Direct Limit Constructions in Infinite Dimensional Lie Theory

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Konstruktion neuer Klassen von unendlichdimensionalen Lie-Gruppen, welche eine Verallgemeinerung bereits bekannter Klassen darstellen. Anschließend soll deren Regularität (nach der Definition John Milnors) bewiesen werden, welche dann insbesondere auch für die bereits bekannten Klassen gilt. Dadurch kann zum Beispiel die offene Frage nach der Regularität der von Pisanelli eingeführten Lie-Gruppen beantwortet werden.

Im ersten Kapitel werden die Voraussetzungen für die weitere Arbeit geschaffen, indem Grundlagen der linearen und nichtlinearen Funktionalanalysis sowie der unendlichdimensionalen Lie-Theorie erläutert werden. Im Einzelnen werden dabei folgende Konzepte definiert:

- Differenzierbare und analytische Abbildungen in lokal-konvexen topologischen Vektorräumen nach Michal-Bastiani
- Direkte Limites von aufsteigenden Folgen von Banachräumen ((LB)-Räume); diese werden im weiteren Verlauf der Arbeit als Modell-Räume der zu konstruierenden Lie-Gruppen verwendet.
- Unendlich-dimensioale Lie-Gruppen sowie der Regularitätsbegriff nach John Milnor

Im zweiten Kapitel beweise ich einen Satz über analytische Abbildungen zwischen (LB)-Räumen. Dieses von mir bereits publizierte Resultat (siehe [5]) dient im Folgenden zur Konstruktion der Lie-Gruppenstrukturen und zum Beweis der Regularität.

Das dritte Kapitel enthält die Konstruktion einer Lie-Gruppe, welche aus Keimen von analytischen Diffeomorphismen um ein Kompaktum in einem reellen oder komplexen Banachraum besteht. Deren Konstruktion war bisher nur möglich, wenn der Banachraum endlichdimensional war. Im Anschluss daran wird die Regularität der so konstruierten Gruppe bewiesen. Dies war selbst im Falle eines eindimensionalen Banachraums mit einpunktigem Kompaktum bisher nicht möglich.

In Kapitel 4 werden aufsteigende Vereinigungen einer Folge von Banach-Liegrupen untersucht und — falls gewisse leicht zu überprüfende Bedingungen erfüllt sind — mit einer (LB)-Lie-Gruppenstruktur versehen. Anschließend wird auch bei diesen (LB)-Lie-Gruppen die Frage nach der Regularität beantwortet. Als Hilfsmittel werden in diesem Kapitel außerdem auch lokale Lie-Gruppen untersucht.

Schließlich werde ich in Kapitel fünf an einigen Beispielen zeigen, wie man mit Hilfe der Ergebnisse aus dem vierten Kapitel neue Klassen regulärer (LB)-Lie-Gruppen erhält, beziehungsweise wie man für bereits bekannte Lie-Gruppen die bislang offene Frage nach der Regularität beantworten kann.

Introduction

Infinite dimensional Lie theory is an area of mathematics, connecting group theoretic questions with (both linear and non-linear) functional analysis. An infinite dimensional Lie group is a group whose elements can be parameterized via elements in an infinite dimensional locally convex topological vector space, the modelling space. So far only some general structure theory is available. There are many open problems.

The most prominent example of an infinite dimensional Lie group is the diffeomorphism group of a compact manifold. Unfortunately, the diffeomorphism group of a *noncompact* manifold carries no canonical Lie group structure (more precisely, it carries no Lie group structure modelled on the Lie algebra of smooth vector fields). However, one can consider the group of compactly supported diffeomorphisms which is a union of Lie groups and can be given a Lie group structure itself. This leads to the question, under which circumstances the union of a family of Lie groups is again a Lie group.

The aim of this dissertation is to construct new, interesting classes of of infinite dimensional Lie groups. Most of these Lie groups are generalizations of known Lie groups. Furthermore, we show their regularity in Milnor's sense. This will imply the regularity of some Lie groups which where already constructed but not known to be regular so far.

In Chapter 1 we fix some notation and discuss the preliminaries concerning our differential calculus, Lie groups, local Lie groups, regularity of Lie groups (in Milnor's sense) and direct limits of locally convex topological vector spaces.

In Chapter 2 we prove our main tool for both constructing new Lie groups and showing regularity of them. Theorem 2.1 is a sufficient criterion for mappings defined on direct limits of normed spaces to be complex analytic.

In 1976, Pisanelli showed in [16] that the germs of holomorphic diffeomorphisms in $\mathbb{C}^n$ form an infinite dimensional Lie group. In Chapter 3, we will generalize this concept to germs of analytic diffeomorphisms around a compact set in a possibly infinite dimensional Banach space. Furthermore, we will show that all these Lie groups obtained in this fashion are regular (in Milnor's sense). In particular this implies regularity of Pisanelli's original example, which has been an open problem before (Problem VI.5 in [15]).

A result by Glöckner (see [8]) shows that the directed union of a sequence of *finite dimensional* Lie groups can always be given an (LB)-Lie group structure. In Chapter 4, we show how to construct regular Lie group structures on ascending unions of a sequence of (possibly infinite dimensional) Banach Lie groups.

As a tool in showing regularity, we will use the concept of *local* Lie groups.

In Chapter 5, we will give some examples of cases where the situation of chapter 4 occurs.

1 Preliminaries

1.1 Infinite dimensional differential calculus

1.1.1 C^k and FC^k mappings

We begin with two different notions of differentiability in infinite dimensional vector spaces: (Details can be found in [7], [10], [13] and in [15].)

Definition 1.1.1 (C^k in the sense of Michal-Bastiani)**.** Let X and Z be Hausdorff locally convex topological vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let Ω be an open nonempty subset of X.

(i) A mapping $\gamma\colon \Omega \longrightarrow Z$ is called $C^0_{\mathbb{K}}$, if it is continuous.

(ii) A mapping $\gamma\colon \Omega \longrightarrow Z$ is called $C^1_{\mathbb{K}}$, if for each $(x, v) \in \Omega \times X$ the directional derivative

$$d\gamma(x, v) := \lim_{\substack{t \to 0 \\ t \in \mathbb{K}}} \frac{\gamma(x + tv) - \gamma(x)}{t}$$

exists and if the map

$$d\gamma\colon \Omega \times X \longrightarrow Z$$

is continuous.

(iii) Inductively, we say that $\gamma\colon \Omega \longrightarrow Z$ is of class $C^k_{\mathbb{K}}$ if it is $C^1_{\mathbb{K}}$ and if $d\gamma\colon \Omega \times X \longrightarrow Z$ is $C^{k-1}_{\mathbb{K}}$. We call γ *smooth* or $C^\infty_{\mathbb{K}}$ if γ is $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}$. If the ground field $\mathbb{K}$ is clear from the context, we write C^k instead of $C^k_{\mathbb{K}}$.

From this definition it follows that if γ is $C^1_{\mathbb{K}}$ and $x \in \Omega$, then the following is a continuous $\mathbb{K}$-linear map:

$$\gamma'(x) := d\gamma(x, \cdot)\colon X \longrightarrow Z : v \mapsto d\gamma(x, v).$$

The following definition of differentiability is more well-known but has the disadvantage that it only works in normed spaces:

Definition 1.1.2 (FC^k-maps)**.** Let X and Z be normed spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let Ω be an open nonempty subset of X.

(i) A mapping $\gamma\colon \Omega \longrightarrow Z$ is called *Fréchet differentiable* at the point $x \in X$ if there exists a bounded linear operator $T \in \mathcal{L}(X, Z)$ such that

$$\lim_{v \to 0} \frac{\gamma(x + tv) - \gamma(x) - Tv}{\|v\|_X} = 0$$

(in this case, this map T is equal to $\gamma'(x) = d\gamma(x, \cdot)$ as defined in Definition 1.1.1).

(ii) The map γ is called $FC^1_{\mathbb{K}}$ if it is everywhere Fréchet differentiable and the map

$$\gamma'\colon \Omega \longrightarrow \left(\mathcal{L}(X, Z), \|\cdot\|_{\mathrm{op}}\right) : x \mapsto \gamma'(x) = d\gamma(x, \cdot)$$

is continuous.

(iii) Inductively, we say that $\gamma\colon \Omega \longrightarrow Z$ is of class $FC^k_{\mathbb{K}}$ if it is $FC^1_{\mathbb{K}}$ and if $\gamma'\colon \Omega \longrightarrow \mathcal{L}(X, Z)$ is $FC^{k-1}_{\mathbb{K}}$. We will use the notation $\gamma^{(1)} := \gamma'$ and

$$\gamma^{(k)}(x)(v_1, \ldots, v_k) := \left(\gamma^{(k-1)}\right)'(x)(v_1)(v_2, \ldots, v_k).$$

Note that each $\gamma^{(k)}(x)\colon X^k \longrightarrow Z$ is a continuous symmetric k-linear map.

Again, if the ground field $\mathbb{K}$ is clear from the context, we write FC^k instead of $FC^k_{\mathbb{K}}$.

These two notions are connected via the following

Lemma 1.1.3 (Criterion of Fréchet Differentiability)**.** *Let X, Z be normed spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\Omega \subseteq X$ open. Then $\gamma\colon \Omega \longrightarrow Z$ is FC^1 if and only if it is C^1 and the map*

$$\gamma'\colon \Omega \longrightarrow \left(\mathcal{L}(X, Z), \|\cdot\|_{\mathrm{op}}\right) : x \mapsto \gamma'(x) = d\gamma(x, \cdot)$$

is continuous.

Proof. If γ is FC^1, it is clearly C^1 and γ' is continuous. Conversely, we assume that γ is C^1 and that γ' is continuous. We will show that γ is Fréchet differentiable at each point. Therefore, let $x \in \Omega$ be fixed and let v so small that the interval $[x, x + v] := \{x + tv : t \in [0, 1]\}$ lies in Ω. Then we define the curve

$$\eta_v\colon [0, 1] \longrightarrow Z : t \mapsto \gamma(x + tv).$$

Since γ is C^1, the curve η_v is also C^1 with

$$\eta_v'(t) = d\gamma(x + tv, v) = \gamma'(x + tv).v.$$

Now, we can write:

$$\begin{aligned}
\left\|\frac{\gamma(x+v)-\gamma(x)-\gamma'(x).v}{\|v\|_X}\right\|_Z &= \frac{1}{\|v\|_X}\left\|\eta_v(1)-\eta_v(0)-\gamma'(x).v\right\|_Z \\
&= \frac{1}{\|v\|_X}\left\|\int_0^1 \eta_v'(t)\; dt-\gamma'(x).v\right\|_Z \\
&= \frac{1}{\|v\|_X}\left\|\int_0^1 \left(\gamma'(x+tv).v-\gamma'(x).v\right) dt\right\|_Z \\
&= \frac{1}{\|v\|_X}\int_0^1 \left\|\left(\gamma'(x+tv)-\gamma'(x)\right).v\right\|_Z \; dt \\
&\leq \int_0^1 \left\|\gamma'(x+tv)-\gamma'(x)\right\|_{\mathrm{op}} dt
\end{aligned}$$

The map $\gamma'\colon \Omega \longrightarrow \mathcal{L}(X,Z)$ is continuous by assumption. Therefore, the integrand on the right hand side of this inequality is continuous in t and in v. So, the theorem of parameter dependent integrals yields that the integral tends to 0, when v converges to 0. This concludes the proof. $\square$

Proposition 1.1.4. *Let X, Z be normed spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $\Omega \subseteq X$ open. Let $\gamma\colon \Omega \longrightarrow Z$ be a C^{k+1} map, $k \geq 0$. Then γ is FC^k. In particular, γ is C^∞ if and only if FC^∞.*

Proof. See e.g. Lemmas A.4.1 and A.4.3 in [18]. $\square$

Remark. Although for mappings between normed spaces FC^∞ and C^∞ are equivalent, the notions C^k and FC^k are not equivalent. An example (due to H. Glöckner and K.-H. Neeb) is the following C^1-map:

$$\begin{aligned}
f : L^2([0,1]) &\longrightarrow L^2([0,1]) \\
\gamma &\longmapsto \sin\circ\gamma.
\end{aligned}$$

If it were FC^1, it would satisfy the hypotheses of the Inverse Function Theorem for Banach spaces (see for example Theorem 1.1.21), so it would be a local diffeomorphism. However, this contradicts the fact that the image of f contains no 0-neighborhood.

Definition 1.1.5 (Differential calculus on non-open subsets of normed spaces)**.** Let X and Z be normed spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Let $A \subseteq X$ be a subset with dense interior A° which is *locally convex* in the sense that for every $a \in A$ there is a convex set $C \subseteq A$ which is a neighborhood of a in the induced topology on A. We say that a map $\gamma\colon A \longrightarrow Z$ is $FC^1_{\mathbb{K}}$ if γ is continuous, $\gamma|_{A^\circ}$ is $FC^1_{\mathbb{K}}$ and if γ' can be extended to continuous map on A. This generalization of Definition 1.1.2 since for open A, the two definitions agree.

Remark. (i) One can also generalize the concepts of FC^k-maps and C^k-maps (also between locally convex spaces) to this setting of locally convex domains with dense interior.

(ii) Most of the statements for C^k- and FC^k-maps which are stated for open sets generalize to this more general setting, e.g. the chain rule. (see [10] for further details).

(iii) We will use this concept only in the case where A is a product of an open set with the unit interval $[0,1]$.

1.1.2 Polynomials between normed spaces

Here and in the rest of the thesis, $\mathrm{B}_r^X(a)$ denotes the open ball of radius r around the point a in the space X.

Definition 1.1.6. (a) Let $f\colon X^k \longrightarrow Y$ be a continuous symmetric k-linear map between normed spaces. Then we define the *operator norm* of f as:

$$\|f\|_{\mathrm{op}} := \left\| f|_{\mathrm{B}_1^X(0)\times\cdots\times\mathrm{B}_1^X(0)} \right\|_\infty = \sup\left\{ \|f(x_1,\ldots,x_k)\|_Y : \|x_j\|_X < 1 \right\}.$$

The space of all k-linear symmetric continuous maps from X^k to Y together with this operator norm will be denoted by $\mathrm{Sym}_c^k(X,Y)$.

(b) Let $p\colon X \longrightarrow Y$ be a continuous homogeneous polynomial, i.e.

$$p(x) = f(x,\ldots,x)$$

for an $f \in \mathrm{Sym}_c^k(X,Y)$. Then we define the *operator norm* of p as:

$$\|p\|_{\mathrm{op}} := \left\| p|_{\mathrm{B}_1^X(0)} \right\|_\infty = \sup\left\{ \|p(x)\|_Y : \|x\|_X < 1 \right\}.$$

The space of all continuous k-homogeneous polynomials from X to Y together with this operator norm will be denoted by $\mathrm{Pow}_c^k(X,Y)$.

(c) Let $\gamma\colon X \longrightarrow Y$ be a continuous polynomial, i.e. a function that can be written as a finite sum of j-homogeneous polynomials for $j \le k$. Then we define the *operator norm* of γ as:

$$\|\gamma\|_{\mathrm{op}} := \left\| \gamma|_{\mathrm{B}_1^X(0)} \right\|_\infty = \sup\left\{ \|\gamma(x)\|_Y : \|x\|_X < 1 \right\}.$$

The space of all continuous polynomials from X to Y of degree at most k together with this operator norm will be denoted by $\mathrm{Pol}_c^k(X,Y)$.

These notions are obviously generalizations of the operator norm of a continuous linear map between normed spaces:

$$(\mathrm{Sym}_c^1(X,Y), \|\cdot\|_{\mathrm{op}}) = (\mathrm{Pow}_c^1(X,Y), \|\cdot\|_{\mathrm{op}}) = (\mathcal{L}(X,Y), \|\cdot\|_{\mathrm{op}}).$$

Remark. The normed space $(\mathrm{Sym}_c^k(X,Y), \|\cdot\|_{\mathrm{op}})$ can be isometrically embedded in the normed space $BC\Big(\big(\mathrm{B}_1^X(0)\big)^k, Y\Big)$ of bounded continuous functions on the k-th power of the open unit ball, endowed with the supremum norm $\|\cdot\|_\infty$. Similarly, the spaces $\mathrm{Pow}_c^k(X,Y)$ and $\mathrm{Pol}_c^k(X,Y)$ can be regarded as subspaces of $BC\big(\mathrm{B}_1^X(0), Y\big)$.

Proposition 1.1.7 (Derivatives of homogeneous polynomials)**.** *Let X and Y be normed vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $k \in \mathbb{N}$, $f \in \mathrm{Sym}_c^k(X,Y)$ be a continuous symmetric k-linear map and*

$$p\colon X \longrightarrow Y : x \mapsto f(x, \ldots, x)$$

be the corresponding k-homogeneous polynomial. Then the Fréchet derivative of p is a $(k-1)$-homogeneous polynomial:

$$\begin{aligned} p' : X &\longrightarrow \mathcal{L}(X,Y) \\ x &\longmapsto p'(x) : (v \mapsto k f(x, \ldots, x, v)) . \end{aligned}$$

Proof. (Sketch) Write the difference quotient

$$\frac{1}{t}\big(f(x+tv, \ldots, x+tv) - f(x, \ldots, x)\big)$$

as a telescoping sum. □

One of the most important formulas when dealing with polynomials between infinite dimensional spaces is the following:

Proposition 1.1.8 (Polarization formula)**.** *Let X and Y be vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $k \in \mathbb{N}_0$ and set $I := \{1, \ldots, k\}$. Let $f\colon X^k \longrightarrow Y$ be a symmetric k-linear map and let*

$$p\colon X \longrightarrow Y : x \mapsto f(x, \ldots, x)$$

be the corresponding k-homogeneous polynomial. Then the values of f can be recovered from p via

$$f(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{F \subseteq I} (-1)^{|I \setminus F|}\, p\Big(\sum_{j \in F} x_j\Big)$$

Proof. See for example Theorem A in [2]. □

Corollary 1.1.9. *Let X and Y be normed vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $k \in \mathbb{N}_0$, let $f \in \mathrm{Sym}_c^k(X,Y)$ be a continuous symmetric k-linear map and*

$$p\colon X \longrightarrow Y : x \mapsto f(x, \ldots, x)$$

be the corresponding k-homogeneous polynomial. Then we have the following estimates:

(a) $\|p\|_{\mathrm{op}} \leq \|f\|_{\mathrm{op}}$.

(b) $\|f\|_{\mathrm{op}} \leq \frac{(2k)^k}{k!} \|p\|_{\mathrm{op}} \leq (2e)^k \|p\|_{\mathrm{op}}$.

(c) $\|p'\|_{\mathrm{op}} \leq k \|f\|_{\mathrm{op}} \leq k(2e)^k \|p\|_{\mathrm{op}}$.

(d) $\|p''\|_{\mathrm{op}} \leq k(k-1)(2e)^{2k-1} \|p\|_{\mathrm{op}}$.

Proof. Part (a) is clear by the definition. Part (b) is a direct consequence of the polarization formula (Proposition 1.1.8) and the well-known formula $k^k/k! \leq e^k$. Part (c) uses Proposition 1.1.7, together with part (b). Part (d) is just part (c) applied twice. □

From Corollary 1.1.9 (a) and (b), it follows that $\mathrm{Sym}_c^K(X,Y)$ and $\mathrm{Pow}_c^k(X,Y)$ are topologically isomorphic. We will now turn to the space $\mathrm{Pol}_c^k(X,Y)$:

Proposition 1.1.10 (Interpolation of Polynomials)**.** *Let X and Z be normed spaces over $\mathbb{K}$ and let $k \in \mathbb{N}_0$ be given. Then the map*

$$\begin{aligned} \prod_{j=0}^{k} \left(\mathrm{Pow}_c^j(X,Z), \|\cdot\|_{\mathrm{op}}\right) &\longrightarrow \left(\mathrm{Pol}_c^k(X,Z), \|\cdot\|_{\mathrm{op}}\right) \\ (\gamma_j)_j &\longmapsto \textstyle\sum_{j=0}^k \gamma_j \end{aligned}$$

is a topological isomorphism.

Proof. The map is clearly bijective and continuous. It remains to show that for every $j_0 \leq k$ the coefficient map

$$\begin{aligned} \left(\mathrm{Pol}_c^k(X,Z), \|\cdot\|_{\mathrm{op}}\right) &\longrightarrow \left(\mathrm{Pow}_c^{j_0}(X,Z), \|\cdot\|_{\mathrm{op}}\right) \\ \gamma = \textstyle\sum_{j=0}^k \gamma_j &\longmapsto \gamma_{j_0} \end{aligned}$$

is continuous.

We fix a subset $F \subseteq]0,1[$ with $k+1$ elements. For every point $\mu \in F$ we define the corresponding Lagrange polynomial:

$$\Lambda_\mu(t) := \prod_{\substack{\nu \in F \\ \nu \neq \mu}} \frac{t-\nu}{\mu-\nu} = \sum_{j=0}^{k} \lambda_{\mu,j}\ t^j \in \mathbb{R}[t].$$

This is the unique polynomial of degree k such that $\Lambda_\mu(\nu) = \delta_{\mu,\nu}$ for $\nu \in F$. The coefficients $\lambda_{\mu,j} \in \mathbb{R}$ depend only on k and F and are therefore considered fixed for the rest of the proof.

Now, suppose that a function $g\colon F \longrightarrow Z$ from the finite set F to the normed space Z is given. Then there is a unique polynomial $\widetilde{g}\colon \mathbb{K} \longrightarrow Z$ such that $\widetilde{g}|_F = g$. This polynomial is given by:

$$\widetilde{g}(t) := \sum_{\mu \in F} g(\mu) \cdot \Lambda_\mu(t) = \sum_{j=0}^{k} \left(\sum_{\mu \in F} g(\mu) \cdot \lambda_{\mu,j}\right) t^j$$

We can estimate the norm of the j-th coefficient of $\widetilde{g}$:

$$\left\| \sum_{\mu \in F} g(\mu) \cdot \lambda_{\mu,j} \right\|_Z \leq \sum_{\mu \in F} |\lambda_{\mu,j}| \, \|g\|_\infty .$$

Now, we consider a continuous polynomial $\gamma = \sum_{j=0}^{k} \gamma_k \colon X \longrightarrow Z$, where each γ_j is a continuous j-homogeneous polynomial. Let $v \in \mathrm{B}_1^X(0)$. Then $\gamma_{j_0}(v)$ is the j_0-th coefficient of the polynomial

$$g_v(t) := \gamma(tv) = \sum_{j=0}^{k} \gamma_j(v)\ t^j$$

and we can estimate its norm by:

$$\|\gamma_{j_0}(v)\|_Z \leq \sum_{\mu \in F} |\lambda_{\mu,j}| \, \|g_v\|_\infty \leq \sum_{\mu \in F} |\lambda_{\mu,j}| \left\| g|_{\mathrm{B}_1^X(0)} \right\|_\infty .$$

Since $v \in \mathrm{B}_1^X(0)$ was arbitrary, this shows

$$\left\| \gamma_j|_{\mathrm{B}_1^X(0)} \right\|_\infty \leq \left(\sum_{\mu \in F} |\lambda_{\mu,j}| \right) \left\| \gamma|_{\mathrm{B}_1^X(0)} \right\|_\infty$$

which finishes the proof. $\square$

Proposition 1.1.11 (Taylor's Formula)**.** *Let X and Z be normed spaces over $\mathbb{K}$ and let Ω be an open convex subset of X and $x_0 \in X$. Assume $\gamma \colon \Omega \longrightarrow Z$ is FC^k with $k \geq 1$. Then we have for all $v \in X$ such that $x_0 + v \in \Omega$:*

$$\text{(a)} \quad \gamma(x_0 + v) = \sum_{j \leq k-1} \frac{\gamma^{(j)}(x_0)(v, \ldots, v)}{j!} + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \gamma^{(k)}(x_0 + tv)(v, \ldots, v) dt.$$

$$\text{(b)} \quad \gamma(x_0 + v) = \sum_{j \leq k} \frac{\gamma^{(j)}(x_0)(v, \ldots, v)}{j!} + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \Big(\gamma^{(k)}(x_0 + tv) - \gamma^{(k)}(x_0) \Big)(v, \ldots, v) dt.$$

Proof. By setting $h \colon]-r, r[\longrightarrow Z : s \mapsto \gamma(x_0 + sv)$ and using continuous linear functionals on F, we can reduce (a) to the classical formula where X and Z are one-dimensional.

Since $\int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} dt = 1/k!$, part (b) follows from (a). $\square$

Proposition 1.1.12 (Complexifications). *(a) Let E be a locally convex topological vector space over $\mathbb{R}$. Then the* complexification

$$E_{\mathbb{C}} := E \times E$$

with the product topology, pointwise addition and the scalar multiplication

$$\begin{aligned} \mathbb{C} \times E_{\mathbb{C}} &\longrightarrow E_{\mathbb{C}} \\ (u + iv, (x, y)) &\longmapsto (ux - vy, vx + uy) \end{aligned}$$

becomes a locally convex topological vector space over $\mathbb{C}$. We will identify $x \in E$ with $(x, 0) \in E_{\mathbb{C}}$ and treat E as a closed real vector subspace of $E_{\mathbb{C}}$. All linear or polynomial mappings between real vector spaces extend uniquely to their complexifications. The extended mappings are continuous if and only if the original mappings were so.

(b) Let $(E, \|\cdot\|_E)$ be a normed space over $\mathbb{R}$. Then the complexification is again a normed space with respect to the norm

$$\|\tilde{x}\|_{\mathbb{C}} := \inf \left\{ \sum_j |z_j| \, \|x_j\| : \tilde{x} = \sum_j z_j x_j \text{ where } x_j \in E, z_j \in \mathbb{C} \right\}.$$

This norm has the property that for all $x + iy \in E_{\mathbb{C}}$, we have:

$$\max(\|x\|_E, \|y\|_E) \leq \|x + iy\|_{\mathbb{C}} \leq \|x\|_E + \|y\|_E .$$

In particular, the norm induces the given norm on the real subspace E.

(c) Let $T \colon E \longrightarrow F$ be a continuous linear operator between real normed spaces. Then the unique extension

$$T_{\mathbb{C}} \colon E_{\mathbb{C}} \longrightarrow F_{\mathbb{C}} : x + iy \mapsto Tx + iTy$$

satisfies $\|T_{\mathbb{C}}\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$.

(c) Let $\beta \colon E \times E \longrightarrow F$ be a bilinear map with

$$\|\beta(x_1, x_2)\|_F \leq \|x_1\|_E \|x_2\|_E .$$

Then the unique extension

$$\beta_{\mathbb{C}} \colon E_{\mathbb{C}} \times E_{\mathbb{C}} \longrightarrow F_{\mathbb{C}}$$

satisfies

$$\|\beta_{\mathbb{C}}(\tilde{x}_1, \tilde{x}_2)\|_{F_{\mathbb{C}}} \leq 4 \|\tilde{x}_1\|_{E_{\mathbb{C}}} \|\tilde{x}_2\|_{E_{\mathbb{C}}} .$$

Proof. See [2] for more details on complexifications of normed spaces. ∎

1.1.3 Analytic mappings between locally convex spaces

Definition 1.1.13 (Complex analytic mappings)**.** Let E and F be Hausdorff locally convex spaces over $\mathbb{C}$, let $f\colon U \longrightarrow F$ be a mapping from an open subset $U \subseteq E$ with values in F. We say that f is *complex analytic* or $C^{\omega}_{\mathbb{C}}$ if it is continuous and if it admits locally a power series expansion around each point $a \in U$, which means there exist continuous homogeneous polynomials p_k of degree k such that

$$f(x) = \sum_{k=0}^{\infty} p_k(x-a)$$

for all x in a neighborhood of a.

Definition 1.1.14 (Real analytic mappings)**.** Let E and F be Hausdorff locally convex spaces over $\mathbb{R}$ and let $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ be their complexifications. A mapping $f\colon U \longrightarrow F$ from an open subset $U \subseteq E$ with values in F is called *real analytic* or $C^{\omega}_{\mathbb{R}}$ if it extends to a complex analytic $F_{\mathbb{C}}$-valued map on an open neighborhood of U in the complexification $E_{\mathbb{C}}$.

There is an easy characterization of complex analyticity, that can be found in [1, Theorem 6.2] :

Lemma 1.1.15 (Gateaux Analyticity)**.** *Let $f\colon U \subseteq E \longrightarrow F$ be a function defined on some open subset of a complex Hausdorff locally convex space E. Then f is $C^{\omega}_{\mathbb{C}}$ if and only if it is continuous and* Gateaux analytic*, which means that for every point $a \in U$ and every vector $b \in E$ there exists an $\varepsilon > 0$ such that the function*

$$\mathrm{B}^{\mathbb{C}}_{\varepsilon}(0) \longrightarrow F : z \mapsto f(a+zb)$$

is complex analytic.

Proposition 1.1.16 ($C^{\omega}_{\mathbb{C}} = C^{\infty}_{\mathbb{C}}$)**.** *Let $f\colon U \subseteq E \longrightarrow F$ be a function defined on some open subset of a complex Hausdorff locally convex space E.*

(a) Then f is $C^{\omega}_{\mathbb{C}}$ if and only if it is $C^{\infty}_{\mathbb{C}}$ in the sense of Michal-Bastiani (Definition 1.1.1).

(b) If F is assumed to be complete, then $C^{1}_{\mathbb{C}}$ already suffices for f to be complex analytic.

Proof of part (b). In view of Lemma 1.1.15, it suffices to show this for $E = \mathbb{C}$ and $U = \mathrm{B}^{\mathbb{C}}_{\varepsilon}(0)$. Without loss of generality, $\varepsilon = 1$. We set $r := \frac{1}{2}$ and define the following function:

$$g\colon \mathrm{B}^{\mathbb{C}}_{r}(0) \longrightarrow F : z \mapsto \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}dw$$

This integral converges in F, since F is complete. By the rule of parameter-dependent integrals, the function is $C^{\infty}_{\mathbb{C}}$. We can show that $f(z) = g(z)$ for all $z \in \mathrm{B}^{\mathbb{C}}_{r}(0)$ by testing with continuous linear functionals (Hahn-Banach) and using the classical one-dimensional Cauchy-formula. ☐

Lemma 1.1.17 (Complex analyticity in Banach spaces)**.** *Let X and Y be complex Banach spaces and let $f\colon U \longrightarrow Y$ be a $FC^1_{\mathbb{R}}$ function defined on an open subset $U \subseteq X$. Then the following are equivalent:*

(a) f is complex analytic

(b) For every $x \in U$, the Fréchet derivative $f'(x)\colon X \longrightarrow Y$ is complex linear.

Lemma 1.1.18 (Complex analyticity with values in a function space)**.** *Let U be an open subset of a complex Hausdorff locally convex space X, let V be a topological space and Z be a complex Banach space. Assume a continuous function $h\colon U \longrightarrow BC(V,Z)$ is given, such that the map*

$$\begin{aligned} h_y : U &\longrightarrow Z \\ u &\longmapsto h(u)(y) \end{aligned}$$

is $C^\omega_{\mathbb{C}}$ for each $y \in V$. Then $h\colon U \longrightarrow BC(V,Z)$ is $C^\omega_{\mathbb{C}}$.

Proof. By Lemma 1.1.15, it suffices to show that h is Gateaux-analytic. Thus, we may assume that $X = \mathbb{C}$ and $U = \mathrm{B}^{\mathbb{C}}_1(0)$. It remains to show that for each $z \in \mathrm{B}^{\mathbb{C}}_{\frac{1}{2}}(0)$ we have

$$h(z) = \int_{|w|=\frac{1}{2}} \frac{h(w)}{w-z} dw.$$

By completeness of $BC(V,Z)$ and continuity of h we know that the integral on the right exists. We check equality by applying the following point evaluations to both sides:

$$\begin{aligned} \pi_y : BC(V,Z) &\longrightarrow Z \\ \gamma &\longmapsto \gamma(y). \end{aligned}$$

Since these operators separate the points of $BC(V,Z)$ and since $\pi_y \circ h$ is analytic by assumption, the assertion follows. □

Lemma 1.1.19 (Absolute convergence of bounded power series)**.** *Let E and F be complex normed vector spaces and let $f\colon \mathrm{B}^E_R(a) \longrightarrow F$ be a* bounded *complex analytic map with the following power series expansion:*

$$f(a+x) = \sum_{k=0}^{\infty} \beta_k(x,\dots,x)$$

where the $\beta_k\colon E^k \longrightarrow F$ are continuous symmetric k-linear maps. Then for all $r < \frac{R}{2e}$ the following series converges and can be estimated as shown:

$$\sum_{k=0}^{\infty} \|\beta_k\|_{\mathrm{op}} r^k \leq \frac{R}{R-2er} \cdot \|f\|_\infty .$$

Here $\|f\|_\infty := \sup\left\{\|f(x)\| : x \in \mathrm{B}^E_R(a)\right\}$ and $e = 2.718281828\dots$

Instead of giving a direct proof of Lemma 1.1.19, we will prove the following generalization:

Lemma 1.1.20 (Absolute convergence of families of bounded power series)**.** *Let $K \subseteq E$ be a nonempty subset of a complex normed vector space E. Let $U := K + \mathrm{B}_R^E(0) = \bigcup_{a\in K} \mathrm{B}_R^E(a)$ be a union of open balls with fixed radius $R > 0$. Now, consider a set M of bounded $C^\omega_\mathbb{C}$-mappings from U to a normed space F such that $\sup_{f\in M} \|f\|_\infty < \infty$. Then we have for all $r < \frac{R}{2e}$ the following estimate:*

$$\sum_{k=0}^{\infty} \sup_{\substack{f\in M\\ a\in K}} \frac{\left\|f^{(k)}(a)\right\|_{\mathrm{op}}}{k!} r^k \leq \frac{R}{R-2er} \cdot \sup_{f\in M} \|f\|_\infty .$$

This clearly implies Lemma 1.1.19 by taking $K := \{a\}$ and $M := \{f\}$.

Proof of Lemma 1.1.20. Without loss of generality, we may assume that F is a Banach space. Let $f \in M$ and $a \in K$ be given. Let $v \in E$ be a vector of norm $\|v\|_E = 1$. Furthermore let $s < R$ be a fixed number. Then we define the following function

$$g \colon \mathrm{B}_R^\mathbb{C}(0) \longrightarrow F : z \mapsto f(a + zv)$$

Note that g depends on the choices of f, a and v. It is possible to expand g into a power series:

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(zv, \dots, zv) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(v, \dots, v) \cdot z^k.$$

Using the Cauchy formula (see e.g. [1, Corollary 3.2]), we can write the coefficients of this power series as a complex integral:

$$\frac{1}{k!} f^{(k)}(a)(v, \dots, v) = \frac{1}{2\pi i} \int_{|z|=s} \frac{g(z)}{z^{k+1}} dz.$$

Multiplying with $k!$ and taking the norm on both sides yields the following estimate of $f^{(k)}(a)(v, \dots, v)$:

$$\begin{aligned}\left\|f^{(k)}(a)(v, \dots, v)\right\|_F &\leq k! \left\|\frac{1}{2\pi i} \int_{|z|=s} \frac{g(z)}{z^{k+1}} dz\right\|_F \leq k! \frac{1}{2\pi} 2\pi s \frac{\|g\|_\infty}{s^{k+1}} \\ &\leq \frac{k!}{s^k} \|f\|_\infty .\end{aligned}$$

Since v was arbitrary, this gives us an estimate for the norm of the homogeneous polynomial

$$\left\|v \mapsto f^{(k)}(a)(v, \dots, v)\right\|_{\mathrm{op}} \leq \frac{k!}{s^k} \|f\|_\infty .$$

By Corollary 1.1.9, this implies an upper bound for the norm of the corresponding k-linear map:

$$\left\|f^{(k)}(a)\right\|_{\mathrm{op}} \le \frac{(2k)^k}{k!} \cdot \frac{k!}{s^k} \|f\|_\infty \le (2e)^k \frac{k!}{s^k} \|f\|_\infty .$$

Since $s < R$, $a \in K$ and $f \in M$ were arbitrary, we obtain the following:

$$\sup_{\substack{f\in M\\ a\in K}} \left\|f^{(k)}(a)\right\|_{\mathrm{op}} \le (2e)^k \frac{k!}{R^k} \cdot \sup_{f\in M} \|f\|_\infty .$$

Now, we multiply both sides of the inequality by $\frac{r^k}{k!}$ and sum up:

$$\sum_{k=0}^{\infty} \sup_{\substack{f\in M\\ a\in K}} \frac{\left\|f^{(k)}(a)\right\|_{\mathrm{op}}}{k!} r^k \le \sum_{k=0}^{\infty} \left(\frac{2er}{R}\right)^k \cdot \sup_{f\in M} \|f\|_\infty .$$

Since r was assumed to be strictly less that $\frac{R}{2e}$, the series $\sum_{k=0}^{\infty} \left(\frac{2er}{R}\right)^k$ converges to $\frac{1}{1-2er/R} = \frac{R}{R-2er}$. This finishes the proof. □

1.1.4 Ordinary differential equations in Banach spaces

In Chapter 3, we will construct and work with Lie groups of germs of diffeomorphisms in a Banach spaces. Here, we collect some tools to deal with diffeomorphisms and differential equations in Banach spaces:

We start with a quantitative version of the Inverse Function Theorem for Banach spaces:

Theorem 1.1.21 (Lipschitz inverse function theorem)**.** *Let X be a Banach space over $\mathbb{K}$ and let $T\colon X \longrightarrow X$ be a continuous linear invertible map. Suppose $f\colon U \longrightarrow X$ is L-Lipschitz continuous with $L > 0$, where U an open neighborhood of 0 in X and $f(0) = 0$, and $\lambda := L \cdot \left\|T^{-1}\right\|_{\mathrm{op}} < 1$. Then $T + f$ is a homeomorphism of U onto an open subset V of X and $(T + f)^{-1}$ is Lipschitz with constant $\frac{1}{1-\lambda}\left\|T^{-1}\right\|_{\mathrm{op}}$. If U contains the ball $\mathrm{B}_r^X(0)$, then V contains the ball $\mathrm{B}_{r'}^X(0)$ with $r' := \frac{r(1-\lambda)}{\|T^{-1}\|_{\mathrm{op}}}$.*

Proof. This can be found at the beginning of [19]. □

Now, we state three tools to work with differential equations:

Theorem 1.1.22 (A Quantitative Version of Picard-Lindelöf)**.** *Let X be a Banach space over $\mathbb{R}$ and let $(t_0, x_0) \in [0,1] \times X$ be given. Let $V \subseteq [0,1] \times X$ be a neighborhood of (t_0, x_0) and let*

$$f \colon V \longrightarrow X$$

be a continuous function which is k-Lipschitz in the second component (for a $k > 0$). Let $\tau, \rho, M > 0$ such that

$$([0,1] \cap [t_0 - \tau, t_0 + \tau]) \times \mathrm{B}_\rho^X(x_0) \subseteq V$$

and that

$$\left\| f|_{([0,1]\cap[t_0-\tau,t_0+\tau])\times \mathrm{B}_\rho^X(x_0)} \right\|_\infty \leq M.$$

Then the initial value problem

$$\frac{dx}{dt} = f(t,x),$$
$$x(t_0) = x_0$$

admits a unique solution on the interval $[0,1] \cap [t_0 - \alpha, t_0 + \alpha]$, where $\alpha := \min\{\tau, \frac{\rho}{M}\}$.

Proof. This is essentially Corollary II.1.7.2 in [4]. ☐

Theorem 1.1.23 (The flow of an ordinary differential equation with parameters)**.** *Let E and L be Banach spaces over $\mathbb{R}$, and let $f \colon U \longrightarrow E$ be a continuous map, defined on an open subset $U \subseteq \mathbb{R} \times E \times L$. Assume the following conditions are satisfied:*

- *The partial differentials $\frac{\partial}{\partial x} f(t,x,\lambda) \in \mathcal{L}(E,E)$ and $\frac{\partial}{\partial \lambda} f(t,x,\lambda) \in \mathcal{L}(L,E)$ exist and are continuous maps on U.*
- *Let $(t_0, x_0, \lambda_0) \in U$ and let $I \subseteq \mathbb{R}$ be a compact interval, containing t_0 on which the initial value problem*

  $$\frac{dx}{dt} = f(t,x,\lambda_0),$$
  $$x(t_0) = x_0$$

  has a unique solution $\phi \colon I \longrightarrow E$.

Then

- *There is a neighborhood V^E of x_0 and a neighborhood V^L of λ_0 such that for all $u \in V^E$ and $\lambda \in V^L$ the initial value problem*

  $$\frac{dx}{dt} = f(t,x,\lambda_0),$$
  $$x(t_0) = u$$

  has a solution $\phi_{u,\lambda} \colon I \longrightarrow E$.

- *The flow* $\Phi\colon I \times V^E \times V^L \longrightarrow E : (t,u,\lambda) \mapsto \phi_{u,\lambda}(t)$ *is of class* $FC^1_{\mathbb{R}}$.
- *Furthermore* $\frac{\partial \Phi}{\partial u}$ *and* $\frac{\partial \Phi}{\partial \lambda}$ *are differentiable with respect to* t *and one has:*

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}\frac{\partial \Phi}{\partial t} \quad \textit{and} \quad \frac{\partial}{\partial t}\frac{\partial \Phi}{\partial \lambda} = \frac{\partial}{\partial \lambda}\frac{\partial \Phi}{\partial t}.$$

- *Let* $u \in V^E$ *and* $\lambda \in V^L$ *be fixed and define:*

$$B_{u,\lambda}(t) = \frac{\partial f}{\partial x}(t, \Phi(t,u,\lambda), \lambda)$$
$$C_{u,\lambda}(t) = \frac{\partial f}{\partial \lambda}(t, \Phi(t,u,\lambda), \lambda).$$

Then the function $\frac{\partial \Phi}{\partial \lambda}(\cdot, u, \lambda)$ *is equal to a solution* $z\colon I \longrightarrow \mathcal{L}(L,E)$ *of the initial value problem*

$$\frac{dz}{dt}(t) = B_{u,\lambda}(t) \circ z(t) + C_{u,\lambda}(t),$$
$$z(t_0) = 0.$$

Proof. This is Theorem II.3.6.1 in [4]. □

Lemma 1.1.24 (Grönwall's inequality)**.** *(a) Let* $b, c \geq 0$ *and let* $h\colon [0,1] \longrightarrow [0,+\infty[$ *be a bounded Lebesgue-measurable function, such that for all* $t \in [0,1]$ *we have*

$$h(t) \leq c + b\int_0^t h(s)ds.$$

Then we have the following estimate:

$$h(t) \leq c \cdot e^{bt}.$$

(b) Let U *be a convex open subset of a Banach space* $(X, \|\cdot\|_X)$ *over* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *and let* $f\colon [0,1] \times U \longrightarrow X$ *be a time dependent vector field on* U*, which is L-Lipschitz. Let* $x_1, x_2\colon [0,1] \longrightarrow U$ *be two solutions of the corresponding differential equation, i.e.* $x_i'(t) = f(t, x_i(t))$ *for all* $t \in [0,1]$*. Then we have the following estimate:*

$$\|x_1(t) - x_2(t)\|_X \leq \|x_1(0) - x_2(0)\|_X \cdot e^{Lt}.$$

Proof. (Sketch) (a) Show the following inequality by induction on $k \in \mathbb{N}_0$ (Fubini):

$$h(t) \leq c \cdot \sum_{j=0}^{k} \frac{(bt)^j}{j!} + b\int_0^t h(s) \cdot \frac{b^k (t-s)^k}{k!} ds.$$

Then, take the limit for $k \to \infty$.

(b) Apply part (a) to the continuous function

$$\begin{aligned} h : [0,1] &\longrightarrow [0,+\infty[\\ t &\longmapsto \|x_1(t) - x_2(t)\|_X . \end{aligned}$$

□

1.1.5 Composition operators

We close this section with facts on composition operators and their differentiability properties:

Lemma 1.1.25 (Derivatives of composition operators)**.** *Let X, Y and Z be normed spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq X$ and $V \subseteq Y$ open subsets and $k, \ell \in \{0, 1, 2, \ldots\}$.*

(a) Then the continuous map

$$g_Z^k : BC_{\mathbb{K}}^{k+\ell+1}(V, Z) \times BC_{\mathbb{K}}^{\partial,k}(U, V) \to BC_{\mathbb{K}}^k(U, Z) : (\gamma, \eta) \mapsto \gamma \circ \eta$$

is a C^ℓ-map. If $\ell \geq 1$, its differential is given by:

$$d\left(g_Z^k\right)(\gamma_0, \eta_0, \gamma, \eta) = \gamma \circ \eta_0 + (\gamma_0' \circ \eta_0) \cdot \eta. \tag{1.1}$$

Here, $BC_{\mathbb{K}}^{\partial,k}(U, V)$ is the set of all $\gamma \in BC_{\mathbb{K}}^k(U, Y)$ whose image is contained in V and has a positive distance to the boundary of V. It is open in $BC_{\mathbb{K}}^k(U, Y)$. The notation $(\gamma_0' \circ \eta_0) \cdot \eta$ should be interpreted as $\varepsilon \circ ((\gamma_0' \circ \eta_0), \eta)$ with $\varepsilon \colon \mathcal{L}(Y, Z) \times Y \longrightarrow Z : (T, y) \mapsto T(y)$.

(b) Let $f \colon V \longrightarrow Z$ be a $BC_{\mathbb{K}}^{k+\ell+1}(V, Z)$ map. Then the map

$$g : BC_{\mathbb{K}}^{\partial,k}(U, V) \to BC_{\mathbb{K}}^k(U, Z) : \eta \mapsto f \circ \eta$$

is $C_{\mathbb{K}}^\ell$ with differential (if $\ell \geq 1$)

$$dg(\eta_0, \eta) = (f' \circ \eta_0) \cdot \eta. \tag{1.2}$$

Proof. Part (a) is Proposition 3.3.10 of [18]. Part (b) is just a special case. □

Proposition 1.1.26 (Evaluations of bounded FC^1-mappings)**.** *Let X and Y be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $U \subseteq X$ be open.*

(a) The map

$$\begin{aligned} \mathrm{ev}_0 : BC(U, Y) \times U &\longrightarrow Y \\ (\gamma, x) &\longmapsto \gamma(x) \end{aligned}$$

is continuous.

(b) The map

$$\begin{aligned} \mathrm{ev}_1 : BC_{\mathbb{K}}^1(U, Y) \times U &\longrightarrow Y \\ (\gamma, x) &\longmapsto \gamma(x) \end{aligned}$$

is an $FC_{\mathbb{K}}^1$ map.

Remark.
- In this thesis, BC^1 always refers to bounded *Fréchet* differentiable mappings with bounded Fréchet derivative.
- The statements (a) and (b) remain true if U is replaced by a non-open locally convex set with dense interior (see Definition 1.1.5).

Proof. (a) Let $(\gamma_0, x_0) \in BC(U,Y) \times U$ and $\varepsilon > 0$ be given. Since $\gamma_0 \colon U \longrightarrow Y$ is continuous at x_0, there is a $\delta > 0$ such that $\gamma_0\left(\mathrm{B}^U_\delta(x_0)\right) \subseteq \mathrm{B}^Y_{\frac{\varepsilon}{2}}(\gamma_0(x_0))$. Let $(\gamma, x) \in \mathrm{B}^{BC(U,Y)}_\delta(\gamma_0) \times \mathrm{B}^U_{\frac{\varepsilon}{2}}(x_0)$. Then we can estimate:

$$\begin{aligned}\|\mathrm{ev}_0(\gamma,x) - \mathrm{ev}_0(\gamma_0,x_0)\| &= \|\gamma(x) - \gamma_0(x_0)\| \\ &\leq \|\gamma(x) - \gamma_0(x)\| + \|\gamma_0(x) - \gamma_0(x_0)\| \\ &\leq \underbrace{\|\gamma - \gamma_0\|_\infty}_{<\frac{\varepsilon}{2}} + \underbrace{\|\gamma_0(x) - \gamma_0(x_0)\|}_{<\frac{\varepsilon}{2}} < \varepsilon.\end{aligned}$$

Hence, ev_0 is continuous.

(b) We start by calculating the directional derivative of ev_1 at the point (γ_0, x_0):

$$\begin{aligned}\frac{1}{t}\big(\mathrm{ev}_1(\gamma_0 + t\eta, x_0 + ty) - \mathrm{ev}_1(\gamma_0, x_0)\big) &= \frac{1}{t}\big((\gamma_0 + t\eta)(x_0 + ty) - \gamma_0(x_0)\big) \\ &= \frac{1}{t}\big(\gamma_0(x_0 + ty) - \gamma_0(x_0)\big) + \eta(x_0 + ty) \\ &\overset{t\to 0}{\longrightarrow} d\gamma_0(x_0, y) + \eta(x_0) \\ &= \gamma_0'(x_0)(y) + \eta(x_0).\end{aligned}$$

This shows that all directional derivatives exist and that

$$d\mathrm{ev}_1\big((\gamma,x),(\eta,y)\big) = \gamma'(x)(y) + \eta(x).$$

So, it remains to show that the map

$$\begin{aligned}\mathrm{ev}_1' : BC^1_{\mathbb{K}}(U,Y) \times U &\longrightarrow \mathcal{L}\left(BC^1_{\mathbb{K}}(U,Y) \times X, Y\right) \\ (\gamma, x) &\longmapsto d\mathrm{ev}_1\big((\gamma,x),\cdot\big)\end{aligned}$$

is continuous.

Let (γ, x) and (γ_0, x_0) in $BC^{k+1}_{\mathbb{K}}(U, Y) \times U$. We may assume that the convex segment $[x_0, x]$ lies in U.

$$
\begin{aligned}
&\left\|\mathrm{ev}_1'(\gamma, x) - \mathrm{ev}_1'(\gamma_0, x_0)\right\|_{\mathrm{op}} \\
&= \sup_{\|\eta\|+\|y\|=1} \left\|\mathrm{ev}_1'(\gamma, x)(\eta, y) - \mathrm{ev}_1'(\gamma_0, x_0)(\eta, y)\right\|_Y \\
&= \sup_{\|\eta\|+\|y\|=1} \left\|\gamma'(x)(y) + \eta(x) - \gamma_0'(x_0)(y) - \eta(x_0)\right\|_Y \\
&\leq \sup_{\|\eta\|+\|y\|=1} \left\|\gamma'(x)(y) - \gamma_0'(x_0)(y)\right\|_Y + \|\eta(x) - \eta(x_0)\|_Y \\
&= \sup_{\|\eta\|+\|y\|=1} \left\|\left(\gamma'(x) - \gamma_0'(x_0)\right)(y)\right\|_Y + \left\|\int_0^1 \eta'(tx + (1-t)x_0)(x - x_0)dt\right\|_Y \\
&\leq \sup_{\|\eta\|+\|y\|=1} \left\|\gamma'(x) - \gamma_0'(x_0)\right\|_{\mathrm{op}} \underbrace{\|y\|_X}_{\leq 1} + \underbrace{\left\|\eta'\right\|_\infty}_{\leq 1} \|x - x_0\|_X \\
&\leq \left\|\gamma'(x) - \gamma_0'(x_0)\right\|_{\mathrm{op}} + \|x - x_0\|_X
\end{aligned}
$$

By part (a), this converges to 0 when (γ, x) tends to (γ_0, x_0). This finishes the proof. □

1.2 Locally convex direct limits of normed spaces

Direct limits will only occur in the following situation: Let $E_1 \subseteq E_2 \subseteq \cdots$ be an increasing sequence of normed $\mathbb{K}$-vector spaces such that all bonding maps $i_n \colon E_n \longrightarrow E_{n+1}$ are continuous (i.e. bounded operators). Then we define the *locally convex direct limit* of the sequence $(E_n)_{n\in\mathbb{N}}$ as the union $E := \bigcup_{n=1}^{\infty} E_n$ together with the locally convex vector topology in which a convex subset U is a 0-neighborhood if and only if $U \cap E_n$ is a 0-neighborhood in E_n, for each $n \in \mathbb{N}$. If each E_n is complete, i.e. a Banach space, the direct limit is called an (LB)-space.

The locally convex direct limit topology satisfies the following universal property: A linear map $f \colon E \longrightarrow F$ to a locally convex space F is continuous if and only if every restriction $f|_{E_n} \colon E_n \longrightarrow F$ is continuous with respect to the topology of E_n.

Note: In general a locally convex direct limit of normed spaces need not be Hausdorff. In the examples of this paper we can show the Hausdorff property directly by constructing an injective continuous map into a suitable Hausdorff space.

Proposition 1.2.1 (Characterization of zero neighborhoods)**.** *Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Then the set*

$$
\mathcal{V}(\delta_1, \delta_2, \ldots) := \bigcup_{n\in\mathbb{N}} \left(\mathrm{B}^{E_1}_{\delta_1}(0) + \cdots + \mathrm{B}^{E_n}_{\delta_n}(0)\right)
$$

is a 0-neighborhood of the locally convex direct limit $E := \bigcup_{n=1}^{\infty} E_n$. Furthermore, the sets of this type form a basis of 0-neighborhoods.

Proof. This is a well-known consequence of the fact that E is a quotient of the direct sum $\bigoplus_n E_n$, equipped with the box topology. (see e.g. [20]) $\square$

1.2.1 Compact regularity and bounded regularity

Definition 1.2.2 (Regularity of Direct Limits)**.** Let $E := \bigcup_{n=1}^{\infty} E_n$ be a direct limit of Banach spaces. We assume E is Hausdorff.

(i) The sequence $(E_n)_{n\in\mathbb{N}}$ is called *compactly regular*, if every compact subset in E is also a compact set in some E_n.

(ii) The sequence $(E_n)_{n\in\mathbb{N}}$ is called *boundedly regular*, if every bounded subset in E is also a bounded set in some E_n.

Lemma 1.2.3. *Let $E := \bigcup_{n=1}^{\infty} E_n$ be a direct limit of normed spaces. We assume E is Hausdorff. Then compact regularity implies bounded regularity.*

Proof. This Statement can also be found in [20], but we also give a nice elementary proof here:

We fix on each Banach space a norm, such that $\mathrm{B}_1^{E_n}(0) \subseteq \mathrm{B}_1^{E_{n+1}}(0)$ for each $n \in \mathbb{N}$. We assume that the direct sequence $(E_n)_{n\in\mathbb{N}}$ is compactly regular and let $A \subseteq E$ be a bounded subset of E. We claim that there is an index $n \in \mathbb{N}$ such that $A \subseteq \mathrm{B}_n^{E_n}(0)$. Since $\mathrm{B}_n^{E_n}(0)$ is a bounded subset of E_n, this claim would imply the statement. We prove the claim by contradiction:

Assume for each $n \in \mathbb{N}$ there is an $a_n \in A$ such that

$$a_n \notin \mathrm{B}_n^{E_n}(0)\,. \tag{$*$}$$

Now, since the set $\{a_n : n \in \mathbb{N}\}$ lies in the bounded set A, the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded in E and hence:

$$\frac{a_n}{n} \to 0 \text{ in } E.$$

So, the set $K := \{0\} \cup \left\{\frac{a_n}{n} : n \in \mathbb{N}\right\}$ is a compact subset of E and by compact regularity, K is a compact subset of one E_m. Since the inclusion map $E_m \longrightarrow E : x \mapsto x$ is continuous, K is compact in E_m and E is Hausdorff, the topologies of E_m and E coincide on the set K. In particular, the sequence $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ converges in E_m to zero. Hence, there is an index $n_0 \in \mathbb{N}$ such that $\frac{a_{n_0}}{n_0} \in \mathrm{B}_1^{E_m}(0)$. We may assume that $n_0 \geq m$. Since the sequence of unit balls $\left(\mathrm{B}_1^{E_n}(0)\right)_{n\in\mathbb{N}}$ is increasing by choice of the norms, this yields:

$$\frac{a_{n_0}}{n_0} \in \mathrm{B}_1^{E_m}(0) \subseteq \mathrm{B}_1^{E_{n_0}}(0)\,.$$

Multiplying by n_0 gives a contradiction to $(*)$. $\square$

Proposition 1.2.4 (Criterion for compact regularity)**.** *Let $E := \bigcup_{n=1}^{\infty} E_n$ be a direct limit of Banach spaces. We assume E is Hausdorff. Consider the following statements:*

(i) For every $n \in \mathbb{N}$, there is a 0-neighborhood $\Omega \subseteq E_n$ and a number $m \geq n$ and such that all the spaces $E_m, E_{m+1}, E_{m+2}, \ldots$ induce the same topology on Ω.

(ii) The sequence $(E_n)_{n\in\mathbb{N}}$ is compactly regular.

(iii) The locally convex vector space E is complete.

Then (i) is equivalent to (ii) and both imply (iii).

Proof. This follows from Theorem 6.4 and corresponding Corollary in [20]. □

1.2.2 Curves in direct limits

The following very useful theorem about continuous mappings into direct limits can be found in [14] or [17].

Theorem 1.2.5 (The Mujica Theorem)**.** *Let X be a compact Hausdorff topological space and let $E := \bigcup_{n=1}^{\infty} E_n$ be a direct limit of locally convex spaces. We assume E is Hausdorff. We denote by $i_n \colon E_n \longrightarrow E$ the inclusion map.*

(a) The map

$$\begin{aligned} \Phi : \textstyle\bigcup_{n\in\mathbb{N}} C(X, E_n) &\longrightarrow C\left(X, \textstyle\bigcup_{n\in\mathbb{N}} E_n\right) \\ \gamma \in C(X, E_n) &\longmapsto i_n \circ \gamma \end{aligned}$$

is a topological isomorphism onto its dense image, with respect to the locally convex direct limit topology on the left hand side.

(b) If the sequence $(E_n)_{n\in\mathbb{N}}$ is compactly regular, then the map Φ is also surjective, hence an isomorphism of topological vector spaces.

Remark. It is easy to verify that the map Φ is continuous, injective and in part (b) surjective. The hard part is to show that it is open.

Definition 1.2.6 (Integral completeness)**.** A locally convex Hausdorff space E is called *integral complete* if every continuous curve $\eta \in C([0,1], E)$ has an antiderivative $\gamma \in C^1([0,1], E)$ with $\gamma(0) = 0$ and if the corresponding operator

$$\begin{aligned} C([0,1], E) &\longrightarrow C^1([0,1], E) \\ \eta &\longmapsto \gamma \end{aligned}$$

is continuous.

Remark.

- If E is complete, then it is also integral complete.
- Integral completeness of E is equivalent to strong C^0-regularity of the Lie group $(E, +)$ (see Definition 1.3.5). So strong C^0-regularity is a generalization of the concept of integral completeness to Lie groups.

Lemma 1.2.7 (Mujica's Theorem for C^k-curves)**.** *Let $k \in \mathbb{N}_0$ be a (finite) number and let $E := \bigcup_{n=1}^{\infty} E_n$ be a direct limit of integral complete locally convex Hausdorff spaces. We assume E is also integral complete and Hausdorff. We denote by $i_n \colon E_n \longrightarrow E$ the inclusion map.*

(a) The map

$$\begin{aligned} \Phi : \textstyle\bigcup_{n\in\mathbb{N}} C^k([0,1], E_n) &\longrightarrow C^k\left([0,1], \textstyle\bigcup_{n\in\mathbb{N}} E_n\right) \\ \gamma &\longmapsto i_n \circ \gamma \end{aligned}$$

is a topological isomorphism onto its dense image.

(b) If the sequence $(E_n)_{n\in\mathbb{N}}$ is compactly regular, then the map Φ is also surjective, hence an isomorphism of topological vector spaces.

Proof. For an integral complete locally convex Hausdorff space F, the map

$$\begin{aligned} \Delta_F : C^k([0,1], F) &\longrightarrow F^k \times C([0,1], F) \\ \gamma &\longmapsto \Big((\gamma(0), \dots, \gamma^{(k-1)}(0)), \gamma^{(k)}\Big) \end{aligned}$$

is an isomorphism of topological vector spaces. Since the maps Δ_{E_n} are compatible with the direct limit structure, we can define

$$\bigcup_{n\in\mathbb{N}} \Delta_{E_n} : \bigcup_{n\in\mathbb{N}} C^k([0,1], E_n) \longrightarrow \bigcup_{n\in\mathbb{N}} E_n^k \times C([0,1], E_n)$$

We get the following commuting diagram, where Ψ is induced by the inclusion maps:

$$\begin{tikzcd} \bigcup_{n\in\mathbb{N}} C^k([0,1], E_n) \arrow[r, "\Phi"] \arrow[d, "\bigcup_{n\in\mathbb{N}} \Delta_{E_n}"] & C^k\left([0,1], \bigcup_{n\in\mathbb{N}} E_n\right) \arrow[d, "\Delta_E"] \\ \bigcup_{n\in\mathbb{N}} E_n^k \times C([0,1], E_n) \arrow[r, "\Psi"] & E^k \times C([0,1], E). \end{tikzcd}$$

It remains to show that the map Ψ is an embedding of topological vector spaces and that it is surjective in the case of (b). But that directly follows from the classical Mujica-Theorem (Theorem 1.2.5) and the fact that locally convex direct limits and finite direct products can be interchanged. $\square$

Remark. The statement of Lemma 1.2.7 no longer holds if one consideres C^∞-curves.

Proposition 1.2.8 (Lipschitz-continuous curves in boundedly regular direct limits)**.** *Let $\gamma \colon [0,1] \longrightarrow \bigcup_{n\in\mathbb{N}} E_n$ be a Lipschitz-continuous curve with values in the Hausdorff locally convex direct limit E of the boundedly regular sequence of Banach spaces $(E_n)_{n\in\mathbb{N}}$. Then there is an $m \in \mathbb{N}$ such that $\gamma([0,1]) \subseteq E_m$ and the corestriction $\gamma \colon [0,1] \longrightarrow E_m$ is Lipschitz continuous.*

Proof. Since the map $\gamma\colon [0,1] \longrightarrow E$ is Lipschitz, the set

$$M := \left\{ \frac{\gamma(s) - \gamma(t)}{s - t} : s, t \in [0,1], s \neq t \right\}$$

is bounded in $E = \bigcup_n E_n$. By the bounded regularity of the sequence $(E_n)_{n \in \mathbb{N}}$, there is an index $m \in \mathbb{N}$ such that M is a bounded subset of E_m. This implies that $\gamma([0,1]) \subseteq E_m$ and that γ is Lipschitz-continuous with values in E_m. $\square$

Remark. See [12] for further discussion of Lipschitz curves in locally convex topological vector spaces.

1.3 Lie Theory

1.3.1 Lie groups and regularity

The definitions of manifolds and Lie groups are analogous to the finite dimensional case:

Definition 1.3.1. (a) Let E be a Hausdorff locally convex $\mathbb{K}$-vector space with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $r \in \{\infty, \omega\}$. A *$C^r_{\mathbb{K}}$-manifold modelled on E* is a Hausdorff space M together with a maximal set *(atlas)* of homeomorphisms (*charts*) $\phi\colon U_\phi \longrightarrow V_\phi$ where $U_\phi \subseteq M$ and $V_\phi \subseteq E$ are open subsets, such that the transition maps $\phi \circ \psi^{-1}\colon \psi(U_\phi \cap U_\psi) \longrightarrow \phi(U_\phi \cap U_\psi)$ are $C^r_{\mathbb{K}}$ and $M = \bigcup_\phi U_\phi$.

(b) Let $k \in \mathbb{N}_0 \cup \{\infty, \omega\}, k \leq r$. Continuous mappings between $C^r_{\mathbb{K}}$-manifolds are called $C^k_{\mathbb{K}}$ if they are $C^k_{\mathbb{K}}$ after composition with suitable charts.

(c) A *$C^r_{\mathbb{K}}$-Lie group* $G$is a group which is at the same time a $C^r_{\mathbb{K}}$-manifold such that the group operations are $C^r_{\mathbb{K}}$.

Remark.
- The definitions of tangent spaces, tangent bundles, vector fields and similar concepts are similar to the finite dimensional case and can be found in [15].
- The *Lie algebra of a Lie group* G, denoted by $\mathbf{L}(G)$, is the tangent space T_1G at the identity element, together with the Lie bracket, obtained from the Lie algebra of left-invariant vector fields on the group G. It is a topological locally convex Lie algebra. Once again, we refer to [15] for details.

Definition 1.3.2 (Left logarithmic derivative)**.** Let $\gamma\colon [0,1] \longrightarrow G$ be a C^1-curve in a locally convex Lie group G. Then the *left logarithmic derivative* of γ is defined as:

$$\begin{aligned} \delta\gamma : [0,1] &\longrightarrow \mathbf{L}(G) \\ t &\longmapsto (\gamma(t))^{-1} \cdot \gamma'(t). \end{aligned}$$

The multiplication on the left is to be understood as the canonical multiplication on the tangent bundle group TG.

Definition 1.3.3 (Left evolution)**.** Let $\eta\colon [0,1] \longrightarrow \mathbf{L}(G)$ be a continuous curve in the Lie algebra of a locally convex Lie group G with identity element 1_G. Then a C^1-curve $\gamma\colon [0,1] \longrightarrow G$ is called a *left evolution* of η if

$$\delta\gamma = \eta \quad \text{and} \quad \gamma(0) = 1_G.$$

The left evolution is unique when it exists and is denoted by $\mathrm{Evol}(\eta)$. We will denote $\mathrm{Evol}(\eta)(1)$ by $\mathrm{evol}(\eta)$.

Definition 1.3.4 (Regular Lie group (in Milnor's sense))**.** A locally convex Lie group G is called *regular* if every $\eta \in C^\infty([0,1], \mathbf{L}(G))$ has a left evolution and if

$$\mathrm{evol}\colon C^\infty([0,1], \mathbf{L}(G)) \longrightarrow G$$

is smooth.

Definition 1.3.5 (Strongly C^k-regular Lie group)**.** Let $k \in \mathbb{N}_0$. A locally convex Lie group G is called *strongly C^k-regular* if every $\eta \in C^k([0,1], \mathbf{L}(G))$ has a left evolution and if

$$\mathrm{evol}\colon C^k([0,1], \mathbf{L}(G)) \longrightarrow G$$

is smooth.

Clearly, strong C^k-regularity implies regularity in Milnor's sense.

Proposition 1.3.6 (Regularity using Right Logarithmic Derivatives)**.** *Instead of using left logarithmic derivatives and left evolutions, one can also use* right logarithmic derivatives *and* right evolutions*. This results in the same concepts of regularity as in Definitions 1.3.4 and 1.3.5.*

Proof. Let $(G, \cdot)$ be a Lie group. Define on the manifold G a new multiplication by $g \odot h := h \cdot g$. Then the Lie groups $(G, \cdot)$ and $(G, \odot)$ are isomorphic via the isomorphism:

$$\begin{aligned} (G, \cdot) &\longrightarrow (G, \odot) \\ g &\longmapsto g^{-1}. \end{aligned}$$

Since the two Lie groups are isomorphic, $(G, \cdot)$ is (strongly C^k-)regular if and only if $(G, \odot)$ is so. Now, the right logarithmic derivative of a $(G, \cdot)$-valued curve is exactly the left logarithmic derivative of the same curve, considered with values in $(G, \odot)$. The assertion follows. □

For further details concerning logarithmic derivatives and regularity, we refer to [13] and [15].

1.3.2 Local Lie groups and regularity

Although we are mostly interested in (global) Lie groups, we will use so called *local Lie groups* as a tool to show regularity of global ones.

Definition 1.3.7 (Local Lie group)**.** Let G be a smooth manifold, $D \subseteq G \times G$ an open subset, $1 \in G$, and let

$$m_G\colon D \longrightarrow G : (x, y) \mapsto x * y,$$

$$\eta_G\colon G \longrightarrow G : x \mapsto x^{-1}$$

be smooth maps. We call $(G, D, m_G, 1_G, \eta_G)$ a *local Lie group* if

(Loc1) Assume that $(x, y), (y, z) \in D$. If $(x*y, z)$ or $(x, y*z) \in D$, then both are contained in D and $(x * y) * z = x * (y * z)$.

(Loc2) For each $x \in G$ we have $(x, 1_G), (1_G, x) \in D$ and $x * 1_G = 1_G * x = x$.

(Loc3) For each $x \in G$ we have $(x, x^{-1}), (x^{-1}, x) \in D$ and $x * x^{-1} = x^{-1} * x = 1$.

(Loc4) If $(x, y) \in D$, then $(y^{-1}, x^{-1}) \in D$.

Remark. Every symmetric open identity neighborhood of a Lie group can be turned into a local Lie group, but not every local Lie group can be enlarged to a Lie group.

Definition 1.3.8 (The Lie algebra of a local Lie group)**.** Let $G = (G, D, m_G, 1_G, \eta_G)$ be a local Lie group. Then there exists a continuous Lie bracket on $\mathbf{L}(G) := T_{1_G}G$. There is also a Lie functor and a left logarithmic derivative (for curves with values in the small manifold D). The definitions are almost literally the same as in the case of global Lie groups.

Definition 1.3.9 (Regularity of a local Lie group)**.** A local Lie group G is called *regular* if there is an open 0-neighborhood $\Omega \subseteq C^\infty([0, 1], \mathbf{L}(G))$ such that every $\eta \in \Omega$ has a left evolution and if

$$\operatorname{evol}\colon \Omega \longrightarrow G$$

is smooth. Consequently, it is called *strongly C^k-regular* if there is an open neighborhood $\Omega \subseteq C^k([0, 1], \mathbf{L}(G))$ such that every $\eta \in \Omega$ has a left evolution and if $\operatorname{evol}\colon \Omega \longrightarrow G$ is smooth.

Remark. Similar to Proposition 1.3.6 one can also use right logarithmic derivatives and evolutions to characterize regularity.

Proposition 1.3.10. *Let G be a Lie group. Then it is (strongly C^k-)regular as Lie group if and only if it is (strongly C^k-)regular as a local Lie group.*

Proof. We assume G to be a strongly C^k-regular local Lie group with $k \in \mathbb{N}_0 \cup \{\infty\}$. The case $k = \infty$ is supposed to mean that G is regular as a local Lie group.

For each continuous seminorm q on $\mathfrak{g}$, and each $\ell \in \mathbb{N}_0$ with $\ell \leq k$ we set

$$\Omega_{\ell,q} := \left\{ \gamma \in C^k([0,1], \mathfrak{g}) : \sup_{t\in[0,1], j\leq \ell} q\left(\gamma^{(j)}(t)\right) < 1 \right\}.$$

By definition of the topology of $C^k([0,1], \mathfrak{g})$ it is clear that

$$\{\Omega_{\ell,q} : \ell \leq k; q \text{ continuous seminorm on } \mathfrak{g}\}$$

is a basis of 0-neighborhoods.

Every curve $\gamma \in C^k([0,1], \mathfrak{g})$ can be split in two parts:

$$L\gamma \colon [0,1] \longrightarrow \mathfrak{g} : t \mapsto \gamma\left(\frac{1}{2}t\right)$$

and

$$R\gamma \colon [0,1] \longrightarrow \mathfrak{g} : t \mapsto \gamma\left(\frac{1}{2} + \frac{1}{2}t\right).$$

This yields two linear mappings:

$$L, R \colon C^k([0,1], \mathfrak{g}) \longrightarrow C^k([0,1], \mathfrak{g}).$$

One checks easily that for every basis 0-neighborhood $\Omega_{\ell,q}$ we have

$$L(\Omega_{\ell,q}) \subseteq \Omega_{\ell,q} \quad \text{and} \quad R(\Omega_{\ell,q}) \subseteq \Omega_{\ell,q}. \tag{$*$}$$

This implies that L and R are continuous.

Now, since G is a strongly C^k-regular local Lie group, there exists an open neighborhood $\Omega \subseteq C^k([0,1], \mathfrak{g})$ such that every $\gamma \in \Omega$ admits a left evolution and such that

$$\mathrm{evol}_\Omega \colon \Omega \longrightarrow G : \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is $C^\infty_{\mathbb{K}}$. We may assume that $\Omega = \Omega_{\ell,q}$ for a suitable $\ell \leq k$ and a continuous seminorm q on $\mathfrak{g}$.

Now, we want to extend the map $\mathrm{evol}_{\Omega_{\ell,q}}$ to the bigger set $2\Omega_{\ell,q}$.

To this end, let $\gamma \in 2\Omega_{\ell,q}$ be given. Then $\frac{1}{2}\gamma \in \Omega_{\ell,q}$ and by $(*)$, we have

$$L\left(\frac{1}{2}\gamma\right) \in \Omega_{\ell,q} \quad \text{and} \quad R\left(\frac{1}{2}\gamma\right) \in \Omega_{\ell,q}.$$

Thus, there exist left evolutions

$$\eta_L := \operatorname{Evol}\left(L\left(\frac{1}{2}\gamma\right)\right) \quad \text{and} \quad \eta_R := \operatorname{Evol}\left(R\left(\frac{1}{2}\gamma\right)\right).$$

Finally, we glue these two curves together and obtain

$$\begin{aligned} \eta : [0,1] &\longrightarrow G \\ t &\longmapsto \begin{cases} \eta_L(2t) & \text{if } t \leq \frac{1}{2} \\ \eta_L(1)\cdot\eta_R(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases} \end{aligned}$$

This curve η is well-defined and piecewise C^1. It maps 0 to 1 and its left-logarithmic derivative is exactly γ, whence η is C^1 in particular. So each $\gamma \in \Omega_{\ell,q}$ has a left evolution. Now, by using the explicit construction of this left evolution, we see that the new evolution map

$$\begin{aligned} \operatorname{evol}_{2\Omega} : 2\Omega_{\ell,q} &\longrightarrow G \\ \gamma &\longmapsto \operatorname{evol}_{\Omega}\left(L\left(\tfrac{1}{2}\gamma\right)\right)\cdot \operatorname{evol}_{\Omega}\left(R\left(\tfrac{1}{2}\gamma\right)\right) \end{aligned}$$

is a composition of $C^\infty_{\mathbb{K}}$ maps, hence smooth.

So, every curve $\gamma \in 2\Omega$ has a left evolution which depends smoothly on γ. By induction we see that every curve in

$$C^k\left([0,1],\mathfrak{g}\right) = \bigcup_{n\in\mathbb{N}} 2^n\Omega_{\ell,q}$$

has an evolution and the evolution map

$$\operatorname{evol}_G = \bigcup_{n\in\mathbb{N}} \operatorname{evol}_{2^n\Omega_{\ell,q}}$$

is smooth.

This finishes the proof that G is strongly C^k-regular. $\square$

Definition 1.3.11 (Baker-Campbell-Hausdorff-Series)**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. The *Baker-Campbell-Hausdorff-Series* (or *BCH*-series) is a formal series of the form

$$x * y = \sum_{n=1}^{\infty} p_n(x,y),$$

where each $p_n : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ is a homogeneous polynomial of degree n, consisting only of linear combinations of iterations of Lie brackets. Its first terms are

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \dots$$

We will not give the explicit formula here, but refer to [3](Definition 1, Ch. II, §6)) or [15](Definition IV.1.3) for a formal definition.

Remark. If $\mathfrak{g}$ is a topological Lie algebra, each p_n is continuous but the series may or may not converge at a given pair $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

For $x, y \in \mathfrak{g}$ with $[x, y] = 0$, we have $x * y = x + y$ since all $p_n = 0$ for $n \geq 2$.

This implies in particular that for all $x \in \mathfrak{g}$ we have

$$x * 0 = 0 * x = x \text{ and } x * (-x) = (-x) * x = 0.$$

Lemma 1.3.12 (*BCH*-series in Banach Lie algebras)**.** *Let $(\mathfrak{g}, \|\cdot\|)$ be a Banach Lie algebra over $K \in \{\mathbb{R}, \mathbb{C}\}$ with compatible norm, i.e. $\|[x, y]\| \leq \|x\| \, \|y\|$.*

(a) The BCH*-series converges on the set*

$$\Omega := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \log 2\}$$

to a $C^{\omega}_{\mathbb{K}}$ function $*\colon \Omega \longrightarrow \mathfrak{g}$.

(b) If x, y, z are elements in $\mathfrak{g}$ with $\|x\| + \|y\| + \|z\| < \log \frac{3}{2}$, then

$$(x, y) \in \Omega, (y, z) \in \Omega, (x * y, z) \in \Omega, (x, y * z) \in \Omega \text{ and } x * (y * z) = (x * y) * z.$$

Proof. This is taken from [3] (Propositions 1 and 2, Ch. II, §7). ☐

Proposition 1.3.13. *Let $(\mathfrak{g}, \|\cdot\|)$ be a Banach Lie algebra with compatible norm, i.e. $\|[x, y]\| \leq \|x\| \, \|y\|$. Set $B := \mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0)$. By construction, $B \times B$ lies in the set Ω, defined in 1.3.12 and therefore two elements in B can always be* BCH*-multiplied. This enables us to define the set*

$$D := \{(x, y) \in B \times B : x * y \in B\} \subseteq B \times B$$

and the map

$$m_B\colon D \longrightarrow B : (x, y) \mapsto x * y.$$

Together with inversion map

$$\eta_B\colon B \longrightarrow B : x \mapsto -x$$

we obtain a local Lie group $(B, D, m_B, 0_{\mathfrak{g}}, \eta_B)$.

Proof. The four properties listed in Definition 1.3.7 follow easily from Lemma 1.3.12. As an example, we prove (Loc1): Let $(x, y), (y, z) \in D$ and assume that $(x * y, z)$ or $(x, y * z) \in D$. Now, since $x, y, z \in B$, we know that

$$\|x\| + \|y\| + \|z\| < \frac{1}{3}\log\frac{3}{2} + \frac{1}{3}\log\frac{3}{2} + \frac{1}{3}\log\frac{3}{2} = \log\frac{3}{2}$$

and by part (b) of Lemma 1.3.12, we get that

$$x * (y * z) = (x * y) * z. \tag{$*$}$$

Since we know that $(x * y, z) \in D$ or $(x, y * z) \in D$, one of the sides of equation $(*)$ lies in B. Hence $(x * y, z) \in D$ and $(x, y * z) \in D$ and $x * (y * z) = (x * y) * z$. ☐

Remark. We will show later, in Theorem 4.2.3, that the local Lie group constructed in Proposition 1.3.13 is always strong C^0-regular in the sense of Definition 1.3.9.

Lemma 1.3.14 (Logarithmic derivative in a local Banach Lie group)**.** *Let* $(\mathfrak{g}, \|\cdot\|)$ *be a Banach Lie algebra with compatible norm and let* $\left(\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0), D, *, 0, -\mathrm{id}\right)$ *be the corespending local Banach Lie group, according to Proposition 1.3.13. For each* C^1*-curve* $\gamma\colon [0,1] \longrightarrow \mathrm{B}^{\mathfrak{g}}_{\log\frac{3}{2}}(0)$*, the left logarithmic derivative can be written as:*

$$\delta\gamma(t) = d\lambda_{-\gamma(t)}(\gamma(t), \gamma'(t)) = d\mu\Big((-\gamma(t), \gamma(t)), (0, \gamma'(t))\Big).$$

Here, $\lambda_{-\gamma(t)}$ *denotes the left-multiplication with the element* $-\gamma(t)$*.*

Proof. This is an easy calculation. □

If a local Lie group can be included into an (abstract) group, there is a possibility to enlarge it to a global Lie group:

Proposition 1.3.15. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*. Let* G *be a group, let* $V \subseteq G$ *be a subset of* G *carrying a* $C^\omega_{\mathbb{K}}$*-manifold structure. Let* $U \subseteq V$ *be an open symmetric subset containing* 1_G *such that* $U \cdot U \subseteq V$*. We assume that*

(i) the multiplication map $U \times U \longrightarrow V : (x, y) \mapsto x \cdot y$ *is analytic,*

(ii) the inversion map $U \longrightarrow U : x \mapsto x^{-1}$ *is analytic,*

(iii) for every $g \in G$ *there is an open subset* $W_g \subseteq U$ *such that* $gW_gg^{-1} \subseteq V$ *and the conjugation map* $W_g \longrightarrow V : x \mapsto gxg^{-1}$ *is analytic.*

Then there exists a unique $C^\omega_{\mathbb{K}}$*-Lie group structure on* G *such that* U *is an open subset of* G *with the given manifold structure. If* G *is generated by* U *then (i) and (ii) imply (iii).*

Proof. See [3, Chapter III, §1.9, Proposition 18] for the case of a Banach Lie group. □

Corollary 1.3.16 (Construction of a Lie group with a given exponential function)**.** *Let* $\mathfrak{g}$ *be a Hausdorff locally convex Lie algebra and let* U *and* V *be symmetric* 0*-neighborhoods in* $\mathfrak{g}$ *such that the* BCH*-series converges on* $U \times U$ *and defines a* $C^\omega_{\mathbb{K}}$*-map* $*\colon U \times U \longrightarrow V$*. Let* $\Phi\colon \mathfrak{g} \longrightarrow H$ *be a map into a group* H *satisfying*

- $\Phi|_V$ *is injective.*
- $\Phi(nx) = (\Phi(x))^n$ *for* $n \in \mathbb{N}, x \in \mathfrak{g}$.
- $\Phi(x * y) = \Phi(x) \cdot \Phi(y)$ *for* $x, y \in U$.

Then there exists a unique $C^\omega_\mathbb{K}$-Lie group structure on $G := \langle \Phi(\mathfrak{g}) \rangle = \langle \Phi(V) \rangle = \langle \Phi(U) \rangle$ such that $\Phi|_U : U \longrightarrow \Phi(U)$ becomes a diffeomorphism onto an open subset.

Furthermore, the topological isomorphism $T_0\Phi : \mathfrak{g} \longrightarrow \mathbf{L}(G)$ is an isomorphism of locally convex Lie algebras and after identifying $\mathfrak{g}$ with $\mathbf{L}(G)$, we obtain that G admits a C^ω exponential function and we have $\exp_G = \Phi$.

Proof. Define $G := \langle \Phi(U) \rangle$. Since for all $x \in \mathfrak{g}$, there is an $n \in \mathbb{N}$ such that $\frac{x}{n} \in U$, we have

$$\Phi(x) = \Phi\left(n\frac{x}{n}\right) = \left(\Phi\left(\frac{x}{n}\right)\right)^n \in G.$$

This shows $G = \langle \Phi(\mathfrak{g}) \rangle = \langle \Phi(V) \rangle = \langle \Phi(U) \rangle$.

Since $\Phi|_V$ is injective, we can use it to define a $C^\omega_\mathbb{K}$ manifold on $\Phi(V)$ such that $\Phi|_V : V \longrightarrow \Phi(V)$ becomes a $C^\omega_\mathbb{K}$-diffeomorphism.

Now, $\Phi(U)$ is an open symmetric subset of the manifold $\Phi(V)$ satisfying the assumptions on Proposition 1.3.15. So, there is a unique Lie group structure on G such that $\Phi|_U$ becomes a $C^\omega_\mathbb{K}$-diffeomorphism.

Let $n \in \mathbb{N}$ be given. Then for all $x \in nU$, we have

$$\Phi(x) = \left(\Phi|_U\left(\frac{x}{n}\right)\right)^n$$

and so the function $\Phi|_{nU} : nU \longrightarrow G$ is $C^\omega_\mathbb{K}$ as a composition of $C^\omega_\mathbb{K}$ maps.

Since $\mathfrak{g} = \bigcup_n nU$, the map Φ is analytic.

Hence, for each $x \in \mathfrak{g}$ the curve

$$\begin{aligned} \gamma_x : \mathbb{R} &\longrightarrow G \\ t &\longmapsto \Phi(tx) \end{aligned}$$

is analytic. It is a group homomorphism from $(\mathbb{R}, +)$ to G since it satisfies $\gamma_x(s+t) = \gamma_x(s)\gamma_x(t)$ for small $s, t \in \mathbb{R}$. Its derivative at time 0 is $\gamma'(0) = x$ since $T_0\Phi = \mathrm{id}_\mathfrak{g}$ after identification of $\mathfrak{g}$ with $\mathbf{L}(G)$. This finishes the proof. □

2 Analytic maps on (LB)-spaces

The following theorem is our main tool for constructing (LB)-Lie groups and showing their regularity:

Theorem 2.1 (Complex analytic mappings defined on (LB)-spaces)**.** *Let E be a $\mathbb{C}$-vector space that is the union of the increasing sequence of subspaces $(E_n)_{n\in\mathbb{N}}$. Assume that a norm $\|\cdot\|_{E_n}$ is given on each E_n such that all bonding maps*

$$i_n \colon E_n \longrightarrow E_{n+1} : x \mapsto x$$

are continuous and have an operator norm at most 1. *We give E the locally convex direct limit topology. Let $R > 0$ and let $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{E_n}(0)$ be the union of all open balls with radius R around* 0. *Let $f \colon U \longrightarrow F$ be a function defined on U with values in a Hausdorff locally convex space F, such that each $f_n := f|_{\mathrm{B}_R^{E_n}(0)}$ is $\mathbb{C}$-analytic and* bounded. *Then f is continuous.*

If in addition the topology on E is Hausdorff then f is even $\mathbb{C}$-analytic.

Note that the statement cannot be generalized to direct limits of Fréchet spaces. There even exist homogeneous polynomials that are continuous on each step E_n but fail to be continuous on the limit. Similar pathologies arise when looking at uncountable direct limits of normed spaces (see e.g. [11] for details).

This theorem and its proof can also be found in [5] (Theorem A).

Proof of Theorem 2.1. We start with some simplifications: Since f restricted to $\mathrm{B}_R^{E_n}(0)$ is analytic and the intersection of each one dimensional affine subspace with U is locally contained in some $\mathrm{B}_R^{E_n}(0)$, the function f is clearly Gateaux analytic. By Lemma 1.1.15 it remains to show the continuity of f. The range space F is Hausdorff and locally convex and can therefore be embedded in a product of Banach spaces. We now use that a function into a product is continuous if and only if the projection onto each factor is continuous. Therefore we can assume without loss of generality that F is a Banach space. Let $p \in U$ be given. It remains to show continuity of f at p. Since $p \in \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{E_n}(0)$, there is an index m such that $p \in \mathrm{B}_R^{E_m}(0)$. We may assume that $m = 1$ since omitting only a finite number of spaces does not change the direct limit. Choose $R' > 0$ such that $\|p\|_{E_1} + R' \leq R$. Then $\mathrm{B}_{R'}^{E_n}(p) \subseteq \mathrm{B}_R^{E_n}(0)$ for all $n \in \mathbb{N}$, using that the inclusion maps have operator norm at most 1. Now, we may restrict f to the subset $\bigcup_{n\in\mathbb{N}} \mathrm{B}_{R'}^{E_n}(p)$ and without loss of generality, we may assume that $R' = R$ and $p = 0$. Therefore we only have to show continuity of f at 0. Furthermore, we may assume that $f(0) = 0$ which can be obtained by a translation in F which is clearly continuous. Let r be a fixed positive real number strictly less that $\frac{R}{2e}$.

We know that each $f_n \colon \mathrm{B}_R^{E_n}(0) \longrightarrow F : x \mapsto \sum_{k=1}^{\infty} \beta_{k,n}(x, \cdots, x)$ is $C_{\mathbb{K}}^{\omega}$ and bounded between normed complex vector spaces. Then by Lemma 1.1.19, we have the estimate:

$$\sum_{k=1}^{\infty} \|\beta_{k,n}\|_{\mathrm{op}} r^k \leq \frac{R}{R - 2er} \|f_n\|_{\infty} =: S_n.$$

Now, let $a_j := \frac{r}{2^j}$ which implies that $r = \sum_{j=1}^{\infty} a_n$. Then we have for every $n \in \mathbb{N}$:

$$\begin{aligned} S_n &\geq \sum_{k=1}^{\infty} \|\beta_{k,n}\|_{\mathrm{op}} r^k = \sum_{k=1}^{\infty} \|\beta_{k,n}\|_{\mathrm{op}} \left(\sum_{j=1}^{\infty} a_j \right)^k \\ &= \sum_{k=1}^{\infty} \|\beta_{k,n}\|_{\mathrm{op}} \sum_{\vec{j} \in \mathbb{N}^k} a_{j_1} a_{j_2} \cdots a_{j_k}. \end{aligned} \tag{$*$}$$

Let $\varepsilon > 0$ be given. We set $b_n := \min\left(1, \frac{\varepsilon}{2^n \cdot (S_n + 1)}\right)$ and $\delta_n := a_n \cdot b_n$. By construction it is clear that $\delta_n \leq a_n$ which will be used later.

To show continuity of f at 0, it suffices to show that the 0-neighborhood $\mathcal{V}(\delta_1, \delta_2, \ldots) \subseteq E$ as defined in Proposition 1.2.1 is a subset of the domain of f and is mapped by f into $\mathrm{B}_{\varepsilon}^{F}(0)$.

Let $x \in \mathcal{V}(\delta_1, \delta_2, \ldots)$. This means that there is a number $m \in \mathbb{N}$ such that $x = \sum_{j=1}^{m} x_j$ with $\|x_j\|_{E_j} < \delta_j$. We can estimate the E_m-norm of x by

$$\left\| \sum_{j=1}^{m} x_j \right\|_{E_m} \leq \sum_{j=1}^{m} \|x_j\|_{E_m} \leq \sum_{j=1}^{m} \|x_j\|_{E_j} < \sum_{j=1}^{m} \delta_j \leq \sum_{j=1}^{m} a_j < r < R.$$

So, $x \in \mathrm{B}_R^{E_m}(0) \subseteq \bigcup_{n \in \mathbb{N}} \mathrm{B}_R^{E_n}(0)$ which is the domain of f. So it makes sense to evaluate $f(x)$:

$$\begin{aligned} f(x) = f_m(x) &= \sum_{k=1}^{\infty} \beta_{k,m}(x, \ldots, x) = \sum_{k=1}^{\infty} \beta_{k,m}\left(\sum_{j_1=1}^{m} x_{j_1}, \ldots, \sum_{j_k=1}^{m} x_{j_k} \right) \\ &= \sum_{k=1}^{\infty} \sum_{\substack{\vec{j} \in \mathbb{N}^k \\ \vec{j} \leq m}} \beta_{k,m}(x_{j_1}, \ldots, x_{j_k}) = \sum_{k=1}^{\infty} \sum_{n=1}^{m} \sum_{\substack{\vec{j} \in \mathbb{N}^k \\ \max \vec{j} = n}} \beta_{k,n}(x_{j_1}, \ldots, x_{j_k}) \end{aligned}$$

Note, that we used in the last line that $\beta_{k,m}$ and $\beta_{k,n}$ coincide when the arguments are elements of E_n. Now, we can estimate the norm:

$$
\begin{aligned}
\|f(x)\|_F &= \Bigg\| \sum_{k=1}^{\infty} \sum_{n=1}^{m} \sum_{\substack{\vec{j}\in\mathbb{N}^k \\ \max \vec{j}=n}} \beta_{k,n}\left(x_{j_1},\dots,x_{j_k}\right) \Bigg\|_F \\
&\leq \sum_{k=1}^{\infty} \sum_{n=1}^{m} \sum_{\substack{\vec{j}\in\mathbb{N}^k \\ \max \vec{j}=n}} \|\beta_{k,n}\|_{\mathrm{op}} \|x_{j_1}\|_{E_n} \cdots \|x_{j_k}\|_{E_n} \\
&\leq \sum_{n=1}^{m} \sum_{k=1}^{\infty} \sum_{\substack{\vec{j}\in\mathbb{N}^k \\ \max \vec{j}=n}} \|\beta_{k,n}\|_{\mathrm{op}} \|x_{j_1}\|_{E_{j_1}} \cdots \|x_{j_k}\|_{E_{j_k}} \\
&\leq \sum_{n=1}^{m} \sum_{k=1}^{\infty} \sum_{\substack{\vec{j}\in\mathbb{N}^k \\ \max \vec{j}=n}} \|\beta_{k,n}\|_{\mathrm{op}} \delta_{j_1} \cdots \delta_{j_k}
\end{aligned}
$$

One of the factors $\delta_{j_1},\dots,\delta_{j_k}$ is equal to $\delta_n = a_n \cdot b_n$, all the others can be estimated by the corresponding a_j:

$$
\begin{aligned}
\|f(x)\|_F &\leq \sum_{n=1}^{m} \sum_{k=1}^{\infty} \sum_{\substack{\vec{j}\in\mathbb{N}^k \\ \max \vec{j}=n}} \|\beta_{k,n}\|_{\mathrm{op}} a_{j_1} \cdots a_{j_k} \cdot b_n \\
&\leq \sum_{n=1}^{m} b_n \sum_{k=1}^{\infty} \|\beta_{k,n}\|_{\mathrm{op}} \sum_{\vec{j}\in\mathbb{N}^k} a_{j_1} a_{j_2} \cdots a_{j_k} \overset{\text{by } (*)}{\leq} \sum_{n=1}^{m} b_n \cdot S_n \\
&\leq \sum_{n=1}^{m} \frac{\varepsilon}{2^n \cdot (S_n + 1)} \cdot S_n < \varepsilon.
\end{aligned}
$$

This finishes the proof. □

Although this result explicitly needs that $\mathbb{K} = \mathbb{C}$, the following easy consequence also holds in the real case:

Corollary 2.2 (Continuity of polynomials)**.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A polynomial function defined on the direct limit $E = \bigcup_{n\in\mathbb{N}} E_n$ of normed $\mathbb{K}$-vector spaces $E_1 \subseteq E_2 \subseteq \cdots$ with values in a Hausdorff locally convex space is continuous if and only if it is continuous on each step.*

Proof. First consider the case $\mathbb{K} = \mathbb{C}$. Let $f\colon E \longrightarrow F$ be an F-valued polynomial map, defined on the locally convex direct limit $E = \bigcup_{n\in\mathbb{N}} E_n$ of normed spaces E_n. It is possible to choose on each E_n an equivalent norm $\|\cdot\|_{E_n}$ such that the continuous bonding maps $i_n\colon E_n \longrightarrow E_{n+1} : x \mapsto x$ have operator norm at most 1. Set $R := 1$ and $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{E_n}(0)$. By hypothesis, each $f_n := f|_{\mathrm{B}_R^{E_n}(0)}$ is continuous. A continuous polynomial is automatically $C^\omega_\mathbb{C}$ and maps bounded sets to bounded sets, in particular, $f_n(\mathrm{B}_R^{E_n}(0))$ is bounded in F. Therefore, we can directly apply Theorem 2.1. and obtain that f is continuous on the 0-neighborhood U. But for polynomial functions this is enough to guarantee continuity on the whole domain E. (see [2, Theorem 1])

Let $\mathbb{K} = \mathbb{R}$ now and let $E_\mathbb{C}$, $(E_n)_\mathbb{C}$ and $F_\mathbb{C}$ denote the complexifications of the $\mathbb{R}$-vector spaces E, E_n and F respectively. Then every polynomial map $f_n\colon E_n \longrightarrow F$ can be extended to a complex polynomial map $(f_n)_\mathbb{C}\colon E_\mathbb{C} \longrightarrow F_\mathbb{C}$. The maps $(f_n)_\mathbb{C}$ are continuous because the maps f_n are so. We now apply the complex case and obtain that $f_\mathbb{C}$ is continuous and hence f is continuous, too. $\square$

3 Germs of diffeomorphisms around a compact set in a Banach space

Throughout this chapter, let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{K}$, and let $K \subseteq X$ be a nonempty compact subset. We are interested in germs of analytic diffeomorphisms around K, i.e. we examine analytic diffeomorphisms $\eta\colon U_\eta \longrightarrow V_\eta$ where U_η and V_η are open subsets of X, both containing K, such that $\eta|_K = \mathrm{id}_K$. We identify two diffeomorphisms if they coincide on an open set $W \subseteq X$, containing K. It is easily checked that these equivalence classes of diffeomorphisms form a group with respect to composition. In the first section we will turn this group into a Lie group modelled on a compactly regular direct limit of Banach spaces, and in the second section, we will prove its regularity.

The content of the first section can also be found in [5].

3.1 Construction of $\mathrm{DiffGerm}(K, X)$

In this section we will prove the following theorem:

Theorem 3.1.1 (Lie group of germs of diffeomorphisms)**.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a Banach space over $\mathbb{K}$. Let $K \subseteq X$ be a non-empty compact subset of X. Consider the group*

$$\mathrm{DiffGerm}(K, X) := \left\{\eta : \begin{array}{c} \eta \textit{ is a } C^\omega_{\mathbb{K}}\textit{-diffeomorphism between open} \\ \textit{neighborhoods of } K \textit{ and } \eta|_K = \mathrm{id}_K \end{array}\right\} / \sim,$$

where two diffeomorphisms η_1, η_2 are considered equivalent, $\eta_1 \sim \eta_2$ if they coincide on a common neighborhood of K. Then $\mathrm{DiffGerm}(K, X)$ *can be turned into a $C^\omega_{\mathbb{K}}$-Lie group modeled on a compactly regular (LB)-space.*

If $X = \mathbb{K}^n$, this is known and can be found in [9]. If in addition $K = \{0\}$, the Lie group structure was first constructed by Pisanelli in [16] and denoted by $\mathrm{Gh}_n(\mathbb{C})$.

We will follow the strategy of [9] to first consider the case $\mathbb{K} = \mathbb{C}$ and reduce the real case to the complex case. The topologies used in [9] do not work when X is infinite dimensional. However, once we have constructed the Lie group structure for $\mathbb{K} = \mathbb{C}$, the proof of the real case can be copied verbatim from [9, Corollary 15.11].

Therefore, from now on, $\mathbb{K} = \mathbb{C}$.

3.1.1 The modelling space

Throughout this section, we will fix the following countable basis of open neighborhoods of K:

$$U_n := K + \mathrm{B}^X_{\frac{1}{n}}(0) = \bigcup_{a \in K} \mathrm{B}^X_{\frac{1}{n}}(a).$$

For every $n \in \mathbb{N}$, we define the following spaces:

$$BC^0_{\mathbb{C}}(U_n, X) := BC(U_n, X) = \{\gamma\colon U_n \longrightarrow X : \gamma \text{ is continuous and bounded }\}$$

$$\mathrm{Hol_b}(U_n, X)_K := \{\gamma\colon U_n \longrightarrow X : \gamma \text{ is } C^\omega_{\mathbb{C}}, \text{ bounded and } \gamma|_K = 0\}.$$

It is well-known that $BC^0_{\mathbb{C}}(U_n, X)$ is a Banach space when equipped with the sup-norm. The space $\mathrm{Hol_b}(U_n, X)_K$ is a closed vector subspace of $BC^0_{\mathbb{C}}(U_n, X)$ and hence becomes a Banach space as well.

For every $k \in \mathbb{N}$ we define

$$BC^k_{\mathbb{C}}(U_n, X) := \left\{\gamma\colon U_n \longrightarrow X : \begin{array}{c}\gamma \text{ is } FC^k_{\mathbb{C}}, \text{ bounded and the} \\ \text{first } k \text{ Fréchet derivates are bounded}\end{array}\right\}.$$

This space becomes a Banach space when endowed with the (finite number of) seminorms $\left(\gamma \mapsto \left\|\gamma^{(\ell)}\right\|_\infty\right)$ where $\ell \in \{0, 1, \ldots, k\}$. Here $\gamma^{(\ell)}\colon U_n \longrightarrow \left(\mathrm{Sym}^\ell_c(X, X), \|\cdot\|_{\mathrm{op}}\right)$ denotes the k-th Fréchet derivative of γ (Definitions 1.1.2 (iii) and 1.1.6).

By Proposition 1.1.16(b) every $C^1_{\mathbb{C}}$-map between complex Banach spaces is automatically $C^\omega_{\mathbb{C}}$ which implies that for $k \geq 1$ all of the elements of $BC^k_{\mathbb{C}}(U_n, X)$ are complex analytic. Therefore the exponent k only refers to the boundedness of the first k derivatives.

Lemma 3.1.2. *Let $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots\}$. Then the linear operator $\mathrm{Hol_b}(U_n, X)_K \longrightarrow BC^k_{\mathbb{C}}(U_{n+1}, X) : \gamma \mapsto \gamma|_{U_{n+1}}$ is continuous.*

Proof. Let $x \in U_{n+1}$ be given. Then there is an $a \in K$ such that $x \in \mathrm{B}^X_{\frac{1}{n+1}}(a)$. Set $R := \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. Then $\mathrm{B}^X_R(x) \subseteq \mathrm{B}^X_{\frac{1}{n}}(a) \subseteq U_n$. For each $\gamma \in \mathrm{Hol_b}(U_n, X)_K$ we obtain a bounded analytic function $\gamma|_{\mathrm{B}^X_R(x)}\colon \mathrm{B}^X_R(x) \longrightarrow X$. We fix a real number $r < \frac{R}{2e}$ and apply Lemma 1.1.19 to get the following estimate:

$$\sum_{\ell=0}^\infty \frac{\left\|\gamma^{(\ell)}(x)\right\|_{\mathrm{op}}}{\ell!} r^\ell \leq \frac{R}{R - 2er} \cdot \|\gamma\|_\infty.$$

In particular we can estimate every summand in the infinite sum by the whole sum and conclude

$$\left\|\gamma^{(\ell)}(x)\right\|_{\mathrm{op}} \leq \frac{\ell!}{r^\ell} \cdot \frac{R}{R - 2er} \cdot \|\gamma\|_\infty.$$

This bound does not depend on the choice of $x \in U_{n+1}$, hence

$$\left\|\gamma^{(\ell)}|_{U_{n+1}}\right\|_\infty = \sup_{x \in U_{n+1}} \left\|\gamma^{(\ell)}(x)\right\|_{\mathrm{op}} \leq \frac{\ell!}{r^\ell} \cdot \frac{R}{R - 2er} \cdot \|\gamma\|_\infty. \qquad \square$$

We also need the space

$$BC^1_{\mathbb{C}}(U_n,X)_K := \left\{\gamma \in BC^1_{\mathbb{C}}(U_n,X) : \gamma|_K = 0\right\}.$$

This is a closed subspace of $BC^1_{\mathbb{C}}(U_n,X)$ and therefore becomes a Banach space with the induced topology.

Lemma 3.1.3. *The topology of the Banach space* $BC^1_{\mathbb{C}}(U_n,X)_K$ *is given by the norm*

$$\|\gamma\|_{\mathrm{D}} := \|\gamma'\|_\infty = \sup_{x\in U_n} \|\gamma'(x)\|_{\mathrm{op}}.$$

Proof. By definition, $\|\cdot\|_{\mathrm{D}}$ is one of the two seminorms generating the topology of $BC^1_{\mathbb{C}}(U_n,X)_K$. It suffices to show that the other, $\|\cdot\|_\infty$, is bounded above by a multiple of $\|\cdot\|_{\mathrm{D}}$.

Let $\gamma \in BC^1_{\mathbb{C}}(U_n,X)_K$ and let $x \in U_n = K + \mathrm{B}^X_{\frac{1}{n}}(0)$. Then $x = a+v$ with $a \in K$ and $\|v\|_X < \frac{1}{n}$. Then

$$\begin{aligned}\|\gamma(x)\|_X = \|\gamma(a+v)\|_X &= \left\|\underbrace{\gamma(a)}_{=0} + \int_0^1 \gamma'(a+tv)(v)dt\right\|_X \\ &\le \int_0^1 \|\gamma'(a+tv)\|_{\mathrm{op}}\|v\|_X\,dt \le \|\gamma\|_{\mathrm{D}}\cdot\frac{1}{n}.\end{aligned}$$

Therefore $\|\gamma\|_\infty \le \frac{1}{n}\|\gamma\|_{\mathrm{D}}$ and this finishes the proof. $\square$

Note that this does not work without the assumption that $\gamma|_K = 0$.

From now on, the space $BC^1_{\mathbb{C}}(U_n,X)_K$ is endowed with the norm $\|\cdot\|_{\mathrm{D}}$.

In the proof of Lemma 3.1.3 we have seen that

$$BC^1_{\mathbb{C}}(U_n,X)_K \longrightarrow \mathrm{Hol_b}(U_n,X)_K : \gamma \mapsto \gamma$$

is a bounded operator of norm at most $\frac{1}{n}$. Hence, we are now in the following situation: All of the arrows in the following diagram are injective bounded operators between Banach spaces:

$$\begin{tikzcd}
\mathrm{Hol_b}(U_n,X)_K \arrow[r] & \mathrm{Hol_b}(U_{n+1},X)_K \arrow[r] & \mathrm{Hol_b}(U_{n+2},X)_K \arrow[r] \arrow[dr] & \cdots \\
BC^1_{\mathbb{C}}(U_n,X)_K \arrow[r] \arrow[u] & BC^1_{\mathbb{C}}(U_{n+1},X)_K \arrow[r] \arrow[u] \arrow[from=ul] & BC^1_{\mathbb{C}}(U_{n+2},X)_K \arrow[r] \arrow[u] \arrow[from=ul] & \cdots
\end{tikzcd}$$

We will now identify a holomorphic function with its image under these injective operators, and thus consider them as germs around K. This allows us to define the following locally convex direct limit:

Proposition 3.1.4. *The direct limit*

$$\mathrm{Germ}(K,X)_K := \bigcup_{n\in\mathbb{N}} \mathrm{Hol_b}\left(U_n,X\right)_K = \bigcup_{n\in\mathbb{N}} BC^1_{\mathbb{C}}\left(U_n,X\right)_K$$

is Hausdorff, compactly regular and complete.

Proof. To simplify notation, let $E_n := \mathrm{Hol_b}\left(U_n,X\right)_K$. To see that the direct limit is Hausdorff, note that every $\gamma \in \mathrm{Hol_b}\left(U_n,X\right)_K$ is uniquely determined if we know its power series expansion at each $a \in K$, since $U_n = \bigcup_{a\in K} \mathrm{B}^X_{\frac{1}{n}}(a)$. Therefore the following mappings are injective:

$$\Phi_n \colon \mathrm{Hol_b}\left(U_n,X\right)_K \longrightarrow \prod_{\substack{a\in K\\ k\in\mathbb{N}}} \mathrm{Sym}^k_c\left(X,X\right) : \gamma \mapsto \left(\gamma^{(k)}(a)\right)_{a\in K, k\in\mathbb{N}}$$

By Lemma 3.1.2, calculating these Fréchet derivatives is continuous with respect to the sup-norm, therefore the mappings above are continuous and linear. Since they are compatible with the bonding maps $\mathrm{Hol_b}\left(U_n,X\right)_K \longrightarrow \mathrm{Hol_b}\left(U_{n+1},X\right)_K : \gamma \mapsto \gamma|_{U_{n+1}}$ we can extend these maps to the direct limit and obtain an injective continuous map into a Hausdorff space. This proves that the direct limit is Hausdorff.

To show compact regularity and completeness, we want to use Proposition 1.2.4. Therefore it suffices to show that for every $n \in \mathbb{N}$, there is a 0-neighborhood $\Omega \subseteq E_n$ and a number $m \geq n$ and such that all the spaces $E_m, E_{m+1}, E_{m+2}, \ldots$ induce the same topology on Ω. We need the following constant: $D := \frac{3}{3-e} \approx 10,6489403$. Let $n \in \mathbb{N}$ be given. Set $\Omega := \mathrm{B}^{E_n}_1(0)$ and $m := 6n$.

Now we can apply Lemma 1.1.20 with $E = F = X$ and $M = 2\Omega = \mathrm{B}^{E_n}_2(0)$, $R = \frac{1}{n}$ and $r = \frac{1}{6n} < \frac{R}{2e}$ and obtain

$$\sum_{k=0}^{\infty} \underbrace{\sup_{\substack{\gamma\in 2\Omega\\ a\in K}} \frac{\left\|\gamma^{(k)}(a)\right\|_{\mathrm{op}}}{k!}}_{=:s_k} \left(\frac{1}{6n}\right)^k \leq \frac{\frac{1}{n}}{\frac{1}{n} - 2e\frac{1}{6n}} \cdot \sup_{\gamma\in 2\Omega} \|\gamma\|_\infty = 2D. \qquad (*)$$

Let $\ell \geq m$. To show that E_ℓ and E_m induce the same topology on Ω, it remains to prove that the inclusion map $\Omega \subseteq E_\ell \longrightarrow E_m : \gamma \mapsto \gamma$ is continuous. Let $\varepsilon > 0$ be given. By $(*)$ the series $\sum_{k=0}^\infty s_k \left(\frac{1}{6n}\right)^k$ converges. Therefore there is a number $k_0 \in \mathbb{N}$ such that $\sum_{k>k_0} s_k \left(\frac{1}{6n}\right)^k < \frac{\varepsilon}{2}$. We set $\delta := \frac{1}{D}\left(\frac{n}{\ell}\right)^{k_0} \cdot \frac{\varepsilon}{2}$.

Now, let $\gamma_1, \gamma_2 \in \Omega$ be two elements with E_ℓ-distance $\|\gamma_1 - \gamma_2\|_{E_\ell} \leq \delta$. For the E_m-distance, we show that $\|\gamma_1 - \gamma_2\|_{E_m} \leq \varepsilon$. With $\gamma_d := \gamma_1 - \gamma_2$, we know that $\gamma_d \in 2\Omega$. It remains to show that $\|\gamma_d\|_{E_m} = \sup_{x\in U_m} \|\gamma_d(x)\|_X \le \varepsilon$. Therefore let $x \in U_m$ be given. By definition of $U_m = K + \mathrm{B}^X_{\frac{1}{m}}(0)$, there is an $a \in K$ such that $x \in \mathrm{B}^X_{\frac{1}{m}}(a)$. Now,

$$\begin{aligned}
\|\gamma_d(x)\|_X &= \left\| \sum_{k=0}^{\infty} \frac{\gamma_d^{(k)}(a)(x-a,\dots,x-a)}{k!} \right\|_X \le \sum_{k=0}^{\infty} \frac{\left\|\gamma_d^{(k)}(a)\right\|_{\mathrm{op}}}{k!} \left(\frac{1}{m}\right)^k \\
&\le \sum_{k\le k_0} \frac{\left\|\gamma_d^{(k)}(a)\right\|_{\mathrm{op}}}{k!} \left(\frac{1}{6n}\right)^k + \sum_{k>k_0} \frac{\left\|\gamma_d^{(k)}(a)\right\|_{\mathrm{op}}}{k!} \left(\frac{1}{6n}\right)^k \\
&\le \sum_{k\le k_0} \frac{\left\|\gamma_d^{(k)}(a)\right\|_{\mathrm{op}}}{k!} \left(\frac{1}{6\ell}\right)^k \left(\frac{\ell}{n}\right)^{k_0} + \sum_{k>k_0} s_k \left(\frac{1}{6n}\right)^k \\
&\le \left(\frac{\ell}{n}\right)^{k_0} \sum_{k=0}^{\infty} \frac{\left\|\gamma_d^{(k)}(a)\right\|_{\mathrm{op}}}{k!} \left(\frac{1}{6\ell}\right)^k + \frac{\varepsilon}{2} \\
&\le \left(\frac{\ell}{n}\right)^{k_0} \cdot D \cdot \underbrace{\|\gamma_d\|_{E_\ell}}_{<\delta} + \frac{\varepsilon}{2} \le \left(\frac{\ell}{n}\right)^{k_0} \cdot D \cdot \frac{1}{D} \left(\frac{n}{\ell}\right)^{k_0} \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that the space

$$\mathrm{Germ}(K,X)_K = \bigcup_{n\in\mathbb{N}} \mathrm{Hol}_{\mathrm{b}}(U_n, X)_K = \bigcup_{n\in\mathbb{N}} BC^1_{\mathbb{C}}(U_n, X)_K$$

is a compactly regular complete (LB)-space. □

Remark. In the proof, we did not use that the range space of the mappings is equal to the Banach space containing K and we did not use that the mappings all vanish on K. So, the same argument also shows that

$$\bigcup_{n\in\mathbb{N}} \mathrm{Hol}_{\mathrm{b}}(U_n, Z)$$

is compactly regular for every Banach space Z. We will not need that in this chapter, but later, in chapter 5 (Theorem 5.1.1).

3.1.2 The monoid

To turn DiffGerm(K,X) into a Lie group modelled on Germ$(K,X)_K$, we first construct an analytic structure on the set

$$\mathrm{EndGerm}(K,X) := \left\{ \eta\colon U_\eta \longrightarrow X : \begin{array}{c} \eta \text{ is a } C^\omega_{\mathbb{C}}\text{-map, } U_\eta \text{ is an open} \\ \text{neighborhood of } K \text{ and } \eta|_K = \mathrm{id}_K \end{array} \right\} /\sim$$

where $\eta_1 \sim \eta_2$ if and only if they coincide on a common neighborhood of K.

The set $\mathrm{EndGerm}(K, X)$ becomes a monoid with the multiplication

$$[\eta_1]_\sim \circ [\eta_2]_\sim := [\eta_1 \circ \eta_2]_\sim .$$

Here, $[\eta]_\sim$ denotes the equivalence class of η with respect to the relation $\sim$.

Since every neighborhood of K contains one of the neighborhoods U_n, it suffices to look at analytic maps of the form $\eta = \mathrm{id}_{U_n} + \gamma$ with $\gamma \in \mathrm{Hol_b}\left(U_n, X\right)_K$ for some $n \in \mathbb{N}$. This implies that the following map is a bijection:

$$\begin{aligned} \Phi : \mathrm{EndGerm}(K, X) &\longrightarrow \mathrm{Germ}(K, X)_K \\ [\gamma + \mathrm{id}_{U_n}]_\sim &\longmapsto \gamma \in \mathrm{Hol_b}\left(U_n, X\right)_K \end{aligned}$$

We use this map as a global chart and define the complex analytic manifold structure on $\mathrm{EndGerm}(K, X)$ such that Φ is a diffeomorphism.

Proposition 3.1.5. *The monoid multiplication of* $\mathrm{EndGerm}(K, X)$*:*

$$\begin{aligned} \mu : \mathrm{EndGerm}(K, X) \times \mathrm{EndGerm}(K, X) &\longrightarrow \mathrm{EndGerm}(K, X) \\ ([\eta_1]_\sim, [\eta_2]_\sim) &\longmapsto [\eta_1 \circ \eta_2]_\sim \end{aligned}$$

is $C^\omega_\mathbb{C}$ *with respect to the manifold structure defined by* Φ*.*

Proof. Using the global chart Φ, this map becomes

$$\begin{aligned} \Phi \circ \mu \circ (\Phi \times \Phi)^{-1} : &\mathrm{Germ}(K, X)_K \times \mathrm{Germ}(K, X)_K \longrightarrow \mathrm{Germ}(K, X)_K \\ &(\gamma_1, \gamma_2) \mapsto (\gamma_1 + \mathrm{id}) \circ (\gamma_2 + \mathrm{id}) - \mathrm{id} = \gamma_1 \circ (\gamma_2 + \mathrm{id}) + \gamma_2. \end{aligned}$$

To show analyticity of that map, it suffices to show that

$$\begin{aligned} f : \mathrm{Germ}(K, X)_K \times \mathrm{Germ}(K, X)_K &\longrightarrow \mathrm{Germ}(K, X)_K \\ (\gamma_1, \gamma_2) &\longmapsto \gamma_1 \circ (\gamma_2 + \mathrm{id}) \end{aligned}$$

is analytic.

For each $n \in \mathbb{N}$, we set

$$\begin{aligned} E_n &:= \left(\mathrm{Hol_b}\left(U_n, X\right)_K, \|\cdot\|_\infty\right), \\ F_n &:= \left(BC^1_\mathbb{C}\left(U_n, X\right)_K, \|\cdot\|_\mathrm{D}\right). \end{aligned}$$

The domain of the map f in question can now be regarded as the following direct limit: $\mathrm{Germ}(K, X)_K \times \mathrm{Germ}(K, X)_K = \bigcup_{n\in\mathbb{N}} E_n \times F_n$. For each $R > 0$ we set $\Omega_R := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{F_n}_R(0)$. One easily checks that

$$\begin{aligned} \bigcup_{R>0} \Omega_R &= \mathrm{Germ}(K, X)_K \\ \bigcup_{n\in\mathbb{N}} \mathrm{B}^{E_n}_R(0) &= \mathrm{Germ}(K, X)_K && \text{for every } R > 0. \end{aligned}$$

Therefore the domain

$$\mathrm{Germ}(K,X)_K \times \mathrm{Germ}(K,X)_K = \bigcup_{R\in\mathbb{N}} (\mathrm{Germ}(K,X)_K \times \Omega_R)$$

can be written as a union of open 0-neighborhoods. This means that f is analytic on $\mathrm{Germ}(K,X)_K \times \mathrm{Germ}(K,X)_K$ if and only if f is analytic on each $\mathrm{Germ}(K,X)_K \times \Omega_R$.

Let $R \in \mathbb{N}$ be given. To simplify notation, we denote the restriction of f to $\mathrm{Germ}(K,X)_K \times \Omega_R$ also by f.

Now, define $\ell_n := (R+1)(n+2) \in \mathbb{N}$. Since $\lim_{n\to\infty} \ell_n = \infty$, the sequence $(F_{\ell_n})_{n\in\mathbb{N}}$ is cofinal in $(F_n)_{n\in\mathbb{N}}$, hence

$$\mathrm{Germ}(K,X)_K \times \Omega_R = \bigcup_{n\in\mathbb{N}} \left(\mathrm{B}_R^{E_n}(0) \times \mathrm{B}_R^{F_{\ell_n}}(0)\right) = \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{H_n}(0)\,.$$

Here, we set $H_n := E_n \times F_{\ell_n}$ with the norm

$$\|(\gamma_1,\gamma_2)\|_{H_n} := \max\{\|\gamma_1\|_\infty, \|\gamma_2\|_{\mathrm{D}}\} = \max\{\|\gamma_1\|_\infty, \|\gamma_2'\|_\infty\}.$$

All bonding maps $i_n \colon H_n \longrightarrow H_{n+1}$ have operator norm at most 1. We now would like to apply Theorem 2.1. To this end, we define

$$f_n \colon \mathrm{B}_R^{H_n}(0) \longrightarrow \mathrm{Germ}(K,X)_K : (\gamma_1,\gamma_2) \mapsto \gamma_1 \circ (\gamma_2 + \mathrm{id}_{U_{\ell_n}}).$$

We claim:

(a) Each f_n makes sense,

(b) Each f_n is $C^\omega_\mathbb{C}$,

(c) Each f_n is bounded.

Once we have this, by Theorem 2.1 the map f is analytic, as we had to show.

(a) Let $(\gamma_1,\gamma_2) \in \mathrm{B}_R^{H_n}(0) = \mathrm{B}_R^{E_n}(0) \times \mathrm{B}_R^{F_{\ell_n}}(0)$. We have to show that γ_1 and $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})$ can be composed, i.e. that $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})(U_{\ell_n}) \subseteq U_n$. In fact, we actually show that $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})(U_{\ell_n}) \subseteq U_{n+2}$.

Therefore, let $x \in U_{\ell_n} = K + \mathrm{B}^X_{\frac{1}{\ell_n}}(0)$ be given. Then x is of the form $x = a + v$ with $a \in K$ and $\|v\|_X < \frac{1}{\ell_n}$. Now, we apply $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})$ to x:

$$(\gamma_2 + \mathrm{id}_{U_{\ell_n}})(x) = \gamma_2(a+v) + a + v = a + w$$

with $w := v + \gamma_2(a+v)$. Now, we estimate the norm of w:

$$\begin{aligned}
\|w\|_X &= \|v + \gamma_2(a+v)\|_X \le \|v\|_X + \|\gamma_2(a+v)\|_X \\
&= \|v\|_X + \|\gamma_2(a+v) - \gamma_2(a)\|_X \\
&= \|v\|_X + \left\|\int_0^1 \gamma_2'(a+tv)(v)dt\right\|_X \\
&< \frac{1}{\ell_n} + \sup_{t\in[0,1]} \left\|\gamma_2'(a+tv)\right\|_{\mathrm{op}} \|v\|_X \le \frac{1}{\ell_n} + \|\gamma_2\|_{\mathrm{D}} \|v\|_X \\
&\le \frac{1}{\ell_n} + R\frac{1}{\ell_n} = \frac{R+1}{(R+1)(n+2)} = \frac{1}{n+2}.
\end{aligned}$$

Therefore $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})(x) \in \mathrm{B}^X_{\frac{1}{n+2}}(a) \subseteq U_{n+2}$.

(b) The image of f_n is a subset of $\mathrm{Hol_b}\,(U_{\ell_n}, X)_K$. The inclusion map:

$$\mathrm{Hol_b}\,(U_{\ell_n}, X)_K \longrightarrow \mathrm{Germ}(K, X)_K$$

is continuous linear and therefore $C^\omega_\mathbb{C}$. It remains to show that the arrow $\clubsuit$ in the following diagram is $C^\omega_\mathbb{C}$:

$$\begin{tikzcd}
\mathrm{B}^{H_n}_R(0) \arrow[r, "\clubsuit"'] \arrow[d, "\diamondsuit"] \arrow[rr, bend left, "f_n"] & \mathrm{Hol_b}\,(U_{\ell_n}, X)_K \arrow[r, hook] \arrow[d, hook, "\spadesuit"'] & \mathrm{Germ}(K, X)_K \\
BC^2_\mathbb{C}(U_{n+1}, X) \times BC^{\partial,0}_\mathbb{C}(U_{\ell_n}, U_{n+1}) \arrow[r, "\heartsuit"] & BC^0_\mathbb{C}(U_{\ell_n}, X) &
\end{tikzcd}$$

The space $\mathrm{Hol_b}\,(U_{\ell_n}, X)_K$ is a closed subspace of $BC^0_\mathbb{C}(U_{\ell_n}, X)$ and $\spadesuit$ is a topological embedding. Therefore f_n will be $C^\omega_\mathbb{C}$ if we are able to show that $\spadesuit \circ \clubsuit$ is so.

Let $(\gamma_1, \gamma_2) \in \mathrm{B}^{H_n}_R(0)$. Then γ_1 is complex analytic and bounded on U_n. We have seen in Lemma 3.1.2 that all derivatives of γ_1 are bounded when restricting γ_1 to the smaller set U_{n+1} and that the inclusion

$$\mathrm{Hol_b}\,(U_n, X)_K \longrightarrow BC^k_\mathbb{C}(U_{n+1}, X) : \gamma_1 \mapsto \gamma_1|_{U_{n+1}}$$

is continuous for every $k \in \mathbb{N}$.

We have just shown in (a) that the image of $(\gamma_2 + \mathrm{id}_{U_{\ell_n}})$ is a subset of U_{n+2}, hence it has a positive distance from the boundary of U_{n+1}. Hence, it lies in the space $BC^{\partial,0}_\mathbb{C}(U_{\ell_n}, U_{n+1})$ as defined in Lemma 1.1.25. The map

$$\diamondsuit\colon \mathrm{B}^{H_n}_R(0) \longrightarrow BC^2_\mathbb{C}(U_{n+1}, X) \times BC^{\partial,0}_\mathbb{C}(U_{\ell_n}, U_{n+1}) : (\gamma_1, \gamma_2) \mapsto (\gamma_1, \gamma_2 + \mathrm{id}_{U_{\ell_n}})$$

in the diagram above is therefore well-defined and continuous. Since it is affine, it is automatically analytic.

To make the diagram commutative, we define the remaining arrow as

$$\begin{aligned} \heartsuit : BC^2_\mathbb{C}(U_{n+1}, X) \times BC^{\partial,0}_\mathbb{C}(U_{\ell_n}, U_{n+1}) &\longrightarrow BC^0_\mathbb{C}(U_{\ell_n}, X) \\ (\gamma, \eta) &\longmapsto \gamma \circ \eta \end{aligned}$$

and this is $C^1_\mathbb{C}$ by Lemma 1.1.25 (with $k = l = 1$). Since we are dealing with mappings between complex Banach spaces, the $C^1_\mathbb{C}$-property implies complex analyticity.

(c) Let $(\gamma_1, \gamma_2) \in \mathrm{B}^{E_n}_R(0) \times \mathrm{B}^{F_{\ell_n}}_R(0)$. Then $f_n(\gamma_1, \gamma_2) = \gamma_1 \circ (\gamma_2 + \mathrm{id}_{U_{\ell_n}})$ is an element of $\mathrm{Hol_b}\,(U_{\ell_n}, X)_K$ of norm $\left\|\gamma_1 \circ (\gamma_2 + \mathrm{id}_{U_{\ell_n}})\right\|_\infty \leq \|\gamma_1\|_\infty < R$. Therefore the image of f_n is a bounded subset of $\mathrm{Hol_b}\,(U_{\ell_n}, X)_K$ and hence a bounded subset of the direct limit $\mathrm{Germ}(K, X)_K$.

Therefore, by Theorem 2.1, f is complex analytic and we have shown that $\mathrm{EndGerm}(K, X)$ is a complex analytic monoid. $\square$

3.1.3 The group

The monoid $\mathrm{Germ}(K, X)_K$ has a $C^\omega_\mathbb{C}$-manifold structure and an analytic multiplication. We now show that the group of invertible elements of the monoid is open and that inversion is analytic.

For the openness, we use a lemma:

Lemma 3.1.6. *Let $\gamma \in BC^1_\mathbb{C}(U_n, X)_K$ with $\|\gamma\|_\mathrm{D} = \|\gamma'\|_\infty < 1$. Then $\eta := \mathrm{id}_{U_{6n}} + \gamma|_{U_{6n}}$ is a $C^\omega_\mathbb{C}$-diffeomorphism onto its open image.*

Proof. Let $x \in U_{6n}$. Then the Fréchet derivative $\eta'(x)$ at x is an element in the Banach algebra $\left(\mathcal{L}(X), \|\cdot\|_{\mathrm{op}}\right)$. The distance between $\eta'(x)$ and the identity of the algebra is $\|\eta'(x) - \mathrm{id}_X\|_{\mathrm{op}} = \|\gamma'(x)\|_{\mathrm{op}} \leq \|\gamma'\|_\infty < 1$. Therefore $\eta'(x) \in \mathrm{B}_1^{\mathcal{L}(X)}(\mathrm{id}_X)$. All elements in an open ball with radius 1 centered around the identity of Banach algebra are invertible (Neumann-series). Hence, $\eta'(x)$ is invertible.

By the Inverse Function Theorem for complex Banach spaces this implies that there is an open neighborhood of x on which η is a diffeomorphism onto its open image. Since $x \in U_{6n}$ was arbitrary, we know that the image $\eta(U_{6n})$ is open. To show that $\eta\colon U_{6n} \longrightarrow \eta(U_{6n})$ is not only a local, but a global diffeomorphism, it remains to show injectivity of η.

Let $x, y \in U_{6n}$ with $\eta(x) = \eta(y)$ be given. We have to show that $x = y$. This is easy once we have shown that the line segment joining x and y lies in U_n. By definition of U_{6n}, there are elements $a, b \in K$ and $v, w \in X$ such that $\|v\|_X, \|w\|_X < \frac{1}{6n}$ and $x = a + v, y = b + w$. Let $[a, x] := \{a + tv : t \in [0, 1]\} \subseteq \mathrm{B}^X_{\frac{1}{6n}}(a) \subseteq U_{6n}$ denote the compact line segment joining a and x. Then

$$\begin{aligned}
\|\eta(x) - \eta(a)\|_X &= \left\| \int_0^1 \eta'(a + tv)(v)dt \right\|_X \leq \max_{t \in [0,1]} \left\|\eta'(a + tv)\right\|_X \cdot \|v\|_X \\
&\leq \underbrace{\left(\|\mathrm{id}\|_{\mathrm{op}} + \left\|\gamma'\right\|_{\mathrm{op}}\right)}_{<2} \cdot \|v\|_X < 2 \cdot \frac{1}{6n} = \frac{1}{3n}.
\end{aligned}$$

Likewise we see that $\|\eta(y) - \eta(b)\|_X < \frac{1}{3n}$. We can now estimate the distance between the points a and b:

$$\begin{aligned}
\|a - b\|_X &= \|\eta(a) - \eta(b)\|_X \\
&\leq \underbrace{\|\eta(a) - \eta(x)\|_X}_{<\frac{1}{3n}} + \underbrace{\|\eta(x) - \eta(y)\|_X}_{=0} + \underbrace{\|\eta(y) - \eta(b)\|_X}_{<\frac{1}{3n}} < \frac{2}{3n}.
\end{aligned}$$

This also allows us to estimate the distance between y and a:

$$\|y - a\|_X \leq \|y - b\|_X + \|b - a\|_X < \frac{1}{6n} + \frac{2}{3n} < \frac{1}{n}.$$

So, $y \in \mathrm{B}^X_{\frac{1}{n}}(a)$. Therefore the two points x and y both lie in the convex set $\mathrm{B}^X_{\frac{1}{n}}(a)$. Therefore also the line segment $[x, y]$ lies in $\mathrm{B}^X_{\frac{1}{n}}(a)$ which is a subset of U_n, and thus

$$\begin{aligned} 0 &= \|\eta(x) - \eta(y)\|_X = \|x - y + \gamma(x) - \gamma(y)\|_X \\ &\geq \|x - y\|_X - \|\gamma(x) - \gamma(y)\|_X \\ &= \|x - y\|_X - \left\| \int_0^1 \gamma'(y + t(x - y))(x - y)dt \right\|_X \\ &\geq \|x - y\|_X - \|\gamma'\|_\infty \|x - y\|_X = \|x - y\|_X \underbrace{(1 - \|\gamma'\|_\infty)}_{>0}. \end{aligned}$$

Therefore $\|x - y\|_X$ has to be zero and so $\eta\colon U_{6n} \longrightarrow X$ is injective. This finishes the proof. $\square$

Proposition 3.1.7. *Let* $\mathrm{EndGerm}(K, X)^\times$ *denote the group of invertible elements of* $\mathrm{EndGerm}(K, X)$ *and let*

$$\mathrm{DiffGerm}(K, X) := \left\{ \eta : \begin{array}{c} \eta \text{ is a } C^\omega_{\mathbb{K}}\text{-diffeomorphism between open} \\ \text{neighborhoods of } K \text{ and } \eta|_K = \mathrm{id}_K \end{array} \right\} /\sim,$$

where two diffeomorphisms η_1, η_2 *are considered equivalent,* $\eta_1 \sim \eta_2$ *if they coincide on a common neighborhood of* K*. Then* $\mathrm{DiffGerm}(K, X) = \mathrm{EndGerm}(K, X)^\times$ *and this is an open subset of* $\mathrm{EndGerm}(K, X)$*.*

Proof. If η is a diffeomorphism between open neighborhoods of K, then $[\eta]_\sim$ is clearly invertible and thus $\mathrm{DiffGerm}(K, X) \subseteq \mathrm{EndGerm}(K, X)^\times$.

Conversely, let $[\eta_1]_\sim \in \mathrm{EndGerm}(K, X)^\times$ be given. This means that there exist $C^\omega_{\mathbb{C}}$-maps $\eta_1\colon V_{\eta_1} \longrightarrow X$ and $\eta_2\colon V_{\eta_1} \longrightarrow X$ on open neighborhoods V_{η_1}, V_{η_2} of K in X such that $\eta_j|_K = \mathrm{id}_K$ and $[\eta_1]_\sim \circ [\eta_2]_\sim = [\mathrm{id}_X]_\sim = [\eta_2]_\sim \circ [\eta_1]_\sim$.

From $[\eta_2]_\sim \circ [\eta_1]_\sim = [\mathrm{id}_X]_\sim$, we get that there is an open K-neighborhood $W_1 \subseteq \eta_1^{-1}(V_{\eta_2})$ such that $\eta_2 \circ \eta_1|_{W_1} = \mathrm{id}_{W_1}$. In particular, $\eta_1|_{W_1}$ is injective.

From $[\eta_2]_\sim \circ [\eta_1]_\sim = [\mathrm{id}_X]_\sim$, we get that there is an open K-neighborhood $W_2 \subseteq \eta_2^{-1}(V_{\eta_1})$ such that $\eta_1 \circ \eta_2|_{W_2} = \mathrm{id}_{W_2}$. By making W_2 smaller, we may assume $W_2 \subseteq \eta_2^{-1}(W_1)$.

Now, $\eta_1(W_1) \supseteq \eta_1(\eta_2(W_2)) = W_2$. Hence the image of $\eta_1|_{W_1}$ contains an open K-neighborhood. so we may restrict η_1 to $\eta_1^{-1}(W_2)$ and obtain an injective $C^\omega_{\mathbb{C}}$-map whose image is the open K-neighborhood W_2 and whose inverse is given by a restriction of the $C^\omega_{\mathbb{C}}$-map η_2. This proves that $\eta_1|_{\eta_1^{-1}(W_2)}$ is a $C^\omega_{\mathbb{C}}$-diffeomorphism onto an open neighborhood of K, fixing K pointwise. Hence, $[\eta_1]_\sim \in \mathrm{DiffGerm}(K, X)$.

This shows the equality of the two sets $\mathrm{DiffGerm}(K, X)$ and $\mathrm{EndGerm}(K, X)^\times$. It remains to show the openness.

The set $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}_1^{BC^1_{\mathbb{C}}(U_n,X)_K}(0) \subseteq \mathrm{Germ}(K,X)_K$ is an open 0-neighborhood in $\mathrm{Germ}(K,X)_K$. Using the global chart, we see that $\Phi^{-1}(U)$ is an open $[\mathrm{id}]_\sim$-neighborhood in $\mathrm{EndGerm}(K,X)$. By Lemma 3.1.6 we know that every $\gamma \in \Phi^{-1}(U)$ is a diffeomorphism onto an open image and thus $\Phi^{-1}(U) \subseteq \mathrm{DiffGerm}(K,X) = \mathrm{EndGerm}(K,X)^\times$. Therefore the unit group of the monoid contains an open identity neighborhood, and hence the whole unit group has to be open. □

From Lemma 3.1.6, we know that the image of $\eta\colon U_{6n} \longrightarrow X$ is an open neighborhood of K and therefore has to contain one of the basic neighborhoods U_m for an $m \in \mathbb{N}$. The next lemma provides quantitative information:

Lemma 3.1.8. *Let $\gamma \in BC^1_{\mathbb{C}}(U_n,X)_K$ with $\|\gamma\|_{\mathrm{D}} = \|\gamma'\|_\infty \leq \frac{1}{2}$ and let $\eta := \mathrm{id}_{U_{6n}} + \gamma|_{U_{6n}}$ be as in Lemma 3.1.6. Then the image of η contains U_{12n} and we have*

$$\left\| \left((\gamma + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \right)^{-1} \Bigg|_{U_{12n}} - \mathrm{id}_{U_{12n}} \right\|_\infty \leq \frac{1}{6n}.$$

To prove this lemma, we need the quantitative version of the Inverse Function Theorem for Banach spaces (1.1.21).

Proof of Lemma 3.1.8: Let $x \in U_{12n} = K + \mathrm{B}^X_{\frac{1}{12n}}(0)$ be given. We have to show that $x \in \eta(U_{6n})$. We know that there is an $a \in K$ such that $x = a + v$ with $v \in \mathrm{B}^X_{\frac{1}{12n}}(0)$. Now, we set $r := \frac{1}{6n}$, $T := \mathrm{id}_X$, $U := \mathrm{B}^X_{\frac{1}{6n}}(0)$ and $f\colon U \longrightarrow X : w \mapsto \gamma(a+w)$. This function satisfies $f(0) = 0$ and is Lipschitz continuous with Lipschitz constant $L := \|f'\|_\infty \leq \|\gamma\|_{\mathrm{D}} \leq \frac{1}{2}$. The number $\lambda := L \cdot \left\|T^{-1}\right\|_{\mathrm{op}} = L \leq \frac{1}{2}$ is strictly less than 1 and therefore all hypotheses of Theorem 1.1.21 are satisfied. Therefore we can conclude that the image of $(\mathrm{id} + f)$ contains the ball $\mathrm{B}^X_{r'}(0)$ with $r' = \frac{r(1-\lambda)}{\|T^{-1}\|_{\mathrm{op}}} = \frac{1}{6n} \cdot (1-\lambda) \geq \frac{1}{6n}(1 - \frac{1}{2}) = \frac{1}{12n}$. So, there exists a $w \in U$ such that $(\mathrm{id} + f)(w) = v$. But this means:

$$x = a + v = a + (\mathrm{id} + f)(w) = a + w + f(w) = a + w + \gamma(a+w) = \eta(a+w).$$

So x is in the image of η. This proves $U_{12n} \subseteq \eta(U_{6n})$.

Since the Fréchet derivative of $\eta := \mathrm{id}_{U_{6n}} + \gamma|_{U_{6n}}$ has distance at most $\frac{1}{2}$ from the identity, the Neumann-series for inverses implies that the Fréchet derivative of η^{-1} has distance at most $\frac{1}{1-\frac{1}{2}} = 2$ from the identity. Therefore:

$$\left\| \left((\gamma + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \right)^{-1} \Bigg|_{U_{12n}} - \mathrm{id}_{U_{12n}} \right\|_{\mathrm{D}} \leq 2.$$

Together with $\|\cdot\|_\infty \leq \frac{1}{12n} \|\cdot\|_{\mathrm{D}}$ the assertion follows. □

So far we showed that $\mathrm{DiffGerm}(K,X)$ is an open subset of the $C^\omega_\mathbb{C}$-manifold $\mathrm{EndGerm}(K,X)$ and therefore has an induced manifold structure. To show complex analyticity of the inversion map, we once again use our global chart Φ and define:

$$\begin{aligned}\mathbf{i}:\Phi(\mathrm{DiffGerm}(K,X)) &\longrightarrow \Phi(\mathrm{DiffGerm}(K,X))\\ \gamma &\longmapsto \Phi\left(\left(\Phi^{-1}(\gamma)\right)^{-1}\right).\end{aligned}$$

It remains to show that $\mathbf{i}$ is analytic.

From now on, we again use the notation: $E_n := \mathrm{Hol_b}\left(U_n,X\right)_K$ and $F_n := BC^1_\mathbb{C}\left(U_n,X\right)_K$. Lemma 3.1.8 allows us to define for every $n\in\mathbb{N}$ the following map:

$$\begin{aligned}\mathbf{i}_n : \mathrm{B}^{F_n}_{\frac{1}{2}}(0) &\longrightarrow \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)\\ \gamma &\longmapsto \left(\left(\gamma+\mathrm{id}_{U_n}\right)\Big|_{U_{6n}}\right)^{-1}\Bigg|_{U_{12n}} - \mathrm{id}_{U_{12n}}.\end{aligned}$$

If we are able to show that every $\mathbf{i}_n$ is $C^\omega_\mathbb{C}$ then we can directly apply Theorem 2.1 and see that the monoid inversion is analytic on an open neighborhood of the identity. Then inversion is everywhere $C^\omega_\mathbb{C}$ and we are done.

Proposition 3.1.9. *(a) The mapping*

$$\begin{aligned}h_n : \mathrm{B}^{F_n}_{\frac{1}{2}}(0)\times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0) &\longrightarrow E_{12n}\\ (\gamma_1,\gamma_2) &\longmapsto (\gamma_1+\mathrm{id}_{U_n})\circ(\gamma_2+\mathrm{id}_{U_{12n}})-\mathrm{id}_{U_{12n}}\end{aligned}$$

is complex analytic.

(b) For every fixed $(\gamma_1,\gamma_2)\in \mathrm{B}^{F_n}_{\frac{1}{2}}(0)\times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)$ *and every* $\widehat{\gamma}_1\in F_n, \widehat{\gamma}_2\in E_{12n}$ *we have*

$$h_n'(\gamma_1,\gamma_2)(\widehat{\gamma}_1,\widehat{\gamma}_2)=\widehat{\gamma}_1\circ(\gamma_2+\mathrm{id}_{U_{12n}})+\gamma_1'\circ(\gamma_2+\mathrm{id}_{U_{12n}})\cdot(\widehat{\gamma}_2)+\widehat{\gamma}_2.$$

(c) For $(\gamma_1,\gamma_2)\in \mathrm{B}^{F_n}_{\frac{1}{2}}(0)\times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)$*, we have the equivalence:*

$$\big(h_n(\gamma_1,\gamma_2)=0\big)\iff\big(\gamma_2=\mathbf{i}_n(\gamma_1)\big).$$

(d) Every $\mathbf{i}_n$ *is complex analytic.*

Proof. (a) The argument is essentially the same as in Proposition 3.1.5. We write $h_n(\gamma_1,\gamma_2)=\clubsuit(\gamma_1,\gamma_2)+\gamma_2$ with $\clubsuit(\gamma_1,\gamma_2)=\gamma_1\circ(\gamma_2+\mathrm{id}_{U_{12n}})$ and have the following commutative diagram:

$$\begin{tikzcd} \mathrm{B}^{F_n}_{\frac{1}{2}}(0)\times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0) \arrow[r, "\clubsuit"] \arrow[d, "\diamondsuit"] & E_{12n} \arrow[d, hook, "\spadesuit"] \\ BC^2_\mathbb{C}(U_{2n},X)\times BC^{\partial,0}_\mathbb{C}(U_{12n},U_{2n}) \arrow[r, "\heartsuit"] & BC^0_\mathbb{C}(U_{12n},X) \end{tikzcd}$$

Once again ♠ is a topological embedding. The map

$$\begin{aligned} \diamondsuit : \mathrm{B}^{F_n}_{\frac{1}{2}}(0) \times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0) &\longrightarrow BC^2_{\mathbb{C}}(U_{2n}, X) \times BC^{\partial,0}_{\mathbb{C}}(U_{12n}, U_{2n}) \\ (\gamma_1, \gamma_2) &\longmapsto (\gamma_1|_{U_{2n}}, \gamma_2 + \mathrm{id}_{U_{12n}}) \end{aligned}$$

is well-defined and continuous. Since it is affine, it is automatically analytic. The last arrow

$$\begin{aligned} \heartsuit : BC^2_{\mathbb{C}}(U_{2n}, X) \times BC^{\partial,0}_{\mathbb{C}}(U_{12n}, U_{2n}) &\longrightarrow BC^0_{\mathbb{C}}(U_{12n}, X) \\ (\gamma, \eta) &\longmapsto \gamma \circ \eta \end{aligned}$$

is $C^\omega_{\mathbb{C}}$ by Lemma 1.1.25 and since the diagram commutes, h_n is analytic.

(b) This follows directly from the formula in Lemma 1.1.25.

(c) Assume that $\gamma_2 = \mathbf{i}_n(\gamma_1)$ holds. Then

$$\begin{aligned} h_n(\gamma_1, \gamma_2) &= (\gamma_1 + \mathrm{id}_{U_n}) \circ (\mathbf{i}_n(\gamma_1) + \mathrm{id}_{U_{12n}}) - \mathrm{id}_{U_{12n}} \\ &= (\gamma_1 + \mathrm{id}_{U_n}) \circ \left((\gamma_1 + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \right)^{-1} \Bigg|_{U_{12n}} - \mathrm{id}_{U_{12n}} \\ &= \mathrm{id}_{U_{12n}} - \mathrm{id}_{U_{12n}} = 0. \end{aligned}$$

Conversely, assume that $(\gamma_1, \gamma_2) \in \mathrm{B}^{F_n}_{\frac{1}{2}}(0) \times \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)$ is given with $h_n(\gamma_1, \gamma_2) = 0$. Then $(\gamma_1 + \mathrm{id}_{U_n}) \circ (\gamma_2 + \mathrm{id}_{U_{12n}}) = \mathrm{id}_{U_{12n}}$. Since $\gamma_2 + \mathrm{id}_{U_{12n}}$ is continuous, $W := (\gamma_2 + \mathrm{id}_{U_{12n}})^{-1}(U_{6n}) \subseteq U_{12n}$ is an open K-neighborhood. Moreover,

$$(\gamma_1 + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \circ (\gamma_2 + \mathrm{id}_{U_{12n}})|_W = \mathrm{id}_W.$$

But since $(\gamma_1 + \mathrm{id}_{U_n}) \Big|_{U_{6n}}$ is a diffeomorphism, we can compose this equality from the left with $\left((\gamma_1 + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \right)^{-1}$ and obtain

$$(\gamma_2 + \mathrm{id}_{U_{12n}})|_W = \left((\gamma_1 + \mathrm{id}_{U_n}) \Big|_{U_{6n}} \right)^{-1} \Bigg|_W .$$

Thus we obtain that γ_2 and $\mathbf{i}_n(\gamma_1)$ coincide on the nonempty set W which is open in the connected set U_{12n}. By the identity theorem for complex analytic maps, this means $\gamma_2 = \mathbf{i}_n(\gamma_1)$.

(d) Let $\gamma_1 \in \mathrm{B}^{F_n}_{\frac{1}{2}}(0)$ and set $\gamma_2 := \mathbf{i}_n(\gamma_1) \in \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)$. By (c) this implies $h_n(\gamma_1, \gamma_2) = 0$. We wish to use the Implicit Function Theorem and therefore examine the following operator, the "partial differential with respect to the second argument":

$$\begin{aligned} T : E_{12n} &\longrightarrow E_{12n} \\ \widehat{\gamma}_2 &\longmapsto h'_n(\gamma_1, \gamma_2)(0, \widehat{\gamma}_2). \end{aligned}$$

By (b) this can be rewritten as $T : \widehat{\gamma}_2 \mapsto \gamma_1' \circ (\gamma_2 + \mathrm{id}_{U_{12n}}) \cdot (\widehat{\gamma}_2) + \widehat{\gamma}_2$.

Let $\widehat{\gamma}_2 \in E_{12n}$ be given. Then we can estimate

$$\begin{aligned}
\left\|\left(T - \mathrm{id}_{E_{12n}}\right)(\widehat{\gamma}_2)\right\|_\infty &= \sup_{x \in U_{12n}} \left\|\left(T(\widehat{\gamma}_2) - \widehat{\gamma}_2\right)(x)\right\|_X \\
&= \sup_{x \in U_{12n}} \left\|\gamma_1'(\gamma_2(x) + \mathrm{id}_{U_{12n}}(x))(\widehat{\gamma}_2(x))\right\|_X \\
&\leq \sup_{x \in U_{12n}} \left\|\gamma_1'(\gamma_2(x) + x)\right\|_{\mathrm{op}} \|\widehat{\gamma}_2(x)\|_X \\
&\leq \left\|\gamma_1'\right\|_\infty \|\widehat{\gamma}_2\|_\infty = \|\gamma_1\|_{\mathrm{D}} \|\widehat{\gamma}_2\|_\infty \leq \frac{1}{2} \|\widehat{\gamma}_2\|_\infty .
\end{aligned}$$

Thus, $\|T - \mathrm{id}_{E_{12n}}\|_{\mathrm{op}} \leq \frac{1}{2} < 1$. Therefore the bounded operator T is invertible, i.e. an isomorphism of Banach spaces.

By the Implicit Function Theorem, there are neighborhoods $\Omega_1 \subseteq F_n$, $\Omega_2 \subseteq E_{12n}$ of γ_1 and γ_2 respectively, such that $h_n^{-1}(\{0\}) \cap (\Omega_1 \times \Omega_2)$ is the graph of a $C^\omega_{\mathbb{C}}$-map from Ω_1 to Ω_2. But by (c), we know that this function has to be a restriction of $\mathbf{i}_n \colon \mathrm{B}^{F_n}_{\frac{1}{2}}(0) \longrightarrow \mathrm{B}^{E_{12n}}_{\frac{1}{6n}}(0)$. Therefore $\mathbf{i}_n$ is $C^\omega_{\mathbb{C}}$ in a neighborhood of γ_1. Since γ_1 was arbitrary, $\mathbf{i}_n$ is $C^\omega_{\mathbb{C}}$. □

This proves Theorem 3.1.1 for $\mathbb{K} = \mathbb{C}$. As mentioned at the beginning of this section, the case $\mathbb{K} = \mathbb{R}$ now follows verbatim as in [9, Corollary 15.11].

3.2 Regularity of $\mathrm{DiffGerm}(K, X)$

Let X be a complex Banach space and $K \subseteq X$ be a nonempty compact subset.

Let $U_r := K + \mathrm{B}^X_r(0)$.

Let $E_r := \mathrm{Hol_b}(U_r, X)_K$, $F_r := BC^1_{\mathbb{C}}(U_r, X)_K$ and $D_r := C^1([0,1], F_r)$. On the space F_r, we use the norm $\|\gamma\|_{\mathrm{D}} := \|\gamma'\|_\infty$, on D_r, we use the norm $\|\Delta\|_{D_r} := \|\Delta\|_\infty + \|\Delta'\|_\infty$. The space E_r is endowed with the usual $\|\cdot\|_\infty$-norm.

Let $\Gamma := \mathrm{Germ}(K, X)_K := \bigcup_{r>0} E_r = \bigcup_{r>0} F_r$ denote the locally convex direct limit.

From now on, we fix two positive real numbers r and R and consider the following function

$$\begin{aligned}
g : [0,1] \times U_r \times \mathrm{B}^{D_r}_R(0) &\longrightarrow X \\
(t, x, \Delta) &\longmapsto \Delta(t)(x)
\end{aligned}$$

and the corresponding ordinary differential equation (with given parameter Δ_0 in the set $\mathrm{B}^{D_r}_R(0)$:

$$\frac{dx}{dt}(t) = g(t, x, \Delta_0).$$

Proposition 3.2.1. *(a) The function g is of class $FC^1_{\mathbb{R}}$.*

(b) The partial derivative

$$\begin{aligned} \tfrac{\partial}{\partial x} g : [0,1] \times U_r \times \mathrm{B}_R^{D_r}(0) &\longrightarrow \mathcal{L}_{\mathbb{R}}(X, Y) \\ (t, x, \Delta) &\longmapsto (\Delta(t))'(x) \end{aligned}$$

is continuous and takes values in the closed subspace $\mathcal{L}_{\mathbb{C}}(X, Y)$.

(c) The partial derivative

$$\begin{aligned} \tfrac{\partial}{\partial \Delta} g : [0,1] \times U_r \times \mathrm{B}_R^{D_r}(0) &\longrightarrow \mathcal{L}_{\mathbb{R}}(D_r, Y) \\ (t, x, \Delta) &\longmapsto g(t, x, \cdot) = (\Sigma \mapsto \Sigma(t)(x)) \end{aligned}$$

is continuous and takes values in the closed subspace $\mathcal{L}_{\mathbb{C}}(D_r, Y)$.

Proof. (a) The mappings

$$D_r \times [0,1] \longrightarrow F_r : (\Delta, t) \mapsto \Delta(t)$$

and

$$F_r \times U_r \longrightarrow X : (\gamma, x) \mapsto \gamma(x)$$

are FC^1 by Proposition 1.1.26. Since $D_r \times [0,1]$ is not an open subset of $D_r \times [0,1]$ we have to use the calculus on locally convex sets with dense interior (Definition 1.1.5).

(b) and (c): The existence and continuity of the partial derivatives follows from (a). It is easily verified that the partial derivatives have the given forms. The complex linearity of the operators is clear from the formulas. □

Proposition 3.2.2 (Local Existence)**.** *Let $(t_0, x_0, \Delta_0) \in [0,1] \times U_r \times \mathrm{B}_R^{D_r}(0)$. Then there exists an $\varepsilon > 0$ such that the initial value problem*

$$\begin{aligned} \frac{dx}{dt}(t) &= g(t, x, \Delta_0) \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution $x\colon]t_0 - \varepsilon, t_0 + \varepsilon[\cap [0,1] \longrightarrow U_r$.

Proof. By Proposition 3.2.1, the function g satisfies a Lipschitz-condition in its second component. Hence, the assertion follows from the classical Picard-Lindelöf-Theorem (see e.g. Theorem 1.1.22). □

Proposition 3.2.3 (Global Existence)**.** *Let $s < \frac{r}{2} e^{-R}$. Let $t_0 := 0, x_0 \in U_s$ and $\Delta_0 \in \mathrm{B}_R^{D_r}(0)$. Then the initial value problem*

$$\begin{aligned} \frac{dx}{dt}(t) &= g(t, x, \Delta_0) = \Delta_0(t)(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x\colon [0,1] \longrightarrow U_r$.

Proof. We know by Proposition 3.2.2 that there exists a unique local solution around each point. We will assume that $x\colon I \longrightarrow U_r$ is a solution where $I = [0,T] \subseteq [0,1]$ is an interval containing 0. The point $x_0 \in U_s = K + \mathrm{B}_s^X(0)$ can be written as $x_0 = a_0 + v_0$ with $a_0 \in K$ and $\|v_0\|_X < s$. We will now show that this solution never gets out of $\overline{\mathrm{B}_{\frac{r}{2}}^X(a_0)}$ and use that fact to conclude that it extends to a global solution:

For every $t \in [0,1]$ the vector field $\Delta_0(t)$ lies in $F_r = BC_{\mathbb{C}}^1(U_r, X)_K$ and therefore vanishes on K. Therefore $\Delta_0(t)(a_0) = 0$ for all $t \in [0,1]$. Hence, the following constant curve

$$y\colon I \longrightarrow U_r : t \mapsto a_0$$

is a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dt}(t) &= g(t, y, \Delta_0) = \Delta_0(t)(y) \\ y(0) &= a_0. \end{aligned}$$

Now, by Grönwall's inequality (Lemma 1.1.24(b)), the difference between two solutions of the same differential equation at time t can be bounded above by

$$\|x(t) - y(t)\|_X \leq \|x(0) - y(0)\|_X \cdot e^{t \cdot \|\Delta_0\|_\infty}.$$

Here, $\max_{t\in[0,1]} \|\Delta_0(t)\|_{\mathrm{D}} = \|\Delta_0\|_\infty \leq \|\Delta_0\|_{C^1} < R$ is a Lipschitz constant for the differential equation. Therefore, we get

$$\begin{aligned} \|x(t) - a_0\|_X = \|x(t) - y(t)\|_X &\leq \|x_0 - a_0\|_X \cdot e^{t \cdot \|\Delta_0\|_\infty} \\ &\leq \underbrace{\|v_0\|_X}_{<s} \cdot e^{1 \cdot R} < \frac{r}{2} \cdot e^{-R} \cdot e^{R} = \frac{r}{2}. \end{aligned}$$

This shows that the maximal solution $x\colon I \longrightarrow U_r$ takes values only in the closed ball $\overline{\mathrm{B}_{\frac{r}{2}}^X(a_0)}$. Hence, its distance to the boundary is always at least $\frac{r}{2}$. We may apply the quantitative version of Picard-Lindelöf (Theorem 1.1.22) with the parameters $\rho := \frac{r}{2}, \tau := 1, M = R$ and see that the solution can be enlarged around point T to a solution on $[0,1] \cap [0, T+\alpha]$ with $\alpha := \frac{\rho}{M}$. Since this value *alpha* does not depend on the number T, we can iterate this argument and obtain after a finite number of steps a global solution on $[0,1]$.

Since the local solutions are unique, the same holds for the global solution. □

Proposition 3.2.4 (Existence of a flow). *As in Proposition 3.2.3, let* $s < \frac{r}{2}e^{-R}$.

(a) There is a unique map

$$\Phi \colon [0,1] \times U_s \times \mathrm{B}_R^{D_r}(0) \longrightarrow U_r$$

with the property that for every fixed $x_0 \in U_s$ *and* $\Delta_0 \in \mathrm{B}_R^{D_r}(0)$*, the map* $\Phi(\cdot, x_0, \Delta_0)$ *is a solution to*

$$\begin{aligned} \frac{dx}{dt}(t) &= g(t, x, \Delta_0) \\ x(0) &= x_0. \end{aligned}$$

(b) The map $\Phi \colon [0,1] \times U_s \times \mathrm{B}_R^{D_r}(0) \longrightarrow U_r$ *is* $FC^1_{\mathbb{R}}$.

(c) For every fixed $t \in [0,1]$*, the map*

$$\Phi(t, \cdot) \colon U_s \times \mathrm{B}_R^{D_r}(0) \longrightarrow U_r$$

is $C^\omega_{\mathbb{C}}$.

(d) For fixed $(t, \Delta) \in [0,1] \times \mathrm{B}_R^{D_r}(0)$*, the map*

$$\begin{aligned} \Phi_{t,\Delta} : U_s &\longrightarrow U_r \\ x &\longmapsto \Phi(t, x, \Delta) \end{aligned}$$

is a $C^\omega_{\mathbb{C}}$*-diffeomorphism onto its open image, fixing* K *pointwise.*

(e) For fixed $t \in [0,1]$*, the map*

$$\begin{aligned} \mathrm{B}_R^{D_r}(0) &\longrightarrow BC(U_s, X) \\ \Delta &\longmapsto \Phi_{t,\Delta} \end{aligned}$$

is $C^\omega_{\mathbb{C}}$.

(f) The curve

$$\begin{aligned} \Sigma : [0,1] &\longrightarrow E_s \\ t &\longmapsto \Phi_{t,\Delta} - \mathrm{id}_{U_s} \end{aligned}$$

makes sense and is $C^1_{\mathbb{R}}$ *with derivative:*

$$\Sigma'(t) = \Delta(t) \circ \Phi_{t,\Delta}$$

Proof. (a) This is exactly what was shown in Proposition 3.2.3.

(b) Using Proposition 3.2.1, we see that our differential equation satisfies the hypotheses of Theorem 1.1.23. Therefore there is an $FC^1_{\mathbb{R}}$-flow around each point. By uniqueness of the flow (Proposition 3.2.2), we get the $FC^1_{\mathbb{R}}$-property of the map $\Phi \colon [0,1] \times U_s \times \mathrm{B}_R^{D_r}(0) \longrightarrow U_r$ constructed in (a).

(c) Let $(t_0, x_0, \Delta_0) \in [0,1] \times U_s \times \mathrm{B}_R^{D_r}(0)$. By Lemma 1.1.17, it suffices to show that the partial derivatives $\frac{\partial}{\partial x}\Phi(t_0, x_0, \Delta_0)$ and $\frac{\partial}{\partial \Delta}\Phi(t_0, x_0, \Delta_0)$ are complex linear.

In the special case $t = 0$, we get $\Phi(0, x_0, \Delta_0) = x_0$. And therefore $\frac{\partial}{\partial x}\Phi(0, x_0, \Delta_0) = \mathrm{id}_X$ and $\frac{\partial}{\partial \Delta}\Phi(0, x_0, \Delta_0) = 0$ are complex linear.

We define the following curve

$$\Omega\colon [0,1] \longrightarrow \mathcal{L}_{\mathbb{R}}(X, X) : t \mapsto \frac{\partial}{\partial x}\Phi(t, x_0, \Delta_0).$$

It suffices to show that it takes values in the closed subspace $\mathcal{L}_{\mathbb{C}}(X, X) \subseteq \mathcal{L}_{\mathbb{R}}(X, X)$.

By Theorem 1.1.23, the curve $\Omega\colon [0,1] \longrightarrow \mathcal{L}_{\mathbb{R}}(X, X)$ is differentiable and we have

$$\begin{aligned}
\Omega'(t) &= \frac{\partial}{\partial t}\frac{\partial}{\partial x}\Phi(t, x_0, \Delta_0) \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial t}\Phi(t, x_0, \Delta_0) \\
&= \frac{\partial}{\partial x}\left(g\big(t, \Phi(t, x_0, \Delta_0), \Delta_0\big)\right) \\
&= \frac{\partial g}{\partial x}\left(t, \Phi(t, x_0, \Delta_0), \Delta_0\right) \cdot \underbrace{\frac{\partial \Phi}{\partial x}(t, x_0, \Delta_0)}_{=\Omega(t)}.
\end{aligned}$$

This shows:

$$\Omega'(t) = \frac{\partial g}{\partial x}\left(t, \Phi(t, x_0, \Delta_0), \Delta_0\right) \cdot \Omega(t).$$

By Proposition 3.2.1, $\frac{\partial g}{\partial x}$ takes only values in $\mathcal{L}_{\mathbb{C}}(X, X)$. Hence, the linear differential equation

$$y' = \frac{\partial g}{\partial x}\left(t, \Phi(t, x_0, \Delta_0), \Delta_0\right) \cdot y.$$

admits a solution Ξ in the closed subspace $\mathcal{L}_{\mathbb{C}}(X, X)$. By uniqueness of solutions of the corresponding differential equation in $\mathcal{L}_{\mathbb{R}}(X, X)$, we get that $\Omega = \Xi$.

The same argument works for $\frac{\partial}{\partial \Delta}\Phi$.

(d) The complex analyticity of Φ_{t_0,Δ_0} follows from part (c). To see injectivity, assume $\Phi_{t_0,\Delta_0}(x_1) = \Phi_{t_0,\Delta_0}(x_2)$. Consider the following initial value problem

$$\begin{aligned}
&\frac{dx}{dt}(t) = g(t, x, \Delta_0) \\
&x(t_0) = \Phi_{t_0,\Delta_0}(x_1).
\end{aligned}$$

It has a unique solution, whence also the value at time 0 is unique. Therefore, $x_1 = x_2$.

Since the vector field $\Delta(t)$ is constantly zero on K, the flow is constant on K.

(e) By part (c) and Lemma 1.1.18, the map

$$\begin{aligned} \mathrm{B}_R^{D_r}(0) &\longrightarrow BC(U_s, X) \\ \Delta &\longmapsto \Phi_{t,\Delta} \end{aligned}$$

is analytic, if it is continuous. Therefore, it remains to show its continuity.

For $(x, \Delta) \in U_s \times \mathrm{B}_R^{D_r}(0)$, set

$$\begin{aligned} B_{x,\Delta}\colon [0,1] &\longrightarrow \mathcal{L}(X, X) : t \mapsto \frac{\partial g}{\partial x}(t, \Phi(t, u, \Delta), \Delta) \\ C_{x,\Delta}\colon [0,1] &\longrightarrow \mathcal{L}(D_r, X) : t \mapsto \frac{\partial g}{\partial \Delta}(t, \Phi(t, u, \Delta), \Delta). \end{aligned}$$

By Theorem 1.1.23, we know that for fixed $(x, \Delta) \in U_s \times \mathrm{B}_R^{D_r}(0)$ the curve

$$\begin{aligned} z_{x,\Delta} : [0,1] &\longrightarrow \mathcal{L}(D_r, X) \\ t &\longmapsto \tfrac{\partial \Phi}{\partial \Delta}(t, x, \Delta) \end{aligned}$$

satisfies the initial value problem

$$\begin{aligned} \frac{dz}{dt}(t) &= B_{x,\Delta}(t) \circ z(t) + C_{x,\Delta}(t), \\ z(0) &= 0. \end{aligned}$$

Therefore, we can estimate:

$$\begin{aligned} \|z_{x,\Delta}(t)\|_{\mathrm{op}} &= \left\| \underbrace{z(0)}_{=0} + \int_0^t \frac{d}{dt} z_{x,\Delta}(s) ds \right\|_{\mathrm{op}} \\ &= \left\| \int_0^t \Big(B_{x,\Delta}(s) \circ z_{x,\Delta}(s) + C_{x,\Delta}(s) \Big) ds \right\|_{\mathrm{op}} \\ &\leq \underbrace{\sup_{s\in[0,1]} \|B_{x,\Delta}(s)\|_{\mathrm{op}}}_{=b_{x,\Delta}} \cdot \int_0^t \|z_{x,\Delta}(s)\|_{\mathrm{op}}\, ds + \underbrace{\sup_{s\in[0,1]} \|C_{x,\Delta}(s)\|_{\mathrm{op}}}_{=c_{x,\Delta}(s)} \end{aligned}$$

So, we can apply Grönwall's inequality (Lemma 1.1.24(a)) and obtain:

$$\left\| \frac{\partial \Phi}{\partial \Delta}(t, x, \Delta) \right\|_{\mathrm{op}} = \|z_{x,\Delta}(t)\|_{\mathrm{op}} \leq c_{x,\Delta} \cdot e^{t \cdot b_{x,\Delta}}. \tag{$*$}$$

Now, we use Proposition 3.2.1 to find estimates for $b_{x,\Delta}$ and $c_{x,\Delta}$, which are independent of x and Δ:

$$\begin{aligned}
b_{x,\Delta} &= \sup_{s\in[0,1]} \|B_{x,\Delta}(s)\|_{\text{op}} \\
&= \sup_{s\in[0,1]} \left\| \frac{\partial g}{\partial x}(s, \Phi(s,u,\Delta), \Delta) \right\|_{\text{op}} \\
&= \sup_{s\in[0,1]} \left\| (\Delta(s))' \left(\Phi(s,u,\Delta)\right) \right\|_{\text{op}} \\
&\le \|\Delta\|_\infty \le \|\Delta\|_{D_r} < R. \\
c_{x,\Delta} &= \sup_{s\in[0,1]} \|C_{x,\Delta}(s)\|_{\text{op}} \\
&= \sup_{s\in[0,1]} \left\| \frac{\partial g}{\partial \Delta}(s, \Phi(s,u,\Delta), \Delta) \right\|_{\text{op}} \\
&= \sup_{s\in[0,1]} \|g(s, \Phi(s,u,\Delta), \cdot)\|_{\text{op}} \\
&\le 1.
\end{aligned}$$

Therefore, $(*)$ reduces to

$$\left\| \frac{\partial \Phi}{\partial \Delta}(t,x,\Delta) \right\|_{\text{op}} \le e^{Rt}. \tag{$**$}$$

Now, we can show Lipschitz-continuity of the map $\Delta \mapsto \Phi_{t,\Delta}$:

$$\begin{aligned}
\|\Phi_{t,\Delta} - \Phi_{t,\Sigma}\|_\infty &= \sup_{x\in U_s} \|\Phi(t,x,\Delta) - \Phi(t,x,\Sigma)\|_X \\
&= \sup_{x\in U_s} \left\| \int_0^1 \frac{\partial \Phi}{\partial \Delta}(t,x,\tau\Delta + (1-\tau)\Sigma).(\Delta - \Sigma)\; d\tau \right\|_X \\
&\le \sup_{x\in U_s} \int_0^1 \left\| \frac{\partial \Phi}{\partial \Delta}(t,x,\tau\Delta + (1-\tau)\Sigma) \right\|_{\text{op}} \|\Delta - \Sigma)\|_{D_r}\; d\tau \\
&\le e^{Rt} \cdot \|\Delta - \Sigma\|_{D_r}.
\end{aligned}$$

(f) The map $\Phi_{t,\Delta} - \mathrm{id}_{U_s} \colon U_s \longrightarrow X$ is complex analytic, bounded and constantly zero on K, therefore it is an element in the Banach space $E_s = \mathrm{Hol_b}\left(U_s, X\right)_K$ and the map

$\Sigma\colon [0,1] \longrightarrow E_s$ makes sense. Now

$$\begin{aligned}
&\left\| \frac{\Phi_{t+\tau,\Delta} - \Phi_{t,\Delta}}{\tau} - \Delta(t) \circ \Phi_{t,\Delta} \right\|_\infty \\
&= \sup_{x \in U_s} \left\| \frac{\Phi(t+\tau, x, \Delta) - \Phi(t, x, \Delta)}{\tau} - \Delta(t) \circ \Phi_{t,\Delta}(x) \right\|_X \\
&= \sup_{x \in U_s} \left\| \int_0^1 \frac{\partial \Phi}{\partial t}(t + u\tau, x, \Delta) du - \Delta(t) \circ \Phi_{t,\Delta}(x) \right\|_X \\
&= \sup_{x \in U_s} \left\| \int_0^1 \big(\Delta(t+u\tau) \circ \Phi(t+u\tau, x, \Delta) - \Delta(t) \circ \Phi_{t,\Delta}(x) \big) du \right\|_X \\
&\le \sup_{\substack{x \in U_s \\ u \in [0,1]}} \left\| \Delta(t+u\tau) \circ \Phi(t+u\tau, x, \Delta) - \Delta(t) \circ \Phi_{t,\Delta}(x) \right\|_X \\
&\le \sup_{\substack{x \in U_s \\ u \in [0,1]}} \left\| \Delta(t+u\tau)\big(\Phi(t+u\tau, x, \Delta)\big) - \Delta(t)\big(\Phi(t+u\tau, x, \Delta)\big) \right\|_X \\
&\quad + \sup_{\substack{x \in U_s \\ u \in [0,1]}} \left\| \Delta(t)\big(\Phi(t+u\tau, x, \Delta)\big) - \Delta(t)\big(\Phi(t, x, \Delta)\big) \right\|_X \\
&\le \sup_{u \in [0,1]} \left\| \Delta(t+u\tau) - \Delta(t) \right\|_\infty \\
&\quad + \left\| (\Delta(t))' \right\|_\infty \sup_{\substack{x \in U_s \\ u \in [0,1]}} \left\| \Phi(t+u\tau, x, \Delta) - \Phi(t, x, \Delta) \right\|_X \\
&\le \sup_{u \in [0,1]} \left\| \Delta(t+u\tau) - \Delta(t) \right\|_\infty + \left\| (\Delta(t))' \right\|_\infty \left\| \frac{\partial}{\partial t} \Phi \right\|_\infty |\tau| \\
&\longrightarrow 0
\end{aligned}$$

This shows that the curve $\Sigma\colon [0,1] \longrightarrow E_s$ is differentiable with derivative $\Sigma'(t) = \Delta(t) \circ \Phi_{t,\Delta}$.

The derivative is continuous by Lemma 1.1.25 . □

Now, we may prove the final theorem of this chapter:

Theorem 3.2.5 (Strong C^1-Regularity of DiffGerm(K, X))**.** *The Lie group $G :=$ DiffGerm(K, X), constructed in Theorem 3.1.1 is strongly C^1-regular, i.e. there is a map*

$$\mathrm{Evol}\colon C^1([0,1], \mathbf{L}(G)) \longrightarrow C^1([0,1], G)$$

such that

(a) *For every* $\Delta \in C^1([0,1], \mathbf{L}(G))$, *we have* $\delta^R(\mathrm{Evol}(\Delta)) = \Delta$.

(b) *The map*

$$\mathrm{evol}\colon C^1([0,1], \mathbf{L}(G)) \longrightarrow G : \Delta \mapsto \mathrm{Evol}(\Delta)(1)$$

is $C^\infty_{\mathbb{R}}$.

Proof. Let $R > 0$ be fixed.

By Proposition 3.2.4(d), we know that for fixed $r > 0$, $0 < s < \frac{r}{2}e^{-R}$ and $\Delta \in \mathrm{B}_R^{D_r}(0)$, the map

$$\begin{aligned} \Phi_{t,\Delta} : U_s &\longrightarrow U_r \\ x &\longmapsto \Phi(t, x, \Delta) \end{aligned}$$

can be considered as an element in the Lie group $\mathrm{DiffGerm}(K, X)$.

By Proposition 3.2.4(f), the curve

$$\begin{aligned} \Sigma : [0,1] &\longrightarrow E_s \\ t &\longmapsto \Phi_{t,\Delta} - \mathrm{id}_{U_s} \end{aligned}$$

is $C^1_{\mathbb{R}}$.

The inclusion map $E_s \longrightarrow \Gamma : \gamma \mapsto \gamma$ is continuous linear. The manifold structure on $\mathrm{DiffGerm}(K, X)$ was constructed such that addition of the identity becomes a diffeomorphism. This means that

$$\begin{aligned} \widetilde{\Sigma} : [0,1] &\longrightarrow \mathrm{DiffGerm}(K, X) \\ t &\longmapsto \Phi_{t,\Delta} \end{aligned}$$

is a $C^1_{\mathbb{R}}$-curve.

Since the global chart is just an affine map, the tangent space at each point on the Lie group can be identified with the vector space Γ.

By Proposition 3.2.4(f), we have

$$\Sigma'(t) = \Delta(t) \circ \Phi_{t,\Delta}$$

This allows us to calculate the right logarithmic derivative of $\widetilde{\Sigma}$:

$$\begin{aligned} \delta^R \widetilde{\Sigma}(t) &= \widetilde{\Sigma}'(t) \circ \left(\widetilde{\Sigma}(t)\right)^{-1} \\ &= \Delta(t) \circ \Phi_{t,\Delta} \circ (\Phi_{t,\Delta})^{-1} = \Delta(t). \end{aligned}$$

So far, we have seen that every curve in the set $\mathrm{B}_R^{D_r}(0)$ has a right evolution.

But since the direct limit $\Gamma = \bigcup_{r>0} F_r$ is compactly regular (Proposition 3.1.4), every C^1-curve $\Delta\colon [0,1] \longrightarrow \Gamma$ can be considered as a C^1-curve with values in F_r for an $r > 0$. By choosing $R > 0$ large enough, we see that every C^1-curve in the Lie algebra has a right evolution.

Now, we claim that for each $r, R > 0$ the map

$$\begin{aligned} \mathrm{evol}_r : \mathrm{B}_R^{D_r}(0) &\longrightarrow \mathrm{DiffGerm}(K, X) \\ \Delta &\longmapsto \mathrm{Evol}(\Delta)(1) = \Phi_{1,\Delta} \end{aligned}$$

is complex analytic. In order to show that, we use the global chart and that the inclusion map into the direct limit is continuous linear. Hence, it suffices to show that

$$\begin{aligned} \mathrm{B}_R^{D_r}(0) &\longrightarrow E_s \\ \Delta &\longmapsto \Phi_{1,\Delta} - \mathrm{id}_{U_s} \end{aligned}$$

is complex analytic, which follows from Proposition 3.2.4(e).

So, we are in the following situation: Each $\mathrm{evol}_r\colon \mathrm{B}_R^{D_r}(0) \longrightarrow \mathrm{DiffGerm}(K, X)$ is complex analytic and bounded (using the global chart). By construction, it is clear that for each $r_2 < r_1$, we have $\mathrm{B}_R^{D_{r_1}}(0) \subseteq \mathrm{B}_R^{D_{r_2}}(0)$ and

$$\mathrm{evol}_{r_2}|_{\mathrm{B}_R^{D_{r_1}}(0)} = \mathrm{evol}_{r_1}$$

By Theorem 2.1, the following map

$$\mathrm{evol} := \bigcup_{r>0} \mathrm{evol}_r\colon \bigcup_{r>0} \mathrm{B}_R^{D_r}(0) \longrightarrow \mathrm{DiffGerm}(K, X)$$

is complex analytic. It is defined on $\bigcup_{r>0} \mathrm{B}_R^{D_r}(0)$, which is an open subset of the locally convex vector space

$$\bigcup_{r>0} D_r = \bigcup_{r>0} C^1\left([0,1], F_r\right),$$

which can be identified with

$$C^1\left([0,1], \bigcup_{r>0} F_r\right)$$

using the Theorem of Mujica for C^k-curves (Lemma 1.2.7). The completeness assumptions of Lemma 1.2.7 are satisfied since every F_r is a Banach space and since $\bigcup_{r>0} F_r$ is complete by Proposition 3.1.4.

This proves that there is a smooth evolution map on a 0-neighborhood of $C^1\left([0,1], \bigcup_{r>0} F_r\right)$, which is by Proposition 1.3.10 sufficient to ensure strong C^1-regularity.

Hence, DiffGerm(K, X) is a strongly C^1-regular Lie group. $\square$

4 Ascending unions of Banach Lie groups

4.1 Construction of the Lie group structure

In the following let $G_1 \subseteq G_2 \subseteq \cdots$ be an increasing sequence of analytic Banach Lie groups, such that the inclusion maps $j_n\colon G_n \longrightarrow G_{n+1}$ are analytic. Our goal is to construct a Lie group structure on the union $G := \bigcup_{n=1}^{\infty} G_n$. But before we can define a manifold structure on G, first we have to construct the modelling locally convex vector space.

For every $n \in \mathbb{N}$ let $\mathfrak{g}_n := \mathbf{L}(G_n)$ be the corresponding Banach Lie algebra. Since every j_n is an injective morphism of Lie groups with exponential function, it is well known that the corresponding morphism of Lie algebras $i_n := \mathbf{L}(j_n)\colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1}$ is injective as well. Therefore we can identify $i_n(\mathfrak{g}_n)$ with $\mathfrak{g}_n$ and we may then assume that the Lie algebras form an increasing sequence. The union of this sequence will be denoted by $\mathfrak{g} := \bigcup_{n=1}^{\infty} \mathfrak{g}_n$. As a directed union of Lie algebras, this is clearly a Lie algebra. We endow it with the locally convex direct limit topology. Since we can only deal with Lie groups modeled on Hausdorff spaces, we have to make the assumption that this direct limit is Hausdorff.

By Corollary 2.2, the Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is continuous and therefore $(\mathfrak{g}, [\cdot,\cdot])$ becomes a locally convex Lie algebra. Note: This would already go wrong in general if we considered direct limits of non-normable Lie algebras $\mathfrak{g}_n$. There are examples where the $\mathfrak{g}_n$ are Fréchet Lie algebras and the resulting Lie bracket fails to be continuous.

Since every group G_n is a Banach Lie group it admits a smooth exponential function. By commutativity of the diagram

$$\begin{tikzcd}
G_n \arrow[r, "j_n"] & G_{n+1} \\
\mathfrak{g}_n \arrow[u, "\exp_{G_n}"'] \arrow[r, "i_n"] & \mathfrak{g}_{n+1} \arrow[u, "\exp_{G_{n+1}}"']
\end{tikzcd}$$

we know that every exponential function $\exp_{G_n}$ can be regarded as the restriction of the exponential function $\exp_{G_{n+1}}$ of the following group. This allows us to define

$$\begin{aligned}
\mathrm{Exp} : \mathfrak{g} &\longrightarrow G \\
x \in \mathfrak{g}_n &\longmapsto \exp_n(x) \in G_n
\end{aligned}$$

(Since we do not have a Lie group structure on G yet, it makes no sense to claim that Exp is the exponential function of G, but it will turn out to be the right exponential function.) So far we did not use the norms on the Banach Lie algebras $\mathfrak{g}_n$. In a Banach Lie algebra one usually expects the bilinear map $[\cdot,\cdot]_n\colon \mathfrak{g}_n \times \mathfrak{g}_n \longrightarrow \mathfrak{g}_n$ to have a norm less than or equal to 1, in which case we call $\|\cdot\|_n$ compatible. This can always be achieved by replacing the norm $\|\cdot\|_n$ by a scalar multiple. For what follows it will be necessary that all bonding maps $i_n\colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1}$ have norm ≤ 1. Unfortunately, in general one cannot have both. There are cases where it is not possible to find equivalent norms such that both, the bonding maps and the Lie brackets, have a norm at most 1. Now we are ready to formulate the main theorem of this section:

Theorem 4.1.1. *Let $G_1 \subseteq G_2 \subseteq \cdots$ be analytic Banach Lie groups over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, such that all inclusion maps $j_n\colon G_n \longrightarrow G_{n+1}$ are analytic group homomorphisms. Set $G := \bigcup_{n\in\mathbb{N}} G_n$. Assume that the following hold:*

(a) For each $n \in \mathbb{N}$ there is a norm $\|\cdot\|_n$ on $\mathfrak{g}_n := \mathbf{L}(G_n)$ defining its topology, such that $\|[x,y]\|_n \leq \|x\|_n \|y\|_n$ for all $x, y \in \mathfrak{g}_n$ and such that the bounded operator $\mathbf{L}(j_n)\colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1}$ has operator norm at most 1.

(b) The locally convex direct limit topology on $\mathfrak{g} := \bigcup_{n\in\mathbb{N}} \mathfrak{g}_n$ is Hausdorff.

(c) The map $\exp_G := \bigcup_{n\in\mathbb{N}} \exp_{G_n}\colon \mathfrak{g} \longrightarrow G$ is injective on some 0-neighborhood

Then there exists a unique $\mathbb{K}$-analytic Lie group structure on G which makes $\exp_G$ a local $C^\omega_{\mathbb{K}}$-diffeomorphism at 0.

This theorem and its proof can also be found in [5] (Theorem C).

Proof. Set $R := \frac{1}{3}\log\frac{3}{2}$ and $C := \log 2$. By Lemma 1.3.12, we know that in a Banach Lie algebra $\mathfrak{g}_n$ with compatible norm $\|\cdot\|_n$, the *BCH*-series converges for all $x, y \in \mathfrak{g}_n$ with $\|x\|_n + \|y\|_n < \log\frac{3}{2}$ and defines an analytic multiplication:

$$*_n\colon \mathrm{B}_R^{\mathfrak{g}_n\times\mathfrak{g}_n}(0) \longrightarrow \mathrm{B}_C^{\mathfrak{g}_n}(0).$$

We give the space $E_n := \mathfrak{g}_n \times \mathfrak{g}_n$ the norm $\|(x,y)\|_{E_n} := \max(\|x\|_n, \|y\|_n)$.

We will show now that this *BCH*-multiplication extends to the direct limit, using Theorem 2.1. Since Theorem 2.1 is only available in the complex case, we need a case distinction:

First, conder $\mathbb{K} = \mathbb{C}$.

The set $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{E_n}(0)$ is an open 0-neighborhood in the direct limit

$$E := \bigcup_{n\in\mathbb{N}} (\mathfrak{g}_n \times \mathfrak{g}_n) \cong \left(\bigcup_{n\in\mathbb{N}} \mathfrak{g}_n\right) \times \left(\bigcup_{n\in\mathbb{N}} \mathfrak{g}_n\right).$$

We are now ready to apply Theorem 2.1, since all hypotheses are satisfied and therefore the map $* = \bigcup_{n\in\mathbb{N}} *_n\colon U \longrightarrow \mathfrak{g}$ is complex analytic.

Now, consider the case $\mathbb{K} = \mathbb{R}$.

If $\mathbb{K} = \mathbb{R}$, we may consider the complexifications $(\mathfrak{g}_n)_{\mathbb{C}}$ of the real Banach spaces $(\mathfrak{g}_n, \|\cdot\|_{\mathfrak{g}_n})$, together with the complex norms, introduced in Proposition 1.1.12. The bonding maps $\mathbf{L}(j_n)\colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1}$ extend to unique continuous $\mathbb{C}$-linear mappings

$$\mathbf{L}(j_n)\colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1},$$

still having operator norm at most 1. The Lie bracket on each $\mathfrak{g}_n$ extends uniquely to a continuous Lie bracket on $(\mathfrak{g}_n)_{\mathbb{C}}$, turning it into a complex Lie algebra. However, it is not clear, if the new norms are compatible with the new Lie bracket. By Proposition 1.1.12, we only have that

$$\|[\tilde{x}_1, \tilde{x}_2]\|_{(\mathfrak{g}_n)_{\mathbb{C}}} \leq 4 \cdot \|\tilde{x}_1\|_{(\mathfrak{g}_n)_{\mathbb{C}}} \|\tilde{x}_2\|_{(\mathfrak{g}_n)_{\mathbb{C}}}.$$

Now, we replace each norm $\|\cdot\|_{(\mathfrak{g}_n)_{\mathbb{C}}}$ by $4\|\cdot\|_{(\mathfrak{g}_n)_{\mathbb{C}}}$. These new norms induce the same topology and are compatible with the Lie bracket. Since we took the same factor for each $(\mathfrak{g}_n)_{\mathbb{C}}$, the bonding maps still have operator norm at most 1.

Now, we proceed like in the complex case and obtain a complex analytic *BCH*-multiplication on a $(0,0)$-neighborhood of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ which then restricts to a $C^\omega_{\mathbb{R}}$-map

$$* = \bigcup_{n\in\mathbb{N}} *_n \colon U \longrightarrow \mathfrak{g},$$

where $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{E_n}_{\frac{R}{4}}(0)$.

Now, we established that the *BCH*-multiplication is $C^\omega_{\mathbb{K}}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and we can now construct the Lie group structure using Corollary 1.3.16:

By hypothesis (c), we know that the exponential function is injective on some neighborhood $V \subseteq \mathfrak{g}$. Since $*$ is continuous, there exists a smaller 0-neighborhood $U' \subseteq U$ such that $U' * U' \subseteq V$. Then, by Corollary 1.3.16 there exists an analytic Lie group structure on the group

$$\langle \exp(\mathfrak{g}) \rangle = \bigcup_{n\in\mathbb{N}} \langle \exp_{G_n}(\mathfrak{g}_n) \rangle = \bigcup_{n\in\mathbb{N}} (G_n)_0$$

which is the union of the identity components of the Banach Lie groups we started with.

We now can extend this manifold structure from $\bigcup_{n\in\mathbb{N}} (G_n)_0$ to the whole group G, using Proposition 1.3.15. In fact, being a subgroup, $\bigcup_{n\in\mathbb{N}} (G_n)_0$ is symmetric and contains 1. As $\bigcup_{n\in\mathbb{N}} (G_n)_0$ already is a Lie group, multiplication and inversion are $C^\omega_{\mathbb{K}}$ as required. It only remains to show that conjugation with elements $g \in G$ is $C^\omega_{\mathbb{K}}$.

Let $g \in G$ be such an element. Then there is an $m \in \mathbb{N}$ such that $g \in G_m$. We have to show that $c_g \colon \bigcup_{n\in\mathbb{N}} (G_n)_0 \longrightarrow \bigcup_{n\in\mathbb{N}} (G_n)_0$ is analytic.

Since $\mathrm{Ad}^G_g := \bigcup_{n\geq m} \mathrm{Ad}^{G_n}_g \colon \bigcup_{n\geq m} \mathfrak{g}_n \longrightarrow \bigcup_{n\geq m} \mathfrak{g}_n$ is continuous by the locally convex direct limit property, exp is a local diffeomorphism at 0 and $c_g \circ \exp = \exp \circ \mathrm{Ad}^G_g$, it follows that c_g is analytic on some identity neighborhood. This is sufficient for a group homomorphism to be analytic everywhere.

This turns G into a $C^\omega_{\mathbb{K}}$-Lie group in which $\bigcup_{n\in\mathbb{N}}(G_n)_0$ is an open connected subgroup, hence the identity component.

The uniqueness of the manifold structure is clear since exp is a local diffeomorphism. $\square$

4.2 Regularity of local Banach Lie groups

Lemma 4.2.1. *Let $\sum_{n=1}^\infty \eta_n X^n$ denote the power series expansion of $-\log(2-\exp(2X))$ around zero. Then the power series*

$$\sum_{n\in\mathbb{N}} n(2e)^n \eta_n X^n \text{ and } \sum_{n\in\mathbb{N}} n(n-1)(2e)^{2n-1}\cdot \eta_n X^n$$

have positive radius of convergence.

Let $\mathfrak{g}$ be a Banach Lie algebra over $\mathbb{K}$ with compatible norm and BCH*-series*

$$\mu(x,y) = x*y = \sum_{n\in\mathbb{N}} p_n(x,y)$$

with continuous homogeneous polynomials $p_n\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ and Fréchet derivatives $p_n'\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$ and $p_n''\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathfrak{g}))\cong \mathrm{Lin}^2_c(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$. On the space $\mathfrak{g}\times\mathfrak{g}$, we use maximum norm. Then we have the following estimates:

(a) $\|p_n\|_{\mathrm{op}}\le \eta_n$.

(b) $\|p_n'\|_{\mathrm{op}}\le n(2e)^n\cdot\eta_n$.

(c) $\|p_n''\|_{\mathrm{op}}\le n(n-1)(2e)^{2n-1}\cdot\eta_n$.

Proof. The convergence of the two series is obvious. The proof of part (a) can be found in [3] (Lemma 1, Ch. II, §7). Part (b) and (c) follow from (a) using Corollary 1.1.9 $\square$

Lemma 4.2.2. *We fix a real number $s_0\in]0,\frac{1}{3}\log\frac{3}{2}[$ such that*

$$\sum_{n\ge 2} n(2e)^n\eta_n\cdot(4s_0)^{n-1}\le\frac{1}{4}$$

and that

$$\sum_{n\ge 2} n(n-1)(2e)^{2n-1}\eta_n\cdot(4s_0)^{n-2}\le\frac{1}{8},$$

using the converging power series introduced in Lemma 4.2.1.

Let $\mathfrak{g}$ be a Banach Lie algebra over $\mathbb{K}$ with compatible norm, addition map $\alpha_{\mathfrak{g}}\colon\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}:(x,y)\mapsto x+y$ and BCH*-multiplication $\mu(x,y)=x*y$. Then for all $a,b\in \mathrm{B}^{\mathfrak{g}}_{4s_0}(0)$, we have the estimates:*

(a) $\|\mu'(a,b) - \alpha_{\mathfrak{g}}\|_{\text{op}} \leq \frac{1}{2}$

(b) $\|\mu''(a,b)\|_{\text{op}} \leq \frac{1}{8}$.

Proof. First of all, a number s_0 with the desired properties exists since the power series introduced in Lemma 4.2.1 have a positive radius of convergence.

(a) We write the BCH-multiplication as in Lemma 4.2.1:

$$\mu(a,b) = a * b = \sum_{n \in \mathbb{N}} p_n(a,b)$$

It is known that $p_1 = \alpha_{\mathfrak{g}}$. Now, we take the Fréchet derivative on both sides:

$$\mu'(a,b) = \sum_{n \in \mathbb{N}} p_n'(a,b).$$

Since p_1 is linear, we have $p_1'(a,b) = p_1 = \alpha_{\mathfrak{g}}$. Hence, we can estimate:

$$\begin{aligned}
\|\mu'(a,b) - \alpha_{\mathfrak{g}}\|_{\text{op}} &= \left\| \sum_{n \in \mathbb{N}} p_n'(a,b) - p_1'(a,b) \right\|_{\text{op}} \\
&= \left\| \sum_{n \geq 2} p_n'(a,b) \right\|_{\text{op}} \\
&\leq \sum_{n \geq 2} \|p_n'(a,b)\|_{\text{op}} \\
&\leq \sum_{n \geq 2} \|p_n'\|_{\text{op}} \|(a,b)\|^{n-1} \\
&\leq \sum_{n \geq 2} n(2e)^n \eta_n (4s_0)^{n-1} \\
&\leq \frac{1}{4}.
\end{aligned}$$

(b) Taking once again the Fréchet derivative of μ', we obtain:

$$\mu''(a,b) = \sum_{n \in \mathbb{N}} p_n''(a,b)$$

with $p_1'' = 0$. Now, we estimate:

$$\begin{aligned}\left\|\mu''(a,b)\right\|_{\mathrm{op}} &= \left\|\sum_{n\geq 2} p_n''(a,b)\right\|_{\mathrm{op}} \\ &\leq \sum_{n\geq 2} \left\|p_n''(a,b)\right\|_{\mathrm{op}} \\ &\leq \sum_{n\geq 2} \left\|p_n''\right\|_{\mathrm{op}} \|(a,b)\|^{n-2} \\ &\leq \sum_{n\geq 2} \left(n(2e)^n\right)^2 \eta_n (4s_0)^{n-2} \\ &\leq \frac{1}{8}.\end{aligned}$$

□

Theorem 4.2.3 (Quantitative Strong C^0-Regularity of local Banach Lie groups)**.** *Let $\left(\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0), D, *, 0_{\mathfrak{g}}, -\mathrm{id}\right)$ be the local Banach Lie group, corresponding to a Banach Lie algebra $(\mathfrak{g}, \|\cdot\|)$ as constructed in Proposition 1.3.13 and let $s_0 > 0$ be the number taken from Lemma 4.2.2 and consider the set $V := \left\{\gamma \in C^1_*([0,1],\mathfrak{g}) : \|\gamma'\|_\infty < 4s_0\right\}$ which is open in the Banach space*

$$C^1_*([0,1],\mathfrak{g}) := \left\{\gamma \in C^1([0,1],\mathfrak{g}) : \gamma(0) = 0\right\}.$$

Then the left logarithmic derivative

$$\begin{aligned}\delta|_V : V &\longrightarrow C([0,1],\mathfrak{g}) \\ \gamma &\longmapsto \delta\gamma\end{aligned}$$

is a diffeomorphism onto its open image $\delta(V)$ which contains $\mathrm{B}^{C([0,1],\mathfrak{g})}_{s_0}(0)$. In particular, the local Lie group $\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0)$ is strongly C^0-regular.

Proof. We set $R := \frac{1}{3}\log\frac{3}{2}$ and $C := \log 2$ and let

$$\mu\colon \mathrm{B}^{\mathfrak{g}}_R(0) \times \mathrm{B}^{\mathfrak{g}}_R(0) \longrightarrow \mathrm{B}^{\mathfrak{g}}_C(0) : (x,y) \mapsto x * y.$$

By Lemma 1.3.14, we know that the left logarithmic derivative of a C^1-curve in $\mathrm{B}^{\mathfrak{g}}_R(0)$ can be written as:

$$\delta\gamma(t) = d\lambda_{-\gamma(t)}(\gamma(t),\gamma'(t)) = d\mu\Big((-\gamma(t),\gamma(t)),(0,\gamma'(t))\Big)$$

First, we will now show that $\delta\colon C^1_*\left([0,1],\mathrm{B}^{\mathfrak{g}}_R(0)\right) \longrightarrow C([0,1],\mathfrak{g})$ is a smooth map.

Since the BCH-multiplication is smooth, the map

$$d\mu\colon (\mathrm{B}^{\mathfrak{g}}_R(0) \times \mathrm{B}^{\mathfrak{g}}_R(0)) \times (\mathfrak{g}\times\mathfrak{g}) \longrightarrow \mathfrak{g}$$

is also smooth. Now, the map $\delta \colon C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_R(0)\right) \longrightarrow C([0,1],\mathfrak{g})$ can be written as $\delta = g \circ \Phi$, with

$$\begin{aligned} \Phi : C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_R(0)\right) &\longrightarrow C\left([0,1], (\mathrm{B}^{\mathfrak{g}}_R(0) \times \mathrm{B}^{\mathfrak{g}}_R(0)) \times (\mathfrak{g} \times \mathfrak{g})\right) \\ \gamma &\longmapsto \Big((-\gamma,\gamma),(0,\gamma')\Big) \end{aligned}$$

and

$$\begin{aligned} g : C\left([0,1], (\mathrm{B}^{\mathfrak{g}}_R(0) \times \mathrm{B}^{\mathfrak{g}}_R(0)) \times (\mathfrak{g} \times \mathfrak{g})\right) &\longrightarrow C([0,1],\mathfrak{g}) \\ \eta &\longmapsto d\mu \circ \eta. \end{aligned}$$

The map Φ is just a restriction of a bounded linear operator, hence smooth. The map g is a composition map and it is smooth by Lemma 1.1.25.

Hence, the left logarithmic derivative $\delta \colon C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_R(0)\right) \longrightarrow C([0,1],\mathfrak{g})$ is a smooth map.

From now on, we fix the following norm on the space $C^1_*([0,1],\mathfrak{g})$:

$$\|\gamma\|_{\mathrm{D}} := \|\gamma'\|_\infty .$$

It generates the usual topology on $C^1_*([0,1],\mathfrak{g})$, because of the estimate:

$$\|\gamma\|_\infty = \sup_{t\in[0,1]} \left\| \int_0^t \gamma'(s)ds \right\| \leq \|\gamma\|_{\mathrm{D}} . \tag{$*$}$$

However, it has the advantage, that the following operator

$$\begin{aligned} T : C^1_*([0,1],\mathfrak{g}) &\longrightarrow C([0,1],\mathfrak{g}) \\ \gamma &\longmapsto \gamma' \end{aligned}$$

becomes an isometric isomorphism.

Now, we define the function

$$\begin{aligned} f : V = \mathrm{B}^{C^1_*([0,1],\mathfrak{g})}_{4s_0}(0) &\longrightarrow C([0,1],\mathfrak{g}) \\ \gamma &\longmapsto \delta\gamma - \gamma'. \end{aligned}$$

Our next goal is to show that f is Lipschitz-continuous by estimating the norm of the Fréchet derivative:

$$f' \colon V \longrightarrow \mathcal{L}\left(C^1_*([0,1],\mathfrak{g}), C([0,1],\mathfrak{g})\right).$$

Let $\gamma \in \mathrm{B}_{4s_0}^{C_*^1([0,1],\mathfrak{g})}(0)$, $\eta \in C_*^1([0,1],\mathfrak{g})$ with $\|\eta\|_{\mathrm{D}} = 1$ and $t \in [0,1]$ be given. By $(*)$, this implies that $\|\gamma\|_\infty < 4s_0$. Now

$$\begin{aligned}
\left\| \left(f'(\gamma).\eta\right)(t) \right\|_{\mathfrak{g}} &= \left\| \left(d\delta(\gamma,\eta) - \eta'\right)(t) \right\|_{\mathfrak{g}} \\
&= \left\| (d\mu)'\left(\Phi(\gamma)(t)\right) \cdot \Phi(\eta)(t) - \eta'(t) \right\|_{\mathfrak{g}} \\
&= \left\| d(d\mu)\left(\Phi(\gamma)(t), \Phi(\eta)(t)\right) - \eta'(t) \right\|_{\mathfrak{g}} \\
&= \left\| d(d\mu)\Big(\left(-\gamma(t), \gamma(t), 0, \gamma'(t)\right), \left(-\eta(t), \eta(t), 0, \eta'(t)\right) \Big) - \eta'(t) \right\|_{\mathfrak{g}} \\
&\leq \left\| d^{(2)}\mu\Big(\left(-\gamma(t), \gamma(t)\right), \left(0, \gamma'(t)\right), \left(-\eta(t), \eta(t)\right) \Big) \right\|_{\mathfrak{g}} \\
&\quad + \left\| d\mu\Big(\left(-\gamma(t), \gamma(t)\right), \left(0, \eta'(t)\right) \Big) - \eta'(t) \right\|_{\mathfrak{g}} \\
&= \left\| \mu''\Big(-\gamma(t), \gamma(t)\Big)\Big(\left(0, \gamma'(t)\right), \left(-\eta(t), \eta(t)\right) \Big) \right\|_{\mathfrak{g}} \\
&\quad + \left\| \mu'\left(-\gamma(t), \gamma(t)\right)\left(0, \eta'(t)\right) - \alpha_{\mathfrak{g}}(0, \eta'(t)) \right\|_{\mathfrak{g}} \\
&\leq \left\| \mu''(-\gamma(t), \gamma(t)) \right\|_{\mathrm{op}} \left\| \left(0, \gamma'(t)\right) \right\| \left\| (-\eta(t), \eta(t)) \right\| \\
&\quad + \left\| \mu'\left(-\gamma(t), \gamma(t)\right) - \alpha_{\mathfrak{g}} \right\|_{\mathrm{op}} \left\| \left(0, \eta'(t)\right) \right\| \\
&\leq \left\| \mu'' \right\|_\infty \|\gamma\|_{\mathrm{D}} \|\eta\|_\infty + \left\| \mu'(-\gamma(t), \gamma(t)) - \alpha_{\mathfrak{g}} \right\|_{\mathrm{op}} \|\eta\|_{\mathrm{D}} \\
&\leq \frac{1}{8} \cdot 4s_0 \cdot 1 + \frac{1}{2} \cdot 1 < \frac{3}{4}.
\end{aligned}$$

This shows that $f' \colon V \longrightarrow \mathcal{L}\left(C_*^1([0,1],\mathfrak{g}), C([0,1],\mathfrak{g})\right)$ is globally bounded by $\frac{3}{4}$ and hence f is $\frac{3}{4}$-Lipschitz.

By the Lipschitz inverse function theorem (Theorem 1.1.21), the map $\delta = T + f$ is a homeomorphism of $\mathrm{B}_{4s_0}^{C_*^1([0,1],\mathfrak{g})}(0)$ onto an open subset of $C([0,1],\mathfrak{g})$, containing the ball $\mathrm{B}_{r'}^{C([0,1],\mathfrak{g})}(0)$ with $r' = 4s_0\left(1 - \frac{3}{4}\right) = s_0$.

For every fixed $\gamma \in \mathrm{B}_{4s_0}^{C_*^1([0,1],\mathfrak{g})}(0)$, we have $\|\delta'(\gamma) - T\|_{\mathrm{op}} = \|f'(\gamma)\|_{\mathrm{op}} \leq \frac{3}{4}$. Therefore the bounded operator $\delta'(\gamma)$ lies in the open ball with radius $\frac{3}{4} < 1$ around an isometric isomorphism and hence, $\delta'(a)$ is invertible. Using the ordinary inverse function theorem for smooth mappings between Banach spaces, we get that δ is a diffeomorphism between V and $\delta(V) \supseteq \mathrm{B}_{s_0}^{C([0,1],\mathfrak{g})}(0)$. This finishes the proof. $\square$

4.3 Regularity of (local and global) (LB)-Lie groups

Theorem 4.3.1 (Regularity of local (LB)-Lie groups). *Let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots$ be an ascending sequence of Banach Lie algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with a Hausdorff locally convex direct limit $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$. Assume furthermore, that all inclusion maps and all Lie brackets have operator norm at most 1. As in Proposition 1.3.13, we have for each $n \in \mathbb{N}$ a local Lie group $\left(\mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0), D_n, *|_{D_n}, 0_{\mathfrak{g}_n}, -\mathrm{id}\right)$, where*

$$D_n := \left\{(x,y) \in \mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0) \times \mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0) : x * y \in \mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0)\right\}.$$

We set $V := \bigcup_n \mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0)$ and $D := \bigcup_n D_n \subseteq V \times V$.

(a) The space $\mathfrak{g}$ is again a topological Lie algebra and $(V, D, \mu, 0, \eta_V)$ becomes a local $C^\omega_{\mathbb{K}}$-Lie group, where μ is the BCH*-multiplication and $\eta_V = -\mathrm{id}_V$.*

(b) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is boundedly regular, then $(V, D, \mu, 0, \eta_V)$ is strongly C^1-regular.

(c) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is compactly regular, then $(V, D, \mu, 0, \eta_V)$ is even strongly C^0-regular.

(d) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is not boundedly regular, then $(V, D, \mu, 0, \eta_V)$ is not even C^∞-regular.

Proof. During this proof, we fix the constants $R := \log\frac{3}{2}$ and $C := \log 2$.

(a) We begin with the case that $\mathbb{K} = \mathbb{C}$. We endow the space $E_n := \mathfrak{g}_n \times \mathfrak{g}_n$ with the norm $\|(x,y)\|_{E_n} := \max\{\|x\|_n, \|y\|_n\}$. Then we have

$$\mathrm{B}^{E_n}_R(0) = \mathrm{B}^{\mathfrak{g}_n}_R(0) \times \mathrm{B}^{\mathfrak{g}_n}_R(0)$$

and the BCH-multiplication maps

$$\tilde{\mu}_n \colon \mathrm{B}^{E_n}_R(0) \longrightarrow \mathrm{B}^{\mathfrak{g}_n}_C(0) : (x,y) \mapsto x * y$$

are complex analytic and bounded (Lemma 1.3.12). It is possible to define the map

$$\tilde{\mu} \colon D := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{E_n}_R(0) \longrightarrow V := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{\mathfrak{g}_n}_C(0) : (x,y) \mapsto x * y,$$

which is complex analytic by Theorem 2.1. Now, we restrict this map to set D:

$$\mu := \tilde{\mu}|_D \colon D \longrightarrow V.$$

One verifies easily that $(V, D, \mu, 0, \eta_V)$ becomes a local Lie group with this multiplication.

The case $\mathbb{K} = \mathbb{R}$ can be reduced to the complex case using the complexifications, $(\mathfrak{g}_n)_C$, together with the norms introduced in Proposition 1.1.12 and proceed like in the proof of Theorem 4.1.1.

(b) We start with the continuous linear map

$$\begin{aligned} \psi : C^1([0,1], \textstyle\bigcup_n \mathfrak{g}_n) &\longrightarrow C([0,1], \textstyle\bigcup_n \mathfrak{g}_n) \\ \gamma &\longmapsto \gamma. \end{aligned}$$

Since the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is boundedly regular, we can apply Proposition 1.2.8 to conclude that the image of ψ is contained in the locally convex direct limit

$$\bigcup_n C([0,1], \mathfrak{g}_n)$$

which is a topological subspace of $C([0,1], \bigcup_n \mathfrak{g}_n)$ by Mujica's Theorem (Theorem 1.2.5).

This yields a continuous linear map

$$\begin{aligned} \widetilde{\psi} : C^1([0,1], \textstyle\bigcup_n \mathfrak{g}_n) &\longrightarrow \textstyle\bigcup_n C([0,1], \mathfrak{g}_n) \\ \gamma &\longmapsto \gamma. \end{aligned}$$

By Theorem 4.2.3, every local group $\left(\mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0), D_n, *|_{D_n}, 0_{\mathfrak{g}_n}, -\mathrm{id}\right)$ is strongly C^0-regular with the smooth evolution map

$$\begin{aligned} \mathrm{Evol}_n : \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0) &\longrightarrow C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) \\ \delta\gamma &\longmapsto \gamma. \end{aligned}$$

The constant $s_0 > 0$ is chosen as in Lemma 4.2.2 and is, in particular, independent of n. If $\mathbb{K} = \mathbb{C}$, this allows us to use Theorem 2.1 again to get the complex analyticity of the map

$$\begin{aligned} \mathrm{Evol} : \textstyle\bigcup_n \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0) &\longrightarrow \textstyle\bigcup_n C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) \\ \delta\gamma &\longmapsto \gamma. \end{aligned}$$

If $\mathbb{K} = \mathbb{R}$, we complexify the Banach Lie algebras $\mathfrak{g}_n$ with the norm introduced in Proposition 1.1.12 and rescale it by 4 to obtain compatibility with the Lie brackets. Then, use Theorem 2.1 to this complex setting and then restrict it to get the real analyticity of $\mathrm{Evol} := \bigcup_n \mathrm{Evol}_n$.

By construction, it is clear that $\widetilde{\psi}$ maps the open neighborhood $\bigcup_n \mathrm{B}^{C^1([0,1],\mathfrak{g}_n)}_{s_0}(0)$ in the neighborhood $\bigcup_n \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0)$. Hence, the composition

$$\begin{aligned} \mathrm{Evol}\circ\widetilde{\psi} : \textstyle\bigcup_n \mathrm{B}^{C^1([0,1],\mathfrak{g}_n)}_{s_0}(0) &\longrightarrow \textstyle\bigcup_n C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) \\ \delta\gamma &\longmapsto \gamma \end{aligned}$$

is complex analytic. Thus, we have shown that each C^1-curve in the 0-neighborhood $\bigcup_n \mathrm{B}_{s_0}^{C^1([0,1],\mathfrak{g}_n)}(0)$ has a left evolution and that the evolution map is complex analytic. Hence, the local Lie group is strongly C^1-regular.

(c) If we assume that the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is even compactly regular, then the Mujica Theorem (Theorem 1.2.5) tells us that

$$\begin{aligned} \phi : C([0,1], \textstyle\bigcup_n \mathfrak{g}_n) &\longrightarrow \textstyle\bigcup_n C([0,1],\mathfrak{g}_n) \\ \gamma &\longmapsto \gamma \end{aligned}$$

is a topological isomorphism and hence we get that

$$\begin{aligned} \mathrm{Evol}\circ\phi : \textstyle\bigcup_n \mathrm{B}_{s_0}^{C([0,1],\mathfrak{g}_n)}(0) &\longrightarrow \textstyle\bigcup_n \mathrm{B}_C^{\mathfrak{g}_n}(0) \\ \delta\gamma &\longmapsto \gamma \end{aligned}$$

is analytic. Hence, the local Lie group is strongly C^0-regular.

(d) By [15], every C^∞-regular local Lie group has a Mackey complete Lie algebra. But by [6] (1.4.(f)), a countable direct limit of Banach spaces is Mackey complete if and only if the sequence of Banach spaces is boundedly regular. The claim follows. □

Let us return to the situation described at the beginning of this chapter:

Theorem 4.3.2 (Regularity of countable unions of Banach Lie groups)**.** *Let $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. We are given an increasing sequence*

$$G_1 \subseteq G_2 \subseteq \cdots$$

of $\mathbb{K}$-analytic Banach Lie groups, such that the inclusion maps $j_n\colon G_n \longrightarrow G_{n+1}$ are analytic group homomorphisms. We fix a norm $\|\cdot\|_n$ on the Banach Lie algebra $\mathfrak{g}_n := \mathbf{L}(G_n)$, defining its topology. We assume, that the locally convex direct limit $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ is Hausdorff and that all inclusion maps and all Lie brackets have operator norm at most 1. Also, we assume that the map

$$\exp_G := \bigcup_n \exp_{G_n} : \mathfrak{g} \longrightarrow \bigcup_n G_n$$

is injective on some 0-neighborhood in $\mathfrak{g}$. Then

- *(a) There exists a unique locally convex Lie group structure on the group $G := \bigcup_n G_n$, such that $\exp_G$ becomes a local diffeomorphism.*
- *(b) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is boundedly regular, then G is strongly C^1-regular.*
- *(c) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is compactly regular, then G is even strongly C^0-regular.*
- *(d) If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is not boundedly regular, then G is not even C^∞-regular.*

Proof. Part (a) is exactly what was shown in Theorem 4.1.1.

The Lie group G is locally exponential which means that there is a 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp_G|_U$ is a diffeomorphism onto the open identity-neighborhood $V := \exp_G(U)$. We may assume that V is symmetric. We set

$$D_V := \{(x,y) \in V \times V : xy \in V\}$$

Then $(V, D_V, \mu, 1_G, \eta_V)$ becomes a local Lie group. After making the 0-neighborhood U smaller, we may assume that the exponential map becomes a local isomorphism between the local Lie groups $W \subseteq G$ and the local Lie group $\left(\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0), D, *, 0_{\mathfrak{g}}, -\mathrm{id}\right)$. By Proposition 1.3.10 strong C^k-regularity of the Lie group G is equivalent to the regularity of the local group $(V, D, \mu, 1_G, \eta_V)$. Therefore, (b), (c) and (d) follow immediately from Theorem 4.3.1. $\square$

5 Examples of ascending unions of Banach Lie groups

5.1 Groups of germs of Lie group-valued mappings

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a Banach space over $\mathbb{K}$. Let $K \subseteq X$ be a non-empty compact subset, let H be a fixed $\mathbb{K}$-Banach Lie group.

Consider the set:

$$\operatorname{Germ}(K,H) := \left\{ \eta \colon U_\eta \longrightarrow H : \begin{array}{c} U_\eta \text{ is a neighborhood of } K \text{ in } X \\ \text{and } \eta \text{ is a } C^\omega_{\mathbb{K}} \text{ map} \end{array} \right\} / \sim,$$

where two maps η_1, η_2 are considered equivalent, $\eta_1 \sim \eta_2$, if they coincide on a common neighborhood of K. This becomes a group with respect to pointwise multiplication of maps.

As in Chapter 3, we fix the following basis of K-neighborhoods: $U_n := K + \mathrm{B}^X_{\frac{1}{n}}(0)$. Consider the following Banach spaces:

$$E_n := \left(\operatorname{Hol}_{\mathrm{b}}(U_n, \mathfrak{h}), \|\cdot\|_\infty\right).$$

Helge Glöckner showed in [9, Section 10] that the group $\operatorname{Germ}(K,H)$ carries the structure of a locally convex Lie group with Lie algebra

$$\operatorname{Germ}(K,\mathfrak{h}) := \bigcup_{n\in\mathbb{N}} E_n,$$

with the following continuous linear mappings as bonding maps:

$$E_n \longrightarrow E_{n+1} : \gamma \mapsto \gamma|_{U_n}.$$

He used different methods for constructing the Lie group structure, which did not depend on Theorem 2.1. However, with the tools developed in this thesis, we can prove the regularity of the above group — a novel result.

Theorem 5.1.1 (Regularity of $\operatorname{Germ}(K,H)$)**.** *The Lie group* $\operatorname{Germ}(K,H)$ *is strongly* C^0*-regular.*

Proof. By Theorem 4.3.2, it suffices to show that the direct limit is compactly regular. But this is the case by Proposition 3.1.4 (and the accompanying remark). ☐

5.2 Lie groups associated to Dirichlet series

In this section we construct more “exotic” examples of Lie groups modelled on Banach and (LB)-spaces. In fact, the discussion of these examples originally led to the discovery of Theorem 2.1. All vector spaces and Lie groups will be over the field $\mathbb{C}$.

5.2.1 Banach spaces of Dirichlet series

Definition 5.2.1.

1. A formal *Dirichlet series* with values in a complex Banach space X is a formal series of the form
$$\sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}},$$
where all a_n are elements in X. Like formal power series, two Dirichlet series are considered equal if and only if all coefficients are equal.
2. For every $s \in \mathbb{R}$, let $\mathbb{H}_s := \{z \in \mathbb{C} : \mathrm{Re}(z) > s\}$ denote the corresponding open half plane and $\overline{\mathbb{H}}_s := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq s\}$ the closed half plane in $\mathbb{C}$.
3. A Dirichlet series is said to *converge absolutely on* $\overline{\mathbb{H}}_s$ if
$$\left\| \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} \right\|_{(s)} := \sum_{n=1}^{\infty} \|a_n\| \, n^{-s} < \infty.$$
The space of all X-valued Dirichlet series that converge absolutely on $\overline{\mathbb{H}}_s$ will be denoted by $D_s(X)$. Together with the norm just defined this vector space becomes a Banach space isomorphic to $\ell^1(\mathbb{N}, X)$ via the isomorphism
$$D_s(X) \longrightarrow \ell^1(\mathbb{N}, X) : \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} \mapsto \left(a_n \cdot n^{-s}\right)_{n\in\mathbb{N}}.$$
4. The Banach space X can be embedded isometrically into $D_s(X)$ via
$$X \longrightarrow D_s(X) : a \mapsto a \cdot 1^{-\mathbf{z}} = \sum_{n\in\mathbb{N}} \delta_{n,1} a \cdot n^{-\mathbf{z}}.$$
All Dirichlet series obtained in this fashion are called *constant*.

Every Dirichlet series in $D_s(X)$ can be viewed as a continuous bounded function from the closed right half plane $\overline{\mathbb{H}}_s$ to X. In fact, this interpretation defines a bounded operator between Banach spaces of norm at most 1:

$$\begin{array}{rccl} \Upsilon_s : & D_s(X) & \longrightarrow & \left(BC(\overline{\mathbb{H}}_s, X), \|\cdot\|_\infty\right) \\ & \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} & \longmapsto & \left(z \mapsto \sum_{n=1}^{\infty} a_n n^{-z}\right) \end{array} \qquad (*)$$

All functions obtained in this fashion are complex analytic on the open half plane $\mathbb{H}_s$. Constant Dirichlet series as defined above are mapped to constant functions. The operator Υ_s is injective which means that it is possible to reconstruct the coefficients $(a_n)_{n\in\mathbb{N}}$ from the function. For example, the first coefficient is $a_1 = \lim_{\mathrm{Re}(z)\to+\infty} \gamma(z)$. Similarly, the other coefficients may be calculated. This means that a continuous function can have at most one Dirichlet series representation.

But there are lots of functions which cannot be written as a Dirichlet series, although they are continuous, bounded on $\overline{\mathbb{H}}_s$ and complex analytic on $\mathbb{H}_s$, e.g. $f(z) = e^{-z} \cdot a$ for an element $a \in X, a \neq 0$. This means that Υ_s is far from being surjective.

5.2.2 (LB)-spaces of Dirichlet series

So far, the number s defining the complex half plane $\overline{\mathbb{H}}_s$ was fixed. Now, we are interested in Dirichlet series which converge absolutely on some half plane. For $s < t$, the vector space $D_s(X)$ is a subspace of $D_t(X)$, when both of them are regarded as vector subspaces of the space of all formal Dirichlet series.

Lemma 5.2.2. *For $s < t$ the bonding maps $i_{t,s} \colon D_s(X) \longrightarrow D_t(X)$ are bounded operators of norm ≤ 1.*

Proof.

$$\left\| \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} \right\|_{(t)} = \sum_{n=1}^{\infty} \|a_n\|_X \, n^{-t} \leq \sum_{n=1}^{\infty} \|a_n\|_X \, n^{-s} = \left\| \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} \right\|_{(s)} . \qquad \square$$

Since $(\mathbb{N}, \leq)$ is cofinal in $(\mathbb{R}, \leq)$ is suffices to look only at $s \in \mathbb{N}$. So, again, we are dealing with a countable direct limit:

Proposition 5.2.3. *The space*

$$D_\infty(X) := \bigcup_{s\in\mathbb{N}} D_s(X)\,,$$

of all formal Dirichlet series which converge absolutely on some half plane, endowed with the locally convex direct limit topology is Hausdorff and compactly regular.

Proof. Let $f_s \colon D_s(X) \longrightarrow X^{\mathbb{N}} : \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}} \mapsto (a_n)_{n\in\mathbb{N}}$ be the map that assigns to every Dirichlet series its sequence of coefficients. This map is continuous since the range space has the product topology and every component of f_s is a continuous functional. The space $X^{\mathbb{N}}$ is locally convex (it is in fact a Fréchet space) and therefore, by the universal property of the locally convex direct limit, there is a continuous extension $f \colon D_\infty(X) \longrightarrow X^{\mathbb{N}}$. Since f is injective by construction and $X^{\mathbb{N}}$ is Hausdorff, it follows that also $D_\infty(X)$ is Hausdorff.

Proposition 1.2.4 guarantees compact regularity of the limit $D_\infty(X)$ if we can show that for every $s \in \mathbb{N}$ there is a $t \geq s$ and an open 0-neighborhood $\Omega \subseteq D_s(X)$ such that $D_t(X), D_{t+1}(X), \ldots$ induce the same topology on Ω.

For every given $s \in \mathbb{N}$ we set $t := s+2$ and $\Omega := \mathrm{B}_1^{D_s(X)}(0)$. Let $u \geq t$. To see that the topologies on Ω induced by $D_t(X)$ and $D_u(X)$ agree, it suffices to show that

$$\Omega \subseteq D_u(X) \longrightarrow D_t(X) : \gamma \mapsto \gamma$$

is continuous. To this end, let $\varepsilon > 0$. Since the positive series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, there is an $n_0 \in \mathbb{N}$ such that $\sum_{n>n_0}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{4}$. Set $\delta := n_0^{t-u} \cdot \frac{\varepsilon}{2}$. We show that, for any Dirichlet series $\gamma_1, \gamma_2 \in \Omega$ with $\|\gamma_1 - \gamma_2\|_{(u)} < \delta$, we have $\|\gamma_1 - \gamma_2\|_{(t)} < \varepsilon$. Since $\gamma_1, \gamma_2 \in \Omega$, we have $\gamma_d := \gamma_1 - \gamma_2 \in 2\Omega$. Therefore $\|\gamma_d\|_{(s)} < 2$ and $\|\gamma_d\|_{(u)} < \delta$. Writing $\gamma_d = \sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}}$, we obtain

$$\begin{aligned}
\|\gamma_d\|_{(t)} &= \sum_{n=1}^{\infty} \|a_n\|\, n^{-t} = \sum_{n\leq n_0} \|a_n\|\, n^{-t} + \sum_{n>n_0} \|a_n\|\, n^{-t} \\
&= \sum_{n\leq n_0} \|a_n\|\, n^{-u} \cdot \underbrace{n^{u-t}}_{\leq n_0^{u-t}} + \sum_{n>n_0} \|a_n\|\, n^{-s} \cdot \underbrace{n^{s-t}}_{=\frac{1}{n^2}} \\
&\leq n_0^{u-t} \sum_{n\leq n_0} \|a_n\|\, n^{-u} + \sum_{n>n_0} \underbrace{\|a_n\|\, n^{-s}}_{\leq \|\gamma_d\|_{(s)}} \cdot \frac{1}{n^2} \\
&\leq n_0^{u-t} \underbrace{\|\gamma_d\|_{(u)}}_{<\delta} + \underbrace{\|\gamma_d\|_{(s)}}_{<2} \underbrace{\sum_{n>n_0} \frac{1}{n^2}}_{<\frac{\varepsilon}{4}} < n_0^{u-t} \cdot \delta + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

This is what we had to show. $\square$

5.2.3 Lie groups associated with Dirichlet series

From now on, let G denote a fixed complex Banach Lie group with Lie algebra $\mathfrak{g}$. As before, $s \in \mathbb{R}$ is a real number. We know that G has an exponential function $\exp_G \colon \mathfrak{g} \longrightarrow G$. Every Dirichlet series $\gamma \in D_s(\mathfrak{g})$ with values in $\mathfrak{g}$ can be composed with the exponential function and yields a continuous function from $\overline{\mathbb{H}}_s$ to G. All these continuous functions generate a group (with respect to pointwise multiplication of functions):

Theorem 5.2.4 (Lie groups associated with Dirichlet series (Banach case))**.** *Let $s \in \mathbb{R}$, and a Banach Lie group G with Lie algebra $\mathfrak{g}$ be given. Then there exists a unique Banach Lie group structure on the group*

$$D_s(G) := \langle \{\exp_G \circ \Upsilon_s(\gamma) : \gamma \in D_s(\mathfrak{g})\} \rangle \leq C\left(\overline{\mathbb{H}}_s, G\right)$$

such that

$$\mathrm{Exp}_s \colon D_s(\mathfrak{g}) \longrightarrow D_s(G) : \gamma \mapsto \exp_G \circ \Upsilon_s(\gamma).$$

becomes a local diffeomorphism around 0*. Like before,* $\Upsilon_s \colon D_s(\mathfrak{g}) \longrightarrow BC\big(\overline{\mathbb{H}}_s, \mathfrak{g}\big)$ *denotes the operator that assigns to a Dirichlet series its continuous function on* $\overline{\mathbb{H}}_s$*.*

Proof. We start by choosing a compatible norm on $\mathfrak{g}$, i.e. $\|[x,y]\|_{\mathfrak{g}} \leq \|x\|_{\mathfrak{g}} \|y\|_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$. Then the space $D_s(\mathfrak{g})$ also carries a continuous bilinear map of operator norm at most 1:

$$\begin{aligned} [\cdot,\cdot] : D_s(\mathfrak{g}) \times D_s(\mathfrak{g}) &\longrightarrow D_s(\mathfrak{g}) \\ \left(\Big(\sum_{n\in\mathbb{N}} a_n \cdot n^{-\mathbf{z}}\Big), \Big(\sum_{n\in\mathbb{N}} b_n \cdot n^{-\mathbf{z}}\Big)\right) &\longmapsto \sum_{N\in\mathbb{N}} \left(\sum_{\substack{(n_1,n_2)\in\mathbb{N}\times\mathbb{N} \\ n_1\cdot n_2 = N}} [a_{n_1}, b_{n_2}] \right) \cdot N^{-\mathbf{z}}, \end{aligned}$$

turning it into a Banach Lie algebra. Note that the inner sum is finite. This Lie bracket corresponds to the pointwise Lie bracket of functions, i.e. the operator

$$\Upsilon_s \colon D_s(\mathfrak{g}) \longrightarrow BC\big(\overline{\mathbb{H}}_s, \mathfrak{g}\big)$$

becomes a morphism of Banach Lie algebras. The Lie algebra $\mathfrak{g}$ becomes a closed Lie subalgebra of $D_s(\mathfrak{g})$ by identifying elements of $\mathfrak{g}$ with constant Dirichlet series.

By Lemma 1.3.12, the BCH-series converges on

$$\Omega_{\mathfrak{g}} := \{(x,y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \log 2\}$$

and defines an analytic multiplication: $* \colon \Omega_{\mathfrak{g}} \longrightarrow \mathfrak{g}$. Since $D_s(\mathfrak{g})$ is a Banach Lie algebra in its own right, we also have a BCH-multiplication there: $* \colon \Omega_{D_s(\mathfrak{g})} \longrightarrow D_s(\mathfrak{g})$. The BCH-series is defined only in terms of iterated Lie brackets. Since addition and Lie bracket of Dirichlet series in $D_s(\mathfrak{g})$ correspond to the pointwise operations in $\mathfrak{g}$, the BCH-multiplication in $D_s(\mathfrak{g})$ corresponds to the pointwise BCH-multiplication of functions in $BC\big(\overline{\mathbb{H}}_s, \mathfrak{g}\big)$.

Since G is a Banach Lie group, it is locally exponential, therefore there is a number $\varepsilon_\circ > 0$ such that $\exp_G|_{\mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)}$ is injective. Since the BCH-multiplication on $\mathfrak{g}$ is continuous, there is a $\delta > 0$ such that $\mathrm{B}^{\mathfrak{g}}_{\delta}(0) \times \mathrm{B}^{\mathfrak{g}}_{\delta}(0) \subseteq \Omega_{\mathfrak{g}}$ and $\mathrm{B}^{\mathfrak{g}}_{\delta}(0) * \mathrm{B}^{\mathfrak{g}}_{\delta}(0) \subseteq \mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)$.

Let $C\left(\overline{\mathbb{H}}_s, G\right)$ be the (abstract) group of all continuous maps from $\overline{\mathbb{H}}_s$ to G with pointwise multiplication. Then we can define the following map

$$\mathrm{Exp}_s \colon D_s(\mathfrak{g}) \longrightarrow C\left(\overline{\mathbb{H}}_s, G\right) : \gamma \mapsto \exp_G \circ \Upsilon_s(\gamma).$$

The restriction of Exp_s to $\mathrm{B}^{D_s(\mathfrak{g})}_{\varepsilon_\circ}(0)$ is injective since $\exp_G|_{\mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)}$ is injective. Here we use that Υ_s is injective and of operator norm at most 1.

Now, all hypotheses for Corollary 1.3.16 are satisfied for $U := \mathrm{B}^{D_s(\mathfrak{g})}_{\delta}(0)$, $V := \mathrm{B}^{D_s(\mathfrak{g})}_{\varepsilon_\circ}(0)$ and $H := C\left(\overline{\mathbb{H}}_s, G\right)$. Therefore, by Corollary 1.3.16, we get a unique $C^{\omega}_{\mathbb{C}}$-Lie group structure on the group $D_s(G) = \langle \mathrm{Exp}_s(D_s(\mathfrak{g}))\rangle$ such that

$$\mathrm{Exp}_s|_U \colon U \subseteq D_s(\mathfrak{g}) \longrightarrow \langle \mathrm{Exp}_s(U)\rangle$$

is a $C^{\omega}_{\mathbb{C}}$-diffeomorphism. □

Since for $s < t$ the bonding maps

$$\begin{aligned} j_{t,s} : D_s(G) &\longrightarrow D_t(G) \\ f &\longmapsto f|_{\overline{\mathbb{H}}_t} \end{aligned}$$

are injective group homomorphisms, we identify $D_s(G)$ with a subgroup of $D_t(G)$ and so we can form the union group, which then consists of germs of functions from a complex half plane to G together with the pointwise multiplication.

Theorem 5.2.5 (Lie groups associated with Dirichlet series ((LB)-case))**.** *(a) On the group*

$$D_\infty(G) := \bigcup_{s\in\mathbb{R}} D_s(G) = \langle \{\exp_G \circ\gamma : \gamma \in D_s(\mathfrak{g})\, , s \in \mathbb{R}\}\rangle$$

there is a unique Lie group structure turning

$$\mathrm{Exp} := \bigcup_{s\in\mathbb{R}} \mathrm{Exp}_s \colon D_\infty(\mathfrak{g}) \longrightarrow D_\infty(G) : \gamma \in D_s(\mathfrak{g}) \mapsto \exp_G \circ\gamma$$

into a local diffeomorphism around 0*.*

(b) This Lie group is strongly C^0*-regular.*

The construction of this Lie group and the proof of part (a) can also be found in [5] (Theorem D).

Proof. (a) We wish to use Theorem 4.1.1. For every $s \in \mathbb{N}$, set $G_s := D_s(G)$. The bonding maps $j_s \colon G_s \longrightarrow G_{s+1}$ are group homomorphisms. Since $j_s \circ \mathrm{Exp}_s = \mathrm{Exp}_s \circ i_s$ with the continuous linear inclusion map $i_s \colon D_s(\mathfrak{g}) \longrightarrow D_{s+1}(\mathfrak{g})$, we see that each j_s is analytic with $\mathbf{L}(j_s) = i_s$.

By construction, the norms on the Lie algebras $D_s(\mathfrak{g})$ and the bounded operators $i_s \colon D_s(\mathfrak{g}) \longrightarrow D_{s+1}(\mathfrak{g})$ have operator norm at most 1. The locally convex direct limit is Hausdorff by Proposition 5.2.3, and the exponential map $\mathrm{Exp} = \bigcup_{s\in\mathbb{N}} \mathrm{Exp}_s$ is injective on the 0-neighborhood $\bigcup_{s\in\mathbb{N}} \mathrm{B}_{\varepsilon_\circ}^{D_s(\mathfrak{g})}(0)$. Hence, by Theorem 4.1.1, there is a unique complex analytic Lie group structure on G such that Exp is a local diffeomorphism at 0.

(b) By Proposition 5.2.3, the modelling space is a compactly regular direct limit. So, we can apply Theorem 4.3.2 and obtain the result. □

5.3 Lie groups associated to Hölder continuous functions

5.3.1 Spaces of Hölder Continuous Functions

Throughout this section, let Ω be a convex bounded non-empty open subset of a *real* Banach space $X \neq 0$.

Definition 5.3.1 (Hölder Spaces)**.** Let Z be a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) We let $BC^{0,0}(\Omega, Z) := BC(\Omega, Z)$ be the vector space of all bounded continuous Z-valued functions on the set Ω. It will always be endowed with the norm $\|\cdot\|_{(0,0)} := p_{(0,0)}(\cdot) := \|\cdot\|_\infty$.

(b) For a real number $s \in]0, 1]$, we set

$$BC^{0,s}(\Omega, Z) := \left\{ \gamma \colon \Omega \longrightarrow Z : p_{(0,s)}(\gamma) := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|\gamma(x) - \gamma(y)\|_Z}{\|x - y\|_X^s} < \infty \right\}.$$

From this definition follows at once that every $\gamma \in BC^{0,s}(\Omega, Z)$ is uniformly continuous and bounded (since Ω is bounded). We endow this vector space with the norm $\|\cdot\|_{(0,s)} := \|\cdot\|_\infty + p_{(0,s)}(\cdot)$.

(c) Recursively, we define

$$BC^{k+1,s}(\Omega, Z) := \left\{ \gamma \in FC^1(\Omega, Z) : \gamma' \in BC^{k,s}(\Omega, \mathcal{L}(X, Z)) \right\}$$

for $k \in \mathbb{N}_0$ and $s \in [0, 1]$. We endow this vector space with the norm $\|\cdot\|_{(k+1,s)} := \|\cdot\|_\infty + p_{(k+1,s)}(\cdot)$, where

$$p_{(k+1,s)}(\gamma) := p_{(k,s)}(\gamma').$$

5.3.2 Inclusion Mappings

In this subsection, we show that the inclusion operators between the above spaces are continuous (Proposition 5.3.5).

We begin with the following special case where the inclusion operator behaves very nicely:

Proposition 5.3.2. *For every $k \in \mathbb{N}_0$ the vector space $BC^{k+1,0}(\Omega, Z)$ is a vector subspace of $BC^{k,1}(\Omega, Z)$ and the inclusion map is an isometric embedding.*

Proof. Since for $(k,s) \neq (0,0)$ the norm $\|\cdot\|_{(k,s)}$ is the sum of the $\|\cdot\|_\infty$-norm and the $p_{(k,s)}(\cdot)$-seminorm, it suffices to show that for every $\gamma \in BC^{k+1,0}(\Omega, Z)$ we have $\gamma \in BC^{k,1}(\Omega, Z)$ and

$$p_{(k,1)}(\gamma) = p_{(k+1,0)}(\gamma).$$

It suffices to show this for $k = 0$. The rest follows immediately by induction on k. Let $\gamma \in BC^{1,0}(\Omega, Z)$ be given. By definition of the Hölder spaces, this means γ is continuously differentiable with bounded Fréchet derivative. Now, we estimate

$$\begin{aligned}\|\gamma(x) - \gamma(y)\|_Z &= \left\| \int_0^1 \gamma'\big(tx + (1-t)y\big)\big(x-y\big)dt \right\|_Z \\ &\leq \|\gamma'\|_\infty \|x-y\|_X \\ &= p_{(1,0)}(\gamma)\, \|x-y\|_X\end{aligned}$$

This shows that $\gamma \in BC^{0,1}(\Omega, Z)$ and

$$p_{(0,1)}(\gamma) \leq p_{(1,0)}(\gamma).$$

But conversely: Let $x_0 \in \Omega, v \in X$ with $\|v\|_Z = 1$ and $t \in \mathbb{R}^\times$ (small enough) be given. Then we can estimate:

$$\begin{aligned}\left\| \frac{1}{t}\big(\gamma(x+tv) - \gamma(x)\big) \right\|_Z &= \frac{1}{|t|}\, \|\gamma(x+tv) - \gamma(x)\|_Z \\ &\leq \frac{1}{|t|} \cdot p_{(0,1)}(\gamma)\, \|(x+tv) - x\|_Z \\ &= p_{(0,1)}(\gamma).\end{aligned}$$

Now, as t tends to zero, the left hand side converges to $\gamma'(x).v$. Since v was arbitrary with norm 1, this yields $\|\gamma'(x)\|_{\mathrm{op}} \leq p_{(0,1)}(\gamma)$ and since x was arbitrary, we finally obtain:

$$p_{(1,0)}(\gamma) \leq p_{(0,1)}(\gamma).$$

Therefore the seminorms are equal and this finishes the proof. ☐

Proposition 5.3.3. *Let $k \in \mathbb{N}_0$ and let $0 < s_1 < s_2 \leq 1$. Then the vector space $BC^{k,s_2}(\Omega, Z)$ is a vector subspace of $BC^{k,s_1}(\Omega, Z)$ and we have for all $\gamma \in BC^{k,s_2}(\Omega, Z)$:*

$$p_{(0,s_1)}(\gamma) \leq (\mathrm{diam}\Omega)^{s_2 - s_1} \cdot p_{(0,s_2)}(\gamma).$$

The inclusion map is continuous with operator norm at most $\max\{1, (\mathrm{diam}\Omega)^{s_2-s_1}\}$.

Proof. First consider the case $k = 0$. Let $\gamma \in BC^{0,s_2}(\Omega, Z)$ be given. Then

$$\begin{aligned}\frac{\|\gamma(x) - \gamma(y)\|_Z}{\|x-y\|_X^{s_1}} &= \frac{\|\gamma(x) - \gamma(y)\|_Z}{\|x-y\|_X^{s_2}} \cdot \|x-y\|_X^{s_2-s_1} \\ &\leq p_{(0,s_2)}(\gamma) \cdot (\mathrm{diam}\Omega)^{s_2-s_1}.\end{aligned}$$

This shows $BC^{0,s_2}(\Omega, Z) \subseteq BC^{0,s_1}(\Omega, Z)$ and

$$p_{(0,s_1)}(\cdot) \leq (\mathrm{diam}\Omega)^{s_2-s_1} \cdot p_{(0,s_2)}(\cdot).$$

This inequality can be generalized for arbitrary $k \in \mathbb{N}_0$ with a simple induction argument. So, we get:

$$BC^{k,s_2}(\Omega, Z) \subseteq BC^{k,s_1}(\Omega, Z) \text{ and } p_{(k,s_1)}(\cdot) \leq (\mathrm{diam}\Omega)^{s_2-s_1} \cdot p_{(k,s_2)}(\cdot).$$

The corresponding inequality for $\|\cdot\|_{(k,s_2)}$ and $\|\cdot\|_{(k,s_1)}$ follows immediately. □

Lemma 5.3.4. *Let $(k, s) \in \mathbb{N}_0 \times]0, 1]$ and $x_0 \in \Omega$ be fixed.*

(a) The linear operator

$$\begin{aligned} BC^{k,s}(\Omega, Z) &\longrightarrow \left(\mathrm{Sym}_c^k(X, Z), \|\cdot\|_{\mathrm{op}}\right) \\ \gamma &\longmapsto \gamma^{(k)}(x_0) \end{aligned}$$

is continuous.

(b) The linear operator

$$\begin{aligned} BC^{k,s}(\Omega, Z) &\longrightarrow BC^{k,0}(\Omega, Z) \\ \gamma &\longmapsto \gamma \end{aligned}$$

is continuous.

The operator norms of these operators may be bounded by constants depending on k, Ω and x_0, but not on Z or s.

Proof. For $k = 0$ both (a) and (b) are trivial. So, we may assume $k \geq 1$.

Before we show (a), we show how (b) follows from (a):

$$\begin{aligned} \|\gamma\|_{(k,0)} &= \|\gamma\|_\infty + p_{(k,0)}(\gamma) \\ &\leq \|\gamma\|_\infty + \left\|\gamma^{(k)}\right\|_\infty \\ &= \|\gamma\|_\infty + \sup_{x\in\Omega} \left\|\gamma^{(k)}(x)\right\|_{\mathrm{op}} \\ &\leq \|\gamma\|_\infty + \sup_{x\in\Omega} \left\|\gamma^{(k)}(x) - \gamma^{(k)}(x_0)\right\|_{\mathrm{op}} + \left\|\gamma^{(k)}(x_0)\right\|_{\mathrm{op}} \\ &\leq \|\gamma\|_\infty + p_{(0,s)}(\gamma^{(k)}) \cdot (\mathrm{diam}\Omega)^s + \left\|\gamma^{(k)}(x_0)\right\|_{\mathrm{op}} \\ &\leq \|\gamma\|_\infty + (\mathrm{diam}\Omega) \cdot p_{(k,s)}(\gamma) + \left\|\gamma^{(k)}(x_0)\right\|_{\mathrm{op}}. \end{aligned}$$

The first two summands are obviously continuous with respect to $\|\cdot\|_{(k,s)}$ and the continuity of the third summand follows from part (a).

Now we prove (a): Since Ω is open, there is an $\varepsilon_0 \in]0,1]$ such that $\overline{\mathrm{B}^X_{\varepsilon_0}(x_0)} \subseteq \Omega$. Let $v \in X$ be a vector with $\|v\|_X \leq 1$. Since $\gamma \in BC^{k,s}(\Omega, Z)$, it is in particular FC^k and therefore we can use Taylor's formula (Proposition 1.1.11 (b)) and obtain:

$$\gamma(x_0 + \varepsilon_0 v) = T^k_{x_0}\gamma(\varepsilon_0 v) + R\gamma(\varepsilon_0 v) \tag{$*$}$$

with

$$T^k_{x_0}\gamma(\varepsilon_0 v) = \sum_{j \leq k} \frac{\gamma^{(j)}(x_0)(v, \ldots, v)\varepsilon_0^j}{j!}$$

and

$$R\gamma(\varepsilon_0 v) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left(\gamma^{(k)}(x_0 + t\varepsilon_0 v) - \gamma^{(k)}(x_0)\right)(v, \ldots, v)\varepsilon_0^k \, dt.$$

First, we look at the remainder part $R\gamma(\varepsilon_0 v)$:

$$\begin{aligned}
\|R\gamma(\varepsilon_0 v)\|_Z &= \left\| \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}\left(\gamma^{(k)}(x_0 + t\varepsilon_0 v) - \gamma^{(k)}(x_0)\right)(v, \ldots, v)\varepsilon_0^k dt \right\|_Z \\
&\leq \int_0^1 \frac{1}{(k-1)!} \left\|\gamma^{(k)}(x_0 + t\varepsilon_0 v) - \gamma^{(k)}(x_0)\right\|_{\mathrm{op}} \|v\|_X^k \, \varepsilon_0^k \, dt \\
&\leq \int_0^1 \frac{1}{(k-1)!} p_{(k,s)}(\gamma) \cdot \|t\varepsilon_0 v\|_X^s \, \varepsilon_0^k \, dt \\
&\leq \frac{\varepsilon_0^{k+s}}{(k-1)!} \|\gamma\|_{(k,s)} \leq \underbrace{\frac{\varepsilon_0^k}{(k-1)!}}_{=:C_1} \|\gamma\|_{(k,s)}.
\end{aligned}$$

Now, we estimate the norm of the Taylor-polynomial:

$$\begin{aligned}
\left\|T^k_{x_0}\gamma(\varepsilon_0 v)\right\|_Z &\overset{(*)}{=} \|\gamma(x_0 + \varepsilon_0 v) - R\gamma(\varepsilon_0 v)\|_Z \\
&\leq \|\gamma(x_0 + \varepsilon_0 v)\|_Z + \|R\gamma(\varepsilon_0 v)\|_Z \\
&\leq \underbrace{\|\gamma\|_\infty}_{\leq \|\gamma\|_{(k,s)}} + C_1 \|\gamma\|_{(k,s)} \\
&\leq C_2 \|\gamma\|_{(k,s)}.
\end{aligned}$$

Since $v \in \overline{\mathrm{B}^X_1(0)}$ was arbitrary, this shows that the sup norm of the Taylor polynomial on the closed unit ball is bounded by a constant times $\|\gamma\|_{(k,s)}$. By Proposition 1.1.10 the norm of every homogeneous part is bounded above by the norm of the polynomial:

$$\left\|\frac{\gamma^{(j)}(x_0)(\cdot)\varepsilon_0^j}{j!}\right\|_{\mathrm{op}} \leq C_3 \|\gamma\|_{(k,s)}.$$

As we saw in the proof of Proposition 1.1.10, this constant does only depend on j and k.

In particular, we have for the case $j = k$:

$$\left\| \gamma^{(k)}(x_0) \right\|_{\mathrm{op}} \leq C_4 \left\| \gamma \right\|_{(k,s)}$$

which is what we had to show. ☐

Proposition 5.3.5. *Let $(k, s), (\ell, t) \in \mathbb{N}_0 \times [0, 1]$ be given. Assume $k + s < \ell + t$. Then*

$$BC^{\ell,t}(\Omega, Z) \subseteq BC^{k,s}(\Omega, Z)$$

and the inclusion map is a continuous operator whose norm can be bounded above by a constant depending only on ℓ, X and Ω.

Proof. This is a immediate consequence of Proposition 5.3.2, Proposition 5.3.3 and Lemma 5.3.4 (b). ☐

5.3.3 Completeness of the Hölder Spaces

Lemma 5.3.6. *Let $s \in [0, 1]$ and $k \in \mathbb{N}_0$ be given. Then the map*

$$\begin{aligned} \kappa : BC^{k+1,s}(\Omega, Z) &\longrightarrow BC(\Omega, Z) \times BC^{k,s}(\Omega, \mathcal{L}(X, Z)) \\ \gamma &\longmapsto (\gamma, \gamma') \end{aligned}$$

is a topological embedding.

Proof. For this proof we endow the product space

$$BC(\Omega, Z) \times BC^{k,s}(\Omega, \mathcal{L}(X, Z))$$

with the norm $\|(\gamma, \eta)\| := \|\gamma\|_\infty + \|\eta\|_{(k,s)}$. The map κ is clearly linear and injective. We show the continuity of κ with the following estimate:

$$\begin{aligned} \|\kappa(\gamma)\| = \|\gamma\|_\infty + \left\|\gamma'\right\|_{(k,s)} &= \|\gamma\|_\infty + p_{(k,s)}(\gamma') + \left\|\gamma'\right\|_\infty \\ &\leq \|\gamma\|_{(k+1,s)} + \|\gamma\|_{(1,0)} . \end{aligned}$$

By Proposition 5.3.5, $\|\cdot\|_{(1,0)}$ is continuous with respect to $\|\cdot\|_{(k+1,s)}$. This implies the continuity of κ.

On the other hand, $\|\gamma\|_{(k+1,s)} = \|\gamma\|_\infty + p_{(k,s)}(\gamma') \leq \|\gamma\|_\infty + \|\gamma'\|_{(k,s)} = \|\kappa(\gamma)\|$. Hence, κ is a topological embedding. ☐

Proposition 5.3.7. *Let $s \in [0, 1]$ and $k \in \mathbb{N}_0$ be given. Then the normed space $\left(BC^{k,s}(\Omega, Z), \|\cdot\|_{(k,s)}\right)$ is complete, hence a Banach space.*

Proof. For $(k, s) = (0, 0)$, this is well known. Now, let $k = 0$ and $s \in]0, 1]$.

For every $\gamma \in BC(\Omega, Z)$, define

$$R\gamma \colon U_\Omega \longrightarrow Z : (x, y) \mapsto \frac{\gamma(x) - \gamma(y)}{\|x - y\|_X^s}.$$

Here, $U_\Omega := \{(x, y) \in \Omega \times \Omega : x \neq y\}$ denotes the complement of the diagonal in $\Omega \times \Omega$.

Now, it is clear that $BC^{0,s}(\Omega, Z) := \{\gamma \in BC(\Omega, Z) : R\gamma \in BC(U_\Omega, Z)\}$ and that

$$\begin{aligned} \iota : BC^{0,s}(\Omega, Z) &\longrightarrow BC(\Omega, Z) \times BC(U_\Omega, Z) \\ \gamma &\longmapsto (\gamma, R\gamma) \end{aligned}$$

is an isometric embedding. Therefore it remains to show that the image of ι is closed in the product of the two Banach spaces $BC(\Omega, Z) \times BC(U_\Omega, Z)$.

Now, let (γ, η) be in the closure of the image of ι. This implies that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in the space $BC^{0,s}(\Omega, Z)$ such that $(\gamma_n)_{n\in\mathbb{N}}$ converges uniformly to $\gamma \in BC(\Omega, Z)$ and that $(R\gamma_n)_{n\in\mathbb{N}}$ converges uniformly to $\eta \in BC(U_\Omega, Z)$. In particular, we have pointwise convergence, hence the following holds for all $(x, y) \in U_\Omega$:

$$\eta(x, y) = \lim_{n\to\infty} \frac{\gamma_n(x) - \gamma_n(y)}{\|x - y\|_X^s}.$$

But the right hand side converges pointwise to $\frac{\gamma(x)-\gamma(y)}{\|x-y\|_X^s}$ since $(\gamma_n)_{n\in\mathbb{N}}$ converges to γ. Therefore $\eta = R\gamma$ and therefore the image of ι is closed and $BC^{0,s}(\Omega, Z)$ is a Banach space.

Now, we show the assertion for $(k+1, s)$, assuming by induction that it holds for $(k, s) \in \mathbb{N}_0 \times [0, 1]$. We use the topological embedding from Lemma 5.3.6:

$$\begin{aligned} \kappa : BC^{k+1,s}(\Omega, Z) &\longrightarrow BC(\Omega, Z) \times BC^{k,s}(\Omega, \mathcal{L}(X, Z)) \\ \gamma &\longmapsto (\gamma, \gamma'). \end{aligned}$$

Again it suffices to show that the image of κ is closed in the Banach space $BC(\Omega, Z) \times BC^{k,s}(\Omega, \mathcal{L}(X, Z))$ which by the inductive hypothesis is a product of two Banach spaces.

Now, let (γ, η) be in the closure of the image of κ. This implies that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in the space $BC^{k+1,s}(\Omega, Z)$ such that $(\gamma_n)_{n\in\mathbb{N}}$ converges to γ in $BC(\Omega, Z)$ and that $(\gamma_n')_{n\in\mathbb{N}}$ converges to $\eta \in BC^{k,s}(\Omega, \mathcal{L}(X, Z))$. We have to show that $\gamma \in BC^{k+1,s}(\Omega, Z)$ and that $\gamma' = \eta$.

To this end, let $x_0 \in \Omega$ and $v \in X$ be given. Since Ω is convex and open, we can write the difference quotient of γ_n at point $x_0 \in \Omega$ in direction $v \in X$ as:

$$\frac{1}{t}\left(\gamma_n(x_0 + tv) - \gamma_n(x_0)\right) = \int_0^1 \gamma_n'(x_0 + stv).v ds$$

if $|t|$ is small enough. Now, we take the pointwise limit as $n \to \infty$ and obtain:

$$\frac{1}{t}\left(\gamma(x_0 + tv) - \gamma(x_0)\right) = \int_0^1 \eta(x_0 + stv).vds$$

For the convergence of the integral, we use that $\|\gamma_n' - \eta\|_\infty \to 0$.

Letting now t tend to 0, the integral on the right hand side converges to $\eta(x_0).v$. So, we have shown that the directional derivative of γ at point x_0 in direction v exists and is equal to $\eta(x_0).v$. Since $v \in X$ was arbitrary, all directional derivatives exist and are of the form

$$d\gamma\colon \Omega \times X \longrightarrow Z : (x, v) \mapsto \eta(x).v.$$

This map is continuous and therefore γ is C^1 the Michal-Bastiani sense. But since

$$\gamma'(x) = d\gamma(x, \cdot) = \eta(x)$$

and $\eta\colon \Omega \longrightarrow \mathcal{L}(X, Z)$ is continuous by hypothesis, we can apply Lemma 1.1.3 and obtain that γ is FC^1. Since $\gamma' = \eta \in BC^{k,s}(\Omega, \mathcal{L}(X, Z))$, this implies that $\gamma \in BC^{k+1,s}(\Omega, Z)$ which finishes the proof. □

5.3.4 Products of Hölder Continuous Functions

Theorem 5.3.8 (Products of Hölder Continuous Functions)**.** *Let Z_1, Z_2, Z be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We assume that* $\operatorname{diam}\Omega \leq 1$*. Let $\bullet\colon Z_1 \times Z_2 \longrightarrow Z$ be a continuous bilinear map. We define the pointwise product of two functions $\gamma_1 \in BC^{k,s}(\Omega, Z_1)$ and $\gamma_2 \in BC^{k,s}(\Omega, Z_2)$ as*

$$\gamma_1 \bullet \gamma_2\colon \Omega \longrightarrow Z : x \mapsto \gamma_1(x) \bullet \gamma_2(x).$$

Then the product is again in $BC^{k,s}(\Omega, Z)$ and we have the following formula:

$$\|\gamma_1 \bullet \gamma_2\|_{(k,s)} \leq C_k \cdot \|\bullet\|_{\mathrm{op}} \cdot \|\gamma_1\|_{(k,s)} \|\gamma_2\|_{(k,s)}$$

Here, $C_k > 0$ is a constant, depending only on k, but not on s or on $\bullet$ (which will important later on).

Proof. By replacing the continuous bilinear map $\bullet$ by its multiple $\frac{1}{\|\bullet\|_{\mathrm{op}}}\bullet$, we may assume that $\|\bullet\|_{\mathrm{op}} = 1$.

The claim is trivial for $(k, s) = (0, 0)$. The case $k = 0$ and $s \in]0, 1]$ is done in the following way:

$$\begin{aligned}
\|\gamma_1 \bullet \gamma_2(x) - \gamma_1 \bullet \gamma_2(y)\|_Z &\leq \|\gamma_1(x) \bullet \gamma_2(x) - \gamma_1(x) \bullet \gamma_2(y)\|_Z \\
&\quad + \|\gamma_1(x) \bullet \gamma_2(y) - \gamma_1(y) \bullet \gamma_2(y)\|_Z \\
&\leq \|\gamma_1(x)\|_Z \|\gamma_2(x) - \gamma_2(y)\|_Z \\
&\quad + \|\gamma_1(x) - \gamma_1(y)\|_Z \|\gamma_2(y)\|_Z \\
&\leq \left(\|\gamma_1\|_\infty p_{(0,s)}(\gamma_2) + p_{(0,s)}(\gamma_1) \|\gamma_2\|_\infty\right) \|x - y\|_X^s .
\end{aligned}$$

Therefore we have

$$p_{(0,s)}(\gamma_1 \bullet \gamma_2) \leq \|\gamma_1\|_\infty \, p_{(0,s)}(\gamma_2) + p_{(0,s)}(\gamma_1) \, \|\gamma_2\|_\infty$$

Now we add the inequality $\|\gamma_1 \bullet \gamma_2\|_\infty \leq \|\gamma_1\|_\infty \|\gamma_2\|_\infty$ and the previous one:

$$\begin{aligned}
\|\gamma_1 \bullet \gamma_2\|_{(0,s)} &\leq \|\gamma_1\|_\infty \, p_{(0,s)}(\gamma_2) + p_{(0,s)}(\gamma_1) \, \|\gamma_2\|_\infty + \|\gamma_1\|_\infty \|\gamma_2\|_\infty \\
&= \|\gamma_1\|_\infty \, p_{(0,s)}(\gamma_2) + \|\gamma_1\|_{(0,s)} \, \|\gamma_2\|_\infty \\
&\leq \|\gamma_1\|_{(0,s)} \, \|\gamma_2\|_{(0,s)} + \|\gamma_1\|_{(0,s)} \, \|\gamma_2\|_{(0,s)} \, .
\end{aligned}$$

This proves the claim for $k = 0$ and $s \in [0, 1]$ for the constant $C_0 := 2$.

Now assume the claim holds for k. We will show it for $k + 1$.

By Proposition 5.3.5, we know that the inclusion maps

$$BC^{k+1,s}(\Omega, Z) \longrightarrow BC^{k,s}(\Omega, Z)$$

and

$$BC^{k+1,s}(\Omega, Z) \longrightarrow BC^{1,0}(\Omega, Z)$$

are continuous and that their operator norms can be bounded by constants D_k and E_k respectively, depending only on k, X and Ω.

Now, let $\gamma_1 \in BC^{k+1,s}(\Omega, Z_1)$ and $\gamma_2 \in BC^{k+1,s}(\Omega, Z_2)$ be given. By definition, this means that γ_1 and γ_2 are FC^1 and

$$\gamma_1' \in BC^{k,s}(\Omega, \mathcal{L}(X, Z_1)) \qquad \text{and} \qquad \gamma_2' \in BC^{k,s}(\Omega, \mathcal{L}(X, Z_2)) \, .$$

Now we define the following bilinear operators:

$$\begin{aligned}
*_1 : Z_1 \times \mathcal{L}(X, Z_2) &\longrightarrow \mathcal{L}(X, Z) \\
(z, T) &\longmapsto (x \mapsto z \bullet (Tx))
\end{aligned}$$

and

$$\begin{aligned}
*_2 : \mathcal{L}(X, Z_1) \times Z_2 &\longrightarrow \mathcal{L}(X, Z) \\
(T, z) &\longmapsto (x \mapsto (Tx) \bullet z) \, .
\end{aligned}$$

It is easy to verify that $\|*_1\|_{\mathrm{op}}, \|*_2\|_{\mathrm{op}} \leq 1$. Therefore, we can use the inductive hypothesis and obtain that $\gamma_1 *_1 \gamma_2'$ and $\gamma_1' *_2 \gamma_2$ belong to $BC^{k,s}(\Omega, \mathcal{L}(X, Z))$ and we have the following estimates:

$$\left\|\gamma_1 *_1 \gamma_2'\right\|_{(k,s)} \leq C_k \, \|\gamma_1\|_{(k,s)} \left\|\gamma_2'\right\|_{(k,s)}$$

and

$$\left\|\gamma_1' *_2 \gamma_2\right\|_{(k,s)} \leq C_k \left\|\gamma_1'\right\|_{(k,s)} \|\gamma_2\|_{(k,s)}$$

By the product rule for Fréchet derivatives, we know that

$$(\gamma_1 \bullet \gamma_2)' = \gamma_1 *_1 \gamma_2' + \gamma_1' *_2 \gamma_2.$$

And hence $(\gamma_1 \bullet \gamma_2)' \in BC^{k,s}(\Omega, \mathcal{L}(X, Z))$ which implies $\gamma_1 \bullet \gamma_2 \in BC^{k+1,s}(\Omega, Z)$.

It remains to show the norm estimate:

$$\begin{aligned}
p_{(k+1,s)}(\gamma_1 \bullet \gamma_2) &= p_{(k,s)}((\gamma_1 \bullet \gamma_2)') \leq p_{(k,s)}(\gamma_1 *_1 \gamma_2') + p_{(k,s)}(\gamma_1' *_2 \gamma_2) \\
&\leq \left\|\gamma_1 *_1 \gamma_2'\right\|_{(k,s)} + \left\|\gamma_1' *_2 \gamma_2\right\|_{(k,s)} \\
&\leq C_k \left(\left\|\gamma_1\right\|_{(k,s)} \left\|\gamma_2'\right\|_{(k,s)} + \left\|\gamma_1'\right\|_{(k,s)} \left\|\gamma_2\right\|_{(k,s)}\right) \\
&= C_k \left\|\gamma_1\right\|_{(k,s)} \left\|\gamma_2'\right\|_{\infty} + C_k \left\|\gamma_1\right\|_{(k,s)} p_{(k,s)}(\gamma_2') \\
&\quad + C_k \left\|\gamma_1'\right\|_{\infty} \left\|\gamma_2\right\|_{(k,s)} + C_k p_{(k,s)}(\gamma_1') \left\|\gamma_2\right\|_{(k,s)} \\
&\leq C_k \left\|\gamma_1\right\|_{(k,s)} \left\|\gamma_2\right\|_{(1,0)} + C_k \left\|\gamma_1\right\|_{(k,s)} \left\|\gamma_2\right\|_{(k+1,s)} \\
&\quad + C_k \left\|\gamma_1\right\|_{(1,0)} \left\|\gamma_2\right\|_{(k,s)} + C_k \left\|\gamma_1\right\|_{(k+1,s)} \left\|\gamma_2\right\|_{(k,s)} \\
&\leq C_k D_k E_k \left\|\gamma_1\right\|_{(k+1,s)} \left\|\gamma_2\right\|_{(k+1,s)} + C_k D_k \left\|\gamma_1\right\|_{(k+1,s)} \left\|\gamma_2\right\|_{(k+1,s)} \\
&\quad + C_k E_k \left\|\gamma_1\right\|_{(k+1,s)} D_k \left\|\gamma_2\right\|_{(k+1,s)} + C_k \left\|\gamma_1\right\|_{(k+1,s)} D_k \left\|\gamma_2\right\|_{(k+1,s)} \\
&= \underbrace{2C_k D_k (E_k + 1)}_{C_{k+1} :=} \left\|\gamma_1\right\|_{(k+1,s)} \left\|\gamma_2\right\|_{(k+1,s)}.
\end{aligned}$$

This finishes the proof. □

5.3.5 Directed Unions of Hölder Spaces

From now on, we will assume that $\operatorname{diam}\Omega \leq 1$. By Proposition 5.3.3, this implies that for a fixed $k \in \mathbb{N}_0$ and $0 < s_1 < s_2 \leq 1$ the inclusion map

$$BC^{k,s_2}(\Omega, Z) \longrightarrow BC^{k,s_1}(\Omega, Z)$$

is continuous with operator norm at most 1. However, this is not really a big restriction since for every bounded nonempty convex open Ω, the scalar multiple $\frac{\Omega}{\operatorname{diam}\Omega}$ has diameter 1 and it is clear that the Banach spaces $BC^{k,s}(\Omega, Z)$ and $BC^{k,s}\left(\frac{\Omega}{\operatorname{diam}\Omega}, Z\right)$ are topologically isomorphic.

Proposition 5.3.9 (Logarithmic Convexity Property for $k = 0$)**.**

(a) Let $0 < s < u \leq 1$. Assume $\gamma \in BC^{0,u}(\Omega, Z)$ and let $\lambda \in]0, 1[$. Then we have

$$p_{(0,\lambda s+(1-\lambda)u)}(\gamma) \leq \left(p_{(0,s)}(\gamma)\right)^{\lambda} \cdot \left(p_{(0,u)}(\gamma)\right)^{1-\lambda}.$$

(b) Let $0 \leq s < u \leq 1$. Assume $\gamma \in BC^{0,u}(\Omega, Z)$ and let $\lambda \in]0, 1[$. Then we have

$$\left\|\gamma\right\|_{(0,\lambda s+(1-\lambda)u)} \leq 2\left(\left\|\gamma\right\|_{(0,s)}\right)^{\lambda} \cdot \left(\left\|\gamma\right\|_{(0,u)}\right)^{1-\lambda}.$$

Proof. (a) We may estimate:

$$\frac{\|\gamma(x)-\gamma(y)\|_Z}{\|x-y\|_X^{\lambda s+(1-\lambda)u}} = \frac{\|\gamma(x)-\gamma(y)\|_Z^{\lambda}\cdot\|\gamma(x)-\gamma(y)\|_Z^{1-\lambda}}{\|x-y\|_X^{\lambda s}\cdot\|x-y\|_X^{(1-\lambda)u}} = \Bigg(\underbrace{\frac{\|\gamma(x)-\gamma(y)\|_Z}{\|x-y\|_X^{s}}}_{\leq p_{(0,s)}(\gamma)}\Bigg)^{\lambda}\cdot\Bigg(\underbrace{\frac{\|\gamma(x)-\gamma(y)\|_Z}{\|x-y\|_X^{u}}}_{\leq p_{(0,u)}(\gamma)}\Bigg)^{1-\lambda}.$$

This shows (a).

Now (b) readily follows:

$$\begin{aligned}\|\gamma\|_{(0,\lambda s+(1-\lambda)u)} &= \|\gamma\|_\infty + p_{(0,\lambda s+(1-\lambda)u)}(\gamma)\\ &\leq \|\gamma\|_\infty^{\lambda}\cdot\|\gamma\|_\infty^{1-\lambda} + \big(p_{(0,s)}(\gamma)\big)^{\lambda}\cdot\big(p_{(0,u)}(\gamma)\big)^{1-\lambda}\\ &\leq \|\gamma\|_{(0,s)}^{\lambda}\cdot\|\gamma\|_{(0,u)}^{1-\lambda} + \|\gamma\|_{(0,s)}^{\lambda}\cdot\|\gamma\|_{(0,u)}^{1-\lambda}.\end{aligned}$$ □

Proposition 5.3.10. *Let $(k,s_0)\in\mathbb{N}_0\times[0,1[$ be given. Then the direct limit space*

$$BC^{k,>s_0}(\Omega,Z) := \bigcup_{t\in]s_0,1]} BC^{k,t}(\Omega,Z)$$

is Hausdorff and compactly regular.

Proof. Since for every $t>s_0$ the inclusion map

$$BC^{k,t}(\Omega,Z)\longrightarrow BC^{k,s_0}(\Omega,Z):\gamma\mapsto\gamma$$

is continuous, it follows from the direct limit property that the inclusion map from the direct limit space into the Banach space

$$BC^{k,>s_0}(\Omega,Z)\longrightarrow BC^{k,s_0}(\Omega,Z):\gamma\mapsto\gamma$$

is also continuous. Since it is injective, we deduce that $BC^{k,>s}(\Omega,Z)$ is Hausdorff.

We prove compact regularity using Proposition 1.2.4. Thus, it suffices to show that for every $u\in]s_0,1]$ there is a $t\in]s_0,u[$ such that every space $BC^{k,s}(\Omega,Z)$ with $s\in]s_0,t]$ induces the same topology on the set $B:=\mathrm{B}_1^{BC^{k,u}(\Omega,Z)}(0)$.

So, let $u\in]s_0,1]$ be given. We choose $t\in]s_0,u[$ arbitrarily. Let $s\in]s_0,t[$. Since t lies between s and u, we can write $t=\lambda s+(1-\lambda)u$. Now, we apply Proposition 5.3.9(a) to $\gamma^{(k)}$ and obtain for every $\gamma\in B$

$$p_{(0,t)}(\gamma^{(k)}) \leq \Big(p_{(0,s)}(\gamma^{(k)})\Big)^{\lambda}\cdot\Big(\underbrace{p_{(0,u)}(\gamma^{(k)})}_{\leq 1}\Big)^{1-\lambda}.$$

This inequality shows that the identity map from $B\subseteq BC^{k,s}(\Omega,Z)$ to $B\subseteq BC^{k,t}(\Omega,Z)$ is continuous. Since the continuity of the inverse map is trivial, we have shown that the topologies coincide. □

5.3.6 Lie groups associated to Hölder continuous functions

In the following, G is an analytic Banach-Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with Lie algebra $\mathfrak{g}$.

As before, $\Omega \subseteq X$ is an open bounded convex subset of a real Banach space X with $\operatorname{diam}\Omega \leq 1$. Let $(k, s) \in \mathbb{N}_0 \times [0, 1]$ be fixed. We may define a pointwise Lie bracket on the function space $BC^{k,s}(\Omega, \mathfrak{g})$ and by Theorem 5.3.8, this bracket is continuous with operator norm at most C_k. Throughout this section, C_k will always denote these constants introduced in Theorem 5.3.8. Note that they do not depend on the space $\mathfrak{g}$.

Now we can compose each $\gamma \in BC^{k,s}(\Omega, \mathfrak{g})$ with the exponential function and obtain the following map:

$$\begin{aligned} \operatorname{Exp}_{(k,s)} : BC^{k,s}(\Omega, \mathfrak{g}) &\longrightarrow C(\Omega, G) \\ \gamma &\longmapsto \exp_G \circ \gamma. \end{aligned}$$

Theorem 5.3.11 (Lie groups associated with Hölder continuous functions (Banach case))**.** *Let $(k, s_0) \in \mathbb{N}_0 \times [0, 1]$ and a Banach-Lie group G with Lie algebra $\mathfrak{g}$ be given. Then there exists a unique Banach-Lie group structure on the group*

$$BC^{k,s_0}(\Omega, G) := \left\langle \left\{ \exp_G \circ \gamma : \gamma \in BC^{k,s_0}(\Omega, \mathfrak{g}) \right\} \right\rangle \leq C(\Omega, G)$$

such that

$$\operatorname{Exp}_{(k,s_0)} \colon BC^{k,s_0}(\Omega, \mathfrak{g}) \longrightarrow BC^{k,s_0}(\Omega, G) : \gamma \mapsto \exp_G \circ \gamma$$

becomes a local diffeomorphism around 0.

Proof. We start by choosing a compatible norm $\|\cdot\|_{\mathfrak{g}}$ on $\mathfrak{g}$ with the additional property that

$$\|[x, y]\|_{\mathfrak{g}} \leq \frac{1}{C_k} \|x\|_{\mathfrak{g}} \|y\|_{\mathfrak{g}}$$

for all $x, y \in \mathfrak{g}$. This means that $\|[\cdot, \cdot]\|_{\mathrm{op}} \leq \frac{1}{C_k}$. Then the space $BC^{k,s_0}(\Omega, \mathfrak{g})$ carries a continuous Lie bracket of operator norm at most 1, due to Theorem 5.3.8:

$$[\cdot, \cdot] \colon BC^{k,s_0}(\Omega, \mathfrak{g}) \times BC^{k,s_0}(\Omega, \mathfrak{g}) \longrightarrow BC^{k,s_0}(\Omega, \mathfrak{g})$$

turning it into a Banach-Lie algebra. The Lie algebra $\mathfrak{g}$ becomes a closed Lie subalgebra of $BC^{k,s_0}(\Omega, \mathfrak{g})$ by identifying elements of $\mathfrak{g}$ with constant functions.

By Lemma 1.3.12, we know that in in a Banach-Lie algebra with compatible norm, the BCH-series converges on

$$U_{\mathfrak{g}} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \log 2\}$$

and defines an analytic multiplication: $* \colon U_{\mathfrak{g}} \longrightarrow \mathfrak{g}$. Since $BC^{k,s_0}(\Omega, \mathfrak{g})$ is a Banach-Lie algebra in its own right, we also have a BCH-multiplication there: $*\colon U_{BC^{k,s_0}(\Omega,\mathfrak{g})} \longrightarrow BC^{k,s_0}(\Omega,\mathfrak{g})$. The BCH-series is defined only in terms of iterated Lie brackets. Since addition and Lie bracket of elements in $BC^{k,s_0}(\Omega,\mathfrak{g})$ correspond to the pointwise operations in $\mathfrak{g}$, the BCH-multiplication in $BC^{k,s_0}(\Omega,\mathfrak{g})$ corresponds to the pointwise BCH-multiplication of functions.

Since G is a Banach-Lie group, it is locally exponential, therefore there is a number $\varepsilon_\circ > 0$ such that $\exp_G|_{\mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)}$ is injective. Since the BCH-multiplication on $\mathfrak{g}$ is continuous, there is a $\delta > 0$ such that $\mathrm{B}^{\mathfrak{g}}_\delta(0) \times \mathrm{B}^{\mathfrak{g}}_\delta(0) \subseteq U_{\mathfrak{g}}$ and $\mathrm{B}^{\mathfrak{g}}_\delta(0) * \mathrm{B}^{\mathfrak{g}}_\delta(0) \subseteq \mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)$.

Let $C(\Omega, G)$ be the (abstract) group of all continuous maps from Ω to G with pointwise multiplication. Then we define the following map

$$\mathrm{Exp}_{(k,s_0)}\colon BC^{k,s_0}(\Omega,\mathfrak{g}) \longrightarrow C(\Omega,G) : \gamma \mapsto \exp_G \circ \gamma.$$

The restriction of $\mathrm{Exp}_{(k,s_0)}$ to $\mathrm{B}^{BC^{k,s_0}(\Omega,\mathfrak{g})}_{\varepsilon_\circ}(0)$ is injective since $\exp_G|_{\mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)}$ is injective.

Now, all hypotheses for Corollary 1.3.16 are satisfied for $U := \mathrm{B}^{BC^{k,s_0}(\Omega,\mathfrak{g})}_{\delta}(0)$, $V := \mathrm{B}^{BC^{k,s_0}(\Omega,\mathfrak{g})}_{\varepsilon_\circ}(0)$ and $H := C(\Omega, G)$. Therefore, by Corollary 1.3.16, we get a unique $C^\omega_{\mathbb{K}}$-Lie group structure on the group $BC^{k,s_0}(\Omega,G) = \left\langle \mathrm{Exp}_{(k,s_0)}(BC^{k,s_0}(\Omega,\mathfrak{g}))\right\rangle$ such that

$$\mathrm{Exp}_{(k,s_0)}|_U\colon U \subseteq BC^{k,s_0}(\Omega,\mathfrak{g}) \longrightarrow \left\langle \mathrm{Exp}_{(k,s_0)}(U)\right\rangle$$

is a $C^\omega_{\mathbb{K}}$-diffeomorphism. □

Theorem 5.3.12 (Lie groups associated with Hölder continuous functions ((LB)-case))**.** *Let $(k,s) \in \mathbb{N}_0 \times [0,1[$ be given.*

(a) There exists a unique Lie group structure on the group

$$BC^{k,>s}(\Omega,G) := \bigcup_{t\in]s,1]} BC^{k,t}(\Omega,G)$$

such that

$$\begin{aligned} \mathrm{Exp}_{(k,>s)} := \textstyle\bigcup_{t\in]s,1]} \mathrm{Exp}_{(k,s)} : BC^{k,>s}(\Omega,\mathfrak{g}) &\longrightarrow BC^{k,>s}(\Omega,G) \\ \gamma &\longmapsto \exp_G \circ \gamma \end{aligned}$$

is a local diffeomorphism around 0*.*

(b) The Lie group $BC^{k,>s}(\Omega,G)$ is strongly C^0-regular.

Proof. (a) We wish to use Theorem 4.1.1. Let $(t_n)_{n\in\mathbb{N}}$ be a strictly decreasing cofinal sequence in $]s,1]$, e. g. $t_n := s + (1-s)\cdot\frac{1}{n}$. For every $n \in \mathbb{N}$, set $G_n := BC^{k,t_n}(\Omega,G)$. The bonding maps $j_n\colon G_n \longrightarrow G_{n+1}$ are group homomorphisms. Since $j_n \circ \mathrm{Exp}_{(k,t_n)} = \mathrm{Exp}_{(k,t_{n+1})} \circ i_n$ with the continuous linear inclusion map $i_n\colon BC^{k,t_n}(\Omega,\mathfrak{g}) \longrightarrow BC^{k,t_{n+1}}(\Omega,\mathfrak{g})$, we see that each j_n is analytic with $\mathbf{L}(j_n) = i_n$.

Like in the proof of Theorem 5.3.11, we choose the norm on $\mathfrak{g}$ such that

$$\|[x,y]\|_{\mathfrak{g}} \leq \frac{1}{C_k} \|x\|_{\mathfrak{g}} \|y\|_{\mathfrak{g}} \text{ for } x,y \in \mathfrak{g}.$$

Note that this is possible because $k \in \mathbb{N}_0$ is fixed and the C_k do not depend on s. This implies that the Lie brackets on the Lie algebras $BC^{k,t_n}(\Omega,\mathfrak{g})$ and the bounded operators $i_n \colon BC^{k,t_n}(\Omega,\mathfrak{g}) \longrightarrow BC^{k,t_{n+1}}(\Omega,\mathfrak{g})$ have operator norm at most 1.

The locally convex direct limit is Hausdorff by Proposition 5.3.10, and the exponential map $\mathrm{Exp} = \bigcup_{t\in]s,1]} \mathrm{Exp}_{(k,t)}$ is injective on the 0-neighborhood $\bigcup_{t\in]s,1]} \mathrm{B}_{\varepsilon_\circ}^{BC^{k,t}(\Omega,\mathfrak{g})}(0)$. Hence, by Theorem 4.1.1, there is a unique complex analytic Lie group structure on G such that Exp is a local diffeomorphism at 0. (b) By Proposition 5.3.10, the modelling space is a compactly regular direct limit. So, we can apply Theorem 4.3.2 and obtain the result. □

5.4 Lie groups associated to ℓ^p-Spaces

This construction follows the same idea as the one in Sections 5.2 and 5.3.

Let G be a Banach Lie group with Lie algebra $\mathfrak{g}$. We fix a compatible norm $\|\cdot\|_{\mathfrak{g}}$ on $\mathfrak{g}$. For every $p \in [1,+\infty[$, we define on the Banach space

$$\ell^p(\mathbb{N},\mathfrak{g}) := \left\{ f\colon \mathbb{N} \longrightarrow \mathfrak{g} : \|f\|_p := \left(\sum_{j\in\mathbb{N}} \|f(j)\|_{\mathfrak{g}}^p \right)^{1/p} < \infty \right\}$$

the pointwise Lie bracket:

$$[f,g](n) := [f(n),g(n)]_{\mathfrak{g}}.$$

In order to include the case $p=\infty$, we set $\ell^\infty(\mathbb{N},\mathfrak{g})$ to be the Banach space of bounded functions from $\mathbb{N}$to $\mathfrak{g}$, together with the ordinary sup-norm and the pointwise Lie bracket.

Lemma 5.4.1 (Inclusion operators). *If $1 \leq p < q \leq \infty$, we have*

$$\ell^p(\mathbb{N},\mathfrak{g}) \subseteq \ell^q(\mathbb{N},\mathfrak{g})$$

and the inclusion operator

$$\ell^p(\mathbb{N},\mathfrak{g}) \longrightarrow \ell^q(\mathbb{N},\mathfrak{g}) : f \mapsto f$$

has operator norm at most 1.

Proof. It is clear that $\|f(j)\|_{\mathfrak{g}} \leq \|f\|_p$ for all $j \in \mathbb{N}$. Hence, $\|f\|_\infty \leq \|f\|_p$ which proves the case $q = \infty$.

For the case $q < \infty$, let $f \in \ell^p(\mathbb{N}, \mathfrak{g})$ with $\|f\|_p = 1$ be given.

$$\begin{aligned}\|f\|_q^q &= \sum_{j\in\mathbb{N}} \|f(j)\|_{\mathfrak{g}}^q \\ &= \sum_{j\in\mathbb{N}} \|f(j)\|_{\mathfrak{g}}^p \cdot (\underbrace{\|f(j)\|_{\mathfrak{g}}}_{\leq \|f\|_p})^{q-p} \\ &\leq \|f\|_p^p \, \|f\|_p^{q-p} \leq 1.\end{aligned}$$ □

Lemma 5.4.2 (Generalized Hölder inequality)**.** *Let $p, q, r \in [1, +\infty[$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $f \in \ell^p(\mathbb{N}, \mathbb{K})$ and $g \in \ell^q(\mathbb{N}, \mathbb{K})$. Then $fg \in \ell^r(\mathbb{N}, \mathbb{K})$ and*

$$\|fg\|_r \leq \|f\|_p \, \|g\|_q \, .$$

Proof. This is just the usual Hölder inequality applied to the functions $|f|^r$ and $|g|^r$. □

Lemma 5.4.3 (Interval lemma)**.** *Let $0 < s < t < u \leq \infty$. Let $f \in \ell^s(\mathbb{N}, \mathbb{K}) \cap \ell^u(\mathbb{N}, \mathbb{K})$. Then $f \in \ell^t(\mathbb{N}, \mathbb{K})$ and*

$$\sum_j \|f(j)\|_{\mathfrak{g}}^t \leq \|f\|_s^\alpha \cdot \|f\|_u^\beta \, .$$

with

$$\alpha := s \cdot \frac{u-t}{u-s} \text{ and } \beta := (t-s) \cdot \frac{u}{u-s}, \text{ where } \frac{\infty}{\infty} := 1.$$

Proof. If $u = \infty$ this is easily verified. For $u < \infty$, the proof becomes more technical: Since $\alpha + \beta = t$, we can write

$$|f|^t = |f|^\alpha \cdot |f|^\beta.$$

If we set

$$p := \frac{s}{\alpha} \text{ and } q := \frac{u}{\beta}$$

one easily checks that $1/p + 1/q = 1$ and that we have

$$\||f|^\alpha\|_p = (\|f\|_s)^\alpha \text{ and } \left\||f|^\beta\right\|_q = (\|f\|_u)^\beta.$$

Thus, we calculate using the ordinary Hölder inequality:

$$\begin{aligned}
\|f\|_t &= \left\||f|^t\right\|_1^{\frac{1}{t}} \\
&= \left\||f|^{\alpha}|f|^{\beta}\right\|_1^{\frac{1}{t}} \\
&\leq \left(\left\||f|^{\alpha}\right\|_p \cdot \left\||f|^{\beta}\right\|_q\right)^{\frac{1}{t}} \\
&= \left((\|f\|_s)^{\alpha} \cdot (\|f\|_u)^{\beta}\right)^{\frac{1}{t}} \\
&= \|f\|_s^{\alpha/t} \cdot \|f\|_u^{\beta/t}.
\end{aligned}$$

□

Remark. More generally, the Lemmas 5.4.2 and 5.4.3 hold also for Banach spaces of the type $L^p(\Omega, \mathfrak{g})$ for a measure space Ω. However, the next lemma is no longer true if one replaces $\ell^p(\mathbb{N}, \mathfrak{g})$ by $L^p(\Omega, \mathfrak{g})$.

Lemma 5.4.4. $(\ell^p(\mathbb{N}, \mathfrak{g}), [,])$ *is a Banach Lie algebra.*

Proof. It is well-known that $\ell^p(\mathbb{N}, \mathfrak{g})$ is a Banach space. It remains to show that for $f, g \in \ell^p(\mathbb{N}, \mathfrak{g})$, we have $[f, g] \in \ell^p(\mathbb{N}, \mathfrak{g})$ and $\|[f, g]\|_p \leq \|f\|_p \|g\|_p$.

$$\begin{aligned}
\|[f, g]\|_p &= \left\|\left(\|[f, g](n)\|_{\mathfrak{g}}\right)_{n\in\mathbb{N}}\right\|_p = \left\|\left(\|[f(n), g(n)]\|_{\mathfrak{g}}\right)_{n\in\mathbb{N}}\right\|_p \\
&\leq \left\|\left(\|f(n)\|_{\mathfrak{g}} \|g(n)\|_{\mathfrak{g}}\right)_{n\in\mathbb{N}}\right\|_p. \\
&\leq \|f\|_{2p} \|g\|_{2p}
\end{aligned}$$

by the generalized Hölder inequality (Lemma 5.4.2). Since the inclusion maps $\ell^p(\mathbb{N}, \mathfrak{g}) \longrightarrow \ell^{2p}(\mathbb{N}, \mathfrak{g}) : f \mapsto f$ have operator norm at most 1 (Lemma 5.4.1), the assertian follows. □

Proposition 5.4.5 (The space $\ell^{<p}$)**.** *Let* $p \in]1, \infty]$. *Then the locally convex direct limit*

$$\ell^{<p}(\mathbb{N}, \mathfrak{g}) := \bigcup_{s<p} \ell^s(\mathbb{N}, \mathfrak{g})$$

is a compactly regular, Hausdorff and becomes a complete locally convex topological Lie algebra with the pointwise Lie bracket.

Proof. By Lemma 5.4.1 each inclusion $\ell^s(\mathbb{N}, \mathfrak{g}) \longrightarrow \ell^p(\mathbb{N}, \mathfrak{g}) : f \mapsto f$ is continuous. Therefore, by the universal property of the locally direct limit, the inclusion operator

$$\ell^{<p}(\mathbb{N}, \mathfrak{g}) \longrightarrow \ell^p(\mathbb{N}, \mathfrak{g}) : f \mapsto f$$

is continuous linear and injective with values in a Banach space. Hence, the space $\ell^{<p}(\mathbb{N}, \mathfrak{g})$ is Hausdorff.

To show compact regularity, we use the criterion in Proposition 1.2.4. Henceforth, let $s < t < u < p$ be given and set $\Omega := \mathrm{B}_1^{\ell^s}(0)$.

Let $f \in \Omega$ and apply Lemma 5.4.3:

$$\|f\|_t \leq \underbrace{\|f\|_s^{\alpha/t}}_{\leq 1} \cdot \|f\|_u^{\beta/t}.$$

This shows that $\ell^t(\mathbb{N}, \mathfrak{g})$ and $\ell^u(\mathbb{N}, \mathfrak{g})$ induce the same topology on Ω. Thus, Proposition 1.2.4 ensures that the direct limit is compactly regular and complete.

Now, Corollary 2.2 ensures the continuity of the Lie bracket on the locally convex direct limit. This finishes the proof. $\square$

We can use the same arguments as in Theorems 5.2.4 and 5.2.5 to obtain the following two theorems:

Theorem 5.4.6 (Lie groups associated with ℓ^p-spaces (Banach case))**.** *For $p \in [1, \infty]$ there exists a unique Banach Lie group structure on the group*

$$\ell^p(\mathbb{N}, G) := \langle \{\exp_G \circ f : f \in \ell^p(\mathbb{N}, \mathfrak{g})\} \rangle \leq G^{\mathbb{N}}$$

such that

$$\mathrm{Exp}_p \colon \ell^p(\mathbb{N}, \mathfrak{g}) \longrightarrow \ell^p(\mathbb{N}, G) : f \mapsto \exp_G \circ f$$

becomes a local diffeomorphism around 0*.*

Theorem 5.4.7 (Lie groups associated with ℓ^p-spaces ((LB)-case))**.** *Let $p \in]1, \infty]$.*

(a) On the group

$$\ell^{<p}(\mathbb{N}, G) := \bigcup_{s<p} \ell^s(\mathbb{N}, G) = \langle \{\exp_G \circ f : f \in \ell^s(\mathbb{N}, \mathfrak{g}), s < p\} \rangle$$

there is a unique Lie group structure turning

$$\mathrm{Exp} := \bigcup_{s<p} \mathrm{Exp}_s \colon \ell^{<p}(\mathbb{N}, \mathfrak{g}) \longrightarrow \ell^{<p}(\mathbb{N}, G) : f \in \ell^{<p}(\mathbb{N}, \mathfrak{g}) \mapsto \exp_G \circ f$$

into a local diffeomorphism around 0*.*

(b) This Lie group is strongly C^0-regular.

Proof of Theorems 5.4.6 and 5.4.7. By Lemma 5.4.4, we know that each space $\ell^p(\mathbb{N}, \mathfrak{g})$ is a Banach Lie algebra with respect to pointwise operations. Therefore, we can copy the proof of Theorem 5.2.4 and obtain the Lie group structure on the $\ell^p(\mathbb{N}, G)$. This proves Theorem 5.4.6.

In order to prove Theorem 5.4.7 (a), we fix a strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to p$ and set $G_n := \ell^{s_n}(\mathbb{N}, G)$.

The inclusion j_n maps between the corresponding Lie algebras are continuous linear maps by Lemma 5.4.1. This implies that the bonding maps between the Lie Groups $j_n \colon G_n \longrightarrow G_{n+1}$ are analytic, since $j_n \circ \mathrm{Exp}_n = \mathrm{Exp}_{n+1} \circ i_n$. The inclusion maps and the Lie algebras have operator norm at most 1 by Lemma 5.4.4 and Lemma 5.4.1. The locally convex direct limit $\ell^{<p}(\mathbb{N}, \mathfrak{g})$ is Hausdorff by Propostion 5.4.5. The exponential map $\mathrm{Exp} = \bigcup_{s<p} \mathrm{Exp}_s$ is injective on the neighborhood $\bigcup_{s<p} \mathrm{B}_{\varepsilon_\circ}^{\ell^s(\mathbb{N},\mathfrak{g})}(0)$, where $\varepsilon_\circ > 0$ is chosen such that $\mathrm{Exp}_G|_{\mathrm{B}^{\mathfrak{g}}_{\varepsilon_\circ}(0)}$ is a diffeomorphism onto its image. Hence, by Theorem 4.1.1, there is a unique complex analytic Lie group structure on $\ell^{<p}(\mathbb{N}, G)$ such that Exp is a local diffeomorphism at 0.

(b) By Proposition 5.4.5, the modelling space is a compactly regular direct limit. So, we can apply Theorem 4.3.2 and obtain the result. $\square$

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Notation