UNIVERSITÄT PADERBORN
Die Universität der Informationsgesellschaft

Existence and Properties of Pure Nash Equilibria in Budget Games

Dissertation

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)
an der Fakultät für Elektrotechnik, Informatik und Mathematik
der Universität Paderborn

vorgelegt von

Maximilian Drees

Paderborn, April 2016

Betreuer:

Jun.-Prof. Dr. Alexander Skopalik	(Universität Paderborn)

Gutachter:

Jun.-Prof. Dr. Alexander Skopalik	(Universität Paderborn)
Prof. Dr. Friedhelm Meyer auf der Heide	(Universität Paderborn)

Weitere Mitglieder der Promotionskommission:

Prof. Dr. Christian Scheideler	(Universität Paderborn)
Prof. Dr. Burkhard Hehenkamp	(Universität Paderborn)
Dr. Rainer Feldmann	(Universität Paderborn)

Datum der mündlichen Prüfung:

27. Juni 2016

Danke!

Ich hatte das große Glück, im Laufe meiner Promotion viel Unterstützung durch alte wie neue Freunde zu erfahren. Diesen Personen möchte ich an dieser Stelle Danken.

Besonderer Dank geht an meinen Betreuer Jun.-Prof. Dr. Alexander Skopalik, welcher mich erst auf das Thema der algorithmischen Spieltheorie gebracht hat. Er stand mir immer mit guten Ratschlägen zur Seite und ermutigte mich stets, mit Ausdauer auch an schwierigen Problemen fest zu halten. Gleichzeitig ließ er mir die Freiheit, meine Arbeit und Ziele selbst zu gestalten.

Ebenfalls danken möchte ich Dr. Patrick Briest, welcher mich ursprünglich ins Team holte, und Prof. Dr. Friedhelm Meyer auf der Heide, der nach Patrick ganz selbstverständlich für einige Monate meine Betreuung übernahm.

Das Arbeitsklima an der Universität Paderborn war immer ausgesprochen angenehm. Nicht nur konnte ich mich jederzeit auf die Unterstützung meiner Kollegen verlassen, mit vielen von ihnen habe habe ich mich auch gerne noch nach Feierabend getroffen. Ich freue mich sehr, in den letzten Jahren so viele neue Freunde kennen gelernt zu haben.

Meinen Eltern Beate und Werner Drees möchte ich für ihre kontinuierliche Unterstützung danken, nicht nur in Bezug auf die Promotion. Das gleiche gilt für meine Freundin Kathrin Alke, die mir gerade in schwierigen und arbeitsreichen Phasen immer den Rücken freigehalten hat.

Maximilian Drees
Paderborn, 28. April 2016

Zusammenfassung

In der vorliegenden Dissertation führen wir das spieltheoretische Modell von Budgetspielen ein und analysieren dessen Eigenschaften in Bezug auf reine Nash Gleichgewichte. In einem Budget Game konkurieren die Spieler um Ressourcen, welche ein begrenztes Budget haben. Als Strategie wählt ein Spieler einen von endlichen vielen Bedarfsvektoren. Jeder Bedarfsvektor enthält einen nicht-negativen Bedarf für jede Ressource. Wenn der Gesamtbedarf aller Spieler an eine einzige Ressource nicht deren Budget überschreitet, so entspricht der Gewinn jedes Spielers durch diese Ressource seinem Bedarf. Andernfalls wird das Budget zwischen allen Spielern proportional zu ihren Bedarf aufgeteilt. Für jede Kombination von Spieler und Ressourcen ist der Bedarf direkt an die Strategie des Spielers gebunden und kann sich daher während der Best-Response Dynamik verändern. Wir zeigen, dass reine Nash Gleichgewicht im Allgemeinen nicht in Budgetspielen existieren und betrachten daher verschiedene alternative Konzepte.

Geordnete Budgetspiele stellen eine Variante von Budgetspielen da, welche die Reihenfolge hervor heben, in der die Spieler sich für ihre Strategie entscheiden. Diese Spiele sind exakte Potentialspiele, für welche sogar die Existenz von superstarken reinen Nash Gleichgewichte garantiert werden kann und für die starke reine Nash Gleichgewichte effizient berechnet werden können.

In einem α-approximativem reine Nash Gleichgewicht existieren keine einseitigen Strategiewechsel, welche den Gewinn des jeweiligen Spielers um mehr als einen konstanten Faktor $\alpha > 1$ erhöhen. Für viele Anwendungen stellen diese eine realistischere Alternative zu reinen Nash Gleichgewicht dar. Wir geben obere und untere Schranken für α an, so dass α-approximative reine Nash Gleichgewichte für Budgetspiele garantiert werden können. Darüber hinaus betrachten wir eine approximative Version der Best-Response Dynamik, welche unter bestimmten Bedingungen schnell konvergiert und welche dazu verwendet werden kann, die optimale Lösung, also das Strategieprofil mit maximalen sozialem Wohlstand, zu approximieren.

Durch die Einschränkung der Struktur der Strategieräume stellen wir reine Nash Gleichgewichte für bestimmte Klassen von Budgetspielen wieder her. Wir konzentrieren uns auf Singleton und Matroid Budgetspiele. In einem Singleton Budgetspiel verwendet ein Spieler zu jedem Zeitpunkt nur eine Ressource. In einem Matroid Budgetspiel kann jeder Strategiewechsel in eine Folge von kleineren zerlegt werden, welche alle gültig sind und jeweils den Gewinn des Spielers durch Austauschen einer einzelnen Ressource erhöhen.

Wir zeigen auch, dass die Berechnung der optimalen Lösung sowohl von Budgetspielen als auch geordneten Budgetspielen äquivalent und in beiden Fällen NP-schwer ist.

Abstract

In this thesis, we introduce the game theoretical model of budget games and analyze their properties regarding pure Nash equilibria. In a budget game, players compete over resources, which have a limited budget. As his strategy, each player decides between a finite number of demand vectors. Each demand vector contains one non-negative demand for every resource. Provided the total demand by all players on a single resource does not exceed its budget, the utility each player receives from that resource equals his demand. Otherwise, the budget is split between all players proportionally to their demands. For any combination of player and resource, the corresponding demand is directly tied to the players strategy and can therefore change during the best-response dynamic. After showing that pure Nash equilibria generally do not exist in budget games, we consider several alternative concepts.

Ordered budget games are a variation of budget games, which emphasize the order in which the players choose their strategies. These games are exact potential games for which even the existence of super-strong pure Nash equilibria can be guaranteed and strong pure Nash equilibria can be computed efficiently.

In an α-approximate pure Nash equilibrium, no unilateral strategy change increases the utility of the corresponding player by more than some constant factor $\alpha > 1$. For many applications, these are a more realistic compared to the concept of pure Nash equilibria. We give upper and lower bound on α such that α-approximate pure Nash equilibria are guaranteed in budget games. In addition, we look at an approximate version of the best-response dynamic, which converges quickly under certain conditions and can also be used to approximate the optimal solution, i.e. the strategy profile which maximizes social welfare.

By restricting the structure of the strategy spaces, we restore pure Nash equilibria to certain classes of budget games. We focus on singleton and matroid budget games. In a singleton budget game, a player uses only one resource at a time. In a matroid budget game, every strategy change can be decomposed into a sequence of smaller ones which are still valid an with each already increasing the utility of the player by swapping only one resource for another.

We also argue that computing the optimal solution is equivalent for both budget games and ordered budget games and NP-hard in both cases.

Contents

CHAPTER 1

Introduction

In the real world, the total payoff obtainable from a system is often independent of the number of its participants. For example, the computational capacity of a server is usually fixed and does not grow with the number of requests. Assume a server is being used by various clients. By increasing their number, the workload on the server grows. If this workload exceeds the capacity of the server, then it is no longer possible to fully satisfy every client. Instead, each client receives a share of the servers capacity, which naturally is at most the amount actually needed to satisfy his requests. We can regard this share as a form of utility which the client receives from the server.

A similar situation can be observed in markets for specific goods or services. A market can provide only a limited amount of sales to its providers before the consumers are satisfied. Even if more providers enter the market, the spending power of the consumers remains the same and if their number grows too large, then some of them can no longer receive the desired revenue. In this context, it becomes even more natural to regard the market share of a provider as his utility.

We study such scenarios in a game theoretic setting. "*Game theory aims to model situations in which multiple participants interact or affect each other's outcomes*" [26]. The participants are usually called players. Regarding the previous examples, these are the clients and providers while the servers and markets can be considered as resources with a limited budget (computational capacity, spending power). We take another look at our initial example. If there are multiple servers and a client can choose the server(s) to handle his requests, then this choice is called his strategy. The important aspect in game theory is that a player can only control his own strategy, but the outcome he experiences depends on the strategies of all players. In our context, a client may wish to use a certain server combination, but if these servers are also working on too many other requests, then he does not receive the full computational power needed and his utility decreases. In general,

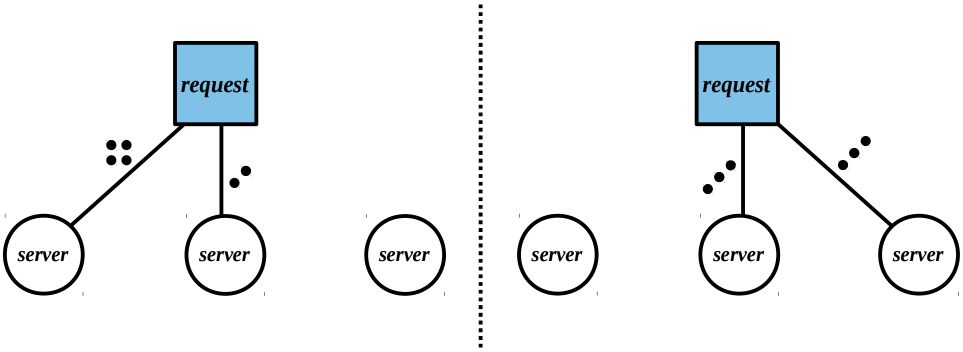

Figure 1.1: Two examples of a request being allocated to a set of three servers. The request is composed of six services. Assuming the computational power needed for each service is the same (e.g. 1), the two allocations impose different demands on the servers: $(4, 2, 0)$ and $(0, 3, 3)$. Due to constraints resulting from the type of the services or the servers, other allocations may not be possible.

we assume a strategy to be a vector of non-negative demands on the resources. Note that not every allocation of the servers may be feasible for a given client. If a request is composed of atomic services, then a client can split these services among several servers, but each individual service has to be located as a whole on a single server. An allocation corresponds to a non-negative demand for computational power on each server and these demands can change with a different strategy. See Figure 1.1 for an example.

The impact of the client on a server depends on his current strategy and can change with it. This property of strategy-dependent demands also holds for our second example concerning markets. Assume there is more than one market and each provider has to make the decision of how much of his goods or services to offer in each of them. While it is technically possible to choose the supply of any individual market freely, this may not be economic due to investment costs. In such cases, a provider may only decide between a small number of different allocations.

We introduce a new game theoretic model called budget games, which captures the two main properties discussed so far: budget-restricted resources and strategy-dependent demands. In a budget game, each player tries to maximize his own utility through the choice of his strategy. This utility is defined as the sum of the individual utilities he receives from the different resources and the utility obtained from a resource may never be larger than his current demand on it. If the total demand of all players on a resource does not exceed its budget, then the utilities of the players are exactly their demands. Otherwise, some or all players receive less.

We are interested in the effects of rational decision making by the individual players. For any given situation, the strategy of one or more players may not be optimal, i.e. a player can increase his utility on his own through a unilateral strategy change. This creates a best-response dynamic in which one strategy change can lead to another. However, there may also be situations in which every player has already optimized his utility, given the strategies of the other players are fixed. Such states are called pure Nash equilibria and their existence and properties in budget games are the central topic of this thesis. Pure Nash equilibria are states in which no player wants to unilaterally deviate from his current strategy, as this would yield no benefit (i.e. an increase in his utility) under the given situation. They can be used to predict the outcome of a game with rational players, since each player will consider the strategy choices of the others before deciding on his own strategy. However, one of our first results will show that in general, there are no pure Nash equilibria in budget games to begin with. Therefore, we consider at three alternative concepts.

1. Ordered Budget Games

The first is a variation of budget games called ordered budget games, which are no longer strategic. In a strategic game, the outcome of multiple strategy changes by different players only depends on the resulting state. Ordered budget games, on the other hand, emphasize the order of the player decisions and the final outcome strongly depends on it. While strategic games are often analyzed as one-shot games, they often do not capture situations like a new provider entering a market having a disadvantage against those already established due to the consumers prioritizing products they already know. In ordered budget games, this is different. When a player changes his strategy, the utility he receives from a resource is at most its remaining budget. As a result, other players using that resource are not affected. To keep the idea of a strategy change attractive, we introduce the concept of tasks, which allow a player to keep his utility from resources being used in both the old and new strategy. We are going to see that these games always have a pure Nash equilibrium.

The way the budget of a resource is shared between the players in our original model (in contrast to ordered budget games) can be called proportional utility sharing. Simply put, the larger the demand of a single player on a resource, the larger his share of that resource's budget. If the total demand on a resource exceeds its budget, we compute the shares of the individual players using this simple formula:

$$\text{budget} \cdot \frac{\text{demand of the player}}{\text{total demand on the resource}}.$$

This approach is not always optimal. There are results for models similar to ours which show that proportional cost sharing does not guarantee the existence of a pure Nash equilibrium (cf. Chapter 3.2). For other cost sharing mechanisms, e.g. based on the Shapley value, this is different. However, we do not approach this from a mechanism design angle. Instead, we consider this system and especially the structure of the utility functions as given and are interested in its strength and weaknesses, mainly regarding pure Nash equilibria. Since they do not exist in general, we weaken this concept, which brings us to the second alternative concept.

2. Approximate Pure Nash Equilibria in Budget Games

In an α-approximate pure Nash equilibrium, no unilateral strategy change yields an increase by more than some constant factor $\alpha > 1$ in the corresponding player's utility. In many applications, this concept is more realistic than the classic notion of (exact) pure Nash equilibria, which can also be considered as α-approximate pure Nash equilibrium for $\alpha = 1$. Usually, players are not interested in marginal increases in their utilities, especially if this could destabilize the current state of the game. In such cases, the guarantee of a certain utility weighs larger than the possibility of some small improvement.

While not every budget game has a pure Nash equilibrium, there is also a large number of instances for which they do exist. To conclude this thesis, we ask which kind of restrictions on budget games are necessary in order to obtain exact equilibria.

3. Pure Nash Equilibria in Singleton and Matroid Budget Games

When restricting the class of budget games, our focus lies on the structure of the strategies and we consider both singleton and matroid budget games. In a singleton budget game, a player uses only a single resource at a time. In other words, every strategy vector contains a demand of 0 for all resources except one. However, this does not necessarily imply that the demand of a player is fixed. Instead, it may vary from resource to resource, modeling differences between the resources aside from their budgets. In matroid budget games, all strategies of a given player use the same number of resources and every strategy change can be decomposed into a sequence of very simple strategy changes, with each switching only one resource for another. The player could then omit every single of these simple changes and the resulting strategy change would still be a valid option for him, implying that each simple strategy change alone already increases his utility. Matroid budget games are an extended form of singleton budget games.

Besides the existence of (approximate) pure Nash equilibria, we are also interested in their properties. Although no player can increase his utility by his own, this does not mean that an equilibrium is also economical and utilizes the resources and obtains as much of the resources' budgets as possible. We measure the efficiency of an equilibrium by comparing its social welfare, i.e. the total utility of all players, with the largest social welfare of all strategy profiles and are mainly interested in the worst-case ratio, called the price of anarchy.

The best-response dynamic already mentioned is a natural way for the game to progress. We consider if it always leads to an equilibrium and how many strategy changes this takes.

Restriction	**Strategy Space Structure** Singleton	Matroid	General
fixed demands	FIP	FIP	no NE
2 resources	WA	WA	?
ordered players & increasing demand ratios	NE	?	no NE
2 demands, 1 budget	WA	?	no NE

Table 1.1: Overview of our results regarding pure Nash equilibria in strategic budget games. FIP stands for *finite improvement property*, WA for *weakly acyclic* and NE for *pure Nash equilibrium.*

We stated that Nash equilibria can be used to predict the outcome of a game. Of course, this only applies if the players are actually able to compute them. Therefore, we also consider the complexity of (approximate) pure Nash equilibria and give efficient algorithms for some of them.

1.1 Result Overview

We introduce the model of budget games and its non-strategic variant called ordered budget games. Our first result shows that pure Nash equilibria generally do not exist in budget games. Ordered budget games, on the other hand, are potential games for which even the existence of super-strong pure Nash equilibria can always be guaranteed. While strong pure Nash equilibria can be found efficiently, the computation of super-strong pure Nash equilibria is NP-hard.

Regarding α-approximate Nash equilibria in general budget games, we introduce an approximate potential function and use it to give both upper and lower bounds on α. Below the lower bounds, finding any approximate pure Nash equilibrium is NP-hard. While these bounds depend on the relative size of the players' demands (compared to the resources' budgets), with smaller demands yielding better bounds, we also give upper bounds for matroid budget games using a different approach. This time, the results get better with decreasing ratios between the demands of each individual player.

For pure Nash equilibria in singleton and matroid budget games, we consider a number of different cases. Our results are summarized in Table 1.1. If the demands of the players are fixed, i.e. do not depend on their strategy, both singleton and matroid games have the finite improvement property. This means that the players will eventually form a pure Nash equilibrium by following the best-response dynamic. If the game consists of only two resources, then it is at least weakly acyclic, implying that the best-response dynamic can always reach an equilibrium if the strategy changes of the players are performed in the right order. Singleton budget games with more than two resources still have pure Nash equilibria if (a) the order over the demands of the players is the same for each resource and the ratios between the different demands are smaller for players with larger demands or (b) all budgets are the same and every demand in the game is one of two values. In the second case, the games are even weakly acyclic.

We also consider the efficiency of (approximate) pure Nash equilibria in budget games. For ordered budget games, the price of stability is 1 while the price of anarchy is 2. In strategic budget games, the price of anarchy of an α-approximate pure Nash equilibrium is at most $\alpha + 1$. In particular, the price of anarchy of a pure Nash equilibrium is 2. All these bounds are tight.

The time it takes the best-response dynamic in ordered budget games to reach a pure Nash equilibrium can be exponential in the games description length. For budget games, we consider an approximate version of this dynamic, which only consists of strategy changes increasing the utility of the corresponding player by more than some constant approximation factor $\alpha > 1$. If α is chosen large enough and the highest demands of the player do not differ too much from each other, this approximate best-response dynamic converges quickly towards approximate pure Nash equilibria.

Computing the strategy profile which maximizes the social welfare is NP-hard for both strategic and ordered budget games. It can be approximated up to the constant factor of $1 - \frac{1}{e}$ for matroid games in linear time and up to a factor of $\mathcal{O}\left(\frac{1-\varepsilon}{\alpha^2 \cdot \log(n)}\right)$ in time $\mathcal{O}\left(n \cdot \alpha \cdot \log \frac{1}{\varepsilon}\right)$ for the general case, with n denoting the number of players.

1.2 Structure of this Thesis

The main focus of this thesis lies on the introduction of budget games and the question under which properties they possess pure Nash equilibria. A formal definition of the model of budget games and other important concepts is given in Chapter 2. Chapter 3 gives an overview of existing literature linked to this thesis. This mainly concerns congestion games, the research field which is has the strongest connection to budget games. In Chapter 4, the foundation for our further research is laid, as we prove that budget games generally do not have pure Nash equilibria. In addition, we show that the problems of computing the strategy profile with the highest social welfare for budget games and ordered budget games are equivalent and NP-hard. The variation of ordered budget games is being analyzed in detail in Chapter 5. Chapter 6 contains our results regarding approximate pure Nash equilibria in budget games. In Chapter 7, we consider the existence of pure Nash equilibria in different forms of singleton and matroid budget games. A last result deals with approximate pure Nash equilibria in such games. The thesis is concluded with Chapter 8, where we reflect our results and discuss some open problems.

CHAPTER 2

Model

This chapter gives a formal introduction into the model of budget games as well as into the other concepts used throughout this thesis. We start with the general notions of game theory. With these fundamentals, we are able to define both budget games and ordered budget games before we conclude with some minor concepts from mathematics.

2.1 General Concepts of Game Theory

We begin this section with a number of general definitions from game theory.

Definition 2.1 (Finite Game)**.** A *finite game* $\mathcal{G}$ is a tuple $(\mathcal{N}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ which consists of the following components:

- a set of players $\mathcal{N} = \{1, \ldots, n\}$
- a finite set $\mathcal{S}_i$ for each player i,

  called strategy space of i with $s_i \in \mathcal{S}_i$ being a strategy of i
- a target function $u_i : \mathcal{S}_1 \times \ldots \times \mathcal{S}_n \to \mathbb{R}$ for each player i

The state of a finite game is described by a combination of strategies, one for each player.

Definition 2.2 (Strategy Profile)**.** For a finite game $\mathcal{G}$, we call $\mathsf{s} \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ a *strategy profile* of $\mathcal{G}$.

The term *finite* comes from the fact that these games only have a finite number of strategy profiles (or states). While there are also games which are not finite (e.g. with $\mathcal{S}_i = \mathbb{R}$ for some or all i), we only consider finite games in this thesis. We assume our players to be both selfish and rational. Here, *selfish* describes the property that a player is only interested in the value of his own target function, which he tries to optimize. His only form of influencing this outcome is the choice of one of his strategies from his own strategy space. However, he cannot directly control the choices made by the other players. Since his target function not only depends on his strategy, but on the strategy profile as a whole, the ability of a single player to determine his individual outcome is limited. Given the strategies of the other players, he can only pick the strategy which optimizes his target function under the present conditions. The players act *rational*, as the value of their target functions is the only decisive factor when choosing a strategy.

The target functions are usually considered to be either cost functions or utility functions. Players try to minimize the first and maximize the latter. In the context of budget games, we consider only utility functions. So if $u_i(\mathsf{s}') < u_i(\mathsf{s})$, then player i prefers the strategy profile s over s'. From now on, we always assume the target functions to be utility functions.

By default, we expect a strategy profile to be complete, i.e. to list a strategy for every player in the game. In some situations, however, it can be helpful to consider only a partial strategy profile. This comes in handy when we want to express the deviation of one or more players from a given strategy profile. For a strategy profile s, we write s_{-i} to denote the same strategy profile excluding the strategy of player i. Furthermore, if s'_i is a strategy of player i, we write (s_{-i}, s'_i) to denote the strategy profile in which player i chooses s'_i and all other players have the same strategy as in s. The same notation applies for any coalition $\mathcal{C} \subseteq \mathcal{N}$ and we use $(\mathsf{s}_{-\mathcal{C}}, \mathsf{s}'_{\mathcal{C}})$ to combine the two strategy profiles $\mathsf{s}_{-\mathcal{C}}$ and $\mathsf{s}'_{\mathcal{C}}$.

An important kind of strategy profiles are pure Nash equilibria, which also represent the central research focus of our work.

Definition 2.3 (Pure Nash Equilibrium)**.** Let s be a strategy profile of a finite game $\mathcal{G}$. If for every player $i \in \mathcal{N}$ and every strategy $s'_i \in \mathcal{S}_i$ it holds that

$$u_i(\mathsf{s}_{-i}, s'_i) \leq u_i(\mathsf{s}),$$

then we call s a *pure Nash equilibrium.*

In a pure Nash equilibrium, no player has an incentive to change his strategy, as this would not increase his utility. Pure Nash equilibria can be considered as stable states of a finite game. Not every finite game has a pure Nash equilibrium, but assuming they exist and that they can be computed efficiently, they can be used to predict the behavior of the players. Being in a pure Nash equilibrium does not necessarily mean that there is no more room for improvement. There may still be a coalition $\mathcal{C}$ of two or more players who would ultimately benefit if they all change their strategies. To further strengthen the stable aspect of the equilibrium, we therefore introduce two more versions of pure Nash equilibria.

Definition 2.4 (Strong Pure Nash Equilibrium)**.** Let s be a strategy profile of a finite game $\mathcal{G}$. If for every coalition $\mathcal{C} \subseteq \mathcal{N}$ and every strategy profile $\mathsf{s}'_{\mathcal{C}} \in \times_{i \in \mathcal{C}} \mathcal{S}_i$, there is at least one player $i \in \mathcal{C}$ with

$$u_i(\mathsf{s}_{-\mathcal{C}}, \mathsf{s}'_{\mathcal{C}}) \leq u_i(\mathsf{s}),$$

then we call s a *strong pure Nash equilibrium.*

Definition 2.5 (Super-Strong Pure Nash Equilibrium)**.** Let s be a strategy profile of a finite game $\mathcal{G}$. If for every coalition $\mathcal{C} \subseteq \mathcal{N}$ and every strategy profile $\mathsf{s}'_{\mathcal{C}} \in \times_{i \in \mathcal{C}} \mathcal{S}_i$, there is at least one player $i \in \mathcal{C}$ with

$$u_i(\mathsf{s}_{-\mathcal{C}}, \mathsf{s}'_{\mathcal{C}}) < u_i(\mathsf{s}),$$

then we call s a *super-strong pure Nash equilibrium.*

For these two kinds of Nash equilibria, we assume that the players are able to form coalitions and cooperate, i.e. change their strategies in mutual agreement, so that they all benefit from it in the end. If a strategy profile is not a strong pure Nash equilibrium, there is a coalition of players such that they can all increase their utility when working together. Additionally, if it is not a super-strong pure Nash equilibrium, then it suffices that at least one player can increase his utility while the others receive the same amount as they do before the collective strategy change.

The central question of this thesis is the existence of pure Nash equilibria in budget games. One important result of game theory is that every finite game has a so-called mixed Nash equilibrium [61]. In a mixed Nash equilibrium, instead of choosing a single strategy deterministically, every player decides on a probability distribution over his strategy space. In this context, the utilities of the players become expected values. If for every player, any unilateral change in his probability distribution does not increase his expected utility, the state is called a mixed Nash equilibrium. We remark that, as we are going to see later in this chapter, budget games are also finite games and therefore always possess mixed Nash equilibria.

The prefix *pure* is used to distinguish pure Nash equilibria from mixed Nash equilibria. As we only consider pure Nash equilibria to begin with, we mostly omit the term *pure* and simply call them Nash equilibria from now on.

If a strategy profile is not a Nash equilibrium, then there is at least one player who can improve his utility through a unilateral strategy change. This action is also called an improving move.

Definition 2.6 (Improving Move)**.** Let $\mathsf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ be a strategy profile of a finite game $\mathcal{G}$ and $s'_i \in \mathcal{S}_i$ such that

$$u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, s'_i).$$

Then deviating from s_i to s'_i is an *improving move* for player i.

In many situations, there may be more than one improving move available to a single player. In this case, we expect the player to choose the one which yields the highest increase in utility (i.e. the *best* improving move).

Definition 2.7 (Best-Response)**.** Let s_{-i} be a strategy profile of a finite game $\mathcal{G}$ excluding player i and $\tilde{s}_i \in \mathcal{S}_i$. If

$$u_i(\mathsf{s}_{-i}, s'_i) \leq u_i(\mathsf{s}_{-i}, \tilde{s}_i)$$

for all $s'_i \in \mathcal{S}_i$, then $\tilde{s}_i$ is a *best-response* of i regarding s_{-i}.

Note that a player always has a best-response to the combined strategies of the other players, even if there is no improving move. In that case, his current strategy already is a best-response. The concept of improving moves induces a dynamic on finite games. Given an initial strategy profile s_0, we choose a single player i who can perform an improving move, which results in the strategy profile s_1. We can then repeat this process over and over, which will either stop when there are no more improving moves available (i.e. in a Nash equilibrium) or may go on for an infinite number of steps. As mentioned above, we expect the players to always play best-response and therefore call this process the *best-response dynamic*.

If multiple players are able to perform an improving move, we choose one of them according to some rule (deterministically or at random). After the improving move of this player, the resulting strategy profile is reevaluated to determine if there are still improving moves left. This tie-breaker rule directly influences the course of the best-response dynamic and it is possible for an initial strategy profile s_0 to both lead to a Nash equilibrium as well as create a cycle of strategy profiles, depending on how this rule operates.

Definition 2.8 (Weakly Acyclic)**.** Let $\mathcal{G}$ be a finite game such that for any strategy profile s_0, there exists a sequence $(\mathsf{s}_0, \ldots, \mathsf{s}_{\mathrm{NE}})$ such that

- for $t \in \{0, \ldots, \mathrm{NE} - 1\}$, s_{t+1} is obtained from s_t by a single improving move of one player
- s_{NE} is a Nash equilibrium

Then $\mathcal{G}$ is said to be *weakly acyclic*.

If a game is weakly acyclic, it is possible (but not guaranteed) to arrive at a Nash equilibrium after a finite number of improving steps. Even if the best-response dynamic enters a cycle, it is always possible to escape from it, thus not being trapped in an inescapable oscillation. Actually computing a Nash equilibrium of a weakly acyclic game, however, can be more difficult this way, since a simple execution of the best-response dynamic may not always terminate. This situation is different when cycles can be ruled out from the start.

Definition 2.9 (Exact Potential Games)**.** Let $\mathcal{G}$ be a finite game. If there is a function $\phi : \mathcal{S} \to \mathbb{R}$ such that for every strategy profile s in which some player i can improve his utility by changing his strategy to $s'_i \in \mathcal{S}_i$, we get

$$\phi(\mathsf{s}_{-i}, s'_i) - \phi(\mathsf{s}) = u_i(\mathsf{s}_{-i}, s'_i) - u_i(\mathsf{s}),$$

then we call $\mathcal{G}$ an *exact potential game* and ϕ its *exact potential function*

An exact potential function assigns a value to every strategy profile in the game. According to its definition, this value grows with every improving step by the same amount as the utility of the corresponding player. Therefore there cannot be any cycles in the best-response dynamic, as the function is strictly increasing. Similar to weakly acyclic games, we can compute Nash equilibria by using the best-response dynamic, but this time it is much easier, since the choice of the next player performing an improving step does not change the fact that we will ultimately reach a Nash equilibrium. This is called the *finite improvement property.* An exact potential function is a strong tool for analyzing games, but in order to prove the finite improvement property, a weaker concept suffices.

Definition 2.10 (Generalized Ordinal Potential Games)**.** Let $\mathcal{G}$ be a finite game. If there is a function $\phi : \mathcal{S} \to \mathbb{R}$ such that for every strategy profile s in which some player i can improve his utility by changing his strategy to $s'_i \in \mathcal{S}_i$, we get

$$u_i(\mathsf{s}_{-i}, s'_i) - u_i(\mathsf{s}) > 0 \Rightarrow \phi(\mathsf{s}_{-i}, s'_i) - \phi(\mathsf{s}) > 0,$$

then we call $\mathcal{G}$ a *generalized ordinal potential game* and ϕ its *generalized ordinal potential function.*

Just as with exact potential games, a generalized ordinal potential function strictly increases with every improving step (although it may not be clear by how much). Still, it directly implies that the underlying game possesses the finite improvement property.

Until now, we only considered the utility of individual players. To evaluate a strategy profile from a global perspective, we look at the total utility of all players combined.

Definition 2.11 (Social Welfare)**.** Let s be a strategy profile of a finite game $\mathcal{G}$. We denote with

$$u(\mathsf{s}) := \sum_{i \in \mathcal{N}} u_i(\mathsf{s})$$

the *social welfare* of s.

Since every finite game has only a finite number of strategy profiles, the function $u(\mathsf{s})$ has a maximum.

Definition 2.12 (Socially Optimal Solution)**.** Let opt be a strategy profile of a finite game $\mathcal{G}$. If for every $\mathsf{s} \in \mathcal{S}$ it holds that

$$u(\mathsf{s}) \leq u(\mathsf{opt}),$$

then we call opt the *(socially) optimal solution* of $\mathcal{G}$.

The social welfare can be regarded as the efficiency of the system which is modeled by the game. The closer it is to the optimal solution, the better the overall situation. This collides with the selfish agendas of the players, who are not interested in the social welfare, but only in their own utilities. As already mentioned, Nash equilibria are used to predict the outcome of a game. so it is natural to compare the social welfare of a Nash equilibrium with that of the optimal solution. As there may be more than one Nash equilibrium, we use two well-established concepts, one for the worst-case, the other for the best-case situation.

Definition 2.13 (Price of Anarchy). Let opt be the optimal solution and $\mathcal{S}_{\mathrm{NE}}$ the set of all Nash equilibria of a finite game $\mathcal{G}$. The *price of anarchy* of $\mathcal{G}$ is defined as

$$PoA(\mathcal{G}) := \max_{\mathsf{s}\in\mathcal{S}_{\mathrm{NE}}} \frac{u(\mathsf{opt})}{u(\mathsf{s})}.$$

Definition 2.14 (Price of Stability). Let opt be the optimal solution and $\mathcal{S}_{\mathrm{NE}}$ the set of all Nash equilibria of a finite game $\mathcal{G}$. The *price of stability* of $\mathcal{G}$ is defined as

$$PoS(\mathcal{G}) := \min_{\mathsf{s}\in\mathcal{S}_{\mathrm{NE}}} \frac{u(\mathsf{opt})}{u(\mathsf{s})}.$$

If s is not a Nash equilibrium, then there is at least one player who can further increase his utility by some margin, no matter how small. When we look at this from a more practical angle, this concept can seem rather inappropriate. If the benefits of a strategy change are negligible, a player may stick to his current strategy, especially if he is aware of the fact that his strategy change can provoke additional strategy changes by other players, which in turn may only hurt him in the end. As an alternative to pure Nash equilibria, we therefore also use the already existing concept of approximate pure Nash equilibria.

Definition 2.15 (Approximate Pure Nash equilibrium). Let $\alpha > 1$ and s be a strategy profile of a finite game $\mathcal{G}$. If for every player $i \in \mathcal{N}$ and every strategy $s_i' \in \mathcal{S}_i$ it holds that

$$u_i(\mathsf{s}_{-i}, s_i') \leq \alpha \cdot u_i(\mathsf{s}),$$

then we call s an *α-approximate pure Nash equilibrium.*

Just like with Nash equilibria, we usually omit the prefix *pure* and only write approximate Nash equilibrium. A lot of the definitions so far are based on the concept of Nash equilibria. In most cases, their adaptation for approximate Nash equilibria is straightforward. Here, we only emphasize the difference between general improving moves (cf. Definition 2.6) and α-improving moves.

Definition 2.16 (α-Improving Move). Let $\alpha > 1$ and $\mathsf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ be a strategy profile of a finite game $\mathcal{G}$ and $s_i' \in \mathcal{S}_i$ such that

$$\alpha \cdot u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, s_i').$$

Then deviating from s_i to s_i' is an *α-improving move* for player i.

In contrast to general improving moves, α-improving moves induce a more restricted kind of (best-response) dynamic, in which the players only perform strategy changes if they yield a sufficient increase in their utility. This dynamic is used several times in Chapter 6 when we are dealing with approximate Nash equilibria.

2.2 Budget Games

In the following two sections, we introduce the central models of this thesis.

Definition 2.17 (Budget Game)**.** A *budget game* $\mathcal{B}$ is a tuple $(\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, which consists of the following components:

- a set of players $\mathcal{N} = \{1, \ldots, n\}$
- a set of resources $\mathcal{R} = \{r_1, \ldots, r_m\}$
- a budget b_r for each resource r
- a strategy space $\mathcal{S}_i = \left\{s_i^1, \ldots, s_i^{k_i}\right\}$ for each player i, with $s_i^k = \left(s_i^k(r_1), \ldots, s_i^k(r_m)\right) \in \mathbb{R}^m_{\geq 0}$ and $s_i^k(r)$ being the demand of strategy s_k on resource r
- a utility function $u_i : \mathcal{S}_1 \times \ldots \times \mathcal{S}_n \to \mathbb{R}$ for each player i defined as $u_i(\mathsf{s}) = \sum_{r\in\mathcal{R}} u_{i,r}(\mathsf{s})$ with

$$u_{i,r}(s_1, \ldots, s_n) = \min\left(s_i(r)\ ,\ s_i(r) \cdot \frac{b_r}{\sum_{j\in\mathcal{N}} s_j(r)}\right)$$

In a budget game, the strategy s_i of a player i is a vector of demands on the resources. If the demand on resource r is greater than 0, we say that r is *used* by s_i. When talking about a specific strategy profile $\mathsf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ in which r is used by s_i, we also say that r is (currently) used by i and that $s_i(r)$ is the demand of i on r. In some cases, we abstract from the actual demands and only distinguish between resources being used and not used. In that context, the players simply choose resources instead of demand vectors.

Different players can have different demands on the same resource and even the demand of a single player may vary depending on his strategy. However, because demands are associated with the strategies, they cannot be picked freely by the players. The only way to change a demand value is to change the whole strategy, which in turn may alter the remaining demands as well.

Given a strategy profile $\mathsf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ of a budget game $\mathcal{B}$, let $T_r(\mathsf{s}) := \sum_{i\in\mathcal{N}} s_i(r)$ be the total demand of a resource r. If $T_r(\mathsf{s}) \leq b_r$, then the utility every player i receives from r equals $s_i(r)$. For $T_r(\mathsf{s}) > b_r$, the budget of r is not sufficient to satisfy the demands of all players. In this case, it is split proportionally, meaning players with a larger demand receive more than players with a smaller one. The total utility of a player is the sum of all the utilities he receives from the resources in $\mathcal{B}$.

In some cases, we want to distinguish between budget games and ordered budget games, which we are going to introduce in the next section. Since budget games are strategic, i.e. the utility of each player only depends on the current strategies, and ordered budget games are not, we often use the term *strategic budget games*. Also, we are sometimes interested in bounding the size of the demands with respect to the corresponding budgets.

Definition 2.18 (δ-share Budget Game)**.** Let $\delta > 0$. A *δ-share budget game* $\mathcal{B}$ is a budget game such that for every player i, every $s \in \mathcal{S}_i$ and every resource r, $s(r) \leq \delta \cdot b_r$.

The two definitions of budget games and δ-share budget games are equivalent. Given a budget game $\mathcal{B}$, we can determine a matching value for δ by simply looking at the demands in $\mathcal{B}$. Once we have this value, we can label $\mathcal{B}$ as a δ-share budget game. Some of the results in this thesis depend on the value of δ, while others are independent of it. Depending on which is the case, we either talk about budget games or δ-share budget games.

We now introduce two special kinds of budget games, defined by the structure of their strategy spaces.

Definition 2.19 (Singleton Budget Game)**.** A *singleton budget game* $\mathcal{B}$ is a budget game with the following properties:

- for all $i \in \mathcal{N}$, all $s, s' \in \mathcal{S}_i$ and all $r \in \mathcal{R}$, it holds that $s(r) \neq 0 \wedge s'(r) \neq 0 \Rightarrow s(r) = s'(r)$
- for all $i \in \mathcal{N}$ and all $s \in \mathcal{S}_i$, there is exactly one resource $r \in \mathcal{R}$ with $s(r) > 0$

In a singleton budget game, each player is using exactly one resource at all times, so one can imagine the players choosing a place in the topographical sense, where they are then located. For the second kind of budget games, we first need an important concept from combinatorics.

Definition 2.20 (Matroid)**.** A matroid is a tuple $\mathcal{M} = (\mathcal{U}, \mathcal{I})$ whereas $\mathcal{U}$ is a finite set and $\mathcal{I} \subseteq 2^{\mathcal{U}}$ with the following properties:

- $\emptyset \in \mathcal{I}$
- $T \in \mathcal{I}$ and $S \subseteq T \Rightarrow S \in \mathcal{I}$
- $S, T \in \mathcal{I}$ with $|S| < |T| \Rightarrow \exists u \in T \setminus S$ such that $S \cup \{u\} \in \mathcal{I}$

If $\mathcal{M} = (\mathcal{U}, \mathcal{I})$ is a matroid, then we call the sets $S \in \mathcal{I}$ *independent sets*. A maximal independent set, i.e. a set $B \in \mathcal{I}$ with $|B| \geq |S|$ for all $S \in \mathcal{I}$, is also called a *basis* of the matroid. By this definition, all bases of a matroid have the same cardinality, which is the rank of $\mathcal{M}$ and denoted by $rk(\mathcal{M})$.

One of the most common examples for matroids from graph theory is the following. Let $G = (V, E)$ be an undirected graph and $\mathcal{I} \subseteq 2^E$ contain all sets of edges without a circle. Then $(E, \mathcal{I})$ is a matroid. If there are no edges, there cannot be a circle, so $\emptyset \in \mathcal{I}$. Furthermore, if a set $T \in \mathcal{I}$ does not contain a circle, then there exists no circle in any $S \subseteq T$, so $S \in \mathcal{I}$. Finally, if both $S, T \in \mathcal{I}$ with $|S| < |T|$, then there is always an edge e in T but not in S which we can add to S without creating a circle. For this example, the bases of $(E, \mathcal{I})$ are the spanning trees of G.

For a strategy s of player i, let $\hat{s} := \{r \in \mathcal{R} \mid s(r) > 0\}$ be the set of all resources used by s and let $\hat{\mathcal{S}}_i = \{\hat{s} \mid s \in \mathcal{S}_i\}$ contain all such sets for a given player i. This is a simplified version of the original strategy space of i, in which we only consider which resources are used by a strategy and which are not, independent of the actual demands.

Definition 2.21 (Matroid Budget Game)**.** A *matroid budget game* $\mathcal{B}$ is a budget game with the following properties:

- for all $i \in \mathcal{N}$, all $s, s' \in \mathcal{S}_i$ and all $r \in \mathcal{R}$, it holds that $s(r) \neq 0 \wedge s'(r) \neq 0 \Rightarrow s(r) = s'(r)$
- $\mathcal{M}_i = (\mathcal{R}, \mathcal{I}_i)$ is a matroid for all $i \in \mathcal{N}$ and $\mathcal{I}_i = \{x \subseteq s \mid s \in \hat{\mathcal{S}}_i\}$.

Matroid budget games are a more general form of singleton budget games. In both, each strategy space $\mathcal{S}_i$ consists of bases of a matroid $\mathcal{M}_i$ over the resources. Therefore, every strategy of a player i uses the same number of resources. The matroids can differ between the players, i.e. $\mathcal{M}_i \neq \mathcal{M}_j$ for $i \neq j$ may hold, which in turn means that the players can choose between different sets of resources. Matroid budget games have the very useful property that every strategy change can be modeled as a sequence of *lazy moves*. A lazy move is a strategy change in which only one resource is exchanged for another. Formally speaking, for $\hat{s}, \hat{s}' \in \hat{\mathcal{S}}_i$ with $|\hat{s} \setminus \hat{s}'| = |\hat{s}' \setminus \hat{s}| = 1$, we call the strategy change from s to s' a lazy move. If the switch from s to s' is not a lazy move in itself, it can be decomposed into a sequence $(s = s_0, s_1, \ldots, s_{m-1}, s_m = s')$ in which the switch from s_t to s_{t+1} is a lazy move for $t = 0, \ldots, m-1$ and $s_t \in \mathcal{S}_i$ for all $t = 0, \ldots, m$. Intuitively speaking, player i is also able to perform the strategy change from s to s' only partially, i.e. exchange only some of the resources. This allows for much more filigree actions by the players.

2.3 Ordered Budget Games

A large part of this thesis deals with so-called ordered budget games. These can be regarded as a variation of the budget games introduced in the last section, as the main principle is the same: if the total demand on a resource exceeds its budget, it has to be shared between the responsible players. Here, temporal aspects are taken into account when deciding how to split the budget. For every resource, there is an order over the demands, which are satisfied sequentially until the budget is spent. These orders are manipulated by the strategy changes of the players and ultimately, the utility of a player not only depends on the current strategies, but also on prior strategy profiles.

Definition 2.22 (Ordered Budget Game)**.** An *ordered budget game* $\mathcal{B}$ is a tuple $(\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, which consists of the following components:

- a set of players $\mathcal{N} = \{1, \ldots, n\}$
- a set of resources $\mathcal{R} = \{1, \ldots, m\}$
- a budget b_r for each resource r
- a strategy space $\mathcal{S}_i \subseteq 2^{\mathcal{T}_i}$ for each player i,
  with $\mathcal{T}_i := \left\{t_i^1, \ldots, t_i^{q_i}\right\}$ being the *tasks* of player i
  and $t_i^k \in \mathbb{R}^m_{\geq 0}$ for $k = 1, \ldots, q_i$
- an ordered utility function

$$u_i : \mathcal{S}_1 \times \ldots \times \mathcal{S}_n \times \left\{ \prec = (\prec_r)_{r\in\mathcal{R}} \;\middle|\; \prec_r \text{ total order on } \mathcal{T} = \bigcup_{i\in\mathcal{N}} \mathcal{T}_i \right\} \to \mathbb{R}$$

  for each player i defined as

$$u_i(s_1, \ldots, s_n, \prec) := \sum_{t\in s_i} \sum_{r\in\mathcal{R}} u_{t,r}(s_1, \ldots, s_n, \prec)$$

  with

$$u_{t,r}(s_1, \ldots, s_n, \prec) := \min\left(t(r), \max\left(0, b_r - \sum_{j\in\mathcal{N}} \sum_{\substack{t'\in s_j \\ t'\prec_r t}} t'(r)\right)\right)$$

The main difference between ordered budget games and budget games is the emphasis on the order of player decisions. How much utility a player receives from a resource also depends on which other players chose the same resource *before* him. The demands of the players are satisfied sequentially according to the first-come, first-served principle. If the remaining budget of a resource is not enough to fulfill the demand of a player, he only receives what is left of the budget and all subsequent players obtain nothing. Since the utility now depends on more than just the current strategies of all players, we extend the definition of a strategy profile.

Definition 2.23 (Strategy Profile)**.** The *strategy profile* of an ordered budget game is a tuple $(\mathsf{s}, \prec)$, whereas $\mathsf{s} \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ and $\prec = (\prec_r)_{r\in\mathcal{R}}$ with $\prec_r$ being a total order on $\mathcal{T}$.

To make sure that a strategy change is still an attractive action to the players, we introduce the concept of tasks. These represent a link between the resources and the strategies as they are defined in budget games. A player chooses a strategy, which is a set of tasks, with each task being a vector of demands. This enables us to handle *small-scale* strategy changes, in which a player does not drastically change his whole strategy, but only adjusts parts of it. Let $(\mathsf{s}, \prec)$ be a strategy profile of an ordered budget game. When player i changes his strategy from s_i to s'_i, this does not only affect s, but $\prec$ as well. All new tasks $t_{\text{new}} \in s'_i \setminus s_i$ are moved to the end of $\prec_r$ for all resources $r \in \mathcal{R}$. If we write $\tau = s'_i \setminus s_i$, the new strategy profile is $((\mathsf{s}_{-i}, s'_i), \prec')$ with $x \prec'_r y$ if and only if $x \prec_r y$ for all $x, y \in \mathcal{T} \setminus \tau$ and $x \prec'_r t_{\text{new}}$ for all $x \in \mathcal{T}$ and $t_{\text{new}} \in \tau$. The order of the new tasks in τ among each other is arbitrary, as it does not change the utility function of player i (it only determines how much of his utility is obtained through which task). All tasks $t_{\text{old}} \in \mathsf{s}'_\mathsf{i} \setminus \tau$ are unaffected by the strategy change regarding their position in $\prec_r$ for any r. Therefore, player i is able to change some of his tasks without loosing the utility generated by those who remain part of this strategy.

2.4 Additional Concepts

To conclude this chapter, we introduce a number of additional concepts which are not directly related to game theory. They still appear in the following chapters, mostly as tools used in the proofs.

Definition 2.24 (Submodular Functions)**.** Let $\mathcal{U}$ be a set and $u \in \mathcal{U}$. A function $f : 2^{\mathcal{U}} \to \mathbb{R}$ is *submodular* if

$$|f(X \cup \{u\}) - f(X)| \geq |f(Y \cup \{u\}) - f(Y)|$$

for all $X, Y \subseteq \mathcal{U}$ with $X \subseteq Y$ and $u \notin Y$.

Suppose we add elements from $\mathcal{U}$ one after another to an initially empty set and evaluate this set under f for each new element. f being submodular states that the effect of any element u on the value of f does not increase over time.

Definition 2.25 (Monotone Functions)**.** Let $\mathcal{U}$ be a set. A function $f : 2^{\mathcal{U}} \to \mathbb{R}$ is *monotone* if

$$f(X) \leq f(Y)$$

for all $X, Y \subseteq \mathcal{U}$ with $X \subseteq Y$.

Under the same process described above for submodular functions, the value of f does not decrease over time if the function is monotone.

Definition 2.26 (Lambert W function)**.** Consider the function $f : \mathbb{R} \to \left[-\frac{1}{e}, \infty\right]$ defined as $f(x) := x \cdot e^x$.
The functions $W_{-1} := (f\mid_{(-\infty,-1]})^{-1}$ and $W_0 := (f\mid_{[-1,\infty)})^{-1}$ together are called the *Lambert W function* and usually denoted by $W(x)$.

The Lambert W function is actually defined over the field of the complex numbers. In our context, however, it suffices to restrict its definition to $\mathbb{R}$. Since f is not injective, W is not an actual function. However, if it is split into W_0 and W_{-1} then these two *branches* of W are functions.

CHAPTER 3

Related Work

In this chapter, we give an overview of the existing literature related to budget games. This mainly concerns congestion games and their variants, which are being researched extensively up to this day. We also reference a number of other subjects connected to the topics of this thesis.

3.1 Pure Nash Equilibria in Congestion Games

In a congestion game, players choose among subsets of resources while trying to minimize personal costs. The cost of a player is the sum of the costs of the chosen resources. In the initial (unweighted) version introduced by Rosenthal [64], the cost of each resource depends only on the number of players choosing that resource and it is the same for each player using that resource. These games are exact potential games [60] and therefore every best-response sequence of strategy profiles is guaranteed to converge to a pure Nash equilibrium. Their exact potential function is known as Rosenthal's potential function [64].

Since then, congestion games and their variations have been researched extensively. In weighted congestion games [58], each player has a fixed weight and the cost of a resource depends on the sum of weights of the players using that resource. This models the aspect that players with different demands, sizes, etc. have a different impact on the workload of a resource. Just like for unweighted congestion games, the actual cost of a resource is the same for each player choosing it. For this larger class of congestion games, Milchtaich [58] showed that pure Nash equilibria can no longer be guaranteed. Ackermann et al. [1] determined that the structure of the strategy spaces is a crucial property for the existence of pure Nash equilibria. While a matroid congestion game always has a pure Nash equilibrium, every non-matroid set system induces a game without it. Besides being a more realistic assumption than unweighted congestion games, weighted congestion games have another, rather remarkable property. Milchtaich [59] discovered that every finite game is isomorph to a weighted network congestion game. In a network congestion game, the strategy space of each player corresponds to the set of all paths between a source and a destination in an underlying graph. By identifying all graphs for which every weighted network congestion game has a pure Nash equilibrium [59], Milchtaich solved the topological equilibrium-existence problem first raised in [53].

Milchtaich also proposed another extension of congestion games with player-specific payoff functions for the resources [58], which only depend on the number of players using a resource, but are different from player to player. For singleton strategy spaces, these games maintain pure Nash equilibria. Ackerman et al. [1] showed that, again, every player-specific matroid congestion game has a Nash equilibrium, while this is also a maximal property. Budget games share traits of both weighted and player-specific congestion games. The impact of a player on a resource is determined by the demand of his strategy, which can be regarded as his current weight. The resulting utility gained from a resource differs between players with different demands. The results by Ackerman et al. show that the structure of the strategy spaces can play a major role in the existence of pure Nash equilibria. For this reason, we take a closer look at both singleton and matroid budget games in Chapter 7. However, there are other restrictions on weighted or player-specific congestion games which restore pure Nash equilibria, namely on the cost functions of the resources. Harks and Klimm [39] gave a complete characterization of the class of cost functions for which every weighted congestion game possesses a pure Nash equilibrium. The cost functions have to be affine transformations of each other as well as be affine or exponential.

Mavronicolas and Monien [56] considered weighted congestion games in which some of the weights are negative. Deciding if a given instance has a pure Nash equilibrium is strongly NP-complete, unless the number of resources is not considered to be part of the input. Finally, Harks and Klimm [40] introduced a model similar to ours in which each player not only chooses his resources, but also his demand on them. Unlike our model, both decisions are independent of each other. The resulting payoff of player is his utility minus the costs of the resources he has chosen. His utility function only depends on his demand and is both non-decreasing and concave, creating an incentive for higher demands. On the other hand, the cost of a resource increase with the total demand on it. The results state that pure Nash equilibria do exists if all cost functions are either exponential or affine. In the latter case, these games also possess the finite improvement property. While this is more of a continuous model (the demand of each player is a value from $\mathbb{R}$), budget games can be seen as a discrete version, as a player can have multiple strategies using the same set of resources, but imposing different demands on them.

There is a huge amount of additional literature related to both weighted or player-specific congestion games not listed here. With so much research dedicated to these games, it was only natural to combine their two models at some point. Mavronicolas et al. [55] introduced weighted congestion games with player-specific constants and gave a detailed overview of the existence of pure Nash equilibria. In these

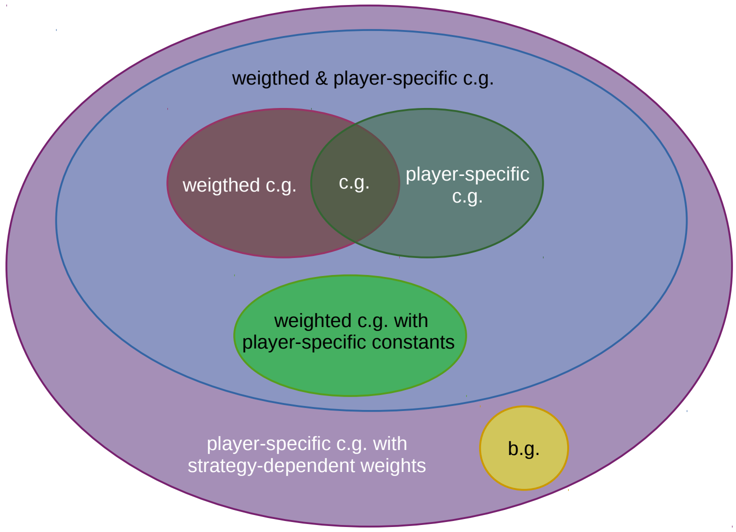

Figure 3.1: A classification of budget games (b.g.) with respect to the field of congestion games (c.g.). The figure illustrates the capacity of our model regarding different impacts of the players. Since our utility functions are fixed, budget games represent only one instance of player-specific games with strategy-dependent weights.

games, the cost function $c_{i,r}$ of player i for resource r consists of a base function c_r, which depends on the weights of all players using r, as well as a constant $k_{i,r}$, both connected by abelian group operations. Later, Gairing and Klimm [30] characterized the conditions for pure Nash equilibria in general player-specific congestion games with weighted players. Pure Nash equilibria exist, if and only if, the cost functions of the resources are affine transformations of each other as well as affine or exponential. The combination of player-specific cost functions and weighted players in particularly emphasizes that the impact of the same strategic choice may vary between the players. As already mentioned, budget games share properties of both types of congestion games. They further extend these models by allowing multiple weights per player, even for the same resource. In contrast to the majority of the existing research mentioned so far, budget game use utility functions instead of cost functions. Omitting this rather important difference, we are better able to embed our model in the field of congestion games. Figure 3.1 shows that budget games are part of the larger class of player-specific congestion games with strategy-dependent weights. In this class, which to the best of our knowledge has not yet been formally introduced, the discrete weights of the players are not fixed but instead are tied directly to their strategies.

Besides the existence of pure Nash equilibria, we also focus on their complexity. While congestion games have the finite improvement property, Fabrikant et al. [27] showed that computing a pure Nash equilibrium is PLS-complete. This implies that the best-response dynamic can perform an exponential number of improving moves until it terminates. In the case of weighted [25] or player-specific [2] congestion games, it is generally NP-hard to decide if a pure Nash equilibrium exists. While computing even strong pure Nash equilibria for ordered budget games is easy, finding a super-strong pure Nash equilibrium is NP-hard. For some restricted instances of strategic budget games, we also give efficient algorithms for finding pure Nash equilibria. Computing the optimal solution of a budget game, strategic or ordered, is again NP-hard.

The negative results regarding both existence and complexity of pure Nash equilibria lead to the study of approximate pure Nash equilibria. Chien and Sinclair [18] showed that in symmetric unweighted congestions games and under a mild assumption on the cost functions every sequence of $(1+\varepsilon)$-improving steps converge to pure $(1+\varepsilon)$-approximate Nash equilibria in polynomial time in the number of players and ε^{-1}. This result cannot be generalized to asymmetric games as Skopalik and Vöcking [68] showed that computing approximate equilibria for these games is still PLS-complete. However, for the case of linear or polynomial cost function Caragiannis et al. [12] presented an algorithm to compute approximate pure Nash equilibria in polynomial time which was slightly improved in [29]. For strategic budget games, a result similar to the one from [18] holds. It was shown that α-approximate pure Nash equilibria with small values of α exist in weighted congestion games [37] and that they can be computed in polynomial time [13], albeit only for a larger values of α. Chen and Roughgarden [16] proved the existence of approximate equilibria in network design games with weighted players. The results have been used by Christodoulou et al. [20] to give tight bounds on the price of anarchy and price of stability of approximate pure Nash equilibria in unweighted congestion games.

To quantify the inefficiency of equilibrium outcomes, the price of anarchy has been thoroughly analyzed for exact equilibria. For unweighted congestion games with linear cost functions [19], the price of anarchy is $5/2$ for symmetric games and $\Theta(\sqrt{n})$ for asymmetric ones. Weighted congestion games have a price of anarchy of $2,681$ for linear cost functions and of $d^{\Theta(d)}$ for polynomial cost functions of degree at most d and with non-negative coefficients [8]. Aland et al. [4] later improved these results. Recently, Awerbuch et al. [11] considered even more general classes of cost functions. Since the price of anarchy is only meaningful if the players form a common pure Nash equilibrium, Roughgarden [65] considered the connection between classic pure Nash equilibria and more general concepts by using smoothness arguments. Christodoulou et al. [20] also investigated the price of anarchy for approximate pure Nash equilibria in (non-) atomic congestion games with linear cost functions. They gave (almost) tight bounds, which encompass the former results regarding exact pure Nash equilibria. We also give almost tight bounds on the price of anarchy in budget games for both pure Nash equilibria as well as approximate pure Nash equilibria.

Instead of bounding the time the best-response dynamic needs to reach a pure Nash equilibrium, recent work also considered the convergence time to states with a social welfare close to the optimum. The concept of smoothness was first introduced by Roughgarden [65]. Several variants such as the concept of semi-smoothness [54] followed. Awerbuch et al. [9] proposed β-niceness, which was reworked in [7]. This serves as the basis for the concept of nice games introduced in [6], which we also use in Chapter 6.

Recently, Paes Leme et al. [49] introduced the sequential price of anarchy, which measures the efficiency of pure Nash equilibria obtained when the players choose their strategy one after another. De Jong et al. [23, 22] gave bounds on the sequential price of anarchy for different forms of congestion games. Although it is actually better than the classic price of anarchy for some game classes, it is also unbounded for linear symmetric routing games, which have a price of anarchy of $5/2$. The sequential price of anarchy differs from what we consider as ordered budget games, as it is still defined for strategic games while the latter are not strategic. Nevertheless, it shows that the impact of introducing a sequential aspect to an existing model can have various outcomes.

3.2 Modifications of the Congestion Games Model

Besides weighted players and player-specific cost functions, there are many other modifications of the classic congestion game model. In this section, we list only a few of them along with the relevant results.

Instead of assigning the whole cost of a resource to each player using it, it can also be shared between those players, so that everyone only pays a part of it. Such games are known as cost sharing games [43]. One method to determine the share of each player is proportional cost sharing, in which the share increases with the weight of a player. This is exactly what we are doing with budget games, but with utilities instead of costs. Under proportional cost sharing, pure Nash equilibria again do not exist in general [5]. Kollias and Roughgarden [48] took a different approach by considering weighted games in which the share of each player is identical to his Shapley value [67]. Using this method, every weighted congestion game yields a weighted potential function. In addition, it minimizes the worst-case price of anarchy [33] for guaranteed pure Nash equilibria. If the condition that a pure Nash equilibrium always exists is dropped, proportional cost sharing is optimal regarding price of anarchy among all cost sharing methods. Gairing et al. [31] further analyzed the effects of the cost functions on the efficiency of cost sharing mechanisms. Further research bounded the price of anarchy for different cost sharing mechanisms based on the structure of the strategy spaces [70] or the underlying cost functions [17, 36]. Usually, these cost functions are anonymous, i.e. they only depend on the weights of the players. Both Roughgarden et al. [66] and Klimm and Schmand [45] gave (tight) bounds on the price of anarchy for non-anonymous cost functions, which are defined over sets of players.

So-called set cover games mainly consist of a universe of elements and a number of subsets. Every set has a cost and a multiplicity for each element, meaning that it can cover an element more than once. Also, every element has a coverage requirement. Similar to budget games, this coverage requirement serves as an upper bound on the availability of the elements. If it is exceeded by the number of covering subsets, the additional subsets are ignored. In this context, different multiplicities can be regarded as different demands. Li et al. [51] considered multiple versions of these games, modeling different practical situations from a cooperative standpoint and gave cost sharing mechanisms with different properties (budget-balanced, core, group-strategyproof). Later [52], they focused on a version where the elements are the players, placing bids on the subsets, and gave truthful mechanisms. Cardinal and Hoefer [14] chose a non-cooperative approach and analyzed both exact and approximate Nash equilibria in set cover games.

Another variation studied is the model of bottleneck congestion games [10]. Instead of considering the sum of all resource costs, the individual cost of a player is given by the maximum of the costs of his resources. This is more suited for modeling data routing over links in series, as the total delay primarily depends on the link with the highest congestion. Budget games, on the other hand, are more concerned with routing over parallel links. Banner and Orda [10] showed that these games always have a pure Nash equilibrium and gave first bounds on their efficiency, which were improved later on [21, 63, 57]. Harks et al. [41] proved the existence of strong pure Nash equilibria through the lexicographical improvement property, which we also use in Chapter 7. Recently, the complexity of (strong) pure Nash equilibria in bottleneck congestion games has been considered [38].

3.3 Additional Literature

To conclude this chapter, we briefly list a number of additional literature connected to the topics of this thesis.

Budget games can be considered as a generalization of market sharing games [34], in which players choose a set of markets in which they offer a service. Each market has a fixed cost and each player a budget. The set of markets a player can service is thus determined by a knapsack constraint. The utility of a player is the sum of utilities that he receives from each market that he services. Each market has a fixed total profit or utility that is evenly distributed among the players that service the market. Market sharing games are potential games and therefore always posses pure Nash equilibria.

Strategic budget games also satisfy the definition of basic utility games [69]. For such games, the social welfare is both submodular and non-decreasing. We use these properties to efficiently approximate the optimal solution in matroid budget games (cf. Chapter 4). Vetta [69] showed that the price of anarchy of any basic utility game is at most 2, which is tight for budget games as well as for ordered budget games.

The field of location games was established by Hotteling [42]. In general, these games are about choosing a location, e.g. in a market, to maximize payoff. The existence of pure Nash equilibria in various forms of this class of games has been analyzed, such as positioning in a graph [46] or in a continuous interval [47]. Similar to ordered budget games, Eiselt and Laporte discussed a sequential execution in which the players choose their location one after another instead of simultaneously. Recently, the focus shifted to facility location games [44], which were mainly considered as cooperative games with cost sharing mechanisms [35, 50, 24]. Vetta [69] defined a potential game in which the players choose a position for their facilities inside a market to offer a service to the customers. Every customer is willing to pay a certain price while serving a costumer induces costs to the provider based on the facility used. Further publications [15, 14] regarding this topic consider fractional location games, in which the consumer can split their demand over multiple facilities. In [3], Ahn et al. studied the Voronoi game in which two players alternately choose their facilities. The space controlled by each player is determined by the nearest-neighbor rule. They give a winning strategy for player 2, although player 1 can ensure that the advantage is only arbitrarily small.

CHAPTER 4

Basic Results

In this chapter, we prove that pure Nash equilibria generally do not exist in budget games. This important result motivates our search for different kinds of equilibria or in different forms of budget games in the later chapters. In addition, we show that finding a strategy profile which maximizes the social welfare in NP-hard, both in strategic as well as in ordered budget games.

4.1 Nonexistence of Pure Nash Equilibria in Budget Games

Our main concern regarding budget games in this thesis is the existence and the properties of Nash equilibria. We start our analysis by showing that in general, a budget game does not have a Nash equilibrium, even if a player may only demand an arbitrarily small part of a resource's budget. We define the budget game $\mathcal{B}_0$, which lacks a Nash equilibrium and is also referred to in some of the following chapters.

Definition 4.1. Let $\delta > 0$ be arbitrary, but fixed. For $\delta < 1$, choose $\sigma > 0$ and $n \in \mathbb{N}_0$ such that $\sigma \leq \delta$ and $n \cdot \sigma + \delta = 1$. Choose $\gamma > 0$ such that $\gamma < \delta$ and $\gamma > 1$ iff $\delta > 1$.
We define the δ-share budget game $\mathcal{B}_0$ as follows:

- $\mathcal{N}_0 = \{1, \ldots, n+2\}$
- $\mathcal{R}_0 = \{r_1, r_2, r_3, r_4\}$
- $b_r = 1$ for $r = r_1, \ldots, r_4$
- $\mathcal{S}_1 = \{\ s_1^1 = (\gamma, \delta, 0, 0),\ s_1^2 = (0, 0, \delta, \gamma)\ \}$
  $\mathcal{S}_2 = \{\ s_2^1 = (\delta, 0, \gamma, 0),\ s_2^2 = (0, \gamma, 0, \delta)\ \}$
  $\mathcal{S}_i = \{\ s_i = (\sigma, \sigma, \sigma, \sigma)\ \}$ for $i \in \{3, \ldots, n+2\}$

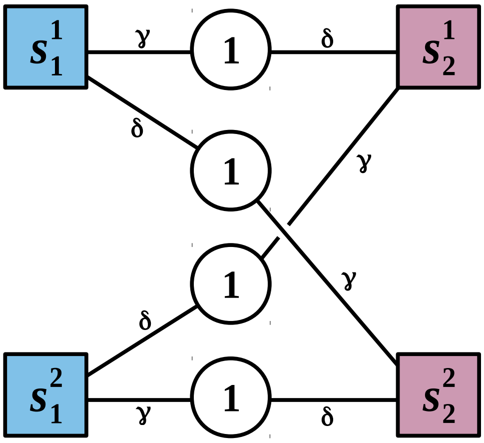

Figure 4.1: The δ-share budget game $\mathcal{B}_0$ without a Nash equilibrium. The auxiliary players are not shown. The circles represent resources, the numbers on the inside their budgets. Strategies are depicted as squares and the demand of a strategy on a resource is written next to an edge between those two. If there is no edge between a strategy and a resource, then the demand is 0.

The budget game $\mathcal{B}_0$ is depicted in Figure 4.1. We discuss an instance of $\mathcal{B}_0$ with actual values for δ and γ in Section 6.1. The players 1 and 2 are the games *actual* players, while the remaining players (if applicable) are *auxiliary players*. They only have one strategy each and therefore do not have to be considered when looking for a Nash equilibrium. These players exists only to guarantee that a resource's budget is not sufficient to satisfy the demands of all players. Due to the small size of the game, it has only four different strategy profiles and we can individually analyze the corresponding utilities of the players for each of them. The results are listed in Table 4.1. There are only two different values for the utilities in the game:

- $u^1 := \min(1, \delta) + \frac{\gamma}{1+\gamma}$
- $u^2 := \min(1, \gamma) + \frac{\delta}{1+\gamma}$.

For $\gamma, \delta > 1$, we immediately see that

$$u^2 - u^1 = 1 + \frac{\delta}{1+\gamma} - 1 - \frac{\gamma}{1+\gamma} = \frac{\delta - \gamma}{1+\gamma} > 0$$

since $\delta > \gamma$. Otherwise we have $\gamma, \delta \leq 1$ and

$$u^1 - u^2 = \delta - \gamma + \frac{1-\delta}{1+\gamma} = (\delta - \gamma)\left(1 - \frac{1}{1+\gamma}\right) > 0$$

players	strategy profiles			
	(s_1^1, s_2^1)	(s_1^1, s_2^2)	(s_1^2, s_2^1)	(s_1^2, s_2^2)
1	$\min(1,\delta) + \frac{\gamma}{1+\gamma}$	$\min(1,\gamma) + \frac{\delta}{1+\gamma}$	$\min(1,\gamma) + \frac{\delta}{1+\gamma}$	$\min(1,\delta) + \frac{\gamma}{1+\gamma}$
2	$\min(1,\gamma) + \frac{\delta}{1+\gamma}$	$\min(1,\delta) + \frac{\gamma}{1+\gamma}$	$\min(1,\delta) + \frac{\gamma}{1+\gamma}$	$\min(1,\gamma) + \frac{\delta}{1+\gamma}$

Table 4.1: Overview of the different strategy profiles and the corresponding utilities of the budget game $\mathcal{B}_0$.

So depending on the value of δ (and ultimately the value of γ), one of the two utilities is always larger than the other one. We define $u^+ := \max(u^1, u^2)$ and $u^- := \min(u^1, u^2)$. In each strategy profile, the utility of one player is u^- while the utility of the other is u^+. Due to game's structure, the player with the smaller utility can always switch his strategy in order to obtain the larger utility instead (cf. Table 4.1). In the process, the utility of the other player is reduced from u^+ to u^-. So in every strategy profile, the player with utility u^- is not in an equilibrium, which shows that $\mathcal{B}_0$ does not have a Nash equilibrium. This conclusion is completely independent of the actual value of δ.

Corollary 4.2. For every $\delta > 0$, there is a δ-share budget game without a Nash equilibrium.

This result shows that we cannot expect any strategic budget game to have a pure Nash equilibrium, thus motivating our further research regarding approximate Nash equilibria and Nash equilibria under additional restrictions in the following chapters.

4.2 Complexity of the Optimal Solution

To conclude this chapter, we prove that computing the optimal solution for both ordered as well as strategic budget games is hard. In fact, these two problems are equivalent. The only real difference between ordered budget games and strategic budget games is the utility sharing mechanism of their utility functions, i.e. how a resource's budget is split if the total demand exceeds it. The concept of tasks, which serves as a link between strategies and resources in ordered budget games was only introduced to model partial strategy changes, which in turn affects the utility of the players. So aside from the utility functions, strategic budget games and ordered budget games are equivalent and one can easily be transformed into the other, as we are going to show now.

Let $\mathcal{B}$ be a strategic budget game, i a player in $\mathcal{B}$ and s_i a strategy of i. By introducing a single task t with $t(r) = s_i(r)$ for every $r \in \mathcal{R}$ and redefining the strategy to $s_i = \{t\}$, we get a strategy which fits the definition of an ordered budget game. If we repeat this process for all players and all strategies, then the only thing which keeps $\mathcal{B}$ from being an ordered budget game is the definition of its utility functions, which we can simply exchange. Note that the demand of player i in s_i remains the same for every resource. Likewise, let $\mathcal{B}$ be an ordered budget game, i a player in $\mathcal{B}$ and s_i a strategy of i. s_i consists of tasks $t_1, \dots, t_q$ for $q \geq 1$. By removing these tasks from s_i and assigning their demands directly to the strategy, i.e. setting $s_i(r) = \sum_{j=1}^{q} t_j(r)$ for every resource r, we turn s_i into the strategy of a strategic budget game. Again, the demand imposed by s_i on r is not changed in the process.

While the differences in their utility functions separate strategic budget games and ordered budget games in most aspects, the concept of social welfare is equivalent for both game variations. The utility functions only differ in the individual utilities of the players, as they state how a resource's budget is split between them if the total demand is too large. However, they are both the same in regards to the social welfare, i.e. how much utility is obtained by all players together. As a result, the problem of computing the optimal solution for any budget game, be it strategic or ordered, is essential the same. Instead of computing the optimal solution for one variation (e.g. strategic budget games), we can always transform the game into one of the other kind (e.g. ordered budget game) as described above and solve the problem there. The optimal solution as well as its social welfare is one and the same for both types of games.

Observation 4.3. Computing the optimal solution in a strategic budget game is equivalent to computing the optimal solution in an ordered budget game.

For the rest of this chapter, we restrict our attention to ordered budget games. However, all of the following results can be transfered over to strategic budget games, as we just explained. Since the orders $\prec$ in a strategy profile $(\mathsf{s}, \prec)$ of an ordered budget game do not affect the social welfare, we abuse notation and omit $\prec$ for the rest of this chapter.

Theorem 4.4. *Computing the optimal solution for an ordered budget game $\mathcal{B}$ is* NP*-hard, even if the strategies and tasks of all players are identical and $\mathcal{B}$ is a singleton ordered budget game.*

Proof. We give a reduction from the maximum set coverage problem. An instance $\mathcal{I} = (\mathcal{U}, \mathcal{W}, w)$ is given by a set $\mathcal{U}$, a set of subsets $\mathcal{W} = \{\mathcal{W}_1, \dots, \mathcal{W}_q\}$ with $\mathcal{W}_i \subseteq \mathcal{U}$ for $i = 1, \dots, q$ and an integer $w \in \mathbb{N}$. Every subset $W \subseteq \mathcal{W}$ with $|W| \leq w$ is a solution for $\mathcal{I}$, but finding the optimal solution, which is the subset such that the number of elements covered by the sets in W is maximized, is known to be NP-hard.

From $\mathcal{I}$, we create an ordered budget game $\mathcal{B}$ as follows:

- $\mathcal{N} = \{1, \dots, w\}$
- $\mathcal{R} = \mathcal{U}$
- $b_r = 1$ for all $r \in \mathcal{R}$

- $\mathcal{S}_i = \mathcal{T}$ for all $i \in \mathcal{N}$ with $\mathcal{T} = \{t_{\mathcal{W}_1}, \dots, t_{\mathcal{W}_q}\}$

The demands are set according to

$$t_{\mathcal{W}_j}(r) = \begin{cases} 1, & \text{if } r \in \mathcal{W}_j \\ 0, & \text{else} \end{cases}$$

for all $r \in \mathcal{R}$.

Given a strategy profile s of $\mathcal{B}$, its social welfare $u(\mathsf{s})$ is the number of resources covered by at least one strategy in s, since all demands and all budgets are 1. Resource r is covered by a strategy $s = \{t_{\mathcal{W}_k}\}$ if and only if there is a set $\mathcal{W}_k$ with $r \in \mathcal{W}_k$. So choosing a strategy for ever player corresponds to choosing w sets from $\{\mathcal{W}_1, \dots, \mathcal{W}_q\}$ and s also describes a solution for $\mathcal{I}$. The social welfare of s is then the number of elements covered by the corresponding solution for $\mathcal{I}$.

On the other hand, every solution of $\mathcal{I}$ can be transformed into a strategy profile of $\mathcal{B}$ by assigning each chosen set $\mathcal{W}_k$ to one player i and setting $s_i = \{t_{\mathcal{W}_k}\}$. Again, the social welfare and the number of covered elements are the same. Therefore, the problems of finding an optimal solution for $\mathcal{B}$ and finding an optimal solution for $\mathcal{I}$ are equivalent. □

As mentioned before, this result carries over to strategic budget games.

Corollary 4.5. Computing the optimal solution for a strategic budget game $\mathcal{B}$ is NP-hard, even if the strategies of all players are identical and $\mathcal{B}$ is a singleton budget game.

We have shown that it can be computationally hard to determine the optimal solution of any budget game. On the other hand, we can can approximate it up to a constant factor if the strategy spaces of the players adhere to a certain structure, namely to the bases of a matroid. In that case, Algorithm 1 yields a $1 - \frac{1}{e}$ approximation of the optimal solution. To prove this, we first introduce a different notation for the social welfare.

Algorithm 1 ApproxOptSolutionOrderedBGs

$s_i \leftarrow \emptyset$ for all $i \in \mathcal{N}$ ▷ strategy of player i
$T_i \leftarrow \mathcal{T}_i$ for all $i \in \mathcal{N}$ ▷ available tasks of player i
$k_i \leftarrow$ number of tasks in each of player i's strategies

while $\bigcup_{i \in \mathcal{N}} T_i \neq \emptyset$ **do**
  choose $i \in \mathcal{N}$ and $t \in T_i$ such that
    $\hat{u}(\bigcup_{i \in \mathcal{N}} s_i \cup \{t\}) \geq \hat{u}(\bigcup_{i \in \mathcal{N}} s_i \cup \{t'\})$ for all $t' \in \bigcup_{i \in \mathcal{N}} T_i$
  $s_i \leftarrow s_i \cup \{t\}$

  $T_i \leftarrow T_i \setminus \{t\} \cup \{t' \in T_i \mid \text{no strategy in } S_i \text{ containing } s_i \cup \{t'\}\}$
return $(s_1, \dots, s_n)$

Definition 4.6. Let $\mathcal{B}$ be an ordered budget game. We define $\hat{u} : 2^{\mathcal{T}} \to \mathbb{R}_{\geq 0}$ with

$$\hat{u}(T) = \sum_{r \in \mathcal{R}} \min \left(\sum_{t \in T} t(r), b_r \right).$$

The function $\hat{u}$ can be regarded as an extension of the social welfare of $\mathcal{B}$. It does not consider how the budgets of the resources are split between the players, but only how much is obtained in total. In addition, $\hat{u}$ is defined for any set of tasks, not just valid strategy profile. Finally, $\hat{u}$ has two important properties.

Lemma 4.7. The function $\hat{u}$ is monotone and submodular.

Proof. We first show that $\hat{u}$ is monotone. Let $T \subseteq \mathcal{T}$ and $t' \in \mathcal{T} \setminus T$.

$$\hat{u}(T \cup \{t'\}) - f(T) = \sum_{r \in \mathcal{R}} \left(\min \left(t'(r) + \sum_{t \in T} t(r), b_r \right) - \min \left(\sum_{t \in T} t(r), b_r \right) \right) \geq 0$$

$\hat{u}$ does not decrease as we add additional tasks, so it is monotone. Now assume that $\hat{u}$ is not submodular. Since $\hat{u}$ is defined as a sum over the resources, there has to be at least one resource r with

$$\begin{aligned} &\min \left(\sum_{t \in S} t(r), b_r \right) - \min \left(\sum_{t \in T} t(r), b_r \right) \\ < &\min \left(t'(r) + \sum_{t \in S} t(r), b_r \right) - \min \left(t'(r) + \sum_{t \in T} t(r), b_r \right) \end{aligned}$$

for $T \subseteq S \subset \mathcal{T}$ and $t' \in \mathcal{T} \setminus S$. Since we add t' to both T and S, the difference between $\min \left(t'(r) + \sum_{t \in S} t(r), b_r \right)$ and $\min \left(t'(r) + \sum_{t \in T} t(r), b_r \right)$ cannot be larger than the difference between $\min \left(\sum_{t \in S} t(r), b_r \right)$ and $\min \left(\sum_{t \in T} t(r), b_r \right)$. Thus, no such resource r exists, which means that f has to be submodular. □

Greedy maximization of a submodular monotone function yields an approximation factor of at least $1 - \frac{1}{e}$ [62]. Algorithm 1 has the same approximation factor, as the resulting strategy profile is constructed greedily by always choosing the task which yields the highest increase in social welfare. For a matroid ordered budget game, in which the strategies of the players are bases of matroids over their tasks, this leads to a valid strategy profile $\mathsf{s} \in \mathcal{S}$. In addition, there can be no better approximation for the maximum set coverage problem than $1 - \frac{1}{e}$ unless $\mathsf{P} = \mathsf{NP}$ [28]. Since this problem can be reduced to finding the optimal solution of a budget game $\mathcal{B}$ (cf. proof of Theorem 4.4), the approximation factor of $1 - \frac{1}{e}$ is tight.

Corollary 4.8. In a matroid ordered budget game, greedy maximization of the social welfare yields a strategy profile $\mathsf{s} \in \mathcal{S}$ with

$$\frac{u(\mathsf{opt})}{u(\mathsf{s})} \geq 1 - \frac{1}{e}$$

This bound is tight, provided $\mathsf{P} \neq \mathsf{NP}$.

As mentioned before, this result holds for matroid strategic budget games, as well. In a matroid strategic budget games, the strategy space of each player consists of bases of a matroid over the resources.

Corollary 4.9. In a matroid strategic budget game, greedy maximization of the social welfare yields a strategy profile $\mathsf{s} \in \mathcal{S}$ with

$$\frac{u(\mathsf{opt})}{u(\mathsf{s})} \geq 1 - \frac{1}{e}$$

This bound is tight, provided $\mathsf{P} \neq \mathsf{NP}$.

So although the computation of the actual optimal solution is not feasible, we can achieve a constant approximation factor for matroid games using a simple greedy approach. Matroid budget games have the property that the players enjoy a lot of freedom when choosing their resources. Instead of deciding between given combinations, a player i can, for example, choose m_i resources from a subset $M_i \subseteq \mathcal{R}$ of all resources. Even if there are multiple subsets $M_i^1, \ldots, M_i^{k_i}$ and he is allowed to choose m_i^l many resources from M_i^l for $l = 1, \ldots, k_i$, then this fits the definition of a matroid budget game.

CHAPTER 5

Pure Nash Equilibria in Ordered Budget Games

In this chapter, we analyze the properties of ordered budget games, especially regarding Nash equilibria. Ordered budget games can be viewed as an extension of strategic budget games, as the major properties (variable demands even on the same resource, social welfare bounded by the budgets) are the same. In addition, ordered budget games also take chronological aspects into account, i.e. which player performed which strategy change *at which point in time*. They are therefore not strategic games. In a strategic game, the current state (and thus the corresponding utilities of the players) only depends on the combined strategies of the players (the strategy profile). This is the case in strategic budget games, for example. When considering ordered budget games, however, this property no longer holds. If two or more players perform a sequence of strategy changes, the outcome also depends on the order in which these strategy changes are executed. The example sketched in Figure 5.1 highlights this aspect. Suppose both players 1 and 2 change their

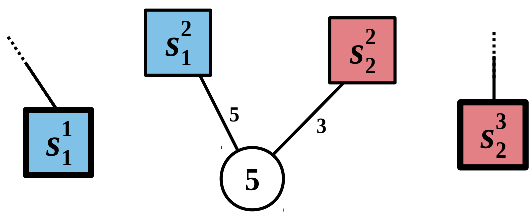

Figure 5.1: Example of an singleton ordered budget game in which the order of the strategy changes determine the resulting utilities. In the current strategy profile, the strategies of player 1 and 2 are s_1^1 and s_2^3, respectively. This is indicated by the thick border around the strategies.

strategy to s_1^2 and s_2^2, respectively. If player 1 is the first to act, then his new utility will be 5, which conforms to the whole budget of the resource. As long as he does not change his strategy a second time, this utility is guaranteed. As he leaves no remaining budget for player 2 to obtain, the new utility of that player would be 0 after his own strategy change. So if $(\mathsf{s}, \prec)$ is the strategy profile after both strategy changes in this order, then $u_1(\mathsf{s}, \prec) = 5$ and $u_2(\mathsf{s}, \prec) = 0$. If, on the other hand, player 1 changes his strategy after player 2, resulting in the strategy profile $(\mathsf{s}, \prec')$, then the resulting utilities will be $u_1(\mathsf{s}, \prec') = 2$ and $u_2(\mathsf{s}, \prec') = 3$, as player 2 only uses a part of the resource's budget. Note that the strategy vector s is the same in both outcomes. What does change is the ordering of the tasks, which in turn leads to the different utilities for the two strategy profiles. Unlike strategic games, the state of an ordered budget game also depends on previous strategy profiles. To put it simply, the player who acts first always has an advantage in ordered budget games, as his tasks have a higher priority regarding the orders in $\prec$. This qualifies ordered budget games to model scenarios with both old and new players. While the old players have already been established their position in the game and secured their utilities, the new ones are unable to obtain any budget shares currently held by other players. Instead, they integrate themselves in the current strategy profile to gain as much of the remaining budgets as possible.

Normally, a strategy can consist of a number of different tasks. In all the figures shown throughout this chapter, we consider only instances with single tasks in each strategy. We can therefore adapt the graphical representation introduced in Figure 4.1 and identify the tasks with their corresponding strategies. In general, having multiple tasks as part of a single strategy enables players to perform strategy changes without loosing all of their current utility. If a player changes his strategy from s to s', then the position of all tasks from $s \cup s'$ in the orders of $\prec$ remain the same, so the utility generated by them does not decrease.

5.1 Existence & Complexity of Pure Nash Equilibria

A nice property of ordered budget games is that they are exact potential games, which in turn means that they always have a Nash equilibrium. This becomes apparent when considering the dynamic between the different players.

Lemma 5.1. In an ordered budget game $\mathcal{B}$, the strategy change of a player i does not decrease the utility of any other player $j \in \mathcal{N} \setminus \{i\}$.

This result follows directly from the definition of ordered budget games. Note that it actually holds for all strategy changes, not only improving ones.

Proof. Assume that the strategy change of player i decreases the utility which player j receives from resource r. If we denote the strategy profile before the strategy change by $(\mathsf{s}, \prec)$ and the one after by $(\mathsf{s}', \prec')$, this means $u_j(\mathsf{s}, \prec) < u_j(\mathsf{s}', \prec')$. So there has to be a task $t_j \in s_j$ with a decrease in its utility from at least one resource r, i.e. $u_{t_j,r}(\mathsf{s}, \prec) < u_{t_j,r}(\mathsf{s}', \prec')$. This can only happen if total demand by all tasks

t on r with $t \prec'_r t_j$ and which are actually part of some players strategy in $(\mathsf{s}', \prec')$ is larger than the total demand in $(\mathsf{s}', \prec')$ of tasks t with $t \prec_r t_j$. However, player i is the only one who is changing his strategy (and therefore his current tasks) and by definition of the ordered utility function, the order of all the tasks by the other players among each other are unaffected. Finally, any new task $t_i \in s'_i \subseteq s_i$ has the property $t_j \prec'_r t_i$. Therefore, we get $u_{t_j,r}(\mathsf{s}, \prec) \geq u_{t_j,r}(\mathsf{s}', \prec')$. □

With this lemma, it becomes clear that the social welfare acts as an exact potential function for ordered budget games.

Theorem 5.2. *Ordered budget games are exact potential games, with the social welfare as an exact potential function.*

Proof. Let $(\mathsf{s}, \prec)$ be a strategy profile of an ordered budget game $\mathcal{B}$. Assume that $(\mathsf{s}, \prec)$ is not a Nash equilibrium, so there is a player i who can still improve his utility. Denote the resulting strategy profile by $((\mathsf{s}_{-i}, s'_i), \prec')$. Since the strategy change is an improving step for i, we get $u_i(\mathsf{s}, \prec) < u_i((\mathsf{s}_{-i}, s'_i), \prec')$ and according to Lemma 5.1, $u_j(\mathsf{s}, \prec) \leq u_j((\mathsf{s}_{-i}, s'_i), \prec')$ holds for all players $j \neq i$. So the social welfare is a generalized ordinal potential function. To see that it is actually an exact potential function, we make a case distinction. First, consider $u_j(\mathsf{s}, \prec) = u_j((\mathsf{s}_{-i}, s'_i), \prec')$ for all $j \neq i$. In that case,

$$\begin{aligned} u((\mathsf{s}_{-i}, s'_i), \prec') - u(\mathsf{s}, \prec) &= \sum_{j \in \mathcal{N}} u_j((\mathsf{s}_{-i}, s'_i), \prec') - u_j(\mathsf{s}, \prec) \\ &= u_i((\mathsf{s}_{-i}, s'_i), \prec') - u_i(\mathsf{s}, \prec) \end{aligned}$$

which proves our point. If $u_j(\mathsf{s}, \prec) < u_j((\mathsf{s}_{-i}, s'_i), \prec')$ holds for some $j \neq i$. Then this additional utility of player j formerly belonged to i, so the increase in their utility does not increase the social welfare, which is why it is again an exact potential function. □

Since they are potential games, ordered budget games always have at least one Nash equilibrium. In addition, because their potential function is also their social welfare, this Nash equilibrium has to be super-strong.

Corollary 5.3. Let $(\mathsf{opt}, \prec)$ be the optimal solution of an ordered budget game $\mathcal{B}$. Then $(\mathsf{opt}, \prec)$ is a super-strong Nash equilibrium of $\mathcal{B}$.

Proof. Assume that $(\mathsf{opt}, \prec)$ is not a super-strong Nash equilibrium. In this case, there is a coalition $\mathcal{C}$ of players such that if all players in $\mathcal{C}$ deviate from their current strategy, at least one player $i \in \mathcal{C}$ increases his utility while the other players $j \in \mathcal{C} \setminus \{i\}$ receive at least the same utility as before. Let $((\mathsf{opt}_{-\mathcal{C}}, \mathsf{s}_{\mathcal{C}}), \prec')$ be the resulting strategy profile after the combined strategy changes of $\mathcal{C}$. We actually do not have to consider the structure of both $\prec$ and $\prec'$, as they do not influence the social welfare. According to Lemma 5.1, no player outside of $\mathcal{C}$ looses any utility through the strategy changes of $\mathcal{C}$.

$$
\begin{aligned}
u(\mathsf{opt}, \prec) &= u_i(\mathsf{opt}, \prec) + \sum_{j \in \mathcal{C}\setminus\{i\}} u_j(\mathsf{opt}, \prec) + \sum_{j \in \mathcal{N}\setminus\mathcal{C}} u_j(\mathsf{opt}, \prec) \\
&< u_i((\mathsf{opt}_{-\mathcal{C}}, \mathsf{s}_{\mathcal{C}}), \prec') + \sum_{j \in \mathcal{C}\setminus\{i\}} u_j((\mathsf{opt}_{-\mathcal{C}}, \mathsf{s}_{\mathcal{C}}), \prec') + \sum_{j \in \mathcal{N}\setminus\mathcal{C}} u_j((\mathsf{opt}_{-\mathcal{C}}, \mathsf{s}_{\mathcal{C}}), \prec') \\
&= u((\mathsf{opt}_{-\mathcal{C}}, \mathsf{s}_{\mathcal{C}}), \prec')
\end{aligned}
$$

This contradicts our assumption that $(\mathsf{opt}, \prec)$ is the optimal solution. So it has to be a super-strong Nash equilibrium. □

According to Theorem 5.2, a Nash equilibrium of an ordered budget game can be computed by simulating the best-response dynamic. As we will see later, this approach is not very efficient, since the number of improving steps can be exponential in the games description length. Nevertheless, there are other, faster methods.

Theorem 5.4. *A strong Nash equilibrium of an ordered budget game $\mathcal{B}$ can be computed in n best-response steps.*

Proof. We start with the initial strategy vector $\mathsf{s}_0 = (\emptyset, \dots, \emptyset)$. Intuitively speaking, the players have not yet chosen any tasks. At this point, we do note care if $\emptyset \notin \mathcal{S}_i$ for some $i \in \mathcal{N}$. We also do not have to consider the initial order $\prec_0$ as the first strategy change by any player only introduces new tasks. The players then choose their best-response one after another according to an arbitrary order. For the sake of simplicity, we assume that this order follows their indices and player $i+1$ chooses his strategy right after player i for $i = 1, \dots, n-1$. As stated by Lemma 5.1, the utility of i is not decreased by the strategy choice of $i+1$ or any other subsequent player.

After every player i has chosen a strategy $s_i \in \mathcal{S}_i$ according to best-response, the result is a valid strategy profile $(\mathsf{s}, \prec) \in \mathcal{S}$. Furthermore, no player has an incentive to deviate from his current strategy, as the utility he received the moment he chose his strategy has not changed. So $(\mathsf{s}, \prec) \in \mathcal{S}$ is a Nash equilibrium. Regarding strong Nash equilibria, note that player i can only increase his utility if a player j with $j < i$ deviates first. Using an induction argument, But it is not possible for player 1 to receive a higher utility in any strategy profile (he was the first player to choose a set of tasks), so he will never participate in any coalition. By using an induction argument, we see that $(\mathsf{s}, \prec)$ has to be a strong Nash equilibrium. □

Computing both Nash equilibria and strong Nash equilibria for ordered budget games can be done in polynomial time. However, if we consider super-strong Nash equilibria, the computation becomes more difficult.

Theorem 5.5. *Computing a super-strong Nash equilibrium for an ordered budget game $\mathcal{B}$ is* NP*-hard, even if the number of strategies and tasks per player is constant.*

Proof. We prove this theorem through reduction from the monotone One-In-Three SAT problem. Instances of this problem have the form $(\mathcal{U}, \mathcal{C})$ with $\mathcal{U} = \{x_1, \ldots, x_n\}$ being a set of binary variables and $\mathcal{C}$ a set of clauses over $\mathcal{U}$ with $c = x_i \vee x_j \vee x_k$, $x_i, x_j, x_k \in \mathcal{U}$ for each $c \in \mathcal{C}$. The clauses in $\mathcal{C}$ are monotone, which means they do not contain any negated literals. We therefore refer to these literals as variables. The question whether there is a truth assignment $\phi : \mathcal{U} \to \{0, 1\}$ for a pair $(\mathcal{U}, \mathcal{C})$ such that every clause $c \in \mathcal{C}$ contains exactly one variable with value 1 is NP-complete and computing such a ϕ is therefore NP-hard [32].

From $(\mathcal{U}, \mathcal{C})$, we construct an ordered budget game $\mathcal{B}$ as follows. For every variable $x_i \in \mathcal{U}$, we add a player $i \in \mathcal{N}$ with $\mathcal{T}_i = \{0_i, 1_i\}$ and $\mathcal{S}_i = \{\{0_i\}, \{1_i\}\}$. Every clause $c_j \in \mathcal{C}$ defines two resources $r_{j,0}, r_{j,1} \in \mathcal{R}$ with $b_{j,0} = 2$ and $b_{j,1} = 1$. The demands of the tasks are set to

$$0_i(r_{j,0}) = \begin{cases} 1, & \text{if } x_i \in c_j \\ 0, & \text{else} \end{cases} \qquad 1_i(r_{j,1}) = \begin{cases} 1, & \text{if } x_i \in c_j \\ 0, & \text{else} \end{cases}$$

All demands not covered by these rules are set to 0. Let k_i be the number of clauses that variable x_i occurs in. Then each of the two tasks of player i has a demand of 1 on k_i many resources and a demand of 0 on all others. The highest utility the player i can obtain is therefore also k_i and the highest possible social welfare is $K = \sum_{i \in \mathcal{N}} k_i$.

Now assume that there is a satisfying truth assignment ϕ for $(\mathcal{U}, \mathcal{C})$ which is also valid (the one-in-three property holds). In that case, every player can obtain his individual maximum utility k_i. If $\phi(x_i) = 0$, let player i choose strategy $\{0_i\}$, otherwise $\{1_i\}$. ϕ has the one-in-three property, which means that in each clause, only one variable is set to 1. Thus, every resource $r_{j,1}$ is covered by exactly one task 1_i and every resource $r_{j,0}$ by exactly two tasks 0_i and $0_{i'}$. No resource experiences a demand higher than its budget, so the order of the tasks has no influence. The resulting strategy profile, simply called s_ϕ, is a super-strong Nash equilibrium, as it is not possible for any player to achieve a higher utility in any strategy profile.

If a satisfying truth assignment ϕ exists, s_ϕ is also the only super-strong Nash equilibrium in $\mathcal{B}$. In any other state, all players could form a coalition and switch to s_ϕ without any player receiving a lower utility. On the other hand, a strategy profile with a social welfare of K is a super-strong Nash equilibrium and can be used to easily construct a satisfying truth assignment ϕ as described above. So by computing any super-strong Nash equilibrium of $\mathcal{B}$ and determining its social welfare, we immediately determine a satisfying truth assignment ϕ, provided it exists, or find out that no such assignment exists at all. So computing a super-strong Nash equilibrium is at least as hard as solving the One-In-Three 3SAT problem. □

We have seen that ordered budget games are potential games, with the social welfare as their potential function. This automatically implies a guaranteed existence of (super-strong) Nash equilibria. While the computation of Nash equilibria and even strong Nash equilibria only takes a linear number of steps, super-strong Nash equilibria are much harder to obtain, provided $\mathsf{P} \neq \mathsf{NP}$.

5.2 Efficiency of Pure Nash Equilibria

Now that we know that even super-strong Nash equilibria always exist, we are interested in their efficiency, namely the price of stability and the price of anarchy. We take another look at Corollary 5.3, which states that the optimal solution is always a super-strong Nash equilibrium. This immediately lets us draw the following conclusion.

Corollary 5.6. Let $\mathcal{B}$ be an ordered budget game. Then super-strong $PoS(\mathcal{B}) = 1$.

We show that the price of anarchy for regular Nash equilibria is at most twice as large as the price of stability, i.e. $PoA(\mathcal{B}) \leq 2$. This bound is *almost* tight in the sense that we can get arbitrarily close to it.

Theorem 5.7. *Let $\mathcal{B}$ be an ordered budget game. Then $PoA(\mathcal{B}) \leq 2$.*

Proof. Let $(\mathsf{s}, \prec)$ be a Nash equilibrium of $\mathcal{B}$ and $(\mathsf{opt}, \prec')$ be its optimal solution. We denote with $\prec^i$ the order vector we obtain if player i switches to opt_i in the strategy profile $(\mathsf{s}, \prec)$. Equation 1 shows in detail how to obtain

$$u(\mathsf{s}, \prec) \geq u(\mathsf{opt}, \prec') - u(\mathsf{s}, \prec),$$

which proves our theorem. We discuss the mathematical conversion step by step.

In (5.1), the social welfare $u(\mathsf{s}, \prec)$ of the Nash equilibrium is split into the individual utilities obtained by the tasks. The Nash inequality (5.2) states that no player i can increase his utility through a unilateral strategy change, especially if he would switch to the strategy opt_i.

For (5.3), we make a case distinction. Let $T_r(\mathsf{s})$ and $T_r(\mathsf{s}_{-i}, \mathsf{opt}_i)$ be the total demand on resource r under the combined strategies s and $(\mathsf{s}_{-i}, \mathsf{opt}_i)$ of all players, respectively. If $T_r(\mathsf{s}_{-i}, \mathsf{opt}_i) \leq b_r$, then $\sum_{t \in s_i} u_{t,r}((\mathsf{s}_{-i}, \mathsf{opt}_i), \prec^i) = \sum_{t \in s_i} t(r)$. For $T_r(\mathsf{s}_{-i}, \mathsf{opt}_i) > b_r$ and $T_r(\mathsf{s}) > b_r$, we get

$$\sum_{t \in s_i} u_{t,r}((\mathsf{s}_{-i}, \mathsf{opt}_i), \prec^i) \geq 0 = b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec)$$

Finally, if $T_r(\mathsf{s}_{-i}, \mathsf{opt}_i) > b_r$ but $T_r(\mathsf{s}) \leq b_r$, note that $u_{t',r}(\mathsf{s}, \prec) = t'(r)$ for all $j \in \mathcal{N}$ and $t' \in s_j$. The utilities cannot be increased any further, so all of the remaining budget b_r under s is allocated to player i and

$$\sum_{t \in s_i} u_{t,r}((\mathsf{s}_{-i}, \mathsf{opt}_i), \prec^i) \geq b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec)$$

Equation (5.4) simply bounds $t(r) \geq u_{t,r}(\mathsf{opt}, \prec')$, as a utility can never exceed its underlying demand. In (5.5), the set of all resources is partitioned into two

$$
\begin{aligned}
\sum_{i \in \mathcal{N}} u_i(\mathsf{s}, \prec) &= \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}} \sum_{t \in s_i} u_{t,r}(\mathsf{s}, \prec) && (5.1)\\
&\geq \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}((\mathsf{s}_{-i}, \mathsf{opt}_i), \prec^i) && (5.2)\\
&\geq \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}} \min \left(\sum_{t \in opt_i} t(r), b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) \right) && (5.3)\\
&\geq \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}} \min \left(\sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec'), b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) \right) && (5.4)\\
&= \sum_{r \in \mathcal{R}_1} \sum_{i \in \mathcal{N}} \min \left(\sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec'), b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) \right)\\
&\quad + \sum_{r \in \mathcal{R}_2} \sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec') && (5.5)\\
&\geq \sum_{r \in \mathcal{R}_1} \left(b_r - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) \right) + \sum_{r \in \mathcal{R}_2} \sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec') && (5.6)\\
&\geq \sum_{r \in \mathcal{R}_1} \left(\sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec') - \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) \right)\\
&\quad + \sum_{r \in \mathcal{R}_2} \sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec') && (5.7)\\
&\geq \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}} \sum_{t \in opt_i} u_{t,r}(\mathsf{opt}, \prec) - \sum_{r \in \mathcal{R}_1} \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) && (5.8)\\
&\geq \sum_{i \in \mathcal{N}} u_i(\mathsf{opt}, \prec') - \sum_{r \in \mathcal{R}} \sum_{j \in \mathcal{N}} \sum_{t' \in s_j} u_{t',r}(\mathsf{s}, \prec) && (5.9)\\
&= \sum_{i \in \mathcal{N}} u_i(\mathsf{opt}, \prec') - \sum_{i \in \mathcal{N}} u_i(\mathsf{s}, \prec)
\end{aligned}
$$

Equation 1: Bounding the social welfare of a Nash equilibrium in ordered budget games, which always has at least half the social welfare of the optimal solution.

disjoint subsets $\mathcal{R}_1$ and $\mathcal{R}_2$. $\mathcal{R}_1$ contains exactly those resources for which at least one player evaluates the minimum statement to the second expression. We omit all other players and restrict the formula to this special player in order to obtain (5.6). The total utility gained from a resource by all players combined can never exceed its budget, so we can lower bound b_r in (5.7). By re-combining the two sums (5.8) and extending the sum over $\mathcal{R}_1$ to all resources in $\mathcal{R}$ (5.9), we obtain our result. □

Although this proof only takes unilateral strategy changes into account (cf. (5.2)), the result is also valid for both strong and super-strong Nash equilibria, since every super-strong Nash equilibrium is also a strong Nash equilibrium and every strong Nash equilibrium is also a Nash equilibrium. It remains to show that this bound is *almost* tight.

Theorem 5.8. *For every $\varepsilon > 0$, there exists an ordered budget game $\mathcal{B}_\varepsilon$ with $PoA(\mathcal{B}_\varepsilon) = 2 - \varepsilon$.*

Proof. Consider the game $\mathcal{B}_\varepsilon$, which is defined as follows:

- $\mathcal{N} = \{1, 2\}$
- $\mathcal{R} = \{r_1, r_2\}$
- $b_1 = 1$ and $b_2 = 1 - \varepsilon$
- $\mathcal{S}_1 = \left\{s_1^1 = \{t_1^1 = (1, 0)\}\ ,\ s_1^2 = \{t_1^2 = (0, 1 - \varepsilon)\}\right\}$
  $\mathcal{S}_2 = \{s_2 = \{t_2 = (1, 0)\}\}$

Player 2 has only one strategy, so we focus on player 1. If the players choose different resources, i.e. if player 1 plays strategy s_1^2, then the social welfare is $u((s_1^1, s_2), \prec) = 1 - \varepsilon + 1 = 2 - \varepsilon$. This is also the optimal solution, as $u(s_1^2, s_2, \prec') = 1$. Both statements are independent of the structure in $\prec$ and $\prec'$.

The strategy profile $((s_1^1, s_2), \prec)$ with $t_1^2 \prec_1 t_2$ (and the rest arbitrary) is a Nash equilibrium, as the utility of player 1 is 1, which also is his personal maximum. This leaves no remaining budget for player 2, so the social welfare is 1, as well. As a result, the price of anarchy is

$$PoA(\mathcal{B}) \leq \frac{2 - \varepsilon}{1} = 2 - \varepsilon$$

Note that we can extend the game $\mathcal{B}$ to n instead of 2 players by using $\frac{n}{2}$ separate instances of this two-player game. □

Our upper bound for the price of anarchy in ordered budget games is tight with respect to any $\varepsilon > 0$ and the social welfare of any Nash equilibrium is at least half as big as that of the optimal value.

5.3 Properties of the Best-Response Dynamic

When simulating the best-response dynamic, it may happen that more than one player is able to perform an improving move. If we want to allow multiple simultaneous strategy changes, we need an additional rule for the case that two or more tasks of different players are newly allocated to the same resource. In strategic games, such rules are not necessary as the outcome only depends on the strategies, but not on the order in which these strategy choices were made. If multiple strategy changes are executed simultaneously in a strategic budget game, the result is identical to any sequential execution. This is different in ordered budget games as already seen in Figure 5.1 at the beginning of this chapter and the reason why they are not considered to be strategic games. On the other hand, one normally assumes that there are no simultaneous strategy changes in strategic games, unless dealing with coalitions of players, as they can easily lead to a cycle. With ordered budget games, we have more possibilities regarding this matter, precisely because the outcome can be influenced by choosing a sequential order for the strategy changes. Since ordered budget games already possess the finite-improvement property, we want make sure that this is retained.

We introduce two tie-breaking rules which guarantee that an ordered budget game still converges towards a Nash equilibrium. Let p be a function that assigns a unique priority to every player i. This priority can depend on the current strategy profile. Whenever simultaneous strategy changes occur, they are executed sequentially in decreasing order of the priorities of the players involved. If both player i and j change their strategy with t_i and t_j being new tasks of i and j, respectively, then we set $t_i \prec_r t_j$ for all resources r if and only if the priority of i is larger than the priority of j before the strategy changes. p is called a *priority function.*

Definition 5.9 (Fixed Priorities)**.** A injective priority function p_{fix} of the form $\mathsf{p}_{\text{fix}} : \mathcal{N} \to \mathbb{N}$ is called a *fixed priority function.*

When using a fixed priority function, the current strategy profile has no influence on the priorities. Instead, they are directly associated with the players and do not change over the course of the game.

Definition 5.10 (Utility-Based Priorities)**.** A priority function $\mathsf{p}_{\max}$ of the form $\mathsf{p}_{\max} : \mathcal{N} \times \mathcal{S} \times \prec \to \mathbb{N}$ is called a *utility-based priority function* if

- $u_i(\mathsf{s}, \prec) < u_j(\mathsf{s}, \prec) \Rightarrow \mathsf{p}_{\max}(i, \mathsf{s}, \prec) < \mathsf{p}_{\max}(j, \mathsf{s}, \prec)$
- $\mathsf{p}_{\max}(i, \mathsf{s}, \prec) = \mathsf{p}_{\max}(j, \mathsf{s}, \prec) \Rightarrow i = j$

The first property states that a player with a higher utility also gets a higher priority. The second property of this definition ensures that for any given strategy profile, the priority function is still injective with respect to the players. If two or more players have the same utility, the priorities among each other can be arbitrary. The next result states that these two rules keep the finite-improvement property of ordered budget games intact.

Theorem 5.11. *Let $\mathcal{B}$ be an ordered budget game which allows multiple simultaneous strategy changes. If $\mathcal{B}$ uses either p_{fix} or p_{max} to determine the priorities of the players, then it reaches a Nash equilibrium after finitely many improvement steps.*

Proof. Let $(\mathsf{s}, \prec)$ be the current strategy profile of $\mathcal{B}$ and $\vec{u}(\mathsf{s}, \prec) \in \mathbb{R}^n_{\geq 0}$ the vector containing the current utilities of all players. We call $\vec{u}(\mathsf{s}, \prec)$ the utility vector of $\mathcal{B}$ under $(\mathsf{s}, \prec)$. $\vec{u}(\mathsf{s})$ is sorted in decreasing order of the player priorities, i.e. the player with his utility at position i has a higher priority than the player with his utility at position $i+1$. For p_{max}, this order may change over time and the utility vector is newly sorted whenever necessary.

Let $N \subseteq \mathcal{N}$ be the set of players who are simultaneously performing a strategy change. Each player would improve his utility if he were the only player in N. Let $(\mathsf{s}', \prec')$ be the resulting strategy profile after all simultaneous strategy changes. Using either p_{fix} or p_{max} only influences $\prec'$. We show that $\vec{u}(\mathsf{s}, \prec) <_{\text{lex}} \vec{u}(\mathsf{s}', \prec')$ for both priority functions, with $<_{\text{lex}}$ being the lexicographical order between the two vectors.

Let $i \in N$ be the player with the highest priority among those in N. For p_{fix}, i receives exactly the increase in utility he expected from the strategy change. From all the players in N, he is also the one with the smallest index in both $\overrightarrow{u}(s)$ and $\overrightarrow{u}(s')$. This already warrants that $\vec{u}(\mathsf{s}, \prec) <_{\text{lex}} \vec{u}(\mathsf{s}', \prec')$.

For p_{max}, the same argumentation holds if the position of i in the two utility vectors does not change. Otherwise, his position in $\vec{u}(\mathsf{s}, \prec)$ is now occupied by a player j with $u_j(\mathsf{s}, \prec) < u_i(\mathsf{s}', \prec') < u_j(\mathsf{s}', \prec')$. Again, we have $\vec{u}(\mathsf{s}, \prec) <_{\text{lex}} \vec{u}(\mathsf{s}', \prec')$. Since the utility vectors are strongly monotonically increasing, but bounded by the vectors containing the maximal utility of each player, a NE is reached after finitely many steps. □

The first section of this chapter has shown that the best-response dynamic yields a strong Nash equilibrium after just n improving steps if we start with an *empty* game and insert the players one after another. However, the situation is different when the initial strategy profile can be arbitrary. To conclude this chapter, we study how fast the best-response dynamic converges in general. The number of improving steps before a Nash equilibrium is reached can be exponential in the number of players, even if the number of strategies per player is constant.

Theorem 5.12. *Let $n \in \mathbb{N}$. There is an ordered budget game $\mathcal{B}_n$ with a description length polynomial in n and a strategy profile $(\mathsf{s}^n_0, \prec^n_0)$ such that the number of best-response improvement steps from $(\mathsf{s}^n_0, \prec^n_0)$ to a Nash equilibrium $(\mathsf{s}^n_{NE}, \prec^n_{NE})$ is exponential in n.*

For the following proof, we assume that p_{fix} is used as a priority function for simultaneous strategy changes, with the indices of the players as their priorities.

Proof. We give a recursive construction of the game $\mathcal{B}_n$. It contains the sub-game $\mathcal{B}_{n-1}$, for which there is exactly one path of best-response improvement steps with length $\mathcal{O}(2^{n-1})$ if $(\mathsf{s}^{n-1}_0, \prec^{n-1}_0)$ is the initial strategy profile. $\mathcal{B}_{n-1}$ is *executed* once, meaning that the players follow this best-response path until they reach a Nash equilibrium. After that, the sub-game is reseted to its original state and then executed once more along the same path. In the end, $\mathcal{B}_n$ has reached a Nash equilibrium after $\mathcal{O}(2^n)$ steps.

Each player i has only two tasks and he can choose exactly one of them, i.e. $\mathcal{S}_i = \{\{t_i^1\}, \{t_i^2\}\}$. By labeling the strategies of player i with 0_i and 1_i, each strategy profile can be written as a binary number. The initial strategy profile $(\mathsf{s}_0^n, \prec_0^n)$ can be regarded as 0 (with leading zeros) and the first execution of $\mathcal{B}_{n-1}$ counts this number up to $2^{n-1} - 1$. The reset of $\mathcal{B}_{n-1}$ then corresponds to increasing that number by 1 to 2^{n-1} and the second execution of $\mathcal{B}_{n-1}$ continues counting it up to $2^n - 1$. In the resulting Nash equilibrium of $\mathcal{B}_n$, every player i plays strategy 1_i.

Since the strategies only contain single tasks, the ordering of these tasks is also an ordering of the players, so we can abuse notation and say $i \prec j$ for players i and j in a strategy profile $((\dots, s_i, \dots, s_j, \dots), \prec)$ if $t_i \prec_r t_j$, $t_i \in s_i$ and $t_j \in s_j$, holds for any (and therefore all) resource r.

In the following construction, the demand of any task t on any resource r is either b_r or 0. Since we use a construction in this proof, we can simply say that t is connected to r in the first or not connected to r in the second case. Due to this property, there is always at most one player who receives a positive utility from a resource.

We also need some additional notations for our proof. The only Nash equilibrium that is reached in our construction of $\mathcal{B}_n$ is the state where every player i plays strategy 1_i and in which the players reach their final state in descending order, i.e. $i \prec j$ for $j < i$. Let p_i^q be the utility of player i in the Nash equilibrium of the sub-game $\mathcal{B}_q$, $q \in \mathbb{N}$. Also, let $(\mathsf{s}_0^n, \prec_0^n)$ be the initial strategy profile of the game. In $\prec_0^n$, let $i \prec j$ for $j < i$. Intuitively, this means that players with a higher index value also have a higher priority in the beginning.

We can now start with the construction of $\mathcal{B}_n$. For $n = 1$, we build an instance $\mathcal{B}_1$ with a single player:

- $\mathcal{N}_1 = \{1\}$
- $\mathcal{R}_1 = \{r_1, r_2\}$
- $b_1 = 1$, $b_2 = 2$
- $\mathcal{S}_1 = \{0_1 = \{t_1^0 = (1, 0)\}$, $1_1 = \{t_1^1 = (0, 2)\}\}$

The initial strategy profile is $\mathsf{s}_0^1 = (0_1, \prec_0^1)$ with only one choice for $\prec_0^1$ as there is only one player. After one improvement step, $\mathcal{B}_1$ has reached a Nash equilibrium.

For $n > 1$, we extend the preexisting game $\mathcal{B}_{n-1}$. This is done in two parts. First, we explain how to reset $\mathcal{B}_{n-1}$ to $(\mathsf{s}_0^{n-1}, \prec_0^{n-1})$. Let m denote the number of all players in $\mathcal{B}_{n-1}$. We introduce a new player n and a number of additional resources:

- $\mathcal{N}'_n = \mathcal{N}_{n-1} \cup \{n\}$

- $\mathcal{R}'_n = \mathcal{R}_{n-1} \cup \{r^0_1, \dots, r^0_m\} \cup \{r^1_1, \dots, r^1_m\} \cup \{r^2_1, \dots, r^2_m\} \cup \{r, r'\}$
- $b_r = 1$ for $r \in \{r^0_1, \dots, r^0_m, r^1_1, \dots, r^1_m\}$
  $b_r = p_i^{n-1} + 2$ for $r \in \{r^2_0, \dots, r^m_0, r^2_1\}$
  $b_r = n - 2$
  $b_{r'} = \sum_{i=1}^m p_i^{n-1} + 2m$
- $\mathcal{S}_n = \{0_n = \{t^0_n\}\ ,\ 1_n = \{t^1_n\}\}$

For $i = 1, \dots, n-1$, we connect r^0_i to task t^0_i and r^1_i to task t^1_i. Therefore, every task in $\mathcal{B}_{n-1}$ gets an (for now) exclusive resource. Since the budgets of these resources are the same, they do not influence the best-response dynamic in $\mathcal{B}_{n-1}$. We also connect all r^0_i to t^1_n and r to t^0_n. The initial strategy profile is $(\mathsf{s}^n_0, \prec^n_0)$ with $\mathsf{s}^n_0 = (\mathsf{s}^{n-1}_0, 0_n)$ and $\prec^n_0$ being identical to $\prec^{n-1}_0$, with player n having the highest priority among all players. His corresponding utility is $u_n(\mathsf{s}^n_0, \prec^n_0) = n - 2$, since his current task is only connected to resource r. If he would change his strategy now, his utility would become 0, since the whole budget of every resource connected to t^1_n is already being obtained by a different player. Only when all other players $i = 1, \dots, n-1$ play strategy $1_i = \{t^1_i\}$ player n can increase his utility by 1 by switching to strategy $1_n = \{t^n_1\}$. This happens exactly when the best-response dynamic has reached a Nash equilibrium for $\mathcal{B}_{n-1}$.

This alone does not yet reset the sub-game $\mathcal{B}_{n-1}$. As the next step, we further extend the current game such that once player n has switched to 1_n, all other players $i = 1, \dots, m$ switch their strategy back to 0_i in order to recreate $(\mathsf{s}^{n-1}_0, \prec^{n-1}_0)$. To achieve this, we introduce some additional connections between already existing tasks and resources. For $i = 1, \dots, m$, connect each task t^0_i to resource r^2_i. Additionally, connect the task t^0_n to every resource r^2_i. Since player n has the highest priority of all players, he receives the total budgets of all resources r^2_i, yielding an additional utility of $\sum_{i=1}^m p_i^{n-1} + 2m =: \Sigma_1$. Finally, we connect task t^n_1 to resource r'. Thus, player n has the same increase in utility for strategy 1_n. No strategy of any other player uses r', which means that player n is guaranteed to receive that resource's budget. Since we increased the potential utility of both 0_n and 1_n, the behavior of player n is not affected. However, as soon as he switches to 1_n, all the budgets of the resources r^2_i become available again. This will cause the players $1, \dots, m$ to change their strategies back to 0_i (just like in the initial strategy profile), increasing their utility from $p_i^{n-1} + 1$ to at least $p_i^{n-1} + 2$. All these strategy changes can occur simultaneously. By using p_{fix} as a priority function, we ensure that the order of the players is again $\prec^{n-1}_0$. The construction up to this point is illustrated in Figure 5.2.

To restart the game $\mathcal{B}_{n-1}$, we apply a similar trick as before. Just like the strategy change of player n caused $\mathcal{B}_{n-1}$ to reset, we introduce an auxiliary player aux_n, whose strategy change will start $\mathcal{B}_{n-1}$ a second time. We extend the current version of $\mathcal{B}_n$ as follows:

- $\mathcal{N}_n = \mathcal{N}'_n \cup \{aux_n\}$
- $\mathcal{R}_n = \mathcal{R}'_n \cup \{r^3_1, \ldots, r^3_n\} \cup \{r^4_1, \ldots, r^4_n\} \cup \{r^5_1, \ldots, r^5_{n-1}\} \cup \{r^{aux_n}\}$
- $b_r = 1$

  $b_r = p^i_{n-1} + 2$

  $b_{r^{aux_n}} = m + \sum_{i=1}^m p^i_{n-1} =: \Sigma_2$
- $\mathcal{S}_{aux_n} = \{0_{aux_n} = \{t^0_{aux_n}\}\ ,\ 1_{aux_n} = \{t^1_{aux_n}\}\}$

We connect t^0_{aux} to all r^5_i, $i = 1, \ldots, n-1$. The initial strategy of aux_n is 0_{aux_n} and we set $aux_n \prec n$ in $(\mathsf{s}^n_0, \prec^n_0)$. Initially, the auxiliary player has a utility of $\Sigma_1 = 2m + \sum_{i=1}^m p^{n-1}_i$. We also connect $t^1_{aux_n}$ to r^{aux_n} and to all resources r^3_i for $i \in \{1, \ldots, m, n\}$. Finally, for every player $i = 1, \ldots, m$, we connect r^4_i to t^0_i, r^3_i to t^1_i and r^5_i to t^1_i. For player n, we establish these connections the other way around, such that r^3_n is connected to t^0_n and r^4_n to t^1_n.

Again, the effects of r^3_i and r^4_i regarding $\mathcal{B}_{n-1}$ cancel out. Only when every player i in $\mathcal{B}_{n-1}$ plays strategy 0_i and player n plays strategy 1_n, the auxiliary player will switch to 1_{aux_n} and obtain a utility of $\sum_{i=1}^m p^i_{n-1} + 2m + 1 = \Sigma_2 + 1$. This frees the budget of all resources r^5_i and the utility of every strategy 1_i in $\mathcal{B}_{n-1}$ is increased by the same amount we increased the utility of 0_i in the first part of our construction. The effects which originally reseted $\mathcal{B}_{n-1}$ are now canceled out by the additional resources and the game $\mathcal{B}_{n-1}$ is executed once more, only player n remains idle. When every players i in $\mathcal{B}_n$ is playing strategy 1_i, the game has reached a Nash equilibrium. Figure 5.3 illustrates this second part of the construction.

By construction, the number of improvement steps between the initial strategy profile $(\mathsf{s}^n_0, \prec^n_0)$ and the Nash equilibrium is doubled when stepping from $\mathcal{B}_{n-1}$ to $\mathcal{B}_n$. Counting the auxiliary players, we see that $|\mathcal{N}_n| = 2 \cdot n - 1$. For every player, we have only two tasks and two strategies. Finally, the number of resources introduced to $\mathcal{B}_{n-1}$ in order to construct $\mathcal{B}_n$ is linear in n. In the end, the description length of $\mathcal{B}_n$ is polynomial in n. □

Since ordered budget games emphasizes the order of strategy changes, letting the players form a Nash equilibria on their own is more complicated than in strategic budget games. Computing a Nash equilibrium is easy according to Theorem 5.4. However, if the game starts in an arbitrary strategy profile s_0, the best-response dynamic may need exponentially long to actually reach it. Due to the structure of the utility functions, it is not sufficient for all players to simple assume their strategies in the corresponding Nash equilibrium all at once, as the outcome also depends on s_0 and on the priority functions.

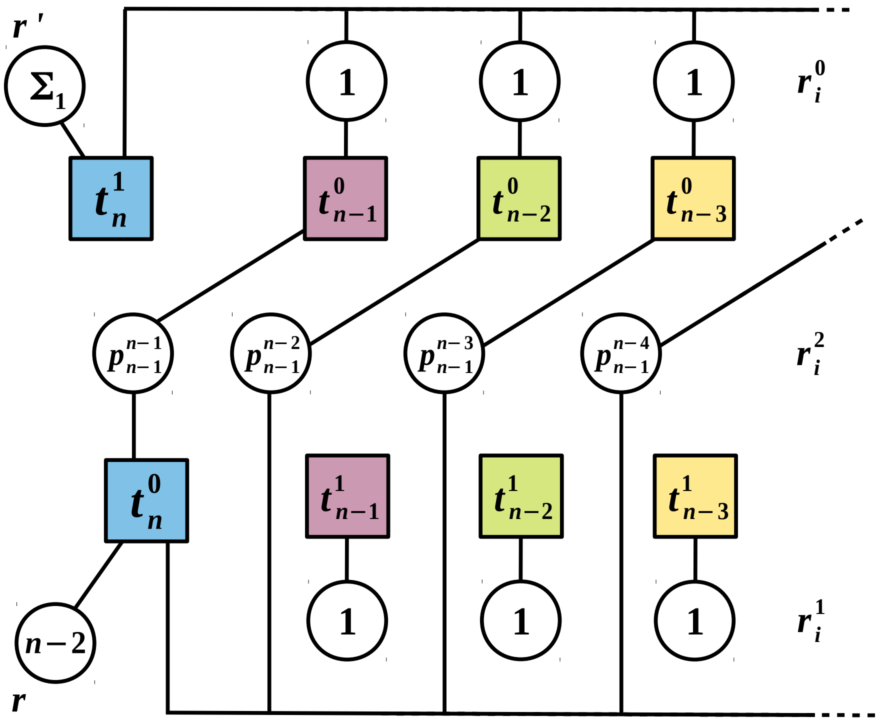

Figure 5.2: First part of the extension of the preexisting game $\mathcal{B}_{n-1}$. Once the players in $\mathcal{B}_{n-1}$ have formed their Nash equilibrium, player n performs a strategy change from t_n^0 to t_n^1, which in turn resets $\mathcal{B}_{n-1}$ to its original strategy profile, i.e. other player i switches back to strategy t_i^0. We define $\Sigma_1 := 2m + \sum_{i=1}^{m} p_{n-1}^i$.

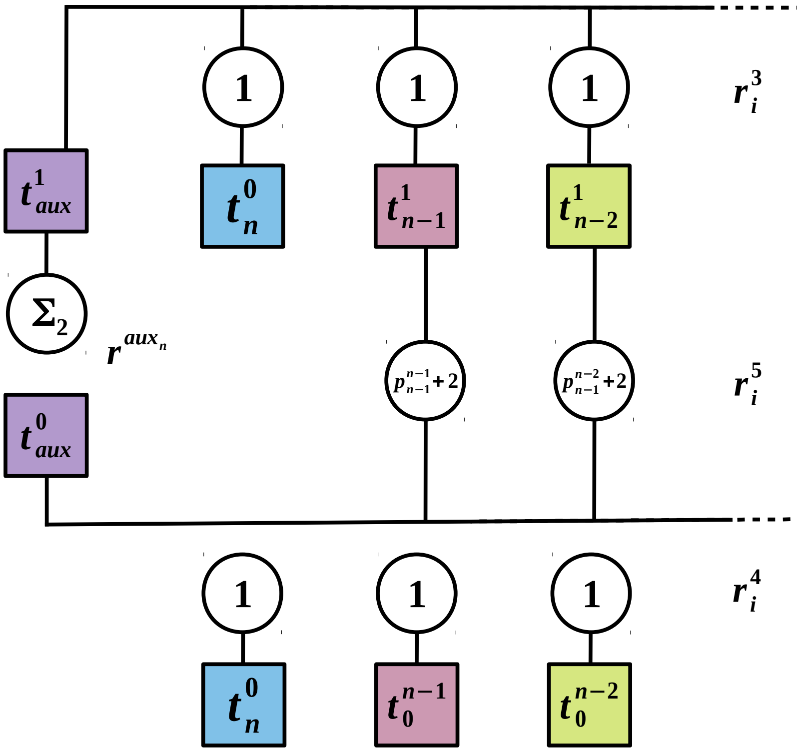

Figure 5.3: Second part of the extension of the preexisting game $\mathcal{B}_{n-1}$. Once player n has switched to t_n^1 and $\mathcal{B}_{n-1}$ has been reseted to its original strategy profile, the player aux switches to t^1_{aux}, which starts another execution of the best-response dynamic of $\mathcal{B}_{n-1}$. We define $\Sigma_2 := m + \sum_{i=1}^{m} p^i_{n-1}$

CHAPTER 6

Approximate Pure Nash Equilibria in Budget Games

By now, we already know that a budget game does not necessarily have a Nash equilibrium (cf. Definition 4.1 and Corollary 4.2). Since we are still interested in finding stable states, we turn towards approximate equilibria. In an α-approximate Nash equilibria, no unilateral strategy change increases the utility of the corresponding player by more than some fixed factor $\alpha \geq 1$. This can be motivated by the fact that players usually do not care for very small improvements, especially if the current strategy profile is somewhat stable. By deviating from his current strategy, a player could destroy this stability and cause additional strategy changes by the other players. This process may then ultimately harm him. When considering approximate Nash equilibria, we expect a player to only change his strategy if this yields a significant improvement.

In this chapter, we focus on the existence and properties of approximate Nash equilibria in (strategic) δ-share budget games. In some cases, the value of δ does not affect our results. We then omit the δ and use only the term budget game.

One of the two main aspects of our model is the distribution of budgets. If the number of players or their demands are too small, we lose this aspect, as then there can then be no situations in which the total demand on a resource actually exceeds its budget. In this case, computing a Nash equilibria is equivalent to computing the optimal solution and both can be done efficiently through greedy maximization of the social welfare.

Lemma 6.1. Let $\mathcal{B}$ be a δ-share budget game with n players. If $n\delta \leq 1$, the optimal solution of $\mathcal{B}$ is its only Nash equilibrium and can be computed in $\mathcal{O}(n \cdot m)$.

Proof. By definition, the demand of player i on resource r is at most δb_r for any strategy s_i. Since this holds for all players, the total demand $T_r(\mathsf{s})$ is at most $n\delta b_r$ for any strategy profile s. The assumption $n\delta \leq 1$ yields $T_r(\mathsf{s}) \leq n\delta b_r \leq b_r$. The total demand on r under s does not exceed its budget, so every player receives exactly his demand. This holds especially for the strategy profile $\mathsf{opt} = (s_1^{\text{opt}}, \dots, s_n^{\text{opt}})$ with $s_i^{\text{opt}} := \max_{s \in \mathcal{S}_i} \left(\sum_{r \in \mathcal{R}} s(r)\right)$, which is both a Nash equilibrium and maximizes social welfare. To compute opt, it therefore suffices to determine s_i^{opt} for every player i. Any other strategy profile where player i receives a smaller utility than in s_i^{opt} cannot be a Nash equilibrium, as a switch to s_i^{opt} would always increase his utility to $\sum_{r \in \mathcal{R}} s_i^{\text{opt}}(r)$. □

From now on, we always assume $n\delta > 1$, as the other case does not require us to resort to approximate Nash equilibria.

6.1 Bounds on the Existence of Approximate Pure Nash Equilibria

To start our analysis of approximate Nash equilibria, we take another look at the δ-share budget game $\mathcal{B}_0$ from Definition 4.1. This time, we set $\delta = 1$ and $\gamma = \frac{1}{2}$. With these values, we do not need any auxiliary players and thus set $\sigma = 0$ and $n = 0$. The resulting game is shown in Figure 6.1. The two possible utilities are $u^- = \frac{7}{6}$ and $u^+ = \frac{4}{3}$, with $u^+ \leq \alpha \cdot u^-$ for $\alpha \geq \frac{8}{7}$. So if we choose α large enough, the player with the smaller utility of u^- is unable to improve his utility by more than this constant factor. In other words, $\mathcal{B}_0$ has an α-approximate Nash equilibrium for every $\alpha > \frac{8}{7}$. In fact, every strategy profile is such a stable state. It becomes apparent that every strategy profile of every budget game can become an α-approximate Nash equilibrium if we just choose α large enough. On the other hand, $\mathcal{B}_0$ also illustrates that there may not be an α-approximate Nash equilibrium if α is too small, in this case $\alpha \in \left[1, \frac{8}{7}\right]$. If we alter the demands and set $\delta = \frac{3}{4}$ and $\gamma = \frac{1}{2}$ as well as introduce auxiliary players as described in the definition of $\mathcal{B}_0$, approximate Nash equilibria already exist for $\alpha > \frac{13}{12}$. So for every budget game $\mathcal{B}$, there has to be a threshold α_δ based on δ such that $\mathcal{B}$ has an α-approximate Nash equilibrium for $\alpha > \alpha_\delta$ and has none for $\alpha \leq \alpha_\delta$.

In this section, we give both upper and lower bounds on α_δ, which both depend on δ. We first introduce an *approximate* potential function for budget games, which will be our main tool throughout this chapter.

Definition 6.2. Let $\phi(\mathsf{s}) := \sum_{r \in \mathcal{R}} \phi_r(\mathsf{s})$ with

$$\phi_r(\mathsf{s}) := \begin{cases} T_r(\mathsf{s}) & \text{if } T_r(\mathsf{s}) \leq b_r \\ b_r + \int_{b_r}^{T_r(\mathsf{s})} \frac{b_r}{x} \, \mathrm{d}x & \text{else.} \end{cases}$$

We call ϕ an *approximate potential function* for budget games.

While $\phi(\mathsf{s})$ stands for the total potential of the game under s, $\phi_r(\mathsf{s})$ is considered to be the potential of the individual resource r. We now use ϕ to give an upper bound on the threshold for the guaranteed existence of approximate Nash equilibria.

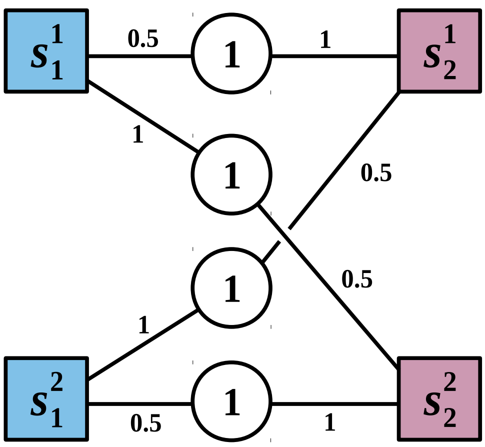

Figure 6.1: A budget game without a Nash equilibrium. For $\alpha \geq \frac{8}{7}$, however, every strategy profile is an α-approximate Nash equilibrium.

Our goal is to find the smallest value for α such that ϕ still increases with every α-move. Then, ϕ is a generalized ordinal potential function for budget games.

Definition 6.3. Let $\delta > 0$ and W_{-1} denote the lower branch of the Lambert W function. For convenience, we define

$$w_\delta := \left(-\frac{1}{2} W_{-1}\left(-2e^{(-\delta)-2}\right)\right).$$

We then define the upper bound α_δ^u on α_δ as

$$\alpha_\delta^u := \frac{w_\delta}{\delta} \cdot \left(\ln(w) - w + \delta + 1\right).$$

α_δ^u will be our upper bound on α_δ, as stated by the following theorem.

Theorem 6.4. *Let $\delta > 0$ and $\mathcal{B}$ be a δ-share budget game. For $\alpha \geq \alpha_\delta^u$, $\mathcal{B}$ reaches an α-approximate Nash equilibrium after a finite number of α-moves.*

Proof. For strategy profile s and resource r, define $T_{-i,r}(\mathsf{s}) := T_r(\mathsf{s}) - s_i(r)$ as the total demand on r under s excluding that of player i. Furthermore, let $\phi_r(\mathsf{s}_{-i})$ be the potential of r omitting the demand of player i, i.e.

$$\phi_r(\mathsf{s}_{-i}) := \begin{cases} T_{-i,r}(\mathsf{s}) & \text{if } T_{-i,r}(\mathsf{s}) \leq b_r \\ b_r + \int_{b_r}^{T_{-i,r}(\mathsf{s})} \frac{b_r}{x}\,\mathrm{d}x & \text{else.} \end{cases}$$

With this definition, $\phi_{i,r}(\mathsf{s}) := \phi_r(\mathsf{s}) - \phi_r(\mathsf{s}_{-i})$ determines how much of r's potential is caused by player i if we regard the impact of the players one by one, with player i being considered last (cf. Figure 6.2).

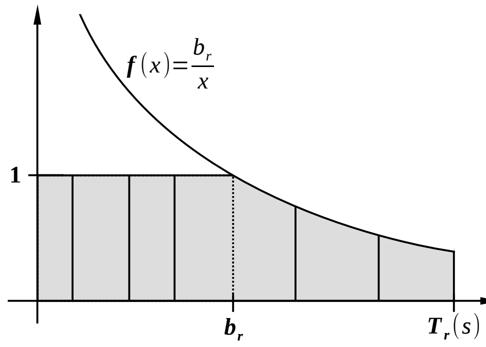

Figure 6.2: The potential of r split over the players. As the potential only depends on the total demand, we can construct it considering player by player. The width of each segment equals the demand of that player.

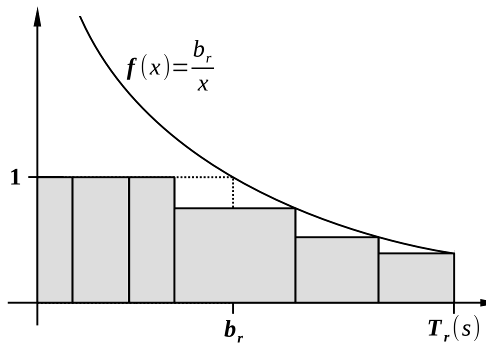

Figure 6.3: Following the same order of the players as in Figure 6.2, the utilities of each player the moment he is added to the resource can be visualized. In this example, the first three players together do not yet exceed the budget of the resource. Once the fourth player is added, the utility of the other three is reduced. However, this is not recorded in the representation above. The same holds for all decreases in utility after a player was added to the resource. Note that the actual utility of the last player added to the resource is also the one shown in the figure.

Note that $u_{i,r}(\mathsf{s}) \leq \phi_{i,r}(\mathsf{s})$ always holds, as can be seen in Figure 6.3. We are going to show that any strategy change of a player i, which improves his utility by a factor of at least α_δ^u, also results in an increase of ϕ This implies that the game does not possess any cycles and thus always reaches an α-approximate Nash equilibrium after finitely many steps.

For now, assume

$$\alpha \geq \max_{\substack{i \in \mathcal{N} \\ r \in \mathcal{R}}} \left(\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} \right), \tag{6.1}$$

which trivially implies $\phi_{i,r}(\mathsf{s}) \leq \alpha \cdot u_{i,r}(\mathsf{s})\ \forall\ i \in \mathcal{N}, r \in \mathcal{R}$. Assume that under the strategy profile s, player i changes his strategy from s_i to s_i', increasing his overall utility by a factor of more than α in the process. We obtain

$$\begin{aligned}
\Delta\phi &= \phi(\mathsf{s}_{-i}, s_i') - \phi(\mathsf{s}) \\
&= \sum_{r \in \mathcal{R}} \phi_{i,r}(\mathsf{s}_{-i}, s_i') - \sum_{r \in \mathcal{R}} \phi_{i,r}(\mathsf{s}) \\
&\geq \sum_{r \in \mathcal{R}} u_{i,r}(\mathsf{s}_{-i}, s_i') - \alpha \cdot \sum_{r \in \mathcal{R}} u_{i,r}(\mathsf{s}) \\
&= u_i(\mathsf{s}_{-i}, s_i') - \alpha \cdot u_i(\mathsf{s}) \\
&> \alpha \cdot u_i(\mathsf{s}) - \alpha \cdot u_i(\mathsf{s}) \\
&= \alpha \cdot (u_i(\mathsf{s}) - u_i(\mathsf{s})) = 0.
\end{aligned}$$

We see that ϕ indeed grows with every α-move. It remains to be shown that our upper bound α_δ^u satisfies Condition 6.1, i.e.

$$\alpha_\delta^u \geq \max_{i \in \mathcal{N}, r \in \mathcal{R}} \left(\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} \right).$$

We make a case distinction based on the size of $T_{-i,r}(\mathsf{s})$ and look at the two cases $T_{-i,r}(\mathsf{s}) < b_r$ and $T_{-i,r}(\mathsf{s}) \geq b_r$. From now on, we consider both r and s to be arbitrary, but fixed, and simply write t_{-i} instead of $T_{-i,r}(\mathsf{s})$ and s_i instead of $s_i(r)$. The first case we look at is $t_{-i} < b_r$. Note that we can assume $t_{-i} + s_i > b_r$, because otherwise the ratio between potential and utility of i at r would be 1. For $t_{-i} + s_i > b_r$, the ratio looks as follows:

$$\begin{aligned}
\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} &= \frac{b_r - t_{-i} + \int_{b_r}^{t_{-i}+s_i} \frac{b_r}{x}\,\mathrm{d}x}{\frac{b_r \cdot s_i}{t_{-i}+s_i}} \\
&= (t_{-i} + s_i) \cdot \frac{b_r - t_{-i} + b_r \left(\ln(t_{-i} + s_i) - \ln(b_r)\right)}{b_r \cdot s_i} \\
&= (t_{-i} + s_i) \cdot \frac{b_r - t_{-i} + b_r \cdot \ln(\frac{t_{-i}+s_i}{b_r})}{b_r \cdot s_i}
\end{aligned}$$

First we show that this ratio does not decrease as s_i grows larger. Its derivative by s_i is

$$\frac{\partial}{\partial s_i}\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} = \frac{b_r(s_i - t_{-i}) - b_r \cdot t_{-i} \cdot \ln\left(\frac{t_{-i}+s_i}{b_r}\right) + t_{-i}^2}{(b_r \cdot s_i)^2}.$$

The numerator can be bounded by

$$\begin{aligned}
&b_r(s_i - t_{-i}) - b_r \cdot t_{-i} \cdot \ln\left(\frac{t_{-i}+s_i}{b_r}\right) + t_{-i}^2\\
&= b_r(s_i - t_{-i}) - b_r \cdot t_{-i} \cdot \ln\left(1 + \frac{t_{-i}+s_i-b_r}{b_r}\right) + t_{-i}^2\\
&\geq b_r(s_i - t_{-i}) - b_r \cdot t_{-i} \cdot \frac{t_{-i}+s_i-b_r}{b_r} + t_{-i}^2\\
&= b_r \cdot s_i - b_r \cdot t_{-i} - t_{-i}^2 - s_i \cdot t_{-i} + b_r \cdot t_{-i} + t_{-i}^2\\
&= b_r \cdot s_i - s_i \cdot t_{-i} = s_i(b_r - t_{-i}) > 0.
\end{aligned}$$

and so the original ratio gets only worse for bigger values of s_i. From now on, we substitute s_i by its upper bound δb_r. Now we determine the worst-case value for t_{-i}. The derivative by t_{-i} is

$$\frac{\partial}{\partial t_{-i}}\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} = \frac{b_r \cdot \ln\left(\frac{t_{-i}+\delta b_r}{b_r}\right) + 2b_r - 2t_{-i} - \delta b_r}{\delta b_r^2}.$$

We are looking for the root of this function.

$$\begin{aligned}
&b_r \cdot \ln\left(\frac{t_{-i}+\delta b_r}{b_r}\right) + 2b_r - 2t_{-i} - \delta b_r = 0\\
&\Leftrightarrow \ln\left(\frac{t_{-i}+\delta b_r}{b_r}\right) = 2\frac{t_{-i}}{b_r} + \delta - 2\\
&\Leftrightarrow \frac{t_{-i}+\delta b_r}{b_r} = e^{2\frac{t_{-i}}{b_r}} \cdot e^{\delta-2}\\
&\Leftrightarrow (-2)\frac{t_{-i}}{b_r} - 2\delta = (-2)e^{2\frac{t_{-i}}{b_r}} \cdot e^{\delta-2}\\
&\Leftrightarrow \left(-2\frac{t_{-i}}{b_r} - 2\delta\right) \cdot e^{(-2)\frac{t_{-i}}{b_r}-2\delta} = (-2)e^{(-\delta)-2}\\
&\Leftrightarrow (-2)\frac{t_{-i}}{b_r} - 2\delta = W_{-1}\left(-2e^{(-\delta)-2}\right)\\
&\Leftrightarrow t_{-i} = \left(-\frac{1}{2}\right) b_r W_{-1}\left(-2e^{(-\delta)-2}\right) - \delta b_r\\
&\Leftrightarrow t_{-i} = b_r(w-\delta) \text{ for } w = \left(-\frac{1}{2}W_{-1}\left(-2e^{(-\delta)-2}\right)\right)
\end{aligned}$$

W_{-1} denotes the lower branch of the Lambert W function, which is used since $b_r < t_{-i} + s_i = t_{-i} + \delta b_r$ and therefore the value $W = (-2)\frac{t_{-i}}{b_r} - 2\delta < (-2)\frac{b_r(1-\delta)}{b_r} -$

$2\delta = -2 < -1$. Using the obtained values for both s_i and t_{-i}, the worst-case ratio between the potential caused at a resource r and the actual utility is

$$\alpha_\delta^u = w \cdot \frac{\ln(w) - w + \delta + 1}{\delta} \text{ for } w = \left(-\frac{1}{2}W_{-1}\left(-2e^{(-\delta)-2}\right)\right).$$

For $t_{-i} > b_r(w - \delta)$, the ratio we seek only becomes smaller as t_{-i} grows. This especially holds for $t_{-i} \geq b_r$, when the ratio between potential and utility becomes

$$\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} = \frac{\int_{t_{-i}}^{t_{-i}+s_i} \frac{b_r}{x}\,\mathrm{d}x}{\frac{b_r \cdot s_i}{t_{-i}+s_i}} = (t_{-i} + s_i)\frac{\ln(1 + \frac{s_i}{t_{-i}})}{s_i}.$$

and which reaches its maximum for $t_{-i} = b_r$ and $s_i = \delta b_r$. By determining the individual derivatives by s_i and t_{-i}, one can see that this function is non-decreasing for growing s_i and non-increasing for growing t_{-i}. So we obtain the worst-case ratio for $s_i = \delta b_r$ and $t_{-i} = b_r$, which is

$$\frac{\phi_{i,r}(\mathsf{s})}{u_{i,r}(\mathsf{s})} = \ln(1 + \delta)\frac{(1 + \delta)}{\delta} \leq \alpha_\delta^u.$$

Therefore, α_δ^u is indeed the worst-case ratio and thus satisfies Condition 6.1. □

Theorem 6.4 gives us an upper bound α_δ^u on the threshold α_δ such that every budget game has an α-approximate Nash equilibrium for $\alpha \geq \alpha_\delta^u$. Furthermore, they also satisfy the finite improvement property above this threshold and an approximate Nash equilibrium can be computed simply by using the best-response dynamic. Note that this does not say anything about the number of α-moves and thus about the complexity of finding an approximate equilibrium. In Table 6.1, we list some rounded values for α_δ^u in relation to the corresponding δ. Before we further discuss this result, we also introduce a lower bound α_δ^l for α_δ. Just like before, we give the bound first and then show that it is actually correct.

Definition 6.5. Let $\delta > 0$. We define the lower bound α_δ^l on α_δ as

$$\alpha_\delta^l := \frac{2\sqrt{\delta^2(\delta + 2)} + \delta - 1}{4\delta - 1}.$$

Theorem 6.6. *Let $\delta > 0$ and $\alpha < \alpha_\delta^l$. There is a δ-share budget game without an α-approximate Nash equilibrium.*

Proof. We once more refer to the δ-share budget game $\mathcal{B}_0$ from Definition 4.1. If we fix δ, the ratio between u^- and u^+ becomes a function f in γ.

$$f(\gamma) := \frac{\delta + \frac{\gamma}{\delta+\gamma+n\cdot\sigma}}{\gamma + \frac{\delta}{\delta+\gamma+n\cdot\sigma}} = \frac{\gamma + \delta(\delta + \gamma + n \cdot \sigma)}{\delta + \gamma(\delta + \gamma + n \cdot \sigma)} = \frac{\gamma + \delta(\gamma + 1)}{\delta + \gamma(\gamma + 1)}$$

δ	α_δ^u	α_δ^l
0.1	1.0485	1.0170
0.2	1.0946	1.0335
0.3	1.1388	1.0497
0.4	1.1816	1.0656
0.5	1.2232	1.0811
0.6	1.2637	1.0964
0.7	1.3033	1.1114
0.8	1.3422	1.1261
0.9	1.3804	1.1405
1	1.4181	1.1547

Table 6.1: Upper and lower bounds for α_δ derived from δ.

Deriving f with respect to γ yields

$$f'(\gamma) = \frac{\delta^2 - \delta\gamma(\gamma+2) - \gamma^2}{(\delta + \gamma^2 + \gamma)^2} = 0 \text{ for } \gamma_0 = \frac{\sqrt{\delta^3 + 2\delta^2} - \delta}{\delta + 1}$$

and this is the only local maximum of f for $\gamma > 0$, so

$$f(\gamma_0) = \frac{2\sqrt{\delta^2(\delta+2)} + \delta - 1}{4\delta - 1} = \alpha_\delta^l.$$

□

Table 6.1 also contains rounded values for α_δ^l. The larger we choose δ, the more the upper bound α_δ^u increases. In addition, the gap between α_δ^u and α_δ^l grows bigger as well. For smaller values of δ, however, our results become better. In fact, we get $\lim_{\delta \to 0} \alpha_\delta^u = 1$ (cf. Figure 6.4). Depending on the situation modeled by budget games, small values for δ can be quite appropriate. Thinking back to our motivation, assume the resources to be servers, with their budget standing for their computational power, i.e. the amount of work they are able to process at a given time. The players represent the clients who want to allocate their jobs on the servers. In a δ-share budget game with $\delta = 1$, we allow a single client to fully occupy a whole server. In reality, however, these are able to handle several ten-thousand of requests at once, which mirrors in a small value for δ which is actually close to 0. Under these considerations, our bounds are *almost* tight and the corresponding approximate Nash equilibria are *almost* exact Nash equilibria.

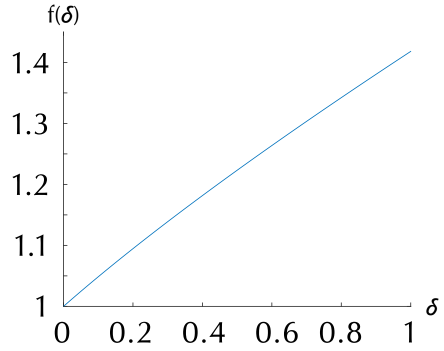

Figure 6.4: Upper bound $f(\delta) := \alpha_\delta^u$ for $\delta \in]0,1]$. The smaller we choose δ, the closer α_δ^u is to 1, i.e. $\lim_{\delta\to 0} \alpha_\delta^u = 1$.

6.2 Efficiency of Approximate Pure Nash Equilibria

In Chapter 4, we explained that the main difference between the two kinds of budget games is how the budgets of the resources are distributed among the players. The social welfare is not affected by these different utility functions. Thus, it seems likely that results regarding the efficiency of Nash equilibria carry over from ordered budget games. In this section, we show that this is indeed the case and moreover how the price of anarchy for α-approximate Nash equilibria relates to the value of α. For the rest of this section, let $PoA_\alpha(\mathcal{B})$ denote the price of anarchy for α-approximate Nash equilibria of a budget game $\mathcal{B}$.

Theorem 6.7. *Let $\mathcal{B}$ be a budget game. Then $PoA_\alpha(\mathcal{B}) \leq \alpha + 1$.*

For this result, we do not need to consider the actual value of δ, as our proof is independent of it. Its structure is similar to the corresponding proof for ordered budget games (cf. Theorem 5.7).

Proof. Let s be an α-approximate Nash equilibrium of a budget game $\mathcal{B}$ and opt be the strategy profile with the maximum social welfare. We lower bound the social welfare of s in Equation 2. Large parts of the transformation are similar to the proof of Theorem 5.7, so we resort to discussing the differences.

Equation (6.2) follows from the Nash inequality with respect to α and (6.3) from the definition of the utility functions in budget games. In (6.4), we modify how the utilities are computed and switch to a rule set akin to ordered budget games. The distributions of the utilities in s are not affected, but if player i would now change his strategy, this would no longer reduce the utility of any other player. If the remaining budget of a resource is not sufficient to satisfy the new demand of i, then he only receive what is left of that budget. In the context of ordered budget game, this is equivalent to i having only one task in each strategy. With this modified utility function, any strategy change by i yields at most as much utility as it would

$$
\begin{aligned}
\sum_{i\in\mathcal{N}} u_i(\mathsf{s}) &= \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{s}) \\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{s}_{-i}, opt_i) && (6.2)\\
&= \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} \min\left(opt_i(r), \frac{b_r\cdot opt_i(r)}{opt_i(r)+\sum_{i'\neq i}\mathsf{s}_{i'}(r)}\right) && (6.3)\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} \min\left(opt_i(r), b_r - \sum_{i'\neq i} u_{i',r}(\mathsf{s})\right) && (6.4)\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} \min\left(opt_i(r), b_r - \sum_{i'\in\mathcal{N}} u_{i',r}(\mathsf{s})\right)\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} \min\left(u_{i,r}(\mathsf{opt}), b_r - \sum_{i'\in\mathcal{N}} u_{i',r}(\mathsf{s})\right)\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_1}\sum_{i\in\mathcal{N}} \min\left(u_{i,r}(\mathsf{opt}), b_r - \sum_{i'\in\mathcal{N}} u_{i',r}(\mathsf{s})\right)\\
&\quad + \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{opt})\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_1}\left(b_r - \sum_{i'\in\mathcal{N}} u_{i',r}(\mathsf{s})\right) + \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{opt})\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_1}\left(\sum_{i\in N} u_{i,r}(\mathsf{opt}) - \sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{s})\right)\\
&\quad + \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{opt})\\
&\geq \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in N} u_{i,r}(\mathsf{opt}) - \alpha^{-1}\cdot\sum_{r\in\mathcal{R}_1}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{s})\\
&\geq \alpha^{-1}\cdot\sum_{i\in\mathcal{N}} u_i(\mathsf{opt}) - \alpha^{-1}\cdot\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} u_{i,r}(\mathsf{s})\\
&\geq \alpha^{-1}\sum_{i\in\mathcal{N}} u_i(\mathsf{opt}) - \alpha^{-1}\sum_{i\in\mathcal{N}} u_i(\mathsf{s})
\end{aligned}
$$

Equation 2: Bounding the social welfare of an α-approximate Nash equilibrium in budget games, which is smaller than the social welfare of the optimal solution by a factor of at most $\alpha + 1$.

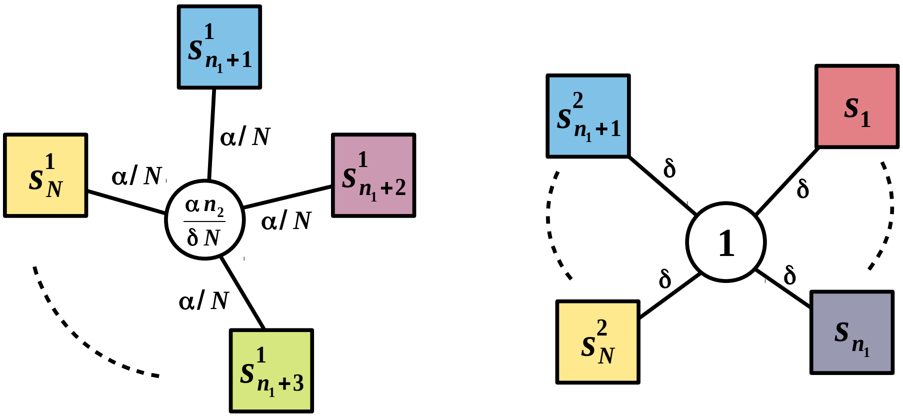

Figure 6.5: While the players $n_1+1,\dots,N$ can choose between the left and the right resource, the other players $1,\dots,n_1$ have to use the right resource, as this is their only strategy.

in our regular setting. From here on, the proof follows the same path as it did for ordered budget games. □

Our result holds for all approximate Nash equilibria. Even if we set $\alpha < \alpha^u_\delta$, a δ-share budget game may still have an α-approximate Nash equilibria and its price of anarchy is at most $\alpha + 1$. Again, this upper bound is almost tight in the sense that there are instances where the actual price of anarchy of a δ-share budget game can come arbitrary close to it.

Theorem 6.8. *For all $\varepsilon > 0$, there is a budget game $\mathcal{B}_\varepsilon$ with $PoA_\alpha(\mathcal{B}_\varepsilon) = \alpha+1-\varepsilon$.*

Proof. A lower bound of α is trivial for any kind of game and any value of α. To see that we can come arbitrarily close to $\alpha + 1$, we consider the budget game $\mathcal{B}_\varepsilon$. Let both δ and α be arbitrary, but fixed. Choose $n_1 \in \mathbb{N}$ such that $\delta n_1 \geq 1$ and $n_2 \in \mathbb{N}$. Let $N := n_1 + n_2$. Then $\mathcal{B}_\varepsilon$ is defined as

- $\mathcal{N}_1 = \{1,\dots,N\}$
- $\mathcal{R}_1 = \{r_1, r_2\}$
- $b_{r_1} = \frac{\alpha n_2}{\delta N}$ and $b_{r_2} = 1$
- $\mathcal{S}_i = \{s_i = (0,\delta)\}$ for $i \in \{1,\dots,n_1\}$

  $\mathcal{S}_i = \{s^1_i = (\frac{\alpha}{N}, 0)\ ,\ s^2_i = (0,\delta)\}$ for $i \in \{n_1+1,\dots,N\}$

The resulting game is a δ-share budget game and shown in Figure 6.5. The players $1,\dots,n_1$ have only one strategy to choose from. Consider the strategy profile $\mathsf{s} = (s_1,\dots,s_{n_1},s^2_{n_1+1},\dots,s^2_N)$. The utility of the players $i = n_1+1,\dots,N$ is $\frac{1}{N}$

each. Strategy s_i^1 yields a fixed utility of $\frac{\alpha}{N}$, so while s is an α-approximate Nash equilibrium with $u(\mathsf{s}) = 1$, $opt = (s_1, \ldots, s_{n_1}, s_{n_1+1}^1, \ldots, s_N^1)$ has a social welfare of $1 + n_2 \cdot \frac{\alpha}{N}$. For n_2 large enough, this comes close to $\alpha + 1$. □

6.3 Complexity of Approximate Pure Nash Equilibria

We are now going to address the complexity of approximate Nash equilibria in budget games. A δ-share budget game can still have an α-approximate Nash equilibrium for $\alpha < \alpha_\delta^l$. However, determining this is NP-hard.

Theorem 6.9. *Let $\delta > 0$ and $\alpha < \alpha_\delta^l$. To decide if a δ-share budget game $\mathcal{B}$ has an α-approximate Nash equilibrium is NP-complete.*

Proof. The problem obviously lies in NP, as the verification of a given strategy profile can be done in polynomial time. To show that it is also NP-hard, we reduce from the exact cover by 3-sets problem, which is also NP-complete. An instance of exact cover by 3-sets problem has the form $\mathcal{I} = (\mathcal{U}, \mathcal{W})$, consisting of a set $\mathcal{U}$ with $|\mathcal{U}| = 3m$ for $m \in \mathbb{N}$ and a collection of subsets $\mathcal{W} = \{\mathcal{W}_1, \ldots, \mathcal{W}_q\} \subseteq \mathcal{U}$ with $|\mathcal{W}_k| = 3$ for every k. Computing an exact cover of $\mathcal{U}$ in which every element is in exactly one subset from $\mathcal{W}$ is NP-hard [32]. For $\delta > 0$, we choose an instance $\mathcal{I}$ with $q - m \geq \frac{1}{\delta}$.

From $\mathcal{I}$, we create a budget game $\mathcal{B}$ by combining two smaller games $\mathcal{B}_0$ and $\mathcal{B}_\mathcal{I}$. $\mathcal{B}_0$ is the budget game introduced in Definition 4.1. We label its two main players as player 1 and 2. For convenience, we define $u' := \frac{\delta}{\delta+\gamma+\sigma} + \gamma$. $\mathcal{B}_\mathcal{I}$ is constructed from $\mathcal{I}$ as follows.

- $\mathcal{N}_{\mathcal{B}_0} = \{3, \ldots, q+2\}$
- $\mathcal{R}_{\mathcal{B}_0} = \mathcal{U} \cup \{r'\}$
- $b_r = 1$ for all $r \in \mathcal{R} \setminus \{r'\}$

  $b_{r'} = 3(q-m)\alpha\delta + \frac{u'}{\alpha}$
- $\mathcal{S}_i = \{s_i^1, s_i^2\}$ for $i = 3, \ldots, q+2$

The demands are defined as

$$s_i^1(r_j) := \begin{cases} \delta & \text{if } j \in \mathcal{W}_{i-2} \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad s_i^2(r') = 3\alpha\delta.$$

All other demands are 0.

For the rest of this proof, we assume that $2\delta > b_r = 1$. If this is not the case, we refer to Definition 4.1 and add auxiliary players with singular strategy spaces to reduce the available capacity of the resources.

We combine the two unrelated games $\mathcal{B}_\mathcal{I}$ and $\mathcal{B}_0$ by creating the union of the corresponding sets and introducing one additional strategy s_2^3 for the second player

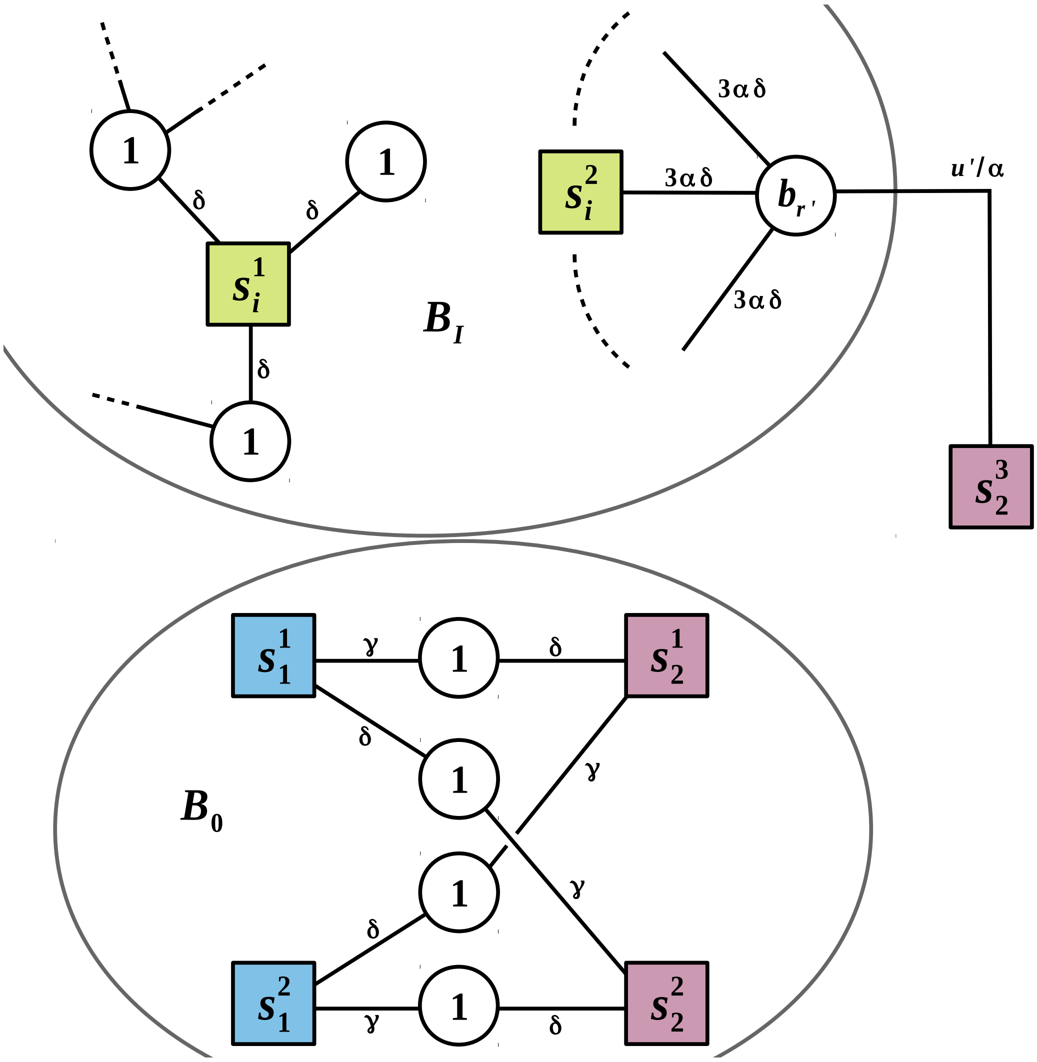

Figure 6.6: The budget game $\mathcal{B}$, created from the budget games $\mathcal{B}_0$ and $\mathcal{B}_\mathcal{I}$. As long as player 2 chooses strategy s_2^1 or s_2^2, $\mathcal{B}_0$ has no α-approximate Nash equilibrium for $\alpha < \alpha_\delta^l$ (cf. Theorem 6.6). So in order for such an equilibrium to exist, player 2 has to choose strategy s_2^3. This only happens if at most $(q-m)$ many players i from $\mathcal{B}_\mathcal{I}$ do not choose strategy s_i^2. These players then form an exact cover over the remaining resources in $\mathcal{B}_\mathcal{I}$.

from $\mathcal{B}_0$. This strategy uses only the resource r' with $s_2^3(r') = \frac{u'}{\alpha}$ and $s_2^3(r) = 0$ for all $r \in \mathcal{R} \setminus \{r'\}$. For the final result, see Figure 6.6.

$\mathcal{B}$ is indeed a δ-share BAG. For the resource r', note that both

$$3(q-m)\alpha\delta + \frac{u'}{\alpha} \geq 3(q-m)\alpha\delta \geq \frac{3\alpha}{\delta}\delta = 3\alpha$$

and

$$(q-m)3\alpha\delta + \frac{u'}{\alpha} \geq 3\alpha + \frac{u'}{\alpha} \geq \frac{3}{\alpha} + \frac{u'}{\alpha} \geq \frac{2\delta(1-\delta)}{\alpha\delta} + \frac{u'}{\alpha} \geq \frac{u'(1-\delta)}{\alpha\delta} + \frac{u'}{\alpha} = \frac{u'}{\alpha\delta},$$

so no demand on r' exceeds $\delta b_{r'}$.

We already know that $\mathcal{B}_0$ has no α-approximate Nash equilibrium for $\alpha < \alpha_\delta^l$. Since the second player now has an additional strategy s_2^3, there cannot be any α-approximate Nash equilibrium $\mathsf{s} = (s_1, s_2, \ldots, s_{q+2})$ of $\mathcal{B}$ in which this player plays a different strategy. However, if $u_{2,r'}(\mathsf{s}) < s_2(r')$, the second player will always choose either s_2^1 or s_2^2. Therefore, at most $(q-m)$ players $i \in \{3, \ldots, q+2\}$ from $\mathcal{B}_\mathcal{I}$ are allowed to play s_i^2. Every player i with strategy s_i^1 and utility below 3δ will switch to s_i^2. Therefore, exactly m players pick s_i^1 in s and they form an exact cover over the resources $r_1, \ldots, r_{3m}$. □

This result is particularly interesting as it also holds for $\alpha = 1$.

Corollary 6.10. To decide if a budget game $\mathcal{B}$ has a Nash equilibrium is NP-complete.

6.4 Properties of the Best-Response Dynamic

In Section 6.1, we have seen that budget games possess the finite improvement property regarding α-approximate Nash equilibria for $\alpha \geq \alpha_\delta^u$. Now, we are going to show that the best-response dynamic also converges quickly towards approximate Nash equilibria if we choose α even larger than α_δ^u and if the utilities of the most-profitable strategies of the players do not differ too much from each other. One example of games with the second property are symmetric games, in which all players share a common strategy space and the best strategy of one player is also the best of all others. In this context, *best* refers to the corresponding utility if the player does not share any budget with another player. The following definition formalizes this notion.

Definition 6.11. Let $\mathcal{B}$ be a budget game. For player $i \in \mathcal{N}$, we define $s_i^{\text{opt}} \in \mathcal{S}_i$ to be the strategy of i with

$$\sum_{r \in \mathcal{R}} \min(s_i^{\text{opt}}(r), b_r) \geq \sum_{r \in \mathcal{R}} \min(s_i(r), b_r)$$

for all $s_i \in \mathcal{S}_i$. The corresponding utility is denoted by

$$u_i^{\text{opt}} := \sum_{r \in \mathcal{R}} \min(s_i^{\text{opt}}(r), b_r).$$

Although it is not guaranteed that s_i^{opt} is actually part of an approximate Nash equilibrium, we are going to use its utility u_i^{opt} to give a general bound on the utility of i for any strategy profile. We need two lemmata in order to pave the way for the main convergence result of this section. The first one states that we can expect that the various demands of a single player do not deviate too much from each other.

Lemma 6.12. Let $\mathcal{B}$ be a δ-share budget game and s a strategy profile of $\mathcal{B}$. For all players $i \in \mathcal{N}$ it holds that

$$u_i(\mathsf{s}) \geq \frac{u_i^{\text{opt}}}{(n\delta)^2}.$$

Proof. First, we show that

$$u_{i,r}(\mathsf{s}_{-i}, s_i^{\text{opt}}) \geq \frac{s_i^{\text{opt}}(r)}{n\delta}$$

holds for all resources r and strategy profiles s_{-i} excluding player i. If $T_r(\mathsf{s}_{-i}, s_i^{\text{opt}}) \leq b_r$, then this is obviously true, as

$$u_{i,r}(\mathsf{s}_{-i}, s_i^{\text{opt}}) = s_i^{\text{opt}}(r) \geq \frac{s_i^{\text{opt}}(r)}{n\delta}$$

for $n\delta > 1$ (for $n\delta \leq 1$, see Lemma 6.1). If $T_r(\mathsf{s}_{-i}, s_i^{\text{opt}}) > b_r$, then

$$u_{i,r}(\mathsf{s}_{-i}, s_i^{\text{opt}}) = \frac{s_i^{\text{opt}}(r) \cdot b_r}{s_i^{\text{opt}} + T_r(\mathsf{s}_{-i})} \geq \frac{s_i^{\text{opt}}(r) \cdot b_r}{n\delta b_r} = \frac{s_i^{\text{opt}}(r)}{n\delta}.$$

By summing up over all resources, we get

$$u_i(\mathsf{s}_{-i}, s_i^{\text{opt}}) \geq \sum_{r \in \mathcal{R}} \frac{s_i^{\text{opt}}(r)}{n\delta} \geq \frac{u_i^{\text{opt}}}{n\delta}.$$

So we assume without loss of generality that for every strategy $s_i \in \mathcal{S}_i$ we get

$$\sum_{r \in \mathcal{R}} \min(s_i(r), b_r) \geq \frac{u_i^{\text{opt}}}{n\delta} \tag{6.5}$$

and ultimately

$$\sum_{r \in \mathcal{R}} s_i(r) \geq \frac{u_i^{\text{opt}}}{n\delta}. \tag{6.6}$$

Otherwise, the strategy s_i^{opt} would yield a higher utility in every strategy profile and s_i would never the part of any Nash equilibrium.

With this bound on the demands of a strategy, we can derive a lower bound on its utility in the same manner. For $T_r(\mathsf{s}) \leq b_r$, we get

$$u_{i,r}(\mathsf{s}) = s_i(r) > \frac{s_i(r)}{n\delta} \text{ for } n\delta > 1.$$

For $T_r(\mathsf{s}) > b_r$, we get

$$u_{i,r}(\mathsf{s}) = \frac{s_i(r) \cdot b_r}{T_r(\mathsf{s})} \geq \frac{s_i(r) \cdot b_r}{n\delta b_r} = \frac{s_i(r)}{n\delta}$$

and in both cases, the result is

$$u_i(\mathsf{s}) = \sum_{r \in \mathcal{R}} u_{i,r}(\mathsf{s}) \geq \sum_{r \in \mathcal{R}} \frac{s_i(r)}{n\delta} \geq \frac{u_i^{\text{opt}}}{(n\delta)^2}.$$

□

This proof is based on the assumption that the players are only considering those strategies which are not too bad compared to their individual optimal strategy s_i^{opt}. This somewhat contradicts our assumption of a best-response dynamic based on α-moves, as a player would be satisfied with any strategy if we just set α large enough. With this approach, we are mainly concerned with whether a strategy could actually be part of an exact Nash equilibrium. Given a game with corresponding strategy spaces, determining a suitable $\alpha \geq 1$ for an α-approximate Nash equilibrium is done afterwards.

The second lemma is more technical and bounds the potential of a strategy profile with respect to its social welfare.

Lemma 6.13. For any δ-share budget game and any strategy profile s, we get

$$(1 + \log(n\delta)) \cdot u(\mathsf{s}) \geq \phi(\mathsf{s}).$$

Proof. Consider a resource r and let $u_r(\mathsf{s})$ be the total utility obtained from r by all players, i.e. $u_r(\mathsf{s}) := \sum_{i \in \mathcal{N}} u_{i,r}(\mathsf{s})$. We first show that $(1 + \log(n\delta)) \cdot u_r(\mathsf{s}) \geq \phi_r(\mathsf{s})$. Assume that $T_r(\mathsf{s}) \geq b_r$, otherwise $\phi_r(\mathsf{s}) = u_r(\mathsf{s})$ by the definition of ϕ_r. So $u_r(\mathsf{s}) = b_r$, while

$$\phi_r(\mathsf{s}) = b_r + \int_{b_r}^{T_r(\mathsf{s})} \frac{b_r}{x} \,\mathrm{d}x = b_r \left(1 + \int_{b_r}^{T_r(\mathsf{s})} \frac{1}{x} \,\mathrm{d}x\right).$$

This yields

$$\begin{aligned}\frac{\phi_r(\mathsf{s})}{u_r(\mathsf{s})} &= 1 + \int_{b_r}^{T_r(\mathsf{s})} \frac{1}{x} \,\mathrm{d}x = 1 + \ln(T_r(\mathsf{s})) - \ln(b_r) \\ &\leq 1 + \ln(n\delta b_r) - \ln(b_r) = 1 + \ln(n\delta).\end{aligned}$$

By summing up over the resources, we get

$$\frac{\phi(\mathsf{s})}{u(\mathsf{s})} = \frac{\sum_{r\in\mathcal{R}}\phi_r(\mathsf{s})}{\sum_{r\in\mathcal{R}}u_r(\mathsf{s})} \le \frac{(1+\ln(n\delta))\cdot\sum_{r\in\mathcal{R}}u_r(\mathsf{s})}{\sum_{r\in\mathcal{R}}u_r(\mathsf{s})} = 1+\ln(n\delta).$$

□

With these two lemmata, we are able to bound the convergence speed of the best-response dynamic using α-moves. We assume that $\alpha \ge \alpha_\delta^u$, otherwise we do not know if there even is an α-approximate Nash equilibrium. Our result states that the convergence speed depends on two factors: the difference between the upper bound α_δ^u and the value of α actually used as well as the ratios between the highest demands of the different players.

Theorem 6.14. *Let $\delta \le 1$, $\varepsilon > 0$ and $\lambda \in]0,1]$. Let $\mathcal{B}$ be a δ-share budget game with the property that for all players $i, j \in \mathcal{N}$, we get $u_i^{opt} \ge \lambda u_j^{opt}$.*

Then the number of $(\alpha_\delta^u+\varepsilon)$-moves until $\mathcal{B}$ reaches an $(\alpha_\delta^u+\varepsilon)$-approximate Nash equilibrium is

$$\mathcal{O}\left(\log(n)\cdot n^5\cdot(\varepsilon\lambda)^{-1}\right).$$

Given two strategies u_i^{opt} and u_j^{opt}, we either get $u_i^{\text{opt}} \ge u_j^{\text{opt}}$ or $u_i^{\text{opt}} < u_j^{\text{opt}}$. For the second case, the scaling factor λ is used to achieve $u_i^{\text{opt}} \ge \lambda u_j^{\text{opt}}$.

Proof. Let i be the player performing an $(\alpha_\delta^u+\varepsilon)$-move under the strategy profile s, leading to the strategy profile s'. We can bound the increase in the potential.

$$\begin{aligned}\Phi(\mathsf{s}') - \Phi(\mathsf{s}) &\ge \varepsilon\cdot u_i(\mathsf{s})\\ &\overset{(1)}{\ge} \frac{\varepsilon}{(n\delta)^2}\cdot u_i^{\text{opt}} \overset{(2)}{\ge} \frac{\varepsilon\cdot\lambda}{n(n\delta)^2}\cdot u(\mathsf{s}) \overset{(3)}{\ge} \frac{\varepsilon\cdot\lambda}{n(1+\log(n))(n\delta)^2}\cdot\Phi(\mathsf{s})\end{aligned}$$

Inequalities (1) and (3) follow by Lemma 6.12 and 6.13 respectively while (2) holds due to

$$u(\mathsf{s}) = \sum_{j\in\mathcal{N}} u_j(\mathsf{s}) \le \sum_{j\in\mathcal{N}} u_j^{\text{opt}} \le \frac{n}{\lambda}u_i^{\text{opt}}.$$

For convenience, we define

$$\beta := \frac{\varepsilon\cdot\lambda}{n(1+\log(n))(n\delta)^2}.$$

Assuming it takes t steps to double the potential the game, then

$$\Phi(\mathsf{s}) = 2\Phi(\mathsf{s}) - \Phi(\mathsf{s}) \ge \beta\cdot t\cdot\Phi(\mathsf{s}) \Leftrightarrow t \le \beta^{-1}.$$

So in order to double the current potential of $\mathcal{B}$, we need at most β^{-1} improving moves. Therefore, the game has to reach a corresponding equilibrium after at most $\log\left(\frac{\Phi_{\max}}{\Phi_{\min}}\right)\cdot\beta^{-1}$ improving moves, with $\Phi_{\max}$ and $\Phi_{\min}$ denoting the maximum and minimum potential of $\mathcal{B}$, respectively. Since $\Phi_{\max} \le \sum_{i\in\mathcal{N}} u_i^{\text{opt}}$ due to $\delta \le 1$ and $\Phi_{\min} \ge \sum_{i\in\mathcal{N}} \frac{u_i^{\text{opt}}}{(n\delta)^2}$, we can bound $\log\left(\frac{\Phi_{\max}}{\Phi_{\min}}\right) \le (n\delta)^2$. □

6.5 Approximate Optimal Solution in General Budget Games

To conclude this chapter, we use the approximate potential function ϕ to show that the optimal solution of a budget game can be approximated by the best-response dynamic. We have already seen that finding the optimal solution is NP-hard (cf. Theorem 4.4) and we have given an approximation algorithm with an approximation factor of $1 - \frac{1}{e}$ (cf. Algorithm 1). However, that algorithm only works for matroid budget games. The method we are going to introduce now, although only yielding an approximation factor polylogarithmic in n, works for every budget game.

Our approach is based on concepts and results found in [6]. First, we show that a certain class of utility-maximization games converge quickly towards socially good states, i.e. strategy profiles with a social welfare close to the optimal solution, if the players keep performing α-moves for a suitable α. We then apply this result to budget games. For the following analysis, we require the notion of *nice games* [6].

Definition 6.15. Let $\lambda, \mu > 0$. A utility-maximization game is (λ, μ)-*nice* if for every strategy profile s, there is a strategy profile s' with

$$\sum_{i \in \mathcal{N}} u_i(\mathsf{s}_{-i}, s'_i) \geq \lambda \cdot u(\mathsf{opt}) - \mu \cdot u(\mathsf{s}).$$

From now on, when talking about the best-response dynamic, we always assume that the players only perform α-moves for $\alpha \geq 1$. If the current strategy s_i of player i is not his best-response s_i^b to the current strategy profile s, he only changes his strategy if $u_i(\mathsf{s}_{-i}, s_i^b) > \alpha \cdot u_i(\mathsf{s})$. The actual value of α will be given later.

Theorem 6.16. *Let $\mathcal{B}$ be a (λ, μ)-nice utility-maximization game with a potential function $\phi(\mathsf{s})$ such that for some $A, B, C \geq 1$, we have that*

$$A \cdot \phi(\mathsf{s}) \geq u(\mathsf{s}) \geq \frac{1}{B} \cdot \phi(\mathsf{s})$$

and

$$\phi(\mathsf{s}_{-i}, s_i^b) - \phi(\mathsf{s}) \geq u_i(\mathsf{s}_{-i}, s_i^b) - C \cdot u_i(\mathsf{s}).$$

Let $\rho = \frac{\lambda}{C+\mu}$. Then, for any $\varepsilon > 0$ and any initial strategy profile s^0, the best-response dynamic reaches a state s^t with

$$u(\mathsf{s}^t) \geq \frac{\rho(1-\varepsilon)}{AB} \cdot u(\mathsf{opt}) \text{ in at most } \mathcal{O}\left(\frac{n}{A(C+\mu)} \log \frac{1}{\varepsilon}\right) \text{ steps.}$$

All future states reached by this best-response dynamic will satisfy this approximation factor as well.

Proof. We adapt a modification of a proof from [6], with the difference that our version does not require an exact potential function, but also works with approximate potential functions like the one from Definition 6.2. For the best-response dynamic, we assume a specific order in which the players perform their strategy changes. For any strategy profile s, the next player i who changes his strategy is chosen such that he maximizes the term $u_i(\mathsf{s}_{-i}, s_i^b) - C \cdot u_i(\mathsf{s})$, i.e.

$$u_i(\mathsf{s}_{-i}, s_i^b) - C \cdot u_i(\mathsf{s}) = \max_{j \in \mathcal{N}} \left\{ u_j(\mathsf{s}_{-j}, s_j^b) - C \cdot u_j(\mathsf{s}) \right\}.$$

We then obtain

$$\begin{aligned}
\phi(\mathsf{s}_{-i}, s_i^b) - \phi(\mathsf{s}) &\geq u_i(\mathsf{s}_{-i}, s_i^b) - C \cdot u_i(\mathsf{s}) \\
&\geq \frac{1}{|\mathcal{N}'|} \left(\sum_{i \in \mathcal{N}'} u_i(s_i^b, s_{-i}) - \alpha \cdot u_i(s) \right) \\
&= \frac{1}{|\mathcal{N}'|} \left(\sum_{i \in \mathcal{N}} u_i(s_i^b, s_{-i}) - \sum_{i \in \mathcal{N}'} \alpha \cdot u_i(s) - \sum_{i \in \mathcal{N} \setminus \mathcal{N}'} u_i(s) \right) \\
&\geq \frac{1}{n} \left(\sum_{j \in \mathcal{N}} u_j(\mathsf{s}_{-j}, s_j^b) - C \cdot u_j(\mathsf{s}) \right) \\
&\overset{(1)}{\geq} \frac{1}{n} \left(\lambda \cdot u(opt) - (C + \mu) u(\mathsf{s}) \right) \\
&\geq \frac{1}{n} \left(\lambda \cdot u(opt) - A(C + \mu) \phi(\mathsf{s}) \right) =: f(\mathsf{s}).
\end{aligned}$$

(1) uses the fact that $\mathcal{B}$ is (λ, μ)-nice. With this definition of $f(s)$, we see that

$$f(\mathsf{s}) - f(\mathsf{s}_{-i}, s_i^b) = \frac{A(C + \mu)}{n} \left(\phi(\mathsf{s}_{-i}, s_i^b) - \phi(\mathsf{s}) \right) \geq \frac{A(C + \mu)}{n} f(\mathsf{s})$$

and therefore $f(\mathsf{s}_{-i}, s_i^b) \leq \left(1 - \frac{A(C+\mu)}{n}\right) f(\mathsf{s})$. So if s^0 is the initial strategy profile, then the best response dynamic converges towards a strategy profile s^t with

$$f(\mathsf{s}^t) \leq \left(1 - \frac{A(C + \mu)}{n}\right)^t f(\mathsf{s}^0).$$

By setting $t = \left\lceil \frac{n}{A(C+\mu)} \log \frac{1}{\varepsilon} \right\rceil$ and using that $\left(1 - \frac{1}{x}\right)^x \leq \frac{1}{e}$, we obtain

$$f(\mathsf{s}^t) \leq e^{(\log 1/\varepsilon)^{-1}} \cdot f(\mathsf{s}^0) = \varepsilon \cdot f(\mathsf{s}^0) \leq \varepsilon \cdot \frac{\lambda \cdot u(opt)}{n}.$$

Using these results and the bounds for the potential function, we obtain

$$\begin{aligned}
u(\mathsf{s}^t) \geq \frac{1}{B} \cdot \phi(\mathsf{s}^t) &= \frac{n}{AB(C + \mu)} \cdot \left(\frac{\lambda \cdot u(opt)}{n} - f(\mathsf{s}^t) \right) \\
&\geq \frac{n}{AB(C + \mu)} \cdot \left((1 - \varepsilon) \frac{\lambda \cdot u(opt)}{n} \right) \\
&\geq \frac{\rho(1 - \varepsilon)}{AB} \cdot u(opt).
\end{aligned}$$

We also see that $\phi(\mathsf{s}^t) \geq \frac{\rho(1-\varepsilon)}{A} \cdot u(opt)$. Since ϕ grows with every strategy change, this bound also holds for all following strategy profiles.

□

When adapting this result for budget games, note that the players have to perform α-moves for $\alpha \geq \alpha_\delta^u$ when following the best-response dynamic. Otherwise, there is no guarantee that the approximate potential function ϕ from Definition 6.2 is strictly monotone.

Corollary 6.17. Let $\mathcal{B}$ be a δ-share budget game and $\alpha \geq \alpha_\delta^u$. For any $\varepsilon > 0$ and any initial strategy profile s^0, the best-response dynamic using only α-moves reaches a state s^t with

$$u(\mathsf{s}^t) \geq \frac{1-\varepsilon}{(\alpha^2+1)(\ln(n\delta)+1)} u(opt)$$

in at most

$$\mathcal{O}\left(\frac{n}{\alpha+\alpha^{-1}} \log \frac{1}{\varepsilon}\right)$$

steps. All future states reached by this best-response dynamic will satisfy this approximation factor as well.

Proof. First we show that any δ-share budget game is $(\alpha^{-1}, \alpha^{-1})$-nice. Let s be an arbitrary strategy profile. We show that $\sum_{i \in \mathcal{N}} u_i(\mathsf{s}_{-i}, s_i^b) \geq \alpha^{-1} \cdot u(opt) - \alpha^{-1} \cdot u(\mathsf{s})$. Note that $u_i(\mathsf{s}_{-i}, s_i^b) \geq u_i(\mathsf{s}_{-i}, opt_i)$ by definition of s_i^b. This implies $u_i(\mathsf{s}_{-i}, s_i^b) \geq \alpha^{-1} \cdot u_i(\mathsf{s}_{-i}, opt_i)$ and we can therefore copy the proof of Theorem 6.7 to show that $\sum_{i \in \mathcal{N}} u_i(\mathsf{s}_{-i}, s_i^b) \geq \alpha^{-1} \cdot u(opt) - \alpha^{-1} \cdot u(\mathsf{s})$.

We now use the approximate potential function $\phi(\mathsf{s})$ from Definition 6.2, for which we know that

$$\phi(\mathsf{s}) \geq u(\mathsf{s}) \geq \frac{1}{1+\ln(n\delta)} \phi(\mathsf{s}) \quad \text{(cf. Lemma 6.13)}$$

and

$$\phi(\mathsf{s}_{-i}, s_i^b) - \phi(\mathsf{s}) \geq u_i(\mathsf{s}_{-i}, s_i^b) - \alpha \cdot u_i(\mathsf{s}) \quad \text{(cf. proof of Theorem 6.4).}$$

So we get $A = 1$, $B = 1 + \ln(n\delta)$ and $C = \alpha$. Using these values together with Theorem 6.16 directly proofs the statement. □

As already stated, this result is valid for all budget games, not just matroid ones. However, it does not say anything about how quickly a utility-maximization game reaches an (approximate) equilibrium if following the best-response dynamic. So while this may take an exponential number of steps, the social welfare approaches its optimum relatively fast.

CHAPTER 7

Nash Equilibria in Budget Games under Restrictions

In the last chapter of this thesis, we approach the non-existence of Nash equilibria in budget games from a different angle. Instead of changing the model or resorting to approximate Nash equilibria, we consider restrictions on budget games which restore equilibria. First, we look at games in which the demand of each player is fixed, i.e. the demand of player i is d_i on every resource in every strategy. For both singleton and matroid budget games, this yields the finite improvement property. We then see that singleton budget games with only two resources are weakly acyclic. By using a similar proof technique, we also show that Nash equilibria still exist for singleton budget games with $m > 2$ resources and additional restrictions on the demands. Singleton budget games are also weakly acyclic if the number of demands is restricted to two, i.e. the demand of any strategy on any resource is either d^+ or d^-, with both values shared by all players and if all budgets are the same. The chapter is concluded with one last result regarding approximate Nash equilibria, this time specifically for matroid budget games. Independent of the size of the demands (the value of δ in a δ-restricted budget games), good approximate equilibria can be obtained by the best-response dynamic if the deviations between the different demands of each individual player are small.

Before we continue with the introduction of this section, we introduce a small technical lemma, which is used in most of the proofs found in this chapter. It states that in a budget game, a player with a larger demand always receives a higher utility from a resource than a player with a smaller demand.

Lemma 7.1. Let $d_1, d_2 \in \mathbb{R}_{>0}$ with $d_1 \leq d_2$ and $b_r, T_r(\mathsf{s}) \in \mathbb{R}_{\geq 0}$ with $T_r(\mathsf{s}) + d_1 \geq b_r$. Then

$$d_1 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s}) + d_1}\right) \leq d_2 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s}) + d_2}\right).$$

Proof. Proof by case distinction. Due to $d_1 \leq d_2$, we only need to consider three cases.

- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = \min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = 1$, the statement becomes trivial.
- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = 1$ and $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = \frac{b_r}{T_r(\mathsf{s})+d_2}$, we get $d_1 \leq b_r - T_r(\mathsf{s})$ while $\frac{b_r - T_r(\mathsf{s})}{d_2} \leq \frac{b_r - T_r(\mathsf{s}) + T_r(\mathsf{s})}{d_2 + T_r(\mathsf{s})} = \frac{b_r}{d_2 + T_r(\mathsf{s})}$, which can be transformed to $b_r - T_r(\mathsf{s}) < d_2 \cdot \frac{b_r}{d_2 + T_r(\mathsf{s})}$.
- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = \frac{b_r}{T_r(\mathsf{s})+d_1}$ and $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = \frac{b_r}{T_r(\mathsf{s})+d_2}$, we get $\frac{d_1}{d_2} \leq \frac{d_1 + T_r(\mathsf{s})}{d_2 + T_r(\mathsf{s})}$ (since $\frac{d_1}{d_2} \leq 1$) and therefore $\frac{d_1}{T_r(\mathsf{s})+d_1} \leq \frac{d_2}{T_r(\mathsf{s})+d_2}$.

□

Large parts of this chapter focus primarily on singleton budget games. In these, every strategy uses exactly one resource, i.e. the demand of the strategy on every resource except one is 0. If we look back to their formal definition (cf. Definition 2.19), we do not allow the same player to have multiple strategies imposing different demands on the same resource. With Lemma 7.1, we see now that a higher demand always yields a higher utility. If two or more strategies by the same player i would use a common resource r (and just that resource), then only the strategy with the highest demand on r can be part of a Nash equilibrium. If i chooses a different strategy using r, then he is always able to increase his own utility through a unilateral strategy change. So the property of having only one demand for each player-resource combination does not really restrict the model of singleton budget games with respect to Nash equilibria.

For matroid budget games, the situation is somewhat different. In the field of congestion games, the matroid property states that every strategy change consists of lazy moves (cf. Definition 2.21), which basically means that every strategy change switches a variable number of old resources for the same number of new ones. One can regard this as a sequence of atomic strategy changes in which only one resource is switched for another and every intermediate step yields a valid strategy for the corresponding player. As a result, every atomic strategy change already increases his utility. Since the weights of the players in a congestion game are usually fixed, this suffices to ensure that the effects of a strategy change only affect a small part of the game, with the congestion of only one resource in- and of another one decreasing. For matroid budget games, we have to be careful that a strategy change which swaps one resource for another does not also change any other demands at the same time. This would no longer fit the intuition behind the lazy moves of a matroid congestion game and a lazy strategy change of a single player could affect every resource in the game. To prevent this from happening, we require each player to a fixed demand for each resource. This is modeled by the first bullet point in Definition 2.21. For congestion games, the matroid property represents an extension of the singleton property. To maintain this for budget games, we also expect a fixed demand for each player-resource combination for singleton budget games. As discussed above, this does not actually restrict the model regarding our concerns.

From now on, we simplify the notation for strategies. Instead of $s_i = (0, \dots, 0, s_i(r), 0, \dots, 0)$, we simply write $s_i = \{r\}$ and denote the corresponding demand by $d_i(r) = s_i(r)$.

7.1 Matroid Budget Games with Fixed Player Demands

The first class of budget games we analyze are those with fixed player demands. In these, every player has a single demand which he imposes on every resource in his strategy. In addition, this value does not change with the strategy.

Definition 7.2 (Budget Games with Fixed Player Demands)**.** Let $\mathcal{B}$ be a budget game. If for every $i \in \mathcal{N}$ there is a $d_i \in \mathbb{R}_{>0}$ such that for every $r \in \mathcal{R}$ and every $s \in \mathcal{S}_i$ it holds that $s(r) = d_i$, then $\mathcal{B}$ is a budget game *with fixed player demands.*

For budget games with fixed player demands, we get

$$u_i(\mathsf{s}) = d_i \cdot \sum_{r \in s_i} \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right).$$

We show that such games have the finite improvement property, provided they are also matroid games. To do this, we need a potential function.

Definition 7.3 (Lexicographical Potential Function)**.** For a budget game $\mathcal{B}$ with resource r and strategy profile s, let

$$c_r(\mathsf{s}) := \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right).$$

The function $\phi : \mathcal{S} \to \mathbb{R}^m_{>0}$ with $\phi(\mathsf{s}) = (c_{r_1}(\mathsf{s}), \dots, c_{r_m}(\mathsf{s}))$ such that the entries of $\phi(\mathsf{s})$ are sorted in ascending order is a *lexicographical potential function* of $\mathcal{B}$.

Using ϕ, we can prove the following theorem.

Theorem 7.4. *A matroid budget game with fixed player demands reaches a Nash equilibrium after a finite number of improving moves.*

Proof. In a matroid game, we can regard every strategy change as a sequence of *lazy* strategy changes. In a lazy strategy change, exactly one resource is switched for another. Every lazy strategy change already increases the utility of the corresponding player. In this proof, we show that it increases ϕ, as well.

Let player i perform a lazy strategy change in strategy profile s, during which he replaces resource r for r'. Let s' be the resulting strategy profile.

$$u_{i,r}(\mathsf{s}) = d_i \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right) < d_i \cdot \min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s}) + d_i}\right) = u_{i,r'}(\mathsf{s}')$$

or simply

$$c_r(\mathsf{s}) = \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right) < \min\left(1, \frac{b_r}{T_r(\mathsf{s}')}\right) = c_{r'}(\mathsf{s}').$$

Since $c_r(\mathsf{s}) < c_r(\mathsf{s}')$ and $c_r(\mathsf{s}) < c_{r'}(\mathsf{s}')$, we get $\phi(\mathsf{s}) \leq_{\text{lex}} \phi(\mathsf{s}')$, so ϕ is strictly increasing regarding the lexicographical order when the only strategy changes are improving moves. □

One might argue that this result is based mainly on the fact that the demands are fixed and not on the structure of the strategy spaces. To see that this is not the case, consider the budget game $\mathcal{B}_1$.

Definition 7.5. Let $d_1 = d_{aux} = 1000$ and $d_2 = 1$. We define the budget game $\mathcal{B}_1$ as follows:

- $\mathcal{N}_1 = \{1, 2\} \cup \{aux_i \mid i = 1, \dots, 999\}$
- $\mathcal{R}_1 = \{r_1, r_2\} \cup \{r_i^1 \mid i = 1, \dots, m\} \cup \{r_i^2 \mid i = 1, \dots, m\}$ for $m = 1000$
- $b_r = 1000$ for all $r \in \mathcal{R}$
- $\mathcal{S}_1 = \{\ s_1^1 = (d_1, 0, \underbrace{d_1, \dots, d_1}_{m}, \underbrace{0, \dots, 0}_{m})\ ,\ s_1^2 = (0, d_1, \underbrace{0, \dots, 0}_{m}, \underbrace{d_1, \dots, d_1}_{m})\ \}$

  $\mathcal{S}_2 = \{\ s_2^1 = (0, d_2, \underbrace{d_2, \dots, d_2}_{m}, \underbrace{0, \dots, 0}_{m})\ ,\ s_2^2 = (d_2, 0, \underbrace{0, \dots, 0}_{m}, \underbrace{d_2, \dots, d_2}_{m})\ \}$

  $\mathcal{S}_{aux_i} = \{\ s = (0, 0, \underbrace{d_{aux}, \dots, d_{aux}}_{2m})\ \}$

The game $\mathcal{B}_1$, which is sketched in Figure 7.1, is not a matroid budget game. The players aux_i for $i = 1, \dots, 999$ are auxiliary players. Just like in the budget game $\mathcal{B}_0$ from Definition 4.1, they only exist so that there is a permanent demand on some of the resources. Each auxiliary player has only one strategy, which imposes a demand of 1000 on every resource except r_1 and r_2. Due to their existence, $\mathcal{B}_1$ is a δ-share budget game for $\delta = 1$. Also similar to $\mathcal{B}_0$, one of the two players 1 or 2 is not in an equilibrium in every strategy profile, of which there are only four. An overview of these strategy profiles and the corresponding utilities is given in Table 7.1. Note that both $\frac{1000^3}{1000^2+1} > \frac{1000^2}{1000+1}$ and $1 + \frac{1000^2}{1000^2+1} < \frac{1000}{1000+1} + \frac{1000^2}{1000 \cdot 999+1}$. Therefore, player 1 prefers strategy s_1^i if and only if player 2 has chosen s_2^i. On the other hand, player 2 prefers strategy s_2^i if and only if player 1 has chosen s_1^j with $i \neq j$.

Observation 7.6. There is a budget game $\mathcal{B}_1$ with fixed player demands, which is not a matroid budget game and does not possess a Nash equilibrium.

For matroid budget games with fixed player demands, the best-response dynamic always results in a Nash equilibrium after a finite number of steps. However, we cannot say anything regarding the number of improving moves (polynomial vs.

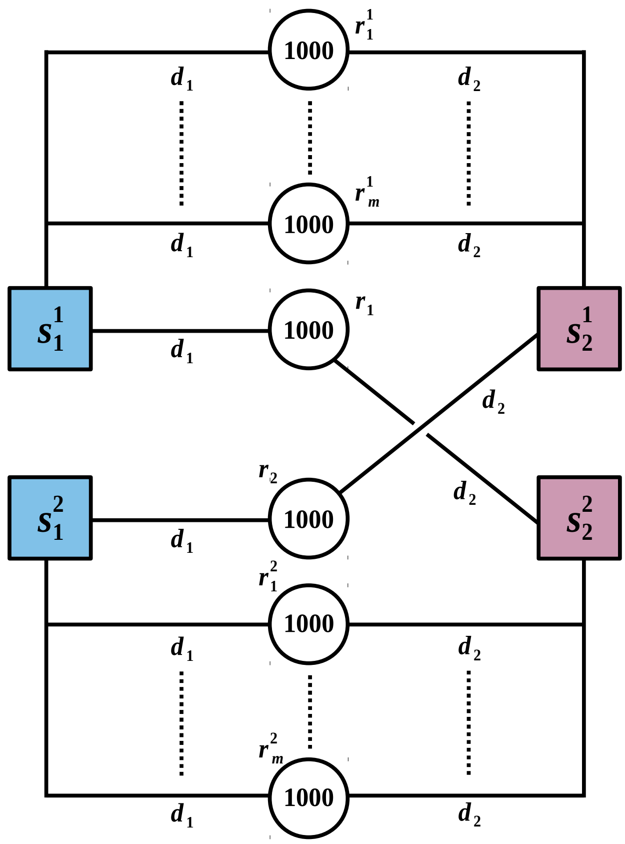

Figure 7.1: The budget game $\mathcal{B}_1$ with fixed demands and no Nash equilibrium. We choose $d_1 = 1000$ and $d_2 = 1$. Note that the auxiliary players, who use all resources expect r_1 and r_2, are not shown. The total number of resources is $2m + 1$: m resources at the top, m resources at the bottom and the two resources r_1 and r_2.

players	strategy profiles			
	(s_1^1, s_2^1)	(s_1^1, s_2^2)	(s_1^2, s_2^1)	(s_1^2, s_2^2)
1	$1000 + \frac{1000^3}{1000^2+1}$	$\frac{1000^2}{1000+1} + 1000$	$\frac{1000^2}{1000+1} + 1000$	$1000 + \frac{1000^3}{1000^2+1}$
2	$1 + \frac{1000^2}{1000^2+1}$	$\frac{1000}{1000+1} + \frac{1000^2}{1000 \cdot 999+1}$	$\frac{1000}{1000+1} + \frac{1000^2}{1000 \cdot 999+1}$	$1 + \frac{1000^2}{1000^2+1}$

Table 7.1: Overview of the different strategy profiles and the corresponding utilities of the budget game $\mathcal{B}_1$. Since the strategies of the auxiliary players are fixed, we abuse notation and restrict the strategy profiles to the strategies of the two players 1 and 2.

exponential) and whether it is a feasible way of determining an equilibrium. We can circumvent this by further restricting the class of games to singleton budget games with fixed player demands. As every singleton game is also a matroid game, our former result still holds. In addition, one can efficiently compute a Nash equilibrium for these kinds of games by using a different approach. To show this, we first introduce the following lemma. It states that if there are two players in a singleton game with fixed demands on the same resource r and the player with the larger demand wants to switch from r to r', then the same holds for the player with the smaller demand.

Lemma 7.7. Let $d_1, d_2 \in \mathbb{R}_{>0}$ with $d_1 \leq d_2$ and $b_r, b_{r'}, T_r(\mathsf{s}), T_{r'}(\mathsf{s}) \in \mathbb{R}_{\geq 0}$. If

$$d_2 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right) \leq d_2 \cdot \min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s}) + d_2}\right)$$

then also

$$d_1 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right) \leq d_1 \cdot \min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s}) + d_1}\right)$$

Proof. Proof by case distinction. We only need to consider two cases.

- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right) < \min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_2}\right) = 1$, the statement becomes trivial.
- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = \frac{b_r}{T_r(\mathsf{s})}$ and $\min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_2}\right) = \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_2}$, the statement $d_2 \cdot \frac{b_r}{T_r(\mathsf{s})} \leq d_2 \cdot \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_2}$ directly implies $d_1 \cdot \frac{b_r}{T_r(\mathsf{s})} \leq d_1 \cdot \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_2} \leq d_1 \cdot \frac{b_{r'}}{T_{r'}(\mathsf{s})+d_1}$

□

Using this lemma, we see that a Nash equilibrium can be computed by inserting the players one after another in descending order of their demands.

Theorem 7.8. *Let $\mathcal{B}$ be a singleton budget game with fixed player demands. Starting with an empty strategy profile and letting the players pick their best-response one after another in descending order of their demands yields a Nash equilibrium in $\mathcal{O}(n)$ steps.*

Proof. We only need to show that a strategy choice made by player i does not change the best-response of any player j who was inserted into the game at some earlier point in time. Let s be the (partial) strategy profile the moment before i is inserted into the game with $s_j = \{r\}$. If i also chooses resource r, then this implies

$$d_i \cdot \min\left(1, \frac{b_{r'}}{T_{r'}(\mathsf{s}) + d_i}\right) \leq d_i \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s}) + d_i}\right)$$

for all $r' \in \mathcal{R}$. By the contraposition of Lemma 7.7, this also holds for all players j on r with $d_i \leq d_j$ who were inserted earlier. This proves our statement. □

7.2 Singleton Budget Games with Two Resources

In this section, we show that any singleton budget game which consists of only two different resources r_1 and r_2 is weakly acyclic. In such a game, every player i has exactly two strategies, $s_i^1 = (s_i^1(r_1), 0)$ and $s_i^2 = (0, s_i^2(r_2))$ with $s_i^1(r_1), s_i^2(r_2) > 0$. This result can then be applied to matroid budget games, as well. We introduce a new variant of budget games, which allow a fixed offset to the total demand on a resource.

Definition 7.9. Let $\mathcal{B}$ be a strategic or ordered budget game for which we introduce an offset $\sigma_r \in \mathbb{R}_{\geq 0}$ to every resource $r \in \mathcal{R}$ such that the utility of player i from r under strategy profile s is defined as

$$u_{i,r}(\mathsf{s}) = \min\left(s_i(r), \frac{s_i(r) \cdot b_r}{T_r(\mathsf{s}) + \sigma_r}\right) = s_i(r) \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s}) + \sigma_r}\right).$$

We then call $\mathcal{B}$ an *offset budget game*.

It is easy to see that by setting $\sigma_r = 0$ for every $r \in \mathcal{R}$, every offset budget game becomes a regular budget game.

Theorem 7.10. *Every singleton offset budget game with two resources is weakly acyclic.*

Proof. For the actual proof, we use an induction over the number of players. For a game with n players, we denote the offset of resource r with σ_r^n.

Induction start ($n = 2$): For any initial strategy profile and any offsets, it takes at most three improving moves by the players to reach a Nash equilibrium. After the first player has chosen his best-response, the second one either picks the other resource, resulting in a Nash equilibrium, or the same as the first player. In the latter case, the first player may change his strategy once more, which only increases the utility of the second player.

Induction hypothesis: Every singleton offset budget game with two resources and $n - 1$ players is weakly acyclic.

Induction step ($n - 1 \to n$): Without loss of generality, we assume that $s_n^1(r_1) \geq s_n^2(r_2)$ and $s_n^1(r_1) \geq s_i^1(r_1)$ for all $i = 1, \ldots, n-1$. We fix the strategy of player n to s_n^1. The resulting game is identical to an offset budget game with $n - 1$ players and $\sigma_{r_1}^{n-1} = \sigma_{r_1}^n + s_n^1(r_1)$, $\sigma_{r_2}^{n-1} = \sigma_{r_2}^n$. By induction hypothesis, this game is weakly acyclic and the remaining players can reach a Nash equilibrium s^1 after a finite number of improving moves. We assume that $u_n(\mathsf{s}^1) < u_n(\mathsf{s}_{-n}^1, s_n^2)$, as otherwise s^1 is still a Nash equilibrium even if the strategy of player n is not fixed. By the same arguments, the game has a Nash equilibrium s^2 if we fix the strategy of player n to s_n^2. This time, we assume $u_n(\mathsf{s}^2) < u_n(\mathsf{s}_{-n}^2, s_n^1)$. We obtain the following two inequalities:

$$\begin{aligned} &s_n^1(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^1) + \sigma_{r_1}^n}\right) \\ &< s_n^2(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_n^2(r_2)}\right) \end{aligned} \tag{7.1}$$

$$\begin{aligned} &s_n^2(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^2) + \sigma_{r_2}^n}\right) \\ &< s_n^1(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_n^1(r_1)}\right) \end{aligned} \tag{7.2}$$

Combining them yields

$$\begin{aligned} &\min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^1) + \sigma_{r_1}^n}\right) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^2) + \sigma_{r_2}^n}\right) \\ &< \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_n^2(r_2)}\right) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_n^1(r_1)}\right). \end{aligned}$$

We now make a crucial observation. There has to be at least one player i who chooses the same resource as player n in both s^1 and s^2. If player n is the only player on resource r_1 in s^1, then all other players are on r_2. Equation 7.1 states that n is still willing to move over to r_2, implying that resource r_1 is never a best-response. So there has to be at least one additional player i on r_1 in s^1. Let $n_r(\mathsf{s})$ be the set of all players located on resource r in strategy profile s. If $n_{r_1}(\mathsf{s}^1) \cap n_{r_2}(\mathsf{s}^2) = \{n\}$, then $n_{r_2}(\mathsf{s}^2) \setminus \{n\} \subseteq n_{r_2}(\mathsf{s}^1)$, contradicting that n is satisfied with r_2 in $(\mathsf{s}_{-n}^1, s_n^2)$ but not in s^2. So at least one player i shares the same resource as n in both s^1 and s^2.

By definition, player i cannot improve his utility in neither s^1 nor s^2, so

$$\begin{aligned} &s_i^1(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^1) + \sigma_{r_1}^n}\right) \geq s_i^2(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_i^2(r_2)}\right) \\ &\Rightarrow \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^1) + \sigma_{r_1}^n}\right) \geq \frac{s_i^2(r_2)}{s_i^1(r_1)} \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_i^2(r_2)}\right) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} &s_i^2(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^2) + \sigma_{r_2}^n}\right) \geq s_i^1(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_i^1(r_1)}\right) \\ &\Rightarrow \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^2) + \sigma_{r_2}^n}\right) \geq \frac{s_i^1(r_1)}{s_i^2(r_2)} \cdot \min\left(1, \frac{b_2}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_i^1(r_1)}\right). \end{aligned} \tag{7.4}$$

At the beginning of the induction step, we assumed that $s_n^1(r_1) \geq s_i^1(r_1)$. We now show that $s_n^2(r_2) \geq s_i^2(r_2)$ has to hold as well. Suppose that $s_i^2(r_2) >$

$s_n^2(r_2)$. According to Lemma 7.1 and Equation 7.3, this would imply that s^1 is also an equilibrium for player n, contradicting our initial assumption. Therefore, $s_n^2(r_2) \geq s_i^2(r_2)$ has to hold. Using the inequalities regarding player i and n, we obtain

$$\min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_i^2(r_2)}\right) \cdot \min\left(1, \frac{b_2}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_i^1(r_1)}\right)$$
$$< \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}^1) + \sigma_{r_2}^n + s_n^2(r_2)}\right) \cdot \min\left(1, \frac{b_2}{T_{r_1}(\mathsf{s}^2) + \sigma_{r_1}^n + s_n^1(r_1)}\right).$$

This is a contradiction, since the demand of player n is larger than that of player i on both resources. Our proof shows that any attempt to construct a cycle of strategy profiles results in a contradiction at some point.

□

Since every budget game is also an offset budget game, we can immediately draw the following conclusion.

Corollary 7.11. Every singleton budget game with two resources is weakly acyclic.

The proof of Theorem 7.10 also yields a trivial recursive algorithm for computing a Nash equilibrium. For the first $n-1$ players, one can compute two Nash equilibria, one for each strategy choice by player n. One of these two states also has to be a Nash equilibrium for all players. We can fix the strategies of an arbitrary number of players, as the offsets of the resources only depend on the total demand, not on the players responsible. However, this approach requires an exponential number of computational steps.

Our result not only holds for singleton, but for matroid budget games, as well. Let i be a player in a matroid budget game with two resources, whose strategy space does not consist of singleton strategies. By definition, he then possesses only one strategy using both resources and his existence only introduces a fixed offset to both resources. This holds for all players with non-singleton strategy spaces. The Nash equilibrium of the remaining players who do possess only singleton strategies can be determined as shown above.

Corollary 7.12. Every matroid (offset) budget game with two resources is weakly acyclic.

7.3 Singleton Budget Games with Ordered Players and Increasing Demand Ratios

We now extend the technique used in the previous section to singleton games with more than two resources. However, we need two additional restrictions: a total order on the players (based on their demands) and increasing demand ratios. The first restriction states that although the demands of the players are no longer fixed, their order is the same for every resource.

Definition 7.13 (Ordered Players)**.** Let $\mathcal{B}$ be a singleton budget game such that for all $i, j \in \mathcal{N}$ and all $r, r' \in \mathcal{R}$ it holds that

$$d_i(r) \leq d_j(r) \Rightarrow d_i(r') \leq d_j(r').$$

Then $\mathcal{B}$ is a budget game with *ordered players*.

In addition, we require the ratios between the demands of a single player to increase with increasing demands.

Definition 7.14 (Increasing Demand Ratios)**.** Let $\mathcal{B}$ be a singleton budget game with ordered players and $i, j \in \mathcal{N}$ such that $d_i(r) \leq d_j(r)$ for all $r \in \mathcal{R}$. If for all $r, r' \in \mathcal{R}$ with $d_i(r) \leq d_i(r')$ it holds that

$$\frac{d_i(r')}{d_i(r)} \leq \frac{d_j(r')}{d_j(r)},$$

then $\mathcal{B}$ has *increasing demand ratios*.

Singleton budget games with increasing demand ratios have similar properties than general singleton budget games with only two resources, which is why Nash equilibria always exist.

Theorem 7.15. *Singleton offset budget games with ordered players and increasing demand ratios always have a Nash equilibrium.*

Proof. Proof by induction over the number of players. For a game with n players, we denote the offset of resource r with σ_r^n.

Induction start ($n = 2$): The induction start is identical to the proof of Theorem 7.10, so we refer to that.

Induction hypothesis: Every singleton offset budget game with $n-1$ ordered players and increasing demand ratios has a Nash equilibrium.

Induction step ($n-1 \to n$): Without loss of generality, we assume that $s_n^1(r_1) \geq s_n^2(r)$ for all $r \in \mathcal{R}$ and also that $s_n^1(r_1) \geq s_i^1(r_1)$ for all $i \in \mathcal{N}$. We fix the strategy of player n to s_n^1. The resulting game is identical to an offset budget game with $n-1$ ordered players and $\sigma_{r_1}^{n-1} = \sigma_{r_1}^n + s_n^1(r_1)$ and $\sigma_r^{n-1} = \sigma_r^n$ for all $r \in \mathcal{R} \setminus \{r_1\}$. By induction hypothesis, this game has a Nash equilibrium s (with $s_n = \{r_1\}$). Now assume that s is not an equilibrium for n. Then we get

$$\begin{aligned} u_n(\mathsf{s}) &= s_n(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}) + \sigma_{r_1}^n}\right) \\ &< s_n(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma_{r_2}^n + s_n(r_2)}\right) = u_n(\mathsf{s}_{-n}, r_2) \end{aligned}$$

for some $r_2 \in \mathcal{R}$. Let i be another player on r_1, i.e. $s_i = \{r_1\}$. Since r_1 is the best-response of i, the following has to hold.

$$\begin{aligned} u_i(\mathsf{s}) &= s_i(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}) + \sigma^n_{r_1}}\right) \\ &\geq s_i(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma^n_{r_2} + s_i(r_2)}\right) = u_i(\mathsf{s}_{-i}, r_2) \end{aligned}$$

By combining these two inequalities, we get

$$\begin{aligned} &\frac{s_i(r_2)}{s_i(r_1)} \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma^n_{r_2} + s_i(r_2)}\right) \\ <&\frac{s_n(r_2)}{s_n(r_1)} \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma^n_{r_2} + s_n(r_2)}\right), \end{aligned}$$

which can be transformed to

$$\begin{aligned} &\min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma^n_{r_2} + s_i(r_2)}\right) \cdot \left(\min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + \sigma^n_{r_2} + s_n(r_2)}\right)\right)^{-1} \\ <&\frac{s_n(r_2)}{s_n(r_1)} \cdot \frac{s_i(r_1)}{s_i(r_2)}. \end{aligned}$$

Since $s_i(r_2) \leq s_n(r_2)$, this can be simplified to

$$1 < \frac{s_n(r_2)}{s_n(r_1)} \cdot \frac{s_i(r_1)}{s_i(r_2)} \Rightarrow \frac{s_i(r_1)}{s_i(r_2)} > \frac{s_n(r_1)}{s_n(r_2)}.$$

This contradicts our restriction that the demand ratios are increasing and implies that i cannot exist. So n is the only player on resource r_1 and since this is his preferred resource, it is also his best-response. We conclude that s is a Nash equilibrium for all n players.

□

Again, this result holds for regular budget games, in particular.

Corollary 7.16. Singleton budget games with ordered players and increasing demand ratios always have a Nash equilibrium.

7.4 Singleton Budget Games with Two Demands and Unified Budgets

In this section, we consider singleton budget games with only two different demands. In other words, there are two constants d^- and d^+ and the demand of any player on any resource equals one of them. In addition, we assume all resources to be identical, with their budgets being all the same. The resulting class of games is called *budget games with two demands and unified budgets*. For this section, we assume that every resource is available to every player. In other words, for every player i and every resource r, there is a strategy $s_i \in \mathcal{S}_i$ using r. Otherwise, our result does not hold.

Definition 7.17 (Budget Games with Two Demands)**.** Let $d^-, d^+ \in \mathbb{R}_{>0}$ and $\mathcal{B}$ be a budget game such that for every player i, every strategy $s_i \in \mathcal{S}_i$ and every resource r, we get $s_i(r) \in \{d^-, d^+\}$. Then we call $\mathcal{B}$ a *budget game with two demands*.

Definition 7.18 (Budget Games with Unified Budgets)**.** Let $\mathcal{B}$ be a budget game such that for all $r, r' \in \mathcal{R}$ it holds that $b_r = b_{r'}$. Then we call $\mathcal{B}$ a *budget game with unified budgets*.

Without loss of generality, we assume $d^- \leq d^+$. In this section, we only consider singleton budget games with two demands and unified budgets, so each player uses exactly one resource in every strategy profile. We introduce Algorithm 2, which computes Nash equilibria for this class of games. The algorithm utilizes the best-response dynamic and only controls the order in which the improving moves are executed. Therefore, the correctness of the algorithm also proves that these games are weakly acyclic. We begin by introducing a number of technical lemmata. Afterwards, we extend the definition of the lexicographical potential function (cf. Definition 7.3) to obtain an *augmented lexicographical potential function* ϕ. During the execution of Algorithm 2, this new kind of potential function is strictly increasing for at least every second strategy change. In other words, if $\mathsf{s}_1, \mathsf{s}_2$ and s_3 are successive strategy profiles during the execution of the algorithm, then either $\phi(\mathsf{s}_1) < \phi(\mathsf{s}_2)$ or $\phi(\mathsf{s}_1) \geq \phi(\mathsf{s}_2)$ but $\phi(\mathsf{s}_1) < \phi(\mathsf{s}_3)$. When discussing the second case, we combine the two strategy changes between $\mathsf{s}_1, \mathsf{s}_2$ and $\mathsf{s}_2, \mathsf{s}_3$ into a single *macro strategy change*, which transforms s_1 directly into s_3.

By assigning a type to every strategy change, depending on the player's demand before and after, we are able to formally analyze its effects. Let player i use resource r_1 in strategy s_i^1 and resource r_2 in s_i^2. We write $d_i^1(r_1) \to d_i^2(r_2)$ to describe the type of the strategy change from s_i^1 to s_i^2. Since there are only two different demands to begin with, there are just four different types of strategy changes.

- $d^+ \to d^+$
- $d^+ \to d^-$
- $d^- \to d^+$
- $d^- \to d^-$

In this section, we analyze how a strategy change by one player affects the others, especially how it can change the best-response of another player who formally was in an equilibrium. We introduce two new notions which distinguish how exactly a new best-response is created.

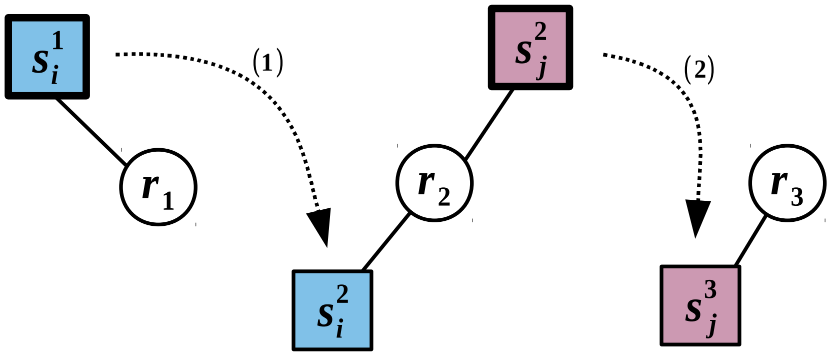

Figure 7.2: Example for a pushing strategy change. Assuming that player j initially is in an equilibrium, the improving move of player i from r_1 to r_2 yields r_3 as the new best-response of j. *j is being pushed by i from r_2 to r_3.*

Definition 7.19 (Pushing Strategy Change)**.** Let $\mathcal{B}$ be a singleton budget game with players $i, j \in \mathcal{N}$, resources $r_1, r_2, r_3 \in \mathcal{R}$ and strategy profile $\mathsf{s} \in \mathcal{S}$. In s, let $s_i = \{r_1\}$ and $s_j = \{r_2\}$ with $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_2)$ and $u_j(\mathsf{s}) \geq u_j(\mathsf{s}_{-j}, r)$ for all $r \in \mathcal{R}$. Denote $\mathsf{s}' = (\mathsf{s}_{-i}, r_2)$.
If $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_3)$, then the strategy change by i from r_1 to r_2 is called a *pushing strategy change.*

The idea behind a pushing strategy change is that i relocates to the same resource as j, increasing the total demand on it. As a result, that resource is then no longer the best-response of j.

Definition 7.20 (Pulling Strategy Change)**.** Let $\mathcal{B}$ be a singleton budget game with players $i, j \in \mathcal{N}$, resources $r_1, r_2, r_3 \in \mathcal{R}$ and strategy profile $\mathsf{s} \in \mathcal{S}$. In s, let $s_i = \{r_2\}$ and $s_j = \{r_1\}$ with $u_j(\mathsf{s}) \geq u_j(\mathsf{s}_{-j}, r)$ for all $r \in \mathcal{R}$ and $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$. Denote $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$.
If $u_j(\mathsf{s}') < u_i(\mathsf{s}'_{-j}, r_2)$, then the strategy change by i from r_2 to r_3 is called a *pulling strategy change.*

A pulling strategy change decreases the total demand on a resource by a large enough margin such that another player becomes interested in it. Both pushing and pulling strategy changes create a new best-response improving move for some player. If a strategy change by i is both pushing and pulling for one and the same player j, we always regard it as the former. In such a case, both players switch their resources.

By combining the concepts of pushing and pulling strategy changes with the four types of strategy changes available, we are able to give a detailed analysis of the best-response dynamic. This gives us enough information to determine a Nash equilibrium only by controlling the order in which the players are allowed to choose their best-response. We introduce Algorithm 2, which computes a Nash equilibrium

Algorithm 2 ComputeNE

$\mathsf{s}_0 \leftarrow$ arbitrary strategy profile
Phase 1:
while there is a player with best-response improving move of type $d^+ \to d^-$ **do**
  perform best-response improving move of type $d^+ \to d^-$
Phase 2:
while current strategy profile s is not a Nash equilibrium **do**
  if there is a player with best-response improving move of type $d^+ \to d^-$ **then**
    perform best-response improving move of type $d^+ \to d^-$
  else if there is a player i with b.r. improving move of type $d^+ \to d^+$ **then**
    $\mathcal{N}' \leftarrow \{j \in \mathcal{N} \mid j$ has best-response improving move of type $d^+ \to d^+\}$
    choose $i \in \mathcal{N}'$ such that $T_{s_i}(\mathsf{s}) \geq T_{s_j}(\mathsf{s})$ for all $j \in \mathcal{N}'$
    perform best-response improving move of i
  else
    perform best-response improving move ▷ $d^- \to d^-$ or $d^- \to d^+$
return current strategy profile s_{NE}

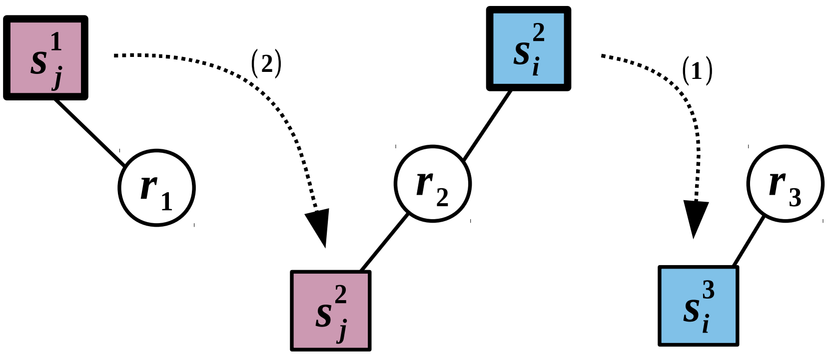

Figure 7.3: Example for a pulling strategy change. Assuming that player j initially is in an equilibrium, the improving move of player i from r_2 to r_3 yields r_2 as the new best-response of j. *j is being pulled by i from r_1 to r_2.*

of any singleton budget game with two demands and unified budgets. Phase 1 of the algorithm is used to create an intermediate strategy profile without any best-response improving move of type $d^+ \to d^-$. With the following three lemmata, we analyze under which circumstances a new best-response move of this type occurs.

Lemma 7.21. Let s be a strategy profile in phase 2 of Algorithm 2. In s, no best-response improving move of type $d^+ \to d^-$ is created by a pushing strategy change.

Proof. Let s be a strategy profile with $s_i = \{r_1\}$ and $s_j = \{r_2\}$ and in which player i can increase his utility by moving to resource r_2.

$$u_i(\mathsf{s}) = d_i(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right) < d_i(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + d_i(r_2)}\right) = u_i(\mathsf{s}_{-i}, r_2)$$

Set $\mathsf{s}' = (\mathsf{s}_{-i}, r_2)$. Now assume that this pushes player j by creating an improving move of type $d^+ \to d^-$.

$$u_j(\mathsf{s}') = d^+ \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) + d_i(r_2)}\right) < d^- \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^-}\right) = u_j(\mathsf{s}'_{-j}, r_3)$$

This implies

$$\begin{aligned} & d_i(r_1) \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right) < d_i(r_2) \cdot \min\left(1, \frac{b_2}{T_{r_1}(\mathsf{s}) + d_i(r_2)}\right) \\ \leq\ & d^+ \cdot \min\left(1, \frac{b_2}{T_{r_1}(\mathsf{s}) + d_i(r_2)}\right) < d^- \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^-}\right) \\ \leq\ & d_i(r_3) \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d_i(r_3)}\right). \end{aligned}$$

or simply

$$u_i(\mathsf{s}_{-i}, r_2) < d_i(r_3) \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d_i(r_3)}\right) = u_i(\mathsf{s}_{-i}, r_3).$$

Therefore, player i would have chosen resource r_3 instead of r_2. If $d_i(r_3) = d^-$, then a strategy change of type $d^+ \to d^-$ would have already existed in s. Note that

$$d_i(r_3) \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d_i(r_3)}\right) < d_i(r_3) \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s})}\right),$$

which covers the case $r_1 = r_3$. □

Lemma 7.22. Let s be a strategy profile in phase 2 of Algorithm 2. In s, no best-response improving move of type $d^+ \to d^-$ is created by a pulling strategy change of type $d^- \to d^-$.

Proof. Proof by contradiction. Let s be a strategy profile with $s_i = \{r_2\}$ and $s_j = \{r_1\}$, $d_i(r_2) = d_i(r_3) = d^-$, $d_j(r_1) = d^+$, $d_j(r_2) = d^-$ and both $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$ and $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ for $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$. According to our pivot rules, s is an equilibrium for j. From

$$u_i(\mathsf{s}) = d^- \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^-}\right) = u_i(\mathsf{s}_{-i}, r_3)$$

and

$$\begin{aligned} u_j(\mathsf{s}) &= d^+ \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right) \\ &< d^- \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) - d^- + d^-}\right) = d^- \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s})}\right) \\ &= u_j(\mathsf{s}'_{-j}, r_2) \end{aligned}$$

we can conclude that

$$u_j(\mathsf{s}) = \min\left(1, d^+ \cdot \frac{b_1}{T_{r_1}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^-}\right) \leq u_j(\mathsf{s}_{-j}, r_3).$$

If r_2 is the best-response of j after the strategy change of i, then r_3 has to be his best-response before, which contradicts our assumption. For both possible values of $d_j(r_3)$, j would have performed this strategy change before i. □

Lemma 7.23. Let s be a strategy profile in phase 2 of Algorithm 2. In s, no best-response improving move of type $d^+ \to d^-$ is created by a pulling strategy change of type $d^- \to d^-$ if the budgets are unified.

Proof. Proof by contradiction. Let s be a strategy profile with $s_i = \{r_2\}$ and $s_j = \{r_1\}$, $d_i(r_2) = d_i(r_3) = d^+$, $d_j(r_1) = d^+$, $d_j(r_2) = d^-$ and both $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$ and $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ for $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$. Since all budgets are equal, we simply write b instead of b_r. According to our pivot rules, the best-response of j, there is no improving move of type $d^+ \to d^-$. We get

$$\begin{aligned} u_i(\mathsf{s}) &= d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) < d^+ \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^+}\right) = u_i(\mathsf{s}_{-i}, r_3) \\ &\Rightarrow T_{r_3}(\mathsf{s}) < T_{r_2}(\mathsf{s}) - d^+ \end{aligned}$$

and

$$u_j(\mathsf{s}) = u_j(\mathsf{s}') = d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right) = u_j(\mathsf{s}'_{-j}, r_2).$$

The rest of the proof is done by case distinction.

- $d_j(r_3) = d^-$
  First we show that

$$u_j(\mathsf{s}) = d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) \geq d^- \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^-}\right) = u_j(\mathsf{s}_{-j}, r_3).$$

  Otherwise, there has to be a resource r_4 with $d_j(r_4) = d^+$ and

$$u_j(\mathsf{s}_{-j}, r_3) < u_j(\mathsf{s}_{-j}, r_4) = d^+ \cdot \min\left(1, \frac{b}{T_{r_4}(\mathsf{s}) + d^+}\right)$$

  or the algorithm would allow j to change his strategy instead of i. For the same reasons, we also get $T_{r_1}(\mathsf{s}) \leq T_{r_2}(\mathsf{s})$, but this contradicts

$$d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) + d^- - d^+}\right) \leq d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right).$$

  So we get

$$\begin{aligned} u_j(\mathsf{s}'_{-j}, r_2) > u_j(\mathsf{s}) &= d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) \\ &\geq d^- \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^-}\right) = u_j(\mathsf{s}_{-j}, r_3) \end{aligned}$$

  or simply

$$\begin{aligned} d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right) &> d^- \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^-}\right) \\ &\geq d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right), \end{aligned}$$

  which is also a contradiction.

- $d_j(r_3) = d^+$

$$\begin{aligned} d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) &< d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right) \\ &< d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^+}\right) \\ &= d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) \end{aligned}$$

  implies both

$$\min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) \Rightarrow T_{r_1}(\mathsf{s}) > T_{r_2}(\mathsf{s})$$

and

$$u_j(\mathsf{s}) = d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right)$$
$$< d^+ \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^+}\right) = u_j(\mathsf{s}_{-j}, r_3).$$

If r_2 is a best-response of j after the strategy change of i, then r_3 would have been a best-response of j before. Both players have the same priority according to the pivot rules, but since $T_{r_1}(\mathsf{s}) > T_{r_2}(\mathsf{s})$ j would have changed his strategy first.

This concludes our proof. □

If the budgets of the resources can be different, then there are examples where this last lemma no longer holds and a strategy change of type $d^+ \to d^+$ pulls a best-response improving move of type $d^+ \to d^-$. By now, we know that as long as there are no improving moves of type $d^+ \to d^-$ to begin with, they will not be created by those of type $d^- \to d^-$ and $d^+ \to d^+$. While it is still possible that an improving move of type $d^- \to d^+$ results in a new improving move of type $d^+ \to d^-$, we circumvent this by introducing *macro strategy changes*.

Definition 7.24 (Macro Strategy Change)**.** For a singleton budget game $\mathcal{B}$ with players $i, j \in \mathcal{N}$ and strategy profile s, let $s_i = \{r_2\}$, $s_j = \{r_1\}$, $d_i(r_2) = d_j(r_2)$ and both $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$ and $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ for $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$. Relocating i from r_2 to r_3 and j from r_1 to r_2 right after another is called a *macro strategy change*.

A macro strategy change is the combination of two regular strategy changes. Although not associated with an actual player in the game, we say that such a strategy change is performed by a *virtual player*. Using the identifiers from the definition, we see that the total demand on r_2 does not change during a macro strategy change while that of r_1 is decreased by $d_j(r_1)$ and that of r_3 is increased by $d_i(r_3)$. The next lemma shows that for $d_j(r_1) = d_i(r_3) = d^+$, the virtual player behind the macro strategy change actually benefits from it.

Lemma 7.25. Let s be a strategy profile with a macro strategy change of type $d^+ \to d^+$ from r_1 to r_3. Then

$$d^+ \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right) < d^+ \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^+}\right).$$

Proof. Due to the underlying strategy changes, we get

$$d^+ \cdot \min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right)$$
$$< d^+ \cdot \min\left(1, \frac{b_2}{T_{r_2}(\mathsf{s}) - d^+ + d^+}\right) < d^- \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^-}\right)$$
$$< d^+ \cdot \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) d^+}\right).$$

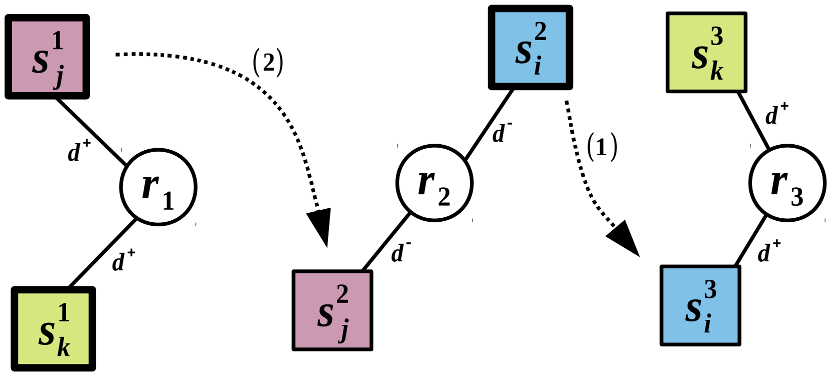

Figure 7.4: Example for a macro strategy change. The improving move of player j from r_1 to r_2 is exectued right after the improving move of player i from r_2 to r_3. After both strategy changes, the total demand on r_2 has not changed. This sequence of strategy changes is equivalent to the strategy change of a virtual player k from r_1 to r_3.

□

The previous lemmata have shown us that once phase 1 of Algorithm 2 has terminated, strategy changes of type $d^+ \to d^-$ can only exist due to a pulling strategy change of type $d^- \to d^+$. By combining these two strategy changes, we get a single macro strategy change of type $d^+ \to d^+$ which is actually an improving move for its virtual player. It therefore shares the main properties of actual strategy changes of type $d^+ \to d^+$, namely the following.

Corollary 7.26. A macro strategy change of type $d^+ \to d^+$ from resource r_1 to r_3 in strategy profile s satisfies the property

$$\begin{aligned} &\min\left(\min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s})}\right), \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s})}\right)\right) \\ <\ &\min\left(\min\left(1, \frac{b_1}{T_{r_1}(\mathsf{s}) - d^+}\right), \min\left(1, \frac{b_3}{T_{r_3}(\mathsf{s}) + d^+}\right)\right). \end{aligned}$$

Thus, a macro strategy change of type $d^+ \to d^+$ strictly increases a lexicographical potential function (cf. Definition 7.3), which is the fundamental property used in the analysis of our algorithm. Since this is generally not the case for strategy changes of type $d^- \to d^+$, we augment the concept of such a potential function.

Definition 7.27 (Augmented Lexicographical Potential Function)**.** For a budget game $\mathcal{B}$ with resource r and strategy profile s, let

$$c_r(\mathsf{s}) := \min\left(1, \frac{b_r}{T_r(\mathsf{s})}\right)$$

and $T(\mathsf{s}) = \sum_{i \in \mathcal{N}} d_i(s_i)$.
The function $\phi : \mathcal{S} \to \mathbb{R}^m_{\geq 0}$ with $\phi(\mathsf{s}) = (T(\mathsf{s}), c_{r_1}(\mathsf{s}), \ldots, c_{r_m}(\mathsf{s}))$ such that the entries $c_r(\mathsf{s})$ are sorted in ascending order is an *augmented lexicographical potential function* of $\mathcal{B}$.

This new potential function is similar to the lexicographical potential function, but it features the additional value $T(\mathsf{s})$, which stands for the total demand on all resources in the current strategy profile. Note that $T(\mathsf{s})$ is always the first entry in the tuple $\phi(\mathsf{s})$ and only the subsequent one are sorted in ascending order. Just like with some of the previous potential functions, we use the lexicographical order $<_{\text{lex}}$ as a relation between two values $\phi(\mathsf{s})$ and $\phi(\mathsf{s}')$, if applicable. We now have all the tools needed to prove the following result.

Theorem 7.28. *For a singleton budget game $\mathcal{B}$ with two demands and unified budgets and a strategy profile s_0 of $\mathcal{B}$, Algorithm 2 returns a Nash equilibrium s_{NE} of $\mathcal{B}$ after a finite number of computational steps.*

Proof. By construction, the output of the algorithm is a Nash equilibrium. It remains to show that the algorithm actually terminates at some point. The number of improving moves in the first phase is at most n, as every player can act at most once.

For the second phase, we use the augmented lexicographical potential function ϕ. This function is strictly increasing regarding $<_{\text{lex}}$ for all strategy changes of type $d^- \to d^-$ and $d^+ \to d^+$. In Section 7.1, we have already seen that strategy changes which leave the current demand of the player unchanged strictly increase the lexicographical potential function. Since such changes do not influence $T(\mathsf{s})$, this holds here as well. For strategy changes of type $d^- \to d^+$, ϕ is also strictly increasing because $T(\mathsf{s})$ grows. ϕ can only decrease for improving moves of type $d^+ \to d^-$.

Let s_1 be the strategy profile right after phase 1 has terminated. Then s_1 contains no best-response improving move of type $d^+ \to d^-$. According to Lemmata 7.21, 7.22 and 7.23, such moves can only appear as the result of a pulling strategy change of type $d^- \to d^+$. In this case, both together can be regarded as a single macro strategy change of type $d^+ \to d^+$. Because of Corollary 7.26 and the fact that it does not change the value of $T(\mathsf{s})$, ϕ grows for such a macro strategy change, too.

The value of ϕ grows with at least every second strategy change, so the algorithm has to terminate at some point. If a pulling strategy change creates multiple best-response improving moves of type $d^+ \to d^-$ to a resource r, then the algorithm executes one of them, chosen by some arbitrary rule. Afterwards, the total demand on r is the same as before the pulling strategy change and the other best-response moves of type $d^+ \to d^-$ cease to exist. □

Although the algorithm decides which strategy changes to prefer over others, it only performs best-response improving moves. We can therefore conclude this section with the following insight.

Corollary 7.29. Every singleton budget game with two demands and unified budgets is weakly acyclic.

We do not know if this result carries over to matroid budget games with the same restrictions. Regarding budget games with two demands and unified budgets but arbitrary strategy space structures, we refer to the game $\mathcal{B}_0$ from Definition 4.1 and the introduction of Chapter 6. We see that there are games of this type without a Nash equilibrium.

Corollary 7.30. There are budget games with two demands and unified budgets without a Nash equilibrium.

7.5 Approximate Pure Nash Equilibria in Matroid Budget Games

To conclude this chapter, we take a last look at approximate Nash equilibria and focus on matroid budget games (all results from Chapter 6 are independent of how the strategy spaces are structured). However, we no longer have to bound the demands with respect to the budgets, i.e. the value of δ from the definition of δ-share budget games does not influence our result. This time, the crucial factor (besides the matroid property) is the largest ratio between any two demands *of the same player*. The closer this ratio is to 1, the better our result. Depending on the instances, this can lead to better upper bounds for the existence of approximate Nash equilibria than the one from Chapter 6.

Once more, the main tool in this section is the lexicographical potential function ϕ introduced in Definition 7.3. We use ϕ to give a new bound on the existence of approximate Nash equilibria, specifically for matroid budget games.

Theorem 7.31. *Let $\alpha \in \mathbb{R}_{\geq 1}$ and $\mathcal{B}$ be a matroid budget game such that for every player i and all resources r_1, r_2 it holds that $\frac{d_i(r_2)}{d_i(r_1)} \leq \alpha$. Then $\mathcal{B}$ has an α-approximate Nash equilibrium.*

Proof. Let $d_i^{\max} := \max\{d_i(r) \mid r \in \mathcal{R}\}$ and $d_i^{\min} := \min\{d_i(r) \mid r \in \mathcal{R}\}$. We use the lexicographical potential function ϕ from Definition 7.3 and let the players only perform improving moves which also increase ϕ. While this may not lead to a Nash equilibrium, it always yields an α-approximate Nash equilibrium for

$$\alpha = \max\left\{\frac{d_i^{\max}}{d_i^{\min}} \mid i \in \mathcal{N}\right\}.$$

Since $\mathcal{B}$ is a matroid budget game, every strategy change can be modeled as a sequence of lazy moves and every lazy move increases the utility of the corresponding player. Otherwise, a lazy move can simply be omitted and the resulting strategy is still valid. Let s be a strategy profile of $\mathcal{B}$ in which player i can switch resource

r_1 for r_2 to increase his utility, resulting in strategy profile s'. We restrict the best-response dynamic such that we only allow this lazy move if it also satisfies

$$d_i(r_1) \cdot \min\left(1, \frac{b_{r_1}}{T_{r_1}(\mathsf{s})}\right) < d_i(r_1) \cdot \min\left(1, \frac{b_{r_2}}{T_{r_2}(\mathsf{s}) + d_i(r_1)}\right).$$

Note that under this restriction, player i would still profit from the strategy change if his demand on both r_1 and r_2 is the same. As shown in the proof of Theorem 7.31, such a lazy move increases ϕ. In addition to the strategy changes where the demand actually does not change, this also includes lazy moves in which the demand decreases (cf. Lemma 7.1) and some of those in which the demand increases.

By following this restricted best-response dynamic, we obtain a strategy profile s^α. Let s' be a strategy profile with $u_i(\mathsf{s}^\alpha) < u_i(\mathsf{s}')$ and which originates from s^α by a unilateral strategy change of i. The strategy change of i from s_i^α to s_i' consists only of lazy moves of the form

$$u_{i,r_1}(\mathsf{s}^\alpha) = d_i(r_1) \cdot \min\left(1, \frac{b_{r_1}}{T_{r_1}(\mathsf{s}^\alpha)}\right) < d_i(r_2) \cdot \min\left(1, \frac{b_{r_2}}{T_{r_2}(\mathsf{s}^\alpha) + d_i(r_2)}\right) = u_{i,r_2}(\mathsf{s}')$$

with $d_i(r_1) < d_i(r_2)$ and $c_{r_1}(\mathsf{s}^\alpha) \geq c_{r_2}(\mathsf{s}')$. We get

$$\begin{aligned}
\frac{u_i(\mathsf{s}')}{u_i(\mathsf{s}^\alpha)} &= \frac{\sum_{r' \in s_i'} u_{i,r'}(\mathsf{s}')}{\sum_{r \in s_i^\alpha} u_{i,r}(\mathsf{s}^\alpha)} = \frac{\sum_{r' \in s_i'} d_i(r') \cdot c_{r'}(\mathsf{s}')}{\sum_{r \in s_i^\alpha} d_i(r) \cdot c_r(\mathsf{s}^\alpha)} \\
&\leq \frac{\sum_{r' \in s_i'} d_i^{\max} \cdot c_{r'}(\mathsf{s}')}{\sum_{r \in s_i^\alpha} d_i^{\min} \cdot c_r(\mathsf{s}^\alpha)} = \frac{d_i^{\max} \cdot \sum_{r' \in s_i'} c_{r'}(\mathsf{s}')}{d_i^{\min} \cdot \sum_{r \in s_i^\alpha} c_r(\mathsf{s}^\alpha)} \\
&\leq \frac{d_i^{\max} \cdot \sum_{r \in s_i^\alpha} c_r(\mathsf{s}^\alpha)}{d_i^{\min} \cdot \sum_{r \in s_i^\alpha} c_r(\mathsf{s}^\alpha)} = \frac{d_i^{\max}}{d_i^{\min}}.
\end{aligned}$$

□

If the demands of each individual player do not differ too much from each other (similar to Section 6.4), then a matroid budget game already has an α-approximate Nash equilibria for small α, even for large values of δ (e.g. $\delta = 1$). If the ratios between the demands of a player are large but δ is small, we can use the upper bound from Section 6.1 and still obtain good results. For matroid budget games, we now have two different upper bounds which depend on different aspects of budget games and are therefore not really compared easily. Based on the actual structure of a budget game, however, we can always use the better of the two bounds.

This concludes the current chapter. We considered a number of different classes of singleton and matroid budget games and proved that they all possess pure Nash equilibria. However, we do not know if this also holds for *all* matroid budget games.

CHAPTER 8

Conclusion

To conclude this thesis, we summarize our results and discuss a number of open problems.

8.1 Synopsis

We introduced a new game theoretical model called budget games, in which the players compete over resources with a finite budget. As a strategy, a player chooses a vector of different demands. In the resulting strategy profile, each player has a non-negative demand on every resource. If the total demand by all players does not exceed the budget of a resource, then the demand of each player on that resource is fully satisfied. Otherwise, the budget is split between the players proportionally, with each player receiving less than what he demands (or receiving nothing). We denote the share of a resources' budget received by a player as the utility he obtains from that resource. The total utility of a player is then the sum of the individual utilities obtained from the resources. We showed that pure Nash equilibria generally do not exist in budget games and therefore turned our attention to a number of alternative concepts.

As a first approach, we changed the way the budgets are distributed among the players. While budget games are strategic games and the utility of each player depends only on the current strategies of all players, the alternative model of ordered budget games emphasizes the order of the player decisions. A strategy change by a player does not decrease the utility of any other player. Instead, if the remaining budget of a resource is not enough to satisfy his demand, then he only receives the part of the budget not yet allocated to another player. Ordered budget games are exact potential games, which in their case even guarantees the existence of super-strong pure Nash equilibria. Although computing a strong pure Nash equilibrium is possible in only linear time, the best-response dynamic may need an exponential number of steps before reaching any equilibrium. Also, the problem of finding a super-strong pure Nash equilibrium is NP-hard. While the price of stability of ordered budget games is 1, their price of anarchy is 2.

We also considered α-approximate pure Nash equilibria in budget games and introduced an approximate potential function, which yielded an upper bound α_δ^u on α such that every budget game has the finite improvement property regarding α-moves for $\alpha > \alpha_\delta^u$. This upper bound depends on the share δ of a resources' budget which a player is at most allowed to demand. A lower bound α_δ^l, also depending on δ, was given as well. Although not tight, the gap between the two bounds decreases with δ. A different approach for matroid budget games yields upper bounds on α which do not rely on δ, but depend on the ratios between each players largest and smallest single demand. There are still instances with α-approximate pure Nash equilibria for $\alpha < \alpha_\delta^l$ (e.g. $\alpha = 1$). However, finding such an equilibrium is NP-hard. If we choose $\alpha = \alpha_\delta^u + \varepsilon$ for $\varepsilon > 0$ and the ratio between the highest-demand strategies of the different players is at most λ, then the best-response dynamic consisting only of α-moves reaches an α-approximate pure Nash equilibrium in time polynomial in λ, ε^{-1} and n. Finally, the optimal solution, i.e. the strategy profile which maximizes social welfare, can be approximated up to a polylogarithmic factor in linear time using α-moves. For matroid budget games, we give an approximation algorithm with an approximation factor of $1 - \frac{1}{e}$. Since finding the optimal solution for budget games and ordered budget games is equivalent, these algorithms also hold for the latter. The problem of computing an exact optimal solution is NP-hard.

Our last approach was to restrict the structure of the strategy spaces and only consider different classes of singleton and matroid budget games. For reasons of clarity, we list the major results in Table 8.1. In singleton budget games with fixed demands, pure Nash equilibria can be computed in linear time.

8.2 Outlook

The central point of this thesis are the properties needed to guarantee the existence of (approximate) pure Nash equilibria in budget games. For general δ-share budget games, we gave both upper and lower bounds regarding the existence of approximate equilibria. While our results improve with decreasing δ, these bounds are never really tight. Considering that the upper bound was derived from an approximated potential function, it seems likely that it can still be improved by using some different approach. The approximated potential function dismisses the fact that the potential of a player at a resource can be larger than the utility which he obtains from it. Instead, when a player performs a strategy change, we assume that these two values are the same for all resources used by the new strategy and we require that the increase in utility alone ensures that the potential of the game grows as well. For further details, we refer to the proof of Theorem 6.4. Since utility and potential of a player for some resource are only the same if his demand

Restriction	Strategy Space Structure: Singleton	Matroid	General
fixed demands	FIP	FIP	no NE
2 resources	WA	WA	?
ordered players & increasing demand ratios	NE	?	no NE
2 demands, 1 budget	WA	?	no NE

Table 8.1: Overview of our results regarding pure Nash equilibria in strategic budget games. FIP stands for *finite improvement property*, WA for *weakly acyclic* and NE for *pure Nash equilibria*. We believe that while not every game with two resources has a pure Nash equilibrium, the results regarding the last two restrictions can be extended to matroid budget games.

is fully satisfied, this can be considered an exception. Usually, the increase in the game's potential exceeds the increase in the player's utility and in many situations, a lesser increase in utility is needed to raise the potential of the game. The fact that better upper bounds exist for approximate equilibria in matroid budget games only reinforces our belief that there is still room for improvement.

In Chapter 7, we analyzed different classes of singleton and matroid budget games, which all possess pure Nash equilibria. This research was motivated by the fact that matroid congestion games always have pure Nash equilibria and that this property is also maximal in the sense that for every non-matroid set system, there is a corresponding congestion game without a pure Nash equilibrium. Backed by our observations from simulations of the dynamics of budget games, we strongly believe this also holds for matroid budget game, i.e. every matroid budget game possesses a pure Nash equilibrium. Sadly, we did find neither a proof nor a counter example for this statement, so the question regarding its correctness remains open.

In the introduction, we motivated the model of budget games via the example of clients choosing servers with limited computational capacity to handle their requests. Every request consists of a number of services, each with his own size. As a strategy, a player allocates the services to the servers. In this context, we expect the total demands of all strategies of a given player to be the same, i.e.

$$\sum_{r \in \mathcal{R}} s(r) = \sum_{r \in \mathcal{R}} s'(r) \tag{8.1}$$

for all strategies s, s' of the same player. This reflects the natural assumption that the size of each request is unaffected by its actual allocation to the servers. However, our model is actually more powerful in the sense that Equation 8.1 is not obligatory. Even if we restrict our model by the limitation imposed by Equation 8.1, pure Nash equilibria still do not exist in general. The δ-share budget game $\mathcal{B}_0$ introduced in Definition 4.1 and used to show the nonexistence of pure Nash equilibria already satisfies this condition. As far as we are aware, we also do not gain any other advantages by restricting the model in this way. When the total demands differ between the strategies of a player, a single player always prefers the strategy with the highest total demand on all resources combined, provided their budget is sufficient to satisfy these demands and assuming that $\delta \leq 1$. Depending on other player's demands on the resources, this preference is subject to change.

We come back to this concept of flexible demands in a moment. First, we consider a natural and interesting extension of our model, the addition of prices to the strategies. Thinking back to the two examples from the introduction, neither the allocation of requests to external servers nor the competition in a market are free. Instead, both induce some costs for the players, which they are going to compare with the resulting utility, the resulting payoff being the difference between utility and costs. In the past, we already considered a basic version of this extension in the form of a bachelor thesis. Simulations suggested that every singleton budget games with fixed prices for the strategies has the finite improvement property. Other possibilities would include prices which depend on the total demands on the resources or which are set functions over the players. It would be interesting to see how our results are affected by the existence of prices alone.

In addition, we can combine prices with flexible demands to model strategies of different quality. We already explained that the higher the demand of a player on a group of resources, the more attractive they are to him. This is independent of the resources' budgets. On the other hand, a lower demand on the same resources can only yield a smaller utility, even if their budgets are sufficient. We can regard this as different options of using the same resources but with varying quality. As an example, consider video streaming, where a higher quality is preferred, but also puts a higher workload on the data connections. By adding different prices to different levels of quality, the players are able to decide for themselves which option is best suited for them. This reflects the basic assumption that quality is expensive and the players balance the option of using cheap strategies against more expensive, but better ones instead of simply choosing the best one available.

Finally, ordered budget games have the ability to model scenarios with old and new players. Providers already established in a market usually have an advantage against new competitors, since their product or service is already well-known. In an ordered budget games, this property is available to some degree, as a new provider cannot obtain any utility currently held by someone else. As a next step, it would be interesting to combine the premises of both strategic and ordered budget games. For example, a new parameter $\beta \in [0, 1]$ could be introduced to determine how much of the other players current utilities could possible be obtained through a strategy change. Here, $\beta = 0$ describes ordered budget games while $\beta = 1$ is akin to strategic budget games. For small β, the new provider would still have a disadvantage against the others, but to a lesser degree. Keeping the concept of strategy-dependent demands in mind, such parameters could also be directly associated with a strategy. Instead of one global value for β, every strategy s would have its own β_s. This leads to different strategies having different levels of (dis)-advantages. Combining this with prices for the strategies as discussed above would enable us to model scenarios in which the new player has the option to choose more expensive investments (e.g. more advertisements) to strengthen his own position among the established players or pay less and only fill out the niches currently left empty. Such a model would be quite robust and could be used to model a multitude of non-strategic scenarios.

List of Figures

List of Tables

Bibliography

[1] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. Pure Nash Equilibria in Player-Specific and Weighted Congestion Games. *Theoretical Computer Science*, 410(17):1552 – 1563, 2009.

[2] Heiner Ackermann and Alexander Skopalik. Complexity of Pure Nash Equilibria in Player-Specific Network Congestion Games. *Internet Mathematics*, 5(4):323–342, 2008.

[3] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai Golin, and Ren van Oostrum. Competitive facility location: the Voronoi game. *Theoretical Computer Science*, 310(13):457 – 467, 2004.

[4] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. Exact Price of Anarchy for Polynomial Congestion Games. *SIAM Journal on Computing*, 40(5):1211–1233, 2011.

[5] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. The Price of Stability for Network Design with Fair Cost Allocation. *SIAM Journal on Computing*, 38(4):1602–1623, 2008.

[6] Elliot Anshelevich, John Postl, and Tom Wexler. Assignment Games with Conflicts: Price of Total Anarchy and Convergence Results via Semi-Smoothness. *Computing Research Repository*, abs/1304.5149, 2013.

[7] John Augustine, Ning Chen, Edith Elkind, Angelo Fanelli, Nick Gravin, and Dmitry Shiryaev. Dynamics of Profit-Sharing Games. *Internet Mathematics*, 11(1):1–22, 2015.

[8] Baruch Awerbuch, Yossi Azar, and Amir Epstein. The Price of Routing Unsplittable Flow. *SIAM Journal on Computing*, 42(1):160–177, 2013.

[9] Baruch Awerbuch, Yossi Azar, Amir Epstein, Vahab Seyed Mirrokni, and Alexander Skopalik. Fast Convergence to Nearly Optimal Solutions in Potential Games. In *Proceedings of the 9th ACM Conference on Electronic Commerce*, pages 264–273. ACM, 2008.

[10] Ron Banner and Ariel Orda. Bottleneck Routing Games in Communication Networks. *IEEE Journal on Selected Areas in Communications*, 25(6):1173–1179, 2007.

[11] Kshipra Bhawalkar, Martin Gairing, and Tim Roughgarden. Weighted Congestion Games: The Price of Anarchy, Universal Worst-Case Examples, and Tightness. *ACM Transactions on Economics and Computation*, 2(4):14, 2014.

[12] Ioannis Caragiannis, Angelo Fanelli, Nick Gravin, and Alexander Skopalik. Efficient Computation of Approximate Pure Nash Equilibria in Congestion Games. In *Proceedings of the 52nd Annual Symposium on Foundations of Computer Science*, pages 532–541. IEEE, 2011.

[13] Ioannis Caragiannis, Angelo Fanelli, Nick Gravin, and Alexander Skopalik. Approximate pure nash equilibria in weighted congestion games: existence, efficient computation, and structure. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 284–301. ACM, 2012.

[14] Jean Cardinal and Martin Hoefer. Non-cooperative Facility Location and Covering Games. *Theoretical Computer Science*, 411(1618):1855 – 1876, 2010.

[15] Chandra Chekuri, Julia Chuzhoy, Liane Lewin-Eytan, Joseph Seffi Naor, and Ariel Orda. Non-Cooperative Multicast and Facility Location Games. *IEEE Journal on Selected Areas in Communications*, 25(6):1193–1206, 2007.

[16] Ho-Lin Chen and Tim Roughgarden. Network Design with Weighted Players. *Theory of Computing Systems*, 45(2):302–324, 2009.

[17] Ho-Lin Chen, Tim Roughgarden, and Gregory Valiant. Designing Network Protocols for Good Equilibria. *SIAM Journal on Computing*, 39(5):1799–1832, 2010.

[18] Steve Chien and Alistair Sinclair. Convergence to approximate Nash equilibria in congestion games. *Games and Economic Behavior*, 71(2):315–327, 2011.

[19] George Christodoulou and Elias Koutsoupias. The Price of Anarchy of Finite Congestion Games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, pages 67–73. ACM, 2005.

[20] George Christodoulou, Elias Koutsoupias, and Paul G. Spirakis. On the Performance of Approximate Equilibria in Congestion Games. *Algorithmica*, 61(1):116–140, 2011.

[21] Richard Cole, Yevgeniy Dodis, and Tim Roughgarden. Bottleneck Links, Variable Demand, and the Tragedy of the Commons. *Networks*, 60(3):194–203, 2012.

[22] José Correa, Jasper de Jong, Bart de Keijzer, and Marc Uetz. The Curse of Sequentiality in Routing Games. In *Proceedings of the 11th International Conference on Web and Internet Economics*, pages 258–271. Springer, 2015.

[23] Jasper de Jong and Marc Uetz. The Sequential Price Of Anarchy for Atomic Congestion Games. In *Proceedings of the 10th International Conference on Web and Internet Economics*, pages 429–434. Springer, 2014.

[24] Nikhil R Devanur, Milena Mihail, and Vijay V Vazirani. Strategyproof cost-sharing mechanisms for set cover and facility location games. *Decision Support Systems*, 39(1):11–22, 2005.

[25] Juliane Dunkel and Andreas S. Schulz. On the Complexity of Pure-Strategy Nash Equilibria in Congestion and Local-Effect Games. In *Proceedings of the 2nd International Workshop on Internet and Network Economics*, pages 62–73. Springer, 2006.

[26] Éva Tardos and Vijay V. Vazirani. Basic Solution Concepts and Computational Issues. *Algorithmic Game Theory*, pages 3–28, 2007.

[27] Alex Fabrikant, Christos Papadimitriou, and Kunal Talwar. The Complexity of Pure Nash Equilibria. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing*, pages 604–612. ACM, 2004.

[28] Uriel Feige. A Threshold of ln n for Approximating Set Cover. *Journal of the ACM*, 45(4):634–652, 1998.

[29] Matthias Feldotto, Martin Gairing, and Alexander Skopalik. Bounding the Potential Function in Congestion Games and Approximate Pure Nash Equilibria. In *Proceedings of the 10th International Conference on Web and Internet Economics*, pages 30–43. Springer, 2014.

[30] Martin Gairing and Max Klimm. Congestion Games with Player-Specific Costs Revisited. In *Proceedings of the 6th International Symposium on Algorithmic Game Theory*, pages 98–109. Springer, 2013.

[31] Martin Gairing, Konstantinos Kollias, and Grammateia Kotsialou. Tight Bounds for Cost-Sharing in Weighted Congestion Games. In *Proceedings of the 42rd International Colloquium on Automata, Languages and Programming*, pages 626–637. Springer, 2015.

[32] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[33] Vasilis Gkatzelis, Konstantinos Kollias, and Tim Roughgarden. Optimal Cost-Sharing in Weighted Congestion Games. In *Proceedings of the 10th International Conference on Web and Internet Economics*, pages 72–88. Springer, 2014.

[34] Michel X Goemans, Li Li, Vahab S. Mirrokni, and Marina Thottan. Market Sharing Games Applied to Content Distribution in Ad Hoc Networks. *IEEE Journal on Selected Areas in Communications*, 24(5):1020–1033, 2006.

[35] Michel X Goemans and Martin Skutella. Cooperative Facility Location Games. *Journal of Algorithms*, 50(2):194–214, 2004.

[36] Ragavendran Gopalakrishnan, Jason R Marden, and Adam Wierman. Potential Games Are Necessary to Ensure Pure Nash Equilibria in Cost Sharing Games. *Mathematics of Operations Research*, 39(4):1252–1296, 2014.

[37] Christoph Hansknecht, Max Klimm, and Alexander Skopalik. Approximate Pure Nash Equilibria in Weighted Congestion Games. In *Proceedings of the 17th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems*, pages 242–257. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2014.

[38] Tobias Harks, Martin Hoefer, Max Klimm, and Alexander Skopalik. Computing pure Nash and Strong Equilibria in Bottleneck Congestion Games. *Mathematical Programming*, 141(1-2):193–215, 2013.

[39] Tobias Harks and Max Klimm. On the Existence of Pure Nash Equilibria in Weighted Congestion Games. In *Proceedings of the 38th International Colloquium on Automata, Languages and Programming*, pages 79–89. Springer, 2010.

[40] Tobias Harks and Max Klimm. Congestion Games with Variable Demands. *Mathematics of Operations Research*, 41(1):255–277, 2015.

[41] Tobias Harks, Max Klimm, and Rolf H. Möhring. Strong Nash Equilibria in Games with the Lexicographical Improvement Property. In *Proceedings of the 5th International Workshop on Internet and Network Economics*, pages 463–470. Springer, 2009.

[42] Harold Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.

[43] Kamal Jain and Mohammad Mahdian. Cost sharing. *Algorithmic Game Theory*, pages 385–410, 2007.

[44] Kamal Jain and Vijay V Vazirani. Approximation Algorithms for Metric Facility Location and k-Median Problems Using the Primal-dual Schema and Lagrangian Relaxation. *Journal of the ACM*, 48(2):274–296, 2001.

[45] Max Klimm and Daniel Schmand. Sharing Non-anonymous Costs of Multiple Resources Optimally. In *Proceedings of the 9th International Conference on Algorithms and Complexity*, pages 274–287. Springer, 2015.

[46] Vicki Knoblauch. Generalizing Location Games to a Graph. *The Journal of Industrial Economics*, 39(6):683–688, 1991.

[47] Vicki Knoblauch. Continuous Strategy Games as Location Games. *Journal of Economic Theory*, 66(1):224 – 237, 1995.

[48] Konstantinos Kollias and Tim Roughgarden. Restoring Pure Equilibria to Weighted Congestion Games. *ACM Transactions on Economics and Computation*, 3(4):21, 2015.

[49] Renato Paes Leme, Vasilis Syrgkanis, and Éva Tardos. The Curse of Simultaneity. In *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, pages 60–67. ACM, 2012.

[50] Stefano Leonardi and Guido Schäfer. Cross-monotonic Cost-sharing Methods for Connected Facility Location Games. In *Proceedings of the 5th ACM Conference on Electronic Commerce*, pages 242–243. ACM, 2004.

[51] Xiang-Yang Li, Zheng Sun, and Weizhao Wang. Cost Sharing and Strategyproof Mechanisms for Set Cover Games. In *Proceedings of the 22nd Symposium on Theoretical Aspects of Computer Science*, pages 218–230. Springer, 2005.

[52] Xiang-Yang Li, Zheng Sun, WeiZhao Wang, Xiaowen Chu, ShaoJie Tang, and Ping Xu. Mechanism design for set cover games with selfish element agents. *Theoretical Computer Science*, 411(1):174 – 187, 2010.

[53] Lavy Libman and Ariel Orda. Atomic Resource Sharing in Noncooperative Networks. *Telecommunication Systems*, 17(4):385–409, 2001.

[54] Brendan Lucier and Renato Paes Leme. GSP Auctions with Correlated Types. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, pages 71–80. ACM, 2011.

[55] Marios Mavronicolas, Igal Milchtaich, Burkhard Monien, and Karsten Tiemann. Congestion games with player-specific constants. In *Proceedings of the 32nd International Conference on Mathematical Foundations of Computer Science*, pages 633–644. Springer, 2007.

[56] Marios Mavronicolas and Burkhard Monien. The complexity of pure equilibria in mix-weighted congestion games on parallel links. *Information Processing Letters*, 115(12):927–931, 2015.

[57] Vladimir Mazalov, Burkhard Monien, Florian Schoppmann, and Karsten Tiemann. Wardrop Equilibria and Price of Stability for Bottleneck Games with Splittable Traffic. In *Proceedings of the 2nd International Workshop on Internet and Network Economics*, pages 331–342. Springer, 2006.

[58] Igal Milchtaich. Congestion Games with Player-Specific Payoff Functions . *Games and Economic Behavior*, 13(1):111 – 124, 1996.

[59] Igal Milchtaich. Network topology and equilibrium existence in weighted network congestion games. *International Journal of Game Theory*, 44(3):515–541, 2014.

[60] Dov Monderer and Lloyd S. Shapley. Potential Games. *Games and Economic Behavior*, 14(1):124–143, 1996.

[61] John Nash. Non-Cooperative Games. *Annals of Mathematics*, 54(2):286–295, 1951.

[62] George L. Nemhauser, Laurence A. Wolsey, and Marshall L. Fisher. An Analysis Of Approximations For Maximizing Submodular Set Functions - I. *Mathematical Programming*, 14(1):265–294, 1978.

[63] Lili Qiu, Yang Richard Yang, Yin Zhang, and Scott Shenker. On Selfish Routing in Internet-Like Environments. *IEEE/ACM Transactions on Networking*, 14(4):725–738, 2006.

[64] Robert W. Rosenthal. A Class of Games Possessing Pure-Strategy Nash Equilibria . *International Journal of Game Theory*, 2(1):65–67, 1973.

[65] Tim Roughgarden. Intrinsic Robustness of the Price of Anarchy. In *Proceedings of the 41st Annual ACM Symposium on Theory of Computing*, pages 513–522. ACM, 2009.

[66] Tim Roughgarden and Okke Schrijvers. Network Cost-Sharing without Anonymity. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory*, pages 134–145. Springer, 2014.

[67] Lloyd S. Shapley. A Value for n-Person Games. Technical report, DTIC Document, 1952.

[68] Alexander Skopalik and Berthold Vöcking. Inapproximability of Pure Nash Equilibria. In *Proceedings of the 40th annual ACM Symposium on Theory of Computing*, pages 355–364. ACM, 2008.

[69] Adrian Vetta. Nash Equilibria in Competitive Societies, with Applications to Facility Location, Traffic Routing and Auctions. In *Proceedings of the 43rd Annual Symposium on Foundations of Computer Science*, pages 416–425. IEEE, 2002.

[70] Philipp von Falkenhausen and Tobias Harks. Optimal Cost Sharing for Resource Selection Games. *Mathematics of Operations Research*, 38(1):184–208, 2013.