New examples and constructions in infinite-dimensional Lie theory

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- *This thesis is typeset with LaTex.*
- *This thesis contains material published before in the author's preprints [Eyn14a], [Eyn14b], [Eyn14c] and [Eyn15].*

Contents

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Abstract

In this thesis, we give new examples and constructions for infinite-dimensional Lie groups. At the beginning, we construct a smooth Lie group structure on the group of real analytic diffeomorphisms of a compact real analytic manifold with corners. In the following part, we examine conditions for the integrability of a given Banach subalgebra of the Lie algebra of a Lie group that is modelled on a locally convex space. For that reason, we elaborate a corresponding Frobenius theorem. In the third part of this thesis, we show that the canonical invariant symmetric bilinear form on the Lie algebra of compactly supported sections of a finite-dimensional perfect Lie algebra bundle is universal in a topological sense. At the end of this thesis, we construct central extensions of Lie groups of compactly supported sections of Lie group bundles over non-compact base manifolds. In addition we show the universality of certain examples of these central extensions.

German translation: In dieser Arbeit stellen wir neue Beispiele und Konstruktionen für unendlich-dimensionale Lie-Gruppen vor. Wir beginnen damit, dass wir eine glatte Lie-Gruppenstruktur auf der Gruppe der reell-analytischen Diffeomorphismen einer kompakten reell-analytischen Mannigfaltigkeit mit Ecken konstruieren. Daran anschließend untersuchen wir Bedingungen für die Integrabilität von Banach-Unteralgebren von Lie-Algebren von Lie-Gruppen, die auf lokal konvexen Räumen modelliert sind. Hierfür zeigen wir einen entsprechenden Frobeniussatz. Im dritten Teil der Arbeit beweisen wir, dass die kanonische invariante symmetrische Bilinearform auf der Lie-Algebra der kompakt getragenen Schnitte eines endlich-dimensionalen perfekten Lie-Algebren-Bündels in einem topologischen Sinn universell ist. Den Schluss der Arbeit bildet ein Kapitel, in dem wir zentrale Erweiterungen von Lie-Gruppen von kompakt getragenen Schnitten von Lie-Gruppen-Bündeln mit nicht kompakter Basis konstruieren. Zusätzlich zeigen wir die Universalität von gewissen Beispielen dieser Erweiterungen.

Introduction and Notations

What is an infinite-dimensional Lie group?

At first we want to introduce our framework for infinite-dimensional Lie groups.[1] Finite-dimensional manifolds "are spaces that locally look like some Euclidean space $\mathbb{R}^n$" ([Lee13, p. 1]). Hence, an infinite-dimensional manifold should be a topological space that looks locally like an open subset of an infinite-dimensional vector space. Of course the infinite-dimensional vector space on which our manifold is modelled has to be a topological space. With the option in mind to define differentiable manifolds the space should even be a topological vector space. One class of infinite-dimensional manifolds are the so-called "Banach manifolds" that are manifolds modelled over Banach spaces. Using the concept of Fréchet differentiability it is clear what a FC^k-Banach manifold should be. A standard reference for Banach manifolds is [Lan01]. Moreover, the definition of a Banach-Lie group is canonical.

Although there are interesting examples of Banach manifolds, there exists no reasonable structure of a Banach-Lie group on the group of smooth diffeomorphisms $\mathrm{Diff}(M)$ for a compact finite-dimensional smooth manifold M ([KM97, p. 457], [Omo78]). Hence, one has to model $\mathrm{Diff}(M)$ over a more general topological vector space. It turns out that locally convex spaces are the right choice.

There are several approaches to differential calculus on locally convex spaces (for details, we recommend [Kel74]). Among the most popular approaches is the convenient setting, invented by Frölicher, Kriegl and Michor (see [KM97]). A map is called smooth in the convenient setting if it is smooth along smooth curves (see [KM97, Definition 3.11]). Of course, this differential calculus is inspired by Boman's Theorem (see [KM97, Theorem 3.4] and [Bom67]). The second popular approach is the differential calculus known as Keller's C^k_c-theory (obviously the name is inspired by [Kel74]) going back to Bastiani (see [Bas64]). In this approach, a continuous map is called continuously differentiable if all directional derivatives $df(x,v)$ exist and the map $(x,v) \mapsto df(x,v)$ is continuous. For details on this approach, we recommend [Mil84], [Ham82] and [GN]. In this thesis, we will always use this differential calculus. Because Milnor used Keller's C^k_c-theory to turn the diffeomorphism group into a Lie group (see [Mil84]), Lie groups constructed within Keller's C^k_c-theory are sometimes called Milnor-Lie groups. One can show that the convenient differential calculus and Keller's C^k_c-theory are equivalent on Fréchet spaces (see [BGN04, p. 270] and [KM97, Theorem 4.11]) but beyond the Fréchet

[1] This subchapter of the introduction contains material published before in the author's preprint [Eyn15].

case this is false. For example, a map that is smooth in the convenient sense need not be continuous (see e.g. [Glo06c, p. 1]).

Diff^{ω}(M) as a Milnor-Lie group for a real analytic manifold M with corners

The prime example of an infinite-dimensional Lie group is the diffeomorphism group $\mathrm{Diff}(M)$ of a finite-dimensional manifold M.[2] First, we categorise different approaches how to construct a Lie group structure on $\mathrm{Diff}(M)$. Then we recall the exact conditions for the existence of a Lie group structure on $\mathrm{Diff}(M)$.

Having chosen a differential calculus, one has to choose a strategy how to turn $\mathrm{Diff}(M)$ into a Lie group. There are basically two different approaches. The first one (and most common one), is to turn the space of (C^r respectively smooth respectively analytic) mappings from M to M (in our notation $C^r(M; M)$ with $r \in \mathbb{N} \cup \{\infty, \omega\}$) into an infinite-dimensional manifold. To this end, one chooses a Riemannian metric on M and obtains a Riemannian exponential function exp. For small $\eta \in \Gamma(TM)$, one can define the map $\Psi_\eta := \exp \circ \eta$. Now it turns out that in many cases it is possible to obtain a manifold structure on the mapping space $C^r(M; M)$ by charts similar to $\Psi\colon \eta \mapsto \Psi_\eta$. The second step in this strategy is to show that $\mathrm{Diff}(M)$ is an open submanifold of $C^r(M; M)$ and that the group operations have the required differential property (e.g. C^r, smooth or real analytic). In the following, we call this strategy the "global approach"(in the table further down we cite articles that used this approach).

The second approach leads to the same Lie group structure on $\mathrm{Diff}(M)$ but its construction is very different. Again one chooses a Riemannian metric on M. With the help of the map $\Psi\colon \eta \mapsto \Psi_\eta$ one obtains a manifold structure on a subset of $\mathrm{Diff}(M)$ that contains the identity id_M. Now one uses the theorem of local description of Lie groups to extend the manifold structure to $\mathrm{Diff}(M)$ and to turn it into a Lie group. In the following, we call this strategy the "local approach". This approach was first used in [Glo06c].

In the following table we cite different articles that constructed Lie group structures on diffeomorphism groups.[3] We emphasize that this list is not comprehensive. The list just contains the cases that are of interest for this thesis.

[2] This subchapter of the introduction consist of material published before in the author's preprint [Eyn15].

[3] We mention that in [Sch15, Remark 5.22] it was stated that the proof of [Sch15] circumvents some problems which remained in [BB08]. Moreover, in [KM90, p.1] it was stated that the proof of [Les85] has a gap.

M	Global, Convenient	Global, Keller-C^∞_c	Local, Keller-C^∞_c
C^∞, compact, no corners		[Mil84]	
C^∞, non compact, no corners	[KM97]		[Glo06c]
C^∞, non compact, with corners		[Mic80]	
orbifold, compact	[BB08]		[Sch15]
orbifold, non compact			[Sch15]
$C^\omega_{\mathbb{R}}$, compact, no corners	[KM90]	[DS15], [Les82]	

Given a compact real analytic manifold without corners, Kriegl and Michor constructed in [KM90] a real analytic Lie group structure on $\mathrm{Diff}^\omega(M)$ in the convenient sense. This structure is modeled over the space of real analytic vector fields $\Gamma^\omega(TM)$ of M. A map defined on an open subset of $\Gamma^\omega(TM)$ is smooth in the convenient setting if and only if it is smooth in the Keller's C^∞_c-theory ([DS15, p.142]). But a map on $\Gamma^\omega(TM)$ that is real analytic in the convenient sense need not be real analytic in the conventional sense as in [Mil84, p. 1028] (see also [DS15, p. 142]). We emphasise that the Lie group structure from [KM90] is only real analytic in the convenient setting (cf. [DS15, Proposition 1.9]). Because a real analytic structure induces a smooth structure, the Lie group structure of [KM90] induces a structure of a smooth Milnor-Lie group on $\mathrm{Diff}^\omega(M)$ as mentioned in [DS15, Proposition 2.9]. One might expect that there also exists a real analytic Lie group structure in the conventional sense on $\mathrm{Diff}^\omega(M)$. But Dahmen and Schmeding showed in [DS15] that there exists no real analytic structure on $\mathrm{Diff}^\omega(\mathbb{S}^1)$ in the conventional sense of Milnor. Therefore we cannot expect that there exists a real analytic structure in the conventional sense on $\mathrm{Diff}^\omega(M)$ for a compact real analytic manifold M with corners. The aim of Chapter 1 of this thesis is to turn the group $\mathrm{Diff}^\omega(M)$ of real analytic diffeomorphisms of a finite-dimensional compact real analytic manifold M with corners into a smooth Milnor-Lie group. This generalises in some sense parts of [KM90] and [DS15]. More precisely we show the following theorem:

Theorem A. *Let M be a finite-dimensional compact real analytic manifold with corners such that there exists a boundary respecting real analytic Riemannian metric on a real analytic enveloping manifold $\tilde{M}$. Then there exists a unique smooth Lie group structure on the group of real analytic diffeomorphisms* $\mathrm{Diff}^\omega(M)$ *modelled over $\Gamma^\omega_{\mathrm{st}}(TM)$ such that for one (and hence each) boundary respecting Riemannian metric on $\tilde{M}$ the map $\eta \mapsto \Psi_\eta$ is a diffeomorphism from an open 0-neighbourhood in $\Gamma^\omega_{\mathrm{st}}(TM)$ onto an open identity neighbourhood in* $\mathrm{Diff}^\omega(M)$.

In this context, an enveloping manifold $\tilde{M}$ of M is a real analytic manifold without boundary that contains M as a submanifold with corners. Moreover, a Riemannian metric on the enveloping manifold $\tilde{M}$ is called boundary respecting, if the strata $\partial^j M$ of M are totally geodesic submanifolds of $\tilde{M}$. The symbol $\Gamma^\omega_{\mathrm{st}}(TM)$ stands for the space of stratified vector fields. These are analytic vector fields on M that restrict to vector fields on the strata $\partial^j M$.

Dahmen and Schmeding ([DS15, Proposition 2.9]) respectively Kriegl and Michor ([KM90]) used the global approach to turn $\mathrm{Diff}^{\omega}(M)$ (with M compact and $\partial M = \varnothing$) into a smooth respectively real analytic Lie group. We instead want to use the local approach for our Theorem A (if M has corners). Hence, to a certain point our Chapter 1 gives also an alternative construction for [KM90] and [DS15]. The local approach was developed by Glöckner in [Glo06c], and we follow the line of thought of [Glo06c]. But Glöckner considered smooth diffeomorphisms on a manifold without corners. Hence, one obvious obstacle is that we cannot use bump functions because we work in the real analytic setting. Moreover, because our manifold M has corners, we will have to model our structure on the space of stratified vector fields as in [Mic80][4].

We also mention [Les82]. In this paper Leslie used the global approach to turn the group of real analytic diffeomorphisms of a compact real analytic manifold without corners into a smooth Lie group. But as pointed out in [KM90, p.1], his proof has a gap.

In the following, we describe our strategy in more detail. Given a manifold with boundary, one can use the double of the manifold to embed it into a manifold without boundary (see, for example, [Lee13, Example 9.32]). However this does not work in the case of a manifold with corners because the boundary of a manifold with corners is not a manifold. If one works with a smooth manifold with corners, one can use a partition of unity to construct a "strictly inner vector field" ([Mic80, p. 21]). With the help of this vector field, one obtains the analogous result (see [Mic80, p. 21] and [DH73, Proposition 3.1]). Obviously this approach does not work if one considers a real analytic manifold with corners. For technical reasons we show the following theorem in Section 1.1:

Theorem B. *Given a compact real analytic finite-dimensional manifold with corners M, there exists an enveloping manifold $\tilde{M}$ of M. If $\tilde{M}_1$ and $\tilde{M}_2$ are enveloping manifolds of M then there exists an open neighbourhood U_1 of M in $\tilde{M}_1$, an open neighbourhood U_2 of M in $\tilde{M}_2$ and a real analytic diffeomorphism $\varphi\colon U_1 \to U_2$ with $\varphi|_M = \mathrm{id}_M$.*

In [BW59, Proposition 1] Bruhat and Whitney show that given a real analytic paracompact manifold M (without corners), there exists a complex analytic manifold $M_{\mathbb{C}}$ that contains M as a totally real submanifold. The manifold $M_{\mathbb{C}}$ is called complexification of M. We can transfer their proof without difficulties to show our Theorem B. In addition, we elaborate some technical properties of real analytic mappings concerning extensions to enveloping manifolds in Section 1.1. The proofs are analogous to the case of extensions of real analytic mappings to complexifications (see e.g. [DGS14, Chapter 2]).

That we use the local approach ([Glo06c] and [Sch15]), is reflected in the structure of Chapter 1 of this thesis: In Section 1.2 we construct a manifold structure on a subset $\mathcal{U}$ of $\mathrm{Diff}(M)$ that contains the identity id_M. The next step is to show

[4] In [Mic80] Michor turned the group of smooth diffeomorphisms of a non-compact manifold with corners into a smooth Lie group. Michor worked with the global approach and as mentioned above this leads to a very different construction.

the smoothness of the group operations. To this end, we elaborate some important preparatory results in Section 1.3. In Section 1.4, we show the smoothness of the multiplication on $\mathcal{U}$ and in Section 1.5 the smoothness of the inversion. The smoothness of the conjugation is proved in Section 1.6. Our proof of the smoothness of the conjugation map follows closely the ideas of [Glo06c, Section 5]: First we show that the Lie group structure on $\mathrm{Diff}^\omega(M)$ is independent of the choice of the Riemannian metric (see [Glo06c, Section 5]). With help of this result, we can show the smoothness of the conjugation map as in [Glo06c, Section 5] (see Lemma 1.87).

Integrability of Banach subalgebras

Given a finite-dimensional Lie group G with Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, the Integral Subgroup Theorem ([HN12, Theorem 9.4.8]) tells us that we find a subgroup H of G that is a Lie group with Lie algebra $\mathfrak{h}$ such that the inclusion is smooth.[5] The analogous result for closed Lie subalgebras $\mathfrak{h}$ is true for so-called Baker-Campbell-Hausdorff Lie groups (see [GN]). These are real analytic Lie groups modelled over locally convex spaces that possess an exponential function that is locally C^ω-diffeomorphic around 0. However as mentioned above $\mathrm{Diff}^\omega(\mathbb{S}^1)$ does not admit the structure of a real analytic Lie group. The same holds for the group of smooth diffeomorphisms (see [Mil84, Corollary 9.2])[6].

Analogously to [Lan01, Chapter VI], [Les68] or [Les92], Frobenius theorems for manifolds that are modelled over locally convex spaces can be used to show generalisations of the Integral Subgroup Theorem (this was mentioned in [Glo08b]).

In 2001, Teichmann showed a Frobenius theorem for finite-dimensional distributions on manifolds that are modelled on locally convex spaces in the convenient sense. It was possible to obtain the analogous result in the author's master's thesis [Eyn12] in the context of manifolds that are modelled over locally convex spaces in the sense of Keller's C^k_c-theory. Moreover, in [Eyn12, Chapter 4] it was shown that if the Lie group G in question has an exponential map then every finite-dimensional Lie subalgebra $\mathfrak{h} \subseteq L(G)$ is integrable[7].

Now, it is a natural question to ask if every Lie subalgebra $\mathfrak{h} \subseteq L(G)$ that is complemented as a topological vector subspace and is a Banach space with the induced topology is integrable as well. The answer is yes. In Chapter 2 we prove the following theorem.

[5] This subchapter of the introduction consist of material published before in the author's preprint [Eyn14a]

[6] In the case where G is the Lie group of smooth diffeomorphisms of a smooth compact manifold without boundary, the Integral Subgroup Theorem has been proved in [Les92]. The special case $G = \mathrm{Diff}^\omega(M)$ has been considered in [Les85].

[7] Frobenius theorems for co-Banach distributions for manifolds that are modelled over locally convex spaces were obtained in [Hil00] respectively [Eyn12]. Other Frobenius theorems have also been elaborated in [Les68] and [Les92] but the more complicated conditions of Leslie's results are of a quite different kind.

Theorem C. *Let G be a Lie group modelled over a locally convex space and $\mathfrak{h} \subseteq L(G)$ be a Lie subalgebra that is complemented as a topological vector subspace and is a Banach space with the induced topology. If G admits an exponential map then we can find a Lie group H that is a subgroup of G and an immersed submanifold of G such that $L(H) = \mathfrak{h}$.*

Although Theorem C can be obtained with the help of [Nee06, Theorem IV.4.9.][8], we will give an alternative proof by using a Frobenius theorem. Hence the main work to prove Theorem C, will be to show a Frobenius theorem for Banach distributions for manifolds that are modelled over locally convex spaces:

Theorem D. *Let M be a C^r-manifold modelled over a locally convex space E with $r \in \mathbb{N} \cup \{\infty\}$, $r \geqslant 4$ and F be a complemented subspace of E such that F is a Banach space with the induced topology from E and D is an involutive subbundle of TM with typical fibre F. Assume that for all $p_0 \in M$ there exists an open p_0-neighbourhood $U \subseteq M$ and a C^{r-1}-vector field $X \colon U \times F \to TU$ with parameters in F such that:*

(a) The map $F \to \Gamma(TU)$, $v \mapsto X(\bullet, v)$ is linear;
(b) We have $\operatorname{im}(X) \subseteq D$;
(c) The map $F \to D_{p_0}$, $v \mapsto X(p_0, v)$ is an isomorphism of topological vector spaces;
(d) The vector field X provides a local flow with parameters of class C^r.
(e) It exists a chart $\varphi \colon U \to V$ of M, such that $\varphi(p_0) = 0$ and $d\varphi(D_{p_0}) = F$.

In this situation D is integrable.

Theorem D will be proved in Section 2.1. Besides new arguments, we use methods from the case where the distribution in question is finite-dimensional ([Tei01] respectively [Eyn12]). Also, we use methods developed in [CS76], where Chillingworth and Stefan work with singular distributions on Banach manifolds. The preceding theorem (details of which will be explained later) will be obtained there. In Theorem D, we have to assume that the vector field admits a local flow because this is not automatic for initial value problems in locally convex spaces. Indeed, it is possible to find linear initial value problems in locally convex spaces that have several solutions, or no solution at all.

Universal bilinear forms for Lie algebras of compactly supported sections

In Chapter 3, we address a further new construction in infinite-dimensional Lie theory[9]. An invariant symmetric bilinear form β on a Lie algebra $\mathfrak{g}$ taking values in a vector space is called algebraically universal if any invariant bilinear form on $\mathfrak{g}$ factorises over β by composition with a unique linear map. Here invariance means

[8] This was mentioned by K. H. Neeb in comments to this thesis.

[9] This subchapter of the introduction consist of material published before in the author's preprint [Eyn14c]

that $\beta([x,y],z) = \beta(x,[y,z])$ for all $x,y,z \in \mathfrak{g}$. Given any Lie algebra $\mathfrak{g}$, such an algebraically universal bilinear form was constructed in [Gun11, Remark 4.1.5]. It is denoted by $\kappa_{\mathfrak{g}}\colon \mathfrak{g}\times\mathfrak{g}\to V_{\mathfrak{g}}$. Analogously, a continuous invariant symmetric bilinear form β on a locally convex Lie algebra $\mathfrak{g}$ taking values in a locally convex space is called topologically universal (we also use the term "universal continuous invariant symmetric bilinear form") if we get all other continuous invariant bilinear forms on $\mathfrak{g}$ by composing β with a unique continuous linear map. In [Gun11], such a topologically universal invariant bilinear form was constructed in the case of Fréchet-Lie algebras (see [Gun11, Proposition 4.5.3]). But the construction works also in the more general case of locally convex Lie algebras (see Remark 3.3). We denote the topologically universal invariant bilinear form on $\mathfrak{g}$ by $\kappa_{\mathfrak{g}}^{ct}\colon \mathfrak{g}\times\mathfrak{g}\to V_{\mathfrak{g}}^{ct}$. Although any two topologically universal invariant symmetric bilinear forms differ only by composition with an isomorphism of topological vector spaces, it is not enough to know the mere existence of universal bilinear forms in general. Often, one would like to use more concrete realisations of universal symmetric invariant bilinear forms. This is the reason why in [Gun11], Gündoğan constructed a concrete universal continuous bilinear form for the Lie algebra $\Gamma(\mathfrak{K})$ of sections for a given Lie algebra bundle $\mathfrak{K}$ with finite-dimensional σ-compact base M. If $\mathfrak{g}$ is the finite-dimensional perfect typical fibre of $\mathfrak{K}$ and $V(\mathfrak{K})$ is the vector bundle with base M and fibres $V(\mathfrak{K}_p)$ for $p\in M$, Gündoğan showed in [Gun11, Theorem 4.6.2] that $\kappa_{\mathfrak{K}}\colon \Gamma(\mathfrak{K})\times\Gamma(\mathfrak{K})\to\Gamma(V(\mathfrak{K}))$, $(\eta,\zeta)\mapsto\kappa_{\mathfrak{K}}\circ(\eta,\zeta)$ is a topologically universal invariant symmetric bilinear form (here $\kappa_{\mathfrak{K}}$ is the fibrewise universal invariant bilinear form).

Moreover, Gündoğan showed that the maps $\kappa_{\mathfrak{K}}\colon \Gamma(\mathfrak{K})\times\Gamma(\mathfrak{K})\to\Gamma(V(\mathfrak{K}))$ and $\kappa_{\mathfrak{g}_*}\colon$: $C^\infty_c(M,\mathfrak{g})\times C^\infty_c(M,\mathfrak{g})\to C^\infty_c(M,V_{\mathfrak{g}})$ are algebraically universal. Summarising we obtain the following table:

	algebraically universal	topologically universal
$\kappa_{\mathfrak{g}_*}\colon C^\infty(M,\mathfrak{g})^2\to C^\infty(M,V_{\mathfrak{g}})$	[Gun11, Prop. 4.3.3]	[Gun11, Theo. 4.6.2]
$\kappa_{\mathfrak{g}_*}\colon C^\infty_c(M,\mathfrak{g})^2\to C^\infty_c(M,V_{\mathfrak{g}})$	[Gun11, Prop. 4.3.3]	
$\kappa_{\mathfrak{K}}\colon \Gamma(\mathfrak{K})^2\to\Gamma(V(\mathfrak{K}))$	[Gun11, Theo. 4.4.4]	[Gun11, Theo. 4.6.2]
$\kappa_{\mathfrak{K}}\colon \Gamma_c(\mathfrak{K})^2\to\Gamma_c(V(\mathfrak{K}))$		

Hence, the first aim of Chapter 3 is to show that the map $\kappa_{\mathfrak{K}}\colon \Gamma_c(\mathfrak{K})^2\to\Gamma_c(V(\mathfrak{K}))$ is topologically universal, by proving:

Theorem E. *For a perfect finite-dimensional Lie-algebra $\mathfrak{g}$, a σ-compact manifold M and a Lie algebra bundle $\mathfrak{K}$ with base M and typical fibre $\mathfrak{g}$, the map $\kappa_{\mathfrak{K}}\colon \Gamma_c(M,\mathfrak{K})^2\to\Gamma_c(V(\mathfrak{K}))$ with $\kappa_{\mathfrak{K}}(\eta,\zeta)(p)=\kappa_{\mathfrak{K}_p}(\eta(p),\zeta(p))$ is topologically universal*[10].

[10] Obviously this shows that $\kappa_{\mathfrak{g}_*}\colon C^\infty_c(M,\mathfrak{g})^2\to C^\infty_c(M,V_{\mathfrak{g}})$ is topologically universal. Moreover it shows that $\kappa_{\mathfrak{K}}\colon \Gamma_c(\mathfrak{K})^2\to\Gamma_c(V(\mathfrak{K}))$ is algebraically universal. In fact given a vector space W and an invariant symmetric bilinear form $\gamma\colon \Gamma_c(\mathfrak{K})^2\to W$, we can equip W with a locally convex topology such that γ is continuous. Hence we obtain the existence of the required linear map. Because of Remark 3.3 the image of $\kappa_{\mathfrak{K}}$ generates $\Gamma_c(V(\mathfrak{K}))$. Therefore we obtain the uniqueness statement.

While the locally convex topology on $\Gamma(\mathfrak{K})$ is a well accessible Fréchet-topology, the locally convex topology on $\Gamma_c(\mathfrak{K})$ is an inductive limit topology. Hence, it is more difficult to handle.

Universal bilinear forms like $\kappa_{\mathfrak{K}}$ from Theorem E play an important role in the extension theory of locally convex Lie algebras. Following [Woc06, Definition A.2.1] respectively [Nee02b, Chapter 1], we define a central extension of a locally convex Lie algebra $\mathfrak{g}$ by a locally convex space V (considered as an abelian Lie algebra) to be a short exact sequence $0 \to V \hookrightarrow \hat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to 0$ of locally convex Lie algebras such that the map $V \hookrightarrow \hat{\mathfrak{g}}$ is a topological embedding, V lies in the center of $\hat{\mathfrak{g}}$ and q has a continuous linear section. A further central extension $V \hookrightarrow \hat{\mathfrak{g}}' \xrightarrow{q'} \mathfrak{g}$ of $\mathfrak{g}$ by V is called equivalent to $V \hookrightarrow \hat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$, if there exists an isomorphism $\varphi\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}'$ of topological Lie algebras such that $\varphi|_V = \mathrm{id}_V$ and $q = q' \circ \varphi$ on $\mathfrak{g}'$ (see [Woc06, Definition A.2.4]). The set of equivalence classes of central extensions of $\mathfrak{g}$ by V is denoted by $\mathrm{Ext}(\mathfrak{g}, V)$. As usual, we can describe central extensions by the Lie algebra cohomology. Hence, we recall the concept of continuous Lie algebra cohomology for example from [Nee02b, Chapter 1] respectively [Gun11, Appendix A] in the following[11]: A continuous anti-symmetric bilinear map $\omega\colon \mathfrak{g}^2 \to V$ is called 2-*cochain* (or simply cochain). It is called cocycle if $0 = \omega([x_1, x_2], x_3) + \omega([x_2, x_3], x_1) + \omega([x_3, x_1], x_2)$ and cobounday if there exists a continous linear map $\gamma\colon \mathfrak{g} \to V$ such that $\omega(x_1, x_2) = \gamma([x_1, x_2])$ for all $x_1, x_2, x_3 \in \mathfrak{g}$. We write $Z^2_{ct}(\mathfrak{g}, V)$ for the space of 2-cochains and $B^2_{ct}(\mathfrak{g}, V)$ for the space of 2-coboundaries. One easily sees $B^2_{ct}(\mathfrak{g}, V) \subseteq Z^2_{ct}(\mathfrak{g}, V)$ and we define $H^2_{ct}(\mathfrak{g}, V) := Z^2_{ct}(\mathfrak{g}, V)/B^2_{ct}(\mathfrak{g}, V)$. Given $\omega \in Z^2_{ct}(\mathfrak{g}, V)$, we obtain the locally convex Lie algebra $V \times_\omega \mathfrak{g}$ with the Lie bracket $[(v_1, x_1), (v_2, x_2)]_\omega = (\omega(x_1, x_2), [x_1, x_2])$ for $v_i \in V$ and $x_i \in \mathfrak{g}$ and the central extension $V \hookrightarrow V \times_\omega \mathfrak{g} \xrightarrow{\mathrm{pr}_2} \mathfrak{g}$. This induces a bijection $H^2_{ct}(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V)$.

In the following, we recall the concept of universality for example form [Nee02b, Chapter 1] respectively [Gun11, Appendix A]: Given two central Lie algebra extensions $V_1 \hookrightarrow \hat{\mathfrak{g}}_1 \xrightarrow{q_1} \mathfrak{g}$ and $V_2 \hookrightarrow \hat{\mathfrak{g}}_2 \xrightarrow{q_2} \mathfrak{g}$ of the locally convex Lie algebra $\mathfrak{g}$, we call a morphism $\varphi\colon \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ of locally convex Lie algebras a *morphism of Lie algebra extensions* if $q_1 = q_2 \circ \varphi$. In this way, one obtains the category of Lie algebra extensions of $\mathfrak{g}$ and an object in this category is called *universal* if it is initial. This definition yields a definition of universality of equivalence classes of central extensions. For now, we say that a cocycle $\omega\colon \mathfrak{g}^2 \to V$ is *universal* if the corresponding central extension is universal[12].

The second aim of Chapter 3 is to show the universality of an extension of certain so-called topological current algebras. In general these are algebras of the form $A \otimes \mathfrak{g}$, where A is a commutative locally convex algebra and $\mathfrak{g}$ is a locally convex Lie algebra. In [Mai02, Theorem 16], Maier constructed a universal continuous central extension for current algebras of the form $A \otimes \mathfrak{g}$, where A is a unital commutative complete locally convex algebra and $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra. The canonical example for such a current algebra is given

[11] Of course the underlying concept of these definitions is the Chevalley-Eilenberg chain-complex presented e.g. in [HN12, Chapter 7.5] in the case without topology and [Gun11] in the case of topological Lie algebras.

[12] We recall the concept of universality in more detail in Chapter 3.

by the smooth functions from a manifold M to $\mathfrak{g}$. To show the universality of the canonical cocycle $\omega\colon \Gamma_c(\mathfrak{K})^2 \to \overline{\Omega}^1_c(M, V(\mathfrak{K}))$ in [JW13, p. 129, (1.1)], Janssens and Wockel used [Mai02, Theorem 16] to show the universality of the canonical cocycle for the compactly supported smooth functions from a σ-compact manifold to a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ in [JW13, Theorem 2.7]. Gündoğan showed in [Gun11, Theorem 5.1.14] that the ideas from [JW13] can be used to show the universality of the canonical cocycle on current algebras $A \otimes \mathfrak{g}$ with pseudo-unital commutative algebras A that are inductive limits of unital Fréchet algebras. But this class of current algebras does not contain the compactly supported smooth maps $C^\infty_c(M, \mathfrak{g})$ on a σ-compact finite-dimensional manifold[13]. So in Section 3.3, we show that the cocycle constructed in [JW13] respectively [Gun11, Theorem 5.1.14] is universal if the algebra A is a complete locally convex commutative pseudo-unital algebra which is the inductive limit of subalgebras $A_n \subseteq A$ such that we can find an element $1_n \in A$ with $1_n \cdot a = a$ for all $a \in A_n$. Obviously, this class of algebras contains the compactly supported smooth functions on a σ-compact manifold. More precisely we prove:

Theorem F. *Let A be a complete locally convex commutative pseudo-unital algebra such that it is the inductive limit of subalgebras $A_n \subseteq A$ with $n \in \mathbb{N}$ such that we find for every $n \in \mathbb{N}$ an element $1_n \in A$ with $1_n \cdot a = a$ for all $a \in A_n$. If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra then $\omega_{\mathfrak{g},A}\colon A \otimes \mathfrak{g} \times A \otimes \mathfrak{g} \to V_{\mathfrak{g},A_1}$ with $(a \otimes x, b \otimes y) \mapsto \kappa_{\mathfrak{g}}(x,y) \otimes [a \cdot d_{A_1}(b)]$ is a universal cocycle for $A \otimes \mathfrak{g}$.*

The proof of Theorem F is based on the ideas from [JW13, Theorem 2.7]. But the discussion of the surjectivity of the map $H^2_{ct}(i)$ in the proof of [JW13, Theorem 2.7] was not complete (see Remark 3.39). Therefore the main work will be to show that this map is actually surjective.

Extensions of groups of compactly supported sections

Having discussed central extensions of Lie algebras in Chapter 3, we will continue with new constructions of central extensions of Lie groups in Chapter 4.[14] Like in [Nee02a], we use the following definition of a central extension of infinite-dimensional Lie groups: Let Z, G and $\hat{G}$ be Lie groups modelled over locally convex spaces. A short exact sequence $0 \to Z \hookrightarrow \hat{G} \xrightarrow{q} G \to 0$ of Lie groups is called central extension of G by Z, if Z lies in the center of $\hat{G}$, and $\hat{G} \xrightarrow{q} G$ is a smooth Z-principal bundle over the basis G. One easily sees that the condition that $\hat{G} \xrightarrow{q} G$ is a Z-principal bundle is equivalent to the existence of a smooth local section of $\hat{G} \xrightarrow{q} G$ that is defined on an open 1-neighbourhood. A further central extension $Z \hookrightarrow \hat{G}' \xrightarrow{q'} G$ of G by Z is called equivalent to

[13] For a compact subset K of M, the algebras $C^\infty_K(M)$ are not unital. Hence, one cannot deduce [Gun11, Corollary 5.2.14] from [Gun11, Theorem 5.2.13] as has been done in [Gun11].

[14] This subchapter of the introduction consist of material published before in the author's preprint [Eyn14b]

$Z \hookrightarrow \hat{G} \xrightarrow{q} G$, if there exists a Lie group isomorphism $\varphi\colon \hat{G} \to \hat{G}'$ such that $\varphi|_Z = \mathrm{id}_Z$ and $q = q' \circ \varphi$ on G' (see [Woc06, Definition A.2.4]). This defines a natural equivalence relation on the set of Lie group extensions of G by Z. We write $\mathrm{Ext}(G, Z)$ for the set of equivalence classes. Now we consider an abelian Lie group Z as a trivial G-module and recall the concept of Lie group cohomology and its relation to central extensions of Lie groups form [Nee02a]. We call a map $f\colon G \times G \to Z$ that is smooth on a $(1,1)$-neighbourhood a cocycle if $f(1, g_1) = f(g_1, 1)$ and $f(g_1, g_2) + f(g_1g_2, g_3) = f(g_1, g_1g_3)f(g_2, g_3)$ for all $g_1, g_2, g_3 \in G$ and write $Z^2_{sm}(G, Z)$ for the group of cocycles. Moreover, f is called coboundary if there exists a map $\varphi\colon G \to Z$ which is smooth on an identity neighbourhood such that $f(g_1, g_2) = \varphi(g_1g_2)\varphi(g_1)^{-1}\varphi(g_2)^{-1}$ and $\varphi(1) = 1$. We write $B^2_{sm}(G, Z)$ for the group of coboundaries. The second group cohomology is defined as $H^2_{sm}(G, Z) := Z^2_{sm}(G, Z)/B^2_{sm}(G, Z)$. Now let G be connected. There exists a canonical bijection $H^2_{sm}(G, Z) \to \mathrm{Ext}(G, Z)$: If $f \in Z^2_{sm}(G, Z)$ we obtain a group $G \times_f Z$ with multiplication $(g_1, z_1) \cdot (g_2, z_2) := (g_1g_2, z_1 + z_2 + f(g_1, g_2))$ and using the theorem of local description of Lie groups one obtains a Lie group structure on $G \times_f Z$ such that $Z \hookrightarrow G \times_f Z \xrightarrow{q} G$ becomes a central extension of Lie groups (see [Nee02a, Proposition 4.2]). Hence, we can describe the Lie group extensions of G by Z with the Lie group cohomology $H^2_{sm}(G, Z)$. A straightforward calculation shows that a central extension of Lie groups $Z \hookrightarrow \hat{G} \xrightarrow{q} G$ induces a central extension of topological Lie algebras $L(Z) \hookrightarrow L(\hat{G}) \xrightarrow{L(q)} L(G)$. We say that a given central extension of topological Lie algebras $V \hookrightarrow \hat{\mathfrak{g}} \to \gamma$ integrates to a central extension of Lie groups $Z \hookrightarrow \hat{G} \to G$, if the derived Lie algebra extension of this Lie group extension is given by $V \hookrightarrow \hat{\mathfrak{g}} \to \gamma$.

Central extensions play an important role in the theory of infinite-dimensional Lie groups. For example, every Banach-Lie algebra $\mathfrak{g}$ is a central extension $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \to \mathrm{ad}(\mathfrak{g})$, where the centre $\mathfrak{z}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ are integrable to a Banach-Lie group; integrability of $\mathfrak{g}$ corresponds to the existence of a corresponding central Lie group extension (see [vK64]).

Inspired by the seminal work of van Est and Korthagen, Neeb elaborated the general theory of central extensions of Lie groups that are modelled over locally convex spaces in 2002 (see [Nee02a]). In particular, Neeb showed that certain central extensions of Lie algebras can be integrated to central extensions of Lie groups: If the central extension of a locally convex Lie algebra $V \hookrightarrow \hat{\mathfrak{g}} \to \mathfrak{g}$ (with a sequentially complete locally convex space V) is represented by a continuous Lie algebra cocycle $\omega\colon \mathfrak{g}^2 \to V$ and G is a Lie group with Lie algebra $\mathfrak{g}$, one considers the so-called period homomorphism

$$\mathrm{per}_\omega\colon \pi_2(G) \to V,\ [\sigma] \mapsto \int_\sigma \omega^l$$

where $\omega^l \in \Omega^2(G, V)$ is the canonical left invariant 2-form on G with $\omega^l_1(v, w) = \omega(v, w)$ and σ is a smooth representative of the homotopy class $[\sigma]$ (the map per_ω is well-defined and a group homomorphism see [Nee02a, Definition 5.8]). One writes Π_ω for the image of the period homomorphism and calls it the period group of ω. The important result from [Nee02a] is that if Π_ω is a discrete subgroup of V and the adjoint action of $\mathfrak{g}$ on $\hat{\mathfrak{g}}$ integrates to a smooth action of G on $\hat{\mathfrak{g}}$ then $V \hookrightarrow \hat{\mathfrak{g}} \to \mathfrak{g}$ integrates to a central extension of Lie groups (see [Nee02a, Proposition 7.6 and Theorem 7.12]).

Given two central Lie group extensions $Z_1 \hookrightarrow \hat{G}_1 \xrightarrow{q_1} G$ and $Z_2 \hookrightarrow \hat{G}_2 \xrightarrow{q_2} G$, we call a Lie group homomorphism $\varphi\colon \hat{G}_1 \to \hat{G}_2$ a morphism of Lie group extensions if $q_1 = q_2 \circ \varphi$. In this way, one obtains a category of Lie group extensions and an object in this category is called universal if it is initial (see [Nee02b, Definition 4.3]). In 2002 Neeb showed that under certain conditions a central extension of a Lie group is universal in the category of Lie group extensions if its corresponding Lie algebra extension is universal in the category of central locally convex Lie algebra extensions (see [Nee02b, Recognition Theorem (Theorem 4.13)]).

The natural next step was to apply the general theory to different types of Lie groups that are modelled over locally convex spaces. Important infinite-dimensional Lie groups are current groups. These are groups of the form $C^\infty(M, G)$ where M is a compact finite-dimensional manifold and G is a Lie group. In 2003 Maier and Neeb constructed universal central extensions for current groups (see [MN03]) by reducing the problem to the case of loop groups $C^\infty(\mathbb{S}^1, G)$.

The compactness of M is a strong condition but it is not possible to equip $C^\infty(M, G)$ with a reasonable Lie group structure if M is non-compact, although one has a natural Lie group structure on the group $C^\infty_c(M, G)$ of compactly supported smooth functions from a σ-compact manifold M to a Lie group G. In this situation, $C^\infty_c(M, G)$ is the inductive limit of the Lie groups $C^\infty_K(M, G) := \{f \in C^\infty(M, G) : \operatorname{supp}(f) \subseteq K\}$ where K runs through a compact exhaustion of M. The Lie algebra of $C^\infty_c(M, G)$ is given by $C^\infty_c(M, \mathfrak{g})$. In this context, $C^\infty_c(M, \mathfrak{g})$ is equipped with the canonical direct limit topology in the category of locally convex spaces. In 2004, Neeb constructed a universal central extension for $C^\infty_c(M, G)$ in important cases (see [Nee04]).

It is possible to turn the group $\Gamma(M, \mathcal{G})$ of sections of a Lie group bundle $\mathcal{G}$ over a compact base manifold M into a Lie group by using the construction of the Lie group structure of the gauge group from [Woc07] (see [NW09, Appendix A]). The Lie algebra of $\Gamma(M, \mathcal{G})$ is the Lie algebra $\Gamma(M, \mathfrak{G})$ of sections of the Lie algebra bundle $\mathfrak{G}$ that corresponds to $\mathcal{G}$. Hence, the question arises if it is possible to construct central extensions for these groups of sections. Under certain conditions this is indeed the case and was done in 2009 by Neeb and Wockel in [NW09].

As mentioned above, one way to show the universality of a Lie group extension is to show the universality of the corresponding locally convex Lie algebra extension and then use the Recognition Theorem from [Nee02b]. Janssens and Wockel constructed a universal central extension of the Lie algebra $\Gamma_c(M, \mathfrak{G})$ of compactly supported smooth sections in a Lie algebra bundle over a σ-compact manifold in the recent paper [JW13] from 2013. They also applied this result to the central extension constructed in [NW09]: By assuming the base manifold M to be compact they obtained a universal Lie algebra extension that corresponds to the Lie group extension described in [NW09][15]; they were able to show the universality of

[15] Analogously to Theorem G a further technical condition about the cardinality of a certain

this Lie group extension.

In 2013, Schütt generalised the construction of the Lie group structure from [Woc07] by endowing the gauge group $\mathrm{Gau}_c(P)$ of compactly supported morphisms of a principal bundle over a not necessary compact base manifold M with a Lie group structure, under mild hypotheses (see [Sch13]). It is clear that we can use an analogous construction to endow the group of compactly supported sections of a Lie group bundle over a σ-compact manifold with a Lie group structure. Similarly, Neeb and Wockel already generalised the construction of the Lie group structure on a gauge group with compact base manifold from [Woc07] to the case of section groups over compact base manifolds.

The principal aim of Chapter 4 is to construct a central extension of the Lie group of compactly supported smooth sections on a non-compact σ-compact manifold such that its corresponding Lie algebra extension is represented by the Lie algebra cocycle described in [JW13] respectively Remark 3.24. This is a complementary result to the ones obtained in [NW09] (compact base manifold). The proof, which combines arguments from [Nee04] and [NW09] with new ideas, is discussed in Section 4.1 and Section 4.2. The main result is Theorem 4.53 where we show:

Theorem G. *Let $\mathfrak{G}$ be a finite-dimensional Lie algebra bundle with non-compact but σ-compact base manifold that is associated to a principal bundle $H \hookrightarrow P \to M$. If the group $\overline{H}$ from Definition 4.3 is finite then the canonical cocycle*

$$\omega\colon \Gamma_c(M,\mathfrak{G})^2 \to \Omega^1_c(M,\mathbb{V})/d\Gamma_c(M,\mathbb{V}),\ (\gamma,\eta)\mapsto [\kappa(\gamma,d\eta)]$$

can be integrated to a cocycle of Lie groups.

In the case of a compact base manifold M, corresponding results were obtained in [NW09, Theorem 4.24 and Theorem 4.26][16]. One step is to show that the period group of ω is a discrete subgroup of $\overline{\Omega}^1_c(M,\mathbb{V}) := \Omega^1_c(M,\mathbb{V})/d\Gamma_c(M,\mathbb{V})$. This will be discussed in Theorem 4.36 and is a complementary result to [NW09, Theorem 4.14]. In Section 4.2, we show that the adjoint action of $\Gamma_c(\mathfrak{G})$ on $\widehat{\Gamma_c(\mathfrak{G})} := \overline{\Omega}^1_c(M,\mathbb{V}) \times_\omega \Gamma_c(\mathfrak{G})$ can be integrated to a Lie group action of $\Gamma_c(\mathcal{G})$ on $\widehat{\Gamma_c(\mathfrak{G})}$. This is a complementary result to [NW09, Theorem 4.25].

In the second part of Chapter 4 (Section 4.3), we turn to the question of universality. Once the central extension is constructed, its universality is not hard to see, mainly because we can use the arguments from the compact case ([JW13]).

We prove:

Theorem H. *Let $G \hookrightarrow \mathcal{G} \to M$ be a finite-dimensional Lie group bundle with a semisimple connected typical fibre G such that it is associated to the frame principal bundle $\mathrm{Aut}(G) \hookrightarrow \mathrm{Fr}(\mathcal{G}) \to M$ and the group $\overline{\mathrm{Aut}(G)}$ is finite. Moreover, let M be non-compact and σ-compact. Then we obtain a universal Lie group cocycle that*

group is needed.

[16] In [NW09] Neeb and Wockel also considered the case where the typical fibre of the Lie group bundle is infinite-dimensional where as we only consider the case of a finite-dimensional typical fibre.

corresponds to the continuous Lie algebra cocycle ω described in Theorem G (or [JW13]).

Notations and conventions

In the following, we fix some general notations and conventions.[17] Notation pertaining to the respective chapters will be introduced there.

- We write $\mathbb{N}$ for the set of integers $\{1, 2, 3, ...\}$.
- All locally convex spaces considered are assumed Hausdorff.
- If E is a locally convex vector space and M a manifold, we write $C^\infty_c(M, E)$ for the space of compactly supported smooth functions from M to E.
- For a fibre bundle $q\colon F \to M$ with total space F, finite-dimensional base manifold M, projection q and typical fibre E we write $E \hookrightarrow F \xrightarrow{q} M$. For the space of smooth sections of such a fibre bundle we write $\Gamma(M, F)$. If it is clear form the context what our base manifold M is, we simply write $\Gamma(F)$. In the case that F is a vector bundle we write $\Gamma_c(M, F)$ respectively $\Gamma_c(F)$ for the space of compactly supported smooth sections.
- Let $\mathbb{V}$ be a finite-dimensional vector bundle over a finite-dimensional σ-compact manifold M. As usual, we write $\Omega^k(M, \mathbb{V})$ for the space of $\mathbb{V}$-valued k-forms on M and $\Omega^k_c(M, \mathbb{V})$ for the space of compactly supported $\mathbb{V}$-valued k-forms on M.
- If V and W are vector spaces, we write $\mathrm{Lin}(V, W)$ for the space of linear maps from V to W and in the case of topological vector spaces we write $\mathcal{L}(V, W)$ for the space of continuous linear maps. As usual, we write $\mathcal{L}(V) := \mathcal{L}(V, V)$ and $\mathcal{L}(V)^*$ for the group of automorphisms of V.
- If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, we write $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$for the space of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{h}$ and in the case of topological Lie algebras we write $\mathrm{Hom}_{ct}(\mathfrak{g}, \mathfrak{h})$ for the space of continuous Lie algebra homomorphisms.
- Given a finite-dimensional vector bundle $V \hookrightarrow \mathbb{V} \xrightarrow{q} M$ over a σ-compact manifold M, a compact set $K \subseteq M$ and $k \in \mathbb{N}_0$ we write $\Omega^k_K(M, \mathbb{V})$ for the space of k-forms on M with values in the vector bundle $\mathbb{V}$ and support in K. Using the identification $\Omega^k(M, \mathbb{V}) \cong \Gamma(M, \Lambda^k T^*M \otimes V)$ we give these spaces the locally convex vector topology described e.g. in [Glo13] and equip $\Omega^k_c(M, \mathbb{V})$ with the canonical locally convex direct limit topology. Especially the spaces $\Gamma(\mathbb{V})$ and $\Gamma_c(\mathbb{V})$ carry the natural locally convex topology described e.g. in [Glo13]. (See also Definition 3.8 for further details).
- If $\mathbb{V}$ and $\mathbb{W}$ are two vector bundles we write $\mathbb{V} \oplus \mathbb{W}$ for their Whitney sum.
- Given a group G and an element $g \in G$, we write $\lambda_g\colon G \to G$, $h \mapsto gh$ for the left multiplication with g and $\varrho_h\colon G \to G$, $h \mapsto hg$ for the right multiplication on G.
- Let $f\colon X \times Y \to Z$ be a map. For $x_0 \in X$ and $y_0 \in Y$, we define the maps $f(x_0, \bullet)\colon Y \to Z$, $y \mapsto f(x_0, y)$ and $f(\bullet, y_0)\colon X \to Z$, $x \mapsto f(x, y_0)$. Moreover,

[17] This subchapter of the introduction consist of material published before in the author's preprints [Eyn14c] and [Eyn14b]

we define the map $\check{f}\colon X \to Z^Y$, $x \mapsto f(x, \bullet)$. If $g\colon X \to Z^Y$ is a map, we define the map $\hat{g}\colon X \times Y \to Z$, $(x, y) \mapsto g(x)(y)$.

1. Diff$^\omega$(M) as a Lie group for a manifold M with corners

The first aim of this thesis is to turn the group of real analytic diffeomorphisms of a compact real analytic manifold with corners into a smooth infinite-dimensional manifold modelled on the locally convex space of real analytic stratified vector fields.[1] As mentioned in the introduction this generalises results of [DS15] respectively [KM90] to the case of a manifold with corners. Moreover, we follow [Glo06c] and use the local approach (see introduction). Hence, we also obtain a new construction for the case of a manifold without boundary.

1.1. Enveloping manifold

Fist, we will prove basic facts about real analytic maps on manifolds with corners and enveloping manifolds. The primary aim of this section is to show that a real analytic manifold with corners can be embedded into a real analytic manifold without corners. As mentioned in the introduction we cannot use a construction like the double of a manifold because the boundary of a manifold with corners is not a manifold. Moreover, we cannot use the construction from the smooth case ([DH73, Proposition 3.1] or [Mic80, p. 21]) because of the lack of real analytic bump functions. Instead we adapt the proof of the existence and uniqueness of complexifications of real analytic manifolds (see [BW59, Proposition 1] or [DGS14, Section 2 and Section 3]). With the help of Lemma 1.11, our proof of the existence of enveloping manifolds (Theorem 1.12) is completely analogous to [BW59, Proposition 1] (see Appendix B). For technical reasons, in this section we work with manifolds that are modelled on a quadrant $[0,\infty[^m$. Of course this definition of a manifold with corners is equivalent to the one where manifolds with corners are modelled on sets of the form $[0,\infty[^k \times \mathbb{R}^{m-k}$ with $k \leqslant m$. In Appendix A we recall some basic definitions and facts concerning manifolds with corners that are used in this chapter.

Convention 1.1. (a) If $x \in \mathbb{R}^m$ and $\varepsilon > 0$ we write $B_\varepsilon(x)$ (respectively $\overline{B}_\varepsilon(x)$) for the open (respectively closed) ball in $\mathbb{R}^m$ with respect to the Euclidean norm. Moreover, we write $B^\infty_\varepsilon(x)$ for the ball in $\mathbb{R}^m$ with respect to the maximum norm.

(b) Let M be an m-dimensional manifold with corners. We write $\partial^j M$ for the set of points in M of index j (like, for instance, in [Mic80]). Therefore, $\partial^j M$ is an $(m-j)$-dimensional manifold. We write $\partial M := \bigcup_{j=1}^m \partial^j M$ and call ∂M

[1] This chapter consist of material published before in the author's preprint [Eyn15]

the boundary of M. The set $\partial^0 M$ is called the interior of M (see Appendix A for more details).

Remark 1.2. *We recall some common definitions and well-known basic facts:*

(a) *Let $U \subseteq \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ be a map. The map f is called real analytic if we can find an open neighbourhood $V \subseteq \mathbb{C}^m$ of U and a complex analytic function $f^*\colon V \to \mathbb{C}^n$ with $f^*|_U = f$ (see e.g. [GN][2]). We write $C^\omega(U;\mathbb{R}^m)$ for the space of real analytic maps from U to $\mathbb{R}^m$. The space of complex analytic maps from V to $\mathbb{C}^m$ is denoted $\mathrm{Hol}(V;\mathbb{C}^m)$; we endow it with the compact-open topology (see [DS15, Lemma A.7]).*

(b) *Given a real analytic map $f\colon \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ and $x \in U$ there exists a 0-neighbourhood $V \subseteq \mathbb{R}^m$ such that $x + V \subseteq U$ and for all $v \in V$ we get $f(x+v) = \sum_{k=0}^{\infty} \frac{\delta_x^k f(v)}{k!}$. In this context, $\delta_x^k f$ is the k-th Gateaux differential of f in x. (See [GN][3]).*

(c) *If U is an open connected subset of $\mathbb{R}^m$, $f\colon U \to \mathbb{R}^n$ is a real analytic map and $x \in U$ with $\delta_x^k f = 0$ for all $k \in \mathbb{N}_0$, then $f = 0$ (see e.g. [GN][4]).*

(d) *Let U be an open subset of $[0,\infty[^m$. We call a map $f\colon U \to \mathbb{R}^n$ real analytic if every $x \in U$ has a neighbourhood $\tilde{U} \subseteq \mathbb{R}^m$ such that there exists a real analytic map $\tilde{f}\colon \tilde{U} \to \mathbb{R}^n$ with $\tilde{f}|_U = f$.*

(e) *A corner-atlas of a Hausdorff space M is a set of homeomorphisms $\varphi\colon U_\varphi \to V_\varphi$ between open subsets U of M and V of $[0,\infty[^m$ such that the transition maps $\psi \circ \varphi^{-1}\colon \varphi(U_\psi \cap U_\varphi) \to V_\psi$ are real analytic. The space M together with a maximal corner-atlas is called real analytic manifold with corners.*

(f) *Let M and N be real analytic manifolds (without corners) and $g_1, g_2\colon M \to N$ be real analytic maps that coincide on a non-empty open subset $V \subseteq M$. If M is connected then $g_1 = g_2$. (See e.g. [DGS14, Lemma 1.7]).*

Lemma 1.3. *If $C \subseteq \mathbb{R}^m$ is convex, $U \subseteq \mathbb{R}^m$ is open, $\mathring{C} \neq \varnothing$ and $C \cap U \neq \varnothing$ then $\mathring{C} \cap U \neq \varnothing$.*

Proof. Let $z \in C \cap U$. If $z \in \mathring{C}$ we are done. Hence, we can assume that $z \in \partial C$. Because C is convex and $\mathring{C} \neq \varnothing$ we have $\overline{\mathring{C}} = \overline{C}$ (see, e.g., [Jar81, p. 104, Theorem 5]) and $\partial C = \overline{C} \backslash \mathring{C} = \overline{\mathring{C}} \backslash \mathring{C} = \partial \mathring{C}$. Hence $z \in \partial \mathring{C}$. Therefore, every z-neighbourhood intersects $\mathring{C}$. Thus $U \cap \mathring{C} \neq \varnothing$. □

Lemma 1.4. *Let U be an open subset of $[0,\infty[^m$ and $f\colon U \to \mathbb{R}^n$ be a real analytic map. There exists an open neighbourhood $\tilde{U} \subseteq \mathbb{R}^m$ of U and a real analytic map $\tilde{f}\colon \tilde{U} \to \mathbb{R}^n$ with $\tilde{f}|_U = f$.*

Proof. Given $x \in U$ we find $\varepsilon_x > 0$ and a real analytic map $\tilde{f}_x\colon \tilde{U}_x \to \mathbb{R}^n$ defined on $\tilde{U}_x := B^\infty_{\varepsilon_x}(x)$ such that $\tilde{U}_x \cap [0,\infty[^m \subseteq U$ and $\tilde{f}_x|_{U \cap \tilde{U}_x} = f$. Let $x, y \in U$ with $\tilde{U}_x \cap \tilde{U}_y \neq \varnothing$ and $z \in \tilde{U}_x \cap \tilde{U}_y$. Now we define $\tilde{z}$ by $\tilde{z}_i := z_i$ if $z_i \geqslant 0$ and $\tilde{z}_i := -z_i$ if $z_i < 0$. Obviously $\tilde{z} \in [0,\infty[^m$. We show that $\tilde{z} \in \tilde{U}_x \cap \tilde{U}_y$. If $z_i \geqslant 0$ we have

[2] Probably this will be stated in Definition 2.2.2.
[3] Probably this will be stated in Lemma 2.2.6.
[4] Probably this will be stated in Theorem 2.2.8.

$|\tilde{z}_i - x_i| < \varepsilon_x$. On the other hand, if $z_i < 0$, we have $x_i \geqslant 0$, $x_i - z_i = |x_i - z_i| < \varepsilon_x$. We calculate $|x_i - \tilde{z}_i| = |x_i + z_i| \leqslant x_i - z_i < \varepsilon_x$. Hence, $\tilde{z} \in \tilde{U}_x$. Analogously one can show that $\tilde{z} \in \tilde{U}_y$. We conclude that $[0,\infty[^m \cap \tilde{U}_x \cap \tilde{U}_y \neq \emptyset$. Using Lemma 1.3, we deduce that $C :=]0,\infty[^m \cap \tilde{U}_x \cap \tilde{U}_y \neq \emptyset$. We have $\tilde{f}_x|_C = f|_C = \tilde{f}_y|_C$. Since $\tilde{U}_x \cap \tilde{U}_y$ is convex, we get $\tilde{f}_x|_{\tilde{U}_x \cap \tilde{U}_y} = \tilde{f}_y|_{\tilde{U}_x \cap \tilde{U}_y}$. Now we define $\tilde{U} := \bigcup_{x \in U} \tilde{U}_x$ and $\tilde{f} \colon \tilde{U} \to \mathbb{R}^n, z \mapsto \tilde{f}_x(z)$ if $z \in \tilde{U}_x$. The construction above ensures that the map $\tilde{f}$ is well-defined. Moreover, $\tilde{f}$ is real analytic and $\tilde{f}|_U = f$. □

Convention 1.5. Given a manifold M we write $\mathcal{A}_x^M$ for the set of charts around a point $x \in M$. If N is a further manifold, $f \colon M \to N$ a map and $\psi \in \mathcal{A}_{f(x)}^N$, we write $f^{\varphi,\psi} := \psi \circ f \circ \varphi^{-1}|_{\varphi(f^{-1}(U_\psi))}$ for the local representative of f in the charts $\varphi \colon U_\varphi \to \mathbb{R}^m$ of M and $\psi \colon U_\psi \to \mathbb{R}^m$ of N.

Definition 1.6. (Cf. [DH73, Proposition 3.1] respectively [Mic80, p. 21]) Let M be a real analytic manifold with corners. A real analytic manifold without corners $\tilde{M}$ is called an *enveloping manifold* for M if $M \subseteq \tilde{M}$ and for every $x \in M$ there exists a chart $\tilde{\varphi} \colon \tilde{U} \to \tilde{V}$ of $\tilde{M}$ around x such that $\tilde{\varphi}(\tilde{U} \cap M) = \tilde{V} \cap [0,\infty[^m$ and $\tilde{\varphi}|_{\tilde{U} \cap M}^{\tilde{U} \cap [0,\infty[^m}$ is a chart of M. In other words: M is an equidimensional submanifold of $\tilde{M}$ with corners such that its submanifold structure coincides with its original manifold structure. The chart $\tilde{\varphi}$ is called an *enveloping chart* of M.

Lemma 1.7. *Let $M \neq \emptyset$ be a real analytic manifold with corners and $\tilde{M}$ an enveloping manifold of M. Moreover, let N be a real analytic manifold without corners and $g_1, g_2 \colon \tilde{M} \to N$ be real analytic maps. If $g_1|_M = g_2|_M$, then there exists an open neighbourhood $V \subseteq \tilde{M}$ of M such that $g_1|_V = g_2|_V$.*

Proof. Let $x \in M$ and $\varphi \in \mathcal{A}_x^{\tilde{M}}$ with $\varphi(M \cap U_\varphi) = V_\varphi \cap [0,\infty[^m$ such that $\varphi|_{M \cap U_\varphi}^{V_\varphi \cap [0,\infty[^m}$ is a chart of M. Let ψ be a chart of N around $g_1(x) = g_2(x)$. Without loss of generality, we can assume that $g_1(U_\varphi), g_2(U_\varphi) \subseteq U_\psi$ and we assume that V_φ is connected. We get $g_1^{\varphi,\psi}|_{V_\varphi \cap [0,\infty[^m} = g_2^{\varphi,\psi}|_{V_\varphi \cap [0,\infty[^m}$. Because of Lemma 1.3 the maps $g_1^{\varphi,\psi}$ and $g_2^{\varphi,\psi}$ coincide on V_φ. Hence, g_1 and g_2 coincide on an open neighbourhood of x. □

The following lemma comes from [DGS14, Lemma 2.1 (a)].

Lemma 1.8. *Let X be a regular topological Hausdorff space, $K \subseteq X$ be a compact subset and $(U_i)_{i \in I}$ be an open cover of K. Then there exists an open cover $(V_j)_{j \in J}$ of K such that, given $j_1, j_2 \in J$ with $V_{j_1} \cap V_{j_2} \neq \emptyset$, there exists $i \in I$ with $V_{j_1} \cup V_{j_2} \subseteq U_i$.*

In the following lemma, we prove an existence result for extensions of real analytic maps on real analytic manifolds with corners. The proof follows the idea of [DGS14, Lemma 2.2 (a)], where Dahmen, Glöckner and Schmeding showed an analogous result for extensions to complexifications.

Lemma 1.9. *Let M and N be real analytic manifolds with corners, M be compact and $\tilde{M}$ (respectively $\tilde{N}$) be an enveloping manifold of M (respectively N). If $f \colon M \to N$ is a real analytic map, then there exists an open neighbourhood $U \subseteq \tilde{M}$ of M and a real analytic map $g \colon U \to \tilde{N}$ with $g|_M = f$.*

Proof. Given $x \in M$, let $\varphi_1 \colon U_1 \to V_1$ be an enveloping chart of M around x and $\varphi_2 \colon U_2 \to V_2$ be an enveloping chart of N around $f(x)$ with $f(U_1 \cap M) \subseteq U_2 \cap N$. In particular, $\varphi_1|_{U_1 \cap M}^{V_1 \cap [0,\infty[^m}$ and $\varphi_2|_{U_2 \cap N}^{V_2 \cap [0,\infty[^n}$ are charts of M and N respectively with $n : \dim(N)$. There exists a real analytic map $\psi \colon V_1 \cap [0,\infty[^m \to V_2 \cap [0,\infty[^n$ such that the diagram

$$\begin{array}{ccc} U_1 \cap M & \xrightarrow{\quad f \quad} & U_2 \cap N \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ V_1 \cap [0,\infty[^m & \xrightarrow{\quad \psi \quad} & V_2 \cap [0,\infty[^n \end{array}$$

commutes. Let $\psi_x \colon V_x \to \mathbb{R}^n$ be a real analytic map defined on an open neighbourhood V_x of $V_1 \cap [0,\infty[^m$ such that $\psi_x|_{V_1 \cap [0,\infty[^m} = \psi$. Without loss of generality, we can assume that $V_x \subseteq V_1$ and $\psi_x(V_x) \subseteq V_2$. Now we define the open set $U_x := \varphi_1^{-1}(V_x)$ and the real analytic map $g_x \colon U_x \to U_2 \subseteq \tilde{N}$ by $g_x := \varphi_2^{-1} \circ \psi_x \circ \varphi_1$. We have $g_x|_{U_x \cap M} = f|_{U_x \cap M}$. By applying [DGS14, Lemma 2.1 (a)] or Lemma 1.8 we can find an open cover $(W_j)_{j \in I}$ of M such that given $j_1, j_2 \in I$ with $W_{j_1} \cap W_{j_2} \neq \emptyset$ there exists $x \in M$ with $W_{j_1} \cup W_{j_2} \subseteq U_x$. Every point $x \in M$ is contained in a set W_i. We can replace W_i with the connected component of x in W_i and hence we can assume all W_i to be connected and intersecting M. For $i \in I$ there exists $x_i \in M$ with $W_i \subseteq U_{x_i}$. Now let $g_i \colon W_i \to \tilde{N}$ be given by $g_i := g_{x_i}|_{W_i}$. If $i, k \in I$ with $W_i \cap W_k \neq \emptyset$, there exists $x \in M$ with $W_i \cup W_k \subseteq U_x$. We have $g_i|_{W_i \cap M} = f|_{W_i \cap M} = g_x|_{W_i \cap M}$. Because W_i is connected, Lemma 1.7 implies that $g_i = g_x|_{W_i}$. Analogously, we get $g_k = g_x|_{W_k}$ and hence, $g_i|_{W_i \cap W_k} = g_k|_{W_i \cap W_k}$. Now, we define the open set $U := \bigcup_{i \in I} U_i$ and the real analytic map $g \colon U \to \tilde{N}$ by $g|_{W_i} := g_i$. By construction g is well-defined and real analytic. □

In the following lemma, we show that a real analytic diffeomorphism of manifolds with corners has a diffeomorphic real analytic extension to open subsets of enveloping manifolds. Our proof is analogous to [DGS14, Lemma 2.2 (e)] (the analogous result for extensions to complexifications).

Lemma 1.10. *Let $f \colon M \to N$ be a real analytic diffeomorphism between real analytic manifolds with corners. Moreover, let $\tilde{M}$ and $\tilde{N}$ be enveloping manifolds of M and N respectively and $U \subseteq \tilde{M}$ be an open neighbourhood of M. Furthermore, let $V \subseteq \tilde{N}$ be an open neighbourhood of N, $\tilde{f} \colon U \to \tilde{N}$ be an extension of f and $\tilde{g} \colon V \to \tilde{M}$ be an extension of $g := f^{-1}$. We can find a neighbourhood $X \subseteq U$ of M such that $\tilde{f}|_X$ is a real analytic diffeomorphism onto its open image $Y := \tilde{f}(X) \subseteq V$ with inverse $\tilde{g}|_Y^X$.*

Proof. Let X be the union of all connected components of $\tilde{f}^{-1}(V)$ that intersect M. Thus $\tilde{g} \circ \tilde{f}|_X \colon X \to \tilde{M}$ is real analytic and $\tilde{g} \circ \tilde{f}|_{X \cap M} = \mathrm{id}_{X \cap M}$. Hence $\tilde{g} \circ \tilde{f}|_X = \mathrm{id}_X$. Let Y be the union of all connected components of $\tilde{g}^{-1}(X)$ that intersect N. As above $\tilde{f}|_X \circ \tilde{g}|_Y \colon Y \to \tilde{M}$ is real analytic and $\tilde{f}|_X \circ \tilde{g}|_Y = \mathrm{id}_Y$. Now we show that $Y = \tilde{f}(X)$. The inclusion "$\subseteq$" follows from $\tilde{f}|_X \circ \tilde{g}|_Y = \mathrm{id}_Y$. It remains to show that $\tilde{f}(X) \subseteq Y$. With $\tilde{g} \circ \tilde{f}|_X = \mathrm{id}_X$ we get $\tilde{f}(X) \subseteq \tilde{g}^{-1}(X)$. If C

is a connected component of $\tilde{f}^{-1}(V)$ that intersects M then $\tilde{f}(C)$ is a connected subset of $\tilde{g}^{-1}(X)$ and intersects N. Hence, $\tilde{f}(C) \subseteq Y$. □

The following technical lemma is a crucial tool for proving Theorem 1.12.

Lemma 1.11. *Let $U \subseteq [0,\infty[^m$ be open. Given an open neighbourhood O of $\overline{U}$ in $\mathbb{R}^m$ there exists an open neighbourhood $\tilde{U}$ of U in $\mathbb{R}^m$ such that $\overline{\tilde{U}} \subseteq O$, $\tilde{U} \cap [0,\infty[^m = U$ and $\overline{\tilde{U}} \cap [0,\infty[^m = \overline{U}$.*

Proof. Given $z \in \mathbb{R}^m$, we define $z_+ \in [0,\infty[^m$ by $(z_+)_i := |z_i|$. Hence, the map $\lambda\colon \mathbb{R}^m \to [0,\infty[^m, z \mapsto z_+$ is continuous and $\lambda^{-1}(U)$ is open in $\mathbb{R}^m$. Given $x \in U$ there exists $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x) \cap [0,\infty[^m \subseteq U$ and $B_{\varepsilon_x}(x) \subseteq \lambda^{-1}(U)$. The set $\tilde{U}_1 := \bigcup_{x\in U} B_{\varepsilon_x}(x)$ is open in $\mathbb{R}^m$. We also have $\tilde{U}_1 \cap [0,\infty[^m = U$ because $U \subseteq \tilde{U}_1$ and $\tilde{U}_1 \cap [0,\infty[^m = \bigcup_{x\in U}(B_{\varepsilon_x}(x) \cap [0,\infty[^m) \subseteq U$. Hence $\overline{U} = \overline{\tilde{U}_1 \cap [0,\infty[^m} \subseteq \overline{\tilde{U}_1} \cap [0,\infty[^m$. Now let $z \in \overline{\tilde{U}_1} \cap [0,\infty[^m$. There exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $\tilde{U}_1$ with $\lim_{n\to\infty} z_n = z$. Hence, $z = \lambda(z) = \lim_{n\to\infty} \lambda(z_n)$. But with $\tilde{U}_1 \subseteq \lambda^{-1}(U)$ we get $\lambda(z_n) \in U$. Thus $z \in \overline{U}$. We conclude that $\overline{\tilde{U}_1} \cap [0,\infty[^m = \overline{U}$. Now let W be a neighbourhood of $\overline{U}$ in $\mathbb{R}^m$ with $\overline{U} \subseteq W \subseteq \overline{W} \subseteq O$. We define $\tilde{U} := \tilde{U}_1 \cap W$ and get $\overline{\tilde{U}} \subseteq \overline{W} \subseteq O$ and $\tilde{U} \cap [0,\infty[^m = U \cap W = U$. Hence $\overline{U} = \overline{\tilde{U} \cap [0,\infty[^m} \subseteq \overline{\tilde{U}} \cap [0,\infty[^m$. Moreover,

$$\overline{\tilde{U}} \cap [0,\infty[^m = \overline{\tilde{U}_1 \cap W} \cap [0,\infty[^m \subseteq \overline{\tilde{U}}_1 \cap \overline{W} \cap [0,\infty[^m = \overline{W} \cap \overline{U} = \overline{U}.$$

Hence, $\overline{\tilde{U}} \cap [0,\infty[^m = \overline{U}$. □

With Lemma 1.11 it is possible to transfer the proof of [BW59, Proposition 1] or [DGS14, Proposition 3.1] (existence of complexifications of real analytic manifolds) to our situation. Making use of Lemma 1.11, our proof is complete analogous to the one of [BW59, Proposition 1] or [DGS14, Proposition 3.1].

Theorem 1.12. *Given a compact real analytic finite-dimensional manifold with corners M, there exists an enveloping manifold $\tilde{M}$ of M. If $\tilde{M}_1$ and $\tilde{M}_2$ are enveloping manifolds of M, then we can find a neighbourhood U_1 of M in $\tilde{M}_1$, a neighbourhood U_2 of M in $\tilde{M}_2$ and a real analytic diffeomorphism $\varphi\colon U_1 \to U_2$ with $\varphi|_M = \mathrm{id}_M$.*

Proof. See Appendix B. □

1.2. Local manifold structure

As mentioned in the introduction we use the "local approach" developed in [Glo06c] and transfer its line of thought to the case of a compact real analytic manifold M with corners. In this section we construct an open subset $\mathcal{V}$ of $\Gamma^{\omega}_{\mathrm{st}}(TM)$ that is small enough such that $\mathcal{U} := \{\exp \circ \eta : \eta \in \mathcal{V}\}$ is a subset of $\mathrm{Diff}^{\omega}(M)$. As in [Glo06c] we control the uniform norm of the vector fields of $\mathcal{V}$ and the norms of the first derivative simultaneously. For the rest of the chapter the manifolds with corners are modelled on spaces of the form $\mathbb{R}^m_k := [0,\infty[^k \times \mathbb{R}^{m-k}$ instead of quadrants $[0,\infty[^m$ (see Section 1.1). Of course, both definitions are equivalent. As in [Mic80], we model groups of diffeomorphisms on spaces of stratified vector fields.

Conventions and notations

First, we fix conventions and our notation.

Definition 1.13. Let M be a Riemannian manifold without boundary and $N \subseteq M$ be a Riemannian submanifold without boundary. We call N *totally geodesic* if all geodesics of N are also geodesics of M (cf. [ONe83, Chapter 4; Definition 12 and Proposition 13]).

Convention 1.14. (a) Let M be an m-dimensional compact real analytic manifold with corners and $\tilde{M}$ be an enveloping manifold of M. Moreover, let $M_\mathbb{C}$ be a complexification of $\tilde{M}$. We assume that there exists a real analytic Riemannian metric g on $\tilde{M}$ such that the submanifolds $\partial^j M$ are totally geodesic for all $j \in \{1,\ldots,m\}$. We call such a metric *boundary respecting*. In this context $\tilde{\Omega} \subseteq T\tilde{M}$ is the maximal domain of definition of the Riemannian exponential map $\exp\colon \tilde{\Omega} \to \tilde{M}$. Analogously let $\Omega_{\partial^j M}$ be the maximal domain of definition of the Riemannian exponential map that comes from induced Riemannian metric on $\partial^j M$ for $j \in \{0,\ldots,m\}$.

(b) We write $B_r(x)$ for balls with radius r in $\mathbb{R}^m$ and $B^\mathbb{C}_r(x)$ for balls with radius r in $\mathbb{C}^m$. Moreover, we define $B^k_r(x) := B_r(x) \cap \mathbb{R}^m_k$ with $\mathbb{R}^m_k = [0,\infty[^k \times \mathbb{R}^{m-k}$.

Example 1.15. *An example of a real analytic manifold with corners is e.g. a tetrahedron as a submanifold of* $\mathbb{R}^3$.

Remark 1.16. *There exist finitely many enveloping charts* $\tilde{\varphi}_i\colon \tilde{U}_{i,6} \to B_6(0)$ *with* $i = 1,\ldots,n$ *and induced* M*-charts* $\varphi_i\colon U_{i,6} \to B^{k_i}_6(0)$ *such that* $M \subseteq \bigcup_{i=1}^n \varphi_i^{-1}(B^{k_i}_1(0))$ *(Compare the smooth case in [Glo06c, 4.1] or [Lan01, Theorem 3.3]). We use the shorthand notation* $K_i := \tilde{\varphi}_i^{-1}(\overline{B}_5(0))$. *There exist an open subset* U_i^* *of* $M_\mathbb{C} := (\tilde{M})_\mathbb{C}$, *an open subset* V_i^* *of* $\mathbb{C}^m$ *and a complex analytic diffeomorphism* $\varphi_i^*\colon U_i^* \to V_i^*$ *such that* $K_i \subseteq U_i^*$, $\overline{B}_5(0) \subseteq V_i^*$ *and* $\varphi_i^*|_{K_i} = \tilde{\varphi}_i|_{K_i}$ *(see [DGS14, Lemma 2.2 (a) and (e)]).*

Convention 1.17.

- On $\mathbb{R}^m$ and $\mathbb{C}^m$ we use the Euclidean norm.
- If f is a differentiable map on an open subset U of $\mathbb{R}^m_k$ (respectively $\mathbb{R}^m$ respectively $\mathbb{C}^m$) then f' always means the first derivative as a map from U to $\mathcal{L}(\mathbb{R}^m)$ (respectively $\mathcal{L}(\mathbb{C}^m)$). We equip $\mathcal{L}(\mathbb{R}^m)$ (respectively $\mathcal{L}(\mathbb{C}^m)$) with the operator norm $\|\bullet\|_{op}$.
- Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. If $f\colon X \to \mathbb{K}^m$ is a map and $Y \subseteq X$, we write $\|f\|^Y_\infty := \sup\{\|f(x)\| : x \in Y\}$ and $\|f\|_\infty := \|f\|^X_\infty$ for the uniform norm. Moreover we write $\|f\|^0_\infty := \|f\|_\infty$ and define $\|g\|_\infty := \sup\{\|g(x)\| : x \in X\}$ for $g\colon X \to \mathcal{L}(\mathbb{K}^m)$. If $X \subseteq \mathbb{K}^m$ and f is differentiable, we write $\|f\|^1_\infty := \max(\|f\|_\infty, \|f'\|_\infty)$ and $\|f\|^1_Y := \sup\{\|f(x)\|, \|f'(x)\|_{op} : x \in Y\}$.

- We fix charts φ_i, $\tilde{\varphi}_i$ and φ_i^* as in Remark 1.16. Moreover, we define
$$\tilde{U}_{i,r} := \tilde{\varphi}_i^{-1}(B_r(0)) \text{ and } U_{i,r} := \varphi_i^{-1}(B_r^{k_i}(0))$$
for $i \in \{1, \ldots, n\}$ and $r \in]0, 6]$.
- If U is a finite-dimensional real analytic manifold, we write $\Gamma^\omega(TU)$ for the space of real analytic vector fields of U.
- If U is a complex finite-dimensional manifold, we write $\Gamma^\infty_{\mathbb{C}}(TU)$ for the space of complex analytic vector fields of U.
- Let g_i be the Riemannian metric on $B_6(0)$ that is induced by g via $\tilde{\varphi}_i$ and let $\exp_i \colon \tilde{\Omega}_i \to B_6(0)$ be the exponential map on $B_6(0)$ that is induced by g_i.
- If $\eta \in \Gamma^\omega(TM)$ we define $\eta_{(i)} := d\varphi_i \circ \eta \circ \varphi_i^{-1} \colon B_6^{k_i}(0) \to \mathbb{R}^m$. If U is an open neighbourhood of M in $\tilde{M}$ (respectively in $M_{\mathbb{C}}$) and $\eta \in \Gamma^\omega(TU)$ (respectively $\eta \in \Gamma^\infty_{\mathbb{C}}(TU)$) we define $\eta_{(i)} := d\tilde{\varphi}_i \circ \eta \circ \tilde{\varphi}_i^{-1} \colon \tilde{\varphi}_i(U_{i,6} \cap U) \subseteq B_6(0) \to \mathbb{R}^m$ (respectively $\eta_{(i)} := d\varphi_i^* \circ \eta \circ \varphi_i^{*-1} \colon \varphi_i^*(U_i^* \cap U) \subseteq V_i^* \to \mathbb{C}^m$).
- If $\eta \in \Gamma^\omega(TM)$ we can use the Lemma 1.7 and Lemma 1.9 to obtain an extension that is a real analytic vector field $\tilde{\eta}$ of a neighbourhood of M in $\tilde{M}$. Analogously, we write η^* for a complex analytic extension to a vector field on an open neighbourhood of M in the complex analytic manifold $M_{\mathbb{C}} = (\tilde{M})_{\mathbb{C}}$.
- If $f \in C^\omega(\overline{B}_R^k(0); \mathbb{R}^m)$ we write $\tilde{f}$ for an real analytic extension to an open neighbourhood of $\overline{B}_R^k(0)$ in $\mathbb{R}^m$ and f^* for an real analytic extension to an open neighbourhood of $\overline{B}_R^k(0)$ in $\mathbb{C}^m$.
- Given a compact connected subset K of a topological space X, we call a sequence of open connected relatively compact subsets $(U_n)_{n\in\mathbb{N}}$ of X a connected *fundamental sequence* of K if $U_n \supseteq \overline{U}_{n+1} \supseteq K$ for all $n \in \mathbb{N}$ and $(U_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of K in X (cf. [DS15, A.9]).
- If U is open in $\mathbb{C}^m$ and $k \in \{0, 1\}$ we define
$$\mathrm{Hol}_b^k(U; \mathbb{C}^m) := \left\{ f \in \mathrm{Hol}(U; \mathbb{C}^m) : \|f\|_\infty^k < \infty \right\}.$$
With $\|\bullet\|_\infty^1 \colon \mathrm{Hol}_b^1(U; \mathbb{C}^m) \to [0, \infty[$, $f \mapsto \max(\|f\|_\infty, \|f'\|_\infty)$ (respectively $\|\bullet\|_\infty^0 := \|\bullet\|_\infty$) the space $\mathrm{Hol}_b^k(U; \mathbb{C}^m)$ becomes a Banach space [5]. We also define
$$\mathrm{Hol}_\varepsilon^k(U; \mathbb{C}^m) := \left\{ f \in \mathrm{Hol}_b^k(U; \mathbb{C}^m) | \|f\|_\infty^k < \varepsilon \right\}$$
for $\varepsilon > 0$ and $k \in \{0, 1\}$.
- If $\mathbb{V}$ is a vector bundle over a manifold M and K is a compact subset of M then write $\Gamma(\mathbb{V}|K)$ for the space of germs of vector fields along K. If K is a compact subset of $\mathbb{C}^m$ then we write $\mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|K)$, for the space of germs of complex analytic $\mathbb{C}^m$-valued functions along K. If $f \colon U \to \mathbb{C}^m$ is a complex analytic map we write $[f]_K$ for the germ of f along K.

[5] This follows easily from [DS15, A7 and A8].

- If $U \subseteq \mathbb{C}^m$ is open, then we define

$$\mathrm{Hol}_b^k(U;\mathbb{C}^m)^{\mathbb{R}} := \left\{ f \in \mathrm{Hol}_b^k(U;\mathbb{C}^m) : f(U \cap \mathbb{R}^m) \subseteq \mathbb{R}^m \right\}.$$

  Analogously, we define $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)^{\mathbb{R}}$.

Definition 1.18. (a) We call a vector field $\eta \in \Gamma^{\omega}(TM)$ *stratified* if

$$p \in \partial^j M \Rightarrow \eta(p) \in T_p\partial^j M$$

for all $p \in M$ and write $\Gamma^{\omega}_{\mathrm{st}}(TM)$ for the subspace of stratified vector fields of M. (Cf. [Mic80, p. 107]).

(b) A map $\eta\colon U \to \mathbb{R}^m$ defined on a subset $U \subseteq \mathbb{R}^m_k$ is called *stratified* if

$$x_j = 0 \Rightarrow \eta(x)_j = 0$$

for all $x \in U$ and $j = 1, \dots, k$. With respect to the canonical identification this definition coincides with the one of (a). If U is open in $\mathbb{R}^m_k$, we write $C^{\omega}(U;\mathbb{R}^m)_{\mathrm{st}}$ for the subspace of stratified real analytic maps. (Cf. [Gor13, Definition 4.0.7])

(c) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq \mathbb{K}^m$ be open and $B^k_r(0) \subseteq U$. A $\mathbb{K}$-analytic map $f\colon U \to \mathbb{K}^m$ is called *stratified along* $B^k_r(0)$ if

$$x_j = 0 \Rightarrow \eta(x)_j = 0$$

for all $x \in B^k_r(0)$ and $j = 1, \dots, k$. We write $C^{\omega}(U;\mathbb{R}^m)_{\mathrm{st}}$ (in the case $\mathbb{K} = \mathbb{R}$) and $\mathrm{Hol}(U;\mathbb{C}^m)_{\mathrm{st}}$ (in the case $\mathbb{K} = \mathbb{C}$) for the subspaces of stratified maps along $B^k_r(0)$ on U.

(d) A germ $[f] \in \mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^k_r(0))$ is called *stratified* if one and hence all representatives are stratified along $B^k_r(0)$. We write $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^k_r(0))_{\mathrm{st}}$ for the subspaces of stratified germs along $B^k_r(0)$. Analogously, we define $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^k_r(0))^{\mathbb{R}}_{\mathrm{st}}$.

Remark 1.19. *A section $\eta \in \Gamma^{\omega}(TM)$ is stratified if and only if for all $i \in \{1, \dots, n\}$ there exists $R \in [1,5]$ such that $\eta_{(i)}\colon B^{k_i}_R(0) \to \mathbb{R}^m$ is stratified.*

Topological considerations

In this subsection, we elaborate on topological foundations for constructions in the sequel. As we want to model $\mathrm{Diff}^{\omega}(M)$ over the space of stratified real analytic vector fields, the so called Silva spaces play an important role.

In the following definition we recall the definition of a Silva space for the convenience of the reader from [Glo11, p. 260] respectively [Les85]:

Definition 1.20. A locally convex space is called a *Silva space* if it is the direct limit of Banach spaces in the category of locally convex spaces over the index set $\mathbb{N}$ such that the transition maps are injective compact operators.

The facts from the following lemma about Silva spaces are direct consequences of [Glo11, p. 261, Proposition 4.5]. See also [Les85] and [Flo71].

Lemma 1.21. *Let E be a Silva space over an inductive system $(E_i, T_{i,j})$ with the canonical morphisms $\varphi_i\colon E_i \to E$. Then the following holds:*

(a) *The space E is Hausdorff.*

(b) *The topology on E coincides with the inductive limit topology in the category of topological spaces.*

(c) *Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of open sets $U_i \subseteq E_i$ such that $(\varphi_i(U_i))$ is an ascending sequence of subsets of E. Then $U := \bigcup_{i\in\mathbb{N}} U_i \subseteq E$ is open in E. Moreover, a map $f\colon U \to F$ to a Hausdorff locally convex space F is smooth if $f \circ \varphi_n\colon U_n \to F$ is smooth for all $n \in \mathbb{N}$.*

Remark 1.22. *Using Lemma 1.21 (b), we can conclude that a closed subspace F of E is a Silva space over the inductive system $(F_i, T_{i,j}|_{F_i})$ with $F_i := \varphi_i^{-1}(F)$.*[6]

As in the proof of [DS15, Appendix A.7], we use the Cauchy integral formula, in the following lemma, to obtain an upper bound of the derivative of a complex analytic function.

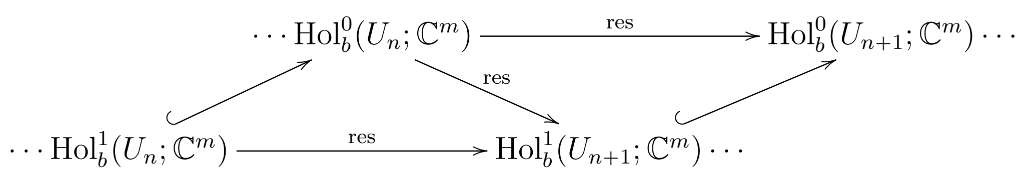
Lemma 1.23. *If U and V are open subsets of $\mathbb{C}^m$ such that V is relatively compact and $V \subseteq \overline{V} \subseteq U$ then the map $\mathrm{res}\colon \mathrm{Hol}_b^0(U;\mathbb{C}^m) \to \mathrm{Hol}_b^1(V;\mathbb{C}^m)$, $f \mapsto f|_V$ is continuous and linear.*

Proof. Let $\mu\colon \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$, $(x,y) \mapsto x + y$ be the addition on $\mathbb{C}^m$. Now $\overline{V} \times \{0\} \subseteq \mu^{-1}(U)$. With Wallace's Lemma we can find $\varepsilon > 0$ with $\overline{V} + \overline{B}_\varepsilon^{\mathbb{C}}(0) \subseteq U$. Hence, for all $p \in V$ we have $\overline{B}_\varepsilon(p) \subseteq U$. If $v \in \mathbb{C}^m$ with $\|v\| = 1$ we can use the Cauchy integral formula and we obtain

$$\|f'(p)(v)\| \leqslant \frac{2}{\varepsilon} \cdot \sup_{q \in \overline{B}_\varepsilon^{\mathbb{C}}(p)} \|f(q)\| \leqslant \frac{2}{\varepsilon} \cdot \|f\|_\infty.$$

Hence, $\|f'\|_\infty^V \leqslant \frac{2}{\varepsilon} \cdot \|f\|_\infty^U$. □

Definition 1.24. Let K be a connected compact subset of $\mathbb{C}^m$ and $(U_n)_{n\in\mathbb{N}}$ a connected fundamental sequence of K. As in [DS15, Appendix A.10] we give $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)$ the direct limit topology induced by the inductive system $\mathrm{Hol}_b^0(U_n;\mathbb{C}^m) \to \mathrm{Hol}_b^0(U_{n+1};\mathbb{C}^m)$, $f \mapsto f|_{U_{n+1}}$ in the category of locally convex spaces. The following commutative diagram



[6] In fact let $X = \bigcup_{i\in\mathbb{N}} X_i$ be an ascending sequence of topological spaces. We give X the inductive limit topology in the category of topological spaces. If $Y \subseteq X$ is closed we write $\mathcal{O}_i$ for the induced topology of X on Y and $\mathcal{O}_l$ for the inductive limit topology of the system $Y = \bigcup_{i\in\mathbb{N}}(X_i \cap Y)$. The map $\mathrm{id}\colon (Y,\mathcal{O}_l) \to (Y,\mathcal{O}_i)$ is obviously continuous and bijective. It is also a closed map because Y is closed.

implies that $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)$ is the direct limit of the inductive system $\mathrm{Hol}^1_b(U_n;\mathbb{C}^m) \to \mathrm{Hol}^1_b(U_{n+1};\mathbb{C}^m)$, $f \mapsto f|_{U_{n+1}}$ in the category of locally convex spaces.

In [DS15, A.10] it was shown that $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)$ is a Silva space realised as the inductive limit $\varinjlim \mathrm{Hol}^0_b(U_n;\mathbb{C}^m)$. In the following we show that $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)$ also is a Silva space realised as the inductive limit $\varinjlim \mathrm{Hol}^1_b(U_n;\mathbb{C}^m)$.

Lemma 1.25. *Let U and V be open subsets of $\mathbb{C}^m$ such that V is relatively compact and $V \subseteq \overline{V} \subseteq U$. Then the restriction $T_1\colon \mathrm{Hol}^1_b(U;\mathbb{C}^m) \to \mathrm{Hol}^1_b(V;\mathbb{C}^m)$, $\eta \mapsto \eta|_V$ is a compact operator. Hence, if $K \subseteq \mathbb{C}^m$ is connected and compact with a connected fundamental sequence $(U_n)_{n\in\mathbb{N}}$ then the space $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|K)$ is a Silva space realised as the inductive limit $\varinjlim \mathrm{Hol}^1_b(U_n;\mathbb{C}^m)$. Moreover, $\mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^k_r(0))^{\mathbb{R}}_{\mathrm{st}}$ is a Silva space realised as the inductive limit $\varinjlim \mathrm{Hol}^1_b(U_n;\mathbb{C}^m)^{\mathbb{R}}_{\mathrm{st}}$ (see Remark 1.22).*

Proof. We choose an open relatively compact subset W of $\mathbb{C}^m$ such that $\overline{V} \subseteq W \subseteq \overline{W} \subseteq U$. From [KM90, Theorem 3.4] and [DS15, A. 10] we deduce that the map $T_0\colon \mathrm{Hol}^0_b(U;\mathbb{C}^m) \to \mathrm{Hol}^0_b(W;\mathbb{C}^m)$, $\eta \mapsto \eta|_W$ is a compact operator. Now we consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hol}^1_b(U;\mathbb{C}^m) & \xrightarrow{T_1} & \mathrm{Hol}^1_b(V;\mathbb{C}^m) \\ \big\downarrow & & \big\uparrow \text{res} \\ \mathrm{Hol}^0_b(U;\mathbb{C}^m) & \xrightarrow{T_0} & \mathrm{Hol}^0_b(W;\mathbb{C}^m) \end{array}$$

where the second vertical arrow is the restriction to V. Because T_0 is compact and both vertical arrows are continuous and linear, T_1 is also compact. □

Definition 1.26. (a) We put a topology on the germs of vector fields around a compact set in the same way as Kriegl and Michor (see [KM90]) or Dahmen and Schmeding (see [DS15]). Hence, we give $\Gamma^\omega(T\tilde{M}|M)$ the subspace topology with respect to $\Gamma^\omega(T\tilde{M}|M)_{\mathbb{C}} = \Gamma^\infty_{\mathbb{C}}(TM_{\mathbb{C}}|M)$. With help of the bijection $\Gamma^\omega(TM) \to \Gamma^\omega(T\tilde{M}|M)$, $\eta \mapsto [\tilde{\eta}]$ we turn $\Gamma^\omega(TM)$ into a locally convex space. Therefore, the closed subspace $\Gamma^\omega_{\mathrm{st}}(TM)$ becomes a locally convex space. Given $R \in [1,6[$ we use [DS15, Lemma A.16] and see that

$$\Gamma^\omega_{\mathrm{st}}(TM) \to \prod_{i=1}^n \mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^{k_i}_R(0))^{\mathbb{R}}_{\mathrm{st}},\ \eta \mapsto [\eta^*_{(i)}]$$

is a linear topological embedding with closed image.

(b) Given $\varepsilon > 0$, $r \in [1,6[$ and $k \in \{0,1\}$ we write

$$\mathcal{B}^k_{r,\varepsilon} := \left\{\eta \in \Gamma^\omega_{\mathrm{st}}(TM) : (\forall i \in \{1,\ldots,n\})\ \|\eta_{(i)}\|^k_{\overline{B}^{k_i}_r(0)} < \varepsilon\right\}.$$

Lemma 1.27. *If $r \in]0,6[$, $\varepsilon > 0$ and $f\colon B^k_6(0) \to \mathbb{R}^m$ is a real analytic map with $\|f\|^1_{\overline{B}^k_r(0)} < \varepsilon$ then there exists a complex analytic map $f^*\colon U \to \mathbb{C}^m$ with $\|f^*\|^1_\infty < \varepsilon$ on an open subset $U \subseteq \mathbb{C}^m$ with $\overline{B}^k_r(0) \subseteq U$ such that $f^*|_{\overline{B}^k_r(0)} = f|_{\overline{B}^k_r(0)}$.*

Proof. We define the real analytic map

$$\varphi\colon \overline{B}_r^k(0) \to B_\varepsilon^{\mathbb{C}^m}(0) \times B_\varepsilon^{\mathcal{L}(\mathbb{C}^m)}(0),\ x \mapsto (f(x), f'(x)),$$

where we consider $\overline{B}_r^k(0)$ as a real analytic manifold with corners. We find a connected open neighbourhood $U \subseteq \mathbb{C}^m$ of $\overline{B}_r^k(0)$ and an extension $\varphi^*\colon U \to B_\varepsilon^{\mathbb{C}^m}(0) \times B_\varepsilon^{\mathcal{L}(\mathbb{C}^m)}(0)$ of φ. If $x \in B_r^{k_i}(0)$ and $v \in \mathbb{R}^m$, we get $\varphi_1^{*\prime}(x)(v) = f'(x)(v) = \varphi_2^*(x)(v)$ ([GN]). Using the linearity over $\mathbb{C}$ we conclude that $\varphi_1^{*\prime}(x) = \varphi_2^*(x)$ for $x \in B_r^{k_i}(0)$. Hence $\varphi_1^{*\prime} = \varphi_2^*$. Therefore, φ_1^* is an extension of f as needed. □

The sets $\mathcal{B}_{r,\varepsilon}^k$ in the following lemma will be the domains of definition of the chart around the identity of $\mathrm{Diff}^\omega(M)$ in the next section.

Lemma 1.28. *Given $\varepsilon > 0$, $r \in [1, 6[$ and $k \in \{0, 1\}$, the set $\mathcal{B}_{r,\varepsilon}^k$ is an open 0-neighbourhood in $\Gamma_{\mathrm{st}}^\omega(TM)$.*

Proof. For $i = 1, \ldots, n$, let $(U_n^i)_{n\in\mathbb{N}}$ be a connected fundamental sequence of $\overline{B}_r^{k_i}(0)$ in $\mathbb{C}^m$. Then

$$\mathcal{U}^i := \mathcal{G}_\varepsilon^k(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_r^{k_i}(0)) := \bigcup_{n\in\mathbb{N}} \left[\mathrm{Hol}_\varepsilon^k(U_n^i; \mathbb{C}^m)\right]_{\overline{B}_r^{k_i}(0)}$$

is open in $\mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_r^{k_i}(0))$ because the right-hand side is an ascending union. Hence, the set

$$(\dagger) := \left\{\eta \in \Gamma_{\mathrm{st}}^\omega(TM) : (\forall i)\ [\eta_{(i)}^*] \in \mathcal{U}^i\right\}$$

is open in $\Gamma_{\mathrm{st}}^\omega(TM)$. Using Lemma 1.27, we find:

$$\begin{aligned}(\dagger) &= \left\{\eta \in \Gamma_{\mathrm{st}}^\omega(TM) : (\forall i)\ (\exists n \in \mathbb{N})\ [(\eta^*)_{(i)}] = [(\eta_{(i)})^*] \in \left[\mathrm{Hol}_\varepsilon^k(U_n^i; \mathbb{C}^m)\right]\right\}\\ &= \left\{\eta \in \Gamma_{\mathrm{st}}^\omega(TM) : (\forall i)\ (\exists n \in \mathbb{N})\ \eta_{(i)} \text{ has an extension } \eta_{(i)}^* \in \mathrm{Hol}_\varepsilon^k(U_n^i; \mathbb{C}^m)\right\}\\ &= \left\{\eta \in \Gamma_{\mathrm{st}}^\omega(TM) : (\forall i)\ \|\eta_{(i)}\|_{\overline{B}_r^{k_i}(0)}^k < \varepsilon\right\} = \mathcal{B}_{r,\varepsilon}^k.\end{aligned}$$

□

A local chart around the identity

In order to apply the theorem about the local description of Lie groups in the sequel, we now endow a special subset of $\mathrm{Diff}^\omega(M)$ that contains the identity with a manifold structure.

The following remark transfers considerations from [Glo06c, p. 11] to the real analytic case.

Remark 1.29. *Obviously we have*

$$\exp_i(T\tilde{\varphi}_i(v)) = \tilde{\varphi}_i(\exp(v))$$

for all $v \in T\tilde{\varphi}_i^{-1}(\tilde{\Omega}_i)$ *and*

$$\exp_i(x, v) = \tilde{\varphi}_i(\exp(T\tilde{\varphi}_i^{-1}(x, v)))$$

for $(x, v) \in \tilde{\Omega}_i$. *Moreover, we have*

$$\exp_i(x, 0) = x \text{ and } d_2 \exp_i(x, 0; \bullet) = \mathrm{id}_{\mathbb{R}^m}$$

for all $x \in B_6(0)$.

Weakening [Glo06a, Theorem 2.3] to our situation, the analogous statement to [Glo06c, Proposition 3.1] in the analytic case is:

Lemma 1.30. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $P \subseteq \mathbb{K}^n$ *and* $U \subseteq \mathbb{K}^m$ *be open and* $f\colon P \times U \to \mathbb{K}^m$ *be a* $\mathbb{K}$*-analytic map. Moreover, let* $(x_0, y_0) \in P \times U$ *and* $d_2 f(x_0, y_0; \bullet) \in \mathrm{GL}(\mathbb{K}^m)$. *There exists an open* y_0*-neighbourhood* $U' \subseteq U$ *and an open* x_0*-neighbourhood* $P' \subseteq P$ *such that:*

- *For all* $x \in P'$ *the map* $f(x, \bullet)\colon U' \to \mathbb{K}^m$ *has open image and is a* $\mathbb{K}$*-analytic diffeomorphism onto its image;*
- *The set* $W := \bigcup_{x \in P'} \{x\} \times f(x, U')$ *is open in* $\mathbb{K}^n \times \mathbb{K}^m$ *and the map* $P' \times U' \to W$, $(x, y) \mapsto (x, f(x, y))$ *is a* $\mathbb{K}$*-analytic diffeomorphism with inverse function* $W \to P' \times U', (x, z) \mapsto (x, f(x, \bullet)^{-1}(z))$;
- *There exists* $\delta > 0$ *such that for all* $x \in P'$ *we have* $B_\delta(f(x, y_0)) \subseteq f(x, U')$ *and* $W' := \bigcup_{x \in P'} \{x\} \times B_\delta(f(x, y_0)) \subseteq W$ *is open.*

The analogous statement to [Glo06c, 3.2] in the real analytic case is:

Lemma 1.31. *There exists an* $\varepsilon_{\exp} > 0$ *such that:*

(a) We have $\overline{B}_5(0) \times \overline{B}_{\varepsilon_{\exp}}(0) \subseteq \tilde{\Omega}_i \subseteq B_6(0) \times \mathbb{R}^m$ *for all* $i \in \{1, ..., n\}$;

(b) For all $x \in \overline{B}_5(0)$ *and* $i \in \{1, ..., n\}$ *the map* $\exp_{i,x} := \exp_i(x, \bullet)\colon B_{\varepsilon_{\exp}}(0) \to \mathbb{R}^m$ *has open image and is a real analytic diffeomorphism onto its image. Moreover, the map* $B_5(0) \times B_{\varepsilon_{\exp}}(0) \to B_5(0) \times \mathbb{R}^m$, $(x, y) \mapsto (x, \exp_i(x, y))$ *has open image and is a real analytic diffeomorphism onto its image.*

Proof. Let $i \in \{1, ..., n\}$. Given $x \in B_6(0)$ we use Lemma 1.30 to find $r_x > 0$ and $\varepsilon_x > 0$ such that $\overline{B}_{r_x}(x) \times \overline{B}_{\varepsilon_x}(0) \subseteq \tilde{\Omega}_i$ and $\exp_i(y, \bullet)\colon B_{\varepsilon_x}(0) \to B_6(0)$ has open image and is a real analytic diffeomorphism onto its image for all $y \in B_{r_x}(x)$ and $B_{r_x}(x) \times B_{\varepsilon_x}(0) \to B_{r_x}(x) \times \mathbb{R}^m$, $(y, v) \mapsto (y, \exp_i(y, v))$ is a real analytic diffeomorphism onto its image. We find finitely many $x_1, \ldots, x_k \in B_6(0)$ such that $\overline{B}_5(0) \subseteq \bigcup_{j=1}^k B_{r_{x_j}}(x_j)$ and set $\varepsilon_{\exp}^i := \min_j \varepsilon_{x_j} > 0$. Now we set $\varepsilon_{\exp} := \min_{i=1,\ldots,n} \varepsilon_{\exp}^i$. Given $i \in \{1, ..., n\}$ and $y \in \overline{B}_5(0)$ we find j such that $y \in B_{r_{x_j}}(x_j)$. Hence, $\{y\} \times \overline{B}_{\varepsilon_{\exp}}(0) \subseteq \tilde{\Omega}_i$ and $\exp_i(y, \bullet)\colon B_{\varepsilon_{\exp}}(0) \to B_5(0)$ is a real analytic diffeomorphism onto its open image. Moreover, the map $B_5(0) \times B_{\varepsilon_{\exp}}(0) \to B_5(0) \times \mathbb{R}^m$, $(x, y) \mapsto (x, \exp_i(x, y))$ is injective and a local diffeomorphism. Hence, it has open image and is a real analytic diffeomorphism onto its image. Therefore, we find $\varepsilon_{\exp}$ as needed. □

Remark 1.32. *Using Remark 1.29, we make the following observation: For all* $i \in \{1, \ldots, n\}$, $x \in B_5(0)$ *and* $w \in B_{\varepsilon_{\exp}}(0)$ *we have* $\exp(T\tilde{\varphi}_i^{-1}(x, w)) \in U_{i,6}$ *and* $\exp(T\tilde{\varphi}_i^{-1}(x, w)) = \tilde{\varphi}_i^{-1}(\exp_i(x, w))$.

In the following definition, we construct a map $M \to \tilde{M}$ from a vector field with the help of the Riemannian metric in the usual way (cf., e.g., [Glo06c, 4.8]).

Definition 1.33. (a) Let $r \in [1, 5]$. If $\eta \in \mathcal{B}^0_{r,\varepsilon_{\exp}}$ then $\operatorname{im}(\eta) \subseteq \tilde{\Omega}$ because $(x, \eta_{(i)}(x)) \in \tilde{\Omega}_i$ for $x \in \overline{B}^{k_i}_r(0)$. In this situation, we define the real analytic map

$$\psi_\eta \colon M \to \tilde{M},\ p \mapsto \exp(\eta(p)).$$

(b) For $i \in \{1, \ldots, n\}$, $U \subseteq B^{k_i}_5(0)$ open and $\eta \in C^\omega(U; \mathbb{R}^m)_{\mathrm{st}}$ with $\|\eta\|^U_\infty < \varepsilon_{\exp}$ we define the real analytic map

$$\psi^i_\eta \colon U \to B_6(0),\ x \mapsto \exp_i(x, \eta(x)).$$

Lemma 1.34. *For* $r \in [1, 5]$ *and* $\eta \in \mathcal{B}^0_{r,\varepsilon_{\exp}}$ *we get* $\psi_\eta(U_{i,r}) \subseteq \tilde{U}_{i,6}$ *and*

$$\psi_\eta|_{U_{i,r}} = \tilde{\varphi}_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \varphi_i|_{U_{i,r}}.$$

Proof. Given $p \in U_{i,r}$ we use Remark 1.32 and calculate

$$\begin{aligned}\psi_\eta(p) &= \psi_\eta(\varphi_i^{-1}(\varphi_i(p))) = \exp(T\varphi_i^{-1} \circ T\varphi_i \circ \eta \circ \varphi_i^{-1} \circ \varphi_i(p))) \\ &= \exp(T\varphi_i^{-1}(\varphi_i(p), \eta_{(i)}(\varphi_i(p))) = \tilde{\varphi}_i^{-1}(\exp_i(\varphi_i(p), \eta_{(i)}(\varphi_i(p)))) = \tilde{\varphi}_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \varphi_i(p)\end{aligned}$$

□

As is [Glo06c] the next step is to choose the vector fields $\eta \in \Gamma^\omega_{st.}(TM)$ small enough, so that $\psi_\eta(M) \subseteq M$ and $\psi_\eta \in \mathrm{Diff}^\omega(M)$.

Remark 1.35. *If* $A \in \mathcal{L}(\mathbb{R}^m)$ *then* $(\mathrm{id}, A) \colon \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ *is linear with* $\|(\mathrm{id}, A)\|_{op} \leqslant 1 + \|A\|_{op}$.

The following lemma is a stronger version of [Glo06c, Lemma 3.7] in the case of open sets with corners and with a variable radius and control of the norms. In order to control the distance of ψ_η to the identity we have to choose η small enough.

Lemma 1.36. *Given* $R \in]0, 5]$, $l \in]0, R[$ *and* $r \in]0, 1[$, *we find* $\varepsilon \in]0, \varepsilon_{\exp}]$ *such that for all* $\eta \in C^\omega(B^{k_i}_R(0); \mathbb{R}^m)_{\mathrm{st}}$ *with* $\|\eta\|^1_{B^{k_i}_l(0)} < \varepsilon$ *and* $i \in \{1, \ldots, n\}$ *the following assertions hold:*

(a) $\|{\psi^i_\eta}'(x) - \mathrm{id}_{\mathbb{R}^m}\|_{op} < r$ *for all* $x \in B^{k_i}_l(0)$;

(b) $\|\psi^i_\eta(x) - x\| < r$ *for all* $x \in B^{k_i}_l(0)$.

Proof. Obviously it is enough to show the lemma for a fixed $i \in \{1, \ldots, n\}$. Hence, let $i \in \{1, \ldots, n\}$ be fixed for the rest of the proof. As in [Glo06c, Lemma 3.7] we define $H\colon B_5(0)\times B_{\varepsilon_{\exp}}(0)\to\mathbb{R}^m$, $(x,y)\mapsto \exp_i(x,y)-x-y$ and $h\colon B_5(0)\times B_{\varepsilon_{\exp}}(0)\to[0,\infty[$, $(x,y)\mapsto\|H'(x,y)\|_{op}$. For all $x\in B_5(0)$ we get $d_1H(x,0;\bullet)=0$ and $d_2H(x,0;\bullet)=0$ and so $H'(x,0)=dH(x,0;\bullet)=0$ in $\mathcal{L}(\mathbb{R}^m\times\mathbb{R}^m;\mathbb{R}^m)$. Hence, $\overline{B}_l(0)\times\{0\}\subseteq h^{-1}([0,\frac{r}{r+10}[)$ and with Wallace's Lemma we can find $\varepsilon\in]0,\min(\varepsilon_{\exp},\frac{r}{2})[$ such that $\|H'(x,y)\|_{op}<\frac{r}{r+10}$ for all $x\in B_l(0)$ and $y\in B_\varepsilon(0)$. Now let $\eta\in C^\omega(B_R^{k_i}(0),\mathbb{R}^m)_{\mathrm{st}}$ with $\|\eta\|^1_{B_l^{k_i}(0)}<\varepsilon$.

(a) We have

$$\psi^i_\eta(x)=H(x,\eta(x))+x+\eta(x) \tag{1.1}$$

for all $x\in B_l^{k_i}(0)$. Hence,

$${\psi^i_\eta}'(x)=H'(x,\eta(x);\bullet)\circ(\mathrm{id}_{\mathbb{R}^m},\eta'(x))+\mathrm{id}_{\mathbb{R}^m}+\eta'(x)$$

for all $x\in B_l^{k_i}(0)$. Using Remark 1.35, we see that

$$\begin{aligned}\|{\psi^i_\eta}'(x)-\mathrm{id}_{\mathbb{R}^m}\|_{op}&\leqslant\|H'(x,\eta(x))\|_{op}\cdot\|(\mathrm{id}_{\mathbb{R}^m},\eta'(x))\|_{op}+\|\eta'(x)\|_{op}\\ <&\frac{r}{r+10}\cdot(1+\varepsilon)+\varepsilon\leqslant\frac{r}{r+2}\cdot\left(1+\frac{r}{2}\right)+\frac{r}{2}=r.\end{aligned} \tag{1.2}$$

(b) Let $x\in B_l(0)$ and $y\in B_\varepsilon(0)$. Then $\|(x,y)\|\leqslant\|x\|+\|y\|<l+\varepsilon\leqslant 5+\frac{r}{2}$. Hence,

$$\begin{aligned}\|H(x,y)\|&=\|H(x,y)-H(0,0)\|=\left\|\int_0^1 dH(tx,ty;x,y)dt\right\|\\ \leqslant&\int_0^1\|H'(tx,ty)\|\cdot\|(x,y)\|dt<\frac{r}{r+10}\cdot\|(x,y)\|\leqslant\frac{r}{2}.\end{aligned}$$

Thus, given $x\in B_l^{k_i}(0)$, (1.1) implies

$$\|\psi^i_\eta(x)-x\|\leqslant\|H(x,\eta(x))\|+\|\eta(x)\|<r. \tag{1.3}$$

□

As in [Glo06c, Lemma 3.7] we will use the following well known fact:

Remark 1.37. *Let $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $U\subseteq\mathbb{K}^m$ be open and convex and $f\colon U\to\mathbb{K}^m$ be $\mathbb{K}$-analytic with $\|df(x)-\mathrm{id}_{\mathbb{K}^m}\|_{op}<1$ for all $x\in U$. In this situation, f is injective and hence, f has open image and is a diffeomorphism onto its image: Let $x\neq y\in U$. Since U is convex we can define*

$$\tau\colon[0,1]\to\mathbb{C}^m,\ t\mapsto df((1-t)x+ty;y-x)-(y-x).$$

We get $\|\tau(t)\|<\|y-x\|$ for $t\in[0,1]$ and so $\|\int_0^1\tau(t)dt\|<\|y-x\|$. With $f(y)-f(x)=y-x+\int_0^1\tau(t)dt$ we get $f(y)\neq f(x)$.

Lemma 1.38. *There exists an $\varepsilon\in]0,\varepsilon_{\exp}[$ such that for all $\eta\in\mathcal{B}^1_{1,\varepsilon}$ the map $\psi_\eta\colon M\to\tilde{M}$ is injective.*

Proof. Because of Lemma 1.36, Remark 1.37 and Remark 1.32, we can find $\varepsilon_1 \in]0, \varepsilon_{\exp}[$ such that $\psi_\eta|_{U_{l,1}} \colon U_{l,1} \to \tilde{M}$ is injective for all $\eta \in \mathcal{B}^1_{1,\varepsilon_1}$ and $l \in \{1, \dots, n\}$[7]. Similar to [Glo06c, 4.10] one can show that given $i, j \in \{1, \dots, n\}$ we can find $\varepsilon_{i,j} \in]0, \varepsilon_1[$ such that ψ_η is injective on $U_{i,1} \cup U_{j,1}$ for all $\eta \in \mathcal{B}^1_{1,\varepsilon_{i,j}}$: Suppose the opposite. Then there exist sequences $(\eta^k)_{k\in\mathbb{N}}$ in $\Gamma^\omega_{\mathrm{st}}(TM)$, $(p_k)_{k\in\mathbb{N}}$ in $U_{i,1}$ and $(q_k)_{k\in\mathbb{N}}$ in $U_{j,1}$ such that for all $k \in \mathbb{N}$ we have $\eta^k \in \mathcal{B}^1_{1,\frac{1}{k}}$, $p_k \neq q_k$ and $\psi_{\eta^k}(p_k) = \psi_{\eta^k}(q_k)$. Because $\overline{U_{i,1}}$ and $\overline{U_{j,1}}$ are compact, we can assume without loss of generality that there exists $p \in \overline{U_{i,1}}$ and $q \in \overline{U_{j,1}}$ such that $p_k \to p$ and $q_k \to q$. Hence, $\eta^k(p_k) \to 0_p$ and $\eta^k(q_k) \to 0_q$ in TM. Therefore, $\psi_{\eta^k}(p_k) \to p$ and $\psi_{\eta^k}(q_k) \to q$. Thus $p = q$. There exists $l \in \{1, \dots, n\}$ such that $p = q \in U_{l,1}$. Hence, there exists k_0 such that $p_{k_0}, q_{k_0} \in U_{l,1}$. The map $\psi_{\eta^{k_0}} \colon U_{l,1} \to M$ is injective. But $p_{k_0} \neq q_{k_0}$ and $\psi_{\eta^{k_0}}(p_{k_0}) = \psi_{\eta^{k_0}}(q_{k_0})$ which is a contradiction. Now $\varepsilon := \min\{\varepsilon_{i,j} : i, j \in \{1, \dots, n\}\}$ is as required. □

In order to show that the functions ψ_η map M to M we need the following definition.

Definition 1.39. For $j \in \{0, \dots, m\}$ and each connected component C of $\partial^j M$ we fix a point $p^j_C \in C$. The submanifold $\partial^j M$ is totally geodesic in $\tilde{M}$. Hence, for fixed $p^j_C \in \partial^j M$ there exists an open neighbourhood U of $0_{p^j_C}$ in $T_{p^j_C} \partial^j M$ such that $\exp_{p^j_C}(U) \subseteq C$ and $\exp_{p^j_C} \colon U \to C$ is continuous. Thus, we find $\varepsilon^j_C \in]0, \varepsilon_{\exp}[$ such that for all $\eta \in \mathcal{B}^0_{1,\varepsilon^j_C}$ we have $\psi_\eta(p^j_C) \in C$.

Remark 1.40. *Let $j \in \{0, \dots, m\}$ and $v \in T_p\partial^j M$ with $[0,1]v \subseteq \tilde{\Omega}$ and $\exp([0,1]v) \subseteq \partial^j M$. We consider the curve $\gamma \colon [0,1] \to \partial^j M$, $t \mapsto \exp(tv)$. Because $\partial^j M$ is totally geodesic, we see that γ is also a geodesic for $\partial^j M$. Hence, v lies in the domain of definition of the exponential map $\exp_{\partial^j M}$ of $\partial^j M$ and $\exp_{\partial^j M}(v) = \exp(v)$.*

Now we can show that the images of the maps ψ_η lie in M.

Lemma 1.41. *There exists $\varepsilon \in]0, \varepsilon_{\exp}[$ such that for all $\eta \in \mathcal{B}^1_{1,\varepsilon}$ the map ψ_η is injective and $\psi_\eta(\partial^j M) = \partial^j M$. Moreover, if C is a connected component of $\partial^j M$ then $\psi_\eta(C) = C$.*

Proof. In this proof we use the following special notation: We write $\exp_{\tilde{M}} \colon \Omega_{\tilde{M}} \to \tilde{M}$ for the exponential map on $\tilde{M}$ and $\exp_{\partial^j M} \colon \Omega_{\partial^j M} \to \partial^j M$ for the exponential map on $\partial^j M$ for $j \in \{0, \dots, m\}$. We use Lemma 1.38 and Definition 1.39 to choose $\varepsilon > 0$ such that for all $\eta \in \mathcal{B}^1_{1,\varepsilon}$ the map $\psi_\eta \colon M \to \tilde{M}$ is injective and $\psi_\eta(p^j_C) \in C$ for all $j \in \{0, \dots, m\}$ and all connected components $C \subseteq \partial^j M$. Note that the strata of M have only finitely many connected components because M is compact. Now we show by induction over j from m to 0 that $\psi_\eta(\partial^j M) = \partial^j M$ for all $\eta \in \mathcal{B}^1_{1,\varepsilon}$. The case $j = m$ is clear. For the inductive step, let C be a

[7] We can find a real analytic extension f of $\psi^l_{\eta(l)}$ on an open convex neighbourhood $U \subseteq \mathbb{R}^m$ of $B^{k_l}_1(0)$ such that $\|df(x) - \mathrm{id}\|_{op} < 1$ for all $x \in U$.

connected component of $\partial^j M$, $\eta \in \mathcal{B}^1_{1,\varepsilon}$ and $Z := \{x \in C : (\forall t \in [0,1])\psi_{t\eta}(x) \in C\}$. Because $p^j_C \in Z$ we get $Z \neq \emptyset$. Now let $p \in Z$. We have $[0,1]\eta|_C(p) \subseteq \Omega_{\tilde{M}}$ and $\exp_{\tilde{M}}(t\eta|_C(p)) \in C$ for all $t \in [0,1]$. Remark 1.40 implies $\eta|_C(p) \in \Omega_{\partial^j M}$ and $\exp_{\partial^j M}(\eta|C(p)) = \exp_{\tilde{M}}(\eta|_C(p)) \in C$. Hence, there exists a p-neighbourhood $V \subseteq C$ such that $\exp_{\partial^j M} \circ \eta|_C(V) \subseteq C$. Let $q \in V$. Because the map $[0,1] \to \partial^j M$, $t \mapsto \exp_{\partial^j M} \circ (t\eta|_C(q))$ makes sense, is continuous and $[0,1]$ is connected, we get $\exp_{\partial^j M}(t\eta|_C(q)) \in C$ for all $t \in [0,1]$. We conclude that Z is open in C. Now let $p \in C \backslash Z$. We find $t \in [0,1]$ with $\psi_{t\eta}(p) \in \tilde{M} \backslash C$. First suppose $\psi_{t\eta}(p) \in \overline{C} \backslash C \subseteq \bigcup_{j<i} \partial^i M$. Because $\psi_{t\eta}$ is injective and $\psi_{t\eta}(\partial^i M) = \partial^i M$ for all $i > j$, we obtain a contradiction. Now suppose $\psi_{t\eta}(p) \in \tilde{M} \backslash \overline{C}$. Then there exists a p-neighbourhood V in M such that $\psi_{t\eta}(V) \subseteq \tilde{M} \backslash \overline{C}$. But $C \cap V$ is a p-neighbourhood in C. Hence, Z is closed. Therefore $Z = C$. We conclude that $\psi_\eta(C) \subseteq C$ and obtain a continuous injective map $\psi_\eta|^C_C \colon C \to C$. From $\psi_\eta(\overline{C} \backslash C) \cap C = \emptyset$, we conclude that $\psi_\eta(C) = \psi_\eta(\overline{C}) \cap C$. Because $\overline{C}$ is compact, we see that $\psi_\eta(C)$ is closed in C. But ψ_η is also an open map because it is injective and continuous (invariance of domain). We conclude that $\psi_\eta(C) = C$. □

The following Lemma 1.42 is a direct consequence of [MO92, Lemma 2.2.3] and a tool in the sequel.

Lemma 1.42. *Let $\tilde{f} \colon \tilde{U} \to \tilde{V}$ be a homeomorphism between open subsets of $\mathbb{R}^m$ and $\tilde{V}$ be convex such that*

$$\tilde{f}(\tilde{U} \cap \partial \mathbb{R}^m_k) \subseteq \tilde{V} \cap \partial \mathbb{R}^m_k \text{ and } \tilde{f}(\tilde{U} \cap \partial^0 \mathbb{R}^m_k) \cap \partial^0 \mathbb{R}^m_k \neq \emptyset;$$

then

$$\tilde{f}(\tilde{U} \backslash \mathbb{R}^m_k) \subseteq \tilde{V} \backslash \mathbb{R}^m_k.$$

In the following lemma we show a qualitative inverse function theorem for open sets with corners in the real analytic case. The proof is analogous to the one of the smooth case see e.g. [MO92, Theorem 2.2.4].

Lemma 1.43. *Let $U \subseteq \mathbb{R}^m_k$ be open, $f \colon U \to \mathbb{R}^m_k$ be a real analytic map and $x_0 \in U$ such that*

$$f(U \cap \partial \mathbb{R}^m_k) \subseteq \partial \mathbb{R}^m_k \text{ and } f'(x_0) \in \mathrm{GL}(\mathbb{R}^m).$$

We can find an open x_0-neighbourhood $U' \subseteq U$ and an open $f(x_0)$-neighbourhood $V \subseteq \mathbb{R}^m_k$ such that $f|^V_{U'} \colon U' \to V$ is a real analytic diffeomorphism.

Proof. Without loss of generality, we can assume that $x_0 \in \partial U := U \cap \partial \mathbb{R}^m_k$ because, otherwise, we can use the standard inverse function theorem. Now let $\tilde{f} \colon \tilde{U} \to \mathbb{R}^m$ be a real analytic extension of f. Without loss of generality we can assume that $\tilde{U} \cap \mathbb{R}^m_k = U$. We have $\tilde{f}|_U = f$ and $\tilde{f}'(x_0) = f'(x_0) \in \mathrm{GL}(\mathbb{R}^m)$. Let $\tilde{U}' \subseteq \tilde{U}$ be a x_0-neighbourhood, $\tilde{V} \subseteq \mathbb{R}^m$ an $f(x_0)$-neighbourhood such that $\tilde{f}|_{\tilde{U}'} \colon \tilde{U}' \to \tilde{V}$ is a real analytic diffeomorphism between open sets of $\mathbb{R}^m$. Without loss of generality we can assume that $\tilde{V}$ is convex. We have

$$\tilde{f}(\tilde{U}' \cap \partial\mathbb{R}^m_k) = f(\tilde{U}' \cap \partial\mathbb{R}^m_k) \subseteq \tilde{V} \cap \partial\mathbb{R}^m_k.$$

On the other hand, $\tilde{f}(\tilde{U}' \cap \partial^0\mathbb{R}^m_k) \subseteq \mathbb{R}^m$ is open in $\mathbb{R}^m$ and non-empty. As it contains $x_0 \in \partial\mathbb{R}^m_k$, it contains a point in $\partial^0\mathbb{R}^m_k$. Lemma 1.42 implies $\tilde{f}(\tilde{U}'\backslash\mathbb{R}^m_k) \subseteq \tilde{V}\backslash\mathbb{R}^m_k$. Hence,

$$\tilde{f}(\tilde{U}' \cap \mathbb{R}^m_k) = \tilde{V} \cap \mathbb{R}^m_k.$$

Now we define $U' := \tilde{U}' \cap \mathbb{R}^m_k \subseteq \tilde{U}'$ and $V := \tilde{V} \cap \mathbb{R}^m_k \subseteq \tilde{V}$. The map $f|^V_{U'} \colon U' \to V$ is bijective real analytic and also $(f|^V_{U'})^{-1} = \tilde{f}^{-1}|_V$ is real analytic. □

Now we come to the first central result in this chapter. If $\eta \in \Gamma^\omega(TM)$ is chosen to be small enough then ψ_η is a diffeomorphism of M.

Theorem 1.44. *There exists $\varepsilon_{diff} \in]0, \varepsilon_{\exp}[$ such that, if $\eta \in \mathcal{B}^1_{1,\varepsilon_{diff}}$ then $\psi_\eta \colon M \to M$, $p \mapsto \psi_\eta(p)$ is a diffeomorphism.*

Proof. We simply use the ε defined in Lemma 1.41. Then ψ_η is a real analytic diffeomorphism because it is bijective and a local real analytic diffeomorphism (see Lemma 1.43). □

The idea of Lemma 1.45 below, is based mainly on [Glo06c, 4.12]. However, since our manifold is compact we can find a single ε and shorten the proof.

Lemma 1.45. *There exists $\varepsilon_{inj} \in]0, \varepsilon_{\exp}[$ such that for all $p \in M$ the map $\exp_p \colon \tilde{\Omega}_p := \tilde{\Omega} \cap \subseteq T_p\tilde{M} \to \tilde{M}$ is injective on*

$$W_pM := \bigcup_{i=1}^{n} T\varphi_i^{-1}(\{\varphi_i(p)\} \times B_{\varepsilon_{inj}}(0)) \subseteq T_pM.$$

Proof. Let $\varepsilon_{inj} \in]0, \varepsilon_{\exp}[$ such that

$$T(\varphi_i \circ \varphi_j^{-1})\left(\{\varphi_j(p)\} \times B_{\varepsilon_{inj}}(0)\right) \subseteq \{\varphi_i(p)\} \times B_{\varepsilon_{\exp}}(0)$$

for all $p \in \overline{U}_{4,i} \cap \overline{U}_{4,j}$ and $i, j \in \{1, \ldots, n\}$ (Wallace's Lemma). For $i \in \{1, \ldots, n\}$ and $p \in M$ let $A'_i := T\varphi_i^{-1}(\{\varphi_i(p)\} \times B_{\varepsilon_{inj}}(0)) \subseteq T_pM$ and $A_i := T\varphi_i^{-1}(\{\varphi_i(p)\} \times B_{\varepsilon_{\exp}}(0)) \subseteq T_pM$. Now let $v, w \in \bigcup_i A'_i$, say $v \in A'_i$ and $w \in A'_j$, with $\exp(v) = \exp(w)$. We know that exp is injective on A_j. But obviously $v, w \in A_j$ by the choice of ε_{inj}. □

In the following we define the canonical local chart around the identity to obtain a local manifold structure (cf. e.g. [Glo06c, 4.13]).

Definition 1.46. We define

$$\varepsilon_{\mathcal{U}} := \min(\varepsilon_{diff}, \varepsilon_{inj}).$$

Moreover, we define $\mathcal{V} := \mathcal{B}^1_{1,\varepsilon_\mathcal{U}} \subseteq \Gamma^\omega_{\text{st}}(TM)$, $\mathcal{U} := \{\psi_\eta : \eta \in \mathcal{V}\} \subseteq \text{Diff}(M)$ and

$$\Psi\colon \mathcal{V} \to \mathcal{U},\ \eta \mapsto \psi_\eta.$$

Given $\alpha \in \mathcal{U}$ we find $\eta \in \mathcal{V}$ with $\alpha = \psi_\eta$. We get $\eta_{(i)}(x) \in B_{\varepsilon_{inj}}(0)$ for all $x \in \overline{B}^{k_i}_1(0)$. Hence $\eta(p) \in W_pM$ for all $p \in U_{i,1}$ and all $i = 1, \dots, n$. Thus $\eta(p) \in W_pM$ for all $p \in M$. Therefore, $\alpha(p) = \psi_\eta(p) = \exp|_{W_pM}(\eta(p)) \in \exp|_{W_pM}(W_pM)$ and $\eta(p) = \exp|^{-1}_{W_pM}(\alpha(p))$. Hence, the map

$$\Phi\colon \mathcal{U} \to \mathcal{V},\ \alpha \mapsto \Phi(\alpha)$$

with $\Phi(\alpha)(p) = \exp|^{-1}_{W_pM}(\alpha(p))$ makes sense and is inverse to Ψ.

1.3. Preparation for results of smoothness

To show the smoothness of the group operations we need some further definitions and results. In particular, we need results concerning extensions of real analytic maps on $B^k_\varepsilon(0)$ to open sets of $\mathbb{C}^m$. In this section we elaborate these foundations.

The following lemma is the standard quantitative inverse function theorem for Lipschitz continuous maps (see [Glo05, Theorem 5.3] and [Wel76]) applied to our setting.

Lemma 1.47. *Let $A\colon \mathbb{R}^m \to \mathbb{R}^m$ be a linear isomorphism, $x_0 \in \mathbb{R}^m$, $r > 0$ and $g\colon B_r(x_0) \to \mathbb{R}^m$ Lipschitz continuous with $\text{Lip}(g) < \frac{1}{\|A^{-1}\|_{op}}$. If we define $a := \frac{1}{\|A^{-1}\|_{op}} - \text{Lip}(g)$, $b := \|A\|_{op} + \text{Lip}(g)$ and $f\colon B_r(x_0) \to \mathbb{R}^m$, $x \mapsto Ax + g(x)$ we have*

$$B_{as}(f(x)) \subseteq f(B_s(x)) \subseteq B_{bs}(f(x))$$

for all $x \in B_r(x_0)$ and $s \in]0, r - \|x - x_0\|]$. Moreover, f has open image and is a homeomorphism onto its image.

In [Gor13] Gorny showed a qualitative inverse function theorem for Lipschitz continuous maps on open sets of half-spaces in Banach spaces ("open sets with boundary" that means the local "boundary"-case). In [Gor13, Remark on p 47] she asks whether there is also a qualitative inverse function theorem for Lipschitz continuous functions on "open sets with corners" (that means the local "corner"-case).

Our Lemma 1.49 is a not only a qualitative but also a quantitative inverse function theorem for Lipschitz continuous maps on open sets with corners in $\mathbb{R}^m$ (the proof can be transferred to the Banach case verbatim by substituting $\mathbb{R}^m$ with a Banach space).

Lemma 1.48. *Let $k \in \{0, \dots, m\}$, $x_0 \in \mathbb{R}^m_k$ and $g\colon B^k_r(x_0) \to \mathbb{R}^m$ be Lipschitz* *continuous with* $\mathrm{Lip}(g) =: L$. *Let* $\|\bullet\|_k\colon B_r(x_0) \to \mathbb{R}^m_k$, $x \mapsto \|x\|_k$ *with*

$$(\|x\|_k)_i = \begin{cases} |x_i| & : i \leqslant k \\ x_i & : \text{otherwise} \end{cases}$$

and $\tilde{g}\colon B_r(x_0) \to \mathbb{R}^m$, $x \mapsto g(\|x\|_k)$. *In this situation* $\tilde{g}|_{B^k_r(x_0)} = g$ *and* $\mathrm{Lip}(\tilde{g}) = \mathrm{Lip}(g)$.

Proof. The map $\|\bullet\|_k\colon B_r(x_0) \to \mathbb{R}^m_k$ is Lipschitz continuous with Lipschitz constant 1. The assertion now follows from $\mathrm{Lip}(\tilde{g}) = \mathrm{Lip}(g \circ \|\bullet\|_k) \leqslant \mathrm{Lip}(g)$. □

Lemma 1.49. *Let* $A\colon \mathbb{R}^m \to \mathbb{R}^m$ *be a linear isomorphism,* $x_0 \in \mathbb{R}^m_k$, $r > 0$ *and* $g\colon B^k_r(x_0) \to \mathbb{R}^m$ *be Lipschitz continuous with* $\mathrm{Lip}(g) < \frac{1}{\|A^{-1}\|_{op}}$. *We define* $a := \frac{1}{\|A^{-1}\|_{op}} - \mathrm{Lip}(g)$, $b := \|A\|_{op} + \mathrm{Lip}(g)$ *and* $f\colon B^k_r(x_0) \to \mathbb{R}^m$, $x \mapsto Ax + g(x)$. *If* $f(\partial B^k_r(x_0)) \subseteq \partial\mathbb{R}^m_k$ *and* $f(B^k_r(x_0)) \subseteq \mathbb{R}^m_k$ *then we have*

$$B^k_{as}(f(x)) \subseteq f(B^k_s(x)) \subseteq B^k_{bs}(f(x)) \tag{1.4}$$

for all $x \in B^k_r(x_0)$ *and* $s \in]0, r - \|x - x_0\|]$. *Moreover,* $f(B^k_r(x_0))$ *is open in* $\mathbb{R}^m_k$ *and* f *is a homeomorphism onto its image.*

Proof. We define $\tilde{g}\colon B_r(x_0) \to \mathbb{R}^m$ as in Lemma 1.48. Then $\mathrm{Lip}(\tilde{g}) = \mathrm{Lip}(g) < \frac{1}{\|A^{-1}\|_{op}}$. Let $\tilde{f}\colon B_r(x_0) \to \mathbb{R}^m$, $x \mapsto Ax + \tilde{g}(x)$ and $a := \frac{1}{\|A^{-1}\|_{op}} - \mathrm{Lip}(\tilde{g}) = \frac{1}{\|A^{-1}\|_{op}} - \mathrm{Lip}(g)$. Using Lemma 1.47 we get:

(a) The map $\tilde{f}\colon B_r(x_0) \to \mathbb{R}^m$ has open image and is a homeomorphism onto its image;
(b) For all $x \in B_r(x_0)$ and $s \in]0, r - \|x - x_0\|[$ we have $B_{as}(\tilde{f}(x)) \subseteq \tilde{f}(B_s(x)) \subseteq B_{bs}(\tilde{f}(x))$;
(c) $\tilde{f}|_{B^k_r(x_0)} = f$.

Let $x \in B^k_r(x_0)$ and $s \in]0, r - \|x - x_0\|[$. The inclusion $f(B^k_s(x)) \subseteq B^k_{bs}(f(x))$ follows directly from (b) and (c). We show that

$$B^k_{as}(f(x)) \subseteq f(B^k_s(x)).$$

From (b) and (c) we get $B^k_{as}(f(x)) \subseteq B_{as}(\tilde{f}(x)) \subseteq \tilde{f}(B_s(x))$. Now let $y \in B^k_{as}(f(x))$. We find $z \in B_s(x)$ such that $y = \tilde{f}(z)$. It remains to show that $z \in \mathbb{R}^m_k$. Let $\tilde{U} := \tilde{f}^{-1}(B_{as}(f(x)))$ and $\tilde{V} := B_{as}(f(x))$. Then $\tilde{U}$ is an open subset of $B_r(x_0)$ and $\tilde{f}\colon \tilde{U} \to \tilde{V} = B_{as}(f(x))$ is a homeomorphism. Because $\tilde{f}(z) = y \in B_{as}(f(x))$ and $\tilde{f}(x) = f(x) \in B_{as}(f(x))$, we get

$$z \in \tilde{U} \text{ and } x \in \tilde{U}.$$

Hence, $\tilde{U} \cap \mathbb{R}^m_k \neq \varnothing$ and so $\tilde{U} \cap \partial^0\mathbb{R}^m_k \neq \varnothing$. We have $\varnothing \neq \tilde{U} \cap \partial^0\mathbb{R}^m_k \subseteq \partial^0 B^k_r(x_0)$. The map $\tilde{f}|_{\partial^0 B^k_r(x_0)}\colon \partial^0 B^k_r(x_0) \to \mathbb{R}^m$ is open. Therefore, $W := \tilde{f}(\tilde{U} \cap \partial^0\mathbb{R}^m_k)$ is an open subset of $\mathbb{R}^m$ and not empty. Because $\tilde{U} \cap \partial^0\mathbb{R}^m_k \subseteq B^k_r(x_0)$ and $f(B^k_r(x_0)) \subseteq \mathbb{R}^m_k$ we get $W \subseteq \mathbb{R}^m_k$. The set $\mathbb{R}^m_k$ is convex. We conclude that $W \cap \partial^0\mathbb{R}^m_k \neq \varnothing$

and hence

$$\tilde{f}(\tilde{U} \cap \partial^0\mathbb{R}^m_k) \cap \partial^0\mathbb{R}^m_k \neq \varnothing. \tag{1.5}$$

Moreover, we get

$$\tilde{f}(\tilde{U} \cap \partial\mathbb{R}^m_k) = f(\tilde{U} \cap \partial\mathbb{R}^m_k) \subseteq \partial\mathbb{R}^m_k. \tag{1.6}$$

Using (1.5), (1.6) and the convexity of $\tilde{V}$, Lemma 1.42 leads to

$$\tilde{f}(\tilde{U}\backslash\mathbb{R}^m_k) \subseteq \tilde{V}\backslash\mathbb{R}^m_k.$$

Suppose $z \notin \mathbb{R}^m_k$, then $\tilde{f}(z) = y \notin \mathbb{R}^m_k$. But this would be a contradiction. Hence $z \in \mathbb{R}^m_k$. Next, we show that $f\colon B^k_r(x_0) \to \mathbb{R}^m_k$ has open image and is a homeomorphism onto its image. Because of (1.4), $f\colon B^k_r(x_0) \to \mathbb{R}^m_k$ has open image. Let $\tilde{X} := B_r(x_0)$, $X := B^k_r(x_0)$, $\tilde{Y} := \tilde{f}(\tilde{X}) \subseteq \mathbb{R}^m$ and $Y := f(X) \subseteq \mathbb{R}^m_k$. The assertion now follows from the fact that $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ is a homeomorphism and $\tilde{f}|^Y_X = f\colon X \to Y$. □

Definition 1.50. Given $r \in]0,1[$ we choose $r^{op} \in]0,1[$ such that for all $A \in \mathcal{L}(\mathbb{C}^m)^*$ we have

$$\|A - \operatorname{id}\|_{op} < r^{op} \Rightarrow \|A^{-1} - \operatorname{id}\|_{op} < r.$$

The observation in the following remark is, of course, well-known.

Remark 1.51. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq \mathbb{K}^n$ and $f\colon U \to \mathbb{K}^n$ be an injective map. Then*

$$\|(f|^{f(U)})^{-1} - \operatorname{id}_{f(U)}\|_\infty = \|f - \operatorname{id}_U\|_\infty.$$

In fact let $g := f - \operatorname{id}_U\colon U \to \mathbb{K}^m$. Then $f(x) = x + g(x)$ for all $x \in U$. Hence, $f^{-1}(y) = y - g(f^{-1}(y))$ for all $y \in f(U)$. Thus $f^{-1} = \operatorname{id}_{f(U)} - g \circ f^{-1}$. Therefore

$$\|f^{-1} - \operatorname{id}_{f(U)}\|_\infty = \|g \circ f^{-1}\|_\infty = \|g\|_\infty = \|f - \operatorname{id}_U\|_\infty.$$

In the following lemma the points (b) and (c) are in some sense a stronger version of [Sch15, Lemma D.4 (b), (c)] (smooth case without boundary).

Lemma 1.52. *Let $l \in]0,\infty[$ and $r \in]0,1[$ such that $l' := (1-r)l - r > 0$. For all $f \in C^\omega(B^k_l(0); \mathbb{R}^m)$ with $\|f - \operatorname{id}\|^1_\infty < \min(r, r^{op})$, $f(\partial(B^k_l(0))) \subseteq \partial\mathbb{R}^m_k$ and $f(B^k_l(0)) \subseteq \mathbb{R}^m_k$ we get:*

(a) $f(B^k_l(0)) \subseteq B^k_{l+r}(0)$;

(b) $f\colon B^k_l(0) \to \mathbb{R}^m_k$ has open image and is a real analytic diffeomorphism onto its image;

(c) $B^k_{l'}(0) \subseteq f(B^k_l(0))$ and the map $f^{-1}\colon B_{l'}(0) \to B^k_l(0)$ has open image and is a real analytic diffeomorphism onto its image;

(d) $\|f^{-1}(x) - x\| < r$ for all $x \in B^k_{l'}(0)$;

(e) $\|(f^{-1})'(x) - \mathrm{id}_{\mathbb{R}^m}\|_{op} < r$ *for all* $x \in B^k_{l'}(0)$.

Proof. (a) Given $x \in B^k_l(0)$, we calculate

$$\|f(x)\| = \|f(x) - x + x\| \leqslant r + l.$$

Hence $f(B^k_l(0)) \subseteq B^k_{l+r}(0)$.

(b) We define $g\colon B^k_l(0) \to \mathbb{R}^m$, $g := f - \mathrm{id}_{\mathbb{R}^m}$. Given $x, y \in B^k_l(0)$, we have

$$\|g(x) - g(y)\| = \left\| \int_0^1 g'((1-t)x + ty; x - y)dt \right\|$$
$$\leqslant \int_0^1 \|f'((1-t)x + ty; x - y) - \mathrm{id}_{\mathbb{R}^m}(x - y)\|dt \leqslant r\|x - y\|.$$

Hence, g is Lipschitz continuous with $\mathrm{Lip}(g) \leqslant r < 1$. Moreover, $f(B^k_l(0)) \subseteq \mathbb{R}^m_k$ and $f(\partial B^k_l(0)) \subseteq \partial\mathbb{R}^m_k$. Therefore, we can apply Lemma 1.49. Hence, $f(B^k_l(0))$ is open in $\mathbb{R}^m_k$ and $f\colon B^k_l(0) \to f(B^k_l(0))$ is a bijection. Because $r^{op} < 1$, the map f is a local real analytic diffeomorphism and so $f\colon B^k_l(0) \to f(B^k_l(0))$ is a real analytic diffeomorphism.

(c) Using Lemma 1.49, we get $B^k_{(1-r)l}(f(0)) \subseteq f(B^k_l(0))$. It remains to prove $B^k_{l'}(0) \subseteq B^k_{(1-r)l}(f(0))$. Given $x \in B^k_{l'}(0)$, we calculate

$$\|f(0) - x\| \leqslant \|f(0) - 0\| + \|x\| < r + (1-r)l - r = (1-r)l.$$

(d) This follows directly from Remark 1.51.

(e) We have $\|f'(x) - \mathrm{id}_{\mathbb{R}^m}\|_{op} < r^{op}$ for all $x \in B^k_l(0)$. Hence, $\|(f'(x))^{-1} - \mathrm{id}_{\mathbb{R}^m}\|_{op} < r$ for all $x \in B^k_l(0)$. Now, let $y \in B^k_{l'}(0)$. We deduce that

$$\|(f^{-1})'(y) - \mathrm{id}_{\mathbb{R}^m}\|_{op} = \|(f'(f^{-1}(y)))^{-1} - \mathrm{id}_{\mathbb{R}^m}\|_{op} < r.$$

□

Definition 1.53. Given $l \in]0, 5[$ and $r \in]0, 1[$ such that $(1-r)l - r > 0$ we write $\varepsilon_{l,r}$ for the ε constructed in Lemma 1.36 with $R = 5$ and $\min(r, r^{op})$ instead of r.

The analogous statement to [Glo06c, 3.3] in the real analytic case is:

Lemma 1.54. *Given* $\varepsilon \in]0, \varepsilon_{\exp}]$ *there exists* $\delta(\varepsilon) \in]0, 1[$ *such that:*

(a) For $x \in \overline{B}_{4+\frac{1}{2}}(0)$ *and* $i \in \{1, ..., n\}$, *we have* $B_{\delta(\varepsilon)}(x) \subseteq \exp_i(x, B_\varepsilon(0))$;

(b) The set $D_\varepsilon := \bigcup_{x \in B_{4+\frac{1}{2}}(0)} \{x\} \times B_{\delta(\varepsilon)}(x)$ *is open in* $B_6(0) \times \mathbb{R}^m$ *and*

$$D_\varepsilon \to B_\varepsilon(0),\ (x, z) \mapsto \exp_i(x, \bullet)^{-1}(z)$$

is real analytic for all $i \in \{1, ..., n\}$.

Proof. (a) Let $i \in \{1, \ldots, n\}$ and consider the map $\exp_i\colon B_5(0) \times B_\varepsilon(0) \to B_6(0)$. Now we use Lemma 1.30. Given $x \in \overline{B}_{4+\frac{1}{2}}(0)$, we can find $r_x \in]0, \frac{1}{2}[$ and $\delta_x > 0$ such that for all $y \in B_{r_x}(x)$ and $i \in \{1, \ldots, n\}$, we have $B_{\delta_x}(y) =$

$B_{\delta_x}(\exp_i(y,0)) \subseteq \exp_i(y, B_\varepsilon(0))$. We find finitely many $x_1, \dots, x_k \in \overline{B}_{4+\frac{1}{2}}(0)$ with $\overline{B}_{4+\frac{1}{2}}(0) \subseteq \bigcup_{j=1}^k B_{r_{x_j}}(x_j)$ and set $\delta(\varepsilon) := \min_j \delta_{x_j}$. Given $i \in \{1, ..., n\}$ and $x \in \overline{B}_{4+\frac{1}{2}}(0)$, we can find j such that $x \in B_{r_{x_j}}(x_j)$. Hence $B_{\delta(\varepsilon)}(x) \subseteq \exp_i(x, B_\varepsilon(0))$.

(b) For $(x_0, y_0) \in D_\varepsilon$, we get $\|y_0 - x_0\| < \delta(\varepsilon)$. Let $\tau := \min\left(4 + \frac{1}{2} - \|x_0\|, \frac{\delta(\varepsilon) - \|y_0 - x_0\|}{2}\right)$. Then $B_\tau(x_0) \times B_\tau(y_0) \subseteq D_\varepsilon$. Hence D_ε is open. Because D_ε is open and contained in the image of $B_5(0) \times B_\varepsilon(0) \to B_5(0) \times \mathbb{R}^m$, $(x, y) \mapsto (x, \exp_i(x, y))$, we conclude that $D_\varepsilon \to B_\varepsilon(0)$, $(x, z) \mapsto \exp_i(x, \bullet)^{-1}(z)$ makes sense and is a real analytic diffeomorphism.

□

In the following, we define open subsets of $\mathbb{C}^m$ that form a connected fundamental sequence of the balls $\overline{B}^k_R(x)$ in $\mathbb{C}^m$ (see Remark 1.56).

Definition 1.55. Let $k \in \{0, \dots, m\}$, $R > 0$, $r > 0$ and $x \in \mathbb{R}^m_k$. We define

$$B^{k,\mathbb{C}}_{R,r}(x) := B^k_R(x) + B^{\mathbb{C}}_r(0) \subseteq \mathbb{C}^m \text{ and } \overline{B}^{k,\mathbb{C}}_{R,r}(x) := \overline{B}^k_R(x) + \overline{B}^{\mathbb{C}}_r(0) = \overline{B^{k,\mathbb{C}}_{R,r}(x)}.$$

Obviously $B^{k,\mathbb{C}}_{R,r}(x)$ is an open neighbourhood of $\overline{B}^k_R(x)$ in $\mathbb{C}^m$. We also write $B^{\mathbb{C}}_{R,r}(x) := B_R(x) + B^{\mathbb{C}}_r(0)$ and $\overline{B}^{\mathbb{C}}_{R,r}(x) := \overline{B}_R(x) + \overline{B}^{\mathbb{C}}_r(0)$ for $x \in \mathbb{R}^m$.

Remark 1.56. *If $U \subseteq \mathbb{C}^m$ is open, $k \in \{0, \dots, m\}$, $x \in \mathbb{R}^m_k$ and $\overline{B}^k_R(x) \subseteq U$ then $\overline{B}^k_R(x) \times \{0\} \subseteq \mu^{-1}(U)$ for $\mu\colon \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$, $(x, y) \mapsto x + y$. Using Wallace's Lemma we can find $r > 0$ such that $\overline{B}^k_R(x) + \overline{B}^{\mathbb{C}}_r(0) = \overline{B}^{k,\mathbb{C}}_{R,r}(x) \subseteq U$.*

Definition 1.57. For $i \in \{1, \dots, n\}$, let $\exp^*_i\colon \Omega^*_i \to B^{\mathbb{C}}_6(0)$ be a complex analytic extension of $\exp_i\colon \tilde{\Omega}_i \to B_6(0)$ along the compact set $\overline{B}_5(0) \times \overline{B}_{\varepsilon_{\exp}}(0) \subseteq \tilde{\Omega}_i$ (see [DGS14, Lemma 2.2 (a)]). Because $\overline{B}_5(0) \times \{0\} \subseteq \Omega^*_i$ we can use Remark 1.56 and find $r_{\exp^*} > 0$ and $\varepsilon^*_1 > 0$ such that $\overline{B}^{\mathbb{C}}_{5,r_{\exp^*}}(0) \times B^{\mathbb{C}}_{\varepsilon^*_1}(0) \subseteq \Omega^*_i$ for all $i \in \{1, \dots, n\}$.

Remark 1.58. *(a) For all $i \in \{1, \dots, n\}$, the map $\exp^*_i(\bullet, 0)\colon B^{\mathbb{C}}_{5,r_{\exp^*}}(0) \to \mathbb{C}^m$, $x \mapsto \exp^*_i(x, 0)$ is an extension of $\exp_i(\bullet, 0)\colon B_5(0) \to \mathbb{R}^m$, $x \mapsto x$. Hence, we have*

$$\exp^*_i(x, 0) = x,$$

for all $x \in B^{\mathbb{C}}_{5,r_{\exp^*}}(0)$, because $B^{\mathbb{C}}_{5,r_{\exp^*}}(0)$ is connected. Therefore, $\exp^*_i(x, 0) = x$ for all $x \in \overline{B}^{\mathbb{C}}_{5,r_{\exp^*}}(0)$.

(b) For all $i \in \{1, \dots, n\}$ the map $d_2 \exp^*_i(\bullet, 0)\colon B^{\mathbb{C}}_{5,r_{\exp^*}}(0) \to \mathcal{L}(\mathbb{C}^m)$, $x \mapsto d_2 \exp^*_i(x, 0)$ is a complex analytic extension of $d_2 \exp_i(\bullet, 0)\colon B_5(0) \to \mathcal{L}(\mathbb{R}^m)$, $x \mapsto d_2 \exp_i(x, 0) = \mathrm{id}_{\mathbb{R}^m}$. Hence, we have

$$d_2 \exp^*_i(x, 0) = \mathrm{id}_{\mathbb{C}^m}$$

for all $x \in B^{\mathbb{C}}_{5,r_{\exp*}}(0)$*, because* $B^{\mathbb{C}}_{5,r_{\exp*}}(0)$ *is connected. Therefore,* $d_2 \exp^*_i(x,0) = \mathrm{id}_{\mathbb{C}^m}$ *for all* $x \in \overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0)$*.*

The following lemma is similar to our Lemma 1.31 as well as [Glo06c, 3.2].

Lemma 1.59. *There exists* $\varepsilon_{\exp*} > 0$ *such that:*

(i) We have

$$\overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0) \times \overline{B}^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \subseteq \Omega^*_i \subseteq B^{\mathbb{C}}_6(0) \times \mathbb{C}^m$$

for all $i \in \{1, ..., n\}$*.*

(ii) For all $x \in \overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0)$ *and* $i \in \{1, ..., n\}$*, the mapping* $\exp^*_{i,x} := \exp^*_i(x, \bullet)\colon B^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \to \mathbb{C}^m$ *has open image and is a complex analytic diffeomorphism onto its image. Moreover, the map* $B^{\mathbb{C}}_{5,r_{\exp*}}(0) \times B^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \to B^{\mathbb{C}}_{5,r_{\exp*}}(0) \times \mathbb{C}^m$*,* $(x, y) \mapsto (x, \exp^*_i(x, y))$ *has open image and is a complex analytic diffeomorphism onto its image.*

Proof. Obviously it is enough to show the assertions for a fixed $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. For every $x \in \overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0)$ we find $r_x > 0$ and $\varepsilon_x > 0$ such that:

- $\overline{B}^{\mathbb{C}}_{r_x}(x) \times \overline{B}^{\mathbb{C}}_{\varepsilon_x}(0) \subseteq \Omega^*_i$;
- For all $y \in B^{\mathbb{C}}_{r_x}(x)$, the map

$$\exp^*_i(y, \bullet)\colon B^{\mathbb{C}}_{\varepsilon_x}(0) \to \mathbb{C}^m$$

  has open image and is a diffeomorphism onto its image;
- The map $B^{\mathbb{C}}_{r_x}(x) \times B^{\mathbb{C}}_{\varepsilon_x}(0) \to B^{\mathbb{C}}_{r_x}(x) \times \mathbb{C}^m$, $(x, y) \mapsto (x, \exp^*_i(x, y))$ has open image and is a diffeomorphism onto its image.

We find finitely many $x_1, \ldots, x_l$ such that $\overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0) \subseteq \bigcup_{j=1}^{l} B_{r_{x_j}}(x_j)$. Let $\varepsilon_{\exp*} := \min_j \varepsilon_{x_j}$. Obviously we have

$$\overline{B}^{\mathbb{C}}_{5,r_{\exp*}}(0) \times \overline{B}^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \subseteq \Omega^*_i.$$

Moreover, the map $B^{\mathbb{C}}_{5,r_{\exp*}}(0) \times B^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \to B^{\mathbb{C}}_{5,r_{\exp*}}(0) \times \mathbb{C}^m$, $(x, y) \mapsto (x, \exp^*_i(x, y))$ is injective and a local diffeomorphism. Hence, it has open image and is a complex analytic diffeomorphism onto its image. Thus, we find $\varepsilon_{\exp*}$ as needed. □

The following lemma is the analogous statement to our Lemma 1.54 as well as [Glo06c, 3.3] in the complex case.

Lemma 1.60. *Let* $r_{\ell*} := r_{\exp*}/2$*. For all* $\varepsilon \in]0, \varepsilon_{\exp*}]$*, there exists* $\delta^{\mathbb{C}}(\varepsilon) \in]0, 1[$ *such that:*

(i) For all $x \in \overline{B}^{\mathbb{C}}_{4,r_{\ell*}}(0)$ *and* $i \in \{1, \ldots, n\}$*, we have* $B^{\mathbb{C}}_{\delta^{\mathbb{C}}(\varepsilon)}(x) \subseteq \exp^*_i(x, B^{\mathbb{C}}_{\varepsilon}(0))$*;*

(ii) The set

$$D_{\varepsilon}^{\mathbb{C}} := \bigcup_{x \in B_{4,r_{\ell *}}^{\mathbb{C}}(0)} \{x\} \times B_{\delta^{\mathbb{C}}(\varepsilon)}^{\mathbb{C}}(x) \subseteq \mathbb{C}^m \times \mathbb{C}^m$$

is open and $D_{\varepsilon}^{\mathbb{C}} \to B_{\varepsilon}^{\mathbb{C}}(0)$, $(x, z) \mapsto \exp_i^*(x, \bullet)^{-1}(z)$ *is complex analytic.*

Proof. (i) We consider the map $\exp_i^* \colon B_{5,r_{\exp *}}^{\mathbb{C}}(0) \times B_{\varepsilon}^{\mathbb{C}}(0) \to \mathbb{C}^m$ for $i = 1, \ldots, n$. Given $x \in \overline{B}_{4,r_{\ell *}}^{\mathbb{C}}(0)$, we use Lemma 1.30 and find $r_x > 0$ and $\delta_x > 0$ such that $B_{r_x}^{\mathbb{C}}(x) \subseteq B_{5,r_{\exp *}}^{\mathbb{C}}(0)$ and, for all $y \in B_{r_x}^{\mathbb{C}}(x)$ and $i \in \{1, \ldots, n\}$

$$B_{\delta_x}^{\mathbb{C}}(y) = B_{\delta_x}^{\mathbb{C}}(\exp_i^*(y, 0)) \subseteq \exp_i^*(y, B_{\varepsilon}^{\mathbb{C}}(0)).$$

We choose finitely many $x_1, \ldots, x_n \in \overline{B}_{4,r_{\ell *}}^{\mathbb{C}}(0)$ such that

$$\overline{B}_{4,r_{\ell *}}^{\mathbb{C}}(0) \subseteq \bigcup_{j=1}^{n} B_{r_{x_j}}(x_j).$$

We set $\delta^{\mathbb{C}}(\varepsilon) := \min_j \delta_{x_j}$. If $i \in \{1, \ldots, n\}$ and $x \in \overline{B}_{4,r_{\ell *}}^{\mathbb{C}}(0)$, then $B_{\delta^{\mathbb{C}}(\varepsilon)}^{\mathbb{C}}(x) \subseteq \exp_i^*(x, B_{\varepsilon}^{\mathbb{C}}(0))$.

(ii) Given $(x_0, y_0) \in D_{\varepsilon}^{\mathbb{C}}$, we find $\sigma > 0$ with $B_{\sigma}^{\mathbb{C}}(x_0) \subseteq B_{4,r_{\ell *}}^{\mathbb{C}}(0)$. Defining $\tau := \min\left(\sigma, \frac{\delta^{\mathbb{C}}(\varepsilon) - \|y_0 - x_0\|}{2}\right)$, we get $B_{\tau}(x_0) \times B_{\tau}(y_0) \subseteq D_{\varepsilon}^{\mathbb{C}}$. The rest of the statement is clear.

□

Definition 1.61. (a) Let $N_0 \in \mathbb{N}$ with $\frac{1}{N_0} < r_{\ell *} < r_{\exp *}$. For $R \in [0, 4]$, $k \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$, we define

$$V_{R,n}^k := B_{R,\frac{1}{n+N_0}}^{k,\mathbb{C}}(0) = B_R^k(0) + B_{\frac{1}{n+N_0}}^{\mathbb{C}}(0)$$

and get a connected fundamental sequence of $\overline{B}_R^k(0)$ in $\mathbb{C}^m$ with $V_{R,n}^k \subseteq B_{4,r_{\exp *}}^{\mathbb{C}}(0)$. Moreover, we define

$$\overline{V}_{R,n}^k := \overline{B}_R^k(0) + \overline{B}_{\frac{1}{n+N_0}}^{\mathbb{C}}(0) = \overline{V_{R,n}^k}.$$

(b) For $i \in \{1, \ldots, n\}$, $R \in [1, 4]$, $n \in \mathbb{N}$ and $f \in \mathrm{Hol}_{\varepsilon_{\exp *}}^0(V_{R,n}; \mathbb{C}^m)$, we define the map $\psi_f^i \colon V_{R,n}^{k_i} \to \mathbb{C}^m$, $x \mapsto \exp_i^*(x, f(x))$.

The following lemma is the analogous statement to our Lemma 1.36 in the complex case and is inspired by [Glo06c, Lemma 3.7].

Lemma 1.62. *For* $r_0 \in]0, 1[$ *there exists* $\varepsilon \in]0, \varepsilon_{\exp *}]$ *such that for all* $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, $R \in \{3, 4\}$ *and* $\eta \in \mathrm{Hol}_b^1(V_{R,n}^{k_i}; \mathbb{C}^m)$ *with* $\|\eta\|_{\infty}^1 < \varepsilon$, *we have* $\|\psi_\eta^i - \mathrm{id}_{\mathbb{C}^m}\|_{V_{R,n}^{k_i}}^1 < r_0$.

Proof. Obviously it is enough to show that for all $r_0 \in]0,1[$ there exists $\varepsilon \in]0, \varepsilon_{\exp*}]$ such that for all $n \in \mathbb{N}$, $i \in \{1,\dots,n\}$, $R \in \{3,4\}$ and $\eta \in \mathrm{Hol}_b^1(V_{R,n}^{k_i};\mathbb{C}^m)$ with $\|\eta\|_\infty^1 < \varepsilon$, we have $\|\psi_\eta^i - \mathrm{id}_{\mathbb{C}^m}\|^1_{V_{R,n}^{k_i}} \leqslant r_0$. In order to shorten the notation, we define $U_{\exp*} := B^{\mathbb{C}}_{5,r_{\exp*}}(0)$. Obviously it is enough to show the lemma for a fixed $i \in \{1,\dots,n\}$. Hence, let $i \in \{1,\dots,n\}$ be fixed for the rest of the proof. We define $H\colon U_{\exp*} \times B^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \to \mathbb{C}^m$, $(x,y) \mapsto \exp_i^*(x,y) - x - y$ and $h\colon U_{\exp*} \times B^{\mathbb{C}}_{\varepsilon_{\exp*}}(0) \to [0,\infty[$, $(x,y) \mapsto \|H'(x,y)\|_{op}$. For all $x \in U_{\exp*}$, we get $d_1H(x,0;\bullet) = 0$ and $d_2H(x,0;\bullet) = 0$ and so $H'(x,0) = dH(x,0;\bullet) = 0$ in $\mathcal{L}(\mathbb{C}^m \times \mathbb{C}^m;\mathbb{C}^m)$. Hence, $\overline{V_{4,1}^{k_i}} \times \{0\} \subseteq h^{-1}([0,\frac{r_0}{r_0+10}[)$ and with Wallace's Lemma we find $\varepsilon \in]0, \min(\varepsilon_{\exp}, \frac{r_0}{2})[$ such that $\|H'(x,y)\|_{op} < \frac{r_0}{r_0+10}$ for all $x \in \overline{V_{4,1}^{k_i}}$ and $y \in B_\varepsilon^{\mathbb{C}}(0)$. Now let $\eta \in \mathrm{Hol}_b^1(V_{R,n}^{k_i};\mathbb{C}^m)$ with $\|\eta\|_\infty^1 < \varepsilon$. We have

$$\psi_\eta^i(x) = H(x,\eta(x)) + x + \eta(x) \tag{1.7}$$

for all $x \in V_{R,n}^{k_i}$. Hence,

$${\psi_\eta^i}'(x) = H'(x,\eta(x);\bullet) \circ (\mathrm{id}_{\mathbb{C}^m}, \eta'(x)) + \mathrm{id}_{\mathbb{C}^m} + \eta'(x)$$

for all $x \in V_{R,n}^{k_i}$. Remark 1.35 implies

$$\begin{aligned} &\|{\psi_\eta^i}'(x) - \mathrm{id}_{\mathbb{C}^m}\|_{op} \leqslant \|H'(x,\eta(x))\|_{op} \cdot \|(\mathrm{id}_{\mathbb{C}^m}, \eta'(x))\|_{op} + \|\eta'(x)\|_{op} \\ <&\frac{r_0}{r_0+10} \cdot (1+\varepsilon) + \varepsilon \leqslant \frac{r_0}{r_0+2} \cdot \left(1 + \frac{r_0}{2}\right) + \frac{r_0}{2} = r_0. \end{aligned} \tag{1.8}$$

Now, let $x \in V_{R,n}^{k_i}$ and $y \in B_\varepsilon^{\mathbb{C}}(0)$. Then $\|(x,y)\| \leqslant \|x\| + \|y\| < 5 + \frac{r_0}{2}$. Hence,

$$\begin{aligned} &\|H(x,y)\| = \|H(x,y) - H(0,0)\| = \left\| \int_0^1 dH(tx,ty;x,y)dt \right\| \\ \leqslant &\int_0^1 \|H'(tx,ty)\| \cdot \|(x,y)\| dt < \frac{r_0}{r_0+10} \cdot \|(x,y)\| \leqslant \frac{r_0}{2}. \end{aligned}$$

Thus, given $x \in V_{R,n}^{k_i}$, we calculate with (1.7)

$$\|\psi_\eta^i(x) - x\| \leqslant \|H(x,\eta(x))\| + \|\eta(x)\| < r_0. \tag{1.9}$$

□

Definition 1.63. Given $r_0 \in]0,1[$, we write $\varepsilon_{r_0}^{\mathbb{C}}$ for the ε constructed in Lemma 1.62 with $\min(r_0, r_0^{op})$ instead of r_0.

The following lemma is the analogous statement to Lemma 1.52 in the complex case.

Lemma 1.64. *Let $R \in]0,\infty[$, $r \in]0,\infty[$ and $r_0 \in]0,1[$ such that $R' := (1-r_0)R - r_0 > 0$. We write $r' := (1-r_0)r$, $R'' := R + r_0$ and $r'' := (1+r_0)r$. For all*

$k \in \{0, \dots, m\}$ and $f \in \mathrm{Hol}^1_b(B^{k,\mathbb{C}}_{R,r}(0); \mathbb{C}^m)$ such that $\|f - \mathrm{id}_{\mathbb{C}^m}\|^1_\infty < \min(r_0, r^{op}_0)$, $f(\partial(B^k_R(0))) \subseteq \partial\mathbb{R}^m_k$ and $f(B^k_R(0)) \subseteq \mathbb{R}^m_k$, we get the following assertions:

(a) *$f\colon B^{k,\mathbb{C}}_{R,r}(0) \to \mathbb{C}^m$ has open image and is a complex analytic diffeomorphism onto its image;*

(b) *$B^{k,\mathbb{C}}_{R',r'}(0) \subseteq f(B^{k,\mathbb{C}}_{R,r}(0))$ and the map $f^{-1}\colon B^{k,\mathbb{C}}_{R',r'}(0) \to B^{k,\mathbb{C}}_{R,r}(0)$ has open image and is a complex analytic diffeomorphism onto its image;*

(c) *$f(B^{k,\mathbb{C}}_{R,r}(0)) \subseteq B^{k,\mathbb{C}}_{R,r+r_0}(0)$ and $f(B^{k,\mathbb{C}}_{R,r}(0)) \subseteq B^{k,\mathbb{C}}_{R'',r''}(0)$;*

(d) *$\|f^{-1}(x) - x\| < r_0$ for all $x \in B^{k,\mathbb{C}}_{R',r'}(0)$;*

(e) *$\|(f^{-1})'(x) - \mathrm{id}_{\mathbb{C}^m}\|_{op} < r_0$ for all $x \in B^{k,\mathbb{C}}_{R',r'}(0)$.*

Proof. (a) From Remark 1.37 and $r_0 < 1$ we get that $f\colon B^{k,\mathbb{C}}_{R,r}(0) \to \mathbb{C}^m$ has open image and is a complex analytic diffeomorphism onto its image.

(b) Let $g := f - \mathrm{id}_{\mathbb{C}^m}\colon B^{k,\mathbb{C}}_{R,r}(0) \to \mathbb{C}^m$. Then, as in Lemma 1.52, we get $\|g(x) - g(y)\| < r_0\|x - y\|$ for all $x, y \in B^{k,\mathbb{C}}_{R,r}(0)$. Hence $\mathrm{Lip}(g) \leqslant r_0 < 1$. Now let $x \in B^k_R(0)$. Then $B^{\mathbb{C}}_r(x) = x + B^{\mathbb{C}}_r(0) \subseteq B^{k,\mathbb{C}}_{R,r}(0)$. We consider the map $f|_{B^{\mathbb{C}}_r(x)}$. Thus, using the identification $\mathbb{C}^m \cong \mathbb{R}^{2m}$ in combination with Lemma 1.47, we get $B^{\mathbb{C}}_{(1-r_0)r}(f(x)) \subseteq f(B^{\mathbb{C}}_r(x)) \subseteq f(B^{k,\mathbb{C}}_{R,r}(0))$. Hence, $B^{\mathbb{C}}_{(1-r_0)r}(0) + f(x) \subseteq f(B^{k,\mathbb{C}}_{R,r}(0))$ for all $x \in B^k_R(0)$. Therefore, $B^{\mathbb{C}}_{(1-r_0)r}(0) + f(B^k_R(0)) \subseteq f(B^{k,\mathbb{C}}_{R,r}(0))$. From Lemma 1.52 we get $B^k_{R'}(0) \subseteq f(B^k_R(0))$ because $f(\partial(B^k_R(0))) \subseteq \partial\mathbb{R}^m_k$ and $f(B^k_R(0))) \subseteq \mathbb{R}^m_k$. Therefore

$$B^{k,\mathbb{C}}_{R',r'}(0) = B^k_{R'}(0) + B^{\mathbb{C}}_{r'}(0) \subseteq f(B^{k,\mathbb{C}}_{R,r}(0)).$$

Hence, the map $f^{-1}\colon B^{k,\mathbb{C}}_{R',r'}(0) \to B^{k,\mathbb{C}}_{R,r}(0)$ makes sense, has open image and is a complex analytic diffeomorphism onto its image.

(c) Let $x \in B^{k,\mathbb{C}}_{R,r}(0)$. Then $f(x) = x + (f(x) - x) \in B^{k,\mathbb{C}}_{R,r}(0) + B^{\mathbb{C}}_{r_0}(0) = B^{k,\mathbb{C}}_{R,r+r_0}(0)$. Now let $x_0 \in B^k_R(0)$. Using Lemma 1.52, we get $f(x_0) \in B^k_{R''}(0)$. Again we consider the map $f|_{B^{\mathbb{C}}_r(x_0)}$. Lemma 1.47 yields

$$f(B^{\mathbb{C}}_r(x_0)) \subseteq B^{\mathbb{C}}_{(1+r_0)r}(f(x_0)) = B^{\mathbb{C}}_{r''}(0) + f(x_0) \subseteq B^{\mathbb{C}}_{r''}(0) + B^k_{R''}(0) = B^{k,\mathbb{C}}_{R'',r''}(0).$$

(d) Because $\|f - \mathrm{id}\|^0_\infty < r_0$, we can use Remark 1.51 and get $\|f^{-1}(x) - x\| < r_0$ for all $x \in B^{k,\mathbb{C}}_{R',r'}(0)$.

(e) Because $\|f'(x) - \mathrm{id}\|_{op} < r^{op}_0$ for all $x \in B^{k,\mathbb{C}}_{R,r}(0)$, we get

$$\|(f^{-1})'(x) - \mathrm{id}_{\mathbb{C}^m}\|_{op} = \|f'(f^{-1}(x))^{-1} - \mathrm{id}_{\mathbb{C}^m}\|_{op} < r_0$$

for all $x \in B^{k,\mathbb{C}}_{R',r'}(0)$.

□

We will use the following technical lemma later to secure that the composition and inversion in the local chart stay in the subspace of stratified vector fields.

Lemma 1.65. *There exists $\varepsilon_\partial > 0$ such that for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, all connected components $C \subseteq \partial^j B^{k_i}_5(0)$, $x, y \in C$ with $\|x - y\| < \frac{\varepsilon_\partial}{2}$ and $v \in B_{\varepsilon_\partial}(0)$ with $y = \exp_i(x, v)$, we get $v \in T_x\partial^j B^{k_i}_5(0)$.*

Proof. There exists $\varepsilon_\partial \in]0, \varepsilon_{\exp}]$ such that $\|d\exp_{i,x}(y) - \mathrm{id}_{\mathbb{R}^m}\| < \frac{1}{2}$ for all $i \in \{1, \ldots, n\}$, $x \in B_4(0)$ and $y \in B_{\varepsilon_\partial}(0)$. Given $x \in \partial^j B_5^k(0)$, we define the subset $J_x \subseteq \{1, \ldots, m\}$ such that $x_l = 0$ if and only if $l \in J_x$. Moreover, we write $I_x := \{1, \ldots, m\} \setminus J_x$ and $\mathbb{R}^m_{I_x} := \mathrm{span}\{e_i : i \in I_x\}$. Obviously, we have $T_x \partial^j B_5^k(0) = \mathbb{R}^m_{I_x}$. Let $v \in B_{\varepsilon_{\exp}}(0) \cap T_x \partial^j B_5^{k_i} = B_{\varepsilon_{\exp}}(0) \cap \mathbb{R}^m_{I_x} =: B^{\mathbb{R}^m_{I_x}}_{\varepsilon_{\exp}}(0)$. Because $\partial^j M$ is totally geodesic there exists $t_0 \in]0, 1[$ such that $d_x \varphi_i^{-1}(tv) \in \Omega_{\partial^j M}$ for all $t \in [0, t_0]$. Hence, for all $t \in [0, t_0]$ we have $\exp_{i,x}(tv) \in \partial^j B_5^{k_i}(0)$. Therefore, $\mathrm{pr}_l \circ \exp_{i,x}(tv) = 0$ for all $t \in [0, t_0]$ and $l \in J_x$ (note that $t \mapsto \exp_{i,x}(tv)$ stays in the connected component of x). Because $\exp i, x$ is real analytic we can use the Identity Theorem and obtain $\mathrm{pr}_l \circ \exp_{i,x}(v) = 0$. Therefore $\exp_{i,x}(B^{\mathbb{R}^m_{I_x}}_{\varepsilon_{\exp}}(0)) \subseteq \mathbb{R}^m_{I_x}$. In particular, we can consider the real analytic map $\exp_{i,x}\colon B^{\mathbb{R}^m_{I_x}}_{\varepsilon_\partial}(0) \to \mathbb{R}^m_{I_x}$. We write C for the connected component of x in $\partial^j B_5^{k_i}(0)$. Because $\|d\exp_{i,x}(y) - \mathrm{id}\| < \frac{1}{2}$, we get

$$C \cap \partial^j B^{k_i}_{\frac{1}{2}\varepsilon_\partial}(x) \subseteq B^{\mathbb{R}^m_{I_x}}_{\frac{1}{2}\varepsilon_\partial}(x) = B^{\mathbb{R}^m_{I_x}}_{\frac{1}{2}\varepsilon_\partial}(\exp_{i,x}(0)) \subseteq \exp_{i,x}(B^{\mathbb{R}^m_{I_x}}_{\varepsilon_\partial}(0))$$

with the quantitative inverse function theorem. Now the assertion follows from the injectivity of $\exp_{i,x}\colon B_{\varepsilon_{\exp}}(0) \to \mathbb{R}^m$. □

Definition 1.66. We use the notation of Lemma 1.54 and define $\delta_{\mathcal{U}} := \delta(\min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial))$, $D_{\mathcal{U}} := D_{\min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)}$ and

$$\ell_i\colon D_{\mathcal{U}} \to B_{\min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)}(0), (x, z) \mapsto \exp_i(x, \bullet)^{-1}(z)$$

for all $i \in \{1, \ldots, n\}$, where $\varepsilon_{\mathcal{U}}$ is as in Definition 1.46 and ε_∂ as in Lemma 1.65.

Remark 1.67. *For all $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$ and all connected components C in $\partial^j B_1^{k_i}(0)$, we choose $p_C^{i,j} \in C$. After shrinking $\varepsilon_{\mathcal{U}}$, we may assume without loss of generality that $\psi^i_{\eta(i)}(p_C^{i,j}) \in C$ for all i, j, C as above and $\eta \in \mathcal{B}^1_{1,\varepsilon_{\mathcal{U}}}$ (see Lemma 1.36).*

Definition 1.68. Let $i \in \{1, \ldots, n\}$. We define the real analytic map

$$\alpha_i\colon B_{4+\frac{1}{2}}(0) \times B_{\delta_{\mathcal{U}}}(0) \to B_{\varepsilon_{\mathcal{U}}}(0),\ (x, y) \mapsto \ell_i(x, x + y)$$

and get $\alpha_i(x, 0) = 0$ for all $x \in B_{4+\frac{1}{2}}(0)$. Hence, $d_1\alpha_i(x, 0) = d\alpha_i(\bullet, 0)(x) = 0$ in $\mathcal{L}(\mathbb{R}^m)$ for all $x \in B_{4+\frac{1}{2}}(0)$. Therefore $\overline{B}_4(0) \times \{0\} \subseteq (d_1\alpha_i)^{-1}(\{0\})$. We find $\nu_\alpha \in]0, \delta_{\mathcal{U}}[$ such that $\|d_1\alpha_i(x, y)\|_{op} < \frac{\varepsilon_{\mathcal{U}}}{4}$ for all $x \in \overline{B}_4(0)$, $y \in B_{\nu_\alpha}(0)$ and $i \in \{1, \ldots, n\}$. Let $K_\alpha \geqslant \sup\left\{\|d_2\alpha_i(x, y)\|_{op} : x \in \overline{B}_4(0), y \in \overline{B}_{\nu_\alpha}(0)\right\}$ for all $i \in \{1, \ldots, n\}$.

1.4. Smoothness of composition

In this section, we show the smoothness of the composition in the local chart Φ.

At first we have to fix a "radius" $\varepsilon_\diamond$ to obtain a 0-neighbourhood on which the composition is smooth.

Definition 1.69. Let $r_\diamond = \min(\frac{\delta_{\mathcal{U}}}{2}, \varepsilon_{\mathcal{U}}, \frac{\varepsilon_{\mathcal{U}}}{16K_\alpha}, \frac{\nu_\alpha}{2}, \frac{\varepsilon_\partial}{4}, \frac{1}{4})$, $r_\diamond^{\mathbb{C}} := \min(\frac{\delta^{\mathbb{C}}(\varepsilon_{\exp*})}{2}, \frac{1}{4})$ and $\varepsilon_\diamond := \min(\varepsilon_{\mathcal{U}}, \varepsilon_{4,r_\diamond}, \varepsilon^{\mathbb{C}}_{r_\diamond^{\mathbb{C}}})$.

The result corresponding to the following lemma, in the case of a non compact smooth manifold without corners, is [Glo06c, 4.17].

Lemma 1.70. *Let $\eta, \zeta \in \mathcal{B}^1_{4,\varepsilon_\diamond}$.*

(a) The maps $\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}} : B_3^{k_i}(0) \to \mathbb{R}^m_{k_i}$ and

$$\eta_{(i)} \diamond \zeta_{(i)} \colon B_3^{k_i}(0) \to \mathbb{R}^m,\ x \mapsto \ell_i(x, \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(x))$$

make sense[8] and are real analytic functions with $\|\eta_{(i)} \diamond \zeta_{(i)}(x)\| < \min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)$ for all $x \in B_3^{k_i}(0)$. Moreover, we have

$$\psi^i_{\eta_{(i)} \diamond \zeta_{(i)}} = \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}} \tag{1.10}$$

on $B_3^{k_i}(0)$ and $\eta_{(i)} \diamond \zeta_{(i)}$ is stratified.

(b) For the map $\psi_\eta \circ \psi_\zeta \colon M \to M$, we have $\psi_\eta \circ \psi_\zeta(U_{i,3}) \subseteq U_{i,4}$ and

$$\varphi_i \circ \psi_\eta \circ \psi_\zeta \circ \varphi_i^{-1}|_{B_3^{k_i}(0)} = \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}|_{B_3^{k_i}(0)}. \tag{1.11}$$

(c) The vector field

$$\eta \diamond \zeta \colon M \to TM,\ p \mapsto \exp|^{-1}_{W_pM}(\psi_\eta \circ \psi_\zeta(p))$$

makes sense and is real analytic and stratified. Moreover, we have

$$(\eta \diamond \zeta)_{(i)}(x) = \eta_{(i)} \diamond \zeta_{(i)}(x)$$

for all $i \in \{1, \dots, n\}$ and $x \in B_3^{k_i}(0)$ and

$$\psi_{\eta \diamond \zeta} = \psi_\eta \circ \psi_\zeta.$$

(d) The vector field $\eta \diamond \zeta$ is in $\mathcal{V}$.

Proof. (a) Using Lemma 1.36 and Lemma 1.52, we get $\psi^i_{\zeta_{(i)}}(B_3^{k_i}(0)) \subseteq B_4^{k_i}(0)$. Hence, $\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}} : B_3^{k_i}(0) \to \mathbb{R}^m_{k_i}$ makes sense. We have $r_\diamond \leqslant \frac{\delta_{\mathcal{U}}}{2}$ and so $\|\psi^i_{\eta_{(i)}} - \mathrm{id}\| < \frac{\delta_{\mathcal{U}}}{2}$ and $\|\psi^i_{\zeta_{(i)}} - \mathrm{id}\| < \frac{\delta_{\mathcal{U}}}{2}$ on $B_4^{k_i}(0)$, by Lemma 1.36. Hence, $\psi^i_{\eta_{(i)}}(\psi^i_{\zeta_{(i)}}(x)) - x \in B_{\delta_{\mathcal{U}}}(0)$. Therefore, $\eta_{(i)} \diamond \zeta_{(i)} \colon B_3^{k_i}(0) \to \mathbb{R}^m$ makes sense and $\|\eta_{(i)} \diamond \zeta_{(i)}(x)\| < \min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)$ for all $x \in B_3^{k_i}(0)$ and so

$$\psi^i_{\eta_{(i)} \diamond \zeta_{(i)}} = \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}$$

on $B_3^{k_i}(0)$ (note $\varepsilon_{\mathcal{U}} < \varepsilon_{\exp}$). Now we show that $\eta_{(i)} \diamond \zeta_{(i)}$ is stratified. With

[8] In this context $\psi^i_{\eta_{(i)}}$ and $\psi^i_{\zeta_{(i)}}$ are defined on $B_4^{k_i}(0)$.

$r_\diamond \leqslant \min(\frac{\nu_\alpha}{2}, \frac{\varepsilon_\partial}{2})$, we calculate for $x \in B_3^{k_i}(0)$:

$$\begin{aligned} &\|\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(x) - x\| \leqslant \|\psi^i_{\eta_{(i)}}(\psi^i_{\zeta_{(i)}}(x)) - \psi^i_{\zeta_{(i)}}(x)\| + \|\psi^i_{\zeta_{(i)}}(x) - x\| \\ &< \min\left(\nu_\alpha, \frac{\varepsilon_\partial}{2}\right). \end{aligned} \tag{1.12}$$

Now let $x \in \partial^j B_3^{k_i}(0)$ and C the connected component of x in $\partial^j B_5^{k_i}(0)$. We define $y := \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(x)$ and $v := \eta_{(i)} \diamond \zeta_{(i)}(x) \in B_{\varepsilon_\partial}(0)$. From Remark 1.67, we deduce that $x, y \in C$ and $y = \exp_{i,x}(v)$. From (1.12) and Lemma 1.65, we deduce that $v \in T_x\partial^j B_5^{k_i}(0)$. Hence, $\eta_{(i)} \diamond \zeta_{(i)}$ is stratified.

(b) The maps ψ_η and ψ_ζ make sense because $r_\diamond < \varepsilon_{\exp}$. From Lemma 1.34 we obtain

$$\begin{aligned} &\psi_\eta(\psi_\zeta(U_{i,3})) = \psi_\eta(\psi_\zeta(\varphi_i^{-1}(B_3^{k_i}(0)))) = \psi_\eta(\varphi_i^{-1}(\psi^i_{\zeta_{(i)}}(B_3^{k_i}(0)))) \\ =&\varphi_i^{-1}(\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(B_3^{k_i}(0))). \end{aligned}$$

With (1.12) and $\nu_\alpha < \delta_{\mathcal{U}} < 1$, the first assertion follows. An analogous calculation shows (1.11).

(c) Let $p \in M$ and $i \in \{1, \ldots, n\}$ with $p \in U_{i,1}$. Let $x := \varphi_i(p) \in B_1^{k_i}(0)$ and $v := T\varphi_i^{-1}(x, \eta_{(i)} \diamond \zeta_{(i)}(x)) \in T_pM$. Thus $v \in \tilde{\Omega}$ and because $\varepsilon_{\mathcal{U}} \leqslant \varepsilon_{inj}$, we get $v \in W_pM$. Now we calculate

$$\exp(v) = \varphi_i^{-1}(\exp_i(x, \eta_{(i)} \diamond \zeta_{(i)}(x))) = \varphi_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}} \circ \varphi_i(p) = \psi_\eta \circ \psi_\zeta(p).$$

Hence, the vector field $\eta \diamond \zeta$ makes sense. Next let $x \in B_3^{k_i}(0)$ and $p := \varphi_i^{-1}(x)$. We calculate

$$\begin{aligned} &(\eta \diamond \zeta)_{(i)}(x) = d\varphi_i \circ \exp|^{-1}_{W_pM} \circ \psi_\eta \circ \psi_\zeta(p) \\ =&d_p\varphi_i \circ \exp|^{-1}_{W_pM} \circ \varphi_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(\varphi_i(p)) \\ =&(\varphi_i \circ \exp|_{W_pM} \circ d_p\varphi_i^{-1})^{-1} \circ \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(x) \\ =&\exp_i(x, \bullet)^{-1}(\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(x)) = \eta_{(i)} \diamond \zeta_{(i)}(x). \end{aligned}$$

Obviously, we have $\psi_{\eta\diamond\zeta} = \psi_\eta \circ \psi_\zeta$. The vector field $\eta \diamond \zeta$ is stratified because its local representation is stratified. For the same reason $\eta \diamond \zeta$ is real analytic.

(d) We show that

$$\|(\eta \diamond \zeta)_{(i)}\|^1_{\overline{B}_1^{k_i}(0)} = \|\eta_{(i)} \diamond \zeta_{(i)}\|^1_{\overline{B}_1^{k_i}(0)} < \varepsilon_{\mathcal{U}}$$

for all $i \in \{1, \ldots, n\}$. From $\psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}}(B_3^{k_i}(0)) \subseteq B_{\delta_{\mathcal{U}}}(0)$, we get $\|\eta_{(i)} \diamond \zeta_{(i)}\|^0_{\overline{B}_1^{k_i}(0)} < \varepsilon_{\mathcal{U}}$. Now, we show that $\|(\eta_{(i)} \diamond \zeta_{(i)})'(x)\|_{op} < \varepsilon_{\mathcal{U}}$ for all $x \in B_3^{k_i}(0)$. We define the auxiliary function $h := \psi^i_{\eta_{(i)}} \circ \psi^i_{\zeta_{(i)}} - \mathrm{id}$ on $B_3^{k_i}(0)$. Let $x \in B_3^{k_i}(0)$. Using (1.12), we see that $\|h(x)\| < \nu_\alpha$. With $r_\diamond \leqslant \min(\frac{\varepsilon_{\mathcal{U}}}{4K_\alpha} \cdot \frac{1}{4}, 1)$, we calculate using Lemma 1.36

$$\begin{aligned}
&\|h'(x)\|_{op} = \|{\psi^i_{\eta(i)}}'(\psi^i_{\zeta(i)}(x)) \circ {\psi^i_{\zeta(i)}}'(x) - \operatorname{id}\|_{op} \\
\leqslant &\|{\psi^i_{\eta(i)}}'(\psi^i_{\zeta(i)}(x)) \circ {\psi^i_{\zeta(i)}}'(x) - \operatorname{id} \circ {\psi^i_{\zeta(i)}}'(x)\|_{op} + \|{\psi^i_{\zeta(i)}}'(x) - \operatorname{id}\|_{op} \\
\leqslant &\|{\psi^i_{\eta(i)}}'(\psi^i_{\zeta(i)}(x)) - \operatorname{id}\|_{op} \cdot \|{\psi^i_{\zeta(i)}}'(x)\|_{op} + \|{\psi^i_{\zeta(i)}}'(x) - \operatorname{id}\|_{op} \\
< &\frac{\varepsilon_{\mathcal{U}}}{4K_\alpha} \cdot \frac{1}{4} \cdot 2 + \frac{\varepsilon_{\mathcal{U}}}{4K_\alpha} \cdot \frac{1}{4} \leqslant \frac{\varepsilon_{\mathcal{U}}}{4K_\alpha}.
\end{aligned}$$

For all $x \in B_3^{k_i}(0)$, we have

$$\eta_{(i)} \diamond \zeta_{(i)}(x) = \alpha_i\left(x, (\psi^i_{\eta(i)} \circ \psi^i_{\zeta(i)} - \operatorname{id})(x)\right) = \alpha_i(x, h(x)).$$

Let $x \in B_3^{k_i}(0)$ and $v \in \mathbb{R}^m$. We show that $\|d(\eta_{(i)} \diamond \zeta_{(i)})(x; v)\| \leqslant \frac{\varepsilon_{\mathcal{U}}}{2}\|v\|$:

$$\begin{aligned}
&\|d(\eta_{(i)} \diamond \zeta_{(i)})(x; v)\| = \|d\alpha_i(x, h(x); v, dh(x, v))\| \\
\leqslant &\|d_1\alpha_i(x, h(x); v)\| + \|d_2\alpha_i(x, h(x); dh(x, v))\| \\
\leqslant &\left(\|d_1\alpha_i(x, h(x); \bullet)\|_{op} + \|d_2\alpha_i(x, h(x); \bullet)\|_{op} \cdot \|dh(x, \bullet))\|_{op}\right) \cdot \|v\| \\
< &\left(\frac{\varepsilon_{\mathcal{U}}}{4} + K_\alpha \cdot \frac{\varepsilon_{\mathcal{U}}}{4K_\alpha}\right)\|v\| = \frac{\varepsilon_{\mathcal{U}}}{2}\|v\|.
\end{aligned}$$

□

Definition 1.71. Let $R \in [1, 5]$, $i \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$. We use the shorthand notation $K_i := \overline{B}_R^{k_i}(0)$ and define the space

$$\mathcal{H}_{R,j} := \left\{(f_i)_i \in \prod_{i=1}^n \operatorname{Hol}_b^1(V_{R,j}^{k_i}; \mathbb{C}^m)_{\mathrm{st}}^{\mathbb{R}} : (\exists \eta \in \Gamma_{\mathrm{st}}^\omega(TM))\ (\forall i)\ f_i|_{K_i} = \eta_{(i)}|_{K_i}\right\}.$$

Moreover, we define the open subset

$$\mathcal{W}_\varepsilon^{R,j} := \left\{(f_i)_i \in \mathcal{H}_{R,j} : (\forall i)\ \|f_i\|_\infty^1 < \varepsilon\right\}$$

for $\varepsilon > 0$.

Remark 1.72. *Let $R \in [1, 5]$. We use the shorthand notation $K_i := \overline{B}_R^{k_i}(0)$. Let $\Phi\colon \Gamma_{\mathrm{st}}^\omega(TM) \hookrightarrow \prod_{i=1}^n \mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|K_i)_{\mathrm{st}}^{\mathbb{R}}$, $\eta \mapsto ([\eta_{(i)}])$ be the canonical embedding (see Definition 1.26). We write $F := \operatorname{im}(\Phi)$ for the closed image of Φ (see Definition 1.26) and $E_j := \prod_{i=1}^n \operatorname{Hol}_b^1(V_{R,j}^{k_i}; \mathbb{C}^m)_{\mathrm{st}}^{\mathbb{R}}$. We identify $\operatorname{Hol}_b^1(V_{R,j}^{k_i}; \mathbb{C}^m)_{\mathrm{st}}^{\mathbb{R}}$ with the corresponding germs in $\mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|K_i)$. As finite products and inductive limits of ascending sequences of locally convex spaces commute, we get*

$$\prod_{i=1}^n \mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|K_i)_{\mathrm{st}}^{\mathbb{R}} = \varinjlim_{j \in \mathbb{N}} \prod_{i=1}^n \operatorname{Hol}_b^1(V_{R,j}^{k_i}; \mathbb{C}^m)_{\mathrm{st}}^{\mathbb{R}} = \varinjlim E_j$$

as Silva spaces (see [Glo11, p. 260, Proposition 4.4 (d)] and Lemma 1.25). As

$\mathcal{H}_{R,j} = E_j \cap F$, we see that $\mathcal{H}_{R,j}$ is a Banach space. From Remark 1.22, we see that $F = \varinjlim E_j \cap F = \varinjlim \mathcal{H}_{R,j}$. Now $F = \bigcup_{j\in\mathbb{N}} \mathcal{H}_{R,j}$ and $\Phi(\mathcal{B}^1_{R,\varepsilon}) \subseteq \bigcup_{j\in\mathbb{N}} \mathcal{W}^{R,j}_{\varepsilon}$ for $\varepsilon > 0$.

Lemma 1.73. *For all $l \in \mathbb{N}$, we define $l' := 2l + N_0$ (with N_0 like in Definition 1.61). Let $j \in \mathbb{N}$, $i_0 \in \{1, \dots, n\}$, $(f_i)_i \in \mathcal{W}^{4,j}_{\varepsilon_\diamond}$, $l \in \mathbb{N}$ with $l \geqslant j$ and $R \in]1, 4[$. Then*

$$\psi^{i_0}_{f_{i_0}}(\overline{V}^{k_{i_0}}_{R-1,l'}) \subseteq V^{k_{i_0}}_{R,l} \text{ and } \|\psi^{i_0}_{f_{i_0}} - \mathrm{id}\|^0_{V^{k_{i_0}}_{4,j}} < r^{\mathbb{C}}_\diamond \leqslant \frac{\delta^{\mathbb{C}}(\varepsilon_{\exp*})}{2}.$$

Proof. This follows from Lemma 1.62 and Lemma 1.64 with $r_0 < 1$. □

The following lemma is a consequence of [AS15, Remark 4.10].

Lemma 1.74. *Let $U \subseteq \mathbb{C}^m$ be open and $f\colon \mathbb{R} \times U \to \mathbb{C}^m$ a map that is complex analytic in the second argument. In this situation, the map f considered as a map between finite dimensional $\mathbb{R}$-vector spaces is smooth if and only if the map $\check{f}\colon \mathbb{R} \to \mathrm{Hol}(U; \mathbb{C}^m)$ considered as a map between locally convex $\mathbb{R}$-vector spaces is smooth.*

Lemma 1.75. *For all $l \in \mathbb{N}$, we define $l' := 2l + N_0$. Let $j \in \mathbb{N}$, $i_0 \in \{1, \dots, n\}$, $(f_i)_i, (g_i)_i \in \mathcal{W}^{4,j}_{\varepsilon_\diamond}$. Then the map*

$$\begin{aligned} f_{i_0} \diamond g_{i_0} \colon V^{k_{i_0}}_{3,j'} &\to \mathbb{C}^m, \\ x &\mapsto \exp^*_{i_0}(x, \bullet)^{-1}(\psi^{i_0}_{f_{i_0}} \circ \psi^{i_0}_{g_{i_0}}(x)) \end{aligned}$$

makes sense and is complex analytic. Moreover, the map $\Phi\colon \mathcal{W}^{4,j}_{\varepsilon_\diamond} \times \mathcal{W}^{4,j}_{\varepsilon_\diamond} \to \mathrm{Hol}(V^{k_{i_0}}_{3,j'}; \mathbb{C}^m)$, $((f_i)_i, (g_i)_i) \mapsto f_{i_0} \diamond g_{i_0}$ is smooth over $\mathbb{R}$ if we consider $\mathrm{Hol}(V^{k_{i_0}}_{3,j'}; \mathbb{C}^m)$ as a vector space over $\mathbb{R}$.

Proof. Lemma 1.73 implies that $f_{i_0} \diamond g_{i_0} \colon V^{k_{i_0}}_{3,j'} \to \mathbb{C}^m$ makes sense and is complex analytic. Now we show the smoothness of Φ. As mentioned above, we consider $\mathrm{Hol}(V^{k_{i_0}}_{3,j'}; \mathbb{C}^m)$ as a vector space over $\mathbb{R}$. In our situation, a map is smooth in the sense of Keller's C^∞_c-theory if and only if it is smooth in the sense of the convenient setting (see [BGN04, p. 270] and [KM97, Theorem 4.11]). Let $\beta, \gamma\colon \mathbb{R} \to \mathcal{W}^{4,j}_{\varepsilon_\diamond}$ be smooth curves. We write $c(t) := (\beta(t), \gamma(t))$ and $\beta_i := \mathrm{pr}_i\, \beta$ respectively $\gamma_i := \mathrm{pr}_i\, \gamma$ for the i-th component. We have to show that $\Phi \circ c\colon \mathbb{R} \to \mathrm{Hol}(V^{k_{i_0}}_{3,j'}; \mathbb{C}^m)$ is smooth over $\mathbb{R}$. Because of Lemma 1.74, it suffices to show that $\mathbb{R} \times V^{k_{i_0}}_{3,j'} \to \mathbb{C}^m$, $(t, x) \mapsto \Phi(c(t))(x)$ is smooth over $\mathbb{R}$. Unwinding the definitions we get

$$\begin{aligned} &\Phi(c(t))(x) \\ &= \exp^*_{i_0}(x, \bullet)^{-1}\Big(\exp^*_{i_0}\Big(\exp^*_{i_0}\big(x, \beta_{i_0}(t)(x)\big), \beta_{i_0}(t)\big(\exp^*_{i_0}(x, \gamma_{i_0}(t)(x))\big)\Big)\Big). \end{aligned} \tag{1.13}$$

The inclusion $\mathrm{Hol}^1_b(V^{k_{i_0}}_{4,j}; \mathbb{C}^m)^{\mathbb{R}}_{\mathrm{st}} \hookrightarrow \mathrm{Hol}(V^{k_{i_0}}_{4,j}; \mathbb{C}^m)$ is continuous linear. Therefore, the maps $\mathbb{R} \times V^{k_{i_0}}_{4,j} \to \mathbb{C}^m$, $(t, x) \mapsto \beta_{i_0}(t, x)$ and $\mathbb{R} \times V^{k_{i_0}}_{4,j} \to \mathbb{C}^m$, $(t, x) \mapsto \gamma_{i_0}(t, x)$ are smooth over $\mathbb{R}$. Now the assertion follows from (1.13). □

Now we come to the central lemma of this subsection.

Lemma 1.76. *The map*

$$\diamond\colon \mathcal{B}^1_{4,\varepsilon_\diamond} \times \mathcal{B}^1_{4,\varepsilon_\diamond} \to \mathcal{B}^1_{1,\varepsilon_\mathcal{U}},\ (\eta,\zeta) \mapsto \eta \diamond \zeta$$

defined in Lemma 1.70 is smooth.

Proof. In this proof, all vector spaces except for $\mathbb{C}^m$ are considered as $\mathbb{R}$ vector spaces. If $k \in \mathbb{N}$ we write $k' := 2k + N_0$. After identification we have $\mathcal{B}^1_{1,\varepsilon_\mathcal{U}} \subseteq \prod_{i=1}^n \mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^{k_i}_1(0))^\mathbb{R}_{\mathrm{st}}$ (see Definition 1.26) and it is left to show the smoothness of

$$\begin{aligned}\mathcal{B}^1_{4,\varepsilon_\diamond} \times \mathcal{B}^1_{4,\varepsilon_\diamond} &\to \mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^{k_{i_0}}_1(0))^\mathbb{R}_{\mathrm{st}},\\ (\eta,\zeta) &\mapsto \left[(\eta\diamond\zeta)^*{}_{(i_0)}\right]_{\overline{B}^{k_{i_0}}_1(0)} = \left[(\eta_{(i_0)}\diamond\zeta_{(i_0)})^*\right]_{\overline{B}^{k_{i_0}}_1(0)}\end{aligned}$$

for all $i_0 \in \{1,\dots,n\}$. Because of Remark 1.72 and Remark 1.22 it suffices to show the smoothness of $\mathcal{W}^{4,j}_{\varepsilon_\diamond} \times \mathcal{W}^{4,j}_{\varepsilon_\diamond} \to \mathcal{G}(\mathbb{C}^m;\mathbb{C}^m|\overline{B}^{k_{i_0}}_1(0))$, $((f_i)_i,(g_i)_i) \mapsto [f_{i_0}\diamond g_{i_0}]_{\overline{B}^{k_{i_0}}_1(0)}$ over $\mathbb{R}$ for all $j \in \mathbb{N}$. But this follows from Lemma 1.75. □

1.5. Smoothness of the inversion

In this section, we prove the smoothness of the inversion in the local chart Φ.

Lemma 1.77. *Let $l \in \mathbb{N}$. As before, we define $l' := 2l + N_0$. There exists $\sigma \in]0,\varepsilon_{\exp*}[$ such that for all $i \in \{1,\dots,n\}$, $x \in V^{k_i}_{2,l'}$ and $y \in B^\mathbb{C}_\sigma(0)$ we have*

(i) $\exp^*_i(x,y) \in V^{k_i}_{3,l}$;

(ii) $\|d_2\exp^*_i(x,y;\bullet) - \mathrm{id}\|_{op} < \frac{1}{2}$.

In particular, we have $\|d_2\exp^*_i(x,y)\|_{op} \leqslant \frac{3}{2}$.

Proof. Let $i \in \{1,\dots,n\}$. We have $\exp^*_i(x,0) \in V^{k_i}_{3,l}$ for all $x \in \overline{V}^{k_i}_{2,l'}$ and so $\overline{V}^{k_i}_{3,l'} \times \{0\} \subseteq (\exp^*_i)^{-1}(V^{k_i}_{3,l})$. Moreover, $d_2\exp^*_i(x,0;\bullet) - \mathrm{id}_{\mathbb{C}^m} = 0$ for all $x \in \overline{V}^{k_i}_{2,l'}$. Let $h\colon V^{k_i}_{3,l} \times B^\mathbb{C}_{\varepsilon_{\exp*}}(0) \to [0,\infty[$, $(x,y) \mapsto \|d_2\exp^*_i(x,y) - \mathrm{id}_{\mathbb{C}^m}\|_{op}$. Then $\overline{V}^{k_i}_{2,l'} \times \{0\} \subseteq h^{-1}([0,\frac{1}{2}[)$. The rest follows from Wallace's Lemma. □

Definition 1.78. We use the constants of Lemma 1.68 and Lemma 1.77. Let $r_\star := \min\left(\frac{1}{4}, \delta_\mathcal{U}, \frac{\varepsilon_\mathcal{U}}{4\cdot K_\alpha}, \nu_\alpha, \frac{\varepsilon_\partial}{2}\right)$, $r^\mathbb{C}_\star := \min(\frac{1}{4}, \delta^\mathbb{C}(\sigma))$ and $\varepsilon_\star := \min(\varepsilon_\mathcal{U}, \varepsilon_{3,r_\star}, \varepsilon^\mathbb{C}_{r^\mathbb{C}_\star})$.

In the case of a smooth manifold without corners, one can use smooth bump functions to show the smoothness of the inversion [Glo06c, Lemma 3.8]. Obviously this is not possible in the real analytic case. As in [Sch15, Lemma D.4], we use a quantitative inverse function theorem to show the smoothness of the inversion. Therefore, it was necessary to show the quantitative inverse function theorem for open sets with corners (Lemma 1.49) to apply it to show Lemma 1.52.

Lemma 1.79. *Let $\eta \in \mathcal{B}^1_{3,\varepsilon_\star}$ and $i \in \{1, \dots, n\}$*

(a) The map ${\psi^i_{\eta_{(i)}}}^{-1}\colon B^{k_i}_2(0) \to \mathbb{R}^m_{k_i}$ makes sense. Moreover, the map

$$(\eta_{(i)})^\star\colon B^{k_i}_2(0) \to \mathbb{R}^m,\ x \mapsto \ell_i\left(x, {\psi^i_{\eta_{(i)}}}^{-1}(x)\right)$$

makes sense and is a stratified real analytic function with $\|(\eta_{(i)})^\star(x)\| < \min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)$ for all $x \in B^{k_i}_2(0)$ and $\psi^i_{(\eta_{(i)})^\star} = (\psi^i_{\eta_{(i)}})^{-1}$ on $B^{k_i}_2(0)$.

(b) The map $\psi_\eta^{-1}\colon M \to M$ makes sense and $\psi_\eta^{-1}(U_{i,2}) \subseteq U_{i,3}$. Moreover, $\psi_\eta^{-1}|_{U_{i,2}} = \varphi_i^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1} \circ \varphi_i|_{U_{i,2}}$.

(c) The vector field

$$\eta^\star\colon M \to TM,\ p \mapsto \exp|^{-1}_{W_pM}(\psi_\eta^{-1}(p))$$

makes sense, is real analytic and stratified. Moreover, we have $\psi_{\eta^\star} = \psi_\eta^{-1}$ and

$$(\eta^\star)_{(i)}|_{B^{k_i}_2(0)} = (\eta_{(i)})^\star. \tag{1.14}$$

(d) We have $\eta^\star \in \mathcal{V} = \mathcal{B}^1_{1,\varepsilon_{\mathcal{U}}}$.

Proof. (a) Because $\varepsilon_\star \leqslant \varepsilon_{3,r_\star}$ and $r_\star \leqslant \min(\delta_U, \frac{1}{4}, \frac{\varepsilon_\partial}{2})$, we can use Lemma 1.36 and see that $(\psi^i_{\eta_{(i)}})^{-1}\colon B^{k_i}_2(0) \to \mathbb{R}^m_{k_i}$ makes sense and that $\|(\psi^i_{\eta_{(i)}})^{-1}(x) - x\| < \min(\delta_{\mathcal{U}}, \frac{\varepsilon_\partial}{2})$ for all $x \in B^{k_i}_2(0)$. Hence, the map $(\eta_{(i)})^\star\colon B^{k_i}_2(0) \to \mathbb{R}^m$, $x \mapsto \ell_i(x, (\psi^i_{\eta_{(i)}})^{-1}(x))$ makes sense and $\|(\eta_{(i)})^\star(x)\| < \min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial) \leqslant \varepsilon_{\exp}$ for all $x \in B^{k_i}_2(0)$. Thus

$$\exp_i(x, \eta^\star_{(i)}(x)) = (\psi^i_{\eta_{(i)}})^{-1}(x) \text{ for all } x \in B^{k_i}_2(0). \tag{1.15}$$

Now let $x \in \partial^j B^{k_i}_2(0)$. Obviously $y := (\psi^i_{\eta_{(i)}})^{-1}(x) \in \partial^j B^{k_i}_3(0)$. Let C be the connected component of x in $\partial^j B^{k_i}_5(0)$. Remark 1.67 implies $y \in C$. We have $v := \eta^\star_{(i)}(x) \in B_{\varepsilon_\partial}(0)$ and $\exp^i_x(v) = y$ and $\|y - x\| < \frac{\varepsilon_\partial}{2}$. From Lemma 1.65, we get $v \in T_x\partial^j B^{k_i}_5(0)$ and so $(\eta_{(i)})^\star \in C^\omega(B^{k_i}_2(0); \mathbb{R}^m)_{\mathrm{st}}$.

(b) Because $\varepsilon_\star \leqslant \varepsilon_{\mathcal{U}}$, the map $\psi_\eta\colon M \to M$ is a real analytic diffeomorphism. Using Lemma 1.34, Lemma 1.36 and $r_\star \leqslant \frac{1}{4}$, we calculate

$$U_{i,2} = \varphi_i^{-1}(B^{k_i}_2(0)) \subseteq \varphi_i^{-1}(\psi^i_{\eta_{(i)}}(B^{k_i}_3(0))) = \varphi_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \varphi_i(U_{i,3}) = \psi_\eta(U_{i,3}).$$

Let $p \in U_{i,2}$. To see $\psi_\eta^{-1}(p) = \varphi_i^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1} \circ \varphi_i(p)$, we use Lemma 1.34 and obtain $\psi_\eta\left(\varphi_i^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1} \circ \varphi_i(p)\right) = p$.

(c) We show that the map $\eta^\star\colon M \to TM$, $p \mapsto \exp|^{-1}_{W_pM}(\psi_\eta^{-1}(p))$ makes sense. Let $p \in M$ and $i \in \{1, \dots, n\}$ such that $p \in U_{i,1}$. We define $x := \varphi_i(p)$ and $v := T\varphi_i^{-1}(x, (\eta_{(i)})^\star(x))$. As $\varepsilon_{\mathcal{U}} \leqslant \varepsilon_{inj}$, we get $v \in W_pM$. Next we calculate

$$\exp(v) = \varphi_i^{-1} \circ \exp_i(x, \eta^\star_{(i)}(x)) = \varphi_i^{-1} \circ \psi^i_{\eta_{(i)}} \circ \varphi_i(p) = \psi_\eta^{-1}(p).$$

Hence, $\eta^\star$ makes sense. Obviously $\psi_{\eta^\star} = \psi_\eta^{-1}$. Now let $x \in B_2^{k_i}(0)$ and $p := \varphi_i^{-1}(x)$. We calculate

$$\begin{aligned}(\eta^\star)_{(i)}(x) &= d_p\varphi_i \circ \exp|_{W_pM}^{-1} \circ \psi_\eta^{-1}(p)\\ &= d_p\varphi_i \circ \exp|_{W_pM}^{-1} \circ \varphi_i^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1}(\varphi_i(p))\\ &= \left(\varphi_i \circ \exp|_{W_pM} \circ d_p\varphi_i^{-1}\right)^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1}(x)\\ &= \exp_i(x, \bullet)^{-1} \circ (\psi^i_{\eta_{(i)}})^{-1}(x) = (\eta_{(i)})^\star(x).\end{aligned}$$

This shows (1.14). We conclude with this local representation and (a) that $\eta^\star$ is real analytic and stratified.

(d) It is enough to show that $\|d\eta^*_{(i)}(x,v)\| \leqslant \frac{\varepsilon_{\mathcal{U}}}{2} \cdot \|v\|$ for all $x \in B_2^{k_i}(0)$ and $v \in \mathbb{R}^m$. Since $r_\star < \delta_U$, we have $(\psi^i_{\eta_{(i)}})^{-1}(x) - x \in B_{\delta_{\mathcal{U}}}(0)$ (Lemma 1.36) and so

$$\eta^\star_{(i)}(x) = \alpha_i\left(x, (\psi^i_{\eta_{(i)}})^{-1}(x) - x\right)$$

for all $x \in B_2^{k_i}(0)$. Now let $x \in B_2^{k_i}(0)$ and $v \in \mathbb{R}^m$. We get

$$\begin{aligned} d\eta^\star_{(i)}(x,v) &= d_1\alpha_i\left(x, (\psi^i_{\eta_{(i)}})^{-1}(x) - x; v\right)\\ &+ d_2\alpha_i\left(x, (\psi^i_{\eta_{(i)}})^{-1}(x) - x; d(\psi^i_{\eta_{(i)}})^{-1}(x;v) - v\right). \end{aligned} \tag{1.16}$$

Using $r_\star \leqslant \nu_\alpha$ and Lemma 1.36, we see that $(\psi^i_{\eta_{(i)}})^{-1}(x) - x \in B_{\nu_\alpha}(0)$ and so $\|d_1\alpha_i(x, (\psi^i_{\eta_{(i)}})^{-1}(x) - x; v)\| \leqslant \frac{\varepsilon_{\mathcal{U}}}{4}\|v\|$. Analogously we get

$$\begin{aligned}&\|d_2\alpha_i(x, (\psi^i_{\eta_{(i)}})^{-1}(x) - x; d(\psi^i_{\eta_{(i)}})^{-1}(x;v) - v)\|\\ \leqslant &K_\alpha \cdot \|d(\psi^i_{\eta_{(i)}})^{-1}(x;\bullet) - \mathrm{id}\,\|_{op} \cdot \|v\| \leqslant \frac{\varepsilon_{\mathcal{U}}}{4}\|v\|.\end{aligned}$$

Now the assertion follows from (1.16).

□

Lemma 1.80. *Let $j \in \mathbb{N}$, $i_0 \in \{1, \ldots, n\}$, $j' := 2j + N_0$ (with N_0 from Definition 1.61) and $(f_i)_i \in \mathcal{W}^{3,j}_{\varepsilon_\star}$.*

(a) We have $V^{k_{i_0}}_{2,j'} \subseteq \psi^{i_0}_{f_{i_0}}(V^{k_{i_0}}_{3,j})$, the map $(\psi^{i_0}_{f_{i_0}})^{-1}\colon V^{k_{i_0}}_{2,j'} \to V^{k_{i_0}}_{3,j}$ is complex analytic and $\|(\psi^{i_0}_{f_{i_0}})^{-1} - \mathrm{id}_{\mathbb{C}^m}\|^1_{V^{k_{i_0}}_{2,j'}} < \delta^{\mathbb{C}}(\sigma)$. Moreover, the map $f^\star_{i_0}\colon V^{k_{i_0}}_{2,j'} \to B^{\mathbb{C}}_\sigma(0)$, $x \mapsto \exp^*_{i_0}(x, \bullet)^{-1}((\psi^{i_0}_{f_{i_0}})^{-1}(x))$ makes sense and is complex analytic.

(b) For all $i_0 \in \{1, \ldots, n\}$, the map $\Phi\colon \mathcal{W}^{3,j}_{\varepsilon_\star} \to \mathrm{Hol}(V^{k_{i_0}}_{2,j'}; \mathbb{C}^m)$, $(f_i)_i \mapsto f^\star_{i_0}$ is smooth over $\mathbb{R}$.

Proof. (a) This follows from Lemma 1.64.

(b) We follow the ideas of [Glo06c, Lemma 3.8]. It suffices to show the assertion in the convenient setting. Let $c\colon \mathbb{R} \to \mathcal{W}^{3,j}_{\varepsilon_\star}$, $t \mapsto c_t$ be smooth. We write $(c_t)_{i_0}$ for the i_0-th component of c_t. We have to show that $\widehat{\Phi \circ c}\colon \mathbb{R} \times V^{k_{i_0}}_{2,j'} \to \mathbb{C}^m$,

$(t,x) \mapsto c_t^\star(x)$ is smooth over $\mathbb{R}$. From (a), we get $c_t^\star \in \mathrm{Hol}^0_\sigma(V^{k_{i_0}}_{2,j'};\mathbb{C}^m)$. Hence, $\psi^{i_0}_{c_t^\star}(V^{k_{i_0}}_{2,j'}) \subseteq V^{k_{i_0}}_{3,j}$. Therefore, $\psi^{i_0}_{(c_t)_{i_0}} \circ \psi^{i_0}_{c_t^\star}$ makes sense and $\psi^{i_0}_{(c_t)_{i_0}} \circ \psi^{i_0}_{c_t^\star}(x) = x$ for all $x \in V^{k_{i_0}}_{2,j'}$. Therefore

$$\exp^*_{i_0}(\exp^*_{i_0}(x, c_t^\star(x)), (c_t)_{i_0}(\exp^*_{i_0}(x, c_t^\star(x)))) - x = 0 \text{ for all } x \in V^{k_{i_0}}_{2,j'} \quad (1.17)$$

for all $x \in V^{k_{i_0}}_{2,j'}$. We define the smooth function

$$\Lambda\colon \mathbb{R} \times V^{k_{i_0}}_{2,j'} \times B^{\mathbb{C}}_\sigma(0) \to \mathbb{C}^m$$
$$(t,x,y) \mapsto \exp^*_{i_0}(\exp^*_{i_0}(x,y), (c_t)_{i_0}(\exp^*_{i_0}(x,y))) - x = \psi^{i_0}_{(c_t)_{i_0}}(\exp^*_{i_0}(x,y)) - x$$

For $(t,x,y) \in \mathbb{R} \times V^{k_{i_0}}_{2,j'} \times B^{\mathbb{C}}_\sigma(0)$, we use Lemma 1.77 and calculate

$$\begin{aligned}
&\|d_3\Lambda(t,x,y;\bullet) - \mathrm{id}\,\|_{op} = \|d\psi_{(c_t)_{i_0}}(\exp^*_{i_0}(x,y); d_2\exp^*_{i_0}(x,y;\bullet)) - \mathrm{id}\,\|_{op} \\
=&\|d\psi_{(c_t)_{i_0}}(\exp^*_{i_0}(x,y);\bullet) \circ d_2\exp^*_{i_0}(x,y;\bullet) - \mathrm{id}\,\|_{op} \\
\leqslant&\|d\psi_{(c_t)_{i_0}}(\exp^*_{i_0}(x,y);\bullet) \circ d_2\exp^*_{i_0}(x,y;\bullet) - \mathrm{id} \circ d_2\exp^*_{i_0}(x,y;\bullet)\|_{op} \\
&+ \|\,\mathrm{id} \circ d_2\exp^*_{i_0}(x,y;\bullet) - \mathrm{id}\,\|_{op} \\
\leqslant&\|d\psi_{(c_t)_{i_0}}(\exp^*_{i_0}(x,y);\bullet) - \mathrm{id}\,\|_{op} \cdot \|d_2\exp^*_{i_0}(x,y;\bullet)\|_{op} + \|d_2\exp^*_{i_0}(x,y;\bullet) - \mathrm{id}\,\|_{op} \\
<&\frac{1}{4}\cdot\frac{3}{2} + \frac{1}{2} = \frac{7}{8}.
\end{aligned}$$

Hence, $d_3\Lambda(t,x,y;\bullet) \in \mathrm{GL}(\mathbb{C}^m)$. Obviously, $\Lambda(t,x,\bullet)\colon B^{\mathbb{C}}_\sigma(0) \to \mathbb{C}^m$ is injective. From the implicit function theorem and (1.17), we see that Φ is smooth.

□

Lemma 1.81. *The map*

$$i_M\colon \mathcal{B}^1_{3,\varepsilon_\star} \to \mathcal{B}^1_{1,\varepsilon_{\mathcal{U}}},\ \eta \mapsto \eta^\star$$

defined in Lemma 1.79 is smooth.

Proof. Analogous to the proof of Lemma 1.76, this follows from Lemma 1.80 (b).

□

1.6. Existence and uniqueness of the Lie group structure

In this section, we follow the strategy of [Glo06c, Section 5]: First, we use the theorem about the local description of Lie groups to obtain a Lie group structure on a subgroup $\mathrm{Diff}^\omega(M)_0$ of $\mathrm{Diff}^\omega(M)$. Then we show that this structure does not depend on the choice of the Riemannian metric (Lemma 1.85). With the help of this result, we show the smoothness of the conjugation map (Lemma 1.87).

The following lemma comes from [Glo06c, Proposition 1.20]:

Lemma 1.82 (Theorem about the local description of Lie groups)**.** *Let G be a group and $U \subseteq G$ a subset that is a smooth manifold such that there exists a symmetric subset $V \subseteq U$ that contains the identity and fulfils $V \cdot V \subseteq U$. If the restriction of the inversion and the multiplication on V are smooth maps then there exists a unique manifold structure on $\langle V \rangle$ such that:*
(i) $\langle V \rangle$ is a Lie group;
(ii) V is open in $\langle V \rangle$;
(iii) U and $\langle V \rangle$ induce the same manifold structure on V.
Moreover, if $\langle V \rangle$ is a normal subgroup of G and for all $g \in G$ the conjugation $\mathrm{int}_g \colon \langle V \rangle \to \langle V \rangle$, $h \mapsto ghg^{-1}$ is smooth, then there exists a unique manifold structure on G such that
(i) G becomes a Lie group;
(ii) V is an open submanifold of G.

In the following we introduce a notational convention (cf. [GN] and [Str06, Definition 9.1]).

Convention 1.83. Let X and Y be sets. If $X' \subseteq X$ and $Y' \subseteq Y$, we write $\lfloor X', Y' \rfloor := \{f \colon X \to Y : f(X') \subseteq Y'\}$.

Adapting [Glo04, Proposition 4.23] to our situation, we obtain the following lemma.

Lemma 1.84. *Let $U, Z, U_g, V_g \subseteq \mathbb{C}^m$ be open subsets such that $\overline{Y} \subseteq U \subseteq U_g$ and $\overline{Y}$ is compact (here the closure of Y is taken in $\mathbb{C}^m$). If $g \colon U_g \times V_g \to \mathbb{C}^m$ is a complex analytic map then*

$$\mathrm{Hol}(U; \mathbb{C}^m) \cap \lfloor \overline{Y}, V_g \rfloor \to \mathrm{Hol}(Y; \mathbb{C}^m),\ \gamma \mapsto g(\bullet, \gamma(\bullet))|_Y$$

is a complex analytic map.

We follow the line of thought of [Glo06c, 4.28, 5.1 and 5.3] in the following lemma.

Lemma 1.85. *(a) Let $i_M \colon \mathcal{B}^1_{3,\varepsilon_\star} \to \mathcal{B}^1_{1,\varepsilon_\mathcal{U}}$, $\eta \mapsto \eta^\star$ be the map defined in Lemma 1.81. There exists $\varepsilon_0 \in]0, \varepsilon_\star[$ such that $\mathcal{B}^1_{4,\varepsilon_0} \subseteq i_M^{-1}(\mathcal{B}^1_{3,\varepsilon_\star})$.*
(b) Let $\mathcal{U}_\star := \Psi(\mathcal{B}^1_{3,\varepsilon_\star})$ and $\iota \colon \mathcal{U}_\star \to \mathcal{U}$ be the inversion of $\mathrm{Diff}^\omega(M)$. The set $\mathcal{U}^1_0 := \Psi(\mathcal{B}^1_{4,\varepsilon_0})$ is an open connected 1-neighbourhood. Moreover, the set $\mathcal{U}_0 := \mathcal{U}^1_0 \cup \iota(\mathcal{U}^1_0) \subseteq \mathcal{U}_\star$ is an open connected symmetric 1-neighbourhood. We define $\mathcal{V}_0 := \Phi^{-1}(\mathcal{U}_0)$ and $\mathcal{V}^1_0 := \mathcal{B}^1_{4,\varepsilon_0}$.
(c) Analogous to [Glo06c, 6.2], we can use Lemma 1.82 to find a unique Lie group structure on $\mathrm{Diff}^\omega(M)_0 := \langle \mathcal{U}_0 \rangle$.
(d) The Lie group structure in (c) is independent of the choice of the atlas $\varphi_i \colon U_{i,5} \to B^{k_i}_5(0)$ (see Lemma 1.16).
(e) The Lie group structure in (c) is independent of the choice of the Riemannian metric g.

Proof. (a) This follows from the same argument as in Lemma 1.81.

(b) Obviously, $\mathcal{U}_0^1 := \Psi(\mathcal{B}^1_{4,\varepsilon_0})$ is an open connected 1-neighbourhood. As in [Glo06c, p. 4.28], the rest of the assertion follows from $\iota^{-1}(\mathcal{U}_0^1) = \iota(\mathcal{U}_0^1)$. In fact $\mathcal{U}_0 := \mathcal{U}_0^1 \cup \iota(\mathcal{U}_0^1)$ is open because of $\iota^{-1}(\mathcal{U}_0^1) = \iota(\mathcal{U}_0^1)$ and obviously it is connected. Moreover $\mathcal{U}_0$ is obviously symmetric.

(c) Clear.

(d) Let $\varphi_i' \colon U'_{i,5} \to B_5^{k_i'}(0)$ with $i \in \{1, \ldots, n'\}$ be another atlas with the same properties. We find an analogous open 0-neighbourhood $V_0' \subseteq \Gamma^\omega_{\mathrm{st}}(TM)$ such that $\mathcal{U}_0' := \Phi(V_0')$ generates a Lie group $\mathrm{Diff}^\omega(M)_0' \subseteq \mathrm{Diff}^\omega(M)$. We define the open 0-neighbourhood $W := \mathcal{V}_0 \cap \mathcal{V}_0'$. Then $X := \Phi(W) \subseteq \mathrm{Diff}^\omega(M)$. Obviously, we get $X \subseteq \mathrm{Diff}^\omega(M)_0 \cap \mathrm{Diff}^\omega(M)_0'$. Moreover, X is open in $\mathrm{Diff}^\omega(M)_0$ and in $\mathrm{Diff}^\omega(M)_0'$. Because both Lie groups are connected, we get $\mathrm{Diff}^\omega(M)_0 = \langle X \rangle = \mathrm{Diff}^\omega(M)_0'$ in the sense of sets and in the sense of Lie groups.

(e) Let g' be another Riemannian metric with the same properties as g. In the following, all objects induced by g' are written with an extra "$'$". Considering Definition 1.68, we find $\nu \in]0, \nu_\alpha[$ such that $\|d_1\alpha_i(x, y; \bullet)\|_{op} < \frac{\varepsilon_0}{4}$ for all $x \in \overline{B}_4(0)$ and $y \in \overline{B}_\nu(0)$. Moreover, we can choose N_0 in Definition 1.61 so large that $\frac{1}{N_0} < \min(r_{\ell*}, r'_{\ell*})$. We choose $\varepsilon > 0$ with

$$\varepsilon < \min(\varepsilon'_{4,\delta_{\mathcal{U}}}, \varepsilon'_{4,\frac{\varepsilon_\partial}{2}}, \varepsilon'_{4,\delta(\frac{\varepsilon_0}{2})}, \varepsilon'_{4,\nu}, \varepsilon'_{\mathcal{U}}, \varepsilon'_{4,\frac{\varepsilon_0}{4K_\alpha}}, \varepsilon'^{\mathbb{C}}_{\delta^{\mathbb{C}}(\exp*)})$$

Let $\eta \in \mathcal{B}^1_{4,\varepsilon}$. For $i \in \{1, \ldots, n\}$, the map $\eta^\dagger_{(i)} \colon B_4^{k_i}(0) \to \mathbb{R}^m$, $x \mapsto \ell^i(x, \exp_i'(x, \eta(x)))$ makes sense and is real analytic. Moreover, the map is stratified: Let $x \in \partial^j B_4^{k_i}(0)$ and C the connected component of x in $\partial^j B_5^{k_i}(0)$. We define $y := (\psi')^i_{\eta_{(i)}}(x)$ and $v := \eta^\dagger_{(i)}(x) \in B_{\varepsilon_\partial}(0)$. We have $\|y - x\| < \frac{\varepsilon_\partial}{2}$. From Remark 1.67 we deduce that $x, y \in C$ and $y = \exp^i_x(v)$. From (1.12) and Lemma 1.65 we deduce that $v \in T_x \partial^j B_5^{k_i}(0)$. Hence, $\eta_{(i)}$ is stratified. Moreover, the map $\eta^\dagger \colon M \to TM$, $p \mapsto \exp|^{-1}_{W_pM}(\exp'(\eta(p)))$ makes sense and is a stratified, real analytic vector field of M. To see this, we choose a chart φ_i around p and define $x := \varphi_i(p)$ and $v := \ell_i(x, \exp_i'(x, \eta_{(i)}(x)))$. Since $v \in B_{\min(\varepsilon_{\mathcal{U}}, \varepsilon_\partial)}(0)$, we get $T\varphi_i^{-1}(x, v) \in W_pM$. The map $\eta^\dagger$ makes sense because $\exp(T\varphi_i^{-1}(x, v)) = \exp'(\eta(p))$. Obviously we have $(\eta^\dagger)_{(i)} = (\eta_{(i)})^\dagger$. Thus $\eta^\dagger$ is stratified and real analytic. We claim that $\eta^\dagger \in \mathcal{B}^1_{4,\varepsilon_0}$. Let $i \in \{1, \ldots, n\}$. We have $\eta^\dagger_{(i)}(x) = \alpha_i(x, \exp_i'(x, \eta_{(i)}(x)) - x)$ for all $x \in B_4^{k_i}(0)$. Hence,

$$d\eta^\dagger_{(i)}(x, v) = d_1\alpha_i(x, {\psi'}^i_{\eta_{(i)}}(x) - x; v) + d_2\alpha_i(x, {\psi'}^i_{\eta_{(i)}}(x) - x; d{\psi'}^i_{\eta_{(i)}}(x, v) - v)$$

and so

$$\|d\eta^\dagger_{(i)}(x, \bullet)\|_{op} \leqslant \|d_1\alpha_i(x, {\psi'}^i_{\eta_{(i)}}(x) - x; \bullet)\|_{op} + \|d_2\alpha_i(x, {\psi'}^i_{\eta_{(i)}}(x) - x; \bullet)\|_{op} \cdot \|d{\psi'}^i_{\eta_{(i)}}(x) - \mathrm{id}\|_{op} < \frac{\varepsilon_0}{4} + K_\alpha \cdot \frac{\varepsilon_0}{K_\alpha \cdot 4} = \frac{\varepsilon_0}{2}.$$

Because $\varepsilon < \varepsilon'_{4,\delta(\frac{\varepsilon_0}{2})}$ we have $\|\eta_{(i)}(x)\| < \frac{\varepsilon_0}{2}$ for all $x \in B_4^{k_i}(0)$. We conclude that $\eta^\dagger \in \mathcal{B}^1_{4,\varepsilon_0}$. Hence, the map

$$\Delta\colon \mathcal{B}^1_{4,\varepsilon} \to \mathcal{B}^1_{4,\varepsilon_0},\ \eta \mapsto \eta^\dagger$$

makes sense. This also shows that $\Psi'(\mathcal{B}^1_{4,\varepsilon}) \subseteq \Psi(\mathcal{B}^1_{4,\varepsilon_0}) \subseteq \mathcal{U}_0$ and that Δ is nothing else than the inclusion in the charts Φ' and Φ. In the following, we use notation of the proof of Lemma 1.28. We want to show that $\Delta\colon \mathcal{B}^1_{4,\varepsilon} \to \Gamma^\omega_{\text{st}}(TM)$ is smooth. For this we have to show the smoothness of the corresponding map between $\mathcal{G}^1_\varepsilon(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_4^{k_i}(0))$ and $\mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_4^{k_i}(0))$. If $n \in \mathbb{N}$ and $f \in \text{Hol}^1_\varepsilon(V^{k_i}_{4,n}; \mathbb{C}^m)$ then $f^\dagger\colon V^{k_i}_{4,n} \to \mathbb{C}^m$, $x \mapsto \exp^*_i(x, \bullet)^{-1}({\psi'}^i_f(x))$ makes sense and is complex analytic (because of Lemma 1.62 we have $\|{\psi'}^i_f(x) - x\| < \delta^{\mathbb{C}}(\varepsilon_{\exp^*})$). We want to show the smoothness of $\text{Hol}^1_\varepsilon(V^{k_i}_{4,n}; \mathbb{C}^m) \to \text{Hol}(V^{k_i}_{4,n+2})$, $f \mapsto f^\dagger$. We can write this map as the following composition

$$\begin{aligned}
&\text{Hol}^1_\varepsilon(V^{k_i}_{4,n}; \mathbb{C}^m) \xrightarrow{\tau_1} \text{Hol}(V^{k_i}_{4,n+1}; \mathbb{C}^m) \cap \Big(\lfloor \overline{V}^{k_i}_{4,n+2}; B^{\mathbb{C}}_{\delta^{\mathbb{C}}(\varepsilon_{\exp^*})}(0) \rfloor + \text{id}\Big)\\
&\xrightarrow{\tau_2} \text{Hol}(V^{k_i}_{4,n+2}; \mathbb{C}^m)\\
&\text{with } \tau_1(\eta) = {\psi'}^i_\eta \text{ and } \tau_2(f)(x) = \exp^*_i(x, \bullet)^{-1}(f(x)).
\end{aligned}$$

Now the smoothness of τ_1 and τ_2 follows from Lemma 1.84. From Lemma 1.21 we deduce that Δ is smooth. The set $\Psi'(\mathcal{B}^1_{4,\varepsilon})$ is an open identity neighbourhood on $\text{Diff}^\omega(M)'_0$ and with $\Psi'(\mathcal{B}^1_{4,\varepsilon}) \subseteq \Psi(\mathcal{B}^1_{4,\varepsilon_0}) \subseteq \mathcal{U}_0$ we get $\text{Diff}^\omega(M)'_0 \subseteq \text{Diff}^\omega(M)_0$. The inclusion $\text{Diff}^\omega(M)'_0 \hookrightarrow \text{Diff}\,\omega(M)_0$ is smooth because Δ is smooth. Analogously we see that $\text{Diff}^\omega(M)_0 \subseteq \text{Diff}^\omega(M)'_0$ and that $\text{Diff}^\omega(M)_0 \hookrightarrow \text{Diff}^\omega(M)'_0$ is smooth.

□

Lemma 1.86. *Given* $f \in \text{Diff}^\omega(M)$ *the map*

$$P_f\colon \Gamma^\omega_{\text{st}}(TM) \to \Gamma^\omega_{\text{st}}(TM),\ \eta \mapsto P_f\eta := Tf \circ \eta \circ f^{-1}$$

is continuous linear.

Proof. Because we can embed $\Gamma^\omega_{\text{st}}(TM)$ into $\prod_{i=1}^n \mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_4^{k_i}(0))$, it suffices to show that $\Gamma^\omega_{\text{st}}(TM) \to \mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_4^{k_i}(0))$, $\eta \mapsto [((P_f\eta)^*)_{(i)}] = [((P_f\eta)_{(i)})^*]$ is continuous. The map $\varphi'_i\colon f^{-1}(U_{i,5}) \to B_5^{k_i}(0)$, $\varphi'_i = \varphi_i \circ f$ is a chart of M. Given $\zeta \in \Gamma^\omega_{\text{st}}(TM)$ we write $\zeta_{\varphi'_i} := d\varphi'_i \circ \zeta \circ {\varphi'_i}^{-1}$ for the local representative. We get

$$(P_f\eta)_{(i)} = d\varphi_i \circ P_f\eta|_{U_{i,5}} \circ \varphi_i^{-1} = \eta_{\varphi'_i}.$$

Hence,

$$[((P_f\eta)_{(i)})^*] = [(\eta_{\varphi'_i})^*] = [(\eta^*)_{{\varphi'_i}^*}].$$

The map $\Gamma^\omega_{\mathrm{st}}(TM) \to \mathcal{G}(\mathbb{C}^m; \mathbb{C}^m|\overline{B}_4^{k_i})$, $\eta \mapsto (\eta^*)_{\varphi_i'^*}$ is continuous because of [DS15, Lemma A.15]. □

The following lemma and its proof are completely analogous to [Glo06c, 5.5,5.6,5.8]. For the convenience of the reader, we recall Glöckners arguments:

Lemma 1.87. *The subgroup* $\mathrm{Diff}^\omega(M)_0$ *of* $\mathrm{Diff}^\omega(M)$ *is normal and for* $f \in \mathrm{Diff}^\omega(M)$ *the conjugation* $\mathrm{int}_f \colon \mathrm{Diff}^\omega(M)_0 \to \mathrm{Diff}^\omega(M)_0$, $h \mapsto f \circ h \circ f^{-1}$ *is smooth.*

Proof. Let $f \in \mathrm{Diff}^\omega(M)$. Then the pullback metric g' from g over f induces a Riemannian exponential function $\exp' \colon \Omega' \to M$, with $\Omega' = Tf(\Omega)$ and $\exp' = f \circ \exp \circ Tf^{-1}|_{\Omega'}$. Hence, for $\eta \in \mathcal{V}_0^1$, we get $f \circ \Psi_\eta \circ f^{-1} = \exp' \circ P_f \eta = \Psi'_{P_f\eta}$. Now we can use Lemma 1.85 and find a 0-neighbourhood $\mathcal{V}_0' \subseteq \Gamma^\omega_{\mathrm{st}}(TM)$ such that $\Phi' \colon \mathcal{V}_0' \to \mathrm{Diff}^\omega(M)_0$ is a diffeomorphism onto an identity neighbourhood. Because the map $P_f \colon \Gamma^\omega_{\mathrm{st}}(TM) \to \Gamma^\omega_{\mathrm{st}}(TM)$ is continuous linear, we can find a 0-neighbourhood $W \subseteq \mathcal{V}_0^1 \subseteq \Gamma^\omega_{\mathrm{st}}(TM)$ such that $P_f(W) \subseteq \mathcal{V}_0'$. Hence, $\mathrm{int}_f \circ \Psi_\eta = \Psi'_{P_f\eta} \in \mathrm{Diff}^\omega(M)_0$ for all $\eta \in W$. Therefore, $\mathrm{int}_f(\Psi(W)) \subseteq \mathrm{Diff}^\omega(M)_0$. Thus $\mathrm{int}_f(\mathrm{Diff}^\omega(M)_0) = \mathrm{int}_f(< \Psi(W) >) \subseteq \mathrm{Diff}^\omega(M)_0$. Moreover, we have $\mathrm{int}_f|_{\Psi(W)} = \Psi' \circ P_f \circ \Phi|_{\Psi(W)}$. Hence, int_f is smooth. □

Now we get the main result of this chapter. As in the case of [Glo06c] we just have to use Lemma 1.82 and the results above.

Theorem 1.88. *There exists a unique smooth Lie group structure on* $\mathrm{Diff}^\omega(M)$ *modelled over* $\Gamma^\omega_{\mathrm{st}}(TM)$ *such that for one (and hence for all) boundary respecting Riemannian metrics on* M *the map* $\eta \mapsto \Psi_\eta$ *is a smooth diffeomorphism from an open* 0*-neighbourhood of* $\Gamma^\omega_{\mathrm{st}}(TM)$ *onto an open identity neighbourhood of* $\mathrm{Diff}^\omega(M)$*.*

2. Integrability of Banach subalgebras

As mentioned in the introduction, Teichmann showed a Frobenius theorem for finite-dimensional vector distributions on convenient manifolds that are modelled over locally convex spaces (see [Tei01, Theorem 2]).[1] A similar result for manifolds that are modelled over locally convex spaces in the sense of Keller's C^k_c-theory was obtained in [Eyn12, Chapter 2; Theorem 2.6][2]. The primary aim of this chapter is to obtain a Frobenius theorem for Banach distributions on manifolds that are modelled over locally convex spaces (see Theorem 2.15). Hence, we obtain a generalisation of [Eyn12, Theorem 2.6] respectively [Tei01, Theorem 2]. In [CS76] Chillingworth and Stefan considered distributions on Banach manifolds that are not necessarily subbundles of the tangent bundle but such that each fibre D_p of the distribution is a Banach space which is complemented in T_pM. Our proof of Theorem 2.15 is inspired by the proofs of [CS76, Section 4] and [Tei01, Theorem 2] respectively [Eyn12, Theorem 2.6]. Whereas Chillingworth and Stefan consider Banach manifolds, we are interested in manifolds that are modelled over locally convex spaces. So one of the main problems will be that we have no solution theory for initial value problems in locally convex spaces. The idea to generalise the methods used in [CS76] was suggested to the author by Glöckner.

In Section 2.2 we apply our Frobenius theorem to obtain Theorem 2.17 concerning the integration of Lie subalgebras of Lie algebras of Lie groups that are modelled over locally convex spaces. It is a standard strategy to show the integrability of Lie subalgebras with the help of a Frobenius theorem (see e.g. [Lan01, Chapter VI, Theorem 5.4], [Les68], [Les92], or [Eyn12, Theorem 4.1]).

2.1. The Frobenius theorem for Banach distributions

Convention 2.1. Throughout this section, E will be a locally convex space, $r \in \mathbb{N} \cup \{\infty\}$ and M a C^r-manifold modelled over E.

Remark 2.2. *Because ordinary differential equations in locally convex spaces do not have a unique solution in general we sometimes assume that certain vector fields admit a local flow.*

At first we recall some standard definitions concerning distributions of manifolds, see e.g. [Lee13], [Eyn12], [Lan01], [Hil00] or [Tei01]:

[1] This chapter consists of material published before in the author's preprint [Eyn14a].

[2] As mentioned in the introduction the Frobenius theorems in [Les68] respectively [Les92] are of a different kind because they require other conditions and their proofs use different methods

Definition 2.3. (a) A subset $D \subseteq TM$ is called *vector distribution* or just *distribution* of M, if for every point $p \in M$ the set $D_p := D \cap T_pM$ is a vector subspace of T_pM. Important examples for vector distributions are subbundles of TM. (See e.g. [Eyn12, Definition 1.7], cf. [Lee13, p. 491])

(b) A subset $N \subseteq M$ is called an *immersed submanifold* of M, if it is a C^r-manifold modelled over a closed complemented vector subspace F of E such that the inclusion $\iota_N^M : N \to M$, $p \mapsto p$ is continuous and given $p \in N$ we find a chart $\varphi\colon U_\varphi \to V_\varphi$ of N around p and a chart $\psi\colon U_\psi \to V_\psi$ of M around p such that $U_\varphi \subseteq U_\psi$ and $\psi \circ \iota_N^M \circ \varphi^{-1} = \iota_F^E|_{V_\varphi}$. (See e.g. [Eyn12, Definition 1.9], cf. [Lee13, p. 108])

(c) Let $F \subseteq E$ be a closed vector subspace of E and $D \subseteq TM$ be a subbundle of TM with typical fibre F. A connected immersed submanifold $N \subseteq M$ is called *integral manifold* for D, if $T_pN = D_p$ for every $p \in N$. Given $p_0 \in M$, we call an integral manifold N containing p_0 *maximal* if every other integral manifold L for D that contains p_0 is a subset of N and the inclusion map $\iota\colon L \hookrightarrow N$, $p \mapsto p$ is of class C^r. (Cf. e.g. [Eyn12, Definition 1.10] or [Lee13, p. 491])

(d) Let $F \subseteq E$ be a closed vector subspace and $D \subseteq TM$ be a subbundle of TM with typical fibre F. Assume that F is complemented in E, say $E = F \oplus H$ topologically with a vector subspace H of E. A chart $\varphi\colon U_\varphi \to V_\varphi$ of M is called a *Frobenius chart* for D, if there are open sets $V_1 \subseteq F$ and $V_2 \subseteq H$ such that $V_\varphi = V_1 \times V_2$ and for $\overline{y} \in V_2$ the submanifold

$$S_{\overline{y}}^{\varphi} := \{\varphi^{-1}(x, \overline{y}) : x \in V_1\} \tag{2.1}$$

is an integral manifold for $D|_{U_\varphi}$. If M admits an atlas of Frobenius charts for D, we call D a *Frobenius distribution*.

(e) If F is a closed vector subspace of E, we call a subbundle $D \subseteq TM$ of TM with typical fibre F *involutive*, if for all C^r-vector fields $X, Y : U \to TM$ on an open set $U \subseteq M$ with $\mathrm{im}(X) \subseteq D$ and $\mathrm{im}(Y) \subseteq D$, also $\mathrm{im}([X, Y]) \subseteq D$. (See [Eyn12, Definition 2.5], cf. [Lee13, p. 492])

The following theorem is a straightforward generalisation of the finite-dimensional case ([War83]), and was proved in [Eyn12, Satz 1.13.].

Theorem 2.4. *Let E be a locally convex space, M be a C^r-manifold with $r \geqslant 2$, F be a complemented vector subspace of E and $D \subseteq TM$ be a subbundle of TM with typical fibre F. If D is a Frobenius distribution then given $p_0 \in M$ there exists a unique maximal integral manifold that contains p_0.*

Remark 2.5. *Let F be a complemented vector subspace of E with vector complement H and $D \subseteq TM$ be a subbundle of TM with typical fibre F. For a chart $\varphi\colon U \to V_1 \times V_2 \subseteq F \oplus H = E$ of M and the inclusion $\iota_{\overline{y}}\colon V_1 \to V_1 \times V_2$, $x \mapsto (x, \overline{y})$, we get the following equivalences:*

$$\begin{aligned} &\varphi \text{ is a Frobenius chart} \\ \Leftrightarrow &(\forall \overline{y} \in V_2)\ S_{\overline{y}}^{\varphi} = \varphi^{-1}(\bullet, \overline{y})(V_1) \text{ is an integral manifold for } D \end{aligned}$$

$$\Leftrightarrow (\forall \overline{y} \in V_2)(\forall x \in V_1)\ T_{\varphi^{-1}(x,\overline{y})} S^{\varphi}_{\overline{y}} = T_x(\varphi^{-1} \circ \iota_{\overline{y}})(\{x\} \times F) = D_{\varphi^{-1}(x,\overline{y})}$$
$$\Leftrightarrow (\forall p \in U_\varphi)\ d\varphi(D_p) = F.$$

Definition 2.6. Let N be a C^r-manifold, $X\colon N \to TN$ be a C^{r-1}-vector field on N and $f\colon M \to N$ be a C^r-diffeomorphism. In this situation we define the C^{r-1}-vector field $f^*X := Tf^{-1} \circ X \circ f$ on M.

The following lemma is a straightforward generalisation of the finite-dimensional case (cf. [GN] or [Lan01, Chaper V, Section 2]).

Lemma 2.7. *Let $r \geqslant 2$. If $X, Y\colon M \to TM$ are C^r-vector fields and X provides a local flow then we have $\frac{d}{ds}\big|_{s=0} \left((\Phi^X_s)^* Y(p)\right) = [X,Y](p)$ for all $p \in M$.*

Proof. It is enough to prove the assertion locally. Let $U \subseteq E$ be an open subset and $f, g\colon U \to E$ be C^r-maps such that f provides a local flow. We write $\Phi\colon \Omega \to U$ for the global flow of f. For $p \in U$, we calculate

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} (\Phi_s^* g(p)) = \frac{d}{ds}\Big|_{s=0} d\Phi_{-s}\left(\Phi_s(p), g(\Phi_s(p))\right) \\
=&\frac{d}{ds}\Big|_{s=0} d\Phi(-s, \Phi(s,p); 0, g(\Phi(s,p))) \\
=&d_1(d\Phi)\left(0, \Phi(0,p), 0, g(\Phi(0,p)); -1, \frac{d}{ds}\Big|_{s=0} \Phi(s,p)\right) \\
&\quad + d\Phi\left(0, \Phi(0,p); 0, \frac{d}{ds}\Big|_{s=0} g(\Phi(s,p))\right) \\
=&d_1(d\Phi)(0,p,0,g(p); -1, 0) + d_1(d\Phi)(0,p,0,g(p); 0, f(p)) \\
&\quad + d\Phi(0,p; 0, dg(p, f(p))) \\
=& - \frac{d}{dt}\Big|_{t=0} d\Phi(t,p;0,g(p)) + \frac{d}{dt}\Big|_{t=0} d\Phi(0, p + tf(p); 0, g(p)) \\
&\quad + d_2\Phi(0,p; dg(p, f(p))) \\
=& - \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \Phi(t, p + sg(p)) + \frac{d}{dt}\Big|_{t=0} d\Phi_0(p + tf(p); g(p)) + dg(p, f(p)) \\
=& - \frac{d}{ds}\Big|_{s=0} f(p + sg(p)) + \frac{d}{dt}\Big|_{t=0} g(p) + dg(p, f(p)) \\
=& - df(p, g(p)) + dg(p, f(p))
\end{aligned}$$

□

The following result comes from [Glo06b, Theorem 2.3].

Theorem 2.8. *Let $r \in \mathbb{N} \cup \{\infty\}$, E be a locally convex space, F be a Banach space, $P \subseteq E$ and $U \subseteq F$ be open sets and $f\colon P \times U \to F$ be a C^r-map with $r \in \mathbb{N}$. We write $f_p := f(p, \bullet)\colon U \to F$ for $p \in P$. Let $p_0 \in P$ and $x_0 \in U$ with $f'_{p_0}(x_0) \in \mathrm{GL}(F)$. If $r \geqslant 2$ or $r = 1$ and*

$$\sup_{(p,x) \in P \times U} \|f'_{p_0}(x_0) - f'_p(x)\|_{op} < \frac{1}{\|f'_{p_0}(x_0)^{-1}\|_{op}}, \tag{2.2}$$

then we find an open p_0-neighbourhood $P_0 \subseteq P$ and an open x_0-neighbourhood $U_0 \subseteq U$ such that

(a) $f_p(U_0)$ *is open in* F *for all* $p \in P$ *and* $f_p|_{U_0}\colon U_0 \to f_p(U_0)$ *is a* C^r-*diffeomorphism.*

(b) $W := \bigcup_{p\in P_0}(\{p\} \times f_p(U_0))$ *is open in* $E \times F$ *and* $g\colon W \to U_0,\ (p, y) \mapsto f_p^{-1}(y)$ *is a* C^r-*map.*

(c) $\Phi\colon P_0 \times U_0 \to W,\ (p, x) \mapsto (p, f_p(x))$ *is a* C^r-*diffeomorphism with inverse* $\Psi\colon W \to P_0 \times U_0,\ (p, z) \to (p, g(p, z))$.

Definition 2.9. Let E and F be locally convex spaces. We write $\mathcal{L}(E, F)_b$ for the space of continuous linear maps equipped with the topology of uniform convergence on bounded sets and $\mathcal{L}(E, F)_{c.o.}$ if we equip the space with the topology of uniform convergence on compact sets (see [Glo, p. 5]).

The following lemma is taken from [Glo, Proposition 2.1] (also cf. [GN]).

Lemma 2.10. *If* E, F *and* H *are locally convex spaces,* $r \in \mathbb{N}$, $U \subseteq E$ *is an open set and* $f\colon U \times F \to H$ *is a* C^r-*map that is linear in the second argument then* $f^\vee : U \to \mathcal{L}(F, H)_{c.o.}$ *is of class* C^r *and* $f^\vee : U \to \mathcal{L}(F, H)_b$ *is of class* C^{r-1}.

Lemma 2.11. *Let* E *be a locally convex space,* F *be a Banach space,* $\mathcal{P}_E$ *be the set of all continuous seminorms on* E *and* B_1 *be the closed unit ball in* F. *If we write* $\|\bullet\|_{B,q}$ *for a typical seminorm on* $\mathcal{L}(F, E)_b$, *where* B *is a bounded set in* F *and* $q \in \mathcal{P}_E$ *then the family of seminorms* $\left(\|\bullet\|_{B_1,q}\right)_{q\in\mathcal{P}_E}$ *defines the locally convex topology of* $\mathcal{L}(F, E)_b$.

Proof. Obviously the topology that comes from $\left(\|\bullet\|_{B_1,q}\right)_{q\in\mathcal{P}_E}$ is coarser than the one of $\mathcal{L}(F, E)_b$. To show that it is also finer let $B \subseteq F$ be bounded and $q \in \mathcal{P}_E$. We find $r > 0$ with $r \cdot B_1 \supseteq B$ and calculate

$$\begin{aligned}\|f\|_{B,q} &\leqslant \|f\|_{rB_1,q} = \sup\{q(f(x)) : x \in rB_1\} = \sup\{r \cdot q(f(x)) : x \in B_1\}\\ &=\|f\|_{B_1,r\cdot q}.\end{aligned}$$

□

Lemma 2.12. *Let* E *be a locally convex space,* F *be a Banach space and* q *be a continuous seminorm on* E. *For* $E_q := E/q^{-1}(0)$ *and* $\pi_q\colon E \to E_q,\ x \mapsto x+q^{-1}(0)$, *the map* $\iota\colon \mathcal{L}(F, E)_b/(\|\bullet\|_{B_1,q})^{-1}(0) \hookrightarrow \mathcal{L}(F, E_q),\ f + (\|\bullet\|^{-1}_{B_1,q}(0)) \mapsto \pi_q \circ f$ *is a well-defined topological embedding. Moreover, for* $f \in \mathcal{L}(F, E)$, $g \in \mathcal{L}(F)$ *and* $\pi_{\|\bullet\|_{B_1,q}}\colon \mathcal{L}(F, E) \to \mathcal{L}(F, E)/(\|\bullet\|^{-1}_{B_1,q}(0)),\ f \mapsto f + \|\bullet\|^{-1}_{B_1,q}(0)$, *we get*

$$\iota \circ \pi_{\|\bullet\|_{B_1,q}}(f \circ g) = (\iota \circ \pi_{\|\bullet\|_{B_1,q}}(f)) \circ g. \tag{2.3}$$

Proof. Let $f \in \mathcal{L}(F, E)$ with $\|f\|_{B_1,q} = 0$. For $x \in F\backslash\{0\}$, we get

$$q \circ f(x) = \|x\| \cdot q \circ f\left(\frac{x}{\|x\|}\right) = 0.$$

Hence ι is well-defined. To show that ι is an isometry we choose $f \in \mathcal{L}(F, E)$ and calculate

$$\begin{aligned}\|\pi_q \circ f\|_{op} &= \sup\{q \circ f(x) : x \in B_1\} = \|f\|_{B_1,q}\\ &=\|f + (\|\bullet\|_{B_1,q})^{-1}(0)\|.\end{aligned}$$

To show (2.3) we calculate

$$\iota \circ \pi_{\|\bullet\|_{B_1,q}}(f \circ g) = \pi_q \circ f \circ g = (\iota \circ \pi_{\|\bullet\|_{B_1,q}}(f)) \circ g.$$

□

If E is a locally convex space one considers the induced Banach space $E_q := E/q^{-1}(\{0\})$ for continuous seminorms q, to obtain information about the existence and uniqueness of initial value problems in E. This is a standard strategy in infinite-dimensional analysis and was shown to the author by Glöckner in a related context (cf. [Omo78] and [DGV16]). We use this method in the following lemma.

Lemma 2.13. *If E is a locally convex space, F is a Banach space, $\lambda\colon I \to \mathcal{L}(F)$ is a C^1-curve and $\mu_0 \in \mathcal{L}(F, E)$ then the initial value problem*

$$\begin{cases}\varphi'(t) &= \varphi(t) \circ \lambda(t)\\ \varphi(0) &= \mu_0\end{cases} \tag{2.4}$$

in $\mathcal{L}(F, E)_b$ has at most one solution.

Proof. Let $\varphi_1, \varphi_2\colon]-\varepsilon, \varepsilon[\to \mathcal{L}(F, E)$ be solutions of the initial value problem (2.4) and q a continuous seminorm of E. Moreover, let $\pi_{\|\bullet\|_{B_1,q}}$ and ι be as in Lemma 2.12. For $i = 1, 2$, we define the map $\varphi_{i,q}\colon]-\varepsilon, \varepsilon[\to \mathcal{L}(F, E_q)$, $t \mapsto \iota \circ \pi_{\|\bullet\|_{B_1,q}} \circ \varphi_i$ and get

$$\begin{aligned}\varphi_{i,q}'(t) &= \iota \circ \pi_{\|\bullet\|_{B_1,q}}(\varphi_i'(t)) = \iota \circ \pi_{\|\bullet\|_{B_1,q}}(\varphi_i(t) \circ \lambda(t))\\ &=\iota \circ \pi_{\|\bullet\|_{B_1,q}}(\varphi_i(t)) \circ \lambda(t) = \varphi_{i,q}(t) \circ \lambda(t)\end{aligned}$$

and $\varphi_{i,q}(0) = \pi_q \circ \mu_0$. Let $\tilde{E}_q$ be a completion of E_q such that $E_q \subseteq \tilde{E}_q$. The composition $\mathcal{L}(F) \times \mathcal{L}(F, \tilde{E}_q) \to \mathcal{L}(F, \tilde{E}_q)$, $(\mu, \psi) \mapsto \psi \circ \mu$ is continuous and bilinear. Hence, $f\colon I \times \mathcal{L}(F, \tilde{E}_q) \to \mathcal{L}(F, \tilde{E}_q)$, $(t, \psi) \mapsto \psi \circ \lambda(t)$ is Fréchet-differentiable of class C^1. Thus, f is continuous and locally Lipschitz-continuous in the second argument. Because $\mathcal{L}(F, \tilde{E}_q)$ is a Banach space, we have $\varphi_{1,q} = \varphi_{2,q}$. Hence $\pi_{\|\bullet\|_{B_1,q}} \circ \varphi_1 = \pi_{\|\bullet\|_{B_1,q}} \circ \varphi_2$. Because q was an arbitrary continuous seminorm of E, we get $\varphi_1 = \varphi_2$. □

In [Eyn12] the author worked with flows (without parameters) of vector fields (without parameters) on infinite-dimensional manifolds (see [Eyn12, Definition 1.19]). For the more general result in this thesis we have to consider flows with parameters of vector fields with parameters. We recall the basic well-known definitions in the following (cf. e.g. [Lan01, Chapter IV, Section 2]).

Definition 2.14. Let P be a locally convex space, $r \in \mathbb{N}$ and M be a C^r-manifold that is modeled over a locally convex space. Moreover let $\Omega \subseteq \mathbb{R} \times M \times F$ be open and $\Phi\colon \Omega \to M$ a C^r-map such that:

(i) $\{0\} \times M \times P \subseteq \Omega$.

(ii) $\Phi(0, x, p) = x$ for all $x \in M$ and $p \in P$.

(iii) For all $x_0 \in M$, $p_0 \in P$ we find a symmetric interval I, a x_0-neighbourhood U in M and a p_0-neighbourhood V in P such that $I \times U \times V \subseteq \Omega$, $I \times \bigcup_{p \in V} \Phi(I \times U \times \{p\}) \times \{p\} \subseteq \Omega$ and $\Phi(t, \Phi(s, x, p), p) = \Phi(t + s, x, p)$ for all $x \in U$, $p \in V$ and $t, s \in I$ with $s + t \in I$.

Then we call Φ a *C^r-$\mathbb{R}$-action on M with parameters*. A C^{r-1}-map $X\colon M \times P \to TM$ is called *vector field with parameters*, if $X(\bullet, p)$ is a C^{r-1}-vector field of M for all $p \in P$. We say that X *provides a local flow with parameters of class C^r*, if we find a local C^r-$\mathbb{R}$-action on M with parameters such that $\frac{\partial}{\partial t}\big|_{t=0}\Phi(t, x, p) = X(x, p)$ for all $x \in M$ and $p \in P$.

As mentioned above, the following Frobenius theorem is inspired by [CS76, Section 4] and [Tei01, Theorem 2] respectively the author's result [Eyn12, Theorem 2.6]. Also in [Les68] and [Les92], Frobenius theorems have been proved. But Leslie's results require different conditions and he used very different methods to prove his statements.

Theorem 2.15. *Let E be a locally convex space, F be a complemented vector subspace of E such that F is a Banach space with the induced topology from E. Moreover, let $r \in \mathbb{N} \cup \{\infty\}$ with $r \geqslant 4$, M be a C^r-manifold modeled over E and D be an involutive subbundle of TM with typical fibre F. Assume that $p_0 \in M$, there exists an open p_0-neighbourhood $U \subseteq M$ and a C^{r-1}-vector field $X\colon U \times F \to TU$ with parameters in F such that:*

(a) The map $F \to \Gamma(TU)$, $v \mapsto X(\bullet, v)$ is linear;

(b) $\mathrm{im}(X) \subseteq D$;

(c) The map $F \to D_{p_0}$, $v \mapsto X(p_0, v)$ is an isomorphism of topological vector spaces;

(d) The C^{r-1}-vector field X provides a local flow with parameters of class C^r.

(e) It exists a chart $\varphi\colon U \to V$ of M, such that $\varphi(p_0) = 0$ and $d\varphi(D_{p_0}) = F$.

In this situation D is a Frobenius distribution.

Proof. Let $p_0 \in M$ and ϕ be a chart around p_0 with $\phi(p_0) = 0_E$, $d\phi(D_{p_0}) = F$ and $d_{p_0}\varphi \circ X(p_0, \bullet)|^{D_{p_0}} = \mathrm{id}_F$. To be a Frobenius distribution is a local property, hence it is enough to show the statement in the local chart ϕ. This means we have the following situation. The set U is an open 0-neighbourhood in E. The vector distribution $D \subseteq U \times E$ is a subbundle of $U \times E$ with typical fibre F. Hence, given $x \in U$ we find a C^r-diffeomorphism $\psi\colon U \times E \to U \times E$ such that $\psi(\{y\} \times E) = \{y\} \times E$, $\mathrm{pr}_2 \circ \psi(y, \bullet)\colon E \to E$ is an isomorphism of topological vector spaces and $\psi(D) = U \times F$. Given $x \in U$, we write D_x for the vector subspace $\mathrm{pr}_2(D \cap (\{x\} \times E))$ of E. By our choice of ϕ we have $D_{0_E} = F$. We write again X for the local representative of X in the chart ϕ. Hence, $X\colon U \times F \to E$ is a C^r-map such that:

(a) The map $\check{X}\colon F \to C^r(U, E)$, $v \mapsto X(\bullet, v)$ is linear;

(b) $X(p,v) \in D_p$ for all $p \in U$ and $v \in F$;
(c) The map $X(0,\bullet)\colon F \to D_0 = F$, $v \mapsto X(0,v)$ is an isomorphism of topological vector spaces (we have $X(0,\bullet) = \mathrm{id}_F$);
(d) X provides a local flow with parameters.

We write $\Phi\colon \Omega \to U$ for the global flow with parameters of X. For convenience we write $X_v := X(\bullet, v)$, $\Phi^v := \Phi(\bullet,\bullet,v)$ and $\Omega^v := \{(t,x) \in \mathbb{R}\times U : (t,x,v) \in \Omega\}$ for $v \in F$. Since $X\colon U \times F \to E$ is a C^1-map, also $\tilde{X}\colon U \times F \to F$, $(x,v) \mapsto \psi(x, X(x,v))$ is of class C^1. This provides the continuity of $\check{\tilde{X}}\colon U \to \mathcal{L}(F)$, $x \mapsto \psi(x, X(x,\bullet))$ because of Lemma 2.10. Since $X(0_E,\bullet)$ is an isomorphism of topological vector spaces, we can assume that $X(x,\bullet)|^{D_x}\colon F \to D_x$ is an isomorphism of topological vector spaces for all $x \in U$. We divide the proof in three steps.

Step 1: Given a vector field $Y\colon U \to E$ with $Y(x) \in D_x$ for all $x \in U$, we show that $\big((\Phi_t^v)^*Y\big)(x) \in D_x$ for all $(t,x,v) \in \Omega$. Moreover, we prove

$$d_2\Phi(t,y,v;\bullet)(D_y) = D_{\Phi(t,y,v)} \tag{2.5}$$

for $(t,y,v) \in \Omega$. This generalises parts of the proof of [CS76, Lemma 4.3].

For $(t,x,v) \in \Omega$, we have to show that $(\Phi_t^v)^*Y(x) \in D_x$. There exists $w \in F$ with $X(\Phi(t,x,v),w) = Y(\Phi(t,x,v))$. Thus

$$\begin{aligned}(\Phi_t^v)^*Y(x) &= (d\Phi_t^v(x,\bullet))^{-1} \circ Y \circ \Phi_t^v(x) = (d\Phi_t^v(x,\bullet))^{-1} \circ X_w \circ \Phi_t^v(x)\\ &=(\Phi_t^v)^*X_w(x).\end{aligned}$$

So we only have to show that $(\Phi_t^v)^*X_w(x) \in D_x$ for all $w \in F$ and $(t,x,v) \in \Omega$. Let $v \in F$ and $x \in U$. On the interval $I_{v,x} := \{t \in \mathbb{R} : (t,x,v) \in \Omega\}$, for all $w \in F$ we have

$$\begin{aligned}\frac{\partial}{\partial t}\Big((\Phi_t^v)^*X_w(x)\Big) &= \frac{\partial}{\partial s}\Big|_{s=0}\big((\Phi_{t+s}^v)^*X_w(x)\big) = \frac{\partial}{\partial s}\Big|_{s=0}((\Phi_s^v)^*(\Phi_t^v)^*X_w(x))\\ &= [X_v, (\Phi_t^v)^*X_w](x) = [(\Phi_t^v)^*X_v, (\Phi_t^v)^*X_w](x) = (\Phi_t^v)^*[X_v, X_w](x)\end{aligned}$$

using Lemma 2.7, [Eyn12, Lemma 2.3] and [Eyn12, Lemma 2.4][3]. Now we define the curve $g_w\colon I_{v,x} \to E$, $g_w(t) := (\Phi_t^v)^*X_w(x)$ for $w \in F$ and write $\lambda_y := X(y,\bullet)|^{D_y}$ for $y \in U$. Moreover, we define $x_t := \Phi_t^v(x)$ for $t \in I_{v,x}$. From $[X_v,X_w](x_t) = X(x_t, \lambda_{x_t}^{-1}([X_v,X_w](x_t)))$ we conclude that

$$g_w'(t) = (\Phi_t^v)^*[X_v,X_w](x) = (\Phi_t^v)^*X_{\lambda_{x_t}^{-1}([X_v,X_w](x_t))}(x) = g_{\lambda_{x_t}^{-1}([X_v,X_w](x_t))}(t).$$

For $t \in I_{v,x}$, we define the maps $A(t)\colon F \to E$, $u \mapsto g_u(t)$ and $B(t)\colon F \to F$, $w \mapsto \lambda_{x_t}^{-1}([X_v,X_w](x_t))$. We also define $A\colon I_{v,x} \to \mathcal{L}(F,E)_b$, $t \mapsto A(t)$ and $B\colon I_{v,x} \to \mathcal{L}(F)$, $t \mapsto B(t)$. The curve A is of class C^1 because $I_{v,x} \times F \to E$, $(t,w) \mapsto g_w(t)$ is of class C^2 (see Lemma 2.10). For $p \in U$, let $\psi_p\colon E \to E$ be the canonical isomorphism that is induced by ψ. We define the map $f\colon I_{v,x} \times F \to F$, $(t,w) \mapsto \psi_{x_t}(X(x_t,w))$. Let $(t_0,w_0) \in I_{x,v} \times F$, $w_1 := \psi_{x_{t_0}}([X_v, X_{w_0}](x_{t_0}))$ and

[3] From [Eyn12, Lemma 2.3] we deduce $X_v = (\Phi_t^v)^*X_v$ and from [Eyn12, Lemma 2.4] we get $[(\Phi_t^v)^*X_v, (\Phi_t^v)^*X_w](x) = (\Phi_t^v)^*[X_v,X_w](x)$ (cf. [Lan01, Chapter V Section 1]).

$z_0 := f(t_0, \bullet)^{-1}(w_1)$. We have $d_2 f(x_0, z_0) \in \mathrm{GL}(F)$ and with Theorem 2.8 we see that $f_2\colon I_{v,x} \times F \to F$, $(t, y) \mapsto f(t, \bullet)^{-1}(y) = \lambda_{x_t}^{-1}(\psi_{x_t}^{-1}(y))$ is C^2 on a (t_0, w_1)-neighbourhood. Hence, $I_{v,x} \times F \to F$, $(t, w) \mapsto \lambda_{x_t}^{-1}([X_v, X_w](x_t))$ is C^2 on a (t_0, w_0)-neighbourhood. Because $(t_0, w_0) \in I_{v,x} \times F$ was arbitrary we see that this map is C^2 and so the curve B is of class C^1 (see Lemma 2.10). For $w \in F$ we write $\varepsilon_w\colon \mathcal{L}(F, E) \to E$, $B \mapsto B(w)$ and get

$$\begin{aligned} A'(t).w = \varepsilon_w(A'(t)) = d(\varepsilon_w \circ A)(t, 1) = \frac{\partial}{\partial t}\big(g_w(t)\big) = g_{\lambda_{x_t}^{-1}([X_v, X_w](x_t))}(t) \\ = A(t).\lambda_{x_t}^{-1}([X_v, X_w](x_t)) = (A(t) \circ B(t))(w). \end{aligned}$$

Hence, A solves the initial value problem

$$\begin{cases} \varphi'(t) & = \varphi(t) \circ B(t) \\ \varphi(0) & = X(x, \bullet) \end{cases} \tag{2.6}$$

in $\mathcal{L}(F, E)_b$. There exists a solution of the initial value problem (2.6) in $\mathcal{L}(F, D_x)$. From Lemma 2.13, we conclude that the image of A lies in $\mathcal{L}(F, D_x)$. It remains to show (2.5). To this end let $(t, y, v) \in \Omega$ and $f\colon U \to E$ be a C^r-map with $f(p) \in D_p$ for all $p \in U$. We define $x := \Phi(t, y, v)$ and get $(-t, x, v) \in \Omega$. Hence, $d\Phi_t^v(\Phi_{-t}^v(x), f(\Phi_{-t}^v(x))) \in D_x$. We conclude that $d\Phi_t^v(y, f(y)) \in D_{\Phi_t^v(y)}$. Because $\Phi(t, \bullet, u)$ is a diffeomorphism, we get $d_2\Phi(t, y, u; \bullet) \in \mathrm{GL}(E)$. This shows $d_2\Phi(t, y, v; \bullet)(D_y) \subseteq D_{\Phi(t,y,v)}$ for all $(t, y, v) \in \Omega$. Again let $(t, y, v) \in \Omega$. With $\Phi(t, \Phi(-t, y, v), v) = y$, we get

$$\big(d_2\Phi(t, y, v; \bullet)\big)^{-1} = d_2\Phi(-t, \Phi(t, y, v), v; \bullet). \tag{2.7}$$

We conclude $d_2\Phi(t, y, v; \bullet)(D_y) = D_{\Phi(t,y,v)}$ for all $(t, y, v) \in \Omega$. Our second aim is to show the following statement.

Step 2: Given $(t, y, u) \in \Omega$, we have

$$d_3\Phi(t, y, u; \bullet)(F) \subseteq D_{\Phi(t,y,u)} \tag{2.8}$$

for the map $d_3\Phi(t, y, u; \bullet)\colon F \to E$. This generalises [CS76, Lemma 4.3].

Indeed, we have

$$d_1\Phi(t, y, u; 1) = X(\Phi(t, y, u), u), \tag{2.9}$$

$$\Phi(0, y, u) = y. \tag{2.10}$$

By differentiating the right-hand side of (2.9) in y in the direction $h \in E$, we get

$$d_y\big(X(\Phi(t, y, u), u)\big)(y, h) = d_1X\big(\Phi(t, y, u), u; d_2\Phi(t, y, u; h)\big).$$

Differentiation of the left-hand side of (2.9) in y in the direction $h \in E$ leads to

$\frac{\partial}{\partial s}\frac{\partial}{\partial t}(\Phi(t, y + sh, u)) = \frac{\partial}{\partial t} d_2\Phi(t, y, u; h)$. We conclude that

$$\frac{\partial}{\partial t} d_2\Phi(t, y, u; h) = d_1 X\big(\Phi(t, y, u), u; d_2\Phi(t, y, u; h)\big) \tag{2.11}$$

$$d_2\Phi(0, y, u; h) = h. \tag{2.12}$$

Now we differentiate the right-hand side of (2.9) in u in the direction $h \in F$ and get

$$\begin{aligned} &d_u\big(X(\Phi(t, y, u), u)\big)(u, h) = dX\big((\Phi(t, y, u), u); (d_3\Phi(t, y, u; h), h)\big) \\ =&d_2 X\big(\Phi(t, y, u), u; h\big) + d_1 X\big(\Phi(t, y, u), u; d_3\Phi(t, y, u; h)\big) \\ =&X\big(\Phi(t, y, u), h\big) + d_1 X\big(\Phi(t, y, u), u; d_3\Phi(t, y, u; h)\big). \end{aligned}$$

Differentiation of the left-hand side of (2.9) leads to

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}(\Phi(t, y, u + sh)) = \frac{\partial}{\partial t} d_3\Phi(t, y, u; h).$$

Hence,

$$\begin{aligned} &\frac{\partial}{\partial t} d_3\Phi(t, y, u; h) = X\big(\Phi(t, y, u), h\big) + d_1 X\big(\Phi(t, y, u), u; d_3\Phi(t, y, u; h)\big), \\ &d_3\Phi(0, y, u; h) = 0. \end{aligned}$$

Thus, $t \mapsto d_3\Phi(t, y, u; \bullet)$ solves the initial value problem

$$\begin{cases} \sigma'(t) &= X\big(\Phi(t, y, u), \bullet\big) + d_1 X\big(\Phi(t, y, u), u; \bullet\big) \circ \sigma(t), \\ \sigma(0) &= 0 \end{cases} \tag{2.13}$$

in $\mathcal{L}(F, E)$. We use the shorthand notation $I := I_{u,y}$. The map $f\colon I \times F \to D_y \subseteq E$, $(t, v) \mapsto d_2\Phi(-t, \Phi(t, y, u), u; X(\Phi(t, y, u), v))$ (see (2.7)) is of class C^2 and $\int_0^t f(s, v)ds = t \cdot \int_0^1 f(ts, v)ds$ (the weak integral exists because D_y is a Banach space). Thus, $f_1\colon I \times F \to E$, $(t, v) \mapsto \int_0^t f(s, v)ds$ is of class C^2. We conclude that $f_2\colon I \times F \to E$, $(t, v) \mapsto d_2\Phi(t, y, u; \int_0^t f(s, v)ds)$ is of class C^2. Hence, $\eta := \check{f}_2\colon I \to \mathcal{L}(F, E)_b$ is a C^1-map. We want to show that η is a solution of the initial value problem (2.13). Given $v \in F$, the evaluation map $\varepsilon\colon \mathcal{L}(F, E) \to E$, $\lambda \mapsto \lambda(v)$ is continuous and linear. Therefore we only need to show that for all $v \in F$ the curve $\tau\colon I \to E$, $t \mapsto d_2\Phi(t, y, u; \int_0^t f(s, v)ds)$ is a solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\sigma(t) &= d_1 X(\Phi(t, y, u), u; \sigma(t)) + X(\Phi(t, y, u), v) \\ \sigma(0) &= 0, \end{cases} \tag{2.14}$$

where σ is a curve in E. We define the map $H\colon I \times E \to E$, $(t, w) \mapsto d_2\Phi(t, y, u; w)$

and get

$$\begin{aligned}\tau'(t) = &\frac{\partial}{\partial t}(H \circ (\mathrm{id}_I(t), f_1(t,v))) = d\big(H \circ (\mathrm{id}_I, f_1(\bullet, v))\big)(t,1)\\ =& dH((t, f_1(t,v)); (1, f(t,v)))\\ =& d_1H\big(t, f_1(t,v); 1\big) + d_2H\big(t, f_1(t,v); f(t,v)\big).\end{aligned}$$

On the one hand we have

$$\begin{aligned}&d_2H(t, f_1(t,v); f(t,v)) = H(t, f(t,v))\\ =& d_2\Phi\big(t,y,u; \underbrace{d_2\Phi(-t, \Phi(t,y,u), u; X(\Phi(s,y,u),v))}_{=(d_2\Phi(t,y,u;\bullet))^{-1}(X(\Phi(s,y,u),v))}\big) = X(\Phi(s,y,u),v)\end{aligned} \tag{2.15}$$

and on the other

$$\begin{aligned}&d_1H(t, f_1(t,v); 1) = \frac{\partial}{\partial h}\Big(d_2\Phi(h,y,u; f_1(t,v))\Big)\Big|_{h=t}\\ =& \frac{\partial}{\partial h'}\Big(\frac{\partial}{\partial h}\Big(\Phi(h, y + h' \cdot f_1(t,v), u)\Big)\Big|_{h=t}\Big)\Big|_{h'=0}\\ =& \frac{\partial}{\partial h'}\Big(X(\Phi(t, y + h' \cdot f_1(t,v), u), u)\Big)\Big|_{h'=0}\\ =& d_1X(\Phi(t,y,u), u; d_2\Phi(t,y,u; f_1(t,v))) = d_1X(\Phi(t,y,u), u; \tau(t)).\end{aligned} \tag{2.16}$$

Thus, τ is a solution of (2.14) and so η solves the initial value problem (2.13). Now we show that the solution of (2.13) is unique. It is enough to show that for every $h \in F$ the initial value problem

$$\begin{cases} g'(t) &= X(\Phi(t,y,u), h) + d_1X(\Phi(t,y,u), u; g(t))\\ g(0) &= 0,\end{cases}$$

where g is a curve in E, has a unique solution. Obviously it is sufficient to show that the initial value problem

$$\begin{cases} g'(t) &= d_1X(\Phi(t,y,u), u; g(t))\\ g(0) &= 0\end{cases} \tag{2.17}$$

has at most one solution. We define $\tilde{\Omega} := \{(t,y) \in \mathbb{R} \times U : (t,y,u) \in \Omega\} \times E$ and consider the map $\tilde{\Phi}\colon \tilde{\Omega} \to U \times E$, $(t,y,w) \mapsto T\Phi^u_t(y,w)$ which is a local C^r-$\mathbb{R}$-action on $U \times E$ because of the chain rule of tangential-maps. The vector field $\tilde{X}\colon U \times E \to E \times E$, $(y,w) \mapsto (X(y,u), d_1X(y,u;w))$ has the local flow $\tilde{\Phi}$ because with (2.11) we get

$$\begin{aligned}&\frac{d}{dt}\big(\tilde{\Phi}(t,y,w)\big)\Big|_{t=0} = \frac{d}{dt}\big(\Phi^u_t(y), d_2\Phi(t,y,u;w)\big)\Big|_{t=0}\\ =& \big(X(y,u), d_1X(y,u;w)\big).\end{aligned}$$

Now let g_1 and g_2 be solutions of (2.17), defined on the interval I. For $i = 1, 2$, the curve $G_i : I \to U \times E$, $t \mapsto (\Phi(t, y, u), g_i(t))$ is a solution of the initial value problem

$$\begin{cases} G_i'(t) = \tilde{X}(G_i(t)) \\ G_i(0) = (y, 0), \end{cases}$$

because

$$G_i'(t) = \big(X(\Phi(t, y, u), u), d_1X(\Phi(t, y, u), u; g_i(t))\big) = \tilde{X}(G_i(t)).$$

Hence, $G_1(t) = G_2(t)$ and so $g_1(t) = g_2(t)$. This implies the uniqueness-statement. Now we conclude that

$$d_3\Phi(t, y, u; \bullet) = \eta(t) = \left(v \mapsto d_2\Phi\left(t, y, u; \int_0^t f(s, v) ds\right)\right). \tag{2.18}$$

Since $\operatorname{im}(f) \subseteq D_y$, with (2.18) and Step 1, we get $d_3\Phi(t, y, u; \bullet)(F) \subseteq d_2\Phi(t, y, u; \bullet).D_y = D_{\Phi(t,y,u)}$.

Step 3: Now we construct a Frobenius chart around 0_E (this construction generalises parts of the proof of [Eyn12, Theorem 2.6] or [Tei01, Theorem 2]). To this end let $\tilde{F}$ be a topological vector complement of F in E. We choose open 0-neighbourhoods $V^{(1)} \subseteq \tilde{F}$, $V^{(2)} \subseteq F$ and a symmetric interval $I \subseteq \mathbb{R}$ such that $V := V^{(1)} \times V^{(2)} \subseteq U$ and $I \times V \times V^{(2)} \subseteq \Omega$. We have $\frac{\partial}{\partial s}\Phi(s, 0, 0) = X(\Phi(t, 0, 0), 0) = 0$ and $\Phi(0, 0, 0) = 0$. Hence, $\Phi(t, 0, 0) = 0$ for all $t \in I$. We have seen $\operatorname{im}(f) \subseteq D_y$ in the calculation above. Taking $y = 0_E$ and $u = 0_F$, we get $d_2\Phi(-s, 0_E, 0_F; v) \in F$ for $s \in I$ and $v \in F$. We define the map $\lambda : I \times F \to F$, $(s, v) \mapsto d_2\Phi(-s, 0_E, 0_F; v)$. Because $\check{\lambda} : I \to \mathcal{L}(F)$, $s \mapsto d_2\Phi(-s, 0, 0, \bullet)$ is continuous and $\check{\lambda}(0) = \operatorname{id}_F$ we find $0 < t < 1$ such that $[-t, t] \subseteq I$ and $\|\check{\lambda}(s) - \operatorname{id}\|_{op} < \frac{1}{2}$ for all $s \in [-t, t]$. We have

$$\|\lambda(s, tv) - v\| \leqslant \|\lambda(s, tv) - tv\| + \|v - tv\| \leqslant \frac{1}{2} \cdot t\|v\| + (1 - t)\|v\| = \left(1 - \frac{t}{2}\right)\|v\|$$

for all $s \in [-t, t]$. We show that

$$d_3\Phi(t, 0_E, 0_F; \bullet) \in \mathcal{L}(F)^* \text{ and } d_3\Phi(-t, 0_E, 0_F; \bullet) \in \mathcal{L}(F)^*. \tag{2.19}$$

With (2.18), we get $d_3\Phi(t, 0, 0; \bullet) = (v \mapsto d_2\Phi(t, 0_E, 0_F; \int_0^t \lambda(s, v) ds))$. The map $d_2\Phi(t, 0, 0; \bullet) : E \to E$ is an isomorphism of topological vector spaces. With (2.7) and Step 1 we see that $d_2\Phi(t, 0, 0; \bullet)|_F^F \in \mathcal{L}(F)^*$. Hence, we have to show that the map $\mu_t : F \to F, v \mapsto \int_0^t \lambda(s, v) ds$ is an isomorphism. To see $\|\mu_t - \operatorname{id}_F\|_{op} < 1$ we choose $v \in F$ and calculate

$$\left\|\int_0^t \lambda(s, v) ds - v\right\| = \left\|\int_0^1 \lambda(ts, v) \cdot t - v ds\right\| \leqslant \int_0^1 \|\lambda(s, tv) - v\| \, ds \leqslant \left(1 - \frac{t}{2}\right)\|v\|. \tag{2.20}$$

Analogously, we get $d_2\Phi(-t,0,0;\bullet)|_F^F \in \mathcal{L}(F)^*$. To show that $d_3\Phi(-t,0_E,0_F;\bullet) \in \mathcal{L}(F)^*$ it is enough to show that $\mu_{-t}\colon F \to F$, $v \mapsto \int_0^{-t} \lambda(s,v)ds$ is an isomorphism. A calculation analogous to (2.20) shows that $-\mu_{-t}$ is an isomorphism. Hence we obtain (2.19). Now we show that $\zeta\colon V^{(1)} \times V^{(2)} \to E$, $(x,w) \mapsto \Phi(-t,x,w)$ has open image and is a diffeomorphism onto its image. To this end we consider the C^r-map $b\colon V^{(1)} \times V^{(2)} \times V^{(2)} \to \tilde{F} \times F$, $(z,w,v) \mapsto \Phi(t,(z,v),w)$. We write $b_2 := \mathrm{pr}_2 \circ B$. We have $\Phi(t,0,0) = 0$. With the information from Step 2, we get $d_3\Phi(t,0,0;\bullet).F \subseteq D_{\Phi(t,0,0)} = F$ and with (2.19) we conclude that

$$d_2b_2(0,0,0;\bullet) = \mathrm{pr}_2(d_3\Phi(t,0_E,0_F;\bullet)) = d_3\Phi(t,0_E,0_F;\bullet)|_F^F \in \mathcal{L}(F)^*.$$

With Theorem 2.8, we get that after shrinking $V^{(1)}$ and $V^{(2)}$ the map $b_2(z,\bullet,v)\colon V^{(2)} \to F$ has open image and is a diffeomorphism onto its image. Moreover, we get that $\Psi\colon V^{(1)} \times V^{(2)} \times V^{(2)} \to E \times F$, $(z,w,v) \mapsto ((z,v),b_2(z,w,v))$ has open image and is a diffeomorphism onto its image. We have $\Phi(t,0,0) = 0$ and so $\Psi(0,0,0) = (0,0)$. We choose 0-neighbourhoods $W^{(1)} \subseteq V^{(1)} \subseteq \tilde{F}$ and $W^{(2)} \subseteq V^{(2)} \subseteq F$ such that $W^{(1)} \times W^{(2)} \times W^{(2)} \subseteq \mathrm{im}(\Psi)$. Hence, $\Psi^{-1}(z,v,0) = (z,b_2(z,\bullet,v)^{-1}(0),v)$ for $(z,v) \in W^{(1)} \times W^{(2)}$. We define $W := W^{(1)} \times W^{(2)}$. We write $(\Psi^{-1})_2 := \mathrm{pr}_2 \circ \Psi^{-1}$. For the map $u\colon W^{(1)} \times W^{(2)} \to V^{(2)}$, $(z,v) \mapsto (\Psi^{-1})_2(z,v,0)$, we get $b_2(z,u(z,v),v) = 0$ because of $(\Psi^{-1})_2(z,v,0) = b_2(z,\bullet,v)^{-1}(0)$. We define the map $\xi\colon W^{(1)} \times W^{(2)} \to E$, $(z,v) \mapsto (b_1(z,u(z,v),v),u(z,v))$. In the following we show that $\xi|_{\xi^{-1}(V)}$ is inverse to $\zeta|_{\zeta^{-1}(W)}$. To this end we calculate

$$\begin{aligned}\zeta \circ \xi(z,v) &= \zeta(b_1(z,u(z,v),v),u(z,v)) = \Phi(-t,\underbrace{b_1(z,u(z,v),v)}_{=b(z,u(z,v),v)},u(z,v))\\ &=\Phi(-t,\Phi(t,(z,v),u(z,v)),u(z,v)) = (z,v).\end{aligned}$$

Given $(x,w) \in \zeta^{-1}(W)$, we have

$$b(\zeta_1(x,w),w,\zeta_2(x,w)) = \Phi(t,\zeta(x,w),w) = x. \tag{2.21}$$

Thus, $b_2(\zeta_1(x,w),w,\zeta_2(x,w)) = 0$ respectively $u(\zeta(x,w)) = w$. Hence

$$\begin{aligned}\xi \circ \zeta(x,w) &= (b_1(\zeta_1(x,w),u(\zeta(x,w)),\zeta_2(x,w)),u(\zeta(x,w)))\\ &=(b_1(\zeta_1(x,w),w,\zeta_2(x,w)),w) = (x,w).\end{aligned}$$

We define $U_\varphi := \xi^{-1}(V)$, $V_\varphi := \zeta^{-1}(W)$ and $\varphi := \xi|_{U_\varphi}^{V_\varphi}$. In particular, we get $\varphi^{-1} = \zeta|_{V_\varphi}$. After shrinking V_φ we assume that $V_\varphi = V_\varphi^{(1)} \times V_\varphi^{(2)}$ with $V_\varphi^{(1)} \subseteq V^{(1)}$ and $V_\varphi^{(2)} \subseteq V^{(2)}$. We show that φ is a Frobenius chart around 0. It is sufficient to show that $d\varphi(\{p\} \times D_p) = F$ respectively $((d\varphi)(p,\bullet))^{-1}(F) = D_p$ for all $p \in U_\varphi$ because of Remark 2.5. This is equivalent to show that $d\varphi^{-1}(x,w;\bullet)(F) = D_{\varphi^{-1}(x,w)}$ respectively $d_2\zeta(x,w;\bullet)(F) = D_{\zeta(x,w)}$ for all $(x,w) \in V_\varphi = V_\varphi^{(1)} \times V_\varphi^{(2)}$. Because of (2.8), the map $\lambda\colon V_\varphi^{(1)} \times V_\varphi^{(2)} \to \mathcal{L}(F)$ $(x,w) \mapsto \psi_2(\Phi(-t,x,w),d_3\Phi(-t,x,w;\bullet))$ is well-defined and continuous. Because of $\lambda(0_{\tilde{F}},0_F) \in \mathcal{L}(F)^*$ we assume that $\lambda(x,w) \in \mathcal{L}(F)^*$ for all $(x,w) \in V_\varphi^{(1)} \times V_\varphi^{(2)}$. Hence, $d_3\Phi(-t,x,w;\bullet) = \psi_2(\Phi(-t,x,w),\bullet)^{-1} \circ \lambda(x,w) \in \mathcal{L}(F, D_{\Phi(-t,x,w)})$ is an isomorphism of topological vector spaces for $x \in V_\varphi^{(1)}$. □

2.2. Application of our Frobenius theorem to Lie theory

Remark 2.16. *From [Lan01, Chapter VI.] respectively [Eyn12, Chapter 4], we get the following facts:*

(a) *Given a Lie group G and a closed Lie subalgebra $\mathfrak{h} \subseteq L(G)$, the vector distribution $D := \bigcup_{g\in G} T\lambda_g(\mathfrak{h})$ is an involutive subbundle of TG with typical fibre $\mathfrak{h}$, if we identify the modelling space of G with $L(G)$ (see e.g. [Eyn12, Lemma 4.6]; cf. [Lan01, Chapter VI]).*

(b) *If the vector bundle D in (a) is a Frobenius distribution then we can find a Lie group H that is an integral manifold for D and a subgroup of G (see e.g. [Eyn12, Lemma 4.7]; cf. [Lan01, Chapter VI]).*

As in [Lan01, Chapter VI, Theorem 5.4], [Les68], [Les92], or [Eyn12, Theorem 4.1] we use a Frobenius theorem to show a result of integrability of Lie subalgebras in the context of infinite-dimensional Lie groups. The following theorem generalises [Eyn12, Theorem 4.1] respectively [Lan01, Chapter VI, Theorem 5.4]. Note also that it is complementary to [Les92, Theorem 4.1] because of the different conditions on the considered Lie groups. However an alternative proof of the following statement can be obtained with the help of [Nee06, Theorem IV.4.9.][4].

Theorem 2.17. *Let G be a Lie group modelled over a locally convex space and $\mathfrak{h} \subseteq L(G)$ be a Lie subalgebra that is complemented as a topological vector subspace and is a Banach space. If G provides an exponential map then we can find a Lie group H that is a subgroup of G and an immersed submanifold of G such that $L(H) = \mathfrak{h}$.*

Proof. Again we define $D := \bigcup_{g\in G} T\lambda_g(\mathfrak{h})$. The vector field with parameters $X\colon G \times \mathfrak{h} \to TG$, $(g,v) \mapsto T\lambda_g(v)$ obviously satisfies the conditions (a)–(c) of Theorem 2.15. Also condition (d) is satisfied because $\Phi\colon \mathbb{R}\times G\times\mathfrak{h} \to G$, $(t,g,v) \mapsto \lambda_g(\exp_G(tv))$ is a local flow with parameters of X which follows from

$$\frac{d}{dt}\Big|_{t=0} \Phi(t,g,v) = T\lambda_g(v) = X(g,v).$$

□

[4]This was mentioned by K. H. Neeb in comments to this thesis.

3. Constructions for Lie algebras of compactly supported sections

Now we turn our attention to topologically universal bilinear forms for Lie algebras of compactly supported sections in Section 3.2. As mentioned in the introduction this a natural continuation of the considerations in [Gun11, Chapter 4]. In Section 3.3 we consider a certain class of pseudo-unital locally convex algebras A that contains the so called CPUSLF-algebras from [Gun11] as well as the algebra of compactly supported smooth functions on a σ-compact manifold. Given such an algebra A, we show the universality of the canonical cocycle on $A \otimes \mathfrak{g}$ (for $\mathfrak{g}$ finite-dimensional and semisimple).

We fix the following specific notation for this chapter[1]:

- Let M be a finite-dimensional σ-compact manifold, $U \subseteq M$ be an open subset, V be a finite-dimensional vector space and $\mathbb{V}$ a vector bundle with base M. For $f \in C_c^\infty(U, V)$, $X \in \Gamma_c(U, \mathbb{V})$ and $\theta \in \Omega_c^k(U, \mathbb{V})$, we write $f_\sim$, $X_\sim$ and $\theta_\sim$, respectively, for the extension of f, X and θ to M by 0 outside of U.
- We write A_1 for the unitalisation of a commutative algebra A.

3.1. Some basic concepts and results

First, we recall the basic concepts of universal continuous invariant symmetric bilinear forms in the following definition. See e.g. [Gun11, Chapter 4].

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra. A pair (V, β) with a vector space V and a symmetric bilinear map $\beta\colon \mathfrak{g} \times \mathfrak{g} \to V$ is called an *invariant symmetric bilinear form* on $\mathfrak{g}$ if $\beta([x, y], z) = \beta(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. The invariant symmetric bilinear form (V, β) is called *algebraically universal* if for every invariant symmetric bilinear form (W, γ) on $\mathfrak{g}$, there exists a unique linear map $\psi\colon V \to W$ such that $\gamma = \psi \circ \beta$. It is clear that if β is algebraically universal then another invariant symmetric bilinear form (W, γ) on $\mathfrak{g}$ is algebraically universal if and only if there exists an isomorphism of vector spaces $\varphi\colon V \to W$ with $\gamma = \varphi \circ \beta$. In the case that $\mathfrak{g}$ is a locally convex Lie algebra and V is a locally convex space, the pair (V, β) is called a *continuous invariant symmetric bilinear form* on $\mathfrak{g}$ if β is continuous and it is called *topologically universal* or a *universal continuous invariant symmetric bilinear form* if for every continuous invariant symmetric bilinear form (W, γ) on $\mathfrak{g}$, there exists a unique continuous linear map $\psi\colon V \to W$ such that $\gamma = \psi \circ \beta$. It is clear that if β is topologically universal then another invariant symmetric

[1] This chapter consist of material published before in the author's preprint [Eyn14c].

bilinear form (W, γ) on $\mathfrak{g}$ is topologically universal if and only if we can find an isomorphism $\varphi\colon V \to W$ of topological vector spaces with $\gamma = \varphi \circ \beta$.

Definition 3.2. If $\mathfrak{g}$ is a Lie algebra then we call

$$\mathrm{Cent}(\mathfrak{g}) := \{f \in \mathrm{Lin}(\mathfrak{g}) : \ (\forall x, y \in \mathfrak{g}) \ f([x,y]) = [f(x), y]\}$$

the *Centroid* of $\mathfrak{g}$. (Cf. [Gun11, Definition 2.1.15]).

Remark 3.3. *[Gun11, Remark 4.1.5 and Proposition 4.1.7] tell us that there always exists an algebraically universal invariant symmetric bilinear form $(V_{\mathfrak{g}}, \kappa_{\mathfrak{g}})$ for a given Lie algebra $\mathfrak{g}$. The same argumentation as in [Gun11, Remark 4.5.6] and [Mai02, Lemma 15] shows the existence of a universal continuous invariant symmetric bilinear form $(V_{\mathfrak{g}}^{ct}, \kappa_{\mathfrak{g}}^{ct})$ for a given locally convex Lie algebra $\mathfrak{g}$. The argument goes as follows: If $\mathfrak{g}$ is a locally convex Lie algebra and $H \subseteq \mathfrak{g} \otimes \mathfrak{g}$ is the subspace that is generated from elements of the form $x \otimes y - y \otimes x$, we write $S^2(\mathfrak{g})$ for the locally convex space $\mathfrak{g} \otimes_\pi \mathfrak{g}/\overline{H}$ (one can show that the space H is a closed subspace of $\mathfrak{g} \otimes \mathfrak{g}$) see [Mai02, p. 63]). We write $\pi_{\overline{H}}\colon \mathfrak{g} \otimes_\pi \mathfrak{g} \to \mathfrak{g} \otimes_\pi \mathfrak{g}/\overline{H}$ for the canonical quotient map and define $\vee\colon \mathfrak{g} \times \mathfrak{g} \to S^2(\mathfrak{g})$, $(x, y) \mapsto x \vee y := \pi_{\overline{H}}(x \otimes y)$. It is well known that for every continuous symmetric bilinear map $\beta\colon \mathfrak{g} \times \mathfrak{g} \to V$ into a locally convex space V, there exists a unique continuous linear map $\varphi\colon S^2(\mathfrak{g}) \to V$ such that $\beta = \varphi \circ \vee$ (see [Mai02, Theorem 3]). Let $D \subseteq S^2(\mathfrak{g})$ be the subspace generated by elements of the form $[x, y] \vee z - x \vee [y, z]$. We define the locally convex space $V_{\mathfrak{g}}^{ct} := (\mathfrak{g} \vee \mathfrak{g})/\overline{D}$ and the continuous invariant symmetric bilinear map $\kappa_{\mathfrak{g}}^{ct}\colon \mathfrak{g} \times \mathfrak{g} \to V_{\mathfrak{g}}^{ct}$, $(x, y) \mapsto [x \vee y] = x \vee y + \overline{D}$. It is clear that the image of $\kappa_{\mathfrak{g}}^{ct}$ generates $V_{\mathfrak{g}}^{ct}$ and that $(\kappa_{\mathfrak{g}}^{ct}, V_{\mathfrak{g}}^{ct})$ is a universal topological invariant symmetric bilinear form. For this result, it is crucial not to take the completion of $\mathfrak{g} \otimes_\pi \mathfrak{g}$ because otherwise $\kappa_{\mathfrak{g}}^{ct}$ would be universal just for complete locally convex spaces. Sometimes we use the notation $V_{ct}(\mathfrak{g}) := V_{\mathfrak{g}}^{ct}$. If $\mathfrak{g}$ is finite-dimensional then the universal continuous invariant symmetric bilinear form and the algebraically universal invariant symmetric bilinear form coincide. Therefore, we write $(V_{\mathfrak{g}}, \kappa_{\mathfrak{g}})$ for $(V_{\mathfrak{g}}^{ct}, \kappa_{\mathfrak{g}}^{ct})$ in this case.*

With [Gun11, Proposition 4.3.3] we get directly the following Lemma 3.4.

Lemma 3.4. *For a σ-compact finite-dimensional manifold M and a finite-dimensional perfect Lie algebra $\mathfrak{g}$, the map*

$$\kappa_{\mathfrak{g}*}\colon C_c^\infty(M, \mathfrak{g}) \times C_c^\infty(M, \mathfrak{g}) \to C_c^\infty(M, V_{\mathfrak{g}}), (f, g) \mapsto \kappa_{\mathfrak{g}} \circ (f, g)$$

is an algebraically universal invariant symmetric bilinear form. Notably the image of $\kappa_{\mathfrak{g}*}$ spans $C_c^\infty(M, V_{\mathfrak{g}})$.

In the case that M is connected, the preceding Lemma 3.4 can be found in [Gun11, Corollary 4.3.4].

The following Lemma 3.5 can be found in [Gun11, Lemma 4.1.6].

Lemma 3.5. *Let $\mathfrak{g}$ be a Lie algebra, W a vector space and $\beta\colon \mathfrak{g} \times \mathfrak{g} \to W$ an invariant symmetric bilinear map. Then $\beta(f(x), y) = \beta(x, f(y))$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$, $y \in \mathfrak{g}$ and $f \in \mathrm{Cent}(\mathfrak{g})$.*

The next Lemma 3.6 comes from [Gun11, Remark 4.2.7].

Lemma 3.6. *The Lie algebra $C_c^\infty(M, \mathfrak{g})$ is perfect for every finite-dimensional σ-compact manifold M and perfect finite-dimensional Lie algebra $\mathfrak{g}$.*

3.2. Topological universal bilinear forms

The aim of this section is to construct a universal invariant continuous bilinear form on the space of compactly supported sections of a Lie algebra bundle. To this end, we first show the "local statement". We construct a universal continuous invariant symmetric bilinear form on the compactly supported smooth functions on a σ-compact manifold with values in a Lie algebra $\mathfrak{g}$ (Theorem 3.12). Afterwards we glue the local constructions together to a global one (Theorem 3.19). This strategy is inspired by [Gun11, Theorem 4.4.4].

In the following definition we recall the well-known concept of a Lie algebra bundle.

Definition 3.7. Let M be a manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\pi\colon \mathfrak{K} \to M$ a vector bundle with typical fibre $\mathfrak{g}$. If for every $x \in M$ the space $\pi^{-1}(\{x\})$ is endowed with a Lie algebra structure and there exists an atlas of local trivialisations $\varphi\colon \pi^{-1}(U_\varphi) \to U_\varphi \times \mathfrak{g}$ of $\mathfrak{K}$ such that for every $p \in U_\varphi$ the map $\varphi(p, \bullet)\colon \mathfrak{K}_p \to \mathfrak{g}$ is a Lie algebra homomorphism then we call $\mathfrak{K}$ a *Lie algebra bundle*.

In Definition 3.8 we endow both the vector space of sections and the space of compactly supported sections of a given vector bundle with a locally convex topology. We follow the definitions from [Glo13, Chapter 3].

Definition 3.8. Let M be a finite-dimensional manifold, V a finite-dimensional vector space and $\pi\colon \mathbb{V} \to M$ a vector bundle with typical fibre V. If $\eta \in \Gamma(\mathbb{V})$ and $\varphi\colon \pi^{-1}(U) \to U_\varphi \times V$ is a local trivialisation of $\mathbb{V}$ we write $\eta_\varphi := \mathrm{pr}_2 \circ \varphi \circ \eta|_{U_\varphi} \in C^\infty(U_\varphi, V)$ for the local representation of η. Let $\mathcal{A}$ be an atlas of $\mathbb{V}$. As mentioned in the introduction of this thesis, we equip $\Gamma(\mathbb{V})$ with the initial topology with respect to the maps $\sigma_\varphi\colon \Gamma(\mathbb{V}) \to C^\infty(U_\varphi, V)$, $\eta \mapsto \eta_\varphi$ as described in [Glo13, Chapter 3]. [Glo13, Lemma 3.9] tells us that this topology does not depend on the choice of the atlas. Moreover, [Glo13, Lemma 3.7] tells us that the topological embedding $\Gamma(\mathbb{V}) \to \prod_{\varphi\in\mathcal{A}} C^\infty(U_\varphi, V)$, $\eta \mapsto (\eta_\varphi)_{\varphi\in\mathcal{A}}$ has closed image and so $\Gamma(\mathbb{V})$ becomes a locally convex space. In particular we see that $\Gamma(\mathbb{V})$ is a Fréchet space if there exists a countable atlas of local trivialisations of $\mathbb{V}$. If $K \subseteq M$ is compact we write $\Gamma_K(\mathbb{V})$ for the closed subspace of sections of $\mathbb{V}$ with support in K. If there exists a countable atlas of local trivialisations of $\mathbb{V}$ then it is clear that $\Gamma_K(\mathbb{V})$ is a Fréchet space. We give $\Gamma_c(\mathbb{V})$ the topology making it the inductive limit of the spaces $\Gamma_K(\mathbb{V})$ in the category of locally convex spaces, where K runs through all compact sets. If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with typical fibre $\mathfrak{g}$, we define the Lie bracket $[\bullet, \bullet]\colon \Gamma(\mathfrak{K}) \times \Gamma(\mathfrak{K}) \to \Gamma(\mathfrak{K})$ by $[\eta, \zeta](p) = [\eta(p), \zeta(p)]$ for $\eta, \zeta \in \Gamma(\mathfrak{K})$, where the latter Lie bracket is taken in

$\mathfrak{K}_p$. Together with this Lie bracket $\Gamma(\mathfrak{K})$ becomes a topological Lie algebra. (The concepts in this Definition 3.8 come from [Glo13, Chapter 3]).

Lemma 3.9. *Let M be a finite-dimensional σ-compact manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with typical fibre $\mathfrak{g}$. Then $\Gamma_c(\mathfrak{K})$ is a topological Lie algebra.*

Proof. The map $\mathfrak{K}\oplus\mathfrak{K} \to \mathfrak{K}$ that maps (v, w) to $[v, w]_{\mathfrak{K}_p}$ for $v, w \in \mathfrak{K}_p$ and $p \in M$ is continuous. With the Ω-Lemma (see, e.g. [Mic80, Theorem 8.7] or [Glo04, F.24]) we see that $\Gamma_c(\mathfrak{K})$ is a topological Lie algebra. □

Lemma 3.10. *Let M be a σ-compact finite-dimensional manifold, E a finite-dimensional vector space and $(\rho_m)_{m\in\mathbb{N}}$ a partition of unity that is subordinate to a locally finite cover $(V_n)_{n\in\mathbb{N}}$ of open relatively compact subsets $V_n \subseteq M$. Then $\Phi\colon \bigoplus_{m\in\mathbb{N}} C^\infty(M, E) \to C^\infty_c(M, E)$, $(f_m)_{m\in\mathbb{N}} \mapsto \sum_{m\in\mathbb{N}} \rho_m \cdot f_m$ is a quotient map.*

Proof. First we show the continuity of Φ. Because Φ is linear, it suffices to show that $C^\infty(M, E) \to C^\infty_c(M, E)$, $f \mapsto \rho_m \cdot f$ is continuous for every $m \in \mathbb{N}$. The locally convex space $C^\infty_c(M, E)$ is the inductive limit of spaces $C^\infty_{K_n}(M, E)$ with $n \in \mathbb{N}$, where $(K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of M. Because the support of ρ_m is compact we can find $n \in \mathbb{N}$ with $\operatorname{supp}(\rho_m) \subseteq K_n$. We see that the map $C^\infty(M, E) \to C^\infty_c(M, E)$, $f \mapsto \rho_m \cdot f$ takes its image in the subspace $C^\infty_{K_n}(M, E)$. Now we conclude that Φ is continuous since $C^\infty(M, E) \to C^\infty_{K_n}(M, E)$, $f \mapsto \rho_m \cdot f$ is continuous. For $n \in \mathbb{N}$, we choose a smooth function $\sigma_n\colon M \to [0, 1]$ such that $\sigma_n|_{\operatorname{supp}(\rho_n)} \equiv 1$ and $\operatorname{supp}(\sigma_n) \subseteq V_n$. Because a compact subset of M is only intersected by a finite number of sets of the cover $(V_n)_{n\in\mathbb{N}}$, we can define the map $\Psi\colon C^\infty_c(M, E) \to \bigoplus_{n=1}^\infty C^\infty(M, E)$, $\gamma \mapsto (\sigma_n \cdot \gamma)_{n\in\mathbb{N}}$, which is obviously a right-inverse for Φ. If $K \subseteq M$ is compact, we find $N \in \mathbb{N}$ such that $K \cap V_n = \emptyset$ for $n \geqslant N$. We conclude that $\Psi(C^\infty_K(M, E)) \subseteq \prod_{n=1}^N C^\infty(M, E)$. Obviously the map $C^\infty(M, E) \to \prod_{n=1}^N C^\infty(M, E)$, $\gamma \mapsto (\sigma_n \cdot \gamma)_{n=1,\dots,N}$ is continuous. We conclude that Ψ is a continuous linear right-inverse for Φ and so we see that Φ is a continuous, open surjective map. □

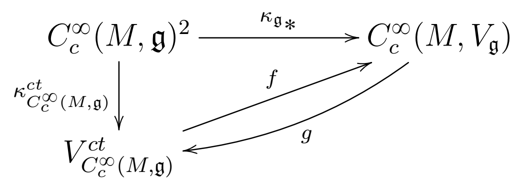
Lemma 3.11. *If M is a finite-dimensional σ-compact manifold and $\mathfrak{g}$ a finite-dimensional Lie algebra then $\kappa_{\mathfrak{g}*}\colon C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_{\mathfrak{g}})$, $(f, g) \mapsto \kappa_{\mathfrak{g}} \circ (f, g)$ is continuous.*

Proof. This follows directly from [Glo02, Lemma 4.12 and Corollary 4.17]. □

As mentioned in the introduction Gündoğan showed that the map $\kappa_{\mathfrak{g}*}$ is universal in the algebraic sense (see Lemma 3.4). We now show that it is also universal in the topological sense.

Theorem 3.12. *Let $\mathfrak{g}$ be a perfect finite-dimensional Lie algebra and M a finite-dimensional σ-compact manifold. Then $\kappa_{\mathfrak{g}*}\colon C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_{\mathfrak{g}})$ is topologically universal.*

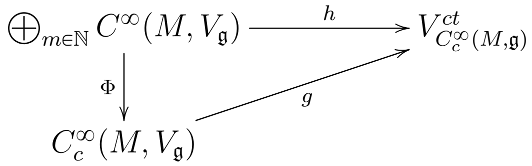
Proof. We know that $\kappa_{\mathfrak{g}*} \colon C_c^\infty(M,\mathfrak{g})^2 \to C_c^\infty(M,V_\mathfrak{g})$ is an algebraically universal invariant symmetric bilinear form. Moreover, $\kappa^{ct}_{C_c^\infty(M,\mathfrak{g})} \colon C_c^\infty(M,\mathfrak{g})^2 \to V^{ct}_{C_c^\infty(M,\mathfrak{g})}$ is a topologically universal invariant symmetric bilinear form. Because $\kappa_{\mathfrak{g}*}$ is a continuous invariant symmetric bilinear map, we find a continuous linear map $f \colon V_{ct}(C_c^\infty(M,\mathfrak{g})) \to C_c^\infty(M,V_\mathfrak{g})$ such that $\kappa_{\mathfrak{g}*} = f \circ \kappa^{ct}_{C_c^\infty(M,\mathfrak{g})}$ and because $\kappa^{ct}_{C_c^\infty(M,\mathfrak{g})}$ is an invariant symmetric bilinear map, we find a linear map $g \colon C_c^\infty(M,V_\mathfrak{g}) \to V^{ct}_{C_c^\infty(M,\mathfrak{g})}$ with $\kappa^{ct}_{C_c^\infty(M,\mathfrak{g})} = g \circ \kappa_{\mathfrak{g}*}$ (as in the proof of [Gun11, Theorem 4.6.2] we use the interplay of algebraic and topologically universality). We get the commutative diagram



With $f \circ g \circ \kappa_{\mathfrak{g}*} = \kappa_{\mathfrak{g}*}$ and the fact that $\kappa_{\mathfrak{g}*}$ is algebraically universal, we get

$$f \circ g = \mathrm{id}_{C_c^\infty(M,V_\mathfrak{g})}. \tag{3.1}$$

Let $(V_n)_{n\in\mathbb{N}}$ be a locally finite cover of M that consists of relatively compact open subsets $V_n \subseteq M$ and $(\rho_m)_{m\in\mathbb{N}}$ be a partition of unity of M that is subordinate to the cover $(V_n)_{n\in\mathbb{N}}$. From Lemma 3.10 we know the quotient map Φ and get the commutative diagram



with $h \colon \bigoplus_{m\in\mathbb{N}} C^\infty(M,V_\mathfrak{g}) \to V^{ct}_{C_c^\infty(M,\mathfrak{g})}$, $(\varphi_m)_{m\in\mathbb{N}} \mapsto \sum_{m\in\mathbb{N}} g(\varphi_m \cdot \rho_m)$. If we can show that h is continuous, we get that also g is continuous. Because h is linear it suffices to show that $C^\infty(M,V_\mathfrak{g}) \to V^{ct}_{C_c^\infty(M,\mathfrak{g})}$, $\varphi \mapsto g(\varphi \cdot \rho_m)$ is continuous for all $m \in \mathbb{N}$. The space $V^{ct}_\mathfrak{g}$ is finite-dimensional because $\mathfrak{g}$ is finite-dimensional. Let $(v_i)_{i=1,\dots,n}$ be a basis of $V_\mathfrak{g}$. We write φ_i for the i-th component of a map $\varphi \in C^\infty(M,V_\mathfrak{g})$. Since $\kappa_{\mathfrak{g}*} \colon C_c^\infty(M,\mathfrak{g})^2 \to C_c^\infty(M,V_\mathfrak{g})$ is algebraically universal, the image $\mathrm{im}(\kappa_{\mathfrak{g}*})$ generates $C_c^\infty(M,V_\mathfrak{g})$. Therefore, we find $\xi_{ij}, \zeta_{ij} \in C_c^\infty(M,\mathfrak{g})$ such that $\rho_m \cdot v_i = \sum_{j=1}^{n_i} \kappa_{\mathfrak{g}*}(\xi_{ij},\zeta_{ij})$. For $\varphi \in C^\infty(M,V_\mathfrak{g})$, we calculate

$$\begin{aligned} g(\varphi\cdot\rho_m) &= \sum_{i=1}^n g(\varphi_i \cdot \rho_m \cdot v_i) = \sum_{i=1}^n \sum_{j=1}^{n_i} g\left(\varphi_i \cdot \kappa_{\mathfrak{g}*}(\xi_{ij},\zeta_{ij})\right) \\ &= \sum_{i=1}^n \sum_{j=1}^{n_i} g\left(\kappa_{\mathfrak{g}*}(\varphi_i\cdot\xi_{ij},\zeta_{ij})\right) = \sum_{i=1}^n \sum_{j=1}^{n_i} \kappa^{ct}_{C_c^\infty(M,\mathfrak{g})}(\varphi_i\cdot\xi_{ij},\zeta_{ij}). \end{aligned}$$

Because $C^\infty(M,\mathbb{R}) \to V^{ct}_{C_c^\infty(M,\mathfrak{g})}$, $\psi \mapsto \kappa^{ct}_{C_c^\infty(M,\mathfrak{g})}(\psi\cdot\xi_{ij},\zeta_{ij})$ is continuous, we see that also g is continuous. We have $g \circ f \circ \kappa^{ct}_{C^\infty_c(M,\mathfrak{g})} = \kappa^{ct}_{C^\infty_c(M,\mathfrak{g})}$. Since g is continuous we get $g \circ f = \mathrm{id}_{V^{ct}_{C^\infty_c(M,\mathfrak{g})}}$. With (3.1), we see that f is an isomorphism of topological vector spaces. □

Remark 3.13. *If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras and $f\colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism then there exists a unique linear map $f_\kappa\colon V_\mathfrak{h} \to V_\mathfrak{g}$ with $f_\kappa(\kappa_\mathfrak{h}(x,y)) = \kappa_\mathfrak{g}(f(x), f(y))$.*

The following definition of the vector bundle $V(\mathfrak{K})$ is equivalent to [Gun11, Definition 4.1.10] and coincides with the definition of $V(\mathfrak{K})$ described in [JW13, p. 1].

Definition 3.14. Let M be a manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\pi\colon \mathfrak{K} \to M$ a Lie algebra bundle with base M and typical fibre $\mathfrak{g}$. If $\mathcal{A}$ is an atlas of local trivialisations of $\mathfrak{K}$, we define $V(\mathfrak{K}) := \bigcup_{x\in M} V(\mathfrak{K}_x)$ and the surjection $\rho\colon V(\mathfrak{K}) \to M$, $v \mapsto x$ for $v \in \mathfrak{K}_x$. For a local trivialisation $\varphi\colon \pi^{-1}(U_\varphi) \to U_\varphi \times \mathfrak{g}$ of $\mathfrak{K}$, we define the map $\tilde{\varphi}\colon \rho^{-1}(U_\varphi) \to U_\varphi \times V_\mathfrak{g}$, $v \mapsto \big(\rho(v), (\mathrm{pr}_2 \circ \varphi|_{\mathfrak{K}|_{\rho(v)}})_\kappa(v)\big)$. Together with the atlas of local trivialisations $\{\tilde{\varphi} : \varphi \in \mathcal{A}\}$ we get a vector bundle[2] $\rho\colon V(\mathfrak{K}) \to M$. In this chapter we will always write $\tilde{\varphi}$ for the trivialisation of $V(\mathfrak{K})$ that comes from a trivialisation φ of $\mathfrak{K}$.

Definition 3.15. (Cf. [Gun11, Definition 4.1.13]) For a finite-dimensional manifold M, a finite-dimensional Lie algebra $\mathfrak{g}$ and a Lie algebra bundle $\mathfrak{K}$ with base M and typical fibre $\mathfrak{g}$, we define the map $\kappa_\mathfrak{K}\colon \Gamma_c(\mathfrak{K})^2 \to \Gamma_c(V(\mathfrak{K}))$ by $\kappa_\mathfrak{K}(X,Y)(x) = \kappa_{\mathfrak{K}_x}(X(x), Y(x))$ for $x \in M$ and $X, Y \in \Gamma_c(\mathfrak{K})$.

Lemma 3.16. *If M is a σ-compact, finite-dimensional manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with base M and typical fibre $\mathfrak{g}$ then $\kappa_\mathfrak{K}\colon \Gamma_c(\mathfrak{K})^2 \to \Gamma_c(V(\mathfrak{K}))$ is continuous.*

Proof. The map $\mathfrak{K} \oplus \mathfrak{K} \to V(\mathfrak{K})$ that maps (v,w) to $\kappa_{\mathfrak{K}_p}(v,w)$ for $v, w \in \mathfrak{K}_p$ and $p \in M$ is continuous. The assertion now follows from the Ω-Lemma (see, e.g. [Mic80, Theorem 8.7] or [Glo04, F.24]). □

Lemma 3.17. *The image of $\kappa_\mathfrak{K}$ spans $\Gamma_c(V(\mathfrak{K}))$, if $\mathfrak{g}$ is a perfect finite-dimensional Lie algebra, M a σ-compact finite-dimensional manifold and $\pi_\mathfrak{K}\colon \mathfrak{K} \to M$ a Lie algebra bundle with base M and typical fibre $\mathfrak{g}$.*

Proof. To show the assertion of the lemma, we only need to show that the global statement can be reduced to the local one because the local statement follows from Lemma 3.4. Let $\eta \in \Gamma_c(V(\mathfrak{K}))$ and $K := \mathrm{supp}(\eta)$. We find local trivialisations $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathfrak{g}$ of $\mathfrak{K}$ for $i = 1, \ldots, k$ with $K \subseteq \bigcup_{i=1}^n U_i$. Let $(\lambda_i)_{i=0,\ldots,k}$ be a partition of unity of M that is subordinate to the open cover that consists of the sets $M \backslash K$ and U_i for $i = 1, \ldots, k$. We get $\eta = \sum_{i=1}^k \lambda_i \cdot \eta$ and $\lambda_i \cdot \eta \in \Gamma_c(V(\mathfrak{K}))$ with $\mathrm{supp}(\lambda_i \cdot \eta) \subseteq U_i$. The assertion now follows from the fact that $\tilde{\varphi}_i\colon \rho^{-1}_{V(\mathfrak{K})}(U_i) \to U_i \times V_\mathfrak{g}$ is a local trivialisation of $V(\mathfrak{K})$. □

[2] In fact given $\psi, \varphi \in \mathcal{A}$ we write $\varphi_x := \mathrm{pr}_2 \circ \varphi(x, \bullet)$ and $\psi_x := \mathrm{pr}_2 \circ \psi(x, \bullet)$ for $x \in M$. We obtain $\tilde{\varphi} \circ \tilde{\psi}^{-1}(x,v) = (x, (\varphi_x \circ \psi_x^{-1})_\kappa(v))$. By choosing a basis $v_1, \ldots, v_n$ of $\mathfrak{g}$ one can construct a basis of $V(\mathfrak{g})$ that consists of vectors of equivalence classes of vectors of the form $v_i \vee v_j$. Now a standard argument shows that the map $(x,v) \mapsto (\varphi_x \circ \psi_x^{-1})_\kappa(v)$ is smooth.

With [JW13, Proposition 2.4.] we get the following lemma:

Lemma 3.18. *Let M be a finite-dimensional σ-compact manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with typical fibre $\mathfrak{g}$. If $\mathfrak{g}$ is perfect then also $\Gamma_c(\mathfrak{K})$ is perfect.*

In [Gun11, Theorem 4.4.4] a local statement for algebraically universal invariant symmetric bilinear forms is used to get an analogous global statement for spaces of sections of a Lie algebra bundle. We now transfer this approach to a topological statement for compactly supported sections of a Lie algebra bundle in Theorem 3.19.

Theorem 3.19. *For a perfect finite-dimensional Lie algebra $\mathfrak{g}$, a σ-compact, finite-dimensional manifold M and a Lie algebra bundle $\mathfrak{K}$ with base M and typical fibre $\mathfrak{g}$, the map $\kappa_{\mathfrak{K}}\colon \Gamma_c(M,\mathfrak{K})^2 \to \Gamma_c(V(\mathfrak{K}))$ is topologically universal.*

Proof. Let $(\psi_i\colon \pi^{-1}(U_{\psi_i}) \to U_{\psi_i} \times \mathfrak{g})_{i\in I}$ be a bundle atlas of $\mathfrak{K}$ with relatively compact subsets $U_{\psi_i} \subseteq M$ such that $(U_{\psi_i})_{i\in I}$ is locally finite and $(\rho_i)_{i\in I}$ be a partition of unity of M with $\operatorname{supp}(\rho_i) \subseteq U_{\psi_i}$. Let $\gamma\colon \Gamma_c(\mathfrak{K})^2 \to W$ be a continuous invariant symmetric bilinear form. For $i \in I$, we define

$$\begin{aligned}\gamma_i\colon C^\infty_c(U_{\psi_i},\mathfrak{g})^2 &\to W\\ (f,g) &\mapsto \gamma\left((\psi_i^{-1}\circ(\mathrm{id},f))_\sim, (\psi_i^{-1}\circ(\mathrm{id},g))_\sim\right).\end{aligned}$$

The bilinear map γ_i is an invariant symmetric bilinear form. We want to show that it is also continuous. Obviously it suffices to show that $\tau\colon C^\infty_c(U_{\psi_i},\mathfrak{g}) \to \Gamma_c(\mathfrak{K})$, $f \mapsto (\psi^{-1}\circ(\mathrm{id},f))_\sim$ is continuous. Given a compact subset $K \subseteq U_{\psi_i}$, we have $\tau(C^\infty_K(U_{\psi_i},\mathfrak{g})) \subseteq \Gamma_K(\mathfrak{K})$. The map $C^\infty_K(U_{\psi_i},\mathfrak{g}) \to \Gamma(\mathfrak{G})$, $f \mapsto (\psi^{-1}\circ(\mathrm{id},f))_\sim$ is continuous because $\Gamma(\mathfrak{K})$ is initial with respect to $\Gamma(\mathfrak{K}) \to C^\infty(U_{\psi_i},\mathfrak{g})$, $X \mapsto X_{\psi_i}$ (see [Glo13, p. 10]). So we can find a continuous linear map $\beta_i\colon C^\infty_c(U_{\psi_i},V_{\mathfrak{g}}) \to W$ such that the diagram

$$\begin{array}{ccc} C^\infty_c(U_{\psi_i},\mathfrak{g})^2 & \xrightarrow{\quad\gamma_i\quad} & W\\ \Big\downarrow{\scriptstyle \kappa_{\mathfrak{g}*}} & \nearrow{\scriptstyle \beta_i} & \\ C^\infty_c(U_{\psi_i},V(\mathfrak{g})) & & \end{array}$$

commutes. For $i \in I$ let $\tilde{\psi}_i$ be the corresponding bundle-chart of $V(\mathfrak{K})$ that comes from ψ_i. We define $\beta\colon \Gamma_c(V(\mathfrak{K})) \to W$, $X \mapsto \sum_{i\in I}\beta_i\big((\rho_i\cdot X)_{\tilde{\psi}_i}\big)$ with $(\rho_i X)_{\tilde{\psi}_i} = \mathrm{pr}_2\circ\tilde{\psi}_i\circ(\rho_i X)|_{U_{\psi_i}}$. Let $K \subseteq M$ be compact. Then $K \cap U_{\psi_i} \neq \emptyset$ only for a finite number of $i \in I$. To show that β is continuous it suffices to show the continuity of $\Gamma_K(V(\mathfrak{K})) \to W$, $X \mapsto \beta_i((\rho_i\cdot X)_{\tilde{\psi}_i})$. Since the map $\Gamma_K(V(\mathfrak{K})) \hookrightarrow C^\infty_{\operatorname{supp}(\rho_i)}(U_{\psi_i},V(\mathfrak{g}))$, $X \mapsto (\rho_i\cdot X)_{\tilde{\psi}_i}$ is continuous, we see that β is continuous. It remains to show that

$$\beta\circ\kappa_{\mathfrak{K}} = \gamma.$$

This can be seen analogously to the second part of the proof of [Gun11, Theorem 4.4.4]: Let $\zeta_i\colon M \to [0,1]$ be a smooth map with $\operatorname{supp}(\zeta_i) \subseteq U_{\psi_i}$ and $\zeta_i|_{\operatorname{supp}(\rho_i)} = 1$ for $i \in I$. With Lemma 3.17 in mind we calculate for $X, Y \in \Gamma_c(\mathfrak{K})$

$$\begin{aligned}\beta(\kappa_{\mathfrak{K}}(X,Y)) &= \sum_{i\in I}\beta_i\left((\rho_i\zeta_i\kappa_{\mathfrak{K}}(X,Y))_{\tilde{\psi_i}}\right) = \sum_{i\in I}\beta_i\left(\kappa_{\mathfrak{K}}(\rho_i X,\zeta_i Y)_{\tilde{\psi_i}}\right)\\ &= \sum_{i\in I}\beta_i\kappa_{\mathfrak{g}_*}\left((\rho_i X)_{\psi_i},(\zeta_i Y)_{\psi_i}\right) = \sum_{i\in I}\gamma_i\left((\rho_i X)_{\psi_i},(\zeta_i Y)_{\psi_i}\right) = \sum_{i\in I}\gamma(\rho_i X,\zeta_i Y)\\ &= \sum_{i\in I}\gamma(\underbrace{\zeta_i\rho_i}_{=\rho_i}X,Y) = \gamma(X,Y).\end{aligned}$$

Here we used that $\Gamma_c(\mathfrak{K}) \to \Gamma_c(\mathfrak{K})$, $X \mapsto \zeta_i \cdot X$ is in $\operatorname{Cent}(\Gamma_c(\mathfrak{K}))$ and that $\Gamma_c(\mathfrak{K})$ is a perfect Lie algebra (see Lemma 3.5). The uniqueness of β follows from Lemma 3.17. □

An application of universal continuous invariant symmetric bilinear forms

In Definition 3.20 we fix our notation concerning k-forms and connections and recall some basic facts. All this is well known, for instance see [Dar94] and [Gun11, Chapter 2.2. and 2.3.].

Definition 3.20. Let M be a finite-dimensional σ-compact manifold, $\mathbb{V}$ be a vector bundle with base M, $\mathfrak{K}$ be a Lie algebra bundle with base M and $k \in \mathbb{N}$.

(a) The space $\Omega_c^k(M,\mathbb{V})$ becomes a $C^\infty(M,\mathbb{R})$-module by the multiplication $C^\infty(M,\mathbb{R}) \times \Omega_c^k(M,\mathbb{V}) \to \Omega_c^k(M,\mathbb{V})$, $(f,\theta) \mapsto f\cdot\theta$ with $(f\cdot\theta)_p = f(p)\cdot\theta_p$.

(b) We get a bilinear map $\Omega_c^k(M,\mathbb{R}) \times \Gamma(\mathbb{V}) \to \Omega_c^k(M,\mathbb{V})$ $(\theta,\eta) \mapsto \theta\cdot\eta$ with $(\theta\cdot\eta)_p(v_1,..,v_k) = \theta_p(v_1,..,v_k)\cdot\eta(p)$ for $v_i \in T_pM$.

(c) We call a $\mathbb{R}$-linear map $d\colon \Gamma_c(\mathbb{V}) \to \Omega_c^1(M,\mathbb{V})$ a *covariant derivation*, if $d(f\eta) = fd\eta + \eta df$ for all $\eta \in \Gamma(\mathbb{V})$ and $f \in C^\infty(M,\mathbb{R})$.

(d) We define the continuous $C^\infty(M,\mathbb{R})$-bilinear map $\Gamma_c(\mathfrak{K}) \times \Omega_c^1(M,\mathfrak{K}) \to \Omega_c^1(M,\mathfrak{K})$, $(\eta,\theta) \mapsto [\eta,\theta]$ with $([\eta,\theta])_p(v) = [\eta(p),\theta_p(v)]$. Moreover, we set $[\theta,\eta] := -[\eta,\theta]$.

(e) We call a covariant derivation $D\colon \Gamma_c(\mathfrak{K}) \to \Omega_c^1(M,\mathfrak{K})$ a *Lie connection*, if $D[\eta,\tau] = [D\eta,\tau] + [\eta,D\tau]$ for $\eta,\tau \in \Gamma_c(\mathfrak{K})$.

In [Gun11, Remark 2.3.14], Gündoğan showed the existence of a Lie connection for a given Lie algebra bundle $\mathfrak{K}$. In the following lemma we use a different argumentation to show the existence of a Lie connection from $\Gamma_c(\mathfrak{K})$ to $\Omega_c^1(M,\mathfrak{K})$ that is also continuous.

Lemma 3.21. *For every finite-dimensional σ-compact manifold M and Lie algebra bundle $\pi\colon \mathfrak{K} \to M$, there exists a continuous Lie connection $D\colon \Gamma_c(\mathfrak{K}) \to \Omega_c^1(M,\mathfrak{K})$.*

Proof. Let $(V_i)_{i\in\mathbb{N}}$ be a locally finite open cover of M and $(\lambda_i)_{i\in\mathbb{N}}$ a partition of unity that is subordinate to V_i. For $i \in I$, we can choose V_i such that we get a continuous Lie connection $d_i\colon \Gamma(\mathfrak{K}|_{V_i}) \to \Omega^1(V_i,\mathfrak{K}|_{V_i})$. It is easily checked that

$d\colon \Gamma_c(\mathfrak{K}) \to \Omega^1_c(M,\mathfrak{K})$, $\eta \mapsto \sum_{i=1}^{\infty}(\lambda_i \cdot d_i(\eta|_{V_i}))_\sim$ is a Lie connection. Because d is local we have $d(\Gamma_K(\mathfrak{K})) \subseteq \Omega^1_K(M,\mathfrak{K})$ and because d is linear it suffices to show that $d\colon \Gamma_K(\mathfrak{K}) \to \Omega^1_K(M,\mathfrak{K})$ is continuous. But the compact set K is only intersected by finitely many V_i, say $V_1,\dots,V_n$. The map $\Gamma_K(\mathfrak{K}) \to \Omega^1_K(M,\mathfrak{K})$, $\eta \mapsto \sum_{i=1}^{n}(\lambda_i \cdot d_i(\eta|_{V_i}))_\sim$ is obviously continuous because the corresponding map from $\Gamma(\mathfrak{K})$ to $\Omega^1(M,\mathfrak{K})$ is continuous. □

Lemma 3.22. *If M is a finite-dimensional σ-compact manifold and $\pi\colon \mathfrak{K} \to M$ a Lie algebra bundle with finite-dimensional typical fibre $\mathfrak{g}$ then we define the map*

$$\tilde{\kappa}_{\mathfrak{K}}\colon \Omega^1_c(M,\mathfrak{K}) \times \Gamma_c(\mathfrak{K}) \to \Omega^1_c(M,V(\mathfrak{K}))$$

by $(\tilde{\kappa}_{\mathfrak{K}}(\theta,\eta))_p(v) = \kappa_{\mathfrak{K}_p}(\theta_p(v),\eta(p))$. The map $\tilde{\kappa}_{\mathfrak{K}}$ is $C^\infty(M,\mathbb{R})$-bilinear and continuous. If moreover $D\colon \Gamma_c(\mathfrak{K}) \to \Omega^1_c(M,\mathfrak{K})$ is a continuous Lie connection then

$$\beta\colon \Gamma_c(\mathfrak{K})^2 \to \Omega^1_c(M,V(\mathfrak{K})),\ (\zeta,\eta) \mapsto \tilde{\kappa}_{\mathfrak{K}}(D\zeta,\eta) + \tilde{\kappa}_{\mathfrak{K}}(D\eta,\zeta)$$

is a continuous, invariant, symmetric bilinear form.

Proof. To show the continuity of β we only have to prove that $\tilde{\kappa}_{\mathfrak{K}}$ is continuous. The map

$$(T^*M \otimes \mathfrak{K}) \oplus \mathfrak{K} \to T^*M \otimes V(\mathfrak{K}), (\lambda \otimes v, w) \mapsto \kappa_{\mathfrak{K}_p}(\lambda(\bullet)\cdot v, w)$$

is continuous. With the identifications $\Omega^1_c(M,\mathfrak{K}) \cong \Gamma_c(T^*M \otimes \mathfrak{K})$ and $\Omega^1_c(M,V(\mathfrak{K})) \cong \Gamma_c(T^*M \otimes V(\mathfrak{K}))$ the continuity follows from the Ω-Lemma (see, e.g. [Mic80, Theorem 8.7] or [Glo04, F.24]). In the following we use the shorthand notation $\kappa := \kappa_{\mathfrak{K}_p}$. We show that β is invariant:

$$\begin{aligned}
&\beta([\eta_1,\eta_2],\eta_3)_p(v) = \kappa((D[\eta_1,\eta_2])_p(v),\eta_3(p)) + \kappa([\eta_1(p),\eta_2(p)],(D\eta_3)_p(v))\\
=&\kappa([(D\eta_1)_p(v),\eta_2(p)],\eta_3(p)) + \kappa([\eta_1(p),(D\eta_2)_p(v)],\eta_3(p))\\
&+\kappa([\eta_1(p),\eta_2(p)],(D\eta_3)_p(v))\\
=&\kappa((D\eta_1)_p(v),[\eta_2(p),\eta_3(p)]) + \kappa(\eta_1(p),[(D\eta_2)_p(v),\eta_3(p)])\\
&+\kappa(\eta_1(p),[\eta_2(p),(D\eta_3)_p(v)])\\
=&\kappa((D\eta_1)_p(v),[\eta_2(p),\eta_3(p)]) + \kappa(\eta_1(p),D[\eta_2,\eta_3]_p(v)) = \beta(\eta_1,[\eta_2,\eta_3])_p(v).
\end{aligned}$$

The rest of the statement is clear. □

In the following remark we use our Theorem 3.19 to argue that the covariant derivative d_∇ constructed in [JW13, p. 129] is actually a continuous map (the continuity of d_∇ was not discussed in [JW13]).[3]

Remark 3.23. *Let M be a finite-dimensional σ-compact manifold, $\pi\colon \mathfrak{K} \to M$ a Lie algebra bundle with perfect, finite-dimensional typical fibre $\mathfrak{g}$, $D\colon \Gamma_c(\mathfrak{K}) \to \Omega^1_c(M,\mathfrak{K})$ a continuous Lie connection and β as in Lemma 3.22. Then there exists*

[3] Note that the continuity of d_∇ is not necessary for the considerations in [JW13].

a unique continuous covariant derivation $d\colon \Gamma_c(V(\mathfrak{K})) \to \Omega_c^1(M, \mathfrak{K})$ *such that the diagram*

$$\begin{array}{ccc} \Gamma_c(\mathfrak{K})^2 & \xrightarrow{\beta} & \Omega_c^1(M, V(\mathfrak{K})) \\ \downarrow{\scriptstyle \kappa_{\mathfrak{K}}} & \nearrow{\scriptstyle d} & \\ \Gamma_c(V(\mathfrak{K})) & & \end{array}$$

commutes.

Proof. With Theorem 3.19 we find a unique continuous $\mathbb{R}$-linear map $d\colon \Gamma_c(V(\mathfrak{K})) \to \Omega_c^1(M, \mathfrak{K})$ such that the above diagram commutes. We show that $d(f \cdot \eta) = df \cdot \eta + f \cdot d\eta$ for $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Gamma_c(V(\mathfrak{K}))$ in the following[4]. Because the image of $\kappa_{\mathfrak{K}}$ spans $\Gamma_c(V(\mathfrak{K}))$, it is sufficient to show the assertion for $\eta = \kappa_{\mathfrak{K}}(\xi, \zeta)$ with $\xi, \zeta \in \Gamma_c(\mathfrak{K})$. We use the shorthand $\kappa := \kappa_{\mathfrak{K}_p}$.

$$\begin{aligned}
&d(f \cdot \kappa_{\mathfrak{K}}(\xi, \zeta))_p(v) = (d(\kappa_{\mathfrak{K}}(f\xi, \zeta)))_p(v) \\
=&\kappa(D(f \cdot \xi)_p(v), \zeta(p)) + \kappa(f(p) \cdot \xi(p), (D\zeta)_p(v)) \\
=&\kappa(df(v) \cdot \xi(p), \zeta(p)) + \kappa(f(p)(D\xi)_p(v), \zeta(p)) + \kappa(f(p)\xi(p), D\zeta_p(v)) \\
=&df(v) \cdot \kappa(\xi(p), \zeta(p)) + f(p)\kappa(D\xi_p(v), \zeta(p)) + f(p)\kappa(\xi(p), D\zeta_p(v)) \\
=&(df \cdot \kappa_{\mathfrak{K}}(\xi, \zeta))_p(v) + f(p) \cdot (\beta(\xi, \zeta))_p(v) \\
=&(df \cdot \kappa_{\mathfrak{K}}(\xi, \zeta))_p(v) + f(p) \cdot (d\kappa_{\mathfrak{K}}(\xi, \zeta))_p(v).
\end{aligned}$$

□

Remark 3.24. *In this remark we recall the construction of the cocycle* ω *from [JW13, p. 129]. Janssens and Wockel used the covariant derivation* d *described in Remark 3.23 to define* $\overline{\Omega}_c^1(M, V(\mathfrak{K})) := \Omega_c^1(M, V(\mathfrak{K}))/(\overline{d\Gamma_c(V(\mathfrak{K}))})$ *(as mentioned above the continuity of* d *is not important for this construction). The cocycle is defined as*

$$\omega\colon \Gamma_c(\mathfrak{K})^2 \to \overline{\Omega}_c^1(M, V(\mathfrak{K})),\ (\eta, \zeta) \mapsto [\tilde{\kappa}_{\mathfrak{K}}(D\eta, \zeta)]$$

with $\tilde{\kappa}_{\mathfrak{K}}$ *as in Lemma 3.23. The continuity of* ω *was not discussed in [JW13] but this follows immediately from Lemma 3.22.*

3.3. Universal continuous extensions of certain current algebras

Maier constructed in [Mai02] a universal cocycle for current algebras with a unital complete locally convex algebra. In [JW13, Theorem II.7] Janssens and Wockel showed that an analogous cocycle is also universal for the algebra of compactly supported functions on a σ-compact finite-dimensional manifold. Gündoğan showed

[4]This easy calculation was not discussed in [JW13]. We give it here for the convenience of the reader.

in [Gun11] that this approach also works for a certain class of locally convex pseudo-unital algebras, the so called CPUSLF-algebras[5]. However this class of locally convex algebras does not contain the compactly supported functions on a σ-compact finite-dimensional manifold. Our aim in this section is to use concepts from [Gun11] to show that the cocycle constructed in [JW13, Theorem 2.7] respectively [Mai02] is universal for a class of locally convex algebras *without unity* that contains the compactly supported smooth functions on a σ-compact algebra. Hence, taking $C_c^\infty(M)$ as the considered algebra, we obtain a more detailed argumentation for [JW13, Theorem 2.7] (see Remark 3.39).

In the following Definitions 3.25 and 3.26 we recall the concept of universality from [Gun11, Chapter A.2] and [Nee02b, Definition 1.9 and Remark 1.10].

Definition 3.25. Let $\mathfrak{g}$ be a locally convex Lie algebra and V a locally convex space considered as a trivial $\mathfrak{g}$-module. Moreover, let $E := V \hookrightarrow \hat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ be a central extension of locally convex Lie algebras and W be a locally convex space considered as a trivial $\mathfrak{g}$-module.

(i) We call the extension E *weakly universal for* W, if for every central extension $E' := W \hookrightarrow \hat{\mathfrak{g}}' \xrightarrow{q'} \mathfrak{g}$ of locally convex Lie algebras there exists a homomorphism of extensions of topological Lie algebras $\varphi\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}'$ from E to E'.

(ii) We call E *weakly universal* if it is weakly universal for every locally convex space.

(iii) We call E *universal for* W, if for every central extension $E' := W \hookrightarrow \hat{\mathfrak{g}}' \xrightarrow{q'} \mathfrak{g}$ of locally convex Lie algebras we can find a unique extension homomorphism $\varphi\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}'$ from E to E'.

(iv) We call E *universal* if it is universal for every locally convex space.

Definition 3.26. Let $\mathfrak{g}$ be a locally convex Lie algebra and V be a locally convex space considered as a trivial $\mathfrak{g}$-module. Moreover, let $\omega \in Z^2_{ct}(\mathfrak{g}, V)$ be a continuous cocycle and W be a locally convex space considered as a trivial $\mathfrak{g}$-module.

(i) We call ω *weakly universal for* W if the map $\delta_W\colon \mathcal{L}(V, W) \to H^2_{ct}(\mathfrak{g}, W)$, $\theta \mapsto [\theta \circ \omega]$ is bijective.

(ii) We call ω *weakly universal* if it is weakly universal for every locally convex space.

(iii) We call ω *universal for* W if it is weakly universal for W and $\mathrm{Hom}_{\mathrm{ct}}(\mathfrak{g}, W) = \{0\}$ (compare [Nee02b, Remark 1.10]).

(iv) We call ω *universal* if it is universal for every locally convex space.

The following theorem is well known.

Theorem 3.27. *Let $\mathfrak{g}$ be a locally convex Lie algebra and V a locally convex space considered as a trivial $\mathfrak{g}$-module, $E := V \hookrightarrow \hat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ a central extension of locally convex Lie algebras, $\omega \in Z^2_{ct}(\mathfrak{g}, V)$ the corresponding cocycle and W a locally*

[5] Although CPUSLF stands for "commutative pseudo-unital strict LF-algebra" (see [Gun11, Definition 5.1.12]) the actual definition is less general. In fact a commutative locally convex algebra is called CPUSLF-algebra if it is the strict inductive limit (in the category of locally convex spaces) of **unital** Fréchet algebras (see [Gun11, Definition 5.1.12]).

convex space considered as a trivial $\mathfrak{g}$-module. Then E is weakly universal for W respectively universal for W, if and only if ω is so.

Proof. This follows from [Nee02b, Remark 1.10 (a)] and [Gun11, Remark A.2.11]. □

The following Lemma 3.28 comes from [Nee02b, Lemma 1.12 (iii)].

Lemma 3.28. *Let $\mathfrak{g}$ be a locally convex Lie algebra, W and V be a locally convex spaces considered as trivial $\mathfrak{g}$-modules and $\omega \in Z^2_{ct}(\mathfrak{g}, V)$ a weakly universal cocycle for W. If $\mathfrak{g}$ is topologically perfect then ω is universal for W.*

Remark 3.29. *Actually the Lemma 1.12 in [Nee02b] requires the condition that the considered extension is weakly universal for the underlying field $\mathbb{K}$ of the vector spaces. But this condition is only used in the proof of statement 1.12 (ii). The proof of statement (iii) neither requires statement 1.12 (ii) nor this condition.*

In Definition 3.30 we recall the basic concept of current algebras for the convenience of the reader (see e.g. [Gun11, Chapter 4]).

Definition 3.30. (a) A commutative algebra A is called *pseudo-unital* if for $x, y \in A$ there exists $z \in A$ with $xz = x$ and $yz = y$ (see e.g. [Gun11, Definition 4.2.3]). If $x_1, \ldots, x_n \in A$ and A is commutative then [Gun11, Remark 4.2.4] tells us that we find $z \in A$ such that $x_i z = x_i$ for all $i = 1, \ldots, n$.

(b) If A is a commutative pseudo-unital algebra and $\mathfrak{g}$ is a finite-dimensional Lie algebra, we endow the tensor product $A \otimes \mathfrak{g}$ with the unique Lie bracket that satisfies $[a \otimes x, b \otimes y] = ab \otimes [x, y]$ for $a, b \in A$ and $x, y \in \mathfrak{g}$.

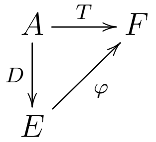
(c) If A is a locally convex $\mathbb{R}$-algebra then we endow $A \otimes \mathfrak{g}$ with the topology of the projective tensor product of locally convex spaces. This Lie algebra is even a locally convex algebra as one can see in [Gun11, Remark 2.1.9.]. Moreover, [Gun11, Remark 4.2.7] tells us that $A \otimes \mathfrak{g}$ is perfect if $\mathfrak{g}$ is so.

In the following definition we remind the reader of the concept of topologically universal differential modules. This concept is for example presented in [Gun11, Chapter 5.2] and [Mai02].

Definition 3.31. Let A be a unital commutative complete locally convex $\mathbb{R}$-algebra.

(a) A continuous $\mathbb{R}$-linear map $D\colon A \to E$ to a complete locally convex A-module E is called a *derivation*, if $D(xy) = xD(y) + yD(x)$ for $x, y \in A$.

(b) For a complete locally convex commutative unital algebra A, a pair (E, D) with a complete locally convex A-module E and a continuous derivation $D\colon A \to E$ of E is called *universal topological differential module* of A if there exists a unique continuous linear map $\varphi\colon E \to F$ such that



commutes for every complete locally convex A-module F and continuous F-derivation $T\colon A \to F$.

(c) [Gun11, Chapter 5.2] or [Mai02] tell us that there always exists a universal topological differential module $(\Omega(A), d_A)$ for a given complete locally convex commutative unital algebra A.

The following definition can be found in [Gun11, Definition 5.29] and [Mai02, p. 73] in the case where the algebra A is unital instead of pseudo-unital. It is also the canonical generalisation of the cocycle $\omega_{M,\mathfrak{g}}$ in [JW13, Theorem 2.7].

Definition 3.32. If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra and A is a commutative pseudo-unital complete locally convex algebra, then we define the cocycle

$$\begin{aligned}\omega_{\mathfrak{g},A}\colon A\otimes\mathfrak{g}\times A\otimes\mathfrak{g} &\to V_{\mathfrak{g}}\otimes(\Omega(A_1)/\overline{d_{A_1}(A_1)})\\ (a\otimes x, b\otimes y) &\mapsto \kappa_{\mathfrak{g}}(x,y)\otimes[a\cdot d_{A_1}(b)].\end{aligned}$$

For convenience we write $V_{\mathfrak{g},A} := V_{\mathfrak{g}}\otimes(\Omega(A_1)/\overline{d_{A_1}(A_1)})$. As mentioned in [Gun11, Definition 5.2.9] the map $\omega_{\mathfrak{g},A}$ satisfies the cocycle condition because $\kappa_{\mathfrak{g}}$ is invariant.

With [Mai02, Theorem 16] we get the following Lemma 3.33.

Lemma 3.33. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and A a commutative unital complete algebra. If W is a complete locally convex space considered as a trivial $\mathfrak{g}$-module then the map $\delta_W\colon \mathcal{L}(V_{\mathfrak{g},A}, W) \to H^2_{ct}(A\otimes\mathfrak{g}, W)$, $\theta\mapsto[\theta\circ\omega_{\mathfrak{g},A}]$ is bijective.*

The following definition is based on [Gun11, Definition 5.1.5 and Lemma 5.1.6].

Definition 3.34. For a locally convex algebra A and a finite-dimensional Lie algebra $\mathfrak{g}$ the Lie algebra $A\otimes\mathfrak{g}$ is locally convex. Let A be a commutative pseudo-unital locally convex $\mathbb{R}$-algebra and $\mathfrak{g}$ be a finite-dimensional Lie algebra. For $y\in\mathfrak{g}$, we get a continuous bilinear map $A\times\mathfrak{g}\to A\otimes\mathfrak{g}$ $(c,x)\mapsto c\otimes[y,x]$ which induces a continuous linear map $\delta_y\colon A\otimes\mathfrak{g}\to A\otimes\mathfrak{g}$ with $\delta_y(c\otimes x) = c\otimes[y,x]$. We get a linear map $\delta\colon\mathfrak{g}\to\mathcal{L}(A\otimes\mathfrak{g})$, $y\mapsto\delta_y$ such that $\mathfrak{g}\times(A\otimes\mathfrak{g})\to A\otimes\mathfrak{g}$, $(y,v)\mapsto\delta_y(v)$ is continuous. Moreover, δ is a Lie algebra homomorphism because $\delta_{[y_1,y_2]}(c\otimes x) = c\otimes[[y_1,y_2],x] = c\otimes[y_1,[y_2,x]] - c\otimes[y_2,[y_1,x]] = \delta_{y_1}\delta_{y_2}(c\otimes x) - \delta_{y_2}\delta_{y_1}(c\otimes x)$. Also we have $\delta_y\in\operatorname{der}(A\otimes\mathfrak{g})$ because $\delta_y([c\otimes x, c'\otimes x']) = cc'\otimes[y,[x,x']] = cc'\otimes[[y,x],x'] + cc'\otimes[x,[y,x']] = [\delta_y(c\otimes x), c'\otimes x'] + [c\otimes x, \delta_y(c'\otimes x')]]$. We define $[y,\bullet] := \delta_y$ for $y\in\mathfrak{g}$. With the Lie algebra homomorphism δ we can define the semidirect product $(A\otimes\mathfrak{g})\rtimes\mathfrak{g}$ with the Lie-bracket $[(z_1,y_1),(z_2,y_2)] = ([z_1,z_2]+\delta_{y_1}(z_2)-\delta_{y_2}(z_1),[y_1,y_2]) = ([z_1,z_2]+[y_1,z_2]-[y_2,z_1],[y_1,y_2])$ for $z_i\in A\otimes\mathfrak{g}$ and $y_i\in\mathfrak{g}$, where we wrote $[y,\bullet] := \delta_y$ for $y\in\mathfrak{g}$. The Lie algebra $A\otimes\mathfrak{g}\rtimes\mathfrak{g}$ is a locally convex Lie algebra. We identify $A\otimes\mathfrak{g}$ with the ideal $\operatorname{im}(i)$, where $i\colon A\otimes\mathfrak{g}\to A\otimes\mathfrak{g}\rtimes\mathfrak{g}$, $z\mapsto(z,0)$ is a topological embedding that is a Lie algebra homomorphism. Obviously the image of i is a closed subspace. Moreover, we identify $\mathfrak{g}$ with the subalgebra $\operatorname{im}(i_{\mathfrak{g}})$, where $i_{\mathfrak{g}}\colon\mathfrak{g}\to(A\otimes\mathfrak{g})\rtimes\mathfrak{g}$, $x\mapsto(0,x)$ is a topological embedding with closed image that is a Lie algebra homomorphism. As usual we write $(z,x) = (z,0)+(0,x) = z+x$ for $(z,x)\in A\otimes\mathfrak{g}\rtimes\mathfrak{g}$.

[Gun11, Remark 5.1.7 and Lemma 5.1.8] lead to the following Lemma 3.35.

Lemma 3.35. *For a pseudo-unital commutative locally convex algebra A, its unitalisation A_1 and a finite-dimensional Lie algebra $\mathfrak{g}$ we have an isomorphism of locally convex Lie algebras $\varphi\colon A_1 \otimes \mathfrak{g} \to (A \otimes \mathfrak{g}) \rtimes \mathfrak{g}$ with $(\lambda, a) \otimes w \mapsto (a \otimes w, \lambda w)$ for all $\lambda \in \mathbb{K}$, $a \in A$ and $w \in \mathfrak{g}$.*

Lemma 3.36. *If $\mathfrak{g}$ and $\mathfrak{h}$ are locally convex Lie algebras, V is a locally convex space considered as a trivial $\mathfrak{g}$-module and $\mathfrak{h}$-module respectively and $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a continuous Lie algebra homomorphism then $H_{ct}(\varphi)\colon H^2_{ct}(\mathfrak{h}, V) \to H^2_{ct}(\mathfrak{g}, V)$, $[\omega] \mapsto [\omega \circ (\varphi, \varphi)]$ is a well-defined and linear map.*

Proof. For $\eta \in \mathcal{L}(\mathfrak{h}, V)$ and $\omega \in Z^2_{ct}(\mathfrak{h}, V)$ with $\omega = \eta \circ [\bullet, \bullet]$ we have $(\varphi, \varphi)^*(\omega) = \eta \circ \varphi \circ [\bullet, \bullet]$. □

We will use the concept of neutral triple from [Gun11, Definition 5.1.3] and recall it in the next Definition 3.37.

Definition 3.37. Let A be a pseudo-unital commutative locally convex algebra A and $\mathfrak{g}$ be a finite-dimensional perfect Lie algebra.

(a) There is an A-module structure $\cdot\colon A \times (A \otimes \mathfrak{g}) \to A \otimes \mathfrak{g}$ with $a \cdot (b \otimes y) = (a \cdot b) \otimes y$ for $a, b \in A$ and $y \in \mathfrak{g}$. Actually $A \otimes \mathfrak{g}$ is an A-module in the category of locally convex spaces because $\mathfrak{g}$ is finite-dimensional. In this situation we call $\nu \in A$ *neutral* for $f \in A \otimes \mathfrak{g}$, if $\nu \cdot f = f$.

(b) For $f \in A \otimes \mathfrak{g}$ (resp. $\varphi \in A$) we call $(\lambda, \nu, \mu) \in A^3$ a *neutral triple* for f (resp. φ), if $\mu \cdot f = f$ (resp. $\mu \cdot \varphi = \varphi$), $\nu \cdot \mu = \mu$ and $\lambda \cdot \nu = \nu$.

(c) Let $(v_i)_{i=1,..,n}$ be a basis of $\mathfrak{g}$. For $f = \sum_{i=1}^n \varphi_i \otimes v_i \in A \otimes \mathfrak{g}$ with $\varphi_i \in A$, we choose $\mu_f \in A$ such that μ_f is neutral for all φ_i. Moreover, we choose ν_f such that it is neutral for μ_f and λ_f such that it is neutral for ν_f. Clearly $(\lambda_f, \nu_f, \mu_f)$ is a neutral triple for f and for all φ_i. We fix this notation for the rest of this section. For $\varphi \in A$, we choose a neutral triple $(\lambda_\varphi, \nu_\varphi, \mu_\varphi)$. Obviously $(\lambda_\varphi, \nu_\varphi, \mu_\varphi)$ is a neutral triple for $\varphi \otimes v$ for all $v \in \mathfrak{g}$.

From the proof of [Gun11, Theorem 5.1.10] we can extract the following lemma.

Lemma 3.38. *Let A be a pseudo-unital commutative locally convex algebra, $\mathfrak{g}$ a finite-dimensional perfect Lie algebra and V a locally convex space considered as a trivial $A \otimes \mathfrak{g}$-module. Moreover let $\omega \in Z^2_{ct}(A \otimes \mathfrak{g}, V)$, $f \in A \otimes \mathfrak{g}$ and $y \in \mathfrak{g}$. If $(\lambda_1, \nu_1, \mu_1)$ and $(\lambda_2, \nu_2, \mu_2)$ are neutral triples for f then $\omega(f, \lambda_1 \otimes y) = \omega(f, \lambda_2 \otimes y)$.*

Remark 3.39. *In the following theorem (Theorem 3.40) we prove that [Gun11, Theorem 5.1.14] also holds for a class of locally convex algebras that contains the compactly supported smooth functions on a σ-compact finite-dimensional manifold[6]. In [JW13, Theorem 2.7] Janssens and Wockel considered the unitalisation $C^\infty_c(M, \mathfrak{g}) \rtimes \mathfrak{g}$ of $C^\infty_c(M, \mathfrak{g})$ to use [Mai02, Theorem 16] (we follow this strategy).*

[6] [Gun11, Corollary 5.2.14] does not follow from [Gun11, Theorem 5.2.13] as claimed in [Gun11]: Given a compact subset $K \subseteq M$, the algebras $C^\infty_K(M)$ are not unital and hence $C^\infty_c(M)$ is not a CPUSLF-algebra in the sense of [Gun11, Defintion 5.1.12].

In this context they showed that the canonical map $H^2_{ct}(i)\colon H^2_{ct}(C^\infty_c(M,\mathfrak{g})\rtimes\mathfrak{g},V)\to H^2_{ct}(C^\infty_c(M,\mathfrak{g}),V)$ is bijective. But the proof of the surjectivity was not complete. It was not discussed whether the constructed cocycle ω that is mapped to ω_0 by $H^2_{ct}(i)$ is actually continuous. Therefore, taking $A = C^\infty_c(M)$ in Theorem 3.40 we obtain a more detailed argument for [JW13, Theorem 2.7.].

Theorem 3.40. *Let A be a locally convex commutative pseudo-unital algebra that is the inductive limit of locally convex subalgebras $A_m \subseteq A$ for which there exists an element $1_m \in A$ with $1_m \cdot a = a$ for all $a \in A_m$. Moreover, let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra, V a locally convex space and $i\colon A\otimes\mathfrak{g}\to A\otimes\mathfrak{g}\rtimes\mathfrak{g}$ the natural inclusion. Then $H^2_{ct}(i)\colon H^2_{ct}(A\otimes\mathfrak{g}\rtimes\mathfrak{g},V)\to H^2_{ct}(A\otimes\mathfrak{g},V)$ is bijective*[7].

Proof. Surjectivity: Let $(v_i)_{i=1,\dots,n}$ be a basis of $\mathfrak{g}$. We use the notation of Definition 3.37. Let $\omega_0 \in Z^2_{ct}(A\otimes\mathfrak{g},V)$. We follow the idea of [Gun11, Theorem 5.1.14] or [JW13, Theorem 2.7] and define $\omega\colon (A\otimes\mathfrak{g})^2\rtimes\mathfrak{g}\to V$ $(f_1,y_1),(f_2,y_2)\mapsto \omega_0(f_1,f_2)+\omega_0(f_1,\lambda_{f_1}\otimes y_2)-\omega_0(f_2,\lambda_{f_2}\otimes y_1)$. The argument that ω is a cocycle works exactly as in the proof [Gun11, Theorem 5.1.14] or [JW13, Theorem 2.7]. For the convenience of the reader we will recall this argument in Appendix C. We show that ω is also continuous. For this, we just have to show that the bilinear map $\psi\colon A\otimes\mathfrak{g}\times\mathfrak{g}\to V$, $(f,y)\mapsto\omega_0(f,\lambda_f\otimes y)$ is continuous. Because we can identify $(A\otimes\mathfrak{g})\otimes\mathfrak{g}$ with $(A\otimes\mathfrak{g})^n$, it is sufficient to prove the continuity of $(A\otimes\mathfrak{g})^n\to V$, $(f_i)_{i=1,\dots,n}\mapsto \sum_{i=1}^n\omega_0(f_i,\lambda_{f_i}\otimes v_i)$. To show the continuity of $A\otimes\mathfrak{g}\to V$, $f\mapsto\omega_0(f,\lambda_f\otimes v)$, we again identify $A\otimes\mathfrak{g}$ with A^n and prove the continuity of $A^n\to V$, $(\varphi_i)_{i=1,\dots,n}\mapsto \sum_{i=1}^n\omega_0(\varphi_i\otimes v_i,\lambda_f\otimes y)$ with $f=\sum_{i=1}^n\varphi_i\otimes v_i$ and an arbitrary $y\in\mathfrak{g}$. Because of the construction of the neutral triple (λ_f,ν_f,μ_f) (see Definition 3.37 (c)) we get $\omega_0(\varphi_i\otimes v_i,\lambda_f\otimes y)=\omega_0(\varphi_i\otimes v_i,\lambda_{\varphi_i}\otimes y)$ for $i\in\{1,\dots,n\}$. It remains to show that the linear map $A\to V$, $\varphi\mapsto\omega_0(\varphi\otimes x,\lambda_\varphi\otimes y)$ is continuous for $x,y\in\mathfrak{g}$. For $m\in\mathbb{N}$, we find an element $1_m\in A$ with $1_m\cdot a=a$ for all $a\in A_m$. We choose an element $\tilde{1}_m\in A$ that is unital for 1_m and an element $\tilde{\tilde{1}}_m$ that is unital for $\tilde{1}_m$ and see that $(\tilde{\tilde{1}}_m,\tilde{1}_m,1_m)$ is a unital triple for all $\varphi\in A_m$. We see that $A_m\to V$, $\varphi\mapsto\omega_0(\varphi\otimes x,\lambda_\varphi\otimes y)=\omega_0(\varphi\otimes x,\tilde{\tilde{1}}_m\otimes y)$ is continuous and conclude that also the map $A\to V$, $\varphi\mapsto\omega_0(\varphi\otimes x\lambda_\varphi\otimes y)$ is continuous because A is the inductive limit of the subalgebras $A_m\subseteq A$. The equation $H^2_{ct}(i)([\omega])=[\omega_0]$ is easily checked because for $f,g\in A\otimes\mathfrak{g}$ we have $\omega\circ(i,i)(f,g)=\omega(f,g)=\omega_0(f,g)$. The argument that $H^2_{ct}(i)$ is injective works exactly as in the proof of [Gun11, Theorem 5.1.14] or [JW13, Theorem 2.7]. For the convenience of the reader, we recall this argument in Appendix C. □

The application of Theorem 3.40 in the following theorem is completely analogous to [JW13, Theorem 2.7].

Theorem 3.41. *Let A be a complete locally convex commutative pseudo-unital algebra such that it is the inductive limit of subalgebras $A_n\subseteq A$ with $n\in\mathbb{N}$ such*

[7] The class of algebras A we consider in this theorem obviously contains the so called CPUSLF-algebras considered in [Gun11, Theorem 5.1.14] as well as the compactly supported smooth functions on M.

that we find for $n \in \mathbb{N}$ *an element* $1_n \in A$ *with* $1_n \cdot a = a$ *for all* $a \in A_n$*. Moreover, let* $\mathfrak{g}$ *be a finite-dimensional semisimple Lie algebra. Then* $\omega_{\mathfrak{g},A} : A \otimes \mathfrak{g} \times A \otimes \mathfrak{g} \to V_{\mathfrak{g},A_1}$ *with* $(a \otimes x, b \otimes y) \mapsto \kappa_{\mathfrak{g}}(x,y) \otimes [a \cdot d_{A_1}(b)]$ *is a universal cocycle for* $A \otimes \mathfrak{g}$*.*

Proof. The assertion follows directly from Lemma 3.28, Lemma 3.33 and Theorem 3.40. □

The transition of the fact stated in Theorem 3.41 for current algebras of the form $A \otimes \mathfrak{g}$ to the special case of Lie algebras of the form $C_c^\infty(M,\mathfrak{g})$ can be done as in [Gun11, Chapter 5.2] or [JW13, Theorem 2.7]. We recall this approach in Remark 3.42.[8]

Remark 3.42. *[Mai02, Theorem 11] tells us that if* M *is a* σ*-compact manifold and* $\Omega(C_c^\infty(M)_1)$ *the universal* $C_c^\infty(M)_1$*-module in the category of complete locally convex spaces, then* $d \circ \mathrm{pr}_1 : C_c^\infty(M)_1 \to \Omega_c^1(M)$ *induces an isomorphism of topological* $C_c^\infty(M)_1$*-modules* $\Omega(C_c^\infty(M)_1) \to \Omega_c^1(M)$*. Let* $\mathfrak{g}$ *be a semisimple Lie algebra. For* $f \in C_c^\infty(M,\mathfrak{g})$ *and* $\eta \in \Omega_c^1(M,\mathfrak{g})$ *we define the* 1*-form* $\kappa_{\mathfrak{g}}(f,dg) \in \Omega_c^1(M,V_{\mathfrak{g}})$ *by* $\kappa_{\mathfrak{g}}(f,\eta)_p(v) := \kappa_{\mathfrak{g}}(f(p),\eta_p(v))$*. Because* $d(C_c^\infty(M,V_{\mathfrak{g}}))$ *is closed in* $\Omega_c^1(M,V_{\mathfrak{g}})$ *(see [Nee04, Lemma 4.11]) the map*

$$\begin{aligned} C_c^\infty(M,\mathfrak{g}) \times C_c^\infty(M,\mathfrak{g}) &\to \Omega_c^1(M,V_{\mathfrak{g}})/d(C_c^\infty(M,V_{\mathfrak{g}})) \\ (f,g) &\mapsto [\kappa_{\mathfrak{g}}(f,dg)]. \end{aligned}$$

is a universal cocycle for all complete locally convex spaces.

[8] As mentioned in [JW13, Theorem 2.7], [Mai02, Corollary 18] does not follow from [Mai02, Theorem 16].

4. Extensions of groups of compactly supported sections

In the previous chapter, we considered extensions of infinite-dimensional Lie algebras. Now, we turn our attention to extensions of infinite-dimensional Lie groups[1]. More precisely, we consider central extensions of groups of compactly supported sections as mentioned in the introduction.

4.1. Construction of the Lie group extension

For this chapter, we fix the following notation:

(a) If $H \hookrightarrow P \xrightarrow{q} M$ is a principal bundle with right action $R\colon P \times H \to P$, we write $VP := \ker(Tq)$ for the vertical bundle of TP and $V_pP := T_pP \cap VP$ for the vertical space in $p \in P$. Analogously, if $HP \subseteq TP$ is a principal connection ($HP \oplus VP = TP$ and $TR_h H_pP = H_{ph}P$), we write $H_pP := T_pP \cap HP$ for the horizontal space in $p \in P$.

(b) Let $H \hookrightarrow P \xrightarrow{q} M$ be a finite-dimensional principal bundle over a connected σ-compact manifold M with right action $R\colon P \times H \to P$ and a principal connection $HP \subseteq TP$. Given a finite-dimensional linear representation $\rho\colon H \to \mathrm{GL}(V)$ and $k \in \mathbb{N}_0$, we write

$$\Omega^k(P,V)_\rho = \left\{\theta \in \Omega^k(P,V) : (\forall g \in H)\ \rho(g) \circ R_g^*\theta = \theta\right\}$$

for the space of H-invariant V-valued k-forms on P and $\Omega^k(P,V)_\rho^{\mathrm{hor}}$ for the space of H-invariant V-valued k-forms that are horizontal with respect to HP ($\exists i :\ v_i \in V_pP \Rightarrow \theta(v_1,\ldots,v_k) = 0$) (see [Bau14, Definition 3.3]). Moreover, given a compact set $K \subseteq M$ we define $\Omega^k_K(P,V)_\rho := \left\{\theta \in \Omega^k_K(P,V)_\rho : \mathrm{supp}(\theta) \subseteq q^{-1}(K)\right\}$ and write $\Omega^k_K(P,V)_\rho^{\mathrm{hor}}$ for the analogous subspace in the horizontal case. We emphasise that these forms are in general not compactly supported on P itself. As mentioned in the introduction, we equip these spaces with the natural Fréchet-topology and write $\Omega^k_c(P,V)_\rho$, respectively $\Omega^k_c(P,V)_\rho^{\mathrm{hor}}$, for the locally convex inductive limit of the spaces $\Omega^k_K(P,V)_\rho$, respectively $\Omega^k_K(P,V)_\rho^{\mathrm{hor}}$. This convention also clarifies what we mean by $C^\infty(P,V)_\rho$, respectively $C^\infty_c(P,V)_\rho$.

(c) In Lemma D.1, we recall that if $\mathbb{V}$ is the vector bundle associated to a principal bundle as in (b) then the canonical isomorphism of chain complexes $\Omega^\bullet_c(P,V)_\rho^{\mathrm{hor}} \cong \Omega^\bullet_c(M,\mathbb{V})$ (see e.g. [Bau14, Theorem 3.5]) induces isomorphisms of locally convex spaces $\Omega^k_c(P,V)_\rho^{\mathrm{hor}} \cong \Omega^k_c(M,\mathbb{V})$.

[1]This chapter consist of material published before in the author's preprint [Eyn14b].

(d) Given a manifold M, we write $C_p^\infty(\mathbb{R}, M)$ for the set of proper smooth maps from $\mathbb{R}$ to M. However, if F is the total space of a fibre bundle $E \hookrightarrow F \xrightarrow{q} M$, we define $C_p^\infty(\mathbb{R}, F) := \{f \in C^\infty(\mathbb{R}, F) : q \circ f \in C_p^\infty(\mathbb{R}, M)\}$.

Now that we have fixed the basic notation, we introduce the following conventions:

Convention 4.1. (a) All finite-dimensional manifolds are assumed to be σ-compact.

(b) Analogously to [NW09, p. 385 and p.388], we consider the following setting[2]: If not defined otherwise, $H \hookrightarrow P \xrightarrow{q} M$ denotes a finite-dimensional principal bundle over a connected non-compact σ-compact manifold M and $\mathfrak{h}$ the Lie algebra of H.[3] Moreover, let G be a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $\kappa_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to V(\mathfrak{g}) =: V$ be the universal invariant symmetric bilinear map on $\mathfrak{g}$ (see Definition 3.1, respectively [Gun11, Chapter 4]). Let $\rho_G\colon H \times G \to G$ be a smooth action of H on G by Lie group automorphisms and $\rho_{\mathfrak{g}}\colon H \times \mathfrak{g} \to \mathfrak{g}$ be the derived action on $\mathfrak{g}$ by Lie algebra automorphisms ($\rho_{\mathfrak{g}}(h, \bullet) = L(\rho_G(h, \bullet)) \in \operatorname{Aut}(\mathfrak{g})$). In view of Remark 3.13, we find a unique map $\rho_V\colon H \times V \to V$ that is linear in the second argument and fulfils $\rho_V(h, \kappa_{\mathfrak{g}}(x, y)) = \kappa_{\mathfrak{g}}(\rho_{\mathfrak{g}}(h, x), \rho_{\mathfrak{g}}(h, y))$ for $x, y \in \mathfrak{g}$ and $h \in H$. The vector space V is generated by elements of the form $\kappa_{\mathfrak{g}}(x, y)$ with $x, y \in \mathfrak{g}$. To see that ρ_V is also a representation, we show that $\rho_V(g, \rho_V(h, \kappa_{\mathfrak{g}}(x, y))) = \rho_V(gh, \kappa_{\mathfrak{g}}(x, y))$ for $x, y \in \mathfrak{g}$ and $g, h \in H$:

$$\begin{aligned}&\rho_V(g, \rho_V(h, \kappa_{\mathfrak{g}}(x, y))) = \rho_V(g, \kappa_{\mathfrak{g}}(\rho_{\mathfrak{g}}(h, x), \rho_{\mathfrak{g}}(h, y)))\\ =&\kappa_{\mathfrak{g}}(\rho_{\mathfrak{g}}(g, \rho_{\mathfrak{g}}(h, x)), \rho_{\mathfrak{g}}(g, \rho_{\mathfrak{g}}(h, y))) = \rho_V(gh, \kappa_{\mathfrak{g}}(x, y)).\end{aligned}$$

Because we can find a basis of V consisting of vectors of the form $\kappa_{\mathfrak{g}}(x, y)$, the smoothness of ρ_V follows. We write $\mathcal{G} := P \times_{\rho_G} G$ for the associated Lie group bundle[4], $\mathfrak{G} := P \times_{\rho_{\mathfrak{g}}} \mathfrak{g}$ for the associated Lie algebra bundle and $\mathbb{V} := P \times_{\rho_V} V$ for the associated vector bundle to $H \hookrightarrow P \to M$. Let VP be the vertical bundle of TP. We fix a principal connection $HP \subseteq TP$ on the principal bundle P and write $\mathrm{pr}_h\colon TP \to HP$ for the projection onto the horizontal bundle. As pointed out in [NW09, p. 385] no generality is lost if we assume that the total space P is connected. Hence, we do so in this chapter.

(c) Let $D_{\rho_{\mathfrak{g}}}\colon C_c^\infty(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \to \Omega_c^1(P, \mathfrak{g})_{\rho_{\mathfrak{g}}}^{\mathrm{hor}}$, $f \mapsto df \circ \mathrm{pr}_h$ and $D_{\rho_V}\colon C_c^\infty(P, V)_{\rho_V} \to \Omega_c^1(P, V)_{\rho_V}^{\mathrm{hor}}$, $f \mapsto df \circ \mathrm{pr}_h$ be the absolute derivatives corresponding to HP (see [Bau14, Definition 3.8]). Moreover, let $d_{\mathfrak{G}}\colon \Gamma_c(M, \mathfrak{G}) \to \Omega_c^1(M, \mathfrak{G})$ and $d_{\mathbb{V}}\colon \Gamma_c(M, \mathbb{V}) \to \Omega_c^1(M, \mathbb{V})$ be the induced covariant derivations on the Lie algebra bundle $\mathfrak{G}$ and the vector bundle $\mathbb{V}$ respectively (see [Bau14, p. 100

[2] In [NW09] Neeb and Wockel also consider situations where the Lie groups H and G can be infinite-dimensional locally exponential Lie groups.

[3] Like in [Nee04] it is crucial for our proof that the manifold M is not compact. Hence, our argumentation is not an alternative for the proof of [NW09]

[4] The definition of a Lie group bundle, respectively associated Lie group bundle, is completely analogous to the definition of a vector bundle, respectively associated vector bundle.

ff] and Lemma D.1).

In [NW09, Appendix A], where M is compact, Neeb and Wockel endowed the group of sections $\Gamma(M,\mathcal{G})$ of a Lie group bundle $\mathcal{G}$ that is associated to a principal bundle P with a Lie group structure. They used the identification $\Gamma(M,\mathcal{G}) \cong C^\infty(P,G)_{\rho_G}$ and endowed the group $C^\infty(P,G)_{\rho_G}$ of G-invariant smooth maps from P to G with a Lie group structure by using the construction of a Lie group structure on the gauge group $\mathrm{Gau}(P)$ described in [Woc07]. To this end, they replaced the conjugation of the structure group on itself by the Lie group action ρ_G. In the following Definition 4.2, we proceed analogously in the case where M is non-compact but σ-compact. As the construction from [NW09] is based on [Woc07], our analogous definition is based on [Sch13, Chapter 4], the generalisation of [Woc07] to the non-compact case.

Definition 4.2. (a) For a set X, a group G with unity 1 and a map $\varphi\colon X \to G$ we define

$$\mathrm{supp}(\varphi) := \overline{\{x \in X : \varphi(x) \neq 1\}}.$$

(b) We equip the group

$$\begin{aligned} C^\infty_c(P,G)_{\rho_G} = \{\varphi \in C^\infty(P,G) : (\exists K \subseteq M \text{ compact})\, \mathrm{supp}(\varphi) \subseteq q^{-1}(K) \\ \text{and } (\forall h \in H, p \in P)\ \rho_G(h) \circ \varphi(ph) = \varphi(p)\} \end{aligned}$$

with the infinite-dimensional Lie group structure described in [Sch13, Chapter 4]. We just replace the conjugation of H on itself by the action ρ_G of H on G. We emphasise that the functions $f \in C^\infty_c(P,G)_{\rho_G}$ are not compactly supported on P itself. The Lie algebra of $C^\infty_c(P,G)_{\rho_G}$ is given by the locally convex Lie algebra

$$\begin{aligned} &C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}} = \{f \in C^\infty(P,\mathfrak{g})_{\rho_\mathfrak{g}} : (\exists K \subseteq M \text{ compact})\, \mathrm{supp}(f) \subseteq q^{-1}(K)\} \\ &= \varinjlim C^\infty_K(P,\mathfrak{g})_{\rho_\mathfrak{g}}, \end{aligned}$$

where K runs through the compact subsets of M.

(c) If $\eta \in \Gamma(M,\mathcal{G})$ we define

$$\mathrm{supp}(\eta) := \overline{\{x \in M : \eta(x) \neq 1\}}$$

and write $\Gamma_c(M,\mathcal{G})$ for the subgroup of sections with compact support in M.

(d) From [Sch13, Chapter 4] (see also [Bau14, Theorem 3.5] and Lemma D.1) we know that $\Gamma_c(M,\mathfrak{G}) \cong C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}}$ in the sense of topological vector spaces. Now, we endow $\Gamma_c(M,\mathcal{G})$ with the Lie group structure that turns the group isomorphism $\Gamma_c(M,\mathcal{G}) \cong C^\infty_c(P,G)_{\rho_G}$ into an isomorphism of Lie groups. Hence, $\Gamma_c(M,\mathcal{G})$ becomes an infinite-dimensional Lie group modelled on the locally convex space $\Gamma_c(M,\mathfrak{G})$.

In the following definition, we fix our notation for the quotient principal bundle. For details on the well-known concept of quotient principal bundles see e.g.

[Gun11, Proposition 2.2.20].

Definition 4.3. Let $N := \ker(\rho_V) \subseteq H$ and $H/N \hookrightarrow P/N \xrightarrow{\overline{q}} M$ be the quotient bundle with projection $\overline{q}\colon P/N \to M$, $pN \mapsto q(p)$ and right action $\overline{R}\colon H/N \times P/N \to P/N$, $([g], pN) \mapsto (pg)N$. We write $\overline{H} := H/N$ and $\overline{P} := P/N$. Let $\overline{\rho}_V\colon H/N \to GL(V)$ be the factorisation of ρ_V over N and $\pi\colon P \to P/N$ the orbit projection. If $\psi\colon q^{-1}(U) \to U \times H$ is a trivialisation of P then $\psi'\colon \overline{q}^{-1}(U) \to U \times \overline{H}$, $pN \mapsto (q(p), [\mathrm{pr}_2 \circ \psi(p)])$ is a typical trivialisation of $\overline{P}$. It is well-known that $\mathbb{V}$ is isomorphic to the associated bundle to $\overline{H} \hookrightarrow \overline{P} \xrightarrow{\overline{q}} M$ via $\overline{\rho}_V$ (see e.g. [Gun11, Remark 2.2.21]). Moreover, we write $H\overline{P} := T\pi(HP)$ for the canonical principal connection on $\overline{P}$ that comes from P (see D.2 (a)). We mention that $\pi^*\colon \Omega^k_c(\overline{P}, V)^{\mathrm{hor}}_{\overline{\rho}} \to \Omega^k_c(P, V)^{\mathrm{hor}}_{\rho_V}$ is an isomorphism of topological vector spaces and induces an isomorphism of chain complexes (see D.2 (c)).

Convention 4.4. Analogously to [NW09], we introduce the following convention. We assume that the identity-neighbourhood of H acts trivially on V by ρ_V (cf. [NW09, p. 385])[5]. Hence, $\overline{H}$ is a discrete Lie group. Moreover, we even assume $\overline{H}$ to be finite (cf. [NW09, p. 386, p.398 f and Theorem 4.14])[6].

Definition 4.5. Let $H \hookrightarrow P \to M$ be a principal bundle with connected total space P and $\rho_V\colon H \times V \to V$ be a linear representation. Moreover, fix a connection HP on TP and let D_{ρ_V} be the induced absolute derivative of the associated vector bundle $\mathbb{V}$.

(a) We define

$$Z^1_{dR,c}(P, V)_{\rho_V} := \left\{\theta \in \Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V} : D_{\rho_V}\theta = 0\right\} \text{ and}$$
$$B^1_{dR,c}(P, V)_{\rho_V} := D_{\rho_V}(C^\infty_c(P, V)_{\rho_V}),$$

and equip these spaces with the induced topology of $\Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V}$.

(b) We define

$$Z^1_{dR,c}(P, V)_{\mathrm{fix}} := Z^1_{dR,c}(P, V) \cap \Omega^1_c(P, V)_{\rho_V} \text{ and}$$
$$B^1_{dR,c}(P, V)_{\mathrm{fix}} := B^1_{dR,c}(P, V) \cap \Omega^1_c(P, V)_{\rho_V}$$

and equip these spaces with the induced topology of $\Omega^1_c(P, V)_{\rho_V}$. In this context $Z^1_{dR,c}(P, V)$ (respectively $B^1_{dR,c}(P, V)$) stands for the closed (respectively exact) compactly supported V-valued 1-forms on P with respect to the compactly supported de Rham cohomology.

Lemma 4.6. *Let $H \hookrightarrow P \to M$ be a principal bundle and $\rho_V\colon H \times V \to V$ be a linear representation. Moreover, fix a connection HP on TP and let D_{ρ_V} be the induced absolute derivative of the associated vector bundle $\mathbb{V}$.*

[5] In Section 4.3 we will see that this is a quite natural assumption.

[6] Even the case $\overline{H} = \{1\}$ is a generalisation of [Nee04] if the typical fibre is finite-dimensional. If $\overline{H}$ is trivial then so is the vector bundle $\mathbb{V}$. However, the Lie algebra bundle $\mathfrak{G}$ and the Lie group bundle $\mathcal{G}$ do not have to be trivial.

(a) If H is discrete, we have

$$Z^1_{dR,c}(P,V)_{\rho_V} = Z^1_{dR,c}(P,V)_{\text{fix}} \text{ and } B^1_{dR,c}(P,V)_{\rho_V} \subseteq B^1_{dR,c}(P,V)_{\text{fix}}.$$

Because in this situation all forms on P are horizontal the topologies on $Z^1_{dR,c}(P,V)_{\rho_V}$ and $Z^1_{dR,c}(P,V)_{\text{fix}}$ coincide.

(b) If H is finite we get $B^1_{dR,c}(P,V)_{\rho_V} = B^1_{dR,c}(P,V)_{\text{fix}}$. Again the topologies on these subspaces coincide, because $\Omega^1_c(P,V)^{\text{hor}}_{\rho_V}$ and $\Omega^1_c(P,V)_{\rho_V}$ are exactly the same topological vector spaces.

Proof. (a) If H is discrete there is only one connection on P, namely $HP = TP$. Hence, in this case, D_{ρ_V} is the ordinary exterior derivative.

(b) Let $n := \#H$ and $\theta \in B^1_{dR,c}(P,V)_{\text{fix}}$ with $\theta = df$ for $f \in C^\infty_c(P,V)$. For $\varphi \in C^\infty_c(P,V)$ and $g \in H$ we write $g.\varphi := \rho_V(g) \circ R^*_g\varphi$ and get $\frac{1}{n} \cdot \sum_{g\in H} g.f \in C^\infty_c(P,V)_{\rho_V}$. Moreover, $d(\frac{1}{n} \cdot \sum_{g\in H} g.f) = \theta$. Hence, $B^1_{dR,c}(P,V)_{\rho_V} = B^1_{dR,c}(P,V)_{\text{fix}}$.

□

Lemma 4.7. *Let $H \hookrightarrow P \xrightarrow{q} M$ be a principal bundle with finite structure group H and connected total space P. Moreover, let $\rho_V : H \times V \to V$ be a finite-dimensional linear representation, HP a connection on TP and D_{ρ_V} be the induced absolute derivative of the associated vector bundle $\mathbb{V}$. Then the following holds:*

(a) The map q is proper. Hence, in this case the forms in $\Omega^k_c(P,V)$ are exactly the compactly supported forms on P.

(b) The space $B^1_{dR,c}(P,V) = dC^\infty_c(P,V)$ is a closed subspace of $\Omega^1_c(P,V)$.

Proof. (a) We see from [NR11, Lemma 10.2.11] that if $F \hookrightarrow \mathbb{F} \xrightarrow{q} M$ is a continuous fibre bundle of finite-dimensional topological manifolds and F is finite then q is a proper map.[7]

(b) From [Nee04, Lemma IV.11] we see that, if M is a connected finite-dimensional manifold and V a finite-dimensional vector space then $B^1_{dR,c}(M,V) = dC^\infty_c(M,V)$ is a closed subspace of $\Omega^1_c(M,V)$.

□

For a corresponding statement of the following lemma in the case of a compact base manifold M, compare [NW09, p. 385 f].

Lemma 4.8. *The subspace $D_{\rho_V} C^\infty_c(P,V)_{\rho_V} \subseteq \Omega^1_c(P,V)^{\text{hor}}_{\rho_V}$ is closed.*

Proof. The lemma simply says that $d\Gamma_c(M,\mathbb{V})$ is closed in $\Omega^1_c(M,\mathbb{V})$. Hence, it is enough to show that the subspace $dC^\infty_c(\overline{P},V)_{\overline{\rho}_V}$ is closed in $\Omega^1_c(\overline{P},V)^{\text{hor}}_{\overline{\rho}_V} = \Omega^1_c(\overline{P},V)_{\overline{\rho}_V}$. We know that $B^1_{dR,c}(\overline{P},V)$ is closed in $\Omega^1_c(\overline{P},V)$. We calculate

$$dC^\infty_c(\overline{P},V)_{\overline{\rho}_V} = B^1_{dR,c}(\overline{P},V)_{\text{fix}} = \bigcap_{g\in\overline{H}} \left\{\theta \in B^1_{dR,c}(\overline{P},V) : \overline{\rho}_V(g) \circ \overline{R}^*_g\theta = \theta\right\}$$

[7] A more general statement in the setting of topological spaces is stated in [Lee13, Exercise A.75.].

$$= \bigcap_{g \in \overline{H}} (\overline{\rho}_V(g) \circ \overline{R}_g^* - \mathrm{id})^{-1} \{0\}$$

and see that $dC_c^\infty(\overline{P}, V)_{\overline{\rho}_V}$ is closed in $\Omega_c^1(\overline{P}, V)$. Because the topology of $\Omega_c^1(\overline{P}, V)_{\rho_V}$ is finer than the induced topology of $\Omega_c^1(\overline{P}, V)$, the space $dC_c^\infty(\overline{P}, V)_{\overline{\rho}_V}$ is also closed in $\Omega_c^1(\overline{P}, V)_{\rho_V}$. □

Definition 4.9. Let $H \hookrightarrow P \to M$ be a principal bundle with connected total space P and $\rho_V \colon H \times V \to V$ be a linear representation. Moreover, fix a connection HP on TP and let D_{ρ_V} be the induced absolute derivative of the associated vector bundle $\mathbb{V}$.

(a) If the quotient group $H/\ker(\rho_V)$ is finite (this of course includes the case where the group H is finite), we define

$$H^1_{dR,c}(P, V)_{\rho_V} := Z^1_{dR,c}(P, V)_{\rho_V} / B^1_{dR,c}(P, V)_{\rho_V} \cong H^1_{dR,c}(M, \mathbb{V}).$$

Because of Lemma 4.8 this is a Hausdorff locally convex space.

(b) We have a canonical H-module structure on $H^1_{dR,c}(P, V)$ given by $H \times H^1_{dR,c}(P, V) \to H^1_{dR,c}(P, V)$, $(h, [\theta]) \mapsto [\rho_V(h) \circ R_h^* \theta]$. As usual, we call the fixed points of this action ρ_V-invariant. If the group H is finite we define

$$H^1_{dR,c}(P, V)_{\mathrm{fix}} := \left\{ [\theta] \in H^1_{dR,c}(P, V) : [\theta] \text{ is } \rho_V\text{-invariant} \right\}$$

and because of Lemma 4.7 the space $H^1_{dR,c}(P, V)_{\mathrm{fix}}$ becomes a Hausdorff locally convex space as a closed subspace of the Hausdorff locally convex space $H^1_{dR,c}(P, V)$.

It is possible to show the following lemma by a more abstract argument using that under certain conditions the fixed point functor is exact like it was done in the compact case in [NW09, Remark 4.12]. Here, we give a more elementary proof.

Lemma 4.10. *Let $H \hookrightarrow P \to M$ be a principal bundle and $\rho_V \colon H \times V \to V$ be a linear representation. If H is finite we get*

$$H^1_{dR,c}(P, V)_{\mathrm{fix}} \cong Z^1_{dR,c}(P, V)_{\mathrm{fix}} / B^1_{dR,c}(P, V)_{\mathrm{fix}},$$

as topological vector spaces.

Proof. Let $n := \#H$. We consider the linear map $\psi \colon Z^1_{dR,c}(P, V)_{\mathrm{fix}} \to H^1_{dR,c}(P, V)_{\mathrm{fix}}$, $\theta \mapsto [\theta]$. The map ψ is continuous because the inclusion $Z_{dR,c}(P, V)_{\mathrm{fix}} \hookrightarrow \Omega_c^1(P, V)$ is continuous and so the canonical map $Z_{dR,c}(P, V)_{\mathrm{fix}} \to H^1_{dR,c}(P, V)$ is continuous. If $[\theta] \in H^1_{dR,c}(P, V)_{\mathrm{fix}}$ with $\theta = df$ for $f \in C_c^\infty(P, V)$ then $[\theta] = [d(\frac{1}{n} \sum_{g \in H} g.f)]$ and $d(\frac{1}{n} \sum_{g \in H} g.f) \in B^1_{dR,c}(P, V)_{\mathrm{fix}}$ so $\ker(\psi) \subseteq B^1_{dR,c}(P, V)_{\mathrm{fix}}$. Obviously, $B^1_{dR,c}(P, V)_{\mathrm{fix}} \subseteq \ker(\psi)$. Now, we show that ψ is surjective. If $[\theta] \in H^1_{dR,c}(P, V)_{\mathrm{fix}}$ then $[\theta] = [\frac{1}{n} \cdot \sum_{g \in H} g.\theta]$ and $\frac{1}{n} \cdot \sum_{g \in H} g.\theta \in Z^1_{dR,c}(P, V)_{\mathrm{fix}}$. Hence, ψ factors through a continuous bijective linear map $\overline{\psi} \colon Z^1_{dR,c}(P, V)_{\mathrm{fix}} / B^1_{dR,c}(P, V)_{\mathrm{fix}} \to H^1_{dR,c}(P, V)_{\mathrm{fix}}$. It is left to show that ψ is also open. We define

$$\tau\colon H^1_{dR,c}(P,V) \to Z^1_{dR,c}(P,V)_{\text{fix}}/B^1_{dR,c}(P,V)_{\text{fix}}, [\theta] \mapsto \left[\frac{1}{n}\cdot\sum_{g\in H} g.\theta\right].$$

Obviously $\tau|_{H^1_{dR,c}(P,V)_{\text{fix}}}$ is inverse to $\overline{\psi}$. The map

$$\Omega^1_c(P,V) \to \Omega^1_c(P,V)_{\rho_V},\ \theta \mapsto \frac{1}{n}\cdot\sum_{g\in H} g.\theta$$

is continuous, because the action $g.\theta = \rho(g)\circ R_g^*\theta$ does not enlarge the support of a given form. □

Corollary 4.11. *Considering the principal bundle $\overline{H} \hookrightarrow \overline{P} \to M$ with the action $\overline{\rho}_V$, we have*

$$H^1_{dR,c}(\overline{P},V)_{\text{fix}} \cong Z^1_{dR,c}(\overline{P},V)_{\text{fix}}/B^1_{dR,c}(\overline{P},V)_{\text{fix}} \cong H^1_{dR,c}(\overline{P},V)_{\overline{\rho}_V}.$$

The following lemma is a generalisation of considerations in [NW09, p. 399 and Remark 4.12] from the compact case to the non-compact case.[8]

Lemma 4.12. *(a) If we endow $H^1_{dR,c}(M,V)$ with the canonical $\overline{H}$-module structure $\overline{H}\times H^1_{dR,c}(M,V) \to H^1_{dR,c}(M,V)$, $(h,[\theta]) \mapsto [\overline{\rho}_V(h)\circ\theta]$, then the map $\overline{q}^*\colon H^1_{dR,c}(M,V) \to H^1_{dR,c}(\overline{P},V)$ becomes an isomorphism of $\overline{H}$-modules such that $H^1_{dR,c}(M,V)_{\text{fix}} \cong_{\overline{q}^*} H^1_{dR,c}(\overline{P},V)_{\text{fix}}$.*

(b) We have

$$H^1_{dR,c}(M,V_{\text{fix}}) \cong H^1_{dR,c}(M,V)_{\text{fix}}, \tag{4.1}$$

where V_{fix} is the subspace of fixed points of the action $\overline{\rho}_V$ in V.

(c) The map

$$H^1_{dR,c}(M,V_{\text{fix}}) \to H^1_{dR,c}(P,V)_{\rho_V},\ [\theta] \mapsto [q^*\theta]$$

is an isomorphism of topological vector spaces.

Proof. (a) For $\overline{h}\in\overline{H}$ we calculate

$$\overline{q}^*[\overline{\rho}(\overline{h})\circ\theta] = [\overline{\rho}(\overline{h})\circ\overline{q}^*\theta] = [\overline{\rho}(\overline{h})\circ(\overline{q}\circ\overline{R}_{\overline{h}})^*\theta] = [\overline{\rho}(\overline{h})\circ\overline{R}^*_{\overline{h}}\overline{q}^*\theta] = \overline{h}.\overline{q}^*[\theta].$$

Hence, $\overline{q}^*$ is an isomorphism of $\overline{H}$-modules. Now the second assertion follows from Lemma D.3.

(b) We exchange P with M and the action $g.\theta = \varphi(g)\circ R_g^*\varphi$ with $g.\theta = \overline{\rho}_V(g)\circ\theta$ in the proof of Lemma 4.10 and get

$$H^1_{dR,c}(M,V)_{\text{fix}} \cong Z^1_{dR,c}(M,V)_{\text{fix}}/B^1_{dR,c}(M,V)_{\text{fix}}.$$

[8] As mentioned above, in [NW09] Neeb and Wockel also consider situations where the Lie groups H and G can be infinite-dimensional locally exponential Lie groups.

Now we show that the isomorphism $\varphi\colon \Omega^1_c(M, V_{\mathrm{fix}}) \to \Omega^1_c(M, V)_{\mathrm{fix}}$, $\theta \mapsto \theta$ is a homeomorphism, where $\Omega^1_c(M, V)_{\mathrm{fix}}$ is equipped with the induced topology from $\Omega^1_c(M, V)$. Given a compact set $K \subseteq M$ the map $\Omega^1_K(M, V_{\mathrm{fix}}) \to \Omega^1_K(M, V)$ is continuous. Hence, $\Omega^1_c(M, V_{\mathrm{fix}}) \to \Omega^1_c(M, V)$ is continuous and therefore φ is continuous. Considering the continuous map $\Omega^1_c(M, V) \to \Omega^1_c(M, V_{\mathrm{fix}})$, $\theta \mapsto \sum_{\overline{h} \in \overline{H}} \overline{h}.\theta$, we see that φ is an isomorphism of topological vector spaces. Now the assertion follows from $Z^1_{dR,c}(M, V)_{\mathrm{fix}} = Z^1_{dR,c}(M, V_{\mathrm{fix}})$ and $B^1_{dR,c}(M, V)_{\mathrm{fix}} = B^1_{dR,c}(M, V_{\mathrm{fix}})$.

(c) We have the commutative diagram

$$\begin{array}{ccccc}
H^1_{dR,c}(M, V_{\mathrm{fix}}) & & \xrightarrow{\quad q^* \quad} & & H^1_{dR,c}(P, V)_{\rho_V} \\
\downarrow & & & & \uparrow{\scriptstyle \pi^*} \\
H^1_{dR,c}(M, V)_{\mathrm{fix}} & \xrightarrow{\overline{q}^*} & H^1_{dR,c}(\overline{P}, V)_{\mathrm{fix}} & \longrightarrow & H^1_{dR,c}(\overline{P}, V)_{\overline{\rho}_V}.
\end{array}$$

The assertion now follows from (a), (b) and Corollary 4.11.

□

Convention 4.13. From now on we write $q_*\colon H^1_{dR,c}(P, V)_{\rho_V} \to H^1_{dR,c}(M, V_{\mathrm{fix}})$ for the inverse of $q^*\colon H^1_{dR,c}(M, V_{\mathrm{fix}}) \to H^1_{dR,c}(P, V)_{\rho_V}$, $[\theta] \mapsto [q^*\theta]$.

Remark 4.14. *Given an infinite-dimensional Lie group G with Lie algebra $\mathfrak{g}$, a trivial locally convex $\mathfrak{g}$-module $\mathfrak{z}$ and a Lie algebra cocycle $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$, [Nee02a, Theorem 7.12] gives us conditions under which we can integrate ω to a Lie group cocycle of the Lie group G. These conditions were recalled in the introduction to this thesis. Theorem 7.12 in [Nee02a] is formulated in the case where $\mathfrak{z}$ is sequentially complete. However, it also holds in a special case when $\mathfrak{z}$ is not sequentially complete: Let E be a Mackey complete space, $F \subseteq E$ be a closed vector subspace, $\mathfrak{z} = E/F$. If ω lifts to a continuous bilinear map $\alpha\colon \mathfrak{g} \times \mathfrak{g} \to E$ then the results of [Nee02a] remain valid. To see this, consider the following: Let ω^l be the left invariant 2 form on G corresponding to ω. The completeness of $\mathfrak{z}$ is only used to guarantee the existence of weak integrals in the following settings:*

(a) $\int_\sigma \omega^l = \int_M \sigma^*\omega^l$ where M is a 2-dimensional manifold (namely $M = \mathbb{S}^1$) or simplex and $\sigma\colon M \to G$ is a smooth map (see [Nee02a, Section 5 and 6]),

(b) $\int_0^1 \omega^l(f(t))dt$ where $f\colon [0,1] \to TG \oplus TG$ is a smooth map into the Whitney sum (see [Nee02a, Section 7]).

The integrals $\int_\sigma \omega^l$ and $\int_0^1 \omega^l(f(t))dt$ are weak integrals but such integrals do not have to exist in arbitrary locally convex spaces. However, they exist in sequentially complete (respectively Mackey complete) locally convex spaces. This is the reason why Neeb assumes $\mathfrak{z}$ to be sequentially complete. Now we consider the situation where $\mathfrak{z}$ is not itself sequentially complete but $\mathfrak{z} = E/F$ with a Mackey complete locally convex space E and a closed subspace F and $\omega = \pi \circ \alpha$ is a Lie algebra cocycle with the canonical projection $\pi\colon E \to E/F$ and a continuous bilinear map $\alpha\colon \mathfrak{g}^2 \to E$. We show the existence of the weak integral $\int_\sigma \omega^l$. We define $\tilde{\alpha}\colon \mathfrak{g}^2 \to E$, $(v, w) \mapsto \frac{1}{2}\alpha(v, w) - \frac{1}{2}\alpha(w, v)$ (see [NW09, Remark 2.2]). It follows that $\pi \circ \tilde{\alpha} = \omega$ with a continuous Lie algebra 2-cochain $\tilde{\alpha}$. Let $\tilde{\alpha}^l \in \Omega^2(G, E)$ be the left invariant *differential form on G that comes from $\tilde{\alpha}$. We get $\omega^l = \pi \circ \tilde{\alpha}^l$ and the weak integral $\int_\sigma \omega^l$ is given by*

$$\int_M \sigma^* \omega^l = \pi \left(\int_M \sigma^* \tilde{\alpha}^l \right).$$

The existence of the weak integral $\int_0^1 \omega^l(f(t))dt$ follows analogously.

Theorem 4.15. *Let $\mathfrak{g}$ be a locally convex Lie algebra, V be a locally convex space, considered as a trivial $\mathfrak{g}$-module and $\omega\colon \mathfrak{g} \times \mathfrak{g} \to V$ be a continuous Lie algebra cocycle, such that there exists a Mackey complete locally convex space E, a closed vector subspace $F \subseteq E$ and a continuous bilinear map $\alpha\colon \mathfrak{g}\times\mathfrak{g} \to E$, with $V = E/F$ and $\omega = \pi \circ \alpha$ (where π is the canonical projection $E \to V$). Moreover, let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If the image of the period map $\mathrm{per}_\omega\colon \pi_2(G) \to V$ (denoted by Π_ω) is discrete and the adjoint action of $\mathfrak{g}$ on $V \times_\omega \mathfrak{g}$ given by $\mathfrak{g} \times (V \times_\omega \mathfrak{g}) \to V \times_\omega \mathfrak{g}$, $(x, w) \mapsto [x, w]_\omega$ integrates to a smooth action of G on $V \times_\omega \mathfrak{g}$, then the central extension of $\mathfrak{g}$ by V represented by ω integrates to a central extension of Lie groups $V/\Pi_\omega \hookrightarrow \widehat{G} \to G$ of the Lie group G by V/Π_ω.*

Proof. This is just [Nee02a, Proposition 7.6] and [Nee02a, Theorem 7.12] combined with Remark 4.14. □

Definition 4.16. We define the locally convex spaces $\overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V} := \Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V}/D_{\rho_V} C^\infty_c(P, V)_{\rho_V}$ and $\overline{\Omega}^1_c(M, \mathbb{V}) := \Omega^1_c(M, \mathbb{V})/d_{\mathbb{V}}\Gamma_c(M, \mathbb{V})$ (see [JW13, p. 129]). With Lemma D.2 and Lemma D.1, we get

$$\overline{\Omega}^1_c(M, \mathbb{V}) \cong \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V} \cong \overline{\Omega}^1_c(\overline{P}, V)^{\mathrm{hor}}_{\overline{\rho}_V}.$$

Remark 4.17. *(a) Considering the vector bundles $V(\mathfrak{G})$ from [JW13] respectively Definition 3.14, we have a vector bundle isomorphism $\mathbb{V} \to V(\mathfrak{G})$ given by*

$$\varphi\colon P \times_{\rho_V} V = \mathbb{V} \to V(\mathfrak{G}) = V(P \times_{\rho_{\mathfrak{g}}} \mathfrak{g}),$$
$$[p, \kappa_{\mathfrak{g}}(x, y)] \mapsto \kappa_{\mathfrak{G}_{q(p)}}([p, x], [p, y]) \text{ for } x, y \in \mathfrak{g}.$$

In fact, φ is well-defined, because for $p \in P$ there exists a unique linear map $\varphi_p\colon V = V(\mathfrak{g}) \to V(\mathfrak{G}_{q(p)})$ given by $\varphi_p(\kappa_{\mathfrak{g}}(x, y)) = \kappa_{\mathfrak{G}_{q(p)}}([p, x], [p, y])$. Furthermore, for $x \in M$ the map $(P \times_{\rho_V} V)_x \to V(\mathfrak{G}_x)$, $[p, v] \mapsto \varphi_p(v)$ is well-defined. The bundle morphism φ is smooth, because locally it has the form $U \times V \to U \times V$, $(x_0, \kappa_{\mathfrak{g}}(x, y)) \mapsto (x_0, \kappa_{\mathfrak{g}}(x, y)$ for a domain $U \subseteq M$ of a trivialisation of P, $x_0 \in U$ and $x, y \in \mathfrak{g}$ in the canonical charts. Hence, φ is locally given by the identity $U \times V \to U \times V$.

(b) Given $\theta \in \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$ and $f \in C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}}$, we have $\kappa_{\mathfrak{g}} \circ (\theta, f) \in \Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V}$. In fact $\kappa_{\mathfrak{g}} \circ (\theta, f)$ is obviously horizontal and compactly supported with respect to the principal bundle $P \xrightarrow{q} M$. Moreover, given $h \in H$, $p \in P$ and $v \in T_pP$, we calculate

$$R^*_h(\kappa_{\mathfrak{g}} \circ (\theta, f))_p(v) = \kappa_{\mathfrak{g}}(\theta_{ph}(TR_h(v)), f(ph))$$

$$=\kappa_{\mathfrak{g}}(\rho_{\mathfrak{g}}(h^{-1}).\theta_p(v), \rho_{\mathfrak{g}}(h^{-1}).f(p)) = \rho_V(h^{-1}).\kappa_{\mathfrak{g}}(\theta_p(v), f(p)).$$

Therefore, the map $\tilde{\kappa}_{\mathfrak{g}} \colon \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}} \times C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \to \Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V}$, $(\theta, f) \mapsto \kappa_{\mathfrak{g}} \circ (\theta, f)$ *makes sense and we obtain the commutative diagram*

$$\begin{array}{ccc}
\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}} \times C^\infty_c(P,\mathfrak{g})_{\rho_{\mathfrak{g}}} & \xrightarrow{\tilde{\kappa}_{\mathfrak{g}}} & \Omega^1_c(P,V)^{\mathrm{hor}}_{\rho_V} \\
\Big\downarrow & & \downarrow \\
 & & \Omega_c(M,\mathbb{V}) \\
 & & \downarrow \\
\Omega^1_c(M,\mathfrak{G}) \times \Gamma_c(\mathfrak{G}) & \xrightarrow{\tilde{\kappa}_{\mathfrak{G}}} & \Omega^1_c(M, V(\mathfrak{G})),
\end{array}$$

where the lower horizontal arrow is given by the map $\tilde{\kappa}_{\mathfrak{G}}$ *described in Lemma 3.22 and the vertical arrows are the canonical isomorphisms of topological vector spaces. In particular,* $\tilde{\kappa}_{\mathfrak{g}}$ *is continuous. We write* $\tilde{\kappa}_{\mathfrak{g}}(\eta, \theta) := \tilde{\kappa}_{\mathfrak{g}}(\theta, \eta)$ *for* $\theta \in \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$ *and* $\eta \in \Gamma_c(\mathfrak{G})$.

(c) The map $C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \times \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}} \to \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$, $(\eta, \theta) \mapsto [\eta, \theta]$ *with* $[\eta, \theta]_p(w) = [\eta(p), \theta_p(w)]$ *for* $p \in P$ *and* $w \in T_pP$ *makes sense. In fact* $[\eta, \theta]$ *is horizontal,* $[\eta, \theta] \in \Omega^1_c(P, \mathfrak{g})$ *and the form* $[\eta, \theta]$ *is* $\rho_{\mathfrak{g}}$*-invariant because* $\rho_{\mathfrak{g}}$ *acts by Lie algebra automorphisms on* $\mathfrak{g}$*. Under the canonical isomorphisms of topological vector spaces* $\Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}} \cong \Omega^1_c(M, \mathfrak{G})$ *and* $C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \cong \Gamma_c(M, \mathfrak{G})$ *this map corresponds to the map* $\Omega^1_c(M, \mathfrak{G}) \times \Gamma_c(M, \mathfrak{G}) \to \Omega_c(M, \mathfrak{G})$, $(\theta, \eta) \mapsto [\theta, \eta]$ *with* $[\theta, \eta]_x(v) = [\theta_x(v), \eta(x)]_{\mathfrak{G}_x}$ *for* $x \in M$ *and* $v \in T_xM$ *(see Definition 3.20). As in Definition 3.20 we define* $[\eta, \theta] := -[\theta, \eta]$ *for* $\theta \in \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$ *and* $\eta \in \Gamma_c(\mathfrak{G})$.

(d) We write $\mathrm{pr}_h \colon TP \to HP$ *for the projection onto the horizontal bundle. In view of the definition of a Lie connection in Definition 3.20, we see directly that* $D_{\rho_{\mathfrak{g}}} \colon C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \to \Omega^1_c(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$, $f \mapsto df \circ \mathrm{pr}_h$ *is a Lie connection.*

(e) We define the map $\beta \colon C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \times C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \to \Omega^1_c(P, V)^{\mathrm{hor}}_{\rho_V}$, $\beta(f, g) = \tilde{\kappa}_{\mathfrak{g}}(D_{\rho_{\mathfrak{g}}} f, g) + \tilde{\kappa}_{\mathfrak{g}}(D_{\rho_{\mathfrak{g}}} g, f)$*. Because* D_{ρ_V} *and* $D_{\rho_{\mathfrak{g}}}$ *are induced by the same principal connection on* P*, we obtain a commutative diagram*

$$\begin{array}{ccc}
C^\infty_c(P,\mathfrak{g})_{\rho_{\mathfrak{g}}} \times C^\infty_c(P,\mathfrak{g})_{\rho_{\mathfrak{g}}} & \xrightarrow{\beta} & \Omega^1_c(P,V)^{\mathrm{hor}}_{\rho_V}, \\
\Big\downarrow{\scriptstyle (\kappa_{\mathfrak{g}})_*} & \nearrow_{D_{\rho_V}} & \\
C^\infty_c(P,V)_{\rho_V} & &
\end{array}$$

where $(\kappa_{\mathfrak{g}})_*(f, g) = \kappa_{\mathfrak{g}} \circ (f, g)$.

Definition 4.18. We define the map

$$\omega_M \colon C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \times C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}} \to \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V},\ (f, g) \mapsto [\kappa_{\mathfrak{g}}(f, D_{\rho_{\mathfrak{g}}} g)],$$

which is analogous to the cocycle ω defined in the compact case in [NW09, Proposition 2.1]. Because $D_{\rho_{\mathfrak{g}}}$ is linear, $D_{\rho_{\mathfrak{g}}}(C^\infty_K(P, \mathfrak{g})_{\rho_{\mathfrak{g}}}) \subseteq \Omega^1_K(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}$ and

$D_{\rho_\mathfrak{g}}(f) = df \circ \mathrm{pr}_h$, we see that $D_{\rho_\mathfrak{g}} \colon C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega_c^1(P, \mathfrak{g})_\mathfrak{g}^{\mathrm{hor}}$ is continuous. Considering Remark 4.17 (b), we see that ω_M is continuous. Repeating the argumentation of the proof of Remark 3.24 with the help of Remark 4.17 (d) and (e), we see that ω_M is anti-symmetric and a cocycle. This is the same argumentation as in [NW09, Proposition 2.1].

Remark 4.19. *Let $\mathfrak{g}$ be perfect in this remark. Because $(\kappa_\mathfrak{g})_* \colon C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \times C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to C_c^\infty(P, V)_{\rho_V}$ corresponds to the universal continuous invariant bilinear form $\kappa_\mathfrak{G} \colon \Gamma_c(\mathfrak{G}) \times \Gamma_c(\mathfrak{G}) \to \Gamma_c(V(\mathfrak{G}))$ from Theorem 3.19, the absolute derivative D_{ρ_V} corresponds to the covariant derivative d constructed in Remark 3.23. In particular, we have $d = d_\mathbb{V}$. Hence, our Lie algebra cocycle ω_M from Definition 4.18 corresponds to the cocycle ω_∇ from [JW13, Chapter 1, (1.1)] respectively Remark 3.24.*

In [NW09] Neeb and Wockel used Lie group homomorphisms that are pull-backs by horizontal lifts of smooth loops $\alpha \colon \mathbb{S}^1 \to M$ to reduce the proof of the discreteness of the period group to the case of $M = \mathbb{S}^1$ (see [NW09, Definition 4.2 and Remark 4.3]). However this approach does not work in the non-compact case. Instead we want to use the results from [Nee04] on current groups on non-compact manifolds. Hence, we use pull-backs by horizontal lifts of proper maps $\alpha \colon \mathbb{R} \to M$ (see the next definition). A corresponding definition in the case of a compact base manifold was given in [NW09, Definition 4.2].

Definition 4.20. We fix $x_0 \in M$, $p_0 \in P_{x_0}$ and $\alpha \in C_p^\infty(\mathbb{R}, M)$ with $\alpha(0) = x_0$. Let $\hat{\alpha} \in C^\infty(\mathbb{R}, P)$ be the unique horizontal lift of α with $\hat{\alpha}(0) = p_0$. We define the group homomorphism

$$\hat{\alpha}_G^* \colon C_c^\infty(P, G)_{\rho_G} \to C_c^\infty(\mathbb{R}, G),\ \varphi \mapsto \varphi \circ \hat{\alpha}$$

and the Lie algebra homomorphism

$$\hat{\alpha}_\mathfrak{g}^* \colon C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to C_c^\infty(\mathbb{R}, \mathfrak{g}),\ f \mapsto f \circ \hat{\alpha}.$$

In this context, the maps in $C_c^\infty(\mathbb{R}, G)$ respectively $C_c^\infty(\mathbb{R}, \mathfrak{g})$ are compactly supported in $\mathbb{R}$ itself. These maps make sense because given $\varphi \in C_c^\infty(P, G)$ we have $\mathrm{supp}(\varphi) \subseteq q^{-1}(L)$ for a compact set $L \subseteq M$. We have $\mathrm{supp}(\varphi \circ \hat{\alpha}) \subseteq \alpha^{-1}(L)$ because if $\varphi(\hat{\alpha}(t)) \neq 1$, we get $\hat{\alpha}(t) \in q^{-1}(L)$ and so $\alpha(t) = q \circ \hat{\alpha}(t) \in L$. Hence, $t \in \alpha^{-1}(L)$. Now, we take the closure. Moreover, we define the integration map

$$I_\alpha \colon \overline{\Omega}_c^1(P, V)_{\rho_V}^{\mathrm{hor}} \to V,\ [\theta] \mapsto \int_\mathbb{R} \hat{\alpha}^* \theta.$$

This map is well-defined: Let $\theta \in \Omega_c^1(P, V)_{\rho_V}$ with $\mathrm{supp}(\theta) \subseteq q^{-1}(L)$ for a compact set $L \subseteq M$. We have $\mathrm{supp}(\hat{\alpha}^*\theta) \subseteq \alpha^{-1}(L)$ because if $(\hat{\alpha}^*\theta)_t \neq 0$, we get $\hat{\alpha}(t) \in q^{-1}(L)$ and so $\alpha(t) = q \circ \hat{\alpha}(t) \in L$. Moreover,

$$(\hat{\alpha}^* D_{\rho_V} f)(t) = (D_{\rho_V} f)_{\hat{\alpha}(t)}(\hat{\alpha}'(t)) = (df)_{\hat{\alpha}(t)}(\hat{\alpha}'(t)) = (f \circ \hat{\alpha})'(t).$$

Hence, $\int_{\mathbb{R}}(\hat{\alpha}^* D_{\rho_V} f) = \int_{\mathbb{R}}(f \circ \hat{\alpha})'(t) = 0$ for $f \in C_c^\infty(P,V)_{\rho_V}$ because $f \circ \hat{\alpha}$ has compact support in $\mathbb{R}$.

The following remark is obvious.

Remark 4.21. *Let $W = \bigcup_{i=1}^n I_i$ be a union of finitely many closed intervals in $\mathbb{R}$. Then W is a submanifold with boundary. In fact, let $W = \bigcup_{j\in J} C_j$ be the disjoint union of the connected components of W. For $j \in J$ and $x \in C_j$, we find i_j with $x \in I_{i_j}$. Hence, $I_{i_j} \subseteq C_j$. If $j_1 \neq j_2$ then $I_{i_{j_1}} \cap I_{i_{j_2}} = \emptyset$ and so $i_{j_1} \neq i_{j_2}$. Therefore $\#J \leqslant n$. Obviously, the sets C_j are intervals.*

The proof of the following lemma is a modification of the proofs of [Sch13, Lemma 3.7 and Corollary 3.10].

Lemma 4.22. *Let $(U_i)_{i\in\mathbb{N}}$ be a relatively compact open cover of $\mathbb{R}$ with $U_i \neq \emptyset$. Then there exists an open cover $(W_i)_{i\in\mathbb{N}}$ of $\mathbb{R}$ such that $W_i \subseteq U_i$, $W_i \neq \emptyset$ and $\overline{W}_i$ is a submanifold with boundary.*

Proof. Let $K_n := [-n, n]$ for $n \in \mathbb{N}$. For all $x \in K_1$ there exists $i_x \in \mathbb{N}$ such that $x \in U_{i_x}$. Let $\overline{B}_{\varepsilon_x}(x) \subseteq U_{i_x}$. We find $x_1, \ldots, x_{N_1}$, such that $K_1 \subseteq \bigcup_{k=1}^{N_1} B_{\varepsilon_{x_k}}(x_k)$. We define $V_{1,k} := B_{\varepsilon_{x_k}}(x_k)$ and $U_{i_{1,k}} := U_{i_{x_k}}$ for $k = 1, \ldots, N_1$. Thus we have $K_1 \subseteq \bigcup_{k=1}^{N_1} V_{1,k}$ and $\overline{V}_{1,k} \subseteq U_{i_{1,k}}$. We can argue analogously for the compact set $K_n \backslash \mathring{K}_{n-1}$ with $n \geqslant 2$ and find open intervals $V_{n,1}, \ldots, V_{n,N_n}$ and indices $i_{n,k}$ such that $K_n \backslash \mathring{K}_{n-1} \subseteq \bigcup_{k=1}^{N_n} V_{n,k}$ and $\overline{V}_{n,k} \subseteq U_{i_{n,k}}$. We obtain $\mathbb{R} \subseteq \bigcup_{n=1}^\infty \bigcup_{k=1}^{N_n} V_{n,k}$. For $i \in \mathbb{N}$, we define $I_i := \{(n,k) : i_{n,k} = i\}$. Then $\#I_i < \infty$ because U_i is relatively compact. Now, we define

$$W_i := \begin{cases} \bigcup_{(n,k)\in I_i} V_{(n,k)} & : I_i \neq \emptyset \\ J, \end{cases}$$

where J is an arbitrary non-degenerated interval that is contained in U_i. We obtain $\bigcup_{i\in\mathbb{N}} W_i = \mathbb{R}$ and $W_i \subseteq U_i$ for all $i \in \mathbb{N}$. Moreover, W_i is a finite union of open intervals. Let $W_i = \bigcup_{j=1}^n J_j$ with intervals J_j. We have $\overline{W}_i = \bigcup_{j=1}^n \overline{J}_j$. Hence, $\overline{W}_i$ is a manifold with boundary (see Remark 4.21). □

In the proof of following lemma, we use the concept of weak direct products of infinite-dimensional Lie groups (cf. [Glo03, Section 7] respectively [Glo07, Section 4]).

Lemma 4.23. *In the situation of Definition 4.20, the group homomorphism*

$$\hat{\alpha}_G^*: C_c^\infty(P,G)_{\rho_G} \to C_c^\infty(\mathbb{R},G),\ \varphi \mapsto \varphi \circ \hat{\alpha}$$

is in fact a Lie group homomorphism such that the corresponding Lie algebra homomorphism is given by $\hat{\alpha}_{\mathfrak{g}}^*: C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}} \to C_c^\infty(\mathbb{R},\mathfrak{g}),\ f \mapsto f \circ \hat{\alpha}$.

Proof. Using the construction of the Lie group structure described in [Sch13, Chapter 4], we can argue in the following way. Let $(\overline{V}_i, \sigma_i)_{i\in\mathbb{N}}$ be a locally finite compact trivialising system of $H \hookrightarrow P \xrightarrow{q} M$ (see [Sch13, Definition 3.6 and Corollary 3.10]). We define $U_i := \alpha^{-1}(V_i)$ for $i \in \mathbb{N}$. The map α is proper. Because $(\alpha^{-1}(\overline{V}_i))_{i\in\mathbb{N}}$ is a compact locally finite cover of $\mathbb{R}$, also $(\overline{U}_i)_{i\in\mathbb{N}}$ is a compact locally finite cover of $\mathbb{R}$. We use Lemma 4.22 and find a cover $(W_i)_{i\in\mathbb{N}}$ of $\mathbb{R}$ such that $W_i \subseteq U_i$ and $(\overline{W}_i)_{i\in\mathbb{N}}$ is a compact locally finite cover of $\mathbb{R}$ by submanifolds with boundaries. Moreover, we have $\overline{W}_i \subseteq \alpha^{-1}(\overline{V}_i)$ for all $i \in \mathbb{N}$. Now $(\overline{W}_i, \mathrm{id}\,|^{\mathbb{R}}_{\overline{W}_i})_{i\in\mathbb{N}}$ is a compact locally finite trivialising system of the trivial principal bundle $\{1\} \hookrightarrow \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R}$ with the trivial action $\{1\} \times G \to G$. We get the following commutative diagram

$$\begin{array}{ccc} C^\infty_c(P,G)_{\rho_G} & \xrightarrow{\hat{\alpha}^*_G} & C^\infty_c(\mathbb{R},G) \\ {\scriptstyle f\mapsto (f\circ\sigma_i)_i}\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto (f|_{W_i})_i} \\ \prod^*_{i\in\mathbb{N}} C^\infty(\overline{V}_i,G) & \xrightarrow{(\psi_i)_{i\in\mathbb{N}}} & \prod^*_{i\in\mathbb{N}} C^\infty(\overline{W}_i,G), \end{array} \tag{4.2}$$

where the group homomorphisms ψ_i are given by the diagram

$$\begin{array}{ccc} C^\infty(\overline{V}_i,G) & \xrightarrow{\psi_i} & C^\infty(\overline{W}_i,G) \\ {\scriptstyle \theta_i}\big\downarrow & & \\ C^\infty(\overline{V}_i \times H,G) & & \\ {\scriptstyle f\mapsto f\circ\varphi_i}\big\downarrow & \nearrow{\scriptstyle f\mapsto f\circ\hat{\alpha}|_{\overline{W}_i}} & \\ C^\infty(P|_{\overline{V}_i},G), & & \end{array}$$

with $\theta_i\colon C^\infty(\overline{V}_i,G) \to C^\infty(\overline{V}_i \times H,G)$, $f \mapsto ((x,h) \mapsto \rho_G(h).f(x))$ and φ_i the inverse of $\overline{V}_i \times H \to P|_{\overline{V}_i}$, $(x,h) \mapsto \sigma_i(x)h$. Defining $\tau^i\colon \overline{W}_i \to \overline{V}_i \times H$, $\tau^i := \varphi_i \circ \hat{\alpha}|_{\overline{W}_i}$ and $\tau^i_j := \mathrm{pr}_j \circ \tau^i$ for $j \in \{1,2\}$, the map $\psi_i\colon C^\infty(\overline{V}_i,G) \to C^\infty(\overline{W}_i,G)$ is given by

$$f \mapsto \rho_G(\mathrm{pr}_2 \circ\varphi_i \circ \hat{\alpha}|_{\overline{W}_i}(\bullet)).(f \circ \mathrm{pr}_1 \circ\varphi_i \circ \hat{\alpha}|_{\overline{W}_i}(\bullet)) = \rho_G(\tau^i_2(\bullet)).(f \circ \tau^i_1(\bullet)).$$

In order to show that (4.2) is commutative let $f \in C^\infty_c(P,G)_{\rho_G}$. Then

$$\rho_G(h).f \circ \sigma_i(x) = f(\sigma_i(x).h) = f(\varphi_i^{-1}(x,h))$$

for all $(x,h) \in \overline{V}_i \times H$. Hence, $\psi_i(f \circ \sigma_i) = f \circ \hat{\alpha}|_{\overline{W}_i}$. To show that ψ_i is a Lie group homomorphism it is enough to show that $C^\infty(\overline{V}_i,G) \times \overline{W}_i \to G$, $(f,x) \mapsto \rho_G(\tau_2(x), f(\tau_1(x)))$ is smooth ([Alz72] respectively [Sch13, Theorem 2.25]). The map $C^\infty(\overline{V}_i,G) \times \overline{V}_i$, $(f,y) \mapsto f(y)$ is smooth (see [Alz72] respectively [Sch13, Theorem 2.26]) and so $C^\infty(\overline{V}_i,G) \times \overline{W}_i \to H \times G$, $(f,x) \mapsto (\tau_2(x), f(\tau_1(x)))$ is smooth. It is left to show that $L(\hat{\alpha}^*_G)$ is given by $C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}} \to C^\infty_c(\mathbb{R},\mathfrak{g})$, $f \mapsto f\circ\hat{\alpha}$. To this end let $f \in C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}}$. We calculate

$$L(\hat{\alpha}^*_G)(f) = \frac{\partial}{\partial t}\bigg|_{t=0} \hat{\alpha}^*_G(\exp(tf)) = \frac{\partial}{\partial t}\bigg|_{t=0} (\exp(tf) \circ \hat{\alpha})$$

$$= \frac{\partial}{\partial t}\bigg|_{t=0} (\exp_G \circ (t \cdot f) \circ \hat{\alpha}) \underset{*}{=} \frac{\partial}{\partial t}\bigg|_{t=0} ((t \cdot f) \circ \hat{\alpha}) = f \circ \hat{\alpha},$$

where $*$ follows from

$$\begin{aligned} \mathrm{ev}_p \left(\frac{\partial}{\partial t}\bigg|_{t=0} (\exp_G \circ (t \cdot f) \circ \hat{\alpha}) \right) &= \frac{\partial}{\partial t}\bigg|_{t=0} (\exp_G (t \cdot f(\hat{\alpha}(p))) \\ = \mathrm{ev}_p \left(\frac{\partial}{\partial t}\bigg|_{t=0} ((t \cdot f) \circ \hat{\alpha}) \right) \end{aligned}$$

for $p \in P$. Now the assertion follows from [Glo07, Proposition 4.5] respectively [Sch13, Corollary 2.38]. $\square$

Definition 4.24 (Cf. Proof of Lemma V.10 in [Nee04])**.** We define the cocycle

$$\begin{aligned} \omega_{\mathbb{R}} \colon C^\infty_c(\mathbb{R}, \mathfrak{g})^2 &\to \overline{\Omega}^1_c(\mathbb{R}, V) = H^1_{dR,c}(\mathbb{R}, V) \to V, \\ (f, g) &\mapsto [\kappa_{\mathfrak{g}}(f, g')] \mapsto \int_{\mathbb{R}} \kappa_{\mathfrak{g}}(f(t), g'(t)) dt. \end{aligned}$$

The following Lemmas 4.25, 4.26 and 4.27 are used to prove Lemma 4.28, which is a generalisation of [Nee04, Lemma V.16] from the case of a current group to the case of a group of sections [9].

In the case of a compact base manifold, a statement corresponding to the following lemma is given by equation (9) in [NW09, Remark 4.3].

Lemma 4.25. *Given $x_0 \in M$, $p_0 \in P_{x_0}$ and $\alpha \in C^\infty_p(\mathbb{R}, M)$ with $\alpha(0) = x_0$, we get*

$$I_\alpha \circ \omega_M = \omega_{\mathbb{R}} \circ (\hat{\alpha}^*_{\mathfrak{g}} \times \hat{\alpha}^*_{\mathfrak{g}}). \tag{4.3}$$

Hence, the following diagram commutes:

$$\begin{array}{ccc} C^\infty_c(P, \mathfrak{g})^2_{\rho_{\mathfrak{g}}} & \xrightarrow{\omega_M} & \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V} \\ \downarrow{\scriptstyle \hat{\alpha}^*_{\mathfrak{g}} \times \hat{\alpha}^*_{\mathfrak{g}}} & & \downarrow{\scriptstyle I_\alpha} \\ C^\infty_c(\mathbb{R}, \mathfrak{g})^2 & \xrightarrow{\omega_{\mathbb{R}}} & V. \end{array}$$

Proof. For $g \in C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}}$, we have

$$(\hat{\alpha}^* D_{\rho_{\mathfrak{g}}} g)(t) = D_{\rho_{\mathfrak{g}}} g(\hat{\alpha}'(t)) = (g \circ \hat{\alpha})'(t)$$

because $\hat{\alpha}$ is a horizontal map. For $f, g \in C^\infty_c(P, \mathfrak{g})_{\rho_{\mathfrak{g}}}$, we get

$$\begin{aligned} I_\alpha(\omega_M(f, g)) = I_\alpha([\kappa_{\mathfrak{g}}(f, D_{\rho_{\mathfrak{g}}} g)]) = \int_{\mathbb{R}} \hat{\alpha}^* \kappa_{\mathfrak{g}}(f, D_{\rho_{\mathfrak{g}}} g) = \int_{\mathbb{R}} \kappa_{\mathfrak{g}}(f \circ \hat{\alpha}, \hat{\alpha}^* D_{\mathfrak{g}} g) \\ = \int_{\mathbb{R}} \kappa_{\mathfrak{g}}(f \circ \hat{\alpha}(t), (g \circ \hat{\alpha})'(t)) dt = \omega_{\mathbb{R}} \circ (\hat{\alpha}^*_{\mathfrak{g}} \times \hat{\alpha}^*_{\mathfrak{g}})(f, g). \end{aligned}$$

[9] In [Nee04, Lemma V.16] Neeb also considers the case of an infinite-dimensional codomain.

□

The following lemma can be found in [NW09, Remark C.2 (a)].

Lemma 4.26. *Let $\varphi\colon G_1 \to G_2$ be a Lie group homomorphism and $\mathfrak{g}_i$ the Lie algebra of G_i for $i \in \{1, 2\}$. Moreover, let V be a trivial $\mathfrak{g}_i$-module and $\omega \in Z^2_c(\mathfrak{g}_2, V)$. Then we get*

$$\mathrm{per}_\omega \circ \pi_2(\varphi) = \mathrm{per}_{L(\varphi)^*\omega} \tag{4.4}$$

as an equation in the set of group homomorphism from $\pi_2(G_1)$ to V.[10]

The following lemma corresponds to the first equation in [NW09, Remark 4.3 (10)].

Lemma 4.27. *Let $x_0 \in M$ and $p_0 \in P_{x_0}$ be base points and $\alpha \in C^\infty_p(\mathbb{R}, M)$. Then*

$$I_\alpha \circ \mathrm{per}_{\omega_M} = \mathrm{per}_{I_\alpha \circ \omega_M} : \pi_2(C^\infty_c(P, G)_{\rho_G}) \to V. \tag{4.5}$$

Proof. Let $\omega^l_M \in \Omega^2(C^\infty_c(P, G)_{\rho_G}, \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V})$ be the corresponding left invariant 2-form of $\omega_M \in Z^2_{ct}(C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}}, \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V})$. Then $I_\alpha \circ \omega^l_M \in \Omega^2_{dR}(C^\infty_c(P, K)_{\rho_K}, V)$ is left invariant and

$$(I_\alpha \circ \omega^l_M)_1(f, g) = I_\alpha((\omega^l_M)_1(f, g)) = I_\alpha \circ \omega_M(f, g)$$

for $f, g \in C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}} = T_1 C^\infty_c(P, G)_{\rho_G}$. Hence, $(I_a \circ \omega_M)^l = I_\alpha \circ \omega^l_M$. For $[\sigma] \in \pi_2(C^\infty_c(P, G)_{\rho_G})$ with a smooth representative σ, we get

$$\begin{aligned}\mathrm{per}_{I_\alpha \circ \omega_M}([\sigma]) &= \int_{\mathbb{S}^2} \sigma^*(I_\alpha \circ \omega^l_M) = \int_{\mathbb{S}^2} I_\alpha \circ \sigma^* \omega^l_M = I_\alpha \circ \int_{\mathbb{S}^2} \sigma^* \omega^l_M \\ &= I_\alpha \circ \mathrm{per}_{\omega_M}([\sigma]).\end{aligned}$$

□

The following lemma is a generalisation of [Nee04, Lemma V.16].[11]

Lemma 4.28. *For a proper map $\alpha \in C^\infty_p(\mathbb{R}, M)$ and the base points $x_0 \in M$ and $p_0 \in P_{x_0}$, the following diagram commutes:*

$$\begin{array}{ccc} \pi_2(C^\infty_c(P, G)_{\rho_G}) & \xrightarrow{\mathrm{per}_{\omega_M}} & \overline{\Omega}^1_c(P, V)^{hor}_{\rho_V} \\ \downarrow{\scriptstyle \pi_2(\hat{\alpha}^*_G)} & & \downarrow{\scriptstyle I_\alpha} \\ \pi_2(C^\infty_c(\mathbb{R}, G)) & \xrightarrow[\mathrm{per}_{\omega_\mathbb{R}}]{} & V. \end{array}$$

[10] The definition of the period map per_ω was recalled in the introduction.
[11] In [Nee04, Lemma V.16] Neeb also considered the case of an infinite-dimensional codomain.

Proof. We calculate

$$I_\alpha \circ \mathrm{per}_{\omega_M} \underset{(4.5)}{=} \mathrm{per}_{I_\alpha \circ \omega_M} \underset{(4.3)}{=} \mathrm{per}_{\omega_{\mathbb{R}} \circ (\hat{\alpha}^* \times \hat{\alpha}^*)} \underset{(4.4)}{=} \mathrm{per}_{\omega_{\mathbb{R}}} \circ \pi_2(\hat{\alpha}_G^*). \tag{4.6}$$

□

Lemma 4.29. *Let $x_0 \in M$ and $p_0 \in P_{x_0}$ be base points and $\alpha \in C_p^\infty(\mathbb{R}, M)$ with $\alpha(0) = x_0$. Moreover, let $\hat{\alpha} \in C^\infty(\mathbb{R}, P)$ be the unique horizontal lift of α to P with $\hat{\alpha}(0) = p_0$.*

(a) We have the commutative diagram

$$\begin{array}{ccc} H^1_{dR,c}(M, V_{\mathrm{fix}}) & \xrightarrow[{[\theta] \mapsto \int_{\mathbb{R}} \alpha^*\theta}]{I_\alpha} & V \\ \cong \Big\downarrow q^* & & \Big\downarrow \mathrm{id} \\ H^1_{dR,c}(P, V)_{\rho_V} & \xrightarrow[{[\theta] \mapsto \int_{\mathbb{R}} \hat{\alpha}^*\theta}]{I_\alpha} & V. \end{array}$$

(b) Given $[\theta] \in H^1_{dR,c}(M, V_{\mathrm{fix}})$, we have $\int_{\mathbb{R}} \hat{a}^* q^*\theta = \int_{\mathbb{R}} \alpha^*\theta$, respectively

$$\int_{\mathbb{R}} \hat{\alpha}^*\theta = \int_{\mathbb{R}} \alpha^* q_*\theta$$

for all $[\theta] \in H^1_{dR,c}(P, V)_{\rho_V}$.

Proof. (a) Given $[\theta] \in H^1_{dR,c}(M, V_{\mathrm{fix}})$, we calculate

$$\int_{\mathbb{R}} \hat{\alpha}^*(q^*\theta) = \int_{\mathbb{R}} (q \circ \hat{\alpha})^*\theta) = \int_{\mathbb{R}} \alpha^*\theta.$$

(b) This is obvious.

□

The following lemma comes from [Nee04, Corollary IV.21].

Lemma 4.30. *If $\Gamma \subseteq V$ is a discrete subgroup then*

$$H^1_{dR,c}(M, \Gamma) := \left\{ [\theta] \in H^1_{dR,c}(M, V) : (\forall \alpha \in C_p^\infty(\mathbb{R}, M)) \int_{\mathbb{R}} \alpha^*\theta \in \Gamma \right\}$$

is a discrete subgroup of $\overline{\Omega}^1_c(M, V)$.

The following statement can be found in the proof of [Nee04, Proposition V.19].

Lemma 4.31. *The group $\Pi_{\omega_{\mathbb{R}}} = \mathrm{im}(\mathrm{per}_{\omega_{\mathbb{R}}})$ is a discrete subgroup of $\overline{\Omega}^1_c(\mathbb{R}, V) = H^1_{dR,c}(\mathbb{R}, V) \cong V$.*

Proof. We argue exactly as in the proof of [Nee04, Proposition V.19] by combining [MN03, Theorem II.9] and [Nee04, Lemma V.11]. □

Remark 4.32. *Because $\overline{q}\colon \overline{P} \to M$ is a finite covering, $\overline{q}$ is a proper map and so a curve $\overline{\alpha}\colon \mathbb{R} \to \overline{P}$ is proper if and only if $\overline{q} \circ \overline{\alpha}\colon \mathbb{R} \to M$ is proper. Hence, the maps in $C^\infty_p(\mathbb{R}, \overline{P})$ are proper in the usual sense.*

Lemma 4.33. *Let $\overline{\alpha}\colon \mathbb{R} \to \overline{P}$ be a proper map. We define $\overline{x}_0 := \overline{\alpha}(0)$, $x_0 := \overline{q}(\overline{x}_0)$ and $\alpha := \overline{q} \circ \overline{\alpha}$. Moreover, let $p_0 \in P$ with $\pi(p_0) = \overline{x}_0$ and $\hat{\alpha}\colon \mathbb{R} \to P$ be the unique horizontal lift of α to P with $\hat{\alpha}(0) = p_0$. In this situation*

$$\begin{array}{ccc} \overline{\Omega}^1_c(\overline{P}, V)_{\overline{\rho}_V} & \xrightarrow[{[\theta] \mapsto \int_{\mathbb{R}} \overline{\alpha}^*\theta}]{} & V \\ \cong \Big\downarrow \pi^* & & \Big\downarrow \mathrm{id} \\ \overline{\Omega}^1_c(P, V)^{\mathrm{hor}}_{\rho_V} & \xrightarrow[{[\theta] \mapsto \int_{\mathbb{R}} \hat{\alpha}^*\theta}]{I_\alpha} & V \end{array}$$

commutes.

Proof. We have $\pi \circ \hat{\alpha} = \overline{\alpha}$ because $\overline{\alpha}$ is the unique horizontal lift of α to $\overline{P}$ with $\overline{\alpha}(0) = \overline{x}_0$ and $\pi \circ \hat{\alpha}$ is also a horizontal lift of α to $\overline{P}$ that maps 0 to $\overline{x}_0$. Hence,

$$\int_{\mathbb{R}} \hat{\alpha}^*(\pi^*\theta) = \int_{\mathbb{R}} (\pi \circ \hat{\alpha})^*\theta = \int_{\mathbb{R}} \overline{\alpha}^*\theta$$

for $\theta \in \Omega^1_c(\overline{P}, V)_{\overline{\rho}_V}$. □

The proof of the following lemma is similar to the proof of [Nee04, Lemma A.1].

Lemma 4.34. *Given a compact set $L \subseteq C^\infty_c(P, G)_{\rho_G}$, we find a compact set $K \subseteq M$ such that $L \subseteq C^\infty_K(P, G)_{\rho_G}$.*

Proof. From [Sch13, Theorem 4.18] we know that the map $\exp_*\colon C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to C^\infty_c(P, G)_{\rho_G}$, $f \mapsto \exp_G \circ f$ is a local diffeomorphism around 0. Given a compact set $K \subseteq M$, we have

$$\exp_*(C^\infty_K(P, \mathfrak{g})_{\rho_\mathfrak{g}}) \subseteq C^\infty_K(P, G)_{\rho_G}. \tag{4.7}$$

Let $U \subseteq C^\infty_c(P, G)_{\rho_G}$ be a 1-neighbourhood and $V \subseteq C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}}$ be a 0-neighbourhood such that $\exp_* |^U_V$ is a diffeomorphism. We write $\Phi := \left(\exp_* |^U_V\right)^{-1}$. If $L \subseteq U$ is a compact set then $\Phi(L)$ is a compact subset of $C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}}$. Because $C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}}$ is a strict LF-space, we find a compact subset $K \subseteq M$ such that $\Phi(L) \subseteq C^\infty_K(P, \mathfrak{g})_{\rho_\mathfrak{g}} \cap V$ (see [Wen03, Theorem 6.4], [Glo08a, Remark 6.2 (d)] or [Bou87, Chapter II Section 4, Proposition 6 and Proposition 9]). Hence, with (4.7) we get $L \subseteq C^\infty_K(P, G)_{\rho_G}$. Now, let $L \subseteq C^\infty_c(P, G)_{\rho_G}$ be an arbitrary compact subset. Let $W \subseteq C^\infty_c(P, G)_{\rho_G}$ be a 1-neighbourhood such that $\overline{W} \subseteq U$. Because L is compact, we find $n \in \mathbb{N}$ and $g_i \in C^\infty_c(P, G)_{\rho_G}$ such that $L \subseteq \bigcup_{i=1}^n g_i \cdot \overline{W}$. Defining the compact set $L_i := L \cap g_i \cdot \overline{W}$, we get $L \subseteq \bigcup_{i=1}^n L_i$. Let $i \in \{1, \ldots, n\}$. It follows that $g_i^{-1} \cdot L_i \subseteq \overline{W} \subseteq U$. Hence, we find a compact set $K_1 \subseteq M$ with $g_i^{-1} \cdot L_i \subseteq C^\infty_{K_1}(P, G)_{\rho_G}$. Let $K_2 \subseteq M$ be compact with $\operatorname{supp}(g_i) \subseteq q^{-1}(K_2)$ and

$K_i := K_1 \cup K_2$ and $K := \bigcup_{i=1}^n K_i$. We have

$$L_i \subseteq g_i \cdot C^\infty_{K_1}(P,G)_{\rho_G} \subseteq C^\infty_{K_i}(P,G)_{\rho_G} \subseteq C^\infty_K(P,G)_{\rho_G}.$$

Hence, $L = \bigcup_{i=1}^n L_i \subseteq C^\infty_K(P,G)_{\rho_G}$. □

In [Nee04, Remark IV.17] (where M is non-compact) Neeb extends a smooth loop $\alpha\colon [0,1] \to M$ by a smooth proper map $\gamma\colon [0,\infty[\to M$ to a proper map $\tilde{\alpha}\colon \mathbb{R} \to M$ such that for all 1-forms θ with compact support one gets $\int_\alpha \theta = \int_{\tilde{\alpha}} \theta$. This construction is also used in the proof of the following theorem. A corresponding result has been proved in the compact case in [NW09, Proposition 4.11].

Theorem 4.35. *If M is non-compact, we have*

$$\operatorname{im}(\operatorname{per}_{\omega_M}) =: \Pi_{\omega_M} \subseteq H^1_{dR,c}(M,\mathbb{V}) \subseteq \overline{\Omega}^1_c(M,\mathbb{V}).$$

This means that all forms in Π_{ω_M} are closed.

Proof. Because $\pi^*\colon \Omega^\bullet_c(\overline{P},V)_{\overline{\rho}_V} \to \Omega^\bullet_c(P,V)^{\mathrm{hor}}_{\rho_V}$ is an isomorphism of chain complexes, it is enough to show that $\Pi_{\omega_M} \subseteq H^1_{dR,c}(\overline{P},V) \subseteq \overline{\Omega}^1_c(\overline{P},V)$. To this end let $[\theta] \in \Pi_{\omega_M}$ and $\overline{\alpha}_0, \overline{\alpha}_1\colon [0,1] \to \overline{P}$ be closed smooth curves in a point $\overline{x}_0 \in \overline{P}$ that are homotopic relative $\{0,1\}$ by a smooth homotopy $\overline{F}\colon [0,1]^2 \to \overline{P}$. From Lemma D.4 we get that it is enough to show that $\int_{\overline{\alpha}_0} \theta = \int_{\overline{\alpha}_1} \theta$. By composing $\overline{\alpha}_i$ respectively $\overline{F}(s,\bullet)$ with a strictly increasing smooth map $\varphi\colon [0,1] \to [0,1]$ whose jet vanishes in 0 and 1, we can assume that in a local chart all derivatives of $\overline{\alpha}_i$ and $\overline{F}(s,\bullet)$ vanish in 0 and 1 because $\int_{\overline{\alpha}_i} \theta = \int_{\overline{\alpha}_i \circ \varphi} \theta$ (forward parametrization does not change line integrals). Because M is non-compact, we find a proper map $\overline{\gamma}\colon [0,\infty[\to \overline{P}$ such that $\gamma(0) = \overline{x}_0$ and in a local chart all derivatives vanish in 0 (see [Nee04, Lemma IV. 5] and composition with a smooth bijection of $[0,\infty[$ that's jet vanishes in 0). For $i \in \{0,1\}$, we define the smooth map

$$\overline{\alpha}_i^{\mathbb{R}}\colon \mathbb{R} \to \overline{P},\ t \mapsto \begin{cases} \overline{\gamma}(-t) & : t < 0 \\ \overline{\alpha}_i(t) & : t \in [0,1] \\ \overline{\gamma}(t-1) & : t > 1. \end{cases}$$

Moreover, we define the smooth homotopy

$$\overline{F}^{\mathbb{R}}\colon [0,1] \times \mathbb{R} \to \overline{P},\ (s,t) \mapsto \begin{cases} \overline{\gamma}(-t) & : t < 0 \\ \overline{F}(s,t) & : t \in [0,1] \\ \overline{\gamma}(t-1) & : t > 1. \end{cases}$$

Hence, we have $\overline{\alpha}_i^{\mathbb{R}}, \overline{F}^{\mathbb{R}}(s,\bullet) \in C^\infty_p(\mathbb{R},\overline{P})$ for $i \in \{0,1\}$ and $s \in [0,1]$ (see Remark 4.32). We define $\alpha_i := \overline{q} \circ \overline{\alpha}_i$, $F := \overline{q} \circ \overline{F}$, $\alpha_i^{\mathbb{R}} := \overline{q} \circ \overline{\alpha}_i^{\mathbb{R}}$, $F^{\mathbb{R}} := \overline{q} \circ \overline{F}^{\mathbb{R}}$, $\gamma := \overline{q} \circ \overline{\gamma}$ and $x_0 := \overline{x}_0$. The curves α_0 and α_1 are closed curves in x_0 and are homotopic relative

$\{0,1\}$ by the homotopy F, because $\alpha_i(j) = \overline{q}(\overline{x}_0) = x_0$ and $F(i,\bullet) = \overline{q} \circ \overline{F}(i,\bullet) = \overline{q} \circ \overline{\alpha}_i = \alpha_i$ for $j,i \in \{0,1\}$. Moreover,

$$\alpha_i^{\mathbb{R}}(t) = \begin{cases} \gamma(-t) & : t < 0 \\ \alpha_i(t) & : t \in [0,1] \\ \gamma(t-1) & : t > 1, \end{cases}$$

$$F^{\mathbb{R}}(s,t) = \begin{cases} \gamma(-t) & : t < 0 \\ F(s,t) & : t \in [0,1] \\ \gamma(t-1) & : t > 1 \end{cases}$$

and $\alpha_i^{\mathbb{R}}, F^{\mathbb{R}}(s,\bullet) \in C_p^\infty(\mathbb{R}, M)$. We choose $p_0 \in \pi^{-1}(\{\overline{x}_0\})$. Now, let $\hat{\alpha}_i^{\mathbb{R}} \colon \mathbb{R} \to P$ be the unique horizontal lift of $\alpha_i^{\mathbb{R}}$ to P with $\hat{\alpha}_i^{\mathbb{R}}(0) = p_0$ and $\hat{F}^{\mathbb{R}} \colon [0,1] \times \mathbb{R} \to P$ be the unique horizontal lift of $F^{\mathbb{R}}$ to P such that $\hat{F}^{\mathbb{R}}(s,0) = p_0$ for all $s \in [0,1]$. The map $\hat{F}^{\mathbb{R}}$ is not a homotopy relative $\{0,1\}$ but we have $\hat{F}^{\mathbb{R}}(0,s) = \hat{\alpha}_0^{\mathbb{R}}(s)$ and $\hat{F}^{\mathbb{R}}(1,s) = \hat{\alpha}_1^{\mathbb{R}}(s)$ for all $s \in [0,1]$. For $i \in \{0,1\}$, we have

$$\int_{\overline{\alpha}_i} \theta = \int_{\overline{\alpha}_i^{\mathbb{R}}} \theta = \int_{\hat{\alpha}_i^{\mathbb{R}}} \pi^*\theta, \tag{4.8}$$

where the last equation follows from Lemma 4.33. Because of (4.8) and Lemma 4.28 it is enough to show that $\pi_2((\hat{\alpha}_0^{\mathbb{R}})^*) = \pi_2((\hat{\alpha}_1^{\mathbb{R}})^*)$ holds as group homomorphisms from $\pi_2(C_c^\infty(P,G)_{\rho_G})$ to $\pi_2(C_c^\infty(\mathbb{R},G))$. From [Nee04, Theorem A.7], we get $\pi_2(C_c^\infty(\mathbb{R},G)) = \pi_2(C_c(\mathbb{R},G))$. We set $I := [0,1]$. Let $\sigma \colon I^2 \to C_c^\infty(P,G)_{\rho_G}$ be continuous with $\sigma|_{\partial I^2} = c_{1_G}$. Because $\pi_2((\hat{\alpha}_i^{\mathbb{R}})^*)([\sigma]) = [\sigma(\bullet) \circ \hat{\alpha}_i^{\mathbb{R}}]$ for $i \in \{0,1\}$, it is enough to show that

$$[\sigma(\bullet) \circ \hat{\alpha}_0^{\mathbb{R}}] = [\sigma(\bullet) \circ \hat{\alpha}_1^{\mathbb{R}}]$$

in $\pi_2(C_c(\mathbb{R},G))$. Hence, we have to construct a continuous map $H \colon [0,1] \times I^2 \to C_c(\mathbb{R},G)$ with $H(0,\bullet) = \sigma(\bullet) \circ \hat{a}_0^{\mathbb{R}}$, $H(1,\bullet) = \sigma(\bullet) \circ \hat{\alpha}_1^{\mathbb{R}}$ and $H(s,x) = c_{1_G}$ for all $s \in [0,1]$ and $x \in \partial I^2$. We define $H(s,x) = \sigma(x) \circ \hat{F}^{\mathbb{R}}(s,\bullet)$ for $s \in [0,1]$ and $x \in I^2$. Because $\sigma|_{\partial I^2} = c_{1_G}$, it is left to show that H is continuous. Let $K \subseteq M$ be compact such that $\operatorname{im}(\sigma) = \sigma(I^2) \subseteq C_K^\infty(P,G)_{\rho_G}$ (see Lemma 4.34). For $f \in C_K^\infty(P,G)_{\rho_G}$ we have $\operatorname{supp}(f \circ \hat{\alpha}_i^{\mathbb{R}}) \subseteq {\alpha_i^{\mathbb{R}}}^{-1}(K)$ as well as $\operatorname{supp}(f \circ \hat{F}^{\mathbb{R}}(s,\bullet)) \subseteq F^{\mathbb{R}}(s,\bullet)^{-1}(K)$ for $s \in [0,1]$. Hence, $\operatorname{supp}(\sigma(x) \circ \hat{F}^{\mathbb{R}}(s,\bullet)) \subseteq F^{\mathbb{R}}(s,\bullet)^{-1}(K)$ for $x \in I^2$ and $s \in [0,1]$. We have

$$\begin{aligned} {F^{\mathbb{R}}}^{-1}(K) &= F^{\mathbb{R}}|_{[0,1]\times[0,1]}^{-1}(K) \cup F^{\mathbb{R}}|_{[0,1]\times]-\infty,0]}^{-1}(K) \cup F^{\mathbb{R}}|_{[0,1]\times[1,\infty[}^{-1}(K) \\ &= F^{\mathbb{R}}|_{[0,1]\times[0,1]}^{-1}(K) \cup ([0,1] \times -\gamma^{-1}(K)) \cup ([0,1] \times \gamma^{-1}(K)+1). \end{aligned}$$

Thus ${F^{\mathbb{R}}}^{-1}(K) \subseteq [0,1] \times \mathbb{R}$ is compact. Therefore,

$$L := \bigcup_{s\in[0,1]} F^{\mathbb{R}}(s,\bullet)^{-1}(K) = \operatorname{pr}_2({F^{\mathbb{R}}}^{-1}(K)) \subseteq \mathbb{R}$$

is compact. We have $\operatorname{supp}(\sigma(x) \circ \hat{F}^{\mathbb{R}}(s, \bullet)) \subseteq L$ for all $x \in I^2$ and $s \in [0,1]$. Thus $\operatorname{im}(H) \subseteq C_L(\mathbb{R}, G)$. Therefore, it is enough to show that $H \colon [0,1] \times I^2 \to C_L(\mathbb{R}, G) \subseteq C(\mathbb{R}, G),\ (s,x) \mapsto \sigma(x) \circ \hat{F}^{\mathbb{R}}(s, \bullet)$ is continuous. We know that $\tau \colon [0,1] \to C(\mathbb{R}, P),\ s \mapsto \hat{F}^{\mathbb{R}}(s, \bullet)$ is continuous and so the assertion follows from the following commutative diagram

$$\begin{array}{ccc}
[0,1] \times I^2 & \xrightarrow{\tau \times \sigma} & C(\mathbb{R}, P) \times C^\infty_K(P, G)_{\rho G} \\
 & \searrow^{H} & \big\downarrow \\
 & & C(\mathbb{R}, P) \times C(P, G) \\
 & & \big\downarrow {\scriptstyle (\alpha, f) \mapsto f \circ \alpha} \\
 & & C(\mathbb{R}, G).
\end{array}$$

□

The following theorem corresponds to the Reduction Theorem [NW09, Theorem 4.14] (where the base manifold M is compact but the principal bundle P and the Lie group G can be infinite-dimensional).

Theorem 4.36. *The period group* $\Pi_{\omega_M} = \operatorname{im} \operatorname{per}_{\omega_M}$ *is discrete in* $\overline{\Omega}^1_c(M, \mathbb{V})$.

Proof. Because $q^* \colon H^1_{dR,c}(M, V_{\text{fix}}) \to H^1_{dR,c}(P, V)_{\rho V}$ is an isomorphism of topological vector spaces and $\Pi_{\omega_M} \subseteq H^1_{dR,c}(M, \mathbb{V}) = H^1_{dR,c}(P, V)_{\rho V}$, it is sufficient to show that Π_{ω_M} is a discrete subgroup of $H^1_{dR,c}(M, V)$ (Lemma 4.29). With Lemma 4.30 and Lemma 4.31, it is enough to show that

$$\Pi_{\omega_M} \subseteq H^1_{dR,c}(M, \Pi_{\mathbb{R}}). \tag{4.9}$$

Let $\beta \in \Pi_{\omega_M}$, $\alpha \in C^\infty_p(\mathbb{R}, M)$ and $[\sigma] \in \pi_2(C^\infty_c(P, G)_{\rho G})$ with $\beta = \operatorname{per}_{\omega_M}([\sigma])$. Using Lemma 4.29 and Lemma 4.28, we get

$$\int_{\mathbb{R}} \alpha^* q_* \beta = \int_{\mathbb{R}} \hat{\alpha}^* \beta = I_\alpha \circ \operatorname{per}_{\omega_M}([\sigma]) = \operatorname{per}_{\omega_{\mathbb{R}}} \circ \pi_2(\hat{\alpha}^*)([\sigma]) \in \Pi_{\omega_{\mathbb{R}}}.$$

Hence (4.9) follows. □

4.2. Integration of the Lie algebra action and the main result

In the case of a compact base manifold ([NW09, Section 4.2 (part about general Lie algebra bundles)]) Neeb and Wockel integrated the adjoint action of $\Gamma(\mathfrak{G})$ on $\widehat{\Gamma(\mathfrak{G})} := \overline{\Omega}^1(M, \mathbb{V}) \times_\omega \Gamma(\mathfrak{G})$ given by

$$\Gamma(\mathfrak{G}) \times \widehat{\Gamma(\mathfrak{G})} \to \widehat{\Gamma(\mathfrak{G})},\ (\eta, ([\alpha], \gamma)) \mapsto [\eta, ([\alpha], \gamma)]_\omega = (\omega(\eta, \gamma), [\eta, \gamma])$$

to a Lie group action of $\Gamma(\mathcal{G})$ on $\widehat{\Gamma(\mathfrak{G})}$.[12] As a first step in their proof, Neeb and Wockel integrated the covariant derivative $d_{\mathfrak{G}}\colon \Gamma(\mathfrak{G}) \to \Omega^1(M,\mathfrak{G})$ to a smooth map from $\Gamma(\mathcal{G})$ to $\Omega^1(M,\mathfrak{G})$. Since the absolute derivative is the sum of $d\colon C^\infty(P,\mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega^1(P,\mathfrak{g})$ and $C^\infty(P,\mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega^1(P,\mathfrak{g})$, $f \mapsto \rho_*(Z) \wedge f$, where $Z\colon TP \to L(H) =: \mathfrak{h}$ is the connection form, $\rho_* = L(\rho_\mathfrak{g})\colon \mathfrak{h} \to \mathrm{der}(\mathfrak{g})$ and $(\rho_*(Z) \wedge f)_p(v) = \rho_*(Z(v)).f(p)$, they integrated these summands separately. The image of the exterior derivative does not lie in $\Omega^1(M,\mathfrak{g})^{\mathrm{hor}}_\mathfrak{g}$ but in the space $\Omega^1(M,\mathfrak{g})_\mathfrak{g}$ and in some sense the summand $f \mapsto \rho_*(Z) \wedge f$ annihilates the vertical parts of df. The exterior derivative $d\colon C^\infty(P,\mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega^1(M,\mathfrak{g})_\mathfrak{g}$ integrates to the left logarithmic derivative $\delta\colon C^\infty(P,G)_{\rho_G} \to \Omega^1(M,\mathfrak{g})_\mathfrak{g}$, $\varphi \mapsto \delta(\varphi)$ with $\delta(\varphi)_p(v) = T\lambda_{\varphi(p)^{-1}} \circ T\varphi(v)$ ([NW09]). The integration of the second summand is more complicated, and Neeb and Wockel assumed the Lie group G to be 1-connected (in the special case of the gauge group they did not need this assumption (see [NW09, Theorem 4.21])). In the second step they used an exponential law to obtain the integrated action. Because our base manifold is not compact, the adjoint action of $\Gamma_c(\mathfrak{G})$ on $\widehat{\Gamma_c(\mathfrak{G})} := \overline{\Omega}^1_c(M,\mathbb{V}) \times_{\omega_M} \Gamma_c(\mathfrak{G})$ is given by

$$\Gamma_c(\mathfrak{G}) \times \widehat{\Gamma_c(\mathfrak{G})} \to \widehat{\Gamma_c(\mathfrak{G})},\ (\eta,([\alpha],\gamma)) \mapsto (\omega_M(\eta,\gamma),[\eta,\gamma]).$$

With the canonical identifications (see Remark 4.17) the adjoint action has the form

$$\begin{aligned} C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}} \times (\overline{\Omega}^1_c(P,V)^{\mathrm{hor}}_{\rho_V} \times_{\omega_M} C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}}) \to \overline{\Omega}^1_c(P,V)^{\mathrm{hor}}_{\rho_V} \times_{\omega_M} C^\infty_c(P,\mathfrak{g})_{\rho_\mathfrak{g}} \\ (g,([\alpha],f)) \mapsto ([\kappa_\mathfrak{g}(g,D_{\rho_\mathfrak{g}}(f))],\mathrm{ad}(g,f)). \end{aligned} \tag{4.10}$$

We have to integrate this action to a Lie group action of $(\Gamma_c(\mathcal{G}))_0$ on $\widehat{\Gamma_c(\mathfrak{G})}$. Like Neeb and Wockel, we have to integrate the covariant derivative $d_{\mathfrak{G}}\colon \Gamma_c(\mathfrak{G}) \to \Omega^1_c(M,\mathfrak{G})$ to a smooth map from $\Gamma_c(\mathcal{G})$ to $\Omega^1_c(M,\mathfrak{G})$. But we will not describe the absolute derivative via the connection form Z as the sum of the exterior derivative d and the map $f \mapsto \rho_*(Z) \wedge f$. Instead, we use the principal connection HP and write $D_{\rho_\mathfrak{g}} = \mathrm{pr}_h^* \circ d$, where pr_h is the projection onto the horizontal bundle and $(\mathrm{pr}_h^* \circ d)(f)(v) = df(\mathrm{pr}_h(v))$. In Theorem 4.51, we show that the map $\Delta := \mathrm{pr}_h^* \circ \delta\colon C^\infty_c(P,G) \to \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ is smooth and its derivative in 1 is given by the absolute derivative $D_{\rho_\mathfrak{g}}$. One could show the smoothness of $\delta\colon C^\infty_c(P,G)_{\rho_G} \to \Omega^1_c(P,\mathfrak{g})_\mathfrak{g}$ and $\mathrm{pr}_h^*\colon \Omega^1_c(P,\mathfrak{g})_\mathfrak{g} \to \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_\mathfrak{g}$ separately but it is more convenient to show the smoothness of Δ directly because we work in the non-compact case and $\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ is an inductive limit (compare Lemma 4.47).

Remark 4.37. *In [NW09, Chapter 4.2 page 408] Neeb and Wockel define* $\chi^Z(f)v := \chi(Z(v),(f(p))$ *for* $f \in \Gamma\mathcal{G} = C^\infty(P,G)_{\rho_G}$, $v \in T_pP$, Z *the connection form of* P *and a smooth map* $\chi\colon \mathfrak{h} \times G \to \mathfrak{g}$ *that is linear in the first argument. If the connection on* P *is not trivial then* $TP \to \mathfrak{g}$, $v \mapsto \chi(Z(v),(f(p)))$ *is not in* $\Omega^1(M,\mathfrak{G}) = \Omega^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ *because it is not horizontal unless it is constantly* 0. *How-*

[12] The Lie algebra structure on $V \times_\omega \mathfrak{g}$ for a continuous cocycle $\omega\colon \mathfrak{g}^2 \to V$ was recalled in the introduction.

ever, the image of the map $\delta^\nabla(f) = \delta(f) + \chi^Z(f^{-1})$ *lies in* $\Omega^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ *because the image of its derivative in* 1 *lies in* $\Omega^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ *and* δ^∇ *is a* 1*-cocycle with respect to the adjoint action of* $C^\infty(P,G)_{\rho_G}$ *on* $\Omega^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}$ *and* $\Omega^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ *is invariant under this action.*

Lemma 4.38. *Let* V *be a locally convex space and* G *a locally convex Lie group. Moreover, let* $\mu\colon G\times V\to V$ *be a map that is continuous linear in the second argument. Let* $f\colon G\to V$ *be a map that is smooth on a* 1*-neighbourhood. If* $f(hg) = f(g) + \mu(g^{-1}, f(h))$ *for* $g,h\in G$*, then* f *is smooth.*

Proof. Let $U\subseteq G$ be a 1-neighbourhood such that $f|_U$ is smooth and $g\in G$. Then Ug is a g-neighbourhood and given $z\in Ug$, we define $h := zg^{-1}\in U$. Hence $z = hg$. Now, we calculate

$$\begin{aligned} f(z) &= f(hg) = f(g) + \mu(g^{-1}, f(h)) = f(g) + \mu(g^{-1}, f(zg^{-1}))\\ &= f(g) + \mu(g^{-1}, f|_U\circ \varrho_{g^{-1}}(z)). \end{aligned}$$

Thus

$$f|_{Ug} = f(g) + \mu(g^{-1},\bullet)\circ f|_U\circ\varrho_{g^{-1}}|_{Ug}.$$

□

Lemma 4.39. *We consider the map*

$$\mu\colon C^\infty_c(P,G)_{\rho_G}\times\Omega^1_c(P,\mathfrak{g})\to\Omega^1_c(P,\mathfrak{g}),\ (\varphi,\theta)\to \mathrm{Ad}^G_\varphi.\theta$$

with $\mathrm{Ad}^G_\varphi.\theta\colon TP\to\mathfrak{g},\ v\mapsto \mathrm{Ad}^G_{\varphi(\pi(v))}.\theta(v)$ *and the canonical projection* $\pi\colon TP\to P$*. The subspace* $\Omega^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ *is* μ*-invariant. Here* Ad^G *denotes the adjoint action of* G *on* $\mathfrak{g}$*.*

Proof. Given $\theta\in\Omega^1_c(P,\mathfrak{g})_{\rho_\mathfrak{g}}$ and $\varphi\in C^\infty_c(P,G)_{\rho_G}$, we show that $\mu(\varphi,\theta)\in\Omega^1_c(P,\mathfrak{g})_{\rho_\mathfrak{g}}$. Let $h\in H$, $p\in P$ and $v\in T_pP$. We calculate

$$\begin{aligned} (R_h^*\mu(\varphi,\theta))_p(v) &= \mathrm{Ad}^G(\varphi(ph),\theta_{ph}(TR_h(v))) = \mathrm{Ad}^G(\rho_G(h^{-1}).\varphi(p),\rho_\mathfrak{g}(h^{-1}).\theta_p(v))\\ &= T\lambda_{\rho_G(h^{-1}).\varphi(p)}\circ T\varrho_{\rho_G(h^{-1}).\varphi(p)^{-1}}\circ T_1\rho_G(h^{-1})(\theta_p(v))\\ &= T_1(\rho_G(h^{-1})(\varphi(p))\cdot\rho_G(h^{-1})(\bullet)\cdot\rho_G(h^{-1})(\varphi(p)^{-1}))(\theta_p(v))\\ &= T_1(\rho_G(h^{-1})\circ I_{\varphi(p)})(\theta_p(v)) = \rho_\mathfrak{g}(h^{-1})\circ\mathrm{Ad}^G_{\varphi(p)}(\theta_p(v)), \end{aligned}$$

where $I_{\varphi(p)}(g) = \varphi(p)g\varphi(p)^{-1}$ is the conjugation on G. Obviously $\mu(\varphi,\theta)$ is horizontal if θ is so. □

Definition 4.40. We define the map

$$\mathrm{Ad}^G_*\colon C^\infty_c(P,G)^{\mathrm{hor}}_{\rho_G}\times\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}\to\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}},\ (\varphi,\theta)\to\mathrm{Ad}^G_\varphi.\theta$$

with $\mathrm{Ad}^G_\varphi.\theta\colon TP\to\mathfrak{g},\ v\mapsto\mathrm{Ad}^G_{\varphi\circ\pi(v)}.\theta(v)$ and the canonical projection $\pi\colon TP\to P$.

Lemma 4.41. *The map* $\mathrm{Ad}_*^G \colon C_c^\infty(P,G)_{\rho_G} \times \Omega_c^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}^{\mathrm{hor}} \to \Omega_c^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}^{\mathrm{hor}}$ *is continuous linear in the second argument.*

Proof. Let $\varphi \in C_c^\infty(P,G)_{\rho_G}$ and $K \subseteq M$ be compact. It is enough to show that

$$\mathrm{Ad}_*^G(\varphi,\bullet) \colon \Omega_K^1(P,\mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega_K^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}$$

is continuous, because $\mathrm{Ad}_*^G(\varphi,\bullet)$ is linear and $\mathrm{Ad}_*^G(\varphi,\bullet)(\Omega_K^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}) \subseteq \Omega_K^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}$. The map $f \colon TP \times \mathfrak{g} \to \mathfrak{g}$, $(v,w) \mapsto \mathrm{Ad}^G(\varphi \circ \pi(v), w)$ is smooth. We know that

$$f_* \colon C^\infty(TP,\mathfrak{g}) \to C^\infty(TP,\mathfrak{g}),\ \theta \mapsto f \circ (\mathrm{id},\theta)$$

is continuous (see e.g. [GN]). We can embed $\Omega_K^1(P,\mathfrak{g})$ into $C^\infty(TP,\mathfrak{g})$. Hence we are done.

□

Definition 4.42. Let $\pi \colon TP \to P$ be the canonical projection and $\mathrm{pr}_h \colon TP \to HP$ the projection onto the horizontal bundle.

(a) We define

$$\delta \colon C_c^\infty(P,G)_{\rho_G} \to \Omega^1(P,\mathfrak{g}),\ \varphi \mapsto \delta(\varphi)$$

with $\delta\varphi(v) = T\lambda_{\varphi(\pi(v))^{-1}} \circ T\varphi(v)$ for $v \in TP$ (cf. [KM97, 38.1]).

(b) We define

$$\mathrm{pr}_h^* \colon \Omega^1(P,\mathfrak{g}) \to \Omega^1(P,\mathfrak{g})^{\mathrm{hor}}, \theta \mapsto \theta \circ \mathrm{pr}_h\,.$$

The statement (b) in the following lemma is well-known and can be found in [KM97, p. 38.1].

Lemma 4.43. *(a) We have*

$$\delta(C_c^\infty(P,G)_{\rho_G}) \subseteq \Omega_c^1(P,\mathfrak{g})_{\rho_\mathfrak{g}}.$$

(b) Given $f, g \in C_c^\infty(P,G)_{\rho_G}$, *we have*

$$\delta(f \cdot g) = \delta(g) + \mathrm{Ad}_*^G(g^{-1}, \delta(f)).$$

Proof. (a) Let $\varphi \in C_c^\infty(P,G)_{\rho_G}$, $h \in H$, $p \in P$ and $w \in T_pP$. We calculate

$$\begin{aligned}&(R_h^*\delta(\varphi))_p(w) = \delta(\varphi)_{ph}(TR_h(w)) = T\lambda_{\varphi(ph)^{-1}}(T\varphi(TR_h(w)))\\ =&T(\lambda_{\varphi(ph)^{-1}} \circ \varphi \circ R_h)(w) =: \dagger.\end{aligned}$$

For $x \in P$, we have

$$\begin{aligned}&\lambda_{\varphi(ph)^{-1}} \circ \varphi \circ R_h(x) = (\rho_G(h^{-1}).\varphi(p))^{-1} \cdot (\rho_G(h^{-1}).\varphi(x))\\ =&\rho_G(h^{-1}).(\varphi(p)^{-1} \cdot \varphi(x)) = \rho_G(h^{-1}) \circ \lambda_{\varphi(p)^{-1}} \circ \varphi(x).\end{aligned}$$

We conclude that

$$\dagger = \rho_{\mathfrak{g}}(h^{-1}) \circ T\lambda_{\varphi(p)^{-1}} \circ T\varphi(w) = \rho_{\mathfrak{g}}(h^{-1}) \circ \delta(\varphi)_p(w).$$

(b) The assertion follows directly from [KM97, p. 38.1].

□

Definition 4.44. Let $\mathrm{pr}_h \colon TP = VP \oplus HP \to HP$, be the projection onto the horizontal bundle HP. We define

$$\Delta \colon C^\infty_c(P, G)_{\rho_G} \to \Omega^1(P, \mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}}, \varphi \mapsto \mathrm{pr}_h^* \circ \delta(\varphi) = \delta(\varphi) \circ \mathrm{pr}_h .$$

As in [Sch13], we use the concept of weak direct products of infinite-dimensional Lie groups described in [Glo03, Section 7] respectively [Glo07, Section 4] in the following considerations. The next lemma is basically [Sch13, Corollary 2.38] but with modified assumptions.

Lemma 4.45. *For $i \in \mathbb{N}$ let G_i be a locally convex Lie group, E_i a locally convex space and $f_i \colon G_i \to E_i$ be a smooth map such that $f_i(1) = 0$. In this situation there exists an open 1-neighbourhood $U \subseteq \prod^*_{i\in\mathbb{N}} G_i$ such that the map $f \colon \prod^*_{i\in\mathbb{N}} G_i \to \bigoplus_{i\in\mathbb{N}} E_i$, $(g_i)_i \mapsto (f_i(g_i))_i$ is smooth on U.*

Proof. Given $i \in \mathbb{N}$, let $\varphi_i \colon U_i \subseteq G_i \to V_i \subseteq \mathfrak{g}_i$ be a chart around 1 with $\varphi_i(1) = 0$. We have the commutative diagram

$$\begin{array}{ccc}
\prod^*_{i\in\mathbb{N}} U_i & \xrightarrow{\;f|_{\prod^*_{i\in\mathbb{N}} U_i} = (f_i|_{U_i})_{i\in\mathbb{N}}\;} & \bigoplus_{i\in\mathbb{N}} E_i \\
{\scriptstyle (\varphi_i)_{i\in\mathbb{N}}}\downarrow & \nearrow{\scriptstyle (f_i\circ\varphi_i^{-1})_{i\in\mathbb{N}}} & \\
\bigoplus_{i\in\mathbb{N}} V_i. & &
\end{array}$$

Now the assertion follows from [Glo03, Proposition 7.1]. □

Remark 4.46. *Let $(\overline{V}_i, \sigma_i)_{i\in\mathbb{N}}$ be a compact locally finite trivializing system of the principal bundle $H \hookrightarrow P \xrightarrow{q} M$ in the sense of [Sch13, Definition 3.6] respectively [Woc07]. We follow [Sch13, Remark 3.5] respectively [Woc07] and define as usual the smooth map $\beta_{\sigma_i} \colon q^{-1}(\overline{V}_i) \to H$ by the equation $\sigma_i(q(p)) \cdot \beta_{\sigma_i}(p) = p$ for all $p \in q^{-1}(\overline{V}_i)$. Obviously, we have $\beta_{\sigma_i}(ph) = \beta_{\sigma_i}(p) \cdot h$ for all $h \in H$. Moreover, we define the smooth cocycle $\beta_{i,j} \colon \overline{V}_i \cap \overline{V}_j \to H$ by the equation $\sigma_i(x) \cdot \beta_{i,j}(x) = \sigma_j(x)$. We have $\beta_{i,j}(x)^{-1} = \beta_{j,i}(x)$ and $\beta_{\sigma_i}(p)^{-1} \cdot \beta_{i,j}(q(p)) = \beta_{\sigma_j}(p)^{-1}$ for $p \in q^{-1}(\overline{V}_i \cap \overline{V}_j)$.*

The proof of the following lemma is similar to the proof of [Sch13, Proposition 4.6] where, beside other results, Schütt constructed a topological embedding from the compactly supported gauge algebra $\mathrm{gau}_c(P, \mathfrak{g})_{\mathfrak{g}}$ to a direct sum $\bigoplus_{i\in\mathbb{N}} C^\infty(\overline{V}_i, \mathfrak{g})$ of locally convex spaces. However, the following lemma differs from [Sch13, Proposition 4.6] because we deal with horizontal differential forms which need some additional considerations.

Lemma 4.47. *Let $(\overline{V_i}, \sigma_i)_{i\in\mathbb{N}}$ be a compact locally finite trivializing system in the sense of [Sch13, Definition 3.6]. The map*

$$\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}} \to \bigoplus_{i\in\mathbb{N}} \Omega^1(\overline{V}_i,\mathfrak{g}),\ \theta \mapsto (\sigma_i^*\theta)_{i\in\mathbb{N}}$$

is a topological embedding.

Proof. We define

$$\Omega_\oplus := \left\{ (\eta_i)_i \in \bigoplus_{i\in\mathbb{N}} \Omega^1(\overline{V}_i,\mathfrak{g}) : (\eta_i)_x = \rho_\mathfrak{g}(\beta_{i,j}(x)) \circ (\eta_j)_x \text{ for } x \in \overline{V}_i \cap \overline{V}_j \right\}$$

and the map

$$\Phi\colon \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}} \to \bigoplus_{i\in\mathbb{N}} \Omega^1(\overline{V}_i,\mathfrak{g}),\ \theta \mapsto (\sigma_i^*\theta)_{i\in\mathbb{N}}.$$

First we show that $\mathrm{im}(\Phi) \subseteq \Omega_\oplus$. For $x \in \overline{V}_i \cap \overline{V}_j, v \in T_xM$, we have $\sigma_j(x) = \sigma_i(x)\beta_{i,j}(x)$ and

$$Tq(T\sigma_j(v) - T(R_{\beta_{i,j}(x)} \circ \sigma_i)(v)) = T(q\circ\sigma_j)(v) - T(q \circ R_{\beta_{i,j}(x)} \circ \sigma_i)(v) = 0. \tag{4.11}$$

Because θ is $\rho_\mathfrak{g}$ invariant and horizontal, we can calculate

$$\begin{aligned}(\sigma_i^*\theta)_x(v) &= \theta_{\sigma_i(x)}(T\sigma_i(v)) = \rho_\mathfrak{g}(\beta_{i,j}(x)) \circ \theta_{\sigma_i(x)\beta_{i,j}(x)}(TR_{\beta_{i,j}(x)}(T\sigma_i(v))) \\ &\underset{(4.11)}{=} \rho_\mathfrak{g}(\beta_{i,j}(x)) \circ \theta_{\sigma_j(x)}(T\sigma_j(v)) = \rho_\mathfrak{g}(\beta_{i,j}(x)) \circ \sigma_j^*\theta_x(v).\end{aligned}$$

Analogously to [Sch13, Proposition 4.6], we can argue as follows: The map Φ is linear, $\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}} = \varinjlim \Omega^1_K(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ and $(\overline{V}_i)_i$ is locally finite, whence the map Φ is continuous. Now, let $(\lambda_i)_{i\in\mathbb{N}}$ be a partition of unity of M subordinate to $(V_i)_i$. Given $\eta \in \Omega^1(\overline{V}_i,\mathfrak{g})$, we define $\widetilde{\lambda_i\eta} \in \Omega^1(P,\mathfrak{g})$ by

$$\widetilde{\lambda_i\eta}_p(w) := \begin{cases} \lambda_i(q(p)) \cdot \rho_\mathfrak{g}(\beta_{\sigma_i}(p)^{-1}).\eta_{q(p)}(Tq(w)) & : p \in q^{-1}(V_i) \\ 0 & : \text{ else.}\end{cases}$$

With Remark 4.46, we get $\widetilde{\lambda_i\eta} \in \Omega^1_{\mathrm{supp}(\lambda_i)}(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ and $\sum_{i\in\mathbb{N}} \widetilde{\lambda_i\eta_i} \in \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ for $(\eta_i)_i \in \bigoplus_{i\in\mathbb{N}} \Omega^1(\overline{V}_i,\mathfrak{g})$. The map $\Psi\colon \bigoplus_{i\in\mathbb{N}} \Omega^1(\overline{V}_i,\mathfrak{g}) \to \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$, $(\eta_i)_i \mapsto \sum_{i\in\mathbb{N}} \widetilde{\lambda_i\eta_i}$ is continuous because it is linear and the inclusions $\Omega^1_{\mathrm{supp}(\lambda_i)}(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}} \hookrightarrow \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$ are continuous. Let $(\eta_i)_i \in \Omega_\oplus$. As in [Sch13, Proposition 4.6], we get

$$\Psi((\eta_i)_i)_p(w) = \rho_\mathfrak{g}(\beta_{\sigma_{i_0}}(p)^{-1}).(\eta_{i_0})_{q(p)}(Tq(w))$$

if $p \in q^{-1}(V_{i_0})$ and $w \in T_pP$. By an abuse of notation, we write $\Phi := \Phi|^{\Omega_\oplus}$ and $\Psi := \Psi|_{\Omega_\oplus}$. One easily sees $\Phi\circ\Psi = \mathrm{id}_{\Omega_\oplus}$. It is left to show that $\Psi\circ\Phi = \mathrm{id}_{\Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}}$.

Let $\theta \in \Omega^1_c(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_\mathfrak{g}}$, $p \in P$, $w \in T_pP$ and $i \in \mathbb{N}$ with $p \in q^{-1}(V_i)$. We calculate

$$Tq(w - TR_{\beta_{\sigma_i}(p)}T\sigma_i Tq(w)) = Tq(w) - T(q \circ R_{\beta_{\sigma_i}(p)} \circ \sigma_i)(Tq(w)) = 0. \tag{4.12}$$

Now, we get

$$\begin{aligned}
&\Psi \circ \Phi(\theta)_p(w) = \rho_\mathfrak{g}(\beta_{\sigma_i}(p)^{-1}).(\sigma_i^*\theta)_{q(p)}(Tq(w))\\
=&\rho_\mathfrak{g}(\beta_{\sigma_i}(p)^{-1}).\theta_{\sigma_i(q(p))}(T\sigma_i Tq(w))\\
=&\rho_\mathfrak{g}(\beta_{\sigma_i}(p)^{-1}).\rho_\mathfrak{g}(\beta_{\sigma_i}(p))\theta_{\sigma_i(q(p))\beta_{\sigma_i}(p)}(TR_{\beta_{\sigma_i}(p)}T\sigma_i Tq(w))\\
\underset{4.12}{=}&\theta_{\sigma_i(q(p))\beta_{\sigma_i}(p)}(w) = \theta_p(w).
\end{aligned}$$

□

Remark 4.48. *Let M be an m-dimensional manifold, $D \subseteq TM$ a d-dimensional subbundle, $p_0 \in M$ and $w_0 \in D_{p_0}$. Then there exists a smooth curve $\gamma\colon [-1,1] \to M$ such that $\gamma(0) = p_0$, $\gamma'(0) = w_0$ and $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [-1,1]$. In fact let $\psi\colon TU \to U \times \mathbb{R}^m$ be a trivialisation with $\psi(D) = U \times \mathbb{R}^d \times \{0\}$ and $v_0 := \mathrm{pr}_2 \circ \psi(w_0) \in \mathbb{R}^d \times \{0\}$. Then $X\colon U \to TU$, $x \mapsto \psi^{-1}(x, v_0)$ is a smooth vector field on U and $\mathrm{im}(X) \subseteq D$. Let $\tilde\gamma\colon [-\varepsilon,\varepsilon] \to U$ be the integral curve of X with $\tilde\gamma(0) = p_0$. Then $\tilde\gamma'(0) = X(p_0) = w_0$ and obviously $\gamma'(t) \in D_{\gamma(t)}$ for all t. Now let $\varphi\colon [-1,1] \to [-\varepsilon,\varepsilon]$ be a diffeomorphism with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then $\gamma := \tilde\gamma \circ \varphi$ is as needed.*

Lemma 4.49. *The pullbacks $\gamma^*\colon \Omega^1(P,\mathfrak{g})^{\mathrm{hor}} \to C^\infty([-1,1],\mathfrak{g})$, $\theta \mapsto \gamma^*\theta$ along horizontal maps $\gamma\colon [-1,1] \to P$ ($\gamma'(t) \in H_{\gamma(t)}P$) separate the points in $\Omega^1(P,\mathfrak{g})^{\mathrm{hor}}$.*

Proof. Let $\theta \in \Omega^1(P,\mathfrak{g})^{\mathrm{hor}}$ and $\gamma^*\theta = 0$ for all horizontal curves $\gamma\colon [-1,1] \to P$. Let $p \in P$ and $w \in T_pP$. We show that $\theta_p(w) = 0$. Because θ is horizontal, we can assume that $w \in H_pP$. We use Remark 4.48 and find a horizontal curve $\gamma\colon [-1,1] \to P$ with $\gamma'(0) = w$. Hence $\theta_p(w) = \theta_p(\gamma'(0)) = \gamma^*\theta(0) = 0$. □

One can easily deduce the following observation from [Woc07, Theorem 1.11] but in the special case of a current group on a compact interval, an easier argument becomes possible.

Remark 4.50. *Let G be a finite-dimensional Lie group and $(U_i)_{i=1,\dots,n}$ be an open cover of the space $[-1,1]$ such that the sets $\overline{U}_i$ are submanifolds with boundary. Then the map $\Phi\colon C^\infty([-1,1],G) \to \prod_{i=1}^n C^\infty(\overline{U}_i,G)$, $\phi \mapsto (\phi|_{\overline{U}_i})_i$ is an injective Lie group morphism whose image is a Lie subgroup of $\prod_{i=1}^n C^\infty(\overline{U}_i,G)$ and $\Phi|^{\mathrm{im}(\Phi)}$ is an isomorphism of Lie groups. We define $\Psi\colon C^\infty([-1,1],\mathfrak{g}) \to \prod_{i=1}^n C^\infty(\overline{U}_i,\mathfrak{g})$, $f \mapsto (f|_{\overline{U}_i})_i$. Let $\exp\colon V_\mathfrak{g} \subseteq \mathfrak{g} \to U_G \subseteq G$ be the exponential function of G restricted to a 0-neighbourhood such that it is a diffeomorphism. We define the open sets $\mathcal{U} := C^\infty([-1,1],U_G) \subseteq C^\infty([-1,1],G)$ and $\mathcal{V} := C^\infty([-1,1],V_\mathfrak{g}) \subseteq C^\infty([-1,1],\mathfrak{g})$. Let $\tau_1\colon C^\infty([-1,1],U_G) \to C^\infty([-1,1],V_\mathfrak{g})$, $\varphi \mapsto (\exp|_{V_\mathfrak{g}}^{U_G})^{-1} \circ \varphi$ and $\tau_2\colon \prod_{i=1}^n C^\infty(\overline{U}_i,U_G) \to \prod_{i=1}^n C^\infty(\overline{U}_i,V_\mathfrak{g})$, $(\varphi_i)_i \mapsto ((\exp|_{V_\mathfrak{g}}^{U_G})^{-1} \circ \varphi_i)_i$ be the* *canonical charts. We obtain the commutative diagram*

$$\begin{array}{ccc} C^\infty([-1,1],U_G) & \xrightarrow{\Phi|_{\mathcal{U}}} & \prod_{i=1}^n C^\infty(\overline{U}_i,U_G) \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ C^\infty([-1,1],V_{\mathfrak{g}}) & \xrightarrow{\Psi|_{\mathcal{V}}} & \prod_{i=1}^n C^\infty(\overline{U}_i,V_{\mathfrak{g}}) \end{array}$$

and calculate

$$\tau_2\left(\operatorname{im}(\Phi)\cap\prod_{i=1}^n C^\infty(\overline{U}_i,U_G)\right)=\tau_2\left(\Phi(C^\infty([-1,1],U_G))\right)$$
$$=\Psi(C^\infty([-1,1],V_{\mathfrak{g}}))=\operatorname{im}(\Psi)\cap\prod_{i=1}^n C^\infty(\overline{U}_i,V_{\mathfrak{g}}).$$

The space

$$\operatorname{im}(\Psi)=\left\{(f_i)_i : f_i(x)=f_j(x) \text{ for } x\in\overline{U}_i\cap\overline{U}_j\right\}$$

is closed in $\prod_{i=1}^n C^\infty(\overline{U}_i,\mathfrak{g})$*. Hence,* $\operatorname{im}(\Phi)$ *is a Lie subgroup of* $\prod_{i=1}^n C^\infty(\overline{U}_i,G)$*. In the commutative diagram*

$$\begin{array}{ccc} C^\infty([-1,1],U_G) & \xrightarrow{\Phi|_{\mathcal{U}}} & \operatorname{im}(\Phi)\cap\prod_{i=1}^n C^\infty(\overline{U}_i,U_G) \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ C^\infty([-1,1],V_{\mathfrak{g}}) & \xrightarrow{\Psi|_{\mathcal{V}}} & \operatorname{im}(\Psi)\cap\prod_{i=1}^n C^\infty(\overline{U}_i,V_{\mathfrak{g}}) \end{array}$$

the lower vertical arrow is a diffeomorphism because $\Psi\colon C^\infty([-1,1],\mathfrak{g})\to\operatorname{im}(\Psi)$ *is a continuous bijective linear map between Fréchet spaces. Now, the assertion follows.*

The following theorem is in some sense a generalisation of [Nee04, Proposition V.7].[13]

Theorem 4.51. *The following holds:*
- *(a) The map* $\Delta\colon C_c^\infty(P,G)_{\rho_G}\to\Omega_c^1(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}^{\mathrm{hor}}$ *is smooth.*
- *(b) We have* $d_1\Delta(f)=D_{\rho_{\mathfrak{g}}}f$ *for* $f\in C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}$*.*

Proof. (a) Because of Lemma 4.38, Lemma 4.41 and Lemma 4.43, it is enough to show the smoothness of Δ on a 1-neighbourhood. Let $(\sigma_i,\overline{V}_i)_{i\in\mathbb{N}}$ be a locally finite compact trivialising system in the sense of [Sch13, Definition 3.6.] (the existence follows from [Sch13, Corollary 3.10]). With the help of Lemma 4.45 and Lemma 4.47 it is enough to construct smooth maps

[13] We consider the case of a finite-dimensional codomain while Neeb additionally considered special infinite-dimensional codomains.

$\psi_i\colon C^\infty(\overline{V}_i, G) \to \Omega^1(\overline{V}_i, \mathfrak{g})$ such that the diagram

$$\begin{array}{ccc} C_c^\infty(P,G)_{\rho_G} & \xrightarrow{\quad\Delta\quad} & \Omega_c^1(P,\mathfrak{g})^{\mathrm{hor}}_{\rho_{\mathfrak{g}}} \\ \downarrow{\scriptstyle \varphi\mapsto(\varphi\circ\sigma_i)_i} & & \downarrow{\scriptstyle \theta\mapsto(\sigma_i^*\theta)_i} \\ \prod^*_{i\in\mathbb{N}} C^\infty(\overline{V}_i, G) & \xrightarrow{\quad\psi_i\quad} & \bigoplus_{i\in\mathbb{N}}\Omega^1(\overline{V}_i,\mathfrak{g}) \end{array}$$

commutes. Let $\tau_i\colon q^{-1}(V_i) \to V_i \times H,\ p \mapsto (q(p), \varphi_i(p))$ be the inverse to $(x,h) \mapsto \sigma_i(x)h$. Then $\sigma_i(x) = \tau_i(x,1)$. For $f \in C^\infty(\overline{V}_i, G)$ we define

$$\tilde{f}\colon q^{-1}(V_i) \to G,\ p \mapsto \rho_G(\varphi_i(p), f(q(p))).$$

If $f \in C_c^\infty(P,G)_{\rho_G}$ then $\widetilde{f\circ\sigma_i} = f|_{q^{-1}(V_i)}$. We define $\psi_i\colon C^\infty(\overline{V}_i, G) \to \Omega^1(\overline{V}_i, \mathfrak{g})$ by

$$\psi_i(f)_x(v) = T\lambda_{f(x)^{-1}} T\tilde{f}(\mathrm{pr}_h(T_x\sigma_i(v))).$$

First we show that the above diagram commutes. We calculate

$$\psi_i(f\circ\sigma_i)_x(v) = T\lambda_{(f_i\circ\sigma(x))^{-1}} Tf(\mathrm{pr}_h \circ T_x\sigma_i(v)) = \sigma_i^*(\delta(f)\circ\mathrm{pr}_h)_x(v).$$

It is left to show the smoothness of ψ_i. Because we can embed $\Omega^1(\overline{V}_i, \mathfrak{g})$ into $C^\infty(T\overline{V}_i, \mathfrak{g})$, we show that

$$C^\infty(\overline{V}_i, G) \times (TV_i) \to \mathfrak{g},\ (f,v) \mapsto T\lambda_{f(x)^{-1}} T\tilde{f}(\mathrm{pr}_h(T_x\sigma_i(v)))$$

is smooth. Let $m\colon G\times G \to G$ be the multiplication on G and $n\colon G \to TG,\ g \mapsto 0_g$ the zero section. Given $f \in C^\infty(\overline{V}_i, G)$ and $v \in T_x\overline{V}_i$, we calculate

$$\psi_i(f)(v) = Tm(n(f(\pi(v))^{-1})), T\tilde{f}(\mathrm{pr}_h T\sigma_i(v))).$$

The map $\mathrm{ev}\colon C^\infty(\overline{V}_i, G) \times \overline{V}_i \to G,\ f,x \mapsto f(x)$ is smooth (see [Alz72, Lemma 121]). Therefore it is left to show the smoothness of $C^\infty(\overline{V}_i, G) \times Tq^{-1}(TV_i) \to TG,\ (f,v) \mapsto T\tilde{f}(v)$. The map $\mathrm{ev}^q\colon C^\infty(\overline{V}_i, G) \times q^{-1}(V_i) \to G, (f,p) \mapsto f\circ q(p)$ is smooth because ev is smooth. We have $T(f\circ q)(v)) = T\,\mathrm{ev}^q(f,\bullet)(v) = T\,\mathrm{ev}^q(n(f),v)$, where n is the zero section of $C^\infty(\overline{V}_i, G)$. Hence,

$$T\,\mathrm{ev}^q \circ(n,\mathrm{id})\colon C^\infty(\overline{V}, G) \times Tq^{-1}(TV) \to TG,\ (f,v) \mapsto Tf \circ Tq(v)$$

is smooth. With $T\tilde{f} = T\rho_G \circ (T\varphi_i, Tf\circ Tq)$ the assertion follows from the smoothness of $T\,\mathrm{ev}^q \circ(n,\mathrm{id})$.

(b) We write $\delta^l\colon C^\infty([-1,1], G) \to C^\infty([-1,1],\mathfrak{g})$ for the classical left logarithmic derivative. It is known that $d_{c_1}\delta^l(f) = f'$ for $f \in C([-1,1],\mathfrak{g})$ (see e.g. [NS13, Proposition 8.4]). Given a horizontal curve $\gamma\colon [-1,1] \to P$, we define the maps

$$\begin{aligned}
&\gamma_G^*\colon C_c^\infty(P,G)_{\rho_G} \to C^\infty([-1,1],G), \varphi \mapsto \varphi \circ \gamma,\\
&\gamma_{\mathfrak{g}}^*\colon C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}} \to C^\infty([-1,1],\mathfrak{g}), f \mapsto f \circ \gamma \text{ and}\\
&\gamma_{\Omega}^*\colon \Omega_c^1(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}^{\mathrm{hor}} \to C^\infty([-1,1],\mathfrak{g}), \theta \mapsto \gamma^*\theta.
\end{aligned}$$

As in Lemma 4.23 one shows that γ_G^* is a smooth Lie group homomorphism with $L(\gamma_G^*) = \gamma_{\mathfrak{g}}^*$ (see Remark 4.50). The diagram

$$\begin{array}{ccc}
C_c^\infty(P,G)_{\rho_G} & \xrightarrow{\quad\Delta\quad} & \Omega_c^1(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}^{\mathrm{hor}}\\
\Big\downarrow{\scriptstyle \gamma_G^*} & & \Big\downarrow{\scriptstyle \gamma_\Omega^*}\\
C^\infty([-1,1],G) & \xrightarrow{\quad\delta^l\quad} & C^\infty([-1,1],\mathfrak{g})
\end{array} \tag{4.13}$$

commutes, because

$$\begin{aligned}
&(\gamma_\Omega^*\Delta(f))(t) = \delta(f)(\mathrm{pr}_h(\gamma'(t))) = \delta(f)(\gamma'(t)) = T\lambda_{f\circ\gamma(t)^{-1}} \circ Tf(\gamma'(t))\\
=&\delta^l(f\circ\gamma).
\end{aligned}$$

Let $f \in C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}$. We want to show that $d_1\Delta(f) = D_{\rho_{\mathfrak{g}}} f$. Since Lemma 4.49 it is enough to show that $\gamma_\Omega^*(d_1\Delta(f)) = \gamma_\Omega^*(D_{\rho_{\mathfrak{g}}} f)$ for an arbitrary horizontal curve $\gamma\colon [-1,1] \to P$. Because γ_Ω^* is continuous linear and the diagram (4.13) commutes, we can calculate

$$\gamma_\Omega^*(d_1\Delta(f)) = d_1(\gamma_\Omega^* \circ \Delta)(f) = d_1(\delta^l \circ \gamma_G^*)(f) = d_1(\delta^l)(L(\gamma_G^*)(f)) = (f\circ\gamma)'.$$

Now, we use that γ is horizontal and obtain

$$\gamma_\Omega^*(D_{\rho_{\mathfrak{g}}} f)_t = D_{\rho_{\mathfrak{g}}} f(\gamma'(t)) = df(\gamma'(t)) = \gamma_\Omega^*(d_1\Delta(f))_t$$

for $t \in [-1,1]$.

□

The proof of the following Lemma 4.52 is analogous to the first part of [MN03, Proposition III.3].

Lemma 4.52. *In the following we write* Ad *for the adjoint action of* $C_c^\infty(P,G)_{\rho_G}$ *on* $C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}$. *The map*

$$\begin{aligned}
\mathcal{A}\colon C_c^\infty(P,G)_{\rho_G} \times (\overline{\Omega}_c^1(P,V)_{\rho_V}^{\mathrm{hor}} \times_{\omega_M} C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}) &\to \overline{\Omega}_c^1(P,V)_{\rho_V}^{\mathrm{hor}} \times_{\omega_M} C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}\\
(\varphi,([\alpha],f)) &\mapsto ([\alpha] - [\kappa_{\mathfrak{g}}(\Delta(\varphi),f)], \mathrm{Ad}(\varphi,f))
\end{aligned}$$

is a smooth group action and its associated Lie algebra action is given by the adjoint action described in (4.2). Hence, the adjoint action of $\Gamma_c(M,\mathfrak{G})$ *on the extension* $\widehat{\Gamma_c(M,\mathfrak{G})} := \overline{\Omega}_c^1(M,\mathbb{V}) \times_{\omega_M} \Gamma_c(M,\mathfrak{G})$ *represented by* ω_M *integrates to a Lie group action of* $\Gamma_c(M,\mathcal{G})$ *on* $\widehat{\Gamma_c(M,\mathfrak{G})}$.

Proof. The smoothness of $\mathcal{A}$ follows from the smoothness of Δ. We show that $\mathcal{A}$ is a group action. For $\varphi, \psi \in C_c^\infty(P,G)_{\rho_G}$, we have

$$\Delta(\varphi\psi) = \Delta(\psi) + \mathrm{Ad}_*^G(\psi^{-1}, \Delta(\varphi))$$

and for $v, w \in \mathfrak{g}$ and $g \in G$, we have

$$\kappa_{\mathfrak{g}}(v, \mathrm{Ad}_g^G w) = \kappa_{\mathfrak{g}}(\mathrm{Ad}_{g^{-1}}^G v, w). \tag{4.14}$$

In this context Ad^G is the adjoint action of G on $\mathfrak{g}$. Now, we calculate for $\alpha \in \Omega_c^1(P,V)_{\rho_V}^{\mathrm{hor}}$

$$\begin{aligned} &\mathcal{A}(\varphi \cdot \psi, ([\alpha], f)) = ([\alpha] - [\kappa_{\mathfrak{g}}(\Delta(\varphi\psi), f)], \mathrm{Ad}_{\varphi\psi} f) \\ =&([\alpha] - [\kappa_{\mathfrak{g}}(\Delta\psi, f)] - [\kappa_{\mathfrak{g}}(\mathrm{Ad}_*^G(\psi^{-1}, \Delta(\varphi)), f)], \mathrm{Ad}_\varphi . \mathrm{Ad}_\psi .f) \\ \underset{(4.14)}{=}& ([\alpha] - [\kappa_{\mathfrak{g}}(\Delta\psi, f)] - [\kappa_{\mathfrak{g}}(\Delta(\varphi), \mathrm{Ad}_*^G(\psi f))], \mathrm{Ad}_\varphi . \mathrm{Ad}_\psi .f) \\ =&\mathcal{A}(\varphi, ([\alpha] - [\kappa_{\mathfrak{g}}(\Delta\psi, f)], \mathrm{Ad}_\psi f)) \\ =&\mathcal{A}(\varphi, (\mathcal{A}(\psi, ([\alpha], f))). \end{aligned}$$

The associated action to $\mathcal{A}$ on $C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}$ is given by the adjoint action described in (4.2) because $(-[\kappa_{\mathfrak{g}}(D_{\rho_{\mathfrak{g}}}(g), f)], \mathrm{ad}(g,f)) = ([\kappa_{\mathfrak{g}}(g, D_{\rho_{\mathfrak{g}}}(f)], \mathrm{ad}(g,f))$ for $f, g \in C_c^\infty(P,\mathfrak{g})_{\rho_{\mathfrak{g}}}$. □

Theorem 4.53. *Let $\overline{H}$ be finite and write $\Gamma := \mathrm{per}_{\omega_M}(\pi_2(\Gamma_c(M;\mathcal{G})_0))$. Then we find a Lie group extension*

$$\overline{\Omega}_c^1(M,\mathbb{V})/\Gamma \hookrightarrow \Gamma_c\widehat{(M,\mathcal{G})}_0 \to \Gamma_c(M,\mathcal{G})_0$$

that corresponds to the central Lie algebra extension that is represented by ω_M.

Proof. We simply need to put Theorem 4.36, Theorem 4.52 and Theorem 4.15 (respectively [Nee02a, Proposition 7.6] and [Nee02a, Theorem 7.12]) together. □

4.3. Universality of the Lie group extension

In this section, we prove [JW13, Theorem 1.2] in the case where M is not compact but σ-compact (as in [JW13, Theorem 1.2] M still has to be connected). In the first part of [JW13] Janssens and Wockel showed that the cocycle $\omega_M \colon \Gamma_c(M,\mathfrak{G})^2 \to \overline{\Omega}_c^1(M,\mathbb{V})$ is universal if $\mathfrak{g}$ is semisimple and M is a σ-compact manifold (see [JW13, p. 129 (1.1)], Remark 4.19 and Remark 3.24). In the second part of the paper they assumed the base manifold M to be compact and got a universal cocycle $\Gamma(M,\mathfrak{G})^2 \to \overline{\Omega}^1(M,\mathbb{V})$. Then they showed that under certain conditions a given Lie group bundle $G \hookrightarrow \mathcal{G} \to M$ with finite-dimensional Lie group G is associated to the principal frame bundle $\mathrm{Aut}(G) \hookrightarrow \mathrm{Fr}(\mathcal{G}) \to M$. Hence, they were able to use [NW09, Theorem 4.24] to integrate the universal Lie algebra cocycle $\Gamma(M,\mathfrak{G})^2 \to$

$\overline{\Omega}^1(M, \mathbb{V})$ to a Lie group cocycle $Z \hookrightarrow \widehat{\Gamma(M, \mathcal{G})_0} \to \Gamma(M, \mathcal{G})_0$. At this point it is crucial that M is compact and connected in order to apply [NW09, Theorem 4.24]. Once the Lie group extension was constructed, Janssens and Wockel proved its universality by using the Recognition Theorem from [Nee02b] (see [JW13, Theorem 1.2]). To generalise [JW13, Theorem 1.2] to the case where M is connected and not compact, many arguments of [JW13] can be transfered to the case of a non-compact base manifold by using Theorem 4.53 instead of [NW09, Theorem 4.24]. However our proof is shorter because Theorem 4.53 holds for section groups and not just for gauge groups, while [NW09, Theorem 4.24] holds only for gauge groups. Hence, unlike the approach in [JW13], we do not have to reduce the statement to the case of gauge groups. We mention that in this section we assume the typical fibre G of the Lie group bundle to be connected, while in [JW13] Janssens and Wockel assume $\pi_0(G)$ to be finitely generated

Convention 4.54. In this section G is a connected semisimple finite-dimensional Lie group. As in the rest of Chapter 4, M still is a connected, non-compact, σ-compact finite-dimensional manifold.

Analogously to [JW13, p. 130] we consider the following setting[14]:

Definition 4.55 (Cf. p. 130 in [JW13])**.** Let G be a connected finite-dimensional semisimple Lie group with Lie algebra $\mathfrak{g}$ and $G \hookrightarrow \mathcal{G} \xrightarrow{q} M$ be a Lie group bundle. As in [HN12, 11.3.1, p. 452], we turn $\mathrm{Aut}(G)$ into a finite-dimensional Lie group. In particular $\mathrm{Aut}(G)$ becomes a Lie group such that $L\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism onto a closed subgroup ([HN12, Lemma 11.3.3]) and $\mathrm{Aut}(G)$ acts smoothly on G.

Lemma 4.56. *The Lie group bundle $G \hookrightarrow \mathcal{G} \xrightarrow{q} M$ is isomorphic to the associated Lie group bundle of the frame principal bundle $\mathrm{Aut}(G) \hookrightarrow \mathrm{Fr}(\mathcal{G}) \to M$ (cf. [JW13, p. 130]). Obviously all manifolds are σ-compact, because M is σ-compact and $\mathrm{Aut}(G)$ is homeomorphic to a closed subgroup of $\mathrm{Aut}(\mathfrak{g})$.*

Definition 4.57. We define $V = V(\mathfrak{g})$. In the situation considered in this subsection the map $\rho_V\colon \mathrm{Aut}(G) \times V \to V,\ (\varphi, \kappa_{\mathfrak{g}}(v, w)) \mapsto \kappa_{\mathfrak{g}}(L(\varphi)(v), L(\varphi)(w))$ is the smooth automorphic action ρ_V described in Convention 4.1.

Lemma 4.58 (Cf. p. 130 in [JW13])**.** *The identity component of $\mathrm{Aut}(G)$ acts trivially on V by the representation $\rho_V\colon \mathrm{Aut}(G) \times V \to V,\ (\varphi, \kappa_{\mathfrak{g}}(v, w)) \mapsto \kappa_{\mathfrak{g}}(L(\varphi)(v), L(\varphi)(w))$.*

Proof. Obviously it is enough to show that $(\mathrm{Aut}(\mathfrak{g}))_0$ acts trivially by $\rho\colon \mathrm{Aut}(\mathfrak{g}) \times V \to V,\ (\varphi, \kappa_{\mathfrak{g}}(x, y)) \mapsto \kappa_{\mathfrak{g}}(\varphi(x), \varphi(y))$. For $\check{\rho}\colon \mathrm{Aut}(\mathfrak{g}) \to GL(V),\ \varphi \mapsto \rho(\varphi, \bullet)$, $x, y \in \mathfrak{g}$ and $f \in \mathrm{der}(\mathfrak{g})$ we have $L(\check{\rho})(f)(\kappa_{\mathfrak{g}}(x, y)) = d_{\mathrm{id}}\rho(\bullet, \kappa_{\mathfrak{g}}(x, y))(f)$. Defining $\mathrm{ev}_x\colon \mathrm{Aut}(\mathfrak{g}) \to \mathfrak{g},\ \varphi \to \varphi(x)$ for $x \in \mathfrak{g}$ we get

$$\rho(\bullet, \kappa_{\mathfrak{g}}(x, y)) = \kappa_{\mathfrak{g}} \circ (\mathrm{ev}_x, \mathrm{ev}_y).$$

[14] In [JW13] the Lie group G is not assumed to be connected. Instead Janssens and Wockel assume $\pi_0(G)$ to be finitely generated.

We have $d_{\mathrm{id}}\,\mathrm{ev}_x(f) = \frac{\partial}{\partial t}|_{t=0}\exp(tf)(x) = f(x)$. Hence

$$\begin{aligned} &d_{\mathrm{id}}\rho(\bullet, \kappa_{\mathfrak{g}}(x,y))(f) = \kappa_{\mathfrak{g}}(\mathrm{ev}_x(\mathrm{id}), d_{\mathrm{id}}\,\mathrm{ev}_y(f)) + \kappa_{\mathfrak{g}}(d_{\mathrm{id}}\,\mathrm{ev}_x(f), \mathrm{ev}_y(\mathrm{id}))\\ =&\kappa_{\mathfrak{g}}(x, f(y)) + \kappa_{\mathfrak{g}}(f(x), y). \end{aligned}$$

Because $\mathfrak{g}$ is semisimple, we have $\mathrm{der}(\mathfrak{g}) = \mathrm{inn}(\mathfrak{g})$. For $z \in \mathfrak{g}$, we calculate

$$L(\check{\rho})(\mathrm{ad}_z)(\kappa_{\mathfrak{g}}(x,y)) = \kappa_{\mathfrak{g}}(x,[y,z]) + \kappa_{\mathfrak{g}}([x,z],y) = \kappa_{\mathfrak{g}}(x,[y,z]) + \kappa_{\mathfrak{g}}(x,[z,y]) = 0.$$

Hence, $\check{\rho}|_{\mathrm{Aut}(\mathfrak{g})_0} = \mathrm{id}_V$. □

Analogously to [JW13, p. 130], we need the following condition:

Convention 4.59. In the following, we assume $\overline{\mathrm{Aut}(G)} := \mathrm{Aut}(G)/\ker(\rho_V)$ to be finite.

Definition 4.60. Combining Convention 4.59, Lemma 4.58 and Theorem 4.53, we find a Lie group extension

$$\overline{\Omega}^1_c(M,\mathbb{V})/\Gamma \hookrightarrow \Gamma_c\widehat{(M,\mathcal{G})}_0 \to \Gamma_c(M,\mathcal{G})_0$$

that corresponds to the central Lie algebra extension that is represented by ω_M (with $\Gamma := \mathrm{per}_{\omega_M}(\pi_2(\Gamma_c(M;\mathcal{G})_0)))$. We write $Z := \overline{\Omega}^1_c(M,\mathbb{V})/\Gamma$. If $\pi\colon \Gamma_c\widetilde{(M,\mathcal{G})}_0 \to \Gamma_c(M,\mathcal{G})_0$ is the universal covering homomorphism and $Z \hookrightarrow H \to \Gamma_c\widetilde{(M,\mathcal{G})}_0$ the pullback extension then [Nee02a, Remark 7.14.] shows that we have a central extension of Lie groups

$$E := Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0) \hookrightarrow H \to \Gamma_c(M,\mathcal{G})_0.$$

Its corresponding Lie algebra extension is represented by ω_M.

The following theorem (case of a non-compact base-manifold and connected typical fibre) corresponds to [JW13, Theorem 1.2.] (case of a compact base-manifold and $\pi_0(G)$ is finitely generated). The proof is analogous as well.

Theorem 4.61. *(a) If W is a locally convex space such that ω_M is universal for W then central Lie group extension $Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0) \hookrightarrow H \to \Gamma_c(M,\mathcal{G})_0$ is universal for all abelian Lie groups modelled over W.*

(b) The central Lie group extension $Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0) \hookrightarrow H \to \Gamma_c(M,\mathcal{G})_0$ is universal for all abelian Lie groups modelled over complete locally convex spaces.

Proof. (a) The statement [Nee02b, Theorem 4.13] and the analogous statement [JW13, Theorem 3.1] are formulated for sequentially complete, respectively Mackey complete, spaces W. However, the completeness is only assumed to guarantee the existence of the period map per_ω and the existence of period maps of the form $\mathrm{per}_{\gamma\circ\omega}$ for continuous linear maps $\gamma\colon \mathfrak{z} \to \mathfrak{a}$. Obviously, the period maps $\mathrm{per}_{\gamma\circ\omega}$ exist if the period map per_ω exists. Hence, with Remark

4.14 we do not need to assume the completeness of the spaces. Therefore, it is left to show that H is simply connected. Using [Nee02a, Remark 5.12], we have the long exact homotopy sequence

$$\pi_2(\Gamma_c(M,\mathcal{G})_0) \xrightarrow{\delta_2} \pi_1(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)) \xrightarrow{i} \pi_1(H) \xrightarrow{p} \pi_1(\Gamma_c(M,\mathcal{G})_0)$$
$$\xrightarrow{\delta_1} \pi_0(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)).$$

We show that $i = 0$. Calculating

$$\pi_1(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)) = \pi_1(\overline{\Omega}_c^1(M,\mathbb{V})/\Pi_{\omega_M}) = \Pi_{\omega_M}$$

and using [Nee02a, Proposition 5.11], we conclude that δ_2 is surjective. Hence $i = 0$. From

$$\pi_0(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)) = \pi_1(\Gamma_c(M,\mathcal{G})_0),$$

we see that δ_1 is injective. Therefore $p = 0$. Thus $\pi_1(H) = 0$.

(b) This is clear, because ω_M is universal for all complete locally convex spaces.

□

A. Basic definitions and results for manifolds with corners

In this section we fix some notation and recall well known basic definitions and results about manifolds with corners for the convenience of the reader.[1] All of the concepts and results in this section are already well known see e.g. [MO92].

Definition A.1. We define $\mathbb{R}^m_l := [0, \infty[^l \times \mathbb{R}^{m-l}$. Given $J \subseteq \{1, ..., l\}$ we define $\mathrm{pr}_J \colon \mathbb{R}^m \to \mathbb{R}^{\#J}$, $v \mapsto (v_j)_{j \in J}$. Moreover we write $X_J := \mathrm{pr}_J(X)$ for $X \subseteq \mathbb{R}^m$ and $v_J := \mathrm{pr}_J(v)$ for $v \in \mathbb{R}^m$.

Definition A.2. We say a point $x \in \mathbb{R}^m_l$ has *index* $k \leqslant l$, if exactly k of the first l components of x equal 0. This means that the maximal index-set $J \subseteq \{1, ..., l\}$ such that $\mathrm{pr}_J(x) = 0$ has the cardinality $\#J = k$.

Lemma A.3. *Let M be a manifold with corners, $p \in M$ and $\varphi \colon U_\varphi \to V_\varphi \subseteq \mathbb{R}^m_{i_1}$ and $\psi \colon U_\psi \to V_\psi \subseteq \mathbb{R}^m_{i_2}$ charts around p. Suppose $\varphi(p)$ has index k, then $\psi(p)$ has index k.*

Proof. Without loss of generality we assume $U_\psi \subseteq U_\varphi$. Let l be the index of $\psi(p)$. First we show $k \leqslant l$. Suppose $l < k$. Let $J \subseteq \{1, \ldots, i_1\}$ be the maximal index-set with $(\varphi(p))_J = 0$ and analogously $I \subseteq \{1, \ldots, i_2\}$ be the set of components of $\psi(p)$ that equal 0. We define the $m - l$-dimensional subspace $E := \mathrm{pr}_I^{-1}(\{0\}) = \bigcap_{i \in I} \{x \in \mathbb{R}^m : x_i = 0\}$ of $\mathbb{R}^m$. We find an open $\psi(p)$-neighbourhood $V \subseteq E$ with $V \subseteq V_\psi$. The map $\eta := \varphi \circ \psi^{-1}|_V \colon V \to V_\varphi$ is an immersion, because $\psi \circ \varphi^{-1} \circ \eta = \mathrm{id}_V$. Therefore $F := d\eta(\psi(p), \bullet)(E)$ is an $m - l$-dimensional subspace of $\mathbb{R}^m$. The subspace F must contain a vector such that one of its J components is not equal to 0, because otherwise F would be contained in the $m - k$-dimensional subspace $\mathrm{pr}_J^{-1}(\{0\})$ and this would contradict $m - l > m - k$. Let $j \in J$ such that $v_j \neq 0$. Without loss of generality we can assume $v_j < 0$. We choose a smooth curve $\gamma \colon] - \epsilon, \epsilon[\to E$ with $\gamma(0) = \psi(p)$, $\mathrm{im}(\gamma) \subseteq V$ and $d\eta(\psi(p), \gamma'(0)) = v$. Let $f := \eta \circ \gamma \colon] - \epsilon, \epsilon[\to V_\varphi$. Then $\mathrm{im}(f) \subseteq V_\varphi \subseteq \mathbb{R}^m_{i_1}$ and so $\mathrm{im}(\mathrm{pr}_j \circ f) \subseteq [0, \infty[$, because $j \leqslant i_1$. But $\mathrm{pr}_j \circ f(0) = \mathrm{pr}_j(\varphi(p)) = 0$ and $(\mathrm{pr}_j \circ f)'(0) = v_j < 0$. Hence we find $t \in]0, \epsilon[$ with $\mathrm{pr}_j \circ f(t) < 0$. Hence $k \leqslant l$. In the analogous way one shows $l \leqslant k$. Hence $k = l$. □

Definition A.4. Let M be a manifold with corners. We say a point $p \in M$ has *index* $j \in \{0, ..., m\}$ if we find a chart $\varphi \colon U_\varphi \to V_\varphi \subseteq \mathbb{R}^m_i$ around p such that $\varphi(p)$ has index j. Because of Lemma A.3 this definition is independent of the choice of the chart φ. We write $\mathrm{ind}(p) := j$ and define the *j-stratum* $\partial^j M := \{x \in M : \mathrm{ind}(x) = j\}$. We call the 0-stratum $\partial^0 M$ the *interior* of M.

[1] This chapter consist of material published before in the author's preprint [Eyn15].

Lemma A.5. *Given an m-dimensional manifold with corners M and $j \in \{0, ..., m\}$ the j-stratum ∂^j is a $m-j$-dimensional submanifold without boundary of M. Obviously we have $M = \bigcup_{j\in\{0,...,m\}} \partial^j M$.*

Proof. To be a submanifold, is a local property and locally $\partial^j M$ looks like $\partial^j \mathbb{R}^m_k$ for a $k \geqslant j$. □

B. Proof of Theorem 1.12

As mentioned in Section 1.1 we can use our Lemma 1.11 to prove Theorem 1.12.[1] The proof is completely analogous to [BW59, Proposition 1].

Proof. First we show the uniqueness result. Using Lemma 1.9 we find a neighbourhood U of M in $\tilde{M}_1$ and a real analytic map $\tilde{f}\colon U \to \tilde{M}_2$ with $\tilde{f}|_M = \mathrm{id}_M$. For the same reason we find a neighbourhood V of M in $\tilde{M}_2$ and a real analytic map $\tilde{g}\colon V \to \tilde{M}_1$ with $\tilde{g}|_M = \mathrm{id}_M$. With Lemma 1.10 we find a neighbourhood U_1 of M in $\tilde{M}_1$, and a neighbourhood U_2 of M in $\tilde{M}_2$ such that $\tilde{f}(U_1) = U_2$ and $\tilde{f}|_{U_1}^{U_2}$ is a real analytic diffeomorphism.

Now we construct the enveloping manifold. For $x \in M$ let $\varphi_x\colon U_x^1 \to V_x^1 \subseteq [0,\infty[^m$ be a chart of M around x. Because M is normal we find relatively compact x-neighbourhoods U_x^3 and U_x^2 in M such that

$$M \supseteq U_x^1 \supseteq \overline{U_x^2} \supseteq U_x^2 \supseteq \overline{U_x^3} \supseteq U_x^3.$$

Here, the closures are taken in the space M and hence coincide with the closures in the topological subspaces. The family $(U_x^3)_{x\in M}$ is an open cover of the compact manifold M, whence we find a finite subcover $(U_{x_i}^3)_{i\in I}$. We define $U_i^1 := U_{x_i}^1$, $V_i^1 := V_{x_i}^1$, $\varphi_i := \varphi_{x_i}$, $U_i^2 := U_{x_i}^2$, $V_i^2 := \varphi_i(U_i^2)$, $U_i^3 := U_{x_i}^3$ and $V_i^3 := \varphi_i(U_i^3)$. Hence we get

$$V_i^1 \supseteq \overline{V_i^2} \supseteq V_i^2 \supseteq \overline{V_i^3} \supseteq V_i^3.$$

Here, the sets $\overline{V_i^2}$ and $\overline{V_i^3}$ are compact and hence the closure in the topological subspace coincides with the closure in $[0,\infty[^m$ respectively $\mathbb{R}^m$. Moreover we define the sets $V_{i,j}^1 := \varphi_i(U_i^1 \cap U_j^1)$, $V_{i,j}^2 := \varphi_i(U_i^2 \cap U_j^2)$ and $V_{i,j}^3 := \varphi_i(U_i^3 \cap U_j^3)$. Given $i,j \in I$ we use Lemma 1.10 to find an open neighbourhood $\tilde{V}_{i,j}^1$ of $V_{i,j}^1$ in $\mathbb{R}^m$ with $\tilde{V}_{i,j}^1 \cap [0,\infty[^m = V_{i,j}^1$ and a real analytic diffeomorphism $\psi_{i,j}\colon \tilde{V}_{i,j}^1 \to \tilde{V}_{j,i}^1$ with $\psi_{i,j}|_{\tilde{V}_{i,j}^1 \cap [0,\infty[^m} = \varphi_j \circ \varphi_i^{-1}|_{V_{i,j}^1}$ with inverse $\psi_{j,i}$. Because $\overline{V_{i,j}^2} \subseteq \overline{V_i^2}$ and V_i^2 is relatively compact, $\overline{V_{i,j}^2}$ is compact and hence the closure in V_i^2 coincides with the closure in $[0,\infty[^m$ respectively $\mathbb{R}^m$. Using Lemma 1.11 we find an open neighbourhood $\tilde{V}_{i,j}^2$ of $V_{i,j}^2$ in $\mathbb{R}^m$ with $\overline{V}_{i,j}^2 \subseteq \tilde{V}_{i,j}^1$, $\tilde{V}_{i,j}^2 \cap [0,\infty[^m = V_{i,j}^2$ and $\overline{\tilde{V}_{i,j}^2} \cap [0,\infty[^m = \overline{V_{i,j}^2}$. We can assume $\psi_{i,j}(\tilde{V}_{i,j}^2) = \tilde{V}_{j,i}^2$, because $\psi_{i,j}(V_{i,j}^2) = V_{j,i}^2$. The set $\overline{V_i^3} \cap \psi_{j,i}^{-1}(\overline{V_j^3} \cap \overline{V_{j,i}^2})$ is compact and contained in $\tilde{V}_{i,j}^2$. Hence we find an open neighbourhood $\tilde{Z}_{i,j}$ of $\overline{V_i^3} \cap \psi_{j,i}^{-1}(\overline{V_j^3} \cap \overline{V_{j,i}^2})$ in $\mathbb{R}^m$ with $\overline{\tilde{Z}}_{i,j} \subseteq \tilde{V}_{i,j}^2$ and $\psi_{i,j}(\tilde{Z}_{i,j}) = \tilde{Z}_{j,i}$. Because $\psi_{j,i}(\overline{V}_j^3 \cap \overline{V_{j,i}^2}) \cap \overline{V_i^3} \subseteq \tilde{Z}_{i,j}$ we get $(\psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2}) \backslash \tilde{Z}_{i,j}) \cap (\overline{V_i^3} \backslash \tilde{Z}_{i,j}) = \varnothing$. The

[1] This chapter consist of material published before in the author's preprint [Eyn15].

sets $\psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2})\backslash\tilde{Z}_{i,j}$ and $\overline{V_i^3}\backslash\tilde{Z}_{i,j}$ are closed, whence we find open disjoint sets $\tilde{X}_{i,j}$ and $\tilde{Y}_{i,j}$ with $\psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2})\backslash\tilde{Z}_{i,j} \subseteq \tilde{X}_{i,j}$ and $\overline{V_i^3}\backslash\tilde{Z}_{i,j} \subseteq \tilde{Y}_{i,j}$. Thus $\psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2}) \subseteq \tilde{Z}_{i,j} \cup \tilde{X}_{i,j}$ and $\overline{V_i^3} \subseteq \tilde{Z}_{i,j} \cup \tilde{Y}_{i,j}$. Because I is finite the set $\bigcap_{j\in I,\overline{V_{i,j}^1}\neq\emptyset} \tilde{Y}_{i,j} \cup \tilde{Z}_{i,j}$ is open. Obviously it contains $\overline{V_i^3}$. Using Lemma 1.11 we find an open set $\hat{V}_i^3$ in $\mathbb{R}^m$ with $\hat{V}_i^3 \cap [0,\infty[^m = V_i^3$, $\overline{\hat{V}_i^3} \cap [0,\infty[^m = \overline{V_i^3}$ and $\hat{V}_i^3 \subseteq \tilde{Y}_{i,j} \cup \tilde{Z}_{i,j}$ for all $j \in I$ with $\tilde{V}_{i,j}^1 \neq \emptyset$. Now we calculate

$$\begin{aligned}&\overline{\psi_{i,j}(\hat{V}_i^3 \cap \tilde{V}_{i,j}^2)} \cap [0,\infty[^m = \psi_{i,j}(\overline{\hat{V}_i^3 \cap \tilde{V}_{i,j}^2}) \cap [0,\infty[^m = \psi_{i,j}(\overline{\hat{V}_i^3 \cap \tilde{V}_{i,j}^2} \cap [0,\infty[^m)\\ \subseteq&\psi_{i,j}(\overline{\hat{V}_i^3} \cap \overline{\tilde{V}_{i,j}^2} \cap [0,\infty[^m) = \psi_{i,j}(\overline{V_i^3} \cap \overline{V_{i,j}^2}).\end{aligned}$$

Given $x \in V_i^2$ we find an open x-neighbourhood $\tilde{V}_{i,x}^2$ with $x \in V_{i,j}^2$ implies $\tilde{V}_{i,x}^2 \subseteq \tilde{V}_{i,j}^2$. This is possible because I is finite and so $\bigcap_{j\in I,x\in\tilde{V}_{i,j}^2} \tilde{V}_{i,j}^2$ is open. Now we shrink $\tilde{V}_{i,x}^2$ such that $x \in \psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2}) \subseteq \tilde{Z}_{i,j} \cup \tilde{X}_{i,j}$ implies $\tilde{V}_{i,x}^2 \subseteq \tilde{Z}_{i,j} \cup \tilde{X}_{i,j}$; again this is possible because I is finite. Suppose $\varphi_i^{-1}(x) \notin \overline{U_j^3}$, then

$$x \notin \varphi_i(\overline{U_j^3} \cap U_i^2) = \varphi_i(\overline{U_j^3} \cap \overline{U_i^2}) \supseteq \varphi_i(\varphi_j^{-1}(\overline{V_j^3} \cap \overline{V_{j,i}^2})) = \psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2}) \supseteq \overline{\psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2)}.$$

Hence we can shrink $\tilde{V}_{i,x}^2$ such that $\varphi_i^{-1}(x) \notin \overline{U_j^3}$ implies $\tilde{V}_{i,x}^2 \cap \psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2) = \emptyset$.

We calculate

$$V_{i,j}^2 \cap V_{i,k}^2 = \varphi_i(U_i^2 \cap U_j^2 \cap U_k^2) = \varphi_i(\varphi_j^{-1}(\varphi_j(U_i^2 \cap U_j^2) \cap \varphi_j(U_j^2 \cap U_k^2))).$$

Let $S \subseteq \mathbb{R}^m$ be open with $V_{i,j}^2 \cap V_{i,k}^2 = S \cap [0,\infty[^m$. Given $x \in V_{i,j}^2 \cap V_{i,k}^2$ we get $x \in \psi_{j,i}(\tilde{V}_{j,i}^2 \cap \tilde{V}_{j,k}^2)$. Analogously we get $x \in \psi_{j,i}(\tilde{V}_{j,i}^2 \cap \tilde{V}_{j,k}^2)$. Therefore we can shrink $\tilde{V}_{i,x}^2$ such that $x \in V_{i,j}^2 \cap V_{i,k}^2$ implies $\tilde{V}_{i,x}^2 \subseteq \psi_{j,i}(\tilde{V}_{j,i}^2 \cap \tilde{V}_{j,k}^2) \cap \psi_{k,i}(\tilde{V}_{k,i}^2 \cap \tilde{V}_{k,j}^2)$. Moreover by replacing $\tilde{V}_{i,x}^2$ with $\tilde{V}_{i,x}^2 \cap S$ we can assume $\tilde{V}_{i,x}^2 \cap [0,\infty[\subseteq V_{i,j}^2 \cap V_{i,k}^2$. Because $x \in S$ we get $x \in \tilde{V}_{i,x}^2$. Now we shrink $\tilde{V}_{i,x}^2$ further by replacing $\tilde{V}_{i,x}^2$ with the connected component of x in $\tilde{V}_{i,x}^2$. The maps $\psi_{i,j}|_{\tilde{V}_{i,x}^2}$ and $\psi_{k,j} \circ \psi_{i,k}|_{\tilde{V}_{i,x}^2}$ are real analytic and coincide on $\tilde{V}_{i,x}^2 \cap [0,\infty[^m \subseteq V_{i,j}^2 \cap V_{i,k}^2$ hence they coincide on the connected set $\tilde{V}_{i,x}^2$. Now we define the open set $\tilde{V}_i^2 := \bigcup_{x\in V_i^2} V_{i,x}^2$ that is a neighbourhood of V_i^2 in $\mathbb{R}^m$ and so $\tilde{V}_i^2$ is also a neighbourhood of $\overline{V_i^3}$. Hence we find an open neighbourhood $\tilde{V}_i^3$ of $\overline{V_i^3}$ with $\tilde{V}_i^3 \subseteq \tilde{V}_i^2 \cap \hat{V}_i^3$ and $\overline{\tilde{V}_i^3} \subseteq \tilde{V}_i^2$. We get $\tilde{V}_i^3 \cap [0,\infty[^m = \tilde{V}_i^2 \cap \hat{V}_i^3 \cap [0,\infty[^m = \tilde{V}_i^2 \cap V_i^3 = V_i^3$ and so $\overline{V_i^3} \subseteq \overline{\tilde{V}_i^3} \cap [0,\infty[^m$. We also get $\overline{\tilde{V}_i^3} \cap [0,\infty[^m \subseteq \overline{\tilde{V}_i^2} \cap \overline{\hat{V}_i^3} \cap [0,\infty[^m = \overline{\tilde{V}_i^2} \cap \overline{V_i^3} = \overline{V_i^3}$. Defining $\tilde{V}_{i,j}^3 := \tilde{V}_i^3 \cap \psi_{j,i}(\tilde{V}_j^3 \cap \tilde{V}_{j,i}^2)$ for $i,j \in I$ we get $\psi_{i,j}(\tilde{V}_{i,j}^3) = \tilde{V}_{j,i}^3$.

Now we define the sets $\tilde{V}_{i,j,k}^3 := \tilde{V}_{i,j}^3 \cap \tilde{V}_{i,k}^3$ and want to show $\psi_{i,j}(\tilde{V}_{i,j,k}^3) = \tilde{V}_{j,i,k}^3$ and $\psi_{i,j}|_{\tilde{V}_{i,j,k}^3} = \psi_{k,j} \circ \psi_{i,k}|_{\tilde{V}_{i,j,k}^3}$. If $y \in \tilde{V}_{i,j,k}^3$ we find $x \in V_i^2$ with $y \in \tilde{V}_{i,x}^2$. Hence $y \in \tilde{V}_{i,x}^2 \cap \psi_{j,i}(\tilde{V}_j^3 \cap \tilde{V}_{j,i}^2) \cap \psi_{k,i}(\tilde{V}_k^3 \cap \tilde{V}_{k,i}^2) \subseteq \tilde{V}_{i,x}^2 \cap \psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2) \cap \psi_{k,i}(\hat{V}_k^3 \cap \tilde{V}_{k,i}^2)$. Since $\varphi_i^{-1}(x) \notin \overline{U_j^3} \Rightarrow \tilde{V}_{i,x}^2 \cap \psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2) = \emptyset$ and $\varphi_i^{-1}(x) \notin \overline{U_k^3} \Rightarrow \tilde{V}_{i,x}^2 \cap \psi_{k,i}(\hat{V}_k^3 \cap \tilde{V}_{k,i}^2) =$

$\varnothing$ we get $\varphi_i^{-1}(x) \in \overline{U_j^3}$. Hence $x \in \varphi_i(U_i^2 \cap U_j^2) \cap \varphi_i(U_i^2 \cap U_k^2) = V_{i,j}^2 \cap V_{i,k}^2$. Therefore $\psi_{i,j}|_{\tilde{V}_{i,x}^2} = \psi_{k,j} \circ \psi_{i,k}|_{\tilde{V}_{i,x}^2}$ and $\tilde{V}_{i,x}^2 \subseteq \psi_{j,i}(\tilde{V}_{j,i}^2 \cap \tilde{V}_{j,k}^2) \cap \psi_{k,i}(\tilde{V}_{k,i}^2 \cap \tilde{V}_{k,j}^2)$. Especially $y \in \psi_{j,i}(\tilde{V}_{j,i}^2 \cap \tilde{V}_{j,k}^2) \cap \psi_{k,i}(\tilde{V}_{k,i}^2 \cap \tilde{V}_{k,j}^2)$ and $\psi_{i,j}(y) = \psi_{k,j} \circ \psi_{i,k}(y)$. Hence $\psi_{i,k}(y) \in \tilde{V}_{k,j}^2$. Because $y \in \tilde{V}_{i,j,k}^3 = \tilde{V}_{i,j}^3 \cap \tilde{V}_{i,k}^3$ we get $\psi_{i,k}(y) \in \tilde{V}_k^3 \cap \tilde{V}_{k,j}^2$. Thus $\psi_{k,j}(\psi_{i,k}(y)) \in \psi_{k,j}(\tilde{V}_k^3 \cap \tilde{V}_{k,j}^2)$. On the other hand $\psi_{i,j}(y) \in \tilde{V}_j^3$, because $y \in \tilde{V}_{i,j}^3$. Hence $\psi_{i,j}(y) = \psi_{k,j}(\psi_{i,k}(y)) \in \tilde{V}_j^3 \cap \psi_{k,j}(\tilde{V}_k^3 \cap \tilde{V}_{k,j}^2) = \tilde{V}_{j,k}^3$. Moreover we have $\psi_{i,j}(y) \in \tilde{V}_{j,i}^3$, because $y \in \tilde{V}_{i,j}^3$. Therefore $\psi_{i,j}(y) \in \tilde{V}_{j,i,k}^3$. We get $\psi_{i,j}(\tilde{V}_{i,j,k}^3) \subseteq \tilde{V}_{j,i,k}^3$ and because $\psi_{i,j}^{-1} = \psi_{j,i}$ we see that $\psi_{i,j}(\tilde{V}_{i,j,k}^3) = \tilde{V}_{j,i,k}^3$.

Now we define the topological space $\tilde{M}^1 := \coprod_{i \in I} \tilde{V}_i^3$ as the disjoint topological union and on $\tilde{M}^1$ we define a relation $\sim$: Let $x, y \in \tilde{M}^1$, say $x \in \tilde{V}_i^3$ and $y \in \tilde{V}_j^3$. We call x equivalent to y if $x \in \tilde{V}_{i,j}^3$, $y \in \tilde{V}_{j,i}^3$ and $y = \psi_{i,j}(x)$. To show that $\sim$ is an equivalence relation on $\tilde{M}^1$ we have to show its transitivity. Let $x \in \tilde{V}_i^3$, $y \in \tilde{V}_j^3$ and $z \in \tilde{V}_k^3$. Moreover let $z \in \tilde{V}_{k,j}^3$, $y \in \tilde{V}_{j,k}^3$, $y \in \tilde{V}_{j,i}^3$, $x \in \tilde{V}_{i,j}^3$, $z = \psi_{j,k}(y)$ and $y = \psi_{i,j}(x)$. Directly we get $y \in \tilde{V}_{j,i}^3 \cap \tilde{V}_{j,k}^3 = \tilde{V}_{j,i,k}^3$. Hence $x = \psi_{j,i}(y) \in \psi_{j,i}(\tilde{V}_{j,i,k}^3) \in \tilde{V}_{i,j,k}^3$ and $z = \psi_{j,k}(y) \in \tilde{V}_{k,j,i}^3$, because $y \in \tilde{V}_{j,i,k}^3 = \tilde{V}_{j,k,i}^3$. Moreover we have $\psi_{i,k}(x) = \psi_{j,k}(\psi_{i,j}(x)) = z$ and so x and z are equivalent. Now we define $\tilde{M} := \tilde{M}^1/\sim$ as the topological quotient. Let $\pi\colon \tilde{M}^1 \to \tilde{M},\ x \mapsto [x]$ be the canonical quotient map. Given $j \in I$ let $\iota_j\colon \tilde{V}_j^3 \hookrightarrow \tilde{M}^1,\ x \mapsto (x, j)$ be the canonical inclusion. The topology on $\tilde{M}$ is final with respect to the maps $\pi \circ \iota_i\colon \tilde{V}_i^3 \to \tilde{M}$ with $i \in I$. We show that the maps $\pi \circ \iota_i\colon \tilde{V}_i^3 \to \tilde{M}$ are open. To this end let $U \subseteq \tilde{V}_i^3$ be open and $j \in I$. We calculate

$$\iota_j^{-1}(\pi^{-1}(\pi(U))) = \iota_j^{-1}\left(\left\{(y,k) \in \tilde{M}^1 : (\exists x \in U \subseteq \tilde{V}_i^3)\ y \sim x\right\}\right) \tag{B.1}$$

$$= \left\{y \in \tilde{V}_j^3 : (\exists x \in U \subseteq \tilde{V}_i^3)\ y \sim x\right\} = \psi_{j,i}^{-1}(U) \subseteq \tilde{V}_{j,i}^3. \tag{B.2}$$

Hence $\pi \circ \iota_i\colon \tilde{V}_i^3 \to \tilde{M}$ is continuous and open. Now we define the maps $\psi_i\colon \pi(\tilde{V}_i^3) \to \tilde{V}_i^3,\ p \mapsto x$ if $\pi(x) = p$ and $x \in \tilde{V}_i^3$. The map ψ_i is well-defined because $\psi_{i,i} = \mathrm{id}_{\tilde{V}_i^3}$. Moreover ψ_i is bijective because its inverse is given by $\psi_i^{-1} = \pi \circ \iota_i\colon \tilde{V}_i^3 \to \pi(\tilde{V}_i^3),\ x \mapsto \pi(x)$. Hence ψ_i is a homeomorphism. To show that the maps ψ_i form a real analytic atlas for $\tilde{M}$ we mention $\psi_i^{-1}(\tilde{V}_j^3) = \tilde{V}_{i,j}^3$ and calculate for $x \in \tilde{V}_{i,j}^3$

$$\psi_j \circ \psi_i^{-1}(x) = \psi_j(\pi(x)) = \psi_{j,i}(x).$$

Now we show that $\tilde{M}$ is a Hausdorff space. To this end, we show $\overline{\tilde{V}_{i,j}^3} \subseteq \tilde{V}_{i,j}^2$. Given $y \in \tilde{V}_{i,j}^3 \subseteq \tilde{V}_i^2$ we find $x \in V_i^2$ with $y \in \tilde{V}_{i,x}^2$. We want to show $x \in \psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2})$. If this was not true, then $\varphi_i^{-1}(x) \notin \overline{U_j^3}$. With $\varphi^{-1}(x) \notin \overline{U_j^3} \Rightarrow \tilde{V}_{i,x}^2 \cap \psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2) = \varnothing$ we get $y \notin \psi_{j,i}(\hat{V}_j^3 \cap \tilde{V}_{j,i}^2) \supseteq \psi_{j,i}(\tilde{V}_j^3 \cap \tilde{V}_{j,i}^2) \supseteq \psi_{j,i}(\tilde{V}_{j,i}^3) = \tilde{V}_{i,j}^3$. But since this is a contradiction we get $x \in \psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2})$. With $x \in \psi_{j,i}(\overline{V_j^3} \cap \overline{V_{j,i}^2}) \Rightarrow \tilde{V}_{i,x}^2 \subseteq \tilde{Z}_{i,j} \cup \tilde{X}_{i,j}$ we get $y \in \tilde{Z}_{i,j} \cup \tilde{X}_{i,j}$. Moreover we have $y \in \tilde{V}_i^3 \subseteq \hat{V}_i^3 \subseteq \tilde{Y}_{i,j} \cup \underline{\tilde{Z}_{i,j}}$ and with $\tilde{Y}_{i,j} \cap \tilde{X}_{i,j} = \varnothing$ we get $y \in \tilde{Z}_{i,j}$. Hence $\tilde{V}_{i,j}^3 \subseteq \tilde{Z}_{i,j}$ and therefore $\overline{\tilde{V}_{i,j}^3} \subseteq \overline{\tilde{Z}}_{i,j} \subseteq \tilde{V}_{i,j}^2$.

Now let $p \neq q \in \tilde{M}$. We choose $x, y \in \tilde{M}^1$ with $\pi(x) = p$ and $\pi(y) = q$. Let $x \in \tilde{V}_i^3$ and $y \in \tilde{V}_j^3$. If there are an open x-neighbourhood $W_x \subseteq \tilde{V}_i^3$ and an open y-neighbourhood $W_y \subseteq \tilde{V}_j^3$ with $\pi(W_x) \cap \pi(W_y) = \varnothing$ then $\tilde{M}$ has to be Hausdorff, because of (B.1). Suppose there do not exist such neighbourhoods W_x and W_y. Then we find a sequence $(x_n)_{n\in\mathbb{N}}$ in $\tilde{V}_i^3$ and a sequence $(y_n)_{n\in\mathbb{N}}$ in $\tilde{V}_j^3$ with $x_n \sim y_n$ for all $n \in \mathbb{N}$. Hence $x_n \in \tilde{V}_{i,j}^3$ and $y_n \in \tilde{V}_{j,i}^3$ and so $x \in \overline{\tilde{V}_{i,j}^3} \subseteq V_{i,j}^2$ and $y \in \overline{\tilde{V}_{j,i}^3} \subseteq V_{j,i}^2$. Since $y_n = \psi_{i,j}(x_n)$ for all $n \in \mathbb{N}$ we get $y = \psi_{i,j}(x)$. Therefore $y \in \tilde{V}_j^3 \cap \psi_{i,j}(\tilde{V}_i^3 \cap \tilde{V}_{i,j}^2) = \tilde{V}_{j,i}^3$. With $x = \psi_{j,i}(y)$ we get $x \in \tilde{V}_{i,j}^3$. We conclude $x \sim y$. But this contradicts $p \neq q$.

We define the map $\varphi\colon M \to \tilde{M}$ by $\varphi|_{U_i^3} := \pi \circ \iota_i \circ \varphi_i$. To see that φ is well-defined choose $p \in U_i^3 \cap U_j^3$. We get $\varphi_i(p) \in \tilde{V}_i^3$ and $\varphi_j(p) \in \tilde{V}_j^3$ moreover we have $\varphi_i(p) \in \varphi_i(U_i^3 \cap U_j^3) = V_{i,j}^3 \subseteq \tilde{V}_{i,j}^3$, $\varphi_j(p) \in \varphi_j(U_i^3 \cap U_j^3) = V_{j,i}^3 \subseteq \tilde{V}_{j,i}^3$ and $\psi_{i,j}(\varphi_i(p)) = \varphi_j(p)$. Hence $\varphi_i(p) \sim \varphi_j(p)$, and so φ is well-defined. Now we show that φ is injective. Let $p_1, p_2 \in M$ with $\varphi(p_1) = \varphi(p_2)$ and $p_1 \in U_i^3$ and $p_2 \in U_j^3$. We conclude $\varphi_i(p_1) \sim \varphi_j(p_2)$ and so $\varphi_i(p_1) \in \tilde{V}_{i,j}^3 \cap [0,\infty[^m$, $\varphi_j(p_2) \in \tilde{V}_{j,i}^3$ and $\psi_{i,j}(\varphi_i(p_1)) = \varphi_j(p_2)$. Hence $\varphi_j(p_1) = \varphi_j(p_2)$ and so $p_1 = p_2$. We give $\varphi(M)$ the real analytic structure such that φ becomes an real analytic diffeomorphism. If we can show that $\tilde{M}$ is an enveloping manifold of $\varphi(M)$ we are done, because we can identify M and $\varphi(M)$. We have $\varphi(M) = \pi(\coprod_{i\in I} V_i^3)$ with $\coprod_{i\in I} V_i^3 \subseteq \coprod_{i\in I} \tilde{V}_i^3$. If $x \in V_i^3$ then $\psi_i\colon \pi(\tilde{V}_i^3) \to \tilde{V}_i^3$ is a chart of $\tilde{M}$ around $\pi(x)$. We show $\psi_i(\pi(\tilde{V}_i^3) \cap \varphi(M))) = V_i^3$. Let $p = \pi(x)$ with $x \in \tilde{V}_i^3$ and $p = \pi(y)$ with $y \in V_j^3$. Then $x \sim y$ and so $x \in V_i^3$ because $x = \psi_{j,i}(y) \in \psi_{j,i}(V_{j,i}^3) = \varphi_i \circ \varphi_j(V_{j,i}^3) \subseteq V_i^3$. Now let $x \in V_i^3$. Obviously $x = \psi_i(\pi(x))$ and $\pi(x) \in \pi(\tilde{V}_i^3) \cap \varphi(M)$. It is left to show that $\psi_i|_{\pi(\tilde{V}_i^3)\cap\varphi(M))}^{V_i^3}$ is a chart of $\varphi(M)$. To this end we show that $\psi_i \circ \varphi|_{\varphi^{-1}(\pi(\tilde{V}_i^3)\cap\varphi(M))}$ is a chart of M. First, we show that $\varphi^{-1}(\pi(\tilde{V}_i^3) \cap \varphi(M)) = U_i^3$. Let $p \in M$ with $\varphi(p) \in \pi(\tilde{V}_i^3) \cap \varphi(M)$. We find $j \in I$ with $p \in U_j^3$. Moreover we find $i \in I$ and $x \in \tilde{V}_i^3$ with $\varphi(p) \sim x$. Hence $\varphi_j(p) \sim x$. Therefore $\psi_{j,i}(\varphi_j(p)) = x$ and so $\varphi_i(p) = x$. We conclude $p \in U_i^3$. Now let $p \in U_i^3$. Then $\varphi(p) = \pi(\varphi_i^{-1}(p)) \in \pi(\tilde{V}_i^3)$, because $\varphi_i^{-1}(p) \in V_i^3$. Now we show $\psi_i \circ \varphi|_{\varphi^{-1}(\pi(\tilde{V}_i^3)\cap\varphi(M))} = \varphi_i$. Let $p \in \varphi^{-1}(\pi(\tilde{V}_i^3) \cap \varphi(M)) = U_i^3$. Then $\psi_i \circ \varphi(p) = \psi_i(\pi(\varphi_i(p))) = \varphi_i(p)$. □

C. Details for the proof of Theorem 3.40

In this chapter we state the rest of the proof of Theorem 3.40[1]. This part of the proof is stated in the appendix and not in Chapter 3, because its arguments are completely analogous to the proof of [Gun11, Theorem 5.1.10] and we just recall them for the convenience of the reader.

Proof. We use the notation from the proof of Theorem 3.40. First, we show that ω is bilinear. For $f, g \in A \otimes \mathfrak{g}$, $r \in \mathbb{R}$ and $y \in \mathfrak{g}$ we can choose a neutral triple (λ, ν, μ) that is neutral for f and for g. Especially this triple is also neutral for $rf + g$. Because we now get $\omega_0(rf + g, \lambda_{rf+g} \otimes y) = \omega_0(rf + g, \lambda \otimes y) = r\omega_0(f, \lambda \otimes y) + \omega_0(g, \lambda \otimes y) = r\omega_0(f, \lambda_f \otimes y) + \omega_0(g, \lambda_g \otimes y)$, one can easily prove that ω is bilinear (see also Lemma 3.38). Obviously ω is anti-symmetric. To show that $\omega \in Z^2(A \otimes \mathfrak{g} \rtimes \mathfrak{g}, V)$ we choose $f, g, h \in A \otimes \mathfrak{g}$ and $x, y, z \in \mathfrak{g}$. First we mention the trivialities $d\omega(f, g, h) = d\omega_0(f, g, h) = 0$ and $d\omega(x, y, z) = 0$. We can choose a triple (λ, ν, μ) that is neutral for f and g, and we can write $f = \sum_{i=1}^n f_i \otimes v_i$ as well as $g = \sum_{j=1}^n g_j \otimes v_j$. We calculate

$$\lambda \cdot [f, g] = \sum_{i,j} \lambda f_i g_j \otimes [v_i, v_j] = [\lambda f, g] = [f, g]$$

and see that (λ, ν, μ) is a neutral triple for $[f, g]$. Now we calculate

$$\begin{aligned} &d\omega(f, g, y) = \omega([f, g], y) + \omega([g, y], f) + \omega([y, f], g) \\ =&\omega_0([f, g], \lambda \otimes y) + \omega_0([g, y], f) + \omega_0([y, f], g) \\ =&\omega_0([f, g], \lambda \otimes y) + \omega_0([g, \lambda \otimes y], f) + \omega_0([\lambda \otimes y, f], g) = \quad d\omega_0(f, g, \lambda \otimes y) = 0. \end{aligned}$$

To check that ω is a cocycle we calculate

$$\begin{aligned} &d\omega((f, x), (g, y)(h, z)) = d\omega(f, g, h) + d\omega(f, g, z) + d\omega(f, y, h) + d\omega(f, y, z) \\ &+ d\omega(x, g, h) + d\omega(x, g, z) + d\omega(x, y, h) + d\omega(x, y, z) = 0. \end{aligned}$$

It remains to show the injectivity of the map $H^2_{ct}(i)$. Let $\omega \in Z^2_{ct}(A \otimes \mathfrak{g} \rtimes \mathfrak{g}, V)$ with $\omega \circ (i, i) = \eta \circ [\bullet, \bullet]$ for $\eta \in \mathcal{L}(A \otimes \mathfrak{g}, V)$. We define the continuous linear map $\eta' \colon A \otimes \mathfrak{g} \rtimes \mathfrak{g} \to V$, $(f, v) \mapsto \eta(f)$. We define the cocycle $\omega' := \omega - \eta' \circ [\bullet, \bullet]$ on $A \otimes \mathfrak{g} \rtimes \mathfrak{g}$. If we can show $[\omega'] = 0$ in $H^2_{ct}(A \otimes \mathfrak{g}, V)$ we are done. First of all we have $\omega'(f, g) = \omega(f, g) - \eta' \circ [f, g] = \omega \circ (i, i)(f, g) - \eta([f, g]) = 0$ for all $f, g \in A \otimes \mathfrak{g}$.

[1] This chapter consist of material published before in the author's preprint [Eyn14c].

For $f, g \in A \otimes \mathfrak{g}$ and $y \in \mathfrak{g}$ we calculate

$$0 = -d\omega'(f, g, y) = \omega'([f, g], y) + \underbrace{\omega'([g, y], f)}_{=0} + \underbrace{\omega'([y, f], g)}_{=0} = \omega'([f, g], y).$$

Because $A \otimes \mathfrak{g}$ is perfect, we get that ω' equals 0 on $A \otimes \mathfrak{g} \times \mathfrak{g}$ in terms of the natural identifications. For $f_1, f_2 \in A \otimes \mathfrak{g}$ and $y_1, y_2 \in \mathfrak{g}$ we have

$$\omega'((f_1, y_1), (f_2, y_2)) = \underbrace{\omega'(f_1, f_2)}_{=0} + \underbrace{\omega'(y_1, f_2)}_{=0} + \underbrace{\omega'(f_1, y_2)}_{=0} + \omega'(y_1, y_2).$$

Because $\mathfrak{g}$ is a subalgebra of $A \otimes \mathfrak{g} \rtimes \mathfrak{g}$ we get $\omega|_{\mathfrak{g} \times \mathfrak{g}} \in Z^2_{ct}(\mathfrak{g}, V)$ and because $\mathfrak{g}$ is semisimple, we get with the Whitehead theorem for locally convex spaces, stated in [Gun11, Corollary A.2.9], that $H^2_{ct}(\mathfrak{g}, V) = \{0\}$. Therefore, we find $\eta'' \in \mathcal{L}(\mathfrak{g}, V)$ with $\omega|_{\mathfrak{g} \times \mathfrak{g}} = \eta'' \circ [\bullet, \bullet]$. Finally we see $\omega' = \eta''' \circ [\bullet, \bullet]$ with $\eta''' \colon A \otimes \mathfrak{g} \rtimes \mathfrak{g} \to V$, $(f, v) \mapsto \eta''(v)$. □

D. Some differential topology

In this chapter we present some topological considerations.[1]

Lemma D.1. *We use the notation introduced at the beginning of Chapter 4.1. Let $H \hookrightarrow P \xrightarrow{q} M$ be a finite-dimensional smooth principal bundle (with σ-compact total space P), $\rho\colon H \times V \to V$ be a finite-dimensional smooth linear representation and $\mathbb{V} := P \times_\rho V$ be the associated vector bundle.*

(a) The canonical isomorphism of vector spaces (see e.g. [Bau14, Satz 3.5]) $\Phi\colon \Omega^k(P,V)^{\mathrm{hor}}_\rho \to \Omega^k(M,\mathbb{V})$, $\omega \mapsto \tilde{\omega}$ (with $\tilde{\omega}_x(v_1,\dots,v_k) = \omega_{\sigma(x)}(T\sigma(v_1),\dots,T\sigma(v_k))$ for a local section ($\sigma\colon U \to P$ of $P \xrightarrow{q} M$ and $x \in U$) is in fact an isomorphism of topological vector spaces.

(b) The isomorphism of vector spaces $\Phi\colon \Omega^k_c(P,V)^{\mathrm{hor}}_\rho \to \Omega^k_c(M,\mathbb{V})$, $\omega \mapsto \tilde{\omega}$ is an isomorphism of topological vector spaces.

Proof. (a) We choose an atlas $\psi_i\colon q^{-1}(U_i) \to U \times H$ of trivialisations of P with $i \in I$. Let $\sigma_i := \psi_i^{-1}(\bullet, 1_H)$ be the canonical section corresponding to ψ_i. As $\Omega^k(P,V)^{\mathrm{hor}}_\rho$ and $\Omega^k(M,\mathbb{V})$ are Fréchet spaces it is enough to show the continuity of Φ (Open mapping theorem). The topology on $\Omega^k(M,\mathbb{V}) = \Gamma(\mathrm{Alt}^k(TM,\mathbb{V}))$ is initial with respect to the maps $\Gamma(\mathrm{Alt}^k(TM,\mathbb{V})) \to \Gamma(\mathrm{Alt}^k(TU_i,\mathbb{V}|_{U_i}))$, $\eta \mapsto \eta|_{U_i}$. Given $\omega \in \Omega^k(P,V)^{\mathrm{hor}}_\rho$, $x \in U_i$ and $v \in T_xU_i$, we have $(\tilde{\omega}|_{U_i})_x(v) = [\sigma_i(x), \sigma_i^*\omega_x(v)]$. Because $\Gamma(\mathrm{Alt}^k(TU_i,\mathbb{V}|_{U_i})) \cong \Gamma(\mathrm{Alt}^k(TU_i,V)) \cong \Omega^k(U_i,V)$ it is enough to show the continuity of $\Omega^k(P,V)^{\mathrm{hor}}_{\rho_V} \to \Omega^k(U_i,V)$, $\omega \mapsto \sigma_i^*\omega$. The map $C^\infty((TP)^k,V) \to C^\infty((TU_i)^k,V)$, $f \mapsto f \circ T\sigma_i \times \ldots \times T\sigma_i$ is continuous (see [GN]). Now, the assertion follows because we can embed $\Omega^k(P,V)^{\mathrm{hor}}_{\rho_V}$ into $C^\infty((TP)^k,V)$.

(b) The analogous map from $\Omega^k(P,V)^{\mathrm{hor}}_\rho$ to $\Omega^k(M,\mathbb{V})$ is continuous. Hence, given a compact set $K \subseteq M$, we get that the corresponding map from $\Omega^k_K(P,V)^{\mathrm{hor}}_\rho$ to $\Omega^k_K(M,\mathbb{V})$ is continuous. Therefore Φ is continuous. The same argument shows that the inverse of Φ is continuous.

□

The basic considerations in the following lemma seem to be part of the folklore.

Lemma D.2. *Given the situation of Definition 4.3 the following holds:*

(a) The vertical bundle of $\overline{H} \hookrightarrow \overline{P} \xrightarrow{\overline{q}} M$ is given by $V\overline{P} = T\pi(VP))$ and $H\overline{P} := T\pi(HP)$ is a principal connection on $\overline{P}$.

(b) Given $k \in \mathbb{N}_0$ the pullback $\pi^*\colon \Omega^k(\overline{P},V)^{\mathrm{hor}}_{\overline{\rho}_V} \to \Omega^k(P,V)^{\mathrm{hor}}_{\rho_V}$, $\theta \mapsto \pi^*\theta$ is an isomorphism of topological vector spaces and an isomorphism of chain complexes.

[1] This chapter consist of material published before in the author's preprint [Eyn14b].

(c) *Given $k \in \mathbb{N}_0$ the pullback $\pi^*\colon \Omega_c^k(\overline{P}, V)_{\overline{\rho}_V}^{\mathrm{hor}} \to \Omega_c^k(P, V)_{\rho_V}^{\mathrm{hor}}$, $\theta \mapsto \pi^*\theta$ is an isomorphism of topological vector spaces and an isomorphism of chain complexes.*

Proof. (a) First we show $T\pi(VP) \subseteq \ker(T\overline{q})$. For $v \in VP$ we get $T\overline{q}(T\pi(v)) = T\overline{q} \circ \pi(v) = Tq(v) = 0$. To see $\ker(T\overline{q}) \subseteq T\pi(VP)$ let $T\overline{q}(w) = 0$. We find $v \in TP$ with $T\pi(v) = w$. Hence $T\overline{q}(T\pi(v)) = T\overline{q} \circ \pi(v) = Tq(v) = 0$. Thus $v \in VP$ and so $w \in T\pi(VP)$. Now, we show that $T\pi(HP)$ is a smooth sub vector bundle of $T\overline{P}$. Let $\overline{x} \in \overline{P}$. Obviously $(T\pi(HP))_{\overline{x}} := T_{\overline{x}}\overline{P} \cap T\pi(HP)$ is closed under scalar multiplication. Let $v, w \in (T\pi(HP))_{\overline{x}} = T_x\overline{P} \cap T\pi(HP)$. We find $p_1, p_2 \in P$, $v_1 \in H_{p_1}P$ and $w_2 \in H_{p_2}P$ with $T_{p_1}\pi(v_1) = v$ and $T_{p_2}\pi(w_2) = w$. Hence $\pi(p_1) = \overline{x} = \pi(p_2)$. Therefore we find $n \in N$ with $p_1 = p_2 \cdot n$ and $\tilde{w} \in T_{p_1}P$ with $TR_n(\tilde{w}) = w_2$. Now, we calculate

$$\begin{aligned} v + w &= T_{p_1}\pi(v_1) + T_{p_2}\pi(w_2) = T_{p_1}\pi(v_1) + T_{p_2}\pi \circ T_{p_1}R_n(\tilde{w}) \\ &= T_{p_1}\pi(v_1) + T_{p_1}\pi \circ TR_n(\tilde{w}) = T_{p_1}\pi(v_1) + T_{p_1}\pi(\tilde{w}) = T_{p_1}\pi(v_1 + \tilde{w}). \end{aligned}$$

Next we show that HP is a smooth sub vector bundle. Let $\overline{p} \in \overline{P}$. Because π is a submersion, we find a smooth local section $\tau\tilde{V} \to P$ of π on an open $\overline{p}$-neighbourhood $\tilde{V} \subseteq \overline{P}$. We define $p := \tau(\overline{p})$ and find a smooth local frame $\sigma_1, \ldots, \sigma_m\colon \tilde{U} \to TP$ of the smooth sub vector bundle HP on a p-neighbourhood $\tilde{U} \subseteq P$. Without loss of generality we can assume $\tau(\tilde{V}) \subseteq \tilde{U}$. Given $i \in \{1, \ldots, m\}$ we define the smooth map

$$\overline{\sigma}_i\colon \tilde{V} \to T\overline{P},\ \overline{x} \mapsto T\pi(\sigma_i \circ \tau(\overline{x})).$$

The map $\overline{\sigma}_i$ is a section for the tangential bundle $T\overline{P}$ because for $\overline{x} \in \tilde{V}$ we have $\sigma_i \circ \tau(\overline{x}) \in T_{\tau(\overline{x})}P$ and thus $\overline{\sigma}_i(\overline{x}) \in T_{\pi(\tau(\overline{x}))}\overline{P} = T_{\overline{x}}\overline{P}$. Let $\overline{x} \in \tilde{V}$. Now, we show that $(\sigma_i(\overline{x}))_{i=1,\ldots,m}$ is a basis of $(T\pi(HP))_{\overline{x}} = T_{\overline{x}}\overline{P} \cap T\pi(HP)$. Let $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^m \lambda_i \cdot \overline{\sigma}(\overline{x}) = 0$. We conclude $T_{\tau(\overline{x})}\pi(\sum_{i=1}^m \lambda_i \cdot \sigma_i(\tau(\overline{x}))) = 0$. Hence

$$Tq\left(\sum_{i=1}^m \lambda_i \cdot \sigma_i(\tau(\overline{x}))\right) = T\overline{q}\left(T\pi\left(\sum_{i=1}^m \lambda_i \cdot \sigma_i(\tau(\overline{x}))\right)\right) = 0.$$

Therefore $\sum_{i=1}^m \lambda_i \cdot \sigma_i(\tau(\overline{x})) \in V_{\tau(\overline{x})}P$ and thus $\lambda_i = 0$ for $i = 1, \ldots, m$. Let $p \in P$ with $\pi(p) = \overline{x}$. One easily sees that the linear map $(T_p\pi)|_{H_pP}\colon H_pP \to (T\pi(HP))_x$ is a surjection (see above). Because $m = \dim(H_pP)$ the linearly independent system $\overline{\sigma}_i(\overline{x})_{i=1,\ldots,m}$ is a basis of $(T\pi(HP))_{\overline{x}}$. Now, we show that $H\overline{P} := T\pi(HP)$ is a principal connection on $\overline{P}$. Because π is a submersion and $T_pP = H_pP \oplus V_pP$ we get $V_{\overline{x}}\overline{P} + H_{\overline{x}}\overline{P} = T_{\overline{x}}\overline{P}$ for $\overline{x} \in \overline{P}$. If $T_p\pi(v) = T_{p'}\pi(w)$ with $v \in V_pP$, $w \in H_{p'}P$ and $\pi(p) = \pi(p') =: \overline{p}$ we get

$$T_{\overline{p}}\overline{q} \circ \pi(v) = T_{\overline{p}}\overline{q} \circ \pi(w).$$

Hence $0 = Tq(v) = Tq(w)$. Thus $w \in V_{p'}P$. Therefore $w = 0$ and so $T_p\pi(v) = T_{p'}(w) = 0$ in $T_{\overline{p}}\overline{P}$. We conclude $V\overline{P} \oplus H\overline{P} = T\overline{P}$. It is left to show that $H\overline{P}$ is invariant under the action of $\overline{H}$. Obviously it is enough to show $T_{\overline{x}}\overline{R}_{[g]}(H_{\overline{x}}P) \subseteq H_{\overline{x}[g]}\overline{P}$ for $\overline{x} \in \overline{P}$ and $[g] \in \overline{H}$. Let $v \in H_{\overline{x}}\overline{P}$. We find $p \in P$ and $w \in H_pP$ with $v = T_p\pi(w)$. With $\overline{R}_{[g]} \circ \pi = \pi \circ R_g$ and $\pi(pg) = \overline{x}.[g]$ we calculate

$$\begin{aligned} &T\overline{R}_{[g]}(v) = T_p(\overline{R}_{[g]} \circ \pi)(w) = T_{pg}\pi \circ T_pR_g(w) \in T_{pg}\pi(H_{pg}P) \\ \subseteq &T_{\overline{x}[g]}\overline{P} \cap T\pi(HP) = H_{\overline{x}[g]}\overline{P}. \end{aligned}$$

(b) First we show that π^* makes sense. Without loss of generality we assume $k = 1$. Let $\theta \in \Omega^1(\overline{P}, V)^{\text{hor}}_{\overline{\rho}_V}$. We have $\pi \circ R_g = \overline{R}_{[g]} \circ \pi$. Hence

$$\begin{aligned} &\rho_V(g) \circ R_g^*\pi^*\theta = \overline{\rho}_V([g]) \circ (\pi \circ R_g)^*\theta = \overline{\rho}_V([g]) \circ (\overline{R}_{[g]} \circ \pi)^*\theta \\ = &\pi^*(\overline{\rho}_V([g]) \circ \overline{R}_{[g]}^*\theta) = \pi^*\theta. \end{aligned}$$

Moreover if $v \in V_pP$ we get $T_p\pi(v) \in V_{\pi(p)}\overline{P}$ and so $\pi^*\theta_p(v) = \theta_{\pi(p)}(T_p\pi(v)) = 0$. We show that π^* is bijective. It is clear that π^* is injective because π is a submersion. To see that π^* is surjective let $\eta \in \Omega^1(P, V)^{\text{hor}}_{\rho_V}$. We define $\theta \in \Omega^1(\overline{P}, V)^{\text{hor}}_{\overline{\rho}_V}$ by $\theta_{\pi(p)}(T_p\pi(v)) := \eta_p(v)$ for $p \in P$ and $v \in T_pP$. To see that this is well-defined, we choose $p, r \in P$, $v \in T_pP$ and $w \in T_rP$ with $\pi(p) = \pi(r)$ and $T_p\pi(v) = T_r\pi(w)$. We find $n \in N$ with $p = r.n$. Because $\eta_{r.n}(T_rR_n(w)) = \eta_r(w)$ ($N = \ker(\rho_V)$), it is enough to show $\eta_p(v) = \eta_p(T_rR_n(w))$. We have $\pi \circ R_n = R_{[n]} \circ \pi = \pi$. Hence $T\pi \circ TR_n = T\pi$. Thus $T_p\pi(T_rR_n(w)) = T_r\pi(w) = T_p\pi(v)$. Therefore we find $x \in \ker(T_p\pi)$ with $T_rR_n(w) + x = v$ in T_pP. Hence $T_pq(x) = 0$ because $Tq = T\overline{q} \circ T\pi$. So $x \in V_pP$ and hence $\eta_p(x) = 0$. The form θ is $\overline{\rho}_V$-invariant because for $g \in H$, $p \in P$ and $v \in T_pP$ we get

$$\begin{aligned} &(\overline{\rho}_V([g]) \circ \overline{R}_{[g]}^*\theta)_{\pi(p)}(T_p\pi(v)) = \overline{\rho}_V([g]) \circ \theta_{\pi(p).[g]}(T\overline{R}_{[g]}(T_p\pi(v))) \\ = &\rho_V(g) \circ \theta_{\pi(p.g)}(T_{p.g}\pi(TR_g(v))) \\ = &\rho_V(g) \circ \eta_{p.g}(TR_g(v)) = \theta_{\pi(p)}(T\pi(v)). \end{aligned}$$

Moreover, θ is horizontal because given $u \in V_{\overline{p}}\overline{P}$ with $\overline{p} \in \overline{P}$, we find $p \in P$ with $\pi(p) = \overline{p}$ and $v \in V_pP$ with $u = T_p\pi(v)$. Hence $\theta_{\overline{p}}(u) = \theta_{\pi(p)}(T\pi(v)) = \eta_p(v) = 0$. Obviously we have $\pi^*\theta = \eta$. In order to show that π^* is an isomorphism of chain complexes we choose $p \in P$ and $v, w \in T_pP$ and calculate

$$\begin{aligned} &(\pi^*D_{\overline{\rho}_V}\theta)_p(v, w) = (D_{\overline{\rho}_V}\theta)_{\pi(p)}(T\pi(v), T\pi(w)) \\ = &(d\theta)_{\pi(p)}(\text{pr}_h \circ T\pi(v), \text{pr}_h \circ T\pi(w)) = (d\theta)_{\pi(p)}(T\pi \circ \text{pr}_h(v), T\pi \circ \text{pr}_h(w)) \\ = &(\pi^*d\theta)_p(\text{pr}_h(v), \text{pr}_h(w)) = (D_{\rho_V}\pi^*\theta)_p(v, w). \end{aligned}$$

It is left to show that π^* is a homeomorphism. Because the corresponding spaces are Fréchet-spaces it is enough to show the continuity of π^*. We can embed $\Omega^k(\overline{P}, V)^{\text{hor}}_{\overline{\rho}_V}$ into $C^\infty(T\overline{P}^k, V)$ and $\Omega^k(P, V)^{\text{hor}}_{\rho_V}$ into $C^\infty(TP^k, V)$. The map $C^\infty(T\overline{P}^k, V) \to C^\infty(TP^k, V)$, $f \mapsto f \circ (T\pi \times \cdots \times T\pi)$ is continuous (see [GN]). Now the assertion follows.

(c) This follows from (b) and the fact that $\pi^*(\Omega^k_K(\overline{P}, V)^{\mathrm{hor}}_{\overline{\rho}_V}) = \Omega^k_K(P, V)^{\mathrm{hor}}_{\rho_V}$ for a compact set $K \subseteq M$.

□

The statement in the following lemma seems to be also well-known, but since we do not have a reference for this exact result, we give a proof in the following. For this we use techniques from the proof of [Ros97, Theorem 1.5]. See also [BT82, Chapter 6].

Lemma D.3. *If* $q\colon \hat{M} \to M$ *is a smooth finite manifold covering then* $q^*\colon \Omega^1_c(M, V) \to \Omega^1_c(\hat{M}, V)$, $\theta \mapsto q^*\theta$ *induces a well-defined isomorphism of topological vector spaces* $H^1_{dR,c}(M, V) \to H^1_{dR,c}(\hat{M}, V)$, $[\theta] \mapsto [q^*\theta]$. *Therefore* $\overline{q}^*\colon H^1_{dR,c}(M, V) \to H^1_{dR,c}(\overline{P}, V)$, $\theta \mapsto \overline{q}^*\theta$ *is an isomorphism of topological vector spaces.*

Proof. As in Chapter 4 we use the notation $\Omega^k_K(\hat{M}, V) := \{\theta \in \Omega^k(\hat{M}, V) \mid \mathrm{supp}(\theta) \subseteq q^{-1}(K)\}$ for a compact subset $K \subseteq M$. Let n be the order of the covering. The first step is to define a continuous linear map $q_*\colon \Omega^k_c(\hat{M}, V) \to \Omega^k_c(M, V)$ for $k \in \mathbb{N}_0$. Without loss of generality let $k = 1$. Let $\theta \in \Omega^1_c(\hat{M}, V)$. Given $y \in M$ we find a y-neighbourhood $V_y \subseteq M$ that is evenly covered by open sets $U_{y,i} \subseteq \hat{M}$ with $i = 1, \ldots, n$. We have diffeomorphisms $q^y_i := q|^{V_y}_{U_{y,i}}$. Then

$$\tilde{\theta}^y := \frac{1}{n} \sum_{i=1}^{n} (q^y_i)_* \theta|_{U_{y,i}}$$

is a form on V_y with $(q^y_i)_*\theta|_{U_{y,i}} = \theta(T q^{y-1}_i(v))$ for $x \in V_y$ and $v \in T_x V_y$. We define $q_*\theta := \tilde{\theta} \in \Omega^1_c(\hat{M}, V)$ by $\tilde{\theta}_x := \tilde{\theta}^y_x$ for $x \in V_y$. Now we show that this is a well-defined map. Let $x \in V_y \cap V_{y'}$ for $y' \in M$ with a y'-neighbourhood $V_{y'}$ that is evenly covered by $(U_{y',i})_{i=1,..,n}$. After renumbering the sets $U_{y',i}$ we get

$$q|^{-1}_{U_{y,i}} = q|^{-1}_{U_{y',i}}$$

on $V_y \cap V_{y'}$ for $i = 1, \ldots, n$. Hence

$$\tilde{\theta}^y_x = \frac{1}{n} \sum_i ((q^y_i)_* \theta|_{U_{y,i}})_x = \frac{1}{n} \sum_i ((q^{y'}_i)_* \theta|_{U_{y',i}})_x = \tilde{\theta}^{y'}_x \text{ for } x \in V_y \cap V_{y'}.$$

We note that q is a proper map because it is a finite covering. Let $\mathrm{supp}(\theta) \subseteq q^{-1}(K)$ for a compact set $K \subseteq M$. If $y \notin K$ then $q^{-1}(\{y\}) \cap q^{-1}(K) = \emptyset$. Hence $q^{-1}(\{y\}) \cap \mathrm{supp}(\theta) = \emptyset$, from which

$$q_*\theta_y = \tilde{\theta}^y_y = \frac{1}{n} \sum_i ((q|_{U^y_i})_* \theta|_{U_{y,i}})_y = \frac{1}{n} \sum_i \theta_{q|^{-1}_{U_i}(y)} = 0$$

follows. Hence $M\backslash K \subseteq M\backslash \{x \in M : q_*\theta_x \neq 0\}$. Therefore $\{x \in M : q_*\theta_x \neq 0\} \subseteq K$ and so $\operatorname{supp}(q_*\theta) \subseteq K$. Obviously q_* is linear. Moreover, q_* is continuous because the analogous map from $\Omega^1(\hat{M}, V)$ to $\Omega^1(M, V)$ is continuous and $q_*(\Omega^1_K(\hat{M}, V)) \subseteq \Omega^1_K(M, V)$. Moreover q_* is a homomorphism of chain complexes: Given $y \in M$, $v, w \in T_yM$ we calculate

$$
\begin{aligned}
(q_*d\theta)_y(v, w) &= \frac{1}{n}\sum_i((q_i^y)_*d\theta|_{U_{y,i}})_y(v, w) = \frac{1}{n}\sum_i(d(q_i^y)_*\theta|_{U_{y,i}})_y(v, w) \\
&= (dq_*\theta)_y(v, w).
\end{aligned}
$$

Now we show

$$
q_* \circ q^* = \mathrm{id}_{\Omega^1_c(M,V)}\,. \tag{D.1}
$$

Given $\theta \in \Omega^1_c(M, V)$, $y \in M$ and $v \in T_yM$ we calculate

$$
\begin{aligned}
(q_*q^*\theta)_y(v) &= \frac{1}{n}\sum_i({q_i^y}_*q^*\theta|_{U_{y,i}})_y(v) = \frac{1}{n}\sum_i(q^*\theta|_{U_{y,i}})_{q_i^{y-1}(y)}(Tq_i^{y-1}(v)) \\
&= \frac{1}{n}\sum_i\theta_{q(q_i^{y-1}(y))}(Tq \circ q_i^{y-1}(v)) = \theta_y(v).
\end{aligned}
$$

Hence $q_* \circ q^* = \mathrm{id}_{\Omega^1_c(M,V)}$. We know that q^* factorises through a continuous linear map $q^*\colon H^1_{dR,c}(M, V) \to H^1_{dR,c}(\hat{M}, V)$ and because q_* is a homomorphism of chain complexes we get a map $q_*\colon H^1_{dR,c}(\hat{M}, V) \to H^1_{dR,c}(M, V)$. With equation (D.1) we see

$$
q_* \circ q^* = \mathrm{id}_{H^1_{dR,c}(M,V)}\,.
$$

Hence q_* is surjective. It remains to show that $q_*\colon H^1_{dR,c}(M, V) \to H^1_{dR,c}(\hat{M}, V)$ is also injective. To this end we show $q_*(B^1_c(\hat{M}, V)) = B^1_c(M, V)$. Given $f \in C^\infty_c(M, V)$ we calculate

$$
q_*(dq^*f) = q_*q^*df = df.
$$

□

The proof of Lemma D.4 is similar to the proof of [Nee04, Lemma II.10 (1)].

Lemma D.4. *Let M be a connected finite-dimensional manifold, E be a finite-dimensional vector space and $\theta \in \Omega^1(M, E)$. If*

$$
\int_{\alpha_0}\theta = \int_{\alpha_1}\theta
$$

for all closed smooth curves $\alpha_0, \alpha_1\colon [0, 1] \to M$ such that α_0 is homotopyic to α_1 relative $\{0, 1\}$, then $\theta \in Z^1_{dR}(M, E)$.

Proof. Let $q\colon \tilde{M} \to M$ be the universal smooth covering of M. First we show that $q^*\theta$ is exact. To this end we show that $q^*\theta$ is conservative. Let $\gamma\colon [0,1] \to \tilde{M}$ be a smooth closed curve in a point $p_0 \in \tilde{M}$ and $q(p_0) =: x_0 \in M$. Because $\tilde{M}$ is simply connected, we find a homotopy H from γ to c_{p_0} relative $\{0,1\}$. Hence $q \circ \gamma$ is homotopy to $c_{x_0} = q \circ c_{p_0}$ relative $\{0,1\}$. Therefore we get

$$\int_\gamma q^*\theta = \int_{[0,1]} \gamma^* q^*\theta = \int_{[0,1]} (q \circ \gamma)^*\theta \underset{(*)}{=} \int_{[0,1]} c_{x_0}^*\theta = 0.$$

Equation $(*)$ follows from the assumptions of the lemma. Because $q^*\theta$ is exact we find $f \in C^\infty(\tilde{M}, E)$ with $q^*\theta = df$. Hence we get

$$q^* d\theta = dq^*\theta = ddf = 0.$$

Therefore $d\theta = 0$ because q is a submersion. □

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List of Symbols

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