamma \exp(uX_{e_1})) \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\frac{u^2}{4}\right) du = 0$$

and

$$\mathcal{O}_\gamma(f_k) = \left(\int_{c_{\gamma_0}} \varphi_j\right) \int_{-\infty}^{\infty} (u^2 + 1)^{-k} {}_2F_1\left(a, b, \frac{1}{2}; -u^2 \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - 1}}\right) du.$$

We will now come to the theorem which shows that the auxiliary zeta function is defined at least for all $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$.

THEOREM 5.4.6. *Let $X = G/K$ be a real hyperbolic space and $\Gamma \subset G$ a uniform lattice. Let φ be a non trivial automorphic eigenfunction and $k \in \mathbb{C}$. The auxiliary zeta function $\mathcal{R}(k; \varphi)$ converges for $\operatorname{Re}(k) > 2\rho_0$.*

PROOF. From Proposition 5.4.3 it follows that $\varphi \cdot \pi_R(f_k)$ is of trace class for $\operatorname{Re}(k) > 2\rho_0$, because if $\pi_R(f_k)$ of trace class then also $\varphi \cdot \pi_R(f)$ as φ is bounded. But then also $\operatorname{Tr}(\varphi \cdot \pi_R(f_k))$ converges for $\operatorname{Re}(k) > 2\rho_0$.

5.5. From $\mathcal{R}(\varphi)$ to $\mathcal{Z}(\varphi)$

From the auxiliary zeta function $\mathcal{R}(\varphi)$ we can also derive the zeta function $\mathcal{Z}(\varphi)$ via the following lemma, which can be found in [AZ07, Lemma 9.3].

LEMMA 5.5.1. *Let $k \in \mathbb{C}$ and $y \in (1, \infty)$ then there exist coefficients $\beta(k; m)$, which tend to zero for $m \rightarrow \infty$, such that*

$$\left(1 - \sqrt{1 - \frac{1}{y}}\right)^k = y^{-k} \sum_{m=0}^{\infty} \beta(k; m) y^{-m}.$$

Further, $\beta(k; 0) = 2^{-k}$ and $k \mapsto \beta(k; m)$ is holomorphic for any $m \in \mathbb{N}_0$.

PROOF. Let $y \in (1, \infty)$. For $k \in \mathbb{C}$ we find by the binomial theorem

$$\begin{aligned} \left(1 - \sqrt{1 - y^{-1}}\right)^k &= \left(1 - \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-1)^m y^{-m}\right)^k \\ &= \left(-\sum_{m=1}^{\infty} \binom{\frac{1}{2}}{m} (-1)^m y^{-m}\right)^k \\ &= \left(\sum_{m=1}^{\infty} \binom{\frac{1}{2}}{m} (-1)^{m-1} y^{-m}\right)^k \\ &= \left(y^{-1} \sum_{m=1}^{\infty} \binom{\frac{1}{2}}{m} (-1)^{m-1} y^{-m+1}\right)^k \\ &= y^{-k} \left(\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m+1} (-1)^m y^{-m}\right)^k. \end{aligned}$$

Now we assume temporarily $k \in \mathbb{N}_0$. Then by induction and Cauchy's product formula we find coefficients $\beta(k; m)$ such that

$$(5.30) \quad \left(\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m+1} (-1)^m y^{-m}\right)^k = \sum_{m=0}^{\infty} \beta(k; m) y^{-m},$$

where

$$\beta(k; m) = (-1)^m \sum_{r_1 + \dots + r_k = m, r_i \in \mathbb{N}_0} \binom{\frac{1}{2}}{r_1 + 1} \cdots \binom{\frac{1}{2}}{r_k + 1}.$$

In particular, $\beta(k; 0) = 2^{-k}$. Let now $z \in (0, \infty)$ and $k \in \mathbb{C}$ arbitrary. Once again the binomial theorem gives

$$(5.31) \quad \begin{aligned} z^k &= (1 + (z - 1))^k \\ &= \sum_{l=0}^{\infty} \binom{k}{l} (z - 1)^l \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{k}{l} \binom{l}{m} (-1)^{l-m} z^m. \end{aligned}$$

So let $k \in \mathbb{C}$ again be arbitrary. We take for z in (5.31) the series $\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{nm+1} (-1)^m y^{-m} (= y - \sqrt{y^2 - y})$. Then by (5.30)

$$\begin{aligned} \left(\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m+1} (-1)^m y^{-m} \right)^k &= \sum_{l=0}^{\infty} \sum_{s=0}^l \binom{k}{l} \binom{l}{s} (-1)^{l-s} \left(\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m+1} (-1)^m y^{-m} \right)^s \\ &\stackrel{(5.30)}{=} \sum_{m=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{s=0}^l \binom{k}{l} \binom{l}{s} \beta(s; m) (-1)^{l-s} \right) y^{-m} \\ &=: \sum_{m=0}^{\infty} \beta(k; m) y^{-m}. \end{aligned}$$

In particular,

$$\begin{aligned} \beta(k; 0) &= \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \binom{k}{l} \binom{l}{s} \beta(s; 0) (-1)^{l-s} \\ &= \sum_{l=0}^{\infty} (-1)^l \sum_{s=0}^l \binom{l}{s} \left(\frac{1}{2} \right)^s (-1)^{-s} \\ &= \sum_{l=0}^{\infty} \binom{k}{l} (-1)^l \sum_{s=0}^l \binom{l}{s} \left(\frac{1}{2} \right)^s (-1)^s \\ &= \sum_{l=0}^{\infty} \binom{k}{l} (-1)^l \left(1 - \frac{1}{2} \right)^l \\ &= \sum_{l=0}^{\infty} \binom{k}{l} (-1)^l \left(\frac{1}{2} \right)^l = \left(\frac{1}{2} \right)^k. \end{aligned}$$

Finally, $k \mapsto \beta(k; m)$ is holomorphic as $k \mapsto \binom{k}{l} = \frac{k(k-1)\dots(k-l+1)}{l!}$ is holomorphic for $l \in \mathbb{N}_0$. □

Now we come to the definition of the *zeta function*

$$(5.32) \quad \mathcal{Z}(k; \varphi) := \sum_{1 \neq [\gamma] \in \text{CT}} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) e^{-(k-\rho_0)L_\gamma}.$$

PROPOSITION 5.5.2. *The zeta function $\mathcal{Z}(k; \varphi)$ converges for $\text{Re}(k) > 2\rho_0$.*

PROOF. This will follow by the next lemma.

LEMMA 5.5.3. *Let $k \in \mathbb{C}$ with $\text{Re}(k) > 2\rho_0$. For any $\gamma \neq 1$ there exists a constant $C(\varphi) > 0$ only depending on φ such that*

$$\left| \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) \right| \leq C(\varphi) \cdot L_\gamma e^{-\rho_0 L_\gamma}.$$

Here $L_\gamma > 0$ is the length of the closed geodesic $[\gamma] \neq 1$.

We start with the orbital integral $\mathcal{O}_\gamma(f_k)$ for $\gamma \neq 1$ which equals by (5.6)

$$\begin{aligned}
\mathcal{O}_\gamma(f_k) &= \int_N f_k(n^{-1}a_\gamma m_\gamma n) I_\gamma(\varphi)(n) dn \\
&= \int_{\mathbb{R}^{l-1}} \left(-(m_\gamma^{m^{-1}})_{1,1} |u|^2 + (1 + |u|^2) \cosh L_\gamma \right)^{-k} I_\gamma(\varphi)(\exp X_u) du \\
&\quad , \text{ where } (m_\gamma)_{1,1} \text{ as in (5.24)} \\
&= \cosh^{-k} L_\gamma \int_{\mathbb{R}^{l-1}} \left(-(m_\gamma^{m^{-1}})_{1,1} \frac{|u|^2}{\cosh L_\gamma} (1 + |u|^2) \right)^{-k} I_\gamma(\varphi)(\exp X_u) du \\
&= \cosh^{-k} L_\gamma \int_{\mathbb{R}^{l-1}} \left(|u|^2 \left(1 - \frac{(m_\gamma^{m^{-1}})_{1,1}}{\cosh L_\gamma} \right) + 1 \right)^{-k} I_\gamma(\varphi)(\exp X_u) du \\
&= \cosh^{-k} L_\gamma \left(1 - \frac{(m_\gamma^{m^{-1}})_{1,1}}{\cosh L_\gamma} \right)^{-\rho_0} \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-k} I_\gamma(\varphi)(\exp X_{T(u)}) du \\
&= \cosh^{\rho_0 - k} L_\gamma (\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{-\rho_0} \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-k} I_\gamma(\varphi)(\exp X_{T(u)}) du \\
&=: (*),
\end{aligned}$$

where $T(u) = ((1 - (m_\gamma^{m^{-1}})_{1,1} \cosh^{-1} L_\gamma)^{-1/2} u_1, \dots, (1 - (m_\gamma^{m^{-1}})_{1,1} \cosh^{-1} L_\gamma)^{-1/2} u_{l-1})^T$.
By comparing the right side of the definition of $\mathcal{R}(k; \varphi)$, (5.29), with the last equation (*) for $\mathcal{O}_\gamma(f_k) = \cosh^{-k + \rho_0} L_\gamma \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k)$ we see that

$$\sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) = \cosh^{k - \rho_0} L_\gamma \cdot \mathcal{O}_\gamma(f_k) \stackrel{(*)}{=} (\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{-\rho_0} \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-k} I_\gamma(\varphi)(\exp X_{T(u)}) du.$$

Now $|(m_\gamma^{m^{-1}})_{1,1}| \leq 1$ and φ is bounded as a continuous function on compact $\Gamma \backslash X$, so

$$\begin{aligned}
|I_\gamma(\varphi)(\exp X_{T(u)})| &\leq \int_{A / \langle a_{\gamma_0} \rangle} |\varphi(\alpha_\gamma^{-1} a \exp X_{T(u)})| da \\
&\leq C \int_{A / \langle a_{\gamma_0} \rangle} = CL_\gamma
\end{aligned}$$

for $C := \max_{x \in X_\Gamma} \{|\varphi(x)|\}$. For $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$ let

$$(5.33) \quad C(\varphi; k) := C \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-\operatorname{Re}(k)} du \geq 0.$$

The integral

$$\int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-\operatorname{Re}(k)} du$$

surely converges for all $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$ and

$$\int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-\operatorname{Re}(k)} \leq \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-2\rho_0} du = \frac{\omega_{l-1}}{2} B(\rho_0, \rho_0)$$

for all $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$. Hence,

$$\begin{aligned}
\left| \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) \right| &= (\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{-\rho_0} \left| \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-k} I_\gamma(\varphi)(\exp X_{T(u)}) du \right| \\
&\leq e^{-\rho_0 L_\gamma} \int_{\mathbb{R}^{l-1}} (|u|^2 + 1)^{-\operatorname{Re}(k)} |I_\gamma(\varphi)(\exp X_{T(u)})| du \\
&\leq C(\varphi) \cdot L_\gamma e^{-\rho_0 L_\gamma}
\end{aligned}$$

with $C(\varphi) := C(\varphi; 2\rho_0)$, see (5.33). \square

We continue with the proof of Proposition 5.5.2. By Lemma 5.5.3 we know that for $\operatorname{Re}(k) > 2\rho_0$

$$|\mathcal{Z}(k; \varphi)| \leq C(\varphi) \sum_{1 \neq [\gamma] \in C\Gamma} L_\gamma e^{-\rho_0 L_\gamma}.$$

But for any $1 \neq \gamma \in C\Gamma$ we can find some natural number n such that $n-1 \leq L_\gamma \leq n$. Hence $L_\gamma e^{-\rho_0 L_\gamma} \leq n e^{-\rho_0(n-1)} = e^{\rho_0} n e^{-\rho_0 n}$. It follows that $\sum_{1 \neq [\gamma] \in C\Gamma} L_\gamma e^{-\rho_0 L_\gamma}$ is convergent as it is bounded from above by

$$e^{\rho_0} \sum_{n=1}^{\infty} n e^{-\rho_0 n}$$

which is convergent as $\rho_0 > 0$. □

REMARK 5.5.4. We follow Anantharaman and Zelditch in calling $\mathcal{R}(\sigma)$ and $\mathcal{Z}(\sigma)$ zeta functions, instead of logarithmic derivatives of zeta functions, which would be more correct in view of the classical Selberg Zeta function

$$Z_S(k) := \prod_{[\gamma]} \prod_{s=0}^{\infty} \left(1 - e^{-(s+k)L_\gamma}\right).$$

Normally one would pass from $\frac{d}{dk} \ln Z_S$ to Z_S by showing that $\frac{d}{dk} \ln Z_S$ has simple poles with integer residue. As we will see in Chapter 7, $\mathcal{Z}(\sigma)$ still has simple poles in some cases, but we can not guarantee that its residues are integers.

THEOREM 5.5.5. *Let $X = G/K$ be a real hyperbolic space and $\Gamma \subset G$ a uniform lattice. Let φ be a non trivial automorphic eigenfunction and $k \in \mathbb{C}$. We have*

$$\mathcal{Z}(k; \varphi) = \sum_{m=0}^{\infty} \beta(k - \rho_0; m) \mathcal{R}(k + 2m; \varphi)$$

for $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$, where the coefficients $\beta(k - \rho_0; m)$ are determined by Lemma 5.5.1.

PROOF. A slight modification of the proof of [AZ07, Lemma 9.3.] works also here. We have $e^{-t} \cdot \cosh t = \frac{1}{2}(1 + e^{-2t})$ for $t \in \mathbb{R}$. Hence,

$$\begin{aligned} (e^{-t})^{k-\rho_0} &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} (1 + e^{-2t})^{k-\rho_0} \\ &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} (1 + (\cosh t - \sinh t)^2)^{k-\rho_0} \\ &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} \left(1 + \left(y - \sqrt{y^2 - 1}\right)^2\right)^{k-\rho_0}, \end{aligned}$$

where $y = \cosh t$ and using $\cosh^2 t - \sinh^2 t = 1$. We continue with

$$\begin{aligned} (e^{-t})^{k-\rho_0} &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} \left(1 + (y^2 - 2y\sqrt{y^2 - 1} + y^2 - 1)\right)^{k-\rho_0} \\ &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} (2y^2 - 2y^2\sqrt{1 - y^{-2}})^{k-\rho_0} \\ &= \left(\frac{1}{2}\right)^{k-\rho_0} (\cosh t)^{-(k-\rho_0)} (2y^2)^{k-\rho_0} \left(1 - \sqrt{1 - y^{-2}}\right)^{k-\rho_0} \\ &= (\cosh t)^{-(k-\rho_0)} (y^2)^{k-\rho_0} \left(1 - \sqrt{1 - y^{-2}}\right)^{k-\rho_0}. \end{aligned}$$

We assume now $t \neq 0$, so that $\cosh t > 1$. By the preceding Lemma 5.5.1 there are coefficients $\beta(k - \rho_0; m)$ such that

$$\left(1 - \sqrt{1 - y^{-2}}\right)^{k - \rho_0} = (y^2)^{-(k - \rho_0)} \sum_{m=0}^{\infty} \beta(k - \rho_0; m) (y^2)^{-m}$$

Hence,

$$(\cosh t)^{-(k - \rho_0)} (y^2)^{k - \rho_0} (y^2)^{-(k - \rho_0)} \sum_{m=0}^{\infty} \beta(k - \rho_0; m) (y^2)^{-m} = \sum_{m=0}^{\infty} \beta(k - \rho_0; m) (\cosh t)^{-(k + 2m - \rho_0)}.$$

Thus, we have shown for any $t \neq 0$

$$(e^{-t})^{k - \rho_0} = \sum_{m=0}^{\infty} \beta(k - \rho_0; m) (\cosh t)^{-(k + 2m - \rho_0)}$$

and it follows

$$\begin{aligned} \sum_{1 \neq \gamma} \sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k) e^{-(k - \rho_0)L_\gamma} &= \sum_{1 \neq \gamma} \sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k) \sum_{m=0}^{\infty} \beta(k - \rho_0; m) (\cosh L_\gamma)^{-(k + 2m - \rho_0)} \\ &= \sum_{m=0}^{\infty} \beta(k - \rho_0; m) \sum_{1 \neq \gamma} \sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k) (\cosh L_\gamma)^{-(k + 2m - \rho_0)}, \end{aligned}$$

where $L_\gamma = \sqrt{2(l-1)}^{-1} l_\gamma > 0$ for $\gamma \neq 1$. \square

5.6. Simplifications - the zeta function for $SO_o(1, 2)$ and $SO_o(1, 3)$

In this section we want to explain how the zeta function

$$\mathcal{R}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) (\cosh L_\gamma)^{-k + \rho_0}$$

resp. $c(\varphi, \gamma, \pi, k)$ simplify if Γ satisfies a special property. Namely, we have seen before that every $\gamma \in \Gamma$ is conjugated in G to some $a_\gamma m_\gamma \in AM$. The assumption we make is that m_γ is always central in M .

This assumption is satisfied for any uniform lattice, if $G = SO_o(1, 2)$ or $SO_o(1, 3)$. In this case M is trivial resp. isomorphic to $SO(2)$, i.e. abelian. Unfortunately, we do not know of any examples in $SO_o(1, l)$, for $l > 3$. The case $SO_o(1, 2)$ has also been considered in [AZ07].

Let $\mathbf{1}$ denote the trivial representation of M on \mathbb{C} and $\varphi \neq 1$ a non trivial automorphic eigenfunction with eigenvalue $\mu = -\frac{1}{4}(\rho_0 + \frac{\nu^2}{\rho_0})$. Further, let us assume $m_\gamma^{m^{-1}} = m_\gamma$ for all $m \in M$ and $\gamma \in \Gamma$. We now look at the definition (5.27) of $c(\varphi, \gamma, \pi, k)$

$$\begin{aligned} c(\varphi, \gamma, \pi, k) &= \omega_{l-1} d_\pi \int_M \frac{1}{(\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{\frac{l-1}{2}}} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} \\ &\quad \cdot (\chi_\pi * I_\gamma(\varphi)) \left(m \cdot \exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}} s X_{e_1} \right) ds dm, \\ &=: (*). \end{aligned}$$

In Proposition 2.2.6 we observed that for $\sigma \in C(\Gamma \backslash G)$, $n \in N$ and $z \in M_{m_\gamma} A$

$$\int_{A / \langle a_{\gamma_0} \rangle} \sigma(\alpha_\gamma^{-1} a z n) da = \int_{\alpha_\gamma \Gamma_\gamma \alpha_\gamma^{-1} \backslash G_{a_\gamma m_\gamma}} \sigma(\alpha_\gamma^{-1} x z n) dx = \int_{\alpha_\gamma \Gamma_\gamma \alpha_\gamma^{-1} \backslash G_{a_\gamma m_\gamma}} \sigma(\alpha_\gamma^{-1} x n) dx.$$

In particular this is true for $z \in M_{m_\gamma} = M$ by assumption. It follows that the weight function

$$I_\gamma(\varphi)(n) = \int_{A/\langle a_{\gamma_0} \rangle} \varphi(\alpha_\gamma^{-1} a n) da = \int_{\alpha_\gamma \Gamma_\gamma \alpha_\gamma^{-1} \backslash G_{a_\gamma m_\gamma}} \varphi(\alpha_\gamma^{-1} x n) dx$$

is also left- M -invariant since $M_{m_\gamma} = M$, i.e. $G_{a_\gamma m_\gamma} = MA$ for every $\gamma \in \Gamma$. Thus, $\chi_\pi * I_\gamma(\varphi) = 0$ for all $\pi \neq 1$ and $c(\varphi, \gamma, \mathbf{1}, k) = 0$ for any non trivial π . Then $\mathcal{R}(k; \varphi)$ simplifies to

$$(5.34) \quad \mathcal{R}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} c(\varphi, \gamma, \mathbf{1}, k) (\cosh L_\gamma)^{-k+\rho_0}.$$

But $\pi = \mathbf{1}$ implies $p = d = 0$, see Theorem 4.3.1, hence we get the following proposition.

PROPOSITION 5.6.1. *For $\varphi \neq 1$, $\gamma \neq 1$ and $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$ the coefficient $c(\varphi, \gamma, \mathbf{1}, k)$ can be computed to*

$$(5.35) \quad \begin{aligned} c(\varphi, \gamma, \mathbf{1}, k) &= \omega_{l-1} \sqrt{2(l-1)} \left(\int_{c_{\gamma_0}} \varphi \right) \frac{1}{(\cosh L_\gamma - (m_\gamma)_{1,1})^{\rho_0}} \\ &\cdot \int_0^\infty s^{l-2} (s^2 + 1)^{-k} {}_2F_1 \left(a, b, \rho_0; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right) ds, \end{aligned}$$

where

$$\frac{1}{(\cosh L_\gamma - (m_\gamma)_{1,1})^{\rho_0}} = \frac{2^{\rho_0}}{e^{\rho_0 L_\gamma} \det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})}.$$

For $l = 2$ and $l = 3$ we thus get

$$\mathcal{R}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} c(\varphi, \gamma, \mathbf{1}, k) (\cosh L_\gamma)^{-k+1/2}, \quad l = 2$$

resp.

$$\mathcal{R}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} c(\varphi, \gamma, \mathbf{1}, k) (\cosh L_\gamma)^{-k+1}, \quad l = 3$$

where

$$c(\varphi, \gamma, \mathbf{1}, k) = \frac{2 \left(\int_{c_{\gamma_0}} \varphi \right)}{\sinh L_\gamma / 2} \int_0^\infty (s^2 + 1)^{-k} {}_2F_1 \left(a, b, \frac{1}{2}; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right) ds$$

for $l = 2$ resp.

$$c(\varphi, \gamma, \mathbf{1}, k) = \frac{2\omega_2 \left(\int_{c_{\gamma_0}} \varphi \right)}{\cosh L_\gamma - (m_\gamma)_{1,1}} \int_0^\infty s (s^2 + 1)^{-k} {}_2F_1 \left(a, b, 1; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right) ds$$

for $l = 3$.

PROOF. First, by the assumptions on m_γ and as $I_\gamma(\varphi)$ is left- M -invariant we get

$$\begin{aligned}
c(\varphi, \gamma, \mathbf{1}, k) &= \omega_{l-1} \int_M \frac{1}{(\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1})^{\frac{l-1}{2}}} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} \\
&\quad \cdot I_\gamma(\varphi) \left(m \cdot \exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}}} s X_{e_1} \right) ds dm, \\
&= \omega_{l-1} \frac{1}{(\cosh L_\gamma - (m_\gamma)_{1,1})^{\frac{l-1}{2}}} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} \\
&\quad \cdot I_\gamma(\varphi) \left(\exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}}} s X_{e_1} \right) ds \\
&=: (+).
\end{aligned}$$

The equality in (5.35) of Proposition 5.6.1 then follows directly from (+) and the following observations

$$\begin{aligned}
I_\gamma(\varphi) \left(\exp \sqrt{\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}}} s X_{e_1} \right) &= I_\gamma(\varphi)(e) {}_2F_1 \left(a, b, \rho_0; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right) \\
&= \sqrt{2(l-1)} \left(\int_{c_{\gamma_0}} \varphi \right) {}_2F_1 \left(a, b, \rho_0; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right),
\end{aligned}$$

see equations (5.19) for $l = 2$ resp. (5.18), $l \geq 3$, with $p = d = 0$ and also (5.11) and Remark 5.4.4. To get the second equality about $(\cosh L_\gamma - (m_\gamma)_{1,1})^{-\rho_0}$ we use two observations.

LEMMA 5.6.2. *Let $a \in A^+$ and $m \in M$. The map $h(ma) : n \mapsto m^{-1}a^{-1}n^{-1}amn$ is a diffeomorphism of N and the Haar measure dn transforms to*

$$dh(ma)(n) = \det(1 - \operatorname{Ad}(ma)^{-1}|_{\mathfrak{n}}) dn.$$

PROOF. See [Hel94, Ch. I Lem.5.4.] or [Wil91, Th.11.24.] \square

COROLLARY 5.6.3. *For integrable functions f on N , $a \in A^+$ and $m \in M$*

$$\int_N f(n) dn = \det(1 - \operatorname{Ad}(ma)^{-1}|_{\mathfrak{n}}) \int_N f(m^{-1}a^{-1}n^{-1}amn) dn$$

resp. if f is also bi- K -invariant, then

$$\int_N f(an) dn = \det(1 - \operatorname{Ad}(ma)^{-1}|_{\mathfrak{n}}) \int_N f(n^{-1}amn) dn.$$

We already know that for $s \in \mathbb{R}$

$$f_k((\exp -sX_{e_1})a_\gamma m (\exp sX_{e_1})) = (-(m_\gamma)_{1,1}s^2 + (1+s^2)\cosh L_\gamma)^{-k},$$

see (5.23). Also for $t \in \mathbb{R}$

$$f_k(\exp tH_0 \exp sX_{e_1}) = f_k(a_t \exp sX_{e_1}) = \left(\cosh t + \frac{s^2}{2}e^t \right)^{-k},$$

see (5.25). We finally recall the polar coordinates formula

$$\int_N f(n) dn = \omega_{l-1} \int_0^\infty s^{l-2} f(\exp sX_{e_1}) ds$$

for bi- M -invariant, integrable functions f on N . Thus for $f = f_k$, $a = a_\gamma = \exp L_\gamma H_0$ and $m = m_\gamma$ we get that

$$\begin{aligned}
& \frac{1}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \int_N f_k(a_\gamma n) dn \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \int_0^\infty \int_M f_k(a_\gamma m \exp sX_{e_1} m^{-1}) dm ds \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \int_0^\infty \int_M f_k(a_\gamma \exp sX_{e_1}) dm ds \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \int_0^\infty s^{l-2} \left(\frac{s^2}{2} e^{L_\gamma} + \cosh L_\gamma \right)^{-k} ds \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \cosh^{-k} L_\gamma \int_0^\infty s^{l-2} \left(\frac{s^2 e^{L_\gamma}}{2 \cosh L_\gamma} + 1 \right)^{-k} ds \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \cosh^{-k} L_\gamma e^{-\frac{l-1}{2} L_\gamma} (2 \cosh L_\gamma)^{\frac{l-1}{2}} \int_0^\infty s^{l-2} (s^2 + 1)^{-k} ds \\
&= \frac{\omega_{l-1}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} 2^{\frac{l-1}{2}} e^{-\frac{l-1}{2} L_\gamma} \cosh^{-k + \frac{l-1}{2}} L_\gamma \int_0^\infty s^{l-2} (s^2 + 1)^{-k} ds \\
&=: (*)
\end{aligned}$$

equals

$$\begin{aligned}
\int_N f_k(n^{-1} m_\gamma a_\gamma n) dn &= \omega_{l-1} \int_0^\infty s^{l-2} \int_M f_k(\exp -sX_{e_1} (m_\gamma^{m^{-1}}) a_\gamma \exp sX_{e_1}) dm ds \\
&= \omega_{l-1} \int_M \int_0^\infty s^{l-2} \left(-(m_\gamma^{m^{-1}})_{1,1} s^2 + (1 + s^2) \cosh L_\gamma \right)^{-k} ds dm, \\
&\quad (m_\gamma^{m^{-1}})_{1,1} = (m^{-1} m m)_{1,1} \\
&= \omega_{l-1} \cosh^{-k} L_\gamma \int_M \int_0^\infty s^{l-2} \left(s^2 \left(1 - \frac{(m^{m'})_{1,1}}{\cosh t} \right) + 1 \right)^{-k} ds dm' \\
&= \omega_{l-1} \cosh^{-k} L_\gamma \int_M \left(1 - \frac{(m_\gamma^{m^{-1}})_{1,1}}{\cosh L_\gamma} \right)^{-\frac{l-1}{2}} dm \int_0^\infty s^{l-2} (1 + s^2)^{-k} ds \\
&= \omega_{l-1} \cosh^{-k + \frac{l-1}{2}} L_\gamma \int_M \left(\frac{1}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}} \right)^{\frac{l-1}{2}} dm \int_0^\infty s^{l-2} (1 + s^2)^{-k} ds \\
&=: (**).
\end{aligned}$$

Therefore, Corollary 5.6.3 and comparing (*) to (**) gives

$$\int_M \left(\frac{1}{\cosh L_\gamma - (m_\gamma^{m^{-1}})_{1,1}} \right)^{\frac{l-1}{2}} dm = \frac{1}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} 2^{\frac{l-1}{2}} e^{-\frac{l-1}{2} L_\gamma}.$$

In particular if m_γ is central then $(m_\gamma^{m^{-1}})_{(1,1)} = (m_\gamma)_{1,1}$ is actually independent of m and as $\int_M = 1$

$$\begin{aligned}
\int_M \left(\frac{1}{\cosh t - (m_\gamma)_{1,1}} \right)^{\frac{l-1}{2}} dm &= \frac{1}{(\cosh t - (m_\gamma)_{1,1})^{\frac{l-1}{2}}} \\
&= \frac{1}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} 2^{\frac{l-1}{2}} e^{-\frac{l-1}{2} L_\gamma}.
\end{aligned}$$

Hence, we can replace

$$\begin{aligned}
\frac{1}{(\cosh L_\gamma - (m_\gamma)_{(1,1)})^{\rho_0}} &= \frac{1}{(\cosh L_\gamma - (m_\gamma)_{(1,1)})^{\frac{l-1}{2}}} \\
&= \frac{1}{\det(1 - \operatorname{Ad}(m_\gamma a_t)^{-1}|_{\mathfrak{n}})} 2^{\frac{l-1}{2}} e^{-\frac{l-1}{2}L_\gamma} \\
(5.36) \qquad \qquad \qquad &= \frac{1}{\det(1 - \operatorname{Ad}(m_\gamma a_t)^{-1}|_{\mathfrak{n}})} 2^{\rho_0} e^{-\rho_0 L_\gamma}
\end{aligned}$$

in the first equality of (5.35). □

We can even get rid off the integrals in the formula (5.35) for $c(\varphi, \gamma, \mathbf{1})$. Putting for the moment $z := \frac{\cosh L_\gamma}{(\cosh L_\gamma - (m_\gamma^{m-1})_{1,1})}$ $\stackrel{(5.36)}{=} \frac{2^{\rho_0} \cosh L_\gamma}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}}) e^{\rho_0 L_\gamma}}$ we have to compute for $\operatorname{Re}(k) > 2\rho_0$ the integral

$$\begin{aligned}
&\int_0^\infty s^{l-2} (s^2 + 1)^{-k} {}_2F_1\left(a, b, \rho_0; -s^2 \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}}\right) ds \\
&= \int_0^\infty s^{l-2} (s^2 + 1)^{-k} {}_2F_1(a, b, \rho_0; -zs^2) ds.
\end{aligned}$$

By the transformation $s \mapsto \sqrt{s}$ this turns into

$$(5.37) \qquad \frac{1}{2} \int_0^\infty s^{\frac{l-2}{2}} s^{-\frac{1}{2}} (s+1)^{-k} {}_2F_1(a, b, \rho_0; -zs) ds.$$

Noting that $\rho_0 = \frac{l-1}{2}$ the transformation $s \mapsto \frac{s}{z}$ yields

$$\begin{aligned}
&\frac{1}{2} \int_0^\infty s^{\frac{l-2}{2}} s^{-\frac{1}{2}} (s+1)^{-k} {}_2F_1(a, b, \rho_0; -zs) ds \\
&= \frac{1}{2} z^{k-\rho_0} \int_0^\infty s^{\rho_0-1} (s+z)^{-k} {}_2F_1(a, b, \rho_0; -s) ds \\
&= \frac{1}{2} z^{k-\rho_0} \cdot \frac{\Gamma(\rho_0)\Gamma(a-\rho_0+k)\Gamma(b-\rho_0+k)}{\Gamma(k)\Gamma(a+b-\rho_0+k)} \\
&\quad \cdot {}_2F_1(a-\rho_0+k, b-\rho_0+k, a+b-\rho_0+k; 1-z) \\
&= \frac{1}{2} z^{k-\rho_0} \cdot \frac{\Gamma(\rho_0)\Gamma(a-c+k)\Gamma(b-\rho_0+k)}{\Gamma(k)^2} \\
&\quad \cdot {}_2F_1(a-\rho_0+k, b-\rho_0+k, a+b-\rho_0+k; 1-z) \\
&= \frac{1}{2} z^{k-\rho_0} \cdot \frac{\Gamma(\rho_0)\Gamma(k-a)\Gamma(k-b)}{\Gamma(k)^2} \\
&\quad \cdot {}_2F_1(a-\rho_0+k, b-\rho_0+k, k; 1-z),
\end{aligned}$$

where we used an integral transform for hypergeometric functions, see [Bat54, 20.2 (10)] which is valid for $\operatorname{Re}(a-\rho_0+k) = \operatorname{Re}(k-b) = \operatorname{Re}(k - \frac{\rho_0}{2} + \frac{ix}{2}) > 0$, $\operatorname{Re}(b-\rho_0+k) = \operatorname{Re}(k-a) = \operatorname{Re}(k - \frac{\rho_0}{2} - \frac{ix}{2}) > 0$ for $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$.

From (5.35) we thus get

$$\begin{aligned}
c(\varphi, \gamma, \mathbf{1}, k) &= \omega_{l-1} \sqrt{2(l-1)} \left(\int_{c_{\gamma_0}} \varphi \right) \frac{1}{(\cosh L_\gamma - (m_\gamma)_{1,1})^{\rho_0}} \\
&\cdot \frac{1}{2} \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right)^{k-\rho_0} \frac{\Gamma(\rho_0) \Gamma(-\frac{1}{2}(\rho_0 - ir) + k) \Gamma(-\frac{1}{2}(\rho_0 + ir) + k)}{\Gamma(k)^2} \\
&\cdot {}_2F_1 \left(-\frac{1}{2}(\rho_0 - ir) + k, -\frac{1}{2}(\rho_0 + ir) + k, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right) \\
&= \frac{\omega_{l-1} \sqrt{2(l-1)} \left(\int_{c_{\gamma_0}} \varphi \right)}{2} \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{(1,1)}} \right)^{k-2\rho_0} \\
&\cdot \frac{\Gamma(\rho_0) \Gamma(-\frac{1}{2}(\rho_0 - ir) + k) \Gamma(-\frac{1}{2}(\rho_0 + ir) + k)}{\Gamma(k)^2} \\
(5.38) \quad &\cdot {}_2F_1 \left(-\frac{1}{2}(\rho_0 - ir) + k, -\frac{1}{2}(\rho_0 + ir) + k, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}} \right)
\end{aligned}$$

where $\frac{2^{\rho_0} \cosh L_\gamma}{\det(1 - \text{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}}) e^{\rho_0 L_\gamma}} = \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma)_{1,1}}$.

REMARK 5.6.4. Recall the definition (5.27) for $c(\varphi, \gamma, \pi, k)$. For $G = SO_o(1, l)$ with $l \geq 4$ the definition of $c(\varphi, \gamma, \pi, k)$ can be simplified at least one more step by using a computer algebra program like MATHEMATICA in order to carry out the ds -integration. Let $z := \frac{\cosh L_\gamma}{\cosh L_\gamma - (m_\gamma^{m-1})_{(1,1)}}$, then

$$\begin{aligned}
&\int_0^\infty s^{l-2} (s^2 + 1)^{-k} s^{2p} {}_2F_1 \left(a + p, b + p, 1 + 2\sqrt{\frac{(\rho_0)^2}{4} - d}; -zs^2 \right) ds \\
&= \frac{1}{2} \left(z^{k-\rho_0-p} \frac{\Gamma(1 + \frac{1}{2}\sqrt{(l-3)^2 - 16d}) \Gamma(k-a) \Gamma(k-b) \Gamma(p-k+\rho_0)}{\Gamma(1+k-\rho_0-p + \frac{1}{2}\sqrt{(l-3)^2 - 16d}) \Gamma(a+p) \Gamma(b+p)} \right. \\
&\quad \cdot {}_3F_2 \left(\{k, k-a, k-b\}, \{k+1-p-\rho_0, 1 + \frac{1}{2}\sqrt{(l-3)^2 - 16d} + k - \rho_0 - p\}; z \right) \\
&\quad \left. + \frac{\Gamma(k-\rho_0-p) \Gamma(\rho_0+p)}{\Gamma(k)} {}_3F_2 \left(\{a+p, b+p, \rho_0+p\}, \{1 + \frac{1}{2}\sqrt{(l-3)^2 - 16d}, \rho_0 - k + p\}; z \right) \right).
\end{aligned}$$

Here

$${}_3F_2(\{\alpha_1, \alpha_2, \alpha_4\}, \{\beta_1, \beta_2\}; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k (\alpha_3)_k z^k}{(\beta_1)_k (\beta_2)_k k!}$$

denotes the generalized hypergeometric function, see for example [Olv74, §11].

5.7. $\sigma \equiv 1$ - the classical Selberg zeta function

In this section we want to compare the (classical) Selberg zeta function as one can find in the book [BO95] with our zeta function $\mathcal{Z}(1)$ derived from $\mathcal{R}(1)$. So we recall that we have considered the auxiliary zeta function from (5.29)

$$\mathcal{R}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) (\cosh L_\gamma)^{-k+\rho_0}$$

and the zeta function from (5.32)

$$\mathcal{Z}(k; \varphi) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\varphi, \gamma, \pi, k) e^{-(k-\rho_0)L_\gamma}.$$

From now on we fix $\varphi \equiv 1$ and go back to Theorem 5.2.4 with $\sigma \equiv \varphi \equiv 1$. Then we note that in this case the weight $I_\gamma(\sigma)$, see (5.5), is a constant which equals

$$\int_{A/\langle a_{\gamma_0} \rangle} da = \int_0^{L_{\gamma_0}} dt = \frac{L_\gamma}{n_\gamma} = L_{\gamma_0} = \frac{l_{\gamma_0}}{\sqrt{2(l-1)}},$$

see Proposition 2.2.3, where l_{γ_0} is the length of the prime closed geodesic belonging to the conjugacy class of γ . In particular, the weight is M -invariant and hence $\chi_\pi * I_\gamma(\sigma) = 0$ for any $\pi \neq \mathbf{1}$. Hence,

$$(5.39) \quad \operatorname{Tr}(\sigma \cdot \pi_R(f)) = \operatorname{Tr}(\pi_R(f)) = f(e)|\Gamma \backslash G| + \sum_{1 \neq [\gamma] \in C\Gamma} L_{\gamma_0} \int_N f(n^{-1}a_\gamma m_\gamma n) dn$$

for any $f \in C^\infty(G//K)$ such that $\pi_R(f)$ is of trace-class. We note that

$$\begin{aligned} \frac{1}{2}B(k - \rho_0, k) &= \int_0^\infty u^{l-2}(u^2 + 1)^{-k} du = \frac{1}{2} \int_0^\infty y^{\frac{n-2}{2}} y^{-\frac{1}{2}} (1+y)^{-k} dy \\ &= \frac{\Gamma(k - \rho_0)\Gamma(\rho_0)}{2\Gamma(k)}, \end{aligned}$$

see (2.24). Furthermore, ω_{l-1} is positive by definition. Now we modify the function f_k from (5.21) by dividing it by the non-zero number, see (6.15),

$$2^{\rho_0} \omega_{l-1} \int_0^\infty s^{l-2}(s^2 + 1)^{-k} ds = 2^{\rho_0-1} \omega_{l-1} B(k - \rho_0, \rho_0).$$

Then replacing f_k by

$$\frac{f_k}{2^{\rho_0-1} \omega_{l-1} B(k - \rho_0, \rho_0)}$$

in $\pi_R(f_k)$ gives still a trace class operator and in the proof of Proposition 5.6.1 we have computed

$$\begin{aligned} \int_N f_k(n^{-1}a_\gamma m_\gamma n) dn &= \frac{1}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \int_N f_k(a_\gamma n) dn \\ &= \frac{2^{\rho_0-1} e^{-\rho_0 t} \omega_{l-1} B(k - \rho_0, \rho_0)}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \cosh^{-k+\rho_0} L_\gamma, \end{aligned}$$

Hence,

$$\frac{1}{2^{\rho_0-1} \omega_{l-1} B(k - \rho_0, \rho_0)} \int_N f_k(n^{-1}a_\gamma m_\gamma n) dn = \frac{e^{-\rho_0 L_\gamma}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} \cosh^{-k+\rho_0} L_\gamma.$$

We define for $\operatorname{Re}(k) > 2\rho_0$

$$\mathcal{R}(k; 1) := \mathcal{R}(k) := \sum_{1 \neq [\gamma] \in C\Gamma} \frac{L_{\gamma_0} e^{-\rho_0 L_\gamma}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} (\cosh L_\gamma)^{(-k+\rho_0)L_\gamma}$$

resp.

$$(5.40) \quad \mathcal{Z}(k; 1) := \mathcal{Z}(k) := \sum_{1 \neq [\gamma] \in C\Gamma} \frac{L_{\gamma_0} e^{-\rho_0 L_\gamma}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)^{-1}|_{\mathfrak{n}})} e^{(-k+\rho_0)L_\gamma}.$$

It follows immediately from (5.39) that

$$\begin{aligned} \mathcal{R}(k) &= \operatorname{Tr} \left(\pi_R \left(\frac{f_k}{\omega_{l-1} 2^{\rho_0-1} B(k - \rho_0, \rho_0)} \right) \right) - \frac{f_k(e)}{\omega_{l-1} 2^{\rho_0-1} B(k - \rho_0, \rho_0)} \cdot |\Gamma \backslash G| \\ &= \operatorname{Tr} \left(\pi_R \left(\frac{f_k}{\omega_{l-1} 2^{\rho_0-1} B(k - \rho_0, \rho_0)} \right) \right) - \frac{1}{\omega_{l-1} 2^{\rho_0-1} B(k - \rho_0, \rho_0)} \cdot |\Gamma \backslash G| \end{aligned}$$

Hence, $\mathcal{R}(k)$ is well-defined at least for $\operatorname{Re}(k) > 2\rho_0$. Also, the proof of 5.5.5 works as before to get

$$\mathcal{Z}(k) = \sum_{m=0}^{\infty} \beta(k - \rho_0; m) \mathcal{R}(k + 2m)$$

for $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 2\rho_0$.

We compare now \mathcal{Z} to the logarithmic derivative $L_{S,\chi}(s, \sigma)$ of the zeta function

$$Z_{S,\mathbf{1}}(k, \mathbf{1}) := \prod_{1 \neq [\gamma] \in C\Gamma} \prod_{r=0}^{\infty} \det \left(1 - (S^r (\operatorname{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})) e^{-(k+\rho_0)l_\gamma} \right)$$

from [BO95, Def. 3.2], when $\chi = \mathbf{1}$ is the trivial character of Γ , $\sigma = \mathbf{1}$ the trivial representation of M and S^r denotes the r^{th} symmetric power of an endomorphism. By Lemma 3.3. in [BO95] $Z_{S,\mathbf{1}}(s, \sigma)$ converges for $\operatorname{Re}(k) > \rho_0$ and $L_{S,\mathbf{1}}(s, \mathbf{1})$ is given by

(5.41)

$$L_{S,\mathbf{1}}(k, \mathbf{1}) := \frac{d}{dk} \ln Z_{S,\mathbf{1}}(k, \mathbf{1}) = 2 \cdot \sum_{1 \neq [\gamma] \in C\Gamma} \frac{l_\gamma}{n_\gamma} \frac{(-1)^{l-1}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)|_{\mathfrak{n}})} e^{(\rho_0 - k)l_\gamma}.$$

Recall that $\gamma_0^{n_\gamma} = \gamma$, that is $l_\gamma/n_\gamma = l_{\gamma_0}$. If we compare (5.40) with (5.41) we see using

$$\det(1 - \operatorname{Ad}(a_t)|_{\mathfrak{n}}) = e^{2\rho_0 t} \det(1 - \operatorname{Ad}(a_{-t})|_{\mathfrak{n}})$$

that up to the constant $2(-1)^{l-1}$, $\mathcal{Z}(k)$ equals

$$(5.42) \quad Z_{S,\mathbf{1}}(k - \rho_0, \mathbf{1}),$$

in the special case that for all $\gamma \in \Gamma$, m_γ is trivial. In particular for $l = 2$, i.e. $G = SO_0(1, 2)$. More general let us define

$$c(a_\gamma m_\gamma) := \frac{l_{\gamma_0}}{\det(1 - \operatorname{Ad}(a_\gamma m_\gamma)^{-1}|_{\mathfrak{n}})}.$$

We recall that for $\gamma \in \Gamma$, $l_\gamma = \sqrt{2(l-1)}l_{\gamma_0}$, thus

$$\mathcal{Z}(k) = (\sqrt{2(l-1)})^{-1} \sum_{1 \neq [\gamma] \in C\Gamma} c(a_\gamma m_\gamma) e^{-kL_\gamma}$$

and

$$Z_{S,\mathbf{1}}(k + \rho_0, \mathbf{1}) = 2(-1)^{l-1} \cdot \sum_{1 \neq [\gamma] \in C\Gamma} c(a_\gamma^{-1} m_\gamma^{-1}) e^{-kL_\gamma}.$$

The spectral trace

In this chapter we want to compute the spectral trace of the trace class operator $\varphi \cdot \pi_R(f_k)$ from Chapter 5. We obtain a general formula involving Wigner distributions which is valid for all rank one symmetric spaces, see Theorem 6.1.3 and a refined version thereof which involves Patterson-Sullivan distributions in Theorem 6.2.18. A careful analysis of this refined trace formula will show that we can define a meromorphic continuation of $\mathcal{R}(\varphi)$ and $\mathcal{Z}(\varphi)$ by Theorem 6.2.18. This will be done in the next chapter.

6.1. First computations on the spectral trace

Let $X = G/K$ be a symmetric space of noncompact type of rank one, $G = NAK$, $M = Z_K(A)$, $\Gamma \subset G$ a uniform lattice and $B = K/M$ the boundary of $X = G/K$. We denote by SX_Γ the unit sphere bundle of $X_\Gamma = \Gamma \backslash X$. It can be identified with $\Gamma \backslash G/M$, see Lemma 2.2.5. We also identify the dual $\mathfrak{a}_\mathbb{C}^*$ of the complexification $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes \mathbb{C}$ with \mathbb{C} by sending a linear functional λ on $\mathfrak{a}_\mathbb{C}$ to $\lambda(H_0)$, where H_0 is defined in (4.9). By Ω we denote the Casimir operator as in (2.2).

Further, let $\{\varphi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(X_\Gamma)$ of automorphic eigenfunctions, in particular

$$\Omega \varphi_j = -(\lambda_j^2 + \rho_0^2) \varphi_j, \quad j \in \mathbb{N}_0$$

for $\lambda_j \in \mathbb{C}$ and $\varphi_j(\gamma g) = \varphi_j(g)$ for $\gamma \in \Gamma$. Then one can assume that

$$0 = \lambda_0^2 + \rho_0^2 \leq \lambda_1^2 + \rho_0^2 \leq \dots,$$

see [Wil91, (12.12)]. Hence, there are the two cases $\lambda_j \in \mathbb{R}$ or $\lambda_j \in i\mathbb{R}$ and there are only finitely many λ_j in the latter case, see [Wil91, Cor. 12.10]. Thus, when we consider asymptotics, we can assume that $\lambda_j \in \mathbb{R}$. If φ_j is an eigenfunction with $\lambda_j \in \mathbb{R}$, we say that φ_j or λ_j lies in the principal series, otherwise we say that φ_j resp. λ_j is in the complementary series. Let $j_0 \in \mathbb{N}$ be such that φ_j is in the complementary series iff $j \leq j_0$. We make the assumption that λ_{j_0} possibly equals 0, while $\lambda_j \neq 0$ for all $j \neq j_0$.

We recall that we defined the horocycle bracket by

$$\langle gK, kM \rangle = A(k^{-1}g),$$

see (2.19), where $A : G \rightarrow \mathfrak{a}$ is the projection belonging to $G = NAK$. One can show that there is for any automorphic eigenfunction φ with eigenvalue $-(\lambda^2 + \rho_0^2)$ some T_φ in the dual $\mathcal{D}'(B)$ of $C^\infty(B)$, such that for $z \in X_\Gamma$

$$(6.1) \quad \varphi(z) = \langle e^{(i\lambda + \rho)\langle z, \cdot \rangle}, T_\varphi \rangle_B =: \int_B e^{(i\lambda + \rho)\langle z, b \rangle} dT_\varphi(b),$$

see [HS] or [Hel01, Ch.II §4 C(iii)]. T_φ is unique up to Weyl group action. That is, if we assume $\operatorname{Re}(\lambda) \geq 0$, then T_φ is unique. We will always assume that $\operatorname{Re}(\lambda_j) \geq 0$ for all eigenvalues μ_j and call T_φ *boundary value* of φ .

We state an analogue of Proposition 2.10 in [Zel89] for the special case of an automorphic eigenfunction. It shows that the operator $\pi_R(f)$, defined in (5.1), is

diagonalized by the orthonormal basis $\{\varphi_j\}_{j \in \mathbb{N}}$. Finally, we recall the definitions of the spherical transform $\mathcal{S}(f, \cdot)$ in (2.21).

PROPOSITION 6.1.1. $\pi_R(f)\varphi_j = \mathcal{S}(f, -\lambda_j)\varphi_j$, where $f \in L^1(G//K)$.

PROOF. [**Gan72**, Th.7, Cor.8 and following remark]

□

By the invariance of the spherical transform under the Weyl group we also have $\mathcal{S}(f, -\lambda_j) = \mathcal{S}(f, \lambda_j)$ for all j .

Let us fix an automorphic eigenfunction φ_k from the orthonormal basis $\{\varphi_j\}$ as we did in Section 5.3. We suppose in addition that $f \in C^\infty(G//K)$ is such that $\pi_R(f)$ is of trace class, for example $f = f_k$ from Section 5.4. We then conclude that, as for Proposition 5.1.1, since φ_k is bounded, that $\varphi_k \cdot \pi_R(f)$ is also of trace class and maps $L^2(\Gamma \backslash X)$ into itself.

In contrast to Chapter 5, where we computed the trace by integrating a kernel for $\varphi_k \cdot \pi_R(f)$ over $\Gamma \backslash X$ to get the trace, we now sum matrix coefficients which we build from pairing $\varphi_k \cdot \pi_R(f)$ with the orthonormal basis $\{\varphi_j\}$. Explicitly,

$$\begin{aligned}
 \text{Tr}(\varphi_k \cdot \pi_R(f)) &= \sum_j \langle \varphi_k \cdot \pi_R(f)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \\
 &\stackrel{\text{Prop.6.1.1}}{=} \sum_j \langle \varphi_k \mathcal{S}(f, -\lambda_j)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \\
 (6.2) \qquad \qquad \qquad &= \sum_j \langle \varphi_k \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f, -\lambda_j).
 \end{aligned}$$

Next we define a map Op from $C(X_\Gamma)$ into the bounded operators on $L^2(X_\Gamma)$ by

$$(6.3) \qquad \qquad \qquad \text{Op}(a)\varphi_j := a \cdot \varphi_j.$$

Then we can infer from (6.2)

$$\begin{aligned}
 \text{Tr}(\varphi_k \cdot \pi_R(f)) &= \sum_j \langle \varphi_k \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f, -\lambda_j). \\
 &= \sum_j \langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f, -\lambda_j).
 \end{aligned}$$

REMARK 6.1.2. This definition of Op is a special instance of a ψ DO-calculus for symbols from [**Sch10**, Def. 4.14.], that is, a map Op from symbols $a \in C^\infty(X \times B)$ into continuous operators from

$$C^\infty(X)_c \rightarrow C^\infty(X),$$

resp. from the dual of $C^\infty(X)$

$$\mathcal{E}'(X) := (C^\infty(X))' \rightarrow \mathcal{D}'(X) := (C_c^\infty(X))'$$

into the dual of $C_c^\infty(X)$, see [**Sch10**, Th. 4.15.]. For $a \in C_c^\infty(X \times B)$, $\text{Op}(a)$ is given by

$$(6.4) \qquad \text{Op}(a)u(x) = \frac{1}{2} \int_{\mathfrak{a}_+^*} \int_{K/M} e^{(i\lambda+\rho)\langle z, b \rangle} \mathcal{F}(u, \lambda, b) a(x, b) db |c(\lambda)|^{-2} d\lambda,$$

where $\mathcal{F}(u, \lambda, b)$ is Helgason's non-euclidean Fouriertransform, see (2.20), $c(\lambda)$ is Harish-Chandra's c -function and we identified $G/M = X \times K/M$, i.e. $a(z) = a(x, b)$ for $z \in G/M$, $x \in X$, $b \in K/M$. Note that this integral is finite, as $\mathcal{F}(u, \lambda, b)$ is rapidly decreasing in λ , see [**Hel94**, Ch.III Th.5.1.]. Furthermore, it satisfies

$$(6.5) \qquad \qquad \qquad \text{Op}(a)e^{(i\lambda+\rho)\langle z, b \rangle} = a(z, b)e^{(i\lambda+\rho)\langle z, b \rangle}$$

for $a \in C(G/M)$, see [HS, (4.2)] or [Sch10, (4.18)].

The linear map $a \mapsto \langle \text{Op}(a)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)}$ is called the *Wigner distribution* associated with φ_j on $C^\infty(SX_\Gamma)$.

With (6.5) we also see that the definition of Op from [Sch10, Def. 4.14.] extends (6.3), since

$$\begin{aligned} \text{Op}(\varphi_k)\varphi_j(x) &= \text{Op}(\varphi_k) \int_B e^{(i\lambda+\rho)\langle x, b \rangle} dT_{\varphi_j}(b), \quad T_{\varphi_j} \in \mathcal{D}'(B) \text{ boundary value, see (6.1)} \\ &= \int_B \text{Op}(\varphi_k) e^{(i\lambda+\rho)\langle x, b \rangle} dT_{\varphi_j}(b), \text{ see [HS, (4.7)]} \\ &= \int_B \varphi_k(x) e^{(i\lambda+\rho)\langle x, b \rangle} dT_{\varphi_j}(b) \\ &\quad \text{with } \text{Op}(a) e^{(i\lambda+\rho)\langle x, b \rangle} = a(x, \lambda, b) e^{(i\lambda+\rho)\langle x, b \rangle}, \text{ see [Sch10, (4.18)]} \\ &= \varphi_k(x) \int_B e^{(i\lambda+\rho)\langle x, b \rangle} dT_{\varphi_j}(b) = \varphi_k(x)\varphi_j(x). \end{aligned}$$

for $x \in X$.

To sum up, we have the following.

PROPOSITION 6.1.3. *Let $X = G/K$ be a noncompact symmetric space of rank one and $\Gamma \subset G$ a uniform lattice. Further, we fix a orthonormal basis $\{\varphi_j\}_j$ of $L^2(\Gamma \backslash X)$ of automorphic Laplace-eigenfunctions with eigenvalues*

$$-(\lambda_j^2 + \rho_0^2)$$

and choose φ_k in $\{\varphi_j\}_j$. Finally, let $f \in C^\infty(G//K)$ be such that $\pi_R(f)$ is of trace class, where π_R is the right-regular representation of G on $L^2(\Gamma \backslash G)$. For example $f \in C_c^\infty(G//K)$ suffices. Then $\varphi_k \cdot \pi_R(f)$ is also of trace class with trace given by

$$(6.6) \quad \text{Tr}(\varphi_k \cdot \pi_R(f)) = \sum_j \langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(SX_\Gamma)} \mathcal{S}(f, \lambda_j).$$

See (6.3) for the definition of $\text{Op}(\varphi_k)$.

We call (6.6) also the spectral trace of the operator $\varphi_k \cdot \pi_R(f)$.

6.2. From Wigner to Patterson-Sullivan distributions

In this section G/K is at first an arbitrary noncompact symmetric space of rank one. From Theorem 6.2.7 on we will specialize to real hyperbolic spaces. The next step then is to relate $\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle$ to the Patterson-Sullivan distributions.

Recall that $B = K/M$. We consider the diagonal action of G on

$$B^{(2)} := (B \times B) - \Delta(B),$$

where $\Delta(B)$ is the diagonal in $B \times B$. One can show that this action is transitive and that the stabilizer of (M, wM) , where w is the non-trivial Weyl group element, is MA , [HS, Prop. 2.4.]. Then we identify $B^{(2)}$ with G/MA and write $g(b, b')$ for an element in G with $g(b, b') \cdot (M, wM) = (g(b, b') \cdot M, g(b, b') \cdot wM) = (b, b')$.

We now recall some definitions from [HHS12], [HS] and [Sch10]. For a function f on G/M the Radon transform on G/M is given by

$$\mathcal{R}f(b, b') := \int_A f(g(b, b')aM) da,$$

see [HHS12, Def. 4.3]. This is well-defined by the unimodularity of A and maps $C_c^\infty(G/M)$ into $C_c^\infty(G/MA)$, see [HHS12, Lem. 4.4]. Next we define the so-called

intertwining operators. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in C(G)$ we set

$$L_{\lambda}a(g) := \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} a(gn)dn,$$

see [HS, (1.3)], [Sch10, (6.35)], whenever the integral exists, where $H(g^{-1}k) = -A(k^{-1}g)$ and w is the non-trivial Weyl group element. One can show that L_{λ} maps $C_c^{\infty}(G/M)$ into $C^{\infty}(G/M)$, [Sch10, Lem. 6.30.].

We continue with the notion of intermediate values. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the *intermediate value* on G/MA

$$d_{\lambda}(gMA) := e^{(i\lambda+\rho)(H(g)+H(gw))},$$

see [HHS12, Def. 4.1]. Then we are in the position to define *Patterson-Sullivan distributions*. We start with distributions on G/M .

DEFINITION 6.2.1. [HHS12, Def. 4.8], [Sch10, Def. 6.12]

For an automorphic Laplace-eigenfunction φ with eigenvalue

$$-(\lambda^2 + \rho_0^2)$$

the Patterson-Sullivan distribution PS_{φ} on G/M is given by

$$\int_{G/M} a(gM) d\text{PS}_{\varphi}(gM) := \langle a, \text{PS}_{\varphi} \rangle := \int_{B^{(2)}} \mathcal{R}a(b, b') d_{\lambda}(b, b') dT_{\varphi}(b) dT_{\varphi}(b'),$$

where T_{φ} is the boundary value of φ .

We call $\chi \in C_c^{\infty}(G/M)$ a *smooth fundamental domain cut-off*, if

$$\sum_{\gamma \in \Gamma} \chi(\gamma g) = 1$$

for all $g \in G$. Smooth fundamental domain cut-off functions exist. If for example $f \in C_c^{\infty}(G/M)$ is such that f equals 1 on a fundamental domain for Γ , then $\chi(g) = \frac{f(g)}{\sum_{\gamma \in \Gamma} f(\gamma g)}$ is a smooth cut-off.

We have the following two Theorems, 6.2.2 and 6.2.10, which are generalizations of Theorem 1.1 and Lemma 6.4 in [AZ07].

THEOREM 6.2.2. [AZ07, Theorem 1.1], [HS, Theorem 1.1], [Sch10, Theorem 6.40.]

Let $a \in C^{\infty}(SX_{\Gamma})$. Then

$$\langle \text{Op}(a)\varphi_{\lambda}, \varphi_{\lambda} \rangle_{L^2(X_{\Gamma})} = \langle L_{\lambda}(\chi a), \text{PS}_{\varphi} \rangle_{G/M},$$

where χ is an arbitrary smooth fundamental domain cut-off.

We will use this theorem to extend Patterson-Sullivan distributions PS_{φ} to the range $L_{\lambda}(C_c^{\infty}(G/M))$ of the intertwiner L_{λ} . The next lemma allows us to define Patterson-Sullivan distributions on $\Gamma \backslash G/M$.

LEMMA 6.2.3. [HHS12, 4.12], [AZ07, 3.5]

Let $T \in \mathcal{D}'(\Gamma \backslash G/M)$, $a \in C^{\infty}(\Gamma \backslash G/M)$ and $a_1, a_2 \in C_c^{\infty}(G/M)$ with $\sum_{\gamma \in \Gamma} a_i(\gamma gM) = a(gM)$ for $i = 1, 2$. Then $\langle a_1, T \rangle_{G/M} = \langle a_2, T \rangle_{G/M}$.

PROOF. We have

$$\begin{aligned}
\langle a_i, T \rangle_{G/M} &= \int_{G/M} a_i(gM) dT(gM) \\
&= \int_{G/M} \sum_{\gamma \in \Gamma} \chi(\gamma gM) a_i(gM) dT(gM) \\
&\stackrel{gM \mapsto \gamma^{-1}gM}{=} \int_{G/M} \sum_{\gamma \in \Gamma} \chi(gM) a_i(\gamma^{-1}gM) dT(\gamma^{-1}gM) \\
&= \int_{G/M} \chi(gM) \sum_{\gamma \in \Gamma} a_i(\gamma^{-1}gM) dT(gM) \\
&= \int_{G/M} \chi(gM) \sum_{\gamma \in \Gamma} a_i(\gamma gM) dT(gM) \\
&= \int_{G/M} \chi(gM) a(gM) dT(gM)
\end{aligned}$$

□

Let now $\chi_1, \chi_2 \in C_c^\infty(G/M)$ with $\sum_{\gamma \in \Gamma} \chi_i(\gamma gM) = 1$ for all $g \in G$ and a in $C^\infty(\Gamma \backslash G/M)$. Set $a_i = a\chi_i$, ($i = 1, 2$). Then

$$\sum_{\gamma \in \Gamma} a_i(\gamma gM) = \sum_{\gamma \in \Gamma} a(\gamma gM) \chi_i(\gamma gM) = a(gM) \sum_{\gamma \in \Gamma} \chi_i(\gamma gM) = a(gM)$$

and hence, see also [AZ07, Lemma 3.5],

$$(6.7) \quad \langle a\chi_1, T \rangle_{G/M} = \langle a\chi_2, T \rangle_{G/M}.$$

One can show that the Patterson-Sullivan distributions are Γ -invariant on G/M , i.e.

$$\langle f \circ \gamma, \text{PS}_\varphi \rangle_{G/M} = \langle f, \text{PS}_\varphi \rangle_{G/M}$$

for all $\gamma \in \Gamma$ and $f \in C_c^\infty(G/M)$, [HHS12, Prop.4.10] or [Sch10, Prop.6.13.]. Here $f \circ \gamma(g) = f(\gamma g)$ for all $g \in G$. Thus they descend to the quotient $\Gamma \backslash G/M$ and define there elements of $\mathcal{D}'(\Gamma \backslash G/M)$, see [HHS12, Prop.4.9].

DEFINITION 6.2.4. [HHS12, Def.4.13], [Sch10, Def.6.19.]

Let φ be an automorphic eigenfunction. On $SX_\Gamma = \Gamma \backslash G/M$ the Patterson-Sullivan distributions PS_φ are given by

$$\int_{\Gamma \backslash G/M} a(\Gamma gM) d\text{PS}_\varphi(\Gamma gM) := \langle a, \text{PS}_\varphi \rangle_{SX_\Gamma} := \langle \chi a, \text{PS}_\varphi \rangle_{G/M},$$

where χ is some smooth fundamental domain cut-off.

By (6.7) this definition is independent of the choice of χ .

REMARK 6.2.5. In [HHS12, Prop. 4.9] it is shown that the Patterson-Sullivan distributions are continuous distributions on $C^\infty(G/M)$ resp. $C^\infty(\Gamma \backslash G/M)$. That is, there exist a constant $C > 0$ depending only on φ and a seminorm $\|\cdot\|$ independent of φ such that

$$|\langle \chi f, \text{PS}_\varphi \rangle_{SX_\Gamma}| \leq C \|\chi f\|$$

for all $f \in C^\infty(G/M)$.

We state now a lemma which connects the Patterson-Sullivan distributions on G/M with those on $\Gamma \backslash G/M$.

LEMMA 6.2.6. *Let $f \in C_c^\infty(G/M)$, then*

$$\langle f, \text{PS}_\varphi \rangle_{G/M} = \left\langle \sum_{\gamma \in \Gamma} f \circ \gamma, \text{PS}_\varphi \right\rangle_{\Gamma \backslash G/M}.$$

PROOF. This follows since PS_φ is Γ -invariant. By Definition 6.2.4 we have for any smooth fundamental domain cut-off χ and any $f \in C_c^\infty(G/M)$

$$\begin{aligned} \left\langle \sum_{\gamma \in \Gamma} f \circ \gamma, \text{PS} \right\rangle_{\Gamma \backslash G/M} &= \left\langle \chi \cdot \sum_{\gamma \in \Gamma} f \circ \gamma, \text{PS}_\varphi \right\rangle_{G/M} \\ &= \left\langle \sum_{\gamma \in \Gamma} \chi \cdot (f \circ \gamma), \text{PS}_\varphi \right\rangle_{G/M} \\ &\stackrel{\Gamma\text{-invariance}}{=} \left\langle f \cdot \sum_{\gamma \in \Gamma} \chi \circ \gamma^{-1}, \text{PS}_\varphi \right\rangle_{G/M} \\ &= \langle f, \text{PS}_\varphi \rangle_{G/M}, \end{aligned}$$

as $\sum_{\gamma \in \Gamma} \chi \circ \gamma^{-1} = 1$, since χ is a fundamental domain cut-off. \square

Let $a \in C^\infty(\Gamma \backslash G/M)$ and let π_M denote the projection from $C(G) \rightarrow C(G/M)$ given by

$$\pi_M(f)(g) := \int_M f(gm) dm.$$

Further, we write $f^n := f \circ r_n$ for $n \in N$ for functions f , i.e. $f^n(g) = f(gn)$ for $g \in G$. Putting $a = \varphi_k$ and applying the theory of Chapter 4 we obtain:

THEOREM 6.2.7. *Let $G = \text{SO}_o(1, l) = \text{ANK}$, $M = Z_K(A)$, $X_{e_1} \in \mathfrak{n}$ as for (2.7) and $\Gamma \subset G$ a uniform lattice. Further, let φ_k be an automorphic Laplace eigenfunction on $\Gamma \backslash G/K$, then $n \mapsto \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}$ is bi- M -invariant and a solution to*

$$\Omega f = -(\lambda_k^2 + \rho_0^2) f = -\frac{1}{4\rho_0}(\rho_0^2 + r_k^2) f$$

on N . On the slice $\exp \mathbb{R}^+ X_{e_1}$ it satisfies for any j and k for $l > 2$

$$\langle \pi_M(\varphi_k^{\exp s X_{e_1}}), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} = \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1(a, b, \rho_0; -s^2),$$

where $a = \frac{1}{2}(\rho_0 + ir_k)$ and $b = \frac{1}{2}(\rho_0 - ir_k)$ as in Section 4.2. For $l = 2$ it is given by

$$\begin{aligned} \langle \pi_M(\varphi_k^{\exp s X_{e_1}}), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} &= \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1\left(a, b, \frac{1}{2}; -s^2\right) \\ &\quad + 2i \langle X_1 \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot s \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -s^2\right). \end{aligned}$$

PROOF. As Ω and translations commute

$$\Omega \varphi_k^n = -\frac{1}{4\rho_0}(\rho_0^2 + r_k^2) \varphi_k^n$$

for all $n \in N$. If we express φ_k^n in local charts of the compact manifold $SX_\Gamma = \Gamma \backslash G/M$ as $\varphi_k^n(x_l)$, then all partial derivatives of $\frac{\partial^\alpha}{\partial x_l^\alpha} \varphi_k^n(x_l)$ in these local coordinates x_l are simultaneously continuous in x_l and n . Thus, we can interchange the

distribution with differentiation using [Sch57, Ch.IV Th.II] to get

$$\begin{aligned} & \Omega \left(n \mapsto \int_{\Gamma \backslash G/M} \int_M \varphi_k^n(\Gamma gm) dm \text{PS}_{\varphi_j}(\Gamma gM) \right) \\ &= \int_{\Gamma \backslash G/M} \int_M \Omega(n \mapsto \varphi_k^n(\Gamma gm)) dm \text{PS}_{\varphi_j}(\Gamma gM) \\ &= n \mapsto -\frac{1}{4\rho_0}(\rho_0^2 + r_k^2) \int_{\Gamma \backslash G/M} \int_M \varphi_k^n(\Gamma gm) dm \text{PS}_{\varphi_j}(\Gamma gM). \end{aligned}$$

Obviously, for all $m_1, m_2 \in M$

$$\begin{aligned} \int_{\Gamma \backslash G/M} \int_M \varphi_k^{m_1 m_2}(\Gamma gm) dm \text{PS}_{\varphi_j}(\Gamma gM) &= \int_{\Gamma \backslash G/M} \int_M \varphi_k(\Gamma gmm_1 m_2) dm \text{PS}_{\varphi_j}(\Gamma gM) \\ &= \int_{\Gamma \backslash G/M} \int_M \varphi_k(\Gamma gmn) dm \text{PS}_{\varphi_j}(\Gamma gM) \end{aligned}$$

by unimodularity of M and right- K -invariance of φ_k . Hence,

$$F(n) := \int_{\Gamma \backslash G/M} \int_M \varphi_k^n(\Gamma gm) dm \text{PS}_{\varphi_j}(\Gamma gM) = \int_{\Gamma \backslash G/M} \int_M \varphi_k(\Gamma gmn) dm \text{PS}_{\varphi_j}(\Gamma gM)$$

is a bi- M -invariant solution on N , in particular at the origin $n = e$, to

$$(6.8) \quad \Omega F + \mu_k F = 0,$$

where $\mu_k = \frac{1}{4\rho_0}(\rho_0^2 + r_k^2)$ and we can restrict F to the slice $S = \exp \mathbb{R}^+ X_1 = \exp \mathbb{R}^+ X_{e_1}$. By Theorem 4.1.7 we know that on the slice S equation (6.8) reads for bi- M -invariant functions F , see equation (4.6),

$$\left((\alpha(H_1)^2 s^2 + 2) \frac{d^2}{ds^2} + \left((\alpha(H_1)^2 + 2\alpha(H_\rho)) s + 2 \frac{l-2}{s} \right) \frac{d}{ds} + \mu_k \right) F(s) = 0.$$

For $l > 3$, by Theorem 4.3.1, the restriction of the function $F(n)$ to the slice is then given by

$$\begin{aligned} F(s) := F(\exp sX_1) &= F(0) \cdot {}_2F_1 \left(a, b, \rho_0; \frac{-s^2}{4(l-1)} \right) \\ &\stackrel{\text{def. of } F}{=} \langle \varphi_k^{\exp 0X_1}, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1 \left(a, b, \rho_0; \frac{-s^2}{4(l-1)} \right) \\ &= \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1 \left(a, b, \rho_0; \frac{-s^2}{4(l-1)} \right) \end{aligned}$$

resp.

$$F(\exp sX_{e_1}) = \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1(a, b, \rho_0; -s^2).$$

The claim for $l = 2$ follows similarly. \square

LEMMA 6.2.8. *For $a \in C^\infty(\Gamma \backslash G/M)$ and any fundamental domain cut-off χ we have*

$$\langle \pi_M(\chi a)^n, \text{PS}_\varphi \rangle_{G/M} = \langle \pi_M(a^n), \text{PS}_\varphi \rangle_{\Gamma \backslash G/M}.$$

PROOF.

$$\begin{aligned}
\langle \pi_M((\chi a)^n), \text{PS}_\varphi \rangle_{G/M} dn &= \int_{G/M} \int_M (\chi a)(gmn) dm \text{PS}_\varphi(gM) \\
&= \int_{\Gamma \backslash G/M} \sum_{\gamma \in \Gamma} \int_M (\chi a)(\gamma gmn) dm \text{PS}_\varphi(\Gamma gM), \text{ as } \int_{G/M} = \int_{\Gamma \backslash G/M} \sum_{\gamma \in \Gamma}, \\
&\text{see Lemma 6.2.6} \\
&= \int_{\Gamma \backslash G/M} \int_M a(gmn) \sum_{\gamma \in \Gamma} \chi(\gamma gmn) dm \text{PS}_\varphi(\Gamma gM), \text{ as } a \text{ is } \Gamma\text{-invariant} \\
&= \int_{\Gamma \backslash G/M} \int_M a(gmn) dm \text{PS}_\varphi(\Gamma gM), \text{ as } \chi \text{ is a fundamental domain cut-off} \\
&= \int_{\Gamma \backslash G/M} \pi_M(a^n)(g) \text{PS}_\varphi(\Gamma gM) \\
&= \langle \pi_M(a^n), \text{PS}_\varphi \rangle_{SX_\Gamma}.
\end{aligned}$$

□

PROPOSITION 6.2.9. *Let φ_j be an automorphic eigenfunction in the principal series and φ_k any automorphic eigenfunction with eigenvalues $-(\rho_0^2 + \lambda_j^2)$ resp. $-\frac{1}{4\rho_0}(\rho_0^2 + r_k^2)$. Then*

$$\int_N e^{-(i\lambda_j + \rho)H(n^{-1}w)} \langle \pi_M(\varphi_k)^n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn$$

converges absolutely.

PROOF. We recall that $G = SO_o(1, l)$ and $M \cong SO(l-1)$. First we note that the integrand $n \mapsto e^{-(i\lambda + \rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_\lambda \rangle_{SX_\Gamma}$ of (6.12) is bi- M -invariant. We find then on the slice $S = \{\exp sX_{e_1} : s > 0\}$ that

$$e^{-(i\lambda_j + \rho)H(\exp -sX_{e_1}w)} = (1 + s^2)^{-(i\lambda_j + \rho_0)},$$

see (2.18). Further, by Theorem 6.2.7 for $l \geq 3$

$$\langle \pi_M(\varphi_k^{\exp sX_{e_1}}), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} = \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot {}_2F_1(a, b, \rho_0; -s^2)$$

on the slice S . Hence, we can use for $l \geq 3$ polar coordinates and we get the following

$$\begin{aligned}
&\int_N e^{-(i\lambda_j + \rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn \\
&= \omega_{l-1} \int_0^\infty s^{l-2} (1 + s^2)^{-(i\lambda_j + \rho_0)} \langle \pi_M(\varphi_k^{\exp sX_{e_1}}), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} ds \\
&\stackrel{\text{Prop. 6.2.7}}{=} \omega_{l-1} \int_0^\infty s^{l-2} (1 + s^2)^{-(i\lambda_j + \rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds \cdot \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} =: (*).
\end{aligned}$$

For $l = 2$ we know that $N = \{\exp sX_{e_1} : s \in \mathbb{R}\}$ and by Theorem 6.2.7 with $a = \varphi_k$ we get

$$\begin{aligned}
& \int_N e^{-(i\lambda_j + \rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn \\
&= \int_{-\infty}^{\infty} (1+s^2)^{-(i\lambda_j + \rho_0)} \langle (\varphi_k^{\exp sX_{e_1}}), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} ds \\
&\stackrel{\text{Prop. 6.2.7}}{=} \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot \int_{-\infty}^{\infty} (1+s^2)^{-(i\lambda_j + \rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds \\
&\quad + 2i \langle X_{e_1} \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot \int_{-\infty}^{\infty} (1+s^2)^{-(i\lambda_j + \rho_0)} s \cdot {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -s^2\right) ds \\
&= \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot \int_{-\infty}^{\infty} (1+s^2)^{-(i\lambda_j + \rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds =: (**).
\end{aligned}$$

Here $a = \frac{1}{2}(\rho_0 + ir_k)$ and $b = \frac{1}{2}(\rho_0 - ir_k)$. We want to use Lemma 6.2.19 from the appendix to this chapter to investigate the convergence of (*) resp. (**). Thus, we have to check whether

$$(6.9) \quad 2\text{Re}(\rho_0 + i\lambda_j) > \rho_0 + \text{Re}(ir_k).$$

As φ_j is in the principal series $2\text{Re}(\rho_0 + i\lambda_j) = 2\rho_0$. Because

$$0 \leq \frac{1}{4\rho_0}(\rho_0^2 + r_k^2)$$

either $r_k \in \mathbb{R}$ or $r_k \in i\mathbb{R}$. In any case $\text{Re}(ir_k) \leq 0$ and (6.9) is true. The integrals in (*) resp. (**) now converge absolutely by Theorem 6.2.19. \square

From now on we will always assume that $G = SO_o(1, l)$. In order to explain the connection between Wigner- and Patterson-Sullivan distributions we need the following continuity property of the latter ones.

PROPOSITION 6.2.10. *Let $a \in C^\infty(\Gamma \backslash G/M)$. Then*

$$\langle L_\lambda(\chi a), \text{PS}_\varphi \rangle_{G/M} = \int_N e^{-(i\lambda + \rho)H(n^{-1}w)} \langle \pi_M((\chi a)^n), \text{PS}_\varphi \rangle_{G/M} dn,$$

where $\chi \in C_c^\infty(G/M)$ is a smooth fundamental domain cut-off and φ an automorphic eigenfunction with eigenvalue $-(\lambda^2 + \rho_0^2)$.

PROOF. By Theorem 6.2.2 we know that $\langle L_\lambda(\chi a), \text{PS}_\varphi \rangle_{G/M}$ is finite. Thus,

$$\begin{aligned}
\langle L_\lambda(\chi a), \text{PS}_\varphi \rangle &\stackrel{\text{def}}{=} \int_{B^2} \mathcal{R}(L_\lambda(\chi a))(b, b') d_\lambda(b, b') dT_\varphi(b) dT_\varphi(b') \\
&\stackrel{\text{Lemma 5.15 in [HS]}}{=} \int_{B^2} \int_X \chi a(z, b) e^{(i\lambda + \rho)(\langle z, b \rangle + \langle z, b' \rangle)} dz dT_\varphi(b) dT_\varphi(b') \\
&=: (*)
\end{aligned}$$

We then identify $G/M = X \times B = G/K \times K/M$ and assume that $\chi(z, b) = \chi(z)$ is independent of b , i.e. $\chi \in C_c^\infty(G/K)$. Further we assume that T_φ equals $D_b^\alpha F_\varphi$ for some differential operator D_b^α on B and some $F_\varphi \in C(B)$ in the distributional sense, see [GO05, Th. 1.3.]. That is,

$$\langle f, T_\varphi \rangle_B = \int_B (D_b^\alpha f)(b) F_\varphi(b) db$$

for all $f \in C^\infty(B)$. Then we also identify $X = AN$, $dx = dn da$. Thus,

$$\begin{aligned}
(*) &= \int_{B^2} \int_{AN} \chi(an) D_b^\alpha D_{b'}^\alpha \left(a(an, b) e^{(i\lambda+\rho)(\langle an, b \rangle + \langle an, b' \rangle)} \right) F_\varphi(b) F_\varphi(b') dndadbdb' \\
&\stackrel{\text{Fubini}}{=} \int_N \int_{B^2} \int_A \chi(an) D_b^\alpha D_{b'}^\alpha \left(a(an, b) e^{(i\lambda+\rho)(\langle an, b \rangle + \langle an, b' \rangle)} \right) F_\varphi(b) F_\varphi(b') dadbdb' dn \\
&= \int_N \int_{B^{(2)}} \int_A \chi(an) D_b^\alpha D_{b'}^\alpha \left(a(an, b) e^{(i\lambda+\rho)(\langle an, b \rangle + \langle an, b' \rangle)} \right) F_\varphi(b) F_\varphi(b') dadbdb' dn \\
&= \int_N \int_{B^{(2)}} \int_A \chi(an) a(an, b) e^{(i\lambda+\rho)(\langle an, b \rangle + \langle an, b' \rangle)} dadT_\varphi(b) dT_\varphi(b') dn =: (**).
\end{aligned}$$

We write χa as a function on G/M , compare [HS, (6.15ff)]. For any $g \in G$ with $(g \cdot M, g \cdot wM) = (b, b')$ we have

$$\chi a(gan \cdot o, b) = \chi a(gan \cdot o, g \cdot M) = \chi a(gan \cdot o, gan \cdot M) = \chi a(ganM).$$

Since $dx = dadn$ is G -invariant, (**) equals

$$\int_N \int_{G/MA} \int_A \chi a(ganM) e^{(i\lambda+\rho)(\langle gan \cdot o, g \cdot M \rangle + \langle gan \cdot o, g \cdot wM \rangle)} dadT_\varphi \otimes dT_\varphi(gMA) dn.$$

Then $\langle gan \cdot o, g \cdot M \rangle = \langle gan \cdot o, gan \cdot M \rangle = H(gan) = H(ga)$, since $g \cdot M = b$ and $P = MAN$, in particular AN fixes M . Further,

$$\begin{aligned}
\langle gan \cdot o, g \cdot wM \rangle &= -H(n^{-1}a^{-1}w) + H(gw) \\
&= -H(n^{-1}w) + H(a) + H(gw) \\
&= -H(n^{-1}w) + H(gaw),
\end{aligned}$$

since $\langle g \cdot x, g \cdot b \rangle = \langle x, b \rangle + \langle g \cdot o, g \cdot b \rangle$, $H(ga) = H(g) + H(a)$, see [Hel94, Ch.II (46)], for all $a \in A$ and $g \in G$. Also $H(waw^{-1}) = H(a^{-1})$ for all $a \in A$. Hence,

$$\begin{aligned}
&\int_N \int_{G/MA} \int_A \chi a(ganM) e^{(i\lambda+\rho)(\langle gan \cdot o, g \cdot M \rangle + \langle gan \cdot o, g \cdot wM \rangle)} dadT_\varphi \otimes dT_\varphi(gMA) dn \\
&= \int_N \int_{G/MA} \int_A \chi a(ganM) e^{(i\lambda+\rho)(H(ga) + H(gaw))} e^{-(i\lambda+\rho)H(n^{-1}w)} dadT_\varphi \otimes dT_\varphi(gMA) dn \\
&= \int_N \int_{G/MA} \int_A \chi a(ganM) e^{-(i\lambda+\rho)H(n^{-1}w)} d_\lambda(gMA) dT_\varphi da \otimes dT_\varphi(gMA) dnda \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \int_{G/MA} d_\lambda(gMA) \int_A (\chi a)^n(gaM) dadT_\varphi \otimes dT_\varphi(gMA) dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \int_{G/M} \chi a(gnM) dPS_\varphi(gM).
\end{aligned}$$

Now H is the projection belonging to $G = KAN$, which is bi- M -invariant, and w normalizes M . Hence, for any $m \in M$

$$H(n^{-1}w) = H(n^{-1}ww^{-1}mw) = H(n^{-1}mw) = H(m^{-1}n^{-1}mw).$$

Furthermore, $\chi a \in C_c^\infty(G/M)$ and $\int_M = 1$, so

$$\begin{aligned}
&\int_N \int_{G/M} e^{-(i\lambda+\rho)H(n^{-1}w)} \chi a(gn) dPS_\varphi(gM) dn \\
&= \int_N \int_{G/M} \int_M e^{-(i\lambda+\rho)H(m^{-1}n^{-1}mw)} \chi a(gnm) dmdPS_\varphi(gM) dn.
\end{aligned}$$

For every $m \in M$ the mapping $n \mapsto mnm^{-1}$ is an automorphism of N fixing dn , thus

$$\begin{aligned}
& \int_N \int_{G/M} \int_M e^{-(i\lambda+\rho)H(m^{-1}n^{-1}mw)} \chi a(gnm) dm d\text{PS}_\varphi(gM) dn \\
&= \int_N \int_{G/M} \int_M e^{-(i\lambda+\rho)H(n^{-1}w)} \chi a(gmn) dm d\text{PS}_\varphi(gM) dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \int_{G/M} \int_M \chi a(gmn) dm d\text{PS}_\varphi(gM) dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \langle \pi_M((\chi a)^n), \text{PS}_\varphi \rangle_{G/M} dn.
\end{aligned}$$

We note that $\pi_M((\chi a)^n)$ is indeed in $C_c^\infty(G/M)$ as M is compact. \square

In view of Lemma 6.2.8 we also get the following corollary.

COROLLARY 6.2.11. *Let $a \in C^\infty(\Gamma \backslash G/M)$. Then*

$$\langle L_\lambda(\chi a), \text{PS}_\varphi \rangle_{G/M} = \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \langle \pi_M(a^n), \text{PS}_\varphi \rangle_{SX_\Gamma} dn.$$

Hence, combining this theorem, Corollary 6.2.11 and Theorem 6.2.2 we get an exact relation between Wigner- and Patterson-Sullivan distributions from the principal series on the level of non-constant automorphic eigenfunctions. For $k \in \mathbb{C}$ with $\text{Re}(k) > \rho_0$ let us set

$$(6.10) \quad I(a, b, \rho_0, k) := \int_0^\infty s^{l-2} (1+s^2)^{-k} {}_2F_1(a, b, \rho_0; -s^2) ds,$$

which depends on k and the eigenfunction φ_j with eigenvalue $\mu_j = -\frac{1}{4\rho_0}(\rho_0^2 + r_j^2)$, $a = \frac{1}{2}(\rho_0 + ir_j)$, $b = \frac{1}{2}(\rho_0 - ir_j)$.

THEOREM 6.2.12. *Let $G = SO_o(1, l)$, φ_j be an automorphic eigenfunction in the principal series and φ_k be any non-constant automorphic eigenfunctions, then*

$$\begin{aligned}
\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} &= \omega_{l-1} \int_0^\infty s^{l-2} (1+s^2)^{-(i\lambda_j+\rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds \cdot \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \\
(6.11) \quad &= \omega_{l-1} I(a, b, \rho_0, i\lambda_j + \rho_0) \cdot \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}
\end{aligned}$$

with $I(a, b, \rho_0, i\lambda_j + \rho_0)$ as in (6.10).

PROOF. For any φ_k we have

$$\begin{aligned}
\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle &\stackrel{6.2.2}{=} \langle L_{\lambda_j}(\chi\varphi_k), \text{PS}_{\varphi_j} \rangle_{G/M} \\
(6.12) \quad &\stackrel{6.2.11}{=} \int_N e^{-(i\lambda_j+\rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn.
\end{aligned}$$

As in the proof of Proposition 6.2.9 we find that for $l \geq 3$

$$\begin{aligned}
& \int_N e^{-(i\lambda_j+\rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn \\
&= \omega_{l-1} \int_0^\infty s^{l-2} (1+s^2)^{-(i\lambda_j+\rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds \cdot \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}.
\end{aligned}$$

For $l = 2$ we have

$$\begin{aligned}
& \int_N e^{-(i\lambda_j+\rho)H(n^{-1}w)} \langle \pi_M(\varphi_k^n), \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} dn \\
&= \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot \int_{-\infty}^\infty (1+s^2)^{-(i\lambda_j+\rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds.
\end{aligned}$$

The last equality of (6.11) follows from the definition of $I(a, b, \rho_0, k)$, see (6.10), and the convergence of the integral $I(a, b, \rho_0, i\lambda_j + \rho_0)$ follows from Theorem 6.2.19. \square

We excluded the case where φ_j is a complementary series eigenfunctions, as we do not know whether the integral

$$\int_0^\infty s^{l-2} (1+s^2)^{-(i\lambda_j+\rho_0)} {}_2F_1\left(\frac{1}{2}(\rho_0+ir_k), \frac{1}{2}(\rho_0-ir_k), \rho_0; -s^2\right) ds$$

converges in this case. It surely converges absolutely if φ_j is in the principal series as Theorem 6.2.19 shows.

But we can still say something about the relation between Wigner- and Patterson-Sullivan distributions in the case of the complementary series. Let us define

$$C_{k,j} = \begin{cases} \frac{\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)}}{\langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}} & , \quad \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \neq 0 \\ 0 & , \quad \text{else} \end{cases}$$

THEOREM 6.2.13. *For any automorphic eigenfunctions φ_k and φ_j we have*

$$\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} = C_{k,j} \cdot \langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}.$$

PROOF. It suffices to consider the case of $\langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} = 0$ and show that it implies $\langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} = 0$. If we assume $\langle \varphi_k, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} = 0$, it follows by Theorem 6.2.7 that for any smooth fundamental domain cut-off χ

$$n \mapsto \langle \pi_M(\chi\varphi_k)^n, \text{PS}_{\varphi_j} \rangle_{G/M}$$

vanishes identically. Thus,

$$\int_N e^{-(i\lambda_j+\rho)H(n^{-1}w)} \langle \pi_M(\chi\varphi_k)^n, \text{PS}_{\varphi_j} \rangle_{G/M} dn = 0.$$

But then it follows by Proposition 6.2.10 and Theorem 6.2.2 that

$$\begin{aligned} \int_N e^{-(i\lambda_j+\rho)H(n^{-1}w)} \langle \pi_M(\chi\varphi_k)^n, \text{PS}_{\varphi_j} \rangle_{G/M} dn & \stackrel{\text{Prop. 6.2.10}}{=} \langle L_{\lambda_j}(\chi\varphi_k), \text{PS}_{\varphi_j} \rangle_{G/M} \\ & \stackrel{\text{Thm. 6.2.2}}{=} \langle \text{Op}(\varphi_k)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} = 0. \end{aligned}$$

\square

Note that by Proposition 6.2.21 from the appendix to this chapter we will know how $C_{k,j}$ decays, if φ_j is in the principal series.

DEFINITION 6.2.14. For an automorphic eigenfunction φ with eigenvalue $-(\lambda^2 + \rho_0^2)$ from the principal series we define the *normalized Patterson-Sullivan distribution* by

$$\langle a, \widehat{\text{PS}}_\varphi \rangle_{SX_\Gamma} := \frac{\langle a, \text{PS}_\varphi \rangle_{SX_\Gamma}}{\langle \mathbf{1}, \text{PS}_\varphi \rangle_{SX_\Gamma}}.$$

It will follow from the next proposition that $\langle \mathbf{1}, \text{PS}_\varphi \rangle_{SX_\Gamma} \neq 0$, if φ is in the principal series. For the normalization of Patterson-Sullivan distributions we have:

PROPOSITION 6.2.15. *Let φ be an automorphic eigenfunction with eigenvalue $-(\lambda^2 + \rho_0^2)$ such that $\text{Re}(\lambda) \in \mathfrak{a}_+^*$, i.e. such that φ_j is in the principal series. Then*

$$\langle L_\lambda(\chi\mathbf{1}), \text{PS}_\varphi \rangle_{G/M} = \langle \mathbf{1}, \text{PS}_\varphi \rangle_{\Gamma \backslash G/M} \cdot \mathcal{C}(\lambda).$$

$$\text{Here } \mathcal{C}(\lambda) = \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} dn = \omega_{l-1} \int_0^\infty s^{l-2} (1+s^2)^{-(i\lambda+\rho_0)} ds.$$

PROOF. It is

$$\begin{aligned}
\langle L\lambda(\chi\mathbf{1}), \text{PS}_\varphi \rangle_{G/M} &= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \langle \pi_M(\chi^n), \text{PS}_\varphi \rangle_G dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \int_{G/M} \int_M \chi^n(gm) dm \text{PS}_\varphi(gM) dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \int_{G/M} \pi_M(\chi^n)(gM) \text{PS}_\varphi(gM) dn \\
&= \int_N e^{-(i\lambda+\rho)H(n^{-1}w)} \langle \mathbf{1}, \text{PS}_\varphi \rangle_{\Gamma \backslash G/M} dn = \mathcal{C}(\lambda) \langle \mathbf{1}, \text{PS}_\varphi \rangle_{\Gamma \backslash G/M}.
\end{aligned}$$

□

REMARK 6.2.16. We recall Harish-Chandra's c -function which is given by

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)H(\bar{n})} d\bar{n},$$

where the measure $d\bar{n}$ on \bar{N} is normalized such that $\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1$, see Theorem 2.4.7. Now with our normalization of dn via $N \cong \bar{N} \cong \mathbb{R}^{l-1}$, $X_u \mapsto \theta X_u \mapsto u$, we compute

$$\begin{aligned}
\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} &= \int_N e^{-2\rho(H(n_{-u}w))} du \\
&= \omega_{l-1} \int_0^\infty (1+s^2)^{-2\rho_0} ds \\
&= \frac{\omega_{l-1}}{2} B(\rho_0, \rho_0).
\end{aligned}$$

Together with Theorem 6.2.2 we get:

COROLLARY 6.2.17. *If φ_j is an automorphic eigenfunction in the principal series with eigenvalue $-(\lambda_j^2 + \rho_0^2)$, then*

$$\langle \mathbf{1}, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot \mathcal{C}(\lambda_j) = 1$$

resp.

$$\langle a, \widehat{\text{PS}}_{\varphi_j} \rangle_{SX_\Gamma} = \mathcal{C}(\lambda_j) \langle a, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}$$

for $a \in C^\infty(SX_\Gamma)$.

Finally, let us state the refinement of Proposition 6.1.3, we obtained in this section by Theorems 6.2.12 and 6.2.13.

THEOREM 6.2.18. *Let $G = SO_o(1, l) = NAK$, $M = Z_K(A)$, $\Gamma \subset G$ a uniform lattice and $SX_\Gamma = \Gamma \backslash G/M$ the unit sphere bundle. We fix an orthonormal basis $\{\varphi_j\}_j$ of $L^2(\Gamma \backslash X)$ of automorphic Laplace-eigenfunctions with eigenvalues $-(\lambda_j^2 + \rho_0^2) = -\frac{1}{4\rho_0} (r_j^2 + \rho_0^2)$ and choose a non-constant φ_k in $\{\varphi_j\}_j$. Finally, let $f \in C^\infty(G//K)$ such that $\pi_R(f)$ is of trace class, where π_R is the right-regular representation of G on $L^2(\Gamma \backslash X)$. For example $f \in C_c^\infty(G//K)$ suffices. Then $\varphi_k \cdot \pi_R(f)$ is also of trace class with trace given by*

$$\begin{aligned}
\mathrm{Tr}(\varphi_k \cdot \pi_R(f)) &= \sum_j \langle \mathrm{Op}(\varphi_k) \varphi_j, \varphi_j \rangle_{L^2(SX_\Gamma)} \mathcal{S}(f, \lambda_j) \\
&= \sum_{j>0}^{j_0} \langle \mathrm{Op}(\varphi_k) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f, \lambda_j) \\
&\quad + \omega_{l-1} \sum_{j>j_0} I(a, b, \rho_0, \rho_0 + i\lambda_j) \mathcal{S}(f, \lambda_j) \cdot \langle \varphi_n, \mathrm{PS}_{\lambda_j} \rangle_{SX_\Gamma} \\
&= \sum_{j>0}^{j_0} C_{k,j} \mathcal{S}(f, \lambda_j) \cdot \langle \varphi_k, \mathrm{PS}_{\varphi_j} \rangle_{SX_\Gamma} \\
&\quad + \omega_{l-1} \sum_{j>j_0} I(a, b, \rho_0, \rho_0 + i\lambda_j) \mathcal{S}(f, \lambda_j) \cdot \langle \varphi_n, \mathrm{PS}_{\lambda_j} \rangle_{SX_\Gamma}
\end{aligned}$$

where $a = \frac{1}{2}(\rho_0 + ir_k)$, $b = \frac{1}{2}(\rho_0 - ir_k)$, $\rho_0 = \frac{l-1}{2}$ and $\mathcal{S}(f)$ denotes the spherical transform.

6.2.1. Convergence and asymptotics of $I(a, b, \rho_0, z)$. In this appendix to Section 6.2 we want to discuss the convergence and asymptotic behaviour of the integrals

$$(6.13) \quad I(a, b, \rho_0, z) = \int_0^\infty s^{l-2} (1+s^2)^{-z} {}_2F_1(a, b, \rho_0; -s^2) ds,$$

we needed in Proposition 6.2.9. The next theorem determines $z \in \mathbb{C}$ for which (6.13) converges absolutely.

LEMMA 6.2.19. *Let φ_k be an automorphic Laplace eigenfunction with eigenvalue $\mu_k = -\frac{1}{4\rho_0}(\rho_0^2 + r_k^2)$, so $a = \frac{1}{2}(\rho_0 + ir_k)$, $b = \frac{1}{2}(\rho_0 - ir_k)$. The integral*

$$\int_0^\infty s^{l-2} (1+s^2)^{-z} {}_2F_1(a, b, \rho_0; -s^2) ds$$

converges absolutely for $\rho_0 + \mathrm{Re}(ir_k) < 2\mathrm{Re}(z)$, more precisely

$$\int_0^\infty |s^{n-2} (1+s^2)^{-z} {}_2F_1(a, b, \rho_0; -s^2)| ds \leq C \int_0^\infty (s+1)^{-2\mathrm{Re}(z) + \rho_0 + \mathrm{Re}(ir_k)} ds$$

for some constant C independent of z and r_k .

PROOF. The proof in [AZ07, Prop. 5.2.] generalizes almost verbatim. First we note that $s^{l-2}(s^2+1)^{-z}$ is asymptotically $s^{2\rho_0-1-2\mathrm{Re}(z)}$, where $\rho_0 = \frac{l-1}{2}$. The hypergeometric factor can be controlled by a formula for hypergeometric functions, see for example [GV88, (4.7.23)],

$$\begin{aligned}
{}_2F_1(\alpha, \beta, \gamma; s) &= \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} |s|^{-\alpha} {}_2F_1(\alpha, 1-\gamma+\alpha, 1-\beta+\alpha; s^{-1}) \\
(6.14) \quad &+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} |s|^{-\beta} {}_2F_1(\beta, 1-\gamma+\beta, 1-\alpha+\beta; s^{-1}).
\end{aligned}$$

Since ${}_2F_1(\alpha, \beta, \gamma; 0) = 1$ it follows that asymptotically ${}_2F_1(a, b, \rho_0; -s^2)$ equals

$$\frac{\Gamma(\rho_0)\Gamma(-ir_k)}{\Gamma(\frac{\rho_0}{2} - \frac{ir_k}{2})^2} \cdot |s|^{-(\rho_0+ir_k)} + \frac{\Gamma(\rho_0)\Gamma(ir_k)}{\Gamma(\frac{\rho_0}{2} + \frac{ir_k}{2})^2} \cdot |s|^{-(\rho_0-ir_k)}.$$

Now $\Gamma(x+iy) \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|y|} |y|^{x-\frac{1}{2}}$ for $|y| \rightarrow \infty$, see (2.25). Hence the ratios $\frac{\Gamma(\rho_0)\Gamma(-ir_k)}{\Gamma(\frac{\rho_0}{2} - \frac{ir_k}{2})^2}$ and $\frac{\Gamma(\rho_0)\Gamma(ir_k)}{\Gamma(\frac{\rho_0}{2} + \frac{ir_k}{2})^2}$ are uniformly bounded in r_k . Thus, for some constant $C > 0$ independent of r_k and for all $s \geq 0$

$$|{}_2F_1(a, b, \rho_0; -s^2)| \leq C(1+s)^{-\rho_0 + \operatorname{Re}(ir_k)}.$$

Multiplying the two terms together we find that the integrand

$$s^{l-2} (s^2 + 1)^{-z} {}_2F_1(a, b, \rho_0; -s^2)$$

is asymptotically bounded by

$$C(1+s)^{\rho_0 - 1 + \operatorname{Re}(ir_k) - 2\operatorname{Re}(z)},$$

which is integrable on $[0, \infty)$ iff

$$\rho_0 - 1 + \operatorname{Re}(ir_k) - 2\operatorname{Re}(z) < -1,$$

i.e. iff

$$\rho_0 + \operatorname{Re}(ir_k) < 2\operatorname{Re}(z).$$

□

In particular $I(a, b, \rho_0, i\lambda + \rho_0)$ converges if $-(\lambda^2 + \rho_0^2)$ is an eigenvalue from the principal series, that is, $\lambda \in \mathbb{R}$. Next, we want to show that $k \mapsto I(a, b, \rho_0, k)$ for fixed a and b can be meromorphically extended to \mathbb{C} .

LEMMA 6.2.20. *Let φ be a Laplacian eigenfunction with eigenvalue $-\frac{1}{4\rho_0}(r^2 + \rho_0^2)$, $a = \frac{1}{2}(\rho_0 + ir)$, $b = \frac{1}{2}(\rho_0 - ir)$.*

$$\begin{aligned} I(a, b, \rho_0, z) &= \frac{1}{2} \frac{\Gamma(\rho_0)\Gamma(a - \rho_0 + z)\Gamma(b - \rho_0 + z)}{\Gamma(z)^2} \\ (6.15) \quad &= \frac{1}{2} \frac{\Gamma(\rho_0)\Gamma(z - a)\Gamma(z - b)}{\Gamma(z)^2} \end{aligned}$$

defines a meromorphic continuation to \mathbb{C} with poles exactly in $a, a-1, a-2, \dots$ and $b, b-1, b-2, \dots$. Furthermore, $I(a, b, \rho_0, k)$ does not vanish for $\operatorname{Re}(k) > 0$.

PROOF. Let $z \in \mathbb{C}$ with $\operatorname{Re}(z) > \rho_0$. By Lemma 6.2.19

$$I(a, b, \rho_0, z) = \int_0^\infty s^{l-2} (s^2 + 1)^{-z} {}_2F_1(a, b, \rho_0, -s^2) ds$$

converges absolutely. We use now an integral transform from [Bat54, 20.2 (9)] to get

$$\begin{aligned} \int_0^\infty s^{l-2} (s^2 + 1)^{-z} {}_2F_1(a, b, \rho_0, -s^2) ds &\stackrel{s \mapsto \sqrt{s}}{=} \frac{1}{2} \int_0^\infty s^{\rho_0-1} (s+1)^{-k} {}_2F_1(a, b, \rho_0; -s) ds \\ &= \frac{1}{2} \frac{\Gamma(\rho_0)\Gamma(z + a - \rho_0)\Gamma(z + b - \rho_0)}{\Gamma(z)^2} \\ (6.16) \quad &= \frac{1}{2} \frac{\Gamma(\rho_0)\Gamma(z - b)\Gamma(z - a)}{\Gamma(z)^2}. \end{aligned}$$

The remaining claims now follow from properties of the Gamma function, see Remark 2.5.1. □

Now we want to examine how

$$\int_0^\infty s^{l-2} (1 + s^2)^{-\rho_0 + i\lambda_j} {}_2F_1(a, b, \rho_0; -s^2) ds,$$

grows as λ_j tends to ∞ . In (6.16) we showed that

$$(6.17) \quad \int_0^\infty s^{l-2} (1 + s^2)^{-(i\lambda_j + \rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds$$

equals

$$\frac{1}{2} \frac{\Gamma(\rho_0) \Gamma(\frac{\rho_0}{2} + i(\lambda_j + \frac{r_k}{2})) \Gamma(\frac{\rho_0}{2} + i(\lambda_j - \frac{r_k}{2}))}{\Gamma(i\lambda_j + \rho_0)^2}.$$

Then we make use of the asymptotic formula for the Γ -function

$$\Gamma(x + iy) \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|y|} |y|^{x-\frac{1}{2}}, \quad y \rightarrow \infty,$$

see (2.25). It follows that

$$\int_0^\infty s^{l-2} (1+s^2)^{-(i\lambda_j+\rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds$$

asymptotically equals, as $\lambda_j \rightarrow \infty$,

$$(6.18) \quad \frac{e^{-\frac{\pi}{2}(\lambda_j + \frac{r_k}{2})} (\lambda_j + \frac{r_k}{2})^{\frac{\rho_0-1}{2}} e^{-\frac{\pi}{2}(\lambda_j - \frac{r_k}{2})} (\lambda_j - \frac{r_k}{2})^{\frac{\rho_0-1}{2}}}{e^{-\pi\lambda_j} \lambda_j^{2\rho_0-1}} \sim \lambda_j^{-\rho_0}.$$

We just have proved the following proposition which will be useful for the meromorphic continuation in Chapter 7.

PROPOSITION 6.2.21. *Let λ_j be in the principal series, i.e. $\lambda_j > 0$. The integral*

$$\int_0^\infty s^{l-2} (s^2 + 1)^{(-i\lambda_j+\rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds$$

decays asymptotically as $\lambda_j^{-\rho_0}$ for $\lambda_j \rightarrow \infty$.

The meromorphic continuation of $\mathcal{R}(\varphi)$ and $\mathcal{Z}(\varphi)$

In this chapter we will at first give meromorphic continuations of $\mathcal{R}(\varphi)$ and $\mathcal{Z}(\varphi)$ which were defined in (5.29) resp. (5.32) on the complex half plane $\{k \in \mathbb{C} : \operatorname{Re}(k) > 2\rho_0\}$ to all of \mathbb{C} . This is done in Section 7.1 and Section 7.2. For the location of possible poles and residues of $\mathcal{R}(\varphi)$ and $\mathcal{Z}(\varphi)$ the focus is on a certain strip \mathcal{S} in \mathbb{C} and the (main) results are summarized in Section 7.3.

In Section 7.4 we will shortly explain how to normalize $\mathcal{Z}(\varphi)$ in order to obtain a simple formula for its residue in the strip \mathcal{S} . The last Section 8.2 compares our results on the zeta function $\mathcal{Z}(\varphi)$ in the surface case with the ones from [AZ07], see also Theorem 1.0.1 and (1.3).

7.1. The meromorphic continuation of $\mathcal{R}(\varphi)$

In this section we will discuss the meromorphic continuation which we obtain by using the formula for the spectral trace from Chapter 6. The case of $G = SO_o(1, 2)$ was dealt with before in [AZ07, Th. 9.1.]. Let us recall that $G = SO_o(1, l) = KAN$ and we fixed a uniform lattice Γ with an orthonormal basis of automorphic Laplace eigenfunctions $\{\varphi_j\}$ in $L^2(\Gamma \backslash G/K)$ with eigenvalues $-(\lambda_j^2 + \rho_0^2) = -\frac{1}{4\rho_0}(\nu_j^2 + \rho_0^2)$. Here $\varphi_0, \dots, \varphi_{j_0}$ are in the complementary series, i.e. $\lambda_0^2, \dots, \lambda_{j_0}^2 \in [-\rho_0^2, 0]$, and φ_j is in the principal series, i.e. $\lambda_j^2 \in (0, \infty)$ for $j > j_0$.

By the results of Chapter 5, see Theorem 5.4.6, we know that the geometric trace of

$$\varphi_n \circ \pi_R(f_k)$$

for any eigenfunction $\varphi_n \in \{\varphi_j\}$ which is orthogonal to constants, is given by the auxiliary zeta function, see (5.29),

$$(7.1) \quad \mathcal{R}(k; \varphi_n) = \sum_{1 \neq [\gamma] \in CT} \sum_{\pi \in \bar{M}} c(\varphi_n, \gamma, \pi, k) (\cosh L_\gamma)^{-k + \rho_0},$$

for $\operatorname{Re}(k) > 2\rho_0$. On the other hand, by Theorem 6.1.3 the spectral trace of $\varphi_n \circ \pi_R(f_k)$ is given for $\operatorname{Re}(k) > 2\rho_0$ by

$$\sum_{j=0}^{\infty} \langle \operatorname{Op}(\varphi_n) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f_k, \lambda_j)$$

which equals

$$\sum_{j=1}^{\infty} \langle \operatorname{Op}(\varphi_n) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \mathcal{S}(f_k, \lambda_j)$$

as φ_n is orthogonal to the constant function by assumption. From Theorem 6.2.18 we then infer that the spectral trace of $\varphi_n \circ \pi_R(f_k)$ equals, if we replace $\langle \operatorname{Op}(\varphi_n) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)}$ by $C_{n,j} \langle \varphi_n, \operatorname{PS}_{\varphi_j} \rangle_{SX_\Gamma}$ and keep in mind that $C_{n,j} = \omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda_j)$, see Theorems 6.2.12 and 6.2.13,

$$\sum_{j=1}^{j_0} C_{n,j} \mathcal{S}(f_k, \lambda_j) \cdot \langle \varphi_n, \operatorname{PS}_{\varphi_j} \rangle_{SX_\Gamma} + \omega_{l-1} \sum_{j=j_0+1}^{\infty} I(a, b, \rho_0, \rho_0 + i\lambda_j) \mathcal{S}(f_k, \lambda_j) \cdot \langle \varphi_n, \operatorname{PS}_{\varphi_j} \rangle_{SX_\Gamma} =: (+).$$

Then we use the formula for the spherical transform of $\mathcal{S}(f_k \cdot \lambda_j)$, see (2.26), to see that (+) equals

$$\begin{aligned} & 2^{k-2} B(k - \rho_0, \rho_0) \omega_{l-1} \sum_{j=1}^{j_0} C_{n,j} B\left(\frac{k - i\lambda_j - \rho_0}{2}, \frac{k + i\lambda_j - \rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} + \\ & 2^{k-2} \omega_{l-1}^2 B(k - \rho_0, \rho_0) \sum_{j=j_0+1}^{\infty} B\left(\frac{k - i\lambda_j - \rho_0}{2}, \frac{k + i\lambda_j - \rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \\ & \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ =: & (*). \end{aligned}$$

Since the operator $\varphi_n \cdot \pi_R(f_k)$ is of trace class for $\text{Re}(k) > 2\rho_0$ we know that (*) coincides with (7.1) for $\text{Re}(k) > 2\rho_0$. To obtain the meromorphic continuation we want to show that (*) converges for any k in \mathbb{C} except from the set

$$(7.2) \quad \begin{aligned} \mathcal{P} & := \{ \rho_0 \pm i\lambda_j, \rho_0 - 2 \pm i\lambda_j, \dots : -(\lambda_j^2 + \rho_0^2) \text{ eigenvalue of the Laplacian} \} \\ & \cup \{ \rho_0, \rho_0 - 1, \rho_0 - 2, \dots \}. \end{aligned}$$

It is clear that

$$k \mapsto 2^{k-2} B(k - \rho_0, \rho_0) \omega_{l-1} \sum_{j=1}^{j_0} C_{n,j} B\left(\frac{k - i\lambda_j - \rho_0}{2}, \frac{k + i\lambda_j - \rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}$$

defines a meromorphic function on \mathbb{C} . The poles correspond to the poles of the Beta functions $B(k - \rho_0, \rho_0)$ and $B\left(\frac{k - i\lambda_j - \rho_0}{2}, \frac{k + i\lambda_j - \rho_0}{2}\right)$, i.e. $k = \rho_0, \rho_0 - 1, \dots$ resp. $k = \rho_0 \pm i\lambda_j, \rho_0 - 2 \pm i\lambda_j, \dots$, where $-(\lambda_j^2 + \rho_0^2)$ is an eigenvalue of the Laplacian.

Thus, for the convergence of (*) we only have to consider the infinite series

$$(7.3) \quad \sum_{j=j_0+1}^{\infty} \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \cdot B\left(\frac{k - i\lambda_j - \rho_0}{2}, \frac{k + i\lambda_j - \rho_0}{2}\right)$$

outside (7.2). Let us check each of the three terms in this series separately. At first, we know that

$$I(a, b, \rho_0, \rho_0 + i\lambda_j) = \int_0^\infty s^{l-2} (1 + s^2)^{-(i\lambda_j + \rho_0)} {}_2F_1(a, b, \rho_0; -s^2) ds$$

behaves for $j \rightarrow \infty$ as $\lambda_j^{-\rho_0}$, see Proposition 6.2.21, because φ_j is in the principal series. For the term $\langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}$ we claim the following:

LEMMA 7.1.1. *For any automorphic eigenfunction φ_n , $\langle \varphi_n, \widehat{\text{PS}}_{\lambda_j} \rangle_{SX_\Gamma}$ is uniformly bounded, i.e. there is some constant $C > 0$ independent of j such that*

$$\left| \langle \varphi_n, \widehat{\text{PS}}_{\lambda_j} \rangle_{SX_\Gamma} \right| \leq C$$

for all j .

PROOF. In [HS, Th. 1.3.] it is shown that

$$\langle \varphi_n, \widehat{\text{PS}}_{\lambda_j} \rangle_{SX_\Gamma}$$

behaves asymptotically for $j \rightarrow \infty$ like

$$\langle \text{Op}(\varphi_n) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} = \int_{X_\Gamma} \varphi_n(x) |\varphi_j(x)|^2 dx.$$

But X_Γ is compact so that

$$\left| \langle \text{Op}(\varphi_n) \varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)} \right| \leq \int_{X_\Gamma} |\varphi_n(x)| |\varphi_j(x)|^2 dx \leq C,$$

where $C := \max\{\varphi_n(x) : x \in X_\Gamma\}$. Hence, $\langle \text{Op}(\varphi_n)\varphi_j, \varphi_j \rangle_{L^2(X_\Gamma)}$ and also $\langle \varphi_n, \widehat{\text{PS}}_{\lambda_j} \rangle_{SX_\Gamma}$ are bounded uniformly in j . \square

Now by definition $\langle \varphi_n, \widehat{\text{PS}}_{\lambda_j} \rangle_{SX_\Gamma} = \mathcal{C}(\lambda_j) \langle \varphi_n, \text{PS}_{\lambda_j} \rangle_{SX_\Gamma}$ and

$$\mathcal{C}(\lambda_j) = \frac{\omega_{l-1}}{2} B(i\lambda_j, \rho_0) = \frac{\omega_{l-1}}{2} \frac{\Gamma(i\lambda_j)\Gamma(\rho_0)}{\Gamma(\rho_0 + i\lambda_j)}$$

which by (2.25) for $\lambda_j \rightarrow \infty$ equals

$$\frac{\sqrt{2\pi}e^{-\pi/2|\lambda_j|}|\lambda_j^{-1/2}|}{\sqrt{2\pi}e^{-\pi/2|\lambda_j|}|\lambda_j^{\rho_0-1/2}|} = |\lambda_j|^{-\rho_0}.$$

Corollary 6.2.17 implies then:

COROLLARY 7.1.2. *For fixed φ_n , $\langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \sim \lambda_j^{\rho_0}$ as $j \rightarrow \infty$.*

Finally, we see:

LEMMA 7.1.3. *Let $k \in \mathbb{C} - \{\rho_0 \pm i\lambda_j, \rho_0 - 2 \pm i\lambda_j, \dots : -(\lambda_j^2 + \rho_0^2) \text{ eigenvalue of the Laplacian}\}$. Then $B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right)$ behaves asymptotically for $j \rightarrow \infty$ like*

$$e^{-\frac{\pi}{4}(|\text{Im}(k)+\lambda_j|+|\text{Im}(k)-\lambda_j|)} |\text{Im}(k) + \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}} |\text{Im}(k) - \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}}.$$

PROOF. Let us write out

$$\begin{aligned} B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right) &= \frac{\Gamma\left(\frac{k-i\lambda_j-\rho_0}{2}\right)\Gamma\left(\frac{k+i\lambda_j-\rho_0}{2}\right)}{\Gamma(k-\rho_0)} \\ &= \frac{\Gamma\left(\frac{\text{Re}(k)-\rho_0+i(\text{Im}(k)-\lambda_j)}{2}\right)\Gamma\left(\frac{\text{Re}(k)-\rho_0+i(\text{Im}(k)+\lambda_j)}{2}\right)}{\Gamma(\text{Re}(k)-\rho_0+i\text{Im}(k))}. \end{aligned}$$

Using the asymptotic formula for the Gamma function, see (2.25) in Remark 2.5.1,

$$(7.4) \quad \Gamma(x+iy) \sim \sqrt{2\pi}e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}, \quad y \rightarrow \infty,$$

we find that $B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right)$ behaves for $j \rightarrow \infty$ as

$$e^{-\frac{\pi}{4}(|\text{Im}(k)+\lambda_j|+|\text{Im}(k)-\lambda_j|)} |\text{Im}(k) + \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}} |\text{Im}(k) - \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}}.$$

\square

Since the term $e^{-\frac{\pi}{4}(|\text{Im}(k)+\lambda_j|+|\text{Im}(k)-\lambda_j|)}$ dominates the other terms

$$|\text{Im}(k) + \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}} |\text{Im}(k) - \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}}$$

and since the series $\sum_{n \in \mathbb{N}} ne^{-\alpha n}$ is absolutely convergent for every $\alpha > 0$, we see that the series

$$\sum_{j=j_0+1}^{\infty} e^{-\frac{\pi}{4}(|\text{Im}(k)+\lambda_j|+|\text{Im}(k)-\lambda_j|)} |\text{Im}(k) + \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}} |\text{Im}(k) - \lambda_j|^{\frac{\text{Re}(k)-\rho_0-1}{2}}$$

is absolute convergent for any $k \in \mathbb{C}$. Thus, by Proposition 6.2.21, Lemma 7.1.1 and Lemma 7.1.3 we find the following proposition,

PROPOSITION 7.1.4. *The infinite series (7.3) converges absolutely for k in \mathbb{C} outside the set \mathcal{P} , see (7.2), and defines there a holomorphic function.*

Hence, we define the meromorphic continuation of $\mathcal{R}(k; \varphi_n)$ by (*). Let us now focus on a certain strip in \mathbb{C} and describe the poles and residues of $\mathcal{R}(k; \varphi_n)$ more precisely. It follows from Proposition 7.1.4 that in the strip

$$\mathcal{S} := \left\{ k \in \mathbb{C} : \rho_0 - \frac{1}{2} < \operatorname{Re}(k) < \rho_0 + \frac{1}{2} \right\}$$

the poles are at

$$\mathcal{P}_{\mathcal{S}} := \{ \rho_0, \rho_0 \pm i\lambda_j : -(\lambda_j^2 + \rho_0^2) \text{ eigenvalue of the Laplacian} \} \cap \mathcal{S}.$$

For computing the residue we make use of the following facts.

LEMMA 7.1.5. See [FB06, Kap.III]

- (1) If f is holomorphic in z_0 , then $\operatorname{Res}_{z=z_0} f = 0$
(2) The residue is linear, i.e. for all functions f, g and $\alpha, \beta \in \mathbb{C}$

$$\operatorname{Res}_{z=z_0} (\alpha f + \beta g) = \alpha \operatorname{Res}_{z=z_0} f + \beta \operatorname{Res}_{z=z_0} g.$$

- (3) If g is holomorphic and bijective on \mathbb{C} , then $\operatorname{Res}_{z=g(z_0)} = \operatorname{Res}_{z=z_0} g' \cdot (f \circ g)$.
(4) If f has a pole of order m at z_0 , then

$$\operatorname{Res}_{z=z_0} f = \left. \frac{d^{m-1}}{dz^{m-1}} \right|_{z=z_0} \left(\frac{(z-z_0)^m}{(m-1)!} f(z) \right).$$

- (5) If f is holomorphic in z_0 and g has a pole of order 1 at z_0 , then

$$\operatorname{Res}_{z=z_0} f \cdot g = f(z_0) \operatorname{Res}_{z=z_0} g.$$

From (4) and (5) we can deduce:

- (6) If f is holomorphic in z_0 and g has a pole of order 2 at z_0 , $g(z) = \sum_{k=-2}^{\infty} a_k (z-z_0)^k$ in a neighbourhood of z_0 , then

$$\operatorname{Res}_{z=z_0} f \cdot g = f'(z_0) \cdot \frac{a_{-2}}{2} + f(z_0) \operatorname{Res}_{z=z_0} g.$$

Let us now compute the residue at $k \in \mathcal{P}_{\mathcal{S}}$, i.e. $k = \rho_0$ or $k = \rho_0 + i\lambda_j$ and there is an eigenfunction φ_j with eigenvalue $-(\lambda_j^2 + \rho_0^2)$. If φ_j is now in the principal series, then $\lambda_j \neq 0$ and $\int_0^{\infty} u^{l-2} (1+u^2)^{-(\rho_0+i\lambda_j)} du = \frac{1}{2} B(i\lambda_j, \rho_0)$ is defined. The residue $\operatorname{Res}_{k=\rho_0+i\lambda_j} \mathcal{R}(k; \varphi_n)$ at $k = \rho_0 + i\lambda_j$ then equals

$$\begin{aligned} & 2^{\rho_0+i\lambda_j-2} \omega_{l-1}^2 B(i\lambda_j, \rho_0) \sum_r \langle \varphi_n, \operatorname{PS}_{\varphi_r} \rangle I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ & \cdot \operatorname{Res}_{k=\rho_0+i\lambda_j} B \left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2} \right), \end{aligned}$$

where the (finite) sum runs over all eigenfunctions φ_r with eigenvalue $-(\lambda_j^2 + \rho_0^2)$.

Now the residue $\operatorname{Res}_{k=\rho_0+i\lambda_j} B \left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2} \right)$ is given by

$$\begin{aligned} \operatorname{Res}_{k=\rho_0+i\lambda_j} B \left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2} \right) &= \operatorname{Res}_{k=\rho_0+i\lambda_j} \frac{\Gamma \left(\frac{k-i\lambda_j-\rho_0}{2} \right) \Gamma \left(\frac{k+i\lambda_j-\rho_0}{2} \right)}{\Gamma(k-\rho_0)} \\ &= \frac{\Gamma(i\lambda_j)}{\Gamma(i\lambda_j)} \cdot \operatorname{Res}_{k=\rho_0+i\lambda_j} \Gamma \left(\frac{k-i\lambda_j-\rho_0}{2} \right) \\ &= \operatorname{Res}_{k=0} \Gamma(k) \\ &= 1 \end{aligned}$$

Consequently, the residue at $k = \rho_0 + i\lambda_j$ in the principal series is

$$\begin{aligned} \text{Res}_{k=\rho_0+i\lambda_j} \mathcal{R}(k; \varphi_n) &= 2^{\rho_0+i\lambda_j-2} \sum_r \langle \varphi_n, \text{PS}_{\varphi_r} \rangle \omega_{l-1}^2 B(i\lambda_j, \rho_0) \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ &= 2^{\rho_0+i\lambda_j-1} \omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda_j) \sum_r \langle \varphi_n, \widehat{\text{PS}}_{\varphi_r} \rangle \\ &=: \text{(I)}, \end{aligned}$$

by Remark 2.5.1 and Corollary 6.2.17, since $2^{-1} \omega_{l-1} B(i\lambda_j, \rho_0) = \mathcal{C}(\lambda_j)$ for $\lambda_j > 0$.

If φ_j is in the complementary series and $j < j_0$, then $\lambda_j \neq 0$ and $B(i\lambda_j, \rho_0)$ is defined. The residue $\text{Res}_{k=\rho_0+i\lambda_j} \mathcal{R}(k; \varphi_n)$ at $k = \rho_0 + i\lambda_j$ is

$$\begin{aligned} &2^{\rho_0+i\lambda_j-2} B(i\lambda_j, \rho_0) \omega_{l-1} \sum_r C_{n,r} \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} \\ &\cdot \text{Res}_{k=\rho_0+i\lambda_j} B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right). \end{aligned}$$

As before $\text{Res}_{k=\rho_0+i\lambda_j} B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right) = 1$ and hence the residue at $k = \rho_0 + i\lambda_j$, $j < j_0$, is

$$\text{Res}_{k=\rho_0+i\lambda_j} \mathcal{R}(k; \varphi_n) = 2^{\rho_0+i\lambda_j-2} B(i\lambda_j, \rho_0) \omega_{l-1} \sum_r C_{n,r} \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} =: \text{(II)},$$

where we again sum over all eigenfunctions φ_r with eigenvalue $-(\lambda_j^2 + \rho_0^2)$.

The same formula (II) is valid for $j = j_0$ and $\lambda_{j_0} \neq 0$. If $\lambda_{j_0} = 0$, then

$$B(k - \rho_0, \rho_0) = \frac{\Gamma(k - \rho_0) \Gamma(\rho_0)}{\Gamma(k)}$$

has a pole of order 1 at $k = \rho_0$, while

$$B\left(\frac{k-i\lambda_{j_0}-\rho_0}{2}, \frac{k+i\lambda_{j_0}-\rho_0}{2}\right) = B\left(\frac{k-\rho_0}{2}, \frac{k-\rho_0}{2}\right) = \frac{\Gamma\left(\frac{k-\rho_0}{2}\right)^2}{\Gamma(k-\rho_0)}$$

has also a pole of order 1 at $k = \rho_0$. The product of both has thus a pole of order 2 at $k = \rho_0$ and the Laurent series around $k = \rho_0$ of the product starts with $(k - \rho_0)^{-2}$. Using Lemma 7.1.5(6) the residue $\text{Res}_{k=\rho_0} \mathcal{R}(k; \varphi_n)$ at $k = \lambda_{j_0} + \rho_0 = \rho_0$ computes to

$$\begin{aligned} &\text{Res}_{k=\rho_0} \mathcal{R}(k; \varphi_n) \\ &= \text{Res}_{k=\rho_0} 2^{k-2} B(k - \rho_0, \rho_0) \omega_{l-1} \sum_s C_{n,s} B\left(\frac{k-i\lambda_s-\rho_0}{2}, \frac{k+i\lambda_s-\rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_s} \rangle_{SX_\Gamma} \\ &\quad + \text{Res}_{k=\rho_0} 2^{k-2} B(k - \rho_0, \rho_0) \omega_{l-1} \sum_r C_{n,r} B\left(\frac{k-i\lambda_r-\rho_0}{2}, \frac{k+i\lambda_r-\rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} \\ &\quad + 2^{\rho_0-2} \omega_{l-1}^2 \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) \sum_{j=j_0+1}^{\infty} B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \\ &\quad \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ &= \omega_{l-1} \sum_s C_{n,s} B\left(\frac{i\lambda_s}{2}, \frac{-i\lambda_s}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_s} \rangle_{SX_\Gamma} 2^{\rho_0-2} \cdot \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) \\ &\quad + \omega_{l-1} \sum_r C_{n,r} \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} \cdot \text{Res}_{k=\rho_0} 2^{k-2} B(k - \rho_0, \rho_0) B\left(\frac{k-i\lambda_j-\rho_0}{2}, \frac{k+i\lambda_j-\rho_0}{2}\right) \\ &\quad + \left(2^{\rho_0-2} \omega_{l-1}^2 \sum_{j=j_0+1}^{\infty} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j)\right) \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0), \end{aligned}$$

where \sum_s runs over all eigenfunctions φ_s with eigenvalue $\mu_s = -(\lambda_s^2 + \rho_0^2) > -\rho_0^2$ and \sum_r runs over all eigenfunctions φ_r with eigenvalue $\mu_r = -(\lambda_r^2 + \rho_0^2) = -\rho_0^2$. Then $B(k - \rho_0, \rho_0) = \frac{\Gamma(k - \rho_0)\Gamma(\rho_0)}{\Gamma(k)}$ has a pole of order 1 at $k = \rho_0$, while

$$B\left(\frac{k - i\lambda_{j_0} - \rho_0}{2}, \frac{k + i\lambda_{j_0} - \rho_0}{2}\right) = B\left(\frac{k - \rho_0}{2}, \frac{k - \rho_0}{2}\right) = \frac{\Gamma\left(\frac{k - \rho_0}{2}\right)^2}{\Gamma(k - \rho_0)}$$

has also a pole of order 1 at $k = \rho_0$. The product of both has thus a pole of order 2 at $k = \rho_0$ and the Laurent series around $k = \rho_0$ of the product starts with $(k - \rho_0)^{-2}$. Using Lemma 7.1.5(6) it follows that

$$\begin{aligned} & \text{Res}_{k=\rho_0} 2^{k-2} B(k - \rho_0, \rho_0) B\left(\frac{k - i\lambda_r - \rho_0}{2}, \frac{k + i\lambda_r - \rho_0}{2}\right) \\ &= \ln(2)2^{\rho_0-3} + 2^{\rho_0-2} \cdot \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) B\left(\frac{k - i\lambda_r - \rho_0}{2}, \frac{k + i\lambda_r - \rho_0}{2}\right) \\ &= \ln(2)2^{\rho_0-3} + 2^{\rho_0-2} \cdot \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) B\left(\frac{k - i\lambda_{j_0} - \rho_0}{2}, \frac{k + i\lambda_{j_0} - \rho_0}{2}\right) \\ &= \ln(2)2^{\rho_0-3} + 2^{\rho_0-2} \cdot \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) B\left(\frac{k - \rho_0}{2}, \frac{k - \rho_0}{2}\right) \end{aligned}$$

and

$$\begin{aligned} & \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) B\left(\frac{k - \rho_0}{2}, \frac{k - \rho_0}{2}\right) \\ &= \text{Res}_{k=\rho_0} \frac{\Gamma\left(\frac{k - \rho_0}{2}\right)^2}{\Gamma(k - \rho_0)} \cdot \frac{\Gamma(k - \rho_0)\Gamma(\rho_0)}{\Gamma(k)} = \frac{\Gamma(\rho_0)}{\Gamma(\rho_0)} \cdot \text{Res}_{k=\rho_0} \frac{\Gamma\left(\frac{k - \rho_0}{2}\right)^2}{\Gamma(k - \rho_0)} \cdot \Gamma(k - \rho_0) \\ &= \text{Res}_{k=\rho_0} \Gamma\left(\frac{k - \rho_0}{2}\right)^2 = \text{Res}_{k=0} \Gamma\left(\frac{k}{2}\right)^2 = 2\text{Res}_{k=0} \Gamma(k)^2 \\ &= 2 \frac{d}{dk} \Big|_{k=0} \frac{k^2 \Gamma(k+1)^2}{2k^2} = 2\Gamma'(1) = -2C_\gamma, \end{aligned}$$

where C_γ is the Euler-Mascheroni constant, see [FB06, Kap.IV]. Also

$$\text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) = \text{Res}_{k=0} = 1.$$

Thus,

$$\begin{aligned} & \text{Res}_{k=\rho_0} \mathcal{R}(k; \varphi_n) \\ &= \omega_{\ell-1} \sum_s C_{n,s} B\left(\frac{i\lambda_s}{2}, \frac{-i\lambda_s}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_s} \rangle_{SX_\Gamma} 2^{\rho_0-2} \\ & \quad + \omega_{\ell-1} \sum_r C_{n,r} \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} \cdot (\ln(2)2^{\rho_0-3} - 2^{\rho_0-1}C_\gamma) \\ & \quad + 2^{\rho_0-2} \omega_{\ell-1}^2 \sum_{j=j_0+1}^{\infty} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ &= \omega_{\ell-1} \sum_s C_{n,s} B\left(\frac{i\lambda_s}{2}, \frac{-i\lambda_s}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_s} \rangle_{SX_\Gamma} 2^{\rho_0-2} \\ & \quad + \omega_{\ell-1} (\ln(2)2^{\rho_0-3} - 2^{\rho_0-1}C_\gamma) \sum_r C_{n,r} \langle \varphi_n, \text{PS}_{\varphi_r} \rangle_{SX_\Gamma} \\ & \quad + 2^{\rho_0-2} \omega_{\ell-1}^2 \sum_{j=j_0+1}^{\infty} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \cdot I(a, b, \rho_0, \rho_0 + i\lambda_j) \\ &=: (III). \end{aligned}$$

Finally, if $\lambda_{j_0} \neq 0$, then $\mathcal{R}(k; \varphi_n)$ has still a pole at $k = \rho_0$ coming from the term $B(k - \rho_0, \rho_0)$. The residue $\text{Res}_{k=\rho_0} \mathcal{R}(k; \varphi_n)$ is

$$\begin{aligned}
& \text{Res}_{k=\rho_0} \mathcal{R}(k; \varphi_n) \\
&= 2^{\rho_0-2} \omega_{\ell-1} \left(\sum_{j=1}^{j_0} C_{n,j} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \right. \\
&\quad \left. + \omega_{\ell-1} \sum_{j=j_0+1}^{\infty} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} I(a, b, \rho_0, \rho_0 + i\lambda_j) \right) \text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) \\
&= 2^{\rho_0-2} \omega_{\ell-1} \left(\sum_{j=1}^{j_0} C_{n,j} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} \right. \\
&\quad \left. + \omega_{\ell-1} \sum_{j=j_0+1}^{\infty} B\left(\frac{i\lambda_j}{2}, \frac{-i\lambda_j}{2}\right) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} I(a, b, \rho_0, \rho_0 + i\lambda_j) \right) \\
&=: (IV),
\end{aligned}$$

since $\text{Res}_{k=\rho_0} B(k - \rho_0, \rho_0) = 1$.

7.2. The meromorphic continuation of $\mathcal{Z}(\varphi)$

In this short section we show how the meromorphic continuation of $\mathcal{Z}(k; \varphi)$ is deduced from the one of $\mathcal{R}(k; \varphi_n)$.

For the zeta function $\mathcal{Z}(k; \varphi_n)$ we note that the poles of $\mathcal{R}(k + 2m; \varphi_n)$ are the $-2m$ -shifted poles of $\mathcal{R}(k; \varphi_n)$. Now

$$(7.5) \quad \mathcal{Z}(k; \varphi_n) = \sum_{m=0}^{\infty} \beta(k - \rho_0; m) \mathcal{R}(k + 2m; \varphi_n) = \beta(k - \rho_0; 0) \mathcal{R}(k; \varphi_n) + \sum_{m=1}^{\infty} \beta(k - \rho_0; m) \mathcal{R}(k + 2m; \varphi_n),$$

see Theorem 5.5.5.

We also know from Lemma 5.5.1 that $\beta(k - \rho_0; m) \rightarrow 0$ for $m \rightarrow \infty$ and that $k \mapsto \beta(k - \rho_0; m)$ is holomorphic for any $m \in \mathbb{N}_0$. Furthermore,

$$\mathcal{R}(k; \varphi_n) = \sum_{1 \neq [\gamma] \in CT} \sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k) (\cosh L_\gamma)^{-k+\rho_0}$$

and by Lemma 5.5.3 we know that

$$\sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k)$$

is absolutely bounded by

$$C(\varphi_n) \cdot L_\gamma e^{-\rho_0 L_\gamma},$$

where L_γ is the length of the closed geodesic $[\gamma]$ and

$$C(\varphi_n) := \frac{\omega_{\ell-1} \max_{X_\Gamma} \{|\varphi_n|\}}{2} B(\rho_0, \rho_0)$$

if $\text{Re}(k) > 2\rho_0$. It follows that for $\text{Re}(k) > 2\rho_0$

$$\begin{aligned}
|\mathcal{R}(k; \varphi_n)| &\leq \sum_{1 \neq [\gamma] \in CT} \left| \sum_{\pi \in \widehat{M}} c(\varphi_n, \gamma, \pi, k) \right| (\cosh L_\gamma)^{-\text{Re}(k)+\rho_0} \\
&\leq C(\varphi_n) \sum_{1 \neq [\gamma] \in CT} e^{-\rho_0 L_\gamma} L_\gamma (\cosh L_\gamma)^{-\text{Re}(k)+\rho_0} \\
&\leq C(\varphi_n) (\cosh L_{\text{inf}})^{-\text{Re}(k)+\rho_0} \sum_{1 \neq [\gamma] \in CT} e^{-\rho_0 L_\gamma} L_\gamma
\end{aligned}$$

for the constant $C(\varphi_n)$ depending only on φ_n . Here $L_{\text{inf}} = \sqrt{2(l-1)}^{-1} l_{\text{inf}}$, where $l_{\text{inf}} > 0$ is the infimum of the set $\{l_\gamma : 1 \neq [\gamma] \in CT\}$, see Proposition 2.2.4. Thus

$$|\mathcal{R}(k; \varphi_n)| \leq C \cdot (\cosh L_{\text{inf}})^{-\text{Re}(k)}$$

for some constant $C > 0$, since

$$C(\varphi_n)(\cosh L_{\text{inf}})^{\rho_0} \sum_{1 \neq [\gamma] \in CT} e^{-\rho_0 L_\gamma} L_\gamma$$

is bounded.

As a consequence, $\mathcal{R}(k; \varphi_n) = O(e^{-\text{Re}(k)})$ for $\text{Re}(k) > 2\rho_0$. In particular, $\mathcal{R}(k + 2m; \varphi_n)$ decays like $e^{-\text{Re}(k)+2m}$ for $m \in \mathbb{N}$ and $m \rightarrow \infty$. Hence, (7.5) converges for k away from the poles and defines there a meromorphic continuation of the zeta function $\mathcal{Z}(k; \varphi_n)$. Since in the strip

$$\mathcal{S} = \rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}$$

the only poles of $\mathcal{R}(k; \varphi_n)$ are at

$$\{\rho_0, \rho_0 \pm i\lambda_j : -(\lambda_j + \rho_0^2) \text{ eigenvalue of the Laplacian}\} \cap \mathcal{S},$$

only the $m = 0$ -term contributes to poles of $\mathcal{Z}(k; \varphi_n)$ in \mathcal{S} . The residues/poles in the strip \mathcal{S} of $\mathcal{Z}(k; \varphi_n)$ are hence the same as the ones of $\mathcal{R}(k; \varphi_n)$ modulo $\beta(k - \rho_0; 0)$ which equals $2^{\rho_0 - k}$ by Lemma 5.5.1. More precisely, we have to distinguish again the 4 cases from (I) to (IV) and to note that

$$\text{Res}_{z=k} \mathcal{Z}(z; \varphi_n) = \beta(k - \rho_0; 0) \text{Res}_{z=k} \mathcal{R}(z; \varphi_n) = 2^{\rho_0 - k} \text{Res}_{z=k} \mathcal{R}(z; \varphi_n)$$

for $k \in \mathbb{C}$ with $\rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}$.

7.3. Summary

We collect the results of the previous sections:

THEOREM 7.3.1. *Let φ_n be a non-constant automorphic eigenfunction. The zeta functions $\mathcal{R}(k; \varphi_n)$ and $\mathcal{Z}(k; \varphi_n)$ can be extended meromorphically to \mathbb{C} by (*) resp. (7.5). In the strip $\rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}$ the only possible poles are at $k \in \{\rho_0, \rho_0 \pm i\lambda_j : -(\lambda_j^2 + \rho_0^2) \text{ eigenvalue of the Laplacian}\}$. If the eigenvalue $-(\lambda_j^2 + \rho_0^2)$ lies in the principal series and (I) $\neq 0$, there is a pole of order 1 at $k = \rho_0 \pm i\lambda_j$, the residue of $\mathcal{R}(k; \varphi_n)$ is then given by (I). If the eigenvalue comes from the complementary series, $\lambda_j \neq 0$ and (II) $\neq 0$, then there is also a pole of order 1 at $k = \rho_0 \pm i\lambda_j$ and residue (II). If $\lambda_j = 0$, there is a pole of order 2 and residue determined by (III). If (I) or (II) vanishes, then $\mathcal{R}(\varphi_n)$ can be holomorphically extended to $k = \rho_0 \pm i\lambda_j$. The residues of $\mathcal{Z}(\varphi_n)$ are the same modulo $\beta(k - \rho_0; 0) = 2^{\rho_0 - k}$.*

7.4. Normalization of $\mathcal{Z}(\varphi)$

In this chapter we have shown so far that $\mathcal{Z}(\varphi_n)$ is a meromorphic function on \mathbb{C} . Let $k_0 := \rho_0 + i\lambda$, where $-(\lambda^2 + \rho_0^2)$ is an eigenvalue of the Laplacian from the principal series. Then $\mathcal{Z}(\varphi_n)$ has a simple pole at k_0 with residue given - up to a non-zero constant - by normalized Patterson-Sullivan distributions, see Equation (I) in Section 7.1. This constant equals

$$\omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda).$$

To be consistent with [AZ07] we can divide $\mathcal{Z}(\varphi_n)$ by this constant, i.e. we consider the function

$$(7.6) \quad k \mapsto \frac{\mathcal{Z}(\varphi_n, k)}{\omega_{l-1} I(a, b, 2^{-1}, k)}.$$

Now

$$k \mapsto \frac{1}{\omega_{l-1}I(a, b, \rho_0, k)} \stackrel{\text{Lem. 6.2.20}}{=} \frac{2\Gamma(k)^2}{\omega_{l-1}\Gamma(k-a)\Gamma(k-b)\Gamma(\rho_0)}$$

is a meromorphic function on \mathbb{C} with poles exactly in $\{0, -1, -2, \dots\}$. Hence (7.6) defines also a meromorphic function on \mathbb{C} . In the strip $\rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}$ the poles of (7.6) equal the poles of $\mathcal{Z}(\varphi_n)$. The residue at k_0 , $\rho_0 + \frac{1}{2} < \text{Re}(k_0) < \rho_0 - \frac{1}{2}$, is

$$\text{Res}_{k=k_0} \frac{\mathcal{Z}(\varphi_n, k)}{\omega_{l-1}I(a, b, \rho_0, k)} = \frac{1}{\omega_{l-1}I(a, b, \rho_0, k_0)} \text{Res}_{k=k_0} \mathcal{Z}(\varphi_n, k).$$

In particular, if $k_0 = \rho_0 + i\lambda$ comes from the principal series, we deduce from equation (I) that

$$\text{Res}_{k=k_0} \frac{\mathcal{Z}(\varphi_n, k)}{\omega_{l-1}I(a, b, \rho_0, k)} = 2^{\rho_0-1} \sum_r \langle \varphi_n, \widehat{\text{PS}}_{\varphi_r} \rangle,$$

where as before the finite sum runs over all eigenfunctions φ_r with eigenvalue $-(\lambda^2 + \rho_0^2)$. We can of course also normalize $\mathcal{R}(\varphi_n)$ by the same constant $(\omega_{l-1}I(a, b, \rho_0, k))^{-1}$.

7.5. Comparison with the zeta function from [AZ07]

Let $G = SO_o(1, 2)$. In this section we want to discuss a difference between the zeta function [AZ07, (1.9ii)] and our zeta function, see (5.32), in the case of a compact hyperbolic surface $X_\Gamma = \Gamma \backslash SO_o(1, 2)/SO(2)$. In [AZ07, (1.9ii)] we find as a definition

$$\mathcal{Z}_1(k; \varphi_n) := \sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-kL_\gamma}}{1 - e^{-L_\gamma}} \left(\int_{c_{\gamma_0}} \varphi_n \right) = \sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-(k-\frac{1}{2})L_\gamma}}{2 \sinh \frac{L_\gamma}{2}} \left(\int_{c_{\gamma_0}} \varphi_n \right)$$

After normalizing our zeta function $\mathcal{Z}(k; \varphi_j) = \sum_{1 \neq [\gamma] \in C\Gamma} c(\varphi_j, \gamma, \mathbf{1}, k) e^{-(k-1/2)L_\gamma}$ from Proposition 5.6.1 by $(4I(a, b, 1/2, k))^{-1}$ the term $(2I(a, b, 1/2, k))^{-1} c(\varphi_j, \gamma, \mathbf{1}, k)$ takes the form, see equation (5.38),

$$\begin{aligned} & (4I(a, b, 1/2, k))^{-1} \cdot c(\varphi_j, \gamma, \mathbf{1}, k) \\ & \stackrel{\text{Lem. 6.2.20}}{=} \frac{2\Gamma(k)^2}{4\Gamma(\frac{1}{2})\Gamma(k-a)\Gamma(k-b)} \cdot \sum_{1 \neq [\gamma] \in C\Gamma} \frac{1}{\sqrt{\cosh L_\gamma - 1}} \sqrt{2} \left(\int_{c_{\gamma_0}} \varphi_n \right) \\ & \cdot \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k-a, k-b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right) \\ & \cdot \frac{\Gamma(\frac{1}{2})\Gamma(k-a)\Gamma(k-b)}{\Gamma(k)^2} \\ & = \sum_{1 \neq [\gamma] \in C\Gamma} \frac{\sqrt{2}}{2\sqrt{2} \sinh \frac{L_\gamma}{2}} \left(\int_{c_{\gamma_0}} \varphi_n \right) \\ & \cdot \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k-a, k-b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{4I(a, b, 1/2, k)} \mathcal{Z}(k; \varphi_n) & = \sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-(k-\frac{1}{2})L_\gamma}}{2 \sinh \frac{L_\gamma}{2}} \left(\int_{c_{\gamma_0}} \varphi_n \right) \\ & \cdot \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k-a, k-b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right). \end{aligned}$$

Here $a = \frac{1}{2}(\rho_0 + ir_n)$ and $b = \frac{1}{2}(\rho_0 - ir_n)$, if the eigenvalue of φ_n is $-\frac{1}{4\rho_0}(\rho_0^2 + r_n^2)$. The difference $\mathcal{Z}_1(k; \varphi_n) - \frac{1}{4I(a, b, 1/2, k)}\mathcal{Z}(k; \varphi_n)$ thus equals on $\{k \in \mathbb{C} : \operatorname{Re}(k) > 1\}$

$$\begin{aligned} \mathcal{D}(k; \varphi_n) &:= \sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-(k-\frac{1}{2})L_\gamma}}{2 \sinh \frac{L_\gamma}{2}} \left(\int_{c_{\gamma_0}} \varphi_n \right) \\ &\cdot \left(1 - \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k - a, k - b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right) \right). \end{aligned}$$

On $\{k \in \mathbb{C} : \operatorname{Re}(k) > 1\}$, the function $\mathcal{D}(\varphi_n)$ is holomorphic as it is the difference of two holomorphic functions. Furthermore, it is also holomorphic on the half plane $\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\}$, because $\left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k - a, k - b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)$ tends to 1 as $L_\gamma \rightarrow \infty$, the argument $1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1}$ lies always in the interval $(-\infty, 1)$ and $k \mapsto {}_2F_1(k - a, k - b, k; x)$ is holomorphic on $\{k \in \mathbb{C} : k \neq 0, -1, -2, \dots\}$ for fixed $x \in (-\infty, 1)$, see [Olv74, Th. 9.1]. It follows that the term

$$\left(1 - \left(\frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right)^{k-1} {}_2F_1 \left(k - a, k - b, k; 1 - \frac{\cosh L_\gamma}{\cosh L_\gamma - 1} \right) \right)$$

is bounded for all L_γ . Also $C \cdot \sum_{n \in \mathbb{N}} \frac{n}{1-e^{-n}} e^{-kn}$ is a summable upper bound to

$$\sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-(k-\frac{1}{2})L_\gamma}}{2 \sinh \frac{L_\gamma}{2}} \left(\int_{c_{\gamma_0}} \varphi_n \right) = \sum_{1 \neq [\gamma] \in C\Gamma} \frac{e^{-kL_\gamma}}{1 - e^{-L_\gamma}} \left(\int_{c_{\gamma_0}} \varphi_n \right)$$

for a suitable constant $C > 0$.

Thus,

$$\frac{1}{4I(a, b, 1/2, k)}\mathcal{Z}(k; \varphi_n) + \mathcal{D}(k; \varphi_n)$$

defines a meromorphic continuation of $\mathcal{Z}_1(k; \varphi_n)$ to the half plane $\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\}$ with the same poles and residues as $\frac{1}{4I(a, b, 1/2, k)}\mathcal{Z}(k; \varphi_n)$.

The zeta function on the spherical spectrum

Let $G = SO_o(1, l) = ANK$ as before. Also let $\Gamma \subset G$ be a uniform lattice. We will use a decomposition of the right-regular representation π_R on $L^2(\Gamma \backslash G)$ in order to extend the definition to zeta function $\mathcal{Z}(\sigma)$, where σ is more general than an automorphic eigenfunction of the Laplacian. More precisely, σ will be an element of phase space $C^\infty(\Gamma \backslash G/M)$, $M = Z_K(A)$, which satisfies a certain finiteness conditions explained in Section 8.1. In Section 8.2 we indicate an approach for general $\sigma \in C^\infty(SX_\Gamma)$.

The case of $SO_o(1, 2)$ is dealt with in [AZ07] and we will basically omit this case as In this chapter we want to show how one can pass in Theorem 5.5.5 and 5.4.6, resp. Theorem 6.2.18 and 7.3.1 from automorphic eigenfunctions φ_n to $\sigma \in C^\infty(\Gamma \backslash G/M)$ lying in the class 1 spectrum with only finitely many nontrivial components.

8.1. Extension to the spherical spectrum

To give a dynamical interpretation of Patterson-Sullivan distributions, one needs to pass from automorphic eigenfunctions φ_n to arbitrary $\sigma \in C^\infty(\Gamma \backslash G/M)$. We make use of the decomposition of the (right-)regular representation π_R of G on $L^2(\Gamma \backslash G)$. So let $G = SO_o(1, l)$, $l > 2$, and Γ a uniform lattice. Since Γ is assumed to be co-compact, the representation decomposes discretely into a direct sum of at most 2 different types, see [Wil91, Th. 2.7.],

$$(8.1) \quad L^2(\Gamma \backslash G) = \bigoplus_{\text{spherical}} V \oplus \bigoplus_{\text{non-spherical}} W.$$

The spherical part is called the *spherical spectrum* or also *class 1 spectrum*. Here we call a representation *spherical*, if it possesses a non-trivial K -invariant vector.

Therefore,

$$(8.2) \quad L^2(\Gamma \backslash G)^M = L^2(\Gamma \backslash G/M) = \bigoplus_{\text{spherical}} V^M \oplus \bigoplus_{\text{non-spherical}} W^M.$$

In this section we want to show how one can pass in Theorem 5.5.5 and 5.4.6, resp. Theorem 6.2.18 and 7.3.1 from automorphic eigenfunctions φ_n to $\sigma \in C^\infty(\Gamma \backslash G/M)$ lying in the class 1 spectrum with only finitely many nontrivial components. That is,

$$\sigma = \sum_{\text{finite}} \sigma_n,$$

where each σ_n lies in some irreducible spherical (principal series) component V and is right- M -invariant.

We show now that the subspace of M -invariant functions in each spherical principal series representation is generated by the spherical vector if we restrict the representation from G to A .

LEMMA 8.1.1. *Let $G = SO_o(1, l)$, $l > 2$, and V be an irreducible spherical representation with spherical vector φ . Then A acts on V^M with cyclic vector φ , i.e. $U(\mathfrak{a})\varphi$ is dense in V^M . If $G = SO_o(1, 2)$, then*

$$U(\mathfrak{a})\varphi \oplus U(\mathfrak{a})X\varphi$$

is dense in $V^M = V$, where $\mathfrak{n} = \mathbb{R}X$.

PROOF. Since V is assumed to be irreducible we have (up to completion)

$$V^M = (U(\mathfrak{g})\varphi)^M = U(\mathfrak{g})^M\varphi,$$

since we can average over the compact group M , i.e. for $X \in \mathfrak{g}$ we can define the M -average \tilde{X} by

$$\tilde{X}f(g) := \int_M \frac{d}{dt} \Big|_{t=0} f(g \exp \text{Ad}(m)tX) dm.$$

Then $\tilde{X}f$ is M -invariant, if f is. Now

$$\begin{aligned} U(\mathfrak{g})^M\varphi &= U(\mathfrak{a} \oplus \mathfrak{n})^M\varphi \\ &= U(\mathfrak{a})U(\mathfrak{n})^M\varphi. \end{aligned}$$

Since $l > 2$, M is not trivial and it follows that $U(\mathfrak{n})^M$ is generated by the Laplacian $\sum_i X_i^2$ on \mathfrak{n} , see [Hel01, Prop. 4.11.]. Further, we know by Chapter 2.1

$$\sum_i X_i^2 = \frac{1}{2}(-\Omega - H^2 + 2H_\rho) \bmod \mathfrak{k}U(\mathfrak{g}).$$

Because the space of K -invariant elements in V is one dimensional and because $\Omega\varphi$ is also K -invariant, we deduce that φ is a Casimir eigenfunction, let us assume $\Omega\varphi = \mu\varphi$. Hence, the effect of applying $\sum_i X_i^2$ to φ can be expressed by elements in $U(\mathfrak{a})$, i.e.

$$\left(\sum_i X_i^2 \right) \varphi = \frac{1}{2}(-\mu - H^2 + 2H_\rho) \varphi,$$

where μ is the eigenvalue of φ .

If $l = 2$, i.e. M is trivial, we just note that $U(\mathfrak{n})^M = U(\mathfrak{n})$ is generated by X , since $\mathfrak{n} = \mathbb{R}X$. \square

If we assume $l > 2$ and V is some irreducible representation in the class 1 spectrum in (8.1), the lemma shows that restricting the irreducible G -representation π_R on V from G to A yields an A -representation on V^M with cyclic vector φ , if φ is the (normalized) K -fixed vector in V . It follows that the induced representation of $L^1(A)$ on V^M is also cyclic with cyclic vector φ , [Dix77, 13.3.5].

Thus, if σ_n is an M -invariant element of the irreducible spherical component V with (normalized) K -fixed vector φ_n , then there is some $\alpha_n \in L^1(A)$ such that

$$(8.3) \quad \sigma_n = \int_A \alpha_n(a) \pi_R(a) \varphi_n da.$$

By invariance under the geodesic flow for any $a \in A$ and any PS_{φ_j}

$$\langle \pi_R(a) \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} = \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}.$$

Hence, as PS_{φ_j} is a continuous functional on $C^\infty(SX_\Gamma)$, see Remark 6.2.5, and as SX_Γ is compact

$$\begin{aligned}
(8.4) \quad \langle \sigma_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} &= \left\langle \int_A \alpha_n \pi_R(a) \varphi_n da, \text{PS}_{\varphi_j} \right\rangle_{SX_\Gamma} \\
&= \int_A \alpha_n(a) \langle \pi_R(a) \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} da \\
&= \int_A \alpha_n(a) \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma} da \\
&= \int_A \alpha_n(a) da \cdot \langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}.
\end{aligned}$$

That is for any PS_{φ_j} , the value $\langle \varphi_n, \text{PS}_{\varphi_j} \rangle_{SX_\Gamma}$ essentially determines PS_{φ_j} on the whole M -invariant part V^M of the irreducible spherical component V associated with φ_n . We also remark that (8.4) remains true if we replace PS_{φ_j} by $\widehat{\text{PS}}_{\varphi_j}$.

For $G = SO_o(1, 2)$ see [AZ07, Th. 9.6.]. Here, for spherical irreducible components V of $L^2(\Gamma \backslash SO_o(1, 2))$ one basically needs knowledge of PS_φ on φ_n and $X\varphi_n$, if φ_n is the (normalized) spherical vector of V . If V is not spherical, in other words, if V is a discrete series representation, then PS_φ on V is determined by its value on the lowest weight vector.

Going back to $G = SO_o(1, l)$, $l > 2$, we define for $\sigma = \sum_n^{\text{finite}} \sigma_n = \sum_n^{\text{finite}} \int_A \alpha_n(a) \pi_R(a) \varphi_n da$, see (8.3),

$$(8.5) \quad \mathcal{Z}(\sigma) := \sum_n^{\text{finite}} \int_A \alpha_n(a) da \cdot \mathcal{Z}(\varphi_n) = \sum_{1 \neq [\gamma] \in C\Gamma} \sum_{\pi \in \widehat{M}} c(\gamma, \sigma, \pi, k) e^{-(k-\rho_0) \log a_\gamma},$$

if φ_n is the normalized K -fixed vector of the component of σ_n in $L^2(\Gamma \backslash G)$, see (8.1). Here

$$c(\gamma, \sigma, \pi, k) := \sum_n^{\text{finite}} \int_A \alpha_n(a) da \cdot c(\gamma, \varphi_n, \pi, k).$$

Now the analogue of Theorem 7.3.1 holds.

PROPOSITION 8.1.2. *Let σ be a function in $C^\infty(\Gamma \backslash G/M)$ with only finitely many nontrivial components in the spherical spectrum (8.1) and no component in the non spherical spectrum. The zeta function $\mathcal{Z}(\sigma)$ defined by (8.5) is a meromorphic function on \mathbb{C} . In the strip $\mathcal{S} = \{k \in \mathbb{C} : \rho_0 - \frac{1}{2} < \text{Re}(k) < \rho_0 + \frac{1}{2}\}$, the poles are at*

$$\{\rho_0, \rho_0 \pm i\lambda : -(\rho_0 + \lambda^2) \text{ eigenvalue of the Laplacian}\} \cap \mathcal{S}.$$

If $-(\lambda^2 + \rho_0^2)$ is an eigenvalue of the Laplacian from the principal series, the residue at $k = \rho_0 + i\lambda$ is up to the non-zero constant $\omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda)$

$$(8.6) \quad \sum_{\text{finite}} \langle \sigma, \widehat{\text{PS}}_\varphi \rangle,$$

where this finite sum ranges over all normalized Patterson-Sullivan distributions $\widehat{\text{PS}}_\varphi$ associated to Laplace eigenfunctions with eigenvalue $-(\lambda^2 + \rho_0^2)$.

PROOF. The meromorphic continuation of $\mathcal{Z}(\sigma)$ follows directly from Theorem 7.3.1 and the fact that σ has only finitely many nontrivial components. In order to compute the residue at $k = \rho + i\lambda \in \mathcal{S}$ we recall that we can assume that there are $\alpha_n \in L^1(A)$ such that

$$\sigma = \sum_n^{\text{finite}} \sigma_n = \sum_n^{\text{finite}} \int_A \alpha_n(a) \pi_R(a) \varphi_n da,$$

see (8.3). This implies, see (8.4),

$$(8.7) \quad \langle \sigma, \widehat{\text{PS}}_{\varphi}, \rangle_{SX_{\Gamma}} = \sum_n^{\text{finite}} \int_A \alpha_n(a) da \langle \varphi_n, \widehat{\text{PS}}_{\varphi} \rangle_{SX_{\Gamma}}.$$

For the residue at $k_0 \in \mathcal{S}$ we have

$$\text{Res}_{k=k_0} \mathcal{Z}(k; \sigma) = \sum_n^{\text{finite}} \int_A \alpha_n(a) da \text{Res}_{k=k_0} \mathcal{Z}(k; \varphi_n).$$

In particular, if $k = \rho_0 + i\lambda \in \mathcal{S}$, $-(\lambda^2 + \rho_0^2)$ an eigenvalue from the principal series, it follows from Theorem 7.3.1 that

$$\begin{aligned} \text{Res}_{k=\rho_0+i\lambda} \mathcal{Z}(k; \sigma) &= \sum_n^{\text{finite}} \int_A \alpha_n(a) da \omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda) \sum_{r:\lambda_r^2=\lambda^2}^{\text{finite}} \langle \varphi_n, \widehat{\text{PS}}_{\varphi_r} \rangle_{SX_{\Gamma}} \\ &= \omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda) \sum_{r:\lambda_r^2=\lambda^2}^{\text{finite}} \sum_n^{\text{finite}} \int_A \alpha_n(a) da \langle \varphi_n, \widehat{\text{PS}}_{\varphi_r} \rangle_{SX_{\Gamma}} \\ &\stackrel{(8.7)}{=} \omega_{l-1} I(a, b, \rho_0, \rho_0 + i\lambda) \sum_{r:\lambda_r^2=\lambda^2}^{\text{finite}} \langle \sigma, \widehat{\text{PS}}_{\varphi_r} \rangle_{SX_{\Gamma}}. \end{aligned}$$

□

For the general case of a K -finite Casimir eigenfunction of type δ , $\delta \in \widehat{K}$ we just state the following.

LEMMA 8.1.3. *Let $l > 2$ and V be any irreducible representation of G with K -finite Ω -eigenfunction v of type δ , $(\delta, V_{\delta}) \in \widehat{K}$, i.e. $\text{span}\{K \cdot v\} = \text{span}\{v_1, \dots, v_{d_{\delta}}\} \cong V_{\delta}$ and $\Omega v = \mu v$ for some $\mu \in \mathbb{C}$. Then $\bigoplus_{i=1}^{d_{\delta}} U(\mathfrak{a})v_i$ contains V^M .*

PROOF. We use the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Then by irreducibility $V = U(\mathfrak{g})v = S(\mathfrak{p})U(\mathfrak{k})v$, where $S(\mathfrak{p})$ is the symmetric algebra of \mathfrak{p} . Since v is K -finite of type δ , we have that $U(\mathfrak{k})v$ is contained in $\text{span}\{v_1, \dots, v_{d_{\delta}}\}$, where the v_i are K -translates which are K -finite of type δ and Ω -eigenfunctions, $\Omega v_i = \mu_i v_i$, since Ω lies in the center of $U(\mathfrak{g})$. Thus,

$$V^M \subset \bigoplus_i S(\mathfrak{p})^M v_i.$$

But $S(\mathfrak{p})^M = U(\mathfrak{a})S(\mathfrak{p})^K$, as $l > 2$, where $U(\mathfrak{p})^K$ is generated by $\sum_j T_j^2$, T_j forming an orthonormal basis of \mathfrak{p} . Now Ω can be expressed as $\Omega = \sum_j T_j^2 - \sum_l W_l^2$, where the W_l form an orthonormal base of \mathfrak{k} . Hence for any i ,

$$\sum_j T_j^2 v_i = \Omega v_i + \sum_l W_l^2 v_i = \mu_i v_i + \sum_l W_l^2 v_i \in \text{span}\{v_1, \dots, v_{d_{\delta}}\}.$$

It follows that

$$V^M \subset \bigoplus_i U(\mathfrak{a})v_i.$$

□

8.2. Outlook

It remains an open problem what are good choices for σ coming from the M -invariant part V^M of a non-spherical representation V in the decomposition (8.2), if $G = SO_o(1, l)$ with $l > 2$. Lemma 8.1.3 shows that if V has a K -finite Ω -eigenfunction v then $\widehat{\text{PS}}_{\varphi_j}$ is basically determined by its value on finitely many K -translates v_i of v . Thus, one would like to associate a zeta function $\mathcal{Z}(v_i)$ to

v_i as one did in the case of an automorphic eigenfunction φ . Here for example the problem occurs how to associate an operator with v_i which maps $L^2(X_\Gamma)$ into itself and how to compute its trace. For $G = SO_o(1, 2)$, [AZ07] gives a solution to these problems, at least if σ has only finitely many nontrivial components in the decomposition (8.1).

An approach would be to consider instead of $\sigma \in C^\infty(X_\Gamma)$ and $f_k \in C^\infty(G//K)$ sections of vector bundles $\Sigma \in C^\infty(X_\Gamma \times_K V_\delta)$ and $F_k \in C^\infty(X \times_K V_{\delta^*})$ for K -types δ , $k \in \mathbb{C}$, so that $\Sigma \cdot \pi_R(F_k)$ maps $L^2(X_\Gamma)$ into itself and is an operator of trace class on $L^2(X_\Gamma)$.

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 $\alpha_\gamma, \alpha_\gamma \gamma \alpha_\gamma^{-1} = a_\gamma m_\gamma$, 58
 α_γ , element in G conjugating γ into δ_γ , 22
 $b = \frac{1}{2}(\rho_0 - ir)$, 50
 $\beta(k; m)$, coefficient, 70
 $c(\lambda)$, Harish-Chandra's c -function, 28
 $c(\varphi, \gamma, \pi, k)$, coefficient in zeta function $\mathcal{Z}(\varphi)$, 68
 $c(a_\gamma m_\gamma)$, coefficient, 81
 $c = \frac{l-1}{2} = \rho_0$, 49
 c_{γ_0} , prime, closed geodesic to γ , 23
 χ , cut-off, 86
 χ_δ , character of δ , 59
 $d = \frac{k_4}{k_1} \sum_j \pi(Z_{Y_j}^2), -V_\pi^{Z_M(S)}$ 49
 $d\dot{x}_\gamma$, Haar measure on $\Gamma_\gamma \backslash G_\gamma$, 21
 d_δ , dimension of δ , 59
 $d_\lambda(gMA)$, intermediate value, 86
 d_π , 59
 d_π , dimension of V_π , 47
 da , Haar measure on A , 55
 $\delta(D)$, 39
 $\delta(D)$, radial part, 34
 $\delta_\gamma, \delta_\gamma = m_\gamma a_\gamma \in MA^+$, 21
 dg , Haar measure on G , 55
 dk , Haar measure on K , 55
 dn , Haar measure on N , 55
 dx_γ , Haar measure on G_γ , 21
 e , neutral element, 64
 f^π , π -projection of f , 59
 f_k , bi- K -invariant function, 28, 65
 φ_λ , spherical function to $\lambda \in \mathfrak{a}_\mathbb{C}^+$, 27
 $\tilde{\phi}, \tilde{\phi}(x) = \phi(x^{-1})$, 56
 γ_0 , primitive element to γ , 21
 $h(ma)$, diffeomorphism on N , 76
 j_0 , index separating principal and complementary series, 83
 l_g , left-translation with g , 62
 l_{\inf} , infimum of $\{l_\gamma : 1 \neq [\gamma] \in C\Gamma\}$, 22
 $\lambda : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$, symmetrizer, 34
 $n_\gamma, \gamma = \gamma_0^{n_\gamma}$, 57
 n_u , element of N , 24
 $o, o = K$, 43
 ω_{l-1} , volume of unit ball, 29
 $p, p = \frac{1-c}{2} + \sqrt{\left(\frac{1-\rho_0}{2}\right)^2 - d}$, 54
 $p_{1,2}$, roots of indicial equation, 51
 $\varphi * f$, convolution, 56
 π -projection, 59, 60
 $\pi(Z)$, derived operator from $\pi(\exp Z)$, 43
 π_1 , projection from $U(\mathfrak{g})$ to $U(\mathfrak{a} \oplus \mathfrak{s})U(\mathfrak{k})$, 34
 π_2 , projection from $U(\mathfrak{g})$ to $U(\mathfrak{a} \oplus \mathfrak{s})$, 35
 π_M , projection $C(G) \rightarrow C(G/M)$, 88
 π_R , right-regular representation of G , 55
 $\pi_R(f)$, Fourier transform w.r.t. π_R , 56
 $\tilde{\pi}$, contragredient representation of π , 47
 $r, \mu = -\frac{1}{4}\left(\rho_0 + \frac{r^2}{\rho_0}\right)$, 62
 r_g , right-translation with g , 62
 $\rho, \rho = \frac{l-1}{2}\alpha$, 24
 ρ , half-sum of positive roots times dimension, 19
 $\rho_0, \rho_0 = \rho(H_0)$, 24
 θ , Cartan involution, 17
 w , 26
 w , nontrivial weyl element, 20
 $n, f^n = f \circ r_n$, 88
(elementary) spherical, 27
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