Weighted diffeomorphism groups of Banach spaces and non-compact manifolds and weighted mapping groups

Dissertation

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Abstract

In this dissertation, we construct and study certain classes of infinite dimensional Lie groups that are modelled on weighted function spaces. In particular, we continue the investigation of the Lie group $\operatorname{Diff}_{\mathcal{W}}(X)$ of diffeomorphisms introduced in [Wal06], where X is a Banach space and $\mathcal{W}$ a set of weights on X containing a constant weight. This construction is now also extended to the case of diffeomorphism groups of manifolds.

We also construct certain types of "weighted mapping groups". These are Lie groups modelled on weighted function spaces of the form $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, where G is a given (finite- or infinite dimensional) Lie group and U an open subset of X. Both the weighted diffeomorphism groups and the weighted mapping groups (when X is a vector space, resp. G is a Banach Lie group) are shown to be regular Lie groups in Milnor's sense.

Further, we discuss semidirect products of the former groups. We study the integrability of Lie algebras of vector fields of the form $\mathcal{C}^\infty_{\mathcal{W}}(X, X) \rtimes \mathbf{L}(G)$, where X is a Banach space and G a Lie group acting smoothly on X.

Zusammenfassung

Gegenstand dieser Dissertation sind die Konstruktion und Untersuchung von unendlichdimensionalen Liegruppen, die auf gewissen Räumen von gewichteten Abbildungen modelliert sind. Im Speziellen fahren wir mit der Untersuchung der Liegruppe $\operatorname{Diff}_{\mathcal{W}}(X)$ von gewichteten Diffeomorphismen auf dem Banachraum X zu geeigneten Gewichtsfunktionen $\mathcal{W}$, die in der Diplomarbeit [Wal06] konstruiert wurde, fort. Wir verallgemeinern die Konstruktion solcher Diffeomorphismengruppen auf Mannigfaltigkeiten.

Weiter werden einige „gewichtete Abbildungsgruppen" zu Liegruppen gemacht. Die zugehörigen Modellräume sind gewichtete Funktionenräume $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, wobei G eine (endlich- oder unendlichdimensionale) Liegruppe und U eine offene Teilmenge von X ist. Wir weisen nach, dass beide Arten von Liegruppen reguläre Liegruppen im Sinne Milnors sind (wenn X ein Vektorraum bzw. G eine Banach-Lie-Gruppe ist).

Wir studieren auch semidirekte Produkte solcher Liegruppen, und beweisen einige Kriterien für die Integrabilität von Liealgebren der Form $\mathcal{C}^\infty_{\mathcal{W}}(X, X) \rtimes \mathbf{L}(G)$, wobei X ein Banachraum und G eine glatt auf X operierende Liegruppe ist.

Contents

1. Introduction

> She tried hard to keep herself a stranger to her poor old father's slight income by the use of the finest production of steel, whose blunt edge eyed the reely covering with marked greed, and offered its sharp dart to faultless fabrics of flaxen fineness.
>
> *(Amanda McKittrick Ros,* Delina Delaney*)*

Diffeomorphism groups of compact manifolds, as well as groups $\mathcal{C}^k(K, G)$ of Lie group-valued mappings on compact manifolds are among the most important and well-studied examples of infinite dimensional Lie groups (see for example [Les67], [Mil84], [Ham82], [Omo97], [PS86] and [KM97]). While the diffeomorphism group $\operatorname{Diff}(K)$ of a compact manifold is modelled on the Fréchet space $\mathcal{C}^\infty(K, \mathbf{T}K)$ of smooth vector fields on K, for a non-compact smooth manifold M, it is not possible to make $\operatorname{Diff}(M)$ a Lie group modelled on the space of all smooth vector fields in a satisfying way (see [Mil82]). We mention that the LF-space $\mathcal{C}^\infty_c(M, \mathbf{T}M)$ of compactly supported smooth vector fields can be used as the modelling space for a Lie group structure on $\operatorname{Diff}(M)$. But the topology on this Lie group is too fine for many purposes; the group $\operatorname{Diff}_c(M)$ of compactly supported diffeomorphisms (those diffeomorphisms that coincide with the identity map outside some compact set) is an open subgroup (see [Mic80] and [Mil82]). Likewise, it is no problem to turn groups $\mathcal{C}^k_c(M, G)$ of compactly supported Lie group-valued maps into Lie groups (cf. [Mil84], [AHM+93], [Glö02b]). However, only in special cases there exists a Lie group structure on $\mathcal{C}^\infty(M, G)$, equipped with its natural group topology, the smooth compact-open topology (see [NW08]).

In view of these limitations, it is natural to look for Lie groups of diffeomorphisms which are larger then $\operatorname{Diff}_c(M)$ and modelled on larger Lie algebras of vector fields than $\mathcal{C}^\infty_c(M, \mathbf{T}M)$. In the same vain, one would like to find mapping groups modelled on larger spaces than $\mathcal{C}^k_c(M, \mathbf{L}(G))$.

In an earlier work [Wal06], the author already constructed such diffeomorphism groups (modelled on weighted function spaces) when M is a finite-dimensional vector space, or a Banach space. In this work, we continue the study of such diffeomorphism groups (including a proof for their regularity) and extend the construction to the case where M is a manifold. We also construct (and study) certain weighted mapping groups.

Diffeomorphims In the vector space case, most of the results are valid even when the space X is a Banach space. The groups we consider are modelled on spaces of weighted functions on X. For example, we discuss a Lie group structure on the group $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ of diffeomorphisms differing from $\mathrm{id}_{\mathbb{R}^n}$ by a rapidly decreasing $\mathbb{R}^n$-valued map. Considered as a topological group, this group has been used in quantum physics ([Gol04]). For $n = 1$, another construction of the Lie group structure (in the setting of convenient differential calculus) has been given by P. Michor ([Mic06, §6.4]), and applied to the Burgers' equation. The general case was treated in the author's unpublished diploma thesis [Wal06] and published in [Wal12]. Results from the diploma thesis will not be reproduced here; we shall only summarize what is needed and refer to [Wal13] for details (a slightly extended preprint version of [Wal12])[1]. After [Wal12] was published, an alternative construction of the Lie group structure on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ and $\mathrm{Diff}_{\{1_{\mathbb{R}^n}\}}(\mathbb{R}^n)$ (within convenient differentiable calculus) was given in [MM13].

To explain our results, let X and Y be Banach spaces, $U \subseteq X$ open and nonempty, $k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathcal{W}$ be a set of functions f on U taking values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ called weights. As usual, we let $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ be the set of all k-times continuously Fréchet-differentiable functions $\gamma : U \to Y$ such that $f \cdot \|D^{(\ell)}\gamma\|_{op}$ is bounded for all integers $\ell \leq k$ and all $f \in \mathcal{W}$. Then $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ is a locally convex topological vector space in a natural way. We prove in Theorem 4.3.11

Theorem. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Consider the Lie group* $\mathrm{Diff}_{\mathcal{W}}(X) := \{\phi \in \mathrm{Diff}(X) : \phi - \mathrm{id}_X,\ \phi^{-1} - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)\}$*, as constructed in [Wal06]. Then* $\mathrm{Diff}_{\mathcal{W}}(X)$ *is a regular Lie group modelled on* $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$.

Replacing $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$ by the subspace of functions γ such that $f(x) \cdot \|D^{(\ell)}\gamma(x)\|_{op} \to 0$ as $\|x\| \to \infty$, we obtain a subgroup $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ of $\mathrm{Diff}_{\mathcal{W}}(X)$ which also is a Lie group (see Proposition 4.2.14).

To explain our results about diffeomorphisms on manifolds, let (M, g) be a Riemannian manifold, and $\mathcal{A}$ an atlas for M that is *adapted* (see Definition 5.3.2 for the precise meaning) and "thin". In Theorem 5.3.6, we prove:

Theorem. *Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ with $1_M \in \mathcal{W}$. Then there exists a Lie group* $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$ *of* weighted diffeomorphisms *that is a subgroup of* $\mathrm{Diff}(M)$ *and modelled on the space* $\mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_{\mathcal{A}}$ *of weighted vector fields with regard to $\mathcal{A}$. Further, the Riemannian logarithm provides a chart for* $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$*. Here $\mathcal{B}$ denotes a suitable subatlas of $\mathcal{A}$, and $\mathcal{W}^e$ denotes a* minimal saturated extension *of $\mathcal{W} \cup \{\omega\}$, where ω is an* adjusted weight.

For the definition of a *minimal saturated extension* and *adjusted weights*, see 5.2.13. The spaces $\mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_{\mathcal{A}}$ are defined in Subsection 5.2.1. They arise as closed vector subspaces of *weighted restricted products* of the weighted functions spaces $\{\mathcal{C}^\infty_{\mathcal{W}_\kappa}(U_\kappa, \mathbb{R}^d) : (\kappa : \widetilde{U_\kappa} \to U_\kappa) \in \mathcal{A}\}$ used above; see Section 5.1 for the technical details concerning these products.
Further, we prove in Proposition 5.3.10 that the groups $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$ contain at least the identity component of $\mathrm{Diff}_c(M)$, provided that $\mathcal{W}$ consists of continuous weights. Finally, we show in Proposition 5.3.11 that if $(M, g) = (\mathbb{R}^d, \langle\cdot,\cdot\rangle)$ and $\mathcal{A}$ consists of

[1] [Wal12] lacks the examination of topologies on spaces of multipliers on page 115ff. Further, the preprint contains a more general treatment of the inversion map of $\mathrm{Diff}_{\mathcal{W}}(X)$ in Subsection 4.2.1 (which can be used to investigate functions that are defined on a subset of X) which uses a Lipschitz inverse function theorem (stated in Subsection A.2.1).

identity mappings, then the connected components and the topology of $\mathrm{Diff}_{\mathcal{W}}(\mathbb{R}^d)$ and $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d, \langle\cdot,\cdot\rangle, 1_{\mathbb{R}^d})$ coincide, giving us plenty of examples for this construction.

Mapping groups For mapping groups, we first consider mappings into Banach Lie groups. In Section 7.1 we show

Theorem. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G be a Banach Lie group. Then there exists a connected Lie group $\mathcal{C}^k_{\mathcal{W}}(U, G) \subseteq G^U$ modelled on $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, and this Lie group is regular.*

Using the natural action of diffeomorphisms on functions, we can always form the semidirect product $\mathcal{C}^\infty_{\mathcal{W}}(X, G) \rtimes \mathrm{Diff}_{\mathcal{W}}(X)$ and make it a Lie group, see Theorem 7.1.19.

In the case of finite-dimensional domains, we can even discuss mappings into arbitrary Lie groups modelled on locally convex spaces. To this end, given a locally convex space Y and an open subset U of a finite-dimensional vector space X we define a certain space $\mathcal{C}^k_{\mathcal{W}}(U, Y)^\bullet$ of $\mathcal{C}^k$-maps which decay as we approach the boundary of U, together with their derivatives (see Definition 3.4.8 for details). We obtain the following result

Theorem. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G be a locally convex Lie group. Then there exists a connected Lie group $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet \subseteq G^U$ modelled on $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))^\bullet$.*

We also discuss certain larger subgroups of G^U admitting Lie group structures that make $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$ an open normal subgroup (see Subsection 7.2.2).

Finally, we consider Lie groups G acting smoothly on a Banach space X. We investigate when the G-action leaves the identity component $\mathrm{Diff}_{\mathcal{W}}(X)_0$ of $\mathrm{Diff}_{\mathcal{W}}(X)$ invariant and whether $\mathrm{Diff}_{\mathcal{W}}(X)_0 \rtimes G$ can be made a Lie group in this case. In particular, we show that $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)_0 \rtimes \mathrm{GL}(\mathbb{R}^n)$ is a Lie group for each n (Example 6.2.4). By contrast, $\mathrm{GL}(\mathbb{R}^n)$ does not leave $\mathrm{Diff}_{\{1_{\mathbb{R}^n}\}}(\mathbb{R}^n)$ invariant (Example 6.2.5).

We mention that certain weighted mapping groups on finite-dimensional spaces (consisting of smooth mappings) have already been discussed in [BCR81, §4.2] assuming additional hypotheses on the range group (cf. Remark 7.2.29). Besides the added generality, we provide a more complete discussion of superposition operators on weighted function spaces.

In the case where $\mathcal{W} = \{1_X\}$, our group $\mathrm{Diff}_{\mathcal{W}}(X)$ also has a counterpart in the studies of Jürgen Eichhorn and collaborators ([Eic96], [ES96], [Eic07]), who studied certain diffeomorphism groups on non-compact manifolds with bounded geometry. While an affine connection is used there to deal with higher derivatives, we are working exclusively with derivatives in local charts.

Semidirect products of diffeomorphism groups and function spaces on compact manifolds arise in Ideal Magnetohydrodynamics (see [KW09, p. II.3.4]). Further, the group $\mathcal{S}(\mathbb{R}^n) \rtimes \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ and its continuous unitary representations are encountered in Quantum Physics (see [Gol04]; cf. also [Ism96, §34] and the references therein).

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2. Preliminaries and notation

We give some notation and basic definitions. More details are provided in the appendix, as is a list of symbols used in this work.

2.1. Notation

We write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$,$\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Further we denote norms by $\|\cdot\|$.

Definition 2.1.1. Let A, B be subsets of the normed space X. As usual, the *distance* of A and B is defined as

$$\operatorname{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\} \in [0, \infty].$$

Thus $\operatorname{dist}(A, B) = \infty$ iff $A = \emptyset$ or $B = \emptyset$.

Further, for $x \in X$ and $r \in \mathbb{R}$ we define

$$B_X(x, r) := \{y \in X : \|y - x\| < r\}$$

Occasionally, we just write $B_r(x)$ instead of $B_X(x, r)$. For the closed ball, we write $\overline{B}_r(x)$ and the like.

Further, we define

$$\mathbb{D} := \overline{B}_{\mathbb{K}}(0, 1),$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. No confusion will arise from this abuse of notation.

2.2. Differential calculus of maps between locally convex spaces

We give basic definitions for the differential calculus for maps between locally convex spaces that is known as Kellers C^k_c-theory. More results can be found in Section A.1.

Definition 2.2.1 (Directional derivatives)**.** Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $u \in U$, $x \in X$ and $f : U \to Y$ a map. The *derivative of f at u in the direction x* is defined as

$$\lim_{\substack{t \to 0 \\ t \in \mathbb{K}^*}} \frac{f(u + tx) - f(u)}{t} =: (D_x f)(u) =: df(u; x),$$

whenever that limit exists.

Definition 2.2.2. Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, and $f : U \to Y$ be a map.

We call f a $\mathcal{C}^1_{\mathbb{K}}$*-map* or just $\mathcal{C}^1_{\mathbb{K}}$ if f is continuous, the derivative $df(u;x)$ exists for all $(u,x) \in U \times X$ and the map $df : U \times X \to Y$ is continuous.

Inductively, for a $k \in \mathbb{N}$ we call f a $\mathcal{C}^k_{\mathbb{K}}$*-map* or just $\mathcal{C}^k_{\mathbb{K}}$ if f is a $\mathcal{C}^1_{\mathbb{K}}$-map and $d^1 f := df : U \times X \to Y$ is a $\mathcal{C}^{k-1}_{\mathbb{K}}$-map. In this case, the *k-th iterated differential* of f is defined by

$$d^k f := d^{k-1}(df) : U \times X^{2^k - 1} \to Y.$$

If f is a $\mathcal{C}^k_{\mathbb{K}}$*-map* for each $k \in \mathbb{N}$, we call f a $\mathcal{C}^\infty_{\mathbb{K}}$*-map* or just $\mathcal{C}^\infty_{\mathbb{K}}$ or *smooth*.

Further, for each $k \in \overline{\mathbb{N}}$ we define

$$\mathcal{C}^k_{\mathbb{K}}(U,Y) := \{f : U \to Y \mid f \text{ is } \mathcal{C}^k_{\mathbb{K}}\}.$$

Often, we shall simply write $\mathcal{C}^k(U,Y)$, $\mathcal{C}^k$ and the like.

It is obvious from the definition of differentiability that iterated directional derivatives exist and depend continuously on the directions. The converse of this assertion also holds.

Proposition 2.2.3. *Let $f : U \to Y$ be a continuous map and $r \in \overline{\mathbb{N}}$. Then $f \in \mathcal{C}^r(U,Y)$ iff for all $u \in U$, $k \in \mathbb{N}$ with $k \leq r$ and $x_1, \dots, x_k \in X$ the iterated directional derivative*

$$d^{(k)} f(u; x_1, \dots, x_k) := (D_{x_k} \cdots D_{x_1} f)(u)$$

exists and the map

$$U \times X^k \to Y : (u, x_1, \dots, x_k) \mapsto d^{(k)} f(u; x_1, \dots, x_k)$$

is continuous. We call $d^{(k)} f$ the k-th derivative *of f.*

2.3. Fréchet differentiability

We give basic definitions for Fréchet differentiability for maps between normed spaces. More results can be found in Section A.2.

Definition 2.3.1 (Fréchet differentiability)**.** Let X and Y be normed spaces and U an open nonempty subset of X. We call a map $\gamma : U \to Y$ *Fréchet differentiable* or $\mathcal{FC}^1$ if it is a $\mathcal{C}^1$-map and the map

$$D\gamma : U \to \mathrm{L}(X,Y) : x \mapsto d\gamma(x; \cdot)$$

is continuous. Inductively, for $k \in \mathbb{N}^*$ we call γ a $\mathcal{FC}^{k+1}$-map if it is Fréchet differentiable and $D\gamma$ is a $\mathcal{FC}^k$-map. We denote the set of all k-times Fréchet differentiable maps from U to Y with $\mathcal{FC}^k(U,Y)$. Additionally, we define the *smooth* maps by

$$\mathcal{FC}^\infty(U,Y) := \bigcap_{k \in \mathbb{N}^*} \mathcal{FC}^k(U,Y)$$

and $\mathcal{FC}^0(U,Y) := \mathcal{C}^0(U,Y)$. The map

$$D : \mathcal{FC}^{k+1}(U,Y) \to \mathcal{FC}^k(U, \mathrm{L}(X,Y)) : \gamma \mapsto D\gamma$$

is called *derivative operator*.

Remark 2.3.2. Let X and Y be normed spaces, U an open nonempty subset of X, $k \in \mathbb{N}^*$ and $\gamma \in \mathcal{FC}^k(U,Y)$. Then for each $\ell \in \mathbb{N}^*$ with $\ell \leq k$ there exists a continuous map

$$D^{(\ell)}\gamma : U \to \mathrm{L}^\ell(X,Y),$$

where $\mathrm{L}^\ell(X,Y)$ denotes the space of ℓ-linear maps $X^\ell \to Y$, endowed with the operator topology. The map $D^{(\ell)}\gamma$ can be described more explicitly. If $\gamma \in \mathcal{FC}^k(U,Y)$, also $\gamma \in \mathcal{C}^k(U,Y)$ holds, and for each $x \in U$ we have the relation

$$D^{(k)}\gamma(x) = d^{(k)}\gamma(x; \cdot).$$

3. Weighted function spaces

In this chapter we give the definition of some locally convex vector spaces consisting of weighted functions. The Lie groups that are constructed in this work will be modelled on these spaces. We first discuss maps between normed spaces. In Section 3.4, we will also look at maps that take values in arbitrary locally convex spaces. The treatment of the latter spaces requires some rather technical effort. Since these function spaces are only needed in Section 7.2, the reader may eventually skip this section.

3.1. Definition and examples

Definition 3.1.1. Let X and Y be normed spaces and $U \subseteq X$ an open nonempty set. For $k \in \mathbb{N}$ and a map $f : U \to \overline{\mathbb{R}}$ we define the quasinorm

$$\|\cdot\|_{f,k} : \mathcal{FC}^k(U,Y) \to [0,\infty] : \phi \mapsto \sup\{|f(x)|\,\|D^{(k)}\phi(x)\|_{op} : x \in U\}.$$

Furthermore, for any nonempty set $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $k \in \overline{\mathbb{N}}$ we define the vector space

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \{\gamma \in \mathcal{FC}^k(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k)\ \|\gamma\|_{f,\ell} < \infty\}$$

and notice that the seminorms $\|\cdot\|_{f,\ell}$ induce a locally convex vector space topology on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.

We call the elements of $\mathcal{W}$ *weights* and $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ a *space of weighted maps* or *space of weighted functions*.

An important example is the space of bounded functions with bounded derivatives:

Example 3.1.2. Let $k \in \overline{\mathbb{N}}$. We define

$$\mathcal{BC}^k(U,Y) := \mathcal{C}^k_{\{1_U\}}(U,Y).$$

Remark 3.1.3. Let U and V be nonempty open subsets of a normed space X and $U \subseteq V$. For a set $\mathcal{W} \subseteq \overline{\mathbb{R}}^V$, we define

$$\mathcal{W}|_U := \{f|_U : f \in \mathcal{W}\}.$$

Further we write with an abuse of notation

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \mathcal{C}^k_{\mathcal{W}|_U}(U,Y).$$

Remark 3.1.4. As is clear, for any set $T \subseteq 2^{\mathcal{W}}$ with $\mathcal{W} = \bigcup_{\mathcal{F}\in T} \mathcal{F}$ we have

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) = \bigcap_{\substack{\mathcal{F}\in T\\ \ell\in\mathbb{N},\ell\leq k}} \mathcal{C}^\ell_{\mathcal{F}}(U,Y).$$

We define some subsets of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$:

Definition 3.1.5. Let X and Y be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. For $k \in \overline{\mathbb{N}}$ we set

$$\mathcal{C}^k_{\mathcal{W}}(U,V) := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : \gamma(U) \subseteq V\}$$

and

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V) : (\exists r > 0)\ \gamma(U) + B_Y(0,r) \subseteq V\}.$$

Obviously

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \subseteq \mathcal{C}^k_{\mathcal{W}}(U,V),$$

and if $1_U \in \mathcal{W}$, then $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. The set $\mathcal{BC}^{\partial,k}(U,V)$ is defined analogously.

If $U \subseteq X$ is an open neighborhood of 0, we set

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)_0 := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : \gamma(0) = 0\}.$$

Analogously, we define $\mathcal{C}^k_{\mathcal{W}}(U,V)_0$, $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)_0$ and $\mathcal{BC}^k(U,V)_0$ as the corresponding sets of functions vanishing at 0.

Furthermore, we define the set of *decreasing weighted maps* as

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)^o := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k, \varepsilon > 0)(\exists r > 0) \|\gamma|_{U\setminus B_r(0)}\|_{f,\ell} < \varepsilon\}.$$

Note that we are primarily interested in the spaces $\mathcal{C}^k_{\mathcal{W}}(X,Y)^o$, but for technical reasons it is useful to have the spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ available for $U \subset X$.

We show that $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ is closed.

Lemma 3.1.6. *$\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ is a closed vector subspace of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.*

Proof. It is obvious from the definition of $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ that it is a vector subspace. It remains to show that it is closed. To this end, let $(\gamma_i)_{i\in I}$ be a net in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ that converges to $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$ in the topology of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. Let $f \in \mathcal{W}$, $\ell \in \mathbb{N}$ with $\ell \leq k$ and $\varepsilon > 0$. Then there exists an $i_\varepsilon \in I$ such that

$$i \geq i_\varepsilon \implies \|\gamma - \gamma_i\|_{f,\ell} < \frac{\varepsilon}{2}.$$

Further there exists an $r > 0$ such that

$$\|\gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} < \frac{\varepsilon}{2}.$$

Hence

$$\|\gamma|_{U\setminus B_r(0)}\|_{f,\ell} \leq \|\gamma|_{U\setminus B_r(0)} - \gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} + \|\gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} < \varepsilon,$$

and this finishes the proof. ☐

Examples involving finite-dimensional spaces Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $n \in \mathbb{N}$. In the following, let U be an open nonempty subset of $\mathbb{K}^n$. For a map $f : U \to \overline{\mathbb{R}}$ and a multiindex $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ we define

$$\|\cdot\|_{f,\alpha} : \mathcal{C}^k_{\mathbb{K}}(U, Y) \to [0, \infty] : \phi \mapsto \sup\{|f(x)| \, \|\partial^\alpha \phi(x)\| : x \in U\}.$$

We conclude from identity (A.3.5.1) in Proposition A.3.5 that for a set $\mathcal{W}$ of maps $U \to \overline{\mathbb{R}}$ and $k \in \overline{\mathbb{N}}$

$$\mathcal{C}^k_{\mathcal{W}}(U, Y) = \{\phi \in \mathcal{C}^k_{\mathbb{K}}(U, Y) : (\forall f \in \mathcal{W}, \alpha \in \mathbb{N}^n_0, |\alpha| \leq k) \; \|\phi\|_{f,\alpha} < \infty\},$$

and the topology defined with the seminorms $\|\cdot\|_{f,\alpha}$ coincides with the one defined above using the seminorms $\|\cdot\|_{f,\ell}$. This characterization of $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ allows us to recover well-known spaces as special cases:

- If $\mathcal{W}$ is the space $\mathcal{C}^0(U, \mathbb{R}^m)$ of all continuous functions, then

  $$\mathcal{C}^\infty_{\mathcal{W}}(U, \mathbb{R}^m) = \mathcal{D}(U, \mathbb{R}^m) = \mathcal{C}^\infty_c(U, \mathbb{R}^m)$$

  where $\mathcal{D}(U, \mathbb{R}^m)$ denotes the space of compactly supported smooth functions from U to $\mathbb{R}^m$; it should be noticed that $\mathcal{C}^\infty_{\mathcal{C}^0(U,\mathbb{R}^m)}(U, \mathbb{R}^m)$ is *not* endowed with the ordinary inductive limit topology $\varinjlim_K \mathcal{D}_K(U, \mathbb{R}^m)$, but instead the (coarser) topology making it the projective limit

  $$\varprojlim_{p \in \mathbb{N}} (\varinjlim_{K} \mathcal{D}^p_K(U, \mathbb{R}^m)) = \varprojlim_{p \in \mathbb{N}} \mathcal{D}^p(U, \mathbb{R}^m),$$

  where $\mathcal{D}^p_K(U, \mathbb{R}^m)$ denotes the $\mathcal{C}^p$-maps with support in the compact set K, endowed with the topology of uniform convergence of derivatives up to order p; and $\mathcal{D}^p(U, \mathbb{R}^m)$ the compactly supported $\mathcal{C}^p$-maps endowed with the inductive limit topology of the sets $\mathcal{D}^p_K(U, \mathbb{R}^m)$.

- The vector-valued *Schwartz space* $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$. Here $U = Y = \mathbb{R}^n$, $k = \infty$ and $\mathcal{W}$ is the set of polynomial functions on $\mathbb{R}^n$.

- The space $\mathcal{BC}^k(U, \mathbb{K}^m)$ of all bounded $\mathcal{C}^k$-functions from $U \subseteq \mathbb{K}^n$ to $\mathbb{K}^m$ whose partial derivatives are bounded (for $\mathcal{W} = \{1_U\}$); see Example 3.1.2.

- If $\mathcal{W} = \{1_X, \infty \cdot 1_{X \setminus U}\}$, then the space $\mathcal{C}^k_{\mathcal{W}}(X, Y)$ consists of $\mathcal{BC}^k(X, Y)$ functions that are defined on X and vanish on the complement of U.

3.2. Topological and uniform structure

We analyze the topology of the weighted function spaces defined above. In Proposition 3.2.3 we shall provide a method that greatly simplifies the treatment of the spaces; it will be used throughout this work. We will also describe the spaces $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ as the projective limits of suitable larger spaces. In particular, this will simplify the treatment of the spaces $\mathcal{C}^\infty_{\mathcal{W}}(U, Y)$. Further we give a sufficient criterion on the set $\mathcal{W}$ which ensures that $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ is complete.

3.2.1. Reduction to lower order

For $\ell > 1$, it is hard to estimate the seminorms $\|\gamma\|_{f,\ell}$ because in most cases the higher order derivatives $D^{(\ell)}\gamma$ can not be computed. We develop a technique that allow us to avoid the computation. First, we show that $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is endowed with the initial topology of the derivative maps. Most of the content of this subsection was already proved in the author's diploma thesis [Wal06, §3.1.1]. We omit the proofs of the older content and some technical lemmas. They also can be found in [Wal13, §3.2.1].

Lemma 3.2.1. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $\gamma \in \mathcal{FC}^k(U,Y)$. Then*

$$\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) \iff (\forall \ell \in \mathbb{N}, \ell \leq k)\, D^{(\ell)}\gamma \in \mathcal{C}^0_{\mathcal{W}}(U, \mathrm{L}^\ell(X,Y)),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) \to \prod_{\substack{\ell\in\mathbb{N}\\ \ell\leq k}} \mathcal{C}^0_{\mathcal{W}}(U, \mathrm{L}^\ell(X,Y)) : \gamma \mapsto (D^{(\ell)}\gamma)_{\ell\in\mathbb{N},\ell\leq k}$$

is a topological embedding.

The next lemma states a relation between the higher order derivatives of γ and those of $D\gamma$.

Lemma 3.2.2. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^{k+1}(U,Y)$. Then*

$$\|D^{(\ell)}D\gamma(x)\|_{op} = \|D^{(\ell+1)}\gamma(x)\|_{op} \tag{3.2.2.1}$$

for each $x \in U$ and $\ell < k$. In particular, for each map $f \in \overline{\mathbb{R}}^U$, $\ell < k$ and subset $V \subseteq U$

$$\|\gamma|_V\|_{f,\ell+1} = \|(D\gamma)|_V\|_{f,\ell}. \tag{3.2.2.2}$$

Proof. In Lemma A.2.14 the identity

$$D^{(\ell+1)}\gamma = \mathcal{E}_{\ell,1} \circ (D^{(\ell)}D\gamma)$$

is proved, where $\mathcal{E}_{\ell,1} : \mathrm{L}(X, \mathrm{L}^\ell(X,Y)) \to \mathrm{L}^{\ell+1}(X,Y)$ is an isometric isomorphism (see Lemma A.2.5). The asserted identities follow immediately. □

We can state the main tool for the treatment of weighted function spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ with $k \geq 1$. It is useful because it allows induction arguments of the following kind: Suppose we want to show that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$. First, we have to show that $\gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Y)$. Then, we suppose $\gamma \in \mathcal{C}^\ell_{\mathcal{W}}(U,Y)$ and show that $D\gamma$ in $\mathcal{C}^\ell_{\mathcal{W}}(U, \mathrm{L}(X,Y))$ by expressing it in terms of γ. This finishes the induction argument.

Proposition 3.2.3 (Reduction to lower order)**.** *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^1(U, Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \iff (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \mathcal{C}^0_{\mathcal{W}}(U, Y).$$

Moreover, the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \mathcal{C}^0_{\mathcal{W}}(U, Y) : \gamma \mapsto (D\gamma, \gamma)$$

is a topological embedding. In particular, the map

$$D : \mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y))$$

is continuous.

The same argument can be made for the vanishing weighted functions.

Corollary 3.2.4. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^1(U, Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Y)^o \iff (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y))^o \times \mathcal{C}^0_{\mathcal{W}}(U, Y)^o.$$

Proof. This is also an immediate consequence of Proposition 3.2.3 and identity (3.2.2.2) in Lemma 3.2.2. □

3.2.2. Projective limits and the topology of $\mathcal{C}^\infty_{\mathcal{W}}(U, Y)$

Sometimes it is useful that $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ can be written as the projective limit of larger weighted functions spaces.

Proposition 3.2.5. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set. Further let $(\mathcal{F}_i)_{i \in I}$ be a directed family of nonempty subsets of $\mathcal{W}$ such that $\bigcup_{i \in I} \mathcal{F}_i = \mathcal{W}$. Consider $I \times \{\ell \in \mathbb{N} : \ell \leq k\}$ as a directed set via*

$$((i_1, \ell_1) \leq (i_2, \ell_2)) \iff i_1 \leq i_2 \text{ and } \ell_1 \leq \ell_2.$$

Then $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ is the projective limit of

$$\{\mathcal{C}^\ell_{\mathcal{F}_i}(U, Y) : \ell \in \mathbb{N}, \ell \leq k, i \in I\}$$

in the category of topological (vector) spaces, with the inclusion maps as morphisms.

Proof. Since

$$\mathcal{C}^k_{\mathcal{W}}(U, Y) = \bigcap_{\substack{i \in I \\ \ell \in \mathbb{N}, \ell \leq k}} \mathcal{C}^\ell_{\mathcal{F}_i}(U, Y),$$

the set $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ is the desired projective limit as a set, and hence also as a vector space. Moreover, it is well known that the initial topology with respect to the limit maps $\mathcal{C}^k_{\mathcal{W}}(U, Y) \to \mathcal{C}^\ell_{\mathcal{F}_i}(U, Y)$ makes $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ the projective limit as a topological space, and also as a topological vector space. But it is clear from the definition that the given topology on $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ coincides with this initial topology. □

Corollary 3.2.6. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. The space $\mathcal{C}^\infty_\mathcal{W}(U,Y)$ is endowed with the initial topology with respect to the inclusion maps*

$$\mathcal{C}^\infty_\mathcal{W}(U,Y) \to \mathcal{C}^k_\mathcal{W}(U,Y).$$

Moreover, $\mathcal{C}^\infty_\mathcal{W}(U,Y)$ is the projective limit of the spaces $\mathcal{C}^k_\mathcal{W}(U,Y)$ with $k \in \mathbb{N}$, together with the inclusion maps.

Proof. This is an immediate consequence of Proposition 3.2.5. □

3.2.3. A completeness criterion

We describe a condition on $\mathcal{W}$ that ensures that $\mathcal{C}^k_\mathcal{W}(U,Y)$ is complete, provided that Y is a Banach space. Most of the content of this subsection was already proved in the author's diploma thesis [Wal06, §3.2], so we omit the proofs and various technical lemmas. They can be also found in [Wal13, §3.2.3].

Proposition 3.2.7. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set and $k \in \mathbb{N}$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ such that for each compact line segment $S \subseteq U$ there exists $f_S \in \mathcal{W}$ with $\inf_{x\in S}|f_S(x)| > 0$. Then the image of $\mathcal{C}^{k+1}_\mathcal{W}(U,Y)$ under the embedding described in Proposition 3.2.3 is closed.*

Corollary 3.2.8. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a Banach space and $k \in \overline{\mathbb{N}}$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ such that for each compact set $K \subseteq U$ there exists $f_K \in \mathcal{W}$ with $\inf_{x\in K}|f_K(x)| > 0$. Then $\mathcal{C}^k_\mathcal{W}(U,Y)$ is complete.*

Corollary 3.2.9. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a Banach space and $k \in \overline{\mathbb{N}}$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ with $1_U \in \mathcal{W}$. Then $\mathcal{C}^k_\mathcal{W}(U,Y)$ is complete; in particular, $\mathcal{BC}^k(U,Y)$ is complete.*

An integrability criterion

The given completeness criterion entails a criterion for the existence of the weak integral of a continuous curve to a space $\mathcal{C}^k_\mathcal{W}(U,Y)$ where Y is not necessarily complete. Note that assertion (a) of the following lemma was already stated and proved in the authors diploma thesis as part of [Wal06, La. 3.3], so we omit the proof. It can also be found in [Wal13, La. 3.2.13].

Lemma 3.2.10. *Let X and Y be normed spaces, $U \subseteq X$ a nonempty open set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $\Gamma : [a,b] \to \mathcal{C}^k_\mathcal{W}(U,Y)$ a map and $R \in \mathcal{C}^k_\mathcal{W}(U,Y)$.*

(a) *Assume that Γ is weakly integrable and that for each $x \in U$ there exists $f_x \in \mathcal{W}$ with $f_x(x) \neq 0$. Then*

$$\int_a^b \Gamma(s)\,ds = R \iff (\forall x \in U)\ \operatorname{ev}_x\left(\int_a^b \Gamma(s)\,ds\right) = R(x),$$

and for each $x \in U$ we have

$$\mathrm{ev}_x\left(\int_a^b \Gamma(s)\, ds\right) = \int_a^b \mathrm{ev}_x(\Gamma(s))\, ds. \tag{$*$}$$

(b) Assume that for each compact set $K \subseteq U$, there exists $f_K \in \mathcal{W}$ with $\inf_{x\in K}|f_K(x)| > 0$, that Γ is continuous and

$$\int_a^b \mathrm{ev}_x(\Gamma(s))\, ds = \mathrm{ev}_x(R) \tag{$**$}$$

holds for all $x \in U$. Then Γ is weakly integrable with

$$\int_a^b \Gamma(s)\, ds = R.$$

Proof. (b) Let $\widetilde{Y}$ be the completion of Y. Then $\mathcal{C}^k_{\mathcal{W}}(U, Y) \subseteq \mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$, and we denote the inclusion map by ι. It is obvious that ι is a topological embedding. In the following, we denote the evaluation of $\mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$ at $x \in U$ with $\widetilde{\mathrm{ev}}_x$.
Since we proved in Corollary 3.2.8 that the condition on $\mathcal{W}$ ensures that $\mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$ is complete, $\iota \circ \Gamma$ is weakly integrable. Since $\widetilde{\mathrm{ev}}_x \circ \iota = \mathrm{ev}_x$ for $x \in U$, we conclude from (a) (using $(*)$ and $(**)$) that

$$\int_a^b (\iota \circ \Gamma)(s)\, ds = \iota(R).$$

This identity ensures the integrability of Γ: By the Hahn-Banach theorem, each $\lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y)'$ extends to a $\widetilde{\lambda} \in \mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})'$, that is $\widetilde{\lambda} \circ \iota = \lambda$. Hence

$$\int_a^b (\lambda \circ \Gamma)(s)\, ds = \int_a^b (\widetilde{\lambda} \circ \iota \circ \Gamma)(s)\, ds = \widetilde{\lambda}(\iota(R)) = \lambda(R),$$

which had to be proved. □

3.3. Composition on weighted functions and superposition operators

In this subsection we discuss the behaviour of weighted functions when they are composed with certain functions. In particular, we show that a continuous multilinear or a suitable analytic map induce a superposition operator between weighted function spaces. Moreover, we examine the composition between bounded functions and between bounded functions mapping 0 to 0 and weighted functions.

3.3.1. Composition with a multilinear map

A slightly less general version of Proposition 3.3.3 was already proved in the author's diploma thesis as [Wal06, Satz 3.15]. Its proof also included the content of Definition 3.3.1 and Lemma 3.3.2. The proofs, which can be found in [Wal13, La. 3.3.2, Prop. 3.3.3], are omitted.

We prove that a continuous multilinear map from a normed space $Y_1 \times \cdots \times Y_m$ to a normed space Z induces a continuous multilinear map from $\mathcal{C}^k_{\mathcal{W}}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}}(U, Y_m)$ to $\mathcal{C}^k_{\mathcal{W}}(U, Z)$. As a preparation, we calculate the differential of a composition of a multilinear map and other differentiable maps. The following definition is quite useful to do that.

Definition 3.3.1. Let $Y_1, \dots, Y_m$, X and Z be normed spaces and $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map. For each $i \in \{1, \dots, m\}$ we define the m-linear continuous map

$$\begin{aligned} b^{(i)} :& Y_1 \times \cdots \times Y_{i-1} \times \mathrm{L}(X, Y_i) \times Y_{i+1} \times \cdots \times Y_m \to \mathrm{L}(X, Z) \\ & (y_1, \dots, y_{i-1}, T, y_{i+1}, \dots, y_m) \mapsto (h \mapsto b(y_1, \dots, y_{i-1}, T \cdot h, y_{i+1}, \dots, y_m)). \end{aligned}$$

Lemma 3.3.2. *Let $Y_1, \dots, Y_m$ and Z be normed spaces, U be an open nonempty subset of the normed space X and $k \in \overline{\mathbb{N}}$. Further let $b : Y_1 \times \cdots \times Y_m \to Z$ be a continuous m-linear map and $\gamma_1 \in \mathcal{FC}^k(U, Y_1), \dots, \gamma_m \in \mathcal{FC}^k(U, Y_m)$. Then*

$$b \circ (\gamma_1, \dots, \gamma_m) \in \mathcal{FC}^k(U, Z)$$

with

$$D(b \circ (\gamma_1, \dots, \gamma_m)) = \sum_{i=1}^{m} b^{(i)} \circ (\gamma_1, \dots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \dots, \gamma_m). \tag{3.3.2.1}$$

Proposition 3.3.3. *Let U be an open nonempty subset of the normed space X. Let $Y_1, \dots, Y_m$ be normed spaces, $k \in \overline{\mathbb{N}}$ and $\mathcal{W}, \mathcal{W}_1, \dots, \mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ nonempty sets such that*

$$(\forall f \in \mathcal{W})(\exists g_{f,1} \in \mathcal{W}_1, \dots, g_{f,m} \in \mathcal{W}_m)\, |f| \leq |g_{f,1}| \cdots |g_{f,m}|.$$

Further let Z be another normed space and $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map. Then

$$b \circ (\gamma_1, \dots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)$$

for all $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_m \in \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m)$. The map

$$M_k(b) : \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : (\gamma_1, \dots, \gamma_m) \mapsto b \circ (\gamma_1, \dots, \gamma_m)$$

is m-linear and continuous.

We prove an analogous result for decreasing functions.

Corollary 3.3.4. *Let $Y_1, \dots, Y_m$ be normed spaces, U an open nonempty subset of the normed space X, $k \in \overline{\mathbb{N}}$ and $\mathcal{W}, \mathcal{W}_1, \dots, \mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ nonempty such that*

$$(\forall f \in \mathcal{W})(\exists g_{f,1} \in \mathcal{W}_1, \dots, g_{f,m} \in \mathcal{W}_m)\, |f| \leq |g_{f,1}| \cdots |g_{f,m}|.$$

Further let Z be another normed space, $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map and $j \in \{1, \dots, m\}$. Then

$$b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^o \tag{†}$$

for all $\gamma_i \in \mathcal{C}^k_{\mathcal{W}_i}(U, Y_i)$ $(i \neq j)$ and $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^o$. Moreover, the map

$$\begin{aligned} M_k(b) : \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^o \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) &\to \mathcal{C}^k_{\mathcal{W}}(U, Z)^o \\ : (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) &\mapsto b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \end{aligned}$$

is m-linear and continuous.

Proof. Using Proposition 3.3.3 and Lemma 3.1.6, we only have to prove that (†) holds. This is done by induction on k (which we may assume finite).

$k = 0$: For $f \in \mathcal{W}$, $x \in U$ and $\gamma_1 \in \mathcal{C}^0_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_j \in \mathcal{C}^0_{\mathcal{W}_j}(U, Y_j)^o, \dots, \gamma_m \in \mathcal{C}^0_{\mathcal{W}_m}(U, Y_m)$ we compute

$$\begin{aligned} &|f(x)|\, \|b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)(x)\| \\ &\qquad \leq \|b\|_{op} \prod_{i=1}^m |g_{f,i}(x)| \|\gamma_i(x)\| \leq \left(\|b\|_{op} \prod_{i \neq j} \|\gamma_i\|_{g_{f,i},0} \right) |g_{f,j}(x)|\, \|\gamma_j(x)\|. \end{aligned}$$

With this estimate we easily see that $b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^0_{\mathcal{W}_j}(U, Z)^o$.

$k \to k+1$: From Corollary 3.2.4 (together with the induction base) we know that for $\gamma_1 \in \mathcal{C}^{k+1}_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_j \in \mathcal{C}^{k+1}_{\mathcal{W}_j}(U, Y_j)^o, \dots, \gamma_m \in \mathcal{C}^{k+1}_{\mathcal{W}_m}(U, Y_m)$

$$b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Z)^o \iff D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))^o.$$

We know from (3.3.2.1) in Lemma 3.3.2 that

$$\begin{aligned} D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) = &\sum_{\substack{i=1 \\ i \neq j}}^m b^{(i)} \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \dots, \gamma_m) \\ &+ b^{(j)} \circ (\gamma_1, \dots, \gamma_{j-1}, D\gamma_j, \gamma_{j+1}, \dots, \gamma_m). \end{aligned}$$

Because $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^o$ and $D\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, \mathrm{L}(X, Y_j))^o$, we can apply the inductive hypothesis to all $b^{(i)}$ and the $\mathcal{C}^k$-maps $\gamma_1, \dots, \gamma_m$ and $D\gamma_1, \dots, D\gamma_m$ to see that this is an element of $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))^o$. □

We list some applications of Proposition 3.3.3. In the following corollaries, $k \in \overline{\mathbb{N}}$, U is an open nonempty subset of the normed space X and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ always contains the constant map 1_U.

Corollary 3.3.5. *Let A be a normed algebra with the continuous multiplication $*$. Then $\mathcal{C}^k_{\mathcal{W}}(U, A)$ is an algebra with the continuous multiplication*

$$\begin{aligned} M(*) :&\mathcal{C}^k_{\mathcal{W}}(U, A) \times \mathcal{C}^k_{\mathcal{W}}(U, A) \to \mathcal{C}^k_{\mathcal{W}}(U, A) \\ &M(*)(\gamma, \eta)(x) = \gamma(x) * \eta(x). \end{aligned}$$

We shall often write $*$ instead of $M(*)$.

Corollary 3.3.6. *If E, F and G are normed spaces, then the composition of linear operators*

$$\cdot : \mathrm{L}(F, G) \times \mathrm{L}(E, F) \to \mathrm{L}(E, G)$$

is bilinear and continuous and therefore induces the continuous bilinear maps

$$\begin{aligned} M(\cdot) :&\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(F, G)) \times \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(E, F)) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(E, G)) \\ &M(\cdot)(\gamma, \eta)(x) = \gamma(x) \cdot \eta(x) \end{aligned}$$

and

$$\begin{aligned} M_{\mathcal{BC}}(\cdot) :&\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(F, G)) \times \mathcal{BC}^k(U, \mathrm{L}(E, F)) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(E, G)) \\ &M_{\mathcal{BC}}(\cdot)(\gamma, \eta)(x) = \gamma(x) \cdot \eta(x). \end{aligned}$$

We shall often denote $M(\cdot)$ just by $\cdot$.

Corollary 3.3.7. *Let E and F be normed spaces. Then the evaluation of linear maps*

$$\cdot : \mathrm{L}(E, F) \times E \to F : (T, w) \mapsto T \cdot w$$

is bilinear und continuous (see Lemma A.2.3) and hence induces the continuous bilinear map

$$\begin{aligned} M(\cdot) :&\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(E, F)) \times \mathcal{C}^k_{\mathcal{W}}(U, E) \to \mathcal{C}^k_{\mathcal{W}}(U, F) \\ &M(\cdot)(\Gamma, \eta)(x) = \Gamma(x) \cdot \eta(x). \end{aligned}$$

Instead of $M(\cdot)$ we will often write $\cdot$.

3.3.2. Composition of weighted functions with bounded functions

We explore the composition between spaces of bounded functions and spaces of weighted functions. One case that is of particular interest is the composition between certain subsets of the spaces $\mathcal{BC}^k(U, Y)$.

Composition of bounded functions

We discuss under which conditions the composition is continuous or differentiable. The next lemma was already stated and proved in [Wal06, La. 3.20], in a slightly less general version. We omit the proof, which also can be found in [Wal13, La. 3.3.8].

Lemma 3.3.8. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty subsets and $k \in \overline{\mathbb{N}}$. Then for $\gamma \in \mathcal{BC}^{k+1}(V, Z)$ and $\eta \in \mathcal{BC}^{\partial,k}(U, V)$*

$$\gamma \circ \eta \in \mathcal{BC}^{k}(U, Z),$$

and the map

$$\mathcal{BC}^{k+1}(V, Z) \times \mathcal{BC}^{\partial,k}(U, V) \to \mathcal{BC}^{k}(U, Z) : (\gamma, \eta) \mapsto \gamma \circ \eta \tag{$*$}$$

is continuous.

As a preparation for discussing the differentiable properties of the composition, we prove a nice identity for its differential quotient.

Lemma 3.3.9. *Let X, Y and Z be normed spaces and $U \subseteq X$, $V \subseteq Y$ be open subsets. Further, let $\gamma \in \mathcal{FC}^{1}(V, Z)$, $\tilde{\gamma} \in \mathcal{C}^{0}(V, Z)$, $\tilde{\eta} \in \mathcal{BC}^{0}(U, Y)$ and $\eta \in \mathcal{C}^{0}(U, V)$ such that $\operatorname{dist}(\eta(U), \partial V) > 0$. Then, for all $x \in U$ and $t \in \mathbb{R}^{*}$ with*

$$|t| \leq \tfrac{\operatorname{dist}(\eta(U), \partial V)}{\|\tilde{\eta}\|_{1_U, 0} + 1},$$

the identity

$$\operatorname{ev}_x\left(\tfrac{(\gamma + t\tilde{\gamma}) \circ (\eta + t\tilde{\eta}) - \gamma \circ \eta}{t}\right) = \operatorname{ev}_x(\tilde{\gamma} \circ (\eta + t\tilde{\eta})) + \int_0^1 \operatorname{ev}_x((D\gamma \circ (\eta + st\tilde{\eta})) \cdot \tilde{\eta})\, ds \tag{3.3.9.1}$$

holds, where ev_x denotes the evaluation at x.

Proof. For t as above the identity

$$(\gamma + t\tilde{\gamma}) \circ (\eta + t\tilde{\eta}) - \gamma \circ \eta = \gamma \circ (\eta + t\tilde{\eta}) + t\tilde{\gamma} \circ (\eta + t\tilde{\eta}) - \gamma \circ \eta$$

holds, and an application of the mean value theorem gives

$$\operatorname{ev}_x(\gamma \circ (\eta + t\tilde{\eta}) - \gamma \circ \eta) = \int_0^1 \operatorname{ev}_x((D\gamma \circ (\eta + st\tilde{\eta})) \cdot t\tilde{\eta})\, ds.$$

Division by t leads to the desired result. □

So we are ready to discuss when the composition is differentiable.

Proposition 3.3.10. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets and $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^{*}$. Then the continuous map*

$$g^{k+\ell+1}_{\mathcal{BC},Z} : \mathcal{BC}^{k+\ell+1}(V, Z) \times \mathcal{BC}^{\partial,k}(U, V) \to \mathcal{BC}^{k}(U, Z) : (\gamma, \eta) \mapsto \gamma \circ \eta$$

(cf. Lemma 3.3.8) is a $\mathcal{C}^{\ell}$-map with

$$dg^{k+\ell+1}_{\mathcal{BC},Z}(\gamma_0, \eta_0; \gamma, \eta) = g^{k+\ell+1}_{\mathcal{BC},Z}(\gamma, \eta_0) + g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0, \eta_0) \cdot \eta. \tag{3.3.10.1}$$

Proof. For $k < \infty$, the proof is by induction on ℓ which we may assume finite because the inclusion maps $\mathcal{BC}^\infty(V,Z) \to \mathcal{BC}^{k+\ell+1}(V,Z)$ are continuous linear (and hence smooth).

$\ell = 1$: Let $\gamma_0, \gamma \in \mathcal{BC}^{k+\ell+1}(V,Z)$, $\eta_0 \in \mathcal{BC}^{\partial,k}(U,V)$ and $\eta \in \mathcal{BC}^k(U,Y)$. From Lemma 3.3.9 and Lemma 3.2.10 we conclude that for $t \in \mathbb{K}$ with $|t| \leq \frac{\operatorname{dist}(\eta_0(U),\partial V)}{\|\eta\|_{1_U,0}+1}$, the integral

$$\int_0^1 (D\gamma_0 \circ (\eta_0 + st\eta)) \cdot \eta \, ds$$

exists in $\mathcal{BC}^k(U,Z)$. Using identity (3.3.9.1) we derive

$$\frac{g^{k+\ell+1}_{\mathcal{BC},Z}(\gamma_0+t\gamma,\eta_0+t\eta)-g^{k+\ell+1}_{\mathcal{BC},Z}(\gamma_0,\eta_0)}{t} = g^{k+\ell+1}_{\mathcal{BC},Z}(\gamma,\eta_0+t\eta) + \int_0^1 g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0,\eta_0+st\eta)\cdot\eta\, ds.$$

We use Proposition A.1.8 and the continuity of $g^{k+\ell+1}_{\mathcal{BC},Z}$, $g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}$ and $\cdot$ (cf. Lemma 3.3.8 and Corollary 3.3.7) to see that the right hand side of this equation converges to

$$g^{k+\ell+1}_{\mathcal{BC},Z}(\gamma,\eta_0) + g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0,\eta_0)\cdot\eta$$

in $\mathcal{BC}^k(U,Z)$ as $t \to 0$. Hence the $g^{k+\ell+1}_{\mathcal{BC},Z}$ is differentiable and its differential is given by (3.3.10.1) and thus continuous.

$\ell - 1 \to \ell$: The map $g^{k+\ell+1}_{\mathcal{BC},Z}$ is $\mathcal{C}^\ell$ if $dg^{k+\ell+1}_{\mathcal{BC},Z}$ is $\mathcal{C}^{\ell-1}$. The latter follows easily from (3.3.10.1), since the inductive hypothesis ensures that $g^{k+\ell+1}_{\mathcal{BC},Z}$ and $g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}$ are $\mathcal{C}^{\ell-1}$; and $\cdot$ and D are smooth.

If $k = \infty$, then in view of Corollary 3.2.6 and Proposition A.1.12, $g^\infty_{\mathcal{BC},Z}$ is smooth as a map to $\mathcal{BC}^\infty(U,Z)$ iff it is smooth as a map to $\mathcal{BC}^j(U,Z)$ for each $j \in \mathbb{N}$. This was already proved in the case where $k = j$ and $\ell = \infty$. □

Composition of weighted functions with bounded functions

Generally, we can not expect that the composition of a bounded function with a weighted function is again a weighted function (to the same weights). As an example, the composition of the constant 1 function and a Schwartz function is not a Schwartz function. However, if we compose a bounded function mapping 0 to 0 with a weighted function, we get good results.

Lemma 3.3.11. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then for $\gamma \in \mathcal{BC}^{k+1}(V,Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$*

$$\gamma \circ \eta \in \mathcal{C}^k_{\mathcal{W}}(U,Z),$$

and the composition map

$$\mathcal{BC}^{k+1}(V,Z)_0 \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma,\eta) \mapsto \gamma\circ\eta \qquad (*)$$

is continuous.

Proof. We distinguish the cases $k < \infty$ and $k = \infty$:
$k < \infty$: To prove that for $\gamma \in \mathcal{BC}^{k+1}(V,Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ the composition $\gamma \circ \eta$ is in $\mathcal{C}^k_{\mathcal{W}}(U,Z)$, in view of Proposition 3.2.3 it suffices to show that

$$\gamma \circ \eta \in \mathcal{C}^0_{\mathcal{W}}(U,Z) \text{ and for } k > 0 \text{ also } D(\gamma \circ \eta) \in \mathcal{C}^{k-1}_{\mathcal{W}}(U, \mathrm{L}(X,Z)).$$

Similarly the continuity of the composition $(*)$, which is denoted by g_k in the remainder of this proof, is equivalent to the continuity of $\iota_0 \circ g_k$ and for $k > 0$ also of $D \circ g_k$, where $\iota_0 : \mathcal{C}^k_{\mathcal{W}}(U,Z) \to \mathcal{C}^0_{\mathcal{W}}(U,Z)$ denotes the inclusion map.

First we show $\gamma \circ \eta \in \mathcal{C}^0_{\mathcal{W}}(U,Z)$. To this end, let $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned} |f(x)|\,\|\gamma(\eta(x))\| &= |f(x)|\,\|\gamma(\eta(x)) - \gamma(0)\| \\ &= |f(x)| \left\| \int_0^1 D\gamma(t\eta(x)) \cdot \eta(x)\,dt \right\| \leq \|D\gamma\|_{1_V,0} \|\eta\|_{f,0}; \end{aligned}$$

here we used that the line segment from 0 to $\eta(x)$ is contained in V. So $\gamma \circ \eta \in \mathcal{C}^0_{\mathcal{W}}(U,Z)$.
To check the continuity of $\iota_0 \circ g_k$, let $\gamma, \gamma_0 \in \mathcal{BC}^{k+1}(V,Z)_0$ and $\eta, \eta_0 \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ such that $\|\eta - \eta_0\|_{1_U,0} < \operatorname{dist}(\eta_0(U), \partial V)$, $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned} &|f(x)|\,\|(\gamma \circ \eta)(x) - (\gamma_0 \circ \eta_0)(x)\| \\ =&|f(x)|\,\|\gamma(\eta(x)) - \gamma(\eta_0(x)) + \gamma(\eta_0(x)) - \gamma_0(\eta_0(x))\| \\ \leq&|f(x)|\,\|\gamma(\eta(x)) - \gamma(\eta_0(x))\| + |f(x)|\,\|(\gamma - \gamma_0)(\eta_0(x))\| \\ =&|f(x)| \left\| \int_0^1 D\gamma(t\eta(x) + (1-t)\eta_0(x)) \cdot (\eta(x) - \eta_0(x))\,dt \right\| \\ &\qquad + |f(x)|\,\|(\gamma - \gamma_0)(\eta_0(x)) - (\gamma - \gamma_0)(0)\| \\ \leq&|f(x)|\,\|D\gamma\|_{1_V,0}\|\eta(x) - \eta_0(x)\| + |f(x)| \left\| \int_0^1 D(\gamma - \gamma_0)(t\eta_0(x)) \cdot \eta_0(x)\,dt \right\| \\ \leq&|f(x)|\,\|D\gamma\|_{1_V,0}\|\eta(x) - \eta_0(x)\| + |f(x)|\,\|D(\gamma - \gamma_0)\|_{1_V,0}\|\eta_0(x)\|. \end{aligned}$$

From this, we easily see that $\iota_0 \circ g_k$ is continuous in (γ_0, η_0).

For $k > 0$, $\gamma \in \mathcal{BC}^{k+1}(V,Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ we apply the chain rule to get

$$(D \circ g_k)(\gamma, \eta) = D(\gamma \circ \eta) = (D\gamma \circ \eta) \cdot D\eta = g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma, \eta) \cdot D\eta; \tag{$**$}$$

here we used that $\eta \in \mathcal{BC}^k(U,V)$ because 1_U is in $\mathcal{W}$. Since $D\eta \in \mathcal{C}^{k-1}_{\mathcal{W}}(U, \mathrm{L}(X,Y))$ and $g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma, \eta) \in \mathcal{BC}^{k-1}(U, \mathrm{L}(Y,Z))$ hold, (see Lemma 3.3.8), $(D \circ g_k)(\gamma, \eta)$ is in $\mathcal{C}^{k-1}_{\mathcal{W}}(U, \mathrm{L}(Y,Z))$ (see Corollary 3.3.6). Using that D, $\cdot$ and $g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}$ are continuous (see Proposition 3.2.3, Corollary 3.3.6 and Lemma 3.3.8, respectively), we deduce the continuity of $D \circ g_k$ from $(**)$.

$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{array}{ccc} \mathcal{BC}^{\infty}(V,Z)_0 \times \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(U,V) & \xrightarrow{\quad g_\infty \quad} & \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(U,Z) \\ \big\downarrow & & \big\downarrow \\ \mathcal{BC}^{n+1}(V,Z)_0 \times \mathcal{C}^{\partial,n}_{\mathcal{W}}(U,V) & \xrightarrow[\quad g_n \quad]{} & \mathcal{C}^{\partial,n}_{\mathcal{W}}(U,Z) \end{array}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent the inclusion maps. With Corollary 3.2.6 we easily deduce the continuity of g_∞ from the one of g_n. $\square$

Proposition 3.3.12. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^*$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the map*

$$g_{\mathcal{W},Z}^{k+\ell+1} : \mathcal{BC}^{k+\ell+1}(V,Z)_0 \times \mathcal{C}_{\mathcal{W}}^{\partial,k}(U,V) \to \mathcal{C}_{\mathcal{W}}^{k}(U,Z) : (\gamma,\eta) \mapsto \gamma \circ \eta$$

whose existence was stated in Lemma 3.3.11 is a $\mathcal{C}^\ell$-map with

$$dg_{\mathcal{W},Z}^{k+\ell+1}(\gamma_0,\eta_0;\gamma,\eta) = g_{\mathcal{W},Z}^{k+\ell+1}(\gamma,\eta_0) + g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0,\eta_0) \cdot \eta. \tag{3.3.12.1}$$

Proof. For $k < \infty$, the proof is by induction on ℓ which we may assume finite because the inclusion maps $\mathcal{BC}^\infty(V,Z)_0 \to \mathcal{BC}^{k+\ell+1}(V,Z)_0$ are continuous linear (and hence smooth).

$\ell = 1$: Let $\gamma_0, \gamma \in \mathcal{BC}^{k+\ell+1}(V,Z)_0$, $\eta_0 \in \mathcal{C}_{\mathcal{W}}^{\partial,k}(U,V)$ and $\eta \in \mathcal{C}_{\mathcal{W}}^{k}(U,Y)$. From Lemma 3.3.9 and Lemma 3.2.10 we conclude that for $t \in \mathbb{K}$ with $|t| \leq \frac{\operatorname{dist}(\eta_0(U),\partial V)}{\|\eta\|_{1_U,0}+1}$, the integral

$$\int_0^1 (D\gamma_0 \circ (\eta_0 + st\eta)) \cdot \eta \, ds$$

exists in $\mathcal{C}_{\mathcal{W}}^{k}(U,Z)$. Using identity (3.3.9.1) we derive

$$\frac{g_{\mathcal{W},Z}^{k+\ell+1}(\gamma_0+t\gamma,\eta_0+t\eta) - g_{\mathcal{W},Z}^{k+\ell+1}(\gamma_0,\eta_0)}{t} = g_{\mathcal{W},Z}^{k+\ell+1}(\gamma,\eta_0+t\eta) + \int_0^1 g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0,\eta_0+st\eta) \cdot \eta \, ds.$$

We use Proposition A.1.8 and the continuity of $g_{\mathcal{W},Z}^{k+\ell+1}$, $g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}$ and $\cdot$ (cf. Lemma 3.3.11, Lemma 3.3.8 and Corollary 3.3.7) to see that the right hand side of this equation converges to

$$g_{\mathcal{W},Z}^{k+\ell+1}(\gamma,\eta_0) + g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0,\eta_0) \cdot \eta$$

in $\mathcal{C}_{\mathcal{W}}^{k}(U,Z)$ as $t \to 0$. Hence the $g_{\mathcal{W},Z}^{k+\ell+1}$ is differentiable and its differential is given by (3.3.12.1) and thus continuous.

$\ell - 1 \to \ell$: The map $g_{\mathcal{W},Z}^{k+\ell+1}$ is $\mathcal{C}^\ell$ if $dg_{\mathcal{W},Z}^{k+\ell+1}$ is $\mathcal{C}^{\ell-1}$. The latter follows easily from (3.3.12.1), since the inductive hypothesis respective Proposition 3.3.10 ensure that $g_{\mathcal{W},Z}^{k+\ell+1}$ and $g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}$ are $\mathcal{C}^{\ell-1}$; and $\cdot$ and D are smooth.

If $k = \infty$, then in view of Corollary 3.2.6 and Proposition A.1.12, $g_{\mathcal{W},Z}^{\infty}$ is smooth as a map to $\mathcal{C}_{\mathcal{W}}^{\infty}(U,Z)$ iff it is smooth as a map to $\mathcal{C}_{\mathcal{W}}^{j}(U,Z)$ for each $j \in \mathbb{N}$. This was already proved in the case where $k = j$ and $\ell = \infty$. $\square$

3.3.3. Composition of weighted functions with an analytic map

We discuss a sufficient criterion for an analytic map to operate on $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ through (covariant) composition. First, we state a result about the superposition on weighted functions that is a direct consequence of Proposition 3.3.12. After that, we have to treat real and complex analytic functions seperately. While the complex case is straightforward, in the real case we have to deal with complexifications.

We begin with a lemma about star-shaped open sets.

Lemma 3.3.13. *Let X be a normed space and $V \subseteq X$ an open set that is star-shaped with center 0. Then for $d := \operatorname{dist}(\{0\}, \partial V)$, there exists an anti-monotone family $(V_r^\partial)_{r\in]0,d[}$ such that*

$$V = \bigcup_{d>r>0} V_r^\partial.$$

Further, each $V^{\partial,r}$ is an open bounded set that is star-shaped with center 0 such that

$$\operatorname{dist}(V^{\partial,r}, \partial V) \geq \tfrac{d-r}{2} \min(1, r^2). \tag{3.3.13.1}$$

Proof. If $V = X$, this is obviously true. Otherwise, $\partial V \neq \emptyset$ and $d \in \mathbb{R}$. We set for $r \in]0, d[$

$$V_r^\partial := [0,1] \cdot \Big(\{y \in V : \operatorname{dist}(\{y\}, \partial V) > r\} \cap B_{\frac{1}{r}}(0)\Big).$$

This set is obviously bounded and star-shaped with center 0. Further, it is open: It is the union of an open set with $\{0\}$, and by the choice of r, it contains $B_{d-r}(0)$. So it remains to show that $\operatorname{dist}(V_r^\partial, \partial V) > 0$. To this end, let $x \in V_r^\partial$. We distinguish two cases.

First case: $x \in B(0, \frac{d-r}{2})$. Then $B(x, \frac{d-r}{2}) \subseteq V$.

Otherwise, there exists $z \in \{y \in V : \operatorname{dist}(\{y\}, \partial V) > r\} \cap B_{\frac{1}{r}}(0)$ and $t \in]0,1]$ with $x = tz$. We show that $B_{tr}(x) \subseteq V$, and use that obviously $B_r(z) \subseteq V$. So let $v \in B_{tr}(x)$. We set $h := \frac{v}{t} - z$. Then $\|h\| < r$, and hence $v = t(z+h) \in V$ since V is star-shaped. Further, we know that $\|z\| < \frac{1}{r}$ and $\|x\| \geq \frac{d-r}{2}$. Hence

$$\tfrac{t}{r} > t\|z\| = \|x\| \geq \tfrac{d-r}{2}.$$

This implies that $B(x, \frac{d-r}{2} r^2) \subseteq V$.

We deduce that estimate (3.3.13.1) holds. □

The proof of the following lemma is somewhat similar to the proof of [Wal06, Folg. 3.24] in the author's diploma thesis. Since it uses a different superposition operator and works for more general weighted function spaces, it is presented here nonetheless.

Lemma 3.3.14. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^*$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Suppose further that $\Phi : V \to Z$ with $\Phi(0) = 0$ satisfies*

$$W \text{ open in } V\text{, bounded and star-shaped with center } 0,\ \operatorname{dist}(W, \partial V) > 0 \\ \implies \Phi|_W \in \mathcal{BC}^{k+\ell+1}(W, Z).$$

Then $\Phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Z)$ *for all* $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$*, and the map*

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : \gamma \mapsto \Phi \circ \gamma$$

is $\mathcal{C}^\ell$*.*

Proof. We let $(V^\partial_r)_{r\in]0,d[}$ as in Lemma 3.3.13. Then for each r, we know from Proposition 3.3.12 that

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V^\partial_r) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : \gamma \mapsto \Phi \circ \gamma$$

is defined and $\mathcal{C}^\ell$ since $\Phi \in \mathcal{BC}^{k+\ell+1}(V^\partial_r, Z)_0$ by our assumption. But

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) = \bigcup_{r>0} \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V^\partial_r),$$

and $1_U \in \mathcal{W}$ implies that each $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V^\partial_r)$ is open in $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$, hence the assertion is proved. □

The following two lemmas are in the author's diploma thesis as [Wal06, La. 3.22]. They are listed here so they can be cited, but their proofs are omitted. The full version can also be found in [Wal13, Las. 3.3.15, 3.3.16].

Lemma 3.3.15. *Let* Y *and* Z *be complex normed spaces,* $V \subseteq Y$ *an open nonempty subset and* $\Phi : V \to Z$ *a complex analytic map. Further, let* $W \subseteq V$ *with* $\operatorname{dist}(W, \partial V) > 0$ *and* $r > 0$ *with* $r < \operatorname{dist}(W, \partial V)$ *such that* $\Phi|_{W+\overline{B}_Y(0,r)} \in \mathcal{BC}^0(W + \overline{B}_Y(0, r), Z)$*. Then* $\Phi|_W \in \mathcal{BC}^\infty(W, Z)$*. More explicitly, for each* $k \in \mathbb{N}$ *we have*

$$\|\Phi\|_{1_W,k} \leq \frac{(2k)^k}{r^k} \|\Phi\|_{1_{W+\overline{B}_Y(0,r)},0}. \tag{3.3.15.1}$$

Lemma 3.3.16. *Let* Y *and* Z *be complex normed spaces,* $V \subseteq Y$ *an open nonempty subset and* $\Phi : V \to Z$ *a complex analytic map that satisfies the following condition:*

$$W \subseteq V,\ W \text{ open in } V,\ \operatorname{dist}(W, \partial V) > 0 \implies \Phi|_W \in \mathcal{BC}^0(W, Z). \tag{3.3.16.1}$$

Then $\Phi|_W \in \mathcal{BC}^\infty(W, Z)$ *for all open subsets* $W \subseteq V$ *with* $\operatorname{dist}(W, \partial V) > 0$*.*

On real analytic maps and good complexifications

The two previous lemmas would allow us to state the desired result concerning covariant composition, but only for complex analytic maps. There are examples of real analytic maps for which the assertion of Lemma 3.3.16 is wrong. We define a class of real analytic maps that is suited to our need. Before that, we state the following small result concerning complexifications.

Lemma 3.3.17. *Let* X *and* Y *be real normed spaces,* $U \subseteq X$ *an open nonempty set,* $k \in \overline{\mathbb{N}}$ *and* $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$*. Further let* $\iota : Y \to Y_{\mathbb{C}}$ *denote the canonical inclusion into* $Y_{\mathbb{C}}$*.*

(a) Then $\mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}})$ *is the complexification of* $\mathcal{C}^k_{\mathcal{W}}(U, Y)$*, and the canonical inclusion map is given by*

$$\mathcal{C}^k_{\mathcal{W}}(U, Y) \to \mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}}) : \gamma \mapsto \iota \circ \gamma.$$

(b) Let $V \subseteq Y$ *be an open nonempty set and* $\widetilde{V} \subseteq Y_{\mathbb{C}}$ *an open neighborhood of* $\iota(V)$ *such that*

$$(\forall M \subseteq V)\ \operatorname{dist}(M, Y \setminus V) > 0 \implies \operatorname{dist}(\iota(M), Y_{\mathbb{C}} \setminus \widetilde{V}) > 0. \tag{3.3.17.1}$$

Then

$$\iota \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V}).$$

Proof. (a) It is a well known fact that $Y_{\mathbb{C}} \cong Y \times Y$ and $\iota(y) = (y, 0)$ for each $y \in Y$. Hence

$$\mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}}) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y \times Y) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y) \times \mathcal{C}^k_{\mathcal{W}}(U, Y)$$

by Lemma 3.4.16 (and Proposition 3.3.3), and

$$\iota \circ \gamma = (\gamma, 0) \in \mathcal{C}^k_{\mathcal{W}}(U, Y) \times \mathcal{C}^k_{\mathcal{W}}(U, Y) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}})$$

for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$.

(b) This is an immediate consequence of (a) and condition (3.3.17.1). □

Definition 3.3.18. Let Y and Z be real normed spaces, $V \subseteq Y$ an open nonempty set, $\Phi : V \to Z$ a real analytic map. We say that Φ has a *good complexification* if there exists a complexification $\widetilde{\Phi} : \widetilde{V} \subseteq Y_{\mathbb{C}} \to Z_{\mathbb{C}}$ of Φ which satisfies condition (3.3.16.1) and whose domain satisfies condition (3.3.17.1). In this case, we call $\widetilde{\Phi}$ a good complexification.

The following lemma states that good complexifications always exist at least locally. It is not needed in the further discussion.

Lemma 3.3.19. *Let* Y *and* Z *be real normed spaces,* $V \subseteq Y$ *an open nonempty set and* $\Phi : V \to Z$ *a real analytic map. Then for each* $x \in V$ *there exists an open neighborhood* $W_x \subseteq Y$ *of* x *such that* $\Phi|_{W_x}$ *has a good complexification.*

Proof. Let $\widetilde{\Phi} : \widetilde{V} \subseteq Y_{\mathbb{C}} \to Z_{\mathbb{C}}$ be a complexification of Φ and $\iota : V \to \widetilde{V}$ the canonical inclusion. Then there exists an $r > 0$ such that $B_{Y_{\mathbb{C}}}(\iota(x), r) \subseteq \widetilde{V}$ and $\widetilde{\Phi}$ is bounded on $B_{Y_{\mathbb{C}}}(\iota(x), r)$. Then it is obvious that $W_x := \iota^{-1}(B_{Y_{\mathbb{C}}}(\iota(x), r)) = B_Y(x, r)$ has the stated property. □

Power series We present a class of analytic maps which have good complexifications: Absolutely convergent power series in Banach algebras. This lemma was in the author's diploma thesis as [Wal06, La. 3.23]. We omit its proof (which can be found in [Wal13, La. 3.3.20]), but present it here so that we can cite it.

Lemma 3.3.20. *Let A be a Banach algebra and $\sum_{\ell=0}^{\infty} a_\ell z^\ell$ a power series with $a_\ell \in \mathbb{K}$ and the radius of convergence $R \in]0, \infty]$. We define for $x \in A$*

$$P_x : B_A(x, R) \to A : y \mapsto \sum_{\ell=0}^{\infty} a_\ell (y - x)^\ell.$$

Then the following assertions hold:

(a) The map P_x is analytic.

(b) If $\mathbb{K} = \mathbb{C}$ then P_x satisfies condition (3.3.16.1).

(c) If $\mathbb{K} = \mathbb{R}$ then P_x has a good complexification.

Main Result

Finally, we state the desired result for complex analytic maps and real analytic maps with good complexifications. The assertion is similar to the one of [Wal06, Folg. 3.24] in the author's diploma thesis, but we present it and the proof since the real case is proved more generally, a wider class of weighted function spaces is treated, and a different superposition operator is used.

Proposition 3.3.21. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Further, let $\Phi : V \to Z$ with $\Phi(0) = 0$ be either a complex analytic map that satisfies condition* (3.3.16.1) *or a real analytic map that has a good complexification. Then for each $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$*

$$\Phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Z),$$

and the map

$$\Phi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : \gamma \mapsto \Phi \circ \gamma$$

is analytic.

Proof. If Φ is complex analytic, this is an immediate consequence of Lemma 3.3.14 and Lemma 3.3.16.

If Φ is real analytic, by our assumptions there exists a good complexification $\widetilde{\Phi} : \widetilde{V} \subseteq Y_\mathbb{C} \to Z$. We know from the first part that $\widetilde{\Phi}$ induces a complex analytic map

$$\widetilde{\Phi}_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V}) \to \mathcal{C}^k_{\mathcal{W}}(U, Z_\mathbb{C}) : \gamma \mapsto \widetilde{\Phi} \circ \gamma.$$

Since $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V})$ by Lemma 3.3.17 and Φ_* coincides with the restriction of $\widetilde{\Phi}_*$ to $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$, it follows that Φ_* is real analytic. $\square$

Quasi-inversion algebras of weighted functions

As an application, we see that for a set $\mathcal{W}$ of weights with $1_U \in \mathcal{W}$ and a Banach algebra A, the space $\mathcal{C}^k_{\mathcal{W}}(U, A)$ can be turned into a topological algebra with continuous quasi-inversion. Details on algebras with quasi-inversion can be found in Appendix C. It was proved as an amalgam of [Wal06, Prop. 3.26, Satz 3.27] in the author's diploma thesis. Since the technique used here is different, the proof is not omitted.

Lemma 3.3.22. *Let A be a Banach algebra, X a normed space, $U \subseteq X$ an open nonempty subset, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the locally convex space $\mathcal{C}^k_{\mathcal{W}}(U, A)$ endowed with the multiplication described in Corollary 3.3.5 becomes a complete topological algebra with continuous quasi-inversion in the sense of Definition C.2.1. For each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, A)^q$*

$$QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = QI_A \circ \gamma,$$

and

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1)) = \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, A) : \|\gamma\|_{1_U,0} < 1\} \subseteq \mathcal{C}^k_{\mathcal{W}}(U, A)^q.$$

Proof. The relation $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = QI_A \circ \gamma$ is an immediate consequence of the definition of the multiplication, so it only remains to show that $\mathcal{C}^k_{\mathcal{W}}(U, A)^q$ is open and $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}$ is continuous. We proved in Lemma C.2.4 that it suffices to find a neighborhood of 0 that consists of quasi-invertible elements such that the restriction of $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}$ to it is continuous. We show that $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1))$ is such a neighborhood. The map

$$G : B_1(0) \to A : x \mapsto \sum_{i=1}^{\infty} x^i$$

is given by a power series and maps 0 to 0, hence we know from Lemma 3.3.20 and Proposition 3.3.21 that the map

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1)) \to \mathcal{C}^k_{\mathcal{W}}(U, A) : \gamma \mapsto G \circ \gamma$$

is defined and analytic. Since

$$G \circ \gamma = \sum_{i=1}^{\infty} \gamma^i$$

for each $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1))$, we conclude from Lemma C.2.5 that γ is quasi-invertible with

$$QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = -G \circ \gamma,$$

so the proof is complete. □

Example 3.3.23. Let Y be a Banach space, $U \subseteq X$ an open nonempty subset, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the locally convex space $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(Y))$ endowed with the multiplication described in Corollary 3.3.6 becomes a complete topological algebra with continuous quasi-inversion.

3.3.4. Superposition with functions defined on a product

We examine whether a function $\Xi : U \times V \to Z$ induces a superposition operation $\gamma \mapsto \Xi \circ (\mathrm{id}_U, \gamma)$ on weighed functions. We show that this is the case if $0 \in V$, Ξ maps $U \times \{0\}$ to 0, and if the size of the derivatives of Ξ can be covered with the weights, see (3.3.26.4) for the precise phrasing.

Estimates for higher derivatives

We give estimates for the higher derivatives of a function of two variables, provided it is linear in its second argument. We also turn to more special cases of such functions.

Lemma 3.3.24. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}^*$ and $\Xi \in \mathcal{FC}^k(U \times Y, Z)$ a map that is linear in its second argument. Further, let $\ell \in \mathbb{N}$ with $\ell \leq k$, $x \in U$ and $y \in Y$.*

(a) *The map $D_1^{(\ell)}\Xi$ is linear in the second argument. Hence $D_1^{(\ell)}\Xi(U \times \{0\}) = \{0\}$ and (if $\ell < k$)*

$$\frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_1, y + th_2) = D_1^{(\ell)}\Xi(x, h_2) + D_1^{(\ell+1)}\Xi(x, y)\neg h_1. \qquad (\dagger)$$

Here, for an $(m+1)$-linear map $b : E_1 \times \cdots \times E_{m+1} \to F$, for $h \in E_{m+1}$ we let $b\neg h$ denote the m-linear map $E_1 \times \cdots \times E_m \to F : (x_1, \ldots, x_m) \mapsto b(x_1, \ldots, x_m, h)$.

(b) *Suppose that $\ell \geq 1$. Let $h^1, \ldots, h^\ell \in X \times Y$ with $h^j = (h_1^j, h_2^j)$. Then the identity*

$$D^{(\ell)}\Xi(x, y) \cdot (h^1, \ldots, h^\ell) = D_1^{(\ell)}\Xi(x, y) \cdot (h_1^1, \ldots, h_1^\ell) + \sum_{j=1}^{\ell} D_1^{(\ell-1)}\Xi(x, h_2^j) \cdot \widehat{h_1^j}$$

holds, where $\widehat{h_1^j} := (h_1^1, \ldots, h_1^{j-1}, h_1^{j+1}, \ldots, h_1^\ell)$. In particular,

$$\|D^{(\ell)}\Xi(x, y)\|_{op} \leq \ell \|D_1^{(\ell-1)}\Xi(x, \cdot)\|_{op} + \|D_1^{(\ell)}\Xi(x, \cdot)\|_{op}\|y\|. \qquad (\dagger\dagger)$$

(c) *Suppose that there exist a normed space $\widetilde{X}$, a map $g \in \mathcal{FC}^k(U, \widetilde{X})$ and a continuous bilinear map $b : \widetilde{X} \times Y \to Z$ such that $\Xi = b \circ (g \times \mathrm{id}_Y)$. Then*

$$D_1^{(\ell)}\Xi(x, y) \cdot (h_1, \ldots, h_\ell) = b(D^{(\ell)}g(x) \cdot (h_1, \ldots, h_\ell), y),$$

for $h_1, \ldots, h_\ell \in X$. In particular,

$$\|D_1^{(\ell)}\Xi(x, \cdot)\|_{op} \leq \|b\|_{op}\|D^{(\ell)}g(x)\|_{op} \qquad (\dagger\,\dagger\,\dagger)$$

and (if $\ell \geq 1$)

$$\|D^{(\ell)}\Xi(x, y)\|_{op} \leq \|b\|_{op}\ell\|D^{(\ell-1)}g(x)\|_{op} + \|b\|_{op}\|y\|\,\|D^{(\ell)}g(x)\|_{op}. \qquad (3.3.24.1)$$

Proof. (a) We prove by induction on ℓ that $d_1^{(\ell)}\Xi$ is linear in its second argument. For $\ell = 0$, this is true by our assumption.

$\ell \to \ell + 1$: Since for $h_1, \dots, h_{\ell+1} \in X$,

$$d_1^{(\ell+1)}\Xi(x, y; h_1, \dots, h_{\ell+1}) = \frac{d}{dt}|_{t=0} d_1^{(\ell)}\Xi(x + th_{\ell+1}, y; h_1, \dots, h_\ell),$$

and $d_1^{(\ell)}\Xi$ is linear in its second argument, also $d_1^{(\ell+1)}\Xi$ is so.

We prove (†). We get using the linearity of $D_1^{(\ell)}\Xi$ in the second argument

$$\frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_1, y + th_2) = \lim_{t\to 0} D_1^{(\ell)}\Xi(x + th_1, h_2) + \frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_1, y)$$

Since $\lim_{t\to 0} D_1^{(\ell)}\Xi(x + th_1, h_2) = D_1^{(\ell)}\Xi(x, h_2)$ and

$$\frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_1, y) \cdot (v_1, \dots, v_\ell) = D_1^{(\ell+1)}\Xi(x, y)(v_1, \dots, v_\ell, h_1),$$

for $v_1, \dots, v_\ell \in X$, the desired identity follows.

(b) We prove the identity for $D^{(\ell)}\Xi$ by induction on ℓ.

$\ell = 1$: This follows directly from (†).

$\ell \to \ell + 1$: We calculate the $(\ell + 1)$-th derivative of Ξ using the inductive hypothesis and (†):

$$\begin{aligned}
&D^{(\ell+1)}\Xi(x, y) \cdot (h^1, \dots, h^{\ell+1}) \\
=&\frac{d}{dt}|_{t=0} D^{(\ell)}\Xi(x + th_1^{\ell+1}, y + th_2^{\ell+1}) \cdot (h^1, \dots, h^\ell) \\
=&\frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_1^{\ell+1}, y + th_2^{\ell+1}) \cdot (h_1^1, \dots, h_1^\ell) + \sum_{j=1}^{\ell} \frac{d}{dt}|_{t=0} D_1^{(\ell-1)}\Xi(x + th_1^{\ell+1}, h_2^j) \cdot \widehat{h_1^j} \\
=&D_1^{(\ell)}\Xi(x, h_2^{\ell+1}) \cdot (h_1^1, \dots, h_1^\ell) + D_1^{(\ell+1)}\Xi(x, y) \cdot (h_1^1, \dots, h_1^\ell, h_1^{\ell+1}) + \sum_{j=1}^{\ell} D_1^{(\ell)}\Xi(x, h_2^j) \cdot \widehat{h_1^j},
\end{aligned}$$

from which we derive the assertion.

The estimate (††) follows directly from this identity.

(c) We first prove the identity by induction on ℓ. The assertion obviously holds for $\ell = 0$.

$\ell \to \ell + 1$: We use the inductive hypothesis to calculate

$$\begin{aligned}
D_1^{(\ell+1)}\Xi(x, y) \cdot (h_1, \dots, h_{\ell+1}) &= \frac{d}{dt}|_{t=0} D_1^{(\ell)}\Xi(x + th_{\ell+1}, y) \cdot (h_1, \dots, h_\ell) \\
&= \frac{d}{dt}|_{t=0} b(D^{(\ell)}g(x + th_{\ell+1}) \cdot (h_1, \dots, h_\ell), y) = b(D^{(\ell+1)}g(x) \cdot (h_1, \dots, h_{\ell+1}), y),
\end{aligned}$$

so the assertion is established.

The estimate († † †) follows directly from this identity. Furthermore, we derive (3.3.24.1) from (††) and († † †). □

Lemma 3.3.25. *Let E, F, X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets, $b : \mathrm{L}(Y, Z) \times E \to F$ continuous bilinear with $\|b\|_{op} \leq 1$ and $\Xi \in \mathcal{FC}^{\infty}(U \times V, Z)$. We define*

$$\Xi_b^{(2)} : U \times V \times E \to F : (x, y, e) \mapsto b(D_2\Xi(x, y), e).$$

Then $\Xi_b^{(2)}(U \times V \times \{0\}) = \{0\}$, and for each $\ell \in \mathbb{N}^*$, we have

$$\|D^{(\ell)}\Xi_b^{(2)}(x, y, e)\|_{op} \leq \ell \|D^{(\ell)}\Xi(x, y)\|_{op} + \|e\| \, \|D^{(\ell+1)}\Xi(x, y)\|_{op}.$$

Moreover, for each $R > 0$,

$$\|\Xi_b^{(2)}\|_{1_{U\times V\times B_E(0,R)},\ell} \leq \ell \|\Xi\|_{1_{U\times V},\ell} + R\|\Xi\|_{1_{U\times V},\ell+1}. \qquad (3.3.25.1)$$

Proof. We get from (3.3.24.1) that

$$\|D^{(\ell)}\Xi_b^{(2)}(x, y, e)\|_{op} \leq \ell \|D^{(\ell-1)}(D_2\Xi)(x, y)\|_{op} + \|e\| \, \|D^{(\ell)}(D_2\Xi)(x, y)\|_{op}.$$

Since

$$\|D^{(\ell)}(D_2\Xi)(x, y)\|_{op} \leq \|D^{(\ell)}(D\Xi)(x, y)\|_{op} = \|D^{(\ell+1)}\Xi(x, y)\|_{op}$$

for all $\ell \in \mathbb{N}^*$, we obtain the first estimate. (3.3.25.1) follows. □

The superposition operator

We prove the above assertion about the superposition, using notation from Lemma 3.3.25. The hardest part of the proof will be the examination of the superposition with $\Xi_M^{(2)}$.

Proposition 3.3.26. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty subset, $V \subseteq Y$ an open neighborhood of 0 that is star-shaped with center 0, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Further, let $\Xi \in \mathcal{FC}^{\infty}(U \times V, Z)$ such that $\Xi(U \times \{0\}) = \{0\}$.*

(a) For maps $\gamma, \eta : U \to V$ such that the line segment $\{t\gamma + (1-t)\eta : t \in [0, 1]\} \subseteq V^U$ and $f \in \mathcal{W}$, the estimate

$$\|\Xi \circ (\mathrm{id}_U, \gamma) - \Xi \circ (\mathrm{id}_U, \eta)\|_{f,0} \leq \|D_2\Xi\|_{1_{U\times V},0} \|\gamma - \eta\|_{f,0} \qquad (3.3.26.1)$$

holds. In particular, for $\eta = 0$ we get

$$\|\Xi \circ (\mathrm{id}_U, \gamma)\|_{f,0} \leq \|D_2\Xi\|_{1_{U\times V},0} \|\gamma\|_{f,0}. \qquad (3.3.26.2)$$

(b) Let $\gamma \in \mathcal{FC}^1(U, V)$. Then

$$D(\Xi \circ (\mathrm{id}_U, \gamma)) = D_1\Xi \circ (\mathrm{id}_U, \gamma) + D_2\Xi \circ (\mathrm{id}_U, \gamma) \cdot D\gamma.$$

The map $D_1\Xi$ maps $U \times \{0\}$ to 0, and for $f \in \mathcal{W}$, we have

$$\|\Xi \circ (\mathrm{id}_U, \gamma)\|_{f,1} \leq \|\Xi\|_{1_{U\times V},2} \|\gamma\|_{f,0} + \|D_2\Xi\|_{1_{U\times V},0} \|\gamma\|_{f,1}. \qquad (3.3.26.3)$$

(c) *Suppose that*

$$(\forall f \in \mathcal{W}, \ell \in \mathbb{N}^*)(\exists g \in \mathcal{W}_{\max}) \, \|\Xi\|_{1_{U\times V},\ell}|f| \leq |g|. \tag{3.3.26.4}$$

Then the map

$$\Xi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \Xi \circ (\mathrm{id}_U, \gamma)$$

is defined and smooth with

$$d\Xi_*(\gamma;\gamma_1) = (d_2\Xi)_*(\gamma,\gamma_1). \tag{3.3.26.5}$$

Proof. (a) For each $x \in U$, we calculate

$$\Xi(x,\gamma(x)) - \Xi(x,\eta(x)) = \int_0^1 d_2\Xi(x, t\gamma(x) + (1-t)\eta(x); \gamma(x) - \eta(x))\, dt.$$

Hence for each $f \in \mathcal{W}$, we have

$$|f(x)| \, \|\Xi(x,\gamma(x)) - \Xi(x,\eta(x))\| \leq \|D_2\Xi\|_{1_{U\times V},0}|f(x)| \, \|\gamma(x) - \eta(x)\|.$$

From this estimate, we conclude that (3.3.26.1) holds.

(b) The identity for $D(\Xi \circ (\mathrm{id}_U,\gamma))$ follows from the Chain Rule. For $x \in U$ and $h \in X$, we have

$$D_1\Xi(x,0)\cdot h = d_1\Xi(x,0;h) = \lim_{t\to 0} \frac{\Xi(x+th,0) - \Xi(x,0)}{t} = 0,$$

whence $D_1\Xi(x,0) = 0$. We then get the estimate by applying (3.3.26.2) to the first summand.

(c) We first prove by induction on k that Ξ_* is defined and continuous.

$k = 0$: We see with (3.3.26.2) that Ξ_* is defined since

$$\|\Xi \circ (\mathrm{id}_U,\gamma)\|_{f,0} \leq \|\Xi\|_{1_{U\times V},1}\|\gamma\|_{f,0} \leq \|\gamma\|_{g,0}.$$

With a similar argument, we see using (3.3.26.1) that Ξ_* continuous since each $\gamma \in \mathcal{C}^{\partial,0}_{\mathcal{W}}(U,V)$ has a convex neighborhood in $\mathcal{C}^{\partial,0}_{\mathcal{W}}(U,V)$.

$k \to k+1$: We use Proposition 3.2.3. So all that remains to show is that $D(\Xi\circ(\mathrm{id}_U,\gamma)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z))$ and $\gamma \mapsto D(\Xi \circ (\mathrm{id}_U,\gamma))$ is continuous. We proved in (b) that

$$D(\Xi \circ (\mathrm{id}_U,\gamma)) = D_1\Xi \circ (\mathrm{id}_U,\gamma) + \Xi^{(2)}_M \circ (\mathrm{id}_U,\gamma, D\gamma),$$

see Lemma 3.3.25 for the definition of $\Xi^{(2)}_M$ (here, M denotes the composition of linear operators). We also proved in (b) that $D_1\Xi(U\times\{0\}) = \{0\}$, and obviously $\|D_1\Xi\|_{1_{U\times V},\ell} \leq \|\Xi\|_{1_{U\times V},\ell+1}$ for all $\ell \in \mathbb{N}$. Hence we can use the inductive hypothesis to see that

$$\mathcal{C}^{\partial,k+1}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z)) : \gamma \mapsto D_1\Xi \circ (\mathrm{id}_U,\gamma)$$

is defined and continuous. We examine $\Xi_M^{(2)}$. To this end, let $R > 0$. We see using (3.3.25.1) that for $\ell \in \mathbb{N}^*$ and $f \in \mathcal{W}$,

$$\|\Xi_M^{(2)}\|_{1_{U\times V\times B_{\mathrm{L}(X,Y)}(0,R)},\ell}|f| \leq \ell\|\Xi\|_{1_{U\times V},\ell}|f| + R\|\Xi\|_{1_{U\times V},\ell+1}|f| \leq \ell|g_\ell| + R|g_{\ell+1}|.$$

Here, $g_\ell, g_{\ell+1} \in \mathcal{W}_{\max}$ exist by our assumptions. Hence in both cases, we can apply the inductive hypothesis to $\Xi_M^{(2)}$ and get (using Lemma 3.4.16 implicitly) that the map

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_{\mathrm{L}(X,Y)}(0,R)) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z)) : (\gamma, \Gamma) \mapsto \Xi_M^{(2)} \circ (\mathrm{id}_U, \gamma, \Gamma)$$

is defined and continuous. Hence for each $\gamma \in \mathcal{C}^{\partial,k+1}_{\mathcal{W}}(U,V)$, the map

$$\{\eta \in \mathcal{C}^{\partial,k+1}_{\mathcal{W}}(U,V) : \|\eta\|_{1_U,1} < \|\gamma\|_{1_U,1} + 1\} \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z)) : \eta \mapsto \Xi_M^{(2)} \circ (\mathrm{id}_U, \eta, D\eta)$$

is defined and continuous. Since $1_U \in \mathcal{W}$, the domain of this map is a neighborhood of γ. This finishes the proof.

We pass on to prove the smoothness of Ξ_*. To do this, we have to examine $d_2\Xi$. Obviously $d_2\Xi = \Xi^{(2)}_{\cdot}$, where $\cdot$ denotes the evaluation of linear operators. Hence we can use a similar argument as above when discussing $\Xi_M^{(2)}$ to see that

$$(d_2\Xi)_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \times \mathcal{C}^k_{\mathcal{W}}(U,Y) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma, \gamma_1) \mapsto d_2\Xi \circ (\mathrm{id}_U, \gamma, \gamma_1)$$

is defined and continuous. Now let $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ and $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$. Since $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ is open, there exists an $r > 0$ such that $\{\gamma + s\gamma_1 : s \in B_{\mathbb{K}}(0,r)\} \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$. We calculate for $x \in U$ and $t \in B_{\mathbb{K}}(0,r) \setminus \{0\}$ (using Lemma 3.4.16 implicitly) that

$$\begin{aligned}\frac{\Xi_*(\gamma + t\gamma_1)(x) - \Xi_*(\gamma)(x)}{t} &= \frac{\Xi(x, \gamma(x) + t\gamma_1(x)) - \Xi(x,\gamma(x))}{t}\\ &= \int_0^1 d_2\Xi(x, \gamma(x) + st\gamma_1(x); \gamma_1(x))\, ds\\ &= \int_0^1 (d_2\Xi)_*(\gamma + st\gamma_1, \gamma_1)(x)\, ds.\end{aligned}$$

Hence we can apply Lemma 3.2.10 to see that

$$\frac{\Xi_*(\gamma + t\gamma_1) - \Xi_*(\gamma)}{t} = \int_0^1 (d_2\Xi)_*(\gamma + st\gamma_1, \gamma_1)\, ds.$$

Using Proposition A.1.8, we derive that Ξ_* is $\mathcal{C}^1$ and (3.3.26.5) holds.

We see with (3.3.25.1) (again, using that $d_2\Xi = \Xi^{(2)}_{\cdot}$) that (3.3.26.4) holds for $d_2\Xi$ on $U \times V \times B_R(0)$ for each $R > 0$. Since $1_U \in \mathcal{W}$, we have that $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V \times Y) = \bigcup_{R>0} \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V \times B_R(0))$. So with an easy induction argument we conclude (using Lemma 3.4.16) from (3.3.26.5) that Ξ_* is $\mathcal{C}^\ell$ for each $\ell \in \mathbb{N}$ and hence smooth. □

Corollary 3.3.27. *Let the data be as in Proposition 3.3.26. Suppose that $D\Xi \in \mathcal{BC}^\infty(U \times V, \mathrm{L}(X \times Y, Z))$. Then*

$$\Xi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \Xi \circ (\mathrm{id}_U, \gamma)$$

is defined and smooth.

Proof. This follows from Proposition 3.3.26 since (3.3.26.4) is obviously satisfied. □

3.4. Weighted maps into locally convex spaces

We define and examine weighted functions with values in arbitrary locally convex spaces. In order to do this, we use tools and definitions that are provided in A.1.2. The material of this section is only needed for latter discussions of weighted mapping groups with values in arbitrary locally convex Lie groups in Section 7.2; readers primarily interested in diffeomorphism groups may want to skip this section.

3.4.1. Definition and topological structure

The definition of weighted function with values in locally convex spaces relies on the one with values in normed spaces.

Definition 3.4.1. Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a locally convex space, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty. We define

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \{\gamma \in \mathcal{C}^k(U,Y) : (\forall p \in \mathcal{N}(Y))\, \pi_p \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y_p)\},$$

using notation as in Definition A.1.29. For $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$,

$$\|\cdot\|_{p,f,\ell} : \mathcal{C}^k_{\mathcal{W}}(U,Y) \to \mathbb{R} : \gamma \mapsto \|\pi_p \circ \gamma\|_{f,\ell}$$

is a seminorm on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. We endow $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ with the locally convex vector space topology that is generated by these seminorms.

We show that the structure of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is already determined by $\{\|\cdot\|_{p,f,\ell} : p \in \mathcal{P}, f \in \mathcal{W}, \ell \in \mathbb{N} \text{ with } \ell \leq k\}$, where $\mathcal{P}$ is just a generator of $\mathcal{N}(Y)$. This can be useful in some cases.

Lemma 3.4.2. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty and $\mathcal{P} \subseteq \mathcal{N}(Y)$ a set that generates $\mathcal{N}(Y)$. Then for $\gamma \in \mathcal{C}^k(U,Y)$*

$$\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) \iff (\forall p \in \mathcal{P})\, \pi_p \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y_p),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) \to \prod_{p \in \mathcal{P}} \mathcal{C}^k_{\mathcal{W}}(U,Y_p) : \gamma \mapsto (\pi_p \circ \gamma)_{p \in \mathcal{P}} \tag{†}$$

is a topological embedding.

Proof. Let $q \in \mathcal{N}(Y)$. Then there exist $p_1, \ldots, p_n \in \mathcal{P}$ and $C > 0$ such that

$$q \leq C \cdot \max_{i=1,\ldots,n} p_i.$$

Further we know that for each $\ell \in \mathbb{N}$ with $\ell \leq k$ and $x \in U$, $h_1, \ldots, h_\ell \in X$

$$d^{(\ell)}(\pi_q \circ \gamma)(x; h_1, \ldots, h_\ell) = (\pi_q \circ d^{(\ell)}\gamma)(x, h_1, \ldots, h_\ell),$$

so for $y \in U$ we get

$$\begin{aligned}
&\|d^{(\ell)}(\pi_q \circ \gamma)(x; h_1, \dots, h_\ell) - d^{(\ell)}(\pi_q \circ \gamma)(y; h_1, \dots, h_\ell)\|_q \\
\leq &\|d^{(\ell)}\gamma(x; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)\|_q \\
\leq &C \cdot \max_{i=1,\dots,n} \|d^{(\ell)}\gamma(x; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)\|_{p_i}.
\end{aligned}$$

Since we assumed that $\pi_{p_i} \circ \gamma \in \mathcal{FC}^k(U, Y_{p_i})$, from this estimate we conclude with Proposition A.3.2 that $\pi_q \circ \gamma \in \mathcal{FC}^k(U, Y_q)$ with

$$\|D^{(\ell)}(\pi_q \circ \gamma)(x)\|_{op} \leq C \cdot \max_{i=1,\dots,n} \|D^{(\ell)}(\pi_{p_i} \circ \gamma)(x)\|_{op}$$

for all $\ell \in \mathbb{N}$ with $\ell \leq k$ and $x \in U$. This implies that

$$\|\gamma\|_{q,f,\ell} \leq C \cdot \max_{i=1,\dots,n} \|\gamma\|_{p_i,f,\ell}$$

for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Hence

$$\pi_q \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_q),$$

and $\|\cdot\|_{q,f,\ell}$ is continuous with respect to the initial topology induced by (†). Since q was arbitrary, the proof is complete. □

An integrability criterion We generalize the assertion of Lemma 3.2.10.

Lemma 3.4.3. *Let X be a normed space, $U \subseteq X$ a nonempty open set, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ such that for each compact set $K \subseteq U$, there exists an $f_K \in \mathcal{W}$ with $\inf_{x \in K} |f_K(x)| > 0$. Further, let $\Gamma : [a, b] \to \mathcal{C}^k_{\mathcal{W}}(U, Y)$ a continuous curve and $R \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$. Assume that*

$$\int_a^b \operatorname{ev}_x(\Gamma(s))\, ds = \operatorname{ev}_x(R) \tag{$*$}$$

holds for all $x \in U$. Then Γ is weakly integrable with

$$\int_a^b \Gamma(s)\, ds = R.$$

Proof. We derive from Lemma 3.4.2 that the dual space of $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ coincides with the set of functionals $\{\lambda \circ \pi_{p_*} : p \in \mathcal{N}(Y), \lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y_p)'\}$. Hence Γ is weakly integrable with the integral R iff

$$\int_a^b \lambda(\pi_p \circ \Gamma)(s)\, ds = \lambda(\pi_p \circ R)$$

holds for all $p \in \mathcal{N}(Y)$ and $\lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y_p)'$; this is clearly equivalent to the weak integrability of $\pi_p \circ \Gamma$ with integral $\pi_p \circ R$ for all $p \in \mathcal{N}(Y)$. But we derive this assertion from identity ($*$) and Lemma 3.2.10. □

Reduction to lower order

We prove a generalization of Proposition 3.2.3. To this end, we need a locally convex topology on $\mathrm{L}(X, Y)$, where X is a normed and Y a locally convex space. We define such a topology and show that it arises as the intial topology with respect to the embedding $\mathrm{L}(X, Y) \to \prod_{p \in \mathcal{N}(Y)} \mathrm{L}(X, Y_p)$.

Topology on linear operators

Definition 3.4.4 (Topology on linear operators)**.** Let X be a normed space and Y a locally convex space. For each $p \in \mathcal{N}(Y)$ and $T \in \mathrm{L}(X, Y)$, we set

$$\|T\|_{op,p} := \sup_{x \neq 0} \frac{\|Tx\|_p}{\|x\|} = \|\pi_p \circ T\|_{op}.$$

This obviously defines a seminorm on $\mathrm{L}(X, Y)$, and henceforth we endow $\mathrm{L}(X, Y)$ with the locally convex topology that is generated by these seminorms. Further we define $\mathrm{L}(X, Y)_{op,p} := \mathrm{L}(X, Y)_{\|\cdot\|_{op,p}}$.

Lemma 3.4.5. *Let X be a normed space, Y a locally convex space and $p \in \mathcal{N}(Y)$. Then the map induced by*

$$(\pi_p)_* : \mathrm{L}(X, Y) \to \mathrm{L}(X, Y_p) : T \mapsto \pi_p \circ T$$

that makes

$$\begin{array}{ccc} (\mathrm{L}(X, Y), \|\cdot\|_{op,p}) & \xrightarrow{(\pi_p)_*} & \mathrm{L}(X, Y_p) \\ & \searrow^{\pi_{op,p}} \quad \nearrow & \\ & \mathrm{L}(X, Y)_{op,p} & \end{array}$$

a commutative diagram is an isometric isomorphism onto the image of $(\pi_p)_*$. The map

$$\mathrm{L}(X, Y) \to \prod_{p \in \mathcal{N}(Y)} \mathrm{L}(X, Y_p) : T \mapsto (\pi_p \circ T)_{p \in \mathcal{N}(Y)}$$

is a topological embedding.

Proof. Since $\|T\|_{op,p} = \|\pi_p \circ T\|_{op}$ for each $T \in \mathrm{L}(X, Y)$, the induced map is an isometry. By the definition of the topology of $\mathrm{L}(X, Y)$,

$$\mathrm{L}(X, Y) \to \prod_{p \in \mathcal{N}(Y)} \mathrm{L}(X, Y)_{op,p} : T \mapsto (\pi_{op,p} \circ T)_{p \in \mathcal{N}(Y)}$$

is an embedding, so by the transitivity of initial topologies, the proof is finished. $\square$

Weighted maps into spaces of linear operators and the main result Before we can prove the main result, we have to take a look at the structure of $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y))$.

Lemma 3.4.6. *Let X be a normed space, Y a locally convex space, $U \subseteq X$ an open nonempty subset and $k \in \overline{\mathbb{N}}$. Then for $\Gamma \in \mathcal{C}^k(U, \mathrm{L}(X, Y))$, nonempty $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $k \in \overline{\mathbb{N}}$ the equivalence*

$$\Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \iff (\forall p \in \mathcal{N}(Y))\, (\pi_p)_* \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p))$$

holds. More precisely, for $\ell \in \mathbb{N}$ with $\ell \leq k$, $f \in \overline{\mathbb{R}}^U$ and $p \in \mathcal{N}(Y)$ we have

$$\|\Gamma\|_{\|\cdot\|_{op,p}, f, \ell} = \|(\pi_p)_* \circ \Gamma\|_{f, \ell}. \tag{3.4.6.1}$$

This induces that the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \to \prod_{p \in \mathcal{N}(Y)} \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)) : \Gamma \mapsto ((\pi_p)_* \circ \Gamma)_{p \in \mathcal{N}(Y)}$$

is a topological embedding.

Proof. Note first that $\pi_{op,p} \circ \Gamma$ is $\mathcal{FC}^k$ iff $(\pi_p)_* \circ \Gamma$ is $\mathcal{FC}^k$ as a consequence of Lemma 3.4.5 and Proposition A.3.2. Using Lemma 3.4.5 it is easy to see that identity (3.4.6.1) is satisfied. This implies that for each $p \in \mathcal{N}(Y)$ the equivalence

$$(\pi_p)_* \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)) \iff \pi_{op,p} \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)_{op,p})$$

holds and that the isometry whose existence was stated in Lemma 3.4.5 induces an embedding

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)_{op,p}) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)).$$

Further we proved in Lemma 3.4.2 that

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \to \prod_{p \in \mathcal{N}(Y)} \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)_{op,p}) : \Gamma \mapsto ((\pi_{op,p})_* \circ \Gamma)_{p \in \mathcal{P}}$$

is an embedding, so we are home. □

Proposition 3.4.7 (Reduction to lower order)**.** *Let X be a normed space, Y a locally convex space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty and $k \in \mathbb{N}$. Let $\gamma \in \mathcal{C}^1(U, Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \iff (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \mathcal{C}^0_{\mathcal{W}}(U, Y).$$

Furthermore, the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \mathcal{C}^0_{\mathcal{W}}(U, Y) : \gamma \mapsto (D\gamma, \gamma)$$

is a topological embedding.

Proof. The definition of $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$, Proposition 3.2.3 and Lemma 3.4.6 give the equivalences

$$\begin{aligned}
\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) &\iff (\forall p \in \mathcal{N}(Y))\, \pi_p \circ \gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p) \\
&\iff (\forall p \in \mathcal{N}(Y))\, (D(\pi_p \circ \gamma), \pi_p \circ \gamma) \in \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y_p)) \times \mathcal{C}^{0}_{\mathcal{W}}(U,Y_p) \\
&\iff (D\gamma, \gamma) \in \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) \times \mathcal{C}^{0}_{\mathcal{W}}(U,Y).
\end{aligned}$$

Furthermore, we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) & \longrightarrow & \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) \times \mathcal{C}^{0}_{\mathcal{W}}(U,Y) \\
\big\downarrow & & \big\downarrow \\
\prod_{p\in\mathcal{N}(Y)} \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p) & \rightarrowtail & \prod_{p\in\mathcal{N}(Y)} \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y_p)) \times \mathcal{C}^{0}_{\mathcal{W}}(U,Y_p)
\end{array}$$

and since the maps represented by the three lower arrows are embeddings, so is the map at the top. □

3.4.2. Weighted decreasing maps

We give another definition for weighted maps that decay at infinity. Here, the domain of the maps is contained in a finite dimensional vector space.

Definition 3.4.8. Let Y be a normed space, U an open nonempty subset of the *finite-dimensional* space X and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty. We define for $k \in \overline{\mathbb{N}}$

$$\begin{aligned}
\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet} := \{\gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k)& \\
(\forall \varepsilon > 0)(\exists K \subseteq U \text{ compact}) \|\gamma|_{U\setminus K}\|_{f,\ell} < \varepsilon\}&.
\end{aligned}$$

For a locally convex space Y we set

$$\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet} := \{\gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U,Y) : (\forall p \in \mathcal{N}(Y))\, \pi_p \circ \gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U,Y_p)^{\bullet}\}.$$

For a subset $V \subseteq Y$, we define

$$\mathcal{C}^{k}_{\mathcal{W}}(U,V)^{\bullet} := \{\gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet} : \gamma(U) \subseteq V\}$$

As in Lemma 3.1.6, we can prove that $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet}$ is closed in $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)$.

Lemma 3.4.9. *Let Y be a locally convex space, U an open nonempty subset of the finite-dimensional space X, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet}$ is a closed vector subspace of $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)$.*

Proof. It is obvious from the definition of $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet}$ that it is a vector subspace. It remains to show that it is closed. To this end, let $(\gamma_i)_{i\in I}$ be a net in $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^{\bullet}$ that converges to $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$ in the topology of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. Let $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$, $\ell \in \mathbb{N}$ with $\ell \leq k$ and $\varepsilon > 0$. Then there exists an $i_\varepsilon \in I$ such that

$$i \geq i_\varepsilon \implies \|\gamma - \gamma_i\|_{p,f,\ell} < \frac{\varepsilon}{2}.$$

Further there exists a compact set K such that

$$\|\gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} < \frac{\varepsilon}{2}.$$

Hence

$$\|\gamma|_{U\setminus K}\|_{p,f,\ell} \leq \|\gamma|_{U\setminus K} - \gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} + \|\gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} < \varepsilon,$$

so $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. □

Further, we prove the following convexity criterion.

Lemma 3.4.10. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, Y a locally convex space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and $V \subseteq Y$ convex. Then the set $\mathcal{C}^\ell_{\mathcal{W}}(U,V)^\bullet$ is convex.*

Proof. It is obvious that $\mathcal{C}^\ell_{\mathcal{W}}(U,V)$ – whose definition is straightforward – is convex since V is so. But then

$$\mathcal{C}^\ell_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^\ell_{\mathcal{W}}(U,V) \cap \mathcal{C}^\ell_{\mathcal{W}}(U,Y)^\bullet$$

is convex as intersection of convex sets. □

As in Corollary 3.2.4, we prove a reduction to lower order for $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet$.

Proposition 3.4.11. *Let X be a finite-dimensional space, Y a locally convex space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty, $k \in \mathbb{N}$ and $\gamma \in \mathcal{C}^1(U,Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet \iff (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y)^\bullet,$$

and the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y)^\bullet : \gamma \mapsto (D\gamma, \gamma)$$

is a topological embedding.

Proof. It is a consequence of identity (3.2.2.2) in Lemma 3.2.2 that for each $p \in \mathcal{N}(Y)$

$$\pi_p \circ \gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p)^\bullet \iff (D(\pi_p \circ \gamma), \pi_p \circ \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y_p)^\bullet.$$

Further it is a consequence of identity (3.4.6.1) in Lemma 3.4.6 that

$$D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \iff (\forall p \in \mathcal{N}(Y)) D(\pi_p \circ \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p))^\bullet,$$

so the equivalence is proved. The assertion on the embedding is a consequence of Proposition 3.4.7 and Lemma 3.4.9. So the proof is finished. □

3.4.3. Composition and Superposition

As in Section 3.3, we examine which kind of maps induce superposition operators on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ or $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. We show that continuous multilinear maps induce superposition operators on both function spaces. For $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$, we can prove a much stronger result: A smooth function mapping 0 on 0 induces a superposition operator between these spaces.

Composition with a multilinear map

The following definition and lemma are mostly the same as in Subsection 3.3.1, but here Z denotes a locally convex space.

Definition 3.4.12. Let X be a normed space, $Y_1,\dots,Y_m$ and Z locally convex spaces and $b : Y_1\times\dots\times Y_m \to Z$ a continuous m-linear map. For each $i\in\{1,\dots,m\}$, we define the m-linear continuous map

$$\begin{aligned} b^{(i)} :&Y_1\times\dots\times Y_{i-1}\times \mathrm{L}(X,Y_i)\times Y_{i+1}\times\dots\times Y_m \to \mathrm{L}(X,Z)\\ :&(y_1,\dots,y_{i-1},T,y_{i+1},\dots,y_m)\mapsto (h\mapsto b(y_1,\dots,y_{i-1},T\cdot h,y_{i+1},\dots,y_m)).\end{aligned}$$

Lemma 3.4.13. *Let $Y_1,\dots,Y_m$ and Z be locally convex spaces, U be an open nonempty subset of the normed space X and $k\in\overline{\mathbb{N}}$. Further let $b : Y_1\times\dots\times Y_m\to Z$ be a continuous m-linear map and $\gamma_1\in\mathcal{C}^k(U,Y_1),\dots,\gamma_m\in\mathcal{C}^k(U,Y_m)$. Then*

$$b\circ(\gamma_1,\dots,\gamma_m)\in\mathcal{C}^k(U,Z)$$

with

$$D(b\circ(\gamma_1,\dots,\gamma_m)) = \sum_{i=1}^m b^{(i)}\circ(\gamma_1,\dots,\gamma_{i-1},D\gamma_i,\gamma_{i+1},\dots,\gamma_m). \tag{3.4.13.1}$$

Proof. To calculate the derivative of $b\circ(\gamma_1,\dots,\gamma_m)$, we apply the chain rule and get

$$\begin{aligned} d(b\circ(\gamma_1,\dots,\gamma_m))(x;h) &= \sum_{i=1}^m b(\gamma_1(x),\dots,\gamma_{i-1}(x),d\gamma_i(x;h),\gamma_{i+1}(x),\dots,\gamma_m(x))\\ &= \sum_{i=1}^m b^{(i)}(\gamma_1(x),\dots,\gamma_{i-1}(x),D\gamma_i(x),\gamma_{i+1}(x),\dots,\gamma_m(x))\cdot h.\end{aligned}$$

This implies (3.4.13.1). □

Now we can prove the results about the multilinear superposition.

Proposition 3.4.14. *Let U be an open nonempty subset of the normed space X. Let $Y_1,\dots,Y_m$ be locally convex spaces, $k\in\overline{\mathbb{N}}$ and $\mathcal{W},\mathcal{W}_1,\dots,\mathcal{W}_m\subseteq\overline{\mathbb{R}}^U$ nonempty sets such that*

$$(\forall f\in\mathcal{W})(\exists g_{f,1}\in\mathcal{W}_1,\dots,g_{f,m}\in\mathcal{W}_m)\,|f|\leq|g_{f,1}|\cdots|g_{f,m}|.$$

Further let Z be another locally convex space and $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map. Then

$$b \circ (\gamma_1, \ldots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)$$

for all $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1), \ldots, \gamma_m \in \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m)$. The map

$$b_* : \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : (\gamma_1, \ldots, \gamma_m) \mapsto b \circ (\gamma_1, \ldots, \gamma_m)$$

is m-linear and continuous.

Proof. Let p be a continuous seminorm on Z. Then there exist $q_1 \in \mathcal{N}(Y_1), \ldots, q_m \in \mathcal{N}(Y_m)$ such that for all $y_1 \in Y_1, \ldots, y_m \in Y_m$,

$$\|b(y_1, \ldots, y_m)\|_p \leq \|y_1\|_{q_1} \cdots \|y_m\|_{q_m}.$$

Hence there exists an m-linear map $\widetilde{b}$ that makes

$$\begin{array}{ccc} Y_1 \times \cdots \times Y_m & \xrightarrow{\quad b \quad} & Z \\ {\scriptstyle \pi_{q_1} \times \cdots \times \pi_{q_m}} \downarrow & & \downarrow {\scriptstyle \pi_p} \\ Y_{1,q_1} \times \cdots \times Y_{m,q_m} & \xrightarrow[\quad \widetilde{b} \quad]{} & Z_p \end{array}$$

a commutative diagram. For $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1), \ldots, \gamma_m \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_m)$ we know from Proposition 3.3.3 that

$$\widetilde{b} \circ (\pi_{q_1} \circ \gamma_1, \ldots, \pi_{q_m} \circ \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z_p)$$

and the map $\widetilde{b}_*$ is continuous. Since

$$\widetilde{b}_* \circ ((\pi_{q_1})_* \times \cdots \times (\pi_{q_m})_*) = (\pi_p)_* \circ b_*$$

and the left hand side is continuous, we conclude using Lemma 3.4.2 that b_* is well-defined and continuous since p was arbitrary. □

Corollary 3.4.15. *Let $Y_1, \ldots, Y_m$ be locally convex spaces, U be an open nonempty subset of the finite-dimensional space X, $k \in \overline{\mathbb{N}}$ and $\mathcal{W}, \mathcal{W}_1, \ldots, \mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ nonempty such that*

$$(\forall f \in \mathcal{W})(\exists g_{f,1} \in \mathcal{W}_1, \ldots, g_{f,m} \in \mathcal{W}_m)\, |f| \leq |g_{f,1}| \cdots |g_{f,m}|.$$

Further let Z be another locally convex space, $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map, and $j \in \{1, \ldots, m\}$. Then

$$b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet \tag{†}$$

for all $\gamma_i \in \mathcal{C}^k_{\mathcal{W}_i}(U, Y_i)$ $(i \neq j)$ and $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^\bullet$. The map

$$\mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^\bullet \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet$$
$$(\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \mapsto b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m)$$

is m-linear and continuous.

Proof. Using Proposition 3.4.14 and Lemma 3.4.9, we only have to prove that (†) holds. This is done by induction on k.

$k = 0$: Let $p \in \mathcal{N}(Z)$. Then there exist $q_1 \in \mathcal{N}(Y_1), \dots, q_m \in \mathcal{N}(Y_m)$ such that

$$\|b(y_1, \dots, y_m)\|_p \leq \|y_1\|_{q_1} \cdots \|y_m\|_{q_m}$$

for all $y_1 \in Y_1, \dots, y_m \in Y_m$. So for $f \in \mathcal{W}$, $x \in U$ and $\gamma_1 \in \mathcal{C}^0_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_j \in \mathcal{C}^0_{\mathcal{W}_j}(U, Y_j)^\bullet, \dots, \gamma_m \in \mathcal{C}^0_{\mathcal{W}_m}(U, Y_m)$ we compute

$$\begin{aligned} &|f(x)|\,\|b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)(x)\|_p \\ &\qquad \leq \prod_{i=1}^{m} |g_{f,i}(x)|\,\|\gamma_i(x)\|_{q_i} \leq \left(\prod_{i \neq j} \|\gamma_i\|_{q_i, g_{f,i}, 0}\right) |g_{f,j}(x)|\,\|\gamma_j(x)\|_{q_j}. \end{aligned}$$

With this estimate we easily deduce that $b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^0_{\mathcal{W}_j}(U, Z)^\bullet$.

$k \to k+1$: From Proposition 3.4.11 (together with the induction base) we know that for $\gamma_1 \in \mathcal{C}^{k+1}_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_j \in \mathcal{C}^{k+1}_{\mathcal{W}_j}(U, Y_j)^\bullet, \dots, \gamma_m \in \mathcal{C}^{k+1}_{\mathcal{W}_m}(U, Y_m)$

$$b\circ(\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Z)^\bullet \iff D(b\circ(\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))^\bullet.$$

We know from (3.4.13.1) in Lemma 3.4.13 that

$$\begin{aligned} D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) &= \sum_{\substack{i=1 \\ i \neq j}}^{m} b^{(i)} \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \dots, \gamma_m) \\ &\quad + b^{(j)} \circ (\gamma_1, \dots, \gamma_{j-1}, D\gamma_j, \gamma_{j+1}, \dots, \gamma_m). \end{aligned}$$

Noticing that $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^\bullet$ and $D\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, \mathrm{L}(X, Y_j))^\bullet$, we can apply the inductive hypothesis to all $b^{(i)}$ and the $\mathcal{C}^k$-maps $\gamma_1, \dots, \gamma_m$ and $D\gamma_1, \dots, D\gamma_m$. Hence $D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))^\bullet$. □

As an application, we prove that the space of weighted functions into a product is canonicly isomorphic to the product of the weighted function spaces.

Lemma 3.4.16. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $(Y_i)_{i \in I}$ a family of locally convex spaces, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty. Then for each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)$ and $j \in I$*

$$\pi_j \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_j),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i) \to \prod_{i \in I} \mathcal{C}^k_{\mathcal{W}}(U, Y_i) : \gamma \mapsto (\pi_i \circ \gamma)_{i \in I} \tag{†}$$

is an isomorphism of locally convex topological vector spaces.

The same statement holds for $\mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)^\bullet$:

$$\mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)^\bullet \to \prod_{i \in I} \mathcal{C}^k_{\mathcal{W}}(U, Y_i)^\bullet : \gamma \mapsto (\pi_i \circ \gamma)_{i \in I} \tag{††}$$

is an isomorphism of locally convex topological vector spaces.

Proof. We proved in Proposition 3.4.14 that for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i\in I} Y_i)$ and $j \in I$, $\pi_j \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_j)$ and the map (†) is linear and continuous. Since a function to a product is determined by its components, the map (†) is also injective. What remains to be shown is the surjectivity, and the continuity of the inverse mapping. To this end, we notice that for each $j \in I$ and $p \in \mathcal{N}(Y_j)$, the map

$$P_{j,p} : \prod_{i\in I} Y_i \to \mathbb{R} : (y_i)_{i\in I} \mapsto \|y_j\|_p$$

is a continuous seminorm, and the set $\{P_{j,p} : j \in I, p \in \mathcal{N}(Y_j)\}$ generates $\mathcal{N}(\prod_{i\in I} Y_i)$. For each $i \in I$, let $\gamma_i \in \mathcal{C}^k_{\mathcal{W}}(U, Y_i)$. We define the map

$$\gamma : U \to \prod_{i\in I} Y_i : x \mapsto (\gamma_i(x))_{i\in I}.$$

Then γ is a $\mathcal{C}^k$-map, and $P_{j,p} \circ \gamma = p \circ \gamma_j$. We see with Proposition A.3.2 that this implies that $\pi_{P_{j,p}} \circ \gamma$ is an $\mathcal{FC}^k$-map, and for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$ we derive the identity

$$\|\pi_{P_{j,p}} \circ \gamma\|_{P_{j,p},f,\ell} = \|\pi_p \circ \gamma_j\|_{p,f,\ell}.$$

We proved in Lemma 3.4.2 that this identity implies that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i\in I} Y_i)$. Further it also proves that the inverse map of (†) is continuous using that it is linear.

The assertions about (††) follow from Corollary 3.4.15 and the assertions proved above about (†). □

Superposition with differentiable functions on weighted decreasing maps

We show that a smooth functions mapping 0 on 0 induces a superposition operator on $\mathcal{C}^k_{\mathcal{W}}(U, Y)^\bullet$, provided that $1_U \in \mathcal{W}$. The proof uses that the image of decreasing maps is (almost) compact, and so the composition with the smooth map can be described in terms of compositions with bounded maps taking values in normed spaces.

On the image of decreasing maps

Lemma 3.4.17. *Let U be an open nonempty subset of the finite-dimensional space X, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)^\bullet$. Then*

$$\gamma(U) \cup \{0\}$$

is compact.

Proof. Since $1_U \in \mathcal{W}$, $\gamma \in \mathcal{C}^0_{\{1_U\}}(U, Y)^\bullet$. By the definition of this space, γ extends to a continuous map $\widetilde{\gamma} : U \cup \{\infty\} \to Y$ defined on the Alexandroff compactification of U by setting $\widetilde{\gamma}(\infty) := 0$. Hence

$$\widetilde{\gamma}(U \cup \{\infty\}) = \gamma(U) \cup \{0\}$$

is compact. □

We describe two easy consequences of the last lemma.

Lemma 3.4.18. *Let U be an open nonempty subset of the finite-dimensional space X, V an open nonempty zero neighborhood of the* normed *space Y, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$.*

Proof. This is an immediate consequence of Lemma 3.4.17. □

Lemma 3.4.19. *Let U be an open nonempty subset of the finite-dimensional space X, Y a normed space, $V \subseteq Y$ an open zero neighborhood, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$.*

Proof. We proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$. Hence $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \cap \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. □

Superposition with a bounded map As a preparation, we prove an analogous version of Lemma 3.3.11 for decreasing functions.

Lemma 3.4.20. *Let U be an open nonempty subset of the finite-dimensional space X, Y and Z normed spaces, $V \subseteq Y$ open and star-shaped with center 0, $k, \ell \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Further let $\phi \in \mathcal{BC}^{k+\ell+1}(V,Z)$ with $\phi(0) = 0$. Then*

$$\phi \circ \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet,$$

and

$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : \gamma \mapsto \phi \circ \gamma$$

is a $\mathcal{C}^\ell$-map.

Proof. We proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$. Hence we can apply Proposition 3.3.12 to see that

$$\phi \circ \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Z)$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \phi \circ \gamma$$

is $\mathcal{C}^\ell$; here we used that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \cap \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Because $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is closed in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ by Lemma 3.4.9, it only remains to show that for each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$, we have $\phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet$. This is done by induction on k:

$k = 0$: Let $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned}|f(x)|\,\|\phi(\gamma(x))\| &= |f(x)|\,\|\phi(\gamma(x)) - \phi(0)\| \\ &= |f(x)| \left\| \int_0^1 D\phi(t\gamma(x)) \cdot \gamma(x)\,dt \right\| \leq \|D\phi\|_{op,\infty} |f(x)|\,\|\gamma(x)\|;\end{aligned}$$

here we used that the line segment from 0 to $\gamma(x)$ is contained in V. From this estimate we conclude that $\phi \circ \gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Z)^\bullet$.

$k \to k+1$: By the chain rule

$$D(\phi \circ \gamma) = (D\phi \circ \gamma) \cdot D\gamma.$$

Now $D\phi \circ \gamma \in \mathcal{BC}^{k+1}(U, \mathrm{L}(Y,Z))$ because of Lemma 3.3.8, since $\gamma \in \mathcal{BC}^{k+1}(U,V)$. Further $D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet$, so we conclude using Corollary 3.4.15 that $(D\phi \circ \gamma) \cdot D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z))^\bullet$. By Proposition 3.4.11, the case $k+1$ follows from the inductive hypothesis. □

We calculate the higher differentials of the superposition map on weighted functions that is induced by a bounded function, see Lemma 3.3.11 where a more general assertion was proved. We will need this later to show that $\mathcal{C}^{k+\ell+2}$-functions induce a superposition operator on the spaces $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$, and that this superposition operator is $\mathcal{C}^\ell$.

Lemma 3.4.21. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $m \in \mathbb{N}^*$, $\phi \in \mathcal{BC}^{k+m+1}(V,Z)_0$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. By Lemma 3.3.11,*

$$\phi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \phi \circ \gamma$$

is defined and $\mathcal{C}^m$. For its ℓ-th differential, the identity

$$d^{(\ell)}\phi_*(\gamma; \gamma_1, \dots, \gamma_\ell) = d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell)$$

holds ($\ell \leq m$).

Proof. Let $x \in U$. Using the identity

$$\mathrm{ev}^Z_x \circ \phi_* = \phi \circ \mathrm{ev}^Y_x$$

(with self-explanatory notation for point evaluations), we calculate

$$\begin{aligned}(\mathrm{ev}^Z_x \circ d^{(\ell)}\phi_*)(\gamma; \gamma_1, \dots, \gamma_\ell) &= d^{(\ell)}(\mathrm{ev}^Z_x \circ \phi_*)(\gamma; \gamma_1, \dots, \gamma_\ell) \\ &= d^{(\ell)}(\phi \circ \mathrm{ev}^Y_x)(\gamma; \gamma_1, \dots, \gamma_\ell) = (d^{(\ell)}\phi \circ (\mathrm{ev}^Y_x)^{\ell+1})(\gamma, \gamma_1, \dots, \gamma_\ell) \\ &= \mathrm{ev}^Z_x (d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell));\end{aligned}$$

here we used Lemma A.1.16 and Lemma A.1.17. □

The main result Before we can prove the main result, we need the following facts concerning compact and star-shaped sets in topological vector spaces.

Lemma 3.4.22. *Let Z be a locally convex space and $K \subseteq Z$ a compact set.*

(a) The set $[0,1] \cdot K$ is compact and star-shaped with center 0.

(b) Let K be star-shaped and V an open neighborhood of K. Then there exists an open star-shaped set W such that $K \subseteq W \subseteq V$.

Proof. (a) $[0,1] \cdot K$ is compact since it is the image of a compact set under a continuous map.

(b) The set $K \times \{0\}$ is compact, hence using the continuity of the addition and the Wallace lemma, we find an open 0-neighborhood U such that $K + U \subseteq V$. We may assume w.l.o.g. that U is absolutely convex. Then $K + U$ is open, star-shaped and contained in V. $\square$

Proposition 3.4.23. *Let U be an open nonempty subset of the finite-dimensional space X, Y and Z locally convex spaces, $V \subseteq Y$ open and star-shaped with center 0, $k, m \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Let $\phi \in \mathcal{C}^{k+m+2}(V, Z)$ with $\phi(0) = 0$. Then for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet$,*

$$\phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet$$

holds, and the map

$$\phi_* : \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet : \gamma \mapsto \phi \circ \gamma$$

is $\mathcal{C}^m$ with

$$d^{(\ell)}\phi_*(\gamma; \gamma_1, \dots, \gamma_\ell) = d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell)$$

for all $\ell \leq m$.

Proof. Let $\widetilde{\gamma} \in \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet$. By Lemma 3.4.17 and Lemma 3.4.22, the set

$$K := [0,1] \cdot (\widetilde{\gamma}(U) \cup \{0\})$$

is compact and star-shaped with center 0. Hence by Lemma A.1.34, for each $p \in \mathcal{N}(Z)$ there exists a $q \in \mathcal{N}(Y)$ and an open set $W \supseteq K$ w.r.t. q such that $\tilde{\phi} \in \mathcal{BC}^{k+m+1}(W_q, Z_p)$. In view of Lemma 3.4.22, we may assume that W (and hence W_q) is star-shaped with center 0. We know from Lemma 3.4.19 that $\mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet$ is a neighborhood of $\pi_q \circ \widetilde{\gamma}$ in $\mathcal{C}^k_{\mathcal{W}}(U, Y_q)^\bullet$. In Lemma 3.4.20 we stated that

$$\tilde{\phi}_* : \mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, Z_p)^\bullet : \gamma \mapsto \tilde{\phi} \circ \gamma$$

is $\mathcal{C}^m$. The diagram

$$\begin{array}{ccc}
\mathcal{C}^k_{\mathcal{W}}(U, W)^\bullet & \xrightarrow{\quad\pi_{q*}\quad} & \mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet \\
& {\scriptstyle (\pi_p \circ \phi)_*} \searrow \quad \swarrow {\scriptstyle \tilde{\phi}_*} & \\
& \mathcal{C}^k_{\mathcal{W}}(U, Z_p)^\bullet &
\end{array}$$

is commutative. This implies that $(\pi_p \circ \phi)_*$ is $\mathcal{C}^m$ on $\mathcal{C}^k_{\mathcal{W}}(U, W)^\bullet$ since it is the composition of $\tilde{\phi}_*$ and the smooth map π_{q*} (see Corollary 3.4.15). Using Lemma A.1.17 and Lemma 3.4.21, we can calculate its higher derivatives:

$$\begin{aligned}
&d^{(\ell)}(\pi_p \circ \phi)_*|_{\mathcal{C}^k_{\mathcal{W}}(U,W)^\bullet}(\gamma; \gamma_1, \dots, \gamma_\ell) \\
&\quad = d^{(\ell)}(\tilde{\phi} \circ \pi_q)_*|_{\mathcal{C}^k_{\mathcal{W}}(U,W)^\bullet}(\gamma; \gamma_1, \dots, \gamma_\ell) = d^{(\ell)}\tilde{\phi}_*(\pi_q \circ \gamma; \pi_q \circ \gamma_1, \dots, \pi_q \circ \gamma_\ell) \\
&\quad = d^{(\ell)}\tilde{\phi} \circ (\pi_q \circ \gamma, \pi_q \circ \gamma_1, \dots, \pi_q \circ \gamma_\ell) = d^{(\ell)}(\tilde{\phi} \circ \pi_q) \circ (\gamma, \gamma_1, \dots, \gamma_\ell) \\
&\quad = d^{(\ell)}(\pi_p \circ \phi) \circ (\gamma, \gamma_1, \dots, \gamma_\ell) = \pi_p \circ d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell)
\end{aligned}$$

for $\ell \in \mathbb{N}$ with $\ell \leq m$.

Since $\widetilde{\gamma}$ and p were arbitrary, we conclude that the map

$$\mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet \to \prod_{p \in \mathcal{N}(Z)} \mathcal{C}^k_{\mathcal{W}}(U, Z_p)^\bullet : \gamma \mapsto (\pi_p \circ \phi \circ \gamma)_{p \in \mathcal{N}(Z)}$$

is $\mathcal{C}^m$. Since its image and all directional derivatives are contained in $\mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet$ (in the sense of Lemma 3.4.2), we conclude that it is $\mathcal{C}^m$ as a map to $\mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet$. □

4. Lie groups of weighted diffeomorphisms on Banach spaces

In this chapter, we prove that for each Banach space X appropriate subgroups of the diffeomorphism group $\mathrm{Diff}(X)$ can be turned into Lie groups that are modelled on some weighted function space described earlier. Further, we show that these Lie groups are regular. Here

$$\mathrm{Diff}(X) := \{\phi \in \mathcal{FC}^\infty(X, X) : \phi \text{ is bijective and } \phi^{-1} \in \mathcal{FC}^\infty(X, X)\};$$

the chain rule ensures that $\mathrm{Diff}(X)$ is actually a group with the composition and inversion of maps as the group operations.

4.1. Weighted diffeomorphisms and endomorphisms

In this section, we define and examine sets of *weighted endomorphisms* $\mathrm{End}_{\mathcal{W}}(X)$ and *weighted diffeomorphisms* $\mathrm{Diff}_{\mathcal{W}}(X)$. We show that if $1_X \in \mathcal{W}$, then $\mathrm{End}_{\mathcal{W}}(X)$ is a smooth monoid and $\mathrm{Diff}_{\mathcal{W}}(X)$ is its group of units that can be turned into a Lie group. Further, we discuss certain subsets of these, the *decreasing weighted diffeomorphisms* respective *endomorphisms*. Most of the results of this subsection were already proved in the author's diploma thesis [Wal06, §4.1, §4.2.1, §4.3.1], mostly in a less general form. We omit some of the proofs and technical results. The results and definitions that follow right now are fairly easy to show, and will remain.

For nonempty $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$, we define

$$\mathrm{Diff}_{\mathcal{W}}(X) := \{\phi \in \mathrm{Diff}(X) : \phi - \mathrm{id}_X,\ \phi^{-1} - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)\}$$

and

$$\mathrm{End}_{\mathcal{W}}(X) := \{\gamma + \mathrm{id}_X : \gamma \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)\}.$$

The set $\mathrm{End}_{\mathcal{W}}(X)$ can be turned into a smooth manifold using the differentiable structure generated by the bijective map

$$\kappa_{\mathcal{W}} : \mathcal{C}^\infty_{\mathcal{W}}(X, X) \to \mathrm{End}_{\mathcal{W}}(X) : \gamma \mapsto \gamma + \mathrm{id}_X\,. \tag{4.1.0.1}$$

We clarify the relation between $\mathrm{End}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$. The following is obvious from the definition:

Lemma 4.1.1. *Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ and $\phi \in \mathrm{Diff}(X)$. Then*

$$\phi \in \mathrm{Diff}_{\mathcal{W}}(X) \iff \phi, \phi^{-1} \in \mathrm{End}_{\mathcal{W}}(X).$$

Furthermore, we have

Lemma 4.1.2. *Let* $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ *such that* $\operatorname{End}_{\mathcal{W}}(X)$ *is a monoid with respect to the composition of maps. Then the group of units is given by*

$$\operatorname{End}_{\mathcal{W}}(X)^\times = \operatorname{Diff}_{\mathcal{W}}(X);$$

in particular $\operatorname{Diff}_{\mathcal{W}}(X)$ *is a subgroup of* $\operatorname{Diff}(X)$.

Proof. Obviously

$$\phi \in \operatorname{End}_{\mathcal{W}}(X)^\times \iff \phi \text{ is bijective and } \phi, \phi^{-1} \in \operatorname{End}_{\mathcal{W}}(X).$$

Since $\operatorname{End}_{\mathcal{W}}(X)$ consists of smooth maps, the assertion follows from Lemma 4.1.1. □

In the rest of this section, we prove that $\operatorname{End}_{\mathcal{W}}(X)$ is a smooth monoid if $1_X \in \mathcal{W}$; thus $\operatorname{Diff}_{\mathcal{W}}(X)$ is a group by Lemma 4.1.2. Further, we define the set of *weighted decreasing endomorphisms* and show that it is a closed submonoid of $\operatorname{End}_{\mathcal{W}}(X)$. The main part is to show that the monoid multiplication

$$\circ : \operatorname{End}_{\mathcal{W}}(X) \times \operatorname{End}_{\mathcal{W}}(X) \to \operatorname{End}_{\mathcal{W}}(X)$$

is defined and smooth, so we elaborate on this.

4.1.1. Composition of weigthed endomorphisms in charts

We study how the composition looks like with respect to the global chart $\kappa_{\mathcal{W}}^{-1}$ (from (4.1.0.1)). For η, $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)$,

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) = (\gamma + \operatorname{id}_X) \circ (\eta + \operatorname{id}_X) = \gamma \circ (\eta + \operatorname{id}_X) + \eta + \operatorname{id}_X . \qquad (4.1.2.1)$$

Obviously $\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) \in \operatorname{End}_{\mathcal{W}}(X)$ if and only if $\gamma \circ (\eta + \operatorname{id}_X) \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)$; and the smoothness of $\circ$ is equivalent to that of

$$\mathcal{C}^\infty_{\mathcal{W}}(X, X) \times \mathcal{C}^\infty_{\mathcal{W}}(X, X) \to \mathcal{C}^\infty_{\mathcal{W}}(X, X) : (\gamma, \eta) \mapsto \gamma \circ (\eta + \operatorname{id}_X).$$

Important maps

For technical reasons we look at more general maps of the form

$$\tilde{\mathfrak{c}} : Y^W \times V^U \to Y^U : (\gamma, \eta) \mapsto \gamma \circ (\eta + \operatorname{id}_U); \qquad (4.1.2.2)$$

here $U, V, W \subseteq X$ are open nonempty subsets with $V + U \subseteq W$ and Y is a normed space. These maps play an important role in further discussions.

Continuity properties We discuss when the restriction of $\tilde{\mathfrak{c}}$ to weighted function spaces has values in a weighted function space and is continuous. We start with the following lemma whose assertion is used as the base case for Lemma 4.1.4. A less general version of both lemmas was implicitely proved in [Wal06, Las. 4.4, 4.5, Prop. 4.6]; there the weights functions had to be defined on the whole vector space, not just open subsets. Since the proofs are mostly unchanged, we omit them. They can also be found in [Wal13, Las. 4.1.3, 4.1.4].

Lemma 4.1.3. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, and $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$.*

(a) For $\gamma \in \mathcal{FC}^1(W, Y)$, $\eta : U \to V$, $f \in \mathcal{W}$ and $x \in U$, the estimate

$$|f(x)| \, \|\gamma \circ (\eta + \mathrm{id}_X)(x)\| \leq |f(x)| \left(\|\gamma\|_{1_{\{x\}+\mathbb{D}\eta(U)},1} \|\eta(x)\| + \|\gamma(x)\|\right) \tag{4.1.3.1}$$

holds. In particular, if $\gamma \in \mathcal{C}^0_{\mathcal{W}}(W, Y) \cap \mathcal{BC}^1(W, Y)$ and $\eta \in \mathcal{C}^0_{\mathcal{W}}(U, V)$, then

$$\tilde{\mathfrak{c}}(\gamma, \eta) = \gamma \circ (\eta + \mathrm{id}_X) \in \mathcal{C}^0_{\mathcal{W}}(U, Y).$$

(b) Let $\gamma, \gamma_0 \in \mathcal{C}^0_{\mathcal{W}}(W, Y) \cap \mathcal{BC}^1(W, Y)$ and $\eta, \eta_0 \in \mathcal{C}^0_{\mathcal{W}}(U, V)$ such that

$$\{t\eta(x) + (1 - t)\eta_0(x) : t \in [0, 1], x \in U\} \subseteq V.$$

Then for each $f \in \mathcal{W}$ the estimate

$$\begin{aligned} \|\tilde{\mathfrak{c}}(\gamma, \eta) - \tilde{\mathfrak{c}}(\gamma_0, \eta_0)\|_{f,0} \leq \|\gamma\|_{1_W,1} \|\eta - \eta_0\|_{f,0} \\ + \|\gamma - \gamma_0\|_{1_W,1} \|\eta_0\|_{f,0} + \|\gamma - \gamma_0\|_{f,0} \end{aligned} \tag{4.1.3.2}$$

holds. In particular, if $1_W \in \mathcal{W}$ then the map

$$\tilde{\mathfrak{c}}^{Y,0}_{\mathcal{W}} : \mathcal{C}^1_{\mathcal{W}}(W, Y) \times \mathcal{C}^{\partial,0}_{\mathcal{W}}(U, V) \to \mathcal{C}^0_{\mathcal{W}}(U, Y) : (\gamma, \eta) \mapsto \tilde{\mathfrak{c}}(\gamma, \eta)$$

is continuous.

Lemma 4.1.4. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$. Then*

$$\tilde{\mathfrak{c}}(\mathcal{C}^{k+1}_{\mathcal{W}}(W, Y) \times \mathcal{C}^k_{\mathcal{W}}(U, V)) \subseteq \mathcal{C}^k_{\mathcal{W}}(U, Y),$$

and the map

$$\tilde{\mathfrak{c}}^{Y,k}_{\mathcal{W}} : \mathcal{C}^{k+1}_{\mathcal{W}}(W, Y) \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \to \mathcal{C}^k_{\mathcal{W}}(U, Y) : (\gamma, \eta) \mapsto \tilde{\mathfrak{c}}(\gamma, \eta)$$

which arises by restricting $\tilde{\mathfrak{c}}$ is continuous.

Restriction to decreasing functions Finally, we study the restriction of $\tilde{\mathfrak{c}}^{Y,k}_{\mathcal{W}}$ to decreasing functions.

Lemma 4.1.5. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then*

$$\tilde{\mathfrak{c}}^{Y,k}_{\mathcal{W}}(\mathcal{C}^{k+1}_{\mathcal{W}}(W,Y)^o \times \mathcal{C}^k_{\mathcal{W}}(U,V)) \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Y)^o.$$

Proof. The proof is by induction on k:

$k = 0$: We use estimate (4.1.3.1) in Lemma 4.1.3:
Let $f \in \mathcal{W}$, $\gamma \in \mathcal{C}^1_{\mathcal{W}}(W,Y)^o$ and $\eta \in \mathcal{C}^0_{\mathcal{W}}(U,V)$. Then for every $\varepsilon > 0$ there exists $r > 0$ such that

$$\|\gamma|_{W \setminus B_r(0)}\|_{f,0} < \frac{\varepsilon}{2}$$

and (as $1_X \in \mathcal{W}$)

$$\|\gamma|_{W \setminus B_r(0)}\|_{1_W,1} < \frac{\varepsilon}{2(\|\eta\|_{f,0} + 1)}.$$

Since $1_X \in \mathcal{W}$, we have $K := \|\eta\|_{1_U,0} < \infty$. Let $R \in \mathbb{R}$ such that $R > r + K$. Then for each $x \in U \setminus \overline{B}_R(0)$, we have

$$x + \mathbb{D}\eta(x) \subseteq W \setminus \overline{B}_r(0),$$

so we conclude from estimate (4.1.3.1) that

$$|f(x)| \, \|\tilde{\mathfrak{c}}^{Y,k}_{\mathcal{W}}(\gamma,\eta)(x)\| \leq \|\gamma\|_{1_{\{x\}+\mathbb{D}\eta(U)},1} \, \|\eta\|_{f,0} + |f(x)| \, \|\gamma(x)\| < \frac{\varepsilon}{2(\|\eta\|_{f,0} + 1)} \|\eta\|_{f,0} + \frac{\varepsilon}{2}.$$

Thus $\tilde{\mathfrak{c}}^{Y,k}_{\mathcal{W}}(\gamma,\eta) \in \mathcal{C}^0_{\mathcal{W}}(U,Y)^o$.

$k \to k+1$: We calculate using the chain rule that

$$(D \circ \tilde{\mathfrak{c}}^{Y,k+1}_{\mathcal{W}})(\gamma,\eta) = \tilde{\mathfrak{c}}^{\mathrm{L}(X,Y),k}_{\mathcal{W}}(D\gamma,\eta) \cdot (D\eta + \mathrm{Id}).$$

Since $D\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(W, \mathrm{L}(X,Y))^o$ (see Corollary 3.2.4),

$$\tilde{\mathfrak{c}}^{\mathrm{L}(X,Y),k}_{\mathcal{W}}(D\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^o$$

by the inductive hypothesis. Further, $D\eta + \mathrm{Id} \in \mathcal{BC}^k(U, \mathrm{L}(X))$, so we conclude with Corollary 3.3.4 that

$$(D \circ \tilde{\mathfrak{c}}^{Y,k+1}_{\mathcal{W}})(\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^o.$$

From this (and the base case $k = 0$) we see with Corollary 3.2.4 that

$$\tilde{\mathfrak{c}}^{Y,k+1}_{\mathcal{W}}(\gamma,\eta) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^o,$$

so the proof is complete. □

Differentiability properties We discuss whether restrictions of $\tilde{\mathfrak{c}}_{\mathcal{W}}^{Y,k}$ to certain weighted function spaces are differentiable. Before we do this, we give the following definitions.

Definition 4.1.6. Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$ and $k, \ell \in \overline{\mathbb{N}}$. Then the map

$$\mathfrak{c}_{\mathcal{W},\ell}^{Y,k} : \mathcal{C}_{\mathcal{W}}^{k+\ell+1}(W, Y) \times \mathcal{C}_{\mathcal{W}}^{\partial,k}(U, V) \to \mathcal{C}_{\mathcal{W}}^{k}(U, Y) : (\gamma, \eta) \mapsto \gamma \circ (\eta + \mathrm{id}_U)$$

is defined by Lemma 4.1.4. Additionally, we set $\mathfrak{c}_{\mathcal{W}}^{Y,k} := \mathfrak{c}_{\mathcal{W},\infty}^{Y,k}$ and $\mathfrak{c}_{\mathcal{W}}^{Y} := \mathfrak{c}_{\mathcal{W},\infty}^{Y,\infty}$.

The smoothness resp. differentiability was already proved in the author's diploma thesis [Wal06, §4.3.1], although in a slightly less general version; the weighted functions were assumed to be smooth and defined on the whole vector space. Since the used techniques are largely the same, we omit the proof and a technical lemma. They can also be found in [Wal13, La. 4.1.7, Prop. 4.1.8].

Proposition 4.1.7. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$ and $k, \ell \in \overline{\mathbb{N}}$. Then $\mathfrak{c}_{\mathcal{W},\ell}^{Y,k}$ is a $\mathcal{C}^\ell$-map. If $\ell > 0$, then it has the directional derivative*

$$d\mathfrak{c}_{\mathcal{W},\ell}^{Y,k}(\gamma, \eta; \gamma_1, \eta_1) = \mathfrak{c}_{\mathcal{W},\ell-1}^{\mathrm{L}(X,Y),k}(D\gamma, \eta) \cdot \eta_1 + \mathfrak{c}_{\mathcal{W},\ell}^{Y,k}(\gamma_1, \eta). \tag{4.1.7.1}$$

In particular, $\mathfrak{c}_{\mathcal{W}}^{Y}$ and $\mathfrak{c}_{\mathcal{W}}^{Y,k}$ are smooth.

Restriction to decreasing functions We examine the restriction of $\mathfrak{c}_{\mathcal{W}}^{Y,k}$ to decreasing functions. We show that it takes values in the decreasing functions and is also smooth.

Corollary 4.1.8. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then*

$$\mathfrak{c}_{\mathcal{W}}^{Y,k}(\mathcal{C}_{\mathcal{W}}^{\infty}(W, Y)^o \times \mathcal{C}_{\mathcal{W}}^{k}(U, V)^o) \subseteq \mathcal{C}_{\mathcal{W}}^{k}(U, Y)^o,$$

and the restriction $\mathfrak{c}_{\mathcal{W}}^{Y,k}|_{\mathcal{C}_{\mathcal{W}}^{\infty}(W,Y)^o \times \mathcal{C}_{\mathcal{W}}^{k}(U,V)^o}^{\mathcal{C}_{\mathcal{W}}^{k}(U,Y)^o}$ is smooth.

Proof. We deduce this from Lemma 4.1.5, the smoothness of the unrestricted map (Proposition 4.1.7) and Proposition A.1.12 that can be used because $\mathcal{C}_{\mathcal{W}}^{k}(U, Y)^o$ is closed by Lemma 3.1.6. □

4.1.2. Smooth monoids of weighted endomorphisms

We are able to prove that $\mathrm{End}_{\mathcal{W}}(X)$ and the set $\mathrm{End}_{\mathcal{W}}(X)^\circ$ – which is defined below – are smooth monoids, provided that $1_X \in \mathcal{W}$. An analogous version of this corollary was proved in the author's diploma thesis in [Wal06, Folg. 4.8, 4.19]. Since the proof given there wasn't entirely correct, and we also treat decreasing weighted functions, the proof is not omitted.

Corollary 4.1.9. *For $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\operatorname{End}_{\mathcal{W}}(X)$ is a smooth monoid with the group of units*

$$\operatorname{End}_{\mathcal{W}}(X)^\times = \operatorname{Diff}_{\mathcal{W}}(X).$$

Further, the set

$$\operatorname{End}_{\mathcal{W}}(X)^\circ := \{\gamma + \operatorname{id}_X : \gamma \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o\} \tag{4.1.9.1}$$

is a closed submonoid of $\operatorname{End}_{\mathcal{W}}(X)$ *that is a smooth monoid.*

Proof. We first show that $\operatorname{End}_{\mathcal{W}}(X)$ is a monoid. Since $\operatorname{id}_X \in \operatorname{End}_{\mathcal{W}}(X)$ is obviously satisfied, it remains to show that it is closed under composition. Since every element of $\operatorname{End}_{\mathcal{W}}(X)$ can uniquely be written as $\phi + \operatorname{id}_X$ with $\phi \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$, we have to show that for arbitrary $\gamma, \eta \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$ the relation

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \operatorname{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$$

holds. But we know from identity (4.1.2.1) that

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \operatorname{id}_X = \mathfrak{c}^X_{\mathcal{W}}(\gamma,\eta) + \eta,$$

which is in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ by Proposition 4.1.7, hence $\operatorname{End}_{\mathcal{W}}(X)$ is a monoid. Further, from this identity we easily conclude the smoothness of the composition from the one of $\mathfrak{c}^X_{\mathcal{W}}$, which was also proved in Proposition 4.1.7.

$\operatorname{End}_{\mathcal{W}}(X)^\circ$ is a closed subset of $\operatorname{End}_{\mathcal{W}}(X)$ since $\kappa_{\mathcal{W}}$ is a homeomorphism and by Lemma 3.1.6, $\mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$ is a closed vector subspace of $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$. We know from Corollary 4.1.8 and the fact that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$ is a vector space that for $\gamma, \eta \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \operatorname{id}_X = \mathfrak{c}^X_{\mathcal{W}}(\gamma,\eta) + \eta \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o.$$

Further, we proved there that the restriction of $\mathfrak{c}^X_{\mathcal{W}}(\gamma,\eta)$ to decreasing maps is smooth, hence $\operatorname{End}_{\mathcal{W}}(X)^\circ$ is a smooth submonoid of $\operatorname{End}_{\mathcal{W}}(X)$.

The relation $\operatorname{End}_{\mathcal{W}}(X)^\times = \operatorname{Diff}_{\mathcal{W}}(X)$ was proved in Lemma 4.1.2. □

4.2. Lie group structures on weighted diffeomorphisms

In this section, we first prove that $\operatorname{Diff}_{\mathcal{W}}(X)$ – which was already shown to be a group in Corollary 4.1.9 – is in fact a Lie group. Also we define and discuss the set $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$ of *decreasing weighted diffeomorphisms.* We show that it is a normal subgroup of $\operatorname{Diff}_{\mathcal{W}}(X)$ that can be turned into a Lie group. Finally, we explain when diffeomorphisms that are weighted endomorphisms are weighted diffeomorphisms.

4.2.1. The Lie group structure of $\operatorname{Diff}_{\mathcal{W}}(X)$

We show that $\operatorname{Diff}_{\mathcal{W}}(X)$ is an open subset of $\operatorname{End}_{\mathcal{W}}(X)$ and the group inversion is smooth, whence $\operatorname{Diff}_{\mathcal{W}}(X)$ is a Lie group. In order to do this, we have to examine the inversion map on $\operatorname{Diff}(X) \cap \operatorname{End}_{\mathcal{W}}(X)$. The results proved in the author's diploma thesis in [Wal06, §4.2.2] can be derived from the results of this subsection; the major change is the treatment of functions that aren't defined on the whole vector space. However, in contrast to the results about the monoid structure it was necessary to turn to other techniques, like the use of Lipschitz inverse function theorems. Some traces of the proofs of [Wal06, 4.11–4.13] can still be found in 4.2.4–4.2.6, but they will not be omitted since they are used in another context, and the results of this subsection are published here for the first time. In particular, we do not omit the proof of Theorem 4.2.10 since it had to be adapted to our more general considerations.

Definition 4.2.1. Let X be a normed space and $U, V \subseteq X$ open nonempty subsets. We define

$$\widetilde{\Omega}_{U,V} := \{\phi \in X^U : \phi + \mathrm{id}_U \text{ injective}, V \subseteq (\phi + \mathrm{id}_U)(U)\}$$

and

$$\tilde{I}_V : \widetilde{\Omega}_{U,V} \to X^V : \phi \mapsto (\phi + \mathrm{id}_U)^{-1}|_V - \mathrm{id}_V . \tag{4.2.1.1}$$

Further, for nonempty $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ we set $\Omega^{U,V}_{\mathcal{W}} := \widetilde{\Omega}_{U,V} \cap \mathcal{C}^\infty_{\mathcal{W}}(U, X)$ and $I^V_{\mathcal{W}} := \tilde{I}_V|_{\Omega^{U,V}_{\mathcal{W}}}$.

Lemma 4.2.2. *Let X be a normed space, $U, V \subseteq X$ open nonempty subsets and $\phi \in \widetilde{\Omega}_{U,V}$. Then*

$$(\tilde{I}_V(\phi) + \mathrm{id}_V) \circ (\phi + \mathrm{id}_U)|_{(\phi+\mathrm{id}_U)^{-1}(V)} = \mathrm{id}_{(\phi+\mathrm{id}_U)^{-1}(V)} \tag{4.2.2.1}$$

$$(\phi + \mathrm{id}_U) \circ (\tilde{I}_V(\phi) + \mathrm{id}_V) = \mathrm{id}_V, \tag{4.2.2.2}$$

and the identities

$$\tilde{I}_V(\phi) \circ (\phi + \mathrm{id}_U)|_{(\phi+\mathrm{id}_U)^{-1}(V)} = -\phi|_{(\phi+\mathrm{id}_U)^{-1}(V)} \tag{4.2.2.3}$$

$$\phi \circ (\tilde{I}_V(\phi) + \mathrm{id}_V) = -\tilde{I}_V(\phi) \tag{4.2.2.4}$$

hold.

Proof. This is obvious. □

On the range of the inversion map

We first discuss whether the range of $I^V_{\mathcal{W}}$ consists of weighted functions, under certain assumptions on U and V.

Lemma 4.2.3. *Let X be a normed space, $U, V \subseteq X$ open nonempty subsets and $\phi \in \widetilde{\Omega}_{U,V}$. Then $\|\tilde{I}_V(\phi)\|_{1_V,0} \leq \|\phi\|_{1_U,0}$.*

Proof. This is an immediate consequence of identity (4.2.2.4). □

We provide a formula for $D\, I^V_{\mathcal{W}}(\phi)$.

Lemma 4.2.4. *Let X be a Banach space, $U, V \subseteq X$ open nonempty subsets, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $\phi \in \Omega^{U,V}_{\mathcal{W}}$.*

(a) Let $x \in (\phi + \mathrm{id}_U)^{-1}(V)$ such that $\|D\phi(x)\|_{op} < 1$. Then

$$D(I^V_{\mathcal{W}}(\phi))((\phi + \mathrm{id}_U)(x)) = D\phi(x) \cdot QI_{\mathrm{L}(X)}(-D\phi(x)) - D\phi(x).$$

(b) Suppose that $\|\phi\|_{1_U,1} < 1$. *Then*

$$D\, I^V_{\mathcal{W}}(\phi) = (D\phi \cdot QI(-D\phi) - D\phi) \circ (I^V_{\mathcal{W}}(\phi) + \mathrm{id}_V). \tag{4.2.4.1}$$

Here $QI_{\mathrm{L}(X)}$ *and* $QI := QI_{\mathcal{C}^\infty_{\mathcal{W}}(U,\mathrm{L}(X))}$ *denote the quasi-inversion (which is discussed in Appendix C).*

Proof. (a) From identity (4.2.2.3) and the chain rule, we get

$$D\, I^V_{\mathcal{W}}(\phi)((\phi + \mathrm{id}_U)(x)) \cdot (D\phi(x) + \mathrm{id}_X) = -D\phi(x).$$

Since $\|D\phi(x)\|_{op} < 1$, the linear map $D\phi(x) + \mathrm{id}_X$ is bijective with

$$(D\phi(x) + \mathrm{id}_X)^{-1} = \sum_{k=0}^{\infty} (-D\phi(x))^k = -QI_{\mathrm{L}(X)}(-D\phi(x)) + \mathrm{id}_X;$$

(c.f. Lemma C.2.6). Using these two identities, we easily derive the one desired.

(b) Since $\|\phi\|_{1_U,1} < 1$, we see with Lemma 3.3.22 that $-D\phi$ is quasi-invertible in $\mathcal{C}^\infty_{\mathcal{W}}(U, \mathrm{L}(X))$ with

$$QI(-D\phi) = QI_{\mathrm{L}(X)} \circ (-D\phi).$$

Hence we get with (a) that

$$D(I^V_{\mathcal{W}}(\phi)) \circ (\phi + \mathrm{id}_U) = D\phi \cdot QI(-D\phi) - D\phi$$

on $(\phi + \mathrm{id}_U)^{-1}(V)$. Composing both sides of this identity with $I^V_{\mathcal{W}}(\phi) + \mathrm{id}_V$ on the right (see identity (4.2.2.2)) gives identity (4.2.4.1). □

Next, we discuss whether $I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(V, X)$.

Proposition 4.2.5. *Let* X *be a Banach space,* $U, V \subseteq X$ *open nonempty subsets such that there exists* $r > 0$ *with* $V + B_r(0) \subseteq U$. *Further, let* $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ *with* $1_U \in \mathcal{W}$ *and* $\phi \in \Omega^{U,V}_{\mathcal{W}}$ *such that* $\|\phi\|_{1_U,1} < 1$ *and* $\|\phi\|_{1_U,0} < r$. *Then* $I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(V, X)$. *In particular, for all* $f \in \mathcal{W}$ *and* $x \in V$, *we have the estimate*

$$|f(x)|\, \|I^V_{\mathcal{W}}(\phi)(x)\| \leq \frac{|f(x)|\, \|\phi(x)\|}{1 - \|\phi\|_{1_U,1}}. \tag{4.2.5.1}$$

Proof. By the inverse function theorem, $I^V_{\mathcal{W}}(\phi)$ is smooth. We prove by induction that $I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^k_{\mathcal{W}}(V, X)$ for all $k \in \mathbb{N}$.

$k = 0$: We compute for $f \in \mathcal{W}$ and $x \in V$ using identity (4.2.2.4) and (4.1.3.1) that

$$|f(x)|\, \|I^V_{\mathcal{W}}(\phi)(x)\| = |f(x)|\, \|\phi(I^V_{\mathcal{W}}(\phi)(x) + x)\| \leq |f(x)|(\|D\phi\|_{1_U,0} \|I^V_{\mathcal{W}}(\phi)(x)\| + \|\phi(x)\|);$$

here we used that $\|I^V_{\mathcal{W}}(\phi)\|_{1_V,0} < r$ by Lemma 4.2.3. From this we can derive (4.2.5.1) since $\|D\phi\|_{1_U,0} = \|\phi\|_{1_U,1} < 1$, and we see that $I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^0_{\mathcal{W}}(V, X)$.

$k \to k + 1$: Using Proposition 3.2.3 (and the induction base), we see that

$$I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^{k+1}_{\mathcal{W}}(V, X) \iff D\, I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^k_{\mathcal{W}}(V, \mathrm{L}(X));$$

the second condition shall be verified now. Remember that we already provided an identity for $D\, I^V_{\mathcal{W}}(\phi)$ in (4.2.4.1). We use Lemma 3.3.22 and Corollary 3.3.6 to see that

$$D\phi \cdot QI(-D\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(U, \mathrm{L}(X)).$$

Since we know from the induction hypothesis that $I^V_{\mathcal{W}}(\phi) \in \mathcal{C}^k_{\mathcal{W}}(V, X)$, we derive from identity (4.2.4.1) and Proposition 4.1.7 (applied on $\mathcal{C}^\infty_{\mathcal{W}}(U, X) \times \mathcal{C}^k_{\mathcal{W}}(V, B_r(0))$) that

$$D\, I^V_{\mathcal{W}}(\phi) = \mathfrak{c}^{\mathrm{L}(X),k}_{\mathcal{W}}(D\phi \cdot QI(-D\phi) - D\phi,\, I^V_{\mathcal{W}}(\phi)) \in \mathcal{C}^k_{\mathcal{W}}(V, \mathrm{L}(X)),$$

which finishes the proof. □

On the domain and the smoothness of the inversion map

We investigate the smoothness of $I^V_{\mathcal{W}}$. Later, we dicuss when $\Omega^{U,V}_{\mathcal{W}}$ is an open 0-neighborhood. Finally, we conclude that the inversion on $\mathrm{Diff}_{\mathcal{W}}(X)$ is smooth.

Smoothness of the inversion map Here, we assume that $\Omega^{U,V}_{\mathcal{W}}$ contains a suitable open set.

Proposition 4.2.6. *Let X be a Banach space, $U, V \subseteq X$ open nonempty subsets such that $V + B_r(0) \subseteq U$ for some $r > 0$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Let $G_{\mathcal{W}} \subseteq \Omega^{U,V}_{\mathcal{W}}$ be an open nonempty set such that for each $\phi \in G_{\mathcal{W}}$, $\|\phi\|_{1_U,0} < r$ and $\|\phi\|_{1_U,1} < 1$.*

(a) Then for $\phi, \psi \in G_{\mathcal{W}}$, the following identity holds:

$$I^V_{\mathcal{W}}(\psi) - I^V_{\mathcal{W}}(\phi) = (\mathrm{Id} + T_{\psi,\phi})^{-1} \cdot \mathfrak{c}^X_{\mathcal{W}}(\phi - \psi, I^V_{\mathcal{W}}(\phi)), \tag{†}$$

where the inversion is the pointwise inversion in $\mathrm{L}(X)$, and

$$T_{\psi,\phi} := \int_0^1 \mathfrak{c}^{\mathrm{L}(X)}_{\mathcal{W}}(D\psi, tI^V_{\mathcal{W}}(\phi) + (1-t)I^V_{\mathcal{W}}(\psi))\, dt \in \mathcal{C}^\infty_{\mathcal{W}}(V, \mathrm{L}(X)).$$

In particular, for $f \in \mathcal{W}$ we have the estimate

$$\|I^V_{\mathcal{W}}(\psi) - I^V_{\mathcal{W}}(\phi)\|_{f,0} \le \tfrac{1}{1-\|\psi\|_{1_U,1}} \left(\|\phi - \psi\|_{1_U,1} \tfrac{\|\phi\|_{f,0}}{1-\|\phi\|_{1_U,1}} + \|\phi - \psi\|_{f,0} \right). \tag{4.2.6.1}$$

(b) $G_{\mathcal{W}} \to \mathcal{C}^\infty_{\mathcal{W}}(V, X) : \phi \mapsto I^V_{\mathcal{W}}(\phi)$ is continuous.

(c) $G_{\mathcal{W}} \to \mathcal{C}^\infty_{\mathcal{W}}(V, X) : \phi \mapsto I^V_{\mathcal{W}}(\phi)$ is smooth with

$$dI^V_{\mathcal{W}}(\phi; \phi_1) = -\mathfrak{c}^X_{\mathcal{W}}(QI(D\phi) \cdot \phi_1 + \phi_1, I^V_{\mathcal{W}}(\phi)). \tag{4.2.6.2}$$

Proof. By Proposition 4.2.5, $I^V_{\mathcal{W}}(G_{\mathcal{W}}) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(V, X)$, which we will use implicitly.

(a) Let $\phi, \psi \in G_{\mathcal{W}}$. We compute for $x \in V$ with identity (4.2.2.4), the mean value theorem (using that $B_r(0)$ is convex) and by adding $0 = \psi(I_{\mathcal{W}}^V(\phi)(x)+x) - \psi(I_{\mathcal{W}}^V(\phi)(x)+x)$ that

$$\begin{aligned}
&I_{\mathcal{W}}^V(\psi)(x) - I_{\mathcal{W}}^V(\phi)(x)\\
=&\psi(I_{\mathcal{W}}^V(\phi)(x) + x) - \psi(I_{\mathcal{W}}^V(\psi)(x) + x) + \phi(I_{\mathcal{W}}^V(\phi)(x) + x) - \psi(I_{\mathcal{W}}^V(\phi)(x) + x)\\
=&\int_0^1 D\psi(tI_{\mathcal{W}}^V(\phi)(x) + (1-t)I_{\mathcal{W}}^V(\psi)(x) + x) \cdot (I_{\mathcal{W}}^V(\phi)(x) - I_{\mathcal{W}}^V(\psi)(x))\, dt\\
&\quad + \mathfrak{c}_{\mathcal{W}}^X(\phi - \psi, I_{\mathcal{W}}^V(\phi))(x);
\end{aligned}$$

note that the identity $\mathfrak{c}_{\mathcal{W}}^X(\phi - \psi, I_{\mathcal{W}}^V(\phi)) = (\phi - \psi) \circ (I_{\mathcal{W}}^V(\phi) + \mathrm{id}_V)$ holds because of Lemma 4.2.3 and the definition of r. From this identity we can derive (†); note that the integral defining $T_{\psi,\phi}$ exists because $\mathcal{C}_{\mathcal{W}}^\infty(V, \mathrm{L}(X))$ is complete. Further

$$\|T_{\psi,\phi}(x)\| \leq \|\psi\|_{1_U,1} < 1$$

for all $x \in V$, hence each $\mathrm{id}_X + T_{\psi,\phi}(x)$ is invertible. Using the Neumann series, we get

$$\|(\mathrm{Id} + T_{\psi,\phi})^{-1}\|_{1_V,0} \leq \frac{1}{1 - \|\psi\|_{1_U,1}}.$$

So we see using (4.1.3.1) and (4.2.5.1) that estimate (4.2.6.1) holds.

(b) By Corollary 3.2.6, $I_{\mathcal{W}}^V$ is continuous iff the corresponding maps

$$I_\ell : G_{\mathcal{W}} \to \mathcal{C}_{\mathcal{W}}^\ell(V, X)$$

are so for each $\ell \in \mathbb{N}$. We shall verify this condition by induction on ℓ.

$\ell = 0$: We use (4.2.6.1) to see that I_0 is continuous in ϕ.

$\ell \to \ell + 1$: Because of Proposition 3.2.3 (and the induction base) $I_{\ell+1}$ is continuous iff $D \circ I_{\ell+1} : G_{\mathcal{W}} \to \mathcal{C}_{\mathcal{W}}^\ell(V, \mathrm{L}(X))$ is so. Using identity (4.2.4.1), we see that for $\phi \in G_{\mathcal{W}}$

$$(D \circ I_{\ell+1})(\phi) = \mathfrak{c}_{\mathcal{W}}^{\mathrm{L}(X),\ell}(D\phi \cdot QI(-D\phi) - D\phi,\, I_\ell(\phi))$$

holds, where $QI := QI_{\mathcal{C}_{\mathcal{W}}^\infty(U,\mathrm{L}(X))}$. Since $\mathfrak{c}_{\mathcal{W}}^{\mathrm{L}(X),\ell}$, D, $\cdot$, QI and I_ℓ are continuous (see Proposition 4.1.7, Proposition 3.2.3, Corollary 3.3.6, Lemma 3.3.22 and the inductive hypothesis, respectively), we conclude that $D \circ I_{\ell+1}$ is continuous.

(c) We prove by induction that $I_{\mathcal{W}}^V$ is a $\mathcal{C}^k$ map for all $k \in \mathbb{N}$.

$k = 1$: Let $\phi \in G_{\mathcal{W}}$, $\phi_1 \in \mathcal{C}_{\mathcal{W}}^\infty(U, X)$ and $t \in \mathbb{K}^*$ such that $\phi + t\phi_1 \in G_{\mathcal{W}}$. We use (†) to see that

$$\frac{I_{\mathcal{W}}^V(\phi + t\phi_1) - I_{\mathcal{W}}^V(\phi)}{t} = (\mathrm{Id} + T_{\phi+t\phi_1,\phi})^{-1} \cdot \mathfrak{c}_{\mathcal{W}}^X(-\phi_1, I_{\mathcal{W}}^V(\phi)).$$

Using Proposition A.1.8, we see that

$$\lim_{t\to 0} T_{\phi+t\phi_1,\phi} = \mathfrak{c}_{\mathcal{W}}^{\mathrm{L}(X)}(D\phi, I_{\mathcal{W}}^V(\phi)).$$

Further, by Lemma C.1.3

$$(\mathrm{Id} + T_{\phi+t\phi_1,\phi})^{-1} - \mathrm{Id} + \mathrm{Id} = QI(T_{\phi+t\phi_1,\phi}) + \mathrm{Id}$$

where $QI := QI_{\mathcal{C}^\infty_{\mathcal{W}}(V,\mathrm{L}(X))}$. Using that QI and $\cdot$ are continuous by Lemma 3.3.22 and Proposition 3.3.3, we therefore get

$$\begin{aligned}\lim_{t\to 0} \frac{I^V_{\mathcal{W}}(\phi + t\phi_1) - I^V_{\mathcal{W}}(\phi)}{t} &= (QI(\mathfrak{c}^{\mathrm{L}(X)}_{\mathcal{W}}(D\phi, I^V_{\mathcal{W}}(\phi))) + \mathrm{Id}) \cdot \mathfrak{c}^X_{\mathcal{W}}(-\phi_1, I^V_{\mathcal{W}}(\phi)) \\ &= -(\mathfrak{c}^{\mathrm{L}(X)}_{\mathcal{W}}(QI(D\phi), I^V_{\mathcal{W}}(\phi))) + \mathrm{Id}) \cdot \mathfrak{c}^X_{\mathcal{W}}(\phi_1, I^V_{\mathcal{W}}(\phi)) \\ &= -\mathfrak{c}^X_{\mathcal{W}}(QI(D\phi) \cdot \phi_1 + \phi_1, I^V_{\mathcal{W}}(\phi));\end{aligned}$$

here we used that $QI(\Phi) = QI_{\mathrm{L}(X)} \circ \Phi$. Since we proved in (b) that $I^V_{\mathcal{W}}$ is continuous, we see from this that $I^V_{\mathcal{W}}$ is $\mathcal{C}^1$ and (4.2.6.2) holds.

$k \to k+1$: Since $I^V_{\mathcal{W}}$ is $\mathcal{C}^k$, we conclude from (4.2.6.2) and the fact that D, $\cdot$, $\mathfrak{c}^X_{\mathcal{W}}$ and QI are smooth (see Proposition 3.2.3, Corollary 3.3.7 (together with Example A.1.15), Proposition 4.1.7 and Lemma 3.3.22, respectively) that $dI^V_{\mathcal{W}}$ is $\mathcal{C}^k$. Hence $I^V_{\mathcal{W}}$ is $\mathcal{C}^{k+1}$ by definition. ☐

$\Omega^{U,V}_{\mathcal{W}}$ contains open sets We show that $\Omega^{U,V}_{\mathcal{W}}$ is a neighborhood of 0 if $V \subseteq U$ and $\operatorname{dist}(V, X \setminus U) > 0$. To this end, we need the following two technical lemmas.

Lemma 4.2.7. *Let X be a Banach space, U a convex open nonempty subset and $\phi \in \mathcal{FC}^1(U,X)$ such that $\|\phi\|_{1_U,1} < 1$. Then the map $\mathrm{id}_U + \phi$ is injective.*

Proof. Let $x, y \in U$. Then

$$(\mathrm{id}_U + \phi)(y) - (\mathrm{id}_U + \phi)(x) = y - x + \int_0^1 D\phi(ty + (1-t)x)(y-x)\, dt.$$

Since $\|\phi\|_{1_U,1} < 1$, the norm of the integral is smaller then $\|y - x\|$. We deduce with the triangle inequality that for $x \neq y$, $(\mathrm{id}_U + \phi)(y) \neq (\mathrm{id}_U + \phi)(x)$. ☐

Lemma 4.2.8. *Let X be a Banach space, $U \subseteq X$ an open nonempty subset and $r > 0$. Let $\phi \in \mathcal{BC}^0(U,X)$ with $\|\phi\|_{1_U,0} < r$. Further, let $y \in (\mathrm{id}_U + \phi)(U)$ such that $B_{2r}(y) \subseteq U$. Then for any $\psi \in \mathcal{FC}^1(U,X)$ with $\|\psi\|_{1_U,1} < 1$ and $\|\psi - \phi\|_{1_U,0} < r(1 - \|\psi\|_{1_U,1})$, $y \in (\mathrm{id}_U + \psi)(U)$.*

Proof. There exists $x \in U$ with $x + \phi(x) = y$. Then $\|y - x\| = \|\phi(x)\| < r$, and hence $B_r(x) \subseteq U$ by the triangle inequality. Further, we derive from the Lipschitz inverse function theorem (Corollary A.2.17) that $B_{r(1-\|\psi\|_{1_U,1})}((\mathrm{id}_U + \psi)(x))$ is contained in the image of $\mathrm{id}_U + \psi$, and since

$$\|y - (x + \psi(x))\| = \|\phi(x) - \psi(x)\| < r(1 - \|\psi\|_{1_U,1}),$$

y is contained in the image of $\mathrm{id}_U + \psi$. ☐

Lemma 4.2.9. *Let X be a Banach space, $U, V \subseteq X$ open nonempty subsets such that U is convex and there exists $r > 0$ with $V + B_r(0) \subseteq U$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $\phi \in \Omega_{\mathcal{W}}^{U,V}$ such that $\|\phi\|_{1_U,1} < 1$ and $\|\phi\|_{1_U,0} < \frac{r}{2}$. Then for any $\varepsilon > 0$ such that $\|\phi\|_{1_U,1} + \varepsilon < 1$,*

$$\left\{\psi \in \mathcal{C}^\infty_{\mathcal{W}}(U, X) : \|\psi - \phi\|_{1_U,1} < \varepsilon \text{ and } \|\psi - \phi\|_{1_U,0} < \frac{r}{2}(1 - \varepsilon - \|\phi\|_{1_U,1})\right\}$$

is a neighborhood of ϕ that is contained in $\Omega_{\mathcal{W}}^{U,V}$ and whose image under $I_{\mathcal{W}}^V$ is contained in $\mathcal{C}^\infty_{\mathcal{W}}(V, X)$.

Proof. Let ψ be an element of the neighborhood. Then $\|\psi\|_{1_U,1} < 1$, hence we can apply Lemma 4.2.7 to see that $\mathrm{id}_U + \psi$ is injective. Further, since

$$\|\psi - \phi\|_{1_U,0} < \frac{r}{2}(1 - \varepsilon - \|\phi\|_{1_U,1}) < \frac{r}{2}(1 - \|\psi\|_{1_U,1}),$$

we see with Lemma 4.2.8 that $V \subseteq (\mathrm{id}_U + \psi)(U)$; hence $\psi \in \Omega_{\mathcal{W}}^{U,V}$. Finally, we can apply Proposition 4.2.5 since

$$\|\psi\|_{1_U,0} \leq \|\phi\|_{1_U,0} + \|\psi - \phi\|_{1_U,0} < r$$

and see that $I_{\mathcal{W}}^V(\psi) \in \mathcal{C}^\infty_{\mathcal{W}}(V, X)$. □

The Lie group $\mathrm{Diff}_{\mathcal{W}}(X)$

We put it all together and see that $\mathrm{Diff}_{\mathcal{W}}(X)$ is a Lie group.

Theorem 4.2.10. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then $\mathrm{Diff}_{\mathcal{W}}(X)$ is an open subset of $\mathrm{End}_{\mathcal{W}}(X)$, and a Lie group when endowed with the canonical differential structure.*

Proof. We established in Corollary 4.1.9 that $\mathrm{End}_{\mathcal{W}}(X)$ is a smooth monoid with the unit group $\mathrm{Diff}_{\mathcal{W}}(X)$. By Lemma C.2.3, $\mathrm{Diff}_{\mathcal{W}}(X)$ is open in $\mathrm{End}_{\mathcal{W}}(X)$ if there exists an open neighborhood of id_X in $\mathrm{End}_{\mathcal{W}}(X)$ that is contained in $\mathrm{Diff}_{\mathcal{W}}(X)$. Moreover, the inversion is smooth if it is so on this neighborhood (The proof for the continuity is in Lemma C.2.3, the smoothness can be derived from Lemma B.2.5). To this end, we set

$$U_{\mathcal{W}} := \{\phi \in \mathcal{C}^\infty_{\mathcal{W}}(X, X) : \|\phi\|_{1_X,1} < 1\}.$$

Then for $\phi \in U_{\mathcal{W}}$, we have that $\kappa_{\mathcal{W}}(\phi) \in \mathrm{Diff}_{\mathcal{W}}(X)$ since we can apply Lemma 4.2.9 ($0 \in \Omega_{\mathcal{W}}^{X,X}$) and see that $\phi \in \Omega_{\mathcal{W}}^{X,X}$ with $(\phi + \mathrm{id}_X)^{-1} - \mathrm{id}_X = I_{\mathcal{W}}^X(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)$; enabling us to use Lemma 4.1.1. Further, we know from Proposition 4.2.6 that $I_{\mathcal{W}}^X$ is smooth on $U_{\mathcal{W}}$. This implies that the inversion map is smooth on $\kappa_{\mathcal{W}}(U_{\mathcal{W}})$, see the commutative diagram

$$\begin{array}{ccc} \kappa_{\mathcal{W}}(U_{\mathcal{W}}) & \xrightarrow{\;-1\;} & \mathrm{Diff}_{\mathcal{W}}(X) \\ {\scriptstyle \kappa_{\mathcal{W}}}\uparrow & & \downarrow{\scriptstyle \kappa_{\mathcal{W}}^{-1}} \\ U_{\mathcal{W}} & \xrightarrow{\;I_{\mathcal{W}}^X\;} & \mathcal{C}^\infty_{\mathcal{W}}(X, X). \end{array}$$

This finishes the proof. □

4.2.2. On decreasing weighted diffeomorphisms and dense subgroups

We define the set $\operatorname{Diff}_{\mathcal{W}}(X)^{\circ}$ of *decreasing weighted diffeomorphisms* and show that it is a closed normal subgroup of $\operatorname{Diff}_{\mathcal{W}}(X)$ which can be turned into a Lie group. Further, we give sufficient conditions on $\mathcal{W}$ ensuring that the group $\operatorname{Diff}_{c}(X)$ of compactly supported diffeomorphisms is dense in $\operatorname{Diff}_{\mathcal{W}}(X)^{\circ}$.

Inversion on weighted diffeomorphisms First, we have to discuss the inversion map restricted to weighted functions.

Lemma 4.2.11. *Let X be a Banach space, $U, V \subseteq X$ open nonempty subsets, $\phi \in \Omega_{\mathcal{W}}^{U,V} \cap \mathcal{BC}^0(U, X)$, $r > 0$ with $r > \|\phi\|_{1_U,0}$ and $\widetilde{U} \subseteq U$ and $\widetilde{V} \subseteq V$ open nonempty subsets such that $\widetilde{V} + B_r(0) \subseteq \widetilde{U}$.*

(a) Then $\phi \in \Omega_{\mathcal{W}}^{\widetilde{U},\widetilde{V}}$.

(b) In particular, if $R > s > 0$, $U = V = X$ and $\|\phi\|_{1_X,0} < R - s$, then $\phi \in \Omega_{\mathcal{W}}^{X\setminus\overline{B}_s(0), X\setminus\overline{B}_R(0)}$.

Proof. (a) Obviously $\phi + \mathrm{id}_U$ is injective on $\widetilde{U}$, so we just need to show that $\widetilde{V} \subseteq (\phi + \mathrm{id}_U)(\widetilde{U})$. To this end, let $y \in \widetilde{V}$. Since $\phi \in \Omega_{\mathcal{W}}^{U,V}$ and $\widetilde{V} \subseteq V$, there exists $x \in U$ with $\phi(x) + x = y$. This implies that $\|y - x\| \leq \|\phi\|_{1_U,0} < r$, and hence

$$x = y + x - y \in \widetilde{V} + B_r(0) \subseteq \widetilde{U}.$$

(b) This is an easy application of (a) since by the triangle inequality $X \setminus \overline{B}_R(0) + B_{R-s}(0) \subseteq X \setminus \overline{B}_s(0)$. □

Lemma 4.2.12. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$ and $\phi \in \Omega_{\mathcal{W}}^{X,X} \cap \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)^o$. Then there exists an $R > 0$ such that*

$$I_{\mathcal{W}}^{X}(\phi)|_{X\setminus\overline{B}_R(0)} \in \mathcal{C}^{\infty}_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)^o.$$

Proof. Since $\phi \in \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)^o$, there exists an $r > 0$ such that $\sup_{x\in X\setminus\overline{B}_r(0)} \|D\phi(x)\|_{op} < 1$. We choose $R > 0$ such that $\|\phi\|_{1_X,0} + r < R$. We see with Lemma 4.2.11 that $\phi \in \Omega_{\mathcal{W}}^{X\setminus\overline{B}_r(0), X\setminus\overline{B}_R(0)}$, and this allows the application of Proposition 4.2.5 to see that $I_{\mathcal{W}}^{X\setminus\overline{B}_R(0)}(\phi) \in \mathcal{C}^{\infty}_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$. Further, by identity (4.2.2.4)

$$I_{\mathcal{W}}^{X\setminus\overline{B}_R(0)}(\phi) = -\phi \circ (I_{\mathcal{W}}^{X\setminus\overline{B}_R(0)}(\phi) + \mathrm{id}_{X\setminus\overline{B}_R(0)}) = \mathfrak{c}_{\mathcal{W}}^{X}(-\phi, I_{\mathcal{W}}^{X\setminus\overline{B}_R(0)}(\phi)),$$

hence an application of Lemma 4.1.5 finishes the proof. □

A normal Lie subgroup To derive the desired result, we need the following technical lemma.

Lemma 4.2.13. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\phi \in \operatorname{End}_{\mathcal{W}}(X)^\circ$ and $\psi \in \operatorname{Diff}_{\mathcal{W}}(X)$. Then $\psi - \psi \circ \phi \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$.*

Proof. We calculate using Lemma 3.2.10 and the mean value theorem that

$$\psi - \psi \circ \phi = \int_0^1 D\psi(\mathrm{id}_X + t(\phi - \mathrm{id}_X)) \cdot (\phi - \mathrm{id}_X)\, dt.$$

Since $D\psi \in \mathcal{BC}^\infty(X, \mathrm{L}(X))$, we conclude with Proposition 4.1.7 that $D\psi(\mathrm{id}_X + t(\phi - \mathrm{id}_X)) \in \mathcal{BC}^\infty(X, \mathrm{L}(X))$. Since $\phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$, the assertion follows from Corollary 3.3.4 and the fact that $\mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$ is closed in $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$. □

Proposition 4.2.14. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. The set*

$$\operatorname{Diff}_{\mathcal{W}}(X)^\circ := \operatorname{Diff}_{\mathcal{W}}(X) \cap \operatorname{End}_{\mathcal{W}}(X)^\circ = \{\phi \in \operatorname{Diff}_{\mathcal{W}}(X) : \phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o\}$$

is a closed normal Lie subgroup of $\operatorname{Diff}_{\mathcal{W}}(X)$*. We call its elements* decreasing weighted diffeomorpisms*.*

Proof. In Corollary 4.1.9 it was proved that $\operatorname{End}_{\mathcal{W}}(X)^\circ$ is a smooth submonoid of $\operatorname{End}_{\mathcal{W}}(X)$ and a closed subset. Since $\operatorname{Diff}_{\mathcal{W}}(X)$ is open in $\operatorname{End}_{\mathcal{W}}(X)$, we conclude that $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$ is a smooth submonoid of $\operatorname{Diff}_{\mathcal{W}}(X)$ that is closed. Further, it is a direct consequence of Lemma 4.2.12 that the inverse function of an element of $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$ is in $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$, whence using Lemma B.1.6 we see that the latter is a closed Lie subgroup of $\operatorname{Diff}_{\mathcal{W}}(X)$.

It remains to show that $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$ is normal. To this end, let $\phi \in \operatorname{Diff}_{\mathcal{W}}(X)^\circ$ and $\psi \in \operatorname{Diff}_{\mathcal{W}}(X)$. Then

$$\psi \circ \phi \circ \psi^{-1} - \mathrm{id}_X = \psi \circ \phi \circ \psi^{-1} - \psi \circ \phi^{-1} \circ \phi \circ \psi^{-1} = (\psi - \psi \circ \phi^{-1}) \circ \phi \circ \psi^{-1},$$

so we derive the assertion from Lemma 4.2.13 and Lemma 4.1.5. □

On the density of compactly supported diffeomorphisms As promised, we give a sufficient criterium on $\mathcal{W}$ that makes $\operatorname{Diff}_c(X)$ a dense subgroup of $\operatorname{Diff}_{\mathcal{W}}(X)$.

Lemma 4.2.15. *Let X and Y be finite-dimensional normed spaces and $U \subseteq X$ an open nonempty set. Further, let $\mathcal{W} \subseteq \mathbb{R}^U$ a set of weights such that*

$$\begin{aligned}
&\bullet\ \mathcal{W} \subseteq \mathcal{C}^\infty(U, [0, \infty[) \\
&\bullet\ (\forall x \in U)(\exists f \in \mathcal{W})\, f(x) > 0 \\
&\bullet\ \begin{array}{l}(\forall f_1, \ldots, f_n \in \mathcal{W})(\forall k_1, \ldots, k_n \in \mathbb{N})(\exists f \in \mathcal{W}, C > 0) \\ (\forall x \in U)\|D^{(k_1)} f_1(x)\|_{op} \cdots \|D^{(k_n)} f_n(x)\|_{op} \leq C f(x).\end{array}
\end{aligned} \tag{4.2.15.1}$$

Then $\mathcal{C}^\infty_c(U, Y)$ is dense in $\mathcal{C}^r_{\mathcal{W}}(U, Y)^o$.

Proof. A proof can be found in [GDS73, §V, 19 b)]. □

Lemma 4.2.16. *Let X be a finite-dimensional normed space, $\mathcal{W} \subseteq \mathbb{R}^X$ such that $1_X \in \mathcal{W}$ and (4.2.15.1) is satisfied (where $U = X$). Then the set of compactly supported diffeomorphisms $\operatorname{Diff}_c(X)$ is dense in $\operatorname{Diff}_{\mathcal{W}}(X)^\circ$.*

Proof. The set $M_{\mathcal{W}}^\circ := \kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_{\mathcal{W}}(X)) \cap \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o = \kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_{\mathcal{W}}(X)^\circ)$ is open in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$, and hence $M_c := \mathcal{C}^\infty_c(X,X) \cap M_{\mathcal{W}}^\circ$ is dense in $M_{\mathcal{W}}^\circ$ by Lemma 4.2.15. But $M_c = \kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_c(X))$, from which the assertion follows. □

4.2.3. On diffeomorphisms that are weighted endomorphisms

It is obvious that the relation

$$\operatorname{Diff}_{\mathcal{W}}(X) \subseteq \operatorname{End}_{\mathcal{W}}(X) \cap \operatorname{Diff}(X)$$

holds. We give a sufficient criterion on $\mathcal{W}$ that ensures that these two sets are identical, provided that X is finite-dimensional. Further we show that $\operatorname{Diff}_{\{1_\mathbb{R}\}}(\mathbb{R}) \neq \operatorname{End}_{\{1_\mathbb{R}\}}(\mathbb{R}) \cap \operatorname{Diff}(X)$.

Proposition 4.2.17. *Let X be a* finite-dimensional *Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. If there exists $\widehat{f} \in \mathcal{W}$ such that*

$$(\forall R > 0)(\exists r > 0)\, \|x\| \geq r \implies |\widehat{f}(x)| \geq R \tag{4.2.17.1}$$

and if each function in $\mathcal{W}$ is bounded on bounded sets, then

$$\operatorname{Diff}_{\mathcal{W}}(X) = \operatorname{End}_{\mathcal{W}}(X) \cap \operatorname{Diff}(X).$$

Proof. We have to show that

$$\operatorname{End}_{\mathcal{W}}(X) \cap \operatorname{Diff}(X) \subseteq \operatorname{Diff}_{\mathcal{W}}(X).$$

So let ψ be in $\operatorname{End}_{\mathcal{W}}(X) \cap \operatorname{Diff}(X)$. Then $\phi := \psi - \operatorname{id}_X \in \Omega_{\mathcal{W}}^{X,X}$, and the equivalences

$$\begin{aligned}\psi \in \operatorname{Diff}_{\mathcal{W}}(X) &\iff \psi^{-1} \in \operatorname{End}_{\mathcal{W}}(X)\\ &\iff \psi^{-1} - \operatorname{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X) \iff I^X_{\mathcal{W}}(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)\end{aligned}$$

hold (see Lemma 4.1.1 and the definition of $I^X_{\mathcal{W}}$ in (4.2.1.1)). The last statement clearly holds iff

$$(\exists R, r > 0)\, I^X_{\mathcal{W}}(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X) \text{ and } I^X_{\mathcal{W}}(\phi)|_{B_{R+r}(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(B_{R+r}(0), X),$$

and this shall be proved now. Obviously $I^X_{\mathcal{W}}(\phi)|_{B_R(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(B_R(0), X)$ for each $R > 0$ because each $f \in \mathcal{W}$ is bounded on bounded sets, the maps $D^{(\ell)} I^X_{\mathcal{W}}(\phi)$ are continuous and each bounded subset of X is relatively compact (as X is finite-dimensional). It remains to show that there exists $R > 0$ such that $I_{\mathcal{W}}^{X}(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^{\infty}_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$. We set $K_\phi := \|\phi\|_{\widehat{f},1} < \infty$ and conclude from (4.2.17.1) that there exists an r_ϕ with

$$\|x\| \geq r_\phi \implies |\widehat{f}(x)| \geq K_\phi + 1.$$

Since $|\widehat{f}(x)| \, \|D\phi(x)\|_{op} \leq K_\phi$ for each $x \in X$, we see that

$$\|\phi|_{X \setminus \overline{B}_{r_\phi}(0)}\|_{1_X,1} \leq \frac{K_\phi}{K_\phi + 1} < 1.$$

We choose $R_\phi > 0$ such that $R_\phi > r_\phi + \|\phi\|_{1_X,0}$. We see with Lemma 4.2.11 that $\phi \in \Omega_{\mathcal{W}}^{X \setminus \overline{B}_{r_\phi}(0), X \setminus \overline{B}_{R_\phi}(0)}$, so we can apply Proposition 4.2.5 to see that

$$I_{\mathcal{W}}^{X}(\phi)|_{X \setminus \overline{B}_{R_\phi}(0)} = I_{\mathcal{W}}^{X \setminus \overline{B}_{R_\phi}(0)}(\phi) \in \mathcal{C}^{\infty}_{\mathcal{W}}(X \setminus \overline{B}_{R_\phi}(0), X),$$

and this finishes the proof. □

We give an affirmative example.

Example 4.2.18. The space $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ satisfies condition (4.2.17.1). We just have to set $\widehat{f}(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$ which clearly is a polynomial function on $\mathbb{R}^n$.

As announced, we give a counterexample. As preparation, we prove the following lemma.

Lemma 4.2.19. *Let $\gamma \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$ be a* bounded *map that satisfies*

$$(\forall x \in \mathbb{R})\ \gamma'(x) > -1. \tag{$*$}$$

Then $\gamma + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$.

Proof. We conclude from $(*)$ that $(\gamma(x) + \mathrm{id}_{\mathbb{R}})'(x) > 0$ for all $x \in \mathbb{R}$, so $\gamma + \mathrm{id}_{\mathbb{R}}$ is strictly monotone and hence injective. Since γ is bounded, $\gamma + \mathrm{id}_{\mathbb{R}}$ is unbounded above and below and hence surjective (by the intermediate value theorem). □

Example 4.2.20. We give an example of a map $\gamma \in \mathcal{BC}^{\infty}(\mathbb{R}, \mathbb{R})$ with the property that $\gamma + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$, but $(\gamma + \mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}} \notin \mathcal{BC}^{\infty}(\mathbb{R}, \mathbb{R})$. To this end, let ϕ be an antiderivative of the function $x \mapsto \frac{2}{\pi} \arctan(x)$ with $\phi(0) = 0$. Then $\sin \circ \phi$ and $\cos \circ \phi$ are in $\mathcal{BC}^{\infty}(\mathbb{R}, \mathbb{R})$ by a simple induction since $\cos, \sin, \arctan \in \mathcal{BC}^{\infty}(\mathbb{R}, \mathbb{R})$,

$$(\sin \circ \phi)'(x) = \frac{2}{\pi} \arctan(x)(\cos \circ \phi)(x), \tag{$*$}$$

and an analogous formula holds for $(\cos \circ \phi)'$. We see with $(*)$ that $(\sin \circ \phi)'(x) > -1$ for all $x \in \mathbb{R}$, so $\sin \circ \phi + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$ by Lemma 4.2.19. But since

$$((\sin \circ \phi + \mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}})'(x) = \frac{1}{(\sin \circ \phi)'((\sin \circ \phi + \mathrm{id}_{\mathbb{R}})^{-1}(x)) + 1} - 1$$

and there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ with

$$\lim_{n \to \infty} \frac{2}{\pi} \arctan(y_n)(\cos \circ \phi)(y_n) = -1,$$

$((\sin \circ \phi + \mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}})'$ clearly is not bounded.

4.3. Regularity

We prove that the Lie groups $\mathrm{Diff}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)^{\circ}$ are regular. For the definition of regularity, see Subsection B.2.2.

4.3.1. The regularity differential equation of $\mathrm{Diff}_{\mathcal{W}}(X)$

We examine the general (right) regularity differential equation (which is stated in initial value problem (B.2.11.1)) and turn it into a differential equation on $\mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$. To this end, we first describe the group multiplication of the tangent group $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ and the right action of $\mathrm{Diff}_{\mathcal{W}}(X)$ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to the chart $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$.

Lemma 4.3.1 (Tangent group of $\mathrm{Diff}_{\mathcal{W}}(X)$)**.** *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. In the following, we denote the multiplication on $\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to the chart $\kappa_{\mathcal{W}}^{-1}$ by $m_{\mathcal{W}}$. Note that the tangent group $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ is canonicly isomorphic to $\mathcal{C}^{\infty}_{\mathcal{W}}(X, X) \rtimes \mathrm{Diff}_{\mathcal{W}}(X)$.*

(a) The group multiplication $\mathbf{T}m_{\mathcal{W}}$ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ (with respect to $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$) is given by

$$\mathbf{T}m_{\mathcal{W}}((\gamma, \gamma_1), (\eta, \eta_1)) = (m_{\mathcal{W}}(\gamma, \eta), D\gamma \circ (\eta + \mathrm{id}_X) \cdot \eta_1 + \gamma_1 \circ (\eta + \mathrm{id}_X) + \eta_1).$$

(b) Let $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)$. Then the right action $\mathbf{T}\rho_{\phi}$ of ϕ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$ is given by

$$\mathbf{T}(\kappa_{\mathcal{W}}^{-1} \circ \rho_{\phi} \circ \kappa_{\mathcal{W}})(\gamma, \gamma_1) = (m_{\mathcal{W}}(\gamma, \kappa_{\mathcal{W}}^{-1}(\phi)), \gamma_1 \circ \phi).$$

Proof. (a) We have

$$m_{\mathcal{W}}(\gamma, \eta) = \gamma \circ (\eta + \mathrm{id}_X) + \eta$$

and the commutative diagram

$$\begin{array}{ccc}
\mathrm{Diff}_{\mathcal{W}}(X) \times \mathrm{Diff}_{\mathcal{W}}(X) & \xrightarrow{\quad\circ\quad} & \mathrm{Diff}_{\mathcal{W}}(X) \\
\uparrow{\scriptstyle \kappa_{\mathcal{W}} \times \kappa_{\mathcal{W}}} & & \uparrow{\scriptstyle \kappa_{\mathcal{W}}} \\
\kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) \times \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) & \xrightarrow{\quad m_{\mathcal{W}}\quad} & \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X))
\end{array}.$$

The group multiplication on the tangent group is given by applying the tangent functor $\mathbf{T}$ to the group multiplication on $\mathrm{Diff}_{\mathcal{W}}(X)$, and therefore we obtain the group multiplication on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ in charts by applying $\mathbf{T}$ to $m_{\mathcal{W}}$ (up to a permutation). Since

$$\mathbf{T}m_{\mathcal{W}}(\gamma, \eta; \gamma_1, \eta_1) = (m_{\mathcal{W}}(\gamma, \eta), D\gamma \circ (\eta + \mathrm{id}_X) \cdot \eta_1 + \gamma_1 \circ (\eta + \mathrm{id}_X) + \eta_1)$$

by (4.1.7.1), the asserted identity holds.

(b) Obviously $(\kappa_{\mathcal{W}}^{-1} \circ \rho_{\phi} \circ \kappa_{\mathcal{W}})(\cdot) = m_{\mathcal{W}}(\cdot, \kappa_{\mathcal{W}}^{-1}(\phi))$, so we derive the assertion if we apply the identity proved in (a) with $\eta = \kappa_{\mathcal{W}}^{-1}(\phi)$ and $\eta_1 = 0$. □

We aim to turn (B.2.11.1) into an ODE on a vector space. Before we can do this, a definition is useful:

Definition 4.3.2. Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and $\mathcal{F}$ be a subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$. By Proposition 4.1.7, the map

$$\begin{aligned} F_{\mathcal{F},k} &: [0,1] \times \mathcal{C}^k_{\mathcal{F}}(X,X) \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^k_{\mathcal{F}}(X,X) \\ &: (t,\gamma,p) \mapsto p(t) \circ (\gamma + \mathrm{id}_X) \end{aligned}$$

is well-defined and smooth (since the evaluation of curves is smooth by Lemma A.1.9). For each parameter curve $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$, we consider the initial value problem

$$\begin{aligned} \Gamma'(t) &= F_{\mathcal{F},k}(t,\Gamma(t),p) \\ \Gamma(0) &= 0, \end{aligned} \tag{4.3.2.1}$$

where $t \in [0,1]$.

Lemma 4.3.3. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$.*

(a) For $\gamma \in \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \operatorname{Diff}_{\mathcal{W}}(X))$, the initial value problem

$$\begin{aligned} \eta'(t) &= \gamma(t) \cdot \eta(t) \\ \eta(0) &= \mathrm{id}_X \end{aligned}$$

has a smooth solution

$$\operatorname{Evol}^\rho_{\operatorname{Diff}_{\mathcal{W}}(X)}(\gamma) : [0,1] \to \operatorname{Diff}_{\mathcal{W}}(X)$$

iff the initial value problem (4.3.2.1) *(in Definition 4.3.2) with $\mathcal{F} = \mathcal{W}$, $k = \infty$ and $p = d\kappa_{\mathcal{W}}^{-1} \circ \gamma$ has a smooth solution*

$$\Gamma_p : [0,1] \to \kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_{\mathcal{W}}(X)).$$

In this case,

$$\operatorname{Evol}^\rho_{\operatorname{Diff}_{\mathcal{W}}(X)}(\gamma) = \kappa_{\mathcal{W}} \circ \Gamma_p.$$

(b) Let $\Omega \subseteq \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \operatorname{Diff}_{\mathcal{W}}(X))$ be an open set such that for each $\gamma \in \Omega$ there exists a right evolution $\operatorname{Evol}^\rho_{\operatorname{Diff}_{\mathcal{W}}(X)}(\gamma)$. Then $\operatorname{evol}^\rho_{\operatorname{Diff}_{\mathcal{W}}(X)}|_\Omega$ is smooth iff the map

$$(d\kappa_{\mathcal{W}}^{-1} \circ \Omega) \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : p \mapsto \Gamma_p(1)$$

is so. As above, Γ_p denotes a solution to (4.3.2.1) *with respect to p.*

Proof. This is an easy computation involving the previous results. □

Solving the differential equation

We show that the regularity differential equation for $\operatorname{Diff}_{\mathcal{W}}(X)$ is solvable. In order to do this, we use that $\mathcal{C}^{\infty}_{\mathcal{W}}(X,X)$ is a projective limit of Banach spaces, see Proposition 3.2.5. We solve the differential equation on each step of the projective limit, see that these solutions are compatible with the bonding morphisms of the projective limit and thus obtain a solution on the limit. Before we do this, we state the following obvious lemma.

Lemma 4.3.4. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F} \subseteq \mathcal{W}$ with $1_X \in \mathcal{F}$ and $k \in \overline{\mathbb{N}}$, $p \in \mathcal{C}^{\infty}([0,1], \mathcal{C}^{\infty}_{\mathcal{W}}(X,X))$ and $\Gamma : I \to \mathcal{C}^{k}_{\mathcal{F}}(X,X)$ a solution to* (4.3.2.1) *corresponding to p. Then Γ solves* (4.3.2.1) *also for all subsets $\mathcal{G} \subseteq \mathcal{F}$ containing 1_X and $\ell \in \overline{\mathbb{N}}$ with $\ell \leq k$.*

Proof. This is an easy calculation since the inclusion map $\mathcal{C}^{k}_{\mathcal{F}}(X,X) \to \mathcal{C}^{\ell}_{\mathcal{G}}(X,X)$ is continuous linear. □

Solving the differential equation on the steps First, we solve (4.3.2.1) on function spaces that are Banach spaces. To this end, we need tools from the theory of ordinary differential equations on Banach spaces. The required facts are described in Section A.4. The hard part will be to show that the solutions are defined on the whole interval $[0,1]$.

The solution on $\mathcal{C}^{0}_{\mathcal{F}}(X,X)$ We start with the function space $\mathcal{C}^{0}_{\mathcal{F}}(X,X)$, where $\mathcal{F} \subseteq \mathcal{W}$ is finite and contains 1_X. Then the initial value problem (4.3.2.1) satisfies a global Lipschitz condition and hence is globally solvable.

Lemma 4.3.5. *Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\mathcal{F} \subseteq \mathcal{W}$ with $1_X \in \mathcal{F}$ and $p \in \mathcal{C}^{\infty}([0,1], \mathcal{C}^{\infty}_{\mathcal{W}}(X,X))$. Then there exists $K > 0$ such that for each $f \in \mathcal{F}$, all $t \in [0,1]$ and $\gamma, \gamma_0 \in \mathcal{C}^{0}_{\mathcal{F}}(X,X)$*

$$\|F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p)\|_{f,0} \leq K \cdot \|\gamma - \gamma_0\|_{f,0}.$$

Proof. We have

$$F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p) = \mathfrak{c}^{X,0}_{\mathcal{F}}(p(t),\gamma) - \mathfrak{c}^{X,0}_{\mathcal{F}}(p(t),\gamma_0),$$

and deduce from estimate (4.1.3.2) in Lemma 4.1.3 that

$$\|F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p)\|_{f,0} \leq \|p(t)\|_{1_X,1} \|\gamma - \gamma_0\|_{f,0}.$$

Since $p([0,1])$ is a compact (and therefore bounded) subset of $\mathcal{C}^{\infty}_{\mathcal{W}}(X,X)$,

$$K := \sup_{t \in [0,1]} \|p(t)\|_{1_X,1}$$

is finite. This proves the assertion. □

Lemma 4.3.6. *Let X be a Banach space, $\mathcal{F}, \mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{F} \subseteq \mathcal{W}$ and $|\mathcal{F}| < \infty$, $p \in \mathcal{C}^{\infty}([0,1], \mathcal{C}^{\infty}_{\mathcal{W}}(X,X))$ and $k = 0$. Then the initial value problem* (4.3.2.1) *corresponding to p has a unique solution which is defined on the whole interval* $[0,1]$.

Proof. We deduce from Lemma 4.3.5 that we can find a norm on $\mathcal{C}^0_{\mathcal{F}}(X,X)$ such that $F_{\mathcal{F},0}(\cdot,\cdot,p)$ satisfies a global Lipschitz condition with respect to the second argument. Since $\mathcal{C}^0_{\mathcal{F}}(X,X)$ is a Banach space, there exists a unique solution

$$\Gamma : [0,1] \to \mathcal{C}^0_{\mathcal{F}}(X,X)$$

of (4.3.2.1) which is defined on the whole interval $[0,1]$; see [Die60, §10.6.1] or Theorem A.4.7 and Lemma A.4.5. □

Solutions in spaces of differentiable functions On the spaces $\mathcal{C}^k_{\mathcal{F}}(X,X)$ with $k \geq 1$, it is harder to show that the maximal solution is defined on the whole of $[0,1]$. To show this, we first verify that the differential curve $D \circ \gamma$ of a solution $\gamma : I \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ to (4.3.2.1) is itself a solution to a linear ODE. We start with the following definition.

Definition 4.3.7. Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F}$ be a subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$, $k \in \overline{\mathbb{N}}$ and $\Gamma : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ and $P : [0,1] \to \mathcal{C}^\infty_{\mathcal{W}}(X, \mathrm{L}(X))$ be continuous curves. We define the continuous map

$$\begin{aligned} G^{\Gamma,P}_{\mathcal{F},k} &: [0,1] \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X)) \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X)) \\ &: (t,\gamma) \mapsto \big(P(t) \circ (\Gamma(t) + \mathrm{id}_X)\big) \cdot (\gamma + \mathrm{Id}) \end{aligned}$$

and consider the initial value problem

$$\begin{aligned} \Phi'(t) &= G^{\Gamma,P}_{\mathcal{F},k}(t, \Phi(t)) \\ \Phi(0) &= 0. \end{aligned} \tag{4.3.7.1}$$

Lemma 4.3.8. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F}$ be a finite subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$, $k \in \mathbb{N}$ and $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$. If*

$$\Gamma_k : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X) \quad \text{and} \quad \Gamma_{k+1} : I \subseteq [0,1] \to \mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$$

are solutions to (4.3.2.1) *corresponding to p, then the curve $D \circ \Gamma_{k+1} : I \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$ is a solution to the initial value problem* (4.3.7.1) *with $\Gamma = \Gamma_k$ and $P = D \circ p$.*

Proof. We have

$$(D \circ \Gamma_{k+1})' = D \circ \Gamma'_{k+1}$$

and therefore for $t \in I$

$$\begin{aligned} (D \circ \Gamma_{k+1})'(t) &= D\, F_{\mathcal{F},k+1}(t, \Gamma_{k+1}(t), p) \\ &= \big(Dp(t) \circ (\Gamma_{k+1}(t) + \mathrm{id}_X)\big) \cdot (D\Gamma_{k+1}(t) + \mathrm{Id}). \\ &= \big((D \circ p)(t) \circ (\Gamma_{k+1}(t) + \mathrm{id}_X)\big) \cdot \big((D \circ \Gamma_{k+1})(t) + \mathrm{Id}\big) \\ &= G^{\Gamma_k, D\circ p}_{\mathcal{F},k}(t, (D \circ \Gamma_{k+1})(t)), \end{aligned}$$

where we used that $\Gamma_k|_I = \Gamma_{k+1}$ by Lemma 4.3.4 since $\mathcal{C}^k_{\mathcal{F}}(X,X)$ is a Banach space. Obviously $(D \circ \Gamma_{k+1})(0) = 0$, so the assertion is proved. □

Now we use the embedding from Proposition 3.2.3 to show that the maximal solution to (4.3.2.1) is defined on $[0,1]$.

Lemma 4.3.9. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\mathcal{F} \subseteq \mathcal{W}$ finite with $1_X \in \mathcal{F}$, $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ and $k \in \mathbb{N}$. Then the initial value problem* (4.3.2.1) *corresponding to p has a unique solution which is defined on the whole interval* $[0,1]$.

Proof. This is proved by induction on k. The case $k = 0$ was treated in Lemma 4.3.6.

$k \to k+1$: We denote the solutions for k and 0 with Γ_k and Γ_0, respectively. Since the function $F_{\mathcal{F},k+1}$ is smooth and $\mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$ is a Banach space, there exists a unique *maximal* solution $\Gamma_{k+1} : I \to \mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$ to (4.3.2.1) (see Proposition A.4.2). Using Lemma 4.3.8, we conclude that $D \circ \Gamma_{k+1}$ is a solution to (4.3.7.1), where $\Gamma = \Gamma_k$ and $P = D \circ p$; here we used that by the induction hypothesis, Γ_k is defined on $[0,1]$. Since the latter ODE is linear, there exists a unique solution

$$S : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$$

that is defined on the whole interval $[0,1]$ (see [Die60, §10.6.3] or Theorem A.4.7). Let

$$\iota : \mathcal{C}^{k+1}_{\mathcal{F}}(X,X) \to \mathcal{C}^0_{\mathcal{F}}(X,X) \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$$

be the embedding from Proposition 3.2.3. By Lemma 4.3.4, Γ_{k+1} is a solution to (4.3.2.1) for the right hand side $F_{\mathcal{F},0}$, so $\Gamma_{k+1} = \Gamma_0|_I$ since solutions to initial value problems in Banach spaces are unique. Hence

$$\Gamma_{k+1}(I) \subseteq \iota^{-1}(\Gamma_0([0,1]) \times S([0,1])).$$

Further, $\Gamma_0([0,1]) \times S([0,1])$ is compact and the image of ι is a closed subset of $\mathcal{C}^0_{\mathcal{F}}(X,X) \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$ (by Proposition 3.2.7). Hence, because ι^{-1} is a homeomorphism, the image of Γ_{k+1} is contained in a compact set. Since Γ_{k+1} is maximal, this implies that Γ_{k+1} must be defined on the whole of $[0,1]$; see Theorem A.4.7. ☐

Smooth dependence on the parameter and taking the solution to the limit We use the constructed solutions on $\mathcal{C}^k_{\mathcal{F}}(X,X)$ and show that there exists a solution to (4.3.2.1) on $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$, depending smoothly on the parameter curve.

Proposition 4.3.10. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. For each $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ there exists a solution Γ_p to* (4.3.2.1) *defined on* $[0,1]$ *which corresponds to p, $\mathcal{W}$ and ∞. The map*

$$[0,1] \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : (t,p) \mapsto \Gamma_p(t) \tag{†}$$

is smooth.

Proof. For $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$, we denote the solution $[0,1] \to \mathcal{C}^0_{\{1_X\}}(X,X)$ to (4.3.2.1) corresponding to p, 0 and $\{1_X\}$ – which exists by Lemma 4.3.9 – with Γ_p. By Lemma 4.3.4, a solution $\Gamma : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ to (4.3.2.1) corresponding to p, a finite set

$\mathcal{F} \subseteq \mathcal{W}$ containing 1_X and $k \in \mathbb{N}$ – which exists by Lemma 4.3.9 – also solves (4.3.2.1) for p, 0 and $\{1_X\}$. Hence, by the uniqueness of solutions to initial value problems for Banach spaces, $\Gamma_p = \Gamma$. Since $\mathcal{F}$ and k were arbitrary, the image of Γ_p is contained in $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$, and we easily calculate that Γ_p is a solution to (4.3.2.1) corresponding to p, $\mathcal{W}$ and ∞.

It remains to show that the map (†) is smooth. The space $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$ is the projective limit of

$$\{\mathcal{C}^k_{\mathcal{F}}(X, X) : k \in \mathbb{N}, \mathcal{F} \subseteq \mathcal{W}, |\mathcal{F}| < \infty, 1_X \in \mathcal{F}\}$$

by Proposition 3.2.5. Hence using the universal property of the projective limit (see Proposition A.1.12), we just have to show that the map

$$[0, 1] \times \mathcal{C}^\infty([0, 1], \mathcal{C}^\infty_{\mathcal{W}}(X, X)) \to \mathcal{C}^k_{\mathcal{F}}(X, X) : (t, p) \mapsto \Gamma_p(t)$$

with a finite set $\mathcal{F} \subseteq \mathcal{W}$ containing 1_X and $k \in \mathbb{N}$ is smooth. We deduce this from Corollary A.4.14 since the map $\mathcal{C}^\infty([0, 1], \mathcal{C}^\infty_{\mathcal{W}}(X, X)) \to \mathcal{C}^k_{\mathcal{F}}(X, X) : p \mapsto 0$ is smooth. Here, we used implicitely that the inclusion map $\mathcal{C}^\infty_{\mathcal{W}}(X, X) \to \mathcal{C}^k_{\mathcal{F}}(X, X)$ is smooth. □

4.3.2. Conclusion and calculation of one-parameter groups

We are ready to prove the regularity of $\mathrm{Diff}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$. After that, we calculate their one-parameter groups and show that these induce flows on certain weighted vector fields.

Theorem 4.3.11. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then the Lie group $\mathrm{Diff}_{\mathcal{W}}(X)$ is regular.*

Proof. We proved in Proposition 4.3.10 that for each smooth curve $p : [0, 1] \to \mathcal{C}^\infty_{\mathcal{W}}(X, X)$ the initial value problem (4.3.2.1) has a solution $\Gamma_p : [0, 1] \to \mathcal{C}^\infty_{\mathcal{W}}(X, X)$ and that the map

$$\Gamma : [0, 1] \times \mathcal{C}^\infty([0, 1], \mathcal{C}^\infty_{\mathcal{W}}(X, X)) \to \mathcal{C}^\infty_{\mathcal{W}}(X, X) : (t, p) \mapsto \Gamma_p(t)$$

is smooth. Obviously, Γ maps $[0, 1] \times \{0\}$ to 0. Since $\kappa^{-1}_{\mathcal{W}}(\mathrm{Diff}_{\mathcal{W}}(X))$ is an open neighborhood of 0 in $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$ (see Theorem 4.2.10) and Γ is continuous, a compactness argument gives a neighborhood U of 0 such that

$$\Gamma([0, 1] \times U) \subseteq \kappa^{-1}_{\mathcal{W}}(\mathrm{Diff}_{\mathcal{W}}(X)).$$

We recorded in Lemma 4.3.3 that this is equivalent to the existence of an open neighborhood V of $0 \in \mathcal{C}^\infty([0, 1], \mathcal{C}^\infty_{\mathcal{W}}(X, X))$ such that for each $\gamma \in V$, there exists a right evolution $\mathrm{Evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}(\gamma)$ and that $\mathrm{evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}|_V$ is smooth. But we know from Lemma B.2.10 that this entails the regularity of $\mathrm{Diff}_{\mathcal{W}}(X)$. □

Corollary 4.3.12. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ is a regular Lie group.*

Proof. Let $\gamma \in \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_\mathcal{W}(X)^\circ)$. Since $\mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_\mathcal{W}(X)^\circ \subseteq \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_\mathcal{W}(X)$ and $\mathrm{Diff}_\mathcal{W}(X)$ is regular by Theorem 4.3.11, there exists a right evolution $\mathrm{Evol}^\rho(\gamma) : [0,1] \to \mathrm{Diff}_\mathcal{W}(X)$. We proved in Lemma 4.3.3 that the curve $\Gamma := \kappa_\mathcal{W} \circ \mathrm{Evol}^\rho(\gamma)$ is a solution to the initial value problem (4.3.2.1), where $\mathcal{F} = \mathcal{W}$, $k = \infty$ and $p = d\kappa_\mathcal{W}^{-1} \circ \gamma$. So for $t \in [0,1]$,

$$\Gamma(t) = \int_0^t \Gamma'(s)\, ds = \int_0^t p(s) \circ (\Gamma(s) + \mathrm{id}_X)\, ds.$$

Hence we see with Lemma 4.1.5 and the fact that $\mathcal{C}^\infty_\mathcal{W}(X,X)^o$ is closed in $\mathcal{C}^\infty_\mathcal{W}(X,X)$ by Lemma 3.1.6 that $\mathrm{Evol}^\rho(\gamma)$ takes its values in $\mathrm{Diff}_\mathcal{W}(X) \cap \mathrm{End}_\mathcal{W}(X)^\circ = \mathrm{Diff}_\mathcal{W}(X)^\circ$. From this and the smoothness of $\mathrm{evol}^\rho_{\mathrm{Diff}_\mathcal{W}(X)}$ we easily conclude that $\mathrm{evol}^\rho_{\mathrm{Diff}_\mathcal{W}(X)^\circ}$ is smooth, and this finishes the proof. □

On the one-parameter groups We calculate the one-parameter groups of $\mathrm{Diff}_\mathcal{W}(X)$ (and hence for $\mathrm{Diff}_\mathcal{W}(X)^\circ$). As excepted, these arise as flows of vector fields.

Lemma 4.3.13. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then for $\gamma \in \mathcal{C}^\infty_\mathcal{W}(X,X)$, the associated flow of the one-parameter subgroup of $\mathrm{Diff}_\mathcal{W}(X)$ with the right logarithmic derivative $\mathbf{T}_0\kappa_\mathcal{W}(\gamma)$ is the flow of γ (as a vector field).*

Proof. We proved in Theorem 4.3.11 that $\mathrm{Diff}_\mathcal{W}(X)$ is regular, hence the one-parameter subgroup $\mathcal{P}$ of $\mathrm{Diff}_\mathcal{W}(X)$ with $\delta_\rho(\mathcal{P})(t) = \mathbf{T}_0\kappa_\mathcal{W}(\gamma)$ for all $t \in \mathbb{R}$ exists. We have to show that for any $x \in X$, the curve $\mathbb{R} \to X : t \mapsto \mathcal{P}(t)(x)$ is the solution to the ODE

$$\begin{aligned} f'(t) &= \gamma(f(t)) \\ f(0) &= x. \end{aligned}$$

Obviously, $\mathcal{P}(0)(x) = \mathrm{id}_X(x) = x$. Further, $\mathcal{P}(t)(x) = (\mathrm{ev}_x \circ \kappa_\mathcal{W} \circ \kappa_\mathcal{W}^{-1} \circ \mathcal{P})(t)$. It is an easy computation to see that $\mathrm{ev}_x \circ \kappa_\mathcal{W}$ is $\mathcal{C}^1$ with

$$d(\mathrm{ev}_x \circ \kappa_\mathcal{W})(\gamma; \gamma_1) = \mathrm{ev}_x(\gamma_1).$$

By our assumptions, for $t \in \mathbb{R}$

$$\mathcal{P}'(t) = \mathbf{T}_0\kappa_\mathcal{W}(\gamma) \cdot \mathcal{P}(t) = \mathbf{T}\rho_{\mathcal{P}(t)}(\mathbf{T}_0\kappa_\mathcal{W}(\gamma)) = \mathbf{T}(\rho_{\mathcal{P}(t)} \circ \kappa_\mathcal{W})(0, \gamma).$$

So by using the last two identities and Lemma 4.3.1, we get

$$\begin{aligned} (\mathrm{ev}_x \circ \mathcal{P})'(t) &= (d(\mathrm{ev}_x \circ \kappa_\mathcal{W}) \circ \mathbf{T}\kappa_\mathcal{W}^{-1})(\mathcal{P}'(t)) \\ &= d(\mathrm{ev}_x \circ \kappa_\mathcal{W})(\kappa_\mathcal{W}^{-1}(\mathcal{P}(t)); \gamma \circ \mathcal{P}(t)) = \gamma(\mathcal{P}(t)(x)). \end{aligned}$$

This proves that the curve $\mathbb{R} \to X : t \mapsto \mathcal{P}(t)(x)$ is the integral curve of γ to the initial value x. □

5. Lie groups of weighted diffeomorphisms on Riemannian manifolds

As the title says, in this chapter we construct weighted diffeomorphisms on Riemannian manifolds, and a Lie group structure on them. To do this, we need a suitable model space of weighted vector fields and a locally convex vector space topology on it. In the first two sections of this chapter, we turn our attention to this problem; in particular, we examine under which conditions the local group operations are smooth. To this end, we need the superposition result obtained in Subsection 3.3.4, some knowledge about Riemannian manifolds presented in Section B.3, and of course the results regarding the composition and inversion maps presented in Subsection 4.1.1 and Subsection 4.2.1, respectively.

Since the local group operations are only smooth under certain conditions on the weights, we have to examine if such weight sets exists. We present our results in Subsection 5.2.3. Before the construction of weighted diffeomorphisms, we need criteria on vector fields X which assure that the map $\exp_g \circ X$ is a diffeomorphism. In Subsection 5.3.1, we derive a criterion that is astonishingly simple.

5.1. Weighted restricted products

In this section, we define and examine some kind of *simultaneously* weighted functions. As a motivation, let M be a manifold, $f : M \to \overline{\mathbb{R}}$ a weight on M and $X : M \to \mathbf{T}M$ a vector field. There is no canonical way to express what it means that X is bounded with respect to f. In contrast, for a chart κ for M we perfectly understand what it means if the function $X_\kappa = d\kappa \circ X \circ \kappa^{-1}$ is bounded with respect to the weight $f \circ \kappa^{-1}$. So we may say that X is bounded with respect to f if all its localizations (with respect to an atlas $\mathcal{A}$) are so, and define seminorms with respect to f and an order of differentiation. This leads to the definition of a topology on a subset of the product $\prod_{\kappa \in \mathcal{A}} \mathcal{C}^\infty_{\mathcal{W}_\kappa}(U_\kappa, \mathbb{R}^d)$ that is finer than the ordinary product topology.

However, we take a more general approach. First, we define such a *restricted product* for a family of locally convex spaces when there exists a set J such that each space has a set of generating seminorms that can be indexed over J, and prove some results about these kind of spaces. After that, we define *weighted restricted products*. These consist of functions that are defined on the disjoint union of open subsets of arbitrary normed spaces, and are bounded w.r.t. weights which also are defined on this union.

Of particular interest is the question of whether operations between these spaces that are defined factorwise are continuous or smooth. We will see that many maps of this type behave quite well.

5.1.1. Restricted products for locally convex spaces with uniformly parameterized seminorms

Definition 5.1.1 (Restricted products)**.** Let I and J be nonempty sets, $(E_i)_{i\in I}$ be a family of locally convex spaces such that for each $i \in I$, there exists a family $(p_{i,j})_{j\in J}$ of seminorms on E_i that defines its topology. For each $j \in J$, we define the quasinorm

$$p_j : \prod_{i\in I} E_i \to [0,\infty] : (x_i)_{i\in I} \mapsto \sup_{i\in I} p_{i,j}(x_i).$$

With these, we define

$$\ell^\infty_{\mathcal{J}}((E_i)_{i\in I}) := \{x \in \prod_{i\in I} E_i : (\forall j \in J)\, p_j(x) < \infty\}.$$

We shall use the same symbol, p_j, for the restriction of p_j to $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I})$. Endowed with the seminorms $\{p_j : j \in J\}$, the latter is a locally convex space. Note that the topology on $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I})$ is finer than the ordinary product topology, and strictly finer if $\{i \in I : E_i \neq \{0\}\}$ is infinite.

On Lipschitz continuous functions to a restricted product

Since the topology of $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I})$ generally is finer than the product topology, a map whose component maps are continuous is not necessarily continuous. But we can give a sufficient criterion for Lipschitz continuity. First, we give the following definition.

Definition 5.1.2. Let X, Y be locally convex spaces, $U \subseteq X$ open, $\phi : U \to Y$ and $p \in \mathcal{N}(Y)$, $q \in \mathcal{N}(X)$. Then we set

$$\mathrm{Lip}^p_q(\phi) := \inf\{L \in [0,\infty] : (\forall x, y \in U)\, \|\phi(x) - \phi(y)\|_p \leq L\|x - y\|_q\}.$$

If $\mathrm{Lip}^p_q(\phi) < \infty$, then $\|\phi(x) - \phi(y)\|_p \leq \mathrm{Lip}^p_q(\phi)\|x - y\|_q$ for all $x, y \in U$.

Lemma 5.1.3. *Let V be a nonempty subset of the locally convex space X. Let $A : V \to \ell^\infty_{\mathcal{J}}((E_i)_{i\in I})$ be a map such that*

$$(\forall j \in J)(\exists p^j \in \mathcal{N}(X)) \sup_{i\in I} \mathrm{Lip}^{p_{i,j}}_{p^j}(\pi_i \circ A) < \infty,$$

where for $i \in I$, $\pi_i : \prod_{j\in I} E_j \to E_i$ denotes the canonical projection. Then A is continuous. In fact, $\mathrm{Lip}^{p_j}_{p^j}(A) \leq \sup_{i\in I} \mathrm{Lip}^{p_{i,j}}_{p^j}(\pi_i \circ A)$ for each $j \in J$.

Proof. Let $x, y \in V$ and $j \in J$. We have

$$\|A(x) - A(y)\|_{p_j} = \sup_{i\in I}\|\pi_i(A(x)) - \pi_i(A(y))\|_{p_{i,j}} \leq \sup_{i\in I} \mathrm{Lip}^{p_{i,j}}_{p^j}(\pi_i \circ A)\|x - y\|_{p^j}.$$

This finishes the proof. □

On the product of restricted products

We turn to the product $\ell^\infty_{\mathcal{J}_E}((E_i)_{i\in I}) \times \ell^\infty_{\mathcal{J}_F}((F_i)_{i\in I})$ of two restricted products. If the seminorms of both spaces are indexed over the same set, it is isomorphic to another restricted product. As a preparation, we make the following remark.

Remark 5.1.4. For the following, note that if the locally convex spaces E and F both have a generating family $(p^E_j)_{j\in J}$ and $(p^F_j)_{j\in J}$ of seminorms indexed over J, then there exists a generating family of seminorms for $E \times F$ that is indexed over J. For example, the family $(\max\circ(p^E_j \times p^F_j))_{j\in J}$ generates the product topology on $E \times F$.

Lemma 5.1.5. *The sets $\ell^\infty_{\mathcal{J}}((E_i \times F_i)_{i\in I})$ and $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I}) \times \ell^\infty_{\mathcal{J}}((F_i)_{i\in I})$ are isomorphic as topological vector spaces. The canonical isomorphism is the map*

$$\ell^\infty_{\mathcal{J}}((E_i \times F_i)_{i\in I}) \to \ell^\infty_{\mathcal{J}}((E_i)_{i\in I}) \times \ell^\infty_{\mathcal{J}}((F_i)_{i\in I}) : (e_i, f_i)_{i\in I} \mapsto ((e_i)_{i\in I}, (f_i)_{i\in I}),$$

and

$$\ell^\infty_{\mathcal{J}}((E_i)_{i\in I}) \times \ell^\infty_{\mathcal{J}}((F_i)_{i\in I}) \to \ell^\infty_{\mathcal{J}}((E_i \times F_i)_{i\in I}) : ((e_i)_{i\in I}, (f_i)_{i\in I}) \mapsto (e_i, f_i)_{i\in I}$$

its inverse.

Proof. We denote the maps defined above by A and B, respectively. Let $j \in J$ and $k \in I$. Then

$$p^E_{k,j}((\pi_k\circ \mathrm{pr}_1\circ A)(e_i, f_i)_{i\in I}) = p^E_{k,j}(e_k) \le \max(p^E_{k,j}(e_k), p^F_{k,j}(f_k)) \le \max(p^E_j \times p^F_j)(e_i, f_i)_{i\in I},$$

independent of k. This shows that $\mathrm{pr}_1 \circ A$ takes values in $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I})$, and since it is linear, we can use Lemma 5.1.3 to see that it is continuous to this space. Since the same argument can be made for the second factor, we see that A is continuous.

On the other hand, we have that

$$\begin{aligned}\max\circ(p^E_{k,j} \times p^F_{k,j})((\pi_k \circ B)((e_i)_{i\in I}, (f_i)_{i\in I})) &= \max(p^E_{k,j}(e_k), p^F_{k,j}(f_k)) \\ &\le p^E_{k,j}(e_k) + p^F_{k,j}(f_k) \le p^E_j(e_i)_{i\in I} + p^F_j(f_i)_{i\in I}.\end{aligned}$$

Since $p^E_j \circ \mathrm{pr}_1 + p^F_j \circ \mathrm{pr}_2$ is a continuous seminorm on $\ell^\infty_{\mathcal{J}}((E_i)_{i\in I}) \times \ell^\infty_{\mathcal{J}}((F_i)_{i\in I})$, this shows that B takes values in $\ell^\infty_{\mathcal{J}}((E_i \times F_i)_{i\in I})$, and since it is linear, we can use Lemma 5.1.3 to see that it is continuous to this space. Nw clearly $B = A^{-1}$. □

On differentiable functions into a restricted product

We give a criterion when a function into a restricted product whose component maps are $\mathcal{C}^1$ is differentiable itself. In order to do this, we give a sufficient condition for the completeness of a restricted product.

Completeness of a restricted product We prove that a restricted product is complete if all factors are so.

Lemma 5.1.6 (Completeness)**.** *Let I and J be nonempty sets, $(E_i)_{i\in I}$ be a family of locally convex spaces and $(p_{i,j})_{j\in J}$ a family of generating seminorms for E_i, for $i \in I$. Further assume that each E_i is complete. Then $\ell^\infty_J((E_i)_{i\in I})$ is complete.*

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a Cauchy net in $\ell^\infty_J((E_i)_{i\in I})$. Then for each $i \in I$, obviously $(\pi_i(x_\alpha))_{\alpha\in A}$ is a Cauchy net in E_i, and since E_i is complete, it converges to some $x_i \in E_i$. We show that $(x_i)_{i\in I} \in \ell^\infty_J((E_i)_{i\in I})$ and that $(x_\alpha)_{\alpha\in A}$ converges to $(x_i)_{i\in I}$. To this end, let $j \in J$. Since $(x_\alpha)_{\alpha\in A}$ is a Cauchy net, for each $\varepsilon > 0$ there exists $\ell \in A$ such that

$$(\forall \alpha, \beta \in A : \alpha, \beta \geq \ell) \sup_{i\in I} \|\pi_i(x_\alpha) - \pi_i(x_\beta)\|_{p_{i,j}} < \varepsilon.$$

We fix α in this estimate, and for each $i \in I$, we take $\pi_i(x_\beta)$ to its limit. Then we get that

$$(\forall \alpha \in A : \alpha \geq \ell) \sup_{i\in I} \|\pi_i(x_\alpha) - x_i\|_{p_{i,j}} \leq \varepsilon.$$

Hence

$$\|(x_i)_{i\in I}\|_{p_j} \leq \|x_\ell\|_{p_j} + \|(x_i)_{i\in I} - x_\ell\|_{p_j} < \infty$$

and thus $(x_i)_{i\in I} \in \ell^\infty_J((E_i)_{i\in I})$. Since $\varepsilon > 0$ was arbitrary, we also see that $(x_\alpha)_{\alpha\in A}$ converges to $(x_i)_{i\in I}$. □

Differentiability criterion The criterion we present is quite useful. The reason for this is that often, we can compute the differentials in terms of the map itself and some well-behaved operations.

Lemma 5.1.7. *Let U be an open nonempty subset of the locally convex space E, I and J nonempty sets, $(F_i)_{i\in I}$ a family of locally convex spaces whose topologies are generated by families of seminorms indexed over J. Let $f : U \to \ell^\infty_J((F_i)_{i\in I})$ be a map such that each component map $f_i : U \to F_i$ is $\mathcal{C}^1$ and the map*

$$(df_i)_{i\in I} : U \times E \to \ell^\infty_J((F_i)_{i\in I})$$

is defined and continuous. Then f is $\mathcal{C}^1$.

Proof. Let $x \in U$ and $h \in E$. Choose $\varepsilon > 0$ so small that $x + B_{\mathbb{K}}(0, \varepsilon)h \subseteq U$. By our assumptions, the map

$$B_{\mathbb{K}}(0, \varepsilon) \times [0, 1] \to \ell^\infty_J((F_i)_{i\in I}) : (t, s) \mapsto (df_i(x + sth; h))_{i\in I}$$

is continuous. Hence we see with Lemma 5.1.6 that for each $t \in B_{\mathbb{K}}(0, \varepsilon)$, $\int_0^1 (df_i(x + sth; h))_{i\in I} \, ds$ exists in $\ell^\infty_J((\widetilde{F_i})_{i\in I})$, where $\widetilde{F_i}$ denotes the completion of F_i. Using the mean value theorem, we conclude that the integral exists in $\ell^\infty_J((F_i)_{i\in I})$ with the value $\frac{1}{t}(f(x + th) - f(x))$, if $t \neq 0$. Hence we see with the continuity of parameter-dependent integrals (Proposition A.1.8) that f is $\mathcal{C}^1$ with $df(x; h) = (df_i(x; h))_{i\in I}$. □

Remark 5.1.8. A similar assertion as in the previous lemma should hold if each component map $f_i : U \to F_i$ is $\mathcal{C}^k$ and the maps

$$(d^{(\ell)} f_i)_{i\in I} : U \times E^\ell \to \ell^\infty_J((F_i)_{i\in I})$$

are defined and continuous for each $\ell \in \mathbb{N}$ with $\ell \leq k$.

On the product of multilinear maps

The last result about the general restricted products is about the continuity of a product of multilinear maps. It assures the continuity if the factors maps are kind of "uniformly bounded" for each generating seminorm of the restricted product.

Lemma 5.1.9 (Multilinear maps). *Let I and J be nonempty sets, $m \in \mathbb{N}$, $E_1, \dots, E_m$ be locally convex spaces and $(F_i)_{i\in I}$ a family of locally convex spaces such that the topology of each F_i is generated by a family $(p_{i,j})_{j\in J}$ of seminorms. Further, for each $i \in I$ let $\beta_i : E_1 \times \dots \times E_m \to F_i$ be an m-linear map such that*

$$\begin{aligned}&(\forall j \in J)(\exists p_1 \in \mathcal{N}(E_1), \dots, p_m \in \mathcal{N}(E_m), C > 0)\\ &\qquad(\forall i \in I, x_1 \in E_1, \dots, x_m \in E_m)\ \|\beta_i(x_1, \dots, x_m)\|_{p_{i,j}} \leq C\|x_1\|_{p_1} \cdots \|x_m\|_{p_m}.\end{aligned} \tag{†}$$

Then the map

$$(\beta_i)_{i\in I} : E_1 \times \ldots \times E_m \to \ell^\infty_J((F_i)_{i\in I})$$

is defined, m-linear and continuous.

Proof. We conclude from (†) that for $j \in J$ and $x_1 \in E_1, \dots, x_m \in E_m$,

$$\|(\beta_i(x_1, \dots, x_m))_{i\in I}\|_{p_j} \leq C\|x_1\|_{p_1} \cdots \|x_m\|_{p_m}.$$

From this estimate, we conclude that $(\beta_i(x_1, \dots, x_m))_{i\in I} \in \ell^\infty_J((F_i)_{i\in I})$. Further, since $(\beta_i)_{i\in I}$ is obviously m-linear, we see that it is continuous in 0 and hence continuous. □

5.1.2. Restricted products of weighted functions

We now turn our attention to special restricted products, where each factor is a weighted function space of the kind examined in Chapter 3. Since we know the topology of these spaces and plenty of operations on and between them very well, we are able to derive more results about them than in the general case. We give the definition and then adapt some previous results about the topological and uniform structure.

Definition, topological and uniform structure

Definition 5.1.10. Let I be a nonempty set, $(U_i)_{i\in I}$ a family such that each U_i is an open nonempty set of a normed space X_i, $(Y_i)_{i\in I}$ another family of normed spaces, $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I} U_i}$ a nonempty family of weights defined on the disjoint union $\cup_{i\in I} U_i$ of $(U_i)_{i\in I}$, and $k \in \overline{\mathbb{N}}$. For $i \in I$ and $f \in \mathcal{W}$, we set $f_i := f|_{U_i}$, and further $\mathcal{W}_i := \{f_i : f \in \mathcal{W}\}$. Then the topology of each space $\mathcal{C}^k_{\mathcal{W}_i}(U_i, Y_i)$ is induced by a family of seminorms indexed over $\mathcal{W} \times \{\ell \in \mathbb{N} : \ell \leq k\}$; for $i \in I$, we map $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$ to $\|\cdot\|_{f_i,\ell}$. We define

$$\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I} := \ell^\infty_{\{\|\cdot\|_{f,\ell} : (f,\ell)\in\mathcal{W}\times\{n\in\mathbb{N}:\, n\leq k\}\}}((\mathcal{C}^k_{\mathcal{W}_i}(U_i, Y_i))_{i\in I}).$$

The seminorms that generate the topology on this space are of the form

$$\|(\phi_i)_{i\in I}\|_{f,\ell} := \sup_{i\in I} \|\phi_i\|_{f_i,\ell},$$

where $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$.

Lemma 5.1.11. *$\mathcal{C}^\infty_{\mathcal{W}}(U_i, Y_i)_{i\in I}$ is endowed with the initial topology of the inclusion maps*

$$\mathcal{C}^\infty_{\mathcal{W}}(U_i, Y_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I},$$

for $k \in \mathbb{N}$. Moreover, $\mathcal{C}^\infty_{\mathcal{W}}(U_i, Y_i)_{i\in I} = \varprojlim_{k\in\mathbb{N}} \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$.

Proof. This is clear from the fact that the seminorms $\|\cdot\|_{f,\ell}$ with $f \in \mathcal{W}$ and $\ell \leq k$ define the topology on the right hand side, while those with $\ell \in \mathbb{N}$ define the topology on the left. □

Proposition 5.1.12. *Let $k \in \mathbb{N}$. Then for $(\phi_i)_{i\in I} \in \prod_{i\in I} \mathcal{FC}^1(U_i, Y_i)$, we have*

$$(\phi_i)_{i\in I} \in \mathcal{C}^{k+1}_{\mathcal{W}}(U_i, Y_i)_{i\in I} \iff (\phi_i)_{i\in I} \in \mathcal{C}^0_{\mathcal{W}}(U_i, Y_i)_{i\in I} \text{ and } (D\phi_i)_{i\in I} \in \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Y_i))_{i\in I}.$$

The map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U_i, Y_i)_{i\in I} \to \mathcal{C}^0_{\mathcal{W}}(U_i, Y_i)_{i\in I} \times \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Y_i))_{i\in I} : ((\phi_i)_{i\in I}) \mapsto ((\phi_i)_{i\in I}, (D\phi_i)_{i\in I})$$

is linear and a topological embedding.

Proof. This is proved in the same way as Proposition 3.2.3 and is a direct consequence of Lemma 3.2.2. □

Lipschitz continuity This is an adaptation of Lemma 5.1.3.

Lemma 5.1.13. *Let V be an open nonempty subset of the locally convex space X. Let $A : V \to \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$ be a map such that*

$$(\forall f \in \mathcal{W}, \ell \in \mathbb{N} : \ell \leq k)(\exists p \in \mathcal{N}(X)) \sup_{i\in I} \mathrm{Lip}^{f_i,\ell}_p(\pi_i \circ A) < \infty.$$

Then A is continuous. In fact, $\mathrm{Lip}^{f,\ell}_p(A) \leq \sup_{i\in I} \mathrm{Lip}^{f_i,\ell}_p(\pi_i \circ A)$.

Proof. This follows from Lemma 5.1.3. □

Adjusting weights and open subsets

Let I be an infinite set and $(r_i)_{i\in I}$ a family of positive real numbers such that $\inf_{i\in I} r_i = 0$. If $\mathcal{W}$ consists only of $1_{\cup_{i\in I}U_i}$, then the set $\prod_{i\in I}\mathcal{C}^0_{\mathcal{W}_i}(U_i, B_{Y_i}(0,r_i))$ is not a neighborhood of 0 in $\mathcal{C}^0_{\mathcal{W}}(U_i,Y_i)_{i\in I}$. But since we later need to discuss such sets, and in particular want functions that are defined on such sets to be differentiable (think of the Riemannian exponential function), we must know under which conditions on $\mathcal{W}$ their interior is not empty.

It turns out that if $\mathcal{W}$ contains a weight ω that is “large enough” on each U_i, then the set $\{(\phi_i)_{i\in I}\in\mathcal{C}^0_{\mathcal{W}}(U_i,Y_i)_{i\in I} : \|(\phi_i)_{i\in I}\|_{\omega,0} < 1\}$ is contained in $\prod_{i\in I}\mathcal{C}^0_{\mathcal{W}_i}(U_i, B_{Y_i}(0,r_i))\cap\mathcal{C}^0_{\mathcal{W}}(U_i,Y_i)_{i\in I}$, so the latter is a neighborhood of 0. We will call ω adjusting to the family $(r_i)_{i\in I}$ since ω adjusts its smallness. We start with some definitions.

Definition 5.1.14. Let $(U_i)_{i\in I}$ and $(r_i)_{i\in I}$ be families such that each U_i is an open nonempty set of the normed space X_i, and each $r_i\in]0,\infty]$. We say that $\omega : \cup_{i\in I}U_i\to\mathbb{R}$ is an *adjusting weight for* $(r_i)_{i\in I}$ if for each $i\in I$, we have that

$$\sup_{x\in U_i}|\omega_i(x)| < \infty \qquad\text{and}\qquad \inf_{x\in U_i}|\omega_i(x)|\geq\max(\tfrac{1}{r_i},1).$$

Notice that generally, ω itself is *not* bounded.

Definition 5.1.15. Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty set of the normed space X_i and each V_i is an open nonempty subset of a normed space Y_i, $\mathcal{W}\subseteq\overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ a nonempty set and $k\in\overline{\mathbb{N}}$. Let $\omega : \cup_{i\in I}U_i\to\mathbb{R}$ with $0\notin\omega(\cup_{i\in I}U_i)$. We set

$$\begin{aligned}&\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}\\ &\quad:= \{(\gamma_i)_{i\in I}\in\mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I} : (\exists r>0)(\forall i\in I, x\in U_i)\,\gamma_i(x)+B_{Y_i}(0,\tfrac{r}{|\omega(x)|})\subseteq V_i\}.\end{aligned}$$

In particular, we define

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I} := \mathcal{C}^{(1_{\cup_{i\in I}U_i})_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}.$$

Additionally, if each V_i is star-shaped with center 0, then ω is called an *adjusting weight for* $(V_i)_{i\in I}$ if it is an adjusting weight for $(\operatorname{dist}(\{0\},\partial V_i))_{i\in I}$. If it is clear to which family ω adjusts, we may call ω just an adjusting weight.

Remark 5.1.16. Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that all U_i and V_i are open nonempty subsets of the normed spaces X_i respectively Y_i, $\mathcal{W}\subseteq\overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ a nonempty set, $k\in\overline{\mathbb{N}}$ and $\omega : \cup_{i\in I}U_i\to\mathbb{R}$ with $0\notin\omega(\cup_{i\in I}U_i)$ such that $\sup_{x\in U_i}|\omega_i(x)|<\infty$ for each $i\in I$. Then $\inf_{x\in U_i}\frac{1}{|\omega_i|(x)}>0$, and hence

$$\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}\subseteq\prod_{i\in I}\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i,V_i).$$

To show that $\prod_{i\in I}\mathcal{C}^0_{\mathcal{W}_i}(U_i,B_{Y_i}(0,r_i))$ contains a neighborhood of the constant 0 function, we estimate the $\|\cdot\|_{1_U,0}$ seminorm with the $\|\cdot\|_{f,0}$ seminorm.

Lemma 5.1.17. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $f : U \to \overline{\mathbb{R}}$ such that $0 \notin f(U)$ and $\phi, \psi : U \to Y$.*

(a) For all $x \in U$, we have $\|\phi(x) - \psi(x)\| \leq \frac{\|\phi-\psi\|_{f,0}}{|f(x)|}$.

(b) Assume that $\inf_{x\in U}|f(x)| > 0$. Then $\|\phi - \psi\|_{1_U,0} \leq \frac{\|\phi-\psi\|_{f,0}}{\inf_{x\in U}|f(x)|}$.

(c) Suppose that $\inf_{x\in U}|f(x)| \geq \max(\frac{1}{d}, 1)$, where $d > 0$. Then

$$\|\phi - \psi\|_{1_U,0} \leq \min(d, 1)\|\phi - \psi\|_{f,0}. \tag{5.1.17.1}$$

Proof. (a) This follows from $|f(x)|\,\|\phi(x) - \psi(x)\| \leq \|\phi - \psi\|_{f,0}$.
(b) This is an easy consequence of (a).
(c) This follows from (b), where we use that $\frac{1}{\max(\frac{1}{d},1)} = \min(d, 1)$. □

Lemma 5.1.18. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty set of a normed space X_i and each V_i is an open nonempty subset of a normed space Y_i, $k \in \overline{\mathbb{N}}$, $f : \cup_{i\in I}U_i \to \mathbb{R}$ with $0 \notin f(\cup_{i\in I}U_i)$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ with $f \in \mathcal{W}$.*

(a) $\mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}$ is open in $\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$. In fact, it is even open in $\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$ when this space is endowed with the topology of $\mathcal{C}^0_{\{f\}}(U_i, Y_i)_{i\in I}$.

(b) Assume that each V_i is star-shaped with center 0 and f is an adjusting weight for $(V_i)_{i\in I}$. Then $\mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}$ is not empty. In particular, for $\tau > 0$ we have

$$\{\eta \in \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I} : \|\eta\|_{f,0} < \tau\} \subseteq \mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, \tau \cdot V_i)_{i\in I}. \tag{5.1.18.1}$$

Proof. (a) Let $\gamma \in \mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}$. Then there exists $r > 0$ such that

$$(\forall i \in I, x \in U_i)\, \gamma_i(x) + B_{Y_i}(0, \tfrac{r}{|f(x)|}) \subseteq V_i.$$

We show that

$$\{\eta \in \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I} : \|\eta - \gamma\|_{f,0} < r\} \subseteq \mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}.$$

To this end, let η be an element of set on the left hand side and $s := r - \|\eta - \gamma\|_{f,0}$. Then for $i \in I$, $x \in U_i$ and $h \in B_{Y_i}(0, \frac{s}{|f(x)|})$, we have with Lemma 5.1.17 and the triangle inequality

$$\|\eta_i(x) - \gamma_i(x) + h\| < \frac{\|\gamma - \eta\|_{f,0}}{|f(x)|} + \frac{s}{|f(x)|} = \frac{r}{|f(x)|}.$$

Hence

$$\eta_i(x) + h = \gamma_i(x) + \eta_i(x) - \gamma_i(x) + h \in V_i.$$

This shows that $\eta \in \mathcal{C}^{f_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}$.

(b) Let η be an element of the set on the left hand side of (5.1.18.1). We set $r := \tau - \|\eta\|_{f,0}$. Let $i \in I$, $x \in U_i$ and $h \in B_{Y_i}(0, \frac{r}{|f(x)|})$. Then we see with (5.1.17.1) that

$$\|\eta_i(x) + h\| \leq \|\eta_i(x)\| + \|h\| < \min(1, d_i)\|\eta\|_{f,0} + \min(1, d_i)(\tau - \|\eta\|_{f,0}),$$

where $d_i := \operatorname{dist}(\{0\}, \partial V_i)$. Hence $\|\eta_i(x) + h\| < \tau d_i$, so $\eta_i(x) + h \in \tau \cdot V_i$. This finishes the proof. □

Remark 5.1.19. Let $(U_i)_{i\in I}$ be a family such that each U_i is an open nonempty set of the normed space X_i. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I} U_i}$ contain ω with $\inf_{x\in U}|\omega(x)| > 0$ (in particular, this holds if ω is an adjusting weight) and $k \in \overline{\mathbb{N}}$. Then for each $\ell \in \mathbb{N}$ with $\ell \leq k$, we see with Lemma 5.1.17 that the seminorm $\|\cdot\|_{1_{\cup_{i\in I} U_i},\ell}$ is continuous on $\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$. In particular, $\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I} = \mathcal{C}^k_{\mathcal{W}\cup\{1_{\cup_{i\in I} U_i}\}}(U_i, Y_i)_{i\in I}$.

5.1.3. Simultaneous superposition and multiplication

In this subsection, we discuss operations between restricted products of weighted functions that consist of operations that are defined on a single factor. The most common operation is the superposition with a family $(\phi_i)_{i\in I}$ of maps of certain characteristics, i.e. linear, analytic etc. In contrast to former results, we often have to take a more quantitative approach, and tailor our assumptions about the permitted weights to $(\phi_i)_{i\in I}$.

Simultaneous multiplication

We begin with simultaneous multiplication. It is pretty straightforward, and (5.1.20.1) provides a good example of the assumptions on the weights that will be made in the following.

Lemma 5.1.20. *Let $(U_i)_{i\in I}$ be a family such that each U_i is an open nonempty set of the normed space X_i, and $(Y_i^1)_{i\in I}$, $(Y_i^2)_{i\in I}$, $(Z_i)_{i\in I}$ be families of normed spaces. Further, for each $i \in I$ let $M_i : U_i \to Y_i^1$ be smooth, and $\beta_i : Y_i^1 \times Y_i^2 \to Z_i$ a bilinear map such that*

$$\sup\{\|\beta_i\|_{op} : i \in I\} < \infty.$$

Assume that $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I} U_i}$ is nonempty and

$$(\forall f \in \mathcal{W}, \ell \in \mathbb{N})(\exists g \in \mathcal{W}_{\max})\,(\forall i \in I)\,\|M_i\|_{1_{U_i},\ell}|f_i| \leq |g_i|. \tag{5.1.20.1}$$

Then for $k \in \overline{\mathbb{N}}$, the map

$$\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i^2)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, Z_i)_{i\in I} : (\gamma_i)_{i\in I} \mapsto (\beta_i \circ (M_i, \gamma_i))_{i\in I}$$

is defined and continuous linear.

Proof. We prove this by induction on k.

$k = 0$: We calculate for $i \in I$, $x \in U_i$, $(\gamma_i)_{i\in I} \in \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i^2)_{i\in I}$ and $f \in \mathcal{W}$ that

$$|f_i(x)| \, \|(\beta_i \circ (M_i, \gamma_i))(x)\| \leq \|\beta_i\|_{op} \, |f_i(x)| \, \|M_i(x)\| \, \|\gamma_i(x)\| \leq \|\beta_i\|_{op} \, \|\gamma_i\|_{g_i,0}.$$

Hence

$$\|(\beta_i \circ (M_i, \gamma_i))_{i\in I}\|_{f,0} \leq \sup_{i\in I} \|\beta_i\|_{op} \, \|(\gamma_i)_{i\in I}\|_{g,0},$$

which shows the assertion.

$k \to k+1$: Using the induction base and Proposition 5.1.12, all we have to show is that for $(\gamma_i)_{i\in I} \in \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i^2)_{i\in I}$, we have $(D(b_i \circ (M_i, \gamma_i)))_{i\in I} \in \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Z_i))_{i\in I}$ and that the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U_i, Y_i^2)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Z_i))_{i\in I} : (\gamma_i)_{i\in I} \mapsto (D(b_i \circ (M_i, \gamma_i)))_{i\in I}$$

is continuous. By Lemma 3.3.2, for each $i \in I$ we have

$$D(\beta_i \circ (M_i, \gamma_i)) = \beta_i^{(1)} \circ (DM_i, \gamma) + \beta_i^{(2)} \circ (M_i, D\gamma_i)$$

(using notation as in Definition 3.3.1). Hence

$$(D(\beta_i \circ (M_i, \gamma_i)))_{i\in I} = (\beta_i^{(1)} \circ (DM_i, \gamma))_{i\in I} + (\beta_i^{(2)} \circ (M_i, D\gamma_i))_{i\in I},$$

and we easily calculate that $\|\beta_i^{(1)}\|_{op}, \|\beta_i^{(2)}\|_{op} \leq \|\beta_i\|_{op}$ for each $i \in I$. Since $\mathcal{W}$ and $(DM_i)_{i\in I}$ satisfy (5.1.20.1), we can apply the inductive hypothesis to both summands and finish the proof. □

Remark 5.1.21. The assertion of Lemma 6.1.6 is similar to the one of Lemma 5.1.20. There, we call maps $(M_i)_i$ for which (5.1.20.1) is satisfied (when $\#I = 1$) *multipliers*.

Simultaneous superposition with multilinear maps

Here, we examine the superpositions with multilinear maps that are uniformly bounded. It is very similar to Proposition 3.3.3, but also involves a result for the more general restricted products defined above.

Lemma 5.1.22. *Let I be a nonempty set, $(X_i)_{i\in I}$, $(X_{i,k})_{(i,k)\in I\times\{1,\dots,n\}}$ and $(Y_i)_{i\in I}$ families of normed spaces, and $U_i \subseteq X_i$ an open nonempty subset for each $i \in I$. Let $\mathcal{W}_1, \dots, \mathcal{W}_n, \mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} U_i}$ be nonempty sets such that*

$$(\forall f \in \mathcal{W})(\exists g^{f,1} \in \mathcal{W}_1, \dots, g^{f,n} \in \mathcal{W}_n)(\forall i \in I) \, |f_i| \leq |g_i^{f,1}| \cdots |g_i^{f,n}|.$$

Further, for each $i \in I$, let $\beta_i : X_{i,1} \times \cdots \times X_{i,n} \to Y_i$ be a continuous n-linear map such that the set

$$\{\|\beta_i\|_{op} : i \in I\}$$

is bounded. Then the map

$$\begin{aligned}\beta : \mathcal{C}^k_{\mathcal{W}_1}(U_i, X_{i,1})_{i\in I} \times \cdots \times \; & \mathcal{C}^k_{\mathcal{W}_n}(U_i, X_{i,n})_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I} \\ & (\gamma_{i,1}, \dots, \gamma_{i,n})_{i\in I} \mapsto (\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n}))_{i\in I}\end{aligned}$$

is defined, n-linear and continuous.

Proof. Using Proposition 3.3.3, we have for each $i \in I$ and $\gamma_{i,1} \in \mathcal{C}^k_{\mathcal{W}}(U_i, X_{i,1})$, $\dots$, $\gamma_{i,n} \in \mathcal{C}^k_{\mathcal{W}}(U_i, X_{i,n})$ that $\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n}) \in \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)$. Further, β is n-linear as map to $\prod_{i\in I} \mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)$. We prove by induction on k that β takes values in $\mathcal{C}^k_{\mathcal{W}}(U_i, Y_i)_{i\in I}$ and is continuous.

$k = 0$: We compute for all $i \in I$, $f \in \mathcal{W}_i$ and $\gamma_{i,1} \in \mathcal{C}^k_{\mathcal{W}_1}(U_i, X_{i,1})$, $\dots$, $\gamma_{i,n} \in \mathcal{C}^k_{\mathcal{W}_n}(U_i, X_{i,n})$ that

$$\|\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n})\|_{f,0} \leq \|\beta_i\|_{op} \prod_{j=1}^{n} \|\gamma_{i,j}\|_{g_i^{f,j},0}.$$

Since i was arbitrary, we can apply Lemma 5.1.9 to derive the assertion.

$k \to k+1$: Using the induction base and Proposition 5.1.12, all we have to show is that for $(\gamma_{i,1})_{i\in I} \in \mathcal{C}^{k+1}_{\mathcal{W}_1}(U_i, X_{i,1})_{i\in I}$, $\dots$, $(\gamma_{i,n})_{i\in I} \in \mathcal{C}^{k+1}_{\mathcal{W}_n}(U_i, X_{i,n})_{i\in I}$,

$$(D(\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n})))_{i\in I} \in \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Y_i))_{i\in I},$$

and that the map

$$\begin{aligned} \mathcal{C}^{k+1}_{\mathcal{W}_1}(U_i, X_{i,1})_{i\in I} \times \dots \times \mathcal{C}^{k+1}_{\mathcal{W}_n}(U_i, X_{i,n})_{i\in I} &\to \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(X_i, Y_i))_{i\in I} \\ (\gamma_{i,1}, \dots, \gamma_{i,n})_{i\in I} &\mapsto (D(\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n})))_{i\in I} \end{aligned}$$

is continuous. By Lemma 3.3.2, for each $i \in I$ we have

$$D(\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n})) = \sum_{j=1}^{n} \beta_i^{(j)} \circ (\gamma_{i,1}, \dots, D\gamma_{i,j}, \dots, \gamma_{i,n})$$

(using notation as in Definition 3.3.1) and hence

$$(D(\beta_i \circ (\gamma_{i,1}, \dots, \gamma_{i,n})))_{i\in I} = \sum_{j=1}^{n} (\beta_i^{(j)} \circ (\gamma_{i,1}, \dots, D\gamma_{i,j}, \dots, \gamma_{i,n}))_{i\in I}.$$

Since we easily calculate that $\|\beta_i^{(j)}\|_{op} \leq \|\beta_i\|_{op}$ for each $i \in I$ and $j \in \{1, \dots, n\}$, we can apply the inductive hypothesis to each summand and get the assertion. □

Simultaneous superposition with differentiable maps

We provide the simultaneous analogue of Proposition 3.3.26. In the proof, we have to use notation introduced in Lemma 3.3.25, as we did in the proof of 3.3.26. Similarly, the technically most challenging part will be the examination of the superposition with $((\beta_i)^{(2)}_{M_i})_{i\in I}$. Another novelty is the use of adjusting weights.

Proposition 5.1.23. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty set of the normed space X_i and each V_i is an open, star-shaped subset with center 0 of a normed space Y_i. Further, let $(Z_i)_{i\in I}$ be another family of normed spaces* *and* $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ *contain an adjusting weight* ω*. For each* $i \in I$*, let* $\beta_i \in \mathcal{FC}^\infty(U_i \times V_i, Z_i)$ *be a map such that* $\beta_i(U_i \times \{0\}) = \{0\}$*. Further, assume that*

$$(\forall f \in \mathcal{W}, \ell \in \mathbb{N}^*)(\exists g \in \mathcal{W}_{\max})\,(\forall i \in I)\,\|\beta_i\|_{1_{U_i\times V_i},\ell}|f_i| \leq |g_i| \tag{5.1.23.1}$$

is satisfied. Then for $k \in \overline{\mathbb{N}}$*, the map*

$$\beta_* := \prod_{i\in I}(\beta_i)_* : \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i,Z_i)_{i\in I} : (\gamma_i)_{i\in I} \mapsto (\beta_i \circ (\mathrm{id}_{U_i},\gamma_i))_{i\in I}$$

is defined and smooth.

Proof. We see with Proposition 3.3.26 (and Remark 5.1.16) that β_* is defined as a map to $\prod_{i\in I}\mathcal{C}^k_{\mathcal{W}}(U_i,Z_i)$. We first prove by induction on k that β_* takes its values in $\mathcal{C}^k_{\mathcal{W}}(U_i,Z_i)_{i\in I}$ and is continuous.

$k = 0$: Let $f \in \mathcal{W}$. Using (3.3.26.2), we see that for $\gamma \in \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}$ and $i \in I$

$$\|\beta_i \circ (\mathrm{id}_{U_i},\gamma_i)\|_{f_i,0} \leq \|D_2\beta_i\|_{1_{U_i\times V_i},0}\|\gamma_i\|_{f_i,0}.$$

Since $\|D_2\beta_i\|_{1_{U_i\times V_i},0} \leq \|\beta_i\|_{1_{U_i\times V_i},1}$, there exists $g \in \mathcal{W}_{\max}$ such that

$$\|(\beta_i \circ (\mathrm{id}_{U_i},\gamma_i))_{i\in I}\|_{f_i,0} \leq \|\gamma\|_{g_i,0}.$$

Hence

$$(\beta_i \circ (\mathrm{id}_{U_i},\gamma_i))_{i\in I} \in \mathcal{C}^0_{\mathcal{W}}(U_i,Z_i)_{i\in I}.$$

With the same reasoning, we see with (3.3.26.1) that for $\eta \in \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}$ in some neighborhood of γ,

$$\|(\beta_i \circ (\mathrm{id}_{U_i},\gamma_i) - \beta_i \circ (\mathrm{id}_{U_i},\eta_i))_{i\in I}\|_{f,0} \leq \|\gamma - \eta\|_{g,0}.$$

So by Lemma 5.1.13, β_* is locally Lipschitz continuous and hence continuous.

$k \to k+1$: We use Proposition 5.1.12. For $(\gamma_i)_{i\in I} \in \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}$, we have by Proposition 3.3.26 using notation from Lemma 3.3.25

$$(D(\beta_i \circ (\mathrm{id}_{U_i},\gamma_i)))_{i\in I} = (D_1\beta_i \circ (\mathrm{id}_{U_i},\gamma_i))_{i\in I} + ((\beta_i)^{(2)}_{M_i} \circ (\mathrm{id}_{U_i},\gamma_i,D\gamma_i))_{i\in I}.$$

(Here, M_i denotes the composition of linear operators). For $i \in I$ and $\ell \in \mathbb{N}^*$,

$$\|D_1\beta_i\|_{1_{U_i\times V_i},\ell} \leq \|\beta_i\|_{1_{U_i\times V_i},\ell+1},$$

and from (3.3.25.1) we get that

$$\|(\beta_i)^{(2)}_{M_i}\|_{1_{U_i\times V_i\times B_{\mathrm{L}(X_i,Y_i)}(0,R)},\ell} \leq \ell\|\beta_i\|_{1_{U_i\times V_i},\ell} + R\|\beta_i\|_{1_{U_i\times V_i},\ell+1}$$

for each $R > 0$. Hence we can apply the inductive hypothesis to see that the maps

$$\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i,\mathrm{L}(X_i,Z_i))_{i\in I} : (\gamma_i)_{i\in I} \mapsto (D_1\beta_i \circ (\mathrm{id}_{U_i},\gamma_i))_{i\in I}$$

and for $R \geq 1$

$$\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i, V_i \times B_{\mathrm{L}(X_i,Y_i)}(0,R))_{i\in I} \to \mathcal{C}^{k}_{\mathcal{W}}(U_i, \mathrm{L}(X_i,Z_i))_{i\in I} : (\gamma_i)_{i\in I} \mapsto ((\beta_i)^{(2)}_M \circ (\mathrm{id}_{U_i}, \gamma_i))_{i\in I}$$

are continuous; here we used that ω is an adjusting weight for $(V_i \times B_{\mathrm{L}(X_i,Y_i)}(0,R))_{i\in I}$ when the product is endowed with the maximum norm of the factor products (and also for $(B_{\mathrm{L}(X_i,Y_i)}(0,R))_{i\in I}$) if $R \geq 1$. From the continuity of the latter map, we deduce using Lemma 3.4.16, Lemma 5.1.22 and Lemma 5.1.5 that

$$\begin{aligned}\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I} \times \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i, B_{\mathrm{L}(X_i,Y_i)}(0,R))_{i\in I} &\to \mathcal{C}^{k}_{\mathcal{W}}(U_i, \mathrm{L}(X_i,Z_i))_{i\in I}\\ ((\gamma_i)_{i\in I}, (\Gamma_i)_{i\in I}) &\mapsto ((\beta_i)^{(2)}_M \circ (\mathrm{id}_{U_i}, \gamma_i, \Gamma_i))_{i\in I}\end{aligned}$$

is continuous. Hence for each $\gamma \in \mathcal{C}^{\omega_\partial,k+1}_{\mathcal{W}}(U_i, V_i)_{i\in I}$, the map

$$\begin{aligned}\{\eta \in \mathcal{C}^{\omega_\partial,k+1}_{\mathcal{W}}(U_i, V_i)_{i\in I} : \|\eta\|_{1_{\cup_{i\in I}U_i},1} < \|\gamma\|_{1_{\cup_{i\in I}U_i},1} + 1\} &\to \mathcal{C}^{k}_{\mathcal{W}}(U_i, \mathrm{L}(X_i,Z_i))_{i\in I}\\ (\eta_i)_{i\in I} &\mapsto (\beta_i)^{(2)}_M \circ (\mathrm{id}_U, \eta_i, D\eta_i)\end{aligned}$$

is defined and continuous. In view of Remark 5.1.19, the domain of this map is a neighborhood of γ. This finishes the inductive proof.

The case $k = \infty$ follows from the case $k < \infty$ by means of Lemma 5.1.11.

Now we prove that β_* is smooth. More exactly, we show by induction on $\ell \in \mathbb{N}^*$ that it is $\mathcal{C}^\ell$.

$\ell = 1$: By Proposition 3.3.26, for any $i \in I$ the map

$$(\beta_i)_* : \mathcal{C}^{\partial,k}_{\mathcal{W}_i}(U_i, V_i) \to \mathcal{C}^{k}_{\mathcal{W}_i}(U_i, Z_i) : \gamma \mapsto \beta_i \circ (\mathrm{id}_{U_i}, \gamma)$$

is $\mathcal{C}^1$. We noted in (3.3.26.5) that its differential is given by

$$d(\beta_i)_*(\gamma; \eta) = (d_2\beta_i)_*(\gamma, \eta).$$

Obviously $d_2\beta_i = (\beta_i)^{(2)}_{\cdot}$, where $\cdot$ denotes the evaluation of linear operators. We see with the same reasoning as above that the map

$$\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I} \times \mathcal{C}^{k}_{\mathcal{W}}(U_i, Y_i)_{i\in I} \to \mathcal{C}^{k}_{\mathcal{W}}(U_i, Z_i)_{i\in I} : (\gamma, \eta) \mapsto ((\beta_i)^{(2)}_{\cdot})_*(\gamma_i, \eta_i))_{i\in I}$$

is defined and continuous. Hence we can apply Lemma 5.1.7 to see that β_* is $\mathcal{C}^1$ with $d\beta_* = \prod_{i\in I}(d_2\beta_i)_*$.

$\ell \to \ell + 1$: We see with the inductive hypothesis that $\prod_{i\in I}(d_2\beta_i)_*$ is $\mathcal{C}^\ell$, and since $d\beta_* = \prod_{i\in I}(d_2\beta_i)_*$, we deduce that β_* is $\mathcal{C}^{\ell+1}$. □

For technical reasons, we show that for a family $(\phi_i)_{i\in I}$ of smooth maps for which (5.1.20.1) is satisfied for their Fréchet differentials $(D\phi_i)_{i\in I}$, the family of their ordinary differentials $(d\phi_i)_{i\in I}$ satisfies (5.1.23.1), at least on bounded subsets.

Lemma 5.1.24. *Let $(U_i)_{i\in I}$ be a family such that each U_i is an open nonempty set of a normed space X_i and $(Y_i)_{i\in I}$ a family of normed spaces. Further, for each $i \in I$ let $\beta_i : U_i \to Y_i$ be a smooth map and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ such that* (5.1.20.1) *is satisfied for $(D\beta_i)_{i\in I}$. Then for each $R > 0$, $(d\beta_i|_{U_i \times B_{X_i}(0,R)})_{i\in I}$ satisfies* (5.1.23.1).

Proof. Let $i \in I$. Then we derive from (3.3.24.1) that for all $\ell \in \mathbb{N}^*$, $x \in U_i$ and $h \in X_i$,

$$\|D^{(\ell)} d\beta_i(x,h)\|_{op} \leq \ell \|D^{(\ell-1)} D\beta_i(x)\|_{op} + \|h\| \, \|D^{(\ell)} D\beta_i(x)\|_{op}.$$

Hence

$$\|d\beta_i\|_{1_{U_i \times B_{X_i}(0,R)},\ell} \leq \ell \|D\beta_i\|_{1_{U_i},\ell-1} + R\|D\beta_i\|_{1_{U_i},\ell},$$

and from this estimate we easily derive that (5.1.23.1) is satisfied when so is (5.1.20.1). □

Simultaneous superposition with uniformly bounded maps As a corollary, we prove a superposition result that is more in the style of Proposition 3.3.12; we examine functions that are not necessarily defined on a product and assume that the norms of the derivatives are uniformly bounded. First, state an obvious fact.

Lemma 5.1.25. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty subset of the normed space X_i and each V_i is an open nonempty subset of a normed space Y_i. Further, let $(Z_i)_{i\in I}$ be another family of normed spaces and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} U_i}$ nonempty. For each $i \in I$, let $\beta_i \in \mathcal{FC}^\infty(U_i \times V_i, Z_i)$ be a map such that for each $\ell \in \mathbb{N}^*$,*

$$K_\ell := \sup_{i\in I}\{\|\beta_i\|_{1_{U_i \times V_i},\ell}\} < \infty.$$

Then (5.1.23.1) *is satisfied.*

Proof. Let $\ell \in \mathbb{N}^*$. For $f \in \mathcal{W}$ and $i \in I$, we have that

$$\|\beta_i\|_{1_{U_i \times V_i},\ell} |f_i| \leq K_\ell |f_i|.$$

Since $K_\ell f \in \mathcal{W}_{\max}$, the assertion is proved. □

We now prove the result. The main difficulty is that in order to use Proposition 5.1.23, we have to adapt its results for functions that are not necessarily defined on a product.

Corollary 5.1.26. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty subset of the normed space X_i and each V_i is an open subset of a normed space Y_i that is star-shaped with center 0. Further, let $(Z_i)_{i\in I}$ be another family of normed spaces and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} U_i}$ contain an adjusting weight ω. For each $i \in I$, let $\beta_i \in \mathcal{FC}^\infty(V_i, Z_i)$ be a map such that $\beta_i(0) = 0$. Further, assume that for each $\ell \in \mathbb{N}^*$, the set*

$$\{\|\beta_i\|_{1_{V_i},\ell} : i \in I\}$$

is bounded. Then for $k \in \overline{\mathbb{N}}$, the map

$$\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, Z_i)_{i\in I} : (\gamma_i)_{i\in I} \mapsto (\beta_i \circ \gamma_i)_{i\in I}$$

is defined and smooth.

Proof. For each $i \in I$, we define $\widetilde{\beta}_i : U_i \times V_i \to Z_i : (x,y) \mapsto \beta_i(y)$. We know from Lemma A.1.17 that $D^{(\ell)}\widetilde{\beta}_i = \mathrm{pr}_2^* \circ (D^{(\ell)}\beta_i \circ \mathrm{pr}_2)$, where $\mathrm{pr}_2 : X_i \times Y_i \to Y_i$ denotes the projection onto the second component. So $\|\widetilde{\beta}_i\|_{1_{U_i\times V_i},\ell} \leq \|\beta_i\|_{1_{V_i},\ell}$ for all $\ell \in \mathbb{N}$. Further $\widetilde{\beta}_i \circ (\mathrm{id}_{U_i}, \gamma_i) = \beta_i \circ \gamma_i$ for each map $\gamma_i : U_i \to V_i$, and $\widetilde{\beta}_i(U_i \times \{0\}) = \{0\}$. Hence we derive the assertion from Proposition 5.1.23 and Lemma 5.1.25. □

Simultaneous superposition with analytic maps We prove a result concerning the superposition with analytic maps. As in Corollary 5.1.26, the results derived here are in the style of Proposition 3.3.21.

We start with simultaneous "good" complexifications.

Lemma 5.1.27. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty set of the normed space X_i, each V_i is an open set of a real normed space Y_i and $(\widetilde{V}_i)_{i\in I}$ a family such that for each $i \in I$, $\widetilde{V}_i$ is an open neighborhood of $\iota_i(V_i)$ in $(Y_i)_\mathbb{C}$, where $\iota_i : Y_i \to (Y_i)_\mathbb{C}$ denotes the canonical inclusion. Assume that*

$$(\forall i \in I, M \subseteq V_i) \ \operatorname{dist}(M, Y_i \setminus V_i) \leq \operatorname{dist}(\iota_i(M), (Y_i)_\mathbb{C} \setminus \widetilde{V}_i). \tag{5.1.27.1}$$

Then

$$\prod_{i\in I}(\iota_i)_*(\mathcal{C}^{\partial,k}_\mathcal{W}(U_i, V_i)_{i\in I}) \subseteq \mathcal{C}^{\partial,k}_\mathcal{W}(U_i, \widetilde{V}_i)_{i\in I}$$

for each $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} U_i}$ containing $1_{\bigcup_{i\in I} U_i}$.

Proof. Note that $\prod_{i\in I}(\iota_i)_*$ is defined by Lemma 5.1.22. Let $\gamma \in \mathcal{C}^{\partial,k}_\mathcal{W}(U_i, V_i)_{i\in I}$. By definition, there exists $r > 0$ such that $\gamma_i(U_i) + B_{Y_i}(0, r) \subseteq V_i$ for all $i \in I$; in particular, $\operatorname{dist}(\gamma_i(U_i), Y_i \setminus V_i) \geq r$. By (5.1.27.1), $\operatorname{dist}(\iota_i(\gamma_i(U_i)), (Y_i)_\mathbb{C} \setminus \widetilde{V}_i) \geq r$ and hence $(\iota_i \circ \gamma_i)(U_i) + B_{(Y_i)_\mathbb{C}}(0, r) \subseteq \widetilde{V}_i$ for each $i \in I$. Thus

$$\prod_{i\in I}(\iota_i)_*(\gamma) = (\iota_i \circ \gamma_i)_{i\in I} \in \mathcal{C}^{\partial,k}_\mathcal{W}(U_i, \widetilde{V}_i)_{i\in I},$$

which finishes the proof. □

We now prove the result. We assume that the domains of the superposition maps do not become arbitrarily small, and that they are uniformly bounded on subsets that have a uniform distance from the domain boundary. This, together with the Cauchy estimates, will enable us to use Proposition 5.1.23.

Corollary 5.1.28. *Let $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ be families such that each U_i is an open nonempty subset of a normed space X_i, each V_i is an open subset of a normed space Y_i that is star-shaped with center 0 such that $\inf_{i\in I} \operatorname{dist}(\{0\}, \partial V_i) > 0$. Further, let $(Z_i)_{i\in I}$ be another family of normed spaces and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} U_i}$ with $1_{\bigcup_{i\in I} U_i} \in \mathcal{W}$. For each $i \in I$, let $\beta_i : V_i \to Z_i$ be a map with $\beta_i(0) = 0$. Further, assume that either all β_i are complex analytic with*

$$(\forall (W_i)_{i\in I} : W_i \subseteq V_i \text{ open and bounded}, \inf_{i\in I} \operatorname{dist}(W_i, \partial V_i) > 0) \ \sup_{i\in I} \|\beta_i\|_{1_{W_i},0} < \infty; \tag{5.1.28.1}$$

or that any β_i is real analytic and has a complexification

$$\widetilde{\beta}_i : \widetilde{V}_i \subseteq (Y_i)_\mathbb{C} \to (Z_i)_\mathbb{C}$$

such that (5.1.28.1) *is satisfied and whose domains* $\widetilde{V}_i$ *are star-shaped with center* 0 *and satisfy* (5.1.27.1). *Then for* $k \in \overline{\mathbb{N}}$*, the map*

$$\beta_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, Z_i)_{i\in I} : (\gamma_i)_{i\in I} \mapsto ((\beta_i)_*(\gamma_i))_{i\in I} = (\beta_i \circ \gamma_i)_{i\in I}$$

is defined and analytic.

Proof. We first assume that all β_i are complex analytic. Let $r \in]0, d[$, where $d := \inf_{i\in I} \operatorname{dist}(\{0\}, \partial V_i)$. We use Lemma 3.3.13 to see that there exists a family $(V_i^{\partial,r})_{i\in I}$ such that each $V_i^{\partial,r}$ is open, bounded and star-shaped with center 0; and furthermore $\inf_{i\in I} \operatorname{dist}(V_i^{\partial,r}, \partial V_i) \geq \frac{d-r}{2} \min(1, r^2)$ and $\bigcup_{r<d} V_i^{\partial,r} = V_i$ for each $i \in I$. Hence we see with estimate (3.3.15.1) that for each $\ell \in \mathbb{N}$, there exists $\widetilde{r} < \frac{d-r}{2} \min(1, r^2)$ such that

$$\|\beta_i\|_{1_{V_i^{\partial,r}},\ell} \leq \frac{(2\ell)^\ell}{(\widetilde{r})^\ell} \|\beta_i\|_{1_{V_i^{\partial,r}+\overline{B}_{Y_i}(0,\widetilde{r})},0}$$

for all $i \in I$. Using (5.1.28.1), we conclude from this that

$$\{\|\beta_i\|_{1_{V_i^{\partial,r}},\ell} : i \in I\}$$

is bounded, so we use Corollary 5.1.26 to see that β_* is defined and smooth (and hence analytic) on $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i^{\partial,r})_{i\in I}$. Since these sets are open in $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I}$ and

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I} = \bigcup_{r\in]0,d[} \mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i^{\partial,r})_{i\in I},$$

we derive the assertion.

Now assume that all β_i are real analytic. We derive from the first part of the proof that $\widetilde{\beta}_* = \prod_i (\widetilde{\beta}_i)_*$ is defined and analytic. Obviously β_* coincides with the restriction of $\widetilde{\beta}_*$ to $\prod_{i\in I}(\iota_i)_*(\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, V_i)_{i\in I})$ (which is contained in the domain of $\widetilde{\beta}_*$ by Lemma 5.1.27), hence β_* is real analytic. □

We provide an application.

Lemma 5.1.29. *Let* $(U_i)_{i\in I}$ *be a family such that each* U_i *is an open nonempty subset of the normed space* X_i*,* $(Y_i)_{i\in I}$ *a family of Banach spaces,* $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I} U_i}$ *with* $1_{\cup_{i\in I} U_i} \in \mathcal{W}$ *and* $k \in \mathbb{N}$*. Then the map*

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, B_{\mathrm{L}(Y_i)}(0,1))_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i, \mathrm{L}(Y_i))_{i\in I} : \gamma \mapsto (QI_{\mathrm{L}(Y_i)} \circ \gamma_i)_{i\in I}$$

is defined and analytic.

Proof. This is simply an application of Corollary 5.1.28 since each $QI_{\mathrm{L}(Y_i)}|_{B_{\mathrm{L}(Y_i)}(0,1)}$ can be written as a (the same) power series, and hence satisfies (5.1.28.1). □

5.1.4. Simultaneous composition and inversion

We examine the simultaneous application of the composition and inversion operations, respectively, that we studied in Proposition 4.1.7 and Proposition 4.2.6.

Simultaneous composition We start with composition. Note that we need the adjusting weight ω to ensure that $\mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}$ is open and not empty.

Proposition 5.1.30. *Let $(U_i)_{i\in I}$, $(V_i)_{i\in I}$ and $(W_i)_{i\in I}$ be families such that for each $i \in I$, U_i, V_i and W_i are open nonempty sets of the normed space X_i with $U_i + V_i \subseteq W_i$, and V_i is balanced. Further, let $(Y_i)_{i\in I}$ be another family of normed spaces and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\bigcup_{i\in I} W_i}$ contain an adjusting weight ω for $(V_i)_{i\in I}$. Then for $k, \ell \in \overline{\mathbb{N}}$, the map*

$$\mathfrak{c}^{Y,k}_{\mathcal{W},\ell} := \prod_{i\in I} \mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,\ell} : \begin{cases} \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W_i,Y_i)_{i\in I} \times \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I} \to \mathcal{C}^{k}_{\mathcal{W}}(U_i,Y_i)_{i\in I} \\ \qquad ((\gamma_i)_{i\in I},(\eta_i)_{i\in I}) \mapsto (\gamma_i \circ (\eta_i + \mathrm{id}_{U_i}))_{i\in I} \end{cases}$$

is defined and $\mathcal{C}^\ell$.

Proof. We see with Proposition 4.1.7 (and Remark 5.1.16) that $\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$ is defined as a map to $\prod_{i\in I} \mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)$. We first prove by induction on k that $\mathfrak{c}^{Y,k}_{\mathcal{W},0}$ takes its values in $\mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I}$ and is continuous.

$k = 0$: We see with estimate (4.1.3.1) that for $f \in \mathcal{W}$, $\gamma \in \mathcal{C}^1_{\mathcal{W}}(W_i,Y_i)_{i\in I}$ and $\eta \in \mathcal{C}^{\omega_\partial,0}_{\mathcal{W}}(U_i,V_i)_{i\in I}$

$$\|\mathfrak{c}^{Y_i,0}_{\mathcal{W}_i,0}(\gamma_i,\eta_i)\|_{f_i,0} \leq \|\gamma_i\|_{1_{U_i},1}\|\eta_i\|_{f_i,0} + \|\gamma_i\|_{f_i,0}$$

for each $i \in I$. So $\mathfrak{c}^{Y,0}_{\mathcal{W},0}$ is defined, taking Remark 5.1.19 into account. Further, we see with the same reasoning – applied to estimate (4.1.3.2) – and Lemma 5.1.13 that $\mathfrak{c}^{Y,0}_{\mathcal{W},0}$ is locally Lipschitz continuous and hence continuous.

$k \to k+1$: We use Proposition 5.1.12. For $\gamma \in \mathcal{C}^{k+2}_{\mathcal{W}}(W_i,Y_i)_{i\in I}$ and $\eta \in \mathcal{C}^{\omega_\partial,k+1}_{\mathcal{W}}(U_i,V_i)_{i\in I}$, for each $i \in I$ we have

$$D(\gamma_i \circ (\eta_i + \mathrm{id}_{U_i})) = D\gamma_i \circ (\eta_i + \mathrm{id}_{U_i}) \cdot (D\eta_i + \mathrm{Id}) = \mathfrak{c}^{\mathrm{L}(X_i,Y_i),k}_{\mathcal{W}_i,0}(D\gamma_i,\eta_i) \cdot (D\eta_i + \mathrm{Id}).$$

By the inductive hypothesis, the map $\mathfrak{c}^{\mathrm{L}(X,Y),k}_{\mathcal{W},0}$ is defined and continuous. Further, we see (noting Remark 5.1.19) that $(D\eta_i + \mathrm{Id})_{i\in I} \in \mathcal{C}^k_{\{1_{\bigcup_{i\in I} U_i}\}}(U_i,\mathrm{L}(X_i))_{i\in I}$. Hence we can apply Lemma 5.1.22 to finish the proof.

The case $k = \infty$ follows from the case $k < \infty$ using Lemma 5.1.11.

Now we prove by induction on $\ell \in \mathbb{N}^*$ that $\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$ is $\mathcal{C}^\ell$.

$\ell = 1$: We know from Proposition 4.1.7 that

$$\mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,1} : \mathcal{C}^{k+2}_{\mathcal{W}_i}(W_i,Y_i) \times \mathcal{C}^{\partial,k}_{\mathcal{W}_i}(U_i,V_i) \to \mathcal{C}^k_{\mathcal{W}_i}(U_i,Y_i) : (\gamma,\eta) \mapsto \gamma \circ (\eta + \mathrm{id}_{U_i})$$

is $\mathcal{C}^1$ for each $i \in I$, and we noted in identity (4.1.7.1) that its differential is given by

$$d\,\mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,1}(\gamma,\eta;\gamma_1,\eta_1) = \mathfrak{c}^{\mathrm{L}(X_i,Y_i),k}_{\mathcal{W}_i,0}(D\gamma,\eta) \cdot \eta_1 + \mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,1}(\gamma_1,\eta).$$

Since we already proved that $\mathfrak{c}^{\mathrm{L}(X,Y),k}_{\mathcal{W},0}$ and $\mathfrak{c}^{Y,k}_{\mathcal{W},1}$ are continuous, we use Lemma 5.1.22 to see that

$$\begin{gathered}\mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W_i,Y_i)_{i\in I} \times \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I} \times \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W_i,Y_i)_{i\in I} \times \mathcal{C}^k_{\mathcal{W}}(U_i,X_i)_{i\in I} \to \mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I} \\ (\gamma,\eta,\gamma^1,\eta^1) \mapsto (\mathfrak{c}^{\mathrm{L}(X_i,Y_i),k}_{\mathcal{W}_i,\ell-1}(D\gamma_i,\eta_i) \cdot \eta^1_i + \mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,\ell}(\gamma^1_i,\eta_i))_{i\in I}\end{gathered}$$

is defined and continuous. Hence we can apply Lemma 5.1.7 to see that $\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$ is $\mathcal{C}^1$ and $d\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$ is given by this map.

$\ell \to \ell+1$: We apply the inductive hypothesis and Lemma 5.1.22 to the identity for $d\mathfrak{c}^{Y,k}_{\mathcal{W},\ell+1}$ derived above to see that $d\mathfrak{c}^{Y,k}_{\mathcal{W},\ell+1}$ is $\mathcal{C}^\ell$, hence $\mathfrak{c}^{Y,k}_{\mathcal{W},\ell+1}$ is $\mathcal{C}^{\ell+1}$. □

Simultaneous inversion We treat inversion. Here an adjusting weight is given explicitly.

Proposition 5.1.31. *Let $(U_i)_{i\in I}$ and $(\widetilde{U}_i)_{i\in I}$ be families such that U_i and $\widetilde{U}_i$ are open nonempty sets of the Banach space X_i. Further assume that there exists $r>0$ such that $\widetilde{U}_i + B_{X_i}(0,r) \subseteq U_i$ for all $i\in I$. Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\cup_{i\in I} U_i}$ with $1_{\cup_{i\in I} U_i} \in \mathcal{W}$ and $\tau \in]0,1[$. Then the map*

$$I^{\widetilde{U}}_{\mathcal{W}} := \prod_{i\in I} I^{\widetilde{U}_i}_{\mathcal{W}_i} : \mathcal{D}^\tau \to \mathcal{C}^\infty_{\mathcal{W}}(\widetilde{U}_i, X_i)_{i\in I} : (\phi_i)_{i\in I} \mapsto ((\phi_i + \mathrm{id}_{U_i})^{-1}|_{\widetilde{U}_i} - \mathrm{id}_{\widetilde{U}_i})_{i\in I}$$

is defined and smooth, where

$$\mathcal{D}^\tau := \left\{\phi \in \mathcal{C}^\infty_{\mathcal{W}}(U_i, X_i)_{i\in I} : \|\phi\|_{1_{\cup_{i\in I} U_i},1} < \tau \text{ and } \|\phi\|_{1_{\cup_{i\in I} U_i},0} < \tfrac{r}{2}(1-\tau)\right\}.$$

Proof. We use Lemma 4.2.9, applied to $\phi = 0$, to see that $I^{\widetilde{U}}_{\mathcal{W}}$ is defined as a map to $\prod_{i\in I} \mathcal{C}^\infty_{\mathcal{W}}(\widetilde{U}_i, X_i)_{i\in I}$. We prove by induction on k that it takes values in $\mathcal{C}^k_{\mathcal{W}}(\widetilde{U}_i, X_i)_{i\in I}$ and is continuous.

$k=0$: By estimate (4.2.5.1), we have for $f\in\mathcal{W}$, $(\phi_i)_{i\in I}\in\mathcal{D}^\tau$ and each $i\in I$ that

$$\|I^{\widetilde{U}_i}_{\mathcal{W}_i}(\phi_i)\|_{f_i,0} \leq \|\phi_i\|_{f_i,0} \frac{1}{1-\|\phi_i\|_{1_{\widetilde{U}_i},1}} \leq \frac{1}{1-\tau}\|\phi_i\|_{f_i,0}.$$

Since $\tau < 1$ and i was arbitrary, $I^{\widetilde{U}}_{\mathcal{W}}$ is defined. In the same manner, we can use estimate (4.2.6.1) to see with Lemma 5.1.13 that $I^{\widetilde{U}}_{\mathcal{W}}$ is locally Lipschitz continuous and hence continuous.

$k\to k+1$: We use Proposition 5.1.12. By identity (4.2.4.1), for $\phi\in\mathcal{D}^\tau$,

$$(D\, I^{\widetilde{U}_i}_{\mathcal{W}_i}(\phi_i))_{i\in I} = (\mathfrak{c}^{\mathrm{L}(X_i)}_{\mathcal{W}_i}(D\phi_i \cdot QI(-D\phi_i) - D\phi_i, I^{\widetilde{U}_i}_{\mathcal{W}_i}(\phi_i)))_{i\in I}.$$

Since $(D\phi_i)_{i\in I} \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i, B_{\mathrm{L}(X_i)}(0,1))_{i\in I}$, we can apply Lemma 5.1.29 and then Lemma 5.1.22, Proposition 5.1.30 and the inductive hypothesis to finish the proof.

The case $k=\infty$ follows from the case $k<\infty$ with Lemma 5.1.11.

Now we prove that $I^{\widetilde{U}}_{\mathcal{W}}$ is smooth. More exactly, we show by induction on $\ell\in\mathbb{N}^*$ that it is $\mathcal{C}^\ell$.

$\ell = 1$: By Proposition 4.2.6 (and Lemma 4.2.9), the map $I^{\widetilde{U}_i}_{\mathcal{W}_i}$ is $\mathcal{C}^1$ on $\pi_i(\mathcal{D}^\tau)$ for each $i\in I$, and by identity (4.2.6.2) its differential is given by

$$d\, I^{\widetilde{U}_i}_{\mathcal{W}_i}(\phi;\phi^1) = \mathfrak{c}^{X_i}_{\mathcal{W}_i}(QI(D\phi)\cdot\phi^1 + \phi^1, I^{\widetilde{U}_i}_{\mathcal{W}_i}(\phi)).$$

We conclude using Lemma 5.1.29, Lemma 5.1.22 Proposition 5.1.30 and the continuity of $I_{\mathcal{W}}^{\widetilde{U}}$ that the map

$$\mathcal{D}^{\tau} \times \mathcal{C}_{\mathcal{W}}^{\infty}(U_i, X_i)_{i\in I} \to \mathcal{C}_{\mathcal{W}}^{\infty}(\widetilde{U}_i, X_i)_{i\in I} : (\phi, \phi^1) \mapsto (\mathfrak{c}_{\mathcal{W}}^{X_i}(QI(D\phi_i) \cdot \phi_i^1 + \phi_i^1, I_{\mathcal{W}_i}^{\widetilde{U}_i}(\phi_i)))_{i\in I}$$

is continuous. So we can apply Lemma 5.1.7 to see that $I_{\mathcal{W}}^{\widetilde{U}}$ is $\mathcal{C}^1$ and its differential is given by this map.

$\ell \to \ell + 1$: We apply the inductive hypothesis, Lemma 5.1.29, Lemma 5.1.22 and Proposition 5.1.30 to the identity for $dI_{\mathcal{W}}^{\widetilde{U}}$ derived above to see that $dI_{\mathcal{W}}^{\widetilde{U}}$ is $\mathcal{C}^{\ell}$, hence $I_{\mathcal{W}}^{\widetilde{U}}$ is $\mathcal{C}^{\ell+1}$. □

Remark 5.1.32. We implicitly used in this subsection that the operator norms of the composition resp. evaluation of linear maps are uniformly bounded.

5.2. Spaces of weighted vector fields on manifolds

We define spaces $\mathcal{C}_{\mathcal{W}}^{k}(M, \mathbf{T}M)_{\mathcal{A}}$ of weighted vector fields on manifolds, where $\mathcal{A}$ is an atlas for M. As discussed in the beginning of Section 5.1, we do this in such a way that the map $\mathcal{C}_{\mathcal{W}}^{k}(M, \mathbf{T}M)_{\mathcal{A}} \to \mathcal{C}_{\mathcal{W}_{\mathcal{A}}}^{k}(U_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}}$ that sends a vector field to the family of its localizations is an embedding. Of particular concern is when $\mathcal{C}_{\mathcal{W}}^{k}(M, \mathbf{T}M)_{\mathcal{A}} = \mathcal{C}_{\mathcal{W}}^{k}(M, \mathbf{T}M)_{\mathcal{B}}$ for another atlas $\mathcal{B}$. We derive a criterion on $\mathcal{W}$ ensuring this.

Further, we will discuss the simultaneous composition and inversion of weighted functions that arise as simultaneous superposition with the localized exponential maps. Again, this will be possible if the weights satisfy certain conditions.

After having made assumptions on the weights, we have to know if there exist weight sets that satisfy them. In particular, we will prove that every set of weights has a "minimal saturated extension".

5.2.1. Definition and properties

We give the definition of weighted vector fields.

Definition 5.2.1 (Weighted vector fields and localizations)**.** Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ nonempty and $k \in \overline{\mathbb{N}}$. For $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$, we define

$$\|\cdot\|_{\mathcal{A},f,\ell} : \prod_{\kappa\in\mathcal{A}} \mathcal{C}^k(U_\kappa, \mathbb{R}^d) \to [0, \infty] : (\gamma_\kappa)_{\kappa\in\mathcal{A}} \mapsto \sup_{\kappa\in\mathcal{A}} \|\gamma_\kappa\|_{f\circ\kappa^{-1},\ell}.$$

For $X \in \mathfrak{X}^k(M)$, we define

$$\|X\|_{\mathcal{A},f,\ell} := \|(X_\kappa)_{\kappa\in\mathcal{A}}\|_{\mathcal{A},f,\ell}$$

and with that

$$\mathcal{C}_{\mathcal{W}}^{k}(M, \mathbf{T}M)_{\mathcal{A}} := \{X \in \mathfrak{X}^k(M) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k)\ \|X\|_{\mathcal{A},f,\ell} < \infty\}.$$

Obviously $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$ is a vector space. We endow it with the locally convex topology induced by the seminorms $\|\cdot\|_{\mathcal{A},f,\ell}$. We call its elements *weighted vector fields.* Furthermore, we set for $f \in \mathcal{W}$ and $\kappa \in \mathcal{A}$

$$f_\kappa := f \circ \kappa^{-1} : U_\kappa \to \overline{\mathbb{R}} \qquad \text{and} \qquad \mathcal{W}_\kappa := \{f_\kappa : f \in \mathcal{W}\}.$$

Finally, we define

$$f_{\mathcal{A}} := \dot\cup_{\kappa\in\mathcal{A}} f_\kappa \in \overline{\mathbb{R}}^{\dot\cup_{\kappa\in\mathcal{A}} U_\kappa} \qquad \text{and} \qquad \mathcal{W}_{\mathcal{A}} := \{f_{\mathcal{A}} : f \in \mathcal{W}\}.$$

Lemma 5.2.2. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ such that for each $p \in M$, there exists $f_p \in \mathcal{W}$ with $f_p(p) \neq 0$. Then the map*

$$\iota^{\mathcal{A}}_{\mathcal{W}} : \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} \to \mathcal{C}^k_{\mathcal{W}_{\mathcal{A}}}(U_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} : \phi \mapsto (\phi_\kappa)_{\kappa\in\mathcal{A}}$$

is a linear topological embedding, with closed image.

Proof. That the map is defined and an embedding is obvious from the definition of $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$ and $\mathcal{C}^k_{\mathcal{W}_{\mathcal{A}}}(U_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}}$. To see that the image is closed, let $(X^i)_{i\in I}$ be a net in $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$ such that $(\iota^{\mathcal{A}}_{\mathcal{W}}(X^i))_{i\in I}$ converges to $(X_\kappa)_{\kappa\in\mathcal{A}}$. We have to show that for $\kappa_1, \kappa_2 \in \mathcal{A}$ with $\widetilde{U}_{\kappa_1} \cap \widetilde{U}_{\kappa_2} \neq \emptyset$,

$$X_{\kappa_1}|_{\kappa_1(\widetilde{U}_{\kappa_1}\cap\widetilde{U}_{\kappa_2})} = d(\kappa_1 \circ \kappa_2^{-1}) \circ (\mathrm{id}_{U_{\kappa_2}}, X_{\kappa_2}) \circ \kappa_2 \circ \kappa_1^{-1}|_{\kappa_1(\widetilde{U}_{\kappa_1}\cap\widetilde{U}_{\kappa_2})}. \tag{†}$$

But since the stated assumption on $\mathcal{W}$ implies that $(X^i_\kappa)_{i\in I}$ converges pointwise to X_κ for each $\kappa \in \mathcal{A}$, and since (†) holds for all $X^i_{\kappa_1}$ and $X^i_{\kappa_2}$, we see that it also holds for X_{κ_1} and X_{κ_2}. □

Comparison of weighted vector fields with regard to different atlases

We examine the relationship between spaces $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$ and $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$ for atlases $\mathcal{A}$ and $\mathcal{B}$. To this end, we define some terminology for atlases.

Definition 5.2.3. Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M. We call $\mathcal{A}$ *locally finite* if $(\widetilde{U}_\kappa)_{\kappa\in\mathcal{A}}$ is a locally finite cover of M. Let $\mathcal{B}$ be another atlas for M. We call $\mathcal{B}$ *subordinate to* $\mathcal{A}$ if for each chart $\kappa : \widetilde{V}_\kappa \to V_\kappa$ of $\mathcal{B}$ there exists $\hat\kappa \in \mathcal{A}$ such that $\kappa = \hat\kappa|^{V_\kappa}_{\widetilde{V}_\kappa}$. Finally, we define

$$\mathcal{A} \otimes \mathcal{B} := \{(\kappa, \phi) \in \mathcal{A} \times \mathcal{B} : \widetilde{U}_\kappa \cap \widetilde{U}_\phi \neq \emptyset\}$$

and

$$\mathcal{A}^{\cap\mathcal{B}} := \{\kappa|_{\widetilde{U}_\kappa\cap\widetilde{U}_\phi} : (\kappa, \phi) \in \mathcal{A} \otimes \mathcal{B}\}.$$

We state two easy results. First, we show that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} = \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}^{\cap\mathcal{B}}}$, and that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$ if $\mathcal{B}$ is subordinate to $\mathcal{A}$.

Lemma 5.2.4. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A}$ and $\mathcal{B}$ atlases for M, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ nonempty and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} = \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}\cap\mathcal{B}}$.*

Proof. This is obvious since for $X \in \mathfrak{X}^k(M)$, $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$, the sets

$$\{|f(x)| \, \|D^{(\ell)} X_\kappa(\kappa(x))\|_{op} : \kappa \in \mathcal{A}, x \in U_\kappa\}$$

and

$$\{|f(x)| \, \|D^{(\ell)} X_\kappa(\kappa(x))\|_{op} : (\kappa, \phi) \in \mathcal{A} \otimes \mathcal{B}, x \in U_\kappa \cap U_\phi\}$$

are the same. □

Lemma 5.2.5. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $\mathcal{B} = \{\kappa : \widetilde{V}_\kappa \to V_\kappa\}$ an atlas subordinate to $\mathcal{A}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ nonempty and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$, and the inclusion map is continuous linear.*

Proof. Let $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Since for each $\kappa \in \mathcal{B}$ there exists $\hat{\kappa} \in \mathcal{A}$ with $\kappa = \hat{\kappa}|_{\widetilde{V}_\kappa}^{V_\kappa}$, we have for $X \in \mathfrak{X}^k(M)$ that

$$\begin{aligned}\|X_\kappa\|_{f_\kappa,\ell} &= \sup_{x \in V_\kappa} |(f \circ \kappa^{-1})(x)| \, \|D^{(\ell)}(d\kappa \circ X \circ \kappa^{-1})(x)\|_{op} \\ &\leq \sup_{x \in U_{\hat{\kappa}}} |(f \circ \hat{\kappa}^{-1})(x)| \, \|D^{(\ell)}(d\hat{\kappa} \circ X \circ \hat{\kappa}^{-1})(x)\|_{op} = \|X_{\hat{\kappa}}\|_{f_{\hat{\kappa}},\ell}.\end{aligned}$$

This shows the assertion. □

Weights with transition maps as multipliers We show the main result of this subsection. If for two atlases $\mathcal{A}$, $\mathcal{B}$, the differentials of the transition maps from $\mathcal{B}$ to $\mathcal{A}$ are "simultaneous multipliers" for $\mathcal{W}$ (that is they satisfy (5.1.20.1)), then $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}\cap\mathcal{B}}$. If additionally $\mathcal{B}$ is subordinate to $\mathcal{A}$, we have that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} = \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$.

Proposition 5.2.6. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ and $\mathcal{B} = \{\phi : \widetilde{V}_\phi \to V_\phi\}$ atlases for M and $k \in \overline{\mathbb{N}}$. Further, let $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ such that* (5.1.20.1) *is satisfied for $\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}$ and $(D(\kappa \circ \phi^{-1})|_{\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}$ and there exists $\omega \in \mathcal{W}$ with $|\omega| \geq 1$. Then the following assertions hold:*

(a) The map

$$\prod_{\substack{(\kappa,\phi)\in\\ \mathcal{A}\otimes\mathcal{B}}} d(\kappa\circ\phi^{-1})_* : \mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa\cap\widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} \to \mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa\cap\widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} \tag{†}$$

is continuous linear.

(b) The map

$$\begin{aligned}\mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} &\to \mathcal{C}^k_{\mathcal{W}_{\mathcal{A}\cap\mathcal{B}}}(\kappa(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} \\ (\gamma_{\kappa,\phi})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} &\mapsto (\gamma_{\kappa,\phi} \circ \phi \circ \kappa^{-1})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}\end{aligned} \tag{††}$$

is defined and continuous.

(c) $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}^{\cap\mathcal{B}}}$*, and the inclusion map is continuous linear.*

Proof. (a) Since $|\omega_\phi| \geq 1 = \max(\frac{1}{1}, 1)$ for each $\phi \in \mathcal{B}$, we can apply (5.1.17.1) to see that for $(\gamma_{\kappa,\phi})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} \in \mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}$,

$$(\forall (\kappa,\phi) \in \mathcal{A}\otimes\mathcal{B})\ \|\gamma_{(\kappa,\phi)}\|_\infty \leq \|\gamma_{(\kappa,\phi)}\|_{\omega_\phi,0} \leq \|(\gamma_{(\kappa,\phi)})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}\|_{\omega_{\mathcal{B}\cap\mathcal{A}},0}.$$

Hence

$$\mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} = \bigcup_{R>1} \mathcal{C}^{\partial,k}_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), B_R(0))_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}},$$

and the sets on the right hand side are open subsets of the space on the left hand side. Using our other assumption on $\mathcal{W}$ and Lemma 5.1.24, we can apply Proposition 5.1.23 to see that (†) is smooth on each set $\mathcal{C}^{\partial,k}_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), B_R(0))_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}$, and hence on $\mathcal{C}^k_{\mathcal{W}_{\mathcal{B}\cap\mathcal{A}}}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}}$. It is obviously linear since each $d(\kappa\circ\phi^{-1})$ is so in its second argument.

(b) We prove this with an induction on k.

$k = 0$: Let $f \in \mathcal{W}$. For all $(\kappa,\phi) \in \mathcal{A}\otimes\mathcal{B}$, we have $f_\kappa = f_\phi \circ \phi \circ \kappa^{-1}$. Hence for $\gamma_{\kappa,\phi} \in \mathcal{C}^k_{\mathcal{W}_\phi}(\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathbb{R}^d)$ and $x \in \kappa(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)$, we have that

$$|f_\kappa(x)|\,\|(\gamma_{\kappa,\phi}\circ\phi\circ\kappa^{-1})(x)\| = |(f_\phi\circ\phi\circ\kappa^{-1})(x)|\,\|(\gamma_{\kappa,\phi}\circ\phi\circ\kappa^{-1})(x)\| \leq \|\gamma_{\kappa,\phi}\|_{f_\phi,0}.$$

Since (††) is linear and $(\kappa,\phi) \in \mathcal{A}\otimes\mathcal{B}$ was arbitrary, we see with this estimate that (††) is defined and continuous.

$k \to k+1$: We use Proposition 5.1.12. We calculate that for $(\kappa,\phi) \in \mathcal{A}\otimes\mathcal{B}$,

$$D(\gamma_{\kappa,\phi}\circ\phi\circ\kappa^{-1}) = D\gamma_{\kappa,\phi}\circ\phi\circ\kappa^{-1}\cdot D(\phi\circ\kappa^{-1}).$$

We see using the inductive hypothesis that

$$(D\gamma_{\kappa,\phi}\circ\phi\circ\kappa^{-1})_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}} \in \mathcal{C}^k_{\mathcal{W}_{\mathcal{A}^{\cap\mathcal{B}}}}(\kappa(\widetilde{U}_\kappa \cap \widetilde{U}_\phi), \mathrm{L}(\mathbb{R}^d))_{(\kappa,\phi)\in\mathcal{A}\otimes\mathcal{B}},$$

and that the corresponding map is continuous. Finally, we get the assertion using Lemma 5.1.20.

(c) Let $(\kappa,\phi) \in \mathcal{A}\otimes\mathcal{B}$. On $\kappa(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)$, we have the identity

$$X_\kappa = d\kappa\circ X\circ\kappa^{-1} = \pi_2\circ\mathbf{T}(\kappa\circ\phi^{-1})\circ\mathbf{T}\phi\circ X\circ\phi^{-1}\circ\phi\circ\kappa^{-1} = (\phi\circ\kappa^{-1})^*(d(\kappa\circ\phi^{-1})_*(X_\phi)).$$

Since κ and ϕ were arbitrary, we can use that the maps (†) and (††) are continuous linear to derive estimates which ensure that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}^{\cap\mathcal{A}}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}^{\cap\mathcal{B}}}$, and that the inclusion map is continuous. Since $\mathcal{B}^{\cap\mathcal{A}}$ is subordinate to $\mathcal{B}$, we derive the assertion using Lemma 5.2.5. □

Corollary 5.2.7. *Let the data be as in Proposition 5.2.6, and additionally assume that* $\mathcal{B}$ *is subordinate to* $\mathcal{A}$*. Then* $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} = \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$ *as topological vector space.*

Proof. We know from Lemma 5.2.5 that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$, and from Proposition 5.2.6 and Lemma 5.2.4 that $\mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} \subseteq \mathcal{C}^k_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{A}}$. □

5.2.2. Simultaneous composition, inversion and superposition with Riemannian exponential map and logarithm

We study simultaneous composition and inversion (see Proposition 5.1.30 and Proposition 5.1.31, respectively) on families of functions that, roughly speaking, arise as the simultaneous application of the superposition with the Riemannian exponential function; and the application of simultaneous superposition with the logarithm after these operations. The result and the techniques we use here are basically those of Subsection 5.1.3 and Subsection 5.1.4, although we also make use of the results presented in Subsection B.3.2.

Rephrasing some previous results We apply some results to the special case of functions that are defined on the disjoint union of chart domains for a manifold. We start with the simultaneous superposition with (slightly modified) Riemannian exponential maps and logarithms.

Definition 5.2.8. Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U and V an open, nonempty, relatively compact set such that $\overline{V} \subseteq U$.

(a) Let $\delta \in]0, R^{E,g}_{V,U}[$. We set

$$\mathcal{E}^g_{V,\delta} : V \times B_\delta(0) \to U : (x, y) \mapsto \exp_g(x, y) - x.$$

(b) Let $\delta \in]0, R^{L,g}_{V,U}[$. We set

$$\mathcal{L}^g_{V,\delta} : V \times B_\delta(0) \to \mathbb{R}^d : (x, y) \mapsto \lg_g(x, x + y)$$

Lemma 5.2.9. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, g a Riemannian metric on M, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ nonempty and $k \in \overline{\mathbb{N}}$. Further, for each $\kappa \in \mathcal{A}$ let V_κ be an open, nonempty, relatively compact set such that $\overline{V_\kappa} \subseteq U_\kappa$, $\mathcal{B} := \{\kappa|^{V_\kappa}_{\kappa^{-1}(V_\kappa)} : \kappa \in \mathcal{A}\}$ is an atlas for M, and $\delta_\kappa > 0$. Further, we assume that $\mathcal{W}_\mathcal{B}$ contains an adjusting weight ω for $(\delta_\kappa)_{\kappa\in\mathcal{A}}$.*

(a) Assume that $\delta_\kappa < R^{E,g_\kappa}_{V_\kappa,U_\kappa}$, and that $\mathcal{W}_\mathcal{B}$ satisfies (5.1.23.1), where $I = \mathcal{A}$ and $\beta_\kappa = \mathcal{E}^{g_\kappa}_{V_\kappa,\delta_\kappa}$ for $\kappa \in \mathcal{A}$. Then the map

$$E^{\mathcal{W}_\mathcal{B},g}_{V,\delta} := \prod_{\kappa\in\mathcal{A}} E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa} : \begin{cases} \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}_\mathcal{B}}(V_\kappa, B_{\delta_\kappa}(0))_{\kappa\in\mathcal{A}} \to \mathcal{C}^k_{\mathcal{W}_\mathcal{B}}(V_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} \\ \phi \mapsto (\exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, \phi_\kappa) - \mathrm{id}_{V_\kappa})_{\kappa\in\mathcal{A}} \end{cases}$$

is defined and smooth.

(b) Assume that $\delta_\kappa < R^{L,g_\kappa}_{V_\kappa,U_\kappa}$, and that $\mathcal{W}_\mathcal{B}$ satisfies (5.1.23.1), where $I = \mathcal{A}$ and $\beta_\kappa = \mathcal{L}^{g_\kappa}_{V_\kappa,\delta_\kappa}$ for $\kappa \in \mathcal{A}$. Then the map

$$L^{\mathcal{W}_\mathcal{B},g}_{V,\delta} := \prod_{\kappa\in\mathcal{A}} L^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa} : \begin{cases} \mathcal{C}^{\omega_\partial,k}_{\mathcal{W}_\mathcal{B}}(V_\kappa, B_{\delta_\kappa}(0))_{\kappa\in\mathcal{A}} \to \mathcal{C}^k_{\mathcal{W}_\mathcal{B}}(V_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} \\ \phi \mapsto (\lg_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, \phi_\kappa + \mathrm{id}_{V_\kappa}))_{\kappa\in\mathcal{A}} \end{cases}$$

is defined and smooth.

Proof. In both cases, we see that β_κ maps $V_\kappa \times \{0\}$ to $\{0\}$, for each $\kappa \in \mathcal{A}$. Hence the assertion follows from Proposition 5.1.23. □

We turn to composition and inversion.

Lemma 5.2.10. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ nonempty and $k, \ell \in \overline{\mathbb{N}}$. Further, let $r > 0$ and for each $\kappa \in \mathcal{A}$ let $W_\kappa, V_\kappa \subseteq U_\kappa$ be open nonempty sets such that $W_\kappa + B_r(0) \subseteq V_\kappa$ and $(\kappa^{-1}(W_\kappa))_{\kappa\in\mathcal{A}}$ is a cover of M. Then the map*

$$\mathfrak{c}^{\mathbb{R}^d,k}_{\mathcal{W}_\mathcal{B},\ell} := \prod_{\kappa\in\mathcal{A}} \mathfrak{c}^{\mathbb{R}^d,k}_{\mathcal{W}_\kappa,\ell} : \begin{cases} \mathcal{C}^{k+\ell+1}_{\mathcal{W}_\mathcal{B}}(V_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} \times \mathcal{C}^{\partial,k}_{\mathcal{W}_\mathcal{B}}(W_\kappa, B_r(0))_{\kappa\in\mathcal{A}} \to \mathcal{C}^{k}_{\mathcal{W}_\mathcal{B}}(W_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} \\ \qquad (\gamma,\eta) \mapsto (\gamma_\kappa \circ (\eta_\kappa + \mathrm{id}_{W_\kappa}))_{\kappa\in\mathcal{A}} \end{cases}$$

is defined and $\mathcal{C}^\ell$, where $\mathcal{B} := \{\kappa|^{V_\kappa}_{\kappa^{-1}(V_\kappa)} : \kappa \in \mathcal{A}\}$.

Proof. This is a direct consequence of Proposition 5.1.30. Note that $1_{\cup_{\kappa\in\mathcal{A}} V_\kappa}$ (eventually multiplied with $\frac{1}{r}$) is an adjusting weight for $(B_r(0))_{\kappa\in\mathcal{A}}$. □

Lemma 5.2.11. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, and $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ containing 1_M. Further, let $r > 0$ and for each $\kappa \in \mathcal{A}$ let $V_\kappa \subseteq U_\kappa$ be an open nonempty set such that $V_\kappa + B_r(0) \subseteq U_\kappa$ and $(\kappa^{-1}(V_\kappa))_{\kappa\in\mathcal{A}}$ is a cover of M. Then for each $\tau \in]0,1[$, the map*

$$I^V_{\mathcal{W}_\mathcal{A}} := \prod_{\kappa\in\mathcal{A}} I^{V_\kappa}_{\mathcal{W}_\kappa} : \mathcal{D}^\tau \to \mathcal{C}^\infty_{\mathcal{W}_\mathcal{A}}(V_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} : \phi \mapsto ((\phi_\kappa + \mathrm{id}_{U_\kappa})^{-1}|_{V_\kappa} - \mathrm{id}_{V_\kappa})_{\kappa\in\mathcal{A}}$$

is defined and smooth, where

$$\mathcal{D}^\tau := \left\{\phi \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{A}}(U_\kappa, \mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\phi\|_{1_{\cup_{\kappa\in\mathcal{A}} U_\kappa},1} < \tau \text{ and } \|\phi\|_{1_{\cup_{\kappa\in\mathcal{A}} U_\kappa},0} < \tfrac{r}{2}(1-\tau)\right\}.$$

Proof. This is a direct consequence of Proposition 5.1.31. □

Composing the operations We compose the maps that were introduced in this subsection. The main difficulty is keeping track of whether the simultaneous operations can be applied.

Lemma 5.2.12. *Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, g a Riemannian metric on M, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M and $r > 0$. For each $\kappa \in \mathcal{A}$, let $V_\kappa, W_\kappa \subseteq U_\kappa$ be open, nonempty, relatively compact sets with $W_\kappa + B_r(0) \subseteq V_\kappa$ such that $\overline{V_\kappa} \subseteq U_\kappa$ and $(\kappa^{-1}(W_\kappa))_{\kappa\in\mathcal{A}}$ is a cover of M. We set $\mathcal{B} := \{\kappa|^{V_\kappa}_{\kappa^{-1}(V_\kappa)} : \kappa \in \mathcal{A}\}$.*

For each $\kappa \in \mathcal{A}$, let $\delta^E_\kappa \in]0, R^E_{V_\kappa,U_\kappa}[$ and $\delta^L_\kappa \in]0, R^L_{W_\kappa,U_\kappa}[$. Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ such that $\mathcal{W}_\mathcal{B}$ contains an adjusting weight ω^E for

$$\left(\min(\delta^E_\kappa, \frac{1}{1 + C^{E,2}_{V_\kappa,\delta^E_\kappa,g_\kappa}}, \frac{1}{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}})\right)_{\kappa\in\mathcal{A}} \tag{†}$$

and ω^L that is adjusting for $(\delta^L_\kappa)_{\kappa\in\mathcal{A}}$, and satisfies

$$(\forall \kappa \in \mathcal{A})\, |\omega^L_\kappa| \leq \frac{1}{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa} C^{L,(1)}_{W_\kappa,\delta^L_\kappa,g_\kappa}} |\omega^E_\kappa|. \tag{5.2.12.1}$$

Additionally, assume that $\mathcal{W}_\mathcal{B}$ satisfies (5.1.23.1) for the families $(\mathcal{E}^{g_\kappa}_{V_\kappa,\delta^E_\kappa})_{\kappa\in\mathcal{A}}$ and $(\mathcal{L}^{g_\kappa}_{W_\kappa,\delta^L_\kappa})_{\kappa\in\mathcal{A}}$, respectively.

(a) Then the map

$$\mathbf{C}^L_E : D_1 \times D_2 \to R : (\gamma,\eta) \mapsto L^{\mathcal{W}_\mathcal{B},g}_{W,\delta^L}(\mathfrak{c}^{\mathbb{R}^d}_{\mathcal{W}_\mathcal{B}}(E^{\mathcal{W}_\mathcal{B},g}_{V,\delta^E}(\gamma), E^{\mathcal{W}_\mathcal{B},g}_{W,\delta^E}(\eta)) + E^{\mathcal{W}_\mathcal{B},g}_{W,\delta^E}(\eta))$$

is defined and smooth. Here

$$D_1 := \{\gamma \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{B}}(V_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\gamma\|_{\omega^E,0} < \tfrac{1}{2} \text{ and } \|\gamma\|_{1_{\cup_{\kappa\in\mathcal{A}} V_\kappa},1} < \tfrac{1}{2}\},$$

$$D_2 := \{\eta \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{B}}(W_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\eta\|_{\omega^E,0} < \min(\tfrac{1}{4}, r)\},$$

and

$$R := \{\phi \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{B}}(W_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\phi\|_{\omega^L,0} < 1\}.$$

Moreover, we have

$$\|\mathbf{C}^L_E(\gamma,\eta)\|_{\omega^L,0} \leq (1 + \|\gamma\|_{\omega^E,0} + \|\gamma\|_{1_{\cup_{\kappa\in\mathcal{A}} V_\kappa},1})\|\eta\|_{\omega^E,0} + \|\gamma\|_{\omega^E,0}. \tag{5.2.12.2}$$

For $X, Y \in \mathfrak{X}(M)$ such that $\iota^\mathcal{B}_\mathcal{W}(X) \in D_1$ and $\iota^\mathcal{B}_\mathcal{W}(Y) \in D_2$, we have for $\kappa \in \mathcal{A}$ that

$$\begin{aligned} L^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(\mathfrak{c}^{\mathbb{R}^d}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa), E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa)) + E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa)) \\ = \lg_{g_\kappa} \circ (\mathrm{id}_{W_\kappa}, \exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, X_\kappa) \circ \exp_{g_\kappa} \circ (\mathrm{id}_{W_\kappa}, Y_\kappa)). \end{aligned} \tag{5.2.12.3}$$

(b) Additionally, let $\rho \in]0,1[$. Then the map

$$\mathbf{I}^L_E : D_\rho \to R_\rho : \phi \mapsto L^{\mathcal{W}_\mathcal{B},g}_{W,\delta^L}(I^W_{\mathcal{W}_\mathcal{B}}(E^{\mathcal{W}_\mathcal{B},g}_{V,\delta^E}(\phi))$$

is defined and smooth. Here

$$D_\rho := \{\phi \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{B}}(V_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\phi\|_{\omega^E,0} < \tfrac{(1-\rho)}{2}\min(\rho, r) \text{ and } \|\phi\|_{1_{\cup_{\kappa\in\mathcal{A}} V_\kappa},1} < \tfrac{\rho}{2}\}$$

and

$$R_\rho := \{\phi \in \mathcal{C}^\infty_{\mathcal{W}_\mathcal{B}}(W_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}} : \|\phi\|_{\omega^L,0} < \tfrac{\min(r,\rho)}{2}\}.$$

Moreover, we have that

$$\|\mathbf{I}^L_E(\phi)\|_{\omega^L,0} \leq \frac{\|\phi\|_{\omega^E,0}}{1 - (\|\phi\|_{\omega^E,0} + \|\phi\|_{1_{\cup_{\kappa\in\mathcal{A}} V_\kappa},1})} \tag{5.2.12.4}$$

For $X \in \mathfrak{X}(M)$ such that $\iota^\mathcal{B}_\mathcal{W}(X) \in D_\rho$, we have for $\kappa \in \mathcal{A}$ that

$$L^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta^L_\kappa}(I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(X_\kappa))) = \lg_{g_\kappa} \circ (\mathrm{id}_{W_\kappa}, (\exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, X_\kappa))^{-1}|_{W_\kappa}). \tag{5.2.12.5}$$

Note that above we occasionally identified maps with their restriction.

Proof. (a) By our previous elaborations, the map $\mathbf{C}_E^L$ is smooth if it is defined, so we shall prove the latter. Let $\gamma \in D_1$ and $\eta \in D_2$. Since ω^E is an adjusting weight for $(\delta_\kappa^E)_{\kappa\in\mathcal{A}}$ and $\|\gamma\|_{\omega^E,0}, \|\eta\|_{\omega^E,0} < 1$ by our assumptions, we see with (5.1.18.1) that $\gamma \in \mathcal{C}_{\mathcal{W}_\mathcal{B}}^{\omega^E\partial,\infty}(V_\kappa, B_{\delta_\kappa^E}(0))_{\kappa\in\mathcal{A}}$ and $\eta \in \mathcal{C}_{\mathcal{W}_\mathcal{B}}^{\omega^E\partial,\infty}(W_\kappa, B_{\delta_\kappa^E}(0))_{\kappa\in\mathcal{A}}$. Hence we can apply $E_{V,\delta^E}^{\mathcal{W}_\mathcal{B},g}$ to γ and $E_{W,\delta^E}^{\mathcal{W}_\mathcal{B},g}$ to η.
We see with Lemma B.3.8 using that ω^E is adjusted to (†) (and (5.1.17.1)) that for $\kappa \in \mathcal{A}$,

$$\|E_{W_\kappa,\delta_\kappa}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)\|_{1_{W_\kappa},0} \leq \frac{C^{E,(1)}_{W_\kappa,\delta_\kappa^E,g_\kappa}}{C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}}\|\eta_\kappa\|_{\omega_\kappa^E,0} < r.$$

This implies that we can apply $\mathfrak{c}_{\mathcal{W}_\mathcal{B}}^{\mathbb{R}^d}$ to $(E_{V,\delta}^{\mathcal{W}_\mathcal{B},g}(\gamma), E_{W,\delta}^{\mathcal{W}_\mathcal{B},g}(\eta))$. Further, we conclude from Lemma B.3.9 using (5.1.17.1) that for each $\kappa \in \mathcal{A}$,

$$\|E_{V_\kappa,\delta_\kappa}^{\mathcal{W}_\kappa,g_\kappa}(\gamma_\kappa)\|_{1_{V_\kappa},1} \leq C^{E,2}_{V_\kappa,\delta_\kappa^E,g_\kappa}\|\gamma_\kappa\|_{1_{V_\kappa},0} + \|\gamma_\kappa\|_{1_{V_\kappa},1} \leq \frac{C^{E,2}_{V_\kappa,\delta_\kappa^E,g_\kappa}}{1+C^{E,2}_{V_\kappa,\delta_\kappa^E,g_\kappa}}\|\gamma_\kappa\|_{\omega_\kappa^E,0} + \|\gamma_\kappa\|_{1_{V_\kappa},1}.$$

Using estimate (4.1.3.1), the last estimate, Lemma B.3.8 (and the triangle inequality) we see that for $f \in \mathcal{W}$ and $\kappa \in \mathcal{A}$,

$$\begin{aligned}
&\|\mathfrak{c}_{\mathcal{W}_\kappa}^{\mathbb{R}^d}(E_{V_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\gamma_\kappa), E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)) + E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)\|_{f_\kappa,0}\\
\leq&\|E_{V_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\gamma_\kappa)\|_{1_{V_\kappa},1}\|E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)\|_{f_\kappa,0} + \|E_{V_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\gamma_\kappa)\|_{f_\kappa,0} + \|E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)\|_{f_\kappa,0} \qquad (*)\\
\leq&C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}((\|\gamma_\kappa\|_{\omega_\kappa^E,0} + \|\gamma_\kappa\|_{1_{V_\kappa},1} + 1)\|\eta_\kappa\|_{f_\kappa,0} + \|\gamma_\kappa\|_{f_\kappa,0}).
\end{aligned}$$

From this estimate we derive using (5.2.12.1) and $\|\gamma_\kappa\|_{\omega_\kappa^E,0}, \|\gamma_\kappa\|_{1_{V_\kappa},1} < \frac{1}{2}$, $\|\eta_\kappa\|_{\omega_\kappa^E,0} < \frac{1}{4}$ that

$$\begin{aligned}
&\|\mathfrak{c}_{\mathcal{W}_\kappa}^{\mathbb{R}^d}(E_{V_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\gamma_\kappa), E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)) + E_{W_\kappa,\delta_\kappa^E}^{\mathcal{W}_\kappa,g_\kappa}(\eta_\kappa)\|_{\omega_\kappa^L,0}\\
&< C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}(2\|\eta_\kappa\|_{\omega_\kappa^L,0} + \|\gamma_\kappa\|_{\omega_\kappa^L,0}) \leq \frac{C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}}{C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}C^{L,(1)}_{W_\kappa,\delta_\kappa^L,g_\kappa}}(2\|\eta_\kappa\|_{\omega_\kappa^E,0} + \|\gamma_\kappa\|_{\omega_\kappa^E,0}) < 1.
\end{aligned}$$

We conclude with (5.1.18.1) that we can apply $L_{W,\delta^L}^{\mathcal{W}_\mathcal{B},g}$ to $\mathfrak{c}_{\mathcal{W}_\mathcal{B}}^{\mathbb{R}^d}(E_{V,\delta^E}^{\mathcal{W}_\mathcal{B},g}(\gamma), E_{W,\delta^E}^{\mathcal{W}_\mathcal{B},g}(\eta)) + E_{W,\delta^E}^{\mathcal{W}_\mathcal{B},g}(\eta)$. Further, for each $\kappa \in \mathcal{A}$ we have (using Lemma B.3.18 and (5.2.12.1)) that

$$\|L_{W_\kappa,\delta_\kappa^L}^{\mathcal{W}_\kappa,g_\kappa}(\phi)\|_{\omega_\kappa^L,0} \leq \frac{C^{L,(1)}_{V_\kappa,\delta_\kappa^L,g_\kappa}}{C^{E,(1)}_{V_\kappa,\delta_\kappa^E,g_\kappa}C^{L,(1)}_{W_\kappa,\delta_\kappa^L,g_\kappa}}\|\phi\|_{\omega_\kappa^E,0}$$

for suitable ϕ. From this and $(*)$ we derive the assertion on the containment in R, and also that (5.2.12.2) holds. It remains to prove (5.2.12.3). To this end, let $p \in W_\kappa$. Then

$$\begin{aligned} &L^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(\mathfrak{c}^{\mathbb{R}^d}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa), E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa)) + E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa))(p)\\ =&\lg_{g_\kappa}(p, \mathfrak{c}^{\mathbb{R}^d}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa), E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa))(p) + E^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(Y_\kappa)(p) + p)\\ =&\lg_{g_\kappa}(p, E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa)(\exp_{g_\kappa}(p, Y_\kappa(p))) + \exp_{g_\kappa}(p, Y_\kappa(p)))\\ =&\lg_{g_\kappa}(p, \exp_{g_\kappa}(\exp_{g_\kappa}(p, Y_\kappa(p)), X_\kappa(\exp_{g_\kappa}(p, Y_\kappa(p)))))\\ =&\lg_{g_\kappa}(p, (\exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, X_\kappa))(\exp_{g_\kappa}(p, Y_\kappa(p))))\\ =&\lg_{g_\kappa}(p, (\exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, X_\kappa) \circ \exp_{g_\kappa} \circ (\mathrm{id}_{W_\kappa}, Y_\kappa))(p)). \end{aligned}$$

This shows that (5.2.12.3) holds.

(b) By our previous elaborations, the map $\mathbf{I}^L_E$ is smooth if it is defined, so we shall prove the latter. Let $\phi \in D_\rho$. Since ω^E is an adjusting weight for $(\delta^E_\kappa)_{\kappa\in\mathcal{A}}$ and $\|\phi\|_{\omega^E,0} < 1$ by our assumptions, we see with (5.1.18.1) that $\phi \in \mathcal{C}^{\omega_\partial,\infty}_{\mathcal{W}_\mathcal{B}}(V_\kappa, B_{\delta^E_\kappa}(0))_{\kappa\in\mathcal{A}}$. Hence we can apply $E^{\mathcal{W}_\mathcal{B},g}_{V,\delta^E}$ to ϕ. We see with Lemma B.3.8 using that ω^E is adjusting for (†) (and (5.1.17.1)) that for each $\kappa \in \mathcal{A}$,

$$\|E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa)\|_{1_{V_\kappa},0} < \frac{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}\|\phi\|_{\omega^E,0} < \tfrac{r}{2}(1-\rho).$$

Similarly, we conclude from Lemma B.3.9 and (5.1.17.1) that for each $\kappa \in \mathcal{A}$,

$$\|E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa)\|_{1_{V_\kappa},1} \leq C^{E,2}_{V_\kappa,\delta^E_\kappa,g_\kappa}\|\phi_\kappa\|_{1_{V_\kappa},0} + \|\phi_\kappa\|_{1_{V_\kappa},1} \leq \frac{C^{E,2}_{V_\kappa,\delta^E_\kappa,g_\kappa}}{C^{E,2}_{V_\kappa,\delta^E_\kappa,g_\kappa}+1}\|\phi_\kappa\|_{\omega^E_\kappa,0} + \|\phi_\kappa\|_{1_{V_\kappa},1}. \tag{$*$}$$

Hence we see using $\|\phi_\kappa\|_{\omega^E_\kappa,0}, \|\phi_\kappa\|_{1_{V_\kappa},1} < \frac{\rho}{2}$ and the last two estimates that we can apply $I^W_{\mathcal{W}_\mathcal{B}}$ to $E^{\mathcal{W}_\mathcal{B},g}_{V,\delta^E}(\phi)$. Further, using estimate (4.2.5.1), Lemma B.3.8 and ($*$) we see that for $f \in \mathcal{W}$ and $\kappa \in \mathcal{A}$,

$$\|I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa))\|_{f_\kappa,0} \leq \frac{\|E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa)\|_{f_\kappa,0}}{1-\|E^{\mathcal{W}_\kappa,g_\kappa}_{V,\delta^E_\kappa}(\phi_\kappa)\|_{1_{W_\kappa},1}} < \frac{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}{1-(\|\phi_\kappa\|_{\omega^E_\kappa,0}+\|\phi_\kappa\|_{1_{V_\kappa},1})}\|\phi_\kappa\|_{f_\kappa,0}. \tag{$**$}$$

From this, we conclude with (5.2.12.1), using $\|\phi_\kappa\|_{\omega^E_\kappa,0} < \frac{\rho(1-\rho)}{2}$ and $\|\phi_\kappa\|_{1_{V_\kappa},1} < \frac{\rho}{2}$, that

$$\|I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa))\|_{\omega^L_\kappa,0} \leq \frac{C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}{(1-\rho)C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}\|\phi_\kappa\|_{\omega^E_\kappa,0} < \frac{\rho}{2}.$$

Since ω^L is adjusting to δ^L, we see from this estimate using (5.1.18.1) that we can apply $L^{\mathcal{W}_\mathcal{B},g}_{W,\delta^L}$ to ϕ. Another application of Lemma B.3.18 and ($**$) shows that

$$\begin{aligned} &\|L^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta^L_\kappa}(I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa)))\|_{\omega^L_\kappa,0} \leq C^{L,(1)}_{W_\kappa,\delta^L_\kappa,g_\kappa}\|I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta^E_\kappa}(\phi_\kappa))\|_{\omega^L_\kappa,0}\\ &\qquad < \frac{C^{L,(1)}_{W_\kappa,\delta^L_\kappa,g_\kappa}C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}{1-(\|\phi_\kappa\|_{\omega^E_\kappa,0}+\|\phi_\kappa\|_{1_{V_\kappa},1})}\|\phi_\kappa\|_{\omega^L_\kappa,0} \leq \frac{\|\phi_\kappa\|_{\omega^E_\kappa,0}}{1-(\|\phi_\kappa\|_{\omega^E_\kappa,0}+\|\phi_\kappa\|_{1_{V_\kappa},1})} \end{aligned}$$

So we derive the assertion on the containment in R_ρ, and that (5.2.12.4) holds. To prove (5.2.12.5), let $\kappa \in \mathcal{A}$ and $p \in W_\kappa$. Then

$$L^{\mathcal{W}_\kappa,g_\kappa}_{W_\kappa,\delta_\kappa}(I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa)))(p) = \mathrm{lg}_{g_\kappa}(p, I^{W_\kappa}_{\mathcal{W}_\kappa}(E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa))(p) + p)$$
$$= \mathrm{lg}_{g_\kappa}(p, (E^{\mathcal{W}_\kappa,g_\kappa}_{V_\kappa,\delta_\kappa}(X_\kappa) + \mathrm{id}_{V_\kappa})^{-1}|_{W_\kappa}(p)) = \mathrm{lg}_{g_\kappa}(p, (\exp_{g_\kappa} \circ (\mathrm{id}_{V_\kappa}, X_\kappa))^{-1}|_{W_\kappa}(p)).$$

This shows that (5.2.12.5) holds. □

5.2.3. Construction of weights on manifolds

We first define the terms saturated resp. adjusted sets of weights, and then show that such weight sets exist.

Definition 5.2.13 (Saturated and adjusted sets of weights)**.** Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M and $\delta_\kappa > 0$ for each $\kappa \in \mathcal{A}$.

(a) We call $\omega : M \to \mathbb{R}$ *adjusted to* $(\mathcal{A}, (\delta_\kappa)_{\kappa\in\mathcal{A}})$ if there exists $K > 0$ such that $K \cdot \omega_{\mathcal{A}}$ is an adjusting weight for $(\delta_\kappa)_{\kappa\in\mathcal{A}}$. We call $\mathcal{W}$ *adjusted to* $(\mathcal{A}, (\delta_\kappa)_{\kappa\in\mathcal{A}})$ if there exists $\omega \in \mathcal{W}$ that is adjusted to this pair.

(b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be atlases for M. We say $\mathcal{W}$ is *saturated with respect to* $(\mathcal{A}_1, \mathcal{A}_2)$ if (5.1.20.1) is satisfied for $\mathcal{W}_{\mathcal{A}_1\cap\mathcal{A}_2}$ and $(D(\kappa \circ \phi^{-1})|_{\phi(\widetilde{U}_\kappa\cap\widetilde{U}_\phi)})_{(\kappa,\phi)\in\mathcal{A}_1\otimes\mathcal{A}_2}$.

(c) Let g be a Riemannian metric on M, and for each $\kappa \in \mathcal{A}$ let $\widetilde{V}_\kappa$ be a relatively compact set with $\overline{\widetilde{V}_\kappa} \subseteq \widetilde{U}_\kappa$ such that $(\widetilde{V}_\kappa)_{\kappa\in\mathcal{A}}$ is a cover of M, $\delta^E_\kappa \in]0, R^{E,g_\kappa}_{V_\kappa,U_\kappa}[$, $\delta^L_\kappa \in]0, R^{L,g_\kappa}_{V_\kappa,U_\kappa}[$ (where $V_\kappa := \kappa(\widetilde{V}_\kappa)$) and $\mathcal{B} := \{\kappa|^{V_\kappa}_{\widetilde{V}_\kappa} : \kappa \in \mathcal{A}\}$. We say $\mathcal{W}$ is *saturated with respect to* $(\mathcal{A}, \mathcal{B}, g)$ if $\mathcal{W}_{\mathcal{B}}$ satisfies (5.1.23.1) for $(\mathcal{E}^{g_\kappa}_{V_\kappa,\delta^E_\kappa})_{\kappa\in\mathcal{A}}$ and $(\mathcal{L}^{g_\kappa}_{V_\kappa,\delta^L_\kappa})_{\kappa\in\mathcal{A}}$ respectively.

If both (b) and (c) hold, we call $\mathcal{W}$ *saturated with respect to* $((\mathcal{A}_1, \mathcal{A}_2), (\mathcal{A}, \mathcal{B}, g))$. Occasionally, we may just say that $\mathcal{W}$ is adjusted or saturated.

Construction of adjusted weights We show that for a locally finite atlas whose chart domains are relatively compact, adjusted weights exist.

Lemma 5.2.14 (Construction of an adjusted weight)**.** *Let $d \in \mathbb{N}^*$, M be a d-dimensional manifold and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ a locally finite atlas for M. For each $\kappa \in \mathcal{A}$, let $\varepsilon_\kappa > 0$ and $\widetilde{V}_\kappa \subseteq \widetilde{U}_\kappa$ be an open, nonempty, relatively compact set such that $(\widetilde{V}_\kappa)_{\kappa\in U}$ is a cover of M. Then there exists a weight $\omega : M \to \mathbb{R}$ adjusted to $(\mathcal{B}, (\varepsilon_\kappa)_{\kappa\in\mathcal{A}})$, where $\mathcal{B} := \{\kappa|^{\kappa(\widetilde{V}_\kappa)}_{\widetilde{V}_\kappa} : \kappa \in \mathcal{A}\}$.*

Proof. For each $\kappa \in \mathcal{A}$, let $f_\kappa : M \to \mathbb{R}$ be a function such that $\mathrm{supp}(f_\kappa) \subseteq \widetilde{U}_\kappa$, $\sup_{x\in M}|f_\kappa|(x) < \infty$ and $\inf_{x\in\widetilde{V}_\kappa}|f_\kappa|(x) \geq \max(\frac{1}{\varepsilon_\kappa}, 1)$. For $x \in M$, we set

$$\omega(x) := \max_{\kappa\in\mathcal{A}}|f_\kappa(x)|;$$

note that this definition is possible because each $x \in M$ is only contained in finitely many sets $\widetilde{U}_\kappa$. Then $|\omega|(x) \geq |f_\kappa|(x) \geq \max(\frac{1}{\varepsilon_\kappa}, 1)$ for each $\kappa \in \mathcal{A}$ and $x \in \widetilde{V}_\kappa$. Further, since each $\widetilde{V}_\kappa$ is relatively compact, it has nonempty intersections with only finitely many sets $\{\widetilde{U}_\kappa : \kappa \in \mathcal{A}\}$. That implies that $\sup_{x\in\widetilde{V}_\kappa} |\omega|(x) < \infty$. Hence $\omega_{\mathcal{B}}$ is an adjusting weight for $(\varepsilon_\kappa)_{\kappa\in\mathcal{A}}$. □

Saturating weights We not only show that saturated weight sets exist, but moreover that each set of weights has a "minimal saturated extension". We first prove a variation of this assertion for a single weight.

Lemma 5.2.15. *Let $d \in \mathbb{N}^*$, M be a d-dimensional manifold, $(\widetilde{U}_\kappa)_{\kappa\in\mathcal{A}}$ a locally finite cover of M, I a nonempty set, $f : M \to \overline{\mathbb{R}}$ and $(B_{\kappa,i})_{(\kappa,i)\in\mathcal{A}\times I}$ a family of nonnegative real numbers. Then there exists a set $\mathcal{M}_{f,B} \subseteq \overline{\mathbb{R}}^M$ such that*

$$(\forall x \in M)(\exists V \in \mathcal{U}(x))(\forall g \in \mathcal{M}_{f,B})(\exists K > 0)\, |g|_V| \leq K \cdot |f|_V|$$

and

$$(\forall i \in I)(\exists g \in \mathcal{M}_{f,B})(\forall \kappa \in \mathcal{A})\, B_{\kappa,i} \cdot \left|f|_{\widetilde{U}_\kappa}\right| \leq \left|g|_{\widetilde{U}_\kappa}\right|. \tag{5.2.15.1}$$

The set $\mathcal{M}_{f,B}$ is minimal in the sense that for any $\mathcal{H} \subseteq \overline{\mathbb{R}}^M$ that also satisfies (5.2.15.1), *we have*

$$(\forall g \in \mathcal{M}_{f,B})(\exists h \in \mathcal{H})\, |g| \leq |h|. \tag{5.2.15.2}$$

Proof. Let $i \in I$ and $x \in M$. Then we define

$$g_i(x) := \max\{B_{\kappa,i} \cdot f(x) : \kappa \in \mathcal{A}, x \in \widetilde{U}_\kappa\}.$$

This definition makes sense since $(\widetilde{U}_\kappa)_{\kappa\in U}$ is locally finite. In particular, for each $x \in M$ there exist $\kappa_1, \ldots, \kappa_n \in \mathcal{A}$ and $V \in \mathcal{U}(x)$ such that for $\kappa \in \mathcal{A}$,

$$\widetilde{U}_\kappa \cap V \neq \emptyset \iff \kappa \in \{\kappa_1, \ldots, \kappa_n\}.$$

Hence

$$|g_i|_V| \leq \max(B_{\kappa_1,i}, \ldots, B_{\kappa_n,i}) \cdot |f|_V|.$$

Further, for $\hat{\kappa} \in \mathcal{A}$ such that $x \in \widetilde{U}_{\hat{\kappa}}$, we have

$$B_{\hat{\kappa},i} \cdot |f(x)| \leq \max\{B_{\kappa,i} : \kappa \in \mathcal{A}, x \in \widetilde{U}_\kappa\} \cdot |f(x)| = |g_i(x)|.$$

So the set

$$\mathcal{M}_{f,B} := \{g_i : i \in I\},$$

has the first two properties. To prove the minimality, let $\mathcal{H} \subseteq \overline{\mathbb{R}}^M$ satisfying (5.2.15.1). Then for each $i \in I$, there exists $h \in \mathcal{H}$ such that $B_{\kappa,i} \cdot \left|f|_{\widetilde{U}_\kappa}\right| \leq \left|h|_{\widetilde{U}_\kappa}\right|$ for all $\kappa \in \mathcal{A}$. So for $x \in M$, we have

$$(\forall \kappa \in \mathcal{A} : x \in \widetilde{U}_\kappa)\, B_{\kappa,i}|f(x)| \leq |h(x)|.$$

Hence

$$|g_i(x)| = \max\{B_{\kappa,i} \cdot |f(x)| : \kappa \in \mathcal{A}, x \in \widetilde{U}_\kappa\} \leq |h(x)|,$$

which finishes the proof. □

Remark 5.2.16. In the last lemma, we proved that $|g|_V| \leq K \cdot |f|_V|$ for every neighborhood V that has nonempty intersection with only finitely many cover sets.

Before we show that each weight set has a minimal saturated superset, we make the following definition.

Definition 5.2.17. Let M be a topological space and $f, g : M \to \overline{\mathbb{R}}$. We call g *locally f-bounded* if

$$(\forall x \in M)(\exists U \in \mathcal{U}(x), K > 0)\, |g|_U| \leq K \cdot |f|_U|.$$

Let $\mathcal{W}_1, \mathcal{W}_2 \subseteq \overline{\mathbb{R}}^M$. We call $\mathcal{W}_2$ *locally $\mathcal{W}_1$-bounded* if for all $g \in \mathcal{W}_2$ there exists $f \in \mathcal{W}_1$ such that g is locally f-bounded. As usual, we call f *locally bounded* if it us locally 1_M-bounded.

In the next lemma, we need the definition of the maximal extension of weights, see Definition 6.1.3.

Lemma 5.2.18 (Minimal saturated extension)**.** *Let $d \in \mathbb{N}^*$, (M, g) a d-dimensional Riemannian manifold, $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$, $\widetilde{\mathcal{A}}$ locally finite atlases for M. For each $\kappa \in \mathcal{A}$, let $\widetilde{V}_\kappa$ a relatively compact set such that $\overline{\widetilde{V}_\kappa} \subseteq \widetilde{U}_\kappa$ and $\mathcal{B} := \{\kappa|_{\widetilde{V}_\kappa}^{V_\kappa} : \kappa \in \mathcal{A}\}$ is an atlas for M (here, $V_\kappa := \kappa(\widetilde{V}_\kappa)$ for $\kappa \in \mathcal{A}$), $\delta_\kappa^E \in]0, R_{V_\kappa, U_\kappa}^{E, g_\kappa}[$ and $\delta_\kappa^L \in]0, R_{V_\kappa, U_\kappa}^{L, g_\kappa}[$. Then there exists $\mathcal{W}^e \subseteq \overline{\mathbb{R}}^M$ that is locally $\mathcal{W}$-bounded and saturated w.r.t. $((\mathcal{A}, \mathcal{B}, g), (\mathcal{A}, \widetilde{\mathcal{A}}))$.*

The set $\mathcal{W}^e$ is minimal in the sense that for any $\mathcal{G} \subseteq \overline{\mathbb{R}}^M$ that is saturated to the same data and contains $\mathcal{W}$, we have $\mathcal{W}^e \subseteq \mathcal{G}_{\max}$. We call $\mathcal{W}^e$ a minimal saturated extension *of $\mathcal{W}$.*

Proof. We define the following three families:

$$B^1 : \mathcal{A} \times \mathbb{N}^* \to [0, \infty[: (\kappa, \ell) \mapsto \|\mathcal{E}_{V_\kappa, \delta_\kappa^E}^{g_\kappa}\|_{1_{V_\kappa \times B_{\delta_\kappa^E}(0)}, \ell},$$

$$B^2 : \mathcal{A} \times \mathbb{N}^* \to [0, \infty[: (\kappa, \ell) \mapsto \|\mathcal{L}_{V_\kappa, \delta_\kappa^L}^{g_\kappa}\|_{1_{V_\kappa \times B_{\delta_\kappa^L}(0)}, \ell},$$

$$B^3 : \mathcal{A} \otimes \widetilde{\mathcal{A}} \times \mathbb{N} \to [0, \infty[: ((\kappa, \phi), \ell) \mapsto \|D(\kappa \circ \phi^{-1})|_{\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)}\|_{1_{\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)}, \ell}.$$

We inductively define $\mathcal{W}_0 := \mathcal{W}$, and if $\mathcal{W}_k$ is defined for $k \in \mathbb{N}$, we set

$$\mathcal{W}_{k+1} := \bigcup_{f \in \mathcal{W}_k} \mathcal{M}_{f, B^1} \cup \bigcup_{f \in \mathcal{W}_k} \mathcal{M}_{f, B^2} \cup \bigcup_{f \in \mathcal{W}_k} \mathcal{M}_{f, B^3};$$

we defined $\mathcal{M}_{f, B^i}$ in Lemma 5.2.15. Finally, we set $\mathcal{W}^e := \bigcup_{k \in \mathbb{N}} \mathcal{W}_k$. Since we can show with an easy induction (using Lemma 5.2.15, of course) that each $\mathcal{W}_k$ is locally $\mathcal{W}$-bounded, so is $\mathcal{W}^e$. Finally, we see with another application of Lemma 5.2.15 that (c) and (b) in Definition 5.2.13 are satisfied.

We prove the minimality condition by induction. More precisely, we prove that $\mathcal{W}_k \subseteq \mathcal{G}_{\max}$ for all $k \in \mathbb{N}$. The case $k = 0$ is satisfied by the assumptions on $\mathcal{G}$. Suppose it holds for $k \in \mathbb{N}$, and let $f \in \mathcal{W}_k$. Because $\mathcal{G}$ is saturated, it satisfies (5.2.15.1) for f and the $B \in \{B^j : j = 1, 2, 3\}$. Hence we derive from (5.2.15.2) that $\mathcal{M}_{f, B^j} \subseteq \mathcal{G}_{\max}$ for $j \in \{1, 2, 3\}$, so obviously $\mathcal{W}_{k+1} \subseteq \mathcal{G}_{\max}$. ∎

5.3. Diffeomorphisms on Riemannian manifolds

We construct *weighted diffeomorphisms* on Riemannian manifolds, and turn them into a Lie group that is modelled on weighted vector fields. In order to do this, we prove a criterion when the composition of the exponential function with a vector field is a diffeomorphism. Then, we can use the local group operations treated in Lemma 5.2.12 to construct Lie group structures from local data. We state the main result concerning these Lie groups in Theorem 5.3.6. Finally, we compare these Lie groups with other well-known Lie groups of diffeomorphisms.

5.3.1. Generating diffeomorphisms from vector fields

In Lemma B.3.10, we established under which conditions the map $\exp_{g_\kappa} \circ X_\kappa$ is a diffeomorphism, where κ is a chart and X_κ a localized vector field. We show that similar assumptions also allow the global behavior of $\exp_g \circ X$ to be controlled.

Proposition 5.3.1. *Let (M, g) be a Riemannian manifold and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M. For each $\kappa \in \mathcal{A}$, let $r_\kappa > 0$ such that $\overline{B}_{r_\kappa}(0) \subseteq U_\kappa$ and $\{\kappa^{-1}(B_{r_\kappa}(0)) : \kappa \in \mathcal{A}\}$ is a cover of M, $\nu_\kappa \in]0, R^{E,g_\kappa}_{B_{r_\kappa}(0),U_\kappa}[$ and $\varepsilon_\kappa \in]0, \frac{1}{2}[$. Further let $k \in \overline{\mathbb{N}}$ with $k \geq 1$ and $X \in \mathfrak{X}^k(M)$ such that for each $\kappa \in \mathcal{A}$,*

$$\|X_\kappa\|_{1_{B_{r_\kappa}(0)},0} < \min\Big(\frac{\varepsilon_\kappa r_\kappa}{2C^{E,(1)}_{B_{r_\kappa}(0),\nu_\kappa,g_\kappa}}, \nu_\kappa, \frac{\varepsilon_\kappa}{4(C^{E,2}_{B_{r_\kappa}(0),\nu_\kappa,g_\kappa}+1)}\Big),$$

and $\|X_\kappa\|_{1_{B_{r_\kappa}(0)},1} < \frac{\varepsilon_\kappa}{4}$. Then the following assertions hold:

- *(a) We have that $\operatorname{im} X \subseteq D^E_g$, the map $\phi_X := \exp_g \circ X$ maps each connected component of M into itself, and for each $\kappa \in \mathcal{A}$, $\kappa \circ \phi_X \circ \kappa^{-1}|_{B_{r_\kappa}(0)}$ is a $\mathcal{C}^k$-diffeomorphism whose image contains $B_{r_\kappa(1-2\varepsilon_\kappa)}(0)$.*
- *(b) For each $y \in M$, $\#(\exp_g \circ X)^{-1}(y) \leq \#\mathcal{A}_y$, where $\mathcal{A}_y := \{\kappa \in \mathcal{A} : y \in \widetilde{U}_\kappa\}$.*

In addition, assume that M is connected, $\{\widetilde{U}_\kappa : \kappa \in \mathcal{A}\}$ is a locally finite cover of M and $\{\kappa^{-1}(B_{\widetilde{r_\kappa}}(0)) : \kappa \in \mathcal{A}\}$ is also a cover of M, where each $\widetilde{r_\kappa} < (1 - \varepsilon_\kappa)r_\kappa$. Then

- *(c) $\exp_g \circ X$ is a proper map.*

Assume that each $\widetilde{r_\kappa} < (1 - 2\varepsilon_\kappa)r_\kappa$. Then

- *(d) $\exp_g \circ X$ is a covering map with finitely many sheets.*

Finally, assume that there exists a point in M that is only contained in one $\widetilde{U}_\kappa$. Then

- *(e) $\exp_g \circ X$ is a diffeomorphism.*

Proof. (a) Since $\|X_\kappa\|_{1_{B_{r_\kappa}(0)},0} < \nu_\kappa$ for each $\kappa \in \mathcal{A}$, we see with Remark B.3.5 that

$$\mathbf{T}\kappa^{-1}((\mathrm{id}_{U_\kappa}, X_\kappa))(B_{r_\kappa}(0)) \subseteq \mathbf{T}\kappa^{-1}(D^E_{g_\kappa}) \subseteq D^E_g.$$

It is obvious from the definition of $\exp_g$ that ϕ_X maps each connected component of M into itself. By our assumptions, for each $\kappa \in \mathcal{A}$ we can apply Lemma B.3.10 to the function X_κ and the exponential function $\exp_{g_\kappa}$ to see that $\phi_{X_\kappa}|_{B_{r_\kappa}(0)}$ is a $\mathcal{C}^k$-diffeomorphism whose image contains $B_{r_\kappa(1-2\varepsilon_\kappa)}(0)$ (here $\phi_{X_\kappa} := \exp_{g_\kappa} \circ (\mathrm{id}_{U_\kappa}, X_\kappa)$). Since $\kappa \circ \phi_X \circ \kappa^{-1}|_{B_{r_\kappa}(0)} = \phi_{X_\kappa}|_{B_{r_\kappa}(0)}$ by Lemma B.3.6, the assertion holds.

(b) Let $y \in M$. For $\kappa \in \mathcal{A}$, we set $W_\kappa := \kappa^{-1}(B_{r_\kappa}(0))$ and define

$$\mathcal{A}_{X,y} := \{\kappa \in \mathcal{A} : (\exists x \in W_\kappa)\, \phi_X(x) = y\}.$$

Since the map $\phi_X|_{W_\kappa}$ is injective for each $\kappa \in \mathcal{A}_{X,y}$, there exists at most one $x_\kappa \in W_\kappa$ with $\phi_X(x_\kappa) = y$. Further $y = \phi_X(x_\kappa) \in \phi_X(W_\kappa) \subseteq \widetilde{U}_\kappa$ (since $\|X_\kappa\|_{1_{B_{r_\kappa}(0)},0} < \nu_\kappa$), hence $\mathcal{A}_{X,y} \subseteq \mathcal{A}_y$. The map $\mathcal{A}_{X,y} \to \phi_X^{-1}(y) : \kappa \mapsto x_\kappa$ is surjective because $\{W_\kappa : \kappa \in \mathcal{A}\}$ is a cover of M, so we derive the assertion.

(c) Let $K \subseteq M$ be a compact set. Since $\{\widetilde{U}_\kappa : \kappa \in \mathcal{A}\}$ is a locally finite cover, using a straightforward compactness argument we can show that there exists a finite set $F \subseteq \mathcal{A}$ such that for $\kappa \in \mathcal{A}$ the equivalence

$$\widetilde{U}_\kappa \cap K \neq \emptyset \iff \kappa \in F$$

holds. We then define

$$\widetilde{K} := \bigcup_{\kappa \in F} \kappa^{-1}\big((\kappa(K) \cap \overline{B}_{\widetilde{r_\kappa} + \frac{\varepsilon_\kappa r_\kappa}{2}}(0)) + \overline{B}_{\frac{\varepsilon_\kappa r_\kappa}{2}}(0)\big).$$

This is a compact set and we prove that it contains $\phi_X^{-1}(K)$. To this end, let $y \in \phi_X^{-1}(K)$. Then there exists $\kappa \in \mathcal{A}$ such that $y \in \kappa^{-1}(B_{\widetilde{r_\kappa}}(0))$, and by our assumptions on X, we have that $\phi_X(y) \in \widetilde{U}_\kappa$, hence $\kappa \in F$. Further, using Lemma B.3.8 we get

$$\|\kappa(\phi_X(y)) - \kappa(y)\| = \|\phi_{X_\kappa}(\kappa(y)) - \kappa(y)\| \leq \frac{\varepsilon_\kappa r_\kappa}{2}.$$

This implies that

$$\|\kappa(\phi_X(y))\| \leq \|\kappa(y)\| + \|\kappa(\phi_X(y)) - \kappa(y)\| < \widetilde{r_\kappa} + \frac{\varepsilon_\kappa r_\kappa}{2}.$$

So we see that

$$\kappa(y) = \kappa(\phi_X(y)) + \kappa(y) - \kappa(\phi_X(y)) \in \kappa(K) \cap \overline{B}_{\widetilde{r_\kappa} + \frac{\varepsilon_\kappa r_\kappa}{2}}(0) + \overline{B}_{\frac{\varepsilon_\kappa r_\kappa}{2}}(0),$$

which shows that $y \in \widetilde{K}$.

(d) ϕ_X is surjective since the image of $\phi_X|_{\kappa^{-1}(B_{r_\kappa}(0))}$ contains $\kappa^{-1}(B_{(1-2\varepsilon_\kappa)r_\kappa}(0))$ by (a), and these sets cover M by assumption. Since we also proved in (a) that ϕ_X is a local homeomorphism and is a proper map by (c), we can use [For81, Theorem 4.22] to see that it is a covering map.

(e) We showed in (a) that ϕ_X is a local diffeomorphism, and by (d) it is a covering map. We see with the hypothesis of (e) and the assertion of (b) that it has only one sheet, so it is a bijection and hence a diffeomorphism. □

5.3.2. Lie groups of weighted diffeomorphisms

We show that on a Riemannian manifold, for each locally finite, *adapted* atlas $\mathcal{A}$ (we will introduce this terminology soon) and each set $\mathcal{W}$ of weights containing 1_M, there exists a Lie group of *weighted diffeomorphisms.* The Lie group is modelled on the space $\mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_{\mathcal{A}}$ of weighted vector fields, where $\mathcal{W}^e$ is a minimal saturated extension of $\mathcal{W} \cup \{\omega\}$, where ω is a suitable adjusted weight.

We then examine under which conditions the compactly supported diffeomorphisms are a subset of the weighted diffeomorphisms, and see that if the manifold is $\mathbb{R}^d$ with the scalar product, the weighted diffeomorphisms constructed here are the same as in [Wal13].

Lie groups modelled on weighted vector fields

We first transfer the results of Lemma 5.2.12 to weighted vector fields. For the inversion, before the introduction of Proposition 5.3.1 this was not possible since we had only developed criteria for local invertibility. Further, we use these results to construct a Lie group modelled on weighted vector fields. Note that we assume the existence of suitable weights, but even with Lemma 5.2.14 it is not clear that adjusting weights that satisfy (5.2.12.1) exist.

Before we begin, we make the following definition.

Definition 5.3.2. Let $d \in \mathbb{N}^*$, M a d-dimensional manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M, $(r_\kappa)_{\kappa\in\mathcal{A}}$, $(\varepsilon_\kappa)_{\kappa\in\mathcal{A}}$ families of positive real numbers and $R > 0$. We call $\mathcal{A}$ *adapted to* $((r_\kappa)_{\kappa\in\mathcal{A}}, (\varepsilon_\kappa)_{\kappa\in\mathcal{A}}, R)$ if $\overline{B}_{r_\kappa+R}(0) \subseteq U_\kappa$ for all $\kappa \in \mathcal{A}$, $(\kappa^{-1}(B_{r_\kappa}(0)))_{\kappa\in\mathcal{A}}$ is a cover of M and $r_\kappa < (\frac{1}{2\varepsilon_\kappa} - 1)R$ for all $\kappa \in \mathcal{A}$. Note that this implies that each $\varepsilon_\kappa < \frac{1}{2}$. Sometimes, we may call such an atlas $\mathcal{A}$ just adapted.

Remark 5.3.3. Note that on a manifold with a countable base every atlas is adapted, see [Lan02, Theorem 3.3].

Lemma 5.3.4. *Let $d \in \mathbb{N}^*$, (M, g) a d-dimensional connected Riemannian manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ a locally finite atlas for M, $R > 0$ and $(r_\kappa)_{\kappa\in\mathcal{A}}$, $(\varepsilon_\kappa)_{\kappa\in\mathcal{A}}$ families of positive real numbers such that $\mathcal{A}$ is adapted to $((r_\kappa)_{\kappa\in\mathcal{A}}, (\varepsilon_\kappa)_{\kappa\in\mathcal{A}}, R)$ and $\varepsilon := \inf_{\kappa\in\mathcal{A}} \varepsilon_\kappa > 0$.*

We then set $V_\kappa := B_{r_\kappa+R}(0)$, $\mathcal{B} := \{\kappa|^{V_\kappa}_{\kappa^{-1}(V_\kappa)} : \kappa \in \mathcal{A}\}$ and $\mathcal{C} := \{\kappa|^{B_{r_\kappa}(0)}_{\kappa^{-1}(B_{r_\kappa}(0))} : \kappa \in \mathcal{A}\}$. Further, for each $\kappa \in \mathcal{A}$, let $\delta^E_\kappa \in]0, R^{E,g_\kappa}_{V_\kappa,U_\kappa}[$ and $\delta^L_\kappa \in]0, R^{L,g_\kappa}_{V_\kappa,U_\kappa}[$. Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ contain weights ω^E, ω^L such that $\omega^E_{\mathcal{B}}$ is an adjusting weight for

$$\left(\min\Big(\delta^E_\kappa, \frac{\min(\varepsilon_\kappa r_\kappa, 1)}{2C^{E,(1)}_{V_\kappa,\delta^E_\kappa,g_\kappa}}, \frac{\varepsilon_\kappa}{4(C^{E,2}_{V_\kappa,\delta^E_\kappa,g_\kappa}+1)}\Big)\right)_{\kappa\in\mathcal{A}}; \tag{†}$$

$\omega^L_{\mathcal{B}}$ is an adjusting weight for $(\delta^L_\kappa)_{\kappa\in\mathcal{A}}$, and (5.2.12.1) *is satisfied for ω^L and ω^E. Further, assume that $\mathcal{W}_{\mathcal{B}}$ satisfies* (5.1.23.1) *for $(\mathcal{E}^{g_\kappa}_{V_\kappa,\delta^E_\kappa})_{\kappa\in\mathcal{A}}$ and $(\mathcal{L}^{g_\kappa}_{V_\kappa,\delta^L_\kappa})_{\kappa\in\mathcal{A}}$, respectively.*

(a) *Then the map*

$$C_{\mathfrak{X}(M)} : D_1^{\mathcal{B}} \times D_2^{\mathcal{B}} \to R^{\mathcal{C}} : (X, Y) \mapsto \log_g \circ(\mathrm{id}_M, (\exp_g \circ X) \circ (\exp_g \circ Y))$$

is defined and smooth, where

$$D_1^{\mathcal{B}} := \{X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} : \|X\|_{\mathcal{B},\omega^E,0} < \tfrac{1}{2} \text{ and } \|X\|_{\mathcal{B},1_M,1} < \tfrac{1}{2}\}$$

and

$$D_2^{\mathcal{B}} := \{X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} : \|X\|_{\mathcal{B},\omega^E,0} < \min(\tfrac{1}{4}, R)\}$$

and

$$R^{\mathcal{C}} := \{X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{C}} : \|X\|_{\mathcal{C},\omega^L,0} < 1\}.$$

Assume that there exists a point in M that is contained in only one $\widetilde{U}_\kappa$.

(b) *Then for each $\rho \in]0, 1[$, the map*

$$I_{\mathfrak{X}(M)} : D_\rho^{\mathcal{B}} \to R_\rho^{\mathcal{C}} : X \mapsto \log_g \circ(\mathrm{id}_M, (\exp_g \circ X)^{-1})$$

is defined and smooth, where

$$D_\rho^{\mathcal{B}} := \{X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} : \|X\|_{\mathcal{B},\omega^E,0} < \tfrac{(1-\rho)\min(\rho,R)}{2}, \|X\|_{\mathcal{B},1_M,1} < \min(\tfrac{\rho}{2}, \tfrac{\varepsilon}{4})\}$$

and

$$R_\rho^{\mathcal{C}} := \{X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{C}} : \|X\|_{\mathcal{C},\omega^L,0} < \tfrac{\min(R,\rho)}{2}\}.$$

We set

$$\mathcal{D}_D := \{\exp_g \circ X : X \in D_1^{\mathcal{B}} \cap D_2^{\mathcal{B}} \cap D_\rho^{\mathcal{B}}\},$$

and assume that (5.1.20.1) *is satisfied for $\mathcal{W}_{\mathcal{C}\cap\mathcal{B}}$ and $(D(\kappa \circ \phi^{-1})|_{\phi(\widetilde{U}_\kappa \cap \widetilde{U}_\phi)})_{(\kappa,\phi)\in\mathcal{B}\otimes\mathcal{C}}$.*

(c) *Then there exists a Lie group structure on the subgroup of* $\mathrm{Diff}(M)$ *generated by*

$$\mathcal{D}_D \cap \mathcal{D}_D^{-1}.$$

The restriction of the map

$$\mathcal{L} : \mathcal{D}_D \to \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} : \phi \mapsto \log_g \circ(\mathrm{id}_M, \phi)$$

is a chart for this set.

Proof. (a) Using Lemma 5.2.12 and (5.2.12.3) (together with Lemma B.3.6), we get the commutative diagram

$$\begin{array}{ccc}
D_1^{\mathcal{B}} \times D_2^{\mathcal{B}} & \xrightarrow{\iota_{\mathcal{W}}^{\mathcal{B}} \times \iota_{\mathcal{W}}^{\mathcal{B}}} & D_1 \times D_2 \\
\downarrow{\scriptstyle C_{\mathfrak{X}(M)}} & & \downarrow{\scriptstyle \mathbf{C}_E^L} \\
R^{\mathcal{C}} & \xrightarrow{\iota_{\mathcal{W}}^{\mathcal{C}}} & R.
\end{array}$$

In particular, $\operatorname{im}\mathbf{C}_E^L \circ (\iota_{\mathcal{W}}^{\mathcal{B}} \times \iota_{\mathcal{W}}^{\mathcal{B}}) \subseteq \operatorname{im}\iota_{\mathcal{W}}^{\mathcal{C}}$, and the corestriction of $\mathbf{C}_E^L \circ (\iota_{\mathcal{W}}^{\mathcal{B}} \times \iota_{\mathcal{W}}^{\mathcal{B}})$ to $\operatorname{im}\iota_{\mathcal{W}}^{\mathcal{C}}$ is smooth by Proposition A.1.12 since we proved in Lemma 5.2.2 the vector fields are a closed subset of the product. Since $\iota_{\mathcal{W}}^{\mathcal{C}}$ is an embedding, this proves our assertion that $C_{\mathfrak{X}(M)}$ is defined and smooth.

(b) We know from Proposition 5.3.1 that for all $X \in \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{C}}$ with $\|X\|_{\mathcal{B},\omega,0} < 1$ and $\|X\|_{\mathcal{B},1_M,1} < \frac{\varepsilon}{4}$, the map $\exp_g \circ X$ is a diffeomorphism. We can apply 5.3.1 since $r_\kappa < (1-2\varepsilon_\kappa)(r_\kappa + R)$ (that is shown with a short calculation), and using our assumptions on ω stated in (†), together with (5.1.17.1).
The rest of the proof follows along the same lines as (a).

(c) We calculate using Lemma B.3.8 and (5.2.12.1) that for all $X \in D_1^{\mathcal{B}} \cap D_2^{\mathcal{B}} \cap D_\rho^{\mathcal{B}}$ and $\kappa \in \mathcal{A}$, we have

$$\|E_{V_\kappa,\delta_\kappa}^{\mathcal{W}_\kappa, g_\kappa}(X_\kappa)\|_{\omega_\kappa^L,0} \le \|X_\kappa\|_{\omega_\kappa^E,0} < 1.$$

Since ω^L is adjusting for δ^L, we know from this estimate that we can apply $\log_g$ to $(\mathrm{id}_M, \exp_g \circ X)$, so $\mathcal{L}$ is well-defined.

At the next step, we show that $\mathcal{D}_D \cap \mathcal{D}_D^{-1} = \mathcal{D}_D \cap \mathcal{L}^{-1}(I_{\mathfrak{X}(M)}^{-1}(\mathcal{L}(\mathcal{D}_D)))$. To this end, let $\phi \in \mathcal{D}_D \cap \mathcal{D}_D^{-1}$. Then there exists $\psi \in \mathcal{D}_D$ such that $\phi^{-1} = \psi$, and $X, Y \in \mathcal{L}(\mathcal{D}_D)$ with $\phi = \exp_g \circ X$, $\psi = \exp_g \circ Y$. Then

$$Y = \log_g \circ (\mathrm{id}_M, \psi) = \log_g \circ (\mathrm{id}_M, (\exp_g \circ X)^{-1}) = I_{\mathfrak{X}(M)}(X).$$

Hence $X \in I_{\mathfrak{X}(M)}^{-1}(\mathcal{L}(\mathcal{D}_D))$ (note that we used that $\mathcal{L}(\mathcal{D}_D) \subseteq D_\rho^{\mathcal{B}} \subseteq R_\rho^{\mathcal{C}}$), and $\phi \in \mathcal{L}^{-1}(I_{\mathfrak{X}(M)}^{-1}(\mathcal{L}(\mathcal{D}_D)))$. On the other hand, if $\phi \in \mathcal{D}_D$ such that $\mathcal{L}(\mathcal{D}_D) \in I_{\mathfrak{X}(M)}^{-1}(\mathcal{L}(\mathcal{D}_D))$, then there exists $X \in \mathcal{L}(\mathcal{D}_D)$ with $X = I_{\mathfrak{X}(M)}(\mathcal{L}(\phi)) = \log_g \circ (\mathrm{id}_M, \phi^{-1})$. Hence $\phi^{-1} = \exp_g \circ X \in \mathcal{D}_D$, so $\phi \in \mathcal{D}_D^{-1}$.

We show that $\mathcal{L}(\mathcal{D}_D \cap \mathcal{D}_D^{-1})$ is open in $\mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$. By the definition of adjusting weights, $|\omega^E| \ge 1$. Hence we can apply Corollary 5.2.7 to see that $\mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}} = \mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{C}}$. Hence $\mathcal{L}(\mathcal{D}_D)$ is open in $\mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{C}}$, and by (b), so is $I_{\mathfrak{X}(M)}^{-1}(\mathcal{L}(\mathcal{D}_D))$ in $\mathcal{C}^\infty_{\mathcal{W}}(M, \mathbf{T}M)_{\mathcal{B}}$.

Since we proved in (b) that $I_{\mathfrak{X}(M)}$ is smooth on $\mathcal{L}(\mathcal{D}_D \cap \mathcal{D}_D^{-1})$, the inversion map is smooth on $\mathcal{D}_D \cap \mathcal{D}_D^{-1}$, with respect to the manifold structure induced by $\mathcal{L}$. Since this set is symmetric and open, and we can deduce from the things we proved in (a) that the composition

$$(\mathcal{D}_D \cap \mathcal{D}_D^{-1}) \times (\mathcal{D}_D \cap \mathcal{D}_D^{-1}) \to \exp_g \circ R^{\mathcal{C}} \cap \mathrm{Diff}(M)$$

is smooth, it is possible to apply the theorem about generation from local data Lemma B.2.5 to get the assertion. □

Restricting the domain of $\exp_g$ We restrict the domain of the exponential function, which allows us to show that adjusting weights satisfying (5.2.12.1) exist. In order to do this, we need the results of Subsection B.3.2, in particular Lemma B.3.17.

Lemma 5.3.5. *Let $d \in \mathbb{N}^*$, (M, g) a d-dimensional Riemannian manifold, $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ an atlas for M and $\sigma \in]0,1[$. Further, for each $\kappa \in \mathcal{A}$ let V_κ be a relatively* *compact set with* $\overline{V_\kappa} \subseteq U_\kappa$*,* $\delta_\kappa \in]0, R^{g_\kappa}_{V_\kappa,\sigma} Q^{g_\kappa}_{V_\kappa}[$ *and* $\omega : M \to \mathbb{R}$ *be adjusted to* $(\frac{(1-\sigma)^2}{1+\sigma}\delta_\kappa)_{\kappa\in\mathcal{A}}$ *such that* $|\omega| \geq \frac{1+\sigma}{1-\sigma}$*. Then* $\omega^L := \frac{1-\sigma}{1+\sigma}\omega$ *is adjusted to* $((1-\sigma)\delta_\kappa)_{\kappa\in\mathcal{A}}$*, we have* $(1-\sigma)\delta_\kappa < R^{L,g_\kappa}_{V_\kappa,U_\kappa}$ *and the weights* ω*,* ω^L *satisfy* (5.2.12.1)*.*

Proof. Let $\kappa \in \mathcal{A}$. Then we have that

$$|\omega^L_\kappa| = \tfrac{1-\sigma}{1+\sigma}|\omega_\kappa| \geq \frac{1}{\frac{1+\sigma}{1-\sigma}} \frac{1}{\frac{(1-\sigma)^2}{1+\sigma}\delta_\kappa} = \frac{1}{(1-\sigma)\delta_\kappa},$$

hence ω^L is adjusted to $((1-\sigma)\delta_\kappa)_{\kappa\in\mathcal{A}}$ since we assumed that $|\omega| \geq \frac{1+\sigma}{1-\sigma}$. Further, we know from Lemma B.3.17 that $(1-\sigma)\delta_\kappa < R^{L,g_\kappa}_{V_\kappa,U_\kappa}$, $C^{E,(1)}_{V_\kappa,\delta_\kappa,g_\kappa} \leq 1+\sigma$ and $C^{L,(1)}_{V_\kappa,(1-\sigma)\delta_\kappa,g_\kappa} \leq \frac{1}{1-\sigma}$. Hence for $\kappa \in \mathcal{A}$,

$$|\omega^L_\kappa| = \tfrac{1-\sigma}{1+\sigma}|\omega_\kappa| \leq \tfrac{1}{C^{E,(1)}_{V_\kappa,\delta_\kappa,g_\kappa} C^{L,(1)}_{V_\kappa,(1-\sigma)\delta_\kappa,g_\kappa}}|\omega_\kappa|.$$

This finishes the proof. □

We are ready to prove the main result.

Theorem 5.3.6. *Let* $d \in \mathbb{N}^*$*,* (M, g) *a* d*-dimensional Riemannian manifold,* $\mathcal{W} \subseteq \overline{\mathbb{R}}^M$ *with* $1_M \in \mathcal{W}$ *and* $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ *a locally finite atlas for* M *such that there exists a point in* M *that is contained in only one* $\widetilde{U}_\kappa$*. Further, for each* $\kappa \in \mathcal{A}$ *let* $\varepsilon_\kappa \in]0, \frac{1}{2}[$ *and* $r_\kappa > 0$ *such that* $\varepsilon := \inf_{\kappa\in\mathcal{A}} \varepsilon_\kappa > 0$ *and* $r := \inf_{\kappa\in\mathcal{A}} r_\kappa > 0$*. Suppose that there exists* $R > 0$ *such that* $\mathcal{A}$ *is adapted to* $((r_\kappa)_{\kappa\in\mathcal{A}}, (\varepsilon_\kappa)_{\kappa\in\mathcal{A}}, R)$*.*

Then there exists a subgroup $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$ *of* $\mathrm{Diff}(M)$ *that is generated by* $\mathcal{D}_D \cap \mathcal{D}_D^{-1}$*, where*

$$\mathcal{D}_D := \{\exp_g \circ X : X \in \mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_{\mathcal{A}}, \|X\|_{\mathcal{B},\omega,0}, \|X\|_{\mathcal{B},1_M,1} < \alpha\}$$

with some suitable $\alpha > 0$*,* $\mathcal{B} := \{\kappa|^{B_{r_\kappa+R}(0)}_{\kappa^{-1}(B_{r_\kappa+R}(0))} : \kappa \in \mathcal{A}\}$ *and* $\omega \in \mathcal{W}^e$ *that is adjusted to* $(\mathcal{B}, (\widetilde{\delta}_\kappa)_{\kappa\in\mathcal{A}})$*, where*

$$\widetilde{\delta}_\kappa := \Big(\min\Big(\tfrac{(1-\sigma)^2}{1+\sigma}\delta_\kappa, \tfrac{\varepsilon_\kappa}{4(C^{E,2}_{V_\kappa,\delta_\kappa,g_\kappa}+1)}\Big)\Big)_{\kappa\in\mathcal{A}}$$

and each $\delta_\kappa \in]0, R^{g_\kappa}_{V_\kappa,\sigma} Q^{g_\kappa}_{V_\kappa}[$ *with some* $\sigma \in]0,1[$*. Further,* $\mathcal{W}^e \subseteq \overline{\mathbb{R}}^M$ *is locally* $\mathcal{W}$*-bounded and a minimal saturated extension of* $\mathcal{W} \cup \{\omega\}$ *with respect to* $((\mathcal{A}, \mathcal{B}, g), (\mathcal{A}, \mathcal{A}))$*. The map*

$$\mathcal{D}_D \cap \mathcal{D}_D^{-1} \to \mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_{\mathcal{B}} : \phi \mapsto \log_g \circ(\mathrm{id}_M, \phi)$$

is a chart for $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$*.*

Proof. We use Lemma 5.2.14 to construct a weight $\omega : M \to \mathbb{R}$ that is adjusted to $(\mathcal{B}, (\widetilde{\delta}_\kappa)_{\kappa\in\mathcal{A}})$. Note that $\omega_{\mathcal{B}}$, after an eventual multiplication of ω with a constant, is also adjusting for $(\frac{\min(1,\varepsilon_\kappa r_\kappa)}{2C^{E,(1)}_{V_\kappa,\delta_\kappa,g_\kappa}})_{\kappa\in\mathcal{A}}$ since $\inf_{\kappa\in\mathcal{A}} r_\kappa\varepsilon_\kappa > 0$ by our assumption, and $C^{E,(1)}_{V_\kappa,\delta_\kappa,g_\kappa} \leq 1+\sigma$ by Lemma B.3.17. Further, we see with Lemma 5.3.5 that there exists an adjusted weight ω^L such that ω and ω^L satisfy (5.2.12.1) (we may assume w.l.o.g. that $|\omega| \geq \frac{1+\sigma}{1-\sigma}$). Since ω is locally 1_M-bounded, $\mathcal{W} \cup \{\omega\}$ is locally $\mathcal{W}$-bounded, and so is the minimal saturated extension $\mathcal{W}^e$ of $\mathcal{W} \cup \{\omega\}$ w.r.t. $((\mathcal{A}, \mathcal{B}, g), (\mathcal{A}, \mathcal{A}))$ that was constructed in Lemma 5.2.18. We get the desired result by applying Lemma 5.3.4. □

Inclusion of compactly supported diffeomorphisms

We want to examine which assumptions on the weight set $\mathcal{W}$ ensure that the group $\mathrm{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(M, g, \omega)$ contains the identity component $\mathrm{Diff}_c(M)_0$ of the group of compactly supported diffeomorphisms. To this end, we need some tools to handle the topology on the compactly supported vector fields, which are the modelling space of $\mathrm{Diff}_c(M)_0$.

Sums and the topology of $\mathcal{C}^\infty_c(M, \mathbf{T}M)$ We use tools provided in the article [Glö04].

Remark 5.3.7. For a d-dimensional manifold M, the smooth vector fields with compact support $\mathcal{C}^\infty_c(M, \mathbf{T}M)$ are usually endowed with the inductive limit topology of the inclusion maps $\mathcal{C}^\infty_K(M, \mathbf{T}M) \to \mathcal{C}^\infty_c(M, \mathbf{T}M)$. Here $\mathcal{C}^\infty_K(M, \mathbf{T}M)$ denotes the smooth vector fields X with $\mathrm{supp}(X) \subseteq K$, and is endowed with the topology of uniform smooth convergence with respect to charts, see [Glö04, Def. F.14 and Def. F.7 & La. F.9] for details.

By [Glö04, Prop. F.19], for a locally finite atlas $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ such that each $\widetilde{U}_\kappa$ is relatively compact, the map

$$\mathcal{C}^\infty_c(M, \mathbf{T}M) \to \bigoplus_{\kappa \in \mathcal{A}} \mathcal{C}^\infty(\widetilde{U}_\kappa, \mathbf{T}\widetilde{U}_\kappa) : X \mapsto (X_\kappa)_{\kappa \in \mathcal{A}}$$

is an embedding. The sum is endowed with the box topology, see [Glö04, 6.1-6.7 and Def. F.7 & La. F9] for the definition of the sum respectively the topology of the summands; We will use these seminorms.

For an easier argument, we relate the sums $\bigoplus_{i \in I} \mathcal{C}^\ell(U_i, Y_i)$ and $\bigoplus_{i \in I} \mathcal{BC}^\ell(V_i, Y_i)$, provided that $V_i \subseteq U_i$ is relatively compact.

Lemma 5.3.8. *Let I a nonempty set and $\ell \in \overline{\mathbb{N}}$. For each $i \in I$, let U_i, V_i be open nonempty subsets of the locally convex space X_i such that $\overline{V_i} \subseteq U_i$ and each V_i is relatively compact, and Y_i a normed space. Then for each $i \in I$, the map*

$$\mathcal{C}^\ell(U_i, Y_i) \to \mathcal{BC}^\ell(V_i, Y_i) : \gamma \mapsto \gamma|_{V_i}$$

is defined and continuous, where each $\mathcal{C}^\ell(U_i, Y_i)$ is endowed with the compact open $\mathcal{C}^\ell$ topology. Consequently, the map

$$\bigoplus_{i \in I} \mathcal{C}^\ell(U_i, Y_i) \to \bigoplus_{i \in I} \mathcal{BC}^\ell(V_i, Y_i) : (\gamma_i)_{i \in I} \mapsto (\gamma_i|_{V_i})_{i \in I}$$

is also defined and continuous.

Proof. According to [Glö04, Rem. 6.7], the spaces $\bigoplus_{i \in I} \mathcal{C}^\ell(U_i, Y_i)$ are the direct sum in the category of locally convex spaces, hence the second assertion follows if the first one is proved. Since we assumed that each V_i is locally compact, each restricted map (and its derivatives) is bounded, and we see using standard compactness arguments that the restriction is continuous. □

We show that function that is locally bounded induces continuous seminorms on the sum $\bigoplus_{\kappa \in \mathcal{A}} \mathcal{BC}^\infty(U_\kappa, \mathbb{R}^d)$.

Lemma 5.3.9. *Let $d \in \mathbb{N}$, M be d-dimensional manifold, $f : M \to \mathbb{R}$ locally bounded, $\ell \in \mathbb{N}$ and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ a locally finite atlas such that each $\widetilde{U}_\kappa$ is relatively compact. Then $\|\cdot\|_{\mathcal{A},f,\ell}$ is a continuous seminorm on $\bigoplus_{\kappa\in\mathcal{A}} \mathcal{BC}^\infty(U_\kappa, \mathbb{R}^d)$.*

Proof. Since f is locally bounded, it is bounded on each compact set, and in consequence on each $\widetilde{U}_\kappa$, which can be proved with a standard compactness argument. So for $\kappa \in \mathcal{A}$ and $\gamma \in \mathcal{BC}^\infty(\widetilde{U}_\kappa, \mathbb{R}^d)$, we have that

$$\|\gamma\|_{f_\kappa,\ell} \leq \|f_\kappa\|_\infty \|\gamma\|_{1_{U_\kappa},\ell}.$$

Hence $\|\cdot\|_{\mathcal{A},f,\ell}$ is continuous since it is so on each summand. □

Inclusion of compactly supported diffeomorphisms We are ready to prove the criterion.

Proposition 5.3.10. *Let $d \in \mathbb{N}^*$, (M, g) a d-dimensional Riemannian manifold and $\mathcal{A} = \{\kappa : \widetilde{U}_\kappa \to U_\kappa\}$ a locally finite atlas for M such that there exists a point in M that is contained in only one $\widetilde{U}_\kappa$ and that is adapted to some $((r_\kappa)_{\kappa\in\mathcal{A}}, (\varepsilon_\kappa)_{\kappa\in\mathcal{A}}, R)$ with $\inf_{\kappa\in\mathcal{A}} \varepsilon_\kappa, \inf_{\kappa\in\mathcal{A}} r_\kappa > 0$. Further, let $\mathcal{W} \subseteq \mathbb{R}^M$ with $1_M \in \mathcal{W}$ such that each $f \in \mathcal{W}$ is bounded on all compact subsets of M. Then $\mathrm{Diff}_c(M)_0 \subseteq \mathrm{Diff}_{\mathcal{W}}^{\mathcal{A},\mathcal{B}}(M, g, \omega)$ for all $\mathcal{B}$ and ω as in Theorem 5.3.6.*

Proof. For relatively compact V_κ such that $\overline{V_\kappa} \subseteq U_\kappa$ and $\overline{B}_{r_\kappa+R}(0) \subseteq V_\kappa$, the map

$$\mathcal{C}^\infty_c(M, \mathbf{T}M) \to \bigoplus_{\kappa\in\mathcal{A}} \mathcal{C}^\infty(V_\kappa, \mathbb{R}^d) : X \mapsto (X_\kappa)_{\kappa\in\mathcal{A}}$$

is an embedding, see Remark 5.3.7. Since $\mathcal{W}^e$ is locally $\mathcal{W}$-bounded and each weight in $\mathcal{W}$ is locally bounded, each $f \in \mathcal{W}^e$ is also locally bounded. Hence we can use Lemma 5.3.9 and Lemma 5.3.8 to see that for $f \in \mathcal{W}^e$ and $\ell \in \mathbb{N}$, $\|\cdot\|_{\mathcal{B},f,\ell}$ is defined and continuous on $\bigoplus_{\kappa\in\mathcal{A}} \mathcal{C}^\infty(V_\kappa, \mathbb{R}^d)$ and hence on $\mathcal{C}^\infty_c(M, \mathbf{T}M)$. This, together with Corollary 5.2.7, implies that $\mathcal{C}^\infty_c(M, \mathbf{T}M) \subseteq \mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_\mathcal{A}$, and that for each $\alpha > 0$,

$$\{X \in \mathcal{C}^\infty_c(M, \mathbf{T}M) : \|X\|_{\mathcal{B},\omega,0}, \|X\|_{\mathcal{B},1_M,1} < \alpha\}$$

is open in $\mathcal{C}^\infty_c(M, \mathbf{T}M)$. We know from Theorem 5.3.6 that $\mathrm{Diff}_{\mathcal{W}}^{\mathcal{A},\mathcal{B}}(M, g, \omega)$ is modelled on $\mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_\mathcal{A}$, and for some $\alpha > 0$, it contains the set

$$\{\exp_g \circ X : X \in \mathcal{C}^\infty_{\mathcal{W}^e}(M, \mathbf{T}M)_\mathcal{A}, \|X\|_{\mathcal{B},\omega,0}, \|X\|_{\mathcal{B},1_M,1} < \alpha\}.$$

Hence $\mathrm{Diff}_{\mathcal{W}}^{\mathcal{A},\mathcal{B}}(M, g, \omega)$ contains an open identity neighborhood of $\mathrm{Diff}_c(M)$, and thus $\mathrm{Diff}_c(M)_0$. □

Comparison with the vector space case

We show that the connected components of the Lie groups $\mathrm{Diff}_{\mathcal{W}}^{\mathcal{A},\mathcal{B}}(\mathbb{R}^d, \langle\cdot,\cdot\rangle, 1_{\mathbb{R}^d})$ that were constructed in Theorem 5.3.6, and of $\mathrm{Diff}_{\mathcal{W}}(\mathbb{R}^d)$ as constructed in Theorem 4.2.10 coincide, if $\mathcal{A}$ consists of identity maps.

Proposition 5.3.11. *Let $d \in \mathbb{N}^*$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^{\mathbb{R}^d}$ with $1_{\mathbb{R}^d} \in \mathcal{W}$. Then $\operatorname{Diff}_{\mathcal{W}}(\mathbb{R}^d)_0 = \operatorname{Diff}_{\mathcal{W}}^{\mathcal{A},\mathcal{B}}(\mathbb{R}^d, \langle\cdot,\cdot\rangle, 1_{\mathbb{R}^d})_0$, where*

$$\mathcal{A} := \{\mathrm{id}_{B_{r1}(x)} : x \in \mathbb{Z}^d\} \text{ and } \mathcal{B} := \{\mathrm{id}_{B_{r_2}(x)} : x \in \mathbb{Z}^d\},$$

with $1 \geq r_1 > r_2 > \frac{1}{2}$, and $\mathbb{R}^d$ is endowed with the supremum norm $\|\cdot\|_\infty$.

Proof. Obviously, $\mathcal{A}$ is a locally finite atlas since $B_{1-r_1}(x)$ has nonempty intersection with at most 2^d chart domains, for all $x \in \mathbb{R}^d$. Further, if we set $R := \frac{1}{2}(r_1 - r_2)$ and choose $\varepsilon \in]0, \frac{1}{2}\frac{r_1-r_2}{r_1+r_2}[$, $\mathcal{A}$ is adapted to (r_2, ε, R).

We have that $D^E_{\langle\cdot,\cdot\rangle} = D^L_{\langle\cdot,\cdot\rangle} = \mathbb{R}^{2d}$, and further that $\exp_{\langle\cdot,\cdot\rangle}(x, y) = x + y$ and $\log_{\langle\cdot,\cdot\rangle}(x, y) = y - x$. Hence $D^{(2)} \exp_{\langle\cdot,\cdot\rangle} = 0$, and for $x \in \mathbb{Z}^d$ and $\sigma \in]0, 1[$,

$$R^{E,\langle\cdot,\cdot\rangle}_{B_{r_2}(x),B_{r_1}(x)} = R^{\langle\cdot,\cdot\rangle}_{B_{r_2}(x),\sigma} = r_1 - r_2$$

and $\mathcal{E}^{\langle\cdot,\cdot\rangle}_{B_{r_2}(x),\delta} = \pi_2$ for all $\delta \in]0, r_1 - r_2[$. For $R^{L,\langle\cdot,\cdot\rangle}_{B_{r_2}(x),B_{r_1}(x)}$ we have

$$R^{L,\langle\cdot,\cdot\rangle}_{B_{r_2}(x),B_{r_1}(x)} = \tfrac{1}{\sqrt{d}}(r_1 - r_2)$$

and $\mathcal{L}^{\langle\cdot,\cdot\rangle}_{B_{r_2}(x),\delta} = \pi_2$ for all $\delta \in]0, \frac{1}{\sqrt{d}}(r_1 - r_2)[$. For $\kappa, \phi \in \mathcal{A}$ with $(\kappa, \phi) \in \mathcal{A} \otimes \mathcal{A}$,

$$D(\kappa \circ \phi^{-1}) = \mathrm{Id}\,.$$

We easily deduce that $\mathcal{W}$ is already saturated, and 1_M is adjusted if we choose the same $\delta < \frac{1}{\sqrt{d}}(r_1 - r_2) = Q^{\langle\cdot,\cdot\rangle}_{B_{r_2}(x)} R^{\langle\cdot,\cdot\rangle}_{B_{r_2}(x),\sigma}$ for all charts. Further, for all $f \in \mathcal{W}$, $\ell \in \mathbb{N}$ and $X \in \mathfrak{X}(\mathbb{R}^d)$,

$$\|\pi_2 \circ X\|_{f,\ell} = \|X\|_{\mathcal{B},f,\ell},$$

and hence $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^d, \mathbf{T}\mathbb{R}^d)_{\mathcal{B}} \cong \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^d, \mathbb{R}^d)$. Since the parameterization maps are also compatible, we see that $\operatorname{Diff}^{\mathcal{A},\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d, \langle\cdot,\cdot\rangle, 1_{\mathbb{R}^d})$ contains an open subset of $\operatorname{Diff}_{\mathcal{W}}(\mathbb{R}^d)$, and vice versa. Hence the assertion holds. □

6. Integration of certain Lie algebras of vector fields

The aim of this chapter is the integration of Lie algebras that arise as the semidirect product of a weighted function space $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$ and $\mathbf{L}(G)$, where G is a subgroup of $\mathrm{Diff}(X)$ which is a Lie group with respect to composition and inversion of functions.

The canonical candidate for this purpose is the semidirect product of $\mathrm{Diff}_{\mathcal{W}}(X)$ and G – if it can be constructed. Hence we need criteria when

$$G \times \mathrm{Diff}_{\mathcal{W}}(X) \to \mathrm{Diff}(X) : (T, \phi) \mapsto T \circ \phi \circ T^{-1}$$

takes its image in $\mathrm{Diff}_{\mathcal{W}}(X)$ and is smooth.

6.1. On the smoothness of the conjugation action on $\mathrm{Diff}_{\mathcal{W}}(X)_0$

We slightly generalize our approach by allowing arbitrary Lie groups to act on $\mathrm{Diff}_{\mathcal{W}}(X)$. We need the following notation.

Definition 6.1.1. Let G be a group and $\omega : G \times M \to M$ an action of G on the set M.

(a) For $g \in G$, we denote the partial map $\omega(g, \cdot) : M \to M$ by ω_g.

(b) Assume that G is a locally convex Lie group with the identity element e, M a smooth manifold and ω is smooth. We define the linear map

$$\dot{\omega} : \mathbf{L}(G) \to \mathfrak{X}(M)$$

by

$$\dot{\omega}(x)(m) = -\mathbf{T}_e\omega(\cdot, m)(x).$$

Note that $\dot{\omega}$ takes its values in the smooth vector fields because ω is smooth.

Now we can state a first criterion for smoothness of the conjugation action – however only on the identity component $\mathrm{Diff}_{\mathcal{W}}(X)_0$ of $\mathrm{Diff}_{\mathcal{W}}(X)$.

Lemma 6.1.2. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, G a Lie group and $\omega : G \times X \to X$ a smooth action. We define the map*

$$\alpha : G \times \mathrm{Diff}_{\mathcal{W}}(X) \to \mathrm{Diff}(X) : (T, \phi) \mapsto \omega_T \circ \phi \circ \omega_{T^{-1}}.$$

Assume that there exists an open set $\Omega \in \mathcal{U}_G(\mathbf{1})$ such that the maps

$$\mathcal{C}^\infty_{\mathcal{W}}(X,X) \times \Omega \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : (\gamma, T) \mapsto \gamma \circ \omega_T \tag{6.1.2.1}$$

and

$$\mathcal{C}^\infty_{\mathcal{W}}(X,X) \times \Omega \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : (\gamma, T) \mapsto D\omega_T \cdot \gamma \tag{6.1.2.2}$$

are well-defined and smooth.

(a) *Then for each open identity neighborhood $U_{\mathcal{W}} \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$ such that $[\phi, \mathrm{id}_X] := \{t\phi + (1-t)\,\mathrm{id}_X : t \in [0,1]\} \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$ for each $\phi \in U_{\mathcal{W}}$, the map*

$$(\Omega \cap \Omega^{-1}) \times U_{\mathcal{W}} \to \mathrm{End}_{\mathcal{W}}(X) : (T, \phi) \mapsto \alpha(T, \phi) \tag{$\dagger$}$$

is well-defined and smooth.

(b) *Suppose that $\Omega = G$. Then the map*

$$G \times \mathrm{Diff}_{\mathcal{W}}(X)_0 \to \mathrm{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto \alpha(T, \phi) \tag{$\dagger\dagger$}$$

is well-defined and smooth.

Proof. (a) Using Proposition 4.1.7, Theorem 4.2.10 and the smoothness of (6.1.2.1) and (6.1.2.2), for each $t \in [0,1]$, $T \in \Omega \cap \Omega^{-1}$ and $\phi \in U_{\mathcal{W}}$ we see that

$$\psi_{t,T,\phi} := (D\omega_T \cdot ((\phi - \mathrm{id}_X) \circ (t\phi + (1-t)\,\mathrm{id}_X)^{-1})) \circ (t\phi + (1-t)\,\mathrm{id}_X) \circ \omega_T^{-1} \in \mathcal{C}^\infty_{\mathcal{W}}(X,X),$$

and $\psi_{t,T,\phi}$ is a smooth map. Further, using that $t\phi + (1-t)\,\mathrm{id}_X$ is a diffeomorphism for each $t \in [0,1]$, we calculate

$$\begin{aligned}
&(\omega_T \circ \phi \circ \omega_{T^{-1}})(x) - x \\
=&(\omega_T \circ \phi \circ \omega_T^{-1})(x) - (\omega_T \circ \omega_T^{-1})(x) \\
=&\int_0^1 D\omega_T \circ (t\phi + (1-t)\,\mathrm{id}_X)(\omega_T^{-1}(x)) \cdot (\phi - \mathrm{id}_X)(\omega_T^{-1}(x))\, dt \\
=&\int_0^1 (D\omega_T \cdot ((\phi - \mathrm{id}_X) \circ (t\phi + (1-t)\,\mathrm{id}_X)^{-1})) \circ (t\phi + (1-t)\,\mathrm{id}_X)(\omega_T^{-1}(x))\, dt.
\end{aligned}$$

Hence $\omega_T \circ \phi \circ \omega_{T^{-1}} - \mathrm{id}_X = \int_0^1 \psi_{t,T,\phi}\, dt \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$ by Proposition A.1.8, using that we proved in Corollary 3.2.9 that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ is complete.

Since $\psi_{t,T,\phi}$ is smooth as a function of t, T and ϕ, we can use Proposition A.1.19 to see that ($\dagger$) is defined and smooth.

(b) Since $\mathrm{Diff}_{\mathcal{W}}(X)$ is locally convex, we find a symmetric open $U_{\mathcal{W}} \in \mathcal{U}(\mathrm{id}_X)$ such that $[U_{\mathcal{W}}, \mathrm{id}_X] \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$. Using the symmetry of $U_{\mathcal{W}}$ and the results of (a), we see that $\alpha(G \times U_{\mathcal{W}}) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)_0$. Since $U_{\mathcal{W}}$ generates $\mathrm{Diff}_{\mathcal{W}}(X)_0$, we can apply Lemma B.2.13 to conclude that $\alpha(G \times \mathrm{Diff}_{\mathcal{W}}(X)_0) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)_0$. Further ($\dagger\dagger$) is smooth by (a) and Lemma B.2.14. □

So all we need are criteria for the smoothness of the maps (6.1.2.1) and (6.1.2.2). This will be the topic of the next two subsections. Before we proceed, the following definition is useful.

Definition 6.1.3. Let X be a normed space, $U \subseteq X$ an open nonempty subset, and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights. We define the *maximal extension* $\mathcal{W}_{\max} \subseteq \overline{\mathbb{R}}^U$ of $\mathcal{W}$ as the set of functions f for which $\|\cdot\|_{f,0}$ is a continuous seminorm on $\mathcal{C}^0_{\mathcal{W}}(U,Y)$, for each normed space Y. Obviously $\mathcal{W} \subseteq \mathcal{W}_{\max}$ and by Lemma 3.2.2, $\|\cdot\|_{f,\ell}$ is a continuous seminorm on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$, provided that $f \in \mathcal{W}_{\max}$ and $\ell \leq k$.

6.1.1. Bilinear action on weighted functions

We first elaborate on the map (6.1.2.2). To this end, we define a class of functions, the *multipliers*. These have the property that for a multiplier M, a weighted function γ and a continuous bilinear map b, the map $b \circ (M, \gamma)$ is a weighted function. Finally, we provide a criterion ensuring that a topology on a set of multipliers makes the map $(M, \gamma) \mapsto b \circ (M, \gamma)$ continuous.

Multipliers

Definition 6.1.4. Let X be a normed space, $U \subseteq X$ an open nonempty set and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights.

(a) A function $g : U \to \mathbb{R}$ is called a *multiplicative weight (for $\mathcal{W}$)* if

$$(\forall f \in \mathcal{W})\, f \cdot g \in \mathcal{W}_{\max}.$$

(b) Let Y be another normed space and $k \in \overline{\mathbb{N}}$. A $\mathcal{C}^k$-map $M : U \to Z$ is called a *k-multiplier (for $\mathcal{W}$)* if $\|D^{(\ell)}M\|_{op}$ is a multiplicative weight for all $\ell \in \mathbb{N}$ with $\ell \leq k$. An ∞-multiplier is also called a *multiplier*.

Lemma 6.1.5. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights and $k \in \overline{\mathbb{N}}$.*

(a) The set of k-multipliers from U to Y is a vector space.

(b) A map $M : U \to Y$ is a $(k+1)$-multiplier iff M is a 0-multiplier and $DM : U \to \mathrm{L}(X,Y)$ is a k-multiplier.

Proof. (a) This is obvious from the definition.

(b) This follows from the identity $\|D^{(\ell)}(DM)\|_{op} = \|D^{(\ell+1)}M\|_{op}$, see Lemma 3.2.2. □

Lemma 6.1.6. *Let X, Y_1, Y_2 and Z be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights and $k \in \overline{\mathbb{N}}$. Further, let $b : Y_1 \times Y_2 \to Z$ be a continuous bilinear map, $M : U \to Y_1$ a k-multiplier and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_2)$. Then*

$$b \circ (M, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, Z).$$

Moreover, the map

$$\mathcal{C}^k_{\mathcal{W}}(U, Y_2) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : \gamma \mapsto b \circ (M, \gamma) \tag{†}$$

is continuous linear and hence smooth.

Proof. For $k < \infty$ the proof is by induction on k:
$k = 0$: We calculate for $x \in U$ and $f \in \mathcal{W}$:

$$|f(x)| \, \|(b \circ (M, \gamma))(x)\| \leq \|b\|_{op} \, |f(x)| \, \|M(x)\| \, \|\gamma(x)\| \leq \|b\|_{op} \, \|\gamma\|_{|f| \cdot \|M\|, 0},$$

and since $\|M\|$ is a multiplicative weight, the right hand side is finite. Hence

$$\|b \circ (M, \gamma)\|_{f,0} \leq \|b\|_{op} \, \|\gamma\|_{|f| \cdot \|M\|, 0},$$

entailing that $b \circ (M, \gamma) \in \mathcal{C}^0_{\mathcal{W}}(U, Z)$ and the linear map (†) is continuous.
$k \to k+1$: By Proposition 3.2.3, we need to prove that $D(b \circ (M, \gamma)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$ and that the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U, Y_2) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z)) : \gamma \mapsto D(b \circ (M, \gamma))$$

is continuous. Using Lemma 3.3.2 we get

$$D(b \circ (M, \gamma)) = b^{(1)} \circ (DM, \gamma) + b^{(2)} \circ (M, D\gamma);$$

for the definition of the maps $b^{(i)}$ see Subsection 3.3.1. So by applying the inductive hypothesis to the maps $b^{(1)} \circ (DM, \gamma)$ and $b^{(2)} \circ (M, D\gamma)$ (by Lemma 6.1.5, DM is a k-multiplier), we see that $D(b \circ (M, \gamma))$ is in $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$ and the map (†) is continuous.
$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty_{\mathcal{W}}(U, Y_2) & \xrightarrow{b(M,\cdot)_{*,\infty}} & \mathcal{C}^\infty_{\mathcal{W}}(U, Z) \\ \downarrow & & \downarrow \\ \mathcal{C}^n_{\mathcal{W}}(U, Y_2) & \xrightarrow[b(M,\cdot)_*]{} & \mathcal{C}^n_{\mathcal{W}}(U, Z) \end{array}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent the inclusion maps. With Corollary 3.2.6 we easily deduce the continuity of $b(M, \cdot)_{*,\infty}$ from the one of $b(M, \cdot)_*$. ☐

Topologies on spaces of multipliers

Lemma 6.1.7. *Let X, Y_1, Y_2 and Z be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights, $k \in \overline{\mathbb{N}}$ and $b : Y_1 \times Y_2 \to Z$ a continuous bilinear map. Further, let $\mathcal{T}$ be a topological space and $(M_T)_{T \in \mathcal{T}}$ a family of k-multipliers such that*

$$\begin{aligned} &(\forall f \in \mathcal{W}, T \in \mathcal{T}, \ell \in \mathbb{N} : \ell \leq k)(\exists g \in \mathcal{W}_{\max}) \\ &\qquad (\forall \varepsilon > 0)(\exists \Omega \in \mathcal{U}_{\mathcal{T}}(T)) \, \forall S \in \Omega : |f| \, \|D^{(\ell)}(M_T - M_S)\| \leq \varepsilon \, |g|. \end{aligned} \tag{6.1.7.1}$$

Then the map

$$\mathcal{T} \times \mathcal{C}^k_{\mathcal{W}}(U, Y_2) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : (T, \gamma) \mapsto b \circ (M_T, \gamma) \tag{†}$$

which is defined by Lemma 6.1.6 is continuous.

Proof. For $k < \infty$. the proof is by induction on k.
$k = 0$: For $S, T \in \mathcal{T}$ and $\gamma, \eta \in \mathcal{C}^0_{\mathcal{W}}(U, Y_2)$, we have

$$b \circ (M_S, \eta) - b \circ (M_T, \gamma) = b \circ (M_S, \eta - \gamma) + b \circ (M_S - M_T, \gamma).$$

We treat each summand separately. To this end, let $f \in \mathcal{W}$ and $x \in U$. Then we calculate for first summand

$$|f(x)|\, \|b(M_S(x), (\gamma - \eta)(x)\| \leq \|b\|_{op} |f(x)|\, \|M_S(x)\|\, \|(\gamma - \eta)(x)\|.$$

For the second summand we get

$$|f(x)|\, \|b \circ (M_S - M_T, \gamma)(x)\| \leq \|b\|_{op} |f(x)|\, \|(M_S - M_T)(x)\|\, \|\gamma(x)\|.$$

Let $g \in \mathcal{W}_{\max}$ as in condition (6.1.7.1). Given $\varepsilon > 0$, let $\Omega \in \mathcal{U}_{\mathcal{T}}(T)$ be as in condition (6.1.7.1). For $S \in \Omega$, we derive from the estimates above that

$$|f(x)|\, \|(b \circ (M_S, \eta) - b \circ (M_T, \gamma))(x)\| \leq \|b\|_{op} (\|\gamma - \eta\|_{f \cdot \|M_S\|, 0} + \varepsilon \|\gamma\|_{g,0}).$$

As the right hand side can be made arbitrarily small, we see that (†) is continuous.

$k \to k+1$: Using Proposition 3.2.3, we just need to prove that for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_2)$ and $T \in \mathcal{T}$, the map $D(b \circ (M_T, \gamma)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$ and that

$$\mathcal{T} \times \mathcal{C}^{k+1}_{\mathcal{W}}(U, Y_2) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z)) : \gamma \mapsto D(b \circ (M_T, \gamma))$$

is continuous. Using Lemma 3.3.2 we get

$$D(b \circ (M_T, \gamma)) = b^{(1)} \circ (DM_T, \gamma) + b^{(2)} \circ (M_T, D\gamma),$$

with $b^{(i)}$ as in Subsection 3.3.1. So by applying the inductive hypothesis to the maps $b^{(1)} \circ (DM_T, \gamma)$ and $b^{(2)} \circ (M_T, D\gamma)$, we see that $D(b \circ (M_T, \gamma))$ is in $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$ and the map (†) is continuous.
$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{C}^\infty_{\mathcal{W}}(U, Y_2) & \xrightarrow{b_{*,\infty}} & \mathcal{C}^\infty_{\mathcal{W}}(U, Z) \\ \downarrow & & \downarrow \\ \mathcal{T} \times \mathcal{C}^n_{\mathcal{W}}(U, Y_2) & \xrightarrow[b_*]{} & \mathcal{C}^n_{\mathcal{W}}(U, Z) \end{array}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent the inclusion maps. With Corollary 3.2.6 we easily deduce the continuity of $b_{*,\infty}$ from the one of b_*. □

6.1.2. Contravariant composition on weighted functions

Here we prove sufficient conditions that make (6.1.2.1) smooth. Since the second factor of the domain of this map in general is not contained in a vector space, we have to wrestle with certain technical difficulties, leading to the definition of a notion of *logarithmically bounded* identity neighborhoods in Lie groups.

Lemma 6.1.8. *Let G be a Lie group and $\omega : G \times M \to M$ a smooth action of G on the smooth manifold M.*

(a) For any $g \in G$, the identity

$$\mathbf{T}\omega = \mathbf{T}\omega_g \circ \mathbf{T}\omega \circ (\mathbf{T}\lambda_{g^{-1}} \times \mathrm{id}_{\mathbf{T}M})$$

holds, where $\lambda_{g^{-1}} : G \to G$ denotes the left multiplication with g^{-1}.

In the following, let $S, T \in G$ and $W : [0,1] \to G$ be a smooth curve with $W(0) = S$ and $W(1) = T$.

(b) Let N be another smooth manifold and $\gamma : M \to N$ a $\mathcal{C}^1-$map. Then for $t \in [0,1]$ and $x \in M$, we have

$$\mathbf{T}(\gamma \circ \omega \circ (W \times \mathrm{id}_M))(t, 1, 0_x) = \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)}(-\dot{\omega}(\delta_\ell(W)(t))(x)). \tag{†}$$

(c) Let X and Y be normed spaces. Assume that M is an open nonempty subset of X. Then for $\gamma, \eta \in \mathcal{C}^1(M, Y)$ and $x \in M$, we have

$$\begin{aligned}&(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x)\\ =&((\gamma - \eta) \circ \omega_T)(x) - \int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt.\end{aligned} \tag{6.1.8.1}$$

Proof. (a) We calculate for $h \in G$ and $m \in M$ that

$$\omega(h, m) = \omega(gg^{-1}h, m) = \omega(g, \omega(g^{-1}h, m)) = \omega_g(\omega(\lambda_{g^{-1}}(h), m)).$$

Applying the tangent functor gives the assertion.

(b) We calculate

$$\begin{aligned}&\mathbf{T}(\gamma \circ \omega \circ (W \times \mathrm{id}_M))(t, 1, 0_x) = \mathbf{T}\gamma \circ \mathbf{T}\omega(W'(t), 0_x)\\ &\quad = \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)} \circ \mathbf{T}\omega(W(t)^{-1} \cdot W'(t), 0_x) = \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)}(-\dot{\omega}(W(t)^{-1}W'(t))(x)).\end{aligned}$$

Here we used (a).

(c) By adding $0 = \eta \circ \omega_T - \eta \circ \omega_T$, we get

$$(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x) = ((\gamma - \eta) \circ \omega_T)(x) + (\eta \circ \omega_T)(x) - (\eta \circ \omega_S)(x)$$

We elaborate on the second summand:

$$\begin{aligned}(\eta \circ \omega_T)(x) - (\eta \circ \omega_S)(x) &= \eta(\omega(W(1), x)) - \eta(\omega(W(0), x))\\ &= \int_0^1 D(\eta \circ \omega \circ (W \times \mathrm{id}_U))(t, x) \cdot (1, 0)\, dt\\ &= -\int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt.\end{aligned}$$

Here we used identity (†). □

Definition 6.1.9. Let G be a Lie group and $U \subseteq G$, $V \subseteq \mathbf{L}(G)$ sets. We call a path $W \in \mathcal{C}^1([0,1], G)$ *V-logarithmically bounded* if $\delta_\ell(W)([0,1]) \subseteq V$. The set U is called *V-logarithmically bounded* if for all $g, h \in U$ there exists an V-logarithmically bounded $W \in \mathcal{C}^\infty([0,1], V)$ with $W(0) = g$ and $W(1) = h$.

Proposition 6.1.10. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights, G a locally convex Lie group and $\omega : G \times U \to U$ a smooth action. Assume that there exists an open neighborhood Ω of $\mathbf{1}$ in G such that*

$$\begin{aligned}&(\forall f \in \mathcal{W}, T \in \Omega)\exists g \in \mathcal{W}_{\max}(\forall \varepsilon > 0)\\ &\qquad \exists V \in \mathcal{U}_{\mathbf{L}(G)}(0), \widetilde{\Omega} \in \mathcal{U}_\Omega(T) \quad V\text{-logarithmically bounded}\\ &\qquad\qquad (\forall S \in \widetilde{\Omega}, v \in V) : |f| \cdot \|D\omega_S \cdot \dot{\omega}(v)\| < \varepsilon |g \circ \omega_S|.\end{aligned} \tag{6.1.10.1}$$

Further assume that $\mathcal{W} \circ \omega_\Omega^{-1} \subseteq \mathcal{W}_{\max}$, and that for all $m \in \mathbb{N}$ with $m < k$ and normed spaces Z, the map

$$\mathcal{C}^m_{\mathcal{W}}(U, \mathrm{L}(X, Z)) \times \Omega \to \mathcal{C}^m_{\mathcal{W}}(U, \mathrm{L}(X, Z)) : (\Gamma, T) \mapsto \Gamma \cdot D\omega_T \tag{6.1.10.2}$$

is defined and continuous.

(a) Then the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U, Y) \times \Omega \to \mathcal{C}^k_{\mathcal{W}}(U, Y) : (\gamma, T) \mapsto \gamma \circ \omega_T$$

is well-defined and continuous.

(b) Let $\ell \in \mathbb{N}^*$. Additionally assume that the maps

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \Omega \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) : (\Gamma, T) \mapsto \Gamma \cdot D\omega_T \tag{6.1.10.3}$$

and

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \times \mathbf{L}(G) \to \mathcal{C}^k_{\mathcal{W}}(U, Y) : (\Gamma, v) \mapsto \Gamma \cdot \dot{\omega}(v) \tag{6.1.10.4}$$

are well-defined and $\mathcal{C}^{\ell-1}$. Then the map

$$\mathfrak{c} : \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(U, Y) \times \Omega \to \mathcal{C}^k_{\mathcal{W}}(U, Y) : (\gamma, T) \mapsto \gamma \circ \omega_T$$

is $\mathcal{C}^\ell$ with the derivative

$$d\mathfrak{c}((\gamma, S); (\gamma_1, S_1)) = -(D\gamma \circ \omega_S) \cdot D\omega_S \cdot \dot{\omega}(S^{-1} \cdot S_1) + \gamma_1 \circ \omega_S. \tag{$\dagger$}$$

Proof. (a) For $k < \infty$, this is proved by induction on k.

$k = 0$: Let $\gamma, \eta \in \mathcal{C}^1_{\mathcal{W}}(U, Y)$, $T \in \Omega$ and $f \in \mathcal{W}$. Let $g \in \mathcal{W}_{\max}$ as in condition (6.1.10.1). Given $\varepsilon > 0$, we find a neighborhood $\widetilde{\Omega}$ of T and $V \in \mathcal{U}_{\mathbf{L}(G)}(0)$ such that condition (6.1.10.1) is satisfied. Using identity (6.1.8.1), we calculate for $S \in \widetilde{\Omega}$, a V-logarithmically bounded path $W : [0,1] \to \widetilde{\Omega}$ connecting S and T, and $x \in U$ that

$$\begin{aligned}
&|f(x)|\,\|(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x)\| \\
\leq &|f(x)| \left(\|((\gamma - \eta) \circ \omega_T)(x)\| + \left\| \int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt \right\| \right) \\
\leq &\|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \int_0^1 |f(x)|\, \|D\eta(\omega_{W(t)}(x))\|_{op} \cdot \|D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\|\, dt \\
\leq &\|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \varepsilon \int_0^1 |(g \circ \omega_{W(t)})(x)|\, \|D\eta(\omega_{W(t)}(x))\|_{op}\, dt \\
\leq &\|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \varepsilon \|\eta\|_{g,1}.
\end{aligned}$$

The continuity at (γ, η) follows from this estimate.

$k \to k+1$: By Proposition 3.2.3 and the inductive hypothesis, we just need to check that the map

$$\mathcal{C}^{k+2}_{\mathcal{W}}(U, Y) \times \Omega \to \mathcal{C}^{k}_{\mathcal{W}}(U, \mathrm{L}(X, Y)) : (\gamma, T) \mapsto D(\gamma \circ \omega_T)$$

is well-defined and continuous. For $\gamma \in \mathcal{C}^{k+2}_{\mathcal{W}}(U, Y)$ and $T \in \Omega$, we have

$$D(\gamma \circ \omega_T) = (D\gamma \circ \omega_T) \cdot D\omega_T.$$

Hence by the inductive hypothesis and the continuity of (6.1.10.2), the induction is finished.

$k = \infty$: This is an easy consequence of the case $k < \infty$ and Corollary 3.2.6.

(b) We prove this by induction on ℓ.

$\ell = 1$: Let $\gamma, \gamma_1 \in \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(U, Y)$, $S \in \Omega$ and $S_1 \in \mathbf{T}_S\Omega$. Further, let $\Gamma :]{-\delta}, \delta[\to \Omega$ be a smooth curve with $\Gamma(0) = S$ and $\Gamma'(0) = S_1$. Then we calculate for a sufficiently small $t \neq 0$:

$$\frac{1}{t}((\gamma + t\gamma_1) \circ \omega_{\Gamma(t)} - \gamma \circ \omega_S) = \frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S) + \gamma_1 \circ \omega_{\Gamma(t)}.$$

Using identity (6.1.8.1) we elaborate on the first summand:

$$\frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S)(x) = -\frac{1}{t} \int_0^1 D\gamma(\omega_{\Gamma(st)}(x)) \cdot D\omega_{\Gamma(st)}(x) \cdot \dot{\omega}(t\delta_\ell(\Gamma)(st))(x)\, ds.$$

Hence

$$\frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S) = -\int_0^1 (D\gamma \circ \omega_{\Gamma(st)}) \cdot D\omega_{\Gamma(st)} \cdot \dot{\omega}(\delta_\ell(\Gamma)(st))\, ds;$$

note that the integral on the right hand side exists by Lemma 3.2.10 since the curve

$$[0,1] \to \mathcal{C}^{k}_{\mathcal{W}}(U, Y) : s \mapsto (D\gamma \circ \omega_{\Gamma(st)}) \cdot D\omega_{\Gamma(st)} \cdot \dot{\omega}(\delta_\ell(\Gamma)(st))$$

is well-defined and continuous by (a) and the continuity of (6.1.10.3) and (6.1.10.4). Hence by Proposition A.1.8,

$$\lim_{t \to 0} \frac{1}{t}((\gamma + t\gamma_1) \circ \omega_{\Gamma(t)} - \gamma \circ \omega_S) = -(D\gamma \circ \omega_S) \cdot D\omega_S \cdot \dot{\omega}(S^{-1} \cdot S_1) + \gamma_1 \circ \omega_S,$$

so the directional derivatives of $\mathfrak{c}$ exist, are of the form (†) and depend continuously on the directions by (a) and the continuity of (6.1.10.3) and (6.1.10.4).

$\ell \to \ell + 1$: Since (6.1.10.3) and (6.1.10.4) are $\mathcal{C}^\ell$ by assumption, we conclude from (†) and the inductive hypothesis that $d\mathfrak{c}$ is $\mathcal{C}^\ell$, whence $\mathfrak{c}$ is $\mathcal{C}^{\ell+1}$. □

6.2. Conclusion and Examples

Finally, we prove a sufficient criterion for the smoothness of the conjugation action of a Lie group G acting on X and $\mathrm{Diff}_{\mathcal{W}}(X)_0$.

Theorem 6.2.1. *Let X be a Banach space, G a Lie group, $\omega : G \times X \to X$ a smooth action and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Assume that $\{f \circ \omega_T : f \in \mathcal{W}, T \in G\} \subseteq \mathcal{W}_{\max}$ (we defined $\mathcal{W}_{\max}$ in Definition 6.1.3), $\{D\omega_T : T \in G\} \subseteq \mathcal{BC}^\infty(X, \mathrm{L}(X))$, the maps*

$$D : G \to \mathcal{BC}^\infty(X, \mathrm{L}(X)) : T \mapsto D\omega_T \tag{†}$$

and (6.1.10.4) *are well-defined and smooth and condition* (6.1.10.1) *is satisfied. Then the map*

$$G \times \mathrm{Diff}_{\mathcal{W}}(X)_0 \to \mathrm{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto \omega_T \circ \phi \circ \omega_T^{-1}$$

is well-defined and smooth.

Proof. Since (†) is well-defined and smooth, we can apply Corollary 3.3.7 to see that (6.1.2.2) is also well-defined and smooth. Similarly, using Corollary 3.3.6, we see that (6.1.10.2) and (6.1.10.3) are well-defined and smooth. Hence Proposition 6.1.10 shows that (6.1.2.1) is smooth. The assertion follows from Lemma 6.1.2. □

Finally, we give a positive and a negative example. The first example shows that we can form the semidirect product $\mathrm{Diff}_{\mathcal{S}}(X)_0 \rtimes \mathrm{GL}(X)$ with respect to the conjugation.

Lemma 6.2.2. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty such that for each $f \in \mathcal{W}$, $f\|\cdot\| \in \mathcal{W}_{\max}$. Further, let $k \in \overline{\mathbb{N}}$ and $b : Y \times X \to Z$ a continuous bilinear map. Then*

$$\mathcal{C}^k_{\mathcal{W}}(U, Y) \times \mathrm{L}(X) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : (\gamma, T) \mapsto b \circ (\gamma, T) \tag{†}$$

is well-defined and smooth.

Proof. The assertion holds for $k = \infty$ if it holds for all $k \in \mathbb{N}$. For $k \neq \infty$, the proof is by induction on k.

$k = 0$: Since (†) is bilinear, it is smooth iff it is continuous in 0. So we only prove that. Let $f \in \mathcal{W}$, $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$, $T \in \mathrm{L}(X)$ and $x \in U$. Then

$$|f(x)|\,\|b(\gamma(x), T(x))\| \leq \|b\|_{op}|f(x)|\,\|x\|\,\|\gamma(x)\|\,\|T\|_{op} \leq \|b\|_{op}\|\gamma\|_{f\|\cdot\|,0}\|T\|_{op}.$$

We conclude that $b \circ (\gamma, T) \in \mathcal{C}^0_{\mathcal{W}}(U, Z)$ and that (†) is continuous in 0.

$k \to k+1$: By Lemma 3.3.2, we have for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$ and $T \in \mathrm{L}(X)$ that

$$D(b \circ (\gamma, T)) = b^{(1)} \circ (D\gamma, T) + b^{(2)} \circ (\gamma, DT).$$

Since $DT \in \mathcal{BC}^\infty(X, \mathrm{L}(X))$, by Proposition 3.3.3 $b^{(2)} \circ (\gamma, DT) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, \mathrm{L}(X,Z))$ and the map $(\gamma, T) \mapsto b^{(2)} \circ (\gamma, DT)$ is smooth (here we use that $\mathrm{L}(X) \to \mathcal{BC}^\infty(X, \mathrm{L}(X)) : T \mapsto DT$ is smooth). By the induction hypothesis, the same holds for $(\gamma, T) \mapsto b^{(1)} \circ (D\gamma, T)$. So using Proposition 3.2.3, the proof is finished. □

Lemma 6.2.3. *Let X be a Banach space and $G := \mathrm{GL}(X)$. We define the action*

$$\omega : G \times X \to X : (g, x) \mapsto g(x)$$

and set $\mathcal{W} := \{x \mapsto \|x\|^n : n \in \mathbb{N}\}$. Then

(a) The map (6.1.10.4) *is smooth.*

(b) The condition (6.1.10.1) *is satisfied.*

Proof. We easily see that $\dot{\omega} = -\operatorname{id}_{\mathrm{L}(X)}$ (since $\mathbf{L}(G) = \mathrm{L}(X)$), and for each $S \in G$ and $x \in X$, $\omega_S = S$ and $DS(x) = S$. For (a), we give two different proofs. The first one uses Lemma 6.2.2, the second uses a topology on the multiplier space $\mathrm{L}(X)$.

(a) *First variant:* Let Y be another normed space. Since for $\Gamma \in \mathcal{C}^k_{\mathcal{W}}(X, \mathrm{L}(X,Y))$ and $S \in \mathbf{L}(G)$, $\Gamma \cdot \dot{\omega}(S) = \mathrm{ev}_{\mathrm{L}(X,Y)} \circ (\Gamma, -S)$ and $\mathrm{ev}_{\mathrm{L}(X,Y)}$ is bilinear and continuous, this is a consequence of Lemma 6.2.2.

Second variant: Obviously $\dot{\omega}(\mathbf{L}(G)) = \mathrm{L}(X)$ consists of multipliers. Further, condition (6.1.7.1) is satisfied (where $\mathcal{T} = \mathrm{L}(X)$ and the family of multipliers is given by $\operatorname{id}_{\mathrm{L}(X)}$) since for $A, B \in \mathrm{L}(X)$ and $x \in X$

$$\|(A-B)(x)\| \le \|A-B\|_{op} \|x\|$$

and

$$\|D(A-B)(x)\| = \|A-B\|_{op}$$

and $\|D^{(k)}(A-B)\| = \|0\| = 0$ for $k > 1$. Hence we can apply Lemma 6.1.7 to see that (6.1.10.4) is smooth.

(b) Let $f = \|\cdot\|^n \in \mathcal{W}$, $T \in G$ and $\varepsilon > 0$. There exists an open convex $U \in \mathcal{U}_G(T)$ such that for all $S \in U$,

- $\|S - T\|_{op} < \varepsilon$
- $\|S^{-1}\|_{op} < 2\|T^{-1}\|_{op}$
- $\|S\|_{op} < 2\|T\|_{op}$.

Then the path $W : [0,1] \to G : t \mapsto tT + (1-t)S$ has the left logarithmic derivative $\delta_\ell(W)(t) = W(t)^{-1}(T-S)$, hence U is $\overline{B}_{\mathrm{L}(X)}(0, 2\|T\|_{op}\varepsilon)$-logarithmically bounded. We calculate for $x \in X$, $S \in U$ and $A \in \overline{B}_{\mathrm{L}(X)}(0, 2\|T\|_{op}\varepsilon)$ that

$$\begin{aligned}|f(x)|\,\|D\omega_S(x)\cdot\dot{\omega}(A)(x)\| = \|x\|^n\|(S\circ A)(x)\| &\leq \|S\|_{op}\|A\|_{op}\|x\|^{n+1}\\ &\leq 4\|T\|_{op}^2\varepsilon\|S^{-1}Sx\|^{n+1} \leq \varepsilon 2^{n+3}\|T\|_{op}^2\|T^{-1}\|_{op}^{n+1}\|Sx\|^{n+1}.\end{aligned}$$

Since $x \mapsto 2^{n+3}\|T\|_{op}^2\|T^{-1}\|_{op}^{n+1}\|x\|^{n+1} \in \mathcal{W}_{\max}$, we see that condition (6.1.10.1) is satisfied. □

Example 6.2.4. Let X, G, ω and $\mathcal{W}$ be as in Lemma 6.2.3. For each $S \in G$ and $x \in X$, $DS(x) = S$. Hence the map

$$D : G \to \mathcal{BC}^\infty(X, \mathrm{L}(X)) : S \mapsto DS$$

is smooth. By Lemma 6.2.3, the assumptions of Theorem 6.2.1 hold (since $\mathcal{W}\circ G \subseteq \mathcal{W}_{\max}$ is obviously true), hence the map

$$\mathrm{GL}(X) \times \mathrm{Diff}_{\mathcal{W}}(X)_0 \to \mathrm{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto T\circ\phi\circ T^{-1}$$

is smooth. So using Lemma B.2.15, we can form the semidirect product

$$\mathrm{Diff}_{\mathcal{W}}(X)_0 \rtimes \mathrm{GL}(X)$$

with respect to the inner automorphisms on $\mathrm{Diff}_{\mathcal{W}}(X)_0$ that are induced by $\mathrm{GL}(X)$.

Finally, we show that the the conjugation of $\mathrm{GL}(\mathbb{R})$ on $\mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$, if it was defined, could not be continuous.

Example 6.2.5. For each $n \in \mathbb{N}$, $\sin((1+\frac{1}{2n})n\pi) = \pm 1$, but $\sin(n\pi) = 0$. Hence

$$\|\sin(t_n\cdot) - \sin\|_{1_\mathbb{R},0} \geq 1$$

for each $n \in \mathbb{N}$, where $t_n := 1 + \frac{1}{2n}$. We see with Lemma 4.2.9 that

$$\frac{1}{2}\sin \in \kappa_{\{1_\mathbb{R}\}}^{-1}(\mathrm{Diff}_{\{1_\mathbb{R}\}}(\mathbb{R})),$$

and obviously $\kappa_{\{1_\mathbb{R}\}}(\frac{1}{2}\sin) \in \mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$. If the conjugation of $\mathrm{GL}(\mathbb{R})$ on $\mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$ was defined and continuous, then the map

$$\mathbb{R}\setminus\{0\} \times \mathcal{BC}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{BC}^\infty(\mathbb{R},\mathbb{R}) : (t,\gamma) \mapsto t^{-1}\gamma(t\cdot)$$

would be continuous in $(1, \frac{1}{2}\sin)$. But it is not since for $t > 0$ and $x \in \mathbb{R}$

$$\begin{aligned}\|t^{-1}\sin(tx) - \sin(x)\| &\geq t^{-1}\|\sin(tx) - \sin(x)\| - \|(t^{-1}-1)\sin(x)\|\\ &\geq t^{-1}\|\sin(tx) - \sin(x)\| - |t^{-1}-1|;\end{aligned}$$

hence we can calculate that for sufficiently large n,

$$\left\|\tfrac{1}{2}t_n^{-1}\sin(t_n\cdot) - \tfrac{1}{2}\sin\right\|_{1_\mathbb{R},0} \geq \tfrac{1}{4}.$$

7. Lie group structures on weighted mapping groups

In this chapter we will use the weighted function spaces discussed in Chapter 3 for the construction of locally convex Lie groups, the *weighted mapping groups.* These groups arise as subgroups of G^U, where G is a suitable Lie group and U is an open nonempty subset of a normed space. First, we give some definitions that are used throughout this chapter.

Definition 7.0.1. Let U be a nonempty set and G be a group with the multiplication map m_G and the inversion map I_G. Then G^U can be endowed with a group structure: The multiplication is given by

$$((g_u)_{u\in U}, (h_u)_{u\in U}) \mapsto (m_G(g_u, h_u))_{u\in U} = m_G \circ ((g_u)_{u\in U}, (h_u)_{u\in U})$$

and the inversion by

$$(g_u)_{u\in U} \mapsto (I_G(g_u))_{u\in U} = I_G \circ (g_u)_{u\in U}.$$

Further we call a set $A \subseteq G$ *symmetric* if

$$A = I_G(A).$$

Inductively, for $n \in \mathbb{N}$ with $n \geq 1$ we define

$$A^{n+1} := m_G(A^n \times A),$$

where $A^1 := A$.

Definition 7.0.2. Let G be a Lie group and $\phi : V \to \mathbf{L}(G)$ a chart. We call the pair (ϕ, V) *centered around* $\mathbf{1}$ or just *centered* if $V \subseteq G$ is an open identity neighborhood and $\phi(\mathbf{1}) = 0$.

7.1. Weighted maps into Banach Lie groups

In this section, we discuss certain subgroups of G^U, where G is a Banach Lie group and U an open subset of a normed space X. We construct a subgroup $\mathcal{C}^k_{\mathcal{W}}(U, G)$ consisting of *weighted mappings* that can be turned into a (connected) Lie group. Its modelling space is $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, where $k \in \overline{\mathbb{N}}$ and $\mathcal{W}$ is a set of weights on U containing 1_U. Later we prove that these groups are regular Lie groups. Finally, we discuss the case when $U = X$. Then $\mathrm{Diff}_{\mathcal{W}}(X)$ acts on $\mathcal{C}^\infty_{\mathcal{W}}(X, G)$, and this we can turn the semidirect product of these groups into a Lie group.

7.1.1. Construction of the Lie group

We construct the Lie group from local data using Lemma B.2.5. For a chart (ϕ, V) of G, we can endow the set $\phi^{-1} \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \phi(V)) \subseteq G^U$ with the manifold structure that turns the superposition operator ϕ_* into a chart. We need to check whether the local multiplication and inversion on this set are smooth with respect to this manifold structure. The group operations on G^U arise as the composition of the corresponding operations on G with the mappings (see Definition 7.0.1). Since the group operations of Banach Lie groups are analytic, we will use the results of Subsection 3.3.3 as our main tools. The use of this tools allows to construct $\mathcal{C}^k_{\mathcal{W}}(U, G)$ when G is an analytic Lie group modelled on an arbitrary normed space.

Remark 7.1.1. We call a Lie group G *normed* if $\mathbf{L}(G)$ is a normable space. A *normed analytic* Lie group is a normed Lie group which is an analytic Lie group.

Local multiplication The treatment of the group multiplication is a simple application of Proposition 3.3.21.

Lemma 7.1.2. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G an normed analytic Lie group with the group multiplication m_G and (ϕ, V) a centered chart of G. Then there exists an open identity neighborhood $W \subseteq V$ such that the map*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \times \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V)) : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta) \quad (\dagger)$$

is defined and analytic.

Proof. By Lemma 3.4.16, the map (†) is defined and analytic iff there exists an open identity neighborhood $W \subseteq G$ such that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_* : \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W) \times \phi(W)) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$$

is so. There exists an open bounded zero neighborhood $\widetilde{W_L} \subseteq \mathbf{L}(G)$ such that $\widetilde{W_L} + \widetilde{W_L} \subseteq \phi(V)$. By the continuity of the multiplication m_G there exists an open $\mathbf{1}$-neighborhood W with $m_G(W \times W) \subseteq \phi^{-1}(\widetilde{W_L})$. We may assume w.l.o.g. that $\phi(W)$ is star-shaped with center 0. Then

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))(\phi(W) \times \phi(W)) \subseteq \widetilde{W_L}.$$

Further the restriction of $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ to $\phi(W) \times \phi(W)$ is analytic, takes $(0, 0)$ to 0 and has bounded image, since ϕ is centered and $\widetilde{W_L}$ is bounded. In the real case, using Lemma 3.3.19 we can choose $\phi(W)$ sufficiently small such that the restriction of $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ to $\phi(W)$ has a good complexification. Hence we can apply Proposition 3.3.21 to see that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})) \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W) \times \phi(W)) \in \mathcal{C}^{\ell}_{\mathcal{W}}(U, \widetilde{W_L})$$

and that the map $(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_*$ is analytic. But

$$\mathcal{C}^{\ell}_{\mathcal{W}}(U, \widetilde{W_L}) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$$

by the definition of $\widetilde{W_L}$, and this gives the assertion. □

Local inversion The discussion of the inversion is more delicate. For a short explanation, let $(\phi, \widetilde{V})$ be a chart for G, $V \subseteq \widetilde{V}$ a symmetric open identity neighborhood and I_G the inversion of G. Then the superposition of $\phi \circ I_G \circ \phi^{-1}$ described in Proposition 3.3.21 does not necessarily map $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$ into itself; hence we have to work to construct symmetrical open subsets.

Lemma 7.1.3. *Let G be a group, $U \subseteq G$ a topological space and $V \subseteq U$ a symmetric subset with $\mathbf{1} \in V^\circ$ such that the inversion $I_G : V \to V$ is continuous. Then*

$$V^\circ \cap I_G(V^\circ)$$

is a symmetric set that is open in U and contains $\mathbf{1}$.

Proof. Let $W := V^\circ \cap I_G(V^\circ)$. Then $\mathbf{1} \in W$, and since

$$W^{-1} = I_G(W) = I_G(V^\circ \cap I_G(V^\circ)) = I_G(V^\circ) \cap I_G(I_G(V^\circ)) = I_G(V^\circ) \cap V^\circ = W,$$

it is a symmetric set. Since I_G is a homeomorphism, $I_G(V^\circ)$ is an open subset of V. Hence $W = I_G(V^\circ) \cap V^\circ$ is an open subset of V° and hence of U. □

Lemma 7.1.4. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G an normed analytic Lie group with the group inversion I_G, (ϕ, V) a centered chart of G such that $\phi(V)$ is bounded and V is symmetric. Then the following statements hold:*

(a) The map

$$I_L := \phi \circ I_G \circ \phi^{-1} : \phi(V) \to \phi(V)$$

is an analytic bijective involution. Hence for any open and star-shaped set $W \subseteq \phi(V)$ with center 0, the map

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \to \mathcal{C}^{\ell}_{\mathcal{W}}(U, \phi(V)) : \gamma \mapsto I_L \circ \gamma$$

is analytic, assuming in the real case that $I_L|_W$ has a good complexification.

(b) Let $\Omega \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$. Then $\phi^{-1} \circ (\Omega \cap I_L \circ \Omega)$ is a symmetric subset of G^U.

(c) For any open zero neighborhood $\widetilde{W} \subseteq \phi(V)$ there exists an open convex zero neighborhood $W \subseteq \widetilde{W}$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}).$$

(d) There exists an open convex zero neighborhood $W \subseteq \phi(V)$ and a zero neighborhood $C^{\ell}_{\mathcal{W}} \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq (C^{\ell}_{\mathcal{W}})^\circ \cap I_L \circ (C^{\ell}_{\mathcal{W}})^\circ,$$

$\phi^{-1} \circ C^{\ell}_{\mathcal{W}}$ is symmetric in G^U, the map

$$C^{\ell}_{\mathcal{W}} \to C^{\ell}_{\mathcal{W}} : \gamma \mapsto I_L \circ \gamma$$

is continuous and its restriction to $(C^{\ell}_{\mathcal{W}})^\circ$ is analytic. The set W can be chosen independently of ℓ and $\mathcal{W}$.

Proof. (a) The assertions concerning I_L follow from the fact that V is symmetric and G is an analytic Lie group.
The assertion on the superposition map of I_L is a consequence of Proposition 3.3.21 since W is star-shaped with center 0 and $\phi(V)$ is bounded.

(b) This is an easy computation.

(c) By the continuity of the addition, we find an open zero neighborhood H with $H + H \subseteq \widetilde{W}$. Since I_L is continuous in 0 there exists an open convex zero neighborhood W with $I_L(W) \subseteq H$ and $W \subseteq \widetilde{W}$. Then

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W})$$

and by (a)

$$I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\ell}_{\mathcal{W}}(U, H) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}).$$

The fact that $I_L \circ I_L = \mathrm{id}_{\phi(V)}$ completes the argument.

(d) Let $W_3 \subseteq \phi(V)$ be an open convex zero neighborhood. Then by (c) we find open convex zero neighborhoods $W_1, W_2 \subseteq \phi(V)$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_i) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_{i+1}) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_{i+1})$$

for $i = 1, 2$. So

$$C^{\ell}_{\mathcal{W}} := \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_3) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_3)$$

is a zero neighborhood, and by (b), $\phi^{-1} \circ C^{\ell}_{\mathcal{W}}$ is symmetric. Hence the superposition of I_L maps $C^{\ell}_{\mathcal{W}}$ into itself and is continuous on $C^{\ell}_{\mathcal{W}}$ and analytic on $(C^{\ell}_{\mathcal{W}})^{\circ}$ (see (a)). Further

$$(C^{\ell}_{\mathcal{W}})^{\circ} \cap I_L \circ (C^{\ell}_{\mathcal{W}})^{\circ} \supseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_2) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_2) \supseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_1),$$

whence (d) is established with $W := W_1$. □

Construction of the Lie group structure After discussing the group operations locally, we turn a subgroup of G^U into a Lie group for each centered chart of G. We will also show that the identity component of this group does not depend on the chart.

Lemma 7.1.5. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and $V \subseteq Y$ convex. Then the set $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, V)$ is convex.*

Proof. It is obvious that $\mathcal{C}^{\ell}_{\mathcal{W}}(U, V)$ is convex since V is so. The set $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, V)$ is the interior of $\mathcal{C}^{\ell}_{\mathcal{W}}(U, V)$ with respect to the norm $\|\cdot\|_{1_U,0}$, hence it is convex. □

Proposition 7.1.6. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G an normed analytic Lie group and (ϕ, V) a centered chart. There exist a subgroup $(G, \phi)^U_{\mathcal{W},\ell}$ of G^U that can be turned into an analytic Lie group which is modelled on $\mathcal{C}^{\ell}_{\mathcal{W}}(U, \mathbf{L}(G))$; and an open $\mathbf{1}$-neighborhood $W \subseteq V$ which is independent of $\mathcal{W}$ and ℓ such that*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to (G, \phi)^U_{\mathcal{W},\ell} : \gamma \mapsto \phi^{-1} \circ \gamma$$

is an analytic embedding onto an open set. Moreover, for any convex open zero neighborhood $\widetilde{W} \subseteq \phi(W)$, the set $\phi^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W})$ generates the identity component of $(G, \phi)^U_{\mathcal{W},\ell}$ as a group.

Proof. Using Lemma 7.1.2 we find an open **1**-neighborhood $\widetilde{W} \subseteq V$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W})) \times \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W})) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V)) : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta)$$

is analytic. We may assume w.l.o.g. that $\widetilde{W}$ is symmetric. With Lemma 7.1.4 (d) and Lemma 7.1.3, we find an open zero neighborhood $H \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W}))$ such that $\phi^{-1} \circ H$ is symmetric, the map

$$H \to H : \gamma \mapsto I_L \circ \gamma$$

is analytic and $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \subseteq H$ for some open **1**-neighborhood $W \subseteq V$, which is independent of $\mathcal{W}$ and ℓ. We endow $\phi^{-1} \circ H$ with the differential structure which turns the bijection

$$\phi^{-1} \circ H \to H : \gamma \mapsto \phi \circ \gamma$$

into an analytic diffeomorphism. Then we can apply Lemma B.2.5 to construct an analytic Lie group structure on the subgroup $(G, \phi)^U_{\mathcal{W},\ell}$ of G^U which is generated by $\phi^{-1} \circ H$ such that $\phi^{-1} \circ H$ becomes an open subset of $(G, \phi)^U_{\mathcal{W},\ell}$.

Since we may assume w.l.o.g. that $\phi(W)$ is convex, $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W))$ is open and convex (see Lemma 7.1.5), hence the set

$$\phi^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W))$$

is connected and open by the construction of the differential structure of $(G, \phi)^U_{\mathcal{W},\ell}$. Furthermore it obviously contains the unit element, whence it generates the identity component. □

Lemma 7.1.7. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G be an normed analytic Lie group. Then for centered charts (ϕ_1, V_1), (ϕ_2, V_2), the identity component of $(G, \phi_1)^U_{\mathcal{W},\ell}$ coincides with the one of $(G, \phi_2)^U_{\mathcal{W},\ell}$, and the identity map between them is an analytic diffeomorphism.*

Proof. We may assume w.l.o.g. that $\phi_1(V_1)$ and $\phi_2(V_2)$ are bounded. Using Proposition 7.1.6, we find open **1**-neighborhoods $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ such that the identity component of $(G, \phi_i)^U_{\mathcal{W},\ell}$ is generated by $\phi_i^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_i(W_i))$ for $i \in \{1, 2\}$. Since $\phi_1 \circ \phi_2^{-1}$ is analytic, we find open zero neighborhoods $\widetilde{W}^L_1 \subseteq \phi_1(W_1)$ and $\widetilde{W}^L_2 \subseteq \phi_2(W_2)$ such that

$$(\phi_1 \circ \phi_2^{-1})(\widetilde{W}^L_2) \subseteq \widetilde{W}^L_1 \text{ and } \widetilde{W}^L_1 + \widetilde{W}^L_1 \subseteq \phi_1(W_1)$$

and $\widetilde{W}^L_2$ is convex. Then by Proposition 7.1.6, the identity component of $(G, \phi_2)^U_{\mathcal{W},\ell}$ is generated by $\phi_2^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}^L_2)$, and in the real case we may assume that $\phi_1 \circ \phi_2^{-1}|_{\widetilde{W}^L_2}$ has a good complexification. By Proposition 3.3.21 the map

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}^L_2) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_1(W_1)) : \gamma \mapsto \phi_1 \circ \phi_2^{-1} \circ \gamma$$

is defined and analytic, and this implies that

$$\phi_2^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}^L_2) \subseteq \phi_1^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_1(W_1)).$$

Hence the identity component of $(G, \phi_2)^U_{\mathcal{W},\ell}$ is contained in the one of $(G, \phi_1)^U_{\mathcal{W},\ell}$, and the inclusion map of the former into the latter is analytic.

Exchanging the roles of ϕ_1 and ϕ_2 in the preceding argument, we get the assertion. □

Definition 7.1.8. Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G be an normed analytic Lie group. We write $\mathcal{C}^\ell_{\mathcal{W}}(U, G)$ for the connected Lie group that was constructed in Proposition 7.1.6. There and in Lemma 7.1.7 it was proved that for any centered chart (ϕ, V) of G and $W \subseteq V$ such that $\phi(W)$ is convex, the inverse map of

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to \mathcal{C}^\ell_{\mathcal{W}}(U, G) : \gamma \mapsto \phi^{-1} \circ \gamma$$

is a chart.

7.1.2. Regularity

We show that for a Banach Lie group G, the Lie group $\mathcal{C}^\ell_{\mathcal{W}}(U, G)$ is regular.

Lemma 7.1.9. *Let G, H be Lie groups and $\phi : G \to H$ a Lie group morphism.*

(a) For each $g \in G$ and $v \in \mathbf{T}_g G$, we have $\mathbf{T}_g\phi(v) = \phi(g) \cdot \mathbf{L}(\phi)(g^{-1} \cdot v)$.

(b) Let $\gamma \in \mathcal{C}^1([0, 1], G)$. Then $\delta_\ell(\phi \circ \gamma) = \mathbf{L}(\phi) \circ \delta_\ell(\gamma)$.

Proof. The proof of (a) being straightforward, we turn to (b). We calculate the derivative of $\phi \circ \gamma$ using (a) and the fact that ϕ is a Lie group morphism:

$$(\phi \circ \gamma)'(t) = \mathbf{T}(\phi \circ \gamma)(t, 1) = \mathbf{T}_{\gamma(t)}\phi(\gamma'(t)) = \phi(\gamma(t)) \cdot \mathbf{L}(\phi)(\gamma(t)^{-1} \cdot \gamma'(t)).$$

From this we derive

$$\delta_\ell(\phi \circ \gamma)(t) = (\phi \circ \gamma)(t)^{-1} \cdot (\phi \circ \gamma)'(t) = \mathbf{L}(\phi)(\gamma(t)^{-1} \cdot \gamma'(t)) = \mathbf{L}(\phi)(\delta_\ell(\gamma)(t)),$$

and the proof is finished. □

The following is well known from the theory of Banach Lie groups.

Lemma 7.1.10. *Let G be a Banach Lie group and $V \in \mathcal{U}(\mathbf{1})$. Then there exists a balanced open $W \in \mathcal{U}_{\mathbf{L}(G)}(0)$ such that*

$$\gamma \in \mathcal{C}^0([0, 1], W) \implies \mathrm{Evol}^\ell_G(\gamma) \in \mathcal{C}^0([0, 1], V). \tag{7.1.10.1}$$

Furthermore, the map $\mathrm{evol}^\ell_G : \mathcal{C}^0([0, 1], W) \to G$ is continuous.

We define some terminology needed for the proof.

Definition 7.1.11. Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G be a Banach Lie group. Further, let $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{W}$ such that $1_U \in \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\ell_1, \ell_2 \in \overline{\mathbb{N}}$ such that $\ell_1 \leq \ell_2 \leq k$. We denote the inclusion

$$\mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U, \mathbf{L}(G)) \to \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, \mathbf{L}(G)).$$

by $\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}$ and the inclusion

$$\mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U, G) \to \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, G)$$

by $\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}$. Further, we define $\iota^L_{\mathcal{F}_1,\ell_1} := \iota^L_{(\mathcal{W},k),(\mathcal{F}_1,\ell_1)}$ and $\iota^G_{\mathcal{F}_1,\ell_1} := \iota^G_{(\mathcal{W},k),(\mathcal{F}_1,\ell_1)}$. Then for a suitable centered chart (ϕ, V) of G, the diagram

$$\begin{array}{ccc}
\mathcal{C}^{\partial,\ell_2}_{\mathcal{F}_2}(U, \phi(V)) & \xrightarrow{\phi_*^{-1}} & \mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U, G) \\
\downarrow{\scriptstyle \iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}} & & \downarrow{\scriptstyle \iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}} \\
\mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U, \phi(V)) & \xrightarrow[\phi_*^{-1}]{} & \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, G)
\end{array}$$

commutes. Hence we derive the identity

$$\mathbf{L}(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}) = \mathbf{T}_0\phi_*^{-1} \circ \mathbf{T}_0\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)} \circ \mathbf{T}_{\mathbf{1}}\phi_*.$$

Let $x \in U$. We let ev^G_x resp. ev^L_x denote the maps

$$\mathrm{ev}^G_x : \mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U, G) \to G : \gamma \mapsto \gamma(x) \qquad \mathrm{ev}^L_x : \mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U, \mathbf{L}(G)) \to \mathbf{L}(G) : \gamma \mapsto \gamma(x).$$

Obviously, the diagram

$$\begin{array}{ccc}
\mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U, \phi(V)) & \xrightarrow{\phi_*^{-1}} & \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, G) \\
\downarrow{\scriptstyle \mathrm{ev}^L_x} & & \downarrow{\scriptstyle \mathrm{ev}^G_x} \\
\phi(V) & \xrightarrow[\phi^{-1}]{} & G
\end{array}$$

commutes, so we derive the identity

$$\mathbf{L}(\mathrm{ev}^G_x) = \mathbf{T}_0\phi^{-1} \circ \mathbf{T}_0 \,\mathrm{ev}^L_x \circ \mathbf{T}_{\mathbf{1}}\phi_*.$$

Remark 7.1.12. In the following, if E is a locally convex vector space, we shall frequently identity $\mathbf{T}_0 E = \{0\} \times E$ with E in the obvious way. Then for a Banach Lie group G and a centered chart (ϕ, V) of G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$, we can identify $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$ with $\mathbf{L}(\mathcal{C}^k_{\mathcal{W}}(U, G))$ via $\mathbf{T}_0\phi_*^{-1}$ and $\mathbf{T}_{\mathbf{1}}\phi_*$, respectively.

Lemma 7.1.13. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ G a Banach Lie group and (ϕ, V) a centered chart for G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$. Further, let $x \in U$ and $\Gamma : [0, 1] \to \mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$ a smooth curve whose left evolution exists. Then $\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\mathbf{T}_0\phi_*^{-1} \circ \Gamma)$ is the left evolution of $\mathrm{ev}^L_x \circ \Gamma$.*

Proof. We set $\eta := \operatorname{Evol}^\ell(\mathbf{T}_0\phi_*^{-1}\circ\Gamma)$ and calculate using Lemma 7.1.9 and Definition 7.1.11 that

$$\delta_\ell(\operatorname{ev}_x^G \circ\eta) = \mathbf{L}(\operatorname{ev}_x^G)\circ\delta_\ell(\eta) = \mathbf{T}_0\phi^{-1}\circ\mathbf{T}_0\operatorname{ev}_x^L\circ\mathbf{T_1}\phi_*\circ\mathbf{T}_0\phi_*^{-1}\circ\Gamma = \operatorname{ev}_x^L\circ\Gamma.$$

This shows the assertion. □

Proposition 7.1.14. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G a Banach Lie group. Then the following assertions hold:*

(a) $\mathcal{C}^k_\mathcal{W}(U, G)$, endowed with the Lie group structure described in Definition 7.1.8, is regular.

(b) The exponential function of $\mathcal{C}^k_\mathcal{W}(U, G)$ is given by

$$\mathcal{C}^k_\mathcal{W}(U, \mathbf{L}(G)) \to \mathcal{C}^k_\mathcal{W}(U, G) : \gamma \mapsto \exp_G\circ\gamma,$$

where we identify $\mathcal{C}^k_\mathcal{W}(U, \mathbf{L}(G))$ with $\mathbf{L}(\mathcal{C}^k_\mathcal{W}(U, G))$.

Proof. (a) Let $(\phi, \widetilde{V})$ be a centered chart of G such that $d\phi|_{\mathbf{L}(G)} = \operatorname{id}_{\mathbf{L}(G)}$. We set

$$\mathbf{F} := \{\mathcal{F} \subseteq \mathcal{W} : 1_U \in \mathcal{F}, |\mathcal{F}| < \infty\}.$$

After shrinking $\widetilde{V}$, we may assume that the inverse map of

$$\mathcal{C}^{\partial,\ell}_\mathcal{F}(U, \widetilde{V}) \to \mathcal{C}^\ell_\mathcal{F}(U, G) : \Gamma \mapsto \phi^{-1}\circ\Gamma$$

is a chart around the identity for $\mathcal{F} \in \mathbf{F}$ and $\ell \in \mathbb{N}$ with $\ell \le k$ (see Definition 7.1.8). Let $V \subseteq \widetilde{V}$ an open $\mathbf{1}$-neighborhood such that $\phi(V) + \phi(V) \subseteq \phi(\widetilde{V})$. We choose an open zero neighborhood $W \subseteq \phi(\widetilde{V})$ such that the implication (7.1.10.1) holds. Let $\Gamma : [0, 1] \to \mathcal{C}^{\partial,k}_\mathcal{W}(U, W)$ be a smooth curve. Then $\Gamma_{\mathcal{F},\ell} := \iota^L_{\mathcal{F},\ell}\circ\Gamma$ is smooth, and since $\mathcal{C}^\ell_\mathcal{F}(U, G)$ is a Banach Lie group, the curve $\mathbf{T}_0\phi_*^{-1}\circ\Gamma_{\mathcal{F},\ell}$ has a smooth left evolution $\eta_{\mathcal{F},\ell} : [0, 1] \to \mathcal{C}^\ell_\mathcal{F}(U, G)$. Then, for each $x \in U$, $\operatorname{ev}_x^G\circ\eta_{\mathcal{F},\ell}$ is the left evolution of $\operatorname{ev}_x^L\circ\Gamma_{\mathcal{F},\ell}$ by Lemma 7.1.13. Since we assumed that (7.1.10.1) holds, we conclude that for each $t \in [0, 1]$, the image of $\eta_{\mathcal{F},\ell}(t)$ is contained in V.

Further, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \le \ell_2 \le k$,

$$\begin{aligned}\delta_\ell(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}\circ\eta_{\mathcal{F}_2,\ell_2}) &= \mathbf{L}(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)})\circ\delta_\ell(\eta_{\mathcal{F}_2,\ell_2})\\ &= \mathbf{T}_0\phi_*^{-1}\circ\mathbf{T}_0\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}\circ\mathbf{T_1}\phi_*\circ\delta_\ell(\eta_{\mathcal{F}_2,\ell_2}) = \mathbf{T}_0\phi_*^{-1}\circ\Gamma_{\mathcal{F}_1,\ell_1} = \delta_\ell(\eta_{\mathcal{F}_1,\ell_1}).\end{aligned}$$

Hence $\eta_{\mathcal{F}_1,\ell_1} = \iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}\circ\eta_{\mathcal{F}_2,\ell_2}$. So the family $(\phi_*\circ\eta_{\mathcal{F},\ell})_{\mathcal{F}\in\mathbf{F},\ell\le k}$ is compatible with the inclusion maps, hence using Proposition 3.2.5 and Proposition A.1.12, we derive a smooth curve $\widetilde{\eta} : [0, 1] \to \mathcal{C}^{\partial,k}_\mathcal{W}(U, \phi(\widetilde{V}))$ such that for all $\mathcal{F} \in \mathbf{F}$ and $\ell \in \mathbb{N}$ with $\ell \le k$, we have $\iota^L_{\mathcal{F},\ell}\circ\widetilde{\eta} = \phi_*\circ\eta_{\mathcal{F},\ell}$. We set $\eta := \phi_*^{-1}\circ\widetilde{\eta}$. Then

$$\mathbf{T}_0\phi_*^{-1}\circ\mathbf{T}_0\iota^L_{\mathcal{F},\ell}\circ\mathbf{T_1}\phi_*\circ\delta_\ell(\eta) = \mathbf{L}(\iota^G_{\mathcal{F},\ell})\circ\delta_\ell(\eta) = \delta_\ell(\eta_{\mathcal{F},\ell}) = \mathbf{T}_0\phi_*^{-1}\circ\Gamma_{\mathcal{F},\ell} = \mathbf{T}_0\phi_*^{-1}\circ\iota^L_{\mathcal{F},\ell}\circ\Gamma,$$

and since $\mathcal{F}$ and ℓ were arbitrary, we conclude (using Proposition 3.2.5) that $\mathbf{T_1}\phi_* \circ \delta_\ell(\eta) = \Gamma$ and thus
$$\delta_\ell(\eta) = \mathbf{T}_0\phi_*^{-1} \circ \Gamma.$$

It remains to show that the left evolution is smooth. To this end, we denote the left evolution of $\mathcal{C}^\ell_\mathcal{F}(U, G)$ with $\mathrm{evol}_{\mathcal{F},\ell}$ and the one of $\mathcal{C}^k_\mathcal{W}(U, G)$ with evol. From our results above and Definition 7.1.11, we derive the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,k}_\mathcal{W}(U, W)) & \xrightarrow{\mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1}} & \phi_*^{-1} \circ \mathcal{C}^{\partial,k}_\mathcal{W}(U, \phi(\widetilde{V})) \\
\downarrow{\scriptstyle \iota^L_{\mathcal{F},\ell}} & & \downarrow{\scriptstyle \iota^G_{\mathcal{F},\ell}} \\
\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,\ell}_\mathcal{F}(U, W)) & \xrightarrow[\mathrm{evol}_{\mathcal{F},\ell} \circ \mathbf{T}_0\phi_*^{-1}]{} & \phi_*^{-1} \circ \mathcal{C}^{\partial,\ell}_\mathcal{F}(U, \phi(\widetilde{V}))
\end{array}$$

Since the three lower arrows represent smooth maps, the map
$$\phi_* \circ \iota^G_{\mathcal{F},\ell} \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1} = \iota^L_{\mathcal{F},\ell} \circ \phi_* \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1}$$
is smooth on $\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,k}_\mathcal{W}(U, W))$. Using Proposition A.1.12 and Subsection 3.2.2, we conclude that $\phi_* \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1}$ is smooth, and since ϕ_* and $\mathbf{T}_0\phi_*^{-1}$ are diffeomorphisms, using Lemma B.2.10 we deduce that evol is smooth.

(b) Let (ϕ, V) be a centered chart of G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$. We denote the exponential function of $\mathcal{C}^k_\mathcal{W}(U, G)$ by $\exp_\mathcal{W}$. Let $x \in U$ and $\gamma \in \mathcal{C}^k_\mathcal{W}(U, \mathbf{L}(G))$. We denote the constant, γ-valued curve from $[0,1]$ to $\mathcal{C}^k_\mathcal{W}(U, \mathbf{L}(G))$ by Γ. We proved in Lemma 7.1.13 that $\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\phi_*^{-1} \circ \Gamma)$ is the left evolution of $\mathrm{ev}^L_x \circ \Gamma$. On the other hand, since Γ is constant, the left evolution of $\mathrm{ev}^L_x \circ \Gamma$ is the restriction of the 1-parameter group $\mathbb{R} \to G : t \mapsto \exp_G(t\, \mathrm{ev}^L_x(\gamma))$. Hence
$$\exp_G(\mathrm{ev}^L_x(\gamma)) = (\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\phi_*^{-1} \circ \Gamma))(1) = \mathrm{ev}^G_x \circ \mathrm{evol}^\ell(\phi_*^{-1} \circ \Gamma) = \mathrm{ev}^G_x \circ \exp_\mathcal{W}(\phi_*^{-1}(\gamma)).$$
Thus $\exp_\mathcal{W}(\phi_*^{-1}(\gamma))(x) = \exp_G(\gamma(x))$, from which we conclude the assertion since $x \in U$ was arbitrary. □

7.1.3. Semidirect products with weighted diffeomorphisms

In this subsection we discuss an action of the diffeomorphism group $\mathrm{Diff}_\mathcal{W}(X)$ on the Lie group $\mathcal{C}^\infty_\mathcal{W}(X, G)$, where G is a Banach Lie group. This action can be used to construct the semidirect product $\mathcal{C}^\infty_\mathcal{W}(X, G) \rtimes \mathrm{Diff}_\mathcal{W}(X)$ and turn it into a Lie group. For technical reasons, we first discuss the following action of $\mathrm{Diff}_\mathcal{W}(X)$ on G^X.

Definition 7.1.15. Let X be a Banach space, G a Banach Lie group and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. We define the map
$$\widetilde{\omega} : \mathrm{Diff}_\mathcal{W}(X) \times G^X \to G^X : (\phi, \gamma) \mapsto \gamma \circ \phi^{-1}.$$

It is easy to see that $\widetilde{\omega}$ is in fact a group action, and moreover that it is a group morphism in its second argument:

Lemma 7.1.16. *(a)* $\widetilde{\omega}$ *is a group action of* $\mathrm{Diff}_{\mathcal{W}}(X)$ *on* G^X.

(b) For each $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)$ *the partial map* $\widetilde{\omega}(\phi, \cdot)$ *is a group homomorphism.*

Proof. These are easy computations. □

We show that this action leaves $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$ invariant. Since we proved in Lemma 7.1.16 that $\widetilde{\omega}$ is a group morphism in its second argument, it suffices to show that it maps a generating set of $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$ into this space.

Lemma 7.1.17. *Let* X *be a Banach space,* G *a Banach Lie group,* $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ *with* $1_X \in \mathcal{W}$, $(\phi, \widetilde{V})$ *a centered chart of* G *and* V *an open identity neighborhood such that* $\phi(V)$ *is convex. Then*

$$\widetilde{\omega}(\mathrm{Diff}_{\mathcal{W}}(X) \times (\phi^{-1} \circ \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)))) \subseteq \phi^{-1} \circ \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)),$$

and the map

$$\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)) \to \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)) : (\psi, \gamma) \mapsto \phi \circ \widetilde{\omega}(\psi, \phi^{-1} \circ \gamma)$$

is smooth. Moreover,

$$\widetilde{\omega}(\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, G)) \subseteq \mathcal{C}^{\infty}_{\mathcal{W}}(X, G).$$

Proof. Let ψ be an element of $\mathrm{Diff}_{\mathcal{W}}(X)$ and $\gamma \in \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V))$. Then

$$\widetilde{\omega}(\psi, \phi^{-1} \circ \gamma) = \phi^{-1} \circ (\gamma \circ \psi^{-1}),$$

and using Proposition 4.1.7 this identity proves the first and the second assertion. The final assertion follows immediately from the first assertion since we proved in Lemma 7.1.16 that $\widetilde{\omega}$ is a group morphism in its second argument, and in Definition 7.1.8 that that $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$ is generated by $\phi^{-1} \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(X, \phi(V))$. □

So by restricting $\widetilde{\omega}$ to $\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$, we get a group action of $\mathrm{Diff}_{\mathcal{W}}(X)$ on $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$.

Definition 7.1.18. We define

$$\omega := \widetilde{\omega}|_{\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X,G)} : \mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \to \mathcal{C}^{\infty}_{\mathcal{W}}(X, G) : (\phi, \gamma) \mapsto \gamma \circ \phi^{-1}.$$

Finally, we are able to turn the semidirect product $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \rtimes_{\omega} \mathrm{Diff}_{\mathcal{W}}(X)$ into a Lie group.

Theorem 7.1.19. *Let* X *be a Banach space,* G *a Banach Lie group and* $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ *with* $1_X \in \mathcal{W}$. *Then* $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \rtimes_{\omega} \mathrm{Diff}_{\mathcal{W}}(X)$ *can be turned into a Lie group modelled on* $\mathcal{C}^{\infty}_{\mathcal{W}}(X, \mathbf{L}(G)) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$.

Proof. We proved in Lemma 7.1.17 that ω is smooth on a neighborhood of $(\mathrm{id}_X, \mathbf{1})$, and since this neighborhood is the product of generators of $\mathrm{Diff}_{\mathcal{W}}(X)$ resp. $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$, we can use Lemma B.2.14 to see that ω is smooth. Hence we can apply Lemma B.2.15 and are home. □

7.2. Weighted maps into locally convex Lie groups

In this section, we discuss certain subgroups of G^U, where G is a Lie group and U an open subset of a finite dimensional space X. We construct a subgroup $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$ consisting of *weighted decreasing mappings* that can be turned into a (connected) Lie group. After that, we extend this group to a Lie group $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$ which contains $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$ as an open normal subgroup, and discuss its relation with "rapidly decreasing mappings".

The modelling space of these groups is $\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))^\bullet$, where $k \in \overline{\mathbb{N}}$ and $\mathcal{W}$ is a set of weights on U containing 1_U. These spaces are introduced in Section 3.4.

7.2.1. Construction of the Lie group

We construct the Lie group from local data using Lemma B.2.5. For a chart (ϕ, V) of G, we can endow the set $\phi^{-1}\circ\mathcal{C}^k_{\mathcal{W}}(U,\phi(V))^\bullet \subseteq G^U$ with the manifold structure that turns the superposition operator ϕ_* into a chart. We then need to check whether the multiplication and inversion on G^U are smooth with respect to this manifold structure. The group operations on G^U arise as the composition of the corresponding group operations on G with the mappings in G^U (see Definition 7.0.1). The main tool used in this subsection is the superposition with smooth maps that we discussed in Proposition 3.4.23.

Local group operations We first discuss the local multiplication.

Lemma 7.2.1. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group with the group multiplication m_G and (ϕ, V) a centered chart of G. Then there exists an open identity neighborhood $W \subseteq V$ such that the map*

$$\mathcal{C}^\ell_{\mathcal{W}}(U,\phi(W))^\bullet \times \mathcal{C}^\ell_{\mathcal{W}}(U,\phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U,\phi(V))^\bullet : (\gamma,\eta) \mapsto \phi\circ m_G \circ (\phi^{-1}\circ\gamma, \phi^{-1}\circ\eta) \quad (\dagger)$$

is defined and smooth.

Proof. By Lemma 3.4.16, the map (†) is defined and smooth iff there exists an open neighborhood $W \subseteq G$ such that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_* : \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W) \times \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U,\phi(V))^\bullet$$

is so. By the continuity of the multiplication m_G there exists an open subset $W \subseteq V$ such that $m_G(W \times W) \subseteq V$. We may assume that $\phi(W)$ is star-shaped with center 0. Since the map $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ is smooth and maps $(0,0)$ to 0, we can apply Proposition 3.4.23 to see that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})) \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W) \times \phi(W))^\bullet \subseteq \mathcal{C}^\ell_{\mathcal{W}}(U,\phi(V))^\bullet$$

and that the map $(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_*$ is smooth. □

Now, we turn to the local inversion.

Lemma 7.2.2. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group with the group inversion I_G and (ϕ, V) a centered chart such that V is symmetric. Further let $W \subseteq V$ be a symmetric open $\mathbf{1}$-neighborhood such that there exists an open star-shaped set W_L with center 0 and $\phi(W) \subseteq W_L \subseteq \phi(V)$. Then for each $\gamma \in \mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet$,*

$$(\phi \circ I_G \circ \phi^{-1}) \circ \gamma \in \mathcal{C}^\ell_\mathcal{W}(U, W)^\bullet,$$

and the map

$$\mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet : \gamma \mapsto (\phi \circ I_G \circ \phi^{-1}) \circ \gamma$$

is smooth.

Proof. Since $I_L := \phi \circ I_G \circ \phi^{-1} : \phi(V) \to \phi(V)$ is smooth and $I_L(0) = 0$, we conclude with Proposition 3.4.23 that

$$\mathcal{C}^\ell_\mathcal{W}(U, W_L)^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U, \phi(V))^\bullet : \gamma \mapsto I_L \circ \gamma$$

is smooth. Since we proved in Lemma 3.4.19 that $\mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet$ is an open subset of $\mathcal{C}^\ell_\mathcal{W}(U, W_L)^\bullet$, the restriction of this map is also smooth, and since W is symmetric, it takes values in this set. ☐

Conclusion We put everything together to obtain a Lie group for each centered chart of G. We show that the identity component does not depend on the used chart.

Lemma 7.2.3. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group and (ϕ, V) a centered chart. Then there exists a subgroup $(G, \phi)^U_{\mathcal{W},\ell}$ of G^U that can be turned into a Lie group. It is modelled on $\mathcal{C}^\ell_\mathcal{W}(U, \mathbf{L}(G))^\bullet$ in such a way that there exists an open $\mathbf{1}$-neighborhood $W \subseteq V$ such that*

$$\mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet \to (G, \phi)^U_{\mathcal{W},\ell} : \gamma \mapsto \phi^{-1} \circ \gamma$$

becomes a smooth embedding and its image is open. Further, for any subset $\widetilde{W} \subseteq W$ such that $\phi(\widetilde{W})$ is an open convex zero neighborhood,

$$\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U, \phi(\widetilde{W}))^\bullet$$

generates the identity component of $(G, \phi)^U_{\mathcal{W},\ell}$.

Proof. Using Lemma 7.2.1 we find an open $\mathbf{1}$-neighborhood $W \subseteq V$ such that

$$\mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet \times \mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U, \phi(V))^\bullet : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta)$$

is smooth. We may assume w.l.o.g. that W is symmetric and that there exists an open convex set H such that $\phi(W) \subseteq H \subseteq \phi(V)$. We know from Lemma 7.2.2 that the set

$$\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U, \phi(W))^\bullet \subseteq G^U$$

is symmetric and

$$\mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet : \gamma \mapsto \phi \circ I_G \circ \phi^{-1} \circ \gamma$$

is smooth. We endow $\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet$ with the differential structure which turns the bijection

$$\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet : \gamma \mapsto \phi \circ \gamma$$

into a smooth diffeomorphism. Then we can apply Lemma B.2.5 to construct a Lie group structure on the subgroup $(G,\phi)^U_{\mathcal{W},\ell}$ of G^U which is generated by $\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet$, such that $\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet$ becomes an open subset.

Moreover, for each open $\mathbf{1}$-neighborhood $\widetilde{W} \subseteq W$ such that $\phi(\widetilde{W})$ is convex, the set $\mathcal{C}^\ell_\mathcal{W}(U,\phi(\widetilde{W}))^\bullet$ is convex (Lemma 3.4.10). Hence $\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi(\widetilde{W}))^\bullet$ is connected, and it is open by the construction of the differential structure of $(G,\phi)^U_{\mathcal{W},\ell}$. Further it obviously contains the unit element, hence it generates the identity component. □

Lemma 7.2.4. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G a locally convex Lie group. Then for centered charts (ϕ_1, V_1) and (ϕ_2, V_2), the identity component of $(G,\phi_1)^U_{\mathcal{W},\ell}$ coincides with the one of $(G,\phi_2)^U_{\mathcal{W},\ell}$, and the identity map between them is a smooth diffeomorphism.*

Proof. Using Lemma 7.2.3, we find open $\mathbf{1}$-neighborhoods $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ such that the identity component of $(G,\phi_i)^U_{\mathcal{W},\ell}$ is generated by $\phi_i^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi_i(W_i))^\bullet$ for $i \in \{1,2\}$. Since $\phi_1 \circ \phi_2^{-1}$ is smooth, we find an open convex zero neighborhood $\widetilde{W}_2^L \subseteq \phi_2(W_1 \cap W_2)$. By Proposition 3.4.23, the map

$$\mathcal{C}^\ell_\mathcal{W}(U,\widetilde{W}_2^L)^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U,\phi_1(W_1))^\bullet : \gamma \mapsto \phi_1 \circ \phi_2^{-1} \circ \gamma$$

is defined and smooth. This implies that

$$\phi_2^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\widetilde{W}_2^L)^\bullet \subseteq \phi_1^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\phi_1(W_1))^\bullet.$$

Hence the identity component of $(G,\phi_2)^U_{\mathcal{W},\ell}$ is contained in the one of $(G,\phi_1)^U_{\mathcal{W},\ell}$, and the inclusion map of the former into the latter is smooth.

Exchanging the roles of ϕ_1 and ϕ_2 in the preceding argument, we get the assertion. □

Definition 7.2.5. Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G a locally convex Lie group. Henceforth, we write $\mathcal{C}^\ell_\mathcal{W}(U,G)^\bullet$ for the connected Lie group that was constructed in Lemma 7.2.3. There and in Lemma 7.2.4 it was proved that for any centered chart (ϕ, V) of G there exists an open $\mathbf{1}$-neighborhood W such that the inverse map of

$$\mathcal{C}^\ell_\mathcal{W}(U,\phi(W))^\bullet \to \mathcal{C}^\ell_\mathcal{W}(U,G) : \gamma \mapsto \phi^{-1} \circ \gamma$$

is a chart, and that for any convex zero neighborhood $\widetilde{W} \subseteq \phi(W)$, the set

$$\phi^{-1} \circ \mathcal{C}^\ell_\mathcal{W}(U,\widetilde{W})^\bullet$$

generates $\mathcal{C}^\ell_\mathcal{W}(U,G)^\bullet$.

7.2.2. A larger Lie group of weighted mappings

We extend the Lie group described in Definition 7.2.5. Generally, it is possible using Lemma B.2.5 to extend a Lie group G that is a subgroup of a larger group H by looking at its "smooth normalizer", that is all $h \in H$ that normalize G and for which the inner automorphism, restricted to suitable **1**-neighborhoods, is smooth. This approach has the disadvantage that we do not really know which maps are contained in the smooth normalizer. So in the following, we will define a subset of G^U and show that it is a group contained in the smooth normalizer of $\mathcal{C}^\ell_{\mathcal{W}}(U,G)^\bullet$.

Further, we show that this bigger group contains certain groups of *rapidly decreasing mappings* constructed in [BCR81] as open subgroups.

A group of mappings

We define a set of mappings.

Definition 7.2.6. Let G be a locally convex Lie group, X a finite-dimensional vector space, $U \subseteq X$ a nonempty open subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty and $k \in \overline{\mathbb{N}}$. Then for any centered chart (ϕ, V_ϕ) of G, compact set $K \subseteq U$ and $h \in \mathcal{C}^\infty_c(U,\mathbb{R})$ with $h \equiv 1_U$ on a neighborhood of K we define $M((\phi, V_\phi), K, h)$ as the set

$$\{\gamma \in \mathcal{C}^k(U,G) : \gamma(U \setminus K) \subseteq V_\phi \text{ and } (1_U - h)\cdot(\phi\circ\gamma)|_{U\setminus K} \in \mathcal{C}^k_{\mathcal{W}}(U\setminus K, \mathbf{L}(G))^\bullet\}.$$

Further we define

$$\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}} := \bigcup_{(\phi,V_\phi),K,h} M((\phi,V_\phi),K,h).$$

In the following, we show that $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$ is a subgroup of G^U. In order to do this, we provide some technical tools. First, we show that we can use a cutoff technique to shrink the domain of a decreasing function.

Lemma 7.2.7. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, Y a locally convex space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty. Let $k \in \overline{\mathbb{N}}$ and $\gamma \in \mathcal{C}^k(U,Y)$.*

(a) Suppose that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Let $A \subseteq U$ be a closed nonempty set such that $\gamma|_{U\setminus A} \equiv 0$ and $V \subseteq U$ an open neighborhood of A. Then $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V,Y)^\bullet$.

(b) Let $K_1 \subseteq K_2 \subseteq U$ be closed sets such that $\gamma|_{U\setminus K_1} \in \mathcal{C}^k_{\mathcal{W}}(U\setminus K_1,Y)^\bullet$ and $h \in \mathcal{BC}^\infty(U,\mathbb{R})$ such that $h \equiv 1$ on a neighborhood of K_2. Then

$$(1_U - h)\cdot\gamma|_{U\setminus K_2} \in \mathcal{C}^k_{\mathcal{W}}(U\setminus K_2, Y)^\bullet.$$

Proof. (a) It is obvious that $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V,Y)$. Let $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. For $\varepsilon > 0$ and $p \in \mathcal{N}(Y)$ there exists a compact set $K \subseteq U$ such that $\|\gamma|_{U\setminus K}\|_{p,f,\ell} < \varepsilon$. The set $\widetilde{K} := K \cap A$ is compact and contained in V. Further $\|\gamma|_{V\setminus\widetilde{K}}\|_{p,f,\ell} < \varepsilon$ since $D^{(\ell)}\gamma|_{U\setminus A} = 0$.

(b) Let $V \supseteq K$ be open in U such that $h|_V \equiv 1$. Then by Corollary 3.4.15,

$$(1_U - h) \cdot \gamma|_{U \setminus K_1} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_1, Y)^\bullet.$$

Further $(1_U - h) \cdot \gamma|_{U \setminus (U \setminus V)} \equiv 0$. Since $U \setminus K_2$ is an open neighborhood of $U \setminus V$, an application of (a) finishes the proof. □

Now we examine $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$. We show that for a mapping in this set, we can change the chart of G, shrink the **1**-neighborhood and enlarge the compact set.

Lemma 7.2.8. *Let X be a finite-dimensional vector space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Further, let $\gamma \in M((\phi, V_\phi), K, h)$.*

*(a) Then for each **1**-neighborhood $V \subseteq V_\phi$, there exists a compact set $K_V \subseteq U$ such that for each map $h_V \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_V \equiv 1$ on a neighborhood of K_V, the map $\gamma \in M((\phi|_V, V), K_V, h_V)$.*

(b) Let (ψ, V_ψ) be a centered chart. Then there exists a compact set $K_\psi \subseteq U$ such that $\gamma \in M((\psi, V_\psi), K_\psi, h_\psi)$ for each $h_\psi \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_\psi \equiv 1$ on a neighborhood of K_ψ.

(c) Let $\eta \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h})$. There exists a compact set L such that for each $g \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $g \equiv 1$ on a neighborhood of L, we have $\gamma, \eta \in M((\phi, V_\phi), L, g)$.

Proof. (a) Since $(1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus K} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G))^\bullet$ and $1_U \in \mathcal{W}$, there exists a compact set $\widetilde{K} \subseteq U$ such that

$$(1_U - h) \cdot (\phi \circ \gamma)((U \setminus K) \setminus \widetilde{K}) \subseteq \phi(V).$$

We define the compact set $K_V := \widetilde{K} \cup \operatorname{supp}(h)$ and choose $h_V \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_V \equiv 1$ on a neighborhood of K_V. Using Lemma 7.2.7 and the fact that $h \equiv 0$ on $U \setminus K_V$, we see that

$$(1_U - h_V) \cdot (\phi \circ \gamma)|_{U \setminus K_V} = (1_U - h_V)(1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus K_V} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_V, \mathbf{L}(G))^\bullet.$$

Further we calculate using again that $h \equiv 0$ on $U \setminus K_V$:

$$(\phi \circ \gamma)(U \setminus K_V) = (1_U - h) \cdot (\phi \circ \gamma)((U \setminus K) \setminus K_V) \subseteq \phi(V).$$

(b) There exists an open **1**-neighborhood $V \subseteq V_\phi \cap V_\psi$ such that $\phi(V)$ is star-shaped with center 0. We know from (a) that there exist a compact set $\widetilde{K} \subseteq U$ and a map $\widetilde{h} \in \mathcal{C}^\infty_c(U, [0,1])$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$ such that

$$\gamma \in M((\phi|_V, V), \widetilde{K}, \widetilde{h}).$$

We conclude with Proposition 3.4.23 that

$$(\psi \circ \phi^{-1}) \circ ((1_U - \widetilde{h}) \cdot (\phi \circ \gamma)|_{U \setminus \widetilde{K}} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet.$$

Let $h_\psi \in \mathcal{C}^\infty_c(U,\mathbb{R})$ such that $h_\psi \equiv 1$ on a neighborhood of K_ψ, where $K_\psi := \widetilde{K} \cup \operatorname{supp}(\widetilde{h})$. We conclude with Lemma 7.2.7 that

$$(1_U - h_\psi) \cdot (\psi \circ \phi^{-1}) \circ ((1_U - \widetilde{h}) \cdot (\phi \circ \gamma)|_{U \setminus K_\psi} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_\psi, \mathbf{L}(G))^\bullet.$$

Since $(1_U - \widetilde{h}) \equiv 1_U$ on $U \setminus K_\psi$, the proof is finished.

(c) We set $L := \operatorname{supp}(h) \cup \operatorname{supp}(\widetilde{h})$. Then

$$\gamma(U \setminus L) \subseteq \gamma(U \setminus K) \subseteq V_\phi,$$

and for $g \in \mathcal{C}^\infty_c(U,\mathbb{R})$ with $g \equiv 1$ on a neighborhood of L we conclude using Lemma 7.2.7 that

$$(1_U - g) \cdot (\phi \circ \gamma)|_{U \setminus L} = (1_U - g) \cdot (1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus L} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus L, \mathbf{L}(G))^\bullet.$$

Since the argument for η is the same, we are home. ☐

Now we are ready to show that $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\text{ex}}$ is a group.

Lemma 7.2.9. *Let X be a finite-dimensional vector space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then the set $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{ex}$ is a subgroup of G^U.*

Proof. Let (ϕ, V_ϕ) be a centered chart for G and $V \subseteq V_\phi$ an open neighborhood of $\mathbf{1}$ such that $m_G(V \times I_G(V)) \subseteq V_\phi$ and $\phi(V)$ is star-shaped. We define the map

$$H_G : V \times V \to V_\phi : (x,y) \mapsto m_G(x, I_G(y)).$$

Let $\gamma, \eta \in \mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\text{ex}}$. Using Lemma 7.2.8 we find a compact set $K \subseteq U$ and a map $h \in \mathcal{C}^\infty_c(U,[0,1])$ with $h \equiv 1_U$ on K such that

$$\gamma, \eta \in M((\phi|_V, V), K, h).$$

We define $H_\phi := \phi \circ H_G \circ (\phi^{-1} \times \phi^{-1})|_{V \times V}$ and want to show that there exists a compact set $\widetilde{K}$ and $\widetilde{h} \in \mathcal{C}^\infty_c(U,\mathbb{R})$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$ such that $H_G \circ (\gamma, \eta) \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h})$. It is obvious that

$$(H_G \circ (\gamma, \eta))(U \setminus K) \subseteq m_G(V \times I_G(V)) \subseteq V_\phi.$$

Since we know with Lemma 3.4.16 that

$$(1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta) = ((1_U - h) \cdot (\phi \circ \gamma), (1_U - h) \cdot (\phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G) \times \mathbf{L}(G))^\bullet,$$

we conclude using Proposition 3.4.23 that

$$H_\phi \circ ((1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G))^\bullet.$$

Further, $\widetilde{K} := K \cup \operatorname{supp}(h)$ is a compact set, so by Lemma 7.2.7

$$(1_U - \widetilde{h}) \cdot H_\phi \circ ((1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet$$

for any $\widetilde{h} \in \mathcal{C}^\infty_c(U,\mathbb{R})$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$. Since $(1_U - h) \equiv 0$ on $U \setminus \widetilde{K}$, $(1_U - \widetilde{h}) \cdot (\phi \circ H_G \circ (\gamma, \eta))|_{U \setminus \widetilde{K}} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet$ and hence

$$H_G \circ (\gamma, \eta) \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h}).$$

The proof is complete. ☐

Inclusion in the smooth normalizer

We show that $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\text{ex}}$ is contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$. To this end, we show that each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\text{ex}}$ can be written as a product of a compactly supported $\mathcal{C}^k$-map and a $\mathcal{C}^k$-map that takes values in a chosen chart domain. After that, we show that these two classes of mappings are contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$.

We start with the following technical lemma about extending decreasing functions.

Lemma 7.2.10. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $A \subseteq U$ a closed subset, Y a locally convex space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U \setminus A, Y)^\bullet$. Then the map*

$$\widetilde{\gamma} : U \to Y : x \mapsto \begin{cases} \gamma(x) & \text{if } x \in U \setminus A, \\ 0 & \text{else} \end{cases}$$

is in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$.

Proof. Obviously, the assertion holds on $U \setminus A$ and A°, since $\widetilde{\gamma}$ and its derivatives vanish on A°. We show that $\widetilde{\gamma}$ is $\mathcal{C}^k$ on ∂A and it and its derivatives also vanish there. Since this is true iff for each $p \in \mathcal{N}(Y)$, the map $\pi_p \circ \widetilde{\gamma}$ is $\mathcal{C}^k$ on ∂A and it and its derivatives vanish there, and the identity $\pi_p \circ \widetilde{\gamma} = \widetilde{\pi_p \circ \gamma}$ holds, we may assume w.l.o.g. that Y is normable.

Since $1_U \in \mathcal{W}$, for each $\ell \in \mathbb{N}$ with $\ell \leq k$, the map $\widetilde{D^{(\ell)}\gamma}$ is continuous and hence

$$\widetilde{D^{(\ell)}\gamma} \in \mathcal{C}^0_{\mathcal{W}}(U, \mathrm{L}^\ell(X,Y))^\bullet.$$

Using Lemma 3.2.1, it remains to show that $\widetilde{\gamma}$ is $\mathcal{C}^k$ with $D^{(\ell)}\widetilde{\gamma} = \widetilde{D^{(\ell)}\gamma}$ for all $\ell \in \mathbb{N}$ with $\ell \leq k$. We show the assertion by an induction over ℓ.

$\ell = 1$: Let $x \in \partial A$ and $h \in X$. If there exists $\delta > 0$ such that $x +]-\delta, 0]\, h \subseteq A$ or $x + [0, \delta[\, h \subseteq A$, then $D_h\widetilde{\gamma}(x) = 0 = \widetilde{D\gamma}(x)h$.
Otherwise, there exists a null sequence $(t_n)_{n\in\mathbb{N}}$ in $]-\infty, 0[$ or $]0,\infty[$ such that for each $n \in \mathbb{N}$, $x + t_n h \in U \setminus A$. After replacing h by $-h$ if necessary, we may assume w.l.o.g. that all t_n are positive. Since $1_U \in \mathcal{W}$, $\widetilde{D\gamma}$ is continuous and $\widetilde{D\gamma}(x) = 0$, given $\varepsilon > 0$ we find $\delta > 0$ such that for all $s \in]-\delta, \delta[$,

$$\|\widetilde{D\gamma}(x + sh)\|_{op} < \varepsilon.$$

We find an $n \in \mathbb{N}$ such that $t_n \in]-\delta, \delta[$. Then we define

$$t := \inf\{\tau > 0 :]\tau, t_n] \subseteq U \setminus A\} > 0.$$

We calculate for $\tau \in]t, t_n[$:

$$\left\| \tfrac{\widetilde{\gamma}(x+t_n h) - \widetilde{\gamma}(x+\tau h)}{t_n} \right\| < \left\| \tfrac{\widetilde{\gamma}(x+t_n h) - \widetilde{\gamma}(x+\tau h)}{t_n - \tau} \right\|$$
$$= \left\| \int_0^1 D\gamma(x + (s t_n + (1-s)\tau)h) \cdot \tfrac{t_n - \tau}{t_n - \tau} h \, ds \right\| < \varepsilon \|h\|.$$

But $\widetilde{\gamma}(x+\tau h) \to 0$ as $\tau \to t$, and hence

$$\left\|\frac{\widetilde{\gamma}(x+t_n h)-\widetilde{\gamma}(x)}{t_n}\right\| = \left\|\frac{\widetilde{\gamma}(x+t_n h)}{t_n}\right\| \leq \varepsilon\|h\|.$$

Since ε was arbitrary, we conclude that $D_h\widetilde{\gamma}(x) = 0 = \widetilde{D\gamma}(x)h$.

$\ell \to \ell+1$: Using the inductive hypothesis, we conclude that $\widetilde{D\gamma}$ is $\mathcal{FC}^\ell$, and $D^{(\ell)}\widetilde{D\gamma} = \widetilde{D^{(\ell)}D\gamma}$. Hence $\widetilde{\gamma}$ is $\mathcal{FC}^{\ell+1}$, so by Lemma A.2.14 $D^{(\ell+1)}\widetilde{\gamma} = \widetilde{D^{(\ell+1)}\gamma}$. □

Proposition 7.2.11. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$, (ϕ, V_ϕ) a centered chart of G and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{ex}$. Then there exist maps $\eta \in M((\phi,V_\phi),\emptyset,0_U)$ and $\chi \in \mathcal{C}^k_c(U,G)$ such that*

$$\gamma = \eta \cdot \chi.$$

Proof. Using Lemma 7.2.8 we find a compact set K and $h \in \mathcal{C}^\infty_c(U,[0,1])$ such that $\gamma \in M((\phi,V_\phi),K,h)$. Using Lemma 7.2.10 we see that

$$\eta := \phi^{-1} \circ \widetilde{(1_U - h) \cdot (\phi \circ \gamma)|_{U\setminus K}} \in M((\phi,V_\phi),\emptyset,0_U),$$

and it is obvious that $\eta|_{U\setminus \operatorname{supp}(h)} = \gamma|_{U\setminus \operatorname{supp}(h)}$. Hence

$$\chi := \eta^{-1} \cdot \gamma \in \mathcal{C}^k_c(U,G),$$

and obviously $\gamma = \eta \cdot \chi$. □

We now show that the weighted maps that take values in a suitable chart domain are contained in the smooth normalizer.

Lemma 7.2.12. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart of G. Further let $W_\phi \subseteq V_\phi$ be an open $\mathbf{1}$-neighborhood such that*

$$W_\phi \cdot W_\phi \cdot W_\phi^{-1} \subseteq V_\phi$$

and $\phi(W_\phi)$ is star-shaped with center 0. Then for each $\eta \in M((\phi,W_\phi),\emptyset,0_U)$, the map

$$\mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\eta \cdot (\phi^{-1}\circ\gamma)\cdot \eta^{-1})$$

is smooth.

Proof. As a consequence of Proposition 3.4.23 and Lemma 3.4.16, the map

$$\begin{aligned}&\mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \times \mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \times \mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\phi(V_\phi))^\bullet\\ &:(\gamma_1,\gamma_2,\gamma_3) \mapsto \phi \circ ((\phi^{-1}\circ\gamma_1)\cdot(\phi^{-1}\circ\gamma_2)\cdot(\phi^{-1}\circ\gamma_3)^{-1})\end{aligned}$$

is smooth. We easily deduce the desired assertion. □

Normalization with compactly supported mappings While the treatment of $\mathcal{C}^k$-maps with values in a suitable chart domain was straightforward, we need to develop other tools to deal with the compactly supported mappings. The main problem is that a compactly supported map may not take values in any chart domain. To get around this problem, we need more technical machinery. As motivation for the following, let $\chi \in \mathcal{C}^k_c(U, G)$ and (ϕ, V_ϕ) be a centered chart of G. Using that $\chi(U)$ is compact, we can find a symmetrical neighborhood O of $\chi(U)$ and an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that $O \cdot W_\phi \cdot O^{-1} \subseteq V_\phi$. Then we can define the "normalization map in charts"

$$N : O \times \phi(W_\phi) \to \phi(V_\phi) : (g, y) \mapsto \phi(g \cdot \phi^{-1}(y) \cdot g^{-1}).$$

We can calculate that for $\gamma \in \phi(W_\phi)^U$, we have the identity

$$\phi \circ (\chi \cdot \gamma \cdot \chi^{-1}) = N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)}) \circ (\mathrm{id}_U, \gamma).$$

In the following two lemmas, we will examine the properties of maps of the form $N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)})$ and whether they induce a kind of superposition operator for decreasing weighted functions.

Lemma 7.2.13. *Let X, Y and Z be locally convex spaces, $U \subseteq X$, $V \subseteq Y$ and $W \subseteq Z$ open nonempty subsets, M a locally convex manifold and $k \in \overline{\mathbb{N}}$. Let $\Gamma \in \mathcal{C}^\infty(M \times V, W)$ and $\eta \in \mathcal{C}^k(U, M)$. Then the map*

$$\Xi := \Gamma \circ (\eta \times \mathrm{id}_V) : U \times V \to W$$

has the following properties:

(a) The second partial derivative of Ξ is

$$d_2\Xi = (\pi_2 \circ \mathbf{T}_2\Gamma) \circ (\eta \times \mathrm{id}_{V \times Y})$$

and if $k \geq 1$, the first partial derivative of Ξ is

$$d_1\Xi = (\pi_2 \circ \mathbf{T}_1\Gamma) \circ (\mathbf{T}\eta \times \mathrm{id}_V) \circ S,$$

where π_2 denotes the projection $W \times Z \to Z$ on the second component, and $S : U \times V \times X \to U \times X \times V : (x, y, h) \mapsto (x, h, y)$ denotes the swap map.

(b) For all $x \in U$, the partial map $\Xi(x, \cdot) : V \to W$ is smooth, and for all $\ell \in \mathbb{N}$ the map $d_2^{(\ell)}\Xi : U \times V \times Y^\ell \to W$ is $\mathcal{C}^k$.

(c) Assume that X has finite dimension. Then for

$$A_1 : U \times V \to \mathrm{L}(X, Z) : (x, y) \mapsto (h \mapsto d_1\Xi(x, y; h))$$

(which is only defined if $k \geq 1$) and

$$A_2 : U \times V \times \mathrm{L}(X, Y) \to \mathrm{L}(X, Z) : (x, y, T) \mapsto (h \mapsto d_2\Xi(x, y; T \cdot h)),$$

all partial maps $A_1(x, \cdot)$ and $A_2(x, \cdot)$ are smooth and all partial derivatives $d_2^{(\ell)}A_1$ and $d_2^{(\ell)}A_2$ are $\mathcal{C}^{k-1}$, respectively $\mathcal{C}^k$.

Proof. (a) We calculate for $x \in U$, $y \in V$ and $h \in Y$ that

$$d_2\Xi((x,y);h) = \lim_{t\to 0} \frac{\Xi(x,y+th)-\Xi(x,y)}{t} = \lim_{t\to 0} \frac{\Gamma(\eta(x),y+th)-\Gamma(\eta(x),y)}{t} = (\pi_2 \circ \mathbf{T}_2\Gamma)(\eta(x),y,h).$$

This shows the desired identity for $d_2\Xi$. If $k > 0$, we get using the chain rule that

$$d\Xi \circ P = \pi_2 \circ \mathbf{T}\Xi \circ P = \pi_2 \circ \mathbf{T}\Gamma \circ (\mathbf{T}\eta \times \mathrm{id}_{\mathbf{T}V}),$$

where $P : U \times X \times V \times Y \to U \times V \times X \times Y$ permutes the middle arguments. Since $d_1\Xi((x,y);h_x) = d\Xi((x,y);(h_x,0))$, we get the assertion for $d_1\Xi$.

(b) It is obvious that the partial maps are smooth. We prove the second assertion by induction on ℓ:

$\ell = 0$: This is obvious.

$\ell \to \ell + 1$: In (a) we proved that $d_2\Xi$ is of the same form as Ξ. By the inductive hypothesis,

$$d_2^{(\ell)}(d_2\Xi) : U \times V \times Y \times (Y \times Y)^\ell \to W$$

is a $\mathcal{C}^k$-map. But

$$d_2^{(\ell+1)}\Xi(x,y;h_1,h_2,\dots,h_{\ell+1}) = d_2^{(\ell)}(d_2\Xi)(x,y,h_1;(h_2,0),\dots,(h_{\ell+1},0)),$$

so $d_2^{(\ell+1)}\Xi$ is $\mathcal{C}^k$.

(c) The partial maps $A_1(x,\cdot)$ and $A_2(x,\cdot)$ are smooth and the maps $d_2^{(\ell)}A_1$ and $d_2^{(\ell)}A_2$ are $\mathcal{C}^{k-1}$ respective $\mathcal{C}^k$ iff for each $h \in X$, the maps $A_1(x,\cdot)\cdot h$ and $A_2(x,\cdot)\cdot h$ have the corresponding properties. By (a),

$$A_1(x,y)\cdot h = d_1\Xi(x,y;h) = (\pi_2 \circ \mathbf{T}_1\Gamma) \circ (\mathbf{T}\eta \times \mathrm{id}_V) \circ S(x,y,h)$$

and

$$\begin{aligned} A_2(x,y,T)\cdot h = d_2\Xi(x,y;T\cdot h) &= (\pi_2 \circ \mathbf{T}_2\Gamma) \circ (\eta \times \mathrm{id}_{V\times Y})(x,y,T\cdot h) \\ &= (\pi_2 \circ \mathbf{T}_2\Gamma \circ S_1) \circ (\eta \times \mathrm{ev}_h \times \mathrm{id}_V) \circ S_2(x,y,T). \end{aligned}$$

Here S_1 and S_2 denote the swap maps

$$M \times Y \times V \to M \times V \times Y,$$

and

$$U \times V \times \mathrm{L}(X,Y) \to U \times \mathrm{L}(X,Y) \times V$$

respectively. Since S, S_1 and S_2 are restrictions of continuous linear maps, (b) applies to both $A_1(x,\cdot)\cdot h$ and $A_2(x,\cdot)\cdot h$. □

Lemma 7.2.14. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, Y and Z locally convex spaces, M a locally convex manifold, $V \subseteq Y$ an open zero neighborhood that is star-shaped with center 0, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Further, let $\Gamma \in \mathcal{C}^\infty(M \times V, Z)$, and $\theta \in \mathcal{C}^k(U,M)$ such that the map*

$$\Xi := \Gamma \circ (\theta \times \mathrm{id}_V) : U \times V \to Z$$

satisfies

- $\Xi(U \times \{0\}) = \{0\}$,
- *There exists a compact set* $K \subseteq U$ *such that* $\Xi((U \setminus K) \times V) = \{0\}$.

Then for any $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$

$$\Xi \circ (\mathrm{id}_U, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet, \tag{†}$$

and the map

$$\Xi_* : \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : \gamma \mapsto \Xi \circ (\mathrm{id}_U, \gamma)$$

is smooth.

Proof. We first prove that Ξ_* is defined and continuous, by induction on k:

$k = 0$: Let $\gamma, \eta \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ such that the line segment $\{t\gamma + (1-t)\eta : t \in [0,1]\} \subseteq \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$. We easily prove using Lemma 3.4.17 that the set

$$\widetilde{K} := \{t\gamma(x) + (1-t)\eta(x) : t \in [0,1], x \in U\}$$

is relatively compact in V. Since $d_2\Xi$ is continuous by Lemma 7.2.13 (b) and satisfies $d_2\Xi(U \times V \times \{0\}) = \{0\}$, we conclude using the Wallace Lemma that for each $p \in \mathcal{N}(Z)$, there exists $q \in \mathcal{N}(Y)$ such that

$$d_2\Xi(K \times \widetilde{K} \times B_q(0,1)) \subseteq B_p(0,1).$$

This relation implies that

$$\forall x \in K, y \in \widetilde{K}, h \in Y : \|d_2\Xi(x,y;h)\|_p \leq \|h\|_q.$$

For each $x \in U$, we calculate

$$\Xi(x,\gamma(x)) - \Xi(x,\eta(x)) = \int_0^1 d_2\Xi(x, t\gamma(x) + (1-t)\eta(x); \gamma(x) - \eta(x))\, dt.$$

Hence for each $f \in \mathcal{W}$, we have

$$|f(x)|\,\|\Xi(x,\gamma(x)) - \Xi(x,\eta(x))\|_p \leq |f(x)|\,\|\gamma(x) - \eta(x)\|_q.$$

Taking $\eta = 0$, this estimate implies (†). Further, since we proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open, γ has a convex neighborhood in $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$; hence the estimate also implies the continuity of Ξ_* in γ.

$k \to k+1$: For each $x \in U$, $h \in X$ and $\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,V)^\bullet$, we calculate

$$\begin{aligned} d(\Xi \circ (\mathrm{id}_U, \gamma))(x;h) &= d\Xi(x,\gamma(x); h, D\gamma(x) \cdot h) \\ &= d_1\Xi(x,\gamma(x);h) + d_2\Xi(x,\gamma(x); D\gamma(x) \cdot h). \end{aligned}$$

Recall the maps A_1 and A_2 defined in Lemma 7.2.13(c). We get the identity

$$D(\Xi \circ (\mathrm{id}_U, \gamma))(x) = (A_1 \circ (\mathrm{id}_U, \gamma))(x) + (A_2 \circ (\mathrm{id}_U, \gamma, D\gamma))(x).$$

We prove that A_1 and A_2 satisfy the same properties as Ξ does: For $x \in U$, $y \in V$, $h \in X$, we have

$$A_1(x,0) \cdot h = d_1\Xi(x,0;h) = \lim_{t\to 0} \frac{\Xi(x+th,0) - \Xi(x,0)}{t} = 0,$$

whence $A_1(x,0) = 0$. Let $x \in U \setminus K$. Then

$$A_1(x,y) \cdot h = d_1\Xi(x,y;h) = \lim_{t\to 0} \frac{\Xi(x+th,y) - \Xi(x,y)}{t} = 0$$

since $U \setminus K$ is open, hence $A_1(x,y) = 0$.
As to A_2, for $x \in U$, $y \in V$ and $h \in X$ we calculate

$$A_2(x,y,0) \cdot h = d_2\Xi(x,y;0\cdot h) = 0,$$

whence $A_2(x,y,0) = 0$. Let $x \in U \setminus K$ and $T \in \mathrm{L}(X,Y)$. Then

$$A_2(x,y,T) \cdot h = d_2\Xi(x,y;T\cdot h) = \lim_{t\to 0} \frac{\Xi(x,y+tT\cdot h) - \Xi(x,y)}{t} = 0,$$

hence $A_2(x,y,T) = 0$.

So we can apply the inductive hypothesis to A_1 and A_2 and conclude that

$$A_1 \circ (\mathrm{id}_X, \gamma),\ A_2 \circ (\mathrm{id}_X, \gamma, D\gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z))^\bullet$$

and the maps $\mathcal{C}^{k+1}_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))^\bullet$

$$\gamma \mapsto A_1 \circ (\mathrm{id}_X, \gamma) \text{ and } \gamma \mapsto A_2 \circ (\mathrm{id}_X, \gamma, D\gamma)$$

are continuous. In view of Proposition 3.4.11, the continuity of Ξ_* is established.

We pass on to prove the smoothness of Ξ_*. In order to do this, we have to examine $d_2\Xi$. By Lemma 7.2.13 (a), $d_2\Xi = \pi_2 \circ \mathbf{T}_2\Gamma \circ (\theta \times \mathrm{id}_{V\times Y})$, and we easily see that

$$d_2\Xi(U \times \{0\} \times \{0\}) = d_2\Xi((U \setminus K) \times V \times Y) = \{0\}.$$

Hence by the results already established, the map

$$(d_2\Xi)_* : \mathcal{C}^k_{\mathcal{W}}(U, V \times Y)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : (\gamma) \mapsto d_2\Xi \circ (\mathrm{id}_U, \gamma)$$

is defined and continuous. Now let $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ and $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Since $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open, there exists an $r > 0$ such that $\{\gamma + s\gamma_1 : s \in B_{\mathbb{K}}(0,r)\} \subseteq \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$. We calculate for $x \in U$ and $t \in B_{\mathbb{K}}(0,r)[\setminus\{0\}$ (using Lemma 3.4.16 implicitly) that

$$\begin{aligned}\frac{\Xi_*(\gamma+t\gamma_1)(x) - \Xi_*(\gamma)(x)}{t} &= \frac{\Xi(x,\gamma(x)+t\gamma_1(x)) - \Xi(x,\gamma(x))}{t} \\ &= \int_0^1 d_2\Xi((x,\gamma(x)+st\gamma_1(x));\gamma_1(x))\,ds = \int_0^1 (d_2\Xi)_*(\gamma+st\gamma_1,\gamma_1)(x)\,ds.\end{aligned}$$

Hence by Lemma 3.4.3 and Proposition A.1.8, Ξ_* is $\mathcal{C}^1$ with

$$d\Xi_*(\gamma;\gamma_1) = (d_2\Xi)_*(\gamma,\gamma_1).$$

So using an easy induction argument we conclude from this identity that Ξ_* is $\mathcal{C}^\ell$ for each $\ell \in \mathbb{N}$ and hence smooth. □

Now we are ready to deal with the inner automorphism induced by a compactly supported map.

Lemma 7.2.15. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart for G. Let $\chi \in \mathcal{C}^k_c(U, G)$. Then there exists an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that the map*

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))^\bullet : \gamma \mapsto \phi \circ (\chi \cdot (\phi^{-1} \circ \gamma) \cdot \chi^{-1}) \qquad (\dagger)$$

is defined and smooth.

Proof. Since $\chi(U)$ is compact, we can find an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ and an open symmetrical neighborhood O of $\chi(U)$ such that

$$O \cdot W_\phi \cdot O^{-1} \subseteq V_\phi;$$

we may assume w.l.o.g. that $\phi(W_\phi)$ is star-shaped with center 0. We define the smooth map

$$N : O \times \phi(W_\phi) \to \mathbf{L}(G) : (g, y) \mapsto \phi(g \cdot \phi^{-1}(y) \cdot g^{-1}) - y.$$

Then it is easy to see that

$$N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)}) : U \times \phi(W_\phi) \to \mathbf{L}(G)$$

satisfies the assumptions of Lemma 7.2.14, and that for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet$

$$(N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)})) \circ (\mathrm{id}_U, \gamma) = \phi \circ (\chi \cdot (\phi^{-1} \circ \gamma) \cdot \chi^{-1}) - \gamma.$$

Hence the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))^\bullet : \gamma \mapsto \phi \circ (\chi \cdot (\phi^{-1} \circ \gamma) \cdot \chi^{-1}) - \gamma$$

is smooth. Since the vector space addition is smooth, (†) is defined and smooth. □

Conclusion and the Lie group structure Finally, we put everything together and show that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$. As mentioned above, we this allows the construction of a Lie group structure on $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$.

Lemma 7.2.16. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart for G. Let $\theta \in \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{ex}$. Then there exists an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that the map*

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\theta \cdot (\phi^{-1} \circ \gamma) \cdot \theta^{-1}) \qquad (\dagger)$$

is defined and smooth.

Proof. Let $\widetilde{V_\phi} \subseteq V_\phi$ be an open **1**-neighborhood such that

$$\widetilde{V_\phi} \cdot \widetilde{V_\phi} \cdot \widetilde{V_\phi}^{-1} \subseteq V_\phi$$

and $\phi(\widetilde{V_\phi})$ is star-shaped with center 0. According to Proposition 7.2.11 there exist $\eta \in M((\phi, \widetilde{V_\phi}), \emptyset, 0_U)$ and $\chi \in \mathcal{C}^k_c(U, G)$ such that $\theta = \eta \cdot \chi$. By Lemma 7.2.15, there exists an open **1**-neighborhood $W_\phi \subseteq V_\phi$ such that

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(\widetilde{V_\phi}))^\bullet : \gamma \mapsto \phi \circ (\chi \cdot (\phi^{-1} \circ \gamma) \cdot \chi^{-1})$$

is smooth, and by Lemma 7.2.12 the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(\widetilde{V_\phi}))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\eta \cdot (\phi^{-1} \circ \gamma) \cdot \eta^{-1})$$

is also smooth. Composing these two maps, we obtain the assertion. □

Theorem 7.2.17. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{ex}$ can be made into a Lie group that contains $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$ as an open normal subgroup.*

Proof. We showed in Definition 7.2.5 that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$ can be turned into a Lie group such that there exists a centered chart (ϕ, V_ϕ) for which

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet : \gamma \mapsto \phi^{-1} \circ \gamma$$

is an embedding and its image generates $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$. Further, we proved in Lemma 7.2.9 and Lemma 7.2.16 that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is a subgroup of G^U and for each $\theta \in \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ there exists an open **1**-neighborhood $W_\phi \subseteq V_\phi$ such that the conjugation operation

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\theta \cdot (\phi^{-1} \circ \gamma) \cdot \theta^{-1})$$

is smooth. Hence Lemma B.2.5 gives the assertion. □

Comparison with groups of *rapidly decreasing mappings*

In the book [BCR81, Section 4.2.1, pages 111-117], for weights that satisfy conditions described below in Definition 7.2.18, certain Γ-*rapidly decreasing functions* with values in locally convex spaces are defined and used to construct Γ-*rapidly decreasing mappings* that take values in Lie groups. We compare these function spaces with our weighted decreasing functions and will see that they coincide. Further, we will show that the Γ-rapidly decreasing mappings are open subgroups of a certain $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$.

$\mathcal{W}$-rapidly decreasing functions We give the definition of the $\mathcal{W}$-rapidly decreasing functions.

Definition 7.2.18 (BCR-weights)**.** Let X be a finite-dimensional vector space and $\mathcal{W} \subseteq [1, \infty]^X$ such that

($\mathcal{W}1$) for all $f, g \in \mathcal{W}$, the sets $f^{-1}(\infty)$ and $g^{-1}(\infty) =: M_\infty$ coincide,

($\mathcal{W}2$) $\mathcal{W}$ is directed upwards and contains a smallest element $f_{\min}$ defined by

$$f_{\min}(x) = \begin{cases} 1 & x \notin M_\infty \\ \infty & \text{else,} \end{cases}$$

($\mathcal{W}3$) and for each $f_1 \in \mathcal{W}$ there exists $f_2 \in \mathcal{W}$ such that

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})\, \|x\| \geq n \text{ or } f_1(x) \geq n \implies f_1(x) \leq \varepsilon \cdot f_2(x).$$

Furthermore each $f \in \mathcal{W}$ has to be continuous on the complement of M_∞.

Definition 7.2.19 ($\mathcal{W}$-rapidly decreasing functions)**.** Let $\mathcal{W}$ be a set of weights as in Definition 7.2.18, $U \subseteq \mathbb{R}^m$ open and nonempty and Y a locally convex space. A smooth function $\gamma : U \to Y$ is called *$\mathcal{W}$-rapidly decreasing* if for each $f \in \mathcal{W}$ and $\beta \in \mathbb{N}^m$ we have $\partial^\beta \gamma|_{U \cap M_\infty} \equiv 0$, and the function

$$f \cdot \partial^\beta \gamma : U \to Y$$

is continuous and bounded, where $\infty \cdot 0 = 0$. The set

$$S(U, Y; \mathcal{W}) := \{\gamma \in \mathcal{C}^\infty(U, Y) : \gamma \text{ is } \mathcal{W}\text{-rapidly decreasing}\}$$

endowed with the seminorms

$$\|\gamma\|_{q,f}^k := \sup\{q(f \cdot \partial^\beta \gamma(x)) : x \in U, |\beta| \leq k\}$$

(where $q \in \mathcal{N}(Y)$, $k \in \mathbb{N}$ and $f \in \mathcal{W}$) becomes a locally convex space.

Comparison of $S(U, Y; \mathcal{W})$ and $\mathcal{C}^\infty_{\mathcal{W}}(U, Y)$ We show that these function spaces coincide as topological vector spaces. To this end, we need the following technical lemma.

Lemma 7.2.20. *Let $\mathcal{W}$ be a set of weights as in Definition 7.2.18, $U \subseteq \mathbb{R}^m$ open and nonempty, F a locally convex space, $\gamma : U \to F$ a smooth function and $\beta \in \mathbb{N}^m$. Suppose that $\partial^\beta \gamma|_{U \cap M_\infty} \equiv 0$ and that for each $f \in \mathcal{W}$ the function*

$$f \cdot \partial^\beta \gamma : U \to F$$

is bounded. Then for each $f \in \mathcal{W}$, the function $f \cdot \partial^\beta \gamma$ is continuous.

Proof. Let $f \in \mathcal{W}$ and $x \in U$. If $x \notin \overline{M_\infty \cap U}$, $f \cdot \partial^\beta \gamma$ is continuous on a suitable neighborhood of x since f is so.
Otherwise, $\partial^\beta \gamma(x) = 0$ because $\partial^\beta \gamma$ is continuous. If there exists $V \in \mathcal{U}(x)$ such that f is bounded on $V \setminus M_\infty$, the map $f \cdot \partial^\beta \gamma$ is continuous on V because for $y \in V \setminus M_\infty$ and $q \in \mathcal{N}(F)$

$$\|f(y)\partial^\beta \gamma(y) - f(x)\partial^\beta \gamma(x)\|_q = \|f(y)\partial^\beta \gamma(y)\|_q \leq \|f|_{V \setminus M_\infty}\|_\infty \|\partial^\beta \gamma(y)\|_q,$$

and this estimate is valid for $y \in M_\infty$.

Otherwise, we choose $g \in \mathcal{W}$ such that ($\mathcal{W}3$) holds. Let $\varepsilon > 0$. There exists an $n \in \mathbb{N}$ such that

$$(\forall y \in U).\, f(y) \geq n \implies f(y) \leq \frac{\varepsilon}{\|\gamma\|_{q,g}^{|\beta|} + 1} g(y).$$

For $q \in \mathcal{N}(F)$ there exists $V \in \mathcal{U}(x)$ such that for $y \in V$

$$\|\partial^\beta \gamma(y)\|_q < \frac{\varepsilon}{n}.$$

Let $y \in V$. If $f(y) \geq n$, we calculate

$$\|f(y)\partial^\beta \gamma(y)\|_q = f(y)\|\partial^\beta \gamma(y)\|_q \leq \frac{\varepsilon}{\|\gamma\|_{q,g}^{|\beta|} + 1} g(y)\|\partial^\beta \gamma(y)\|_q < \varepsilon.$$

Otherwise

$$\|f(y)\partial^\beta \gamma(y)\|_q \leq n\|\partial^\beta \gamma(y)\|_q < \varepsilon.$$

So the assertion holds in all cases. □

Lemma 7.2.21. *Let $\mathcal{W}$ be a set of weights as in Definition 7.2.18, $U \subseteq \mathbb{R}^m$ open and nonempty and F a locally convex space. Then $\mathcal{C}^\infty_\mathcal{W}(U, Y) = S(U, Y; \mathcal{W})$ as a topological vector space.*

Proof. We first prove that $\mathcal{C}^\infty_\mathcal{W}(U, Y) = S(U, Y; \mathcal{W})$ as set. To this end, let $\gamma \in \mathcal{C}^\infty_\mathcal{W}(U, Y)$, $f \in \mathcal{W}$ and $\beta \in \mathbb{N}^m$. We set $k := |\beta|$. We know that for $p \in \mathcal{N}(Y)$, the map $D^{(k)}(\pi_p \circ \gamma)$ vanishes on M_∞, and

$$f \cdot D^{(k)}(\pi_p \circ \gamma) : U \to \mathrm{L}^k(\mathbb{R}^m, Y_p)$$

is bounded. Since the evaluation $\mathrm{L}^k(\mathbb{R}^m, Y_p) \to Y_p$ at a fixed point is continuous linear, the map $f \cdot \partial^\beta(\pi_p \circ \gamma) = \pi_p \circ (f \cdot \partial^\beta \gamma) : U \to Y_p$ is also bounded. Hence $f \cdot \partial^\beta \gamma$ is bounded, so an application of Lemma 7.2.20 gives $\gamma \in S(U, Y; \mathcal{W})$.

On the other hand, let $\gamma \in S(U, Y; \mathcal{W})$ and $k \in \mathbb{N}$. For each $p \in \mathcal{N}(Y)$, we get with identity (A.3.5.1)

$$D^{(k)}(\pi_p \circ \gamma) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = k}} S_\alpha \cdot \partial^\alpha(\pi_p \circ \gamma) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = k}} S_\alpha \cdot (\pi_p \circ \partial^\alpha \gamma)$$

Hence for $f \in \mathcal{W}$

$$\|\gamma\|_{p,f,k} \leq \|\gamma\|_{p,f}^k \cdot \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \|S_\alpha\|_{op} < \infty. \tag{†}$$

So $\gamma \in \mathcal{C}^\infty_\mathcal{W}(U, Y)$.

We see from (†) that for each $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$ and $k \in \mathbb{N}$ the seminorm $\|\cdot\|_{p,f,k}$ is continuous on $S(U, Y; \mathcal{W})$. Since the seminorms $\|\cdot\|_{p,f}^k$ are obviously continuous on $\mathcal{C}^\infty_\mathcal{W}(U, Y)$, the spaces are the same as topological vector spaces. □

Remark 7.2.22. Let $\mathcal{W}$ be a set of weights as in Definition 7.2.18. Then $1_U \in \mathcal{W} \iff M_\infty = \emptyset$. But obviously $\mathcal{C}^k_\mathcal{W}(U, Y) = \mathcal{C}^k_{\mathcal{W} \cup \{1_U\}}(U, Y)$ and $\mathcal{C}^k_\mathcal{W}(U, Y)^\bullet = \mathcal{C}^k_{\mathcal{W} \cup \{1_U\}}(U, Y)^\bullet$ as topological vector spaces.

Rapidly decreasing mappings In [BCR81, Section 4.2.1, page 117–118], the set of Γ-*rapidly decreasing mappings* is defined. We will show that these mappings are open subgroups of $\mathcal{C}^\infty_\mathcal{W}(\mathbb{R}^m, G)^\bullet_{\text{ex}}$.

Definition 7.2.23 ($\mathcal{W}$-rapidly decreasing mappings)**.** Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition 7.2.18. We define $S(\mathbb{R}^m, G; \mathcal{W})$ as the set of smooth functions $\gamma : \mathbb{R}^m \to G$ such that

- $\gamma(x) = \mathbf{1}$ for each $x \in M_\infty$, and $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$.
- For any centered chart $(\phi, \widetilde{V})$ of G and each open $\mathbf{1}$-neighborhood V with $\overline{V} \subseteq \widetilde{V}$, $\phi \circ \gamma|_{\gamma^{-1}(V)} \in S(\gamma^{-1}(V), \mathbf{L}(G); \mathcal{W})$.

In the next lemmas, we provide tools needed for the further discussion. First, we show that for weights as in Definition 7.2.18, the product of a weighted function with an suitable cutoff function is a weighted decreasing function. We use this result to prove a superposition lemma for the spaces $\mathcal{C}^k_\mathcal{W}(U, Y)$.

Lemma 7.2.24. *Let K be a compact subset of the finite-dimensional vector space X, Y a locally convex space, $k \in \mathbb{N}$, $\mathcal{W}$ a set of weights as in Definition 7.2.18, $\gamma \in \mathcal{C}^k_\mathcal{W}(U, Y)$ (where $U := X \setminus K$) and $h \in \mathcal{C}^\infty_c(X, \mathbb{R})$ such that $h \equiv 1$ on a neighborhood V of K. Then*

$$(1 - h)|_U \cdot \gamma \in \mathcal{C}^k_\mathcal{W}(U, Y)^\bullet.$$

Proof. We prove this by induction on k.

$k = 0$: Let $f \in \mathcal{W}$, $p \in \mathcal{N}(Y)$ and $\varepsilon > 0$. We use ($\mathcal{W}$3) to see that there exists a $n \in \mathbb{N}$ such that

$$\|\gamma|_{U \setminus \overline{B}_n(0)}\|_{p,f,0} < \frac{\varepsilon}{1 + \|1 - h\|_\infty}.$$

Further, the set

$$A := \left\{ x \in X : |(1 - h)(x)| \geq \frac{\varepsilon}{\|\gamma\|_{p,f,0} + 1} \right\} \cap \overline{B}_n(0)$$

is compact and contained in U since $(1 - h) \equiv 0$ on V. Using this two estimates, we easily calculate that $\|(1 - h) \cdot \gamma|_{U \setminus A}\|_{p,f,0} < \varepsilon$.

$k \to k + 1$: We have

$$D((1 - h)|_U \cdot \gamma) = (1 - h)|_U \cdot D\gamma - Dh|_U \cdot \gamma.$$

By the inductive hypothesis, $(1 - h)|_U \cdot D\gamma \in \mathcal{C}^k_\mathcal{W}(U, \mathrm{L}(X, Y))^\bullet$, and since $Dh|_U \in \mathcal{C}^\infty_c(U, \mathrm{L}(X, \mathbb{R}))$, we use Corollary 3.4.15 and Proposition 3.4.11 to finish the proof. □

Lemma 7.2.25. *Let $m \in \mathbb{N}$, $k \in \overline{\mathbb{N}}$, $\mathcal{W}$ a set of weights as in Definition 7.2.18, Y and Z locally convex spaces, $\Omega \subseteq Y$ open and balanced, $\phi : \Omega \to Z$ a smooth map with $\phi(0) = 0$ and $U \subseteq \mathbb{R}^m$ open and nonempty such that $\mathbb{R}^m \setminus U$ is compact and $\overline{M_\infty} \subseteq U$. Further, let $\gamma \in \mathcal{C}^k_\mathcal{W}(U, Y)$ such that $\gamma(U) \subseteq \Omega$. Then there exists an open set $V \subseteq U$ such that $\mathbb{R}^m \setminus V$ is compact, $\overline{M_\infty} \subseteq V$ and $\phi \circ \gamma|_V \in \mathcal{C}^k_\mathcal{W}(V, Z)$.*

Proof. By our assumptions, there exists $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, [0,1])$ with $h \equiv 1$ on a neighborhood of $\mathbb{R}^m \backslash U$ and $h \equiv 0$ on a neighborhood of $\overline{M_\infty}$. Using Lemma 7.2.24 and Proposition 3.4.23 we see that

$$\phi \circ ((1-h) \cdot \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet,$$

so $\phi \circ \gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Z)$, where $V := \mathbb{R}^m \setminus \operatorname{supp}(h)$. Further, $\mathbb{R}^m \setminus V$ is compact and $\overline{M_\infty} \subseteq V$, so the proof is finished. □

To complete our preparations, we prove a kind of extension lemma for weighted functions.

Lemma 7.2.26. *Let $m \in \mathbb{N}$, $k \in \overline{\mathbb{N}}$, $\mathcal{W}$ a set of weights as in Definition 7.2.18, Y a locally convex space, $V \subseteq U$ open and nonempty subsets of $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus V$ is compact and $\overline{M_\infty} \subseteq V$. Further, let $\gamma \in \mathcal{C}^k(U, Y)$ such that $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Y)$. Then for any open set W with $\overline{W} \subseteq U$, the map $\gamma|_W$ is in $\mathcal{C}^k_{\mathcal{W}}(W, Y)$.*

Proof. Obviously $\overline{W \setminus V} \subseteq \overline{W} \cap (\mathbb{R}^m \setminus V)$, hence $\overline{W \setminus V}$ is compact and does not meet M_∞. So for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$, the map $f \cdot D^{(\ell)}\gamma$ is bounded on $\overline{W \setminus V}$ since f is continuous on this set. But $f \cdot D^{(\ell)}\gamma$ is bounded on V by our assumption. Hence $f \cdot D^{(\ell)}\gamma$ is bounded on all of W and the proof is finished. □

Now we are able to prove the main results.

Proposition 7.2.27. *Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition 7.2.18. Then the following assertions hold:*

(a) $S(\mathbb{R}^m, G; \mathcal{W})$ is a group.

(b) $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet \subseteq S(\mathbb{R}^m, G; \mathcal{W})$.

(c) $S(\mathbb{R}^m, G; \mathcal{W}) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{ex}$.

Proof. (a) Let $\gamma_1, \gamma_2 \in S(\mathbb{R}^m, G; \mathcal{W})$. We set $\gamma := \gamma_1 \cdot \gamma_2^{-1}$. Then for $x \in M_\infty$, we have $\gamma(x) = \gamma_1(x) \cdot \gamma_2^{-1}(x) = \mathbf{1}$, and it is easy to see that $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$.
Let $(\phi, \widetilde{V})$ be a centered chart of G and $V \subseteq \widetilde{V}$ an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. There exist centered charts (ϕ_1, V_1) and (ϕ_2, V_2) such that $\phi_i \circ \gamma_i \in S(\gamma_i^{-1}(V_i), \mathbf{L}(G); \mathcal{W})$, where $i \in \{1,2\}$; we may assume w.l.o.g. that $V_1 \cdot V_2^{-1} \subseteq V$, $V_2 \subseteq V$ and $\phi_1(V_1)$ and $\phi_2(V_2)$ are balanced. We define $W := \bigcap_{i \in \{1,2\}} \gamma_i^{-1}(V_i)$. Then by Lemma 3.4.16 and Lemma 7.2.21

$$(\phi_1 \circ \gamma_1|_W, \phi_2 \circ \gamma_2|_W) \in \mathcal{C}^\infty_{\mathcal{W}}(W, \phi_1(V_1) \times \phi_2(V_2)).$$

Further $\mathbb{R}^m \setminus W$ is compact, and since there exist closed $A_i \in \mathcal{U}_G(\mathbf{1})$ with $A_i \subseteq V_i$ ($i \in \{1,2\}$), we have $\overline{M_\infty} \subseteq \bigcap_{i \in \{1,2\}} \gamma_i^{-1}(A_i) \subseteq W$. We now apply Lemma 7.2.25 to $(\phi_1 \circ \gamma_1|_W, \phi_2 \circ \gamma_2|_W)$ and the map

$$\phi \circ \widetilde{m_G} \circ (\phi_1^{-1} \times \phi_2^{-1}) : \phi_1(V_1) \times \phi_2(V_2) \to \mathbf{L}(G)$$

(where $\widetilde{m_G}$ denotes the map $G \times G \to G : (g, h) \mapsto g \cdot h^{-1}$) and find an open set $W' \subseteq W$ such that $\overline{M_\infty} \subseteq W'$, $\mathbb{R}^m \setminus W'$ is compact and $\phi \circ \gamma|_{W'} \in \mathcal{C}^\infty_{\mathcal{W}}(W', \mathbf{L}(G))$. Applying Lemma 7.2.26 with the open sets $W' \subseteq \gamma^{-1}(\widetilde{V})$ and $\gamma^{-1}(V) \subseteq \gamma^{-1}(\widetilde{V})$, we obtain

$$\phi \circ \gamma|_{\gamma^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}(\gamma^{-1}(V), \mathbf{L}(G)) = S(\gamma^{-1}(V), \mathbf{L}(G); \mathcal{W}).$$

(b) Since we proved that $S(\mathbb{R}^m, G; \mathcal{W})$ is a group, we just have to show that it contains a generating set of $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet$. We know from Definition 7.2.5 that $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet$ is generated by $\phi^{-1} \circ \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, W)^\bullet$, where $(\phi, \widetilde{W})$ is a centered chart of G and $W \subseteq \phi(\widetilde{W})$ is an open convex zero neighborhood. Let $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, W)^\bullet$. Then $\gamma|_{M_\infty} \equiv 0$, hence $\phi^{-1} \circ \gamma|_{M_\infty} \equiv \mathbf{1}$. Further, since $1_{\mathbb{R}^m} \in \mathcal{W}$, $\gamma(x) \to 0$ if $\|x\| \to \infty$, and thus $(\phi^{-1} \circ \gamma)(x) \to \mathbf{1}$ if $\|x\| \to \infty$. Now let $(\psi, \widetilde{V})$ be a centered chart of G and $V \subseteq \widetilde{V}$ an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. There exists an open balanced set $\Omega \subseteq W$ such that $\phi^{-1}(\Omega) \subseteq V$. We set $U := \gamma^{-1}(\Omega)$. Then $\gamma|_U \in \mathcal{C}^\infty_{\mathcal{W}}(U, \mathbf{L}(G))$, $\mathbb{R}^m \setminus U$ is compact, and $\overline{M_\infty} \subseteq \gamma^{-1}(\{0\}) \subseteq U$. Hence we can apply Lemma 7.2.25 to $\gamma|_U$ and $\psi \circ \phi^{-1}|_\Omega$ to see that $\psi \circ \phi^{-1} \circ \gamma|_U \in \mathcal{C}^\infty_{\mathcal{W}}(U, \mathbf{L}(G))$ Applying Lemma 7.2.26 with the open sets $U \subseteq (\psi \circ \phi^{-1} \circ \gamma)^{-1}(\widetilde{V})$ and $(\psi \circ \phi^{-1} \circ \gamma)^{-1}(V) \subseteq (\psi \circ \phi^{-1} \circ \gamma)^{-1}(\widetilde{V})$, we obtain

$$\psi \circ \phi^{-1} \circ \gamma|_{(\psi \circ \phi^{-1} \circ \gamma)^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}((\psi \circ \phi^{-1} \circ \gamma)^{-1}(V), \mathbf{L}(G)) = S((\psi \circ \phi^{-1} \circ \gamma)^{-1}(V), \mathbf{L}(G); \mathcal{W}).$$

(c) Let $\gamma \in S(\mathbb{R}^m, G; \mathcal{W})$, $(\phi, \widetilde{V})$ be a centered chart of G and V an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. Then the set $K := \mathbb{R}^m \setminus \gamma^{-1}(V)$ is closed and bounded, hence compact, and

$$\phi \circ \gamma|_{\mathbb{R}^m \setminus K} \in S(\mathbb{R}^m \setminus K, \mathbf{L}(G); \mathcal{W}) = \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G));$$

the last identity is by Lemma 7.2.21. Let $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R})$ such that $h \equiv 1$ on a neighborhood of K. Then by Lemma 7.2.24

$$(1_{\mathbb{R}^m} - h) \cdot \phi \circ \gamma|_{\mathbb{R}^m \setminus K} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G))^\bullet.$$

Hence $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$. □

We characterize when $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$ consists entirely of $\mathcal{W}$-rapidly decreasing mappings.

Lemma 7.2.28. *Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition 7.2.18. The following equivalence holds:*

$$\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{ex} = S(\mathbb{R}^m, G; \mathcal{W}) \iff M_\infty = \emptyset.$$

Proof. Suppose that $M_\infty = \emptyset$. Let $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$, $(\psi, \widetilde{V})$ a centered chart of G and V a $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. By Lemma 7.2.8, there exist a compact set $K \subseteq \mathbb{R}^m$ and $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R})$ with $h \equiv 1$ on a neighborhood of K such that $\gamma(\mathbb{R}^m \setminus K) \subseteq \widetilde{V}$ and $(1 - h) \cdot (\psi \circ \gamma)|_{\mathbb{R}^m \setminus K} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G))^\bullet$. Since $1_{\mathbb{R}^m} \in \mathcal{W}$ and K and $\mathrm{supp}(h)$ are compact, $(\psi \circ \gamma)(x) \to 0$ if $\|x\| \to \infty$, hence $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$. Further $\psi \circ \gamma|_{\mathbb{R}^m \setminus \mathrm{supp}(h)} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus \mathrm{supp}(h), \mathbf{L}(G))$, so we apply Lemma 7.2.26 with the open sets $\mathbb{R}^m \setminus \mathrm{supp}(h) \subseteq \gamma^{-1}(\widetilde{V})$ and $\gamma^{-1}(V) \subseteq \gamma^{-1}(\widetilde{V})$ and get $\psi \circ \gamma|_{\gamma^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}(\gamma^{-1}(V), \mathbf{L}(G))$. Hence $\gamma \in S(\mathbb{R}^m, G; \mathcal{W})$, so in view of Proposition 7.2.27, the implication holds.

Now let $M_\infty \neq \emptyset$. By definition, $\mathcal{C}^\infty_c(\mathbb{R}^m, G) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$, so there exists a $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$ such that $\gamma \not\equiv \mathbf{1}$ on M_∞. Then $\gamma \notin S(\mathbb{R}^m, G; \mathcal{W})$. □

Remark 7.2.29. In the book [BCR81], the groups $S(\mathbb{R}^m, G; \mathcal{W})$ are only defined if G is a so-called *LE-Lie group*. Since we do not need this concept, we do not discuss it further. In Proposition 7.2.27 we proved that $S(\mathbb{R}^m, G; \mathcal{W})$ is an open subgroup of $\mathcal{C}^\infty_\mathcal{W}(\mathbb{R}^m, G)^\bullet_{\text{ex}}$ and hence a Lie group. Further, for a set $\mathcal{W}$ of weights as in Definition 7.2.18 obviously $\mathcal{C}^\infty_\mathcal{W}(\mathbb{R}^m, \mathbf{L}(G))^\bullet = \mathcal{C}^\infty_\mathcal{W}(\mathbb{R}^m, \mathbf{L}(G))$, whence the results derived by [BCR81] concerning the Lie group structure of $S(\mathbb{R}^m, G; \mathcal{W})$ are special cases of our more general construction.

It should be noted that the proof of [BCR81, Lemma 4.2.1.9] (whose assertion resembles Proposition 3.4.23) is not really complete: The boundedness of $\gamma \cdot \partial^\beta(g \circ f)$, where $|\beta| > 0$, is hardly discussed. In the finite-dimensional case, compactness arguments simular to the one in Lemma 3.4.17 and the Faà di Bruno-formula should save the day, but the infinite-dimensional case requires more work.

A. Differential calculus

In this chapter, we present the tools of Michal-Bastiani and Fréchet differential calculus used in this work. For proofs of the assertions, we refer the reader to [Mil84], [Ham82], or [Mic80]. Further, we state some facts about ordinary differential equations.

In the following, let X, Y and Z denote locally convex topological vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

A.1. Differential calculus of maps between locally convex spaces

A.1.1. Curves and integrals

Definition A.1.1 (Curves). A continuous map $\gamma : I \to X$ that is defined on a proper interval $I \subseteq \mathbb{R}$ is called a $\mathcal{C}^0$-curve. A $\mathcal{C}^0$-Kurve $\gamma : I \to X$ is called a $\mathcal{C}^1$-curve if the limit

$$\gamma^{(1)}(s) := \lim_{t \to 0} \frac{\gamma(s+t) - \gamma(s)}{t}$$

exists for all $s \in I$ and the map $\gamma^{(1)} : I \to X$ is a $\mathcal{C}^0$-curve.

Inductively, for $k \in \mathbb{N}$ a map $\gamma : I \to X$ is called a $\mathcal{C}^k$-curve if it is a $\mathcal{C}^1$-curve and the map $\gamma^{(1)}$ is a $\mathcal{C}^{k-1}$-curve. We then define $\gamma^{(k)} := (\gamma^{(1)})^{(k-1)}$.

If γ is a $\mathcal{C}^k$-curve for each $k \in \mathbb{N}$, we call γ a $\mathcal{C}^\infty$- or *smooth* curve.

Definition A.1.2 (Weak integral). Let $\gamma : [a,b] \to X$ be a map. If there exists $x \in X$ such that

$$\lambda(x) = \int_a^b (\lambda \circ \gamma)(t)\, dt \qquad \text{for all } \lambda \in X',$$

we call γ *weakly integrable* with the *weak integral* x and write

$$\int_a^b \gamma(t)\, dt := x.$$

Definition A.1.3 (Line integral). Let $\gamma : [a,b] \to X$ be a $\mathcal{C}^1$-curve and $f : \gamma([a,b]) \to Y$ a continuous map. We define the line integral of f on γ by

$$\int_\gamma f(\zeta)\, d\zeta := \int_a^b f(\gamma(t)) \cdot \gamma^{(1)}(t)\, dt$$

if the weak integral on the right hand side exists.

We record some properties of weak integrals.

Lemma A.1.4. *Let $\gamma : [a,b] \to X$ be a weakly integrable curve and $A : X \to Y$ a continuous linear map. Then the map $A \circ \gamma$ is weakly integrable with the integral*

$$\int_a^b (A \circ \gamma)(t)\,dt = A\left(\int_a^b \gamma(t)\,dt\right).$$

Proposition A.1.5 (Fundamental theorem of calculus)**.** *Let $\gamma : [a,b] \to X$ be a $\mathcal{C}^1$-curve. Then $\gamma^{(1)}$ is weakly integrable with the integral*

$$\int_a^b \gamma^{(1)}(t)\,dt = \gamma(b) - \gamma(a).$$

Lemma A.1.6. *If X is sequentially complete, each continuous curve in X is weakly integrable.*

Lemma A.1.7. *We endow the set of weakly integrable continuous curves from $[a,b]$ to X with the topology of uniform convergence. The weak integral defines a continuous linear map between this space and X. In particular, for each continuous seminorm $p : X \to \mathbb{R}$ and each weakly integrable continuous curve $\gamma : [a,b] \to X$*

$$\left\| \int_a^b \gamma(t)\,dt \right\|_p \leq \int_a^b \|\gamma(t)\|_p\,dt,$$

where we define $\|\cdot\|_p := p$.

Proposition A.1.8 (Continuity of parameter-dependent integrals)**.** *Let P be a topological space, $I \subseteq \mathbb{R}$ a proper interval and $a, b \in I$. Further, let $f : P \times I \to X$ be a continuous map such that the weak integral*

$$\int_a^b f(p,t)\,dt =: g(p)$$

exists for all $p \in P$. Then the map $g : P \to X$ is continuous.

Evaluation of curves We prove that the (simultaneous) evaluation of smooth curves is smooth.

Lemma A.1.9. *Let Y be a locally convex topological vector space and $m \in \overline{\mathbb{N}}$. Then the* evaluation function

$$\mathrm{ev} : \mathcal{C}^m([0,1], Y) \times [0,1] \to Y : (\Gamma, t) \mapsto \Gamma(t)$$

is a $\mathcal{C}^m$-map. For $m \geq 1$, we have

$$d\,\mathrm{ev}((\Gamma,t);(\Gamma_1,s)) = s \cdot \mathrm{ev}(\Gamma',t) + \mathrm{ev}(\Gamma_1,t) \tag{†}$$

(using the same symbol, ev*, for the evaluation of $\mathcal{C}^{m-1}$-curves).*

Proof. The proof is by induction:

$m = 0$: Let $\Gamma \in \mathcal{C}^0([0,1], Y)$ and $t \in [0,1]$. For a continuous seminorm $\|\cdot\|$ on Y and $\varepsilon > 0$ let U be a neighborhood of Γ in $\mathcal{C}^0([0,1], Y)$ such that for all $\Phi \in U$

$$\|\Phi - \Gamma\|_\infty < \frac{\varepsilon}{2},$$

where $\|\cdot\|_\infty$ is defined by

$$\mathcal{C}^0([0,1], Y) \to \mathbb{R} : \Phi \mapsto \sup_{t \in [0,1]} \|\Phi(t)\|.$$

By the continuity of Γ, there exists $\delta > 0$ such that for all $s \in [0,1]$ with $|s - t| < \delta$ the estimate

$$\|\Gamma(s) - \Gamma(t)\| < \frac{\varepsilon}{2}$$

holds. Then

$$\|\mathrm{ev}(\Gamma, t) - \mathrm{ev}(\Phi, s)\| \leq \|\Gamma(t) - \Gamma(s)\| + \|\Gamma(s) - \Phi(s)\| < \varepsilon,$$

whence ev is continuous in (Γ, t).

$m = 1$: Let $\Gamma, \Gamma_1 \in \mathcal{C}^1([0,1], Y)$, $t \in]0,1[$, $h \in \mathbb{R}^*$ and $s \in \mathbb{R}$ such that $t + hs \in [0,1]$. Then

$$\frac{\mathrm{ev}((\Gamma, t) + h(\Gamma_1, s)) - \mathrm{ev}(\Gamma, t)}{h} = \frac{\Gamma(t + hs) - \Gamma(t)}{h} + \mathrm{ev}(\Gamma_1, t + hs),$$

and because Γ is differentiable and ev is continuous, this term converges to

$$s \cdot \mathrm{ev}(\Gamma', t) + \mathrm{ev}(\Gamma_1, t)$$

for $h \to 0$. Since this term has an obvious continuous extension to $\mathcal{C}^1([0,1], Y) \times [0,1] \times \mathcal{C}^1([0,1], Y) \times \mathbb{R}$, ev is differentiable with the directional derivative (†), which is continuous.

$m \to m+1$: The map

$$\mathcal{C}^{m+1}([0,1], Y) \to \mathcal{C}^m([0,1], Y) : \Gamma \mapsto \Gamma'$$

is continuous linear and thus smooth. Using the inductive hypothesis, we therefore deduce from (†) that $d\,\mathrm{ev}$ is $\mathcal{C}^m$. Hence ev is $\mathcal{C}^{m+1}$. □

A.1.2. Differentiable maps

We give a short introduction on a differential calculus for maps between locally convex spaces. It was first developed by A. Bastiani in the work [Bas64] and is also known as Keller's C^k_c-theory.

Recall the definitions given in Section 2.2. In the following, let X and Y be locally convex spaces and $U \subseteq X$ an open nonempty set.

Proposition A.1.10 (Mean value theorem)**.** *Let $f \in \mathcal{C}^1(U, Y)$ and $v, u \in U$ such that the line segment $\{tu + (1-t)v : t \in [0,1]\}$ is contained in U. Then*

$$f(v) - f(u) = \int_0^1 df(u + t(v-u); v - u)\, dt.$$

Proposition A.1.11 (Chain rule)**.** *Let $k \in \overline{\mathbb{N}}$, $f \in \mathcal{C}^k(U,Y)$ and $g \in \mathcal{C}^k(V,Z)$ such that $f(U) \subseteq V$. Then the composition $g \circ f : U \to Z$ is a $\mathcal{C}^k$-map with*

$$d(g \circ f)(u;x) = dg(f(u); df(u;x)) \qquad \text{for all } (u,x) \in U \times X.$$

Proposition A.1.12. *Let X and Y be locally convex spaces, $U \subseteq X$ be open and nonempty and $k \in \overline{\mathbb{N}}$.*

(a) A map

$$f = (f_i)_{i \in I} : U \to \prod_{i \in I} Y_i$$

to a direct product of locally convex spaces ist $\mathcal{C}^k$ iff each component f_i is $\mathcal{C}^k$.

(b) A map $f : U \to Y$ with values in a closed vector subspace Z is $\mathcal{C}^k$ iff $f|^Z : U \to Z$ is $\mathcal{C}^k$.

(c) If Y is the projective limit of locally convex spaces $\{Y_i : i \in I\}$ with limit maps $\pi_i : Y \to Y_i$, then a map $f : U \to Y$ is $\mathcal{C}^k$ iff $\pi_i \circ f : U \to Y_i$ is $\mathcal{C}^k$ for all $i \in I$.

Characterization of differentiability of higher order In Proposition 2.2.3, we stated that a map is $\mathcal{C}^k$ iff all iterated directional derivatives up to order k exist and depend continuously on the directions. Here, we present some facts about the iterated directional derivatives.

Remark A.1.13. We give a more explicit formula for the k-th derivative. Obviously, $d^{(1)}f(u;x_1) = df(u;x_1)$ and

$$d^{(k)}f(u;x_1,\dots,x_k) = \lim_{t \to 0} \frac{d^{(k-1)}f(u+tx_k;x_1,\dots,x_{k-1}) - d^{(k-1)}f(u;x_1,\dots,x_{k-1}))}{t}.$$

The Schwarz theorem extends to the present situation:

Proposition A.1.14 (Schwarz' theorem)**.** *Let $r \in \overline{\mathbb{N}}$, $f \in \mathcal{C}^r_{\mathbb{K}}(U,Y)$, $k \in \mathbb{N}$ with $k \leq r$ and $u \in U$. The map*

$$d^{(k)}f(u;\cdot) : X^k \to Y : (x_1,\dots,x_k) \mapsto d^{(k)}f(u;x_1,\dots,x_k)$$

is continuous, symmetric and k-linear (over the field $\mathbb{K}$).

Examples We give some examples of $\mathcal{C}^k$-maps and calculate the higher-order differentials of some maps.

Example A.1.15. (a) A map $\gamma : I \to X$ is a $\mathcal{C}^k$-curve iff it is a $\mathcal{C}^k_{\mathbb{R}}$-map, and $d\gamma(x;h) = h \cdot \gamma^{(1)}(x)$.

(b) A continuous linear map $A : X \to Y$ is smooth with $dA(x;h) = A \cdot h$.

(c) More general, a k-linear continuous map $b : X_1 \times \cdots \times X_k \to Y$ is smooth with

$$db(x_1, \ldots, x_k; h_1, \ldots, h_k) = \sum_{i=1}^{k} b(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k).$$

We can calculate higher differentials of $f \circ g$ if one of the maps is linear.

Lemma A.1.16. *Let X, Y and Z be locally convex topological vector spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $A : Y \to Z$ a continuous linear map. Then for $\gamma \in \mathcal{C}^k(U, Y)$*

$$A \circ \gamma \in \mathcal{C}^k(U, Z).$$

Moreover, for each $\ell \in \mathbb{N}$ with $\ell \leq k$

$$d^{(\ell)}(A \circ \gamma) = A \circ d^{(\ell)}\gamma. \tag{†}$$

Proof. This is proved by induction on ℓ:
The chain rule (Proposition A.1.11) assures $A \circ \gamma \in \mathcal{C}^k(U, Z)$ and

$$d(A \circ \gamma)(x; h) = dA(\gamma(x); d\gamma(x; h)) = A(d\gamma(x; h))$$

for $x \in U$ and $h \in X$, hence (†) is satisfied for $\ell = 1$.
If we assume that (†) holds for an $\ell \in \mathbb{N}$, we conclude for $x \in U$ and $h_1, \ldots, h_\ell, h_{\ell+1} \in X$

$$\begin{aligned}
&d^{(\ell+1)}(A \circ \gamma)(x; h_1, \ldots, h_\ell, h_{\ell+1}) \\
=&\lim_{t \to 0} \frac{d^{(\ell)}(A \circ \gamma)(x + th_{\ell+1}; h_1, \ldots, h_\ell) - d^{(\ell)}(A \circ \gamma)(x; h_1, \ldots, h_\ell)}{t} \\
=&\lim_{t \to 0} \frac{A(d^{(\ell)}\gamma(x + th_{\ell+1}; h_1, \ldots, h_\ell)) - A(d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell))}{t} \\
=&A\left(\lim_{t \to 0} \frac{d^{(\ell)}\gamma(x + th_{\ell+1}; h_1, \ldots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell)}{t}\right) \\
=&(A \circ d^{(\ell+1)}\gamma)(x; h_1, \ldots, h_\ell, h_{\ell+1}),
\end{aligned}$$

so (†) holds for $\ell + 1$ as well. □

Lemma A.1.17. *Let X, Y and Z be locally convex topological vector spaces, $k \in \overline{\mathbb{N}}$ and $A : X \to Y$ a continuous linear map. Then for $\gamma \in \mathcal{C}^k(Y, Z)$*

$$\gamma \circ A \in \mathcal{C}^k(X, Z).$$

Moreover, for each $\ell \in \mathbb{N}$ with $\ell \leq k$

$$d^{(\ell)}(\gamma \circ A) = d^{(\ell)}\gamma \circ \prod_{j=1}^{\ell+1} A. \tag{†}$$

Proof. This is proved by induction on ℓ:
The chain rule (Proposition A.1.11) assures $\gamma \circ A \in \mathcal{C}^k(U, Z)$ and

$$d(\gamma \circ A)(x; h) = d\gamma(A(x); dA(x; h)) = d\gamma(A(x); A(h))$$

for $x \in X$ and $h \in X$, hence (†) is satisfied for $\ell = 1$.
If we assume that (†) holds for an arbitrary $\ell \in \mathbb{N}$, we conclude that for $x \in X$ and $h_1, \dots, h_\ell, h_{\ell+1} \in X$

$$\begin{aligned}
&d^{(\ell+1)}(\gamma \circ A)(x; h_1, \dots, h_\ell, h_{\ell+1})\\
=&\lim_{t\to 0} \frac{d^{(\ell)}(\gamma \circ A)(x + th_{\ell+1}; h_1, \dots, h_\ell) - d^{(\ell)}(\gamma \circ A)(x; h_1, \dots, h_\ell)}{t}\\
=&\lim_{t\to 0} \frac{d^{(\ell)}\gamma(A(x + th_{\ell+1}); A \cdot h_1, \dots, A \cdot h_\ell) - d^{(\ell)}\gamma(A(x); A \cdot h_1, \dots, A \cdot h_\ell)}{t}\\
=&\lim_{t\to 0} \frac{1}{t} \int_0^1 d^{(\ell+1)}\gamma(A(x) + stA(h_{\ell+1}); A \cdot h_1, \dots, A \cdot h_\ell, tA \cdot h_{\ell+1})\, ds\\
=&d^{(\ell+1)}\gamma(A(x); A \cdot h_1, \dots, A \cdot h_\ell, A \cdot h_{\ell+1})
\end{aligned}$$

so (†) holds for $\ell + 1$ as well. □

Another example for the computation of directional derivatives follows.

Lemma A.1.18. *Let X, Y and Z be locally convex spaces, $V \subseteq Y$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\gamma : V \to Z$ a map and $A \in \mathrm{L}(X, Y)$ surjective such that*

$$\gamma \circ A \in \mathcal{C}^k(U, Z),$$

where $U := A^{-1}(V)$. Then all directional derivatives of γ up to order k exist and satisfy the identity

$$d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A = d^{(\ell)}(\gamma \circ A)$$

for all $\ell \in \mathbb{N}$ with $\ell \leq k$.

Proof. This is proved by induction on ℓ:
$\ell = 0$: This is obvious.
$\ell \to \ell + 1$: Let $y \in V$ and $h_1, \dots, h_\ell, h_{\ell+1} \in Y$. By the surjectivity of A there exist $x \in U$ and $v_1, \dots, v_\ell, v_{\ell+1} \in X$ with $A \cdot x = y$ and $A \cdot v_i = h_i$ for $i = 1, \dots, \ell, \ell + 1$. Then for all suitable $t \neq 0$

$$\begin{aligned}
&\lim_{t\to 0} \frac{d^{(\ell)}\gamma(y + th_{\ell+1}; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)}{t}\\
=&\lim_{t\to 0} \frac{d^{(\ell)}\gamma(A(x + tv_{\ell+1}); A \cdot v_1, \dots, A \cdot v_\ell) - d^{(\ell)}\gamma(A \cdot x; A \cdot v_1, \dots, A \cdot v_\ell)}{t}\\
=&\lim_{t\to 0} \frac{(d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A)(x + tv_{\ell+1}, v_1, \dots, v_\ell) - (d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A)(x, v_1, \dots, v_\ell)}{t}\\
=&d^{(\ell+1)}(\gamma \circ A)(x; v_1, \dots, v_\ell, v_{\ell+1}),
\end{aligned}$$

and this completes the proof. □

We give a specialization of Proposition A.1.8.

Proposition A.1.19 (Differentiability of parameter-dependent integrals)**.** *Let P be an open subset of a locally convex space, $I \subseteq \mathbb{R}$ a proper interval, $a, b \in I$ and $k \in \overline{\mathbb{N}}$. Further, let $f : P \times I \to X$ be a $\mathcal{C}^k$-map such that the weak integral*

$$\int_a^b f(p,t)\,dt =: g(p)$$

exists for all $p \in P$. Then the map $g : P \to X$ is $\mathcal{C}^k$.

Analytic maps

Complex analytic maps will be defined as maps which can locally be approximated by polynomials. Real analytic maps are maps that have a *complexification.*

Polynomials and symmetric multilinear maps For the definition of complex analytic maps we need to define polynomials.

Definition A.1.20. Let $k \in \mathbb{N}$. A *homogenous polynomial of degree k* from X to Y is a map for which there exists a k-linear map $\beta : X^k \to Y$ such that

$$p(x) = \beta(\underbrace{x, \dots, x}_{k})$$

for all $x \in X$. In particular, a homogenous polynomial of degree 0 is a constant map.

A *polynomial of degree* $\leq k$ is a sum of homogenous polynomials of degree $\leq k$.

There is a bijection between the set of homogenous polynomials and that of symmetric multilinear maps. In this article, we just need that one can reconstruct a symmetric multilinear map from its homogenous polynomial.

Proposition A.1.21 (Polarization formula)**.** *Let $\beta : X^k \to Y$ be a symmetric k-linear map, $p : X \to Y : x \mapsto \beta(x, \dots, x)$ its homogenous polynomial and $x_0 \in X$. Then*

$$\beta(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\varepsilon_1, \dots, \varepsilon_k = 0}^{1} (-1)^{k-(\varepsilon_1 + \dots + \varepsilon_k)} p(x_0 + \varepsilon_1 x_1 + \dots + \varepsilon_k x_k)$$

for all $x_1, \dots, x_k \in X$.

Complex analytic maps Now we can define complex analytic maps.

Definition A.1.22 (Complex analytic maps)**.** Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. A map $f : U \to Y$ is called *complex analytic* if it is continuous and, for each $x \in U$ there exists a sequence $(p_k)_{k \in \mathbb{N}}$ of continuous homogenous polynomials $p_k : X \to Y$ of degree k such that

$$f(x + v) = \sum_{k=0}^{\infty} p_k(v)$$

for all v in some zero neighborhood V such that $x + V \subseteq U$.

Definition A.1.23. Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. A map $f : U \to Y$ is called *Gateaux analytic* if its restriction on each affine line is complex analytic; that is, for each $x \in U$ and $v \in X$ the map

$$\{z \in \mathbb{C} : x + zv \in U\} \to Y : z \mapsto f(x + zv)$$

is complex analytic.

Theorem A.1.24. *Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. Then for a map $f : U \to Y$ the following assertions are equivalent:*

(a) f is $\mathcal{C}^\infty_\mathbb{C}$,

(b) f is complex analytic,

(c) f is continuous and Gateaux analytic.

We state a few results concerning analytic curves. These share many properties with holomorphic functions. Using Theorem A.1.24, we see that some of these properties carry over to general analytic functions.

Definition A.1.25. Let Y be a complex locally convex topological vector space and $U \subseteq \mathbb{C}$ an open nonempty set. A continuous map $f : U \to Y$ is called a $\mathcal{C}^0_\mathbb{C}$*-curve.* A $\mathcal{C}^0_\mathbb{C}$-curve $f : U \to Y$ is called a $\mathcal{C}^1_\mathbb{C}$-curve if for all $z \in U$ the limit

$$f^{(1)}(z) := \lim_{w \to 0} \frac{f(z + w) - f(z)}{w}$$

exists and the curve $f^{(1)} : U \to X$ is a $\mathcal{C}^0_\mathbb{C}$-curve.
Inductively, for $k \in \mathbb{N}$ a curve f is called a $\mathcal{C}^k_\mathbb{C}$-curve if it is a $\mathcal{C}^1_\mathbb{C}$-curve and $f^{(1)}$ is a $\mathcal{C}^{k-1}_\mathbb{C}$-curve. In this case, we define $f^{(k)} := (f^{(1)})^{(k-1)}$.
If f is a $\mathcal{C}^k_\mathbb{C}$-curve for all $k \in \mathbb{N}$, f is called a $\mathcal{C}^\infty_\mathbb{C}$-curve.

Lemma A.1.26 (Cauchy integral formula)**.** *Let Y be a complex locally convex topological vector space, $U \subseteq \mathbb{C}$ an open nonempty set and $f : U \to Y$ a map. Then*

$$f \text{ is a } \mathcal{C}^k_\mathbb{C}\text{-curve} \iff f \in \mathcal{C}^k_\mathbb{C}(U, Y)$$

and furthermore

$$d^{(k)} f(x; h_1, \ldots, h_k) = h_1 \cdot \cdots \cdot h_k \cdot f^{(k)}(x).$$

A $\mathcal{C}^\infty_\mathbb{C}$-curve is complex analytic, and for each $x \in U$, $k \in \mathbb{N}_0$ and $r > 0$ with $\overline{B}_r(x) \subseteq U$ the Cauchy integral formula

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta - x| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta$$

holds, where $z \in B_r(x)$.

The Cauchy integral formula implies the Cauchy estimates.

Corollary A.1.27. *Let Y be a complex locally convex topological vector space, $U \subseteq \mathbb{C}$ an open nonempty set, $f : U \to Y$ a complex analytic map, $x \in U$, $r > 0$ such that $\overline{B}_r(x) \subseteq U$, $\sigma \in]0,1[$ and p a continuous seminorm on Y. Then for each $k \in \mathbb{N}$ and $z \in B_r(x)$ with $|z - x| = \sigma r$, we get the estimate*

$$\|f^{(k)}(z)\|_p \leq \frac{k!}{(1-\sigma)^{k+1} r^k} \sup_{|\zeta - x| = r} \|f(\zeta)\|_p.$$

Real analytic maps

Definition A.1.28 (Real analytic maps)**.** Let X, Y be real locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. Let $X_{\mathbb{C}}$ resp. $Y_{\mathbb{C}}$ denote the complexifications of X resp. Y. A map $f : U \to Y$ is called *real analytic* if there is an extension $\widetilde{f} : V \to Y_{\mathbb{C}}$ of f to an open neighborhood V of U in $X_{\mathbb{C}}$ that is complex analytic. Such a map $\widetilde{f}$ will be refered to as a *complexification* of f.

Lipschitz continuous maps between locally convex spaces and induced maps on normed spaces

We define and discuss Lipschitz continuous maps between locally convex spaces. To this end, we define some terms concerning seminorms and the quotient maps they induce.

Definition A.1.29. Let X be a locally convex space and $p : X \to \mathbb{R}$ a continuous seminorm. We denote the Hausdorff space $X/p^{-1}(0)$ with X_p and the quotient map with $\pi_p : X \to X_p$. More general, for any subset $A \subseteq X$ we set $A_p := \pi_p(A)$.
Further, we let $\mathcal{N}(X)$ denote the set of continuous seminorms on X.
Let $p \in \mathcal{N}(X)$. We call $U \subseteq X$ *open with respect to* p if for each $x \in U$ there exists $r > 0$ such that $\{y \in X : \|y - x\|_p < r\} \subseteq U$.

Remark A.1.30. For any locally convex space X and each $p \in \mathcal{N}(X)$, the norm induced by p on X_p will also be denoted by p. Note that this leads to the identity $p = \pi_p \circ p$, in particular p is a norm and generates the topology on X_p. No confusion will arise.

Definition A.1.31 (Lipschitz continuous maps)**.** Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $p \in \mathcal{N}(Y)$ and $q \in \mathcal{N}(X)$. We call $\gamma : U \to Y$ *Lipschitz up to order k with respect to p and q* if $\gamma \in \mathcal{C}^k(U, Y)$,

$$\|d^{(\ell)}\gamma(y; h_1, \ldots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell)\|_p \leq \|y - x\|_q \prod_{i=1}^{\ell} \|h_i\|_q \tag{A.1.31.1}$$

and

$$\|d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell)\|_p \leq \prod_{i=1}^{\ell} \|h_i\|_q. \tag{A.1.31.2}$$

for all $\ell \in \mathbb{N}$ with $\ell \leq k$, $x, y \in U$ and $h_1, \ldots, h_\ell \in X$. We write $\mathcal{LC}^k_{q,p}(U, Y)$ for the set of maps that are Lipschitz up to order k with respect to p and q.

As for differentiable maps between normed spaces, differentiable maps always are at least locally Lipschitz.

Lemma A.1.32. *Let X, Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\gamma \in \mathcal{C}^{k+1}(U, Y)$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Then for each $p \in \mathcal{N}(Y)$ and $x_0 \in U$ there exist $q \in \mathcal{N}(X)$ and a convex neighborhood $U_{x_0} \subseteq U$ of x with respect to q such that $\gamma|_{U_{x_0}} \in \mathcal{LC}^k_{q,p}(U_{x_0}, Y)$.*

Proof. Since $d^{(\ell)}\gamma$ and $d^{(\ell+1)}\gamma$ are continuous in $(x_0, 0, \dots, 0)$ and multilinear in their last ℓ resp. $\ell + 1$ arguments, for each $p \in \mathcal{N}(Y)$ there exist $q \in \mathcal{N}(X)$ and an open ball $U_{x_0} := B_q(x_0, r) \subseteq U$ such that

$$1 \geq \sup\{\|d^{(\ell+1)}\gamma(y; h_1, \dots, h_{\ell+1})\|_p : y \in B_q(x_0, r), \|h_1\|_q, \dots, \|h_{\ell+1}\|_q \leq 1\}$$

and

$$1 \geq \sup\{\|d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)\|_p : y \in B_q(x_0, r), \|h_1\|_q, \dots, \|h_\ell\|_q \leq 1\}.$$

This implies that for each $y \in B_q(x_0, r)$ and $h_1, \dots, h_n \in X$

$$\|d^{(n)}\gamma(y; h_1, \dots, h_n)\|_p \leq 1 \cdot \prod_{i=1}^{n} \|h_i\|_q, \tag{†}$$

where $n \in \{\ell, \ell + 1\}$; this proves estimate (A.1.31.2).
To prove estimate (A.1.31.1), we see that for $x, y \in B_q(x_0, r)$ and $h_1, \dots, h_{\ell+1} \in X$

$$d^{(\ell)}\gamma(y; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \dots, h_\ell) = \int_0^1 d^{(\ell+1)}\gamma(ty + (1-t)x; h_1, \dots, h_\ell, y - x)\, dt.$$

We apply Lemma A.1.7 to the right hand side and get using (†) with $n = \ell + 1$.

$$\|d^{(\ell)}\gamma(y; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \dots, h_\ell)\|_p \leq \|h_1\|_q \cdots \|h_\ell\|_q \cdot \|y - x\|_q$$

which finishes the proof. ∎

We show that each Lipschitz map induces another Lipschitz map between the respective (normed) quotient spaces.

Lemma A.1.33. *Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $p \in \mathcal{N}(Y)$, $q \in \mathcal{N}(X)$ and $\gamma \in \mathcal{LC}^k_{q,p}(U, Y)$. Then there exists a map $\tilde{\gamma} \in \mathcal{LC}^k_{q,p}(U_q, Y_p)$ that makes the diagram*

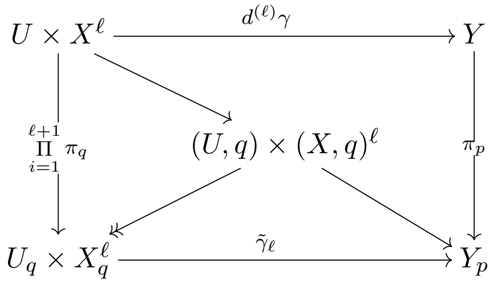
$$\begin{array}{ccc} U & \xrightarrow{\gamma} & Y \\ \downarrow{\scriptstyle \pi_q} & & \downarrow{\scriptstyle \pi_p} \\ U_q & \xrightarrow{\tilde{\gamma}} & Y_p \end{array}$$

commutative (using notation as in Definition A.1.29).

Proof. Let $\ell \in \mathbb{N}$ with $\ell \leq k$. Since $\gamma \in \mathcal{LC}^k_{q,p}(U, Y)$, the map

$$\pi_p \circ d^{(\ell)}\gamma : (U, q) \times (X, q)^\ell \to Y_p$$

is continuous. Hence by the universal property of the separation there exists a continuous map $\tilde{\gamma}_\ell$ such that the diagram



commutes, where we denote $\pi_q|_U$ with π_q. The diagram for $\ell = 0$ implies that $\tilde{\gamma} \circ \pi_q = \pi_p \circ \gamma \in \mathcal{C}^k(U, Y_p)$, where $\tilde{\gamma} := \tilde{\gamma}_0$. We proved in Lemma A.1.18 that the ℓ-th directional derivative of $\tilde{\gamma}$ exists and satisfies the identity

$$d^{(\ell)}\tilde{\gamma} \circ \prod_{i=1}^{\ell+1} \pi_q = d^{(\ell)}(\tilde{\gamma} \circ \pi_q) = d^{(\ell)}(\pi_p \circ \gamma) = \pi_p \circ d^{(\ell)}\gamma = \tilde{\gamma}_\ell \circ \prod_{i=1}^{\ell+1} \pi_q.$$

Since $\prod_{i=1}^{\ell+1} \pi_q$ is surjective, this implies that $d^{(\ell)}\tilde{\gamma} = \tilde{\gamma}_\ell$, so the former is continuous. From this we conclude that $\tilde{\gamma} \in \mathcal{C}^k(U_q, Y_p)$ and that the estimates (A.1.31.1) and (A.1.31.2) are satisfied by $\tilde{\gamma}$. □

Finally, we prove that for each compact set, each $\mathcal{C}^{k+1}$-map defined on it and each seminorm on the image there exists a seminorm on the domain such that the quotient map, and its differentials, are bounded. For that, we need to use a lemma about the relationship between differentiability and Fréchet differentiability that is proved later in Section A.2.

Lemma A.1.34. *Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \mathbb{N}$, $\gamma \in \mathcal{C}^{k+1}(U, Y)$, $p \in \mathcal{N}(Y)$ and K a compact subset of U. Then there exists a seminorm $q \in \mathcal{N}(X)$ and an open set V w.r.t. q such that $K \subseteq V \subseteq U$ and $\tilde{\gamma} \in \mathcal{BC}^k(V_q, Y_p)$ (For the definition of $\tilde{\gamma}$ see Lemma A.1.33).*

Proof. Using Lemma A.1.32 and standard compactness arguments, we find $q \in \mathcal{N}(X)$ and a neighborhood $\widetilde{V}$ w.r.t. q of K in U such that estimate (A.1.31.1) and estimate (A.1.31.2) hold for γ on $\widetilde{V}$ and all $\ell \in \mathbb{N}$ with $\ell \leq k$. We proved in Lemma A.1.33 that this implies that $\tilde{\gamma} \in \mathcal{LC}^k_{q,p}(\widetilde{V}_q, Y_p)$, and with Proposition A.3.2 we can conclude that $\tilde{\gamma} \in \mathcal{FC}^k(\widetilde{V}_q, Y_p)$. Further, since $D^{(\ell)}\tilde{\gamma}(K_q)$ is compact for all $\ell \leq k$, there exists a neighborhood V_q of K_q such that $\tilde{\gamma}$ and all its derivatives up to degree k are bounded on V_q. □

A.2. Fréchet differentiability

For maps between normed spaces, there is the classical notion of *Fréchet differentiability*. This concept relies on the existence of a well-behaved topology on the space of (k-)linear maps between normed spaces.

Spaces of multilinear maps between normed spaces We provide the details about the norm topology of multilinear operators.

Definition A.2.1. Let X, Y be normed spaces. For each $k \in \mathbb{N}^*$ we define

$$\mathrm{L}^k(X,Y) := \{\Xi : X^k \to Y : \Xi \text{ is } k\text{-linear and continuous}\}.$$

For $k = 1$ we define

$$\mathrm{L}(X,Y) := \mathrm{L}^1(X,Y) \text{ and } \mathrm{L}(X) := \mathrm{L}^1(X,X),$$

and furthermore

$$\mathrm{L}^0(X,Y) := Y.$$

The set of multilinear continuous maps can be turned into a normed vector space:

Proposition A.2.2. *Let X, Y be normed spaces and $k \in \mathbb{N}^*$. A k-linear map $\Xi : X^k \to Y$ is continuous iff*

$$\|\Xi\|_{op} := \sup\{\|\Xi(v_1,\dots,v_k)\| : \|v_1\|,\dots,\|v_k\| \leq 1\} < \infty.$$

$\|\Xi\|_{op}$ is called the operator norm *of Ξ. $\|\cdot\|_{op}$ is a norm on $\mathrm{L}^k(X,Y)$. The space $\mathrm{L}^k(X,Y)$, endowed with this norm, is complete if Y is so.*

Proof. The (elementary) proof can be found in [Die60, Chapter V, §7]. □

Lemma A.2.3. *Let X, Y be normed spaces and $k \in \mathbb{N}^*$. Then the evaluation map*

$$\mathrm{L}^k(X,Y) \times X^k : (\Xi, v_1,\dots,v_k) \mapsto \Xi(v_1,\dots,v_k)$$

is $(k+1)$-linear and continuous.

Proof. This is trivial. □

Lemma A.2.4. *Let X and Y be normed spaces, $k \in \mathbb{N}^*$, $\Xi \in \mathrm{L}^k(X,Y)$ and $h_1,\dots,h_k$, $v_1,\dots,v_k \in X$. Then*

$$\|\Xi(h_1,\dots,h_n) - \Xi(v_1,\dots,v_k)\| \leq \sum_{i=1}^{k} \|\Xi(v_1,\dots,v_{i-1},h_i - v_i,h_{i+1},\dots,h_k)\|.$$

Proof. This estimate is derived by an iterated application of the triangle inequality. □

The following lemma helps to deal with higher derivatives of Fréchet-differentiable maps.

Lemma A.2.5. *Let X, Y be normed spaces and $n, k \in \mathbb{N}^*$. Then the map*

$$\mathcal{E}_{k,n} : \mathrm{L}^k(X, \mathrm{L}^n(X,Y)) \to \mathrm{L}^{k+n}(X,Y)$$
$$\mathcal{E}_{k,n}(\Xi)(h_1, \dots, h_n, v_1, \dots, v_k) := \Xi(v_1, \dots, v_k)(h_1, \dots, h_n)$$

is an isometric isomorphism. In some cases, we will denote $\mathcal{E}_{k,n}$ by $\mathcal{E}^Y_{k,n}$.

Proof. Obviously $\mathcal{E}_{k,n}$ is linear and injective. Furthermore

$$\|\mathcal{E}_{k,n}(\Xi)(h_1, \dots, h_n, v_1, \dots, v_k)\| = \|\Xi(v_1, \dots, v_k)(h_1, \dots, h_n)\|$$
$$\leq \|\Xi(v_1, \dots, v_k)\|_{op} \prod_{i=1}^{n} \|h_i\| \leq \|\Xi\|_{op} \prod_{i=1}^{k} \|v_i\| \prod_{i=1}^{n} \|h_i\|,$$

and hence

$$\|\mathcal{E}_{k,n}(\Xi)\|_{op} \leq \|\Xi\|_{op}.$$

On the other hand, for $\|v_1\|, \dots, \|v_k\|, \|h_1\|, \dots, \|h_n\| \leq 1$ we have

$$\|\Xi(v_1, \dots, v_k)(h_1, \dots, h_n)\| \leq \|\mathcal{E}_{k,n}(\Xi)\|_{op}.$$

Hence

$$\|\Xi(v_1, \dots, v_k)\|_{op} \leq \|\mathcal{E}_{k,n}(\Xi)\|_{op},$$

which leads to

$$\|\Xi\|_{op} \leq \|\mathcal{E}_{k,n}(\Xi)\|_{op},$$

so $\mathcal{E}_{k,n}$ is an isometry. It remains to show that $\mathcal{E}_{k,n}$ is surjective. To this end, for a $M \in \mathrm{L}^{k+n}(X,Y)$ we define the map $\overline{M} \in \mathrm{L}^k(X, \mathrm{L}^n(X,Y))$ by

$$\overline{M}(v_1, \dots, v_k)(h_1, \dots, h_n) := M(h_1, \dots, h_n, v_1, \dots, v_k).$$

Clearly, $\mathcal{E}_{k,n}(\overline{M}) = M$. Since M was arbitrary, $\mathcal{E}_{k,n}$ is surjective. □

Lemma A.2.6. *Let X, Y and Z be normed spaces and $k \in \mathbb{N}$. Then the map*

$$\mathrm{L}^k(X, Y \times Z) \to \mathrm{L}^k(X,Y) \times \mathrm{L}^k(X,Z) : \Xi \mapsto (\pi_Y \circ \Xi, \pi_Z \circ \Xi), \tag{A.2.6.1}$$

where π_Y respective π_Z denotes the canonical projection from $Y \times Z$ to Y respective Z, is an isomorphism of topological vector spaces.

Proof. The map in (A.2.6.1) is linear since its component maps $\Xi \mapsto \pi_Y \circ \Xi$ and $\Xi \mapsto \pi_Z \circ \Xi$ are so. The injectivity of (A.2.6.1) is clear, and the surjectivity can also be shown by an easy computation.
To see that (A.2.6.1) is an isomorphism we denote it by $\mathfrak{i}$ and compute for $x_1, \dots, x_k \in X$

$$\big((\pi_{\mathrm{L}^k(X,Y)} \circ \mathfrak{i})(\Xi)(x_1, \dots, x_k), (\pi_{\mathrm{L}^k(X,Z)} \circ \mathfrak{i})(\Xi)(x_1, \dots, x_k)\big)$$
$$= \big((\pi_Y \circ \Xi)(x_1, \dots, x_k), (\pi_Z \circ \Xi)(x_1, \dots, x_k)\big) = \Xi(x_1, \dots, x_k).$$

From this one can easily derive that $\mathfrak{i}$ and its inverse are continuous since depending on the norm we chose on the products, $\mathfrak{i}$ is an isometry. □

The calculus In the following, let X, Y and Z denote normed spaces and U be an open nonempty subset of X. Recall the definition of Fréchet differentiability given in Definition 2.3.1.

We give some examples of Fréchet differentiable maps.

Example A.2.7. (a) A continuous linear map $A : X \to Y$ is smooth with $DA(x) = A$.

(b) More generally, a continuous k-linear map $b : X_1 \times \cdots \times X_k \to Y$ is smooth with

$$Db(x_1, \ldots, x_k)(h_1, \ldots, h_k) = \sum_{i=1}^{k} b(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k).$$

We prove the Chain Rule and the Mean Value Theorem for Fréchet differentiable maps. Beforehand, we need the following

Lemma A.2.8. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $A : Y \to Z$ a continuous linear map. Then for $\gamma \in \mathcal{FC}^k(U, Y)$*

$$A \circ \gamma \in \mathcal{FC}^k(U, Z).$$

Proof. We prove this by induction over k. The assertion is obviously true for $k = 0$. If $k = 1$, then $A \circ \gamma$ is $\mathcal{C}^1$ by Proposition A.1.11 with

$$d(A \circ \gamma)(x; \cdot) = dA(\gamma(x); \cdot) \cdot d\gamma(x; \cdot) = A \circ d\gamma(x; \cdot).$$

Since the composition of linear maps is continuous, we conclude that $A \circ \gamma$ is $\mathcal{FC}^1$ with $D(A \circ \gamma) = A \circ D\gamma$.

$k \to k+1$: The map $D\gamma$ is $\mathcal{FC}^k$, hence by the induction hypothesis, so is $A \circ D\gamma = D(A \circ \gamma)$. Hence $A \circ \gamma$ is $\mathcal{FC}^{k+1}$, which finishes the induction. ☐

Lemma A.2.9. *Let $k \in \overline{\mathbb{N}}$, $\eta \in \mathcal{FC}^k(U, Y)$ and $\gamma \in \mathcal{FC}^k(U, Z)$. Then the map*

$$(\gamma, \eta) : U \to Y \times Z : x \mapsto (\gamma(x), \eta(x))$$

is contained in $\mathcal{FC}^k(U, Y \times Z)$.

Proof. For $k = 0$ the assertion is obviously true. If $k = 1$, we easily calculate that (γ, η) is $\mathcal{C}^1$ with

$$d(\gamma, \eta)(x; h) = (d\gamma(x; h), d\eta(x; h)).$$

Hence

$$d(\gamma, \eta)(x; \cdot) = \mathfrak{i}^{-1}(d\gamma(x; \cdot), d\eta(x; \cdot)),$$

where $\mathfrak{i}$ denotes the isomorphism (A.2.6.1) from Lemma A.2.6. We conclude that (γ, η) is $\mathcal{FC}^1$.

For $k > 1$, the assertion is proved with an easy induction using Lemma A.2.8. ☐

Proposition A.2.10 (Chain Rule)**.** *Let $k \in \overline{\mathbb{N}}$, $\eta \in \mathcal{FC}^k(U,Y)$ and $\gamma \in \mathcal{FC}^k(V,Z)$ such that $\eta(U) \subseteq V$. Then $\gamma \circ \eta \in \mathcal{FC}^k(U,Z)$ and*

$$D(\gamma \circ \eta)(u) = (D\gamma \circ \eta)(u) \cdot D\eta(u) \tag{$*$}$$

for all $u \in U$.

Proof. The proof is by induction on k:
$k = 1$: We apply the chain rule for $\mathcal{C}^1$-maps (Proposition A.1.11) to see that $\gamma \circ \eta$ is $\mathcal{C}^1$, and for $(u,x) \in U \times X$ we have

$$d(\gamma \circ \eta)(u;x) = d\gamma(\eta(u); d\eta(u;x)).$$

From this identity we conclude that $(*)$ holds. Finally we obtain the continuity of $D(\gamma \circ \eta)$ from the one of $\cdot$, $D\gamma$, $D\eta$ and η.

$k \to k+1$: By the inductive hypothesis, the maps $D\gamma$ and $D\eta$ are $\mathcal{FC}^k$. We already proved in the case $k = 1$ that $(*)$ holds. By the inductive hypothesis, $D\gamma \circ \eta \in \mathcal{FC}^k$. Since $\cdot$ is smooth (see Example A.2.7), we conclude using Lemma A.2.9 and the inductive hypothesis that $D(\gamma \circ \eta)$ is $\mathcal{FC}^k$. Hence $\gamma \circ \eta$ is $\mathcal{FC}^{k+1}$. □

Proposition A.2.11 (Mean Value Theorem)**.** *Let $f \in \mathcal{FC}^1(U,Y)$. Then*

$$f(v) - f(u) = \int_0^1 Df(u + t(v-u)) \cdot (v-u)\, dt$$

for all $v, u \in U$ such that the line segment $\{tu + (1-t)v : t \in [0,1]\}$ is contained in U. In particular

$$\|f(v) - f(u)\| \leq \sup_{t \in [0,1]} \|Df(u + t(v-u))\|_{op} \|v - u\|.$$

Proof. The identity is a reformulation of Proposition A.1.10, hence the estimate is a direct consequence of Lemma A.1.7. □

The isomorphisms provided by Lemma A.2.5 can be used to characterize Fréchet differentiability of higher order.

Remark A.2.12. We define inductively

$$L^0_{X,Y} := Y \text{ and } L^{k+1}_{X,Y} := \mathrm{L}(X, L^k_{X,Y}).$$

Definition A.2.13 (Higher derivatives)**.** Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$ with $k \leq n$ we define a linear map

$$D^{(k)} : \mathcal{FC}^n(U,Y) \to \mathcal{FC}^{n-k}(U, \mathrm{L}^k(X,Y))$$

by $D^{(0)} := \mathrm{id}_{\mathcal{FC}^n(U,Y)}$ for $k = 0$, $D^{(1)} := D$ for $k = 1$ and for $1 < k \leq n$ by

$$D^{(k)}\gamma := \mathcal{E}^Y_{k-1,1} \circ \cdots \circ \mathcal{E}^{L^{k-3}_{X,Y}}_{2,1} \circ \mathcal{E}^{L^{k-2}_{X,Y}}_{1,1} \circ (\underbrace{D \circ \cdots \circ D}_{k \text{ times}})(\gamma).$$

Here we used the notations introduced in Remark A.2.12. Note that the image of $D^{(k)}$ is contained in $\mathcal{FC}^{n-k}(U, \mathrm{L}^k(X,Y))$ because the maps $\mathcal{E}^{Y}_{k-1,1}$, …, $\mathcal{E}^{L^{k-3}_{X,Y}}_{2,1}$, $\mathcal{E}^{L^{k-2}_{X,Y}}_{1,1}$ are continuous linear maps and hence smooth (see Example A.2.7); so the chain rule (Proposition A.2.10) gives the result.
We call $D^{(k)}$ the *k-th derivative operator.*

The $(k+1)$-st derivative of a map γ is closely related to the k-th derivative of $D\gamma$:

Lemma A.2.14. *Let $n \in \overline{\mathbb{N}}^*$, $\gamma \in \mathcal{FC}^n(U,Y)$ and $k \in \mathbb{N}$ with $k < n$. Then*

$$D^{(k+1)}\gamma = \mathcal{E}^{Y}_{k,1} \circ (D^{(k)}(D\gamma)).$$

Proof. This follows directly from the definition of $D^{(k+1)}\gamma$. □

A.2.1. The Lipschitz inverse function theorem

We discuss an inverse function theorem for functions of the sort $T+\eta$, where T is a linear Operator and η is a "small" perturbation map. We derive an estimate for the Lipschitz constant of $(T+\eta)^{-1}$, and consequently another for the size of the image of $T+\eta$. Further we discuss families of such functions to derive a parametrized inverse function theorem. The next lemma discusses a special case. The main tool for proving it is a parameterized version of the Banach fixed point theorem which can be found in [Irw80, Appendix C]

Lemma A.2.15. *Let X be a normed space, $T \in \mathrm{L}(X)$ a linear homeomorphism, $D \subseteq X$ a nonempty set, and $\eta : D \to X$ Lipschitz with a constant L such that $L\|T^{-1}\|_{op} < 1$.*

(a) The map $T+\eta$ is injective.

Suppose that $D = \overline{B}_r(0)$ with $r > 0$ and $\eta(0) = 0$. Further, we set $r' := r(\frac{1-L\|T^{-1}\|_{op}}{\|T^{-1}\|_{op}})$.

(b) The map

$$H : \overline{B}_r(0) \times \overline{B}_{r'}(0) \to \overline{B}_r(0) : (x,y) \mapsto T^{-1}(y - \eta(x))$$

is defined and a contraction in the first argument. For $y \in \overline{B}_{r'}(0)$ and $x \in D$, we have

$$y = (T+\eta)(x) \iff x = H(x,y).$$

Suppose that X is a Banach space.

(c) Then $\operatorname{im}(T+\eta) = \overline{B}_{r'}(0)$, and $(T+\eta)^{-1}$ is Lipschitz with constant $\frac{\|T^{-1}\|_{op}}{1-L\|T^{-1}\|_{op}}$. In particular, $B_{r'}(0) \subseteq (T+\eta)(B_r(0))$.

(d) Additionally, let Y be a normed space, $U \subseteq Y$ an open nonempty set and $k \in \overline{\mathbb{N}}$. Further, let $\Xi \in \mathcal{FC}^k(U \times D, X)$ such that $\Xi(U \times \{0\}) = \{0\}$ and for each $p \in U$, the map $\Xi_p := \Xi(p,\cdot) : D \to X$ is L-Lipschitz. Then

$$U \times B_{r'}(0) \to B_r(0) : (p,y) \mapsto (T+\Xi_p)^{-1}(y) \qquad (\dagger)$$

is $\mathcal{FC}^k$.

Proof. (a) Let $x, y \in D$ such that $(T+\eta)(x) = (T+\eta)(y)$. Then

$$\|x-y\| = \|T^{-1}(\eta(y)-\eta(x))\| \leq \|T^{-1}\|_{op} L \|y-x\|.$$

Since $\|T^{-1}\|_{op} L < 1$, we conclude that $\|x-y\| \leq 0$, and hence $x = y$.

(b) Let $x \in \overline{B}_r(0)$ and $y \in \overline{B}_{r'}(0)$. We calculate

$$\begin{aligned}\|T^{-1}(y-\eta(x))\| \leq \|T^{-1}\|_{op}\|y-\eta(x)\| &\leq \|T^{-1}\|_{op}(\|y\| + \|\eta(x)-\eta(0)\|) \\ &\leq \|T^{-1}\|_{op}(\|y\| + L\|x\|) \leq r\|T^{-1}\|_{op}\left(\tfrac{1-L\|T^{-1}\|_{op}}{\|T^{-1}\|_{op}} + L\right) = r.\end{aligned}$$

Thus H is defined. We show that H is a contraction in the first argument. To this end, let $x, z \in \overline{B}_r(0)$ and $y \in \overline{B}_{r'}(0)$. Then

$$\|H(x,y) - H(z,y)\| = \|T^{-1}(\eta(z) - \eta(x)\| \leq \|T^{-1}\|_{op} L \|z-x\|.$$

Hence the map $H(\cdot, y) : \overline{B}_r(0) \to \overline{B}_r(0)$ is a contraction. The stated characterization is proved by an easy calculation.

(c) Since $\overline{B}_r(0)$ is complete, by the Banach Fixed Point Theorem $H(\cdot, y)$ has a fixed point $g(y)$. We can use [Irw80, Theorem (C.7), p. 241-242] to see that g is continuous, and moreover, we see that g is Lipschitz with a constant not greater then $\frac{\|T^{-1}\|_{op}}{1-L\|T^{-1}\|_{op}}$ since for $y, z \in \overline{B}_{r'}(0)$ and $x \in \overline{B}_r(0)$,

$$\|H(x,y) - H(x,z)\| = \|T^{-1}(y-z)\| \leq \|T^{-1}\|_{op}\|y-z\|.$$

(Notice that the Lipschitz constant of the fixed point map is implicitly calculated in the proof of [Irw80, Theorem (C.7)]). Furthermore, we calculate for $y \in \overline{B}_{r'}(0)$ that

$$g(y) = H(g(y), y) = T^{-1}(y - \eta(g(y))),$$

and hence $y = (T+\eta)(g(y))$. This shows that $(T+\eta)|_{\overline{B}_r(0)}^{-1} = g$ since we proved that $T+\eta$ is injective. To prove the last assertion, let $y \in B_{r'}(0)$. Then

$$\|g(y)\| = \|g(y) - g(0)\| \leq \frac{\|T^{-1}\|_{op}}{1 - L\|T^{-1}\|_{op}} r' < r.$$

Hence $B_{r'}(0) \subseteq (T+\eta)(B_r(0))$.

(d) The map (†) is defined by (c), and we see with (b) that it arises as the restriction of the fixed point map for

$$\widetilde{H} : \overline{B}_r(0) \times \overline{B}_{r'}(0) \times U \to \overline{B}_r(0) : (x, y, p) \mapsto T^{-1}(y - \Xi(p, x)).$$

Hence we derive the assertion from [Irw80, Theorem (C.7)]. □

We use this lemma to prove two theorems on inverse functions that are better suited for citation.

Theorem A.2.16 (Parameterized Lipschitz inverse function theorem)**.** *Let X be a Banach space, Y a normed space, $T \in \mathrm{L}(X)$ invertible, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets, $k \in \overline{\mathbb{N}}$ and $\Xi \in \mathcal{FC}^k(V \times U, X)$ such that for each $p \in V$, the map $\Xi_p := \Xi(p, \cdot) : U \to X$ is L-Lipschitz, where $L\|T^{-1}\|_{op} < 1$. Then for each $p \in V$, $T + \Xi_p$ is a homeomorphism on its image, which is an open subset of X. More precisely, for each $x \in U$ and $r > 0$ such that $B_r(x) \subseteq U$, we have that $B_{r'}((T+\Xi_p)(x)) \subseteq (T+\Xi_p)(B_r(x))$, where $r' := r(\frac{1-L\|T^{-1}\|_{op}}{\|T^{-1}\|_{op}})$. Further, $(T+\Xi_p)^{-1}|_{B_{r'}((T+\Xi_p)(x))}$ is Lipschitz with constant $\frac{\|T^{-1}\|_{op}}{1-L\|T^{-1}\|_{op}}$, and the map*

$$\bigcup_{p \in V} \{p\} \times B_{r'}((T+\Xi_p)(x)) \to B_r(x) : (p, y) \mapsto (T+\Xi_p)^{-1}(y)$$

is $\mathcal{FC}^k$.

Proof. By Lemma A.2.15, for each $p \in V$ the map $T + \Xi_p$ is injective. To prove the other assertions, let $x \in U$ and $r > 0$ such that $B_r(x) \subseteq U$. Since each Ξ_p is uniformly continuous, it can be extended to $\overline{B}_r(0)$; and the extension also is L-Lipschitz. Then we can apply Lemma A.2.15 to T and the map

$$\Xi_p^x : \overline{B}_r(0) \to X : y \mapsto \Xi_p(x+y) - \Xi_p(x) = (\tau_{-\Xi_p(x)} \circ \eta \circ \tau_x)(y).$$

We derive that $T + \Xi_p^x$ is a homeomorphism, $B_{r'}(0) \subseteq (T + \Xi_p^x)(B_r(0))$, and its inverse map is Lipschitz with constant $\frac{\|T^{-1}\|_{op}}{1-L\|T^{-1}\|_{op}}$. Thus using the identity

$$(T + \Xi_p^x)^{-1} = \tau_{-x} \circ (T + \Xi_p)^{-1} \circ \tau_{(\Xi_p + T)(x)},$$

we derive all assertions. ☐

Corollary A.2.17 (Lipschitz inverse function theorem)**.** *Let X be a Banach space, $T \in \mathrm{L}(X)$ invertible, $U \subseteq X$ an open nonempty set, and $\eta : U \to X$ Lipschitz with constant L such that $L\|T^{-1}\|_{op} < 1$. Then $T + \eta$ is a homeomorphism on its image, which is an open subset of X. If η is $\mathcal{FC}^k$, so is $(T+\eta)^{-1}$. More precisely, for each $x \in U$ and $r > 0$ such that $B_r(x) \subseteq U$, we have that $B_{r'}((T+\eta)(x)) \subseteq (T+\eta)(B_r(x))$, where $r' := r(\frac{1-L\|T^{-1}\|_{op}}{\|T^{-1}\|_{op}})$. Further, $(T+\eta)^{-1}|_{B_{r'}((T+\eta)(x))}$ is Lipschitz with constant $\frac{\|T^{-1}\|_{op}}{1-L\|T^{-1}\|_{op}}$.*

Proof. The assertions follow immediately from Theorem A.2.16. ☐

Application to the classical case

We apply the Lipschitz inverse function theorems we derived to a more familiar case, and derive a quantitative version of the classic inverse function theorem, and a parameterized version.

Theorem A.2.18 (Quantitative version of the inverse function theorem)**.** *Let X be a Banach space, $U \subseteq X$ open and convex, $k \in \overline{\mathbb{N}}^*$, $g \in \mathcal{FC}^k(U,X)$ and $x \in U$ such that $Dg(x)$ is invertible. Further, let $\sup_{y\in U}\|Dg(y) - Dg(x)\|_{op} < \delta$ with $\delta < \frac{1}{\|Dg(x)^{-1}\|_{op}}$. Then g is a homeomorphism of U onto an open subset of X, and g^{-1} is $\mathcal{FC}^k$. Further, if U contains the ball $B_r(x')$, then $g(U)$ contains the ball $B_{r'}(g(x'))$, where $r' := \frac{r(1-\delta\cdot\|Dg(x)^{-1}\|_{op})}{\|Dg(x)^{-1}\|_{op}}$. Further, $g^{-1}|_{B_{r'}(g(x'))}$ is $\frac{\|Dg(x)^{-1}\|_{op}}{1-\delta\cdot\|Dg(x)^{-1}\|_{op}}$-Lipschitz.*

Proof. We set $\eta : U \to X : y \mapsto g(y) - Dg(x)\cdot y$. Then η is Lipschitz with constant δ since

$$\eta(y)-\eta(z) = g(y)-g(z)-Dg(x)\cdot(y-z) = \int_0^1 Dg(ty+(1-t)z)\cdot(y-z)\,dt - Dg(x)\cdot(y-z)$$

and hence

$$\|\eta(y) - \eta(z)\| \leq \delta\|y - z\|.$$

So we derive the assertion from Corollary A.2.17, applied to $Dg(x)$ and η. □

Proposition A.2.19 (Parameterized quantitative version of the inverse function theorem)**.** *Let X be a normed space, Y a Banach space, $U \subseteq X$ open and $V \subseteq Y$ open and convex, $k \in \overline{\mathbb{N}}^*$, $g \in \mathcal{FC}^k(U \times V, Y)$ and $(x_0, y_0) \in U \times V$ such that $D_2g(x_0, y_0)$ is invertible. Further, let*

$$\sup_{(x,y)\in U\times V} \|D_2g(x, y) - D_2g(x_0, y_0)\|_{op} < \delta$$

with $\delta < \frac{1}{\|D_2g(x_0,y_0)^{-1}\|_{op}}$. Then for each $x \in U$, $g_x := g(x,\cdot) : V \to Y$ is a homeomorphism onto an open subset of Y. Further,

$$B_Y(y, r) \subseteq V \implies B_{r'}(g_x(y)) \subseteq g_x(B_Y(y, r)),$$

where $r' := \frac{r(1-\delta\cdot\|D_2g(x_0,y_0)^{-1}\|_{op})}{\|D_2g(x_0,y_0)^{-1}\|_{op}}$. Further, $g_x^{-1}|_{B_{r'}(g(x,y))}$ is $\frac{\|D_2g(x_0,y_0)^{-1}\|_{op}}{1-\delta\cdot\|D_2g(x_0,y_0)^{-1}\|_{op}}$-Lipschitz, and the map

$$\bigcup_{x\in U} \{x\} \times B_{r'}(g(x, y)) \to B_Y(y, r) : (x, z) \mapsto g_x^{-1}(z)$$

is $\mathcal{FC}^k$.

Proof. We define

$$\eta : U \times V \to Y : (x, y) \mapsto g(x, y) - D_2g(x_0, y_0) \cdot y.$$

Then for each $x \in U$, the map $\eta_x := \eta(x,\cdot)$ is Lipschitz with constant δ since

$$\begin{aligned}\eta_x(y) - \eta_x(z) &= g(x, y) - g(x, z) - D_2g(x_0, y_0) \cdot (y - z)\\ &= \int_0^1 D_2g(x, ty + (1 - t)z) \cdot (y - z)\,dt - D_2g(x_0, y_0) \cdot (y - z)\end{aligned}$$

and hence

$$\|\eta_x(y) - \eta_x(z)\| \leq \delta\|y - z\|.$$

So we derive the assertion from Theorem A.2.16, applied to $D_2g(x_0, y_0)$ and η. □

A.3. Relation between the differential calculi

We show that the two calculi presented are closely related. First we prove that each $\mathcal{FC}^k$-map is a $\mathcal{C}^k$-map and that the higher differentials are in a close relation.

Lemma A.3.1. *Let $k \in \mathbb{N}^*$ and $\gamma \in \mathcal{FC}^k(U, Y)$. Then γ is a $\mathcal{C}^k$-map (in the sense of Section A.1), and for each $x \in U$ we have*

$$D^{(k)}\gamma(x) = d^{(k)}\gamma(x; \cdot).$$

Proof. We prove this by induction.
$k = 1$: It follows directly from Definition 2.3.1 that γ is a $\mathcal{C}^1$ map and that the identity

$$D^{(1)}\gamma(x) = D\gamma(x) = d\gamma(x; \cdot) = d^{(1)}\gamma(x; \cdot)$$

holds.

$k \to k + 1$: Let $x \in U$ and $h_1, \ldots, h_{k+1} \in X$. We know from Lemma A.2.14 that

$$\begin{aligned}
&(D^{(k+1)}\gamma)(x)(h_1, \ldots, h_{k+1})\\
=&(\mathcal{E}_{k,1} \circ (D^{(k)}D\gamma))(x)(h_1, \ldots, h_{k+1})\\
=&(D^{(k)}D\gamma(x)(h_2, \ldots, h_{k+1})) \cdot h_1.
\end{aligned}$$

The inductive hypothesis gives

$$\begin{aligned}
=&(d^{(k)}D\gamma(x; h_2, \ldots, h_{k+1})) \cdot h_1\\
=&\left(\lim_{t\to 0} \frac{d^{(k-1)}(D\gamma)(x + th_{k+1}; h_2, \ldots, h_k) - d^{(k-1)}(D\gamma)(x; h_2, \ldots, h_k)}{t}\right) \cdot h_1.
\end{aligned}$$

Another application of the inductive hypothesis, together with the continuity of the evaluation of linear maps (Lemma A.2.3) and Lemma A.2.14, gives

$$\begin{aligned}
=&\lim_{t\to 0} \frac{D^{(k-1)}(D\gamma)(x + th_{k+1})(h_2, \ldots, h_k) \cdot h_1 - D^{(k-1)}(D\gamma)(x)(h_2, \ldots, h_k) \cdot h_1}{t}\\
=&\lim_{t\to 0} \frac{(\mathcal{E}_{k-1,1} \circ D^{(k-1)}(D\gamma))(x + th_{k+1})(h_1, \ldots, h_k) - (\mathcal{E}_{k-1,1} \circ D^{(k-1)}(D\gamma))(x)(h_1, \ldots, h_k)}{t}\\
=&\lim_{t\to 0} \frac{D^{(k)}\gamma(x + th_{k+1})(h_1, \ldots, h_k) - D^{(k)}\gamma(x)(h_1, \ldots, h_k)}{t}.
\end{aligned}$$

Another application of the inductive hypothesis finally gives

$$=\lim_{t\to 0} \frac{d^{(k)}\gamma(x + th_{k+1}; h_1, \ldots, h_k) - d^{(k)}\gamma(x; h_1, \ldots, h_k)}{t}.$$

Hence $d^{(k+1)}\gamma$ exists and satisfies the identity

$$d^{(k+1)}\gamma(x; h_1, \ldots, h_{k+1}) = D^{(k+1)}\gamma(x)(h_1, \ldots, h_{k+1}).$$

Since $D^{(k+1)}\gamma$ and the evalution of multilinear maps are continuous (see Lemma A.2.3), $d^{(k+1)}\gamma$ is so. In Proposition 2.2.3 we stated that this (and the inductive hypothesis) assure that γ is a $\mathcal{C}^{k+1}$ map. $\square$

The preceding can be used to give a characterization of Fréchet differentiable maps.

Proposition A.3.2. *Let $\gamma : U \to Y$ be a continuous map. Then $\gamma \in \mathcal{FC}^k(U,Y)$ iff γ is a $\mathcal{C}^k$-map and the map*

$$U \to \mathrm{L}^\ell(X,Y) : x \mapsto d^{(\ell)}\gamma(x;\cdot) \qquad (*_k)$$

is continuous for each $\ell \in \mathbb{N}$ with $\ell \leq k$.

Proof. For $\gamma \in \mathcal{FC}^k(U,Y)$ we stated in Lemma A.3.1 that $\gamma \in \mathcal{C}^k(U,Y)$ and

$$d^{(\ell)}\gamma(x;\cdot) = D^{(\ell)}\gamma(x)$$

for each $x \in U$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Since $D^{(\ell)}\gamma$ is continuous by its definition (A.2.13), $(*_k)$ is satisfied.

We have to prove the other direction. This is done by induction on k:
$k = 1$: This follows directly from the definition of $\mathcal{FC}^1(U,Y)$.
$k \to k+1$: We have to show that $\gamma \in \mathcal{FC}^{k+1}(U,Y)$, and this is clearly the case if $D\gamma \in \mathcal{FC}^k(U,\mathrm{L}(X,Y))$. By the inductive hypothesis this is the case if $D\gamma \in \mathcal{C}^k(U,\mathrm{L}(X,Y))$ and it satisfies $(*_k)$. Since $\gamma \in \mathcal{FC}^k(U,Y)$ by the inductive hypothesis and hence $D\gamma \in \mathcal{FC}^{k-1}(U,\mathrm{L}(X,Y))$, we just have to show that $D\gamma$ is $\mathcal{C}^k$ and

$$U \to \mathrm{L}^k(X,\mathrm{L}(X,Y)) : x \mapsto d^{(k)}(D\gamma)(x;\cdot)$$

is continuous. To this end, let $x \in U$, $h, v_1, \ldots, v_{k-1}, v_k \in X$ and $t \in \mathbb{K}$ such that the line segment $\{x + stv_k : s \in [0,1]\} \subseteq U$. We calculate using Lemma A.2.14, the mean value theorem and two applications of Lemma A.3.1:

$$\begin{aligned}&\left(\frac{d^{(k-1)}(D\gamma)(x+tv_k;v_1,\ldots,v_{k-1}) - d^{(k-1)}(D\gamma)(x;v_1,\ldots,v_{k-1})}{t}\right)\cdot h\\ =&\frac{d^{(k)}\gamma(x+tv_k;h,v_1,\ldots,v_{k-1}) - d^{(k)}\gamma(x;h,v_1,\ldots,v_{k-1})}{t}\\ =&\int_0^1 d^{(k+1)}\gamma(x+stv_k;h,v_1,\ldots,v_{k-1},v_k)\,ds.\end{aligned}$$

Since $x \mapsto d^{(k+1)}\gamma(x;\cdot)$ is continuous by hypothesis, the left hand side of this identity converges for $t \to 0$ with respect to the topology of uniform convergence on bounded sets to the linear map

$$h \mapsto d^{(k+1)}\gamma(x;h,v_1,\ldots,v_{k-1},v_k).$$

Hence $D\gamma$ is $\mathcal{C}^k$ with

$$d^{(k)}(D\gamma)(x;v_1,\ldots,v_{k-1},v_k) = \mathcal{E}_{k,1}^{-1}(d^{(k+1)}\gamma(x;\cdot))(v_1,\ldots,v_{k-1},v_k),$$

and since $x \mapsto d^{(k+1)}\gamma(x;\cdot)$ and $\mathcal{E}_{k,1}^{-1}$ are continuous (by hypothesis resp. Lemma A.2.5), $x \mapsto d^{(k)}(D\gamma)(x;\cdot)$ is so, too. □

We show that a $\mathcal{C}^{k+1}$ map is $\mathcal{FC}^k(U,Y)$.

Lemma A.3.3. *Let $f : U \to Y$ be a $\mathcal{C}^{k+1}$ map. Then $f \in \mathcal{FC}^k(U, Y)$.*

Proof. We stated in Proposition A.3.2 that f is in $\mathcal{FC}^k(U, Y)$ iff for each $\ell \in \mathbb{N}$ with $\ell \leq k$ the map

$$U \to \mathrm{L}^\ell(X, Y) : x \mapsto d^{(\ell)} f(x; \cdot)$$

is continuous; but this is a direct consequence of Lemma A.1.32 since it implies that estimate (A.1.31.1) holds. □

Differential calculus on finite-dimensional spaces We show that the three definitions of differentiability for maps that are defined on a finite-dimensional space (Fréchet-differentiability, Kellers C^k_c theory and continuous partial differentiability) are equivalent.

Definition A.3.4. Let $n, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}_0^n$ a multiindex with $|\alpha| = k$. We set

$$I_\alpha := \{(i_1, \dots, i_k) \in \{1, \dots, n\}^k : (\forall \ell \in \{1, \dots, n\})\ \alpha_\ell = |\{j : i_j = \ell\}|\}$$

and use this set to define the continuous k-linear map

$$S_\alpha : (\mathbb{K}^n)^k \to \mathbb{K} : (h_1, \dots, h_k) \mapsto \sum_{(i_1,\dots,i_k) \in I_\alpha} h_{1,i_1} \cdots h_{k,i_k},$$

where $h_j = (h_{j,1}, \dots, h_{j,n})$ for $j = 1, \dots, k$.

Proposition A.3.5. *Let $U \subseteq \mathbb{K}^n$ be open and nonempty and $\gamma : U \to Y$ a map. Then the following conditions are equivalent:*

(a) $\gamma \in \mathcal{FC}^k(U, Y)$

(b) $\gamma \in \mathcal{C}^k(U, Y)$

(c) γ *is k-times continuously partially differentiable.*

If one of these conditions is satisfied, then

$$D^{(k)}\gamma(x)(h_1, \dots, h_k) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = k}} S_\alpha(h_1, \dots, h_k) \cdot \partial^\alpha \gamma(x) \tag{A.3.5.1}$$

for all $x \in U$ and $h_1, \dots, h_k \in \mathbb{K}^n$.

Proof. The assertion $(a) \implies (b)$ is a consequence of Lemma A.3.1; and since

$$\frac{\partial^k \gamma}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) = d^{(k)}\gamma(x; e_{i_k}, \dots, e_{i_1})$$

and $d^{(k)}\gamma$ is continuous (Proposition 2.2.3), the implication $(b) \implies (c)$ also holds.

It remains to show that $(c) \implies (a)$. It is well known from calculus that $D_h\gamma = \sum_{i=1}^n h_i \frac{\partial\gamma}{\partial x_i}$. Hence $d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell)$ exists and is given by

$$\begin{aligned}
d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell) &= \sum_{i_1=1,\ldots,i_\ell=1}^{n} h_{1,i_1} \cdots h_{\ell,i_\ell} \cdot \frac{\partial^k \gamma}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \\
&= \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=\ell}} \Big(\sum_{(i_1,\ldots,i_\ell)\in I_\alpha} h_{1,i_1} \cdots h_{\ell,i_\ell} \Big) \cdot \partial^\alpha \gamma(x) \\
&= \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=\ell}} S_\alpha(h_1, \ldots, h_\ell) \cdot \partial^\alpha \gamma(x).
\end{aligned}$$

From this identity we derive the continuity of $x \mapsto d^{(\ell)}\gamma(x; \cdot)$, and can conclude using Proposition A.3.2 that $\gamma \in \mathcal{FC}^k(U, Y)$ and (A.3.5.1) is satisfied. □

A.4. Some facts concerning ordinary differential equations

We state some facts about the global solvabilty of initial value problems and the dependence of solution on parameters.

A.4.1. Maximal solutions of ODEs

In the following, we let $J \subseteq \mathbb{R}$ be a nondegenerate interval and U an open subset of the Banach space X. For a continuous function $f : J \times U \to X$, $x_0 \in U$ and $t_0 \in J$ we consider the initial value problem

$$\begin{aligned} \gamma'(t) &= f(t, \gamma(t)) \\ \gamma(t_0) &= x_0. \end{aligned} \tag{A.4.0.2}$$

We state the famous theorem of Picard and Lindelöf:

Theorem A.4.1. *Let f satisfy a local Lipschitz condition with respect to the second argument, that is, for each $(t_0, x_0) \in J \times U$ there exist a neighborhood W of (t_0, x_0) in $J \times U$ and an $K \in \mathbb{R}$ such that for all $(t, x), (t, \tilde{x}) \in W$*

$$\|f(t, x) - f(t, \tilde{x})\| \leq K\|x - \tilde{x}\|.$$

Then, for each $(t_0, x_0) \in J \times U$ there exists a neighborhood I of t_0 in J such that the initial value problem (A.4.0.2) *corresponding to t_0 and x_0 has a unique solution that is defined on I.*

It is well-known that the local theorem of Picard and Lindelöf can be used to ensure that there exists a maximal solution.

Proposition A.4.2. *Let f satisfy a local Lipschitz condition with respect to the second argument and let $(t_0, x_0) \in J \times U$. Then there exists an interval $I \subseteq J$ and a function $\phi : I \to U$ that is a maximal solution to* (A.4.0.2)*; that is, if $\gamma : D(\gamma) \to U$ is a solution to* (A.4.0.2) *defined an a connected set, $D(\gamma) \subseteq I$ and $\gamma = \phi|_{D(\gamma)}$.*

A criterion on global solvabilty

Linearly bounded vector fields One class of ODEs that can be globally solved is that of linear vector fields. This solvabilty property can be generalized to linearly bounded vector fields.

Definition A.4.3. We call f *linearly bounded* if there exist continuous functions $a, b : J \to \mathbb{R}$ such that

$$\|f(t,x)\| \leq a(t)\|x\| + b(t)$$

for all $(t,x) \in J \times U$.

To prove that this condition on f ensures globally defined solutions, we first need to prove some lemmas.

Lemma A.4.4. *Let f be a linearly bounded map that satisfies a local Lipschitz condition with respect to the second argument. Let $\phi : I \to U$ be an integral curve of f. Then the following assertions hold:*

(a) If ϕ is bounded, $\overline{I} \subseteq J$ and $\overline{I}$ is compact, then f is bounded on the graph of ϕ.

(b) If $\beta := \sup I \neq \sup J$, then ϕ is bounded on $[t_0, \beta[$ for each $t_0 \in J$. The analogous result for $\inf I$ *also holds.*

Proof. (a) Let $t \in I$. Then

$$\|f(t,\phi(t))\| \leq a(t)\|\phi(t)\| + b(t)$$

since f is linearly bounded. Because a and b are continuous and defined on $\overline{I}$, they are clearly bounded on I.

(b) For each $t \in [t_0, \beta[$ we have

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(s,\phi(s))\, ds,$$

and from this we deduce using that f is linearly bounded:

$$\begin{aligned}
\|\phi(t)\| &\leq \|\phi(t_0)\| + \left\| \int_{t_0}^{t} f(s,\phi(s))\, ds \right\| \\
&\leq \|\phi(t_0)\| + \left| \int_{t_0}^{t} a(s)\|\phi(s)\| + b(s)\, ds \right| \\
&\leq \|a|_{[t_0,\beta]}\|_\infty \left| \int_{t_0}^{t} \|\phi(s)\|\, ds \right| + \|\phi(t_0)\| + \|b\|_{\infty,[t_0,\beta]}|\beta - t_0|.
\end{aligned}$$

The assertion is proved with an application of Groenwall's lemma. □

Lemma A.4.5. *Assume that f satisfies a global Lipschitz condition with respect to the second argument. Then f is linearly bounded.*

Proof. Let $(t,x) \in J \times U$ and $x_0 \in U$. Then

$$\begin{aligned}
\|f(t,x)\| &\leq \|f(t,x) - f(t,x_0)\| + \|f(t,x_0)\| \\
&\leq L\|x - x_0\| + \|f(t,x_0)\| \leq L\|x\| + L\|x_0\| + \|f(t,x_0)\|.
\end{aligned}$$

Defining $a(t) := L$ and $b(t) := L\|x_0\| + \|f(t,x_0)\|$ gives the assertion. □

The criterion We give a sufficent condition on when an integral curve is uniformly continuous. This can be used to extend solutions to larger domains of definition.

Lemma A.4.6. *Let f satisfy a local Lipschitz condition with respect to the second argument and let $\phi : I \to U$ be an integral curve of f such that f is bounded on the graph of ϕ. Then ϕ is Lipschitz continuous and hence uniformly continuous.*

Proof. Let $t_1, t_2 \in I$. Then

$$\|\phi(t_2) - \phi(t_1)\| = \left\| \int_{t_1}^{t_2} \phi'(s)\, ds \right\| = \left\| \int_{t_1}^{t_2} f(s, \phi(s))\, ds \right\| \leq K|t_2 - t_1|,$$

where $K := \sup_{s \in I} \|f(s, \phi(s))\| < \infty$. □

Theorem A.4.7. *Assume that f satisfies a local Lipschitz condition with respect to the second argument. Let $\phi : I \to U$ be a maximal integral curve of f. Assume further that*

(a) The image of ϕ is contained in a compact subset of U or

(b) f is linearly bounded.

Then ϕ is a global solution, that is $I = J$.

Proof. We prove this by contradiction. To this end, we may assume w.l.o.g. that $\beta := \sup I \neq \sup J$. We choose $t_0 \in I$. In both cases, f is bounded on the graph of $\phi|_{[t_0, \beta[}$: If the image of ϕ is contained in a compact set, we easily see that the graph of $\phi|_{[t_0, \beta[}$ is contained in a compact subset. If f is linearly bounded, we use Lemma A.4.4.

We apply Lemma A.4.6 to see that $\phi|_{[t_0, \beta[}$ is uniformly continuous, and thus has a continuous extension $\widetilde{\phi}$ to $[t_0, \beta]$. We easily calculate that $\widetilde{\phi}$ is a solution to (A.4.0.2) using the integral represention of an ODE. Since $\widetilde{\phi}$ extends ϕ, we get a contradiction to the maximality of ϕ. □

A.4.2. Flows and dependence on parameters and initial values

For the purpose of full generality, we need a definition.

Definition A.4.8. Let X be a locally convex space. We call $P \subseteq X$ a *locally convex subset with dense interior* if for each $x \in P$, there exists a convex neighborhood $U \subseteq P$ of x and if $P \subseteq \overline{P^\circ}$.

In the following, we let $J \subseteq \mathbb{R}$ be a nondegenerate interval, U an open subset of the Banach space X, P be a locally convex subset with dense interior of a locally convex space and $k \in \overline{\mathbb{N}}$ with $k \geq 1$. Further, let f be in $\mathcal{C}^k(J \times U \times P, X)$. We consider the initial value problem

$$\begin{aligned} \gamma'(t) &= f(t, \gamma(t), p) \\ \gamma(t_0) &= x_0 \end{aligned} \tag{A.4.8.1}$$

for $t_0 \in J$, $x_0 \in U$ and $p \in P$.

Definition A.4.9. Let $\Omega \subseteq J \times J \times U \times P$. We call a map

$$\phi : \Omega \to U$$

a *flow* for f if for all $t_0 \in J$, $x_0 \in U$ and $p \in P$ the set

$$\Omega_{t_0,x_0,p} := \{t \in J : (t_0, t, x_0, p) \in \Omega\}$$

is connected and the partial map

$$\phi(t_0, \cdot, x_0, p) : \Omega_{t_0,x_0,p} \to U$$

is a solution to (A.4.8.1) corresponding to the initial values t_0, x_0 and p.

A flow is called *maximal* if each other flow is a restriction of it.

Remark A.4.10. In [Glö06, Theorem 10.3] it was stated that for each $t_0 \in J$, $x_0 \in U$ and $p_0 \in P$ there exist neighborhoods J_0 of t_0, U_0 of x_0 and P_0 of p_0 such that for every $s \in J_0$, $x \in U_0$ and $p \in P_0$ the corresponding initial value problem (A.4.8.1) has a unique solution $\Gamma_{s,x,p} : J_0 \to U$ and the map

$$\Gamma : J_0 \times J_0 \times U_0 \times P_0 \to U : (s, t, x, p) \mapsto \Gamma_{s,x,p}(t)$$

is $\mathcal{C}^k$. Therefore $\mathcal{C}^k$-flows exist.

The following lemma shows that two related flows can be glued together:

Lemma A.4.11. *Let $I \subseteq J$ be a connected set with nonempty interior and $\gamma : I \to U$ a solution to* (A.4.8.1) *corresponding to $t_\gamma \in J$, $x_\gamma \in U$ and $p_\gamma \in P$. Further let*

$$\phi_0 : J_0 \times I_0 \times U_0 \times P_0 \to U \text{ and } \phi_1 : I_1 \times I_1 \times U_1 \times P_1 \to U$$

be $\mathcal{C}^k$-flows for f such that U_1 is open in X and

$$I = I_0 \cup I_1,\ I_0 \cap I_1 \neq \emptyset,\ p_\gamma \in P_0 \cap P_1,\ (t_\gamma, x_\gamma) \in J_0 \times U_0 \text{ and } \gamma(I_1) \subseteq U_1.$$

Then there exist neighborhoods J_γ of t_γ, U_γ of x_γ, P_γ of p_γ and a $\mathcal{C}^k$-flow

$$\phi : J_\gamma \times I \times U_\gamma \times P_\gamma \to U$$

for f.

Proof. We choose $t_1 \in I_0 \cap I_1$. Since ϕ_0 is continuous in $(t_\gamma, t_1, x_\gamma, p_\gamma)$ and

$$\phi_0(t_\gamma, t_1, x_\gamma, p_\gamma) = \gamma(t_1) \in U_1,$$

there exist neighborhoods J_γ of t_γ in J_0, U_γ of x_γ in U_0 and $P_\gamma \subseteq P_0 \cap P_1$ of p_γ such that

$$\phi_0(J_\gamma \times \{t_1\} \times U_\gamma \times P_\gamma) \subseteq U_1.$$

Then the map

$$\phi : J_\gamma \times I \times U_\gamma \times P_\gamma \to U : (t_0, x_0, p, t) \mapsto \begin{cases} \phi_0(t_0, t, x_0, p) & \text{if } t \in I_0 \\ \phi_1(t_1, t, \phi_0(t_0, t_1, x_0, p), p) & \text{if } t \in I_1 \end{cases}$$

is well defined since the curves $\phi_0(t_0, \cdot, x_0, p)$ and $\phi_1(t_1, \cdot, \phi_0(t_0, t_1, x_0, p), p)$ are both solutions to the ODE (A.4.8.1) that coincide in t_1 and hence on $I_0 \cap I_1$. Since both ϕ_0 and ϕ_1 are $\mathcal{C}^k$-flows for f, so is ϕ. □

Lemma A.4.12. *Let $I \subseteq J$ be a connected set with nonempty interior, $t_1 \in I$ and $\gamma : I \to U$ a solution to* (A.4.8.1) *corresponding to $t_\gamma \in J$, $x_\gamma \in U$ and $p_\gamma \in P$. Then there exist neighborhoods J_γ of t_γ, U_γ of x_γ, P_γ of p_γ, an interval $\widetilde{I} \subseteq I$ with $t_\gamma, t_1 \in \widetilde{I}$ such that $\widetilde{I}$ is a neighborhood of t_1 in I, and a $\mathcal{C}^k$-flow*

$$\phi : J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \to U$$

for f.

Proof. We use [Glö06, Theorem 10.3] to see that for each $s \in I$ there exist neighborhoods J_s of s in J, U_s of $\gamma(s)$ in U, P_s of p_0 in P and a $\mathcal{C}^k$-flow

$$\phi_s : J_s \times J_s \times U_s \times P_s \to U$$

for f; we may assume w.l.o.g. that $\gamma(J_s) \subseteq U_s$ since γ is continuous and that J_s is open in I. Since I is connected and $\{J_s\}_{s\in I}$ is an open cover of I, there exist finitely many sets $J_{s_1}, \dots, J_{s_n}$ such that $t_\gamma \in J_{s_1}$, $t_1 \in J_{s_n}$ and $J_{s_m} \cap J_{s_\ell} \neq \emptyset \iff |m - \ell| \leq 1$. Applying Lemma A.4.11 to ϕ_{s_1} and ϕ_{s_2} we find neighborhoods I_1 of t_γ, V_1 of x_γ, P_1 of p_γ and a $\mathcal{C}^k$-flow

$$\phi_1 : I_1 \times (J_{s_1} \cup J_{s_2}) \times V_1 \times P_1 \to U$$

for f. Likewise, ϕ_1 and ϕ_{s_3} lead to ϕ_2, and iterating the argument, we find a $\mathcal{C}^k$-flow

$$\phi_{n-1} : I_{n-1} \times \bigcup_{k=1}^{n} J_{s_k} \times V_{n-1} \times P_{n-1} \to U$$

for f. □

Concerning maximal flows, we can state the following

Theorem A.4.13. *For each ODE* (A.4.8.1) *there exists a maximal flow*

$$\phi : J \times J \times U \times P \supseteq \Omega \to U.$$

Ω is an open subset of $J \times J \times U \times P$ and ϕ is a $\mathcal{C}^k$-map.

Proof. The existence of a maximal flow is a direct consequence of the existence of maximal solutions to ODEs without parameters, see Proposition A.4.2. Now let $(t_0, t, x_0, p) \in \Omega$ and $\gamma : I \subseteq J \to U$ the maximal solution corresponding to t_0, x_0 and p. Then $t_0, t \in I$, and according to Lemma A.4.12, there exists a $\mathcal{C}^k$-flow

$$\Gamma : J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \to U$$

for f that is defined on a neighborhood of (t_0, t, x_0, p). Since ϕ is maximal,

$$J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \subseteq \Omega$$

and

$$\phi|_{J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma} = \Gamma.$$

This gives the assertion. □

We examine the situation that an initial time is fixed and the initial values depend on the parameters.

Corollary A.4.14. *Let $\alpha : P \to U$ be a $\mathcal{C}^k$-map. Further, let $I \subseteq J$ be a nonempty interval and $t_0 \in I$ such that for every $p \in P$ there exists a solution*

$$\gamma_p : I \to U$$

to the initial value problem (A.4.8.1) *corresponding to p, t_0 and the initial value $\alpha(p)$. Then the map*

$$\Gamma : I \times P \to U : (t, p) \mapsto \gamma_p(t)$$

is $\mathcal{C}^k$.

Proof. We consider a maximal flow $\phi : \Omega \to U$ for f. Since ϕ is maximal,

$$\{t_0\} \times I \times \{(\alpha(p), p) : p \in P\} \subseteq \Omega,$$

and for each $p \in P$

$$\phi(t_0, \cdot, \alpha(p), p) = \gamma_p.$$

Hence Γ is the composition of ϕ and the $\mathcal{C}^k$-map

$$I \times P \to J \times I \times U \times P : (t, p) \mapsto (t_0, t, \alpha(p), p),$$

and this gives the assertion. □

B. Locally convex Lie groups and manifolds

The goal of this appendix mainly is to fix our conventions and notation concerning manifolds and Lie groups modelled on locally convex spaces. For further information see the articles [Mil84], [Nee06] and [BGN04].

B.1. Locally convex manifolds

Locally convex manifolds are essentially like finite-dimensional ones, replacing the finite-dimensional modelling space by a locally convex space.

Definition B.1.1 (Locally convex manifolds)**.** Let M be a Hausdorff topological space, $k \in \overline{\mathbb{N}}$ and X a locally convex space. A $\mathcal{C}^k$*-atlas* for M is a set $\mathcal{A}$ of homeomorphisms $\phi : U \to V$ from an open subset $U \subseteq M$ onto an open set $V \subseteq X$ whose domains cover M and which are $\mathcal{C}^k$-compatible in the sense that $\phi \circ \psi^{-1}$ is $\mathcal{C}^k$ for all $\phi, \psi \in \mathcal{A}$. A maximal $\mathcal{C}^k$-atlas $\mathcal{A}$ on M is called a *differentiable structure* of class $\mathcal{C}^k$. In this case, the pair $(M, \mathcal{A})$ is called (locally convex) $\mathcal{C}^k$-manifold modelled on X.

Direct products of locally convex $\mathcal{C}^k$-manifolds are defined as expected.

Definition B.1.2 (Tangent space and tangent bundle)**.** Let $(M, \mathcal{A})$ be a $\mathcal{C}^k$-manifold modelled on X, where $k \geq 1$. Given $x \in M$, let $\mathcal{A}_x$ be the set of all charts around x (i.e. whose domain contains x). A *tangent vector* of M at x is a family $y = (y_\phi)_{\phi \in \mathcal{A}_x}$ of vectors $y_\phi \in X$ such that $y_\psi = d(\psi \circ \phi^{-1})(\phi(x); y_\phi)$ for all $\phi, \psi \in \mathcal{A}_x$.

The *tangent space* of M at x is the set $\mathbf{T}_x M$ of all tangent vectors of M at x. It has a unique structure of locally convex space such that the map $d\psi|_{\mathbf{T}_x M} : \mathbf{T}_x M \to X : (y_\phi)_{\phi \in \mathcal{A}_x} \mapsto y_\psi$ is an isomorphism of topological vector spaces for any $\psi \in \mathcal{A}_x$.

The *tangent bundle* $\mathbf{T}M$ of M is the union of the (disjoint) tangent spaces $\mathbf{T}_x M$ for all $x \in M$. It admits a unique structure as a $\mathcal{C}^{k-1}$-manifold modelled on $X \times X$ such that $\mathbf{T}\phi := (\phi, d\phi)$ is chart for each $\phi \in \mathcal{A}$. We let $\pi_M : \mathbf{T}M \to M$ be the map taking tangent vectors at x to x for any $x \in M$.

Definition B.1.3. A continuous map $f : M \to N$ between $\mathcal{C}^k$-manifolds is called $\mathcal{C}^k$ if the map $\psi \circ f \circ \phi^{-1}$ is so for all charts ψ of N and ϕ of M.

If $k \geq 1$, then we define the *tangent map* of f as the $\mathcal{C}^{k-1}$-map $\mathbf{T}f : \mathbf{T}M \to \mathbf{T}N$ determined by $d\psi \circ \mathbf{T}f \circ (\mathbf{T}\phi)^{-1} = d(\psi \circ f \circ \phi^{-1})$ for all charts ψ of N and ϕ of M.

Given $x \in M$, we define $\mathbf{T}_x f := \mathbf{T}f|_[\mathbf{T}_{f(x)} N]\mathbf{T}_x M : \mathbf{T}_x M \to \mathbf{T}_{f(x)} N$.

Definition B.1.4. Let $k > 0$, M, N and P be $\mathcal{C}^k$-manifolds, and $f : M \times N \to P$ a $\mathcal{C}^k$-map. We define

$$\mathbf{T}_1 f : \mathbf{T}M \times N \to \mathbf{T}P : (v, n) \mapsto \mathbf{T}\Gamma(v, 0_n)$$

and

$$\mathbf{T}_2 f : M \times \mathbf{T}N \to \mathbf{T}P : (m, v) \mapsto \mathbf{T}\Gamma(0_m, v).$$

Definition B.1.5 (Submanifolds)**.** Let M be a $\mathcal{C}^k$-manifold modelled on the locally convex space X and $Y \subseteq X$ be a sequentially closed vector subspace. A *submanifold of M modelled on Y* is a subset $N \subseteq M$ such that for each $x \in N$, there exists a chart $\phi : U \to V$ around x such that $\phi(U \cap N) = V \cap Y$. It is easy to see that a submanifold is also a $\mathcal{C}^k$-manifold.

The following lemma states that submanifolds are initial:

Lemma B.1.6. *Let M be a $\mathcal{C}^k$-manifold and N a submanifold of M. Then the inclusion $\iota : N \to M$ is $\mathcal{C}^k$. Moreover, a map $f : P \to N$ from a $\mathcal{C}^k$-manifold is $\mathcal{C}^k$ iff the map $\iota \circ f : P \to M$ is so.*

Definition B.1.7 (Vector fields)**.** A *vector field* on a manifold M is a map $\xi : M \to \mathbf{T}M$ such that $\pi_M \circ \xi = \mathrm{id}_M$. We denote the set of $\mathcal{C}^k$ vector fields on M by and set $\mathfrak{X}^\infty(M) := \mathfrak{X}(M)$.

A vector field ξ is determined by its local representations $\xi_\phi := d\phi \circ \xi \circ \phi^{-1} : V \to X$ for each chart $\phi : U \to V$ of M. Given vector fields ξ and η on M, there is a unique vector field $[\xi, \eta]$ on M such that $[\xi, \eta]_\phi = d\eta_\phi \circ (\mathrm{id}_V, \xi_\phi) - d\xi_\phi \circ (\mathrm{id}_V, \eta_\phi)$ for all charts $\phi : U \to V$ of M.

Remark B.1.8 (Analytic manifolds)*.* The definition of analytic manifolds and analytic maps between them is literally the same as above, except that the term $\mathcal{C}^k$-map has to be replaced by analytic map.

B.2. Lie groups

Definition B.2.1 (Lie groups)**.** A (locally convex) *Lie group* is a group G equipped with a smooth manifold structure turning the group operations into smooth maps.

An analytic Lie group is a group G equipped with an analytic manifold structure turning the group operations into analytic maps.

Lemma B.2.2 (Tangent group, action of group on TG)**.** *Let G be a Lie group with the group multiplication m and the inversion i. Then $\mathbf{T}G$ is a Lie group with the group multiplication*

$$\mathbf{T}m : \mathbf{T}(G \times G) \cong \mathbf{T}G \times \mathbf{T}G \to \mathbf{T}G$$

and the inversion $\mathbf{T}i$. Identifying G with the zero section of $\mathbf{T}G$, we obtain a smooth right action

$$\mathbf{T}G \times G \to \mathbf{T}G : (v, g) \mapsto v.g := \mathbf{T}m(v, 0_g)$$

and a smooth left action

$$G \times \mathbf{T}G \to \mathbf{T}G : (g, v) \mapsto g.v := \mathbf{T}m(0_g, v).$$

Definition B.2.3 (Left invariant vector fields)**.** A vector field V on a Lie group G is called *left invariant* if $g.V(h) = V(gh)$ for all $g, h \in G$. The set $\mathfrak{X}(G)_\ell$ of left invariant vector fields is a Lie algebra under the bracket of vector fields defined above.

Definition B.2.4 (Lie algebra functor)**.** Let G and H be Lie groups. Using the isomorphism $\mathfrak{X}(G)_\ell \to \mathbf{T_1}G : V \mapsto V(\mathbf{1})$ we transport the Lie algebra structure on $\mathfrak{X}(G)_\ell$ to $\mathbf{L}(G) := \mathbf{T_1}G$. If $\phi : G \to H$ is a smooth homomorphism, then the map $\mathbf{L}(\phi) : \mathbf{L}(G) \to \mathbf{L}(H)$ defined as $\mathbf{T}\phi|_{\mathbf{L}(G)}$ is a Lie algebra homomorphism.

B.2.1. Generation of Lie groups

We need the following result concerning the construction of Lie groups from local data (compare [Bou89, Chapter III, §1.9, Proposition 18] for the case of Banach Lie groups; the general proof follows the same pattern).

Lemma B.2.5 (Local description of Lie groups)**.** *Let G be a group, $U \subseteq G$ a subset which is equipped with a smooth manifold structure, and $V \subseteq U$ an open symmetric subset such that $\mathbf{1} \in V$ and $V \cdot V \subseteq U$. Consider the conditions*

(a) The group inversion restricts to a smooth self map of V.

(b) The group multiplication restricts to a smooth map $V \times V \to U$.

(c) For each $g \in G$, there exists an open $\mathbf{1}$-neighborhood $W \subseteq U$ such that $g \cdot W \cdot g^{-1} \subseteq U$, and the map

$$W \to U : w \mapsto g \cdot w \cdot g^{-1}$$

is smooth.

If (a)–(c) hold, then there exists a unique smooth manifold structure on G which makes G a Lie group such that V is an open submanifold of G. If (a) and (b) hold, then there exists a unique smooth manifold structure on the subgroup $\langle V \rangle$ generated by V which makes $\langle V \rangle$ a Lie group such that V is an open submanifold of $\langle V \rangle$.

B.2.2. Regularity

We recall the notion of regularity (see [Mil84] for further information). To this end, we define left evolutions of smooth curves. As a tool, we use the group multiplication on the tangent bundle $\mathbf{T}G$ of a Lie group G.

Definition B.2.6 (Left logarithmic derivative)**.** Let G be a Lie group, $k \in \mathbb{N}$ and $\eta : [0,1] \to G$ a $\mathcal{C}^{k+1}$-curve. We define the *left logarithmic derivative* of η as

$$\delta_\ell(\eta) : [0,1] \to \mathbf{L}(G) : t \mapsto \eta(t)^{-1} \cdot \eta'(t).$$

The curve $\delta_\ell(\eta)$ is obviously $\mathcal{C}^k$.

Definition B.2.7 (Left evolutions)**.** Let G be a Lie group and $\gamma : [0,1] \to \mathbf{L}(G)$ a smooth curve. A smooth curve $\eta : [0,1] \to G$ is called a *left evolution* of γ and denoted by $\operatorname{Evol}_G^{\ell}(\gamma)$ if $\delta_\ell(\eta) = \gamma$ and $\eta(0) = \mathbf{1}$. One can show that in case of its existence, a left evolution is uniquely determined.

The existence of a left evolution is equivalent to the existence of a solution to a certain initial value problem:

Lemma B.2.8. *Let G be a Lie group and $\gamma : [0,1] \to \mathbf{L}(G)$ a smooth curve. Then there exists a left evolution $\operatorname{Evol}^{\ell}(G)\gamma : [0,1] \to G$ iff the initial value problem*

$$\begin{aligned} \eta'(t) &= \eta(t) \cdot \gamma(t) \\ \eta(0) &= \mathbf{1} \end{aligned} \tag{B.2.8.1}$$

has a solution η. In this case, $\eta = \operatorname{Evol}_G^{\ell}(\gamma)$.

Now we give the definition of regularity:

Definition B.2.9 (Regularity)**.** A Lie group G is called *regular* if for each smooth curve $\gamma : [0,1] \to \mathbf{L}(G)$ there exists a left evolution and the map

$$\operatorname{evol}_G^{\ell} : \mathcal{C}^\infty([0,1], \mathbf{L}(G)) \to G : \gamma \mapsto \operatorname{Evol}_G^{\ell}(\gamma)(1)$$

is smooth.

Lemma B.2.10. *Let G be a Lie group. Suppose there exists a zero neighborhood $\Omega \subseteq \mathcal{C}^\infty([0,1], \mathbf{L}(G))$ such that for each $\gamma \in \Omega$ the left evolution $\operatorname{Evol}_G^{\ell}(\gamma)$ exists and the map*

$$\Omega \to G : \gamma \mapsto \operatorname{Evol}_G^{\ell}(\gamma)(1)$$

is smooth. Then G is regular.

Remark B.2.11. We can define *right logarithmic derivatives* and *right evolutions* in the analogous way as we did above for the left ones. We denote the right logarithmic derivative by δ_ρ, the right evolution map by $\operatorname{Evol}^{\rho}$ and the endpoint of the right evolution by $\operatorname{evol}^{\rho}$. One can show that a Lie group is left-regular iff it is right-regular. Also the equivalent of Lemma B.2.10 holds. In particular, initial value problem (B.2.8.1) becomes

$$\begin{aligned} \eta'(t) &= \gamma(t) \cdot \eta(t) \\ \eta(0) &= \mathbf{1} \end{aligned}. \tag{B.2.11.1}$$

Definition B.2.12. Let G be a Lie group. A smooth map $\exp_G : \mathbf{L}(G) \to G$ is called an *exponential map* for G if $\mathbf{T}_0 \exp_G = \operatorname{id}_{\mathbf{L}(G)}$ and $\exp_G((s+t)v) = \exp_G(sv) \cdot \exp_G(tv)$ for all $s, t \in \mathbb{R}$ and $v \in \mathbf{L}(G)$.

B.2.3. Group actions

Lemma B.2.13. *Let G and H be groups and $\alpha : G \times H \to H$ a group action that is a group morphism in its second argument. Further, let $\widetilde{H}$ be a subgroup of H that is generated by U. Then*

$$\alpha(G \times \widetilde{H}) \subseteq \widetilde{H} \iff \alpha(G \times U) \subseteq \widetilde{H}.$$

Proof. By our assumption, $\widetilde{H} = \bigcup_{n\in\mathbb{N}}(U \cup U^{-1})^n$. So we calculate

$$\begin{aligned}\alpha(G \times \widetilde{H}) &= \alpha(G \times \bigcup_{n\in\mathbb{N}}(U \cup U^{-1})^n) = \bigcup_{n\in\mathbb{N}} \alpha(G \times (U \cup U^{-1})^n)\\ &= \bigcup_{n\in\mathbb{N}} \alpha(G \times (U \cup U^{-1}))^n = \bigcup_{n\in\mathbb{N}} (\alpha(G \times U) \cup \alpha(G \times U)^{-1})^n \subseteq \widetilde{H}.\end{aligned}$$

That's it. □

Lemma B.2.14. *Let G and H be Lie groups and $\alpha : G \times H \to H$ a group action that is a group morphism in its second argument. Then α is smooth iff the following assertions hold:*

(a) it is smooth on $U \times V$, where U and V are open unit neighborhoods, respectively.

(b) for each $h \in H$, there exists an open unit neighborhood W such that the map $\alpha(\cdot, h) : W \to H$ is smooth.

(c) for each $g \in G$ the map $\alpha(g, \cdot) : H \to H$ is smooth.

If U generates G, (b) follows from (a). If V generates H, (c) follows from (a).

Proof. We first show that by our assumptions, α is smooth. To this end, let $(g, h) \in G\times H$. Choose W as in (b). Then $U' := U \cap W \in \mathcal{U}_G(\mathbf{1})$. We show that $\alpha|_{gU'\times Vh}$ is smooth. Since the map $U' \times V \to gU' \times Vh : (u, v) \mapsto (gu, vh)$ is a smooth diffeomorphism, we only need to show that the map

$$U' \times V \to H : (u, v) \mapsto \alpha(gu, hv)$$

is smooth. But

$$\alpha(gu, hv) = \alpha_g(\alpha(u, vh)) = \alpha_g(\alpha(u, v)\alpha(u, h)) = \alpha_g(\alpha(u, v)\alpha^h(u)),$$

where we denote $\alpha(\cdot, h)$ by α^h and $\alpha(g, \cdot)$ by α_g. Since the right hand side is obviously smooth, we are home.

Now we prove the other two assertions. We suppose that (a) holds. We let $S \subseteq H$ be the set of all $h \in H$ such that (b) holds. Then $V \subseteq S$; and since $\alpha^{h^{-1}}(g) = \alpha^h(g)^{-1}$ and $\alpha^{hh'}(g) = \alpha^h(g)\alpha^{h'}(g)$ for all $g \in G$ and $h, h' \in H$, we easily see that S is a subgroup of H. Since V is a generator, $S = H$.
Since U generates G, for each $g \in G$ we find $g_1, \ldots, g_n \in U \cup U^{-1}$ such that

$$\alpha_g = \alpha_{g_n} \circ \cdots \circ \alpha_{g_1}.$$

Further, for $g' \in G$ and $h \in H$, $\alpha_{g'^{-1}}(h) = \alpha_{g'}(h)^{-1}$, so each α_{g_k} is smooth by our assumption. Hence α_g is smooth. □

Lemma B.2.15. *Let G and H be Lie groups and $\omega : G \times H \to H$ a smooth group action that is a group morphism in its second argument. Then the semidirect product $H \rtimes_\omega G$ can be turned into a Lie group that is modelled on $\mathbf{L}(H) \times \mathbf{L}(G)$.*

Proof. The semidirect product $H \rtimes_\omega G$ is endowed with the multiplication

$$(H \times G) \times (H \times G) \to H \times G : ((h_1, g_1), (h_2, g_2)) \mapsto (h_1 \cdot \omega(g_1, h_2), g_1 \cdot g_2)$$

and the inversion

$$H \times G \to H \times G : (h, g) \mapsto (\omega(g^{-1}, h^{-1}), g^{-1}),$$

so the smoothness of the group operations follows from the one of ω. □

B.3. Riemannian geometry and manifolds

We introduce notation and prove some results involving Riemannian geometry.

B.3.1. Definitions and elementary results

We need the following, well-known facts about Riemannian geometry:

Definition B.3.1 (Riemannian exponential function)**.** Let $d \in \mathbb{N}^*$ and (M, g) be a d-dimensional Riemannian manifold. Then the (maximal) domain D_g^E of the *Riemannian exponential map* $\exp_g : D_g^E \to M$ is an open subset of $\mathbf{T}M$. D_g^E is an open neighborhood of the zero section in $\mathbf{T}M$, and for each $x \in M$, we have $[0,1].(D_g^E \cap \mathbf{T}_x M) \subseteq D_g^E \cap \mathbf{T}_x M$.

For each $x \in M$, we define $\exp_x$ as $\exp_g|_{\mathbf{T}_x M \cap D_g^E}$. If M is an open subset of $\mathbb{R}^d$, then for each $x \in M$ and $v, w \in \mathbb{R}^d$, we have the identity

$$d \exp_g(x, 0; v, w) = v + w. \tag{B.3.1.1}$$

In order to define the logarithm, we need the following definition.

Definition B.3.2. Let $d \in \mathbb{N}^*$ and (M, g) a Riemannian manifold. For x and $h \in \mathbf{T}_x M$, we define

$$\|h\|_{g_x} := \sqrt{g(h, h)}.$$

Obviously, each $\|\cdot\|_{g_x}$ is a norm on $\mathbf{T}_x M$. We also define

$$B_r^{g_x}(0) := B_{(\mathbf{T}_x M, \|\cdot\|_{g_x})}(0, r).$$

If M is an open subset of $\mathbb{R}^d$, we set for $h \in \mathbb{R}^d$

$$\|h\|_{g_x} := \sqrt{g((x, h), (x, h))}.$$

Obviously, each $\|\cdot\|_{g_x}$ is a norm on $\mathbb{R}^d$. In particular, we define

$$B_r^{g_x}(0) := B_{(\mathbb{R}^d, \|\cdot\|_{g_x})}(0, r).$$

Definition B.3.3 (Riemannian logarithm map)**.** Let $d \in \mathbb{N}^*$ and (M, g) be a d-dimensional Riemannian manifold. For all $x \in M$ there exists an open neighborhood $V_x \subseteq \mathbf{T}_x M$ of 0_x such that $\exp_x|_{V_x}$ is a diffeomorphism onto its image, which is an open subset of M. So the *Riemannian logarithm map* $\log_g : D_g^L \to \mathbf{T}M$ can be defined, where

$$D_g^L := \bigcup_{x \in M} \{x\} \times \exp_x(B_{r_x}^{g_x}(0)) \subseteq M^2$$

and $\log_g(x, y) := \exp_x|_{B_{r_x}^{g_x}(0)}^{-1}$; here $r_x := \sup\{r > 0 : \exp_x|_{B_r^{g_x}(0)} \text{ is injective}\}$. Further, for $x \in M$ we set $\log_x(y) := \log_g(x, y)$ for $y \in M$ such that $(x, y) \in D_g^L$.

Let M be an open subset of $\mathbb{R}^d$. We define $\lg_g := \pi_2 \circ \log_g$, where $\pi_2 : \mathbb{R}^{2d} \to \mathbb{R}^d$ denotes the projection on the second factor. For each $x \in M$ and $v, w \in \mathbb{R}^d$, the identity $d \lg_g(x, x; v, w) = w - v$ holds. This is an immediate consequence of (B.3.1.1) and the chain rule since $\log_g$ and $(\pi_1, \exp_g)$ are inverse functions.

Definition B.3.4 (Localizations)**.** Let M be a d-dimensional manifold, $\kappa : U \to V$ a chart of M, $k \in \overline{\mathbb{N}}$ and $X \in \mathfrak{X}^k(U)$. Then we set $X_\kappa := d\kappa \circ X \circ \kappa^{-1} : V \to \mathbb{R}^d$. If g is a metric on M, we set $g_\kappa := g \circ (\mathbf{T}\kappa^{-1} \oplus \mathbf{T}\kappa^{-1})$.

Remark B.3.5. Let (M, g) be a Riemannian manifold and $\kappa : U \to V$ a chart of M. Then $\mathbf{T}\kappa^{-1}(D_{g_\kappa}^E) \subseteq D_g^E$, and $\exp_{g_\kappa} = \kappa \circ \exp_g \circ \mathbf{T}\kappa^{-1}|_{D_{g_\kappa}^E}$. Further, $(\kappa^{-1} \times \kappa^{-1})(D_{g_\kappa}^L) \subseteq D_g^L$, and $\log_{g_\kappa} = \mathbf{T}\kappa \circ \log_g \circ (\kappa^{-1} \times \kappa^{-1})|_{D_{g_\kappa}^L}$.

Lemma B.3.6. *Let (M, g) be a Riemannian manifold and $\kappa : \widetilde{U_\kappa} \to U_\kappa$, $\phi : \widetilde{U_\phi} \to U_\phi$ charts for M such that $\widetilde{U_\kappa} \cap \widetilde{U_\phi} \neq \emptyset$. Then the following identities hold:*

(a) On $D^E_{g_\kappa|_{\kappa(\widetilde{U_\kappa} \cap \widetilde{U_\phi})}}$, we have $\phi \circ \kappa^{-1} \circ \exp_{g_\kappa} = \exp_{g_\phi} \circ \mathbf{T}(\phi \circ \kappa^{-1})$.

(b) On $D^L_{g_\kappa|_{\kappa(\widetilde{U_\kappa} \cap \widetilde{U_\phi})}}$, we have $\mathbf{T}(\phi \circ \kappa^{-1}) \circ \log_{g_\kappa} = \log_{g_\phi} \circ (\phi \circ \kappa^{-1} \times \phi \circ \kappa^{-1})$.

Let $X \in \mathfrak{X}^0(M)$.

(c) $\mathbf{T}(\phi \circ \kappa^{-1}) \circ (\mathrm{id}_{\kappa(\widetilde{U_\kappa} \cap \widetilde{U_\phi})}, X_\kappa) = (\mathrm{id}_{\phi(\widetilde{U_\kappa} \cap \widetilde{U_\phi})}, X_\phi) \circ \phi \circ \kappa^{-1}$.

Additionally, let $V \subseteq U_\kappa$ such that $\mathrm{im}\,(\mathbf{T}\kappa \circ X|_V) \subseteq D_{g_\kappa}^E$.

(d) Then $\kappa \circ \exp_g \circ X \circ \kappa^{-1} = \exp_{g_\kappa} \circ (\mathrm{id}_{\kappa(V)}, X_\kappa)$ on $\kappa(V)$.

Proof. These are easy computations involving Remark B.3.5. □

B.3.2. Riemannian exponential function and logarithm on open subsets of $\mathbb{R}^d$

We examine functions that arise as the composition of the second component $\lg_g$ of the Riemannian logarithm or the exponential map $\exp_g$ with functions of the form (id, X).

Of particular interest are estimates for the function values and the values of the first derivatives of such functions.

We also derive a sufficient criterion on a vector field X that ensures that $\exp_g \circ X$ is injective, and gives a lower bound for the size of its image. Further, we use that $\lg_g$ is the fiberwise inverse function to $\exp_g$, and apply the parameterized inverse function theorem Proposition A.2.19. We will get estimates for the domain and the first partial derivative of $\lg_g$ in terms of those numbers for $\exp_g$.

For open nonempty $U \subseteq \mathbb{R}^d$, we will tacitly identify $\mathbf{T}U$ with $U \times \mathbb{R}^d$ and, for $x \in U$, $\mathbf{T}_x U$ with $\mathbb{R}^d$.

Superposition with the Riemannian exponential map

We start with the exponential map.

Estimates for function values and the first derivatives We derive estimates for the function values and first derivatives of $\exp_g \circ(\mathrm{id}, X)$. This is mostly done using the mean value theorem and the triangle inequality.

Definition B.3.7. Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ an open nonempty subset, g a Riemannian metric on U and $K \subseteq U$ a relatively compact set. Using standard compactness arguments, we see that there exists $\tau > 0$ such that $\overline{K} \times B_\tau(0) \subseteq D_g^E$ (note that this implies $\exp_g(\overline{K} \times B_\tau(0)) \subseteq U$). We denote the supremum of such τ by $R_{K,U}^{E,g}$. If the metric discussed is obvious, we may omit it and write $R_{K,U}^{E}$. Now let $0 < \delta < R_{K,U}^{E}$. We define

$$C_{K,\delta,g}^{E,(1)} := \sup_{x \in \overline{K}} \|\exp_x\|_{1_{\overline{B}_\delta(0)},1} = \|D_2 \exp_g\|_{1_{\overline{K} \times \overline{B}_\delta(0)},0}$$

and

$$C_{K,\delta,g}^{E,2} := \|\exp_g\|_{1_{\overline{K} \times \overline{B}_\delta(0)},2}.$$

As above, if the metric discussed is clear, we may omit it and just write $C_{K,\delta}^{E,(1)}$ or $C_{K,\delta}^{E,2}$, respectively. Note that $\|\exp_g\|_{1_{K \times \overline{B}_\delta(0)},2}$ relates to the norm $\|(v, w)\| = \max(\|v\|, \|w\|)$ on $\mathbb{R}^{2d}$.

Lemma B.3.8. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ an open nonempty subset, g a Riemannian metric on U, $K \subseteq U$ a relatively compact set and $0 < \delta < R_{K,U}^{E}$.*

(a) Then for all $x \in K$ and $y \in \overline{B}_\delta(0)$ the following estimate holds:

$$\|\exp_g(x, y) - x\| \leq C_{K,\delta}^{E,(1)} \|y\|.$$

(b) Let $X : K \to \mathbb{R}^d$ with $\|X\|_{1_K,0} \leq \delta$. Then for all $x \in K$, the following estimate holds:

$$\|(\exp_g \circ(\mathrm{id}_K, X))(x) - \mathrm{id}_K(x)\| \leq C_{K,\delta}^{E,(1)} \|X(x)\|$$

Proof. (a) We calculate using the mean value theorem

$$\exp_g(x,y) - x = \exp_g(x,y) - \exp_g(x,0) = \int_0^1 D\exp_x(ty)\cdot y\,dt.$$

From this and the definition of $C^{E,(1)}_{K,\delta}$, we easily derive the assertion.

(b) This is an easy consequence of (a). □

Lemma B.3.9. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ an open nonempty subset and g a Riemannian metric on U. Further, let $W \subseteq U$ be an open, nonempty, relatively compact subset and $\delta \in]0, R^E_{W,U}[$. Then for each $X \in \mathcal{C}^1(W,\mathbb{R}^d)$ with $\|X\|_{1_W,0} \leq \delta$, we have*

$$\begin{aligned}&\|D((\exp_g \circ(\mathrm{id}_W, X)) - \mathrm{id}_W)(x)\cdot v\| \\ &\qquad\qquad \leq \|\exp_g\|_{1_{W\times \overline{B}_\delta(0)},2}\|(0,X(x))\|\,\|(v,DX(x)\cdot v)\| + \|DX(x)\cdot v\|\end{aligned}$$

for $x \in W$ and $v \in \mathbb{R}^d$. In particular, if we endow $\mathbb{R}^{2d}$ with the norm $\|(v,w)\| = \max(\|v\|,\|w\|)$ and assume that $\|X\|_{1_W,1} \leq 1$, we get the estimate

$$\|D((\exp_g \circ(\mathrm{id}_W, X)) - \mathrm{id}_W)(x)\|_{op} \leq C^{E,2}_{W,\delta}\|X(x)\| + \|DX(x)\|_{op}.$$

Proof. Let $x \in W$ and $v \in \mathbb{R}^d$. Then we calculate using that $v = D\exp_g(x,0)\cdot(v,0) = D\exp_g(x,0)\cdot(0,v)$ and $0 = D\exp_g(x,0)\cdot(DX(x)\cdot v, -DX(x)\cdot v)$

$$\begin{aligned}&D(\exp_g \circ(\mathrm{id}_W, X) - \mathrm{id}_W)(x)\cdot v \\ =&D\exp_g(x,X(x))\cdot(v,DX(x)\cdot v) - D\exp_g(x,0)\cdot(v,0) \\ &\quad + D\exp_g(x,0)\cdot(DX(x)\cdot v, -DX(x)\cdot v) \\ =&D\exp_g(x,X(x))\cdot(v,DX(x)\cdot v) - D\exp_g(x,0)\cdot(v,DX(x)\cdot v) + DX(x)\cdot v.\end{aligned}$$

For the difference we derive using the mean value theorem

$$\begin{aligned}&(D\exp_g(x,X(x)) - D\exp_g(x,0))\cdot(v,DX(x)\cdot v) \\ &\qquad\qquad = \int_0^1 D(D\exp_g)(x,tX(x))\cdot(0,X(x))\,dt\cdot(v,DX(x)\cdot v).\end{aligned}$$

From this, the assertion follows. □

On invertibility and the size of the image Having established the estimates, we can give a criterion on when $\exp_g\circ(\mathrm{id},X)$ is injective, and how large its image is. The main tool used is a quantitative, parameterized version of the inverse function theorem that is provided in Theorem A.2.18.

Lemma B.3.10. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U, $r > 0$ such that $\overline{B}_r(0) \subseteq U$ and $k \in \overline{\mathbb{N}}$ with $k \geq 1$. Further, let $\varepsilon \in]0,\frac{1}{2}[$, $\nu \in]0, R^E_{B_r(0),U}[$ and $\delta > 0$ with $\delta < \min(\frac{\varepsilon r}{2C^{E,(1)}_{B_r(0),\nu}}, \nu, \frac{\varepsilon}{4(C^{E,2}_{B_r(0),\nu}+1)})$. Then for $X \in \mathcal{C}^k(B_r(0),\mathbb{R}^d)$ such that $\|X\|_{1_{B_r(0)},0} < \delta$ and $\|X\|_{1_{B_r(0)},1} < \frac{\varepsilon}{4}$, the map $\exp_g \circ(\mathrm{id}_{B_r(0)}, X)$ is a $\mathcal{C}^k$-diffeomorphism onto its image, which is an open subset of $\mathbb{R}^d$ and contains $B_{r(1-2\varepsilon)}(0)$.*

Proof. By Lemma B.3.8, for a function X with $\|X\|_{1_{B_r(0)},0} < \min(\nu, \frac{\varepsilon r}{2C^{E,(1)}_{B_r(0),\nu}})$, we have

$$\|\exp_g(0, X(0))\| < \frac{\varepsilon r}{2}. \tag{†}$$

We set $W := B_r(0)$. Since $\|X\|_{1_{B_r(0)},0} < \frac{\varepsilon}{4(C^{E,2}_{B_r(0),\nu}+1)}$ and $\|X\|_{1_{B_r(0)},1} < \frac{\varepsilon}{4} < 1$, we see with Lemma B.3.9 that $\|\exp_g \circ(\mathrm{id}_W, X) - \mathrm{id}_W\|_{1_{B_r(0)},1} < \frac{\varepsilon}{2}$. This implies that

$$\|D(\exp_g \circ(\mathrm{id}_W, X))(y) - D(\exp_g \circ(\mathrm{id}_W, X))(x)\|_{op} < \varepsilon$$

for all $x, y \in B_r(0)$, and that $D(\exp_g \circ(\mathrm{id}_W, X))(0)$ is invertible with

$$\|D(\exp_g \circ(\mathrm{id}_W, X))(0)^{-1}\|_{op} < \frac{1}{1-\frac{\varepsilon}{2}}. \tag{††}$$

Since $\varepsilon < \frac{2}{3}$, we conclude that $\varepsilon < 1 - \frac{\varepsilon}{2} < \frac{1}{\|D(\exp_g \circ(\mathrm{id}_W,X))(0)^{-1}\|_{op}}$. Hence we can apply Theorem A.2.18 to see that $\exp_g \circ(\mathrm{id}_W, X)$ is a diffeomorphism onto its image and that the image contains $B_{r'}(\exp_g(0, X(0)))$, where $r' = r\left(\frac{1}{\|D(\exp_g \circ(\mathrm{id}_W,X))(0)^{-1}\|_{op}} - \varepsilon\right)$. From this we deduce using (†), (††) and the triangle inequality (where we need $\varepsilon < \frac{1}{2}$) that the image of $\exp_g \circ(\mathrm{id}_W, X)$ contains $B_{r(1-2\varepsilon)}(0)$. □

Superposition with the Riemannian logarithm

We examine $\lg_g$. In particular, we use that $\lg_g$ is the fiberwise inverse function to $\exp_g$. We show that its domain D^L_g is a neighborhood of the diagonal, and that we can quantify what is contained in it; and we give estimates for its first derivative.
Further, we examine maps that arise as the composition of $\lg_g$ with maps of the form $(\mathrm{id}, X+\mathrm{id})$. Of particular interest are estimates for the function values and the derivatives of these maps.

Uniform estimates for Riemannian norms We start by establishing estimates for the Riemannian norms and a given norm on $\mathbb{R}^d$.

Lemma B.3.11. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U. Then for each $x \in U$, there exist $V \in \mathcal{U}^\circ(x)$ and $c, C > 0$ such that*

$$c\|\cdot\| \le \|\cdot\|_{g_y} \le C\|\cdot\|$$

for all $y \in V$.

Proof. In the proof, for $x \in U$ we let G_x denote the matrix $(g((x, e_i), (x, e_j)))_{1\le i,j\le d}$.

There exists $\widetilde{C} > 0$ such that $\|\cdot\|_{g_x} \le \widetilde{C}\|\cdot\|$. Further, for $\varepsilon > 0$ there exists $V \in \mathcal{U}^\circ(x)$ such that

$$\|G_y - G_x\|_{op} < \varepsilon$$

for all $y \in V$. Hence for $y \in V$ and $h \in \mathbb{R}^d$,

$$\|h\|_{g_y}^2 = \langle h, G_y \cdot h\rangle - \langle h, G_x \cdot h\rangle + \langle h, G_x \cdot h\rangle = \langle h, (G_y - G_x)\cdot h\rangle + \|h\|_{g_x}^2 \leq \varepsilon\|h\|^2 + \widetilde{C}^2\|h\|^2.$$

From this, we easily deduce the first estimate.

For the second estimate, we have for $y \in U$ and $h \in \mathbb{R}^d$ that

$$\|h\|_{g_y}^2 = \langle h, G_y \cdot h\rangle = \langle A_y \cdot h, A_y \cdot h\rangle = \|A_y \cdot h\|_2^2 \geq \frac{1}{\|A_y^{-1}\|_{op}^2}\|h\|_2^2;$$

where $A_y = \sqrt{G_y}$. Since the map $y \mapsto \|\sqrt{G_y}^{-1}\|_{op}$ is continuous, and there exists $\widetilde{c} > 0$ such that $\|\cdot\|_2 \geq \widetilde{c}\|\cdot\|$, we see that the assertion holds. □

Definition B.3.12. Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ an open nonempty subset, g a Riemannian metric on U and $K \subseteq U$ a relatively compact set. We define

$$Q_K^g := \frac{\sup\{c > 0 : (\forall x \in K)\, c\|\cdot\| \leq \|\cdot\|_{g_x}\}}{\inf\{C > 0 : (\forall x \in K)\, \|\cdot\|_{g_x} \leq C\|\cdot\|\}}.$$

Note that because of Lemma B.3.11, $Q_K^g \in]0,1]$.

Applying the parametrized inverse function theorem We use Proposition A.2.19 to derive estimates for the domain and the first derivatives of $\lg_g$, under a certain condition on the partial differentials of $\exp_g$. Further, we show that D_g^L is a neighborhood of the diagonal.

Definition B.3.13. Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, (U, g) a Riemannian manifold, V a nonempty relatively compact set with $\overline{V} \subseteq U$ and $\sigma \in]0,1[$. We define

$$R_{V,\sigma}^g := \sup\{r \in]0, R_{V,U}^E[: (\forall x \in \overline{V})\, \|\exp_x - \mathrm{id}_{\mathbb{R}^d}\|_{1_{\overline{B}_r(0)},1} < \sigma\}.$$

If the metric can not be confused, we may omit it in the notation and just write $R_{V,\sigma}$. Note that $R_{V,\sigma} > 0$ as one can prove using compactness arguments and (B.3.1.1).

Lemma B.3.14. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U, $V \subseteq U$ an open, nonempty, relatively compact set, $\sigma \in]0,1[$ and $\tau \in]0, R_{V,\sigma}^g[$. Then the following assertions hold:*

(a) $C_{V,\tau}^{E,(1)} \leq 1 + \sigma$.

Let $x \in V$. Then

(b) $\exp_x|_{B_\tau(0)}$ *is a diffeomorphism onto its image,*

(c) $B_{(1-\sigma)r}(x) \subseteq \exp_x(B_r(0))$ *for all* $r \in [0, \tau]$, *and*

(d) $(\exp_x|_{B_\tau(0)}^{-1})|_{B_{(1-\sigma)\tau}(x)}$ *is* $\frac{1}{1-\sigma}$*-Lipschitz.*

Finally, assume that $\tau < Q_V^g R_{V,\sigma}^g$. *Then* *(e)* $\bigcup_{x\in V}\{x\}\times\exp_x(B_\tau(0))\subseteq D_g^L$.

Proof. The assertion about $C^{E,(1)}_{V,\tau}$ follows from a simple application of the triangle inequality to $\|D_2\exp_g\pm\mathrm{id}_{\mathbb{R}^d}\|_{op}$. To prove (b)-(d), let $x,y\in V$. Then for each $z\in B_\tau(0)$, we have

$$\|D_2\exp_g(x,z)-D_2\exp_g(x,0)\|_{op}=\|D_2\exp_g(x,z)-\mathrm{id}_{\mathbb{R}^d}\|_{op}<\sigma.$$

Further, $\sigma<1=\frac{1}{\|(D_2\exp_g(x,0))^{-1}\|_{op}}$. So we can apply Proposition A.2.19 to derive the assertions about $\exp_x|_{B_\tau(0)}$ and $(\exp_x|^{-1}_{B_\tau(0)})|_{B_{(1-\sigma)\tau}(x)}$.

(e) We see using Lemma B.3.11 and standard compactness arguments that for each $x\in V$,

$$B_\tau(0)\subseteq B^{g_x}_{\tau C}(0)\subseteq B_{\frac{\tau}{Q^g_V}}(0);$$

here C denotes the denominator in the definition of Q^g_V. Since $\frac{\tau}{Q^g_V}<R^g_{V,\sigma}$ by our assumption, we see with (b) that each map $\exp_x|_{B^{g_x}_{\tau C}(0)}$ is injective, and can conclude that $\{x\}\times\exp_x(B_\tau(0))\subseteq D_g^L$. □

Estimates for function values and first derivatives Before we establish the estimates, we make the following definitions.

Definition B.3.15. Let X be a normed space, $S\subseteq X$ and $\tau>0$. We set

$$S^{\times\tau}:=\bigcup_{x\in S}\{x\}\times B_\tau(x)\qquad\text{and}\qquad S^{\overline{\times}\tau}:=\bigcup_{x\in\overline{S}}\{x\}\times\overline{B}_\tau(x).$$

Definition B.3.16. Let $d\in\mathbb{N}^*$, $U\subseteq\mathbb{R}^d$ an open nonempty subset, g a Riemannian metric on U and $K\subseteq U$ a relatively compact set. By Lemma B.3.14 (more precisely, (c) and (e)), there exists $\tau>0$ such that $\overline{K}^{\times\tau}\subseteq D_g^L$. We denote the supremum of such τ by $R^{L,g}_{K,U}$. If the metric discussed is obvious, we may omit it and just write $R^L_{K,U}$.

Now let $0<\delta<R^L_{K,U}$. We define

$$C^{L,(1)}_{K,\delta,g}:=\sup_{x\in\overline{K}}\|\pi_2\circ\log_x\|_{1_{\overline{B}_\delta(x)},1}=\|D_2\,\mathrm{lg}_g\|_{1_{\overline{K}^{\times\delta}},0}$$

and

$$C^{L,2}_{K,\delta,g}:=\|\mathrm{lg}_g\|_{1_{\overline{K}^{\times\delta}},2}.$$

As above, if the metric discussed is clear, we may omit it in the notation and just write $C^{L,(1)}_{K,\delta}$ or $C^{L,2}_{K,\delta}$, respectively. Note that $\|\cdot\|_{1_{\overline{K}^{\times\delta}},2}$ relates to the norm $\|(v,w)\|=\max(\|v\|,\|w\|)$ on $\mathbb{R}^{2d}$.

We rephrase some results of Lemma B.3.14.

Lemma B.3.17. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U, $W \subseteq U$ an open, nonempty, relatively compact set, $\sigma \in]0,1[$ and $\tau \in]0, R^g_{W,\sigma} Q^g_W[$. Then $C^{E,(1)}_{W,\tau} \leq 1+\sigma$, $(1-\sigma)\tau < R^L_{W,U}$ and $C^{L,(1)}_{W,(1-\sigma)\tau} \leq \frac{1}{1-\sigma}$.*

Proof. The assertions follow from Lemma B.3.14. ☐

Now we prove the estimates.

Lemma B.3.18. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ an open nonempty subset, g a Riemannian metric on U, $K \subseteq U$ a relatively compact set and $\delta \in]0, R^L_{K,U}[$.*

(a) Then for all $x \in K$ and $y \in \overline{B}_\delta(x)$ the following estimate holds:

$$\|\mathrm{lg}_g(x, y+x)\| \leq C^{L,(1)}_{K,\delta} \|y\|.$$

(b) Let $X : K \to \mathbb{R}^d$ with $\|X\|_{1_K,0} \leq \delta$. Then for all $x \in K$, the following estimate holds:

$$\|(\mathrm{lg}_g \circ (\mathrm{id}_K, X + \mathrm{id}_K))(x)\| \leq C^{L,(1)}_{K,\delta} \|X(x)\|.$$

Proof. (a) We calculate using the mean value theorem and $\mathrm{lg}_g(x,x) = 0$:

$$\mathrm{lg}_g(x, y+x) = \mathrm{lg}_g(x, y+x) - \mathrm{lg}_g(x,x) = \int_0^1 D_2 \,\mathrm{lg}_g(x, x+ty) \cdot y \, dt.$$

From this and the definition of $C^{L,(1)}_{K,\delta}$, we easily derive the assertion.

(b) This is an easy consequence of (a). ☐

Lemma B.3.19. *Let $d \in \mathbb{N}^*$, $U \subseteq \mathbb{R}^d$ open, g a Riemannian metric on U, $W \subseteq U$ an open, nonempty, relatively compact set, $\tau \in]0, R^L_{W,U}[$ and $X \in \mathcal{C}^1(W, B_\tau(0))$. Then for $x \in W$*

$$\|D(\mathrm{lg}_g \circ (\mathrm{id}_W, X + \mathrm{id}_W))(x)\|_{op} \leq C^{L,2}_{W,\tau,g} \|X(x)\| + C^{L,(1)}_{W,\tau,g} \|DX(x)\|_{op} \qquad \text{(B.3.19.1)}$$

Proof. We get with the Chain Rule that

$$\begin{aligned} D(\mathrm{lg}_g \circ (\mathrm{id}_W, X + \mathrm{id}_W))(x) &= D\,\mathrm{lg}_g \circ (\mathrm{id}_W, X + \mathrm{id}_W)(x) \cdot (\mathrm{Id}, DX(x) + \mathrm{Id}) \\ &= D\,\mathrm{lg}_g \circ (\mathrm{id}_W, X + \mathrm{id}_W)(x) \cdot (\mathrm{Id}, \mathrm{Id}) + D_2\,\mathrm{lg}_g \circ (\mathrm{id}_W, X + \mathrm{id}_W)(x) \cdot DX(x). \end{aligned}$$

We get the desired estimate for the second summand, and now treat the first. To this end, let $v \in \mathbb{R}^d$. Then we get, using that $D\,\mathrm{lg}_g(x,x) \cdot (v,v) = v - v = 0$:

$$\begin{aligned} &D\,\mathrm{lg}_g \circ (x, X(x)+x) \cdot (v,v) - D\,\mathrm{lg}_g(x,x) \cdot (v,v) \\ &\qquad = \int_0^1 D(D\,\mathrm{lg}_g)(x, x+tX(x)) \cdot (0, X(x)) \, dt \cdot (v,v). \end{aligned}$$

From this, we also get the desired estimate. ☐

C. Quasi-inversion in algebras

We give a short introduction to the concept of *quasi-inversion*. It is a useful tool for the treatment of algebras without a unit, where it serves as a replacement for the ordinary inversion. Many of the algebras we treat are without a unit. Unless the contrary is stated, all algebras are assumed associative.

C.1. Definition

Definition C.1.1 (Quasi-Inversion)**.** Let A denote a $\mathbb{K}$-algebra with the multiplication $*$. An $x \in A$ is called *quasi-invertible* if there exists a $y \in A$ such that

$$x + y - x * y = y + x - y * x = 0.$$

In this case, we call $QI_A(x) := y$ the *quasi-inverse* of x. The set that consists of all quasi-invertible elements of A is denoted by A^q. The map $A^q \to A^q : x \mapsto QI_A(x)$ is called the *quasi-inversion* of A. Often we will denote QI_A just by QI.

An interesting characterization of quasi-inversion is

Lemma C.1.2. *Let A be a $\mathbb{K}$-Algebra with the multiplication $*$. Then A, endowed with the operation*

$$A \times A \to A : (x, y) \mapsto x \diamond y := x + y - x * y,$$

is a monoid with the unit 0 and the unit group A^q. The inversion map is given by QI_A.

Proof. This is shown by an easy computation. □

In unital algebras there is a close relationship between inversion and quasi-inversion.

Lemma C.1.3. *Let A be an algebra with multiplication $*$ and unit e. Then $x \in A$ is quasi-invertible iff $x - e$ is invertible. In this case*

$$QI_A(x) = (x - e)^{-1} + e.$$

Proof. One easily computes that

$$(A, \diamond) \to (A, *) : x \mapsto e - x$$

is an isomorphism of monoids ($\diamond$ was introduced in Lemma C.1.2), and from this we easily deduce the assertion. □

C.2. Topological monoids and algebras with continuous quasi-inversion

In this section, we examine algebras that are endowed with a topology. For technical reasons we also examine monoids.

Definition C.2.1. An algebra A is called a *topological algebra* if it is a topological vector space and the multiplication is continuous.

A topological algebra A is called *algebra with continuous quasi-inversion* if the set A^q is open and the quasi-inversion QI is continuous.

A monoid, endowed with a topology, is called a *topological monoid* if the monoid multiplication is continuous.

A monoid, endowed with a differential structure, is called a *smooth monoid* if the monoid multiplication is smooth.

Remark C.2.2. If A is an algebra with continuous quasi-inversion, then QI is not only continuous, but automatically analytic, see [Glö02a].

In topological monoids the unit group is open and the inversion continuous if they are so near the unit element:

Lemma C.2.3. *Let M be a topological monoid with unit e and the multiplication $*$. Then the unit group $M^\times$ is open iff there exists a neighborhood of e that consists of invertible elements. The inversion map*

$$I : M^\times \to M^\times : x \mapsto x^{-1}$$

is continuous iff it is so in e.

Proof. Let U be a neighborhood of e that consists of invertible elements and $m \in M^\times$. Since the map

$$\ell_m : M \to M : x \mapsto m * x$$

is a homeomorphism, $\ell_m(U)$ is open; and it is clear that $\ell_m(U) \subseteq M^\times$. Hence $M^\times = \cup_{m \in M^\times} \ell_m(U)$ is open.

Let I be continuous in e. We show it is so in $x \in M^\times$. For $m \in M^\times$, we have

$$I(m) = m^{-1} = m^{-1} * x * x^{-1} = (x^{-1} * m)^{-1} * x^{-1} = (\rho_{x^{-1}} \circ I \circ \ell_{x^{-1}})(m), \qquad (\dagger)$$

where $\rho_{x^{-1}}$ denotes the right multiplication by x^{-1}. Since I is continuous in e and $\ell_{x^{-1}}(x) = e$, we can derive the continuity of I in x from (†). □

For algebras with a continuous multiplication we can deduce

Lemma C.2.4. *Let A be an algebra with the continuous multiplication $*$. Then A^q is open if there exists a neighborhood of 0 that consists of invertible elements. The quasi-inversion QI_A is continuous if it is so in 0.*

Proof. Since the map

$$A \times A \to A : (x, y) \mapsto x + y - x * y$$

is continuous, we derive the assertions from Lemma C.1.2 and Lemma C.2.3. □

A criterion for quasi-invertibility We give an criterion that ensures that an element of an algebra is quasi-invertible. It turns out that it is quite useful in certain algebras, namely Banach algebras.

Lemma C.2.5. *Let A be a topological algebra and $x \in A$. If $\sum_{i=1}^{\infty} x^i$ exists, then x is quasi-invertible with*

$$QI_A(x) = -\sum_{i=1}^{\infty} x^i.$$

Proof. We just compute that x is quasi-invertible:

$$x + \left(-\sum_{i=1}^{\infty} x^i\right) - x * \left(-\sum_{i=1}^{\infty} x^i\right) = -\sum_{i=2}^{\infty} x^i + \sum_{i=2}^{\infty} x^i = 0.$$

The identity $(-\sum_{i=1}^{\infty} x^i) + x - (-\sum_{i=1}^{\infty} x^i) * x = 0$ is computed in the same way. So the quasi-invertibility of x follows direct from the definition. □

Quasi-inversion in Banach algebras

Lemma C.2.6. *Let A be a Banach algebra. Then $B_1(0) \subseteq A^q$. Moreover, for $x \in B_1(0)$*

$$QI_A(x) = -\sum_{i=1}^{\infty} x^i.$$

Proof. For $x \in B_1(0)$ the series $\sum_{i=1}^{\infty} x^i$ exists since it is absolutely convergent and A is complete. So the assertion follows from Lemma C.2.5. □

Lemma C.2.7. *Let A be a Banach algebra. Then A^q is open in A and the quasi-inversion QI_A is continuous.*

Proof. This is an immediate consequence of Lemma C.2.6 and Lemma C.2.4 since

$$x \mapsto \sum_{i=1}^{\infty} x^i$$

is analytic (see [Bou67, §3.2.9]) and hence continuous. □

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Notation

The following list contains the symbols that are used on several occasions, together with a short explanation of their meaning and the page number where the respective symbol is defined. For better overview, the entries are arranged into the categories basic notation, spaces of weighted functions, Lie groups and manifolds, groups and monoids of functions and further notation.

Basic notation

$B_r(x)$	Open ball with radius r around x	9
$B_X(x, r)$	Like $B_r(x)$, here with indication of the space X	9
$\overline{B}_r(x)$	Closed ball with radius r around x	9
$\mathcal{C}^k(U, Y)$	The set of all k times differentiable functions from U to Y	10
$\mathcal{FC}^k(U, Y)$	The set of all k times Fréchet differentiable functions from U to Y	10
$d^{(k)}f$	k-th iterated derivative of f	10
$D^{(k)}\gamma$	k-th Fréchet derivative of γ	11
$\mathbb{D}$	The closed unit disk in $\mathbb{R}$ or $\mathbb{C}$	9
$\operatorname{dist}(A, B)$	Distance between A and B	9
$\mathbin{\dot\cup}_{i \in I} U_i$	The disjoint union $\bigcup_{i \in I} \{i\} \times U_i$	77
$\mathbb{K}$	The field $\mathbb{R}$ or $\mathbb{C}$	9
$\operatorname{Lip}^q_p(\phi)$	The Lipschitz constant for a map ϕ with respect to the seminorms p, q	74
$\overline{\mathbb{N}}$	$\mathbb{N} \cup \{\infty\} = \{\infty, 0, 1, \dots\}$	9
$\mathbb{N}^*$	$\mathbb{N} \setminus \{0\}$	9
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, \infty\}$	9

Spaces of weighted functions, and related notation

$\mathcal{BC}^k(U, Y)$	The set of k-times differentiable functions from U to Y that and whose derivatives are bounded	12
$\mathcal{BC}^{\partial,k}(U, V)$	The functions $\gamma \in \mathcal{BC}^k(U, Y)$ such that $\operatorname{dist}(\gamma(U), Y \setminus V) > 0$	13
$\mathcal{BC}^k(U, V)_0$	The subset of functions in $\mathcal{BC}^k(U, Y)$ mapping 0 to 0	13
$\mathcal{C}^\infty_c(U, V)$	The compactly supported smooth functions defined on U taking values in V	14
$\mathcal{C}^k_{\mathcal{W}}(U, Y)$	The set of k-times differentiable functions from U to Y that and whose derivatives are $\mathcal{W}$-bounded	12, 36
$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$	The functions $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$ such that $\operatorname{dist}(\gamma(U), Y \setminus V) > 0$	13
$\mathcal{C}^k_{\mathcal{W}}(U, Y)^o$	The set of functions in $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ whose seminorms decay outside of bounded sets	13

$\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$	The set of functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ whose seminorms decay outside of compact sets	40
$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$	The set of functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ taking values in V	40
$\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$	The map $(\gamma,\eta)\mapsto\gamma\circ(\eta+\mathrm{id})$, restricted to certain weighted function spaces	54
$\mathcal{D}(U,V)$	See $\mathcal{C}^\infty_c(U,V)$	14, *see* $\mathcal{C}^\infty_c(U,V)$
$I^V_{\mathcal{W}}$	The map $\phi\mapsto(\phi+\mathrm{id}_U)^{-1}-\mathrm{id}_V$, defined on $\Omega^{U,V}_{\mathcal{W}}$	56
$\lVert\cdot\rVert_{f,k}$	Supremum of the operator norm of the k-th Fréchet derivative multiplied with f. These quasinorms define the spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)$	12
$\Omega^{U,V}_{\mathcal{W}}$	The maps $\phi\in\mathcal{C}^\infty_{\mathcal{W}}(U,X)$ for which $\phi+\mathrm{id}_U$ is injective, and its image contains V	56
$\mathcal{W}_{\max}$	For $\mathcal{W}\subseteq\overline{\mathbb{R}}^U$, the set of functions $f:U\to\overline{R}$ such that $\lVert\cdot\rVert_{f,0}$ is continuous on $\mathcal{C}^0_{\mathcal{W}}(U,Y)$, for each normed space Y. Called the *maximal extension* of $\mathcal{W}$.	114

Lie groups and manifolds

Evol^ℓ_G	The left evolution	184
evol^ℓ_G	The endpoint of the left evolution	184
Evol^ρ_G	The right evolution	184
evol^ρ_G	The endpoint of the right evolution	184
$\exp_G$	The exponential function of the Lie group G	184
$\dot{\omega}$	For a group action ω, this denotes some kind of "derivation" at the unital element	112
$\mathbf{L}(\cdot)$	The Lie algebra functor	183
$\delta_\ell(\cdot)$	The left logarithmic derivative	183
$\delta_\rho(\cdot)$	The right logarithmic derivative	184
$\mathbf{T}M$	The tangent bundle of the manifold M	181
$\mathbf{T}f$	For a differentiable map $f:M\to N$ between the manifolds M and N, this denotes the tangent map $\mathbf{T}M\to\mathbf{T}N$	181
$\mathbf{T}_xf$	The restriction of $\mathbf{T}f$ to $\mathbf{T}_xM$ and $\mathbf{T}_{f(x)}N$, respectively	181
$\mathbf{T}_1f$, $\mathbf{T}_2f$	For a $\mathcal{C}^1$-map f defined on a product $M\times N$, these denote the partial tangent maps	181
$\mathbf{T}_xM$	The tangent space at the point x of the manifold M	181
$\mathfrak{X}(M)$	The set of smooth vector fields of the manifold M	182

Restricted products

$\mathfrak{c}^{Y,k}_{\mathcal{W},\ell}$	Has two meanings. If $Y=(Y_i)_{i\in I}$ is a family and the weights in $\mathcal{W}$ have their domain in a disjoint union (also indexed over I), this is the product $\prod_{i\in I}\mathfrak{c}^{Y_i,k}_{\mathcal{W}_i,\ell}:(\gamma_i,\eta_i)_{i\in I}\mapsto(\gamma_i\circ(\eta_i+\mathrm{id}))_{i\in I}$	89, 96
$\mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I}$	For families $(U_i)_{i\in I}$, $(Y_i)_{i\in I}$ of open sets resp. normed spaces and $\mathcal{W}\subseteq\overline{\mathbb{R}}^{\bigcup_{i\in I}U_i}$, the space $\ell^\infty_{\mathcal{J}}((\mathcal{C}^k_{\mathcal{W}_i}(U_i,Y_i))_{i\in I})$. Here J consists of the seminorms $\lVert\cdot\rVert_{f,\ell}$	78

$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U_i,Y_i)_{i\in I}$	This is $\mathcal{C}_{\mathcal{W}}^{(1_{\cup_{i\in I}U_i})_\partial,k}(U_i,V_i)_{i\in I}$	79
$\mathcal{C}^k_{\mathcal{W}}(M,\mathbf{T}M)_{\mathcal{A}}$	The subspace of $\mathcal{C}^k_{\mathcal{W}_{\mathcal{A}}}(U_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}}$ that consists of vector fields	91
$\mathcal{C}^{\omega\partial,k}_{\mathcal{W}}(U_i,V_i)_{i\in I}$	The functions in $\mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I}$ which not only take their values in $(V_i)_{i\in I}$, but whose image also has a distance from $(Y_i\setminus V_i)_{i\in I}$ which is adjusted by ω (which must not take 0 as value)	79
$\iota^{\mathcal{A}}_{\mathcal{W}}$	The inclusion map from $\mathcal{C}^k_{\mathcal{W}}(M,\mathbf{T}M)_{\mathcal{A}}$ to $\mathcal{C}^k_{\mathcal{W}_{\mathcal{A}}}(U_\kappa,\mathbb{R}^d)_{\kappa\in\mathcal{A}}$	92
$I^V_{\mathcal{W}}$	Has two meanings. If $V=(V_i)_{i\in I}$ is a family and the weights in $\mathcal{W}$ have their domain in a disjoint union (also indexed over I), this is the product $\prod_{i\in I} I^{V_i}_{\mathcal{W}_i} : (\phi_i)_{i\in I}\mapsto ((\phi_i+\mathrm{id}_{U_i})^{-1}\lvert_{V_i} - \mathrm{id}_{V_i})_{i\in I}$, where each U_i is a certain superset of V_i	90, 96
$\ell^\infty_J((E_i)_{i\in I})$	For a family $(E_i)_{i\in I}$ of locally convex spaces such that for each space there exists a family $(p^i_j)_{j\in J}$ of generating seminorms, the subset of $\prod_{i\in I} E_i$ to the seminorms $(\sup_{i\in I} p^j_i\circ\pi_i)_{j\in J}$	74
$\lVert(\phi_i)_{i\in I}\rVert_{f,\ell}$	The ℓ^∞ quasinorm on $\prod_{i\in I}\mathcal{C}^k_{\mathcal{W}_i}(U_i,Y_i)$ for $f\in\overline{\mathbb{R}}^{\cup_{i\in I}U_i}$ and $\ell\in\mathbb{N}$ with $\ell\leq k$. Defines $\mathcal{C}^k_{\mathcal{W}}(U_i,Y_i)_{i\in I}$	78
$\lVert\cdot\rVert_{\mathcal{A},f,\ell}$	Has two meanings. The first is as the quasinorm for $\prod_{\kappa\in\mathcal{A}}\mathcal{C}^k(U_\kappa,\mathbb{R}^d)$ (where U_κ is the image of κ and $\ell\leq k$). The second is as quasinorm on vector fields, which is defined by the applying the quasinorm to the family $(X_\kappa)_{\kappa\in\mathcal{A}}$ of localizations. With its second meaning, defines the space $\mathcal{C}^k_{\mathcal{W}}(M,\mathbf{T}M)_{\mathcal{A}}$	91
$E^{\mathcal{W}_{\mathcal{B}},g}_{V,\delta}$	(Simultaneous) Superposition with $\mathcal{E}^g_{V,\delta}$	95
$E^{\mathcal{W}_{\mathcal{B}},g}_{V,\delta}$	(Simultaneous) Superposition with $\mathcal{L}^g_{V,\delta}$	95

Riemannian geometry and manifolds

$\mathcal{A}^{\cap\mathcal{B}}$	For two atlases $\mathcal{A}$, $\mathcal{B}$ for the manifold M, this is the atlas that consists of the charts of $\mathcal{A}$ whose domains have been intersected with the chart domains of $\mathcal{B}$. Can be indexed over $\mathcal{A}\otimes\mathcal{B}$	92
$\mathcal{A}\otimes\mathcal{B}$	For two atlases $\mathcal{A}$, $\mathcal{B}$ for the manifold M, this denotes the subset of the product $\mathcal{A}\times\mathcal{B}$ where the two chart domains have nonempty intersection	92
$f_\kappa, g_\kappa, X_\kappa$	Certain "localizations" for objects defined on a manifold and a chart κ for the manifold. Defined for vector fields, metrics, functions	92, 187
$f_{\mathcal{A}}$	For $f\in\overline{\mathbb{R}}^M$ and an atlas $\mathcal{A}$ for M, this is the family of localizations $(f_\kappa)_{\kappa\in\mathcal{A}}$	92
$\mathcal{W}_\kappa$	For $\mathcal{W}\subseteq\overline{\mathbb{R}}^M$ and a chart κ for M, this is the set of localizations $\{f_\kappa : f\in\mathcal{W}\}$	92
$\mathcal{W}_{\mathcal{A}}$	For $\mathcal{W}\subseteq\overline{\mathbb{R}}^M$ and an atlas $\mathcal{A}$ for M, this is the set $\{f_{\mathcal{A}} : f\in\mathcal{W}\}$ of localized families	92
Q^g_K	For a relative compact subset $K\subseteq U\subseteq\mathbb{R}^d$ and a Riemannian metric g on U, this term compares a given norm on $\mathbb{R}^d$ with the norms defined by g	191

$\exp_g$	The exponential function to the Riemannian metric g. Defined on D_g^E	186
D_g^E	The maximal domain for the Riemannian exponential function to the Riemannian metric g	186
$\log_g$	The logarithmic function to the Riemannian metric g. Defined on D_g^L	187
D_g^L	The domain for the Riemannian logarithm to the Riemannian metric g	187
$\lg_g$	For a Riemannian manifold (U,g), where $U \subseteq \mathbb{R}^d$ is open, this is $\mathrm{pr}_2 \circ \log_g$	187

Further notation involving Riemannian exponential function and logarithm

$C^{E,(1)}_{K,\delta,g}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{E,g}_{K,U}$, this is the number $\|D_2 \exp_g\|_{1_{\overline{K}\times\overline{B}_\delta(0)},0}$	188
$C^{E,2}_{K,\delta,g}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{E,g}_{K,U}$, this is the number $\|\exp_g\|_{1_{\overline{K}\times\overline{B}_\delta(0)},2}$	188
$C^{L,(1)}_{K,\delta,g}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{L,g}_{K,U}$, the number $\|D_2 \lg_g\|_{1_{\overline{K}\times\delta},0}$	192
$C^{L,2}_{K,\delta,g}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{L,g}_{K,U}$, the number $\|\lg_g\|_{1_{\overline{K}\times\delta},2}$	192
$\mathcal{E}^g_{K,\delta}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{E,g}_{K,U}$, , the function $K \times B_\delta(0) \to U : (x,y) \mapsto \exp_g(x,y) - x$	95
$\mathcal{L}^g_{K,\delta}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open, relatively compact $K \subseteq U$ and $\delta < R^{L,g}_{K,U}$, , the function $K \times B_\delta(0) \to \mathbb{R}^d : (x,y) \mapsto \lg_g(x, x+y)$	95
$R^{E,g}_{K,U}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open and relatively compact $K \subseteq U$, the supremum of all τ such that $\overline{K} \times B_\tau(0) \subseteq D_g^E$	188
$R^g_{K,\sigma}$	For a Riemannian manifold (U,g), where $U \subseteq \mathbb{R}^d$ is open, relatively compact $K \subseteq U$ and $\sigma \in]0,1[$, this is the supremum of all $\tau < R^{E,g}_{K,U}$ such that $\|\exp_x - \mathrm{id}_{\mathbb{R}^d}\|_{1_{\overline{B}_\tau(0)},1} < \sigma$ for $x \in \overline{K}$	191
$R^{L,g}_{K,U}$	For a Riemannian manifold (U,g) with $U \subseteq \mathbb{R}^d$ open and relatively compact $K \subseteq U$, the supremum of $\tau > 0$ such that $\bigcup_{x\in\overline{K}} \{x\} \times B_\tau(x) \subseteq D_g^L$	192

Groups and monoids of functions

$\kappa_{\mathcal{W}}$	The inverse of the canonical chart for $\mathrm{End}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$	50
$\mathcal{C}^\ell_{\mathcal{W}}(U,G)$	Lie group of weighted mappings that take values in a Banach Lie group, modelled on $\mathcal{C}^\ell_{\mathcal{W}}(U, \mathbf{L}(G))$	128

$\mathcal{C}^\ell_{\mathcal{W}}(U,G)^\bullet$	Lie group of decaying weighted mappings that take values in a Lie group, modelled on $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$	135
$\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$	Lie group normalizing $\mathcal{C}^\ell_{\mathcal{W}}(U,G)^\bullet$	136
$\mathrm{Diff}(X)$	The set of all diffeomorphisms of the Banach space X	50
$\mathrm{Diff}_c(M)$	The diffeomorphisms of a manifold M that coincide with the identity outside some compact set	6
$\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$	The set of diffeomorphisms of $\mathbb{R}^n$ differing from $\mathrm{id}_{\mathbb{R}^n}$ by a rapidly decreasing $\mathbb{R}^n$-valued map	6
$\mathrm{Diff}_{\mathcal{W}}(X)$	The set of weighted diffeomorphisms of the Banach space X to the weights $\mathcal{W}$. Is a Lie group modelled on $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$	50
$\mathrm{Diff}_{\mathcal{W}}(X)^\circ$	$\mathrm{Diff}_{\mathcal{W}}(X) \cap \mathrm{End}_{\mathcal{W}}(X)^\circ$	63
$\mathrm{Diff}_{\mathcal{W}}(X)_0$	The identity component of $\mathrm{Diff}_{\mathcal{W}}(X)$	112
$\mathrm{End}_{\mathcal{W}}(X)$	The set of weighted endomorphisms of the Banach space X to the weights $\mathcal{W}$	50
$\mathrm{End}_{\mathcal{W}}(X)^\circ$	The functions $\phi \in \mathrm{End}_{\mathcal{W}}(X)$ such that $\phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$	55

Further notation

$\mathrm{L}^k(X,Y)$	Space of k-linear maps from the normed space X to the normed space Y, endowed with the topology induced by the operator norm	164
$\mathcal{N}(X)$	For a locally convex space X, this denotes the set of continuous seminorms on X	161
X_p	For a locally convex space X and $p \in \mathcal{N}(X)$, this denotes the quotient space $X/p^{-1}(0)$	161
π_p	For a locally convex space X and $p \in \mathcal{N}(X)$, this denotes the quotient map $X \to X_p$	161
$\lVert\gamma\rVert_{p,f,k}$	For a suitable map γ taking values in the locally convex space Y and $p \in \mathcal{N}(Y)$, this is $\lVert\pi_p\rVert_{f,k}$	36
$\lVert\cdot\rVert_{op}$	The operator norm of a k-linear map between normed spaces	164
$\lVert T\rVert_{op,p}$	The operator norm of the linear operator $T : X \to Y_p$, where X is a normed space and Y a locally convex space, with respect to $p \in \mathcal{N}(Y)$	38
QI_A	Quasi-Inversion map $A^q \to A^q$ of the algebra A	194
A^q	The set of quasi-invertible elements of the algebra A	194

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